PSEUDOCOLIMITS OF SMALL FILTERED DIAGRAMS OF INTERNAL CATEGORIES

by

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Submitted in partial fulfillment of the requirements
for the degree of Master of Science

at

Dalhousie University
Halifax, Nova Scotia
August 2022

For Nanuq and my fellow House Kweens.

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Abstract

Pseudocolimits are formal gluing constructions that combine objects in a category indexed by a pseudofunctor. When the objects are categories and the domain of the pseudofunctor is small and filtered it is known [1, Exposé 6] that the pseudocolimit can be computed by taking the Grothendieck construction of the pseudofunctor and inverting the class of cartesian arrows with respect to the canonical fibration. In this thesis we present a set of conditions on an ambient category $\mathcal{E}$ for defining the Grothendieck construction as an oplax colimit and another set of conditions on $\mathcal{E}$ along with conditions on an internal category, $\mathbb{C}$, in $\mathbf{Cat}(\mathcal{E})$ and a map $w : W \to \mathbb{C}_1$ that allow us to translate the axioms for a category of (right) fractions, and construct an internal category of (right) fractions. We combine these results in a suitable context to compute the pseudocolimit of a small filtered diagram of internal categories.

Acknowledgements

Special thanks to my supervisor Dr. Pronk for her invaluable feedback and support over the last two years, to Dr. Pronk, Dr. Paré, and Dr. Sellinger for their additional feedback and interesting discussions, and to Dr. Faridi for chairing my defence. I'd also like to acknowledge that this two-year research project was funded by the Natural Sciences and Engineering Research Council (NSERC) Canadian Graduate Scholarship - Master's program as well as the Nova Scotia Graduate Scholarship - Master's program. Finally I'd like to thank my family and friends for their patience and support over the last two years.

Chapter 1

Introduction

The term ‘Grothendieck construction’ is used to describe a correspondence between pseudofunctors $\mathcal{A} \to \mathbf{Cat}$, and fibrations over $\mathcal{A}$. For a given pseudofunctor $\mathcal{A} \to \mathbf{Cat}$ the domain of the corresponding fibration is often called ‘the Grothendieck construction’ (of the pseudofunctor) or the ‘category of elements.’ The ‘construction’ aspect of this naming is fitting because the category of elements is the *oplax colimit* of the pseudofunctor [6], which we can think of as a category constructed from the diagram in **Cat** determined by the pseudofunctor. The *pseudocolimit* of the pseudofunctor can be computed by a bit of renovation to the Grothendieck construction; more precisely, by localizing with respect to a suitable class of arrows. When the indexing category, $\mathcal{A}$, is filtered the Grothendieck construction satisfies the Gabriel-Zisman axioms for a (left) category of fractions [5] and the pseudocolimit is given by a category of (left) fractions. A weaker set of axioms is given in [14] which allows for localization with respect to a smaller class of arrows.

Having colimits in a category is important for understanding how to glue or combine things in that category. The usual Grothendieck construction, as an oplax colimit, is a gluing construction for categories indexed by a pseudofunctor and this has been translated in various other settings such as $(\infty, 1)$-cats [6], enriched categories [3], and bicategories by [14]. The geometric realization of the Grothendieck construction for a diagram of small categories has also been studied as a homotopy colimit by Thomason in [12] and for a diagram of quasi categories in [10]. In the first part of this thesis we will translate the usual Grothendieck construction into the language of internal category theory in order to compute an oplax colimit of a small diagram of internal categories. The fibration perspective of the Grothendieck construction has already been developed for monoidal categories [9], 2-cats and bicategories [4], and $(\infty, 1)$-categories [6] as well but we focus more on the colimit perspective because it is not possible to view the indexing category as an internal category in general.

A different setting in which this would be useful is to describe the tom Dieck fundamental group for a space equipped with a group action. This is a category enriched in topological spaces that is the oplax colimit of fundamental groups of fixed point sets of all the subgroup actions [13]. This is also useful for computing atlas groupoids for orbifolds, which are pseudocolimits of categories internal to **Top** [11]. Yet another relevant setting is for double categories, which are internal categories in **Cat**. Such a construction here would allow us to compute oplax colimits of category-indexed diagrams of double categories.

To replicate these colimit constructions we need an ambient category, $\mathcal{E}$, with sufficient structure. In particular, to define an internal Grothendieck construction/category of elements for a pseudofunctor $D : \mathcal{A}^{op} \to \mathbf{Cat}(\mathcal{E})$ we need that $\mathcal{E}$ has pullbacks along certain source and target maps of the internal categories in the image of D, has disjoint coproducts of these pullbacks and of the objects of objects for the internal categories in the diagram, and that these commute with one another. This allows us to construct an oplax colimit of D, by Theorem 19 which we restate here:

Theorem (The internal Grothendieck Construction, $\mathbb{D}$, as an oplax colimit)**.** *Let $\mathcal{E}$ admit an internal Grothendieck construction of $D : \mathcal{A} \to \mathbf{Cat}(\mathcal{E})$, as in Definition 2.Let $\mathbb{D}$ denote the internal Grothendieck construction. Then for every internal category $\mathbb{X} \in \mathbf{Cat}(\mathcal{E})$, the category of lax natural transformations $D \implies \Delta\mathbb{X}$ and their modifications is isomorphic to the category of internal functors $\mathbb{D} \to \mathbb{X}$ and their internal natural transformations.*

$$[D, \Delta\mathbb{X}]_\ell \cong \mathbf{Cat}(\mathcal{E})(\mathbb{D}, \mathbb{X})$$

The pullback and coproduct commutativity is an extensivity property and is relied on heavily to define the internal category structure and prove the required properties are satisfied and makes the construction unlikely to work for arbitrary diagrams of categories internal to non-extensive categories such as the category of vector spaces. The internal category of fractions requires a special class of epimorphisms in the ambient category to locally witness internalized versions of category of fractions axioms in the sense of admitting lifts that define local sections. Part of what makes these epimorphisms special is that the local data they witness (on their codomains) can be combined to give global definitions of structure on their codomains provided the pieces of local data satisfy a kind of compatibility/descent condition.

In the contexts we describe we show that the constructions used and the result stated in [1] can be translated into the language of internal categories. The class of epimorphisms we require for our internal category of fractions are coequalizers of their kernel pairs, stable under pullback, and closed under composition. Such a class always exists in any category, namely the identity arrows, but it is not always possible to get an internal fractions construction with this class. The Internal Fractions Axioms are described in Definition 34 in terms of certain lifts of these epimorphisms and a section of an induced target structure map. Asking for sections in settings where continuity is important can be a strong condition. For example, when working with an arbitrary internal category in **Top** asking for continuous global sections is generally too much when the axiom of choice is being used, but there are nice classes of effective epimorphisms given by open surjections or étale surjections which give us local sections instead of global sections. In Section 5.1 we show that the internal Grothendieck construction, when it exists in a suitable context, satisfies the Internal Fractions Axioms with respect to the object representing the canonical cleavage of the cartesian arrows by global sections, meaning it only requires identities for the class of epimorphisms in Definition 33. We prove this in the main theorem of this thesis which we restate here:

Theorem. *Let* $D : \mathcal{A}^{op} \to \mathbf{Cat}(\mathcal{E})$ *be a pseudofunctor for which* $\mathcal{E}$ *admits an internal Grothendieck construction 2 and let* $w : W \to \mathbb{D}_1$ *be the object of a canonical cleavage of the internal Grothendieck construction as defined in Section 5.1.1. If the pair* $(\mathbb{D}, W)$ *satisfies the Internal Fractions Axioms in Definition 34, then for any* $\mathbb{X}$ *in* $\mathbf{Cat}(\mathcal{E})$ *there is an isomorphism*

$$[D, \Delta\mathbb{X}]_{ps} \cong [\mathbb{D}[W^{-1}], \mathbb{X}]^{\mathcal{E}}$$

between the category of pseudonatural transformations $D \implies \Delta\mathbb{X}$ *(and their modifications) and the category of internal functors* $\mathbb{D}[W^{-1}] \to \mathbb{X}$.

We begin in Chapter 2 with a few words on notation and internal categories. In Chapter 3 we present a context, $\mathcal{E}$, in which we define an internal Grothendieck construction for a pseudofunctor $\mathcal{A} \to \mathcal{E}$. We then define the internal Grothendieck construction, $\mathbb{D}$, and show it is a lax colimit. The context for an internal (right) category of fractions and its definition are given in Chapter 4 and an isomorphism of categories that describes the universal property of the internal localization is proven at the end in Section 4.6. Chapter 5 describes a setting in which we can compute the pseudocolimit of D as the localization of the internal Grothendieck construction with respect to the canonical cleavage of the cartesian arrows object, $w : W \to \mathbb{D}_1$.

Chapter 2

Notation and A Word on Internal Categories

Composition is written diagrammatically, so that 'f followed by g' is written by juxtaposition as 'fg.' Internal categories are denoted with blackboard bold font, $\mathbb{C}, \mathbb{D}, \mathbb{X}$, and we use $\mathcal{A}$ to denote the indexing category for pseudofunctors we will consider. We assume $\mathcal{A}$ is small for Chapter 3 and will assume it is also cofiltered in Chapter 5.

The main point of this thesis is to take a technique for computing pseudocolimits of small filtered diagrams of categories, give an internal category theoretic version of it, and show that it satisfies the universal property of a pseudocolimit of a small filtered diagram of internal categories. Internal categories are defined in Section B2.3 in [7] when working in an ambient category with all pullbacks. Some of the ambient categories we wish to consider in future applications of this thesis, such as the category of smooth manifolds, do not have all pullbacks however. In this thesis, we do note assume the existence of all pullbacks in an ambient category, rather we make the existence of the necessary pullbacks and structure maps part of the definition of an internal category.

Definition 1. An internal category, $\mathbb{C}$ in $\mathcal{E}$, consists of the following data.

- An object of objects, $\mathbb{C}_0 \in \mathcal{E}_0$.
- An object of arrows, $\mathbb{C}_1 \in \mathcal{E}_0$.
- Structure maps

$$\begin{tikzcd} \mathbb{C}_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathbb{C}_0 \arrow[r, "e"] & \mathbb{C}_1 \end{tikzcd}$$

  in $\mathcal{E}_1$ such that e is a common section of s and t.

