VOLTAGE CONSTRUCTION OF HIGHLY-CONNECTED CUBIC GRAPHS

by

Raziyeh Shafiee

Submitted in partial fulfillment of the requirements
for the degree of Master of Science

at

Dalhousie University
Halifax, Nova Scotia
Summer 2023

This thesis dedicated to all great and knowledgeable professors and teachers.

Table of Contents

Abstract . . . . **iv**

Acknowledgements . . . . **v**

Chapter 1 Introduction **1**

Chapter 2 Voltage Construction **3**

2.1 History 11

Chapter 3 Classification of Base Graphs with at most 4 Vertices for Cubic Derived Graphs **12**

3.1 Cubic Circulant Graphs 13

3.2 Cubic Theta Graphs 16

3.3 Cubic Ears Graphs 23

3.4 Base graphs with 3 vertices 27

3.5 Base graphs with 4 vertices 29
3.5.1 Base graphs with 4 vertices without single sided loops 29
3.5.2 Base graphs with 4 vertices and single sided loops 34
3.5.3 The base graphs with 4 vertices and single and double sided loops 34

3.6 Summary 35

3.7 Comparison with known graphs 38

Chapter 4 Discussion and future work **42**

Bibliography **43**

Abstract

In this thesis we consider the spectrum and Algebraic Connectivity (AC) of cubic graphs that have representation as voltage graphs. These graphs have relatively high symmetry and often turn out to have high AC. We first discuss how to compute the full spectrum of a general voltage graph over the group $\mathbb{Z}_N$. This includes, for example, the Tutte-Coxeter graph. We then use voltage construction to search for cubic graphs with high AC and fixed number of vertices n, constructed from a base graph having at most four vertices. We were able to reproduce known records for $n \leq 10$ and $n = 14, 16, 18, 24, 26, 30, 40, 48, 50, 60$. Moreover, we found a new record of a high-AC graph when $n = 36, 46$ and 52. In particular, the record for $n = 36$ gives a counter-example to conjecture 6.1 proposed in [1] which states that the graph with the maximal AC also has the highest girth.

Acknowledgements

I am grateful for this opportunity to study at Dalhousie University. I appreciate all efforts, kindness, and support of professors Jeannette Janssen and Theodore Kolokolnikov. I am honored that I have their guidance and supervision. I am grateful for all efforts of professors David Iron and Peter Selinger as graduate coordinator and member of committee. I owe my success to my lovely family and friends especially my open mind lovely dad who is my first role model.

Chapter 1

Introduction

The Algebraic connectivity (AC) or spectral gap of a graph G defined by the second smallest eigenvalue of the Laplacian matrix $L = D - A$. D is a diagonal matrix where its diagonal entries are degree of each vertex in G and A the adjacency matrix of the graph G. By using AC, it is possible to measure connectivity of graphs. AC is important to extensively examine and study because of its applications [1,3,4,10,11]. Graphs with high AC such as Ramanujan and expander graphs play a key role in Graph Theory, so that constructing them is interesting [7, 7, 12–14]. In particular, constructing an infinite family of them. In this thesis, we focus on constructing cubic (i.e., 3-regular) family of graphs with high AC using Voltage construction. We explore all possible derived cubic graphs from different base graphs with at most four vertices. We find what is the structure of the base graph whose derived graphs have large algebraic connectivity. For example, based on House of Graphs (HOG) there are about 10^{18} cubic graphs with 36 vertices [18]. It is therefore impractical to search through all such graphs. Some restrictions such as high AC and girth are required to efficiently construct 3-regular graphs by Voltage construction [2, 8].

There is Boyd algorithm in [9] to construct graphs with high AC heuristically. This greedy algorithm works efficiently for constructing a graph with high AC which is better than random cases. Paper [1] uses a combination of stochastic algorithms and exhaustive search to find graphs with high girth and AC. Their stochastic algorithms are better than Boyd algorithm but much slower. On the other hand, voltage construction restricts search space to very symmetric graphs. Despite this restriction, some of these turn out to have high AC. Voltage graphs that we study in this thesis turns out to be an appropriate restriction which can produce graphs with high AC. For example, we find a new record high AC for cubic graphs on 36 vertices, using voltage graphs with base graph with 4 vertices. We also compare all results of large AC from voltage construction with record result for cubic graphs with high AC. We also found a method of computing spectrum of the voltage graphs from the base graph.

The summary of this thesis is as follows. Chapter 2 is about voltage construction. We show how to compute the spectrum of a voltage graph from its base graph, and apply it to some well known examples, including the Tutte-Coxeter graph. Chapter 3 is about voltage graphs constructed by base graphs with at most four vertices. We found a new result for a voltage graph with 36 vertices and the largest known AC. In Chapter 4, we will see future works and open problems related to voltage graphs. Any theorem and lemma without references is new.

Chapter 2

Voltage Construction

In Voltage construction, the voltage graph is built from a finite directed base graph B having k vertices; whose edges are labeled using elements of the group L and N is the order or size of the group L. We denote the vertex, edge, and arc sets of B by $V(B)$, $E(B)$, and $D(B)$, respectively. We note arc set $D(B)$ contains directed edges. So, any edge such as e=uv in $E(B)$ appeared in $D(G)$ with directed edges uv and vu. In the following, if we show edges or loops of the base graph without direction, that means they have double directions. There are other graphs that will be replaced by labeled directed edges of B, called lifts, to derive a graph with $n = k * N$ vertices. We see the algorithm of the Voltage construction in the following.

Definition 2.0.1. *The base graph B is a directed graph whose edges are labeled by elements of a given group L; we only consider $L = Z_N$.*

□

Definition 2.0.2. *The Voltage graph G has the vertex set contains N copies of vertices of B; we note the size of the group L is N, i.e.,*

$$V(G) = \{(x, l)|x \in V(B), l \in L\}$$

and the edge set

$$E(G) = \{((x, l), (x', l'))|(x, x') \in E(B), l' - l \textit{ is the label of the edge } (x, x') \textit{ in } B\}.$$

□

As a motivation, it is worth mentioning that by proper labeling of edges of the base graph, the Voltage construction can construct the sparse highly connected graphs like Expander graphs, in particular Ramanujan graphs. In the following parts, we see some examples.

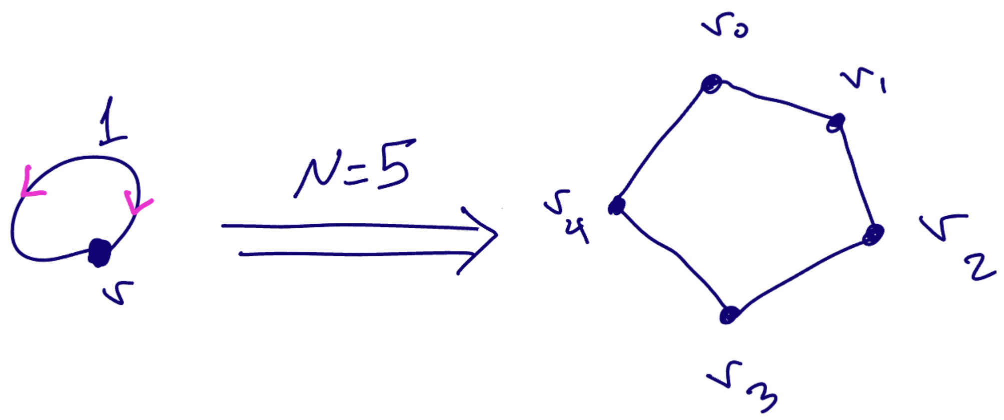

Figure 2.1: The base graph B and its voltage derived graph.

Example 2.0.3 The first example is a cycle(Circulant) graph which is the derived graph from the base graph B with one vertex and one double sided loop. As we see in figure 2.1, the derived graph from the base graph B is a cycle of length five with vertex set $\{v_0, v_1, v_2, v_3, v_4\}$. The adjacency matrix is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Thus, $\lambda v_j = v_{j-1} + v_{j+1}$ where indices are taken mod 5. In other words,

$$Av = v\lambda \Rightarrow \begin{cases} \lambda v_1 = v_5 + v_2, \\ \lambda v_2 = v_1 + v_3, \\ \lambda v_3 = v_2 + v_4, \\ \lambda v_4 = v_3 + v_5, \\ \lambda v_5 = v_4 + v_1. \end{cases}$$

Solutions of these equations are $v_j = z^j = e^{\frac{2j\pi i}{5}}$ for $j \in \{1, \ldots, 5\}$ because by replacing $v_j = z^j$

for $j \in \{1, \cdots, 5\}$ in above equations, we have

$$\begin{cases} \lambda z^1 = z^5 + z^2 \text{ or } \lambda = z^4 + z^1, \\ \lambda z^2 = z^1 + z^3 \text{ or } \lambda = z^{-1} + z^1, \\ \lambda z^3 = z^2 + z^4 \text{ or } \lambda = z^{-1} + z^1, \\ \lambda z^4 = z^3 + z^5 \text{ or } \lambda = z^{-1} + z^1, \\ \lambda z^5 = z^4 + z^1 \text{ or } \lambda = z^{-1} + z^{-4}. \end{cases} \tag{2.1}$$

If $z^5 = 1$, then all above equations in (2.1) become $\lambda = z^{-1} + z^1$. Otherwise, they are inconsistent. This implies $z = e^{\frac{2m\pi i}{5}}$ for $m \in \{0, 1, \ldots, 4\}$. Then, the eigenvalues are

$$\lambda_m = z^{-1} + z^1 = e^{\frac{2m\pi i}{5}} + e^{\frac{-2m\pi i}{5}} = 2\cos\left(\frac{2m\pi i}{5}\right)$$

for $m \in \{0, 1, \ldots, 4\}$.

Note 2.0.4. *We note that the power of z change mod 5 in $z = e^{\frac{2m\pi i}{5}}$ for $m \in \{0, 1, \ldots, 4\}$, i.e., $z^{5+i} = z^i$. But this property is not true for any other values of z except $z = e^{\frac{2m\pi i}{5}}$ for $m \in \{0, 1, \ldots, 4\}$.*

□

Therefore, we can say the eigenvalues of any Circulant graph with N vertices are

$$\lambda_m = 2\cos\left(\frac{2m\pi i}{N}\right) \tag{2.2}$$

for $m \in \{0, 1, \ldots, N-1\}$.

Example 2.0.5 The second example is the Peterson graph, we construct the Peterson graph with the Voltage construction. Consider the base graph B in figure 2.2. The derived graph or Peterson graph is lifts of this base graph (see figure 2.2, taken from [2]). The base graph B can be represented by the following table.

Tail	head	label
1	1	1
1	2	0
2	2	2

Table 2.1: The table correspond to the base graph B.

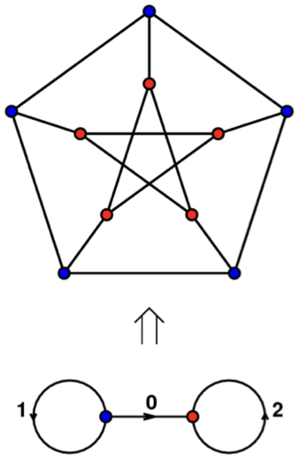

Figure 2.2: The base graph B and its derived(Peterson) graph, taken from [2].

