

STATISTICAL INFERENCES USING DIFFERENT LIFETIME DATA

by

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Abstract

This thesis focuses on statistical inferences for reliability models based on various types of lifetime data. We begin by introducing the progressively hybrid censoring scheme (PHC), which combines Type-I and progressively Type-II censoring schemes. The aim of this method is to save time and cost in data collection. Maximum likelihood and Bayesian estimations are employed using the Sarhan-Tadj-Hamilton (STH) distribution. Real data sets are analyzed to compare parameter estimations between complete and PHC samples. Simulation studies assess the performance of the PHC method with the STH distribution. Additionally, we investigate the expected experimentation time using progressively Type-II censoring under the STH distribution. Overall, this research explores statistical inferences, censoring methods, and estimation techniques to enhance reliability modeling using lifetime data. Next, we examine step stress partially accelerated life testing (SSPALT), a methodology that accelerates time to failure by subjecting experimental units to progressively harsher conditions. We apply maximum likelihood and Bayesian estimations for SSPALT using the Sarhan-Tadj-Hamilton (STH) distribution. Additionally, we explore the optimal change time for SSPALT under the STH distribution across various model parameter values. A real data set is analyzed using the STH distribution and compared to the Power Lindley and Weibull distributions. To assess the method's performance, we conduct simulation studies. Through these investigations, we contribute to the understanding and application of SSPALT and provide insights into the suitability of the STH distribution compared to alternative distributions. In life testing, a competing risks model (CRM) is applicable when multiple causes contribute to failure. This model allows estimation of specific cause risks among other factors. The third problem of this thesis focuses on independent (ICRM) and dependent competing risks (DCRM) within CRM. Multivariate lifetime data commonly occur in practical scenarios, necessitating consideration of appropriate distributional models. We explore the Bivariate Modified Weibull Extension (BMWE) distribution and its properties. DCRM is discussed, and maximum likelihood estimation is applied to bivariate and DCR data using the BMWE distribution. Data generation and simulations evaluate the method's performance. Two real data sets are analyzed using the BMWE distribution and compared with other approaches.

List of Abbreviations

i.i.d Identically and Independently Distributed

CDF Cumulative Distribution Function

PDF Probability Density Function

MLE Maximum Likelihood Estimation

BE Bayes Estimator

CI Confidence interval

CP Coverage Percentage

MSE Mean Squared Error

MCMC Markov Chain Monte Carlo

PHC Progressively Hybrid Censoring

EET Expected Experimentation Time

REET The Ratio of the Expected Experimentation Time

ALT Accelerated Life Testing

PALT Partially Accelerated Life Testing

SSPALT Step Stress Partially Accelerated Life testing

GAV Generalized Asymptotic Variance

CRM Competing Risks Model

DCRM Dependent Competing Risks Model

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Chapter 1

Introduction

1.1 Introduction

Product reliability, and more broadly, quality assurance, are of fundamental importance to the manufacturing sector, and one of the primary aspects of reliability analysis is the estimation of product lifetime. Balakrishnan et al. (2000) provide a good review of statistical methods for the design and analysis of experiments to assess reliability. Lifetime data often exhibit censoring, and in the industrial setting, this is most often right censoring. Censoring may be out of the experimenter's control, for example due to accidental breakage of an experimental unit, or, in the medical context, due to subject drop out or loss to followup Aggarwala (2001). In the industrial setting, experimental units may be removed from study prior to failure in order to expedite completion of an experiment and/or, to free testing units for further experimentation. In this case, the experimenter might apply a censoring scheme using preassigned removals. One such censoring scheme is progressively hybrid censoring (PHC), in which censoring times are preassigned to the experimental units. The use of PHC will typically allow for inferences on the underlying survival time distribution while reducing the overall time to complete the experiment, although one must be careful to assess the properties of the associated estimators, particularly when the proportion of censored units is high. Kundu and Joarder (2006), discuss PHC with an underlying exponential failure time distribution, and Kundu (2007) extend this to PHC with a two parameter Weibull distribution.

Sarhan et al. (2014b) proposed two new distributions, named Sarhan-Tadj-Hamilton (STH), the first a one-parameter distribution similar to the Lindley distribution Lindley (1958), and the second, a power transformation of the first, referred to as the $STH(\alpha, \gamma)$ distribution. In chapter 2 we explore the use of PHC with an underlying $STH(\alpha, \gamma)$ distribution.

In addition to PHC, a number of design methods have been proposed to reduce the time required to carry out an experiment where the outcome is time to event. Chapter 2 focused

on one such method - progressive hybrid censoring. In Chapter 3, we consider two other methods - accelerated life testing (ALT) and partially accelerated life testing (PALT) - each method designed to reduce time to failure by testing experimental units under increasingly severe conditions, for example, increased temperature, voltage, or humidity. One type of accelerated life testing is step stress accelerated life testing (SSALT), in which the stress on each experimental unit is increased at some point during an experiment. In the step stress partially accelerated design (SSPALT), all experimental items are put on test under a fixed stress, and units not failing by a pre-assigned time τ are subject to an increased stress. The goal of the SSPALT experiment is to quickly acquire reliable information on the failure time distribution Ismail and Aly (2014). An extensive coverage of accelerated testing with constant stress with a variety of parametric failure time distributions is provided by Nelson (2009). DeGroot and Goel (1979) introduced Bayesian estimation and optimal design in SSPALT with an underlying exponential distribution, and Bhattacharyya and Soejoeti (1989) described a tampered failure rate for SSPALT using Weibull distribution. Ismail and Aly (2014) proposed an optimal plan for SSPLAT under Type-II censoring. In chapter 3 we will investigate the SSPALT method with more applications.

Another statistical method which concerns the type of failure of an item is investigated in this dissertation. In life testing, when the failure time has many reasons or causes, we could assign a competing risks model CRM. Typical examples for a CRM include time to death associated with any cause or time to cause-specific death. Using this model, we can estimate a specific cause risk in the presence of other causes risk factors. Here we assume T is the time of failure and C is the specific cause of failure. A competing risks model generalizes standard survival analysis of a single endpoint as failure to investigate multiple first event types Beyersmann et al. (2011). Moreover, competing risks refer to the study of the time to an event with more than one type of failure event. Competing risks arise in many fields such as medical, criminological, financial, engineering, and many other contexts when the death or failure of an individual or unit is classified into one of a variety of types or causes Maller and Zhou (2002). Crowder (2001) stated, "For example, in Medicine, C might be the cause of death and T the age at death. In Reliability, C might identify the faulty component in a system and T the running time from start-up to breakdown. In Engineering, our subject is more likely to be referred to as non-repairable series systems' reliability. In Economics, T might be the time spent on the unemployment register and C the reason for de-registering.

In Manufacturing, T might be the usage (e.g., mileage) and C the cause of the breakdown of a machine (e.g., vehicle). Therefore, CRM can be applied to vast fields of science. Studying the risks of many outcomes could be valuable information for the following step procedure. For example, in a medical trial, not only can it inform the patients what risks they are facing, but also it helps to select the appropriate treatment for a particular patient. Larson and Dinse (1985) authored a book focused on analyzing competing risks using regression structure. Crowder (2001) wrote a book about the classical model of competing risks, their studies based on the likelihood function. Lin (1997) proposed a non-parametric inference for cumulative incidence functions in competing risk studies. Sarhan (2007) studied the censored data analysis with competing risks using the generalized exponential distribution. In this dissertation, we will address independent competing risks, and dependent competing risks; we will give more information in chapter 4.

1.2 Background

Survival Analysis is used to estimate the lifetime of a certain population under study. It is also called time to failure analysis, as the goal is to estimate the time for an individual or a group of individuals to experience a failure of interest. Time to failure experiment has gained more importance in recent years in many fields, such as medicine, economy, and engineering. Studying the lifetime of persons, individuals, units, and components significantly develops their lifetimes. For instance, in medicine, exploring the lifetime of patients with cancer under specific treatments can provide valuable insights into the effectiveness and potential complications of those treatments. By studying the outcomes of different treatment approaches in a large sample of patients, researchers can identify patterns and factors that contribute to improved patient outcomes. In the vehicle industry, analyzing the lifetime of spare parts will enhance the mean lifetime. Medical researchers were the first to use survival analysis to estimate the lifetimes of a certain population. Survival analysis has many examples when we record the birth and the death events of a person, individual, or unit. The birth event can be thought of as when a customer starts their membership with a company, and the death event can be considered the customer leaving the company. The assumption must be made that the population has the same failure time distribution for all units. Assuming n independent identical distributed units (will give more information about order statistics in a further section), inferences about the population can be made using the joint probability density

function. However, some experiments need to be ceased to save time and cost. Since time and cost are important, censoring schemes can be used to limit them. Censoring is defined when the exact failure time of units is known for some units but not all units. There is an interval, left and right censoring, but the most widely used censoring is the right censoring. In right censoring, for example, Y is a random variable for the time to failure, and C is a random variable for the time to a censoring event. What we observe is $X = \min\{Y, C\}$ and a censoring indicator $\delta = I[Y < C]$ Moore (2016). That is, δ is 0 or 1 according to whether X is a censored or observed failure time. A right-censored example is when patients enter a cancer study, and some patients were still alive at the end of the study; these patients will be right-censored observations. Through all the dissertation, we will use the right censoring. In addition, some known accelerated life testing censoring schemes can be addressed here as Type-I censoring and progressively type-II censoring. We will give some details in further sections. Combing Type-I censoring, and progressively Type-II censoring, will give us progressively hybrid censoring (PHC). Given these conditions, the challenging question is, what is the best scheme to make an inference? Sometimes the lifetime experiment takes a very long time to end. Due to the high technology and quality of products, their lifetimes get longer and longer. Therefore, another way to accelerate the life test experiment is to increase the stress level to a higher one to obtain information quickly. It might take a long time for a lifetime unit, such as several years, to fail, making it difficult or even impossible to obtain the failure information under normal use conditions for such a highly reliable unit. While running at a higher stress level shortens the products' life, the accelerated life testing (ALT) induces more failures and then derives the survival information under normal use conditions. This method accelerates the failure when the occurrence of failure time is infrequent. Many factories worldwide apply ALT to quickly obtain information on the lifetimes' distribution or product performance under normal use stress. For instance, satellite equipment that could serve for many years must be tested at higher stress quickly to spot its weaknesses. Cell phones are always tested at higher stress (voltage, temperature, or solidity) to improve performance. Step stress ALT (SSALT) is one type of ALT where we increase the stress on all lifetime units. Another type of ALT partially steps stress accelerated life testing (SSPALT), where we increase stress on the part of lifetime units. Such testing could save much time and cost. In a partially step-stress accelerated life test (SSPALT), the stress for survival units is changed to a higher stress level at a pre-determined time τ . Determination of the stress

change times is one of the most important design problems in SSPALT. The method applies normal use condition stress on each unit; if the unit didn't fail at pre-specified time τ , the method applies higher than usual stress using the acceleration factor $\beta > 1$. We will give more information about the method in chapter 3. Li (2009) discussed the optimal change time τ for SSALT when the unit has Weibull distribution because of its broad application in industry.

Another method will be addressed here, which is the CRM. When the failure time has many causes in survival analysis, we could assign a CRM. The observation here is a pair of the time and the cause of the failure. Using this model, we can estimate a specific risk in the presence of other risk factors. Competing risks arise in many fields, such as medical, engineering, and many other contexts, when the death or failure of an individual or unit is classified into one of a variety of types or causes Maller and Zhou (2002). This model will be addressed in Chapter 4.

1.3 Basic Definitions

1.3.1 Order Statistics

Consider observing n independent identical distributed as x_1, x_2, \dots, x_n . As ascending order, the observations became, $x_{1:n} \leq x_{2:n} \leq \dots \leq x_{n:n}$, where $x_{i:n}$ is the i -th order statistics. All these observations have a continuous and non negative distribution as a failure time distribution, with probability density function $f(x)$, and cumulative distribution function $F(x)$. The joint density of the order statistics of an *iid* sample of size n is:

$$f(x_1, \dots, x_n) = n! \prod_{i=1}^n f(x_i). \quad (1.1)$$

In the case of usual order statistics from continuous populations, the marginal probability density function of the i^{th} order statistic takes the simple form:

$$f_{X_{i:n}}(x_i) = \frac{n!}{(i-1)!(n-i)!} \{F(x_i)\}^{i-1} \{1-F(x_i)\}^{n-i} f(x_i).$$

The joint probability density function of the i^{th} and j^{th} order statistics takes the form:

$$f_{X_{i:n}, X_{j:n}}(x_i, x_j) = C \times F(x_i)^{i-1} [F(x_j) - F(x_i)]^{j-i-1} [1 - F(x_j)]^{n-j} f(x_i) f(x_j) \quad (1.2)$$

where,

$$C = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}, x_i < x_j.$$

Equations from 1.1 to 1.2 apply when the cumulative distribution is continuous. For more information about order statistics we refer to Balakrishnan et al. (2000), Arnold et al. (1992), and David and Nagaraja (2004).

1.3.2 The Hazard and Survival Functions

Survival analysis methods relies on studying the survival distribution by obtaining the survival and the hazard functions. The survival function defines the probability of surviving up to a point x :

$$S(x) = P(X > x), 0 < x < \infty.$$

The range of survival function will have a value of 1 at time 0 or at the beginning of the study. Then, the survival function will be decreasing over time. We can define the hazard function as:

$$h(x) = \lim_{\delta \rightarrow 0} \frac{P(x < X < x + \delta | X > x)}{\delta}.$$

This function is also known as the force of mortality. In addition to the survival and hazard functions, there are several other ways to define a survival distribution. The cumulative distribution function (*CDF*), which is commonly used outside of survival analysis, is given by:

$$F(x) = P(X \leq x), 0 < x < \infty.$$

This function is the complement of the survival function, and both functions are continuous only when the survival time is a continuous r.v. The probability density function:

$$f(x) = \frac{d}{dx}F(x) = -\frac{d}{dx}S(x).$$

The probability density functions is the rate of change of the cumulative distribution function, or minus the rate of change of the survival function. The hazard function is related to the probability density and survival functions by:

$$h(x) = \frac{f(x)}{S(x)}.$$

The hazard at time x is the probability that an event occurs in the neighborhood of time x divided by the probability that the subject is alive at time x . The cumulative hazard function is defined as the area under the hazard function up to time x :

$$H(x) = \int_0^x h(u)du.$$

The survival function could be defined in terms of the hazard and the cumulative hazard functions by:

$$S(x) = \exp\left(-\int_0^x h(u)du\right) = \exp(-H(x)).$$

1.4 Types of Censoring and Acceleration Methods

Censoring of data can arise naturally due to the nature of the sampling or experimental model; however, sometimes censoring can be an efficient method of obtaining information with regards to cost and time Volterman (2011). Several censoring methods available to experimenters will describe the most well-known two schemes, type-I and type-II censoring schemes. Next sections, we will give information on the left, right, and interval censoring schemes.

1.4.1 Interval, left, and right censoring

In life testing, we could face analyzing data that involved censored duration of time called interval censoring. The interval censoring is combining the right and the left censoring schemes. For more illustration, consider a sample where we observe failures only in a pre-determined interval (L, U) , where $L < U$. Therefore, the exact failure time will be known only if an item fails in the given interval. However, if the item fails in the interval $(-\infty, L]$ we call this as left-censored observation, and if the item fails in the interval, $[U, \infty)$ we call this as a right-censored observation. In interval-censored observation, we only know the interval that it fails in. The most used among these censoring schemes in the field is the right censoring as it is a special case of interval censoring. Besides, because the interval censoring has a more complex structure, it provides less relevant information than the right-censored data Klein et al. (2016). The interval censoring is applied when an event occurs in an interval of time, but the exact time is unknown. The left censoring is used when the censoring time occurred before the failure time.

The likelihood function for the interval censoring is:

$$\prod_{i=1}^n [S(l_i) - S(r_i)].$$

Here, we observe the interval in which the i th failure occurs, say $I_i = (l_i, r_i)$, $i = 1, 2, \dots, n$.

The likelihood function for left censoring is:

$$\prod_{i=1}^n [f(x_i)]^{\delta_i} [1 - S(x_i)]^{1-\delta_i}.$$

On the other hand, the right censoring applied when the event of interest occurred after the failure time.

The likelihood function for right censoring is:

$$\prod_{i=1}^n [f(x_i)]^{\delta_i} [S(x_i)]^{1-\delta_i}.$$

Let $\delta_i = 1$ if the failure time occurred, and $\delta_i = 0$ if the observation is right or left censored. Therefore, in left censoring $x_i = \max \{Censoring\ Time, Failure\ Time\}$, on the other hand, in right censoring $x_i = \min \{Censoring\ Time, Failure\ Time\}$. In all the following schemes, we will use the right censoring data for all chapters of this dissertation.

1.4.2 Type I Censoring

The Type-I censoring was first introduced by Epstein (1954). In Type-I, consider n independent units placed on a life-test at time 0 with identically distributed failure times X_1, \dots, X_n with cumulative distribution function $F(x)$ and probability density function $f(x)$. A time T which is independent of the failure times is pre-assigned. Beyond this time, no failures will be observed; therefore, the test terminates at time T . Thus, the number of the complete lifetimes observed is a random variable. The likelihood function for Type – I is:

$$L(\theta; x, r) = \left\{ \prod_{i=1}^J f(x_i) \right\} [S(T)]^{n-J}. \quad (1.3)$$

1.4.3 Type II Censoring

Consider n independent and identical units are placed on a life-test at time 0. An integer $m < n$ is pre-assigned as the number of the observed failures. The test will be terminated at the m th failure at which all remaining $n - m$ surviving units will be removed from the test. The likelihood function for Type – II is:

$$L(\theta; x, r) = c \left[\prod_{i=1}^m f(x_i) \right] [S(x_m)]^{n-m}, \quad (1.4)$$

where $c = n(n - 1) \cdots (n - m + 1)$.

1.4.4 Progressively Type II Censoring

In Type – II censoring, n independent units are placed on a life-test with identically distributed failure times X_1, \dots, X_n with cumulative distribution function $F(x)$ and probability density function $f(x)$. In addition, the preassigned number of failures to be observed is m , and hence, only m are completely observed until failure. In such cases, the progressively Type-II right censoring scheme with random removals (R_1, \dots, R_m) is taking place. For more illustration, this censoring scheme works as follows; at the first failure (so the total number of unit now is $n - 1$), R_1 units are withdrawn from the test. At the second failure (so the total number of unit now is $n - 2 - R_1$) R_2 units are withdrawn from the life-testing experiment or censored (i.e. no more information about it), and so on. At the time of m th failure, all the $R_m = n - m - R_1 - \dots - R_{m-1}$ surviving units are censored. Therefore, the i^{th} observed data for the Type-II Right Censoring with random removals is a pair of the failure time and the fixed removals units (X_i, R_i) . Where X_1, X_2, \dots, X_n is a random sample of size n independent and identically distributed variables from a population with cumulative distribution function $F(x)$ Balakrishnan et al. (2000), the likelihood function for the Progressively Type – II is:

$$L(\theta; x, r) = c \prod_{i=1}^m f(x_i) [S(x_i)]^{R_i}, \quad (1.5)$$

where $c = n(n - R_1 - 1) \cdots (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1)$.

Note, we can see that Type-II censoring is a special case of progressively Type-II censoring where $R_i = 0$, for $i = 1, \dots, m - 1$, and $R_m = n - m$.

Algorithm - Generating PT-II Data

Balakrishnan et al. (2000, p. 32) efficiently generate a progressively Type-II right censored sample from any continuous distribution using the following simple algorithm:

1. Generate m independent *Uniform* $(0, 1)$ observations W_1, W_2, \dots, W_m .
2. Set $V_i = W_i^{\frac{1}{i + \sum_{j=m-i+1}^m R_j}}$ for $i = 1, 2, \dots, m$.
3. Set $U_{i:m:n} = 1 - V_m V_{m-1} \cdots V_{m-i+1}$ for $i = 1, 2, \dots, m$.
4. Set $X_{i:m:n} = F^{-1}(U_{i:m:n})$ for $i = 1, 2, \dots, m$. where F^{-1} is the inverse cumulative distribution function of the lifetime distribution under consideration.

Binomial Removal Units

The number of removed items at each failure time are considered to follow a binomial distribution such that:

$$P(R, p) = \frac{(n - m)!}{(n - m - \sum_{i=1}^m r_i)! \prod_{i=1}^{m-1} r_i!} p^{\sum_{i=1}^{m-1} r_i} (1 - p)^{(m-1)(n-m) - \sum_{i=1}^{m-1} (m-i)r_i}. \quad (1.6)$$

We will generate random removals R and then apply them to the generated failure times from $STH(\alpha, \gamma)$ distribution. For more information about the removals we refer to Sarhan et al. (2008)

1.4.5 Accelerated Life Testing

In the traditional life data analysis, times-to-failure data of a product obtained under usage conditions are analyzed to identify the life characteristics of the product, and to make predictions about the product's performance. In many situations, and for many reasons, obtaining such life data may be difficult or even impossible. The longer lifetimes of today's products, the shorter period between product design and release, and the more challenges in testing products that are used continuously under usage conditions are among the difficulties. Therefore, ALTs are developed to observe product failures to analyze their failure modes and understand their life characteristics in a short time. Nelson (2009) provides accelerated life tests (ALTs) or partially accelerated life tests (PALTs) resulting in shorter lives than would be observed under normal operating conditions. In ALTs, the items are run only at accelerated conditions (stress), whereas in PALTs they are run at both use and accelerated conditions. Stress can be applied in various ways; commonly used methods are step-stress and constant-stress. Under step-stress PALTs, a test item is first run at use condition and, if it does not fail for a specified time, then it is run at accelerated condition until failure occurs. But the constant-stress PALTs run each item at either use condition or accelerated condition only; that is, each unit is run at a constant-stress level until the test is terminated. Accelerated test stresses involve higher than usual temperature, voltage, pressure, load, humidity, etc., or some combination of these. The objective of PALTs is to collect more failure data in a limited time without necessarily using high stresses to all test units. ALTs are often used for reliability analysis. According to step-stress ALTs scheme, a test unit is subjected to successively higher levels of stress. ALTs can be applied only if the relation that relates

between the life and stress is known or can be assumed; if not, ALTs cannot be applied and PALTs are a good alternative method to use in reliability analysis via a tampered random variable model proposed by DeGroot and Goel (1979). This model is described as $Y = T$, if $T \leq \tau$; and $\tau + \beta^{-1}(T - \tau)$, if $T > \tau$; where T is the lifetime of an item at use condition, Y is its total lifetime, τ is the stress change time, and β is the acceleration factor. The intent of such experiments is to collect more failure data in a limited time without necessarily using a high stress to all test units. Bhattacharyya and Soejoeti (1989) indicated, step-stress PALTs are practical for many problems of life testing where the test process requires a long time if the test is simply carried out under the normal condition.

1.5 Competing Risks Models

1.5.1 Independent Competing Risks

Two competing risks are considered independent if information about a subject's risk of experiencing one type of event provides no information about the subject's risk of experiencing the other type of event. One cannot formally test whether competing events are independent of one another. In medical/epidemiological applications, biology often suggests at least some dependence between competing risks. Hence, we might assume the dependence between causes through a bivariate distribution to study their dependency. Independent competing risks data is performed under the assumption that the causes of failure are independent. The sample n in this model consisted of a failure time x_i and an indicator δ_i for the cause of failure, where $i = 1, \dots, n$ and hence, our data will be a pair of (x_i, δ_i) . Hence, all that we considered is a time and a cause for each i observation, $i = 1, \dots, n$. The failure time x is taken to be a continuous variable, and the indicator δ_i can take one of a fixed number of values labelled $1, \dots, j$ as a discrete variable, where j is the specific cause of failure Crowder (2001). This dissertation will focus on two causes with our proposed failure time distribution.

1.5.2 Dependent Competing Risks

For independent competing risks, the failure times are often restricted to follow a univariate known distribution. However, the independence assumption among the causes of failure may be inappropriate. For example, the breakdown of one airplane engine will cause additional pressure on other remaining engines and thus increase the risk of the airplane itself. Another

example is if a patient suffering from severe eye disease goes blind in one eye, the other eye may face a higher risk of blindness. These two examples imply that dependence between risks may exist. Hence, it is more reasonable to assume the dependence among the causes of failure Shen and Xu (2018). Also, we can see from these two examples that the cause of failure could be any cause or both causes, and hence, we need to study their dependency. Bivariate lifetime distributions play an essential role in modelling dependent competing risks when there are two dependent causes of failure. The data here will be the same in the independent case, and also, we will assume a known bivariate multivariate distribution to study their effects on each other. In further sections, we will give more information about the bivariate multivariate distribution and how to apply it to the dependent competing risks model.

1.6 Useful lifetime distributions

1.6.1 STH distribution

Sarhan et al. (2014b) proposed the STH distribution. Let X be a random variable which have failure times from the $STH(2)$ with the parameters α, γ . The probability density function, and the survival function, take the following forms:

$$f(x) = \frac{\alpha\gamma x^{\alpha-1}}{1+\gamma} [\gamma + (1 + 2\gamma x^\alpha) \exp\{-\gamma x^\alpha\}] \exp\{-\gamma x^\alpha\}; x \geq 0; \alpha, \gamma > 0, \quad (1.7)$$

$$S(x) = \frac{1}{1+\gamma} [\gamma + (1 + \gamma x^\alpha) \exp\{-\gamma x^\alpha\}] \exp\{-\gamma x^\alpha\}; x \geq 0; \alpha, \gamma > 0. \quad (1.8)$$

The probability density function of the STH distribution is a mixture density of two mixture components. The first follows Weibull(α, γ) and the second is also a two independent component series system with Weibull(α) and *Power Gamma* ($\alpha, 2, \gamma$), with mixture weights:

$$a_1 = \frac{\gamma}{(1+\gamma)}, \quad a_2 = \frac{1}{(1+\gamma)}. \quad (1.9)$$

Therefore, $STH(\alpha, \gamma)$ can be expressed in term of Weibull(α) and *Power Gamma* ($\alpha, 2, \gamma$), however we could also generate by using the previous algorithm for $STH(\gamma)$ Sarhan et al. (2014b), and then, we use the transformation:

$$Y = X^{\frac{1}{\alpha}}. \quad (1.10)$$

1.6.2 Chen Distribution

Let X be a random variable, which have *Chen* (η, λ) distribution. The density function, the survival function, and the cumulative distribution function takes the following forms: (η, λ) are all shape parameters.

$$f(x) = \eta \lambda x^{\lambda-1} \exp \left\{ \eta \left(1 - e^{x^\lambda} \right) + x^\lambda \right\}; x \geq 0, \eta, \lambda > 0, \quad (1.11)$$

$$S(x) = \exp \left\{ \eta \left(1 - e^{x^\lambda} \right) \right\}; x \geq 0, \eta, \lambda > 0, \quad (1.12)$$

$$F(x) = 1 - \exp \left\{ \eta \left(1 - e^{x^\lambda} \right) \right\}; x \geq 0, \eta, \lambda > 0. \quad (1.13)$$

To generate failure times X from the cumulative distribution function $F(\cdot)$, we can generate U from Uniform(0, 1), then set $X = F^{-1}(U)$ to be a random variate from $F(\cdot)$. In Chen distribution, we set:

$$\begin{aligned} u &= 1 - \exp \left\{ \eta \left(1 - e^{x^\lambda} \right) \right\} \rightarrow \exp \left\{ \eta \left(1 - e^{x^\lambda} \right) \right\} = 1 - u, \\ \rightarrow x &= \sqrt[\lambda]{\log \left\{ 1 - \frac{\log u}{\eta} \right\}} \rightarrow x = \left(\log \left\{ 1 - \frac{\log u}{\eta} \right\} \right)^{\frac{1}{\lambda}}. \end{aligned}$$

1.6.3 Weibull Distribution – Two parameters

Let X be a random variable which have a two parameters Weibull distribution with parameters γ the shape parameter, α the scale parameter, and $\mu = 0$. The density function, the survival function, and the cumulative distribution function takes the following forms:

$$f(x) = \frac{\gamma}{\alpha} \left(\frac{x}{\alpha} \right)^{\gamma-1} \exp \left\{ - \left(\frac{x}{\alpha} \right)^\gamma \right\}; x \geq 0, \alpha, \gamma > 0, \quad (1.14)$$

$$S(x) = \exp \left\{ - \left(\frac{x}{\alpha} \right)^\gamma \right\}; x \geq 0, \gamma, \alpha > 0, \quad (1.15)$$

$$F(x) = 1 - \exp \left\{ - \left(\frac{x}{\alpha} \right)^\gamma \right\}; x \geq 0, \gamma, \alpha > 0. \quad (1.16)$$

To generate failure times X from the cumulative distribution function $F(\cdot)$, we can generate U from Uniform(0, 1), then set $X = F^{-1}(U)$ to be a random variate from $F(\cdot)$. In the two parameters Weibull distribution, we set:

$$u = 1 - \exp\left\{-\left(\frac{x}{\alpha}\right)^\gamma\right\} \rightarrow \exp\left\{-\left(\frac{x}{\alpha}\right)^\gamma\right\} = 1 - u$$

$$\rightarrow x = (-\alpha^\gamma \log u)^{\frac{1}{\gamma}}.$$

1.6.4 Gamma Distribution

Let X be a random variable which have the general formula for the three parameters gamma distribution α, μ, γ (shape, location, and scale parameters respectively). The density function takes the following form:

$$f(x) = \frac{\left(\frac{x-\mu}{\gamma}\right)^{\alpha-1} \exp\left\{-\frac{x-\mu}{\gamma}\right\}}{\gamma \Gamma(\alpha)} \quad x \geq \mu, \alpha, \gamma > 0. \quad (1.17)$$

When $\mu = 0$, and the scale parameter $\gamma = 1$, we reduced the previous density to the standard gamma distribution. The density function, the survival function, and the cumulative distribution function takes the following forms:

$$f(x) = \frac{x^{\alpha-1} \exp\{-x\}}{\Gamma(\alpha)} \quad ; \quad x \geq 0, \alpha > 0, \quad (1.18)$$

$$F(x) = \frac{\Gamma_x(\alpha)}{\Gamma(\alpha)} \quad ; \quad x \geq 0, \alpha > 0, \quad (1.19)$$

$$S(x) = 1 - \frac{\Gamma_x(\alpha)}{\Gamma(\alpha)} \quad ; \quad x \geq 0, \alpha > 0, \quad (1.20)$$

where

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} \exp\{-y\} dy, \text{ and}$$

$$\Gamma_x(\alpha) = \int_0^x y^{\alpha-1} \exp\{-y\} dy.$$

1.6.5 Modified Weibull Distribution

Let X be a random variable which have the general formula for the three parameters modified Weibull distribution α, β, γ , where α is scale parameter, β and γ are shape parameters. The density function, the survival function, and the cumulative distribution function takes the following forms:

$$f(x) = (\alpha + \beta\gamma x^{\gamma-1}) \exp\{-\alpha x - \beta x^\gamma\}; x \geq 0, \alpha, \beta, \gamma > 0, \quad (1.21)$$

$$S(x) = \exp\{-\alpha x - \beta x^\gamma\}; x \geq 0, \alpha, \beta, \gamma > 0, \quad (1.22)$$

$$F(x) = 1 - \exp\{-\alpha x - \beta x^\gamma\}; x \geq 0, \alpha, \beta, \gamma > 0, \quad (1.23)$$

where $\alpha > 0, \alpha, \beta \geq 0$ such as $\alpha + \beta > 0$.

1.6.6 Modified Weibull Extension Distribution

Let X is a random variable which have $MWE(\lambda, \alpha, \gamma)$ distribution. Here α and λ are scale parameters while γ is a shape parameter. The density function, the survival function, and the cumulative distribution function takes the following forms:

$$f(x) = \lambda\gamma \left(\frac{x}{\alpha}\right)^{\gamma-1} \exp\left\{\alpha\lambda \left(1 - \exp\left\{\left(\frac{x}{\alpha}\right)^\gamma\right\}\right) + \left(\frac{x}{\alpha}\right)^\gamma\right\} x \geq 0, \alpha, \gamma, \lambda > 0, \quad (1.24)$$

$$S(x) = \exp\left\{\lambda\alpha \left[1 - e^{\left(\frac{x}{\alpha}\right)^\gamma}\right]\right\}; x \geq 0, \alpha, \gamma, \lambda > 0, \quad (1.25)$$

$$F(x) = 1 - \exp\left\{\lambda\alpha \left[1 - e^{\left(\frac{x}{\alpha}\right)^\gamma}\right]\right\}; x \geq 0, \alpha, \gamma, \lambda > 0. \quad (1.26)$$

To generate failure times X from the cumulative distribution function $F(\cdot)$, we can generate U from Uniform(0, 1), then set $X = F^{-1}(U)$ to be a random variate from $F(\cdot)$. In MWE distribution, we set:

$$\begin{aligned} u &= 1 - \exp\left\{\lambda\alpha \left[1 - e^{\left(\frac{x}{\alpha}\right)^\gamma}\right]\right\} \rightarrow \exp\left\{\lambda\alpha \left[1 - e^{\left(\frac{x}{\alpha}\right)^\gamma}\right]\right\} = 1 - u, \\ &\rightarrow \left(\log\left\{1 - \frac{\log(u)}{\lambda\alpha}\right\}\right)^{\frac{1}{\gamma}} = \frac{x}{\alpha} \rightarrow x = \alpha \left(\log\left\{1 - \frac{\log(u)}{\lambda\alpha}\right\}\right)^{\frac{1}{\gamma}}. \end{aligned}$$

1.7 Structure of the Dissertation

This dissertation will investigate various inferential aspects for single and multiple samples through simulations under PHC, SSPALT and competing risks. Throughout most of this dissertation, it will be assumed that the underlying distribution is *STH*. Some methods for point estimation and confidence intervals are discussed in the following chapters.

Chapter 2 will give a detailed introduction to the progressively hybrid censoring PHC method. We describe how to construct a point estimate using a numerical technique for the maximum likelihood and Bayesian estimations since there is no closed form of STH distribution. Then, we have considered the problem of estimation the model parameter of two parameters STH distribution. The maximum likelihood estimation and Bayesian estimation briefly discuss the likelihood for all possible cases. Then, we analyze actual data using PHC with STH distribution and other distributions for illustrative purposes. Simulations have been done to test the method's performance with different parameter values. The expected experimentation time was discussed and performed under the STH distribution. We provide some algorithms regarding the simulation to estimate the parameters and to calculate the expected experimentation time under different censoring schemes.

Chapter 3 will describe step stress partially accelerated life testing (SSPALT) with STH distribution. A general design for SSPALT is first presented, followed by a summary of the literature related to this dissertation. Maximum likelihood and Bayesian estimations were obtained for SSPALT using STH distribution. Next, we determined the optimal change stress time τ , using different parameters' values. In addition, we have combined the SSPALT with the PHC in a simulation after determining the optimal change time τ . A further study for the optimal change time τ was conducted. For illustration, to precisely estimate the product survival at increased stress, we have to use a criterion to minimize the generalized asymptotic variance (GAV) of the maximum likelihood estimates of the model parameters and the acceleration factor. This criterion will lead to the test's optimal design, yielding the most accurate estimates of a lifetime at the increased stress. Algorithms on how to perform SSPALT simulation or SSPALT – PHC simulation were presented. A separate algorithm for the optimal change time τ is concluded. Real data is investigated using three different distributions. We estimate the parameters and the acceleration factor for each distribution.

In Chapter 4, we will address competing risks model CRM. A complete introduction to the CRM and the Modified Weibull Extension distribution is introduced. We discuss the

Bivariate Modified Weibull Extension BMWE distribution and some of its basic properties. The dependent competing risks model is presented. The maximum likelihood method using bivariate data and dependent competing risks data is discussed. Data generation and simulation results are obtained to assess the method's performance. Two real data sets are investigated using the new proposed distribution and compared with the bivariate Chen distribution BCD.

Chapter 2

Progressively Hybrid Censoring

2.1 Introduction

Product reliability, and more broadly, quality assurance, are of fundamental importance to the manufacturing sector, and one of the primary aspects of reliability analysis is the estimation of product lifetime. Balakrishnan et al. (2000) provide a good review of statistical methods for the design and analysis of experiments to assess reliability. Balakrishnan et al. (2023) gave on the topic of hybrid censoring, with a focus on recent advancements.

Lifetime data often exhibit censoring, and in the industrial setting, this is most often right censoring. Censoring may be out of the experimenter's control, for example due to accidental breakage of an experimental unit, or, in the medical context, due to subject drop out or loss to followup Aggarwala (2001). In the industrial setting, experimental units may be removed from study prior to failure in order to expedite completion of an experiment and/or, to free testing units for further experimentation. In this case, the experimenter might apply a censoring scheme using preassigned removals. One such censoring scheme is progressively hybrid censoring (PHC), in which censoring times are preassigned to the experimental units. The use of PHC will typically allow for inferences on the underlying survival time distribution while reducing the overall time to complete the experiment, although one must be careful to assess the properties of the associated estimators, particularly when proportion of censored units is high. Kundu and Joarder (2006), discuss PHC with an underlying exponential failure time distribution, and Kundu (2007) extends this to PHC with a two parameter Weibull distribution.

Statistical distributions are used to model lifetime data. Exponential and Weibull distributions are the commonly used distributions in the literature. They are flexible to model survival/reliability data. However, they have limitations since the hazard rate function (HRF) for exponential can only be constant, and the HRF for Weibull can be constant or decreasing or increasing. Therefore they can not be suitable for data sets that show non monotonic hazard shape such as unimodal or bathtub or increasing-decreasing-increasing hazard shapes.

Thus, numerous researchers developed new distributions that utilize non-monotonic hazard function. Among those distribution, for example, the exponentiated Weibull distribution Mudholkar and Srivastava (1993b), the generalized linear failure rate distribution Sarhan and Kundu (2009), the generalized quadratic hazard rate distribution Sarhan (2009b), new generalized Weibull distribution Zaindin and Sarhan (2011), the exponentiated generalized linear exponential distribution Sarhan et al. (2013b), the exponentiated modified Weibull extension distribution Sarhan and Apaloo (2013a), Power Lindley distribution Ghitany et al. (2013), Sarhan-Tadj-Hamilton distribution Sarhan et al. (2014b), and the odd generalized exponential two-parameter bathtub shaped distribution Sarhan and Mustafa (2022b).

The main aim of this chapter is to use progressively hybrid censored samples under the assumption that the lifetime of tested units follow Weibull, power Lindley (PL), and Sarhan-Tadj-Hamilton (STH). The reasons of using these distributions are, they are flexible in analyzing different types of lifetime data that display non-monotonic and monotonic hazard rate shapes and also they have some similarities in their properties. Similarly we can extend the results to any other lifetime distribution.

The rest of the chapter is organized as follows. In Section 2.2, we present the model description and assumptions. Section 2.3 discusses how to generate random samples. Estimation methods are discussed in Section 2.4. Applications on three real data sets are discussed in Section 2.5. Simulation study is given in Section 2.6. Section 2.7 discusses the expected experimentation time. The chapter is concluded in Section 2.8. Finally, some derivatives are given in the Appendix.

2.2 Assumptions

The following assumptions are adopted throughout this study. Suppose that n independent and identical items are put on the life test at time 0. The test is terminated either at the m th failure ($m \leq n$, where m is predetermined) or at a prefixed termination time $T > 0$, whichever comes first. At the 1st failure, say $X_{1:m:n}$, we randomly remove $R_1 \geq 0$ of the $n - 1$ remaining items on the test, at the 2nd failure, say $X_{2:m:n}$, we randomly remove $R_2 \geq 0$ of the $n - 2 - R_1$ remaining items on the test. We keep doing the same until the experiment is terminated. If the m^{th} failure $X_{m:m:n}$ occurs before the termination time T , the experiment terminates at $X_{m:m:n}$, and all remaining $R_m = n - R_1 - \dots - R_{m-1} - m$ units will be removed from the test. However, if the m^{th} failure does not occur before T and only at most J failures

have occurred before T , $J < m$, in this case all the remaining $R_j^* = n - (R_1 + \dots + R_J) - J$ items will be removed from the test, and the experiment is terminated at T .

This experiment produces two cases of data sets, Case-I and Case-II, respectively. Figure 2.1 displays a chart that summarizes this lifetime experiment. The available observations obtained in the two cases are summarized below.

Case-I:

$$\{(X_{1:m:n}, R_1), \dots, (X_{m:m:n}, R_m)\} \quad \text{and} \quad X_{m:m:n} < T. \quad (2.1)$$

Case II:

$$\{(X_{1:m:n}, R_1), \dots, (X_{J:m:n}, R_J)\} \quad \text{and} \quad X_{J:m:n} < T < X_{J+1:m:n}. \quad (2.2)$$

The number of removals, R_j , $j = 1, 2, \dots, m$ (Case-I) or J (Case-II), is assumed to be random from binomial distribution with number of trials $n_j = n - j - \sum_{i=1}^{j-1} R_i$ and probability of success p .