- The iterated pullbacks of composable chains of arrows, $\mathbb{C}_n = \mathbb{C}_1 \, {}_t\!\times_s \cdots \, {}_t\!\times_s \mathbb{C}_1$ in $\mathcal{E}_1$.

- A composition structure map $c : \mathbb{C}_2 \to \mathbb{C}_1$ such that the squares

  $$\begin{tikzcd}
  \mathbb{C}_2 \arrow[r, "c"] \arrow[d, "\pi_0"'] & \mathbb{C}_1 \arrow[d, "s"] \\
  \mathbb{C}_1 \arrow[r, "s"'] & \mathbb{C}_1
  \end{tikzcd}
  \qquad
  \begin{tikzcd}
  \mathbb{C}_2 \arrow[r, "c"] \arrow[d, "\pi_1"'] & \mathbb{C}_1 \arrow[d, "t"] \\
  \mathbb{C}_1 \arrow[r, "t"'] & \mathbb{C}_1
  \end{tikzcd}$$

  commute in $\mathcal{E}$, along with the associativity diagram,

  $$\begin{tikzcd}
  \mathbb{C}_3 \arrow[r, "1\times c"] \arrow[d, "c\times 1"'] & \mathbb{C}_2 \arrow[d, "c"] \\
  \mathbb{C}_2 \arrow[r, "c"'] & \mathbb{C}_1
  \end{tikzcd} ,$$

  and the identity law diagrams

  $$\begin{tikzcd}
  \mathbb{C}_1 \arrow[r, "(1_{\mathbb{C}_1}{,}te)"] \arrow[dr, "1_{\mathbb{C}_1}"'] & \mathbb{C}_2 \arrow[d, "c"] & \mathbb{C}_1 \arrow[l, "(se{,}1_{\mathbb{C}_1})"'] \arrow[dl, "1_{\mathbb{C}_1}"] \\
  & \mathbb{C}_1 &
  \end{tikzcd} .$$

Chapter 3

Internal Grothendieck Construction

In this chapter we define (the category of elements for) the (internal) Grothendieck construction of a pseudofunctor $D : \mathcal{A} \to \mathbf{Cat}(\mathcal{E})$ where $\mathcal{E}$ is an extensive category and show that it is the oplax colimit of D. Section 3.1 defines the internal category structure of the (internal) Grothendieck construction, $\mathbb{D}$, of D. Section 3.2 proves the associativity and identity laws for composition and shows it is an internal category. Section 3.3 shows the existence of a canonical lax natural transformation from $D \implies \Delta\mathbb{D}$, and then proves the universal properties that show $\mathbb{D}$ is the oplax colimit of D. We often consider how the usual proofs and definitions look in the case $\mathcal{E} = \mathbf{Set}$ to help our readers follow the internalized definitions and results for an arbitrary extensive category $\mathcal{E}$.

3.1 Internal Category Structure

3.1.1 Classical

Let $\mathcal{A}$ be a small category and let $\mathbf{Cat}(\mathbf{Set})$ denote the 2-category of small categories, strict functors, and natural transformations. This is a fully faithful subcategory of $\mathbf{Cat}$ so for every pseudofunctor

$$\mathcal{A} \xrightarrow{D} \mathbf{Cat}(\mathbf{Set}) \ ,$$

the *Grothendieck construction* of D is the strict 2-pullback of D along the canonical projection, π, from the lax-pointed 2-category of categories, $\mathbf{Cat}_{*,\ell}$ [7].

$$\begin{tikzcd}
\int F \arrow[r] \arrow[d, "P"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cat}_{*,\ell} \arrow[d, "\pi"] \\
\mathcal{A} \arrow[r, "D"'] & \mathbf{Cat}
\end{tikzcd}$$

The objects of $\int D$ are pairs (A, a), where $A \in \mathcal{A}_0$ and $a \in (DA)_0$. The morphisms are pairs $(\varphi, f) : (A, a) \to (B, b)$ where $\varphi : A \to B$ is an arrow in $\mathcal{A}$ and $f : \varphi(f)(a) \to$

b is an arrow in $D(B)$. Let $\delta_A : D(1_A) \implies 1_{D(A)}$ and $\delta_{\varphi;\psi} : D(\varphi\psi) \implies D(\varphi)D(\psi)$ denote the identity and composition natural isomorphisms associated to $A \in \mathcal{A}_0$ and an arbitrary composable pair φ, ψ in $\mathcal{A}_1$ respectively. The identity morphisms in $\mathbb{D}$ are the pairs $(1_A, \delta_{A,a}) : (A, a) \to (A, a)$ where $\delta_{A,a} \in$ is the a-indexed component the natural transformaiton δ_A. Two morphisms $(\varphi, f), (\psi.g)$ are composable when $\mathrm{cod}(D(\psi)_1(f)) = \mathrm{dom}(g)$ in $D(C)$. Then

$$f : D(\varphi)_0(a) \to b \quad , \quad g : D(\psi)_0(b) \to d$$

and the following diagram

$$\begin{tikzcd}
D(\psi)_0\,(D(\varphi_0)(a))) \arrow[d, "D(\psi)_1(f)"'] & D(\varphi\psi)_0(a) \arrow[l, "\delta_{\varphi;\psi,a}"', "\cong"] \arrow[d, dotted] \\
D(\psi)_0(b) \arrow[r, "g"'] & d
\end{tikzcd}$$

defines the composite

$$(\varphi, f)(\psi, g) := (\varphi\psi, \delta_{\psi,a} D(\psi)_1(f) g).$$

The so-called category of elements, $\int D$ is an oplax colimit of D in **Cat** [6]. Next we translate this into the language of internal categories.

3.1.2 Internal

Let $D : \mathcal{A} \to \mathbf{Cat}(\mathcal{E})$ be a pseudofunctor. In this section we give a suitable context, $\mathcal{E}$, for defining an internal category of elements, $\mathbb{D}$, in $\mathbf{Cat}(\mathcal{E})$ for a pseudofunctor, $\mathcal{A} \to \mathbf{Cat}(\mathcal{E})$, that is inspired by the usual category of elements. The context given in the following definition can be thought of as a kind of 'local extensivity' condition on $\mathcal{E}$ with respect to the pseudofunctor D.

Definition 2. We say $\mathcal{E}$ *admits an internal Grothendieck construction of* $D : \mathcal{A} \to \mathbf{Cat}(\mathcal{E})$ if

1. for every $\varphi : A \to B$ in $\mathcal{A}$, the pullback

$$\begin{tikzcd}
D_\varphi \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & D(B)_1 \arrow[d, "s"] \\
D(A)_0 \arrow[r, "D(\varphi)_0"'] & D(B)_0
\end{tikzcd}$$

exists in $\mathcal{E}$.

2. For any composable chain of maps $\varphi_i : A_i \to A_{i+1}$ in $\mathcal{A}$, where $1 \leq i \leq n$ for an arbitrary $n \in \mathbb{N}$, the pullback

$$D_{\varphi_1;\ldots;\varphi_n} = D_{\varphi_1} \ {}_{\pi_1 t}{\times}_{\pi_0} \cdots \ {}_{\pi_1 t}{\times}_{\pi_0} D_{\varphi_{n+1}}$$

exists in $\mathcal{E}$.

3. Let $\mathcal{A}_0$ denote the objects of $\mathcal{A}$ and let $\mathcal{A}_n$ denote the composable paths of length $n \geq 1$ in $\mathcal{A}$. The coproducts

$$\mathbb{D}_0 = \coprod_{A \in \mathcal{A}_0} D(A)_0$$

and

$$\mathbb{D}_{\coprod(n)} = \coprod_{(\varphi_i)_{i=1}^n \in \mathcal{A}_n} D_{\varphi_1;\ldots;\varphi_n}$$

exist for all $n \geq 1$ and are disjoint, with coprojections:

$$\iota_{\varphi_1;\ldots;\varphi_n} : D_{\varphi_1;\ldots;\varphi_n} \to \mathbb{D}_{\coprod(n)}$$

4. The coproducts, $\mathbb{D}_{\coprod(n)}$, are stable under pullbacks of source and target in the sense that

$$\mathbb{D}_{\coprod(n)} \cong \mathbb{D}_1 \ {}_t{\times}_s \ \mathbb{D}_1 \ {}_t{\times}_s \cdots \ {}_t{\times}_s \ \mathbb{D}_1 = \mathbb{D}_n$$

where $s, t : \mathbb{D}_1 \to \mathbb{D}_0$ are uniquely induced by the source and target maps.

$$\begin{tikzcd}
\mathbb{D}_2 \arrow[r, "\rho_1"] \arrow[d, "\rho_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{D}_1 \arrow[d, dotted, "s"'] & D_\varphi \arrow[l, tail, "\iota_\varphi"'] \arrow[dl, "\pi_0 \iota_A"] \\
\mathbb{D}_1 \arrow[r, dotted, "t"] & \mathbb{D}_0 & \\
D_\varphi \arrow[u, tail, "\iota_\varphi"] \arrow[ur, "\pi_1 t \iota_B"'] & &
\end{tikzcd}$$

These conditions allow us to define the objects and structure maps of our internal Grothendieck construction, $\mathbb{D}$. The last condition above should be thought of as an extensivity condition that allows us to to define the composition structure and prove $\mathbb{D}$ is an internal category.

For the rest of this section we will assume that $\mathcal{E}$ admits an internal Grothendieck construction.

Define the object of objects to be

$$\mathbb{D}_0 := \coprod_{A \in \mathcal{A}_0} D(A)_0.$$

Remark 3. When $\mathcal{E} = \mathbf{Set}$, we can think of the elements of $\mathbb{D}_0$ as elements $a \in D(A)_0$ for each $A \in \mathcal{A}_0$. This implies that every element of $\mathbb{D}_0$ can be represented as a pair (A, a) where $A \in \mathcal{A}_0$ and $a \in D(A)_0$.

For any $\varphi : A \to B$ in $\mathcal{A}_1$, we have the pullback

$$\begin{tikzcd}
D_\varphi \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & D(B)_1 \arrow[d, "s"] \\
D(A)_0 \arrow[r, "D(\varphi)_0"'] & D(B)_0
\end{tikzcd}$$

which is used to define the object of arrows:

$$\mathbb{D}_1 := \coprod_{\varphi \in \mathcal{A}_1} D_\varphi$$

Remark 4. When $\mathcal{E} = \mathbf{Set}$ an arbitrary element of $\mathbb{D}_1$ is an element of D_φ for some unique $\varphi \in \mathcal{A}_1$. In this case elements of D_φ are pairs (x, f) where $x \in D(A)_0$, $f \in D(B)_1$ and $D(\varphi)_0(x) = s(f)$. In this way every element of $\mathbb{D}_1$ can be represented by a pair (φ, f) where $f : D(\varphi)(x) \to y$ in $D(B)$.