Corresponding eigenvalues solves

$$Av = v\lambda \Rightarrow \begin{cases} \lambda v_j = w_j + v_{j+1} + v_{j-1}, & j \in \{0,1,2,3,4\}, \\ \lambda w_j = v_j + w_{j+2} + w_{j-2}, & j \in \{0,1,2,3,4\}, \end{cases}$$

where A is the adjacency matrix of the Peterson graph. Now it is enough to assume $v_j = \phi z^j$ and $w_j = \psi z^j$, therefore, by replacing them in above equations, we have

$$\begin{cases} \lambda\phi z^j = \psi z^j + \phi z^{j+1} + \phi z^{j-1}, & j \in \{0,1,2,3,4\}, \\ \lambda\psi z^j = \phi z^j + \psi z^{j+2} + \psi z^{j-2}, & j \in \{0,1,2,3,4\}. \end{cases}$$

By cancelling z^j from both sides of these equations, we have

$$\begin{cases} \lambda\phi = \psi + \phi z^1 + \phi z^{-1}, \\ \lambda\psi = \phi + \psi z^2 + \psi z^{-2}, \end{cases}$$

or

$$\lambda \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} z + z^{-1} & 1 \\ 1 & z^2 + z^{-2} \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix}.$$

Thus, the eigenvalues of the above matrix for $z = e^{\frac{2m\pi i}{5}}$ for $m \in \{0,1,2,3,4\}$ are eigenvalues of the Peterson graph. In better words, the characteristic polynomial of the matrix

$$\begin{bmatrix} z + z^{-1} & 1 \\ 1 & z^2 + z^{-2} \end{bmatrix}$$

is

$$P(\lambda) = \lambda^2 - (z + z^{-1} + z^2 + z^{-2})\lambda + (-1 + z + z^2 + z^3 + z^4)$$

$$= \lambda^2 - (z^4 + z^3 + z^2 + z)\lambda + (z^4 + z^3 + z^2 + z - 1).$$

If $z \neq 1$, then we use

$$1 + z + z^2 + z^3 + z^4 = \frac{z^5 - 1}{z - 1}$$

to obtain

$$P(\lambda) = \lambda^2 - z(\frac{z^4 - 1}{z - 1})\lambda + (\frac{z^5 - 1}{z - 1} - 2) = \lambda^2 - (\frac{z^5 - z}{z - 1})\lambda + (\frac{z^5 - 1}{z - 1} - 2).$$

Otherwise, $P(\lambda)$ becomes $\lambda^2 - 4\lambda + 3$. In summary, we obtain

$$\begin{cases} \lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda = 1, \lambda = -2 & if\ z \neq 1, \\ \lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda = 1, \lambda = 3 & if\ z = 1. \end{cases}$$

□

Example 2.0.6 The third example is Tutte–Coxeter graph or Tutte eight-cage or Cremona–Richmond graph. Its base graph B (the left graph) as well as the Tutte–Coxeter graph(the right graph) itself is shown in figure 2.3.

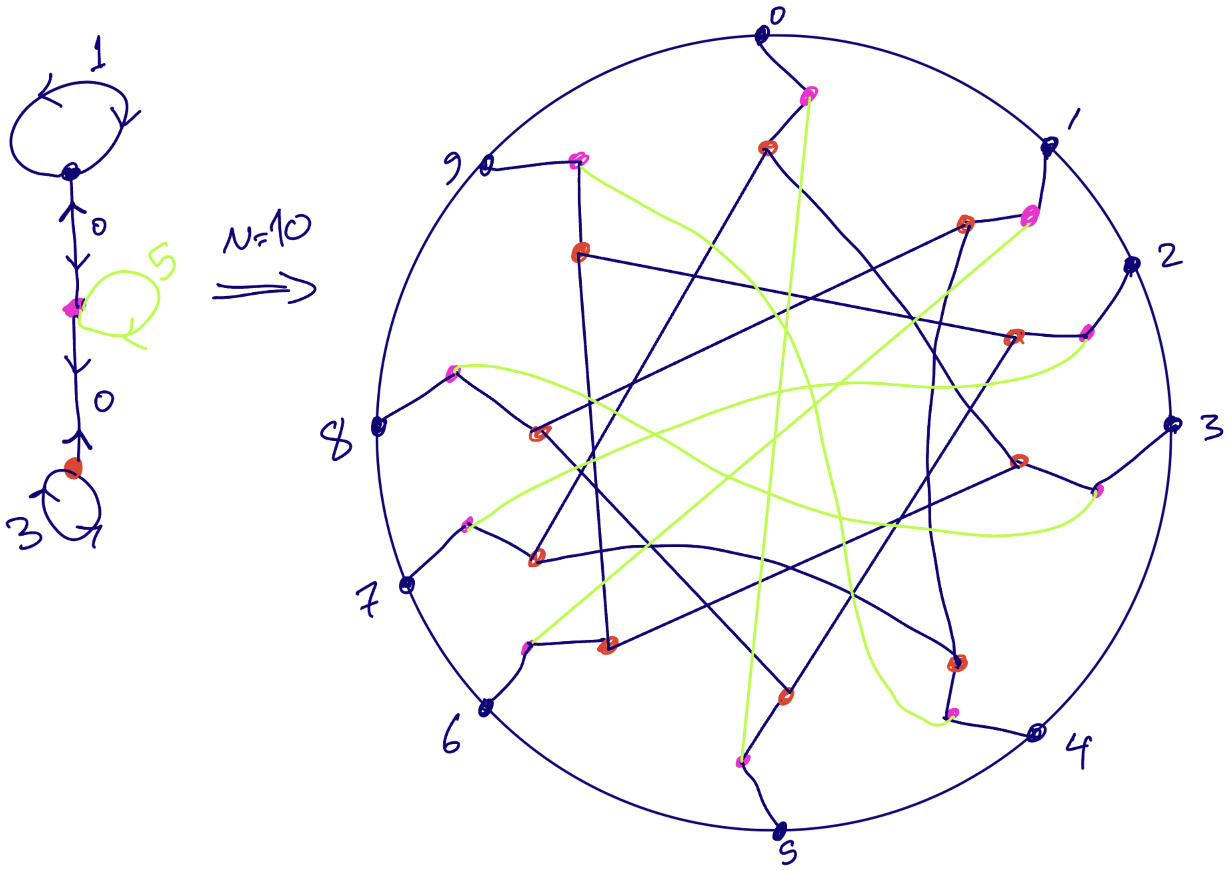

Figure 2.3: The base graph B for the Tutte-Coxeter graph.

The base graph B can be represented as a table; see table 2.2.

Tail	head	label
1	1	1
1	2	0
2	2	5
2	3	0
3	3	3

Table 2.2: The table correspond to B of the Tutte-Coxeter graph.

Now by considering the adjacency matrix of the Tutte-Coxeter graph or the table corresponding to the base graph B, we have

$$\begin{cases} \lambda u_j = v_j + u_{j+1} + u_{j-1}, \\ \lambda v_j = u_j + w_j + v_{j+5}, \qquad j \in \{0,1,2,\cdots,9\}, \\ \lambda w_j = v_j + w_{j+3} + w_{j-3}. \end{cases}$$

Now solve these equations by replacing

$$\begin{cases} u_j = \phi z^j, \\ v_j = \psi z^j, \\ w_j = \gamma z^j, \end{cases}$$

where $z^{10} = 1$. We then obtain

$$\begin{cases} \lambda\phi = \psi + \phi z^1 + \phi z^{-1}, \\ \lambda\psi = \phi + \gamma + \psi z^5, \\ \lambda\gamma = \psi + \gamma z^3 + \gamma z^{-3}, \end{cases}$$

or

$$\lambda \begin{bmatrix} \phi \\ \psi \\ \gamma \end{bmatrix} = \begin{bmatrix} z+z^{-1} & 1 & 0 \\ 1 & z^5 & 1 \\ 0 & 1 & z^3+z^{-3} \end{bmatrix} \begin{bmatrix} \phi \\ \psi \\ \gamma \end{bmatrix}. \tag{2.3}$$

Thus, the eigenvalues of the Tutte-Coxeter graph correspond to the eigenvalues of the matrix in equation 2.3 for $z = e^{\frac{2m\pi i}{10}}$ with $m \in \{0,1,\cdots,9\}$. The characteristic polynomial of this matrix is

$$\begin{aligned} P(\lambda) &= (\lambda - z - z^{-1})(\lambda - z^5)(\lambda - z^3 - z^{-3}) - (\lambda - z^3 - z^{-3}) - (\lambda - z - z^{-1}) \\ &= \lambda^3 - \lambda^2(z^9 + z^7 + z^5 + z^3 + z) + \lambda(2z^8 + 2z^6 + 2z^4 + 2z^2 - 2)) + 0. \end{aligned}$$

By using

$$1 + z^2 + (z^2)^2 + (z^2)^3 + (z^2)^4 = 2(\frac{z^{10}-1}{z^2-1}),$$

and assuming $z^2 \neq 1$, we obtain

$$\begin{aligned} P(\lambda) &= \lambda^3 - \lambda^2(z^9 + z^7 + z^5 + z^3 + z) + \lambda(-4 + 2(\frac{z^{10}-1}{z-1})) \\ &= \lambda^3 - \lambda^2(z\frac{z^{10}-1}{z^2-1}) + \lambda(-4 + 2(\frac{z^{10}-1}{z^2-1})) \end{aligned}$$

$$= \lambda^3 - 4\lambda = \lambda(\lambda^2 - 4) = \lambda(\lambda - 2)(\lambda + 2).$$

If $z^2 = 1$, then, we have two following cases:

- if z=1, then

$$P(\lambda) = \lambda^3 - 5\lambda^2 + 6\lambda = \lambda(\lambda^2 - 5\lambda + 6) = \lambda(\lambda - 2)(\lambda - 3).$$

- if $z = -1$, then

$$P(\lambda) = \lambda^3 + 5\lambda^2 + 6\lambda = \lambda(\lambda^2 + 5\lambda + 6) = \lambda(\lambda + 2)(\lambda + 3).$$

Hence, the spectrum of the Tutte-Coxeter graph is $3, 2^9, 0^{10}, (-2)^9, -3$.

□

More generally, for any base graph B with k vertices and the $\mathbb{Z}_N$ group to find spectrum of the derived graph G, eigenspace of G has dimension $n = Nk$. It decomposes into N eigenspaces, each of dimension k. In other words, we have N matrices k by k corresponding to the base graph B. In these matrices, we note that the entry corresponds to a loop with label a in Z_N is $z^a + z^{-a}$ and the entry corresponds to a directed edge(arc) uv with label b in the group is z^b, and the entry corresponds to arc vu is z^{-b}. We saw in above examples, how to compute the spectrum of derived graphs by these N matrices. In the following theorem, we see that some derived graphs have a base graph with different labels, but the derived graphs are isomorphic to each other.

The following two theorems simplify search for the derived graphs by exploring the isomorphisms of derived graphs from the same base graph with different labels.

Theorem 2.0.7. *The derived graph from the base graph with labels* $(l_1, l_2, \cdots, l_t)$ *is isomorphic to the derived graph from the base graph with labels* $(l_1a, l_2a, \cdots, l_ta)$ *for any invertible* a *in* $\mathbb{Z}_N$.