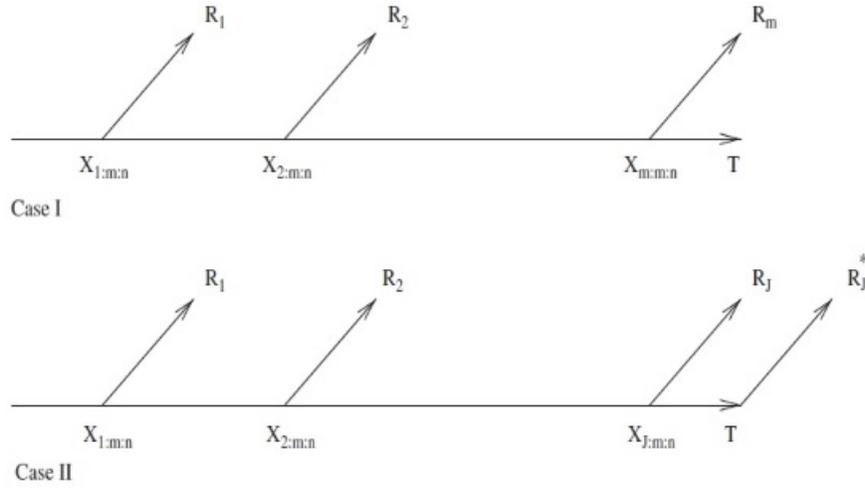


Figure 2.1: Case-I and Case-II Progressively Hybrid Censoring scheme.

Finally, let $f(x; \theta)$ and $S(x; \theta)$ be the probability density function (PDF) and the survival function (SF) of the underlying item. Table 2.1 shows the PDF and SF for the list of distributions used in this research.

Model	$S(x; \theta)$	$f(x; \theta)$
Weibull(W)	$e^{-\alpha x^\beta}$	$\alpha \beta x^{\beta-1} e^{-\alpha x^\beta}$
PL	$\frac{1 + \beta(1 + x^\alpha)}{\beta + 1} e^{-\beta x^\alpha}$	$\frac{\alpha \beta^2}{\beta + 1} (1 + x^\alpha) x^{\alpha-1} e^{-\beta x^\alpha}$
STH	$\frac{\beta + (1 + \beta x^\alpha) e^{-\beta x^\alpha}}{1 + \beta} e^{-\beta x^\alpha}$	$\frac{\alpha \beta x^{\alpha-1} [\beta + (1 + 2\beta x^\alpha) e^{-\beta x^\alpha}]}{1 + \beta} e^{-\beta x^\alpha}$

Table 2.1: List of the lifetime distributions implemented here.

The main reasons of using these distributions are: (1) each of them has two parameters; (2) their hazard rate function takes different shapes. The hazard rate function of Weibull distribution can be monotonic (increasing or decreasing) or constant. The hazard rate function of each of the PL and STH can be monotonic or bathtub shape of upside down bathtub shape and other non-monotonic shapes, which make these two distributions more flexible to fit different types of lifetime data sets.

2.3 Random Sample generation

In this section we explain how to generate progressively hybrid censored random samples. We start with an algorithm that uses a complete sample to generate a progressively Type-II censored sample with random removals. Then we present an alternative algorithm that generate a progressively Type-II censoring sample without having the original complete version. Then we discuss another algorithm to use the generated progressively Type-II censoring sample to generate a progressively hybrid censored sample.

Algorithm 1:

1. Given a complete sample Y_1, Y_2, \dots, Y_n , with size n .
2. Specify m , pre-determined number of failures, and the probability of removal of any item from the test, p .
3. Generate the random removals as follows:
 - (a) Generate R_1 from binomial distribution with number of trials equals $n - m$ and probability of success equals p . That is, $R_1 \sim \text{Binom}(n - m, p)$,
 - (b) For $i = 2, 3, \dots, m - 1$, generate $R_i \sim \text{Binom}\left(n - m - \sum_{j=1}^{i-1} R_j, p\right)$,

- (c) Set $R_m = n - m - \sum_{j=1}^{m-1} R_j$.
4. For $i = 1, 2, \dots, m$:
- (a) Set $X_i = \min\{Y\}$ and update Y by removing X_i from it.
- (b) If $R_i > 0$ do the following:
- i. Randomly generate R_i locations from 1 to the size of Y ,
 - ii. Remove those values of Y that are located in the generated locations from to get an update Y .

This algorithm will produce a progressively Type II censored sample $(X_1, R_1), (X_2, R_2), \dots, (X_m, R_m)$ of the complete sample Y_1, Y_2, \dots, Y_n .

The following Algorithm can be used to generate a progressively Type-II censoring sample $(X_1, R_1), (X_2, R_2), \dots, (X_m, R_m)$ from a specific continuous distribution, with a cdf $F(\cdot)$, directly without have a complete sample, Balakrishnan et al. (2000).

Algorithm 1.A:

1. Specify the values of n, m , and p .
2. Generate the random removals R_1, R_2, \dots, R_m as we did in Step 3 of Algorithm 1.
3. Generate the m time to failures, X_1, X_2, \dots, X_m , as follows:
 - (a) Generate m independent values W_1, W_2, \dots, W_m from Uniform(0, 1) distribution.
 - (b) Set $V_i = W_i \frac{1}{i + \sum_{j=m-i+1}^m R_j}$ for $i = 1, 2, \dots, m$.
 - (c) Set $U_{i:m:n} = 1 - V_m V_{m-1} \dots V_{m-i+1}$ for $i = 1, 2, \dots, m$.
 - (d) Set $X_{i:m:n} = F^{-1}(U_{i:m:n})$ for $i = 1, 2, \dots, m$, where F^{-1} is the inverse function of $F(\cdot)$.

Algorithm 2:

1. Given a progressively Type II censored sample: $(X_1, R_1), (X_2, R_2), \dots, (X_m, R_m)$. Set $n = m + \sum_{i=1}^m R_i$.
2. Specify the termination time T .

3. Set $D = 0$; for $i = 1, 2, \dots, m$, do

(a) If $X_i < T$,

i. set: $D = D + 1$;

ii. Keep (X_i, R_i) ,

else stop.

4. If $D = m$, then we have Case I, else set $J = D$ and $R_J^* = n - J - \sum_{j=1}^J R_j$ and we have Case II.

For simplicity, from now on, we will use either X_i or $X_{(i)}$ instead of $X_{i:m:n}$, and (x_i, R_i) as a realization of (X_i, R_i) .

2.4 Estimation Methods

In this section, we discuss how to implement the maximum likelihood and Bayesian methods to estimate the model parameters based on progressive hybrid censoring (PHC) samples.

2.4.1 Maximum Likelihood Estimation

Maximum likelihood estimation (MLE) is one of the most commonly used methods for estimating the parameters in statistical literature.

Based on the given data from progressively hybrid censoring samples of Case I, $(X_{i:m:n}, R_i) = (x_i, R_i)$, $i = 1, 2, \dots, m$, the likelihood function is:

$$L_I(\theta; \text{data}) = c \prod_{i=1}^m f(x_i; \theta) S(x_i; \theta)^{R_i}, \quad (2.3)$$

where $c = n(n - R_1 - 1) \dots (n - R_1 - R_2 - \dots - R_{m-1} - m + 1)$.

Based on the given data from progressively hybrid censoring samples of Case II, T, R_J^* , $(X_{i:m:n}, R_i) = (x_i, R_i)$, $i = 1, 2, \dots, J < m$, the likelihood function is:

$$L_{II}(\theta; \text{data}) = k \prod_{i=1}^J f(x_i; \theta) S(x_i; \theta)^{R_i} S(T; \theta)^{R_J^*}, \quad (2.4)$$

where $k = n(n - R_1 - 1) \dots (n - R_1 - R_2 - \dots - R_{J-1} - J + 1) R_J^*/(n - J)$.

In the following, we will present the log-likelihood functions when the lifetime of tested units follow STH distribution as a main model. Similarly, we can derive the log-likelihood

functions under the assumption that the lifetime of the test units follow either Weibull or PL distribution.

Substituting the pdf and sf of $\text{STH}(\alpha, \beta)$ distribution, given in Table 2.1, into equation 2.3, we get the log-likelihood function for progressively hybrid censoring data, Case I, as:

$$\begin{aligned} \mathcal{L}_I(\theta; \text{data}) \propto & m \log(\alpha) + m \log(\beta) - \left[m + \sum_{i=1}^m R_i \right] \log(1 + \beta) + (\alpha - 1) \sum_{i=1}^m \log(x_i) + \\ & -\beta \sum_{i=1}^m (1 + R_i) x_i^\alpha + \sum_{i=1}^m \log A_i(\alpha, \beta) + \sum_{i=1}^m R_i \log C_i(\alpha, \beta), \end{aligned} \quad (2.5)$$

where $A_i(\alpha, \beta) = \beta + (1 + 2\beta x_i^\alpha) e^{-\beta x_i^\alpha}$ and $C_i(\alpha, \beta) = \beta + (1 + \beta x_i^\alpha) e^{-\beta x_i^\alpha}$.

Substituting the pdf and sf of $\text{STH}(\alpha, \beta)$ distribution into equation 2.4, we get the log-likelihood function for progressively hybrid censoring data, Case II, as

$$\begin{aligned} \mathcal{L}_{II}(\theta; \text{data}) \propto & J \log(\alpha) + J \log(\beta) - \left[J + R_J^* + \sum_{i=1}^J R_i \right] \log(1 + \beta) + \\ & (\alpha - 1) \sum_{i=1}^J \log(x_i) - \beta \left[\sum_{i=1}^J (1 + R_i) x_i^\alpha + R_J^* T^\alpha \right] + \sum_{i=1}^J \log A_i(\alpha, \beta) + \\ & \sum_{i=1}^J R_i \log C_i(\alpha, \beta) + R_J^* \log D(\alpha, \beta), \end{aligned} \quad (2.6)$$

where $D(\alpha, \beta) = \beta + (1 + \beta T^\alpha) e^{-\beta T^\alpha}$.

As it is well known, to get the maximum likelihood point estimates (MLE) for the parameters α and β , we should maximize the log-likelihood function with respect to α and β . This can be done by solving the likelihood equations that can be obtained by setting the first partial derivatives of the log-likelihood function with respect to the unknown parameters equal to zero. The MLE's are the solution of the likelihood equations at which the Fisher information matrix is positive definite. The Fisher information matrix consists of the second partial derivatives of the log-likelihood function with respect α and β . This system of likelihood equations has no analytic solution, so numerical methods using R code shall be used here. We will present the first and second partial derivatives of the log-likelihood functions \mathcal{L}_I and \mathcal{L}_{II} in the Appendix.

2.4.2 Bayesian Estimation

In this section, we will use the Bayesian method to infer the unknown parameters α and β of the STH distribution under progressively hybrid censored samples. It is assumed here that α and β are independent random variables that follow gamma prior distributions with hyperparameters (a_1, a_2) and (b_1, b_2) , respectively. The joint posterior density function of $\theta = (\alpha, \beta)$, up to a normalized constant, using PHC data of case I and case II can be expressed as

$$g(\theta|\text{data}) \propto \exp \{ \mathcal{L}(\theta; \text{data}) + (a_1 - 1) \log \alpha + (b_1 - 1) \log \beta - a_2 \alpha - b_2 \beta \}, \quad (2.7)$$

where \mathcal{L} is replaced with \mathcal{L}_I and \mathcal{L}_{II} , for the PHC cases I and II, respectively.

Under the squared error loss function, the Bayes estimate of α and β are the posterior expected values, given respectively by

$$\tilde{\alpha} = \frac{\int_0^{\infty} \int_0^{\infty} \exp \{ \mathcal{L}(\theta; \text{data}) + a_1 \log \alpha + (b_1 - 1) \log \beta - a_2 \alpha - b_2 \beta \} d\alpha d\beta}{\int_0^{\infty} \int_0^{\infty} \exp \{ \mathcal{L}(\theta; \text{data}) + (a_1 - 1) \log \alpha + (b_1 - 1) \log \beta - a_2 \alpha - b_2 \beta \} d\alpha d\beta}, \quad (2.8)$$

and

$$\tilde{\beta} = \frac{\int_0^{\infty} \int_0^{\infty} \exp \{ \mathcal{L}(\theta; \text{data}) + (a_1 - 1) \log \alpha + b_1 \log \beta - a_2 \alpha - b_2 \beta \} d\alpha d\beta}{\int_0^{\infty} \int_0^{\infty} \exp \{ \mathcal{L}(\theta; \text{data}) + (a_1 - 1) \log \alpha + (b_1 - 1) \log \beta - a_2 \alpha - b_2 \beta \} d\alpha d\beta}. \quad (2.9)$$

The integrals in equations 2.8 and 2.9 do not have analytic solutions. Therefore, numerical approximation methods should be applied to obtain the Bayes point and interval estimations of the model parameters. Markov chain Monte Carlo (MCMC) is one of the most powerful techniques that can be used to approximate the joint posterior distributions without computing such integrals. The following subsection discusses the MCMC.

Markov Chain Monte Carlo

Instead of approximating the integrals stated above, we can use Markov chain Monte Carlo (MCMC) method to simulate random draws from the joint posterior distribution of $\theta = (\alpha, \beta)$, without computing the normalizing constant in equation 2.7. The main idea of MCMC is to draw random samples from a proposal distribution that mimics the posterior

distribution. The simulated draws from the proposal function can be accepted under certain acceptance criteria to be simulated draws from the posterior distribution. For more information on the MCMC, we refer to Albert (2009). There are different versions of algorithms that can be used to implement the MCMC. In this chapter, we apply Metropolis-Hastings algorithm that is summarized in the following subsection.

Metropolis- Hastings algorithm

This algorithm can be described as follows:

1. Chose a candidate function to simulate from.
2. Set an initial starting point $\theta_0 = (\alpha^{(0)}, \beta^{(0)})$ and an integer number N .
3. For $t = 1, 2, \dots, N$:
 - (a) Simulate a candidate value $\theta^* = (\alpha^*, \beta^*)$ from the proposal function $p(\theta^*|\theta_{t-1})$.
 - (b) Compute the ratio:

$$R = \frac{g(\theta^*|\text{data}) p(\theta_{t-1}|\theta^*)}{g(\theta_{t-1}|\text{data}) p(\theta^*|\theta_{t-1})}.$$
 - (c) Compute the acceptance probability $P = \min\{1, R\}$.
 - (d) Set θ_t equals θ^* with probability P , and θ_{t-1} with probability $1 - P$.

Under some regularity conditions on the proposal function $p(\theta^*|\theta_{t-1})$, this algorithm produces a sequence of non-independent random draws $\theta_1, \theta_2, \dots, \theta_N$ from the posterior distribution in equation 2.7. Ignoring the early $N - M$ burn-in elements in the sequence $\{\theta_t\}_{t=1}^N$ will produce a sequence of M draws $\{\theta_t\}_{t=N-M+1}^N$ from the posterior distribution. This sequence can be used to draw Bayesian inference of any function of the model parameters $\theta = (\alpha, \beta)$, say $v(\theta)$. Under the squared error loss function, the Bayes estimate of $v(\theta)$ is the posterior mean of $v(\theta)$, that can be approximated as

$$\tilde{v}(\theta) = E[v(\theta)|\text{data}] = \frac{\sum_{i=N-M+1}^N v(\theta_i)}{M}.$$

The two-sided $(1 - \vartheta)100\%$ probability intervals for $v(\theta)$ can be approximated as the $\frac{\vartheta}{2}$ 100th and $(1 - \frac{\vartheta}{2})$ 100th percentiles of $v(\theta_{N-M+1}), v(\theta_{N-M+2}), \dots, v(\theta_N)$. Note, to estimate the parameters α and β , we use $v(\theta) = \alpha$ and $v(\theta) = \beta$, respectively.

Choosing the proposal distribution in MCMC is one of the challenges. Such distribution should satisfy basic conditions. Among them: (1) it should mimic the posterior distribution; and (2) it is easy to simulate from. Since the posterior distribution is not of a well known form, then we can approximate condition (1) by looking for a candidate that has the same support as the model parameters. For the problem discussed in this chapter, $\alpha, \beta > 0$ can be mapped to the real line by using their log-transformation. That is, $\log \alpha$ and $\log \beta \in (-\infty, \infty)$. Therefore, we can use the bivariate normal distribution of $\theta = (\log \alpha, \log \beta)$ as the proposal distribution which satisfies the above two conditions. We will use the MLE and the corresponding Fisher information matrix of $\theta = (\log \alpha, \log \beta)$ as the mean and variance matrix of the bivariate normal proposal distribution.

2.5 Data Analysis

This section, we analyze three real-life data sets. We begin the analysis of every data set by checking the shape of its hazard rate. This can be done by using the total time on test transform plot, as a non-parametric test. In the context of a lifetime distribution characterized by a survival function denoted as $S(x) = 1 - F(x)$, the computation of the scaled TTT-transform involves the formula $\phi(u) = \frac{H^{-1}(u)}{\mu}$, where $H^{-1}(u) = \int_0^{F^{-1}(u)} S(x)dx$ is valid for $0 < u < 1$, and $\mu = H^{-1}(1)$. When considering empirical scenarios, the approximation of the scaled TTT-transform takes the form

$$\hat{\phi}_n(i/n) = \frac{H_n^{-1}(i/n)}{H_n^{-1}(1)} = \left[\left(\sum_{j=1}^i x_{j:n} \right) + (n-i)x_{i:n} \right] / \left(\sum_{j=1}^n x_{j:n} \right)$$

where i varies from 1 to n , and $x_{i:n}, i = 1, \dots, n$ represents the i -th order statistics of the sample. A seminal contribution by Aarset (1987) demonstrates that the scaled TTT-transform displays convexity (concavity) in the case of a declining (increasing) hazard rate. Moreover, for distributions with bathtub (unimodal) hazard rates, the scaled TTT-transform exhibits an initial convex (concave) pattern followed by a concave (convex) shape. For more information on the TTT-Transform plot we refer to Bergman (1977). Figure 2.2 shows the TTT-Transform plots for the three data sets.

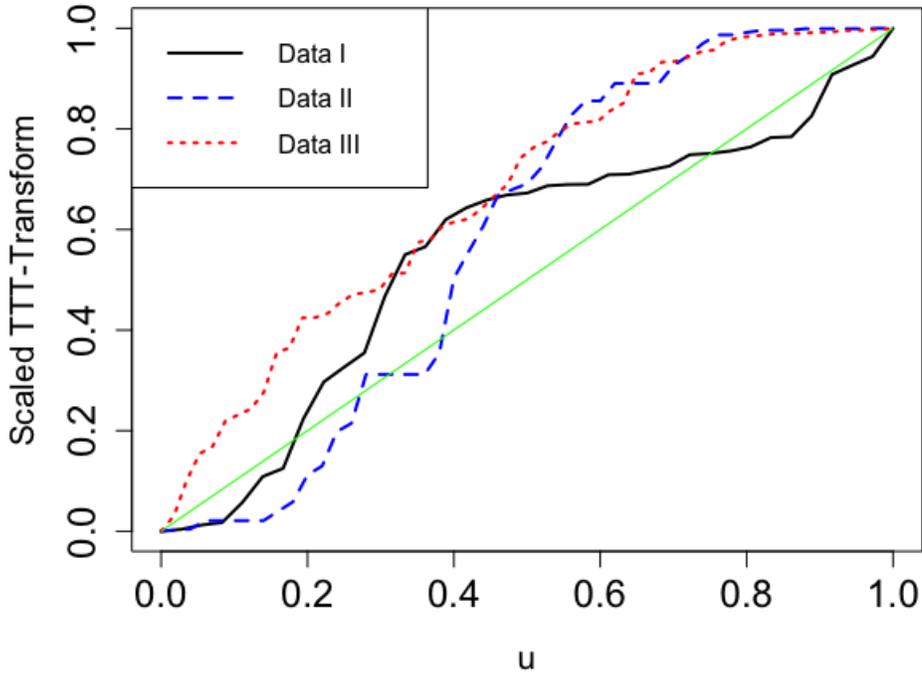


Figure 2.2: The TTT-transform plots for the three data sets I, II and III.

We use three proposed distributions Weibull(W), Power Lindley (PL) and Sarhan-Tadj-Hamilton (STH) to fit these three data sets under different scenarios (complete and progressively hybrid censored versions). The three underlying distributions do not belong to the same family of distributions, therefore to compare their performance, we can use the Akaike information criterion (AIC) Akaike (1974) and/or the Bayesian Information Criterion (BIC) Schwarz (1978) defined, respectively, as

$$AIC = -2\mathcal{L} + 2k,$$

$$BIC = -2\mathcal{L} + k \log(n),$$

where \mathcal{L} is the log likelihood function, k is the number of model parameters, and n is the sample size. The best model in the list of distributions compared is the one that minimizes the values of these criteria, in both cases. We used maximum likelihood method to compute the point estimates of each model's parameters. We will see in the results for all three data sets that STH distribution is the best fit among the three candidate distributions. Because of

this, we only explored the behaviour of the Bayes method to estimate the STH distribution parameters.

Table 2.2 displays the MLE of the three models parameters, the corresponding \mathcal{L} , AIC and BIC using the three data sets.

Model	MLE	\mathcal{L}	AIC	BIC
Data - I				
Weibull	$\hat{\alpha} = 0.02701, \hat{\beta} = 0.9492$	-241.002	486.004	489.828
PL	$\hat{\alpha} = 0.664, \hat{\beta} = 0.161$	-242.087	488.175	491.999
STH	$\hat{\alpha} = 0.892, \hat{\beta} = 0.026$	-239.838	483.676	487.500
Data - II				
Weibull	$\hat{\alpha} = 0.00028, \hat{\beta} = 1.0312$	-321.159	646.3187	649.4858
PL	$\hat{\alpha} = 0.6639, \hat{\beta} = 0.0113$	-322.508	649.0153	652.1823
STH	$\hat{\alpha} = 0.9352, \hat{\beta} = 0.00046$	-320.595	645.1907	648.3577
Data - III				
Weibull	$\hat{\alpha} = 0.01819282, \hat{\beta} = 1.793059$	-167.153	338.3054	342.3915
PL	$\hat{\alpha} = 1.2523, \hat{\beta} = 0.1256$	-168.056	340.1116	344.1977
STH	$\hat{\alpha} = 1.638032, \hat{\beta} = 0.019898$	-166.638	337.2753	341.3615

Table 2.2: Values of MLE's, \mathcal{L} , AIC and BIC for the models used based on the the complete samples.

Data set I: This data set, was originally given by Aarset (1987), contains the lifetimes (in weeks) of 50 devices, and is given below:

0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

The TTT-Transform plot for this data set is shown in Figure 2.2, which shows that the data have a bathtub hazard shape. Therefore, we expect that PL and STH distributions can fit this data set better than Weibull distribution. Then we used the three models (W, PL, and STH) to fit this complete data set and two generated samples of PHC that are given in Table 2.3. The two PHC samples of the data set are obtained from the complete sample when: (1) $m = 25, p = 0.3$ and $T = 86$; and (2) $m = 25, p = 0.3, T = 80, J = 20$ and $R_J^* = 5$. Note, sample 1 is of case I, and sample 2 is of case II of progressive hybrid censoring. The results of the maximum likelihood method for these samples (complete and progressively hybrid) are shown in Tables 2.2 and 2.4. Based on the AIC and BIC test statistic values, we can see that STH is the best fit among the above mentioned models for

all versions of this sample.

Progressive hybrid sample 1: $m = 25, T = 86$ (Case-I)								
(0.1, 10)	(0.2, 2)	(1, 3)	(1, 2)	(1, 1)	(1, 3)	(11, 0)	(12, 0)	(18, 0)
(18, 2)	(18, 0)	(18, 0)	(32, 0)	(36, 2)	(40, 0)	(47, 0)	(60, 0)	(63, 0)
(67, 0)	(67, 0)	(67, 0)	(82, 0)	(84, 0)	(84, 0)	(85, 0)		
Progressive hybrid sample 2: $m = 25, T = 80, J = 20, R_J^* = 5$ (Case-II)								
(0.1, 7)	(0.2, 5)	(1, 5)	(1, 1)	(1, 0)	(3, 4)	(6, 2)	(7, 0)	(12, 0)
(18, 0)	(18, 1)	(18, 0)	(21, 0)	(36, 0)	(40, 0)	(45, 0)	(46, 0)	(47, 0)
(67, 0)	(79, 0)							

Table 2.3: The progressively hybrid samples $(X_{i:m:n}, R_i), i = 1, 2, \dots, m(J)$, of data I.

Model	MLE	\mathcal{L}	AIC	BIC
Progressively hybrid sample 1				
Weibull	$\hat{\alpha} = 0.043, \hat{\beta} = 0.849$	-117.453	238.907	242.731
PL	$\hat{\alpha} = 0.619, \hat{\beta} = 0.199$	-117.811	239.623	243.447
STH	$\hat{\alpha} = 0.798, \hat{\beta} = 0.041$	-116.913	237.826	241.65
Progressively hybrid sample 2				
Weibull	$\hat{\alpha} = 0.084, \hat{\beta} = 0.646$	-96.213	196.422	200.244
PL	$\hat{\alpha} = 0.498, \hat{\beta} = 0.284$	-96.361	196.721	200.544
STH	$\hat{\alpha} = 0.598, \hat{\beta} = 0.081$	-96.141	196.294	200.118

Table 2.4: Values of MLE's, \mathcal{L} , AIC and BIC for the models used based on the PHC versions of data I.

We also used the PHC samples for this data set, to compute the Bayes estimates of the STH distribution parameters using the Bayes method when the hyperparameters are all equal and equal to 0.001 (that reflects non-informative prior distribution of the two parameters). We applied MCMC with $N = 10000$ and the proposal as a bivariate normal distribution of $\theta = (\theta_1, \theta_2) = (\log \alpha, \log \beta)$ with mean equals the MLE of (θ_1, θ_2) and variance equals the variance-covariance matrix of the MLE of (θ_1, θ_2) . For PHC sample 1 of data I, the proposal distribution is bivariate normal with mean $(\log(0.798), \log(0.0405)) = (-0.225, -3.204)$ and variance-covariance

$$\mathcal{F} = \begin{pmatrix} 0.02624706 & -0.08181659 \\ -0.0818165 & 0.28649884 \end{pmatrix}$$

We discarded the early 2000 draws and we used the remaining 8000 draws to compute the Bayes point estimates and the 95% Bayes probability intervals of $\alpha = e^{\theta_1}$ and $\beta = e^{\theta_2}$ as

shown in Table 2.5. We followed the same procedure of Bayes method for PHC sample 2 of data I and all PHC samples of data II and data III.

Parameters	Bayes PE	95% PI	Bayes PE	95% PI
	Sample 1		Sample 2	
α	0.7805	(0.55618, 1.06849)	0.5862	(0.40544, 0.81602)
β	0.0407	(0.01261, 0.11122)	0.0789	(0.03144, 0.16445)

Table 2.5: Bayes results for STH parameters using the two generated progressive hybrid samples of data I.

As diagnostic tests for the MCMC applied here, we provide the autocorrelation and the scatter plots for the random draws of (θ_1, θ_2) as shown in figures 2.3 and 2.4. Also in Figure 2.4, we provide the the marginal posterior density functions of the original parameters α and β .

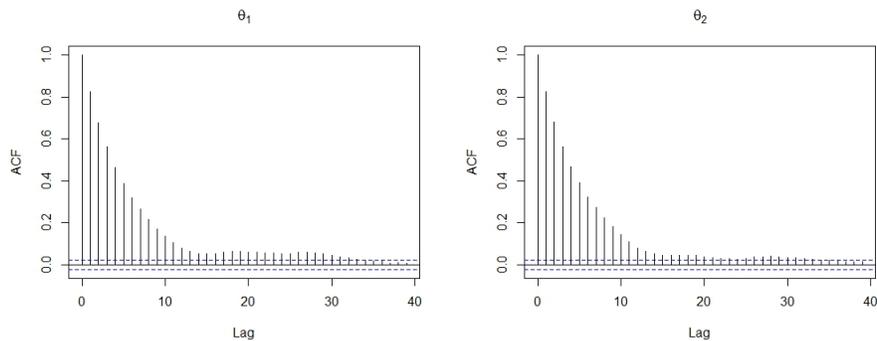


Figure 2.3: The autocorrelation plots for the random draws from the joint posterior distribution of $(\theta_1, \theta_2) = (\log \alpha, \log \beta)$ using the PHC sample 1 shown in Table 2.3.

From Figure 2.3, we can see that the autocorrelation of the drawn samples goes to zero at lag of 22, this indicates that the random draws become more independent very quickly. From trace plots in Figure 2.4, we can see that the draws do not show any pattern that indicate that the accepted draws are randomly generated from the posterior distribution. The marginal posterior density functions of α and β are also approximated in Figure 2.4, from which we can see that the marginal posterior distribution of α is approximately symmetric while that of β is positively skewed.

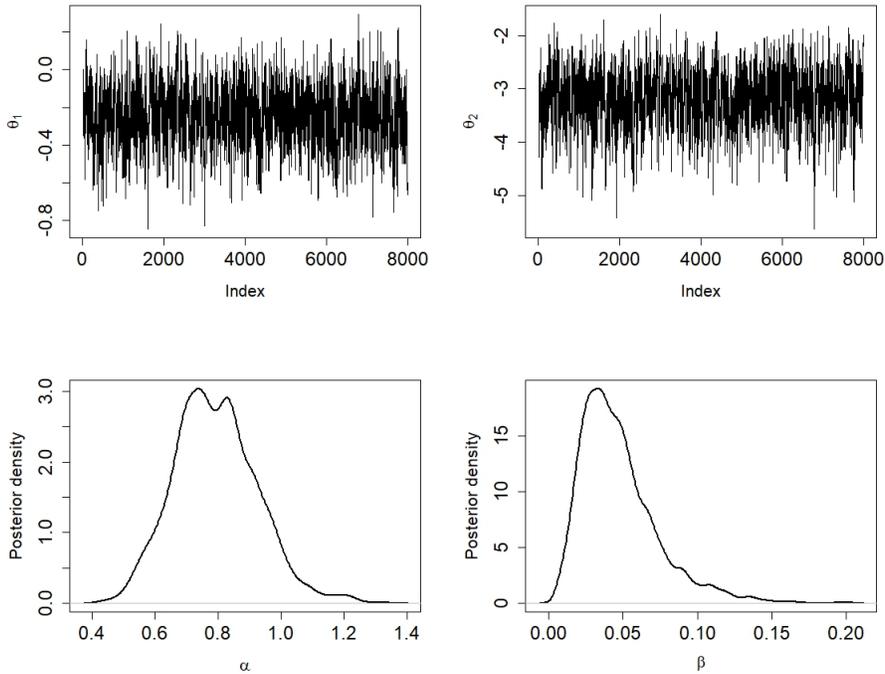


Figure 2.4: The trace plots for the random draws from the joint posterior distribution of (θ_1, θ_2) using the PHC sample 1 shown in Table 2.3, and the approximated marginal posterior density functions of α and β .

Data set II: This data set consists of times to failure (in days) for 36 appliances subjected to an automatic life test. The data are given below Lawless (2011):

11, 35, 49, 170, 329, 381, 708, 958, 1062, 1167, 1594, 1925, 1990, 2223, 2327, 2400, 2451, 2471, 2551, 2565, 2568, 2694, 2702, 2761, 2831, 3034, 3059, 3112, 3214, 3478, 3504, 4329, 6367, 6976, 7846, 13403.

From Figure 2.2, we can see that the hazard rate shape of this data set is decreasing-increasing-decreasing. Therefore, we expect that the STH distribution can be a good fit for this data set. Using this data set as is (complete sample), we applied the three underlying distributions and computed the MLE of each model parameter, the log-likelihood value, AIC and BIC as shown in Table 2.2. Based on the values of AIC and BIC scores, we can see that STH distribution fits this data set better than Weibull and PL distributions.

We generated two progressively hybrid samples by setting $m = 18$, $p = 0.3$ and (1) $T = 10000$; and (2) $T = 5000$, $J = 16$ and $R_j^* = 2$. Table 2.6 shows the two generated samples. We used these two generated samples to compute the MLE, log-likelihood function

value, AIC and BIC scores for the three proposed models as shown in Table 2.7. Based on the AIC and BIC, shown in Table 2.7, we can conclude that STH distribution fits the two generated progressively hybrid samples better than Weibull and PL distributions.

Progressively hybrid sample 1: $m = 18, T = 10000$ (Case-I)						
(11, 2)	(35, 4)	(49, 1)	(329, 5)	(381, 2)	(958, 0)	(1062, 1)
(1594, 0)	(1990, 1)	(2223, 0)	(2471, 0)	(2565, 0)	(2694, 1)	(2761, 1)
(2831, 0)	(3112, 0)	(6367, 0)	(6976, 0)			
Progressively hybrid sample 2: $m = 18, T = 5000, J = 16, R_J^* = 2$ (Case-II)						
(11, 5)	(49, 2)	(329, 3)	(381, 1)	(708, 0)	(958, 0)	(1167, 2)
(1594, 2)	(1925, 2)	(1990, 0)	(2327, 0)	(2451, 0)	(2551, 0)	(2702, 0)
(2761,1)	(3112, 0)					

Table 2.6: The hybrid progressively samples $(X_{i:m:n}, R_i), i = 1, 2, \dots, m(J)$, of data II

Model	MLE	\mathcal{L}	AIC	BIC
Progressively hybrid sample 1				
Weibull	$\hat{\alpha} = 0.00044, \hat{\beta} = 0.97567$	-160.551	325.103	328.27
PL	$\hat{\alpha} = 0.59468, \hat{\beta} = 0.01912$	-161.515	327.03	330.197
STH	$\hat{\alpha} = 0.90541, \hat{\beta} = 0.000602$	-160.111	324.222	327.389
Progressively hybrid sample 2				
Weibull	$\hat{\alpha} = 0.00088, \hat{\beta} = 0.87903$	-152.105	308.211	311.378
PL	$\hat{\alpha} = 0.535179, \hat{\beta} = 0.028607$	-152.753	309.506	312.673
STH	$\hat{\alpha} = 0.824089, \hat{\beta} = 0.001098$	-151.719	307.439	310.606

Table 2.7: Values of MLE's, \mathcal{L} , AIC and BIC for the models parameters using the two generated progressively hybrid samples of data II.

We used the progressively hybrid samples to compute Bayes point and interval estimations of the STH parameters under the assumption that the parameters are independent and follow gamma prior distributions with hyperparameters $a = b = 0.001$. The results are shown in Table 2.8.

Parameters	Bayes PE	95% PI	Bayes PE	95% PI
	Sample 1		Sample 2	
α	0.882610	(0.58010, 1.27188)	0.805884	(0.54394, 1.18467)
β	0.000604	(0.00003, 0.00711)	0.001089	(0.00006, 0.00968)

Table 2.8: Bayes results for STH parameters using the two generated progressively hybrid samples of data II.

Data III: This data set consists of 57 times (in thousands of operating hours) of unscheduled maintenance actions for the number 4 diesel engine of the U.S. Grampus, up to 16 thousand hours of operation Meeker and Escobar (1998). Below are the observations:

0.860, 1.258, 1.317, 1.442, 1.897, 2.011, 2.122, 2.439, 3.203, 3.298, 3.902, 3.910, 4.000, 4.247, 4.411, 4.456, 4.517, 4.899, 4.910, 5.676, 5.755, 6.137, 6.221, 6.311, 6.613, 6.975, 7.335, 8.158, 8.498, 8.690, 9.042, 9.330, 9.394, 9.426, 9.872, 10.191, 11.511, 11.575, 12.100, 12.126, 12.368, 12.681, 12.795, 13.399, 13.668, 13.780, 13.877, 14.007, 14.028, 14.035, 14.173, 14.173, 14.449, 14.587, 14.610, 15.070, 16.000

Based on the total time on test transform plot for this data set, shown in Figure 2.2, we can see that it has an increasing failure rate, therefore anyone of the three models (Weibull, PL, and STH) can be a good fit for this data set.

We used the three models (PL, STH, and Weibull) to fit this data set. We generated two progressively hybrid samples by setting $m = 28$, $p = 0.3$ and (1) $T = 15.5$; and (2) $T = 14$, $J = 25$ and $R_j^* = 3$. Table 2.9 shows the two generated samples. We used these two generated samples to compute the MLE, log-likelihood function value, AIC and BIC scores for the three proposed models as shown in Table 2.10. Tables 2.2 and 2.10 display the MLEs of the models parameters, the values of AIC and BIC for each model using the original complete sample and for the two of its progressively hybrid samples. Based on these values, we can see that STH distribution provides a better fit than the Weibull distribution for this data set.

Similar to what we have done for data sets I and II, we used Bayes method to estimate the parameters of STH distribution using the generated progressively hybrid samples, see Table 2.11. The Bayesian results are similar to the MLE results.

Progressively hybrid sample 1: $m = 28, T = 15.5$ (Case-I)						
(0.86, 9)	(1.258, 6)	(1.317, 1)	(1.897, 2)	(2.011, 2)	(2.122, 0)	(2.439, 3)
(3.91, 1)	(4, 2)	(4.456, 1)	(4.91, 2)	(5.755, 0)	(6.137, 0)	(6.221, 0)
(6.975, 0)	(7.335, 0)	(8.158, 0)	(8.498, 0)	(8.69, 0)	(9.33, 0)	(11.511, 0)
(12.1, 0)	(12.126, 0)	(13.399, 0)	(13.668, 0)	(13.78, 0)	(14.173, 0)	(15.07, 0)
Progressively hybrid sample 2: $m = 28, T = 14, J = 25, R_j^* = 3$ (Case-II)						
(0.86, 5)	(1.258, 5)	(1.442, 3)	(1.897, 4)	(2.011, 5)	(3.203, 2)	(3.298, 1)
(3.902, 1)	(3.91, 1)	(4.456, 0)	(5.676, 0)	(6.137, 1)	(6.975, 0)	(7.335, 0)
(8.498, 1)	(8.69, 0)	(9.042, 0)	(9.394, 0)	(9.426, 0)	(9.872, 0)	(10.191, 0)
(11.575, 0)	(12.126, 0)	(12.795, 0)	(13.877, 0)			

Table 2.9: The progressively hybrid samples $(X_{i:m:n}, R_i)$ of data III.

Model	MLE	\mathcal{L}	AIC	BIC
Progressively hybrid sample 1				
Weibull	$\hat{\alpha} = 0.019132, \hat{\beta} = 1.823568$	-82.616	169.231	173.318
PL	$\hat{\alpha} = 1.27671, \hat{\beta} = 0.12921$	-83.183	170.366	174.452
STH	$\hat{\alpha} = 1.665687, \hat{\beta} = 0.02088$	-82.548	169.097	173.183
Progressively hybrid sample 2				
Weibull	$\hat{\alpha} = 0.024172, \hat{\beta} = 1.658276$	-82.365	168.731	172.816
PL	$\hat{\alpha} = 1.172351, \hat{\beta} = 0.145955$	-82.724	169.448	173.534
STH	$\hat{\alpha} = 1.522226, \hat{\beta} = 0.025755$	-82.304	168.608	172.694

Table 2.10: Values of MLE's, \mathcal{L} , AIC and BIC for the models used using the progressively hybrid censored versions of data III.

Parameters	Bayes PE	95% PI	Bayes PE	95% PI
	Sample 1		Sample 2	
α	1.637536	(1.23407, 2.13292)	1.493579	(1.10639, 1.96969)
β	0.021073	(0.00639, 0.05768)	0.025727	(0.00874, 0.06569)

Table 2.11: Bayes results for STH parameters using the two generated progressively hybrid samples of data III.

2.6 Simulation Studies

To investigate the performance of the estimation methods applied in this chapter, we devote this section to carry out a simulation study using progressively hybrid censored samples from STH distribution. The following algorithm is used in this study:

1. Specify the values of the model parameters α and β .
2. Specify the sample information n, m, T , and p .
3. Apply Algorithm 1.A, to generate a progressively Type-II censoring sample from $\text{STH}(\alpha, \beta)$.
4. Use the sample in step 3, to compute the point estimations and a 95% confidence/probability intervals of the parameters α and β using both maximum likelihood and Bayes methods.

5. Repeat steps 3 - 4, K times. Then, for each parameter and using each method (maximum likelihood and Bayes), compute the average of the point estimates, the mean squared error, and the coverage probability.

We run the above algorithm when $p = 0.4$ and $K = 1000$, assuming different values of n , m with six sets of values of α, β and T . Tables 2.12 - 2.17 summarize the results. In these tables, we used the following abbreviations:

- APE_{MLE} , APE_{BE} to denote the average of point estimates using maximum likelihood and Bayes methods, respectively.
- MSE_{MLE} , MSE_{BE} to denote the mean squared errors associated with the point estimates using maximum likelihood and Bayes methods, respectively.
- CP_{MLE} , CP_{BE} to denote the coverage probability of the 95% CI for the parameter using maximum likelihood and Bayes methods, respectively.

Furthermore, to see the effect of the percentage of removals on the accuracy of the estimation process using maximum likelihood and Bayes method, we provide plots for the MSE against the percentage of the original sample which are observed failures as shown in figures 2.5 to 2.8.