To define source and target maps for $\mathbb{D}$, it suffices to define them on the components, D_φ, for each $\varphi \in \mathcal{A}_1$. Let

$$s_\varphi, t_\varphi : D_\varphi \to \mathbb{D}_0$$

be defined as the composites on the top and left in the following diagram.

$$\begin{tikzcd}
D_{\varphi} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[rrr, bend left, "t_{\varphi}"] \arrow[dd, bend right=60, "s_{\varphi}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & D(B)_1 \arrow[r, "t"] \arrow[d, "s"] & D(B)_0 \arrow[r, tail, "\iota_B"] & \mathbb{D}_0 \\
D(A)_0 \arrow[r, "D(\varphi)_0"'] \arrow[d, tail, "\iota_A"] & D(B)_0 & & \\
\mathbb{D}_0 & & &
\end{tikzcd}$$

These induce the source and target maps $s, t : \mathbb{D}_1 \to \mathbb{D}_0$ by the universal property of the coproduct $\mathbb{D}_1$. Their pullback defines the object of composable arrows, $\mathbb{D}_2$.

$$\begin{tikzcd}
\mathbb{D}_2 \arrow[r, "\rho_1"] \arrow[d, "\rho_0"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbb{D}_1 \arrow[d, dotted, "s"'] & D_{\varphi} \arrow[l, tail, "\iota_{\varphi}"'] \arrow[dl, "s_{\varphi}"] \\
\mathbb{D}_1 \arrow[r, dotted, "t"] & \mathbb{D}_0 & \\
D_{\varphi} \arrow[u, tail, "\iota_{\varphi}"] \arrow[ur, "t_{\varphi}"'] & &
\end{tikzcd}$$

For any $\varphi, \psi \in \mathcal{A}_1$ we also pull t_{φ} back along s_{ψ} and denote the object $D_{\varphi;\psi}$. If φ and ψ are not composable in $\mathcal{A}$ then s_{ψ} and t_{φ} land in different components of $\mathbb{D}_0$. In that case $D_{\varphi;\psi}$ is trivial because coproducts are disjoint in $\mathcal{E}$.

Using the universal property of the coproduct $\mathbb{D}_2$ we describe composition in $\mathbb{D}$ by defining composition on the cofibers $D_{\varphi;\psi}$. Suppose $\varphi \in \mathcal{A}(A, B)$ and $\psi \in \mathcal{A}(B, C)$. Then $\varphi\psi \in \mathcal{A}(A, C)$ and the outsides of the following diagrams commute

$$\begin{tikzcd}
D_{\varphi;\psi} \arrow[r, "p_0"] \arrow[d, "p_0"'] \arrow[dr, dotted, "c'_{\delta;(\varphi;\psi)}"] & D_{\varphi} \arrow[r, "\pi_1"] & D(B)_1 \arrow[d, "D(\psi)_1"] \\
D_{\varphi} \arrow[d, "\pi_0"'] & D(C)_2 \arrow[r, "q_1"] \arrow[d, "q_0"'] & D(C)_1 \arrow[d, "s"] \\
D(A)_0 \arrow[r, "\delta_{\varphi,\psi}"'] & D(C)_1 \arrow[r, "t"'] & D(C)_0
\end{tikzcd} \tag{$\star\star$}$$

and

$$\begin{tikzcd}
D_{\varphi;\psi} \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[rd, dotted, "c'_{\varphi;\psi}"] & D_{\psi} \arrow[rd, bend left, "\pi_1"] & \\
D_{\varphi} \arrow[d, "\pi_1"'] & D(C)_2 \arrow[r, "q_1"] \arrow[d, "q_0"'] & D(C)_1 \arrow[d, "s"] \\
D(B)_1 \arrow[r, "D(\psi)_1"'] & D(C)_1 \arrow[r, "t"'] & D(C)_0
\end{tikzcd} \cdot \tag{$\star\star$}$$

To see the first diagram commutes we can check directly that

$$\begin{aligned}
p_0\pi_0\delta_{\varphi,\psi}t &= p_0\pi_0 D(\varphi)_0 D(\psi)_0 && \text{(Def. } \delta_{\varphi,\psi}\text{)} \\
&= p_0\pi_1 s D(\psi)_0 && \text{(Def. } D_{\varphi}\text{)} \\
&= p_0\pi_1 D(\psi)_1 s && (D(\psi) \text{ an internal functor)}
\end{aligned}$$

and to see the second square commutes it suffices to show that the canonical monic $\iota_C : D(C)_0 \rightarrowtail \mathbb{D}_0$ coequalizes both sides of the diagram.

$$\begin{aligned}
p_0\pi_1 D(\psi)_1 t\iota_C = &= p_0\pi_1 t D(\psi)_0\iota_C \\
&= p_0\pi_1 t\iota_B\chi_\psi \\
&= p_0 t_\varphi \chi_\psi \\
&= p_1 s_\psi \chi_\psi \\
&= p_1\pi_0\iota_B\chi_\psi \\
&= p_1\pi_0 D(\psi)_0\iota_C \\
&= p_1\pi_1 s\iota_C
\end{aligned}$$

Since ι_C is monic, we can conclude that the outer squares above commute and induce the maps $c'_{\varphi;\psi}$ and $c'_{\delta;(\varphi;\psi)}$ by diagrams ($\star\star$) and ($\star\star$). Let $q_{01}, q_{12} : D(C)_3 \to D(C)_2$ denote the pullback projections of $D(C)_3$. Notice that

$$c'_{\varphi;\psi}q_0 = c'_{\delta;(\varphi;\psi)}q_1$$

so there is a unique map

$$\begin{tikzcd}
& D_{\varphi;\psi} \arrow[ld, bend right, "c'_{\delta;(\varphi;\psi)}"'] \arrow[d, dotted, "c'_{\delta;\varphi;\psi}"] \arrow[rd, bend left, "c'_{\varphi;\psi}"] & \\
D(C)_2 & D(C)_3 \arrow[l, "q_{01}"] \arrow[r, "q_{12}"'] & D(C)_2
\end{tikzcd}$$

which we can postcompose with triple-composition in $D(C)$ (given by associativity). Notice that

$$\begin{aligned}
p_0\pi_0 D(\varphi\psi)_0 &= p_0\pi_0\delta_{\varphi,\psi}s && \text{(Def. } \delta_{\varphi,\psi}\text{)}\\
&= c'_{\delta;(\varphi;\psi)}q_0 s && \text{(Def. } c'_{\delta;(\varphi;\psi)}\text{)}\\
&= c'_{\delta;\varphi;\psi}q_{01}q_0 s && \text{(Def. } c'_{\delta;\varphi;\psi}\text{)}\\
&= c'_{\delta;\varphi;\psi}cs && \text{(source-composite law in } D(C)\text{)}
\end{aligned}$$

so there exists a unique ‘cofiber-wise composition’ map as shown in the following diagram.

$$\begin{tikzcd}
D_{\varphi;\psi} \arrow[r, "c'_{\delta;\varphi;\psi}"] \arrow[d, "p_0"'] \arrow[rd, dotted, "c_{\varphi;\psi}"] & D(C)_3 \arrow[rd, bend left, "c"] & \\
D_{\varphi} \arrow[rd, bend right, "\pi_0"'] & D_{\varphi\psi} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[rd, phantom, very near start, "\lrcorner"] & D(C)_1 \arrow[d, "s"] \\
& D(A)_0 \arrow[r, "D(\varphi\psi)_0"'] & D(C)_0
\end{tikzcd}$$

Define composition in $\mathbb{D}$ as the universal map out of the coproduct $\mathbb{D}_2$ induced by the family of maps $\{c_{\varphi;\psi}\iota_{\varphi\psi}\}_{(\varphi,\psi)\in\mathcal{A}_2}$.

$$\begin{tikzcd}
\mathbb{D}_2 \arrow[r, dotted, "c"] & \mathbb{D}_1 \\
D_{\varphi;\psi} \arrow[u, tail, "\iota_{\varphi;\psi}"] \arrow[r, "c_{\varphi,\psi}"'] & D_{\varphi\psi} \arrow[u, tail, "\iota_{\varphi\psi}"']
\end{tikzcd}$$

The identity structure map

$$\epsilon : \mathbb{D}_0 \to \mathbb{D}_1$$

is defined as the universal map out of the coproduct $\mathbb{D}_0$ induced by a family of unique maps ϵ_A

$$\begin{tikzcd}
\mathbb{D}_0 \arrow[r, dotted, "\epsilon"] & \mathbb{D}_1 \\
D(A)_0 \arrow[u, tail, "\iota_A"] \arrow[r, "\epsilon_A"'] & D_{1_A} \arrow[u, tail, "\iota_{1_A}"']
\end{tikzcd}$$

where $\epsilon_A = \langle 1_{D(A)_0}, \delta_A \rangle$ is induced by the components $\delta_A : D(A)_0 \to D(A)_1$ of the natural isomorphism $D(1_A) \cong 1_{D(A)}$ and the identity map $1_{D(A)_0}$. This is well-defined because

$$\delta_A s = D(1_A)_0$$

by definition of δ_A:

$$\begin{tikzcd}
D(A)_0 \arrow[rrd, bend left, "\delta_A"] \arrow[rd, dotted, "\epsilon_A"] \arrow[rdd, equals, bend right] & & \\
& D_{1_A} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & D(A)_1 \arrow[d, "s"] \\
& D(A)_0 \arrow[r, "D(1_A)_0"'] & D(A)_0
\end{tikzcd}$$

The following commuting diagram shows that the source and target maps of $\mathbb{D}$ are compatible with these identity maps.