Theorem 2.0.8. *Every derived graph from the base graph* B *is isomorphic to the derived graph from the same base graph whose spanning tree gets identity of* $\mathbb{Z}_N$ *as labels of its edges [8].*

2.1 History

Now we know what is voltage construction. So, we want to shortly review its history. One of the first papers that use voltage construction for constructing dense graphs or highly connected graphs is [23] where they did not use voltage construction as the name of the construction. The voltage construction appeared in Joel Friedman's papers as "covering" [20]. There are more papers that used covering as the name of voltage construction [24,25]. From algebraic approach, the paper [21] explained the voltage construction as "lifts of digraphs." It also appeared in [22] as a type of "cylindrical construction."

Chapter 3

Classification of Base Graphs with at most 4 Vertices for Cubic Derived Graphs

In this chapter, we find the cubic voltage graphs with large AC whose base graph has at most four vertices. Some of these graphs have both maximum girth and AC with same labels for their base graph. But we will see some exceptions derived graphs where labels of maximum AC and maximum girth are not same. Section 3.1 is about the base graphs with one vertex as shown in figure 3.1. It can be used to construct cubic circulant graphs. Its base graph B has one sided loop with label $\frac{N}{2}$ (with N which is even) and a double sided loop with arbitrary label (but not $\frac{N}{2}$ because of degree) in $\mathbb{Z}_N$. Note that a double sided loop increases the degree of each vertex of G by 2, but the one sided loop with label $\frac{N}{2}$ increases degree of each vertex by one, which is required for some base graphs in order to have a cubic (or odd regular) derived graph. Section 3.2 is about Theta or Ears graphs whose base graphs have two vertices. We will see they are good candidates of having girth 5 or 6. Section 3.3 is about the base graphs with three vertices. One of the well known examples is the Tutte-Coxeter graph. It has a representation as a voltage graph with the base graph B as in figure 2.3 with three vertices where one of them has a single side loop with label 5 in $\mathbb{Z}_{10}$. Section 3.5 is for the base graphs with four vertices. The well known example of the this section would be the Coxeter graph.

A related question is, "What is the maximum girth or the maximum AC for derived graphs from the base graph B?" For cubic circulant graphs, the maximum girth is 4 by theorem 3.1.2 as we will show in section 3.1. Aside from finding derived cubic graphs with large AC, exploring for the derived cubic graphs with large girth is interesting. We can see in table 3.11, there is a derived graph(from the Base4(b)(1,2,4)) with 36 vertices and maximum girth 7 whose AC is larger than all cubic derived graph with girth 8. This graph is a counter example for conjecture 6.1 in [1]. In this chapter, we will see tables with maximum AC for derived voltage graphs. The interesting remark is in most cases (but not all) the graph with Max-AC also maximizes the girth in those derived voltage graphs.

3.1 Cubic Circulant Graphs

A circulant graph is a voltage graph whose base consists of a single vertex such as shown in figure 3.1. We note to the base graph as shown in figure 3.1(a) for derived circulant graphs in general not just a derived cubic circulant graph. We start with the following theorem about connectivity of circulant graphs.

Figure 3.1: (a) The base graph B of circulant graphs.
(b) The base graph for cubic circulant graphs.

Theorem 3.1.1. *The derived circulant graphs whose base graph has labels* $l_1, l_2, \cdots, l_t$ *is connected if and only if* $GCD(l_1, l_2, \cdots, l_t, N) = 1$.

Proof. If $GCD(l_1, l_2, \cdots, l_t, N) = 1$, then by Bezout lemma there exists a_i for $1 \leq i \leq t+1$ such that

$$l_1a_1 + l_2a_2 + \cdots + l_ta_t + Na_{t+1} = 1$$

or

$$l_1a_1 + l_2a_2 + \cdots + l_ta_t = 1 \text{ mod N}.$$

This gives a path. This path consists of following edges with label l_1, a_1 times, l_2, a_2 and so on. It is from an arbitrary vertex v of the derived graph which connects the vertex v to the vertex $v+1$ mod N in $\mathbb{Z}_N$ when we marked vertices of the derived graph by elements of $\mathbb{Z}_N$. Since circulant graphs are symmetric, if we rotate this path, then we can connect the vertex $v+1$ to vertex $v+2$ (addition is mod N) of the graph and so on. Hence, the derived circulant graph is connected since there is a path between any two vertices of the graph.

To show the converse, suppose the graph is connected. Then, there is a path between an arbitrary vertex v to the vertex $v+1$ in the graph. This means there exists a_i for $1 \leq i \leq t$ such that

$$l_1a_1 + l_2a_2 + \cdots + l_ta_t = 1 \text{ mod N}.$$

By Bezout lemma, we have $GCD(l_1, l_2, \cdots, l_t, N) = 1$. $\square$

Theorem 3.1.2. *The maximum girth of derived graphs with the base graph as shown in figure 3.1(a) is four.*

Proof. Consider the path

$$l_1 \to l_2 \to -l_1 \to -l_2.$$

This path starts and ends at the same vertex. It has no repeated edges(assuming $l_1 \neq \pm l_2 \bmod N$). Therefore, the girth would be at most four. $\square$

In the following figure, we can see an example of the circulant graph with $\mathbb{Z}_6$ where we show the girth is 4 by drawing a directed cycle of length 4 based on the proof of the above theorem.

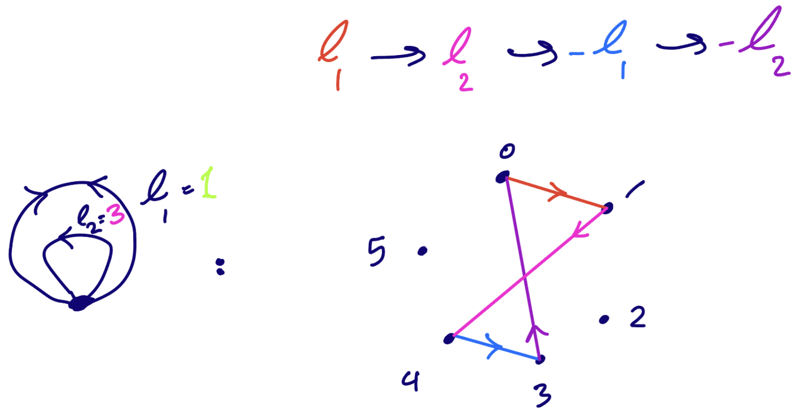

Figure 3.2: An example of the girth of a circulant graph.

Because of theorem 3.1.2, circulant graphs are not a good candidates for graphs with high girth. However, they may still have high AC. Now we show that cubic circulant graphs have the following AC.

Theorem 3.1.3. *The AC of any connected derived cubic circulant graph from the base graph in figure 3.1(b) is*

$$2 - 2\cos\frac{4\pi}{N}.$$

Proof. The spectrum of a derived cubic circulant graph from the base graph in fig 3.1(b) whose labels are l and $\frac{N}{2}$ is (see (2.2))

$$\lambda_m = 2\cos\left(\frac{2\pi m l}{N}\right) + (-1)^m, \quad \text{where} \quad m \in \{0, 1, \cdots, N-1\}.$$

Note that $\lambda_m = 3$ when $m = 0$. Therefore,

$$AC = \min_{m \neq 0}(3 - \lambda_m). \tag{3.1}$$

We want to have maximum value for λ_m which is dependent on $\cos(\frac{2\pi ml}{N})$ and $(-1)^m$. The value of AC in (3.1) is the smallest value of

$$3 - 2(\max_{m\neq 0, m \text{ even}}(\cos \frac{2\pi ml}{N})) - 1 = 2 - 2(\max_{m\neq 0, m \text{ even}}(\cos \frac{2\pi ml}{N}))$$

or

$$3 - 2(\max_{m\neq 0, m \text{ odd}}(\cos \frac{2\pi ml}{N})) + 1 = 4 - 2(\max_{m\neq 0, m \text{ odd}}(\cos \frac{2\pi ml}{N})).$$

So, m should be even since $|\cos\theta| \leq 1$ for any θ. One of the interval that cos gets its maximum value is $[0, \frac{\pi}{2})$. We claim there exist such m to satisfy

$$ml \equiv 2 \mod N \tag{3.2}$$

Such m maximizes

$$\max_{m\neq 0, m \text{ even}}(\cos \frac{2\pi ml}{N}).$$

Now m and N are assumed to be even, so we can write $m = 2\hat{m}$ and $N = 2\hat{N}$. If $ml \equiv 2$ mod N, then (3.2) is equivalent to

$$2l\hat{m} = 2 + aN = 2 + 2a\hat{N} \tag{3.3}$$

for some a and $\hat{m}$, or

$$l\hat{m} = 1 + a\hat{N} \quad \text{or} \quad l\hat{m} - a\hat{N} = 1 \tag{3.4}$$

for some a and $\hat{m}$. Since the circulant graph is connected, by the theorem 3.1.1, $GCD(l, \hat{N}) = 1$. Then, by Bezout identity, the equation 3.4 has solution for a and $\hat{m}$. □

The following table gives maximum AC for different N for derived circulant graphs.

Derived Cubic Circulant Graphs via base graph B													
N	4	6	8	10	12	14	16	18	20	22	24	26	126
Max AC	4	3	2	1.382	1	0.753	0.5858	0.4679	0.382	0.3175	0.2679	0.2291	0.0099

3.2 Cubic Theta Graphs

The base graph B has two vertices and multiple edges with different labels as we can see in figure 3.3(a). In order to have a cubic derived graph G, we use the base graph B with $t = 3$ labels as in figure 3.3(b).

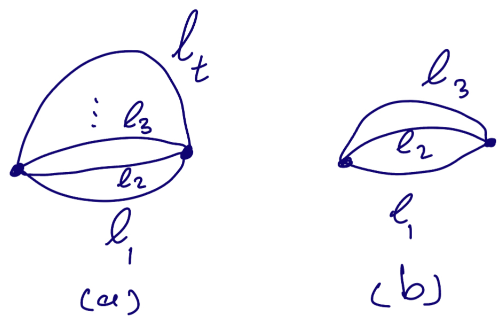

Figure 3.3: (a) The base graph B of derived t-regular Theta graphs.
(b) The base graph B of derived cubic Theta graphs.

Definition 3.2.1. *The derived graph G from the base graph shown in 3.3(a) with labels*

$$l_1, l_2, \cdots, l_t$$

will be denoted by $(l_1, l_2, \cdots, l_t)$.

□

Lemma 3.2.2. 1. *The graph $(l_1, l_2, \cdots, l_t)$ is isomorphic to the graph $(l_1{+}a, l_2{+}a, \cdots, l_t{+}a)$ for any a in $\mathbb{Z}_N$,*

2. *the graph $(l_1, l_2, \cdots, l_t)$ is isomorphic to the graph $(l_1 a, l_2 a, \cdots, l_t a)$ for any invertible a in $\mathbb{Z}_N$.*

Proof. Part 1: Geometrically, figure 3.4 shows that the only difference between labeling $l_1, l_2, \cdots, l_t$ and $l_1{+}a, l_2{+}a, \cdots, l_t{+}a$ (where a=1) is just a permutation of the outer ring in $\mathbb{Z}_N$ since all labels are mod N and the derived graph is symmetric. The following automorphism ϕ shows that if we label vertices by

$$\{u_0, u_1, \cdots, u_{N-1}\} \bigcup \{v_0, v_1, \cdots, v_{N-1}\},$$

then those two graphs are isomorphic by fixing the vertices of inner ring, that is,

$$\{u_0, u_1, \cdots, u_{N-1}\}$$

and their label in $\mathbb{Z}_N$ and relabeling vertices of outer ring where indices changed mod N, that is,

$$\{v_0, v_1, \cdots, v_{N-1}\} \to \{v_a, v_{1+a}, \cdots, v_{N-1+a}\}$$

by a proper permutation in $\mathbb{Z}_N$.

$$\phi : \mathbb{Z}_N \to \mathbb{Z}_N$$

$$\phi(v_i) = v_{i+a}, \quad \phi(u_i) = u_i$$

where $i \in \{0, 1, \cdots, N-1\}$ and the indices are taken mod N. So, ϕ preserves the structure of graph.