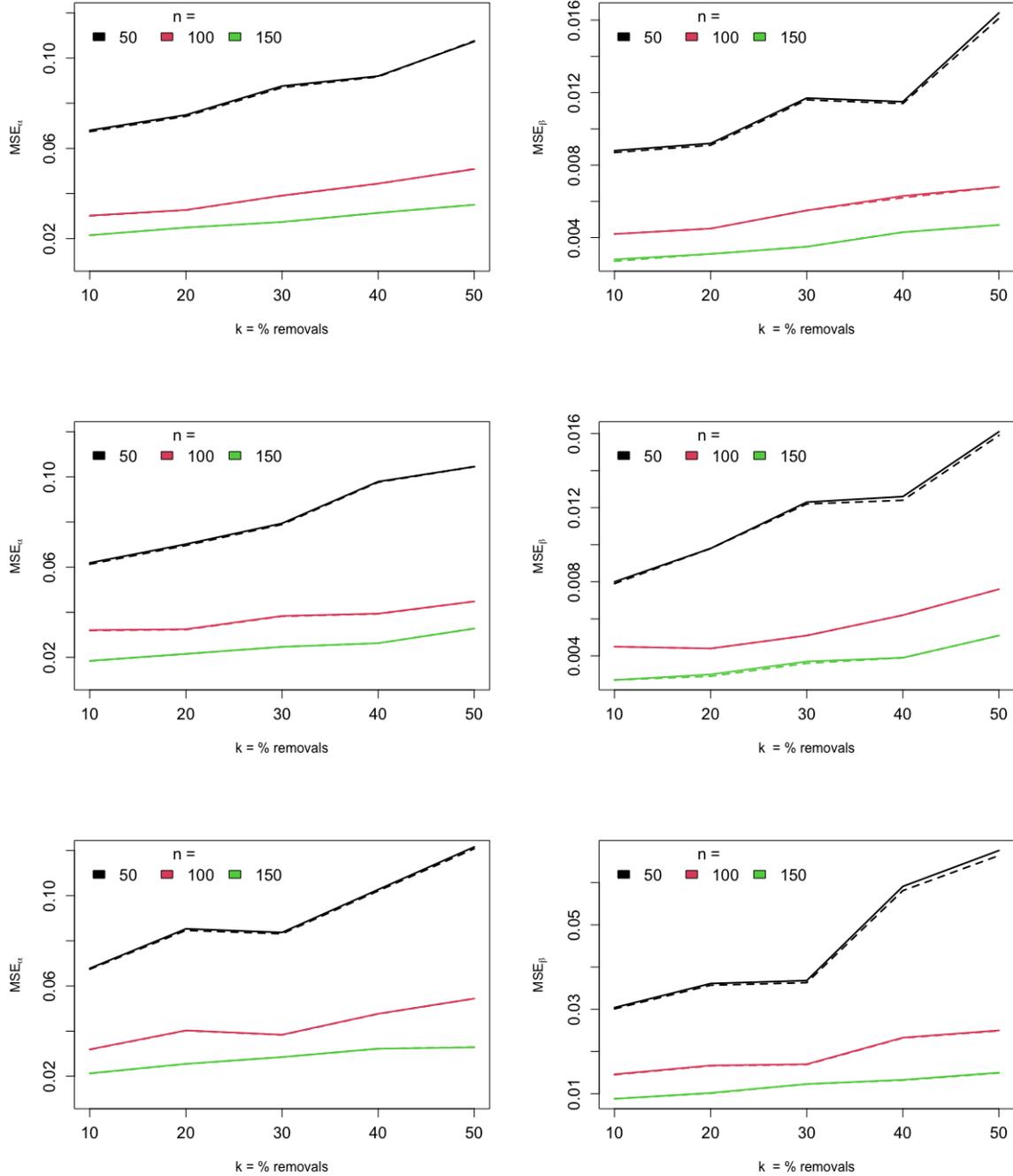


Figure 2.5: The MSE associated with MLE (solid lines) and Bayes estimate (dashed lines) versus m ($m = n - k\%n$, $k = 10, 20, \dots, 50$) when $n = 50, 100, 150$; and (1) $\alpha = 2$, $\beta = 0.5$ and $T = 2.18$ (top row); (2) $\alpha = 2$, $\beta = 0.5$ and $T = 2.72$ (2nd top row) ; (3) $\alpha = 2$, $\beta = 1$ and $T = 1.59$ (bottom row). The hyper parameters for the Bayesian Estimations are $a_1 = a_2 = b_1 = b_2 = 0.001$.

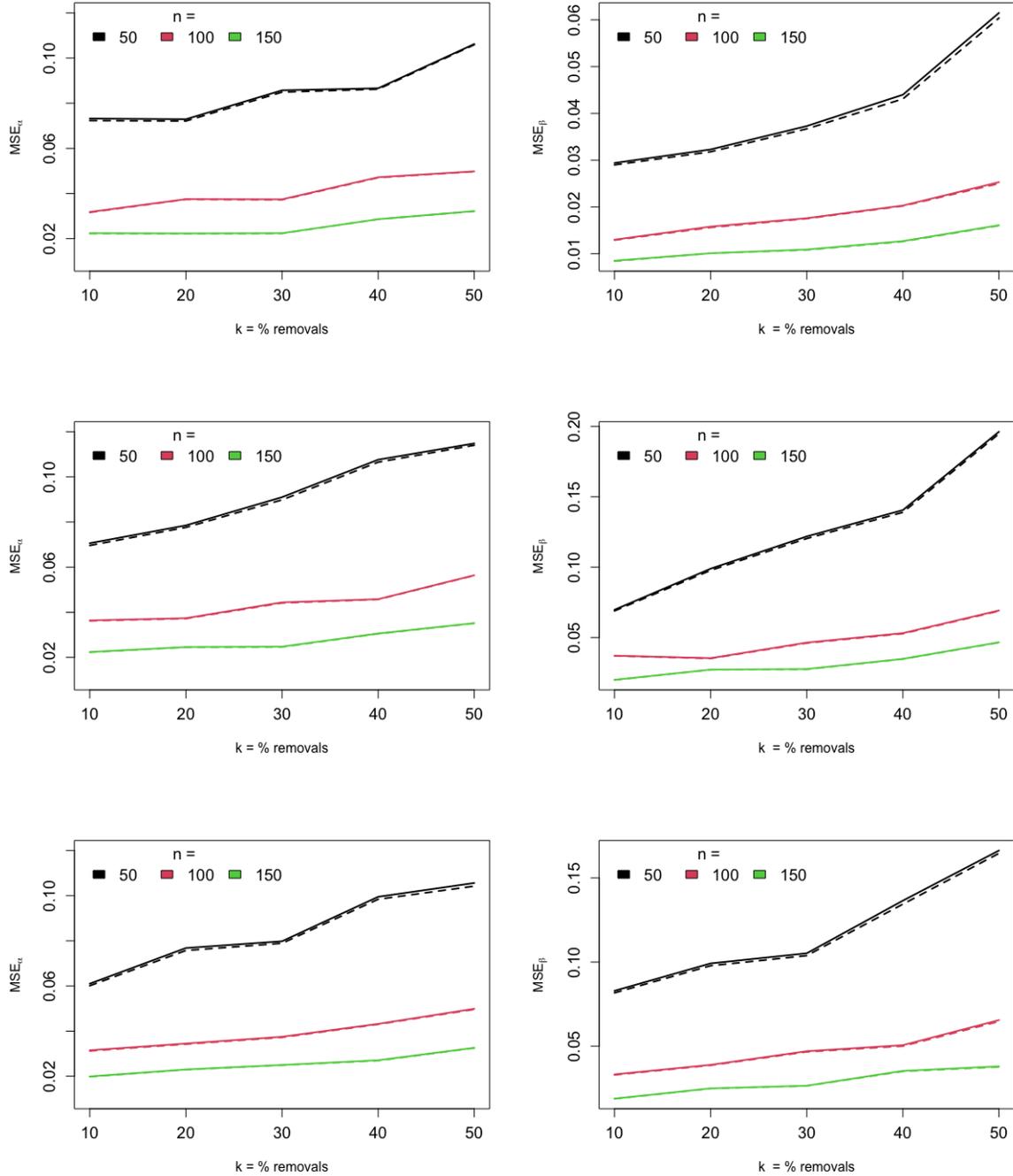


Figure 2.6: The MSE associated with MLE (solid lines) and Bayes estimate (dashed lines) versus m ($m = n - k\%n$, $k = 10, 20, \dots, 50$) when $n = 50, 100, 150$; and (1) $\alpha = 2$, $\beta = 1$ and $T = 1.99$ (top row); (2) $\alpha = 2$, $\beta = 1.5$ and $T = 1.32$ (2nd top row) ; (3) $\alpha = 2$, $\beta = 1.5$ and $T = 1.66$ (bottom row). The hyper parameters for the Bayesian Estimations are $a_1 = a_2 = b_1 = b_2 = 0.001$.

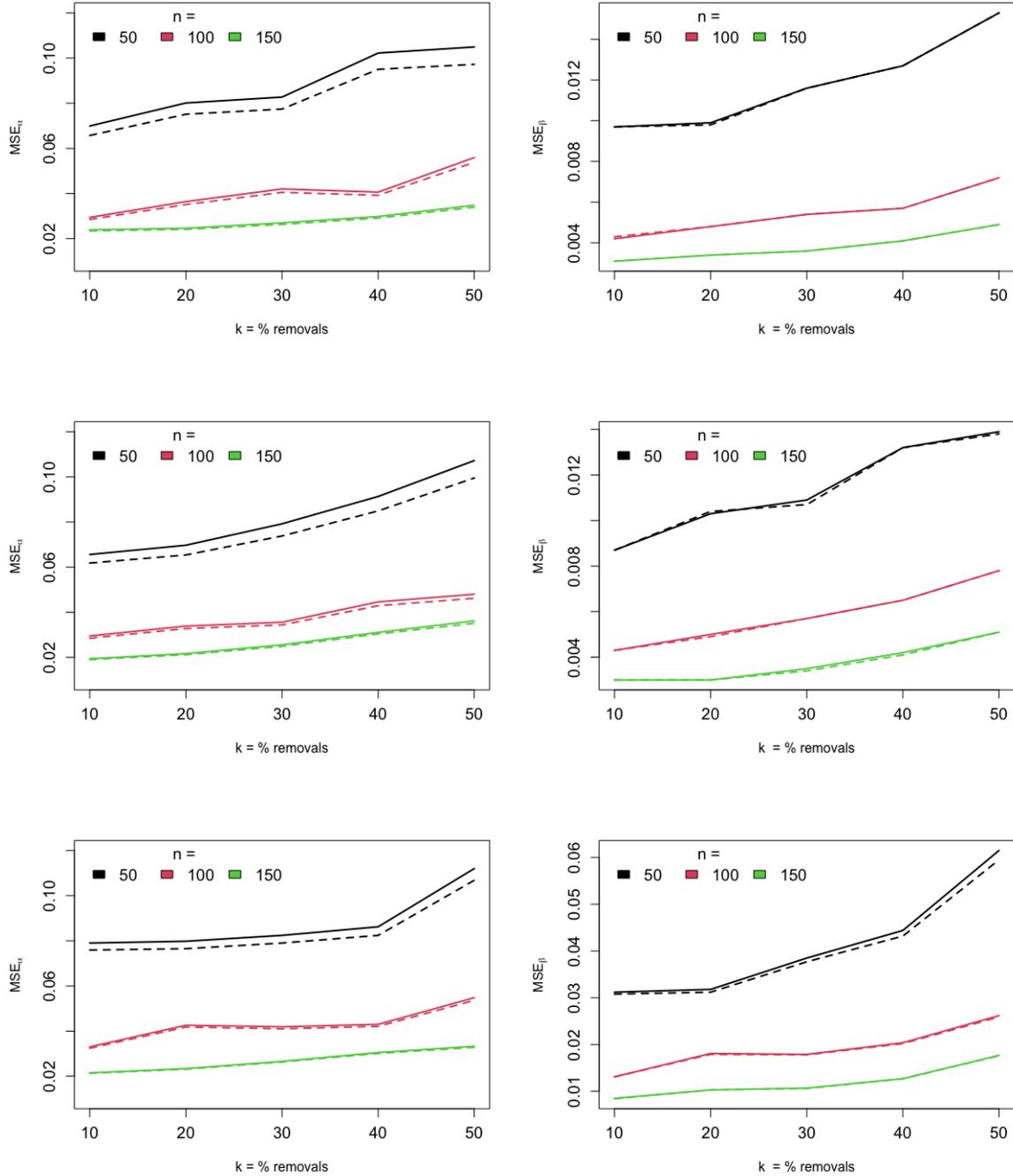


Figure 2.7: The MSE associated with MLE (solid lines) and Bayes estimate (dashed lines) versus m ($m = n - k\%n$, $k = 10, 20, \dots, 50$) when $n = 50, 100, 150$; and (1) $\alpha = 2$, $\beta = 0.5$ and $T = 2.18$ (top row); (2) $\alpha = 2$, $\beta = 0.5$ and $T = 2.72$ (2nd top row) ; (3) $\alpha = 2$, $\beta = 1$ and $T = 1.59$ (bottom row). The hyper parameters for the Bayesian Estimations are $a_1 = 0.8, a_2 = 0.4, b_1 = 0.45, b_2 = 0.3$.

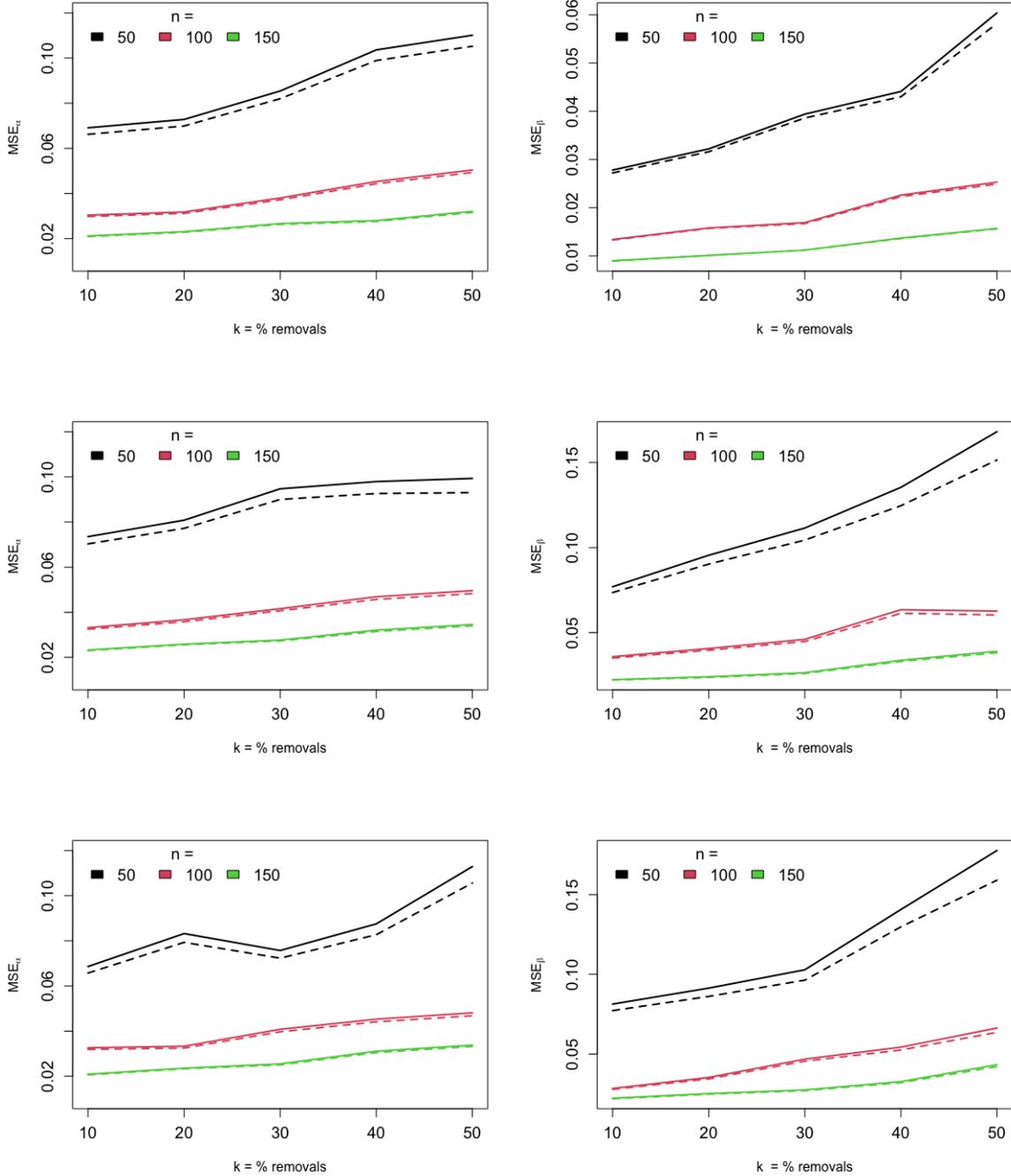


Figure 2.8: The MSE associated with MLE (solid lines) and Bayes estimate (dashed lines) versus m ($m = n - k\%n, k = 10, 20, \dots, 50$) when $n = 50, 100, 150$; and (1) $\alpha = 2, \beta = 1$ and $T = 1.99$ (top row); (2) $\alpha = 2, \beta = 1.5$ and $T = 1.32$ (2nd top row) ; (3) $\alpha = 2, \beta = 1.5$ and $T = 1.66$ (bottom row). The hyper parameters for the Bayesian Estimations are $a_1 = 0.8, a_2 = 0.4, b_1 = 0.45, b_2 = 0.3$.

Parameter	n	m	APE_{MLE}	APE_{BE}	MSE_{MLE}	MSE_{BE}	CP_{MLE}	CP_{BE}	
α	50	45	2.0543	2.0512	0.0680	0.0674	0.951	0.947	
β			0.5048	0.5083	0.0088	0.0087	0.947	0.953	
α		40	2.0403	2.0375	0.0748	0.0742	0.942	0.937	
β			0.5034	0.5066	0.0092	0.0091	0.941	0.945	
α		35	2.0605	2.0585	0.0876	0.0869	0.944	0.941	
β			0.5068	0.5099	0.0117	0.0116	0.931	0.938	
α		30	2.0741	2.0734	0.0920	0.0917	0.957	0.949	
β			0.5066	0.5094	0.0115	0.0114	0.952	0.948	
α		25	2.0789	2.0799	0.1073	0.1076	0.948	0.943	
β			0.5111	0.5127	0.0164	0.0161	0.931	0.932	
α		100	90	2.0236	2.0220	0.0302	0.0301	0.953	0.954
β				0.5049	0.5067	0.0042	0.0042	0.959	0.954
α			80	2.0215	2.0204	0.0327	0.0326	0.958	0.955
β				0.4997	0.5016	0.0045	0.0045	0.940	0.953
α			70	2.0265	2.0258	0.0391	0.0390	0.947	0.946
β				0.5062	0.5080	0.0055	0.0055	0.940	0.941
α			60	2.0307	2.0305	0.0444	0.0443	0.940	0.943
β				0.5041	0.5057	0.0063	0.0062	0.944	0.943
α	50		2.0311	2.0324	0.0508	0.0509	0.957	0.951	
β			0.5056	0.5069	0.0068	0.0068	0.955	0.955	
α	150		135	2.0188	2.0178	0.0215	0.0215	0.948	0.945
β				0.5007	0.5020	0.0028	0.0027	0.951	0.954
α			120	2.0205	2.0196	0.0249	0.0249	0.944	0.943
β				0.5031	0.5044	0.0031	0.0031	0.962	0.958
α			105	2.0221	2.0216	0.0274	0.0273	0.947	0.946
β				0.5006	0.5018	0.0035	0.0035	0.945	0.952
α			90	2.0278	2.0278	0.0314	0.0314	0.945	0.944
β				0.5005	0.5017	0.0043	0.0043	0.937	0.944
α		75	2.0323	2.0330	0.0350	0.0350	0.953	0.949	
β			0.5033	0.5044	0.0047	0.0047	0.952	0.953	

Table 2.12: Simulation results when $\alpha = 2$, $\beta = 0.5$, and $T = 2.18$. The hyper parameters for the Bayesian Estimations are $a_1 = a_2 = b_1 = b_2 = 0.001$.

Parameter	n	m	APE_{MLE}	APE_{BE}	MSE_{MLE}	MSE_{BE}	CP_{MLE}	CP_{BE}	
α	50	45	2.0472	2.0439	0.0619	0.0613	0.947	0.947	
β			0.5000	0.5038	0.0080	0.0079	0.949	0.957	
α		40	2.0565	2.0537	0.0702	0.0696	0.959	0.956	
β			0.5022	0.5060	0.0098	0.0098	0.940	0.943	
α		35	2.0708	2.0686	0.0794	0.0789	0.953	0.952	
β			0.5084	0.5119	0.0123	0.0122	0.929	0.940	
α		30	2.0684	2.0675	0.0979	0.0976	0.947	0.950	
β			0.5137	0.5170	0.0126	0.0124	0.956	0.953	
α		25	2.0894	2.0898	0.1045	0.1045	0.950	0.947	
β			0.5162	0.5186	0.0161	0.0159	0.945	0.941	
α		100	90	2.0345	2.0328	0.0321	0.0319	0.937	0.942
β				0.4987	0.5006	0.0045	0.0045	0.937	0.941
α			80	2.0264	2.0249	0.0325	0.0323	0.957	0.951
β				0.5018	0.5039	0.0044	0.0044	0.958	0.962
α			70	2.0331	2.0322	0.0384	0.0382	0.950	0.950
β				0.5011	0.5032	0.0051	0.0051	0.952	0.957
α			60	2.0377	2.0373	0.0394	0.0393	0.952	0.950
β				0.5040	0.5059	0.0062	0.0062	0.943	0.947
α	50		2.0333	2.0339	0.0448	0.0448	0.951	0.943	
β			0.5083	0.5099	0.0076	0.0076	0.952	0.950	
α	150		135	2.0161	2.0150	0.0185	0.0184	0.958	0.957
β				0.5013	0.5026	0.0027	0.0027	0.951	0.953
α			120	2.0186	2.0176	0.0216	0.0215	0.951	0.948
β				0.5007	0.5021	0.0030	0.0029	0.954	0.960
α			105	2.0178	2.0171	0.0247	0.0247	0.944	0.939
β				0.5006	0.5019	0.0037	0.0036	0.941	0.947
α			90	2.0269	2.0267	0.0263	0.0263	0.961	0.960
β				0.5043	0.5057	0.0039	0.0039	0.950	0.958
α		75	2.0304	2.0307	0.0328	0.0328	0.949	0.947	
β			0.5034	0.5047	0.0051	0.0051	0.946	0.951	

Table 2.13: Simulation results when $\alpha = 2$, $\beta = 0.5$, and $T = 2.72$. The hyper parameters for the Bayesian Estimations are $a_1 = a_2 = b_1 = b_2 = 0.001$.

Parameter	n	m	APE_{MLE}	APE_{BE}	MSE_{MLE}	MSE_{BE}	CP_{MLE}	CP_{BE}	
α	50	45	2.0353	2.0326	0.0678	0.0674	0.952	0.954	
β			1.0277	1.0282	0.0304	0.0301	0.937	0.941	
α		40	2.0644	2.0620	0.0853	0.0846	0.944	0.942	
β			1.0275	1.0274	0.0361	0.0357	0.930	0.931	
α		35	2.0595	2.0578	0.0837	0.0831	0.948	0.947	
β			1.0253	1.0248	0.0368	0.0363	0.947	0.950	
α		30	2.0863	2.0850	0.1027	0.1021	0.945	0.942	
β			1.0506	1.0488	0.0591	0.0581	0.937	0.926	
α		25	2.0786	2.0785	0.1216	0.1208	0.944	0.937	
β			1.0490	1.0460	0.0676	0.0664	0.939	0.931	
α		100	90	2.0207	2.0193	0.0319	0.0318	0.955	0.952
β				1.0107	1.0110	0.0146	0.0145	0.945	0.941
α			80	2.0248	2.0239	0.0403	0.0402	0.933	0.933
β				1.0200	1.0203	0.0167	0.0166	0.945	0.941
α			70	2.0177	2.0174	0.0384	0.0383	0.948	0.946
β				1.0165	1.0164	0.0170	0.0169	0.963	0.960
α			60	2.0351	2.0355	0.0477	0.0476	0.941	0.935
β				1.0231	1.0228	0.0233	0.0232	0.949	0.939
α	50		2.0403	2.0417	0.0544	0.0544	0.948	0.944	
β			1.0175	1.0170	0.0250	0.0249	0.953	0.953	
α	150		135	2.0114	2.0105	0.0213	0.0212	0.948	0.948
β				1.0078	1.0081	0.0088	0.0088	0.945	0.951
α			120	2.0188	2.0182	0.0255	0.0254	0.943	0.936
β				1.0151	1.0153	0.0102	0.0101	0.952	0.949
α			105	2.0249	2.0247	0.0285	0.0285	0.930	0.932
β				1.0083	1.0084	0.0123	0.0123	0.941	0.940
α			90	2.0209	2.0211	0.0322	0.0322	0.942	0.940
β				1.0104	1.0104	0.0133	0.0132	0.955	0.956
α		75	2.0245	2.0255	0.0328	0.0329	0.961	0.959	
β			1.0083	1.0080	0.0150	0.0149	0.956	0.956	

Table 2.14: Simulation results when $\alpha = 2$, $\beta = 1$, and $T = 1.59$. The hyper parameters for the Bayesian Estimations are $a_1 = a_2 = b_1 = b_2 = 0.001$.

Parameter	n	m	APE_{MLE}	APE_{BE}	MSE_{MLE}	MSE_{BE}	CP_{MLE}	CP_{BE}	
α	50	45	2.0616	2.0576	0.0732	0.0723	0.945	0.947	
β			1.0261	1.0264	0.0294	0.0290	0.951	0.952	
α		40	2.0521	2.0485	0.0729	0.0721	0.948	0.941	
β			1.0230	1.0228	0.0323	0.0318	0.949	0.944	
α		35	2.0700	2.0672	0.0857	0.0849	0.955	0.944	
β			1.0336	1.0329	0.0373	0.0367	0.955	0.953	
α		30	2.0568	2.0550	0.0866	0.0862	0.946	0.947	
β			1.0354	1.0337	0.0440	0.0431	0.955	0.951	
α		25	2.0839	2.0837	0.1062	0.1059	0.946	0.941	
β			1.0361	1.0332	0.0615	0.0604	0.939	0.934	
α		100	90	2.0278	2.0259	0.0318	0.0316	0.949	0.944
β				1.0074	1.0077	0.0130	0.0129	0.949	0.949
α			80	2.0201	2.0183	0.0375	0.0374	0.928	0.931
β				1.0152	1.0152	0.0158	0.0156	0.951	0.945
α			70	2.0315	2.0302	0.0374	0.0372	0.956	0.954
β				1.0123	1.0121	0.0176	0.0175	0.942	0.944
α			60	2.0457	2.0450	0.0472	0.0470	0.943	0.942
β				1.0079	1.0075	0.0203	0.0202	0.938	0.944
α	50		2.0359	2.0361	0.0498	0.0497	0.954	0.953	
β			1.0242	1.0232	0.0253	0.0250	0.949	0.954	
α	150		135	2.0214	2.0201	0.0224	0.0223	0.941	0.944
β				1.0036	1.0038	0.0085	0.0084	0.946	0.943
α			120	2.0157	2.0145	0.0223	0.0222	0.950	0.948
β				1.0094	1.0095	0.0101	0.0101	0.942	0.944
α			105	2.0193	2.0183	0.0224	0.0223	0.966	0.965
β				1.0046	1.0047	0.0109	0.0108	0.944	0.944
α			90	2.0229	2.0224	0.0286	0.0286	0.948	0.942
β				1.0148	1.0147	0.0127	0.0126	0.954	0.949
α		75	2.0291	2.0291	0.0322	0.0321	0.959	0.957	
β			1.0140	1.0135	0.0161	0.0160	0.948	0.945	

Table 2.15: Simulation results when $\alpha = 2$, $\beta = 1$, and $T = 1.99$. The hyper parameters for the Bayesian Estimations are $a_1 = a_2 = b_1 = b_2 = 0.001$.

Parameter	n	m	APE_{MLE}	APE_{BE}	MSE_{MLE}	MSE_{BE}	CP_{MLE}	CP_{BE}	
α	50	45	2.0592	2.0555	0.0706	0.0696	0.948	0.951	
β			1.5389	1.5374	0.0697	0.0689	0.951	0.945	
α		40	2.0521	2.0492	0.0785	0.0776	0.951	0.949	
β			1.5639	1.5620	0.0989	0.0977	0.943	0.941	
α		35	2.0714	2.0688	0.0910	0.0898	0.944	0.939	
β			1.5678	1.5651	0.1218	0.1203	0.937	0.939	
α		30	2.0800	2.0782	0.1076	0.1065	0.946	0.942	
β			1.5919	1.5886	0.1405	0.1389	0.954	0.939	
α		25	2.0857	2.0851	0.1148	0.1140	0.957	0.953	
β			1.6088	1.6047	0.1962	0.1947	0.959	0.951	
α		100	90	2.0274	2.0258	0.0364	0.0362	0.946	0.944
β				1.5286	1.5282	0.0371	0.0369	0.947	0.944
α			80	2.0217	2.0203	0.0374	0.0372	0.951	0.947
β				1.5232	1.5226	0.0353	0.0351	0.950	0.954
α			70	2.0352	2.0340	0.0444	0.0441	0.951	0.950
β				1.5248	1.5240	0.0464	0.0460	0.941	0.941
α			60	2.0325	2.0323	0.0458	0.0457	0.948	0.946
β				1.5351	1.5340	0.0531	0.0527	0.958	0.953
α	50		2.0325	2.0331	0.0564	0.0563	0.945	0.944	
β			1.5450	1.5435	0.0692	0.0688	0.951	0.952	
α	150		135	2.0181	2.0170	0.0224	0.0223	0.951	0.948
β				1.5176	1.5173	0.0199	0.0198	0.966	0.963
α			120	2.0213	2.0203	0.0246	0.0245	0.952	0.949
β				1.5223	1.5220	0.0272	0.0271	0.937	0.937
α			105	2.0239	2.0233	0.0248	0.0246	0.960	0.959
β				1.5220	1.5214	0.0276	0.0274	0.955	0.959
α			90	2.0208	2.0207	0.0306	0.0305	0.944	0.939
β				1.5362	1.5356	0.0348	0.0347	0.956	0.950
α		75	2.0320	2.0325	0.0352	0.0351	0.951	0.947	
β			1.5291	1.5282	0.0466	0.0464	0.938	0.938	

Table 2.16: Simulation results when $\alpha = 2$, $\beta = 1.5$, and $T = 1.32$. The hyper parameters for the Bayesian Estimations are $a_1 = a_2 = b_1 = b_2 = 0.001$.

Parameter	n	m	APE_{MLE}	APE_{BE}	MSE_{MLE}	MSE_{BE}	CP_{MLE}	CP_{BE}	
α	50	45	2.0502	2.0451	0.0610	0.0601	0.955	0.955	
β			1.5500	1.5472	0.0829	0.0817	0.942	0.933	
α		40	2.0723	2.0675	0.0768	0.0757	0.950	0.945	
β			1.5558	1.5523	0.0992	0.0978	0.938	0.932	
α		35	2.0557	2.0519	0.0798	0.0789	0.954	0.950	
β			1.5632	1.5594	0.1052	0.1038	0.948	0.943	
α		30	2.0850	2.0818	0.0995	0.0984	0.949	0.944	
β			1.5836	1.5786	0.1365	0.1344	0.945	0.937	
α		25	2.0904	2.0876	0.1056	0.1042	0.950	0.948	
β			1.6001	1.5936	0.1663	0.1646	0.964	0.953	
α		100	90	2.0361	2.0335	0.0315	0.0312	0.961	0.962
β				1.5251	1.5236	0.0332	0.0329	0.955	0.954
α			80	2.0266	2.0243	0.0345	0.0342	0.951	0.951
β				1.5270	1.5254	0.0389	0.0386	0.939	0.933
α			70	2.0304	2.0283	0.0375	0.0372	0.954	0.953
β				1.5430	1.5410	0.0470	0.0466	0.953	0.945
α			60	2.0347	2.0332	0.0432	0.0430	0.956	0.958
β				1.5476	1.5451	0.0506	0.0500	0.954	0.951
α	50		2.0481	2.0476	0.0499	0.0496	0.952	0.952	
β			1.5546	1.5517	0.0655	0.0647	0.952	0.951	
α	150		135	2.0155	2.0138	0.0199	0.0198	0.950	0.951
β				1.5126	1.5117	0.0188	0.0187	0.957	0.955
α			120	2.0170	2.0155	0.0230	0.0229	0.953	0.949
β				1.5132	1.5123	0.0249	0.0248	0.942	0.940
α			105	2.0183	2.0167	0.0250	0.0249	0.942	0.940
β				1.5159	1.5148	0.0265	0.0264	0.946	0.945
α			90	2.0200	2.0190	0.0271	0.0269	0.954	0.951
β				1.5337	1.5322	0.0353	0.0350	0.941	0.937
α		75	2.0245	2.0240	0.0326	0.0325	0.958	0.960	
β			1.5214	1.5196	0.0380	0.0376	0.961	0.956	

Table 2.17: Simulation results when $\alpha = 2$, $\beta = 1.5$, and $T = 1.66$. The hyper parameters for the Bayesian Estimations are $a_1 = a_2 = b_1 = b_2 = 0.001$.

Parameter	n	m	APE_{MLE}	APE_{BE}	MSE_{MLE}	MSE_{BE}	CP_{MLE}	CP_{BE}	
α	50	45	2.0434	2.0321	0.0699	0.0657	0.943	0.945	
β			0.5041	0.5126	0.0097	0.0097	0.933	0.938	
α		40	2.0514	2.0396	0.0801	0.0751	0.946	0.945	
β			0.5024	0.5114	0.0099	0.0098	0.936	0.945	
α		35	2.0568	2.0456	0.0827	0.0774	0.949	0.952	
β			0.5079	0.5169	0.0116	0.0116	0.942	0.950	
α		30	2.0743	2.0626	0.1022	0.0950	0.943	0.944	
β			0.5093	0.5191	0.0127	0.0127	0.948	0.949	
α		25	2.0779	2.0664	0.1049	0.0972	0.951	0.956	
β			0.5126	0.5223	0.0153	0.0153	0.938	0.947	
α		100	90	2.0204	2.0150	0.0294	0.0285	0.958	0.961
β				0.5044	0.5087	0.0042	0.0043	0.951	0.949
α			80	2.0368	2.0311	0.0364	0.0351	0.953	0.950
β				0.4977	0.5024	0.0048	0.0048	0.942	0.949
α			70	2.0386	2.0328	0.0420	0.0405	0.941	0.946
β				0.5019	0.5069	0.0054	0.0054	0.945	0.949
α			60	2.0246	2.0196	0.0406	0.0392	0.960	0.961
β				0.5030	0.5081	0.0057	0.0057	0.962	0.962
α	50		2.0455	2.0406	0.0559	0.0538	0.955	0.951	
β			0.5046	0.5101	0.0072	0.0072	0.946	0.951	
α	150		135	2.0179	2.0143	0.0239	0.0234	0.938	0.938
β				0.5029	0.5059	0.0031	0.0031	0.941	0.940
α			120	2.0175	2.0139	0.0246	0.0241	0.945	0.942
β				0.4996	0.5027	0.0034	0.0034	0.940	0.943
α			105	2.0279	2.0242	0.0270	0.0263	0.950	0.947
β				0.4999	0.5033	0.0036	0.0036	0.950	0.953
α			90	2.0207	2.0169	0.0298	0.0291	0.945	0.947
β				0.5019	0.5056	0.0041	0.0041	0.945	0.948
α		75	2.0200	2.0169	0.0348	0.0339	0.949	0.947	
β			0.5034	0.5072	0.0049	0.0049	0.949	0.949	

Table 2.18: Simulation results when $\alpha = 2$, $\beta = 0.5$, and $T = 2.18$. The hyper parameters for the Bayesian Estimations are $a_1 = 0.8$, $a_2 = 0.4$, $b_1 = 0.45$, $b_2 = 0.3$.

Parameter	n	m	APE_{MLE}	APE_{BE}	MSE_{MLE}	MSE_{BE}	CP_{MLE}	CP_{BE}
α	50	45	2.0562	2.0445	0.0656	0.0618	0.952	0.950
β			0.5051	0.5139	0.0087	0.0087	0.941	0.951
α		40	2.0534	2.0419	0.0697	0.0654	0.949	0.953
β			0.5128	0.5219	0.0103	0.0104	0.939	0.952
α		35	2.0726	2.0603	0.0792	0.0738	0.949	0.949
β			0.5018	0.5117	0.0109	0.0107	0.940	0.952
α		30	2.0731	2.0611	0.0913	0.0849	0.957	0.957
β			0.5086	0.5190	0.0132	0.0132	0.943	0.946
α		25	2.1023	2.0907	0.1072	0.0995	0.951	0.950
β			0.5078	0.5181	0.0139	0.0138	0.948	0.965
α	100	90	2.0284	2.0227	0.0295	0.0285	0.955	0.953
β			0.5002	0.5047	0.0043	0.0043	0.944	0.950
α		80	2.0298	2.0240	0.0339	0.0328	0.948	0.954
β			0.5017	0.5065	0.0050	0.0049	0.937	0.945
α		70	2.0243	2.0187	0.0356	0.0344	0.951	0.946
β			0.5037	0.5087	0.0057	0.0057	0.950	0.945
α		60	2.0436	2.0377	0.0446	0.0429	0.941	0.941
β			0.5007	0.5062	0.0065	0.0065	0.930	0.943
α		50	2.0407	2.0355	0.0480	0.0462	0.951	0.948
β			0.5072	0.5128	0.0078	0.0078	0.947	0.944
α	150	135	2.0252	2.0215	0.0194	0.0190	0.957	0.957
β			0.5015	0.5045	0.0030	0.0030	0.939	0.944
α		120	2.0228	2.0189	0.0217	0.0212	0.956	0.957
β			0.4984	0.5017	0.0030	0.0030	0.954	0.959
α		105	2.0281	2.0242	0.0256	0.0249	0.947	0.947
β			0.4987	0.5022	0.0035	0.0034	0.937	0.954
α		90	2.0301	2.0262	0.0311	0.0304	0.943	0.939
β			0.5004	0.5042	0.0042	0.0041	0.945	0.946
α		75	2.0358	2.0321	0.0362	0.0351	0.931	0.927
β			0.5026	0.5067	0.0051	0.0051	0.945	0.948

Table 2.19: Simulation results when $\alpha = 2$, $\beta = 0.5$, and $T = 2.72$. The hyper parameters for the Bayesian Estimations are $a_1 = 0.8$, $a_2 = 0.4$, $b_1 = 0.45$, $b_2 = 0.3$.

Parameter	n	m	APE_{MLE}	APE_{BE}	MSE_{MLE}	MSE_{BE}	CP_{MLE}	CP_{BE}	
α	50	45	2.0538	2.0494	0.0790	0.0759	0.942	0.937	
β			1.0221	1.0262	0.0312	0.0308	0.929	0.937	
α		40	2.0622	2.0584	0.0798	0.0765	0.956	0.952	
β			1.0153	1.0194	0.0318	0.0312	0.948	0.950	
α		35	2.0578	2.0545	0.0824	0.0790	0.955	0.952	
β			1.0237	1.0278	0.0385	0.0377	0.945	0.940	
α		30	2.0566	2.0541	0.0862	0.0824	0.959	0.962	
β			1.0322	1.0359	0.0444	0.0432	0.945	0.949	
α		25	2.0875	2.0844	0.1120	0.1068	0.955	0.946	
β			1.0535	1.0559	0.0615	0.0595	0.951	0.938	
α		100	90	2.0298	2.0280	0.0330	0.0324	0.959	0.957
β				1.0108	1.0130	0.0131	0.0131	0.957	0.957
α			80	2.0368	2.0351	0.0426	0.0418	0.928	0.926
β				1.0138	1.0161	0.0181	0.0179	0.929	0.934
α			70	2.0415	2.0404	0.0419	0.0410	0.951	0.954
β				1.0157	1.0180	0.0179	0.0178	0.956	0.956
α			60	2.0277	2.0273	0.0430	0.0421	0.955	0.953
β				1.0093	1.0118	0.0204	0.0202	0.945	0.943
α	50		2.0450	2.0454	0.0548	0.0536	0.961	0.960	
β			1.0292	1.0317	0.0262	0.0259	0.960	0.952	
α	150		135	2.0103	2.0093	0.0215	0.0213	0.954	0.952
β				1.0097	1.0112	0.0085	0.0084	0.953	0.959
α			120	2.0180	2.0171	0.0234	0.0231	0.959	0.961
β				1.0115	1.0132	0.0103	0.0103	0.947	0.953
α			105	2.0223	2.0216	0.0266	0.0263	0.953	0.955
β				1.0052	1.0069	0.0107	0.0106	0.951	0.952
α			90	2.0230	2.0229	0.0305	0.0301	0.948	0.950
β				1.0059	1.0078	0.0127	0.0127	0.949	0.952
α		75	2.0170	2.0177	0.0333	0.0328	0.951	0.956	
β			1.0143	1.0161	0.0177	0.0176	0.942	0.939	

Table 2.20: Simulation results when $\alpha = 2$, $\beta = 1$, and $T = 1.59$. The hyper parameters for the Bayesian Estimations are $a_1 = 0.8$, $a_2 = 0.4$, $b_1 = 0.45$, $b_2 = 0.3$.