$$\begin{tikzcd}
& & \mathbb{D}_0 \arrow[r, "\epsilon"] \arrow[dr, equals] & \mathbb{D}_1 \arrow[d, "s"] \\
& & & \mathbb{D}_0 \\
D(A)_0 \arrow[uurr, tail, "\iota_A"] \arrow[r, "\epsilon_A"] \arrow[dr, equals] & D_{1_A} \arrow[uurr, tail, "\iota_{1_A}"] \arrow[d, "\pi_0"] & & \\
& D(A)_0 \arrow[uurr, tail, "\iota_A"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
& & \mathbb{D}_0 \arrow[r, "\epsilon"] \arrow[dr, equals] & \mathbb{D}_1 \arrow[d, "t"] \\
& & & \mathbb{D}_0 \\
D(A)_0 \arrow[uurr, tail, "\iota_A"] \arrow[r, "\epsilon_A"] \arrow[dr, equals] & D_{1_A} \arrow[uurr, tail, "\iota_{1_A}"] \arrow[d, "\pi_1 t"] & & \\
& D(A)_0 \arrow[uurr, tail, "\iota_A"'] & &
\end{tikzcd}$$

The top squares commute by definition of ϵ, the bottom squares commute trivially, and the squares on the right commute by definition of s_{1_A} and t_{1_A} respectively. The left front triangle commutes by definition of ϵ_A and the right front triangle commutes by definition of δ_A. More precisely, on the left we have

$$\epsilon_A \pi_1 t = \delta_A t = 1_{D(A)_0}.$$

3.2 Associativity and Identity Laws

3.2.1 Classical

When $\mathcal{E} = \mathbf{Set}$, the identity arrows of the usual Grothendieck construction are pairs $(1_A, \delta_{A,a})$ for each object (A, a) where $A \in \mathcal{A}$ and $a \in D(A)_0$. The coherence law between the natural isomorphisms, $\delta_{1_A;\varphi}, \delta_A$ and $\delta_{\varphi;1_B}, \delta_B$ respectively, says that pasting the 2-cells $\delta_{1_A;\varphi}$ and δ_A and the 2-cells $\delta_{\varphi;1_B}$ and δ_B is equal to $1_{D(\varphi)}$ respectively. This means that at the level of components we have a commuting diagram

$$\begin{tikzcd}
D(\varphi)(a) \arrow[r, "\delta_{\varphi;1_B,a}"] \arrow[d, "\delta_{1_A;\varphi,a}"'] \arrow[dr, equal] & \left(D(\varphi)D(1_B)\right)(a) \arrow[d, "\delta_{B,D(\varphi)(a)}"] \\
\left(D(1_A)D(\varphi)\right)(a) \arrow[r, "D(\varphi)(\delta_{A,a})"'] & D(\varphi)(a)
\end{tikzcd}$$

in $D(B)$ for each $a \in D(A)_0$. The upper and lower triangles are necessary for proving the right and left identity laws for the Grothendieck construction respectively. For example, the lower coherence precisely cancels the isomorphisms we pick up in our definitions of identity and composition in order for the left identity law to hold.

$$(1_A, \delta_{A,a})(\varphi, f) = (1_A\varphi, \delta_{1_A;\varphi,a} D(\varphi)(\delta_{A,a}) f) = (\varphi, f)$$

The associativity law relies on the other coherence law for D that gives the following commutating squares for every composable triple, φ, ψ, and γ, involving the components

$$\begin{tikzcd}
D(\varphi\psi\gamma)(a) \arrow[r, "\delta_{\varphi;\psi\gamma,a}"] \arrow[d, "\delta_{\varphi\psi;\gamma,a}"'] & D(\varphi)D(\psi\gamma)(a) \arrow[d, "\delta_{\psi;\gamma,D(\varphi)(a))}"] \\
D(\varphi\psi)D(\gamma)(a) \arrow[r, "D(\gamma)(\delta_{\varphi;\psi,a})"'] & D(\varphi)D(\psi)D(\gamma)(a)
\end{tikzcd}.$$

3.2.2 Internal

Now we give internal translations of the proofs of the associativity and identity laws for the Grothendieck construction. The following proposition states that the identity map $\epsilon : \mathbb{D}_1 \to \mathbb{D}_0$ of the internal Grothendieck construction satisfies the identity laws.

Proposition 5 (Identity Laws for $\mathbb{D}$). *Given the following pullbacks,*

$$\begin{tikzcd}
 & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_0 \arrow[r] \arrow[d, "\rho_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{D}_0 \arrow[d, equal] \\
\mathbb{D}_0 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "\rho_1"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{D}_1 \arrow[r, "t"'] \arrow[d, "s"] & \mathbb{D}_0 \\
\mathbb{D}_0 \arrow[r, equal] & \mathbb{D}_0 &
\end{tikzcd}$$

let $\langle\rho_0\epsilon,\rho_1\rangle$ and $\langle\rho_0,\rho_1\epsilon\rangle$ be the universal maps induced by the pairs of pullback projections with ϵ postcomposed respectively. Then the following diagram commutes.

$$\begin{tikzcd}
\mathbb{D}_0 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "\langle\rho_0\epsilon{,}\rho_1\rangle"] \arrow[dr, "\rho_1"'] & \mathbb{D}_2 \arrow[d, "c"] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_0 \arrow[l, "\langle\rho_0{,}\rho_1\epsilon\rangle"'] \arrow[dl, "\rho_0"] \\
 & \mathbb{D}_1 &
\end{tikzcd}$$

Proof. For each $A \in \mathcal{A}_0$ and each $\varphi \in \mathcal{A}_1$, we have the following commuting diagram

$$\begin{tikzcd}
 & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_0 \arrow[r] \arrow[d, "\rho_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{D}_0 \arrow[r, "\epsilon"] \arrow[d, equal] & \mathbb{D}_1 & \\
\mathbb{D}_0 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "\rho_1"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{D}_1 \arrow[r, "t"'] \arrow[d, "s"] & \mathbb{D}_0 & & \\
\mathbb{D}_0 \arrow[r, equal] \arrow[d, "\epsilon"'] & \mathbb{D}_0 & & & \\
\mathbb{D}_1 & & D_\varphi \,{}_{\pi_1 t}\!\times_1 D(B)_0 \arrow[r] \arrow[d, "p_0"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[uuul, dotted, tail] & D(B)_0 \arrow[r, "\epsilon_A"] \arrow[d, equal] \arrow[uuul, dotted, tail] & D_{1_B} \arrow[uuul, dotted, tail] \\
 & D(A)_0 \,{}_1\!\times_{\pi_0} D_\varphi \arrow[r, "p_1"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[uuul, dotted, tail] & D_\varphi \arrow[r, "\pi_1 t"'] \arrow[d, "\pi_0"] \arrow[uuul, dotted, tail] & D(B)_0 \arrow[uuul, dotted, tail] & \\
 & D(A)_0 \arrow[r, equal] \arrow[d, "\epsilon_A"] \arrow[uuul, dotted, tail] & D(A)_0 \arrow[uuul, dotted, tail] & & \\
 & D_{1_A} \arrow[uuul, dotted, tail] & & &
\end{tikzcd}$$

where the dotted arrows are all coproduct monos by stability of coproducts under pullback. Let $\langle p_0\epsilon_A, p_1\rangle$ and $\langle p_1, p_0\epsilon_A\rangle$ be universal maps out of $D_{1_A;\varphi}$ and $D_{\varphi;1_B}$ induced by the pairs $p_0\epsilon_A, p_1$ and $p_0, p_1\epsilon_B$ respectively. We have a similar diagrams for each of the triangles in the proposition.

$$\begin{tikzcd}
 & & \mathbb{D}_0 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "\langle \rho_0\epsilon, \rho_1\rangle"] \arrow[dr, "\rho_1"'] & \mathbb{D}_2 \arrow[d, "c"] \\
 & & & \mathbb{D}_1 \\
D(A)_0 \times_{D(A)_0} D_\varphi \arrow[r, "\langle p_0\epsilon_A, p_1\rangle"] \arrow[dr, "p_1"'] \arrow[uurr, dotted, tail] & D_{1_A;\varphi} \arrow[d, "c_{1_A;\varphi}"] \arrow[uurr, dotted, tail] & & \\
 & D_\varphi \arrow[uurr, dotted, tail] & &
\end{tikzcd}$$

and so it suffices to show the component triangles commute. Each case similarly follows by the universal property of the pullback D_φ so we only show the proof for the diagram above. By the pullback square defining $D(A)_0 \times_{D(A)_0} D_\varphi$ above we have that

$$p_0 D(\varphi)_0 = p_1\pi_0 D(\varphi)_0 = p_1\pi_1 s.$$

This induces the unique map in the following commuting diagram.

$$\begin{tikzcd}
D(A)_0 \times_{D(A)_0} D_\varphi \arrow[rr, bend left, "p_1"] \arrow[dr, dotted] \arrow[ddr, bend right, "p_0"'] & & D_\varphi \arrow[d, bend left, "\pi_1"] \\
& D_\varphi \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dd, bend right=60, "s_\varphi"'] \arrow[dr, phantom, "\lrcorner", very near start] & D(B)_1 \arrow[d, "s"] \\
& D(A)_0 \arrow[r, "D(\varphi)_0"'] \arrow[d, tail, "\iota_A"] & D(B)_0 \arrow[d, tail, "\iota_B"] \\
& \mathbb{D}_0 \arrow[r, "\chi_\varphi"'] & \mathbb{D}_9
\end{tikzcd}$$