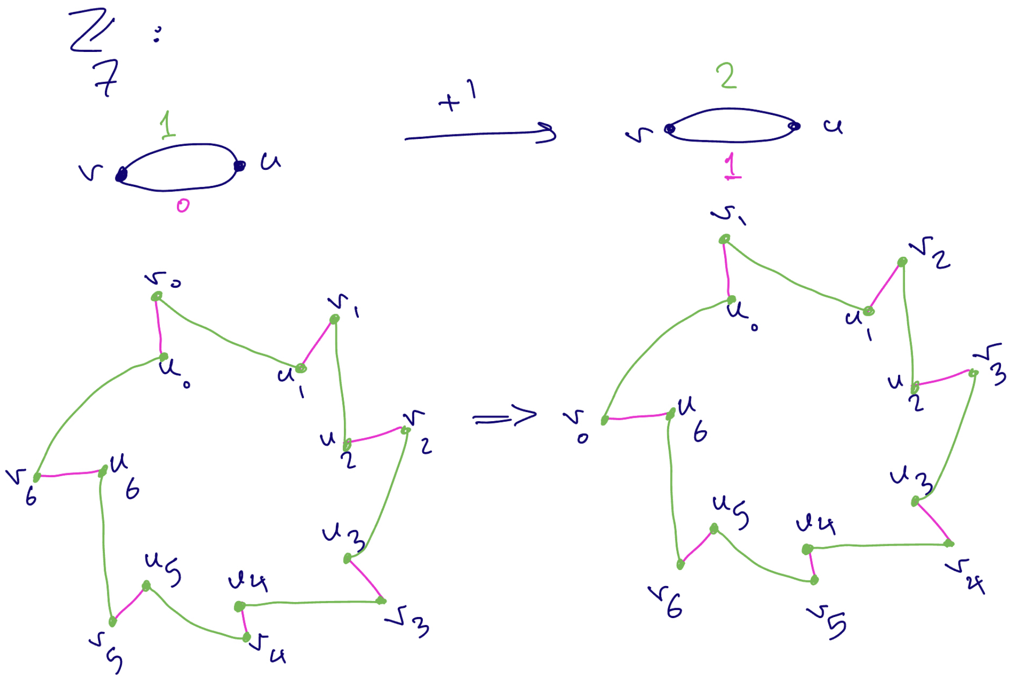

Figure 3.4: Changing labels of the base graph of derived Theta graph.

Part 2: this is a consequence of theorem 2.0.7. □

Corollary 3.2.3. *Let (a,b,c) be a connected theta-graph as shown in figure 3.3. Then, the specturm of (a,b,c) is same as*

1. *the spectrum of (a+g,b+g,c+g) for any* $g \in \mathbb{Z}_N$,
2. *the spectrum of (ag,bg,cg) for any invertible* $g \in \mathbb{Z}_N$.

Based on the table 3.2, the first observation would be the first two labels should be 0 and 1, that is $l_0 = 0$ and $l_1 = 1$ for the base graph of any derived theta graph. But we note that not all derived theta graphs whose base graph is $(l_0, l_1, \cdots, l_t)$ are isomorphic to the derived graph from $(0, 1, l_2, \cdots, l_t)$. The smallest counter example is the base graph $(0, 2, 5)$ with $N = 30$. We will see the following theorem which is true for any prime N.

Theorem 3.2.4. *If N is prime, then there exist $l \in \mathbb{Z}_N$ such that (l_0, l_1, l_2) (for any l_0, l_1, and l_2 in $\mathbb{Z}_N$) is isomorphic to $(0, 1, l)$.*

Proof. First of all, if all of labels (l_0, l_1, l_2) are non zero, we choose an arbitrary label such as l_0. Then, subtract all labels by l_0. We get $(0, l_1 - l_0, l_2 - l_0)$. Now both of $l_1 - l_0$ and $l_2 - l_0$ are invertible in $\mathbb{Z}_N$. We arbitrarily choose $l_1 - l_0$ which is invertible in $\mathbb{Z}_N$. Now by lemma 3.2.2 we can multiply the labels by $(l_1 - l_0)^{-1}$ to get $(o, 1, l)$ which is isomorphic to (l_0, l_1, l_2). □

Remark 3.2.5. In above theorem, N can be also a product of two prime numbers as well. We omit the proof. ▲

The following figures show the comparison(relationship) between different l in $(0, 1, l)$ with maximum AC. For example, when $N = 17$, all $(0, 1, 4)$, $(0, 1, 5)$, $(0, 1, 7)$, $(0, 1, 11)$, $(0, 1, 13)$, and $(0, 1, 14)$ achieve the maximum Ac, 0.5626. It is interesting that all of the derived graphs of these base graphs are isomorphic to each other. Additionally, when $N = 111$, $(0, 1, 11)$ and $(0, 1, 101)$ achieve the maximum Ac, 0.1174. $(0, 1, 11)$ and $(0, 1, 101)$ are also isomorphic. We see that the label l in $(0, 1, l)$ that gets the maximum AC is not unique and there is not any clear relation between N and the label l that maximize AC.

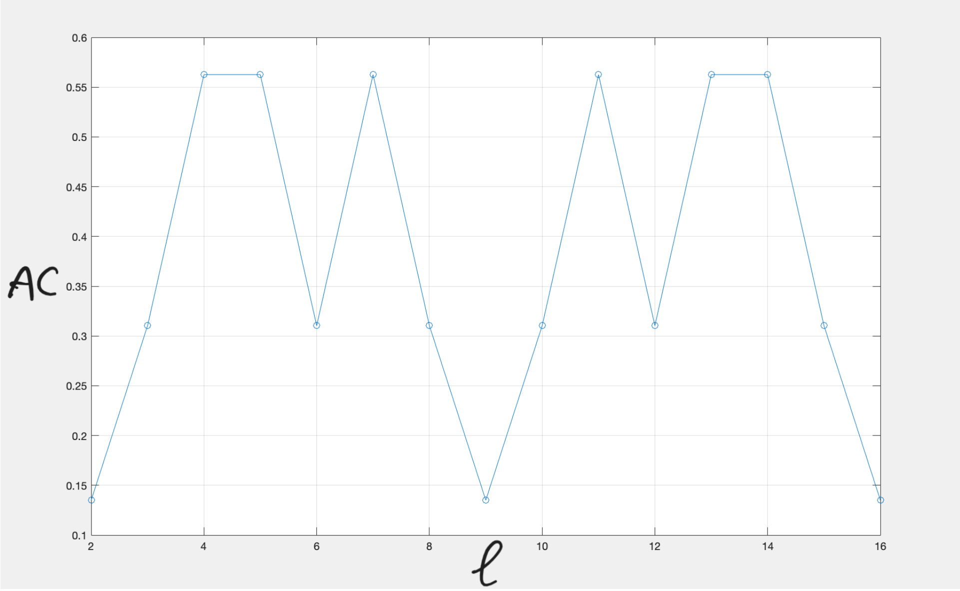

Figure 3.5: Comparison of different labels $(0, 1, l)$ with maximum Ac for $N = 17$ of the derived Theta graph.

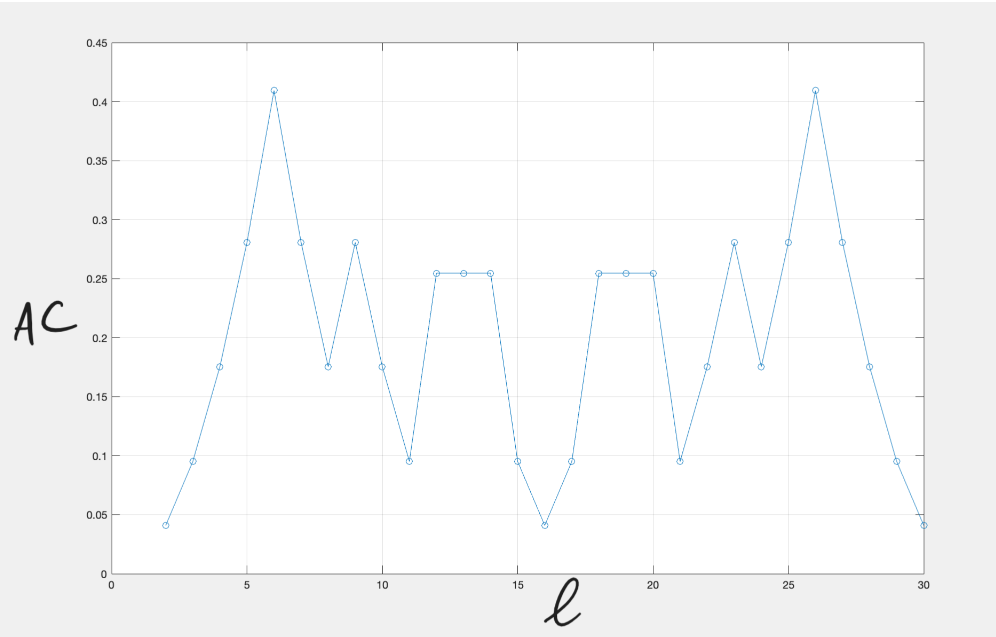

Figure 3.6: Comparison of different labels $(0, 1, l)$ with maximum AC for $N = 31$ of the derived Theta graph.

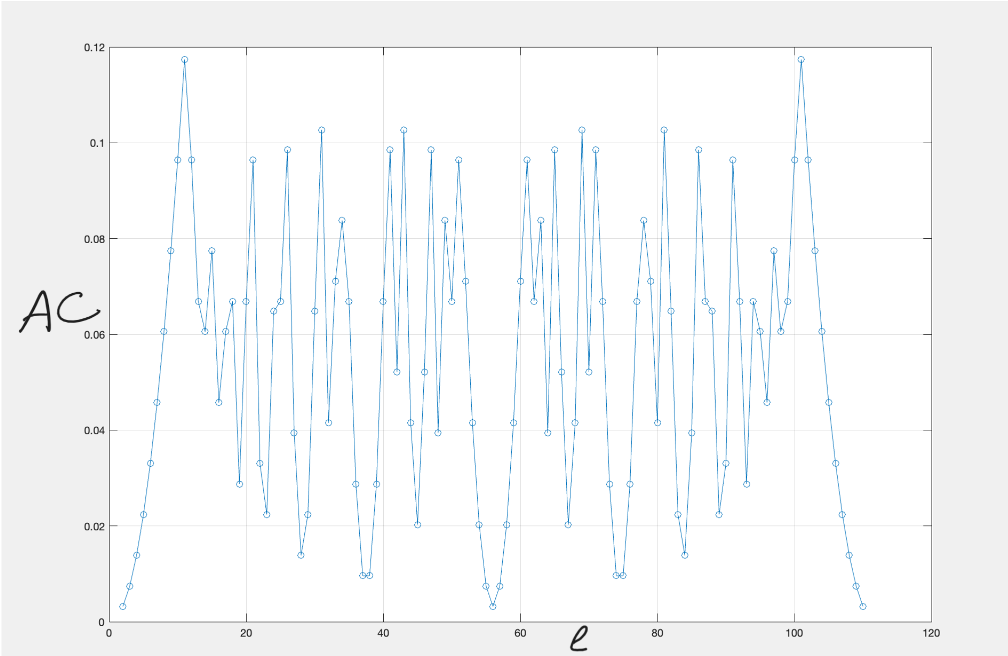

Figure 3.7: Comparison of different labels $(0, 1, l)$ with maximum AC for $N = 111$ of the derived Theta graph.