Parameter	n	m	APE_{MLE}	APE_{BE}	MSE_{MLE}	MSE_{BE}	CP_{MLE}	CP_{BE}	
α	50	45	2.0630	2.0572	0.0691	0.0662	0.950	0.950	
β			1.0135	1.0175	0.0278	0.0272	0.953	0.957	
α		40	2.0507	2.0458	0.0728	0.0699	0.950	0.955	
β			1.0249	1.0286	0.0322	0.0316	0.959	0.955	
α		35	2.0716	2.0670	0.0854	0.0820	0.954	0.950	
β			1.0454	1.0486	0.0394	0.0386	0.950	0.952	
α		30	2.0788	2.0746	0.1036	0.0989	0.943	0.943	
β			1.0399	1.0432	0.0441	0.0430	0.950	0.947	
α		25	2.0852	2.0822	0.1101	0.1052	0.938	0.937	
β			1.0586	1.0610	0.0604	0.0582	0.963	0.956	
α		100	90	2.0351	2.0324	0.0304	0.0298	0.949	0.945
β				1.0069	1.0089	0.0134	0.0133	0.948	0.950
α			80	2.0212	2.0190	0.0318	0.0312	0.963	0.958
β				1.0148	1.0169	0.0158	0.0157	0.957	0.950
α			70	2.0318	2.0296	0.0380	0.0372	0.952	0.952
β				1.0155	1.0179	0.0169	0.0167	0.960	0.960
α			60	2.0398	2.0381	0.0453	0.0443	0.952	0.948
β				1.0192	1.0214	0.0226	0.0223	0.940	0.942
α	50		2.0449	2.0440	0.0504	0.0493	0.955	0.952	
β			1.0295	1.0315	0.0253	0.0249	0.951	0.952	
α	150		135	2.0229	2.0212	0.0212	0.0209	0.941	0.936
β				1.0083	1.0096	0.0090	0.0089	0.950	0.952
α			120	2.0196	2.0180	0.0231	0.0228	0.954	0.954
β				1.0098	1.0113	0.0101	0.0101	0.955	0.954
α			105	2.0205	2.0191	0.0267	0.0263	0.938	0.940
β				1.0152	1.0168	0.0112	0.0112	0.955	0.955
α			90	2.0209	2.0198	0.0280	0.0276	0.952	0.950
β				1.0142	1.0157	0.0137	0.0136	0.952	0.954
α		75	2.0302	2.0296	0.0321	0.0316	0.947	0.945	
β			1.0152	1.0168	0.0157	0.0156	0.952	0.953	

Table 2.21: Simulation results when $\alpha = 2$, $\beta = 1$, and $T = 1.99$. The hyper parameters for the Bayesian Estimations are $a_1 = 0.8$, $a_2 = 0.4$, $b_1 = 0.45$, $b_2 = 0.3$.

Parameter	n	m	APE_{MLE}	APE_{BE}	MSE_{MLE}	MSE_{BE}	CP_{MLE}	CP_{BE}	
α	50	45	2.0578	2.0527	0.0735	0.0703	0.945	0.944	
β			1.5497	1.5467	0.0770	0.0736	0.949	0.948	
α		40	2.0647	2.0599	0.0808	0.0772	0.944	0.943	
β			1.5558	1.5519	0.0955	0.0903	0.945	0.943	
α		35	2.0730	2.0680	0.0947	0.0900	0.939	0.942	
β			1.5666	1.5614	0.1115	0.1044	0.958	0.954	
α		30	2.0537	2.0498	0.0979	0.0926	0.940	0.941	
β			1.5810	1.5736	0.1354	0.1246	0.955	0.951	
α		25	2.0497	2.0463	0.0993	0.0930	0.965	0.964	
β			1.5995	1.5891	0.1681	0.1515	0.954	0.955	
α		100	90	2.0313	2.0292	0.0332	0.0325	0.951	0.950
β				1.5271	1.5261	0.0359	0.0351	0.943	0.938
α			80	2.0351	2.0332	0.0367	0.0358	0.956	0.953
β				1.5253	1.5240	0.0407	0.0397	0.941	0.942
α			70	2.0283	2.0269	0.0416	0.0407	0.954	0.952
β				1.5330	1.5315	0.0461	0.0448	0.949	0.953
α			60	2.0397	2.0386	0.0469	0.0457	0.944	0.944
β				1.5536	1.5513	0.0635	0.0614	0.942	0.938
α	50		2.0249	2.0249	0.0496	0.0483	0.953	0.957	
β			1.5376	1.5352	0.0627	0.0604	0.961	0.957	
α	150		135	2.0185	2.0172	0.0233	0.0230	0.942	0.940
β				1.5120	1.5115	0.0224	0.0222	0.942	0.944
α			120	2.0233	2.0222	0.0259	0.0256	0.937	0.939
β				1.5151	1.5146	0.0241	0.0237	0.958	0.951
α			105	2.0144	2.0136	0.0277	0.0273	0.943	0.944
β				1.5079	1.5073	0.0265	0.0260	0.950	0.947
α			90	2.0092	2.0089	0.0321	0.0315	0.935	0.938
β				1.5162	1.5154	0.0338	0.0331	0.949	0.947
α		75	2.0222	2.0224	0.0346	0.0340	0.959	0.954	
β			1.5227	1.5214	0.0391	0.0382	0.953	0.959	

Table 2.22: Simulation results when $\alpha = 2$, $\beta = 1.5$, and $T = 1.32$. The hyper parameters for the Bayesian Estimations are $a_1 = 0.8$, $a_2 = 0.4$, $b_1 = 0.45$, $b_2 = 0.3$.

Parameter	n	m	APE_{MLE}	APE_{BE}	MSE_{MLE}	MSE_{BE}	CP_{MLE}	CP_{BE}	
α	50	45	2.0557	2.0494	0.0686	0.0657	0.944	0.942	
β			1.5636	1.5590	0.0813	0.0772	0.953	0.947	
α		40	2.0743	2.0679	0.0832	0.0793	0.944	0.939	
β			1.5682	1.5627	0.0913	0.0861	0.953	0.943	
α		35	2.0523	2.0467	0.0757	0.0723	0.962	0.954	
β			1.5822	1.5753	0.1028	0.0963	0.961	0.948	
α		30	2.0621	2.0562	0.0875	0.0827	0.957	0.956	
β			1.5727	1.5636	0.1406	0.1297	0.944	0.935	
α		25	2.0862	2.0798	0.1129	0.1056	0.939	0.947	
β			1.6140	1.6009	0.1776	0.1590	0.961	0.956	
α	100	90	2.0291	2.0263	0.0326	0.0319	0.952	0.954	
β			1.5206	1.5189	0.0284	0.0278	0.958	0.957	
α		80	2.0319	2.0292	0.0333	0.0325	0.966	0.963	
β			1.5236	1.5217	0.0353	0.0344	0.956	0.957	
α		70	2.0476	2.0450	0.0408	0.0397	0.950	0.951	
β			1.5283	1.5257	0.0468	0.0455	0.929	0.933	
α		60	2.0521	2.0497	0.0453	0.0441	0.951	0.949	
β			1.5388	1.5357	0.0543	0.0525	0.942	0.935	
α		50	2.0269	2.0250	0.0481	0.0468	0.955	0.956	
β			1.5528	1.5485	0.0663	0.0636	0.957	0.953	
α		150	135	2.0258	2.0239	0.0209	0.0206	0.952	0.953
β				1.5164	1.5153	0.0223	0.0219	0.944	0.936
α			120	2.0217	2.0198	0.0236	0.0233	0.939	0.939
β				1.5257	1.5244	0.0252	0.0248	0.945	0.943
α			105	2.0299	2.0283	0.0255	0.0250	0.961	0.957
β				1.5295	1.5281	0.0275	0.0270	0.952	0.955
α	90		2.0247	2.0234	0.0311	0.0305	0.939	0.938	
β			1.5272	1.5254	0.0327	0.0320	0.950	0.947	
α	75		2.0251	2.0242	0.0338	0.0332	0.953	0.947	
β			1.5343	1.5320	0.0433	0.0421	0.946	0.939	

Table 2.23: Simulation results when $\alpha = 2$, $\beta = 1.5$, and $T = 1.66$. The hyper parameters for the Bayesian Estimations are $a_1 = 0.8$, $a_2 = 0.4$, $b_1 = 0.45$, $b_2 = 0.3$.

Based on the results in tables 2.12 - 2.23 and figures 2.5 to 2.8, we can observe that: (1) for fixed n , as m increases, the MSE decreases using either maximum likelihood or Bayes method; (2) for fixed m , as n increases, the MSE decreases using either maximum likelihood or Bayes method; (3) better selections of hyperparameter values result in improved Bayesian estimates; (4) for all cases, the coverage probability is very close to the confidence level used to compute the confidence/probability intervals. This indicates that we could have only used the maximum likelihood method to save time and effort that we spent on applying the

Bayesian method.

2.7 Expected Experimentation Time

The experiment's termination time plays an important rule in lifetime experiments. The shorter the time the less the cost of the experiment. In this section, we discuss the expected experimentation time, which may help to make better choices. Such information can be important for the experimenter to choose an appropriate sampling plan because it helps to know the time required to complete the test which is highly related to the cost. For a progressively Type-II censoring sample $(X_1, R_1), (X_2, R_2), \dots, (X_m, R_m)$ with fixed probability of removals, from a distribution with a probability density function $f(x)$ and survival function $S(x)$, the expected experimentation time is given as, Balakrishnan et al. (2000),

$$E(X_{m:m:n}) = \int_0^{\infty} x f_{X_{m:m:n}}(x) dx, \quad (2.10)$$

where $f_{X_{m:m:n}}(x)$ is the probability density function at the m^{th} failure $X_{m:m:n}$, is given by

$$f_{X_{m:m:n}}(x) = c_{m-1} f(x) \sum_{j=1}^m a_{j,m} S(x)^{w_j-1}, \quad m = 1, 2, \dots, n, \quad (2.11)$$

$c_{m-1} = \prod_{i=1}^m \beta_i$, $a_{j,m} = \prod_{i=1, i \neq j}^m \frac{1}{w_i - w_j}$, and $w_i = \sum_{j=i}^m (R_j + 1)$, $i, j = 1, 2, \dots, m$.

The expected experimentation time of the complete sample with size n is the expected value of the n^{th} order statistic, that can be obtained from 2.10 by setting $m = n$.

The ratio of the expected experiment time, REET, under progressively Type-II censoring is defined as

$$\text{REET} = \frac{E(X_{m:m:n})}{E(X_n)} \in (0, 1). \quad (2.12)$$

REET can be used to test time efficiency of the experiment. The smaller the value of REET, the more the efficient the experimental time.

Under the assumption that the tested item lifetime follows STH distribution, REET cannot be obtained in closed form solution. Therefore, to compute it, we should use numeral integration methods. We will use R software to compute REET under some values of the model parameters and different scenarios of progressively Type-II censoring. To perform this computation, we used the following simulation algorithm:

1. Specify the values of the parameters α and β for STH distribution.

2. Specify the values of n, m, p , where n is the sample size, m is the number of observed failures, and p is the probability of removal.

3. Generate a progressively Type II censored sample from STH distribution.

4. Calculate: $E(X_{m:m:n})$, $E(X_{(n)})$, and $REET = \frac{E(X_{m:m:n})}{E(X_{(n)})}$.

5. Replicate the steps 3-4, N times.

6. Compute the average of the N values of $E(X_{m:m:n})$ and REET.

We carried out the above algorithm at different values of n, m, p and two sets of parameter values: (1) set 1: $(\alpha, \beta) = (5, 1)$, and (2) set 2: $(\alpha, \beta) = (3, 2)$, when $N = 1000$. With the purpose of analyzing how parameters α, β , the probability of removal p , sample size n , and the observed failures m affect the ratio of expected experiment time REET, we take $n = 10$ with $m = 3, 5, 7$, $n = 30$ with $m = 9, 15, 21$, and $n = 50$ with $m = 15, 25, 35$ as the sample sizes and observed failures with the probability of removal $p = 0.1, 0.5, 0.9$ for each case to perform the REET simulations. Table 2.24 summarizes the results. For fixed n and m , as p increases, the value of REET increases. Also, for fixed n , as m increases, the value of REET increases. As expected, for fixed n , as m increases, we'll have more information about the sample size n , and hence, the value of REET increases too. However, the results show that the removal probability p has a significant influence on the ratio of expected experiment time REET as p increases in all cases. Both sets of the parameter values show similar responses to n, m , and p .

n	m	p	$(\alpha, \beta) = (5, 1)$			$(\alpha, \beta) = (3, 2)$		
			$E(X_n)$	$E(X_{m:m:n})$	REET	$E(X_n)$	$E(X_{m:m:n})$	REET
10	3	0.1	1.1863	0.7799	0.6574	1.0794	0.5236	0.4851
		0.5		0.8808	0.7424		0.5496	0.5092
		0.9		1.0295	0.8678		0.7218	0.6686
	5	0.1		0.9571	0.8067		0.7131	0.6606
		0.5		1.0710	0.9028		0.9033	0.8368
		0.9		1.1094	0.9351		0.9582	0.8877
	7	0.1		1.0522	0.8870		0.8878	0.8225
		0.5		1.1497	0.9691		0.8905	0.8249
		0.9		1.1497	0.9691		1.0215	0.9463
30	9	0.1	1.2714	0.8601	0.6764	1.2101	0.6394	0.5284
		0.5		1.1492	0.9038		1.0411	0.8604
		0.9		1.1684	0.9189		1.0486	0.8666
	15	0.1		1.0231	0.8046		0.8168	0.6751
		0.5		1.2148	0.9554		1.1143	0.9209
		0.9		1.2152	0.9557		1.1261	0.9307
	21	0.1		1.1422	0.8983		0.9958	0.8229
		0.5		1.2430	0.9776		1.1651	0.9629
		0.9		1.2445	0.9787		1.1681	0.9653
50	15	0.1	1.3081	0.9531	0.7285	1.2631	0.7357	0.5821
		0.5		1.2009	0.9179		1.0917	0.8638
		0.9		1.2153	0.9289		1.1188	0.8852
	25	0.1		1.1565	0.8841		1.0346	0.8185
		0.5		1.2547	0.9591		1.1834	0.9362
		0.9		1.2564	0.9603		1.1872	0.9393
	35	0.1		1.1437	0.8742		1.1001	0.8704
		0.5		1.2811	0.9791		1.2245	0.9688
		0.9		1.2814	0.9794		1.2256	0.9696

Table 2.24: The expected time for the complete experiment, the average of the expected time of the progressively Type II censoring experiment and the average of REET.

2.8 Conclusion

In this chapter, we discussed how to use progressively hybrid censoring (PHC) samples to estimate the unknown parameters of Sarhan-Tadj-Hamilton (STH) distribution. We used the maximum likelihood and Bayes methods to estimate those unknown parameters. For the Bayes method, we assumed that the model parameters are independent and gamma prior distributions with hyperparameters all equal to 0.001. This choice of the hyperparameters' values reflect the lack of prior information on the model parameters.

There is no closed-form solution for either the maximum likelihood estimate or Bayes estimate for the parameters. Hence, numerical methods are applied. We used R software to apply such numerical methods. For the MLE, we used the R optim function to maximize the log-likelihood function. For Bayes method, we used MCMC to approximate the Bayesian analysis.

We applied the theoretical methods discussed in this chapter on three real data sets and we compared the STH distribution with Weibull and power Lindely distributions. For each of these real data sets, we generated two PHC samples then we used them to estimate the model parameters for the three mentioned distributions using maximum likelihood and Bayes methods. Based on the log-likelihood, AIC and BIC values, we concluded that STH distribution fits those three real data sets better than the other two distributions for the original samples and the PHC samples.

To investigate the performance of the estimation methods and compare them, a large simulation study was performed using STH distribution at different values of the model parameters and at different values of the sampling scheme (n, m, p and T).

Also, we discussed the expected experimentation time, through the relative expected experimentation time (REET), for the progressively Type-II censoring samples using the STH distribution. We performed a simulation study to investigate how REET is affected by the values of the model parameters and progressively censoring values (n, m, p).

Further investigations for this method as future work. We could also consider the PHC samples with unknown probability of removal. Also, Bayesian methods with informative prior information on the unknown model parameters can be investigated. This chapter was published in Sarhan et al. (2023).

Chapter 3

Step Stress Partially Accelerated Life Testing

3.1 Introduction

A number of design methods have been proposed to reduce the time required to carry out an experiment where the outcome is time to event. Chapter two focused on one such method - progressive hybrid censoring. In this chapter, we consider two other methods - accelerated life testing (ALT) and partially accelerated life testing (PALT) - each method designed to reduce time to failure by testing experimental units under increasingly severe conditions, for example, increased temperature, voltage, or humidity.

One type of accelerated life testing is step stress accelerated life testing (SSALT), in which the stress on each experimental unit is increased at some point during an experiment. In the partially accelerated step stress design (SSPALT), all experimental items are put on test under a fixed stress, and units not failing by a pre-assigned time τ are subject to an increased stress. The goal of the SSPALT experiment is to quickly acquire reliable information on the failure time distribution Ismail and Aly (2014).

An extensive coverage of accelerated testing with constant stress with a variety of parametric failure time distributions is provided by Nelson (2009). DeGroot and Goel (1979) introduced Bayesian estimation and optimal design in SSPALT with an underlying exponential distribution, and Bhattacharyya and Soejoeti (1989) described a tampered failure rate for SSPALT using Weibull distribution. Ismail and Aly (2014) proposed an optimal plan for SSPLAT under Type-II censoring.

The rest of the chapter is organized as follows. Section 3.2 introduces the assumptions of the SSPALT, and section 3.3 introduces likelihood based and Bayesian estimation methods for unknown parameters. Section 3.4 addresses the important issue of choosing τ in an optimal fashion, in order to minimize the variability of estimated model parameters. The behavior of the methods is addressed with a simulation study in Section 3.5. Some real data sets were investigated in section 3.6.

3.2 Assumptions and Testing Schemes

The following assumptions are adopted throughout this chapter:

A) Step stress partially accelerated life testing (SSPALT):

In SSPALT we assume that:

1. The experimenter can switch the test item from the normal (standard) use (stress) condition to higher stress condition.
2. The experiment starts the test under a normal use condition, and then if the test unit does not fail by a pre-specified time $\tau > 0$, they put the test unit under a higher stress use condition.
3. Let T be the lifetime of the unit that runs under the normal use condition, with a pdf $f_T(t)$ and a survival function $S_T(t)$.
4. Let X be the life time of the testing unit under the SSPALT, with a pdf $f(x)$ and survival function $S(x)$. X is related to T according to the following relationship:

$$X = \begin{cases} T & \text{if } T \leq \tau, \\ \tau + \frac{1}{\beta}(T - \tau) & \text{if } T > \tau, \end{cases} \quad (3.1)$$

where $\beta > 1$ is the acceleration factor of the stress. Using equation 3.1, we get

$$f(x) = \begin{cases} f_T(x) & \text{if } x \leq \tau, \\ \beta f_T(\tau + \beta(x - \tau)) & \text{if } x > \tau, \end{cases} \quad (3.2)$$

and

$$S(x) = \begin{cases} S_T(x) & \text{if } x \leq \tau, \\ S_T(\tau + \beta(x - \tau)) & \text{if } x > \tau. \end{cases} \quad (3.3)$$

B) Type-II progressively hybrid censoring scheme with SSPALT:

In this scheme, we assume that:

1. n independent and identical units are put on an SSPALT.

2. An integer m is fixed before the experiment starts, such that $1 \leq m \leq n$. Also, m fixed integers, r_1, r_2, \dots, r_m are pre-determined such that $n = m + \sum_{i=1}^m r_i$. Moreover, a time point $\eta > \tau$ is determined before the experiment starts.
3. The Type-II progressively hybrid censoring scheme is described as follows:
 - (a) At the time of first failure X_1 , r_1 of the surviving $(n - 1)$ units are randomly removed from the test.
 - (b) At the time of second failure X_2 , r_2 of the surviving $(n - 2 - r_1)$ units are randomly removed from the test, and so on.
 - (c) If the time of m^{th} failure X_m occurs before the time point η , the experiment terminates at the m^{th} failure time X_m . This case is referred to as Case-I of Type-II progressively hybrid censoring scheme.
 - (d) While, if the time of m^{th} failure X_m does not occur before time point η and only J failures occur before η , where $0 \leq J < m$, then the experiment terminates at the time η and all the remaining $r_J^* = n - J - \sum_{i=1}^J r_i$ units are removed. This case is referred to as Case-II of Type-II progressively hybrid censoring scheme.

As a result of Type-II progressively hybrid censoring scheme with SSPALT, we will have one of the following cases of data sets:

$$\text{Case-I: } (X_1, r_1, \delta_1), (X_2, r_2, \delta_2), \dots, (X_m, r_m, \delta_m) \quad \text{if } X_m < \eta, \quad (3.4)$$

$$\text{Case-II: } (X_1, r_1, \delta_1), (X_2, r_2, \delta_2), \dots, (X_J, r_J, \delta_J) \quad \text{if } X_J < \eta < X_{J+1}, \quad (3.5)$$

where

$$\delta_i = \begin{cases} 1 & \text{if } X_i \leq \tau, \\ 2 & \text{if } X_i > \tau. \end{cases}$$

Note that: (1) in both two cases, $X_1 < X_2 < \dots < X_m$; (2) in Case-II, the test terminates at the time point η , where there are J failures and therefore $X_{J+1}, X_{J+2}, \dots, X_m$ are not observed; (3) when $m = n$, then $r_1 = r_2 = \dots, r_m = 0$, therefore the Type-II progressively censoring samples turns to a complete SSPALT sample.

We will mainly assume that the life time of the testing unit, under the normal use condition follows Sarhan-Tadj-Hamilton (STH) distribution with the following pdf and sf

Sarhan et al. (2014b)

$$f_T(t) = \frac{\alpha \gamma t^{\alpha-1}}{1 + \gamma} [\gamma + (1 + 2\gamma t^\alpha) e^{-\gamma t^\alpha}] e^{-\gamma t^\alpha}; t \geq 0; \alpha, \gamma > 0, \quad (3.6)$$

$$S_T(t) = \frac{1}{1 + \gamma} [\gamma + (1 + \gamma t^\alpha) e^{-\gamma t^\alpha}] e^{-\gamma t^\alpha}; t \geq 0; \alpha, \gamma > 0. \quad (3.7)$$

3.3 Estimation Methods

We will use the maximum likelihood and Bayesian methods to estimate the model parameters using the available SSPALT samples (complete and Type-II progressively hybrid censoring with SSPALT). Also, we will discuss how to determine the optimal change time τ .

3.3.1 Maximum Likelihood Estimation

Given the random sample obtained from Case-I, the log-likelihood function can be written as

$$\begin{aligned} \mathcal{L}_I(\theta) = & \sum_{i=1}^m \left\{ I(\delta_i = 1) \left[\log(f_T(x_i)) + r_i \log(S_T(x_i)) \right] + \right. \\ & \left. I(\delta_i = 2) \left[\log(\beta) + \log(f_T(\tau + \beta(x_i - \tau))) + r_i \log(S_T(\tau + \beta(x_i - \tau))) \right] \right\}, \end{aligned} \quad (3.8)$$

where $I(A) = 1$, if A is true, and 0 otherwise.

Setting $m = n$, and $r_1 = r_2 = \dots, r_m = 0$, we can derive the log-likelihood function using a complete SSPALT sample, from (3.8), as

$$\mathcal{L}(\theta) = \sum_{i=1}^n \left\{ I(\delta_i = 1) \log(f_T(x_i)) + I(\delta_i = 2) \left[\log(\beta) + \log(f_T(\tau + \beta(x_i - \tau))) \right] \right\}. \quad (3.9)$$

Using the random sample obtained from Case-II, the log-likelihood function can be written as

$$\begin{aligned} \mathcal{L}_{II}(\theta) = & r_J^* \log(S_T(\tau + \beta(\eta - \tau))) + \\ & \sum_{i=1}^J \left\{ I(\delta_i = 1) \left[\log(f_T(x_i)) + r_i \log(S_T(x_i)) \right] + \right. \\ & \left. I(\delta_i = 2) \left[\log(\beta) + \log(f_T(\tau + \beta(x_i - \tau))) + r_i \log(S_T(\tau + \beta(x_i - \tau))) \right] \right\}. \end{aligned} \quad (3.10)$$

Substituting equations (3.6) and (3.7) into (3.8), (3.9), and (3.10), we can derive the log-likelihood functions of the three underlying cases stated above as follows.

For Case-I, the log-likelihood function can be expressed as

$$\begin{aligned}
\mathcal{L}_I(\theta) \propto & m(\log \alpha + \log \gamma) - \left(m + \sum_{i=1}^m r_i \right) \log(1 + \gamma) + n_2 \log \beta + \\
& + (\alpha - 1) \sum_{i=1}^m \log \left(x_i^{I(\delta_i=1)} (\tau + \beta(x_i - \tau))^{I(\delta_i=2)} \right) + \\
& - \gamma \sum_{i=1}^m \left[I(\delta_i = 1) x_i^\alpha + I(\delta_i = 2) (\tau + \beta(x_i - \tau))^\alpha \right] + \\
& + \sum_{i=1}^m r_i \left[I(\delta_i = 1) \log \left[\frac{1}{1 + \gamma} (\gamma + (1 + \gamma x_i^\alpha) e^{-\gamma x_i^\alpha}) e^{-\gamma x_i^\alpha} \right] \right. \\
& \left. + I(\delta_i = 2) \log \left[\frac{1}{1 + \gamma} (\gamma + (1 + \gamma (\psi_i(\beta))^\alpha) e^{-\gamma (\psi_i(\beta))^\alpha}) e^{-\gamma (\psi_i(\beta))^\alpha} \right] \right] \\
& + \sum_{i=1}^m \left[I(\delta_i = 1) \log (\gamma + (1 + 2\gamma x_i^\alpha) e^{-\gamma x_i^\alpha}) \right. \\
& \left. + I(\delta_i = 2) \log (\gamma + (1 + 2\gamma (\psi_i(\beta))^\alpha) e^{-\gamma (\psi_i(\beta))^\alpha}) \right].
\end{aligned} \tag{3.11}$$

Here $n_1 = \sum_{i=1}^m I(\delta_i = 1)$ is the number of failures under the normal use condition, and $n_2 = \sum_{i=1}^m I(\delta_i = 2)$ is the number of failures at the higher stress use condition, $m = n_1 + n_2$, and $\theta = (\alpha, \gamma, \beta)$ is the vector of the unknown parameters.

For the complete SSPALT sample, the log-likelihood function becomes:

$$\begin{aligned}
\mathcal{L}(\theta) = & n \log(\alpha) + n \log(\gamma) - n \log(1 + \gamma) + n_2 \log \beta + (\alpha - 1) \sum_{i=1}^{n_1} \log(x_i) + \\
& - \gamma \sum_{i=1}^{n_1} x_i^\alpha + \sum_{i=1}^{n_1} \log \left(\gamma + (1 + 2\gamma x_i^\alpha) \exp \left\{ -\gamma \sum_{i=1}^{n_1} x_i^\alpha \right\} \right) + \\
& - \gamma \sum_{i=n_1+1}^n [\tau + \beta(x_i - \tau)]^\alpha + (\alpha - 1) \sum_{i=n_1+1}^n \log(\tau + \beta(x_i - \tau)) + \\
& + \sum_{i=n_1+1}^n \log \left(\gamma + (1 + 2\gamma [\tau + \beta(x_i - \tau)]^\alpha) \exp \left\{ -\gamma \sum_{i=n_1+1}^n [\tau + \beta(x_i - \tau)]^\alpha \right\} \right),
\end{aligned}$$

where $n_1 = \sum_{i=1}^n I(\delta_i = 1)$, $n_2 = \sum_{i=1}^n I(\delta_i = 2)$ and $n = n_1 + n_2$.

For Case-II, the log-likelihood function can be expressed as

$$\begin{aligned}
\mathcal{L}_{II}(\theta) \propto & J (\log \alpha + \log \gamma) - \left(r_J^* + J + \sum_{i=1}^J r_i \right) \log(1 + \gamma) + n_2 \log \beta + \quad (3.12) \\
& + (\alpha - 1) \sum_{i=1}^J \log \left(x_i^{I(\delta_i=1)} (\tau + \beta(x_i - \tau))^{I(\delta_i=2)} \right) + \\
& + \sum_{i=1}^J r_i \left[I(\delta_i = 1) \log \left[\frac{1}{1 + \gamma} (\gamma + (1 + \gamma x_i^\alpha) e^{-\gamma x_i^\alpha}) e^{-\gamma x_i^\alpha} \right] \right. \\
& + I(\delta_i = 2) \log \left[\frac{1}{1 + \gamma} (\gamma + (1 + \gamma (\psi_i(\beta))^\alpha) e^{-\gamma (\psi_i(\beta))^\alpha}) e^{-\gamma (\psi_i(\beta))^\alpha} \right] \left. \right] \\
& + \sum_{i=1}^J \left[I(\delta_i = 1) \log (\gamma + (1 + 2\gamma x_i^\alpha) e^{-\gamma x_i^\alpha}) \right. \\
& \quad \left. + I(\delta_i = 2) \log (\gamma + (1 + 2\gamma (\psi_i(\beta))^\alpha) e^{-\gamma (\psi_i(\beta))^\alpha}) \right] . \\
& + r_J^* \log \left[\frac{1}{1 + \gamma} (\gamma + (1 + \gamma (\eta(\beta))^\alpha) e^{-\gamma (\eta(\beta))^\alpha}) e^{-\gamma (\eta(\beta))^\alpha} \right] .
\end{aligned}$$

Here $n_1 = \sum_{i=1}^J I(\delta_i = 1)$, $n_2 = \sum_{i=1}^J I(\delta_i = 2)$, and $J = n_1 + n_2$.

To get the MLE of the parameters, we need to maximize the log-likelihood function. This can be done by solving a system of three non-linear equations with respect to the model parameters. These non-linear equation are called the likelihood equations that can be obtained by setting the first partial derivatives of $\mathcal{L}(\theta)$, with respect to α , γ and β , say \mathcal{L}_α , \mathcal{L}_γ , and \mathcal{L}_β , equal to zero. Note that the MLE of the parameters is the solution of the likelihood equations such that the corresponding information matrix is positive definite. The first and second partial derivatives of $\mathcal{L}(\theta)$ are given in the Appendix. There is no analytic solution of the likelihood equations. Therefore, numerical approximation methods can be applied using R software to obtain the MLE of the parameters and the corresponding standard errors.

3.3.2 Bayesian Estimation

In this section, we will consider the Bayesian estimation of the unknown parameters. In the context of $STH(\alpha, \gamma)$ lifetimes, may be reasonably modeled by the gamma priors for each parameter because they are positive parameters. We assume that α , γ , and β are following $gamma(a_1, a_2)$, $gamma(g_1, g_2)$, and $gamma(b_1, b_2)$ respectively, where the gamma hyper-parameters a_1 , a_2 , g_1 , g_2 , b_1 , and b_2 are assumed to be positive. Then, the posterior

distribution follows:

$$g(\theta|x) \propto L(\theta; x) g(\alpha, \gamma, \beta),$$

where $L(\theta; x)$ is the likelihood function given in (3.1), (3.2), or (3.3) depending on which case we have. It is helpful to reparameterize all parameters so that they are all unconstrained.

$$\theta_1 = \log(\alpha) \quad , \quad \theta_2 = \log(\gamma) \quad , \quad \theta_3 = \log(\beta) \quad , \quad \theta_1, \theta_2, \theta_3 \in (-\infty, \infty),$$

$$\rightarrow \log g_T(\theta|x, r) = \log g(\alpha = e^{\theta_1}, \gamma = e^{\theta_2}, \beta = 1 + e^{\theta_3}|x, r) + (\theta_1 + \theta_2 + \theta_3).$$

Here $g_T(\theta|x, r)$ is the posterior function of the transformed parameters. To carry out Bayesian inference, we will use Markov Chain Monte Carlo to sample from the posterior distribution. In addition, we will use noninformative priors for the unknown parameters. We will use MCMC - Metropolis- Hastings algorithm as in Chapter 2.

3.4 The Optimal Stress Change Time

Survival prediction for product design and high technology development plays a significant role in the industry. To precisely estimate the product survival at increased stress, we use a criterion defined to minimize the generalized asymptotic variance (GAV) of the maximum likelihood estimates of the model parameters and the acceleration factor. This criterion will lead to the test's optimal design, yielding the most accurate estimates of a lifetime at the increased stress. This method is called the D-Optimality, which is based on the determinant of Fisher's information matrix. It can be constructed in terms of the (GAV) of the MLEs of the model parameters. As we mentioned earlier, determining the optimal change stress time τ is an important design problem in SSPALT. Determining τ will lead us to the most accurate estimates of our parameters, which is the optimal plan's primary purpose. Many authors have considered such optimal plans for their designs. DeGroot and Goel (1979) were among the first to use optimal plans for their SSPALT designs. Bai et al. (1993) used such optimal methods with Lognormal and Weibull distributions. The optimal τ^* is found by minimizing the asymptotic variance of the MLE's of all three parameters. The formula to get GAV:

$$GAV = \frac{1}{|F|}, \quad (3.13)$$

Here F is the information matrix. We consider different τ with $\alpha = 2$, $\gamma = (0.5, 1, 1.5)$ and $\beta = 1.5$. We replicate the process 1000 times in each case and calculate the average MLE's, mean squared errors (MSEs), and the average of GAV values in Tables 3.1-3.3. At the (0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8) quantiles for τ , we investigated this process when $n = 50, 100, 150, 200, 250$. Figure 3.1 shows the MSE and GAV plots for different percentile of τ for $\alpha = 2$, $\beta = 1.5$, and $\gamma = (0.5, 1, 1.5)$ respectively. We can see the optimal stress change time at the 60% percentile since it has the lowest GAV using all the sample sizes. The following are the algorithms to generate and apply SSPALT model.

3.4.1 Generating SSPALT Data Algorithm

1. Specify the values of n , and τ .
2. Specify the values of the parameters α , γ , β .
3. Generate failure times from a given distribution T_i as in section 1.2 for $i = 1, \dots, n$, using the specified sample size n , and the parameters values.
4. Use the generated failure times with τ , and β to generate SSPALT data using the following relationship:

$$X_i = \begin{cases} T_i & \text{if } T_i \leq \tau, \\ \tau + \frac{1}{\beta}(T_i - \tau) & \text{if } T_i > \tau, \end{cases}$$

$i = 1, \dots, n$. Then, for $i = 1, \dots, J$ the test under the normal stress S_1 , and for $i = J + 1, \dots, n$ the test under a higher stress S_2 , where x_J is the failure time before the pre-specified τ , and x_{J+1} is the failure time after τ .

3.4.2 The Optimal change Time τ - Simulation Algorithm

1. Specify the value of Q , where Q is the number of iterations.
2. Apply Algorithm 3.4.1 to generate SSPALT Data.
3. Get the estimations of the generated data for the parameters using maximum likelihood method.
4. Compute the GAV as in (3.2).
5. Replicate the Steps 1-4, Q times.
6. Compute the mean of the GAV's for the specified τ .
7. Do Steps 1–6 with different τ .
8. Compare the GAV values between all τ 's, the optimal τ is associated with the minimum GAV value.
9. Do Steps 1-8 with different sample size n .