It suffices to check that plugging in $\langle p_0\epsilon_A, p_1\rangle c_{1_A;\varphi}$ and p_1 as the dotted arrow both make the triangles above commute. By definition of $D(A)_0 \times_{D(A)_0} D_\varphi$ we have that

$$p_1\pi_0 = p_0$$

and we tautologically know $p_1\pi_1 = p_1\pi_1$ so we only need to check what happens when postcomposing $\langle p_0\epsilon_A, p_1\rangle c_{1_A;\varphi}$ with π_0 and π_1. First notice that

$$\begin{aligned}
\langle \epsilon_A, 1_{D_\varphi}\rangle c_{1_A;\varphi}\pi_0\iota_A &= \langle \epsilon_A, 1_{D_\varphi}\rangle c_{1_A;\varphi}s_\varphi && (\text{Def}: \ s_\varphi)\\
&= \langle \epsilon_A, 1_{D_\varphi}\rangle p_0 s_{1_A} && (\text{Lemma } 81)\\
&= p_0\epsilon_A s_{1_A} && (\text{Def}: \ \langle \epsilon_A, 1_{D_\varphi}\rangle)\\
&= p_0\epsilon_A\pi_0\iota_A && (\text{Def}: \ s_\varphi)\\
&= p_0\iota_A && (\text{Def}: \ \epsilon_A)
\end{aligned}$$

implies that

$$\langle p_0\epsilon_A, p_1\rangle c_{1_A;\varphi}\pi_0 = p_0$$

since ι_A is monic. Now by definition of e_B

$$se_Bt = s$$

and this induces a unique map $\langle se_B, 1\rangle D(B)_1 \to D(B)_2$ which factors uniquely as

$$\langle se_B, 1\rangle = \langle s, 1\rangle\langle e_B, 1\rangle.$$

Recall that composition in $\mathbb{D}$ is defined cofiber-wise and is in part induced by composition in the internal category associated to the codomain of a composable pair in $\mathcal{A}$. It's reasonable to think that the family of composable pairs of arrows in $D(B)$ indexed by the composite $\langle p_0\epsilon_A, p_1\rangle c'_{1_A;\varphi}$ should simplify in terms of identity maps, e_B, at the internal level. More precisely, we claim that

$$\langle p_0\epsilon_A, p_1\rangle c'_{1_A;\varphi} = p_1\pi_1\langle se_B, 1\rangle$$

and for this we use the universal property of the pullback $D(B)_2$. By pasting commuting squares we have that

$$\begin{tikzcd}
D(A)_0 \times_{D(A))} D_\varphi \arrow[rrr, bend left, "p_1"] \arrow[rrd, dotted, "\langle p_0\epsilon_A, p_1\rangle"] \arrow[dd, "p_0"'] & & & D_\varphi \arrow[rdd, bend left, "\pi_1"] & \\
& & D_{1_A;\varphi} \arrow[ru, "p_1"] \arrow[rd, dotted, "c'_{1_A;\varphi}"] \arrow[lldd, "p_0"'] & & \\
D(A)_0 \arrow[d, "\epsilon_A"'] & & & D(B)_2 \arrow[r, "q_1"] \arrow[d, "q_0"'] & D(B)_1 \arrow[d, "s"] \\
D_{1_A} \arrow[rr, "\pi_1"'] & & D(A)_1 \arrow[r, "D(\varphi)_1"'] & D(B)_1 \arrow[r, "t"'] & D(B)_0
\end{tikzcd}$$

commutes. Now

$$p_1\pi_1\langle se_B, 1\rangle q_1 = p_1\pi_1$$

and

$$
\begin{aligned}
p_1\pi_1\langle se_B, 1\rangle q_0 &= p_1\pi_1 se_B && (\text{Def } \langle se_B, 1\rangle)\\
&= p_1\pi_0 D(\varphi)_0 e_B && (\text{Def } D_\varphi)\\
&= p_0 D(\varphi)_0 e_B && (\text{Def } D(A)_0 \times_{D(A)_0} D_\varphi)\\
&= p_0 e_A D(\varphi)_1 && ((\text{internal) functoriality})\\
&= p_0 \epsilon_A \pi_1 D(\varphi)_1 && (\text{Def } \epsilon_A)
\end{aligned}
$$

so by uniqueness we have that

$$\langle p_0\epsilon_A, p_1\rangle c'_{1_A;\varphi} = p_1\pi_1\langle se_B, 1\rangle.$$

This allows us to consider the following cone

$$\begin{tikzcd}
D(A)_0 \times_{D(A)_{)}} D_\varphi \arrow[rrr, bend left, "p_1"] \arrow[dd, "p_0"'] \arrow[dr, dotted, "\langle p_0\epsilon_A, p_1\rangle"] & & & D_\varphi \arrow[ddr, bend left, "\pi_1"] & \\
& D_{1_A;\varphi} \arrow[dl, "p_0\pi_0"'] \arrow[drr, bend left, "c'_{1_A;\varphi}"] \arrow[dr, dotted, "c'_{\delta;1_A;\varphi}"] \arrow[ddr, bend right, "c'_{\delta;(1_A;\varphi)}"'] & & & \\
D(A)_0 \arrow[ddrr, bend right, "\delta_{1_A;\varphi}"'] & & D(B)_3 \arrow[r, "q_{12}"] \arrow[d, "q_{01}"'] & D(B)_2 \arrow[r, "q_1"] \arrow[d, "q_0"] & D(B)_1 \arrow[d, "s"] \\
& & D(B)_2 \arrow[r, "q_1"'] \arrow[d, "q_0"'] & D(B)_1 \arrow[r, "t"'] \arrow[d, "s"] & D(B)_0 \\
& & D(B)_1 \arrow[r, "t"'] & D(B)_0 &
\end{tikzcd}$$

which is construcucted by pasting commuting squares and triangles. Notice that

$$\langle p_0\epsilon_A, p_1\rangle c'_{\delta_A;1_A;\varphi} = \langle p_0\delta_{1_A;\varphi}, p_0\delta_A D(\varphi)_1, p_1\pi_1\rangle$$

where the left and right components can be seen in the commuting diagram above and the middle component is verified by checking that

$$\langle p_0\epsilon_A, p_1\rangle c'_{\delta;(1_A;\varphi)} q_1 = \langle p_0\epsilon_A, p_1\rangle p_0\pi_1 D(\varphi)_1 = \langle p_0\epsilon_A, p_1\rangle c'_{1_A;\varphi} q_0.$$

In fact

$$\langle p_0\epsilon_A, p_1\rangle p_0\pi_1 D(\varphi)_1 = p_0\epsilon_A\pi_1 D(\varphi)_1 = p_0\delta_A D(\varphi)_1$$

shows what the middle component must be in the composable triple. After forming the composite of this triple in $D(B)$ we should have the coherence isomorphisms canceling by the coherence law for $\delta_{1_A;\varphi}$ and δ_A, and we formalize this internally by using associativity in $D(B)$ first along with the coherence law for the structure isomorphisms of D that say

$$\langle \delta_{1_A;\varphi}, \delta_A D(\varphi)_1\rangle c = e_A D(\varphi)_1$$

and

$$\begin{aligned}
\langle p_0\delta_{1_A;\varphi}, \delta_A D(\varphi)_1, p_1\pi_1\rangle c &= \langle\langle p_0\delta_{1_A;\varphi}, p_0\delta_A D(\varphi)_1\rangle c, p_1\pi_1\rangle c\\
&= \langle p_0\langle \delta_{1_A;\varphi}, \delta_A D(\varphi)_1\rangle c, p_1\pi_1\rangle c\\
&= \langle p_0 e_A D(\varphi)_1, p_1\pi_1\rangle c\\
&= \langle p_0 D(\varphi)_0 e_B, p_1\pi_1\rangle c\\
&= \langle p_1\pi_0 D(\varphi)_0 e_B, p_1\pi_1\rangle c && (\text{Def. } D(A)_0 \times_{D(A)_0} D_\varphi)\\
&= \langle p_1\pi_1 s e_B, p_1\pi_1\rangle c\\
&= p_1\pi_1\langle s e_B, 1\rangle c\\
&= p_1\pi_1\langle s, 1\rangle\langle e_B, 1\rangle c\\
&= p_1\pi_1\langle s, 1\rangle\\
&= p_1\pi_1.
\end{aligned}$$

and now we can put our calculations above together to see that

$$\begin{aligned}
\langle p_0\epsilon_A, p_1\rangle c_{1_A;\varphi}\pi_1 &= \langle p_0\epsilon_A, p_1\rangle c'_{\delta;1_A;\varphi} c && (\text{Def: } c_{1_A;\varphi})\\
&= \langle p_0\delta_{1_A;\varphi}, p_1\pi_1 e_B, p_1\pi_1\rangle c && (\text{above})\\
&= p_1\pi_1 && (\text{above})
\end{aligned}$$

and by the universal property of D_φ we can conclude

$$\langle p_0 \epsilon_A, p_1 \rangle c_{1_A;\varphi} = p_1.$$

□

To see that $\mathbb{D}$ is an internal category in $\mathcal{E}$ with the structure defined above it only remains to show that composition is associative. This proof is long and technical, follows by a similar pattern to the proof for the identity laws, and ultimately relies on proving associativity on the cofibers of the composable triples coproduct and using the universal property of coproducts.

Proposition 6. *Composition in $\mathbb{D}$ is associative.*

Proof. By extensivity of $\mathcal{E}$ it suffices to show that cofiber composition is associative and this is shown using several lemmas along with the universal property of each cofiber of $\mathbb{D}_1$. A complete proof can be seen in the appendix, precisely in Proposition 85. □

This brings us to the main theorem of this section.

Theorem 7 (The Internal Category $\mathbb{D}$)**.** *The objects, $(\mathbb{D}_0, \mathbb{D}_1)$, along with the structure maps $s, t : \mathbb{D}_1 \to \mathbb{D}_0$ and $c : \mathbb{D}_2 \to \mathbb{D}_1$ defined above form an internal category in $\mathcal{E}$.*

Proof. The required objects, structure maps, and pullbacks exist by definition of $\mathcal{E}$ admitting an internal Grothendieck construction. The associativity and identity laws follow from Propositions 6 and 5. □

3.3 Internal Grothendieck Construction as an Oplax Colimit

In Sections 3.3.1 and A.3 we define the 1-cells and 2-cells of a canonical lax natural transformation

$$\ell : D \implies \Delta\mathbb{D}$$

respectively. In Section 3.3.3 we prove that a lax transformation $D \implies \Delta X$ corresponds uniquely to an internal functor $\mathbb{D} \to X$. Section 3.3.4 shows modifications of lax transformations $D \implies \Delta X$ correspond uniquely to internal natural transformations of internal functors $\mathbb{D} \to X$. Section 3.3.5 combines these results with functoriality to give an equivalence of categories that establishes $\mathbb{D}$ as the oplax colimit of D.