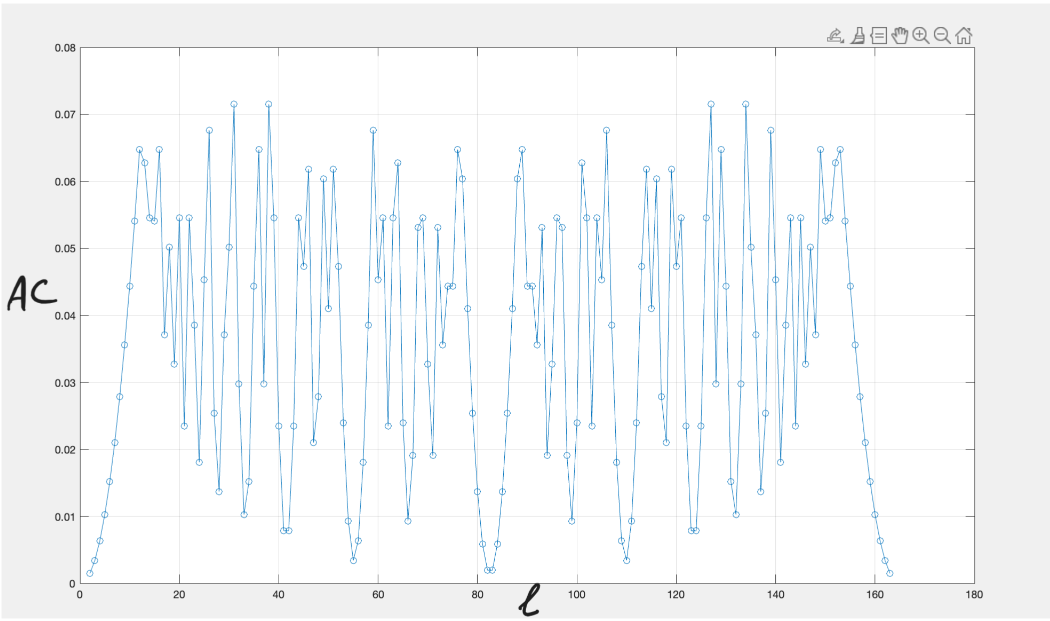

Figure 3.8: Comparison of different labels $(0, 1, l)$ with maximum AC for $N = 164$ of the derived Theta graph.

In the following theorem, we see that any derived theta graph can have a cycle of length at least six.

Theorem 3.2.6. *The maximum girth of the derived graphs from the base graph as shown in figure 3.3 is six.*

Proof. Consider the following path

$$a \to -b \to c \to -a \to b \to -c,$$

without repeated vertices. Then, this corresponds to a cycle of length at most six. □

The following table contains the maximum AC of the derived theta graph from the base graph (a, b, c) for different N. In addition to theorem 3.2.6, the table shows that the maximum girth is 6.

Table 3.2: Derived Cubic Theta Graphs

N	Max AC	Max AC Maximizer Labels(a,b,c)	Girth
3	3	0,1,2	4
4	2	0,1,3	4
5	1.382	0,1,4	4
6	1.2679	0,1,4	4
7	1.5858	0,1,3	6
8	1.2679	0,1,6	6
9	1.0304	0,1,3	6
10	0.8510	0,1,3	6
11	0.7134	0,1,3	6
12	0.7639	0,1,4	6
13	0.9257	0,1,4	6
14	0.7530	0,1,4	6
15	0.7118	0,1,4	6
16	0.6308	0,1,4	6
17	0.5626	0,1,4	6
18	0.5047	0,1,4	6

19	0.6533	0,1,8	6
20	0.5028	0,1,5	6
21	0.5935	0,1,5	6
22	0.4946	0,1,5	6
23	0.4542	0,1,5	6
24	0.4604	0,1,5	6
25	0.4257	0,1,5	6
26	0.3948	0,1,5	6
27	0.3670	0,1,5	6
28	0.3421	0,1,5	6
29	0.3195	0,1,5	6
30	0.3820	0,1,9	6
31	0.4093	0,1,6	6
32	0.3491	0,1,6	6
33	0.3175	0,1,6	6
34	0.3103	0,1,6	6
35	0.3237	0,1,6	6
63	0.1860	0,1,8	6

As we see in above table, there is not strictly decreasing MaxAC when N is increasing; we show it in table with blue. We can see its pattern in the following image.

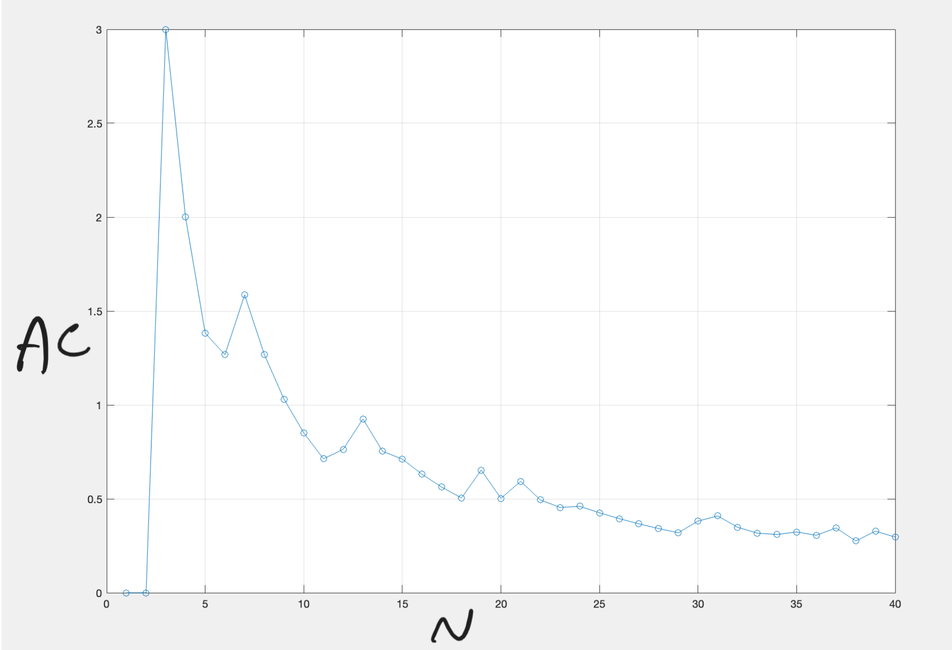

Figure 3.9: (a) The base graph B.

3.3 Cubic Ears Graphs

An Ears graph is defined to be the derived graph whose base graph B is as shown in figure 3.10. The difference is single or double sided loops. In figure 3.10(a), $a, b \neq \frac{N}{2}$ and in figure 3.10(b), N should be even. We note that the edges of the spanning tree of each graph in figure 3.10 get 0 as their label by using theorem 2.0.8.

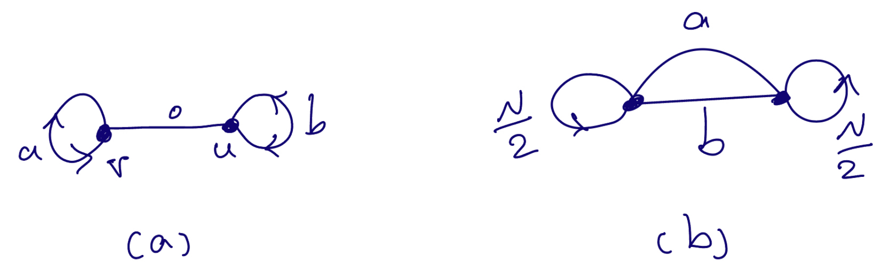

Figure 3.10: (a) The base graph B of cubic Ears graphs,
(b) the base graph B with multiple edges of cubic Ears graphs.

Now one of the interesting questions is what is the formula of spectrum of the derived graph of the base graph in figure 3.10(a)? The answer is the eigenvalues of the derived graph is given by λ_m, where λ_m is the eigenvalues of

$$M = \begin{bmatrix} z^a + z^{-a} & 1 \\ 1 & z^b + z^{-b} \end{bmatrix},$$

with $z = \exp\left(\frac{2\pi i m}{N}\right)$ and $m \in \{0, 1, \cdots, N-1\}$. Equivalently, λ_m satisfies

$$\lambda^2 + 2(\cos\frac{2m\pi a}{N} + \cos\frac{2m\pi b}{N})\lambda + 2\cos\frac{2m\pi(a+b)}{N} + 2\cos\frac{2m\pi(a-b)}{N} - 1 = 0.$$

Similarly, the spectrum of the derived graph of the base graph in figure 3.10(b) is given by the eigenvalues of λ_m, where λ_m is the eigenvalues of

$$M = \begin{bmatrix} z^{\frac{N}{2}} & z^a + z^b \\ z^{-a} + z^{-b} & z^{\frac{N}{2}} \end{bmatrix},$$

where $z = \exp\left(\frac{2\pi i m}{N}\right)$, and $m \in \{0, 1, \cdots, N-1\}$. Equivalently, λ_m satisfies

$$\lambda^2 - 2\lambda(-1)^m - 2\cos\frac{2\pi m(a-b)}{N} - 1 = 0.$$

The following table compares maximum AC and the girth of derived Ears graphs whose base graph are shown in figure 3.10. The first three columns of the table corresponds to the base graph in figure 3.10(a) and the last three columns are correspond to the base graph in figure 3.10(b). For the rest of this chapter we should note to the following remark.

Remark 3.3.1. For any base graph B where the maximizer labels of AC are not same as the maximizer labels of girth and the maximum girth is g with labels (i, j), we use MaxG(B)=g(i,j) to show that the maximum girth corresponds to this base. ▲

We note that for all N in this table, the maximum AC of derived graphs from the base graph 3.10(a) are larger than maximum AC of the derived graph from the base graph 3.10(b).