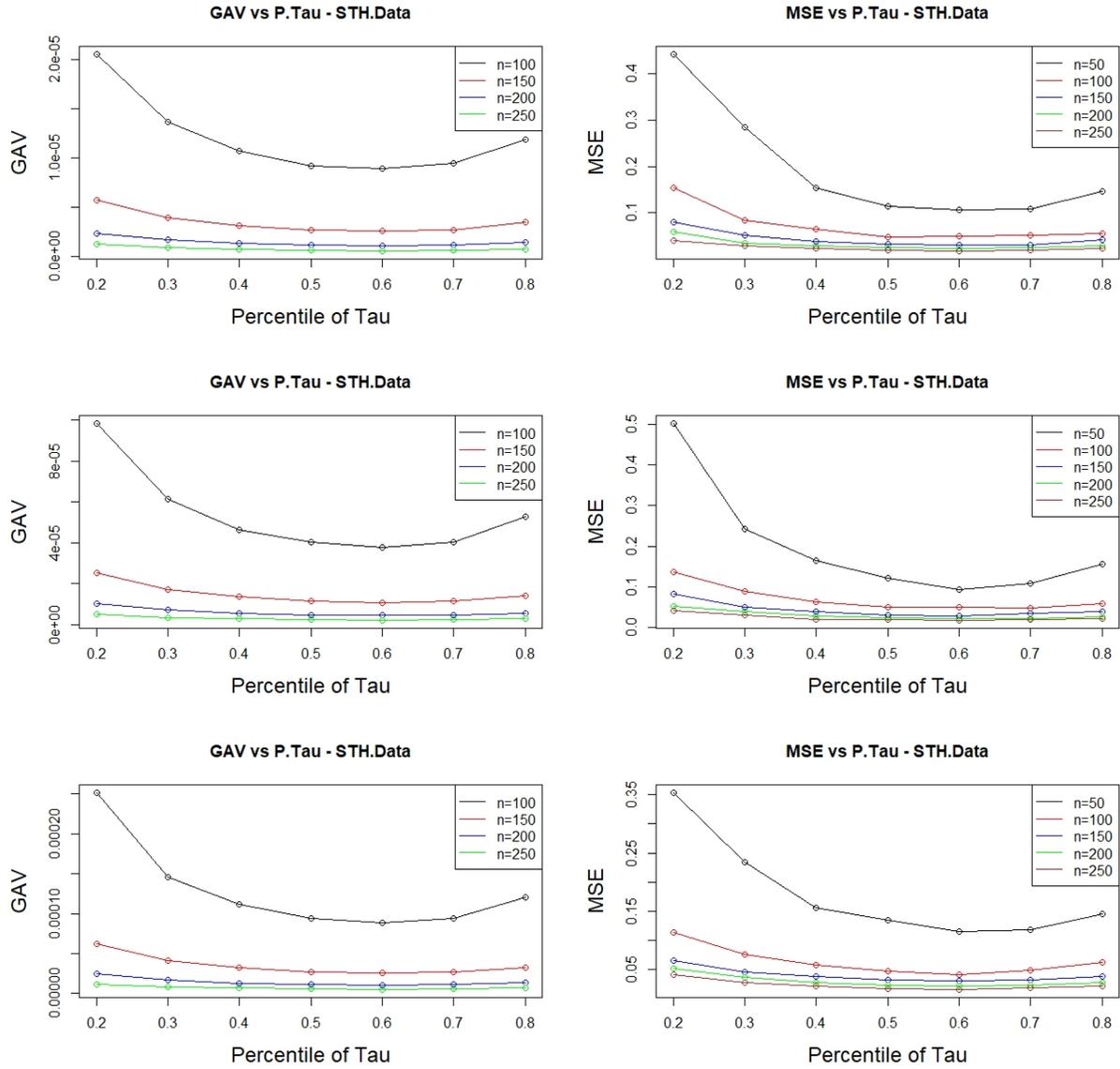


Figure 3.1: MSE and Gav plots for different percentile of τ .

n	$P.\tau$	τ	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\beta}$	MSE	GAV
50	0.2	0.648	2.20	0.58	1.69	0.47898	1.911896e-04
	0.3	0.809	2.17	0.53	1.58	0.20934	1.150338e-04
	0.4	0.957	2.12	0.51	1.61	0.15723	9.157544e-05
	0.5	1.102	2.10	0.51	1.57	0.11900	7.865766e-05
	0.6	1.252	2.08	0.50	1.60	0.10854	7.779076e-05
	0.7	1.420	2.06	0.50	1.61	0.11387	8.550826e-05
	0.8	1.623	2.06	0.50	1.67	0.16687	1.414721e-04
100	0.2	0.648	2.08	0.54	1.58	0.11919	1.973898e-05
	0.3	0.809	2.08	0.51	1.55	0.08039	1.354833e-05
	0.4	0.957	2.06	0.51	1.54	0.05890	1.073601e-05
	0.5	1.102	2.05	0.50	1.53	0.05155	9.153258e-06
	0.6	1.252	2.06	0.50	1.53	0.04660	8.805506e-06
	0.7	1.420	2.02	0.51	1.55	0.05364	9.595894e-06
	0.8	1.623	2.03	0.50	1.58	0.06558	1.246408e-05
150	0.2	0.648	2.06	0.51	1.57	0.07367	5.710029e-06
	0.3	0.809	2.05	0.51	1.53	0.05047	3.958314e-06
	0.4	0.957	2.04	0.51	1.52	0.03755	3.087832e-06
	0.5	1.102	2.02	0.50	1.54	0.03453	2.700987e-06
	0.6	1.252	2.03	0.50	1.52	0.02800	2.529647e-06
	0.7	1.420	2.02	0.50	1.54	0.03127	2.690847e-06
	0.8	1.623	2.01	0.50	1.54	0.04277	3.246602e-06
200	0.2	0.648	2.05	0.51	1.55	0.05590	2.333784e-06
	0.3	0.809	2.03	0.50	1.53	0.03716	1.640692e-06
	0.4	0.957	2.03	0.50	1.52	0.02757	1.295455e-06
	0.5	1.102	2.02	0.50	1.52	0.02576	1.119775e-06
	0.6	1.252	2.01	0.50	1.53	0.02324	1.074275e-06
	0.7	1.420	2.02	0.50	1.52	0.02454	1.089795e-06
	0.8	1.623	2.01	0.50	1.53	0.02761	1.365280e-06
250	0.2	0.648	2.02	0.51	1.56	0.04395	1.175938e-06
	0.3	0.809	2.03	0.50	1.52	0.02996	8.340445e-07
	0.4	0.957	2.02	0.50	1.52	0.02363	6.624911e-07
	0.5	1.102	2.02	0.50	1.51	0.01866	5.648149e-07
	0.6	1.252	2.01	0.50	1.52	0.01784	5.404634e-07
	0.7	1.420	2.01	0.50	1.52	0.01786	5.549227e-07
	0.8	1.623	2.01	0.50	1.52	0.02030	6.716327e-07

Table 3.1: SSPALT - Simulation: $\alpha = 2$, $\gamma = 0.5$, and $\beta = 1.5$ with different values of τ and sample size n .

n	$P.\tau$	τ	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\beta}$	MSE	GAV
50	0.2	0.462	2.19	1.45	1.72	0.41411	0.0029687134
	0.3	0.578	2.16	1.17	1.65	0.25935	0.0006225235
	0.4	0.685	2.12	1.07	1.62	0.17832	0.0004180005
	0.5	0.791	2.11	1.06	1.56	0.11550	0.0003491258
	0.6	0.902	2.09	1.04	1.56	0.10234	0.0003294564
	0.7	1.025	2.07	1.02	1.59	0.10673	0.0003661194
	0.8	1.177	2.06	1.01	1.67	0.14795	0.0005515513
100	0.2	0.462	2.08	1.15	1.60	0.13223	9.668716e-05
	0.3	0.578	2.08	1.07	1.56	0.08115	6.255126e-05
	0.4	0.685	2.05	1.04	1.55	0.05735	4.728748e-05
	0.5	0.791	2.04	1.03	1.53	0.04983	4.023429e-05
	0.6	0.902	2.04	1.01	1.56	0.04925	3.873934e-05
	0.7	1.025	2.03	1.01	1.55	0.05147	4.008498e-05
	0.8	1.177	2.02	1.01	1.61	0.06993	5.378453e-05
150	0.2	0.462	2.04	1.08	1.58	0.07659	2.529776e-05
	0.3	0.578	2.04	1.04	1.54	0.04684	1.733810e-05
	0.4	0.685	2.04	1.03	1.53	0.03748	1.361023e-05
	0.5	0.791	2.03	1.01	1.53	0.03469	1.161658e-05
	0.6	0.902	2.04	1.02	1.51	0.03024	1.101644e-05
	0.7	1.025	2.02	1.01	1.53	0.03066	1.141203e-05
	0.8	1.177	2.02	1.00	1.56	0.03942	1.454987e-05
200	0.2	0.462	2.04	1.06	1.55	0.05264	1.021429e-05
	0.3	0.578	2.04	1.04	1.53	0.03630	7.167071e-06
	0.4	0.685	2.03	1.02	1.52	0.02715	5.649523e-06
	0.5	0.791	2.02	1.01	1.53	0.02453	4.841240e-06
	0.6	0.902	2.04	1.01	1.51	0.02341	4.565923e-06
	0.7	1.025	2.01	1.01	1.53	0.02311	4.765799e-06
	0.8	1.177	2.01	1.00	1.54	0.02898	5.779970e-06
250	0.2	0.462	2.03	1.05	1.54	0.04101	5.150974e-06
	0.3	0.578	2.03	1.03	1.52	0.02806	3.652422e-06
	0.4	0.685	2.02	1.01	1.53	0.02335	2.870746e-06
	0.5	0.791	2.02	1.01	1.52	0.02016	2.479516e-06
	0.6	0.902	2.01	1.00	1.52	0.01804	2.294910e-06
	0.7	1.025	2.01	1.00	1.53	0.01852	2.422474e-06
	0.8	1.177	2.01	1.00	1.54	0.02119	2.971482e-06

Table 3.2: SSPALT - Simulation: $\alpha = 2$, $\gamma = 1$, and $\beta = 1.5$ with different values of τ and sample size n .

n	$P.\tau$	τ	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\beta}$	MSE	GAV
50	0.2	0379	2.17	2.34	1.76	0.49135	0.0042562886
	0.3	0.475	2.12	1.85	1.63	0.21467	0.0014854273
	0.4	0.564	2.14	1.74	1.58	0.14072	0.0010843031
	0.5	0.652	2.08	1.61	1.60	0.12828	0.0008455402
	0.6	0.745	2.08	1.58	1.58	0.10316	0.0007778507
	0.7	0.848	2.06	1.55	1.62	0.11143	0.0008610171
	0.8	0.975	2.07	1.56	1.67	0.16126	0.0012952226
100	0.2	0379	2.10	1.84	1.55	0.10309	2.435789e-04
	0.3	0.475	2.09	1.69	1.55	0.09858	1.514914e-04
	0.4	0.564	2.05	1.59	1.56	0.06305	1.121168e-04
	0.5	0.652	2.04	1.55	1.54	0.04743	9.339482e-05
	0.6	0.745	2.04	1.55	1.53	0.04632	8.859848e-05
	0.7	0.848	2.04	1.53	1.55	0.04720	9.373412e-05
	0.8	0.975	2.03	1.53	1.57	0.06014	1.183638e-04
150	0.2	0379	2.05	1.66	1.55	0.06563	5.987418e-05
	0.3	0.475	2.05	1.61	1.53	0.04776	4.137015e-05
	0.4	0.564	2.04	1.57	1.53	0.03739	3.177870e-05
	0.5	0.652	2.04	1.54	1.52	0.03263	2.710885e-05
	0.6	0.745	2.02	1.53	1.53	0.03289	2.528512e-05
	0.7	0.848	2.02	1.52	1.54	0.03069	2.674450e-05
	0.8	0.975	2.02	1.51	1.54	0.03763	3.279174e-05
200	0.2	0379	2.05	1.65	1.54	0.05396	2.468398e-05
	0.3	0.475	2.04	1.59	1.53	0.03813	1.710715e-05
	0.4	0.564	2.03	1.55	1.51	0.02658	1.312830e-05
	0.5	0.652	2.01	1.52	1.53	0.02635	1.121636e-05
	0.6	0.745	2.02	1.52	1.51	0.02043	1.042439e-05
	0.7	0.848	2.02	1.52	1.51	0.02173	1.098469e-05
	0.8	0.975	2.01	1.51	1.53	0.02740	1.334999e-05
250	0.2	0379	2.03	1.61	1.54	0.04143	1.200040e-05
	0.3	0.475	2.03	1.58	1.51	0.02798	8.419210e-06
	0.4	0.564	2.02	1.54	1.51	0.02201	6.626313e-06
	0.5	0.652	2.02	1.52	1.52	0.01888	5.692853e-06
	0.6	0.745	2.02	1.51	1.51	0.01780	5.329162e-06
	0.7	0.848	2.01	1.52	1.52	0.01903	5.557288e-06
	0.8	0.975	2.01	1.50	1.52	0.02078	6.613481e-06

Table 3.3: SSPALT - Simulation: $\alpha = 2$, $\gamma = 1.5$, and $\beta = 1.5$ with different values of τ and sample size n .

3.5 Simulation Studies

3.5.1 SSPALT - STH

In this section, we will give an algorithm for the simulations conducted with different values of n and percentile of τ to test the performance of the MLE and BE's estimations in terms of their mean square error and coverage probabilities using a given distribution. Algorithm 3.5.2 is to conduct a simulation using SSPALT. We consider different n and τ percentile with $\alpha = 2$, $\gamma = (0.5, 1, 1.5)$ and $\beta = 1.5$. We got the MLE and the 95% confidence interval, and the Bayesian estimation with its credible intervals. We replicate the process 1000 times in each case and report average MLE estimation, the standard error, mean squared errors (MSEs), and the coverage probabilities. We have the results obtained using the gamma prior for the Bayesian estimation. For fixed n as τ percentile increases, the MSEs increase too as we start from the median. For fixed τ percentile, as n increases, the MSEs decrease. The result is summarized in Tables 3.4-3.6:

3.5.2 SSPALT - Simulation Algorithm

1. Specify the value of Q , where Q is the number of iterations.
2. Apply Algorithm 3.4.1 to generate SSPALT Data.
3. Get the estimations of the generated data for the parameters using MLE's and Bayesian methods.
4. Compute the confidence intervals with confidence levels 95% for the two parameters of the model for ML's and BE's estimations.
5. Replicate the Steps 1-4, Q times.
6. Compute the mean square error and the coverage probabilities of our parameters.
7. repeat steps 1-6 with different values of n and change stress time τ .

n	q_τ	Method	Point Estimate			MSE			CP		
			α	γ	β	α	γ	β	α	γ	β
50	0.5	MLE	2.10	0.51	1.58	0.1744	0.0106	0.4033	0.954	0.939	0.909
		Bayes	2.09	0.51	1.75	0.1703	0.0106	0.6102	0.953	0.945	0.935
	0.6	MLE	2.08	0.50	1.59	0.1331	0.0086	0.3195	0.945	0.943	0.914
		Bayes	2.07	0.51	1.72	0.1310	0.0086	0.4414	0.943	0.948	0.943
	0.7	MLE	2.06	0.50	1.62	0.0974	0.0082	0.3539	0.952	0.935	0.932
		Bayes	2.05	0.50	1.73	0.0965	0.0081	0.4604	0.952	0.949	0.948
100	0.5	MLE	2.05	0.50	1.55	0.0775	0.0047	0.1589	0.953	0.938	0.926
		Bayes	2.04	0.50	1.62	0.0766	0.0047	0.1974	0.944	0.933	0.951
	0.6	MLE	2.03	0.50	1.53	0.0624	0.0042	0.1320	0.941	0.942	0.938
		Bayes	2.03	0.50	1.59	0.0619	0.0042	0.1558	0.940	0.948	0.955
	0.7	MLE	2.04	0.50	1.53	0.0529	0.0038	0.1442	0.949	0.934	0.936
		Bayes	2.04	0.50	1.58	0.0526	0.0037	0.1620	0.940	0.942	0.948
150	0.5	MLE	2.03	0.50	1.53	0.0484	0.0032	0.0986	0.946	0.948	0.942
		Bayes	2.03	0.50	1.57	0.0480	0.0032	0.1129	0.947	0.945	0.952
	0.6	MLE	2.02	0.50	1.54	0.0391	0.0028	0.0985	0.942	0.950	0.935
		Bayes	2.02	0.50	1.58	0.0389	0.0028	0.1102	0.943	0.947	0.940
	0.7	MLE	2.02	0.50	1.54	0.0341	0.0029	0.0915	0.953	0.941	0.957
		Bayes	2.02	0.50	1.57	0.0339	0.0029	0.0999	0.951	0.942	0.953
200	0.5	MLE	2.02	0.50	1.52	0.0358	0.0022	0.0730	0.950	0.956	0.950
		Bayes	2.02	0.50	1.56	0.0357	0.0022	0.0809	0.949	0.965	0.949
	0.6	MLE	2.03	0.50	1.51	0.0287	0.0021	0.0634	0.954	0.936	0.952
		Bayes	2.02	0.50	1.54	0.0285	0.0021	0.0686	0.951	0.938	0.961
	0.7	MLE	2.01	0.50	1.53	0.0232	0.0021	0.0708	0.952	0.933	0.949
		Bayes	2.01	0.50	1.55	0.0232	0.0021	0.0758	0.954	0.935	0.952

Table 3.4: SSPALT - Simulation: $\alpha = 2$, $\gamma = 0.5$, and $\beta = 1.5$ with different values of τ , sample size n , and gamma priors for the Bayesian estimation.

n	q_τ	Method	Point Estimate			MSE			CP		
			α	γ	β	α	γ	β	α	γ	β
50	0.5	MLE	2.11	1.06	1.57	0.1916	0.0750	0.3760	0.946	0.944	0.927
		Bayes	2.10	1.07	1.74	0.1865	0.0781	0.5604	0.945	0.946	0.946
	0.6	MLE	2.09	1.04	1.58	0.1336	0.0474	0.3660	0.956	0.943	0.925
		Bayes	2.08	1.04	1.71	0.1316	0.0482	0.5181	0.951	0.946	0.941
	0.7	MLE	2.07	1.03	1.61	0.1099	0.0338	0.3308	0.955	0.950	0.938
		Bayes	2.07	1.04	1.70	0.1089	0.0340	0.4257	0.949	0.952	0.951
100	0.5	MLE	2.04	1.03	1.55	0.0790	0.0304	0.1575	0.941	0.956	0.948
		Bayes	2.03	1.03	1.63	0.0783	0.0311	0.1945	0.941	0.958	0.946
	0.6	MLE	2.05	1.02	1.52	0.0661	0.0216	0.1299	0.956	0.950	0.932
		Bayes	2.05	1.02	1.58	0.0654	0.0217	0.1511	0.952	0.950	0.957
	0.7	MLE	2.04	1.01	1.55	0.0488	0.0150	0.1419	0.950	0.956	0.940
		Bayes	2.03	1.01	1.60	0.0485	0.0150	0.1614	0.948	0.955	0.954
150	0.5	MLE	2.01	1.00	1.56	0.0458	0.0196	0.1039	0.963	0.946	0.951
		Bayes	2.01	1.01	1.61	0.0456	0.0198	0.1219	0.963	0.943	0.952
	0.6	MLE	2.03	1.00	1.53	0.0420	0.0128	0.0888	0.946	0.950	0.944
		Bayes	2.03	1.00	1.57	0.0418	0.0129	0.0993	0.951	0.946	0.948
	0.7	MLE	2.02	1.01	1.54	0.0315	0.0110	0.0997	0.952	0.944	0.939
		Bayes	2.02	1.01	1.57	0.0314	0.0110	0.1085	0.951	0.942	0.946
200	0.5	MLE	2.03	1.01	1.51	0.0375	0.0145	0.0691	0.950	0.947	0.942
		Bayes	2.03	1.02	1.54	0.0373	0.0147	0.0761	0.946	0.942	0.953
	0.6	MLE	2.02	1.01	1.51	0.0288	0.0092	0.0630	0.956	0.958	0.950
		Bayes	2.02	1.01	1.54	0.0287	0.0093	0.0678	0.954	0.962	0.955
	0.7	MLE	2.01	1.00	1.52	0.0243	0.0082	0.0649	0.953	0.947	0.940
		Bayes	2.01	1.01	1.54	0.0243	0.0082	0.0692	0.953	0.948	0.950

Table 3.5: SSPALT - Simulation: $\alpha = 2$, $\gamma = 1$, and $\beta = 1.5$ with different values of τ , sample size n , and gamma priors for the Bayesian estimation.

n	q_τ	Method	Point Estimate			MSE			CP		
			α	γ	β	α	γ	β	α	γ	β
50	0.5	MLE	2.11	1.64	1.57	0.1775	0.2713	0.3044	0.960	0.957	0.919
		Bayes	2.09	1.68	1.73	0.1720	0.3018	0.4586	0.955	0.955	0.943
	0.6	MLE	2.09	1.59	1.58	0.1379	0.1579	0.2991	0.957	0.954	0.931
		Bayes	2.07	1.60	1.71	0.1342	0.1653	0.4145	0.957	0.948	0.952
	0.7	MLE	2.08	1.57	1.60	0.1079	0.1132	0.3194	0.957	0.944	0.942
		Bayes	2.07	1.58	1.70	0.1064	0.1156	0.4128	0.957	0.948	0.946
100	0.5	MLE	2.04	1.56	1.55	0.0727	0.1054	0.1487	0.951	0.963	0.942
		Bayes	2.04	1.57	1.62	0.0716	0.1102	0.1826	0.953	0.960	0.952
	0.6	MLE	2.02	1.53	1.57	0.0607	0.0734	0.1470	0.949	0.948	0.942
		Bayes	2.02	1.54	1.62	0.0602	0.0749	0.1752	0.951	0.940	0.951
	0.7	MLE	2.02	1.52	1.56	0.0492	0.0451	0.1475	0.953	0.955	0.942
		Bayes	2.01	1.52	1.61	0.0490	0.0454	0.1681	0.952	0.953	0.947
150	0.5	MLE	2.03	1.54	1.53	0.0499	0.0690	0.0920	0.949	0.968	0.940
		Bayes	2.03	1.55	1.58	0.0495	0.0713	0.1060	0.949	0.958	0.953
	0.6	MLE	2.02	1.52	1.54	0.0413	0.0448	0.0898	0.946	0.952	0.949
		Bayes	2.02	1.53	1.57	0.0411	0.0455	0.1006	0.944	0.944	0.948
	0.7	MLE	2.02	1.52	1.53	0.0350	0.0340	0.0923	0.946	0.939	0.943
		Bayes	2.02	1.52	1.56	0.0348	0.0342	0.1002	0.940	0.936	0.953
200	0.5	MLE	2.03	1.53	1.51	0.0380	0.0518	0.0735	0.941	0.944	0.934
		Bayes	2.02	1.54	1.55	0.0376	0.0529	0.0807	0.945	0.949	0.942
	0.6	MLE	2.02	1.52	1.53	0.0291	0.0334	0.0648	0.954	0.952	0.947
		Bayes	2.02	1.52	1.55	0.0290	0.0338	0.0707	0.955	0.956	0.957
	0.7	MLE	2.02	1.51	1.52	0.0266	0.0230	0.0682	0.938	0.954	0.941
		Bayes	2.01	1.51	1.54	0.0265	0.0231	0.0727	0.940	0.952	0.948

Table 3.6: SSPALT - Simulation: $\alpha = 2$, $\gamma = 1.5$, and $\beta = 1.5$ with different values of τ , sample size n , and gamma priors for the Bayesian estimation.

3.5.3 SSPALT - PHC - STH

After generating the data as in Section 1.2.1 and using a certain percentile of the sample as τ to generate accelerated data, we applied the PHC scheme as in Chapter 2 and then applied the R code to get the result of this simulation. q here is the percentage of the observed failures, $q = 0.9, 0.8, 0.7$, and Cases is the percentage of Case-I and Case-II in each 1000 sample. We used $n = 50, 100, 150, 200$, $\alpha = 2$, $\gamma = (0.5, 1, 1.5)$ and $\beta = 1.5$ for the generated data with the 0.6 percentile of the generated sample as τ the optimal change stress time, and $Q(0.95)$ for T as the preassigned stop time. The results are shown in Tables 3.7-3.9. Besides, we tested the measure GAV with six percentiles of τ , and different percentage of observed failures q for the PHC. Table 3.10 shows that the optimal percentile at 0.6 has the lowest GAV for all given q 's compare to the other percentiles. The following algorithm will give a full description of the simulation:

3.5.4 SSPALT - PHC - Simulation Algorithm

1. Specify the values of Q , n , τ , m , p , and T , where $m = q \times n$ is the observed failures, Q the number of iterations, and p is the removal probability.
2. Specify the values of the parameters α , γ , β .
3. Apply Algorithm 2.5.1 to generate PHC Data.
4. Apply Algorithm 3.4.1 on the data in the previous step to generate SSPALT Data.
5. Get the estimations of the generated data for the parameters using MLE's and Bayesian methods.
6. Compute the confidence intervals with confidence levels 95% for the two parameters of the model for ML's and BE's estimations.
7. Replicate the Steps 2-6, Q times.
8. Compute the mean square error and the coverage probabilities of our parameters.
9. repeat steps 1-8 with different values of n , m , τ , and β .

n	m	$Method$	$PointEstimate$			MSE			CP		
			α	γ	β	α	γ	β	α	γ	β
50	45	MLE	2.08	0.50	1.57	0.1493	0.0096	0.3333	0.939	0.943	0.929
		Bayes	2.07	0.50	1.71	0.1475	0.0096	0.4808	0.938	0.943	0.955
	40	MLE	2.07	0.51	1.62	0.1617	0.0105	0.5311	0.939	0.938	0.905
		Bayes	2.06	0.51	1.79	0.1607	0.0106	0.8116	0.935	0.942	0.926
	35	MLE	2.07	0.51	1.63	0.1513	0.0128	0.5195	0.957	0.935	0.932
		Bayes	2.06	0.51	1.81	0.1503	0.0128	0.8288	0.958	0.944	0.945
100	90	MLE	2.04	0.50	1.55	0.0614	0.0048	0.1573	0.960	0.937	0.946
		Bayes	2.03	0.50	1.61	0.0611	0.0048	0.1881	0.956	0.934	0.955
	80	MLE	2.03	0.50	1.58	0.0726	0.0050	0.2180	0.944	0.945	0.936
		Bayes	2.02	0.50	1.66	0.0726	0.0050	0.2677	0.946	0.948	0.937
	70	MLE	2.04	0.50	1.56	0.0733	0.0058	0.2254	0.954	0.943	0.941
		Bayes	2.03	0.51	1.64	0.0732	0.0059	0.2767	0.954	0.950	0.943
150	135	MLE	2.03	0.50	1.52	0.0482	0.0030	0.1146	0.937	0.949	0.925
		Bayes	2.03	0.50	1.56	0.0480	0.0030	0.1277	0.938	0.952	0.937
	120	MLE	2.02	0.50	1.54	0.0487	0.0035	0.1211	0.942	0.936	0.952
		Bayes	2.01	0.50	1.59	0.0486	0.0035	0.1378	0.941	0.946	0.942
	105	MLE	2.03	0.50	1.54	0.0473	0.0037	0.1291	0.957	0.951	0.949
		Bayes	2.03	0.50	1.59	0.0472	0.0037	0.1484	0.954	0.956	0.948
200	180	MLE	2.02	0.50	1.52	0.0330	0.0024	0.0771	0.942	0.932	0.948
		Bayes	2.02	0.50	1.55	0.0329	0.0024	0.0838	0.937	0.940	0.951
	160	MLE	2.02	0.50	1.53	0.0348	0.0026	0.0904	0.947	0.942	0.946
		Bayes	2.01	0.50	1.56	0.0347	0.0026	0.0990	0.941	0.948	0.946
	140	MLE	2.02	0.50	1.52	0.0380	0.0028	0.0957	0.951	0.953	0.947
		Bayes	2.02	0.50	1.55	0.0380	0.0028	0.1053	0.952	0.953	0.945

Table 3.7: SSPALT - HPC - Simulation: $\alpha = 2$, $\gamma = 0.5$, $\beta = 1.5$, $T = 1.87$, and $q_\tau = 0.6$ with different sample size n , and observed failure m using gamma priors for the Bayesian estimations.

n	m	$Method$	$PointEstimate$			MSE			CP		
			α	γ	β	α	γ	β	α	γ	β
50	45	MLE	2.08	1.04	1.56	0.1405	0.0501	0.3586	0.964	0.946	0.916
		Bayes	2.07	1.04	1.70	0.1395	0.0512	0.5127	0.957	0.953	0.939
	40	MLE	2.08	1.03	1.62	0.1502	0.0594	0.4862	0.961	0.938	0.918
		Bayes	2.07	1.04	1.78	0.1495	0.0607	0.7386	0.959	0.932	0.941
	35	MLE	2.07	1.05	1.62	0.1549	0.0641	0.4796	0.956	0.949	0.915
		Bayes	2.07	1.05	1.80	0.1553	0.0660	0.7429	0.953	0.954	0.945
100	90	MLE	2.03	1.01	1.55	0.0669	0.0219	0.1579	0.947	0.950	0.944
		Bayes	2.02	1.01	1.61	0.0667	0.0221	0.1886	0.943	0.948	0.955
	80	MLE	2.04	1.01	1.55	0.0716	0.0259	0.1827	0.954	0.944	0.942
		Bayes	2.03	1.01	1.62	0.0716	0.0263	0.2213	0.946	0.940	0.950
	70	MLE	2.06	1.03	1.55	0.0852	0.0340	0.2227	0.941	0.937	0.929
		Bayes	2.05	1.04	1.62	0.0851	0.0346	0.2701	0.942	0.938	0.939
150	135	MLE	2.03	1.01	1.53	0.0444	0.0145	0.1053	0.949	0.949	0.943
		Bayes	2.02	1.01	1.57	0.0444	0.0147	0.1174	0.951	0.951	0.954
	120	MLE	2.03	1.01	1.53	0.0451	0.0167	0.1062	0.962	0.944	0.952
		Bayes	2.02	1.01	1.57	0.0451	0.0168	0.1201	0.962	0.949	0.957
	105	MLE	2.02	1.01	1.53	0.0507	0.0187	0.1354	0.954	0.948	0.940
		Bayes	2.02	1.02	1.58	0.0507	0.0189	0.1541	0.950	0.951	0.951
200	180	MLE	2.01	1.01	1.53	0.0319	0.0116	0.0768	0.939	0.951	0.951
		Bayes	2.01	1.01	1.56	0.0318	0.0117	0.0840	0.940	0.943	0.951
	160	MLE	2.01	1.01	1.52	0.0324	0.0134	0.0790	0.955	0.942	0.940
		Bayes	2.01	1.01	1.55	0.0324	0.0135	0.0863	0.959	0.941	0.954
	140	MLE	2.03	1.01	1.52	0.0378	0.0144	0.0943	0.942	0.958	0.938
		Bayes	2.03	1.01	1.55	0.0378	0.0145	0.1032	0.946	0.949	0.946

Table 3.8: SSPALT - HPC - Simulation: $\alpha = 2$, $\gamma = 1$, $\beta = 1.5$, $T = 1.36$, and $q_r = 0.6$ with different sample size n , and observed failure m using gamma priors for the Bayesian estimations.

n	m	$Method$	$PointEstimate$			MSE			CP		
			α	γ	β	α	γ	β	α	γ	β
50	45	MLE	2.08	1.60	1.60	0.1450	0.1836	0.3703	0.954	0.948	0.938
		Bayes	2.07	1.62	1.74	0.1433	0.1940	0.5359	0.944	0.948	0.948
	40	MLE	2.08	1.60	1.57	0.1507	0.2206	0.3630	0.956	0.956	0.926
		Bayes	2.07	1.62	1.72	0.1489	0.2349	0.5297	0.957	0.947	0.949
	35	MLE	2.09	1.62	1.64	0.1807	0.2679	0.4971	0.958	0.946	0.913
		Bayes	2.08	1.64	1.82	0.1797	0.2866	0.8410	0.950	0.954	0.942
100	90	MLE	2.04	1.54	1.53	0.0674	0.0711	0.1371	0.955	0.958	0.959
		Bayes	2.03	1.55	1.59	0.0671	0.0733	0.1621	0.950	0.964	0.967
	80	MLE	2.04	1.54	1.54	0.0727	0.0859	0.1638	0.954	0.955	0.934
		Bayes	2.03	1.55	1.61	0.0724	0.0885	0.1957	0.949	0.951	0.947
	70	MLE	2.04	1.56	1.55	0.0844	0.1055	0.2257	0.944	0.951	0.929
		Bayes	2.04	1.57	1.63	0.0841	0.1093	0.2758	0.942	0.947	0.946
150	135	MLE	2.02	1.52	1.53	0.0419	0.0484	0.0989	0.955	0.938	0.952
		Bayes	2.01	1.52	1.57	0.0418	0.0492	0.1107	0.951	0.946	0.959
	120	MLE	2.01	1.53	1.54	0.0442	0.0554	0.1058	0.950	0.954	0.955
		Bayes	2.01	1.53	1.58	0.0442	0.0565	0.1204	0.954	0.954	0.956
	105	MLE	2.02	1.53	1.55	0.0523	0.0640	0.1278	0.954	0.947	0.947
		Bayes	2.02	1.54	1.60	0.0523	0.0655	0.1480	0.955	0.944	0.958
200	180	MLE	2.01	1.52	1.53	0.0310	0.0358	0.0763	0.959	0.949	0.943
		Bayes	2.01	1.52	1.56	0.0310	0.0362	0.0835	0.962	0.948	0.957
	160	MLE	2.00	1.50	1.55	0.0352	0.0409	0.0943	0.951	0.934	0.950
		Bayes	2.00	1.51	1.58	0.0352	0.0414	0.1048	0.948	0.931	0.949
	140	MLE	2.03	1.54	1.52	0.0404	0.0487	0.1007	0.960	0.962	0.927
		Bayes	2.03	1.55	1.55	0.0404	0.0498	0.1105	0.956	0.956	0.944

Table 3.9: SSPALT - HPC - Simulation: $\alpha = 2$, $\gamma = 1.5$, $\beta = 1.5$, $T = 1.13$, and $q_\tau = 0.6$ with different sample size n , and observed failure m using gamma priors for the Bayesian estimations.

n	q_τ	m	GAV
100	0.3	90	0.000581
		80	0.000812
		70	0.001118
	0.4	90	0.000394
		80	0.000537
		70	0.000787
	0.5	90	0.000338
		80	0.000464
		70	0.000642
	0.6	90	0.000321
		80	0.000442
		70	0.000642
	0.7	90	0.000369
		80	0.000514
		70	0.000731
0.8	90	0.000522	
	80	0.000761	
	70	0.001047	

Table 3.10: SSPALT - HPC - GAV - Simulation: $\alpha = 3$, $\gamma = 2$, $\beta = 1.2$, $T = 1.06$, and $n = 100$ with different values of τ , and different values of observed failure m .

For fixed n as q decreases, the MSEs increase. In addition, as n increases, the MSEs decrease for all parameters. As we saw from this simulation by using a preassigned stop time T , not all the cases end up with the stop time T as Case-II, we had also some situations end up with Case-I as they had observed the m th failure.

3.6 Data Analysis

In this section, we consider a real-life data set. The data set was investigated using the Modified Weibull distribution by Sarhan and Zaindin (2009b). The data set was given by Aarset (1987), which contains the lifetimes (in months) of 50 devices. The data are given below:

0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86

We estimated the MLEs using the complete and the accelerated samples. We assumed that the life time of the distribution has the STH, Weibull, and PL distributions, as shown in Table 3.11.

Model	$\hat{\alpha}_{MLE}$	SD_{α}	$\hat{\gamma}_{MLE}$	SD_{γ}	ll	AIC
STH	0.8924	0.1118	0.0257	0.0125	-91.36	186.721
Weibull	0.9488	0.1195	0.0223	0.0034	-92.52	189.048
PL	0.6638	0.0685	0.1612	0.0447	-93.61	191.219

Table 3.11: Data Analysis - STH and Weibull distributions - Complete Sample.

The MLEs for the accelerated samples were obtained using the optimal change time τ for each distribution. To find the optimal τ , we perform the following:

1. Generate a sequence of the percentiles;
2. Based on the percentile, compute the corresponding τ for the underlying distribution.
3. Generate an SSPLAT sample, from the complete sample, using τ at a selected value of $\beta > 1$.
4. Compute the MLE of the parameters, using the SSPALT sample.
5. Compute the mean squared error using the following formula.

$$MSE = \frac{(\hat{\alpha}_S - \hat{\alpha}_C)^2 + (\hat{\gamma}_S - \hat{\gamma}_C)^2 + (\hat{\beta}_S - \beta)^2}{3},$$

where $\hat{\alpha}_S, \hat{\gamma}_S, \hat{\beta}$ are the MLE of α, γ, β using the SSPALT sample, while $\hat{\alpha}_C$ and $\hat{\gamma}_C$ are the MLE of α and γ using the complete sample.

Figure 3.2 shows the MSE versus τ , as the percentile of the distribution, when $\beta = 1.5$. Based on the results in Figure 3.2, We found the changing time τ occurs at: (1) the 35% quantile ($\tau^* = 20.99$) for the STH distribution, (2) the 39% quantile ($\tau^* = 20.13$) for the Weibull distribution, and (3) the 40% quantile ($\tau^* = 21.49$) for the PL distribution.

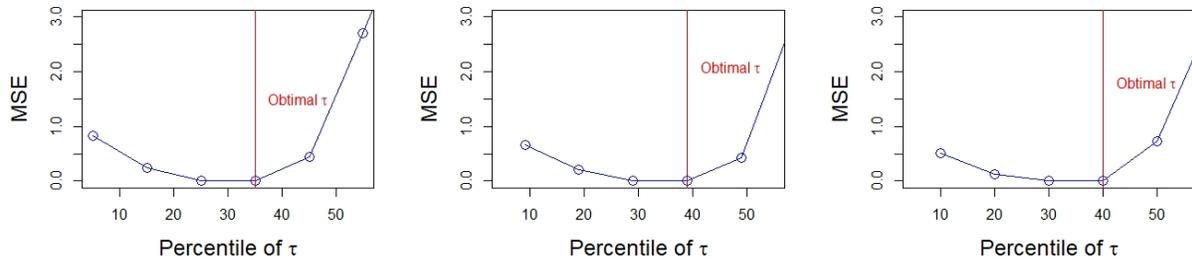


Figure 3.2: MSE plots for different percentile of τ for STH (Left Panel), Weibull (Middle Panel), and PL (Right Panel) distributions.

Then we used the optimal τ for each distribution to estimate the parameters using the SSPALT samples as shown in Table 3.12. In this Table, we include the AIC for each model for comparison purposes.

Model	$\hat{\alpha}_{MLE}$	$\hat{\gamma}_{MLE}$	$\hat{\beta}_{MLE}$	MSE	$P.\tau$	τ	AIC
STH	0.8915	0.0257	1.507	1.8e-05	35%	20.99	459.727
Weibull	0.9517	0.0224	1.4839	9.2e-05	39%	20.13	462.053
PL	0.6652	0.1611	1.4854	7.2e-05	40%	21.49	465.035

Table 3.12: The MLE of the model parameters, the MSE, percentile, optimal τ , and AIC for STH, Weibull, and PL distributions using the accelerated samples.

We can see that the $STH(\alpha, \gamma)$ has the smallest AIC in the accelerated samples, and hence, this distribution provides a better fit of the data.

Now, using all the optimal change time $\tau = (20.99, 20.13, 21.49)$ with all three distributions in Table 3.13:

Model	τ^*	$\hat{\alpha}_{MLE}$	$\hat{\gamma}_{MLE}$	$\hat{\beta}_{MLE}$	MSE	AIC
STH		0.8915	0.0258	1.506	1.2e-05	459.727
Weibull	20.99	0.9329	0.0214	1.6094	0.0041	462.018
PL		0.6523	0.1635	1.621	0.0049	464.181
STH		0.9105	0.0249	1.387	0.0043	460.112
Weibull	20.13	0.9519	0.0224	1.482	0.00019	462.053
PL		0.6644	0.1611	1.494	9.8e-06	464.225
STH		0.9115	0.0251	1.378	0.00505	460.483
Weibull	21.49	0.9534	0.0225	1.471	0.00029	462.862
PL		0.6654	0.1609	1.485	7.3e-05	465.035

Table 3.13: Data Analysis - STH, Weibull, and PL distributions - Accelerated Sample.

Using the average of the optimal τ 's, and applying it to each of the three distributions, leads to Table 3.14.

Model	τ^*	$\hat{\alpha}_{MLE}$	$\hat{\gamma}_{MLE}$	$\hat{\beta}_{MLE}$	MSE	AIC
STH		0.8943	0.0256	1.489	4.3e-05	460.304
Weibull	20.87	0.9358	0.0215	1.591	0.00279	462.029
PL		0.6541	0.1631	1.603	0.00361	464.192

Table 3.14: Data Analysis - STH, Weibull, and PL distributions - Accelerated Sample, using the averages.

In all cases the STH distribution has the best fit to the Aarset data using SSPALT method.

3.7 Conclusion

In this chapter, we discussed the SSPALT scheme using a given distribution. We obtain the maximum likelihood estimators of the unknown parameters for SSPALT and SSPALT - PHC. A Bayesian estimate of the unknown parameters is also considered using gamma priors for both parameters. There is no closed-form solution for both parameters; hence, a simulation is conducted using specified parameter values. MLEs and BE's show promising results for various sample sizes and acceleration factors β . Next, we apply the PHC schemes to the SSPALT method. Next, we considered the suggested transformation in Balakrishnan et al. (2000) on the generated failure times, accelerated failure times using τ and β , and removals. An optimal stress change time τ is considered and obtained using different values of the parameters. We encountered one problem with this method: determining the exact

value of the change stress time τ for generated failure times and accelerated failure times before estimating the MLEs. We assigned different τ percentiles of specified parameters' values to overcome this. Then, we get the MLEs for all τ percentiles, and according to the minimum GAV, we pick the optimal τ as mentioned in section 3.4. We applied this method using an actual data set using three distributions. We can determine the optimal change time stress using STH, PL, and Weibull distribution for this data set. All distributions provide a good fit of the data, but the STH distribution has a better fit. A further investigation of other optimal plan criteria and comparing this optimal plan is targeted for future work. As a project goal, we can extend from one stress to k stresses for SSPALT - PHC. We could also investigate the sensitivity analysis performed to examine the effect of the preliminary estimates of parameters on the optimal values of change time stress τ . It will provide information about the robustness of the optimal design. This will also identify the sensitive parameters, which need to be estimated with special care to reduce estimation error.

Chapter 4

Competing Risks

4.1 Introduction

The Weibull distribution is one of the most commonly used life-time distributions in reliability and lifetime data analysis. It is flexible for modeling failure time data, as the corresponding hazard rate function can be increasing, constant or decreasing. But in many applications in reliability and survival analysis, an appropriate hazard rate function should be of bathtub shape. The hazard rate function plays a central role to the work of reliability, see Bebbington et al. (2007b), Bebbington et al. (2007a) and the references therein. Models with a bathtub hazard rate function are often needed in reliability analysis and decision-making when the system's lifetime is to be modeled. Many parametric probability distributions have been introduced to analyze real data sets with bathtub shaped hazard functions. The bathtub shaped hazard function provides an appropriate conceptual model for some electronic and mechanical products as well as the lifetime of humans Sarhan and Apaloo (2013b). An extensive amount of work has been done related to bathtub shaped hazard functions, see for example, Sarhan et al. (2014a), Sarhan and Mustafa (2022a), Sarhan and Zaindin (2009a), Sarhan (2009a), Sarhan and Kundu (2009), Sarhan et al. (2013a), Smith and Bain (1975), Leemis (1986), Gaver and Acar (1979), Hjorth (1980), Mudholkar and Srivastava (1993a), and Lemonte (2013). Xie et al. (2002) proposed a new modified extension of the Weibull distribution with a bathtub-shaped hazard rate function. We refer to this extension as the Modified Weibull Extension (MWE) distribution. The probability density function (pdf), the survival function (sf), and the hazard rate function (hrf) of the $MWE(\lambda, \alpha, \gamma)$ distribution are, respectively:

$$f_{MWE}(x; \lambda, \alpha, \gamma) = \lambda\gamma \left(\frac{x}{\alpha}\right)^{\gamma-1} \exp\left\{\alpha\lambda \left(1 - \exp\left\{\left(\frac{x}{\alpha}\right)^\gamma\right\}\right) + \left(\frac{x}{\alpha}\right)^\gamma\right\}, \quad (4.1)$$

$$S_{MWE}(x; \lambda, \alpha, \gamma) = \exp\left\{\lambda\alpha \left[1 - e^{\left(\frac{x}{\alpha}\right)^\gamma}\right]\right\}, \text{ and} \quad (4.2)$$

$$h_{MWE}(x; \lambda, \alpha, \gamma) = \lambda\gamma \left(\frac{x}{\alpha}\right)^{\gamma-1} \exp\left\{\left(\frac{x}{\alpha}\right)^\gamma\right\}, \quad (4.3)$$

where $x \geq 0$ and $\alpha, \gamma, \lambda > 0$. Here α and λ are scale parameters while γ is a shape parameter. The Chen distribution Chen (2000) with parameters λ and γ is a special case of the MWE with $\alpha = 1$.

In many practical situations, multivariate lifetime data arise, and it is important to consider different distributions that could be used to model such data. Sarhan and Balakrishnan (2007), Sarhan et al. (2009), Sarhan et al. (2011), Kundu et al. (2012) and Sarhan et al. (2022) used independent shock models to propose bivariate/multivariate lifetime distributions. These studies used the bivariate distributions mainly to fit bivariate data sets. Such distributions can also be used to analyze dependent competing risks data. Recently, Sarhan et al. (2022) used the same idea with Chen distribution to introduce a new bivariate distribution, abbreviated as BCD, to fit bivariate data and dependent competing risks data sets.

In this chapter, we will first use the independent shock model concept to introduce a new bivariate distribution using the MWE distribution, named as the BMWWE distribution. The BMWWE distribution generalizes the BCD of Sarhan et al. (2022). We study the properties of the BMWWE distribution. We discuss how to use this new proposed distribution as a dependent competing risks model. The maximum likelihood method is used to estimate the unknown parameters of the BMWWE distribution using both bivariate data and competing risks data.

The rest of the chapter is organized as follows. In Section 4.2, we introduce the BMWWE distribution and discuss some of its basic properties. In Section 4.3, we discuss the dependent competing risks model. The maximum likelihood method using bivariate data and dependent competing risks data is discussed in Section 4.4. Data generation and some simulation results are presented in Section 4.5. In Section 4.6, we provide the analysis of two real data sets using the new proposed distribution and compare it with the BCD. The chapter is concluded in Section 4.7.