3.3.1 Canonical Transformation 1-cells

Classical

For a diagram of small categories $D : \mathcal{A} \to \mathbf{Cat}(\mathbf{Set})$, for each $A \in \mathcal{A}$ there is a functor $\ell_A : D(A) \to \mathbb{D}$. On an arbitrary object $a \in D(A)_0$, it is defined as

$$\ell_A(a) = (A, a).$$

For an arrow $f \in D(A)(a, b)$, it is defined as

$$\ell_A(f) = (1_A, \delta_{A,a} f)$$

because $s(\delta_{A,a} f) = s(\delta_{A,a}) = D(1_A)(a)$.

$$\begin{tikzcd}
D(1_A)(a) \arrow[r, "\delta_{A,a}", "\cong"'] \arrow[dr] & a \arrow[d, "f ."] \\
& b
\end{tikzcd}$$

Identities are preserved by the identity law in $\mathcal{A}$ in the left component along with coherence in the right component, for any $\varphi : A \to B$ in $\mathcal{A}$ and any $f : D(\varphi)(a) \to b$ in $D(B)_1$

$$\begin{aligned}
\ell_A(1_a)(\varphi, f) &= (1_A, \delta_{A,a})(\varphi, f) \\
&= (1_A\varphi, \delta_{1_A;\varphi,a} D(\varphi)(\delta_{A,a}) f) && \text{Def.} \\
&= (\varphi, f) && \text{Coherence} \\
&= (\varphi 1_B, \delta_{\varphi;1_B,a} \delta_{B,D(\varphi)(a)} f && \text{Coherence} \\
&= (\varphi 1_B, \delta_{\varphi;1_B,a} D(1_B)(f) \delta_{B,b}) && \text{Naturality} \\
&= (\varphi, f)(1_B, \delta_{B,b}) && \text{Def.} \\
&= (\varphi, f)\ell_B(1_b).
\end{aligned}$$

Similarly, composition is preserved in the left component because it is defined as composition in $\mathcal{A}$, and in the right component we only need naturality of the identity coherence isomorphism. For any $f : a \to b$ and $g : b \to c$ in $D(A)_1$, we have that

$$\begin{aligned}
\ell_A(f)\ell_A(g) &= (1_A, \delta_{A,a}f)(1_A, \delta_{A,b}g) && \text{Def.}\\
&= (1_A 1_A, \delta_{1_A;1_A,a}D(1_A)(\delta_{A,a}f)\delta_{A,b}g) && \text{Def.}\\
&= (1_A, \delta_{1_A;1_A,a}D(1_A)(\delta_A)D(1_A)(f)\delta_{B,b}g) && \text{Functoriality}\\
&= (1_A, \delta_{1_A;1_A,a}D(1_A)(\delta_A)\delta_{A,a}fg) && \text{Naturality } \delta_A\\
&= \big(1_A, \delta_{1_A;1_A,a}\delta_{A,D(1_A)(a)}\delta_{A,a}fg\big) && \text{Naturality}\delta_A\\
&= (1_A, \delta_{A,a}fg) && \text{Coherence}\\
&= (\ell_A)(fg) && \text{Def.}
\end{aligned}$$

Internal

The definitions and proofs above can be internalized within an arbitrary extensive category $\mathcal{E}$ as follows. For each $A \in \mathcal{A}_0$, notice that

$$s\delta_A t = s1_{D(A)_0} = s = 1_{D(A)_1}s$$

so there exists a unique map $\langle s\delta_A, 1_{D(A)_1}\rangle : D(A)_1 \to D(A)_2$ in $\mathcal{E}$. Now

$$\langle s\delta_A, 1_{D(A)_1}\rangle cs = \langle s\delta_A, 1_{D(A)_1}\rangle q_0 s = s\delta_A s = sD(1_A)_0$$

induces a unique map $(\ell_A)'_1 := \langle s, \langle s\delta_A, 1_{D(A)_1}\rangle c\rangle$ which we can use to define $\ell_A = ((\ell_A)_0, (\ell_A)_1)$:

$$\begin{tikzcd}
D(A)_0 \arrow[r, "(\ell_A)_0 := \iota_A"] & \mathbb{D}_0
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
D(A)_1 \arrow[r, "(\ell_A)'_1"] \arrow[dr, "(\ell_A)_1"'] & D_{1_A} \arrow[d, tail, "\iota_A"] \\
& \mathbb{D}_1
\end{tikzcd} .$$

Lemma 8. *Identities are preserved by ℓ_A. That is, the diagram*

$$\begin{tikzcd}
D(A)_0 \arrow[r, "e_A"] \arrow[d, "(\ell_A)_0"'] & D(A)_1 \arrow[d, "(\ell_A)_1"] \\
\mathbb{D}_0 \arrow[r, "\epsilon"'] & \mathbb{D}_1
\end{tikzcd}$$

commutes in $\mathcal{E}$.

Proof. First compute

$$e_A(\ell_A)'_1\pi_0 = e_A s = 1_{D(A)_0}$$

and

$$\begin{aligned} e_A(\ell_A)'_1\pi_1 &= e_A\langle s\delta_A, 1_{D(A)_1}\rangle c \\ &= \langle e_A s\delta_A, e_A 1_{D(A)_1}\rangle c \\ &= \langle \delta_A, e_A\rangle c \\ &= \delta_A \end{aligned}$$

by the identity law in D(A) and then see

$$e_A(\ell_A)'_1 = \langle 1_{D(A)_0}, \delta_A\rangle = \epsilon_A,$$

by definition of ϵ_A. Now post-composing with ι_{1_A} and using the equality above along with the definitions of ϵ_A and ϵ gives

$$e_A(\ell_A)_1 = e_A(\ell_A)'_1\iota_{1_A} = \epsilon_A\iota_{1_A} = \iota_A\epsilon$$

as required. □

Due to extensivity and our definition of $\mathbb{D}$ involving coproducts, in order to prove composition is preserved by ℓ_A we need to prove that composition is preserved by ℓ'_A at the level of cofibers. This is done in Lemma 86 in the appendix.

Lemma 9. *For each $A \in \mathcal{A}_0$, composition is preserved by ℓ_A. That is, the diagram*

$$\begin{tikzcd} D(A)_2 \arrow[r, "c"] \arrow[d, "\langle q_0(\ell_A)_1, q_1(\ell_A)_1\rangle"'] & D(A)_1 \arrow[d, "(\ell_A)_1"] \\ \mathbb{D}_2 \arrow[r, "c"'] & \mathbb{D}_1 \end{tikzcd}$$

commutes in $\mathcal{E}$.

Proof. First notice that

$$\begin{aligned}\langle q_0(\ell_A)'_1, q_1(\ell_A)'_1\rangle \iota_{1_A;1_A}\rho_0 &= \langle q_0(\ell_A)'_1, q_1(\ell_A)'_1\rangle p_0\iota_{1_A} && \text{Def. } D_{1_A;1_A}\\ &= q_0(\ell_A)'_1\iota_{1_A}\\ &= q_0(\ell_A)_1 && \text{Def.}\end{aligned}$$

and

$$\begin{aligned}\langle q_0(\ell_A)'_1, q_1(\ell_A)'_1\rangle \iota_{1_A;1_A}\rho_1 &= \langle q_0(\ell_A)'_1, q_1(\ell_A)'_1\rangle p_1\iota_{1_A}\\ &= q_1(\ell_A)'_1\iota_{1_A}\\ &= q_1(\ell_A)_1\end{aligned}$$

so by the universal property of $\mathbb{D}_2$,

$$\langle q_0(\ell_A)_1, q_1(\ell_A)_1\rangle = \langle q_0(\ell_A)'_1, q_1(\ell_A)'_1\rangle \iota_{1_A;1_A}.$$

Use the equation above along with Lemma 86,

$$\begin{aligned}\langle q_0(\ell_A)_1, q_1(\ell_A)_1\rangle c &= \langle q_0(\ell_A)'_1, q_1(\ell_A)'_1\rangle \iota_{1_A;1_A}c\\ &= \langle q_0(\ell_A)'_1, q_1(\ell_A)'_1\rangle c_{1_A;1_A}\iota_{1_A} && \text{Def. } c\\ &= c(\ell_A)'_1\iota_{1_A} && \text{Lemma 86}\\ &= c(\ell_A)_1 && \text{Def. } (\ell_A)_1\end{aligned}$$

to see the square in question commutes. □

The following proposition is the main result of this subsection.

Proposition 10. *For each* $A \in \mathcal{A}_0$, $\ell_A : D(A) \to \mathbb{D}$ *is an internal functor.*

Proof. It preserves identities by Lemma 8 and it preserves composition by Lemma 9. □

3.3.2 2-cells of Canonical Lax Transformation

Classical

When $\mathcal{E} = \mathbf{Set}$, for each $\varphi \in \mathcal{A}(A, B)$ the natural transformation ℓ_φ is defined with components

$$\ell_{\varphi,a} := (\varphi, 1_{D(\varphi)(a)})$$

such that for any $f : a \to b$ in $D(A)$, the square

$$\begin{tikzcd}
a \arrow[r, "{(\varphi,1_{D(\varphi)(a)})}"] \arrow[d, "{(1_A,\delta_{A,a}f)}"'] & D(\varphi)(a) \arrow[d, "{(1_B,\delta_{B,D(\varphi)(a)}D(\varphi)(f))}"] \\
b \arrow[r, "{(\varphi,1_{D(\varphi)(b)})}"'] & D(\varphi)(b)
\end{tikzcd}$$

commutes. This calculation looks like

$$\begin{aligned}
&(1_A, \delta_{A,a}f)(\varphi, 1_{D(\varphi)(b)}) && \\
&= \big(\varphi, \delta_{1_A;\varphi,a} D(\varphi)(\delta_{A,a}f) 1_{D(\varphi)(b)}\big) && \\
&= (\varphi, \delta_{1_A;\varphi,a} D(\varphi)(\delta_{A,a}) D(\varphi)(f)) && \text{Functoriality} \\
&= \big(\varphi 1_B, \delta_{\varphi;1_B,a} \delta_{B,D(\varphi)(a)} D(\varphi)(f)\big) && \text{Coherence} \\
&= \big(\varphi 1_B, \delta_{\varphi;1_B,a} D(1_B)(1_{D(\varphi)(a)}) \delta_{B,D(\varphi)(a)} D(\varphi)(f)\big) && \text{Functoriality} \\
&= (\varphi, 1_{D(\varphi)(a)})(1_B, \delta_{B,D(\varphi)(a)} D(\varphi)(f)) && \text{Def.}
\end{aligned}$$

and can all be internalized to an arbitrary extensive category $\mathcal{E}$. Note that the class of cartesian arrows in the usual Grothendieck contruction are pairs $(\varphi, f) : (A, a) \to (B, b)$ such that f is an isomorphism. The components of ℓ_φ are a special subclass of these which are actually a set when $\mathcal{A}$ is small and these are typically called the canonical cleavage of the cartesian arrows.