Derived Cubic Graphs via Ears the base graph in figure 3.10							
N	Max AC(a)	(a,b)	Girth	Max AC(b)	(a,b)	Girth	Comment
2				4	0,1	3	
3	2	1,1	3				

4	2	1,1	4	2	0,1	4	
5	2	1,2	5				
6	1	1,1	4	1	0,1	4	
7	1.2892	1,2	5				
8	1.2679	1,3	6	0.5858	0,1	4	
9	0.8980	1,2	5				
10	1	1,3	6	0.3820	0,1	4	
11	0.8984	1,3	6				
12	1	1,5	6	0.2679	0,1	4	
13	0.9474	1,5	7				
14	0.7639	1,4	7	0.1981	0,1	4	
15	0.7639	1,4	7				
16	0.7639	1,6	7	0.1522	0,1	4	
17	0.7731	1,5	7				
18	0.7375	1,5	8	0.1206	0,1	4	
19	0.6350	1,7	7				
20	0.5858	1,8	5	0.0979	0,1	4	
21	0.7530	1,8	8				
22	0.6564	1,6	8	0.0810	0,1	4	
23	0.5779	1,5	8				
24	0.5858	1,7	8	0.0681	0,1	4	
25	0.6903	1,7	8				
26	0.6278	1,10	8	0.0581	0,1	4	
27	0.5731	1,6	8				
28	0.5822	1,8	7	0.0501	0,1	4	MaxG(a)=8(1,5)
29	0.5455	1,8	8				
30	0.5121	2,9	8	0.0437	0,1	4	
31	0.4816	1,7	8				
32	0.5524	1,7	8	0.0384	0,1	4	
33	0.4533	1,6	8				
34	0.5170	1,13	8	0.0341	0,1	4	

35	0.4755	1,10	7				MaxG(a)=8(1,6)
63	0.3005	1,14	8				

In the following table, we compare all results of the derived theta graphs with the derived ears graphs from the figure 3.10(a). We see for $3 \leq N \leq 9$ and $N \neq 5$, the theta graphs have larger AC. The larger maximum AC of these graphs shown in red. We note when $N = 5$, the derived graph is the Peterson graph with maximum AC=2.

Table 3.4: Comparison of Derived Cubic Theta Graphs with Derived Ears graphs

N	Max AC(Ears-111)	Max AC(theta)
3	2	3
4	2	2
5	2	1.382
6	1	1.2679
7	1.2892	1.5858
8	1.2679	1.2679
9	0.8980	1.0304
10	1	0.8510
11	0.8984	0.7134
12	1	0.7639
13	0.9474	0.9257
14	0.7639	0.7530
15	0.7639	0.7118
16	0.7639	0.6308
17	0.7731	0.5626
18	0.7375	0.5047
19	0.6350	0.6533

20	0.5858	0.5028
21	0.7530	0.5935
22	0.6564	0.4946
23	0.5779	0.4542
24	0.5858	0.4604
25	0.6903	0.4257
26	0.6278	0.3948
27	0.5731	0.3670
28	0.5822	0.3421
29	0.5455	0.3195
30	0.5121	0.3820
31	0.4816	0.4093
32	0.5524	0.3491
33	0.4533	0.3157
34	0.5170	0.3103
35	0.4755	0.3237
63	0.3005	0.1860

3.4 Base graphs with 3 vertices

There are four possible base graphs with three vertices and at least one single side loop, see figure 3.11. In these base graphs, the edges of the spanning tree get 0 as their labels by using theorem 2.0.8. Note that N must be even in order to have a cubic derived graph.

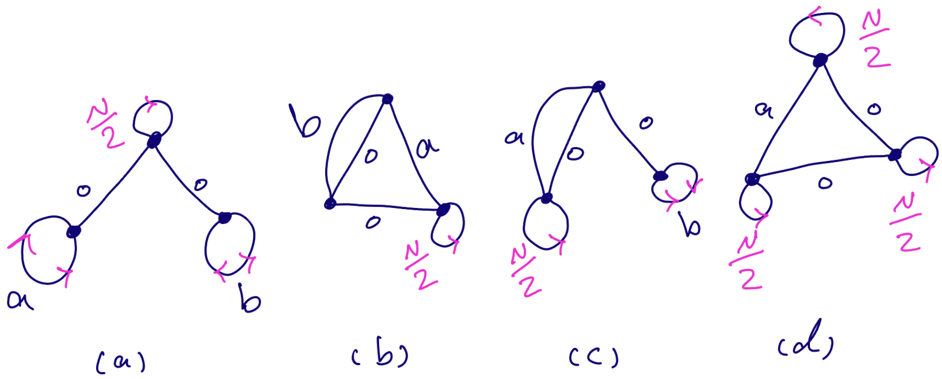

Figure 3.11: The base graph B with 3 vertices.

In the following table, we see maximum AC of the derived graphs from all possible base graphs with three vertices for $2 \leq N \leq 34$. The maximum AC for these derived graphs for any N shown in red.

Derived Cubic Circulant Graphs via base graphs in figure 3.11								
N	Max AC(a)	Labels(a,b)	Max AC(b)	Labels(a,b)	Max AC(c)	Labels(a,b)	Max AC(d)	Labels(a)
2	0.4384	2,2	2	1,1	0.5858	1,2	3	1
4	1	1,1	1	1,1	1.2679	1,1	1	1
6	1	1,1	1	1,4	1.2679	2,1	0.4679	1
8	1	1,3	1	1,2	0.8299	1,3	0.2679	1
10	1	1,3	0.7639	1,3	0.7118	1,3	0.1729	1
12	0.5858	1,1	0.6971	1,2	0.5858	1,2	0.1206	1
14	0.6931	1,3	0.7530	1,11	0.5405	1,3	0.0889	1
16	0.6571	1,3	0.6343	1,2	0.4877	1,3	0.0681	1
18	0.5010	1,2	0.5505	1,3	0.4679	1,4	0.0539	1
20	0.5188	1,3	0.5064	1,7	0.4406	1,4	0.0437	1
22	0.4679	1,3	0.4321	1,4	0.4169	1,4	0.0361	1
24	0.4859	1,5	0.4384	1,3	0.4145	1,5	0.0304	1
26	0.4571	1,5	0.4055	1,3	0.3996	1,5	0.0259	1
28	0.3566	1,3	0.3622	1,3	0.3590	1,4	0.0223	1
30	0.3820	1,4	0.3820	1,3	0.3886	1,6	0.0195	1

32	0.3636	1,7	0.3636	1,6	0.3788	1,6	0.0171	1
34	0.3729	1,5	0.3437	1,10	0.3480	1,5	0.0152	1
42	0.3682	1,8	0.2712	1,12	0.3386	1,8	0.0099	1

3.5 Base graphs with 4 vertices

In this section, we have three categories of base graphs with four vertices. The first category is as subsection 3.5.1 with base graphs that have single sided loops. The second subsection 3.5.2 is based on base graphs that have both single and double sided loops. The third subsection 3.5.3 have base graphs that have not any single sided loops.

3.5.1 Base graphs with 4 vertices without single sided loops

There are two base graphs without any loop, shown in figure 3.12. By using theorem 2.0.8, all edges of their spanning tree get 0 as their labels. In the following table, we summarize all results of the maximum AC of derived graphs from these base graphs. In addition, there are some exceptional derived graphs that have different labels for maximum AC and the maximum girth in the base graph. This property shown by red color in comments column by Max G(B)=g(a,b,c) which means the maximum girth is g by labels a,b, and c in the base graph B.

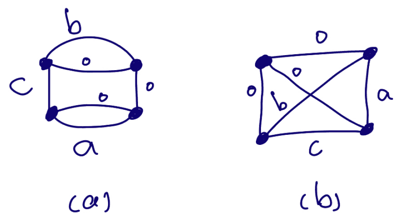

Figure 3.12: The base graph B with 4 vertices without loop.

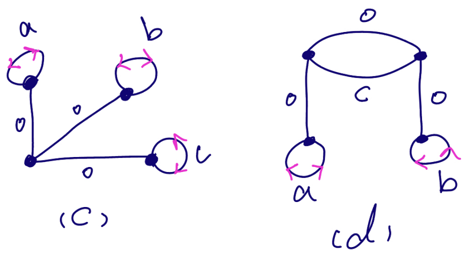

Figure 3.13: The base graph B with 4 vertices Base with 4 vertices with double side loops.

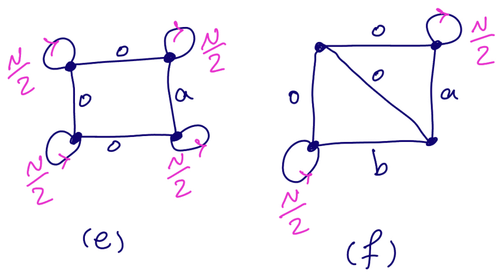

Figure 3.14: The base graph B with 4 vertices Base with 4 vertices with single side loops.

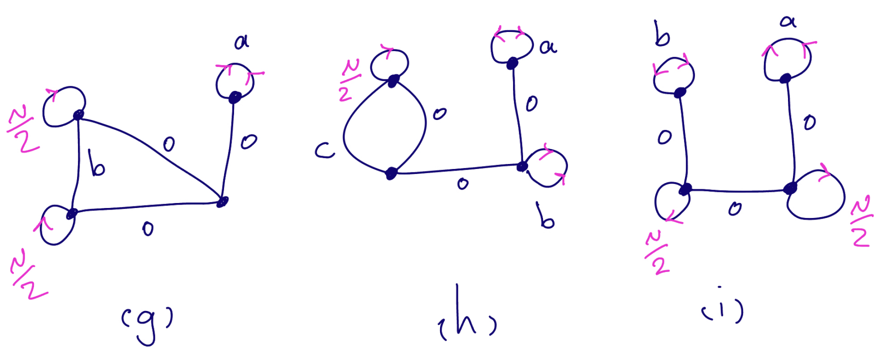

Figure 3.15: The base graph with 4 vertices with single and double side loops.

Derived Cubic Graphs from the base graph in figure 3.12							
N	Max AC(a)	Labels(a,b,c)	Girth	Max AC(b)	Labels(a,b,c)	Girth	Comments
2	2	1,1,1	4	2	1,1,1	4	
3	1.2679	1,1,1	4	1	1,1,1	4	
4	1.2679	1,1,1	6	1.2679	1,3,1	6	
5	1	1,2,1	6	0.9316	1,2,3	6	
6	1	1,1,2	6	0.7639	1,2,3	6	
7	0.7530	1,1,2	6	0.7928	1,2,3	6	
8	0.7639	1,3,1	6	0.7639	1,2,4	6	
9	0.7375	1,2,3	8	0.7766	1,2,4	7	
10	0.7639	2,4,1	8	0.6554	1,2,4	7	
11	0.6805	1,2,3	8	0.6030	1,5,1	6	
12	0.5858	1,2,3	8	0.6239	1,2,5	7	
13	0.6433	1,3,4	8	0.5948	1,2,5	7	
14	0.5858	1,6,3	8	0.6086	1,3,7	6	Max G(b)=7(1,2,4)
15	0.5408	1,3,4	8	0.6506	1,7,2	9	
16	0.5858	1,3,4	8	0.5858	1,3,7	8	Max G(b)=9(1,7,2)
17	0.5302	1,3,4	8	0.5287	1,3,7	9	
18	0.5047	1,8,4	8	0.5703	1,3,7	10	
19	0.5262	1,3,5	8	0.5193	1,3,7	9	
20	0.5505	2,6,1	10	0.5002	1,8,2	9	Max G(b)=10(1,3,7)
21	0.4851	1,7,4	6	0.4859	1,3,8	9	Max G(a)=8(1,2,3)
22	0.4946	1,10,4	10	0.4352	1,4,9	9	Max G(b)=10(1,3,7)
23	0.4933	1,7,3	10	0.4264	1,3,9	9	Max G(b) =10(1,3,7)
24	0.4938	1,4,6	10	0.4859	2,3,9	9	Max G(b)=10(1,3,7)
25	0.4593	1,9,4	10	0.4188	1,3,10	10	
26	0.4450	1,3,7	10	0.4389	1,4,9	10	
27	0.4424	1,12,5	10	0.3982	1,5,12	10	

28	0.4549	1,3,8	10	0.4292	1,6,15	10	
29	0.4134	1,3,8	10	0.3819	1,3,9	10	
30	0.4373	2,3,9	10	0.4026	2,3,9	10	
31	0.3937	1,4,10	10	0.3551	1,3,13	10	
32	0.4013	1,4,10	10	0.3575	1,3,10	10	
33	0.4297	1,15,6	10	0.3672	1,3,10	10	
34	0.4213	1,12,26	10	0.3718	1,4,14	10	
35	0.3999	1,15,8	10	0.3654	1,5,12	10	

The base graphs in figure 3.13 have four vertices and double side loops. So, N should be even in order to have the cubic derived graph. The labels of the edges of the spanning tree of these base graphs get 0 as their labels by using theorem 2.0.8. The following table has all results of the maximum AC of the derived graphs from these base graphs for $2 \leq N \leq 35$.