4.2 The BMWWE distribution

The BMWWE distribution can be formulated as follows. Let U_1, U_2 , and U_3 be independent random variables such that $U_j \sim MWE(\lambda_j, \alpha, \gamma)$, $j = 1, 2, 3$. Let $X_1 = \min(U_1, U_3)$ and $X_2 = \min(U_2, U_3)$. The distribution of the random vector (X_1, X_2) follows the bivariate modified Weibull extension distribution, and it will be denoted by $BMWWE(\lambda_1, \lambda_2, \lambda_3, \alpha, \gamma)$.

This distribution can be interpreted in a reliability context as follows. Consider a reliability system consisting of two units. The system units are subject to three independent fatal sources of shocks. A shock from the first source destroys unit 1, a shock from the second source destroys unit 2, while a shock from the third source destroys both units. Let X_1 and X_2 be the lifetime of the system units 1 and 2, respectively. Then, $X_1 = \min(U_1, U_3)$ and $X_2 = \min(U_2, U_3)$. Under the assumption that time at which the shocks occur (U_1 , U_2 and U_3) follow MWE distributions, as described above, the joint distribution of (X_1, X_2) will follow the BMWE distribution. We can easily show that the marginal distributions of X_1 and X_2 follow $MWE(\lambda_1 + \lambda_3, \alpha, \gamma)$ and $MWE(\lambda_2 + \lambda_3, \alpha, \gamma)$, respectively.

Note that due to the fact that $P(X_1 = X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}$, while the Lebesgue measure of the set $A = (x_1, x_2) | x_1 = x_2 > 0$ is zero, the BMWE distribution will have both absolutely continuous and singular parts. This means that the BMWE distribution is continuous but not absolutely continuous with respect to ordinary Lebesgue measure on $(0, \infty) \times (0, \infty)$.

The following theorem gives the joint survival and joint probability density functions of the BMWE distribution, and its marginal distributions.

Theorem 1 *Let (X_1, X_2) follow the BMWE($\lambda_1, \lambda_2, \lambda_3, \alpha, \gamma$), then*

1. *The joint survival function of (X_1, X_2) is*

$$S_{X_1, X_2}(x_1, x_2) = \begin{cases} \exp \left\{ \alpha (\lambda_1 + \lambda_3) \left[1 - e^{\left(\frac{x_1}{\alpha}\right)^\gamma} \right] + \alpha \lambda_2 \left[1 - e^{\left(\frac{x_2}{\alpha}\right)^\gamma} \right] \right\}, & x_1 > x_2 \\ \exp \left\{ \alpha \lambda_1 \left[1 - e^{\left(\frac{x_1}{\alpha}\right)^\gamma} \right] + \alpha (\lambda_2 + \lambda_3) \left[1 - e^{\left(\frac{x_2}{\alpha}\right)^\gamma} \right] \right\}, & x_1 < x_2 \\ \exp \left\{ \alpha \left[1 - e^{\left(\frac{x}{\alpha}\right)^\gamma} \right] (\lambda_1 + \lambda_2 + \lambda_3) \right\}, & x_1 = x_2 = x \end{cases} \quad (4.4)$$

2. *The joint probability density function of (X_1, X_2) is*

$$f_{X_1, X_2}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & \text{if } x_1 > x_2 > 0, \\ f_2(x_1, x_2) & \text{if } x_2 > x_1 > 0, \\ f_3(x) & \text{if } x_1 = x_2 = x > 0, \end{cases} \quad (4.5)$$

where

$$\begin{aligned} f_1(x_1, x_2) &= f_{MWE}(x_1; \lambda_1 + \lambda_3, \alpha, \gamma) f_{MWE}(x_2; \lambda_2, \alpha, \gamma), \\ f_2(x_1, x_2) &= f_{MWE}(x_1; \lambda_1, \alpha, \gamma) f_{MWE}(x_2; \lambda_2 + \lambda_3, \alpha, \gamma), \\ f_3(x) &= \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} f_{MWE}(x; \lambda_1 + \lambda_2 + \lambda_3, \alpha, \gamma). \end{aligned} \quad (4.6)$$

Note: $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ are the continuous parts and $f_3(x)$ is the singular part of the joint pdf of the BMWE distribution.

3. The marginal distributions are MWE. That is, $X_j \sim MWE(\lambda_j + \lambda_3, \alpha, \gamma)$, $j = 1, 2$.

Proof:

1. Using the definition of the joint survival function of (X_1, X_2) , $S_{X_1, X_2}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$, we get

$$\begin{aligned} S_{X_1, X_2}(x_1, x_2) &= P(U_1, x_1, U_3 > x_1, U_2 > x_2, U_3 > x_2) \\ &= (U_1 > x_1) P(U_2 > x_2) P(U_3 > \max(x_1, x_2)) \\ &= \prod_{j=1}^3 S_{MWE}(x_j; \lambda_j, \alpha, \gamma), \quad x_3 = \max(x_1, x_2). \end{aligned}$$

Substituting from (4.1) into the above equation, we get $S_{X_1, X_2}(x_1, x_2)$ as given in (4.4).

2. The absolutely continuous parts of the joint pdf of (X_1, X_2) , $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ can be easily obtained by using the relationship between the joint pdf and joint sf given by $f_{X_1, X_2}(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} S_{X_1, X_2}(x_1, x_2)$. The singular part $f_3(x)$ can be obtained by using the following well know identity:

$$1 = \int_0^\infty f_3(x) dx + \iint_{x_1 > x_2} f_{X_1, X_2}(x_1, x_2) dx_1 dx_2 + \iint_{x_1 < x_2} f_{X_1, X_2}(x_1, x_2) dx_1 dx_2.$$

3. It is straightforward using the definition of $X_j = \min(U_j, U_3)$, $j = 1, 2$.

Special Case: The bivariate Chen distribution (BCD), that was proposed in Sarhan et al. (2022), can be derived from the BMWE distribution as a special case from the BMWE distribution when $\alpha = 1$.

Figure 4.1 depicts the continuous components of the joint probability density function (jpdf) for (X_1, X_2) , along with corresponding contour plots showcasing various parameter values of the model. Simultaneously, figures 4.2 illustrate the singular portions of the jpdf, while 4.3 showcases the probability density function (pdf) and hazard rate function (hrf) of the marginal distribution for X_1 , utilizing three distinct sets of parameter values to highlight diverse hazard function shapes. These figures were generated using Matlab. These figures reveal that the distribution's support fluctuates in response to the model parameters' values and the profiles of the probability density function (pdf) and hazard rate function (hrf)..

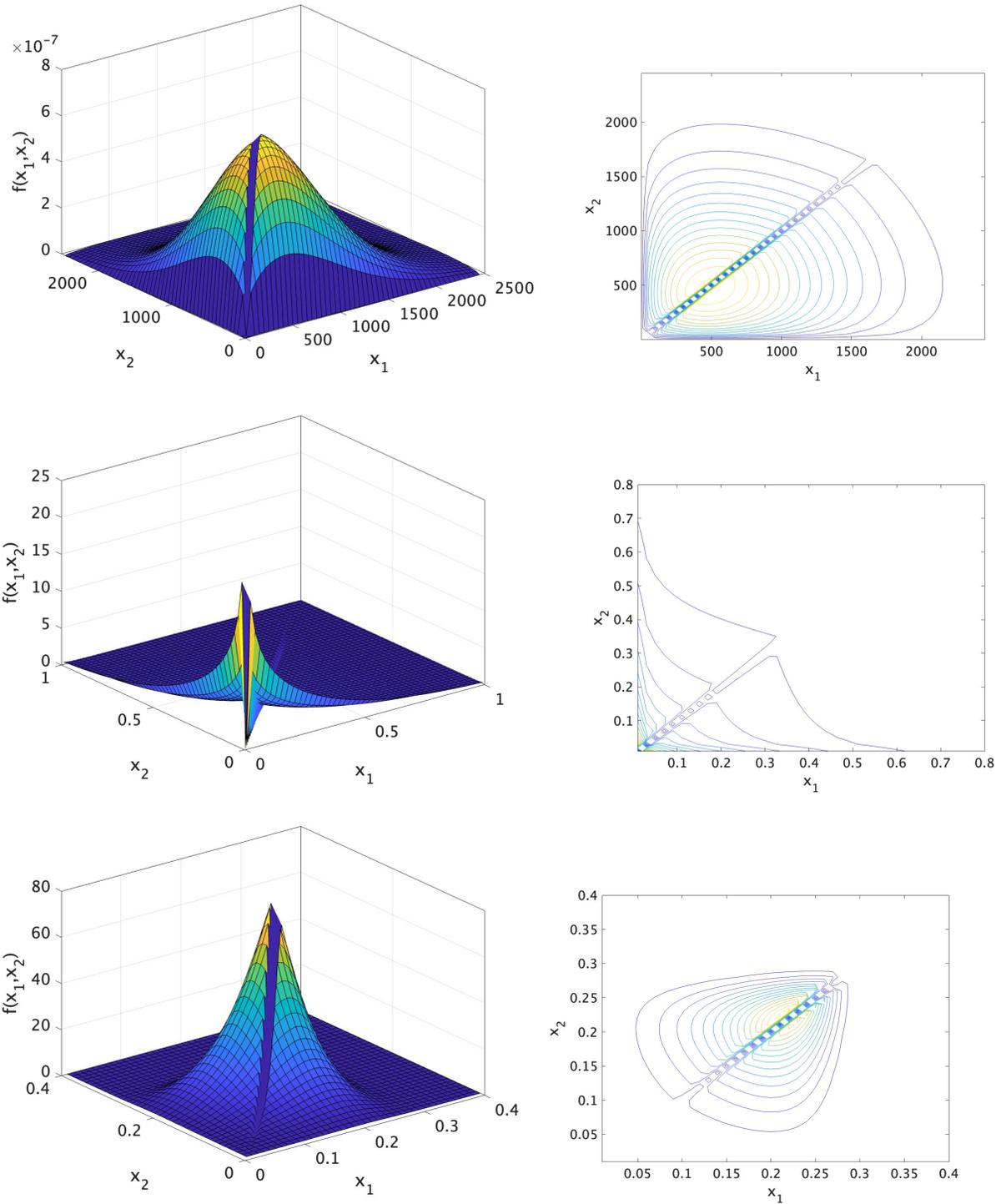


Figure 4.1: The absolutely continuous parts of the joint pdf and the corresponding contour plots of the BMWE distribution at different values for the model parameters: (1) Set 1: $\lambda_1 = 0.0031, \lambda_2 = 0.0036, \lambda_3 = 0.0011, \alpha = 9905.2, \gamma = 1.547$; (2) Set 2: $\lambda_1 = 1.2, \lambda_2 = 1.0, \lambda_3 = 1.7, \alpha = 0.2, \text{ and } \gamma = 0.5$; (2) Set 3: $\lambda_1 = 1.2, \lambda_2 = 1.0, \lambda_3 = 1.7, \alpha = 0.2, \text{ and } \gamma = 2.5$ (from top to bottom).

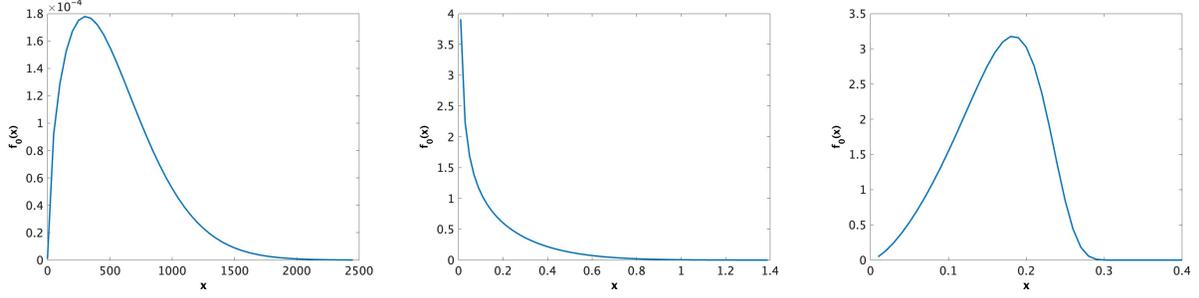


Figure 4.2: The singular part of the BMWE($\lambda_1, \lambda_2, \lambda_3, \alpha, \gamma$) when: (1) Set 1: $\lambda_1 = 0.0031, \lambda_2 = 0.0036, \lambda_3 = 0.0011, \alpha = 9905.2, \gamma = 1.547$; (2) Set 2: $\lambda_1 = 1.2, \lambda_2 = 1.0, \lambda_3 = 1.7, \alpha = 0.2$, and $\gamma = 0.5$; (2) Set 3: $\lambda_1 = 1.2, \lambda_2 = 1.0, \lambda_3 = 1.7, \alpha = 0.2$, and $\gamma = 2.5$.

Substituting (4.1) into (4.6) we can get the explicit forms of f_1, f_2 , and f_0 as follows:

$$f_1(x_1, x_2) = (\lambda_1 + \lambda_3) \gamma \left(\frac{x_1}{\alpha}\right)^{\gamma-1} \exp\left\{(\lambda_1 + \lambda_3) \alpha \left[1 - e^{\left(\frac{x_1}{\alpha}\right)^\gamma}\right] + \left(\frac{x_1}{\alpha}\right)^\gamma\right\} \quad (4.7)$$

$$\times \lambda_2 \gamma \left(\frac{x_2}{\alpha}\right)^{\gamma-1} \exp\left\{\lambda_2 \alpha \left[1 - e^{\left(\frac{x_2}{\alpha}\right)^\gamma}\right] + \left(\frac{x_2}{\alpha}\right)^\gamma\right\},$$

$$f_2(x_1, x_2) = (\lambda_2 + \lambda_3) \gamma \left(\frac{x_2}{\alpha}\right)^{\gamma-1} \exp\left\{(\lambda_2 + \lambda_3) \alpha \left[1 - e^{\left(\frac{x_2}{\alpha}\right)^\gamma}\right] + \left(\frac{x_2}{\alpha}\right)^\gamma\right\} \quad (4.8)$$

$$\times \lambda_1 \gamma \left(\frac{x_1}{\alpha}\right)^{\gamma-1} \exp\left\{\lambda_1 \alpha \left[1 - e^{\left(\frac{x_1}{\alpha}\right)^\gamma}\right] + \left(\frac{x_1}{\alpha}\right)^\gamma\right\},$$

and

$$f_3(x) = \frac{\lambda_3}{\sum_{i=1}^3 \lambda_i} \sum_{i=1}^3 \lambda_i \gamma \left(\frac{x}{\alpha}\right)^{\gamma-1} \exp\left\{\left(\sum_{i=1}^3 \lambda_i\right) \alpha \left(1 - e^{\left(\frac{x}{\alpha}\right)^\gamma}\right) + \left(\frac{x}{\alpha}\right)^\gamma\right\} \quad (4.9)$$

$$= \lambda_3 \gamma \left(\frac{x}{\alpha}\right)^{\gamma-1} \exp\left\{\left(\sum_{i=1}^3 \lambda_i\right) \alpha \left(1 - e^{\left(\frac{x}{\alpha}\right)^\gamma}\right) + \left(\frac{x}{\alpha}\right)^\gamma\right\}.$$

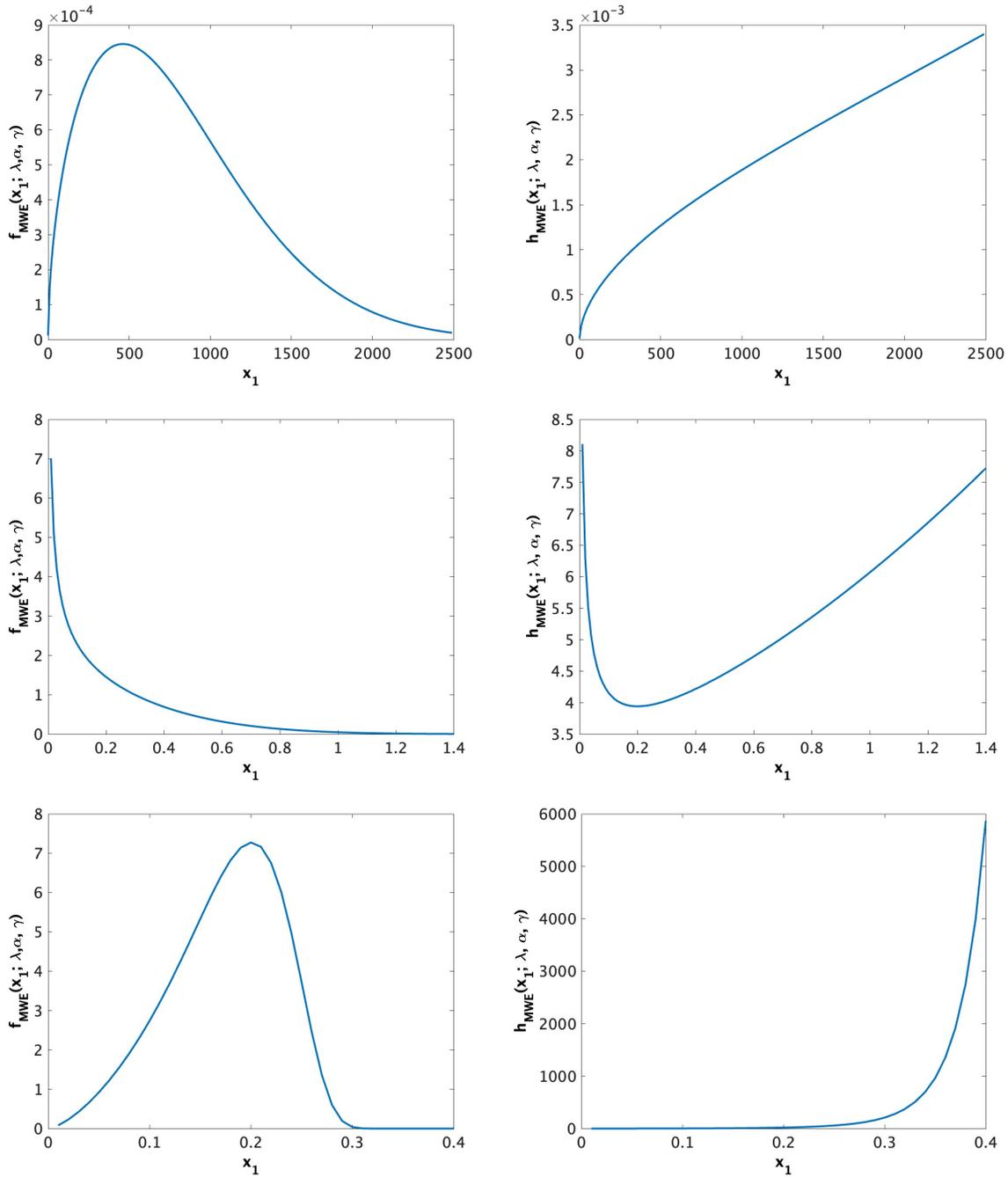


Figure 4.3: The pdf and the hrf of the marginal distribution of X_1 when: (1) Set 1: $\lambda_1 = 0.0031, \lambda_2 = 0.0036, \lambda_3 = 0.0011, \alpha = 9905.2, \gamma = 1.547$; (2) Set 2: $\lambda_1 = 1.2, \lambda_2 = 1.0, \lambda_3 = 1.7, \alpha = 0.2$, and $\gamma = 0.5$; (2) Set 3: $\lambda_1 = 1.2, \lambda_2 = 1.0, \lambda_3 = 1.7, \alpha = 0.2$, and $\gamma = 2.5$.

4.3 Dependent Competing Risks Model

In survival and reliability analysis one is often interested in the assessment of one risk in the presence of other risk factors. In statistics, this is known as the competing risks problem. The competing risks model assumes that the data consists of the failure time of the underlying object and the associated cause of failure. An extensive amount of study has been carried out under both parametric and non-parametric assumptions, see for example the monograph by Crowder (2001) in this respect. In this chapter, we adopt the parametric setup along with the latent failure time assumption. Furthermore, it is assumed here that there are two possible causes of death (competing risks), although the method can be easily generalized to any fixed number of competing risks.

It is assumed in the competing risks model that the underlying experimental object (system, or a human, or an animal) is under fatal attack from two competing risks. The object will be destroyed if it receives at least one attack from the two risks. The attack from risk j ($j = 1, 2$) occurs at time X_j . When the object fails, we observe two quantities (T, δ) , where T represents the lifetime of the object, and δ is an indicator that describes the cause of failure. That is, $T = \min \{X_1, X_2\}$, and

$$\delta = \begin{cases} 1 & \text{if risk 1 causes the failure,} \\ 2 & \text{if risk 2 causes the failure,} \\ 3 & \text{if both risks cause the failure.} \end{cases}$$

Crowder (2001) includes an extensive literature dealing with the analysis of competing risks data based on the specific continuous parametric distribution assumption on X_1 and X_2 , assuming X_1 and X_2 to be independent random variables. In many applications, the independence assumption will not hold. Therefore, competing risks models with dependent causes of failure are needed in reliability/survival analysis.

Recently, to analyze the dependent competing risks data, some work adopted the assumption that (X_1, X_2) has a specific bivariate distribution, see for example Feizjavadian and Hashemi (2015), Shen and Xu (2018), Samanta and Kundu (2021) and Sarhan et al. (2022). In this chapter, we assume that the joint distribution of (X_1, X_2) is the BMWE distribution. This assumption produces a dependent competing risks model with risks that follow modified Weibull distributions. Under this assumption, the risks will have either increasing, decreasing or bathtub shaped hazard rate functions. It follows that the BMWE

distribution can be applied to the analysis of a variety different types of lifetime data of dependent competing risks type.

4.4 Maximum Likelihood Estimation

In this section, we estimate the model parameters $\theta = (\lambda_1, \lambda_2, \lambda_3, \alpha, \gamma)$ using two different types of data. Type I Data consists of simple random samples of the bivariate vector (X_1, X_2) that follows the new bivariate distribution. Type II Data consists of dependent competing risks data (T, δ) , with $T = \min\{X_1, X_2\}$, where (X_1, X_2) follows a bivariate lifetime distribution and δ represents the cause of failure. The maximum likelihood method is used to estimate model parameters.

Type I Data (Bivariate data): Let us assume that $(X_{11}, X_{21}), (X_{12}, X_{22}), \dots, (X_{1n}, X_{2n})$ be an independent and identical random sample of (X_1, X_2) that follows the BMWWE($\lambda_1, \lambda_2, \lambda_3, \alpha, \gamma$). For simplicity, let us introduce an indicator variable $\delta_i, i = 1, 2, \dots, n$, where:

$$\delta_i = \begin{cases} 3 & \text{if } X_{i1} = X_{i2}, \\ 1 & \text{if } X_{i1} < X_{i2}, \\ 2 & \text{if } X_{i1} > X_{i2}. \end{cases}$$

Using the above sample, the likelihood function can be expressed as

$$L_1(\theta) = \prod_{i=1}^n [f_2(x_{1i}, x_{2i})]^{I[\delta_i=1]} [f_1(x_{1i}, x_{2i})]^{I[\delta_i=2]} [f_3(x_{1i})]^{I[\delta_i=3]}. \quad (4.10)$$

where $f_1(x_{1i}, x_{2i})$, $f_2(x_{1i}, x_{2i})$, and $f_3(x_{1i})$ are given in (4.7), (4.8), and (4.9). Substituting from (4.5) into (4.10), we get the log-likelihood function as

$$\begin{aligned} \mathcal{L}_1(\theta) &= -2(n_1 + n_2)(\gamma - 1) \log \alpha + (2n - n_3) \log \gamma + \sum_{j=1}^2 n_j \log(\lambda_j + \lambda_3) + \\ &(\gamma - 1) \sum_{i=1}^n \log \left(\frac{x_{1i} x_{2i}}{x_{2i}^{I(\delta_i=3)}} \right) + \frac{1}{\alpha^\gamma} \sum_{i=1}^n [x_{1i}^\gamma + x_{2i}^\gamma - I(\delta_i = 3)x_{2i}^\gamma] + \\ &\alpha \sum_{j=1}^2 \lambda_j \sum_{i=1}^n (1 - e^{(x_{ji}/\alpha)^\gamma}) + \alpha \lambda_3 \sum_{i=1}^n I(\delta_i = j) (1 - e^{(x_{ji}/\alpha)^\gamma}) + \\ &\alpha \lambda_3 \sum_{j=1}^2 \sum_{i=1}^n I(\delta_i = j) (1 - e^{(x_{ji}/\alpha)^\gamma}) + \sum_{j=1}^3 n_j \log \lambda_j, \end{aligned} \quad (4.11)$$

where $n_j = \sum_{i=1}^n I(\delta_i = j)$, $j = 0, 1, 2$, with $I(A) = 1$ if A is true and 0 otherwise.

In order to get the MLE of the model parameters, we need to maximize the log-likelihood function of the parameters, for the given data, with respect to those unknown parameters. Equivalently, we need to solve the likelihood equations derived by setting the first partial derivatives of the log-likelihood function with respect to the unknown parameters to zero. The MLE's are the solution of obtained likelihood equations, at which the Fisher information matrix should be positive definite. The Fisher information matrix consists of the second partial derivatives of the log-likelihood function with respect to the parameters. For the distribution in question, the likelihood equations cannot be solved analytically, and we have used numerical routines in R software to solve the system of five non-linear equations in five unknowns. The likelihood equations and the second partial derivatives of f_1 , f_2 , and f_0 are given in Appendix C..

Type II Data (Dependent Competing Risks Data): We observe a pair of quantities: T , the system time to failure, and δ an indicator of the cause of failure. This means that the observations in this case are (T_i, δ_i) , $i = 1, 2, \dots, n$. The lifetime experiment that produces this type of data can be illustrated as follows: (1) we put n independent and identical devices (system/objects) on the life test, (2) each system is under the attack from two dependent competing risks which occur at times X_1 and X_2 , the system will fail once it receives one of the two attacks, (3) (X_1, X_2) follows the BMW distribution; (4) we observe (T_i, δ_i) , $i = 1, 2, \dots, n$, where $T_i = \min(X_{1i}, X_{2i})$, and δ_i is defined as $\delta_i = 1$, if risk 1 causes the failure ($X_{1i} < X_{2i}$); $\delta_i = 2$, if risk 2 causes the failure ($X_{2i} < X_{1i}$) and $\delta_i = 3$, if the cause of failure is due to both causes ($X_{1i} = X_{2i}$). The likelihood function using such dependent competing risks data is

$$L_2(\theta) = \prod_{i=1}^n [f_2^*(t_i)]^{I[\delta_i=1]} [f_1^*(t_i)]^{I[\delta_i=2]} [f_3^*(t_i)]^{I[\delta_i=3]}, \quad (4.12)$$

where

$$\begin{aligned} f_j^*(t) &= h_{MWE}(t; \lambda_j, \alpha, \gamma) \cdot S_{MWE}(t; \lambda_1 + \lambda_2 + \lambda_3, \alpha, \gamma) \\ &= \lambda_j \gamma \left(\frac{t}{\alpha}\right)^{\gamma-1} e^{-\left(\frac{t}{\alpha}\right)^\gamma} e^{-(\lambda_1 + \lambda_2 + \lambda_3)\alpha \left(1 - e^{-\left(\frac{t}{\alpha}\right)^\gamma}\right)}, \quad j = 1, 2, 3; \end{aligned} \quad (4.13)$$

See Appendix C for more details. Substituting (4.13) into (4.12), we can get logarithm of the likelihood function $L_2(\theta)$, say $\mathcal{L}_2(\theta)$, as

$$\begin{aligned} \mathcal{L}_2(\theta) = & n_3 \log \lambda_3 + n_1 \log \lambda_1 + n_2 \log \lambda_2 + n \log \gamma + (\gamma - 1) \sum_{i=1}^n \log t_i + \\ & -n(\gamma - 1) \log \alpha + (\lambda_1 + \lambda_2 + \lambda_3) \alpha \sum_{i=1}^n \left(1 - e^{\left(\frac{t_i}{\alpha}\right)^\gamma}\right) + \sum_{i=1}^n \left(\frac{t_i}{\alpha}\right)^\gamma. \end{aligned} \quad (4.14)$$

To get the MLE of the model parameters, using dependent competing risks data, we set the first partial derivatives of \mathcal{L}_2 with respect to the five parameters $\lambda_1, \lambda_2, \lambda_3, \alpha, \gamma$ equal to zero, we get a system of five non-linear equations in five unknowns. Solving the first three equations, in $\lambda_j, j = 1, 2, 3$, we get λ_j as a function of (α, γ) , as

$$\lambda_j(\alpha, \gamma) = \frac{-n_j}{\alpha \sum_{i=1}^n \left(1 - \exp\left\{\left(\frac{t_i}{\alpha}\right)^\gamma\right\}\right)}, \quad j = 1, 2, 3. \quad (4.15)$$

Substituting (4.15) into (4.14), we can express the log-likelihood function as a function of two parameters (α, γ) , say $\mathcal{L}_2^*(\alpha, \gamma)$:

$$\begin{aligned} \mathcal{L}_2^*(\alpha, \gamma) = & \sum_{j=1}^3 n_j \log \frac{-n_j}{\alpha \sum_{i=1}^n \left(1 - e^{\left(\frac{t_i}{\alpha}\right)^\gamma}\right)} + n \log \gamma \\ & + (\gamma - 1) \sum_{i=1}^n \log t_i - n(\gamma - 1) \log \alpha - n + \sum_{i=1}^n \left(\frac{t_i}{\alpha}\right)^\gamma. \end{aligned} \quad (4.16)$$

This will make the optimization of the log-likelihood function much easier, since we will deal only with two parameters instead of five. Once we get the MLE of α and γ , we can use (4.15) to get the MLE of $\lambda_j, j = 1, 2, 3$.

Since the MLE of the parameters using either type of data are not obtained in closed form, we can not get the explicit sampling distributions of the MLEs of these parameters. Therefore, we cannot obtain exact confidence intervals for the model parameters. Alternatively, we can use the large sample distribution of the MLE and the corresponding observed Fisher information matrix to derive approximate confidence intervals for the model parameters.

4.5 Data generation and simulation results

In this section, we present how to generate random samples from the proposed BMW model. We then show how to use the bivariate sample to generate dependent competing risks data from the underlying distribution.

4.5.1 Random sample generation from the BMWWE distribution

The following algorithm can be applied to generate random samples of bivariate variables (X_1, X_2) from the BMWWE($\lambda_1, \lambda_2, \lambda_3, \alpha, \gamma$), "Type I Data":

Algorithm 1:

1. Specify the sample size n .
2. Specify the model parameters: $\lambda_1, \lambda_2, \lambda_3, \alpha, \gamma$.
3. Generate three independent random values v_1, v_2, v_3 from $(0, 1)$ uniform distribution.
4. Compute the following three random values that follow MWE($\lambda_i, \alpha, \lambda$):

$$u_i = \alpha \left(\log \left\{ 1 - \frac{\log(v_i)}{\lambda \alpha} \right\} \right)^{\frac{1}{\gamma}} \quad i = 1, 2, 3.$$

5. Compute $x_{1i} = \min(u_1, u_3)$ and $x_{2i} = \min(u_2, u_3)$.
6. Repeat steps 3-5 n -times, we obtain a simple random sample $(x_{1i}, x_{2i}), i = 1, 2, \dots, n$, for (X_1, X_2) that follows BMWWE($\lambda_1, \lambda_2, \lambda_3, \alpha, \gamma$).

Given a random sample of the bivariate variables (X_1, X_2) , say $(x_{11}, x_{21}), (x_{12}, x_{22}), \dots, (x_{1n}, x_{2n})$ from BMWWE distribution, we can generate a dependent competing risks data set, "Type II Data", with two risks, by applying the following algorithm:

Algorithm 2:

1. Compute the time to failure of the i^{th} object as $t_i = \min\{x_{1i}, x_{2i}\}, i = 1, 2, \dots, n$,
2. Determine the cause of the i^{th} failure, δ_i , as

$$\delta_i = \begin{cases} 1 & \text{if } x_{i1} < x_{i2} \text{ (if risk 1 causes the failure),} \\ 2 & \text{if } x_{i1} > x_{i2} \text{ (if risk 2 causes the failure),} \\ 3 & \text{if } x_{i1} = x_{i2} \text{ (if both risks cause the failure).} \end{cases}$$

We used algorithms 1 and 2, to generate a random sample of 100 dependent competing risk observations from the BMWWE distribution when $\lambda_1 = 0.0031, \lambda_2 = 0.0036, \lambda_3 = 0.0011, \alpha = 9905.2, \gamma = 1.547$, as shown in Table 4.1. The reason for using these values of the model

parameters is that we get similar values when we analyze a real data set, as will be discussed later in Section 6.

Using the generated sample, we computed the MLE of all model parameters, the corresponding standard error (SE), the 95% CI (lower and upper limits) and the percentage error (PE) as given in Table 4.2. Based on these results, we conclude that the random number generation algorithms and the estimation process used in this chapter work very well.

We used R software to do the simulation and all the computations for this example, the simulations in Section 5 and data analysis in Section 6.

<i>i</i>	1	2	3	4	5	6	7	8	9	10
<i>t_i</i>	219.1	297.5	456.1	331.0	594.8	472.3	904.2	226.4	1120.4	342.4
<i>δ_i</i>	2	1	1	2	1	1	3	1	2	1
<i>i</i>	11	12	13	14	15	16	17	18	19	20
<i>t_i</i>	69.6	1668.3	607.5	1048.1	240.5	1453.2	1690.1	710.5	438.3	931.9
<i>δ_i</i>	1	2	3	1	3	1	2	2	3	2
<i>i</i>	21	22	23	24	25	26	27	28	29	30
<i>t_i</i>	582.3	217.5	900.5	382.1	117.5	882.2	215.6	223.3	648.7	669.3
<i>δ_i</i>	2	3	1	1	1	1	1	1	2	2
<i>i</i>	31	32	33	34	35	36	37	38	39	40
<i>t_i</i>	44.8	482.9	699.8	769.3	438.3	1124.3	202.4	539.8	292.7	298.5
<i>δ_i</i>	1	2	2	3	1	1	2	2	2	2
<i>i</i>	41	42	43	44	45	46	47	48	49	50
<i>t_i</i>	457.2	312.8	1131.4	891.8	219.1	343.3	1474.1	767.5	683.1	749.1
<i>δ_i</i>	3	2	2	1	2	1	3	2	1	1
<i>i</i>	51	52	53	54	55	56	57	58	59	60
<i>t_i</i>	228.2	107.9	564.8	225.7	427.1	735.9	235.1	764.5	550.6	883.6
<i>δ_i</i>	1	1	3	2	2	3	3	2	2	1
<i>i</i>	61	62	63	64	65	66	67	68	69	70
<i>t_i</i>	288.1	667.3	225.5	58.5	870.9	729.2	272.5	1398.9	943.1	98.5
<i>δ_i</i>	3	2	2	1	3	2	1	2	1	3
<i>i</i>	71	72	73	74	75	76	77	78	79	80
<i>t_i</i>	536.8	987.3	649.3	183.2	1034.5	953.9	871.3	804.2	384.9	114.4
<i>δ_i</i>	2	2	2	1	1	2	1	1	1	1
<i>i</i>	81	82	83	84	85	86	87	88	89	90
<i>t_i</i>	250.2	297.5	175.4	523.2	530.4	113.9	664.9	180.5	918.7	291.5
<i>δ_i</i>	2	2	2	2	2	2	1	2	1	1
<i>i</i>	91	92	93	94	95	96	97	98	99	100
<i>t_i</i>	708.4	473.7	345.6	438.1	278.2	121.5	172.4	765.1	156.3	372.7
<i>δ_i</i>	2	2	1	1	3	2	1	2	2	2

Table 4.1: Dependent competing risks data set generated from the BMW distribution.

Parameter	λ_1	λ_2	λ_3	α	γ
Actual Values	0.0031	0.0036	0.0011	9905.2	1.547
Estimation	0.003139	0.003532	0.001178	10891.5	1.5514
PE	%1.26	%1.87	%7.06	%9.95	%0.28

Table 4.2: Parameters estimation using the generated sample in Table 4.1.

4.5.2 Simulation Study

In this section we present a simulation study to show how the proposed methods work for different sample sizes and different parameter values. In this simulation study we used the two algorithms 1 and 2 presented in the previous section to generate bivariate and dependent competing risks samples with different sample sizes $n = 100, 150, 200, 250, 1000$ using two different sets of the model parameter values. For every sample, we perform the following: (1) generate 2000 samples; (2) for each sample, we find the MLE, its squared deviation from the actual value, and a 95% confidence interval for each parameter are computed; (3) using the 2000 MLEs and 95% CIs, we computed the average value of the MLEs (AMLE); the mean squared error (MSE), and the coverage probability (CP) for each parameter:

$$AMLE_{\theta} = \frac{\sum_{\ell=1}^{2000} \hat{\theta}^{(\ell)}}{2000} , \quad MSE_{\theta} = \frac{\sum_{\ell=1}^{2000} (\theta - \hat{\theta}^{(\ell)})^2}{2000} , \quad CP_{\theta} = \frac{\sum_{\ell=1}^{2000} I(\hat{\theta}^{(\ell)} \in CI)}{2000} .$$

where θ is the actual value of the parameter; $\hat{\theta}^{(\ell)}$ is the MLE of θ using sample number ℓ ; CI is the confidence interval using sample number ℓ . This strategy for the simulation is followed for the bivariate data sets and for the dependent competing risks data sets. Tables 4.3 and 4.4 show the results for bivariate data and depending competing risks data, respectively. In this study, three sets of parameter values were used. In set 1, it is assumed that $\lambda_1 = 0.3, \lambda_2 = 0.4, \lambda_3 = 0.7, \alpha = 0.2, \gamma = 2.5$; in set 2, it is assumed that $\lambda_1 = 1.2, \lambda_2 = 1, \lambda_3 = 1.7, \alpha = 0.2, \gamma = 2.5$; and in set 3, it is assumed that $\lambda_1 = 1.2, \lambda_2 = 1, \lambda_3 = 1.7, \alpha = 0.2, \gamma = 0.5$.

θ	n	Set 1			Set 2			Set 3		
		$AMLE_{\theta}$	MSE_{θ}	CP_{θ}	$AMLE_{\theta}$	MSE_{θ}	CP_{θ}	$AMLE_{\theta}$	MSE_{θ}	CP_{θ}
λ_1	100	0.43	0.1455	0.9165	1.51	1.3780	0.9130	0.36	0.0356	0.9155
λ_2		0.56	0.2249	0.9145	1.26	1.0183	0.9130	0.48	0.0655	0.9110
λ_3		1.00	0.7142	0.9170	2.14	2.9819	0.9175	0.83	0.1714	0.9520
α		0.21	0.0018	0.9115	0.20	0.0017	0.9695	0.36	0.1784	0.9095
γ		2.68	0.4233	0.9125	2.55	0.2582	0.9285	0.54	0.0205	0.9080
λ_1	150	0.39	0.0802	0.9220	1.41	0.7389	0.9325	0.34	0.0185	0.9110
λ_2		0.53	0.1413	0.9270	1.19	0.5258	0.9285	0.45	0.0285	0.9150
λ_3		0.92	0.4420	0.9215	2.00	1.4055	0.9310	0.79	0.0860	0.9395
α		0.21	0.0014	0.9095	0.20	0.0011	0.9705	0.32	0.1047	0.9045
γ		2.64	0.3172	0.9140	2.54	0.1859	0.9275	0.53	0.0150	0.9065
λ_1	200	0.37	0.0500	0.9215	1.36	0.5095	0.9180	0.33	0.0144	0.9200
λ_2		0.49	0.0840	0.9235	1.13	0.3599	0.9170	0.45	0.0255	0.9160
λ_3		0.87	0.2710	0.9175	1.93	1.0448	0.9255	0.78	0.0740	0.9300
α		0.20	0.0012	0.9055	0.20	0.0010	0.9625	0.28	0.0593	0.9080
γ		2.61	0.2527	0.9050	2.52	0.1469	0.9330	0.52	0.0111	0.9035
λ_1	250	0.35	0.0418	0.9115	1.32	0.3559	0.9325	0.34	0.0131	0.9175
λ_2		0.47	0.0712	0.9145	1.10	0.2531	0.9335	0.45	0.0219	0.9255
λ_3		0.83	0.2226	0.9190	1.89	0.7613	0.9380	0.79	0.0627	0.9165
α		0.20	0.0010	0.9080	0.20	0.0008	0.9650	0.26	0.0480	0.9010
γ		2.57	0.2112	0.9060	2.52	0.1184	0.9455	0.51	0.0101	0.9040
λ_1	1000	0.32	0.0103	0.9075	1.22	0.0734	0.9540	0.31	0.0030	0.9460
λ_2		0.42	0.0181	0.9075	1.02	0.0532	0.9520	0.42	0.0050	0.9275
λ_3		0.74	0.0538	0.9080	1.74	0.1510	0.9535	0.73	0.0145	0.9190
α		0.20	0.0004	0.9025	0.20	0.0002	0.9565	0.21	0.0106	0.9140
γ		2.52	0.0755	0.9060	2.50	0.0317	0.9485	0.50	0.0032	0.9160

Table 4.3: Simulation results for the bivariate data from BMWED using the three sets of parameters' values. Set 1: $\lambda_1 = 0.3, \lambda_2 = 0.4, \lambda_3 = 0.7, \alpha = 0.2, \gamma = 2.5$; set 2: $\lambda_1 = 1.2, \lambda_2 = 1, \lambda_3 = 1.7, \alpha = 0.2, \gamma = 2.5$ and set 3: $\lambda_1 = 0.3, \lambda_2 = 0.4, \lambda_3 = 0.7, \alpha = 0.2, \gamma = 0.5$.