Internal

Define the internal natural transformation $\ell_\varphi : \ell_A \implies D(\varphi)\ell_B$ as the composite

$$\begin{tikzcd}
D(A)_0 \arrow[r, "\langle 1_{D(A)_0}{,}D(\varphi)_0 e_B\rangle"] \arrow[dr, "\ell_\varphi"'] & D_\varphi \arrow[d, tail, "\iota_\varphi"] \\
& \mathbb{D}_1
\end{tikzcd}$$

and we can immediately check

$$\begin{aligned}
\ell_\varphi s &= \langle 1_{D(A)_0}, D(\varphi)_0 e_B\rangle \iota_\varphi s \\
&= \langle 1_{D(A)_0}, D(\varphi)_0 e_B\rangle \pi_0 \iota_A && \text{Def. } s_\varphi \\
&= 1_{D(A)_0} \iota_A \\
&= (\ell_A)_0
\end{aligned}$$

and

$$\begin{aligned}
\ell_\varphi t &= \langle 1_{D(A)_0}, D(\varphi)_0 e_B\rangle \iota_\varphi t \\
&= \langle 1_{D(A)_0}, D(\varphi)_0 e_B\rangle \pi_1 t \iota_B && \text{Def. } t_\varphi \\
&= D(\varphi)_0 e_B t \iota_B \\
&= D(\varphi)_0 1_{D(B)_0} \iota_B] && \text{Def. } e_B \\
&= D(\varphi)_0 (\ell_B)_0 && \text{Def. } \ell_B \\
&= (D(\varphi)\ell_B)_0 && \text{Functoriality.}
\end{aligned}$$

This shows us that ℓ_φ is well-defined in terms of its source and target. Now we need to check that it satisfies the naturality square. This is done in the proof of the following proposition as a big calculation that involves manipulating pairing maps of pullbacks. References to a few side calculations appearing as lemmas in Section A.3 of the appendix are included on the side along with references to definitions, internal category structure laws, functoriality, and coherences.

Proposition 11 ((Internal) Naturality of ℓ_φ)**.** *For each $\varphi : A \to B$, the map $\ell_\varphi : D(A)_1 \to \mathbb{D}_1$ defines an internal natural transformation, $\ell_A \implies D(\varphi)\ell_B$ in the sense that the diagram,*

$$\begin{tikzcd}
D(A)_1 \arrow[r, "\langle s\ell_\varphi{,}D(\varphi)_1(\ell_B)_1\rangle"] \arrow[d, "\langle(\ell_A)_1{,}t\ell_\varphi\rangle"'] & \mathbb{D}_2 \arrow[d, "c"] \\
\mathbb{D}_2 \arrow[r, "c"'] & \mathbb{D}_1
\end{tikzcd}$$

commutes in $\mathcal{E}$.

Proof.

$$
\begin{aligned}
&\langle (\ell_A)_1, t\ell_\varphi \rangle c && \\
&= \langle (\ell_A)'_1 \iota_{1_A} \ , \ t\langle 1_{D(A)_0}, D(\varphi)_0 e_B \rangle \iota_\varphi \rangle c && \\
&= \langle (\ell_A)'_1 \ , \ t\langle 1_{D(A)_0}, D(\varphi)_0 e_B \rangle \rangle \iota_{1_A;\varphi} c && \text{Lemma 87} \\
&= \langle (\ell_A)'_1 \ , \ t\langle 1_{D(A)_0}, D(\varphi)_0 e_B \rangle c_{1_A;\varphi} \iota_\varphi && \text{Def. } c \\
&= \langle s \ , \ \langle s\delta_{1_A;\varphi}, s\delta_A D(\varphi)_1, D(\varphi)_1 \rangle c \rangle \iota_\varphi && \text{Lemma 89} \\
&= \langle s \ , \ \langle s\langle \delta_{1_A;\varphi}, \delta_A D(\varphi)_1 \rangle c, D(\varphi)_1 \rangle c \rangle \iota_\varphi && \text{Assoc.} \\
&= \langle s \ , \ \langle s\langle \delta_{\varphi;1_B}, D(\varphi)_0 \delta_B \rangle c, D(\varphi)_1 \rangle c \rangle \iota_\varphi && \text{Coherence.} \\
&= \langle s \ , \ \langle s\delta_{\varphi;1_B}, sD(\varphi)_0 \delta_B, D(\varphi)_1 \rangle c \rangle \iota_\varphi && \text{Assoc.} \\
&= \langle s \ , \ \langle s\delta_{\varphi;1_B}, sD(\varphi)_0 \langle D(1_B)_0, \delta_B \rangle p_1, D(\varphi)_1 \rangle c \rangle \iota_\varphi && \\
&= \langle s \ , \ \langle s\delta_{\varphi;1_B}, sD(\varphi)_0 \langle D(1_B)_0, \delta_B \rangle \langle p_0 e_B, p_1 \rangle c, D(\varphi)_1 \rangle c \rangle \iota_\varphi && \text{Id.-Law} \\
&= \langle s \ , \ \langle s\delta_{\varphi;1_B}, sD(\varphi)_0 \langle D(1_B)_0 e_B, \delta_B \rangle c, D(\varphi)_1 \rangle c \rangle \iota_\varphi && \\
&= \langle s \ , \ \langle s\delta_{\varphi;1_B}, sD(\varphi)_0 \langle e_B D(1_B)_1, \delta_B \rangle c, D(\varphi)_1 \rangle c \rangle \iota_\varphi && \text{Func'y } D(1_B) \\
&= \langle s \ , \ \langle s\delta_{\varphi;1_B}, sD(\varphi)_0 e_B D(1_B)_1, \langle sD(\varphi)_0 \delta_B, D(\varphi)_1 \rangle c \rangle c \rangle \iota_\varphi && \text{Assoc.} \\
&= \langle s \ , \ \langle s\delta_{\varphi;1_B}, sD(\varphi)_0 e_B D(1_B)_1, \langle D(\varphi)_1 s\delta_B, D(\varphi)_1 \rangle c \rangle c \rangle \iota_\varphi && \text{Def. } D(\varphi) \\
&= \langle s \ , \ \langle s\delta_{\varphi;1_B}, sD(\varphi)_0 e_B D(1_B)_1, D(\varphi)_1 \langle s\delta_B, 1_{D(B)_1} \rangle c \rangle c \rangle \iota_\varphi && \text{Factor} \\
&= \langle s \ , \ \langle s\delta_{\varphi;1_B}, sD(\varphi)_0 e_B D(1_B)_1, D(\varphi)_1 (\ell_B)'_1 \pi_1 \rangle c \rangle \iota_\varphi && \text{Def. } (\ell_B)'_1 \\
&= \langle s\langle 1_{D(A)_0}, e_A D(\varphi)_1 \rangle \ , \ \langle sD(\varphi)_0, D(\varphi)_1 (\ell_B)'_1 \pi_1 \rangle \rangle c_{\varphi;1_B} \iota_\varphi && \text{Lemma 91} \\
&= \langle s\langle 1_{D(A)_0}, e_A D(\varphi)_1 \rangle \ , \ \langle D(\varphi)_1 s, D(\varphi)_1 (\ell_B)'_1 \pi_1 \rangle \rangle c_{\varphi;1_B} \iota_\varphi && \text{Def. } D(\varphi) \\
&= \langle s\langle 1_{D(A)_0}, e_A D(\varphi)_1 \rangle \ , \ D(\varphi)_1 \langle s, (\ell_B)'_1 \pi_1 \rangle \rangle c_{\varphi;1_B} \iota_\varphi && \text{Factor} \\
&= \langle s\langle 1_{D(A)_0}, e_A D(\varphi)_1 \rangle \ , \ D(\varphi)_1 (\ell_B)'_1 \rangle c_{\varphi;1_B} \iota_\varphi && \text{Uniqueness} \\
&= \langle s\langle 1_{D(A)_0}, e_A D(\varphi)_1 \rangle \ , \ D(\varphi)_1 (\ell_B)'_1 \rangle \iota_{\varphi;1_B} c && \text{Def. } c \\
&= \langle s\langle 1_{D(A)_0}, e_A D(\varphi)_1 \rangle \iota_\varphi \ , \ D(\varphi)_1 (\ell_B)'_1 \iota_{1_B} \rangle c && \text{Lemma92} \\
&= \langle s\ell_\varphi, D(\varphi)_1 (\ell_B)_1 \rangle && \text{Def. } \ell_\varphi, (\ell_B)_1
\end{aligned}
$$

□

3.3.3 Universal Property for 1-cells

Classical

Suppose $\mathcal{E} = \mathbf{Set}$ and $x : D \implies \Delta X$ is a lax natural transformation. That is, there are functors

$$x_A : D(A) \to X$$

for each $A \in \mathcal{A}_0$ and natural transformations

$$x_\varphi : x_A \to D(\varphi)x_B$$

for each $\varphi : A \to B$ in $\mathcal{A}$ that are coherent with respect to the pseudofunctor's structure isomorphisms. More concretely, for each $a \in D(A)_0$ and each composable $\varphi, \psi \in \mathcal{A}$ where $A = \mathrm{dom}(\varphi)$ we have

$$x_{1_A,a} = x_A(\delta^{-1}_{A,a}) \quad , \quad x_{\varphi\psi,a} x_C(\delta_{\varphi;\psi,a}) = x_{\varphi,a} x_{\psi,D(\varphi)(a)}.$$

Then we can define a functor $\theta : \mathbb{D} \to X$ on an object (A, a) in $\mathbb{D}$ as

$$\theta\left((A, a)\right) = x_A\left((A, a)\right)$$

and on a morphism $(\varphi, f) : (A, a) \to (B, b)$ in $\mathbb{D}$ as

$$\theta\left((\varphi, f)\right) = x_{\varphi,a} x_B(f).$$

Identities are preserved by this assignment because the diagram

$$\begin{tikzcd}
x_A(a) \arrow[r, "x_A(\delta^{-1}_{A,a})"] \arrow[rr, bend left, "x_A(\delta^{-1}_{A,a}\delta_{A,a}) = x_A(1_a) = 1_{x_A(a)}"] \arrow[rr, bend right, "\theta((1_A, \delta_{A,a}))"'] & x_A(D(1_A)(a)) \arrow[r, "x_A(\delta_{A,a})"] & x_A(a)
\end{tikzcd}$$

commutes in X and composition is preserved by the following commuting diagram in X.