Derived Cubic Graphs from the base graph in figure 3.13.							
N	Max AC(c)	Labels(a,b,c)(skip N/2)	Girth	Max AC(d)	Labels(a,b,c)(skip N/2)	Girth	Comments
2				0.5858	2,2,1	1	
3	1	1,1,1	3	0.7639	1,1,1	3	
4	1	1,1,1	4	0.7639	1,1,1	4	
5	1	1,1,2	5	0.7639	1,1,1	5	
6	0.6972	1,1,1	6	0.7639	1,1,2	6	
7	1	1,2,3	7	0.7639	1,1,3	7	
8	0.6340	1,1,3	6	0.7639	1,1,3	7	
9	0.7766	1,2,4	7	0.6972	1,1,3	6	MaxG(d)=7(1,1,4)
10	0.5107	1,2,3	5	0.7639	1,1,4	8	MaxG(c)=6(1,1,1)
11	0.6283	1,2,4	7	0.6564	1,1,4	8	

12	0.5107	1,2,5	6	0.5935	1,5,3	7	MaxG(d)=8(1,1,5)
13	0.6066	1,3,4	8	0.6531	1,2,5	8	
14	0.5858	1,3,5	8	0.5858	1,2,5	7	MaxG(d)=8(1,1,4)
15	0.4859	1,3,4	5	0.6972	1,4,6	9	MAxG(c)=7(1,2,4)
16	0.5218	1,3,5	8	0.5858	1,7,6	8	
17	0.4938	1,3,5	8	0.5437	1,2,7	9	
18	0.5703	1,5,7	10	0.5376	1,5,7	9	
19	0.5849	1,7,8	9	0.5304	1,2,7	9	
20	0.4971	1,6,9	8	0.5858	2,6,5	8	MaxG(d)=10(1,9,6)
21	0.4859	1,6,8	7	0.4845	1,3,8	7	MaxG(c)=9(1,4,5), MaxG(d)=9(1,2,5)
22	0.5107	1,5,8	9	0.4944	1,5,8	10	
23	0.4758	1,4,10	9	0.4849	1,2,9	9	
24	0.4384	1,4,7	6	0.4390	2,3,7	8	MaxG(c)=8(1,3,5), MAxG(d)=9(1,2,5)
25	0.4536	1,4,11	9	0.4755	1,4,10	10	
26	0.5070	1,6,11	10	0.4460	2,6,5	10	
27	0.4786	1,6,8	9	0.4679	1,5,8	10	MAxG(c)=10(1,8,10)
28	0.4384	1,5,12	7	0.4410	2,6,7	8	MaxG(c)=9(1,5,6), MAxG(d)=10(1,3,8)
29	0.4511	1,5,11	10	0.4129	1,5,11	9	MaxG(d)=10(1,3,8)
30	0.4444	2,5,9	6	0.4414	2,8,9	10	MAxG(c)=10(1,7,11)
31	0.4278	1,7,9	10	0.3822	1,5,9	9	MaxG(d)=10(1,3,8)
32	0.4054	1,7,12	8	0.4206	1,5,13	10	MaxG(c)=10(1,5,7)
33	0.4384	1,6,10	10	0.3898	1,6,8	9	MAxG(d)=10(1,3,8)
34	0.4226	1,6,15	10	0.4007	1,14,9	10	
35	0.4312	1,8,15	7	0.4355	1,6,10	10	MaxG(c)=10(1,6,8)

3.5.2 Base graphs with 4 vertices and single sided loops

In figure 3.14, two base graphs have only single side loops. We note the labels of the edges of their spanning trees is 0 (see theorem 2.0.8). By noting to the following table, we can compare maximum AC of the derived graphs of the base graphs as shown in figure 3.14. The maximum AC of the derived graphs from the base graph 3.14(e) (or 3.14(f)) showed by Max AC(e)(or Max AC(f)). The next two columns to the maximum AC is the corresponding labels in the base graph and the girth of the derived graph from these labels.

Derived Cubic Graphs from the base graph B in figure 3.14						
N	Max AC(e)	Labels(a)	Girth	Max AC(f)	Labels(a,b)	Girth
2	2	1	4	2	1,1	4
4	0.5858	1	4	0.7639	1,1	5
6	0.2679	1	4	1	1,1	6
8	0.1522	1	4	0.5858	1,1	6
10	0.0979	1	4	0.4755	1,2	6
12	0.0681	1	4	0.4859	1,2	6
14	0.0501	1	4	0.4474	1,2	6
16	0.0384	1	4	0.4071	1,5	6
18	0.0304	1	4	0.3663	1,3	6
20	0.0246	1	4	0.3104	1,6	6
22	0.0204	1	4	0.2600	1,3	6
24	0.0171	1	4	0.2907	1,3	6
26	0.0146	1	4	0.2518	1,3	6
28	0.0126	1	4	0.2514	1,10	6
30	0.0110	1	4	0.2212	1,11	6
32	0.0096	1	4	0.2183	1,4	6
34	0.0085	1	4	0.2087	1,10	6

3.5.3 The base graphs with 4 vertices and single and double sided loops

In figure 3.15, the base graphs have both single and double side loops. The label of the edges of their spanning tree is 0 by using theorem 2.0.8. By noting to the following table, we can compare maximum AC of the derived graphs of the base graphs as shown in figure 3.15. The maximum AC of the derived graphs from the base graph 3.14(g) (or other base graphs) showed by Max AC(g)(or Max AC(h) or Max AC(i)). The next two columns to this maximum AC is the corresponding labels in the base graph and the girth of the derived graph from these labels.

Derived Cubic Graphs from the base graph in figure 3.15										
N	Max AC(g)	(a,b)	Girth	Max AC(h)	(a,b,c)	Girth	Max AC(i)	(a,b)	Girth	Comments
2	0.5107	2,1	1	0.2679	2,2,1	1	0.5858	2,2	1	
4	1	1,1	4	0.6571	1,1,1	4	0.5858	1,1	4	
6	0.6472	1,1	4	0.6571	1,1,1	4	0.5858	1,1	4	
8	0.4384	1,1	4	0.6571	1,1,1	4	0.5858	1,1	4	
10	0.3820	1,2	4	0.6571	1,2,1	5	0.5858	1,2	4	MaxG(h)=6(1,3,1)
12	0.3236	2,1	4	0.6236	2,3,1	4	0.3820	1,1	4	MAxG(h)=6(1,5,1)
14	0.3043	1,2	4	0.5544	1,6,2	6	0.4329	1,2	4	
16	0.2799	1,3	4	0.5129	1,4,3	4	0.4071	1,3	4	MAxG(h)=6(1,3,1)
18	0.2662	1,2	4	0.4803	3,1,2	6	0.3238	1,2	4	
20	0.2591	2,3	4	0.4597	1,5,3	4	0.3249	1,3	4	MaxG(h)=6(1,3,1)
22	0.2479	1,3	4	0.4431	1,4,3	6	0.2940	1,3	4	
24	0.2477	1,5	4	0.4283	1,11,5	6	0.2955	1,5	4	
26	0.2406	1,5	4	0.4151	1,3,11	6	0.2769	1,5	4	
28	0.2215	1,4	4	0.4116	1,4,3	6	0.2266	1,3	4	
30	0.2344	3,2	4	0.4022	5,3,2	6	0.2346	1,4	4	
32	0.2312	2,3	4	0.3946	1,6,7	6	0.2459	1,7	4	
34	0.2098	1,7	4	0.3945	1,4,14	6	0.2242	1,5	4	

3.6 Summary

In this section, we see which base graphs resulted in maximum AC among all of the base graphs with at most four vertices. We see that the largest AC for graphs with 10 vertices is for the

Peterson graph. We also found that our result mathced with the maximum AC for the graph with 48 vertices based on HOG.

The largest AC of the derived graphs from the base graphs with at most 4 vertices								
n	Max AC(1)	Base graph	Max AC(2)	Base graph	Max AC(3)	Base graph	Max AC(4)	Base graph
4	4	Cir(1,2)	4	Ears121(0,1)				
6	3	Cir(1,3)	3	Theta(0,1,2)	3	Base3(d)(1)		
8	2	Cir(1,4)	2	Theta(0,1,3), Ears111(1,1), Ears121(0,1)			2	Base4(a)(1,1,1), Base4(b)(1,1,1), Base4(e)(1), Base4(f)(1,1)
10	1.382	Cir(1,5)	2	Ears111(1,2)				
12	1	Cir(1,6)	1.2679	Theta(0,1,4)	1.2679	Base3(c)	1.2679	Base4(a)
14	0.753	Cir(1,7)	1.5858	Theta(0,1,3)				
16	0.5858	Cir(1,8)	1.2679	Theta(0,1,6), Ears111(1,3)			1.2679	Base4(a), Base4(b)
18	0.4679	Cir(1,9)	1.0304	Theta(0,1,3)	1.2679	Base3(c)(2,1)		
20	0.382	Cir(1,10)	1	Ears111(1,3)			1	Base4(a)(1,2,1), Base4(c)(1,1,2)
22	0.3175	Cir(1,11)	0.8984	Ears111(1,3)				
24	0.2679	Cir(1,12)	1	Ears111(1,5)	1	Base3(a)(1,3), Base3(b)(1,2)	1	Base4(a)(1,1,2), Base4(f)(1,1)
26	0.2291	Cir(1,13)	0.9474	Ears111(1,5)				
28	0.1981	Cir(1,14)	0.7639	Ears111(1,4)			1	Base4(c)(1,2,3)
30	0.1729	Cir(1,15)	0.7639	Ears111(1,4)	1	Base3(a)(1,3)		
32	0.1522	Cir(1,16)	0.7639	Ears111(1,6)			0.7639	Base4(a)(1,3,1), Base4(b)(1,2,4), Base4(d)(1,1,3)
34	0.1351	Cir(1,17)	0.7731	Ears111(1,5)				