θ	n	Set 1			Set 2			Set 3		
		$AMLE_{\theta}$	MSE_{θ}	CP_{θ}	$AMLE_{\theta}$	MSE_{θ}	CP_{θ}	$AMLE_{\theta}$	MSE_{θ}	CP_{θ}
λ_1	100	0.45	0.1639	0.9215	1.74	4.5560	0.8960	0.37	0.0418	0.9170
λ_2		0.59	0.2785	0.9180	1.45	3.0209	0.9000	0.48	0.0623	0.9145
λ_3		1.03	0.7791	0.9255	2.49	11.094	0.9010	0.85	0.1898	0.9385
α		0.21	0.0022	0.9480	0.20	0.0028	0.9750	0.40	0.3428	0.9030
γ		2.68	0.4469	0.9195	2.54	0.2924	0.9365	0.53	0.0208	0.9000
λ_1	150	0.41	0.1198	0.9145	1.52	1.9245	0.9075	0.35	0.0280	0.9360
λ_2		0.55	0.2109	0.9135	1.28	1.2720	0.9075	0.47	0.0473	0.9230
λ_3		0.97	0.6338	0.9210	2.17	4.2002	0.9155	0.82	0.1369	0.9385
α		0.21	0.0018	0.9305	0.20	0.0018	0.9530	0.33	0.1156	0.9040
γ		2.63	0.3655	0.9055	2.53	0.1991	0.9345	0.52	0.0155	0.9060
λ_1	200	0.38	0.0773	0.9090	1.45	1.1631	0.9215	0.34	0.0201	0.9230
λ_2		0.51	0.1370	0.9025	1.22	0.8122	0.9205	0.46	0.0339	0.9250
λ_3		0.89	0.3892	0.9095	2.07	2.2416	0.9235	0.81	0.0985	0.9350
α		0.20	0.0015	0.9225	0.20	0.0015	0.9550	0.31	0.1034	0.9065
γ		2.58	0.2942	0.9140	2.52	0.1689	0.9340	0.52	0.0141	0.9015
λ_1	250	0.37	0.0637	0.9120	1.39	0.6535	0.9250	0.34	0.0126	0.9110
λ_2		0.49	0.1152	0.9140	1.16	0.4627	0.9250	0.46	0.0294	0.9290
λ_3		0.86	0.3510	0.9170	1.97	1.3191	0.9305	0.80	0.0850	0.9315
α		0.20	0.0013	0.9135	0.20	0.0011	0.9510	0.28	0.0721	0.9060
γ		2.57	0.2550	0.9040	2.52	0.1367	0.9425	0.51	0.0112	0.9040
λ_1	1000	0.32	0.0110	0.9300	1.23	0.1056	0.9450	0.31	0.0023	0.9460
λ_2		0.42	0.0192	0.9340	1.03	0.0774	0.9405	0.42	0.0039	0.9480
λ_3		0.74	0.0586	0.9320	1.75	0.2084	0.9450	0.73	0.0119	0.9190
α		0.20	0.0004	0.9210	0.20	0.0003	0.9595	0.22	0.0125	0.9325
γ		2.52	0.0822	0.9225	2.50	0.0364	0.9515	0.50	0.0032	0.9335

Table 4.4: Simulation results for the dependent competing risks data from BMWED using the three sets of parameters' values. Set 1: $\lambda_1 = 0.3, \lambda_2 = 0.4, \lambda_3 = 0.7, \alpha = 0.2, \gamma = 2.5$; set 2: $\lambda_1 = 1.2, \lambda_2 = 1, \lambda_3 = 1.7, \alpha = 0.2, \gamma = 2.5$ and set 3: $\lambda_1 = 0.3, \lambda_2 = 0.4, \lambda_3 = 0.7, \alpha = 0.2, \gamma = 0.5$.

Based on the simulation results, it is seen that in all cases, as the sample size increases, the AMLE of all parameters approach their actual values, the MSE decreases and the coverage probability approaches the nominal confidence level. This indicates consistency of the maximum likelihood estimate of the parameters.

4.6 Data Analysis

This section illustrates how the proposed BMW distribution can be applied to real data sets. We analyze two real data sets. The first is a bivariate data set and the second is a dependent competing risks data set.

4.6.1 Bivariate real data set

In this subsection we analyze the Union of European Football Associations (UEFA) data set. This data was introduced in Meintanis (2007). It consists of bivariate observations (X_{1i}, X_{2i}) , $i = 1, 2, \dots, 37$. Here X_1 measures the time in minutes of the first kick goal scored by any team, while X_2 is the time in minutes of the first goal of any type scored by the home team. For more information about this data, we refer to Meintanis (2007).

This data was analyzed by different researchers using different bivariate distributions. For example, Meintanis (2007) used the Marshall-Olkin bivariate exponential (MOBE), Kundu and Gupta (2009) used the bivariate generalized exponential (BVGE), Sarhan et al. (2011) used the bivariate generalized linear failure rate (BGLFR) distribution, and recently Sarhan et al. (2022) used the bivariate Chen distribution (BCD). In this chapter, we use the BMW distribution to fit this data set.

To compare the performance of different non-nested distributions (BVGE, BGLFR, BCD, and BMW) to fit the underlying data set, we use Akaike Information Criterion Akaike (1969), $AIC = -2\hat{\mathcal{L}} + 2k$, where k is the number of the model parameters. Table 4.5 shows the MLE, the value of the log-likelihood function and AIC for all the models stated above. The results for BVGE, BGLFR and BCD are taken from Sarhan et al. (2022).

Sarhan et al. (2011) showed that BVGE should be rejected versus the BGLFR distribution at any significance level greater than or equal to 0.05. Sarhan et al. (2022) showed that BCD fits the data better than BGLFR model. Since BCD is a special case from BMW, we can use the likelihood ratio test statistic Λ that follows a chi square distribution with one degree of freedom χ_1 , to test the following hypothesis:

$$H_0 : \text{the data follow BCD, } (\alpha = 1) \text{ vs } H_1 : \text{the data follow BMW, } (\alpha \neq 1).$$

Using the results in Table 4.5, we get $\Lambda = -2(\hat{\mathcal{L}}_{H_0} - \hat{\mathcal{L}}_{H_1}) = 4.08$, and the corresponding p-value is $P(\chi_1 \geq 4.08) = 0.04496$. Therefore, BCD is rejected to fit this data set, in favor of BMW distribution at any significance level of 0.04496 or larger.

Model	MLEs	$\hat{\mathcal{L}}$	AIC
BVGE	$\hat{\alpha}_1 = 1.351, \hat{\alpha}_2 = 0.465, \hat{\alpha}_3 = 1.153,$ $\hat{\beta} = 0.039$	-296.94	601.87
BGLFR	$\hat{\alpha}_1 = 0.492, \hat{\alpha}_2 = 0.166, \hat{\alpha}_3 = 0.411,$ $\hat{\beta} = 2.013 \times 10^{-4}, \hat{\gamma} = 8.051 \times 10^{-4}$	-293.38	596.76
BCD	$\hat{\alpha}_1 = 2.817 \times 10^{-3}, \hat{\alpha}_2 = 6.298 \times 10^{-3},$ $\hat{\alpha}_3 = 6.006 \times 10^{-3}, \hat{\beta} = 0.4035$	-288.23	584.47
BMWE	$\hat{\lambda}_1 = 0.0039, \hat{\lambda}_2 = 0.0086, \hat{\lambda}_3 = 0.0082,$ $\hat{\alpha} = 47.55, \hat{\gamma} = 1.192$	-286.19	582.38

Table 4.5: The MLEs of the models parameters, corresponding log-likelihood function values and the values of AIC.

4.6.2 Diabetic retinopathy data analysis

Diabetic retinopathy is a major eye condition that can cause vision loss and blindness in diabetic people. It affects blood vessels in the retina (the light-sensitive layer of tissue in the back of the eye). The National Eye Institute in Bethesda, Maryland conducted an experiment on 71 patients, to study the effect of laser treatment in reducing the risk of blindness. For each patient, the laser treatment was given to a randomly selected eye. The time to blindness and the indicator whether treated or untreated or both eyes became blind were recorded. The main aim of the experiment is to test whether the laser treatment has any effect in delaying the onset of blindness in the diabetic retinopathy patients. The data set provided from this experiment can be treated as a dependent competing risks data with two causes of failure (blindness). The data are presented in Table 4.6, and the BMWE distribution may be a reasonable model to analyze this data set. Sarhan et al. (2022) used BCD to analyze this data and reported, based on the Kolmogorov-Smirnov test statistic, that the BCD is a good fit for this data. In this chapter, we use the BMWE distribution to fit this data set and compare it with the BCD. Table 4.7 shows the MLE, the corresponding log-likelihood function and AIC for BCD and BMWE distribution.

i	1	2	3	4	5	6	7	8	9	10	11	12
t_i	266	91	154	285	583	547	79	622	707	469	93	1313
δ_i	1	2	2	3	1	2	1	3	2	2	1	2
i	13	14	15	16	17	18	19	20	21	22	23	24
t_i	805	344	790	125	777	306	415	307	637	577	178	517
δ_i	1	1	2	2	2	1	1	2	2	2	1	2
i	25	26	27	28	29	30	31	32	33	34	35	36
t_i	272	1137	1484	315	287	1252	717	642	141	407	356	1653
δ_i	3	3	1	1	2	1	2	1	2	1	1	3
i	37	38	39	40	41	42	43	44	45	46	47	48
t_i	427	699	36	667	588	471	126	350	350	663	567	966
δ_i	2	1	2	1	2	3	1	2	1	3	2	3
i	49	50	51	52	53	54	55	56	57	58	59	60
t_i	203	84	392	1140	901	1247	448	904	276	520	485	248
δ_i	3	1	1	2	1	3	2	2	1	1	2	2
i	61	62	63	64	65	66	67	68	69	70	71	
t_i	503	423	285	315	727	210	409	584	355	1302	227	
δ_i	1	2	2	2	2	2	2	1	1	1	2	

Table 4.6: Diabetic Retinopathy Data.

Based on the AIC, we can conclude that BMWE distribution fits this data set better than BCD. Moreover, performing the likelihood ratio test, the value of the test statistic is 6.54 and the corresponding p-value is 0.0105. Therefore, the BCD is rejected in favor of the BMWE distribution at a level of significance larger than or equal to 0.0105.

Model	MLE	$\hat{\mathcal{L}}$	AIC
BCD	$\hat{\lambda}_1 = 0.00125, \hat{\lambda}_2 = 0.001472, \hat{\lambda}_3 = 0.00044, \hat{\gamma} = 0.27193$	-581.29	1170.58
BMWE	$\hat{\lambda}_1 = 0.003059, \hat{\lambda}_2 = 0.003612, \hat{\lambda}_3 = 0.00109, \hat{\alpha} = 9905.17, \hat{\gamma} = 1.54657$	-578.05	1166.09

Table 4.7: MLE, log-likelihood values, and AIC for BCD and BMWE distributions, using the Diabetic Retinopathy Data.

Finally, we can investigate if the laser treatment has a significant effect on delaying the blindness of the Diabetic Retinopathy patients. To do so, we can test the following hypothesis

$$H_0 : \lambda_1 = \lambda_2 \text{ (laser is not effective)} \text{ vs } H_1 : \lambda_1 \neq \lambda_2 \text{ (laser is effective)}$$

To use the likelihood ratio test, we need the the log-likelihood function, under H_0 , which

can be obtained from (4.14), by assuming that $\lambda_1 = \lambda_2 = \lambda$, as

$$\begin{aligned} \mathcal{L}_{02}(\theta) = & n_3 \log \lambda_3 + (n_1 + n_2) \log \lambda + n \log \gamma + (\gamma - 1) \sum_{i=1}^n \log t_i + \\ & -n(\gamma - 1) \log \alpha + (2\lambda + \lambda_3) \alpha \sum_{i=1}^n \left(1 - e^{\left(\frac{t_i}{\alpha}\right)^\gamma}\right) + \sum_{i=1}^n \left(\frac{t_i}{\alpha}\right)^\gamma. \end{aligned} \quad (4.17)$$

Under the null hypothesis, the MLE of the unknown parameters are $\hat{\lambda} = 0.00545$, $\hat{\lambda}_3 = 0.00179$, $\hat{\alpha} = 22857.73$, $\hat{\gamma} = 1.55443$, the corresponding log-likelihood value is -578.23 . Thus, the value of the test statistic is $\Lambda = -2(-578.2249 + 578.0463) = 0.357$, and the associated p-value is 0.550. Therefore, we cannot reject H_0 , in favor of H_1 , at any reasonable significance level, which indicates that the laser treatment does not have a significant effect in delaying blindness.

4.7 Conclusion

In this chapter, we used the fatal shock model concept to propose a new bivariate distribution of Marshall-Olkin type, using three independent univariate random variables that follow the modified Weibull extension distribution. This distribution is referred to as BMW distribution. The BMW distribution generalizes the bivariate Chen distribution that was recently proposed by Sarhan et al. (2022). We illustrated how to use the BMW distribution to fit dependent competing risks data as well as bivariate data. We estimated the unknown parameters included in the BMW distribution using the maximum likelihood method based on two different types of data sets: a bivariate data set and a dependent competing risks data set. We performed a simulation study to investigate the consistency of the maximum likelihood method. Finally, we used the new proposed distribution to fit two real data sets (bivariate data and dependent competing risks data) and compared it with the bivariate Chen distribution and some other related distributions and reported that the BMW fits those data sets better than the other distributions.

As a future plan, we can use either step-stress acceleration life test plan and/or progressively type-II censored data to estimate the parameters of BMW in the presence of either bivariate data or dependent competing risks data. This chapter was published in Manshi et al. (2023).

Chapter 5

Conclusion

In chapter 2, we discussed how to use progressively hybrid censoring (PHC) samples to estimate the unknown parameters of Sarhan-Tadj-Hamilton (STH) distribution. We used the maximum likelihood and Bayes methods to estimate those unknown parameters. For the Bayes method, we assumed that the model parameters are independent and gamma prior distributions with hyperparameters all equal to 0.001. This choice of the hyperparameters' values reflect the lack of prior information on the model parameters. There is no closed-form solution for either the maximum likelihood estimate or Bayes estimate for the parameters. Hence, numerical methods are applied. We used R software to apply such numerical methods. For the MLE, we used the R optim function to maximize the log-likelihood function. For Bayes method, we used MCMC to approximate the Bayesian analysis. We applied the theoretical methods discussed in this chapter on three real data sets and we compared the STH distribution with Weibull and power Lindely distributions. For each of these real data sets, we generated two PHC samples then we used them to estimate the model parameters for the three mentioned distributions using maximum likelihood and Bayes methods. Based on the log-likelihood, AIC and BIC values, we concluded that STH distribution fits those three real data sets better than the other two distributions for the original samples and the PHC samples. To investigate the performance of the estimation methods and compare them, a large simulation study was performed using STH distribution at different values of the model parameters and at different values of the sampling scheme (n, m, p and T). Also, we discussed the expected experimentation time, through the relative expected experimentation time (REET), for the progressively Type-II censoring samples using the STH distribution. We performed a simulation study to investigate how REET is affected by the values of the model parameters and progressively censoring values (n, m, p). Further investigations for this method as future work. We could also consider the PHC samples with unknown probability of removal. Also, Bayesian methods with informative prior information on the unknown model parameters can be investigated. This chapter was published in Sarhan et al.

(2023)

In Chapter 3, we discussed the SSPALT scheme using a given distribution. We obtain the maximum likelihood estimators of the unknown parameters for SSPALT and SSPALT - PHC. A Bayesian estimate of the unknown parameters is also considered using gamma priors for both parameters. There is no closed-form solution for both parameters; hence, a simulation is conducted using specified parameter values. MLEs and BE's show promising results for various sample sizes and acceleration factors β . Next, we apply the PHC schemes to the SSPALT method. Next, we considered the suggested transformation in Balakrishnan et al. (2000) on the generated failure times, accelerated failure times using τ and β , and removals. An optimal stress change time τ is considered and obtained using different values of the parameters. We encountered one problem with this method: determining the exact value of the change stress time τ for generated failure times and accelerated failure times before estimating the MLEs. We assigned different τ percentiles of specified parameters' values to overcome this. Then, we get the MLEs for all τ percentiles, and according to the minimum GAV, we pick the optimal τ as mentioned in section 3.4. We applied this method using an actual data set using three distributions. We can determine the optimal change time stress using STH, PL, and Weibull distribution for this data set. All distributions provide a good fit of the data, but the STH distribution has a better fit. A further investigation of other optimal plan criteria and comparing this optimal plan is targeted for future work. As a project goal, we can extend from one stress to k stresses for SSPALT - PHC. We could also investigate the sensitivity analysis performed to examine the effect of the preliminary estimates of parameters on the optimal values of change time stress τ . It will provide information about the robustness of the optimal design. This will also identify the sensitive parameters, which need to be estimated with special care to reduce estimation error.

In Chapter 4, we used the fatal shock model concept to propose a new bivariate distribution of Marshall-Olkin type, using three independent univariate random variables that follow the modified Weibull extension distribution. This distribution is referred to as BMWE distribution. The BMWE distribution generalizes the bivariate Chen distribution that was recently proposed by Sarhan et al. (2022). We illustrated how to use the BMWE distribution to fit dependent competing risks data as well as bivariate data. We estimated the unknown parameters included in the BMWE distribution using the maximum likelihood method based on two different types of data sets: a bivariate data set and a dependent competing risks

data set. We performed a simulation study to investigate the consistency of the maximum likelihood method. Finally, we used the new proposed distribution to fit two real data sets (bivariate data and dependent competing risks data) and compared it with the bivariate Chen distribution and some other related distributions and reported that the BMWWE fits those data sets better than the other distributions. As a future plan, we can use either step-stress acceleration life test plan and/or progressively type-II censored data to estimate the parameters of BMWWE in the presence of either bivariate data or dependent competing risks data. This chapter was published in Manshi et al. (2023).

Appendix A

Chapter 2

The derivatives of the log likelihood function for PHC

The following are the first and second partial derivatives of the log-likelihood given in equation 2.5, for PHC Case I. The first partial derivatives of \mathcal{L}_I , with respect to α and β , respectively are:

$$\mathcal{L}_{I,\alpha} = \frac{m}{\alpha} + \sum_{i=1}^m \log(x_i) - \beta \sum_{i=1}^m (1 + r_i) x_i^\alpha \log x_i + \sum_{i=1}^m \frac{A_{i,\alpha}(\alpha, \beta)}{A_i(\alpha, \beta)} + \sum_{i=1}^m r_i \frac{C_{i,\alpha}(\alpha, \beta)}{C_i(\alpha, \beta)},$$

and

$$\mathcal{L}_{I,\beta} = \frac{m}{\beta} - \frac{[m + \sum_{i=1}^m r_i]}{(1 + \beta)} - \sum_{i=1}^m (1 + r_i) x_i^\alpha + \sum_{i=1}^m \frac{A_{i,\beta}(\alpha, \beta)}{A_i(\alpha, \beta)} + \sum_{i=1}^m r_i \frac{C_{i,\beta}(\alpha, \beta)}{C_i(\alpha, \beta)},$$

where

$$A_{i,\alpha}(\alpha, \beta) = \frac{\partial A_i(\alpha, \beta)}{\partial \alpha} = \beta x_i^\alpha (1 - 2\beta x_i^\alpha) \log x_i \exp\{-\beta x_i^\alpha\},$$

$$C_{i,\alpha}(\alpha, \beta) = \frac{\partial C_i(\alpha, \beta)}{\partial \alpha} = -x_i^{2\alpha} \log(x_i) \beta^2 \exp\{-\beta x_i^\alpha\},$$

$$A_{i,\beta}(\alpha, \beta) = \frac{\partial A_i(\alpha, \beta)}{\partial \beta} = 1 + (1 - 2\beta x_i^\alpha) x_i^\alpha \exp\{-\beta x_i^\alpha\}, \text{ and}$$

$$C_{i,\beta}(\alpha, \beta) = \frac{\partial C_i(\alpha, \beta)}{\partial \beta} = 1 - x_i^{2\alpha} \beta \exp\{-\beta x_i^\alpha\}.$$

The second partial derivatives of \mathcal{L}_I , with respect to α and β , respectively are:

$$\begin{aligned} \mathcal{L}_{I,\alpha^2} &= -\frac{m}{\alpha^2} - \beta \sum_{i=1}^m (1 + r_i) x_i^\alpha (\log x_i)^2 + \sum_{i=1}^m \frac{A_i(\alpha, \beta) A_{i,\alpha^2}(\alpha, \beta) - [A_{i,\alpha}(\alpha, \beta)]^2}{[A_i(\alpha, \beta)]^2} \\ &\quad + \sum_{i=1}^m r_i \frac{C_i(\alpha, \beta) C_{i,\alpha^2}(\alpha, \beta) - [C_{i,\alpha}(\alpha, \beta)]^2}{[C_i(\alpha, \beta)]^2}, \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{I,\beta^2} &= -\frac{m}{\beta^2} + \frac{[m + \sum_{i=1}^m r_i]}{(1 + \beta)^2} + \sum_{i=1}^m \frac{A_i(\alpha, \beta) A_{i,\beta^2}(\alpha, \beta) - [A_{i,\beta}(\alpha, \beta)]^2}{[A_i(\alpha, \beta)]^2} \\ &\quad + \sum_{i=1}^m r_i \frac{C_i(\alpha, \beta) C_{i,\beta^2}(\alpha, \beta) - [C_{i,\beta}(\alpha, \beta)]^2}{[C_i(\alpha, \beta)]^2}, \end{aligned}$$

$$\begin{aligned}\mathcal{L}_{I,\alpha,\beta} &= -\sum_{i=1}^m (1+r_i) x_i^\alpha \log x_i + \sum_{i=1}^m \frac{A_i(\alpha, \beta) A_{i,\alpha\beta}(\alpha, \beta) - A_{i,\alpha}(\alpha, \beta) A_{i,\beta}(\alpha, \beta)}{[A_i(\alpha, \beta)]^2} \\ &\quad + \sum_{i=1}^m r_i \frac{C_i(\alpha, \beta) C_{i,\alpha\beta}(\alpha, \beta) - C_{i,\alpha}(\alpha, \beta) C_{i,\beta}(\alpha, \beta)}{[C_i(\alpha, \beta)]^2},\end{aligned}$$

where

$$A_{i,\alpha^2}(\alpha, \beta) = \beta x_i^\alpha (\log(x_i))^2 \exp\{-\beta x_i^\alpha\} (1 - 5\beta x_i^\alpha + 2\beta^2 x_i^{2\alpha}),$$

$$C_{i,\alpha^2}(\alpha, \beta) = \beta^2 x_i^{2\alpha} (\log(x_i))^2 (\beta x_i^\alpha - 2) \exp\{-\beta x_i^\alpha\},$$

$$A_{i,\alpha\beta}(\alpha, \beta) = x_i^\alpha \log(x_i) \exp\{-\beta x_i^\alpha\} (1 - 5\beta x_i^\alpha + 2\beta^2 x_i^{2\alpha}),$$

$$C_{i,\alpha\beta}(\alpha, \beta) = \alpha \beta x_i^{2\alpha} \log(x_i) \exp\{-\beta x_i^\alpha\} (\beta x_i^\alpha - 2),$$

$$A_{i,\beta^2}(\alpha, \beta) = x_i^{2\alpha} \exp\{-\beta x_i^\alpha\} (3 - 2\beta x_i^\alpha), \text{ and}$$

$$C_{i,\beta^2}(\alpha, \beta) = x_i^{2\alpha} (\beta x_i^\alpha - 1) \exp\{-\beta x_i^\alpha\}.$$

Below are the first and second partial derivatives of the log-likelihood given in equation 2.6, for PHC Case II. The first partial derivatives of \mathcal{L}_{II} , with respect to α and β , respectively are:

$$\begin{aligned}\mathcal{L}_{II,\alpha} &= \frac{J}{\alpha} + \sum_{i=1}^J \log(x_i) - \beta \left[\sum_{i=1}^J (1+r_i) x_i^\alpha \log x_i + R_j^* T^\alpha \log T \right] + \sum_{i=1}^J \frac{A_{i,\alpha}(\alpha, \beta)}{A_i(\alpha, \beta)} \\ &\quad + \sum_{i=1}^J r_i \frac{C_{i,\alpha}(\alpha, \beta)}{C_i(\alpha, \beta)} + R_j^* \frac{D_\alpha(\alpha, \beta)}{D(\alpha, \beta)},\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{II,\beta} &= \frac{J}{\beta} - \frac{[J - R_j^* + \sum_{i=1}^J r_i]}{(1+\beta)} - \left[\sum_{i=1}^J (1+r_i) x_i^\alpha + R_j^* T^\alpha \right] + \sum_{i=1}^J \frac{A_{i,\beta}(\alpha, \beta)}{A_i(\alpha, \beta)} \\ &\quad + \sum_{i=1}^J r_i \frac{C_{i,\beta}(\alpha, \beta)}{C_i(\alpha, \beta)} + R_j^* \frac{D_\beta(\alpha, \beta)}{D(\alpha, \beta)},\end{aligned}$$

where

$$D_\alpha(\alpha, \beta) = \frac{\partial D(\alpha, \beta)}{\partial \alpha} = -T^{2\alpha} \log(T) \beta^2 \exp\{-\beta T^\alpha\} \text{ and}$$

$$D_\beta(\alpha, \beta) = \frac{\partial D(\alpha, \beta)}{\partial \beta} = 1 - T^{2\alpha} \beta \exp\{-\beta T^\alpha\}.$$

The second derivative of \mathcal{L}_{II} with respect to α and β are:

$$\begin{aligned}\mathcal{L}_{II,\alpha^2} &= -\frac{J}{\alpha^2} - \beta \left[\sum_{i=1}^m (1+r_i) x_i^\alpha (\log x_i)^2 + R_J^* T^\alpha (\log T)^2 \right] \\ &+ \sum_{i=1}^J \frac{A_i(\alpha, \beta) A_{i,\alpha^2}(\alpha, \beta) - [A_{i,\alpha}(\alpha, \beta)]^2}{[A_i(\alpha, \beta)]^2} \\ &+ \sum_{i=1}^J r_i \frac{C_i(\alpha, \beta) C_{i,\alpha^2}(\alpha, \beta) - [C_{i,\alpha}(\alpha, \beta)]^2}{[C_i(\alpha, \beta)]^2} + \\ &+ R_J^* \frac{D(\alpha, \beta) D_{\alpha^2}(\alpha, \beta) - [D_\alpha(\alpha, \beta)]^2}{[D(\alpha, \beta)]^2},\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{II,\beta^2} &= -\frac{J}{\beta^2} + \frac{[J - R_J^* + \sum_{i=1}^J r_i]}{(1+\beta)^2} \\ &+ \sum_{i=1}^J \frac{A_i(\alpha, \beta) A_{i,\beta^2}(\alpha, \beta) - [A_{i,\beta}(\alpha, \beta)]^2}{[A_i(\alpha, \beta)]^2} + \\ &+ \sum_{i=1}^J r_i \frac{C_i(\alpha, \beta) C_{i,\beta^2}(\alpha, \beta) - [C_{i,\beta}(\alpha, \beta)]^2}{[C_i(\alpha, \beta)]^2} \\ &+ R_J^* \frac{D(\alpha, \beta) D_{\beta^2}(\alpha, \beta) - [D_\beta(\alpha, \beta)]^2}{[D(\alpha, \beta)]^2},\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{II,\alpha\beta} &= - \left[\sum_{i=1}^J (1+r_i) x_i^\alpha \log x_i + R_J^* T^\alpha \log T \right] \\ &+ \sum_{i=1}^J \frac{A_i(\alpha, \beta) A_{i,\alpha\beta}(\alpha, \beta) - A_{i,\alpha}(\alpha, \beta) A_{i,\beta}(\alpha, \beta)}{[A_i(\alpha, \beta)]^2} \\ &+ \sum_{i=1}^J r_i \frac{C_i(\alpha, \beta) C_{i,\alpha\beta}(\alpha, \beta) - C_{i,\alpha}(\alpha, \beta) C_{i,\beta}(\alpha, \beta)}{[C_i(\alpha, \beta)]^2} \\ &+ R_J^* \frac{D(\alpha, \beta) D_{\alpha\beta}(\alpha, \beta) - D_\alpha(\alpha, \beta) D_\beta(\alpha, \beta)}{[D(\alpha, \beta)]^2},\end{aligned}$$

where

$$D_{\alpha^2}(\alpha, \beta) = \beta^2 T^{2\alpha} (\log(T))^2 (\beta T^\alpha - 2) \exp\{-\beta T^\alpha\},$$

$$D_{\beta^2}(\alpha, \beta) = T^{2\alpha} (\beta T^\alpha + 3) \exp\{-\beta T^\alpha\}, \text{ and}$$

$$D_{\alpha\beta}(\alpha, \beta) = \beta T^{2\alpha} \log(T) (\beta T^\alpha - 2) \exp\{-\beta T^\alpha\}.$$

Appendix B

Chapter 3

The derivatives of the log likelihood function for SSPALT

For Case-I, the log-likelihood function can be expressed as follows:

$$\begin{aligned}\mathcal{L}_I(\theta) \propto & m(\log \alpha + \log \gamma) - \left(m + \sum_{i=1}^m r_i \right) \log(1 + \gamma) + n_2 \log \beta + \\ & + (\alpha - 1) \sum_{i=1}^m \log \left(x_i^{I(\delta_i=1)} (\tau + \beta(x_i - \tau))^{I(\delta_i=2)} \right) + \\ & - \gamma \sum_{i=1}^m \left[I(\delta_i = 1) x_i^\alpha + I(\delta_i = 2) (\tau + \beta(x_i - \tau))^\alpha \right] + \\ & + \sum_{i=1}^m r_i \left[I(\delta_i = 1) \log \left[\frac{1}{1 + \gamma} (\gamma + (1 + \gamma x_i^\alpha) e^{-\gamma x_i^\alpha}) e^{-\gamma x_i^\alpha} \right] \right. \\ & \left. + I(\delta_i = 2) \log \left[\frac{1}{1 + \gamma} (\gamma + (1 + \gamma (\psi_i(\beta))^\alpha) e^{-\gamma (\psi_i(\beta))^\alpha}) e^{-\gamma (\psi_i(\beta))^\alpha} \right] \right] \\ & + \sum_{i=1}^m \left[I(\delta_i = 1) \log (\gamma + (1 + 2\gamma x_i^\alpha) e^{-\gamma x_i^\alpha}) \right. \\ & \left. + I(\delta_i = 2) \log (\gamma + (1 + 2\gamma (\psi_i(\beta))^\alpha) e^{-\gamma (\psi_i(\beta))^\alpha}) \right]\end{aligned}$$

This equation can be expressed as:

$$\begin{aligned}
\mathcal{L}_I(\theta) \propto & m(\log \alpha + \log \gamma) - \left(m + \sum_{i=1}^m r_i \right) \log(1 + \gamma) + n_2 \log \beta + \\
& (\alpha - 1) \sum_{i=1}^{n_1} \log(x_i) + (\alpha - 1) \sum_{i=n_1+1}^m \log(\psi_i(\beta)) \\
& - \gamma \sum_{i=1}^{n_1} x_i^\alpha - \gamma \sum_{i=n_1+1}^m [\psi_i(\beta)]^\alpha + \\
& + \sum_{i=1}^m r_i \left[I(\delta_i = 1) \log \left[\frac{1}{1 + \gamma} \left(\gamma + (1 + \gamma x_i^\alpha) e^{-\gamma x_i^\alpha} \right) e^{-\gamma x_i^\alpha} \right] \right. \\
& \left. + I(\delta_i = 2) \log \left[\frac{1}{1 + \gamma} \left(\gamma + (1 + \gamma (\psi_i(\beta))^\alpha) e^{-\gamma (\psi_i(\beta))^\alpha} \right) e^{-\gamma (\psi_i(\beta))^\alpha} \right] \right] \\
& + \sum_{i=1}^m \left[I(\delta_i = 1) \log \left(\gamma + (1 + 2\gamma x_i^\alpha) e^{-\gamma x_i^\alpha} \right) \right. \\
& \left. + I(\delta_i = 2) \log \left(\gamma + (1 + 2\gamma (\psi_i(\beta))^\alpha) e^{-\gamma (\psi_i(\beta))^\alpha} \right) \right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{L}_I(\theta) \propto & m \log(\alpha) + m \log(\gamma) - \left(m + \sum_{i=1}^m r_i \right) \log(1 + \gamma) + n_2 \log \beta \\
& + (\alpha - 1) \sum_{i=1}^{n_1} \log(x_i) - \gamma \sum_{i=1}^{n_1} x_i^\alpha + \sum_{i=1}^{n_1} \log A_i(\alpha, \gamma) + \sum_{i=1}^{n_1} r_i \log C_i(\alpha, \gamma) \\
& + (\alpha - 1) \sum_{i=n_1+1}^m \log(\psi_i(\beta)) - \gamma \sum_{i=n_1+1}^m [\psi_i(\beta)]^\alpha + \sum_{i=n_1+1}^m \log D_i(\alpha, \gamma, \beta) \\
& + \sum_{i=n_1+1}^m r_i \log G_i(\alpha, \gamma, \beta),
\end{aligned}$$

where

$$\psi_i(\beta) = \tau + \beta(x_i - \tau),$$

$$A_i(\alpha, \gamma) = \gamma + (1 + 2\gamma x_i^\alpha) \exp \left\{ -\gamma \sum_{i=1}^{n_1} x_i^\alpha \right\},$$

$$C_i(\alpha, \gamma) = \frac{1}{1+\gamma} \left[\gamma + (1 + \gamma x_i^\alpha) \exp \left\{ -\gamma \sum_{i=1}^{n_1} x_i^\alpha \right\} \right] \exp \left\{ -\gamma \sum_{i=1}^{n_1} x_i^\alpha \right\},$$

$$D_i(\alpha, \gamma, \beta) = \gamma + (1 + 2\gamma [\psi_i(\beta)]^\alpha) \exp \left\{ -\gamma \sum_{i=n_1+1}^m [\psi_i(\beta)]^\alpha \right\},$$

$$G_i(\alpha, \gamma, \beta) = \frac{1}{1+\gamma} \left[\gamma + (1 + \gamma [\psi_i(\beta)]^\alpha) \exp \left\{ -\gamma \sum_{i=n_1+1}^m [\psi_i(\beta)]^\alpha \right\} \right]$$

$$\times \exp \left\{ -\gamma \sum_{i=n_1+1}^m [\psi_i(\beta)]^\alpha \right\},$$

here $n_1 = \sum_{i=1}^m I(\delta_i = 1)$ is the number of failures under the normal use condition, and $n_2 = \sum_{i=1}^m I(\delta_i = 2)$ is the number of failures at the higher stress use condition, $m = n_1 + n_2$, and $\theta = (\alpha, \gamma, \beta)$ is the vector of the unknown parameters.