36	0.1206	Cir(1,18)	0.7375	Ears111(1,5)	0.6971	Base3(b)(1,2)	0.7766	Base4(b)(1,2,4), Base4(c)(1,2,4)
38	0.1084	Cir(1,19)	0.6533	Theta(0,1,8)				
40	0.0979	Cir(1,20)	0.5858	Ears111(1,8)			0.7639	Base4(a)(2,4,1), Base4(d)(1,1,4)
42	0.0889	Cir(1,21)	0.7530	Ears111(1,8)	0.7530	Base3(b)(1,11)		
44	0.0810	Cir(1,22)	0.6564	Ears111(1,6)			0.6805	Base4(a)(1,2,3)
46	0.0742	Cir(1,23)	0.5779	Ears111(1,5)				
48	0.0681	Cir(1,24)	0.5858	Ears111(1,7)	0.6571	Base3(a)(1,3)	0.6239	Base4(b)(1,2,5)
50	0.0628	Cir(1,25)	0.6903	Ears111(1,7)				
52	0.0581	Cir(1,26)	0.6278	Ears111(1,10)			0.6531	Base4(d)(1,2,5)
54	0.0539	Cir(1,27)	0.5731	Ears111(1,6)	0.5505	Base3(b)(1,3)		
56	0.0501	Cir(1,28)	0.5822	Ears111(1,8)			0.6086	Base4(b)(1,3,7)
58	0.0468	Cir(1,29)	0.5455	Ears111(1,8)				
60	0.0437	Cir(1,30)	0.5121	Ears111(2,9)	0.5188	Base3(a)(1,3)	0.6972	Base4(d)(1,4,6)
62	0.0409	Cir(1,31)	0.4816	Ears111(1,7)				
64	0.0384	Cir(1,32)	0.5524	Ears111(1,7)			0.5858	Base4(d)(1,7,6), Base4(a)(1,3,7), Base4(b)(1,3,4)
66	0.0361	Cir(1,33)	0.4533	Ears111(1,6)	0.4679	Base3(a)(1,3)		
68	0.0341	Cir(1,34)	0.5170	Ears111(1,13)			0.5437	Base4(d)(1,2,7)
70	0.0321	Cir(1,35)	0.4755	Ears111(1,10)				

126	0.0099	Cir(1,63)	0.3005	Ears111(1,14)	0.3682	Base3(a)(1,8)		

3.7 Comparison with known graphs

In this section we compare our search for maximum AC based on voltage graphs with known results for cubic graphs.

Cubic graphs up to around 64 vertices are well studied. House of Graphs (HOG) website tabulates cubic graphs of high girth up to $n = 64$ vertices [19], as computed by Nauty and other programs [15–17]. For the cases where the count for a fixed girth and order n is sufficiently small (up to about 10^6), the corresponding maxiumum AC was also tabulated in [1]. This allows a relatively exhaustive comparison up to 44 vertices, when restricted to higher-girth graphs.

For $44 < n < 52$, the knowledge of cubic graphs is incomplete as there are too many graphs even when restricted to the highest possible girth: there are, for example, around 2×10^7 cubic graphs of girth 8 when $n = 46$ (see HOG, table of cubic graphs according to girth, [19]). For these cases, we compare our results with a search of HOG database, which is rather sparse. The situation changes again for $n = 58$ when girth 9 graphs first appear; all girth 9 cubic graphs are known up to $n = 64$.

In the table below, we compare our search for maximum AC based on voltage graphs with the known graphs from HOG as described above. In the comments of second row, we indicate whether the record comes from the table of all cubic graphs for a given girth ("Max for girth X"), or HOG database.

Comparison with known records				
n	Max AC (HOG)	Comment	Max AC (voltage, $k \leq 4$)	Base graph
4	4	Max for girth 3	4	Cir(1,2), Ears121(0,1)

6	3	Max for girth 4	3	Cir(1,3), Theta(0,1,2)
8	2	Max for girth 4	2	Cir(1,4), Theta(0,1,3), Ears111(1,1), Ears121(0,1), Base4(a)(1,1,1), Base4(b)(1,1,1), Base4(e)(1), Base4(f)(1,1)
10	2	Max for girth 5	2	Ears111(1,2)
12	1.467911	Max for girth 5	1.2679	Theta(0,1,4), Base3(c), Base4(a)
14	1.585786	Max for girth 6	1.5858	Theta(0,1,3)
16	1.267949	Max for girth 6	1.2679	Theta(0,1,6), Ears111(1,3), Base4(a), Base4(b)
18	1.267949	Max for girth 6	1.2679	Base3(c)(2,1)
20	1.064568	Max for girth 6	1	Ears111(1,3), Base4(a)(1,2,1), Base4(c)(1,1,2)
22	1	Max for girth 6	0.8984	Ears111(1,3)
24	1	Max for girth 7 and 6	1	Ears111(1,5), Base3(a)(1,3), Base3(b)(1,2), Base4(a)(1,1,2), Base4(f)(1,1)
26	0.94737	Max for girth 7	0.9474	Ears111(1,5)

28	1	Max for girth 7	0.7639	Ears111(1,4)
30	1	Max for girth 8, the Tutte-Coxter graph	1	Base3(a)(1,3)
32	0.864221	Max for girth 7	0.7639	Ears111(1,6), Base4(a)(1,3,1), Base4(b)(1,2,4), Base4(d)(1,1,3)
34	0.78568	Max for girth 8	0.7731	Ears111(1,5)
36	0.763932	Max for girth 8	0.7766	Has girth 7: Base4(b)(1,2,4), Base4(c)(1,2,4), these graphs are isomorphic
38	0.763932	Max for girth 8	0.6533	Theta(0,1,8)
40	0.763932	Max for girth 8	0.7639	Base4(a)(2,4,1), Base4(d)(1,1,4)
42	0.763932	Max for girth 8	0.7530	Ears111(1,8), Base3(b)(1,11)
44	0.7067	Max for girth 8	0.6805	Base4(a)(1,2,3)
46	0.45425	HOG database	0.5779	Ears111(1,5)
48	0.65708	HOG database	0.6571	Base3 (a)(1,3)
50	0.69032	HOG database	0.6903	Ears111(1,7)
52	0.49977	HOG database	0.6531	Based4(d)(1,2,5)
54	0.68213	HOG database	0.5731	Ears111(1,6)
56	0.61752	HOG database (girth 7)	0.6086	Base4(b)(1,3,7)
58	0.63766	Max for girth 9	0.5455	Ears111(1,8)
60	0.697224	Max for girth 9	0.6972	Base4(d)(1,4,6)
62	0.603671	Max for girth 9	0.4816	Ears111(1,7)

64	0.633832	Max for girth 9	0.5858	Base4(d)(1,7,6), Base4(a)(1,3,7), Base4(b)(1,3,4)
126	0.5505	Girth 12, Benson graph	0.3682	Base3(a)(1,8)

We were able to reproduce known records for $n \leq 10$ and $n = 14, 16, 18, 24, 26, 30, 40, 48, 50, 60$. (indicated in red in column 4) Moreover, we found a new record of a high-AC graph when $n = 36, 46$ and 52 (indicated in green).

Of particular interest is the case $n = 36$. For this case, HOG exhaustively lists 3 cubic graphs of girth 8 and around 95 million cubic graphs of girth 7. Unlike all other cases we looked at, the record AC has girth 7 rather than the maximum possible girth 8. This gives a counter-example to conjecture 6.1 proposed in [1] which states that the graph with the maximal AC also has the highest possible girth.

Chapter 4

Discussion and future work

The main motivation of this research is the open problem of constructing an infinite family of 7-regular Ramanujan graphs. But the main problem is finding a base graph and a group to make an infinite family of 7-regular Ramanujan graphs just by changing N. In this thesis, we found among base graphs with at most four vertices, which one has voltage graphs with the large AC. So, we did not have enough time to see what will happen for base graphs with more than four vertices. As we could find a new graphs with 36 vertices with the largest AC (bigger than known result of AC), we can say having a base graph with more than 4 vertices can have interesting voltage graphs with large girth.

Bibliography

[1] Geoffrey Exoo, Theodore Kolokolnikov, Jeanette Janssen, and Timothy Salamon, *Attainable bounds for algebraic connectivity and maximally-connected regular graphs*, arXiv, 2023.

[2] Exoo, Geoffrey and Jajcay, Robert, *Dynamic cage survey*, The electronic journal of combinatorics, 2012.

[3] Kolokolnikov, Theodore, *It is better to be semi-regular when you have a low degree*, arXiv preprint arXiv:2112.14289, 2021.

[4] Bilu, Yonatan and Linial, Nathan, *Lifts, discrepancy and nearly optimal spectral gap*,Combinatorica, 2006.

[5] Hoory, Shlomo and Linial, Nathan and Wigderson, Avi, *Expander graphs and their applications*, 2006.

[6] Exoo, Geoffrey and Jajcay, Robert, *On the girth of voltage graph lifts*, Elsevier, 2011.

[7] Hoory, Shlomo and Linial, Nathan and Wigderson, Avi, *Expander graphs and their applications*, 2006.

[8] Exoo, Geoffrey and Jajcay, Robert, *On the girth of voltage graph lifts*, Elsevier, 2011.

[9] Ghosh, Arpita and Boyd, Stephen, *Growing well-connected graphs*, IEEE, 2006.

[10] Fiedler, Miroslav, *Algebraic connectivity of graphs*, Institute of Mathematics, Academy of Sciences of the Czech Republic, 1973.

[11] Kolokolnikov, Theodore, *Maximizing algebraic connectivity for certain families of graphs*, Elsevier, 2015.

[12] Marcus, Adam and Spielman, Daniel A and Srivastava, Nikhil, *Interlacing families I: Bipartite Ramanujan graphs of all degrees*, IEEE, 2013.

[13] Marcus, Adam W and Spielman, Daniel A and Srivastava, Nikhil, *Interlacing families IV: Bipartite Ramanujan graphs of all sizes*, SIAM, 2018.

[14] Lubotzky, Alexander, *Expander graphs in pure and applied mathematics*, Bulletin of the American Mathematical Society, 2012.

[15] McKay, Brendan D, *Nauty user's guide (version 2.4)*, Computer Science Dept., Australian National University, 2007.

[16] B. D. McKay, W. Myrvold and J. Nadon, *Fast backtracking principles applied to find new cages*, 9th Annual ACM-SIAM Symposium on Discrete Algorithms, 188-191, 1998.

[17] G. Brinkmann, B. D. McKay and C. Saager, *The smallest cubic graphs of girth 9—, Combinatorics, Probability and Computing, 5:1-13, 1995.*

[18] Coolsaet, Kris and D'hondt, Sven and Goedgebeur, Jan, House of Graphs 2.0: A database of interesting graphs and more, Elsevier, 2023. https://houseofgraphs.org

[19] House of graphs, table of all connected cubic graphs by girth and order, see https://houseofgraphs.org/meta-directory/cubic

[20] Angel, Omer and Friedman, Joel and Hoory, Shlomo, *The non-backtracking spectrum of the universal cover of a graph*, Transactions of the American Mathematical Society, 2015.

[21] Dalfó, Cristina and Fiol, Miguel Angel and Miller, Mirka and Ryan, Joe and Širáň, J, *An algebraic approach to lifts of digraphs*, Elsevier, 2019.

[22] Dalfó, Cristina and Fiol, Miguel Angel and Miller, Mirka and Ryan, Joe and Širáň, J, *Cylindrical Graph Construction (definition and basic properties)*, arXiv preprint arXiv:1406.3767, 2014.

[23] Alegre, Ignacio and Fiol, Miguel Angel and Yebra, J Luis A, *Some large graphs with given degree and diameter*, Journal of graph theory, 1986.

[24] Gross, Jonathan L and Alpert, Seth R, *Branched coverings of graph imbeddings*, 1973.

[25] Gross, Jonathan L and Alpert, Seth R, *The topological theory of current graphs*, Elsevier, 1974.