The first derivative with respect to α :

$$\begin{aligned} \mathcal{L}_{I,\alpha} &= \frac{m}{\alpha} + \sum_{i=1}^{n_1} \log(x_i) - \gamma \sum_{i=1}^{n_1} x_i^\alpha \log x_i + \sum_{i=1}^{n_1} \frac{A_{i,\alpha}(\alpha, \gamma)}{A_i(\alpha, \gamma)} + \sum_{i=1}^{n_1} \frac{r_i C_{i,\alpha}(\alpha, \gamma)}{C_i(\alpha, \gamma)} \\ &+ \sum_{i=n_1+1}^m \log(\psi_i(\beta)) - \gamma \sum_{i=n_1+1}^m [\psi_i(\beta)]^\alpha \log \psi_i(\beta) + \sum_{i=n_1+1}^m \frac{D_{i,\alpha}(\alpha, \gamma, \beta)}{D_i(\alpha, \gamma, \beta)} \\ &\quad \sum_{i=n_1+1}^m \frac{r_i G_{i,\alpha}(\alpha, \gamma, \beta)}{G_i(\alpha, \gamma, \beta)}. \end{aligned}$$

The first derivative with respect to γ :

$$\begin{aligned} \mathcal{L}_{I,\gamma} &= \frac{m}{\gamma} - \frac{(m + \sum_{i=1}^m r_i)}{(1 + \gamma)} - \sum_{i=1}^{n_1} x_i^\alpha + \sum_{i=1}^{n_1} \frac{A_{i,\gamma}(\alpha, \gamma)}{A_i(\alpha, \gamma)} + \sum_{i=1}^{n_1} \frac{r_i C_{i,\gamma}(\alpha, \gamma)}{C_i(\alpha, \gamma)} \\ &- \sum_{i=n_1+1}^m [\psi_i(\beta)]^\alpha + \sum_{i=n_1+1}^m \frac{D_{i,\gamma}(\alpha, \gamma, \beta)}{D_i(\alpha, \gamma, \beta)} + \sum_{i=n_1+1}^m \frac{r_i G_{i,\gamma}(\alpha, \gamma, \beta)}{G_i(\alpha, \gamma, \beta)}. \end{aligned}$$

The first derivative with respect to β :

$$\begin{aligned} \mathcal{L}_{I,\beta} &= \frac{n_2}{\beta} - (\alpha - 1) \sum_{i=n_1+1}^m \frac{(x_i - \tau)}{\psi_i(\beta)} - \gamma \alpha \sum_{i=n_1+1}^m (x_i - \tau) [\psi_i(\beta)]^{\alpha-1} \\ &+ \sum_{i=n_1+1}^m \frac{D_{i,\beta}(\alpha, \gamma, \beta)}{D_i(\alpha, \gamma, \beta)} + \sum_{i=n_1+1}^m \frac{r_i G_{i,\beta}(\alpha, \gamma, \beta)}{G_i(\alpha, \gamma, \beta)}. \end{aligned}$$

For the complete SSPALT sample, the log-likelihood function becomes:

$$\begin{aligned} \mathcal{L}(\theta) &= n \log(\alpha) + n \log(\gamma) - n \log(1 + \gamma) + n_2 \log \beta + (\alpha - 1) \sum_{i=1}^{n_1} \log(x_i) + \\ &- \gamma \sum_{i=1}^{n_1} x_i^\alpha + \sum_{i=1}^{n_1} \log \left(\gamma + (1 + 2\gamma x_i^\alpha) \exp \left\{ -\gamma \sum_{i=1}^{n_1} x_i^\alpha \right\} \right) + \\ &- \gamma \sum_{i=n_1+1}^n [\tau + \beta(x_i - \tau)]^\alpha + (\alpha - 1) \sum_{i=n_1+1}^n \log(\tau + \beta(x_i - \tau)) + \\ &+ \sum_{i=n_1+1}^n \log \left(\gamma + (1 + 2\gamma [\tau + \beta(x_i - \tau)]^\alpha) \exp \left\{ -\gamma \sum_{i=n_1+1}^n [\tau + \beta(x_i - \tau)]^\alpha \right\} \right), \end{aligned}$$

$$\begin{aligned}
\mathcal{L}(\theta) &= n \log(\alpha) + n \log(\gamma) - n \log(1 + \gamma) + n_2 \log \beta + (\alpha - 1) \sum_{i=1}^{n_1} \log(x_i) + \\
&\quad - \gamma \sum_{i=1}^{n_1} x_i^\alpha + \sum_{i=1}^{n_1} \log A_i(\alpha, \gamma) - \gamma \sum_{i=n_1+1}^n [\psi_i(\beta)]^\alpha + (\alpha - 1) \sum_{i=n_1+1}^n \log(\psi_i(\beta)) \\
&\quad + \sum_{i=n_1+1}^n \log C_i(\alpha, \gamma, \beta),
\end{aligned}$$

where $n_1 = \sum_{i=1}^n I(\delta_i = 1)$, $n_2 = \sum_{i=1}^n I(\delta_i = 2)$ and $n = n_1 + n_2$,

$$\psi_i(\beta) = \tau + \beta(x_i - \tau),$$

$$A(\alpha, \gamma) = \gamma + (1 + 2\gamma x_i^\alpha) \exp\{-\gamma \sum_{i=1}^{n_1} x_i^\alpha\}, \text{ and}$$

$$C(\alpha, \gamma, \beta) = \gamma + (1 + 2\gamma [\psi_i(\beta)]^\alpha) \exp\{-\gamma \sum_{i=n_1+1}^n [\psi_i(\beta)]^\alpha\}.$$

The first derivative with respect to α :

$$\begin{aligned}
\mathcal{L}_\alpha &= \frac{n}{\alpha} + \sum_{i=1}^{n_1} \log(x_i) - \gamma \sum_{i=1}^{n_1} x_i^\alpha \log x_i + \sum_{i=1}^{n_1} \frac{A_{i,\alpha}(\alpha, \gamma)}{A_i(\alpha, \gamma)} + \\
&\quad + \sum_{i=n_1+1}^n \log(\psi_i(\beta)) - \gamma \sum_{i=n_1+1}^n [\psi_i(\beta)]^\alpha \log \psi_i(\beta) + \sum_{i=n_1+1}^n \frac{C_{i,\alpha}(\alpha, \gamma, \beta)}{C_i(\alpha, \gamma, \beta)}.
\end{aligned}$$

The first derivative with respect to γ :

$$\mathcal{L}_\gamma = \frac{n}{\gamma} - \frac{n}{(1 + \gamma)} - \sum_{i=1}^{n_1} x_i^\alpha + \sum_{i=1}^{n_1} \frac{A_{i,\gamma}(\alpha, \gamma)}{A_i(\alpha, \gamma)} - \sum_{i=n_1+1}^n [\psi_i(\beta)]^\alpha + \sum_{i=n_1+1}^n \frac{C_{i,\gamma}(\alpha, \gamma, \beta)}{C_i(\alpha, \gamma, \beta)}.$$

The first derivative with respect to β :

$$\mathcal{L}_\beta = \frac{n_2}{\beta} - (\alpha - 1) \sum_{i=n_1+1}^n \frac{(x_i - \tau)}{\psi_i(\beta)} - \gamma \alpha \sum_{i=n_1+1}^n (x_i - \tau) [\psi_i(\beta)]^{\alpha-1} + \sum_{i=n_1+1}^n \frac{C_{i,\beta}(\alpha, \gamma, \beta)}{C_i(\alpha, \gamma, \beta)}.$$

where

$$A_{i,\alpha}(\alpha, \gamma) = \frac{dA_i(\alpha, \gamma)}{d\alpha} = \gamma x_i^\alpha (1 - 2\gamma x_i^\alpha) \log x_i \exp\{-\gamma \sum_{i=1}^{n_1} x_i^\alpha\},$$

$$A_{i,\gamma}(\alpha, \gamma) = \frac{dA_i(\alpha, \gamma)}{d\gamma} = 1 + (1 - 2\gamma x_i^\alpha) \log x_i \exp\{-\gamma \sum_{i=1}^{n_1} x_i^\alpha\},$$

$$C_{i,\alpha}(\alpha, \gamma, \beta) = \frac{dC_i(\alpha, \gamma, \beta)}{d\alpha} = -\psi_i(\beta)^{2\alpha} \log(\psi_i(\beta)) \gamma^2 \exp\{-\gamma \sum_{i=n_1+1}^n [\psi_i(\beta)]^\alpha\},$$

$$C_{i,\gamma}(\alpha, \gamma, \beta) = \frac{dC_i(\alpha, \gamma, \beta)}{d\gamma} = 1 - \psi_i(\beta)^{2\alpha} \gamma \exp\{-\gamma \sum_{i=n_1+1}^n [\psi_i(\beta)]^\alpha\}, \text{ and}$$

$$C_{i,\beta}(\alpha, \gamma, \beta) = \frac{dC_i(\alpha, \gamma, \beta)}{d\beta} = -\alpha \gamma (1 + \gamma) (x_i - \tau) (\psi_i(\beta))^{\alpha-1} \exp\{-\gamma \sum_{i=n_1+1}^n [\psi_i(\beta)]^\alpha\}.$$

For Case-II, the log-likelihood function can be expressed as

$$\begin{aligned}
\mathcal{L}_{II}(\theta) \propto & J(\log \alpha + \log \gamma) - \left(r_J^* + J + \sum_{i=1}^J r_i \right) \log(1 + \gamma) + n_2 \log \beta + \\
& + (\alpha - 1) \sum_{i=1}^J \log \left(x_i^{I(\delta_i=1)} (\tau + \beta(x_i - \tau))^{I(\delta_i=2)} \right) + \\
& + \sum_{i=1}^J r_i \left[I(\delta_i = 1) \log \left[\frac{1}{1 + \gamma} (\gamma + (1 + \gamma x_i^\alpha) e^{-\gamma x_i^\alpha}) e^{-\gamma x_i^\alpha} \right] \right. \\
& + I(\delta_i = 2) \log \left[\frac{1}{1 + \gamma} (\gamma + (1 + \gamma (\psi_i(\beta))^\alpha) e^{-\gamma (\psi_i(\beta))^\alpha}) e^{-\gamma (\psi_i(\beta))^\alpha} \right] \left. \right] \\
& + \sum_{i=1}^J \left[I(\delta_i = 1) \log (\gamma + (1 + 2\gamma x_i^\alpha) e^{-\gamma x_i^\alpha}) \right. \\
& \quad \left. + I(\delta_i = 2) \log (\gamma + (1 + 2\gamma (\psi_i(\beta))^\alpha) e^{-\gamma (\psi_i(\beta))^\alpha}) \right] . \\
& + r_J^* \log \left[\frac{1}{1 + \gamma} (\gamma + (1 + \gamma (\eta(\beta))^\alpha) e^{-\gamma (\eta(\beta))^\alpha}) e^{-\gamma (\eta(\beta))^\alpha} \right],
\end{aligned}$$

This equation can be expressed as:

$$\begin{aligned}
\mathcal{L}_{II}(\theta) \propto & J(\log \alpha + \log \gamma) - \left(r_J^* + J + \sum_{i=1}^J r_i \right) \log(1 + \gamma) + n_2 \log \beta + \\
& (\alpha - 1) \sum_{i=1}^{n_1} \log(x_i) + (\alpha - 1) \sum_{i=n_1+1}^J \log(\psi_i(\beta)) \\
& - \gamma \sum_{i=1}^{n_1} x_i^\alpha - \gamma \sum_{i=n_1+1}^J [\psi_i(\beta)]^\alpha + \\
& + \sum_{i=1}^J r_i \left[I(\delta_i = 1) \log \left[\frac{1}{1 + \gamma} (\gamma + (1 + \gamma x_i^\alpha) e^{-\gamma x_i^\alpha}) e^{-\gamma x_i^\alpha} \right] \right. \\
& + I(\delta_i = 2) \log \left[\frac{1}{1 + \gamma} (\gamma + (1 + \gamma (\psi_i(\beta))^\alpha) e^{-\gamma (\psi_i(\beta))^\alpha}) e^{-\gamma (\psi_i(\beta))^\alpha} \right] \left. \right] \\
& + \sum_{i=1}^J \left[I(\delta_i = 1) \log (\gamma + (1 + 2\gamma x_i^\alpha) e^{-\gamma x_i^\alpha}) \right. \\
& \quad \left. + I(\delta_i = 2) \log (\gamma + (1 + 2\gamma (\psi_i(\beta))^\alpha) e^{-\gamma (\psi_i(\beta))^\alpha}) \right] \\
& + r_J^* \log \left[\frac{1}{1 + \gamma} (\gamma + (1 + \gamma (\eta(\beta))^\alpha) e^{-\gamma (\eta(\beta))^\alpha}) e^{-\gamma (\eta(\beta))^\alpha} \right],
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{L}_{II}(\theta) &\propto J \log(\alpha) + J \log(\gamma) - \left(r_J^* + J + \sum_{i=1}^J r_i \right) \log(1 + \gamma) + n_2 \log \beta \\
&+ (\alpha - 1) \sum_{i=1}^{n_1} \log(x_i) - \gamma \sum_{i=1}^{n_1} x_i^\alpha + \sum_{i=1}^{n_1} \log A_i(\alpha, \gamma) + \sum_{i=1}^{n_1} r_i \log C_i(\alpha, \gamma) \\
&+ (\alpha - 1) \sum_{i=n_1+1}^J \log(\psi_i(\beta)) - \gamma \sum_{i=n_1+1}^J [\psi_i(\beta)]^\alpha + \sum_{i=n_1+1}^J \log D_i(\alpha, \gamma, \beta) \\
&+ \sum_{i=n_1+1}^J r_i \log G_i(\alpha, \gamma, \beta) + r_J^* \log H(\alpha, \gamma, \beta),
\end{aligned}$$

where

$$\eta(\beta) = \tau + \beta(T - \tau),$$

$$A_i(\alpha, \gamma) = \gamma + (1 + 2\gamma x_i^\alpha) \exp\{-\gamma \sum_{i=1}^{n_1} x_i^\alpha\},$$

$$C_i(\alpha, \gamma) = \frac{1}{1+\gamma} [\gamma + (1 + \gamma x_i^\alpha) \exp\{-\gamma \sum_{i=1}^{n_1} x_i^\alpha\}] \exp\{-\gamma \sum_{i=1}^{n_1} x_i^\alpha\},$$

$$D_i(\alpha, \gamma, \beta) = \gamma + (1 + 2\gamma [\psi_i(\beta)]^\alpha) \exp\left\{-\gamma \sum_{i=n_1+1}^J [\psi_i(\beta)]^\alpha\right\},$$

$$G_i(\alpha, \gamma, \beta) = \frac{1}{1+\gamma} \left[\gamma + (1 + \gamma [\psi_i(\beta)]^\alpha) \exp\left\{-\gamma \sum_{i=n_1+1}^J [\psi_i(\beta)]^\alpha\right\} \right]$$

$$\times \exp\left\{-\gamma \sum_{i=n_1+1}^J [\psi_i(\beta)]^\alpha\right\}, \text{ and}$$

$$H(\alpha, \gamma, \beta) = \frac{1}{1+\gamma} [\gamma + (1 + \gamma [\eta(\beta)]^\alpha) \exp\{-\gamma [\eta(\beta)]^\alpha\}] \exp\{-\gamma [\eta(\beta)]^\alpha\}.$$

The first derivative with respect to α :

$$\begin{aligned}
\mathcal{L}_{II,\alpha} &= \frac{J}{\alpha} + \sum_{i=1}^{n_1} \log(x_i) - \gamma \sum_{i=1}^{n_1} x_i^\alpha \log x_i + \sum_{i=1}^{n_1} \frac{A_{i,\alpha}(\alpha, \gamma)}{A_i(\alpha, \gamma)} + \sum_{i=1}^{n_1} \frac{r_i C_{i,\alpha}(\alpha, \gamma)}{C_i(\alpha, \gamma)} \\
&+ \sum_{i=n_1+1}^J \log(\psi_i(\beta)) - \gamma \sum_{i=n_1+1}^J [\psi_i(\beta)]^\alpha \log \psi_i(\beta) + \sum_{i=n_1+1}^J \frac{D_{i,\alpha}(\alpha, \gamma, \beta)}{D_i(\alpha, \gamma, \beta)} \\
&\quad \sum_{i=n_1+1}^J \frac{r_i G_{i,\alpha}(\alpha, \gamma, \beta)}{G_i(\alpha, \gamma, \beta)} + \frac{r_J^* H_\alpha(\alpha, \gamma, \beta)}{H(\alpha, \gamma, \beta)},
\end{aligned}$$

The first derivative with respect to γ :

$$\begin{aligned}
\mathcal{L}_{II,\gamma} &= \frac{J}{\gamma} - \frac{\left(r_J^* + J + \sum_{i=1}^J r_i \right)}{(1 + \gamma)} - \sum_{i=1}^{n_1} x_i^\alpha + \sum_{i=1}^{n_1} \frac{A_{i,\gamma}(\alpha, \gamma)}{A_i(\alpha, \gamma)} + \sum_{i=1}^{n_1} \frac{r_i C_{i,\gamma}(\alpha, \gamma)}{C_i(\alpha, \gamma)} \\
&- \sum_{i=n_1+1}^J [\psi_i(\beta)]^\alpha + \sum_{i=n_1+1}^J \frac{D_{i,\gamma}(\alpha, \gamma, \beta)}{D_i(\alpha, \gamma, \beta)} + \sum_{i=n_1+1}^J \frac{r_i G_{i,\gamma}(\alpha, \gamma, \beta)}{G_i(\alpha, \gamma, \beta)} \\
&+ \frac{r_J^* H_\gamma(\alpha, \gamma, \beta)}{H(\alpha, \gamma, \beta)},
\end{aligned}$$

The first derivative with respect to β :

$$\begin{aligned} \mathcal{L}_{II,\beta} = & \frac{n_2}{\beta} - (\alpha - 1) \sum_{i=n_1+1}^J \frac{(x_i - \tau)}{\psi_i(\beta)} - \gamma\alpha \sum_{i=n_1+1}^J (x_i - \tau) [\psi_i(\beta)]^{\alpha-1} \\ & + \sum_{i=n_1+1}^J \frac{D_{i,\beta}(\alpha, \gamma, \beta)}{D_i(\alpha, \gamma, \beta)} + \sum_{i=n_1+1}^J \frac{r_i G_{i,\beta}(\alpha, \gamma, \beta)}{G_i(\alpha, \gamma, \beta)} + \frac{r_J^* H_\beta(\alpha, \gamma, \beta)}{H(\alpha, \gamma, \beta)}. \end{aligned}$$

Since the system has no analytic solutions for α , γ , and β , and hence, we will use numerical technique such as Newton Raphson by R code.

Appendix C

Chapter 4

C.1 Proof of Theorem 1

The joint survival of (X_1, X_2) is:

$$S_{X_1, X_2}(x_1, x_2) = \begin{cases} S_{MWE}(x_1; (\lambda_1 + \lambda_3), \alpha, \gamma) S_{MWE}(x_2; \lambda_2, \alpha, \gamma) & \text{if } x_1 > x_2 \\ S_{MWE}(x_1; \lambda_1, \alpha, \gamma) S_{MWE}(x_2; (\lambda_2 + \lambda_3), \alpha, \gamma) & \text{if } x_1 < x_2 \\ S_{MWE}(x; (\lambda_1 + \lambda_2 + \lambda_3), \alpha, \gamma) & \text{if } x_1 = x_2 = x \end{cases}$$

Using the MWE functions, we get:

$$S_{X_1, X_2}(x_1, x_2) = \begin{cases} \exp \left\{ \alpha \left[1 - e^{\left(\frac{x_1}{\alpha}\right)^\gamma} \right] (\lambda_1 + \lambda_3) \right\} \exp \left\{ \lambda_2 \alpha \left[1 - e^{\left(\frac{x_2}{\alpha}\right)^\gamma} \right] \right\} & \text{if } x_1 > x_2 \\ \exp \left\{ \lambda_1 \alpha \left[1 - e^{\left(\frac{x_1}{\alpha}\right)^\gamma} \right] \right\} \exp \left\{ \alpha \left[1 - e^{\left(\frac{x_2}{\alpha}\right)^\gamma} \right] (\lambda_2 + \lambda_3) \right\} & \text{if } x_1 < x_2 \\ \exp \left\{ \alpha \left[1 - e^{\left(\frac{x}{\alpha}\right)^\gamma} \right] (\lambda_1 + \lambda_2 + \lambda_3) \right\} & \text{if } x_1 = x_2 = x \end{cases}$$

For $x_1 \neq x_2$, the expressions for $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ can be obtained from $S_{X_1, X_2}(x_1, x_2)$ using $\frac{\partial^2}{\partial x_1 \partial x_2} S_{X_1, X_2}(x_1, x_2)$ and $\frac{\partial^2}{\partial x_2 \partial x_1} S_{X_1, X_2}(x_1, x_2)$ respectively. Finding $\frac{\partial^2}{\partial x_1 \partial x_2} S_{X_1, X_2}(x_1, x_2)$:

$$\begin{aligned} \frac{\partial}{\partial x_1} S_{X_1, X_2}(x_1, x_2) &= -(\lambda_1 + \lambda_3) \gamma \left(\frac{x_1}{\alpha}\right)^{\gamma-1} \exp \left\{ (\lambda_1 + \lambda_3) \alpha \left[1 - e^{\left(\frac{x_1}{\alpha}\right)^\gamma} \right] + \left(\frac{x_1}{\alpha}\right)^\gamma \right\} \\ &\quad \times \exp \left\{ \lambda_2 \alpha \left[1 - e^{\left(\frac{x_2}{\alpha}\right)^\gamma} \right] \right\}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2}{\partial x_1 \partial x_2} S_{X_1, X_2}(x_1, x_2) &= (\lambda_1 + \lambda_3) \gamma \left(\frac{x_1}{\alpha}\right)^{\gamma-1} \exp \left\{ (\lambda_1 + \lambda_3) \alpha \left[1 - e^{\left(\frac{x_1}{\alpha}\right)^\gamma} \right] + \left(\frac{x_1}{\alpha}\right)^\gamma \right\} \\ &\quad \times \lambda_2 \gamma \left(\frac{x_2}{\alpha}\right)^{\gamma-1} \exp \left\{ \lambda_2 \alpha \left[1 - e^{\left(\frac{x_2}{\alpha}\right)^\gamma} \right] + \left(\frac{x_2}{\alpha}\right)^\gamma \right\}, \end{aligned}$$

which can be written as:

$$f_{MWE}(x_1; (\lambda_1 + \lambda_3), \alpha, \gamma) f_{MWE}(x_2; \lambda_2, \alpha, \gamma) = f_1(x_1, x_2).$$

Likewise, finding $\frac{\partial^2}{\partial x_2 \partial x_1} S_{X_1, X_2}(x_1, x_2)$:

$$\begin{aligned} \frac{\partial}{\partial x_2} S_{X_1, X_2}(x_1, x_2) &= -(\lambda_2 + \lambda_3) \gamma \left(\frac{x_2}{\alpha}\right)^{\gamma-1} \exp \left\{ (\lambda_2 + \lambda_3) \alpha \left[1 - e^{\left(\frac{x_2}{\alpha}\right)^\gamma} \right] + \left(\frac{x_2}{\alpha}\right)^\gamma \right\} \\ &\quad \times \exp \left\{ \lambda_1 \alpha \left[1 - e^{\left(\frac{x_1}{\alpha}\right)^\gamma} \right] \right\}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2}{\partial x_2 \partial x_1} S_{X_1, X_2}(x_1, x_2) &= (\lambda_2 + \lambda_3) \gamma \left(\frac{x_2}{\alpha}\right)^{\gamma-1} \exp\left\{(\lambda_2 + \lambda_3) \alpha \left[1 - e^{\left(\frac{x_2}{\alpha}\right)^\gamma}\right] + \left(\frac{x_2}{\alpha}\right)^\gamma\right\} \\ &\quad \lambda_1 \gamma \left(\frac{x_1}{\alpha}\right)^{\gamma-1} \exp\left\{\lambda_1 \alpha \left[1 - e^{\left(\frac{x_1}{\alpha}\right)^\gamma}\right] + \left(\frac{x_1}{\alpha}\right)^\gamma\right\}, \end{aligned}$$

which can be written as:

$$f_{MWE}(x_2; (\lambda_2 + \lambda_3), \alpha, \gamma) f_{MWE}(x_1; \lambda_1, \alpha, \gamma) = f_2(x_1, x_2).$$

While, when $x_1 = x_2 = x$, we can get the expression of $f_3(x_1, x_2)$ using the following fact:

$$\begin{aligned} \int_0^\infty \int_0^{x_2} f_2(x_1, x_2) dx_1 dx_2 + \int_0^\infty \int_0^{x_1} f_1(x_1, x_2) dx_2 dx_1 + \int_0^\infty f_3(x) dx &= 1 \\ \int_0^\infty f_3(x) dx = 1 - \left[\int_0^\infty \int_0^{x_2} f_2(x_1, x_2) dx_1 dx_2 + \int_0^\infty \int_0^{x_1} f_1(x_1, x_2) dx_2 dx_1 \right] \\ \int_0^\infty \int_0^{x_2} f_2(x_1, x_2) dx_1 dx_2 &= \\ \int_0^\infty \int_0^{x_2} f_{MWE}(x_1; \lambda_1, \alpha, \gamma) f_{MWE}(x_2; \lambda_2 + \lambda_3, \alpha, \gamma) dx_1 dx_2 &= \\ = \int_0^\infty F_{MWE}(x_2; \lambda_1, \alpha, \gamma) f_{MWE}(x_2; (\lambda_2 + \lambda_3), \alpha, \gamma) dx_2 \end{aligned}$$

where, $F_{MWE}(x_2; \lambda_1, \alpha, \gamma) = 1 - \exp\left\{\lambda_1 \alpha \left[1 - e^{\left(\frac{x_2}{\alpha}\right)^\gamma}\right]\right\}$

$$\begin{aligned} &= 1 - \int_0^\infty S_{MWE}(x_2; \lambda_1, \alpha, \gamma) f_{MWE}(x_2; \lambda_2 + \lambda_3, \alpha, \gamma) dx_2 \\ \int_0^\infty \int_0^{x_2} f_2(x_1, x_2) dx_1 dx_2 &= \int_0^\infty f_{MWE}(x_2; (\lambda_2 + \lambda_3), \alpha, \gamma) dx_2 \\ &\quad - \int_0^\infty f_{MWE}(x_2; (\lambda_2 + \lambda_3), \alpha, \gamma) S_{MWE}(x_2; \lambda_1, \alpha, \gamma) dx_2 \\ &= 1 - \int_0^\infty (\lambda_2 + \lambda_3) \gamma \left(\frac{x_2}{\alpha}\right)^{\gamma-1} \exp\left\{\left(\frac{x_2}{\alpha}\right)^\gamma\right\} \exp\left\{\left(\sum_{i=1}^3 \lambda_i\right) \alpha \left[1 - e^{\left(\frac{x_2}{\alpha}\right)^\gamma}\right]\right\} dx_2 \end{aligned}$$

The same way for $\int_0^\infty \int_0^{x_1} f_1(x_1, x_2) dx_2 dx_1$

$$\begin{aligned} &\int_0^\infty \int_0^{x_1} f_1(x_1, x_2) dx_2 dx_1 \\ &= \int_0^\infty \int_0^{x_1} f_{MWE}(x_2; \lambda_2, \alpha, \gamma) f_{MWE}(x_1; (\lambda_1 + \lambda_3), \alpha, \gamma) dx_2 dx_1 = \\ &\int_0^\infty F_{MWE}(x_1; \lambda_2, \alpha, \gamma) f_{MWE}(x_1; (\lambda_1 + \lambda_3), \alpha, \gamma) dx_1 \end{aligned}$$

where, $F_{MWE}(x_1; \lambda_2, \alpha, \gamma) = 1 - \exp\left\{\lambda_2 \alpha \left[1 - e^{\left(\frac{x_1}{\alpha}\right)^\gamma}\right]\right\}$

$$\begin{aligned}
&= 1 - \int_0^\infty S_{MWE}(x_1; \lambda_2, \alpha, \gamma) f_{MWE}(x_1; \lambda_1 + \lambda_3, \alpha, \gamma) dx_1 \\
&\int_0^\infty \int_0^{x_1} f_1(x_1, x_2) dx_2 dx_1 = \int_0^\infty f_{MWE}(x_1; (\lambda_1 + \lambda_3), \alpha, \gamma) dx_1 \\
&\quad - \int_0^\infty f_{MWE}(x_1; (\lambda_1 + \lambda_3), \alpha, \gamma) S_{MWE}(x_1; \lambda_2, \alpha, \gamma) dx_1 \\
&= 1 - \int_0^\infty (\lambda_1 + \lambda_3) \gamma \left(\frac{x_1}{\alpha}\right)^{\gamma-1} \exp\left\{\left(\frac{x_1}{\alpha}\right)^\gamma\right\} \exp\left\{\left(\sum_{i=1}^3 \lambda_i\right) \alpha \left[1 - e^{\left(\frac{x_1}{\alpha}\right)^\gamma}\right]\right\} dx_1
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_0^\infty \int_0^{x_2} f_2(x_1, x_2) dx_1 dx_2 + \int_0^\infty \int_0^{x_1} f_1(x_1, x_2) dx_2 dx_1 = \\
&= 2 - \int_0^\infty \left(\sum_{i=1}^3 \lambda_i + \lambda_3\right) \gamma \left(\frac{x}{\alpha}\right)^{\gamma-1} \exp\left\{\left(\frac{x}{\alpha}\right)^\gamma\right\} \exp\left\{\left(\sum_{i=1}^3 \lambda_i\right) \alpha \left[1 - e^{\left(\frac{x}{\alpha}\right)^\gamma}\right]\right\} dx \\
&= 2 - 1 - \lambda_3 \int_0^\infty \gamma \left(\frac{x}{\alpha}\right)^{\gamma-1} \exp\left\{\left(\frac{x}{\alpha}\right)^\gamma\right\} \exp\left\{\left(\sum_{i=1}^3 \lambda_i\right) \alpha \left[1 - e^{\left(\frac{x}{\alpha}\right)^\gamma}\right]\right\} dx \\
&= 1 - \lambda_3 \int_0^\infty \gamma \left(\frac{x}{\alpha}\right)^{\gamma-1} \exp\left\{\left(\frac{x}{\alpha}\right)^\gamma\right\} \exp\left\{\left(\sum_{i=1}^3 \lambda_i\right) \alpha \left[1 - e^{\left(\frac{x}{\alpha}\right)^\gamma}\right]\right\} dx \\
&\rightarrow \int_0^\infty f_3(x) dx = 1 - \left[\int_0^\infty \int_0^{x_2} f_2(x_1, x_2) dx_1 dx_2 + \int_0^\infty \int_0^{x_1} f_1(x_1, x_2) dx_2 dx_1\right] \\
&\int_0^\infty f_3(x) dx = 1 - \left[1 - \lambda_3 \int_0^\infty \gamma \left(\frac{x}{\alpha}\right)^{\gamma-1} \exp\left\{\left(\frac{x}{\alpha}\right)^\gamma\right\} \exp\left\{\left(\sum_{i=1}^3 \lambda_i\right) \alpha \left[1 - e^{\left(\frac{x}{\alpha}\right)^\gamma}\right]\right\} dx\right] \\
&\int_0^\infty f_3(x) dx = \lambda_3 \int_0^\infty \gamma \left(\frac{x}{\alpha}\right)^{\gamma-1} \exp\left\{\left(\frac{x}{\alpha}\right)^\gamma\right\} \exp\left\{\left(\sum_{i=1}^3 \lambda_i\right) \alpha \left[1 - e^{\left(\frac{x}{\alpha}\right)^\gamma}\right]\right\} dx \\
&\rightarrow f_3(x) = \frac{\lambda_3}{\sum_{i=1}^3 \lambda_i} f_{MWE}\left(x; \left(\sum_{i=1}^3 \lambda_i\right), \alpha, \gamma\right).
\end{aligned}$$

C.2 The Derivatives of $f_j^*(t)$ and $f_3^*(t)$

Using $T = \min(X_1, X_2)$ and $X_i = \min(U_i, U_3)$, $i = 1, 2$; $U_j = MWE(\lambda_j, \alpha, \gamma)$, $j = 1, 2, 3$.

1. If $X_1 < X_2$ ($\delta = 1$):

$$\begin{aligned} f_1^*(t) &= f_{MWE}(t; \lambda_1, \alpha, \gamma) \cdot S_{MWE}(t; (\lambda_2 + \lambda_3), \alpha, \gamma) \\ &= h_{MWE}(t; \lambda_1, \alpha, \gamma) \cdot S_{MWE}(t; \lambda_1 + \lambda_2 + \lambda_3, \alpha, \gamma) \\ &= \lambda_1 \gamma \left(\frac{t}{\alpha}\right)^{\gamma-1} \exp \left\{ \left(\sum_{i=1}^3 \lambda_i \right) \alpha \left(1 - e^{\left(\frac{t}{\alpha}\right)^\gamma} \right) + \left(\frac{t}{\alpha}\right)^\gamma \right\}, \end{aligned}$$

2. If $X_1 > X_2$ ($\delta = 2$):

$$\begin{aligned} f_2^*(t) &= f_{MWE}(t; \lambda_2, \alpha, \gamma) \cdot S_{MWE}(t; (\lambda_1 + \lambda_3), \alpha, \gamma) \\ &= h_{MWE}(t; \lambda_2, \alpha, \gamma) \cdot S_{MWE}(t; \lambda_1 + \lambda_2 + \lambda_3, \alpha, \gamma) \\ &= \lambda_2 \gamma \left(\frac{t}{\alpha}\right)^{\gamma-1} \exp \left\{ \left(\sum_{i=1}^3 \lambda_i \right) \alpha \left(1 - e^{\left(\frac{t}{\alpha}\right)^\gamma} \right) + \left(\frac{t}{\alpha}\right)^\gamma \right\}, \end{aligned}$$

3. If $X_1 = X_2$ ($\delta = 3$):

$$\begin{aligned} f_3^*(t) &= \frac{\lambda_3}{\sum_{i=1}^3 \lambda_i} f_{MWE} \left(t; \sum_{i=1}^3 \lambda_i, \alpha, \gamma \right) \\ &= \frac{\lambda_3}{\sum_{i=1}^3 \lambda_i} \sum_{i=1}^3 \lambda_i \gamma \left(\frac{t}{\alpha}\right)^{\gamma-1} \exp \left\{ \left(\sum_{i=1}^3 \lambda_i \right) \alpha \left(1 - e^{\left(\frac{t}{\alpha}\right)^\gamma} \right) + \left(\frac{t}{\alpha}\right)^\gamma \right\} \\ &= \lambda_3 \gamma \left(\frac{t}{\alpha}\right)^{\gamma-1} \exp \left\{ \left(\sum_{i=1}^3 \lambda_i \right) \alpha \left(1 - e^{\left(\frac{t}{\alpha}\right)^\gamma} \right) + \left(\frac{t}{\alpha}\right)^\gamma \right\}. \end{aligned}$$

Bibliography

- Magne Vollan Aarset. How to identify a bathtub hazard rate. *IEEE Transactions on Reliability*, 36(1):106–108, 1987.
- Rita Aggarwala. Ch. 13. progressive censoring: A review. *Handbook of statistics*, 20:373–429, 2001.
- H Akaike. Fitting autoregressive model for regression. *Annals of the Institute of Statistical Mathematics*, 21:243–247, 1969.
- H Akaike. A new look at the statistical model identification. *IEEE Transactions on Automatic Control*, AC-19:716–723., 1974.
- Jim Albert. *Bayesian computation with R*. Springer Science & Business Media, 2009.
- Barry C Arnold, Narayanaswamy Balakrishnan, and Haikady Navada Nagaraja. *A first course in order statistics*, volume 54. Siam, 1992.
- Do Sun Bai, JG Kim, and YR Chun. Design of failure-censored accelerated life-test sampling plans for lognormal and weibull distributions. *Engineering optimization*, 21(3):197–212, 1993.
- Narayanaswamy Balakrishnan, Narayanaswamy Balakrishnan, and Rita Aggarwala. *Progressive censoring: theory, methods, and applications*. Springer Science & Business Media, 2000.
- Narayanaswamy Balakrishnan, Erhard Cramer, and Debasis Kundu. *Hybrid censoring know-how: Designs and implementations*. Academic Press, 2023.
- Mark Bebbington, Chin-Diew Lai, and Ričardas Zitikis. Bathtub-type curves in reliability and beyond. *Australian & New Zealand Journal of Statistics*, 49(3):251–265, 2007a.
- Mark Bebbington, Chin-Diew Lai, and Ričardas Zitikis. Modeling human mortality using mixtures of bathtub shaped failure distributions. *Journal of Theoretical Biology*, 245(3): 528–538, 2007b.
- Bo Bergman. Crossings in the total time on test plot. *Scandinavian Journal of Statistics*, pages 171–177, 1977.
- Jan Beyersmann, Arthur Allignol, and Martin Schumacher. *Competing risks and multistate models with R*. Springer Science & Business Media, 2011.
- GK Bhattacharyya and Zanzawi Soejoeti. A tampered failure rate model for step-stress accelerated life test. *Communications in statistics-Theory and methods*, 18(5):1627–1643, 1989.

- Z. Chen. A new two parameter lifetime distribution with bathtub shaped or increasing failure rate function. *Statist. Probab. Lett.*, 49:155–161, 2000.
- Martin J Crowder. *Classical competing risks*. Chapman and Hall/CRC, 2001.
- Herbert Aron David and Haikady Navada Nagaraja. Order statistics. *Encyclopedia of Statistical Sciences*, 2004.
- Morris H DeGroot and Prem K Goel. Bayesian estimation and optimal designs in partially accelerated life testing. *Naval research logistics quarterly*, 26(2):223–235, 1979.
- Benjamin Epstein. Truncated life tests in the exponential case. *The Annals of Mathematical Statistics*, pages 555–564, 1954.
- SH Feizjavadian and R Hashemi. Analysis of dependent competing risks in the presence of progressive hybrid censoring using marshall–olkin bivariate weibull distribution. *Computational Statistics & Data Analysis*, 82:19–34, 2015.
- Donald Paul Gaver and Mustafa Acar. Analytical hazard representations for use in reliability, mortality, and simulation studies. *Communications in Statistics-Simulation and Computation*, 8(2):91–111, 1979.
- M.E. Ghitany, D.K. Al-Mutairi, N. Balakrishnan, and L.J. Al-Enezi. Power lindley distribution and associated inference. *Computational Statistics & Data Analysis*, 64:20–33, 2013.
- Urban Hjorth. A reliability distribution with increasing, decreasing, constant and bathtub-shaped failure rates. *Technometrics*, 22(1):99–107, 1980.
- Ali A Ismail and HM Aly. Optimal planning of failure-step stress partially accelerated life tests under type-ii censoring (retraction of vol 80, pg 1335, 2010). *JOURNAL OF STATISTICAL COMPUTATION AND SIMULATION*, 84(12):2759–2759, 2014.
- John P Klein, Hans C Van Houwelingen, Joseph G Ibrahim, and Thomas H Scheike. *Handbook of survival analysis*. CRC Press, 2016.
- Debasis Kundu. On hybrid censored weibull distribution. *Journal of Statistical Planning and Inference*, 137(7):2127–2142, 2007.
- Debasis Kundu and Rameshwar D Gupta. Bivariate generalized exponential distribution. *Journal of multivariate analysis*, 100(4):581–593, 2009.
- Debasis Kundu and Avijit Joarder. Analysis of type-ii progressively hybrid censored data. *Computational Statistics & Data Analysis*, 50(10):2509–2528, 2006.
- Debasis Kundu, A M Sarhan, and R Gupta. On sarhan-balakrishnan bivariate distribution. *Journal of Statistics Applications & Probability*, 1:163–170, 2012.

- Martin G Larson and Gregg E Dinse. A mixture model for the regression analysis of competing risks data. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 34(3):201–211, 1985.
- Jerald F Lawless. *Statistical models and methods for lifetime data*, volume 362. John Wiley & Sons, 2011.
- Lawrence M Leemis. Relationships among common univariate distributions. *The American Statistician*, 40(2):143–146, 1986.
- Artur J Lemonte. A new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. *Computational Statistics & Data Analysis*, 62:149–170, 2013.
- Chenhua Li. *Optimal Step-Stress plans for accelerated life testing considering reliability/life prediction*. PhD thesis, Northeastern University, 2009.
- DY Lin. Non-parametric inference for cumulative incidence functions in competing risks studies. *Statistics in medicine*, 16(8):901–910, 1997.
- Dennis V Lindley. Fiducial distributions and bayes’ theorem. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 102–107, 1958.
- Ross A Maller and Xian Zhou. Analysis of parametric models for competing risks. *Statistica Sinica*, 12(3):725–750, 2002.
- Thamer Manshi, Ammar M. Sarhan, and B. Smith. A new bivariate distribution with applications on dependent competing risks data. *International Journal of Statistics and Probability*, 12(3), 2023.
- O. Meeker, W. and A. Escobar, L. *Statistical Methods for Reliability Data*. ohn Wiley & Sons, New York, NY, USAa, 1998.
- G. Simos Meintanis. Test of fit for marshall–olkin distributions with applications. *Journal of Statistical Planning and Inference*, 137:3954–3963, 2007.
- Dirk F Moore. *Applied survival analysis using R*. Springer, 2016.
- Govind S Mudholkar and Deo Kumar Srivastava. Exponentiated weibull family for analyzing bathtub failure-rate data. *IEEE transactions on reliability*, 42(2):299–302, 1993a.
- G.S. Mudholkar and D.K. Srivastava. Exponentiated weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability*, 42(2):299–302, 1993b. doi: 10.1109/24.229504.
- Wayne B Nelson. *Accelerated testing: statistical models, test plans, and data analysis*, volume 344. John Wiley & Sons, 2009.

- Debashis Samanta and Debasis Kundu. Bayesian inference of a dependent competing risk data. *Journal of Statistical Computation and Simulation*, 91(15):3069–3086, 2021.
- A M Sarhan and N Balakrishnan. A new class of bivariate distributions and its mixture. *Journal of Multivariate Analysis*, 98(7):1508–1527, 2007.
- A M Sarhan and D Kundu. Generalized linear failure rate distribution. *Communications in Statistics-Theory and Methods*, 38(5):642–660, 2009.
- A M Sarhan and M Zaindin. Modified weibull distribution. *Applied Sciences (APPS)*, pages 123–136, 2009a.
- A M Sarhan, D C Hamilton, B Smith, and D Kundu. The bivariate generalized linear failure rate distribution and its multivariate extension. *Computational statistics & data analysis*, 55(1):644–654, 2011.
- A M Sarhan, A A Abd EL-Baset, and I A Alasbahi. Exponentiated generalized linear exponential distribution. *Applied Mathematical Modelling*, 37(5):2838–2849, 2013a.
- Ammar M Sarhan. Analysis of incomplete, censored data in competing risks models with generalized exponential distributions. *IEEE Transactions on Reliability*, 56(1):132–138, 2007.
- Ammar M. Sarhan. Generalized quadratic hazard rate distribution. *International Journal of Applied Mathematics & Statistics*, 14 (S09):94–106, 2009a.
- Ammar M Sarhan. Generalized quadratic hazard rate distribution. *International Journal of Applied Mathematics & Statistics*, 14(S09):94–106, 2009b.
- Ammar M. Sarhan and Joseph Apaloo. Exponentiated modified weibull extension distribution. *Reliability Engineering & System Safety*, 112:137–144, 2013a.
- Ammar M Sarhan and Joseph Apaloo. Exponentiated modified weibull extension distribution. *Reliability Engineering & System Safety*, 112:137–144, 2013b.
- Ammar M. Sarhan and Abdelfattah Mustafa. A new extension of the two-parameter bathtub hazard shaped distribution. *Scientific African*, 17:e01345, 2022a. ISSN 2468-2276. doi: <https://doi.org/10.1016/j.sciaf.2022.e01345>. URL <https://www.sciencedirect.com/science/article/pii/S2468227622002526>.
- Ammar M. Sarhan and Abdelfattah Mustafa. A new extension of the two-parameter bathtub hazard shaped distribution. *Scientific African*, 17:e01345, 2022b.
- Ammar M Sarhan and Mazen Zaindin. Modified weibull distribution. *APPS. Applied Sciences*, 11:123–136, 2009b.
- Ammar M Sarhan, M Alameri, and I Al-Wasel. Analysis of progressive censoring competing risks data with binomial removals. *International Journal of Mathematics Analysis*, 2(20): 965–976, 2008.

- Ammar M. Sarhan, A El-Gohary, A H El-Bassiouny, and N Balakrishnan. A class of bivariate linear failure rate distributions and their mixtures. *International Journal of Reliability and Applications*, 10:63–79, 2009.
- Ammar M. Sarhan, Abd EL-Baset A. Ahmad, and Ibtesam A. Alasbahi. Exponentiated generalized linear exponential distribution. *Applied Mathematical Modelling*, 37(5):2838–2849, 2013b.
- Ammar M. Sarhan, Lotfi Tadj, and David C. Hamilton. A new extension of the two-parameter bathtub hazard shaped distribution. *Scientific African*, 17:14 pages, 2014a. URL <https://doi.org/10.1155/2014/532024>.
- Ammar M Sarhan, Lotfi Tadj, and David C Hamilton. A new lifetime distribution and its power transformation. *Journal of Probability and Statistics*, 2014, 2014b.
- Ammar M Sarhan, Joseph Apaloo, and Debasis Kundu. A new bivariate lifetime distribution: properties, estimations and its extension. *Communications in Statistics-Simulation and Computation*, pages 1–18, 2022.
- Ammar M. Sarhan, Thamer Manshi, and B. Smith. Statistical inference of reliability distributions based on progressively hybrid censoring data. *Scientific African*, Submitted, 2023.
- G Schwarz. Estimating the dimension of a model. *Annals of Statistics*, 6:461–464, 1978.
- Yan Shen and Ancha Xu. On the dependent competing risks using marshall–olkin bivariate weibull model: Parameter estimation with different methods. *Communications in Statistics-Theory and Methods*, 47(22):5558–5572, 2018.
- Robert M Smith and Lee J Bain. An exponential power life-testing distribution. *Communications in Statistics-Theory and Methods*, 4(5):469–481, 1975.
- Shan Pou Tsai and CP Wen. The impact of competing risks on relative risk in occupational cohort studies. *International journal of epidemiology*, 13(4):518–525, 1984.
- William D Volterman. *On some inferential aspects for Type-II and progressive Type-II censoring*. PhD thesis, 2011.
- Min Xie, Y Tang, and Thong Ngee Goh. A modified weibull extension with bathtub-shaped failure rate function. *Reliability Engineering & System Safety*, 76(3):279–285, 2002.
- M Zaindin and Ammar M Sarhan. New generalized weibull distribution. *Pakistan Journal of Statistics*, 27:13–30, 2011.