$$\begin{tikzcd}
 & x_C\left(D(\varphi\psi)(a)\right) \arrow[dr, bend left, "x_C(\delta_{\varphi;\psi,a})"] & \\
x_A(a) \arrow[ur, bend left, "x_{\varphi\psi,a}"] \arrow[r, "x_{\varphi,a}"] \arrow[d, "\theta((\varphi,f))"'] & x_B(D(\varphi)(a)) \arrow[r, "x_{\psi,D(\varphi)(a)}"] \arrow[dl, "x_B(f)"] & x_C\left(\left(D(\varphi)D(\psi)\right)(a)\right) \arrow[dl, "x_C(D(\psi)(f))"] \arrow[ddll, bend left, "x_C(D(\psi)(f)g)"] \\
x_A(b) \arrow[r, "x_{\psi,a}"] \arrow[d, "\theta((\psi,g))"'] & x_C\left(D(\psi)(b)\right) \arrow[dl, "x_C(g)"] & \\
x_A(c) & &
\end{tikzcd}$$

The top square commutes by coherence, the triangles on the left commute by definition of θ, the middle square commutes by naturality of x_ψ and by functoriality of x_C we know the bottom right triangle commutes so we can see

$$\begin{aligned}
\theta((\varphi,f))\theta((\psi,g)) &= x_{\varphi\psi,a}x_C(\delta_{\varphi;\psi,a})x_C(D(\psi)(f)g) && \text{Def.}\theta \\
&= x_{\varphi\psi,a}x_C(\delta_{\varphi;\psi,a}D(\psi)(f)g) && \text{Functoriality } x_C \\
&= \theta((\varphi\psi,\delta_{\varphi;\psi,a}D(\psi)(f)g)) && \text{Def. } \theta \\
&= \theta((\varphi,f)(\psi,g)).
\end{aligned}$$

Notice that

$$\theta(\ell_A(a)) = \theta((A,a)) = x_A(a)$$

and

$$\begin{aligned}
\theta(\ell_A(f)) &= \theta((1_A,\delta_{A,a}f)) \\
&= x_{1_A,a}x_A(\delta_{A,a}f) \\
&= x_A(\delta_{A,a}^{-1})x_A(\delta_{A,a}f) && \text{Coherence} \\
&= x_A(\delta_{A,a}^{-1}\delta_{A,a}f) && \text{Functoriality} \\
&= x_A(f) && \text{Functoriality}
\end{aligned}$$

so $\ell_A\theta = x_A$ for each A in $\mathcal{A}$. Moreover, for any functor $\omega : \mathbb{D} \to X$ and any $\varphi : A \to B$ in $\mathcal{A}$ one can get a natural transformation

$$\omega_\varphi : \ell_A\omega \implies D(\varphi)\ell_B\omega$$

by whiskering. Componentwise this amounts to defining

$$\omega_{\varphi,a} := \omega(\ell_{\varphi,a}).$$

Functoriality of ω makes ω_φ coherent with respect to composition in $\mathcal{A}$,

$$\begin{aligned}\omega_{\varphi\psi,a} &= \omega(\ell_{\varphi\psi,a}) \\ &= \omega(\ell_{\varphi,a}\ell_{\psi,D(\varphi)(a)}\ell_C(\delta^{-1}_{\varphi;\psi,a})) \\ &= \omega(\ell_{\varphi,a})\omega(\ell_{\psi,D(\varphi)(a)})\omega(\ell_C(\delta^{1}_{\varphi;\psi,a})) \\ &= \omega_{\varphi,a}\omega_{\psi,D(\varphi)(a)}\left((\ell_C\omega)(\delta_{\varphi;\psi,a})\right)^{-1},\end{aligned}$$

and with respect to identities in $\mathcal{A}$

$$\omega_{1_A,a} = \omega(\ell_{1_A,a}) = \omega(\ell_A(\delta^{-1}_{A,a})) = (\ell_A\omega)(\delta^{-1}_{A,a}).$$

The assignments above are inverses to one another. On one hand if we start with a family of natural transformations $\{x_\varphi : x_A \implies D(\varphi)x_B\}_{\varphi\in\mathcal{A}(A,B)}$, consider the induced functor θ and its induced lax natural transformation. For each $\varphi : A \to B$ in $\mathcal{A}$, the natural transformation

$$\theta_\varphi : \ell_A\omega \implies D(\varphi)\ell_B\omega$$

has precisely the same components as the natural transformation x_φ from the family we started with since $\ell_A\omega = x_A$ for each A in $\mathcal{A}$.

$$\theta_{\varphi,a} = \theta(\ell_{\varphi,a}) = \theta((\varphi, 1_{D(\varphi)(a)})) = x_{\varphi,a}x_B(1_{D(\varphi)(a)}) = x_{\varphi,a}$$

Now if we start with a functor $\omega : \mathbb{D} \to X$, get the natural transformations $\omega_\varphi : \ell_A\omega \implies D(\varphi)\ell_B\omega$, and then let $\theta_\omega : \mathbb{D} \to X$ be the induced functor, we have that for each $(A, a) \in D(A)$,

$$\theta_\omega((A,a)) := (\ell_A\omega)((A,a)) = (\omega(\ell_A(a)) = \omega((A,a))$$

as well as

$$\begin{aligned}\theta_\omega((\varphi,f)) &= \omega_{\varphi,a}\left((\ell_B\omega)(f)\right)\\ &= \omega(\ell_{\varphi,a})\left((\ell_B\omega)(f)\right)\\ &= \omega\left(\ell_{\varphi,a}\ell_B(f)\right)\\ &= \omega\left((\varphi,1_{D(\varphi)(a)})(1_B,\delta_{B,D(\varphi)(a)}f)\right)\\ &= \omega\left((\varphi 1_B,\delta_{\varphi;1_B,a}D(1_B)(1_{D(\varphi)(a)})\delta_{B,D(\varphi)(a)}f)\right)\\ &= \omega\left((\varphi,\delta_{\varphi;1_B,a}\delta_{B,D(\varphi)(a)}f)\right)\\ &= \omega\left((\varphi f)\right)\end{aligned}$$

where the last equality is by coherence. This shows that $\theta_\omega = \omega$ and it follows that the assignments

$$\{x_\varphi : x_A \implies D(\varphi)x_B\}_{\varphi\in\mathcal{A}(A,B)} \mapsto (\theta : \mathbb{D} \to X)$$

and

$$(\omega : \mathbb{D} \to X) \mapsto \{\omega_\varphi : \ell_A\omega \implies D(\varphi)\ell_B\omega\}_{\varphi\in\mathcal{A}(A,B)}$$

are inverses of one another. In particular, every functor $\mathbb{D} \to X$ corresponds uniquely to a lax natural transformation $D \implies \Delta X$.

Internal

Let $\mathcal{E}$ be an extensive category and let $\mathbb{X}$ be an arbitary internal category in $\mathcal{E}$ with $\{x_A : D(A) \to \mathbb{D}\}$ an $\mathcal{A}_0$-indexed family of internal functors and $\{x_\varphi : x_A \implies D(\varphi)x_B\}$ an $\mathcal{A}_1$-indexed family of internal natural transformations that satisfy the following (internalized) coherences with respect to the pseudofunctor isomorphisms

$$\langle x_{1_A}, \delta_A(x_A)_1\rangle c = e_A(x_A)_1 \qquad , \qquad \langle x_{\varphi\psi}, \delta_{\varphi;\psi}x_C\rangle c = \langle x_\varphi, D(\varphi)_0 x_\psi\rangle c.$$

Define θ_0 to be uniquely induced by the maps $\{x_A\}_{A\in\mathcal{A}_0}$

$$\begin{tikzcd}
\mathbb{D}_0 \arrow[rr, dotted, "\theta_0"] & & \mathbb{X}_0 \\
& D(A)_0 \arrow[ul, "\iota_A"] \arrow[ur, "(x_A)_0"'] &
\end{tikzcd}$$

and define θ_1 to be uniquely induced by the family of composites

$$\begin{tikzcd}
\mathbb{D}_1 \arrow[r, dotted, "\theta_1"] & \mathbb{X}_1 \\
D_\varphi \arrow[u, "\iota_\varphi"] \arrow[r, "\langle \pi_0 x_\varphi, \pi_1 (x_B)_1\rangle"'] & \mathbb{X}_2 \arrow[u, "c"']
\end{tikzcd} .$$

on each component. The following lemma shows $\theta = (\theta_0, \theta_1)$ preserves identities and will be used later to conclude θ is an internal functor.

Lemma 12. *The assignment* $(\theta_0, \theta_1) : \mathbb{D} \to \mathbb{X}$ *preserves identities.*

Proof. By the universal property of the coproduct, $\mathbb{D}_1$, it suffices to see the following diagram commutes.

$$\begin{tikzcd}
\mathbb{D}_0 \arrow[r, dotted, "\theta_0"] \arrow[rrrr, dotted, bend left=30] & \mathbb{X}_0 \arrow[r, "e"] \arrow[rr, equal, bend left] & \mathbb{X}_1 \arrow[r, "s"] & \mathbb{X}_0 \arrow[r, "e"] & \mathbb{X}_1 \\
D(A)_0 \arrow[u, "\iota_A"] \arrow[d, "\iota_A"'] \arrow[ur, "(x_A)_0"] \arrow[r, "e_A"] \arrow[rr, equal, bend right] & D(A)_1 \arrow[ur, "(x_A)_1"] \arrow[r, "s"] & D(A)_0 \arrow[ur, "(x_A)_0"] \arrow[r, "\langle 1_{D(A)_0}, \delta_A\rangle"'] \arrow[rr, bend left, "\langle x_{1_A}, \delta_A (x_A)_1\rangle"] & D_{1_A} \arrow[r, "\langle \pi_0 x_{1_A}, \pi_1 (x_A)_1\rangle"'] \arrow[dl, "\iota_{1_A}"] & \mathbb{X}_2 \arrow[u, "c"'] \arrow[d, "c"] \\
\mathbb{D}_0 \arrow[rr, dotted, "\epsilon"] \arrow[rrrr, dotted, bend right=30] & & \mathbb{D}_1 \arrow[rr, "\theta_1"] & & \mathbb{X}_1
\end{tikzcd}$$

The top left two squares commute by definition of θ_0 and e. They show that the composite $\theta_0 e$ is uniquely induced by the family of maps $\{e_A (x_A)_1\}_{A\in\mathcal{A}_0}$. The other two squares on the top commute by functoriality of x_A and coherence. In the middle on the left we have the source-identity coherence in $D(A)$, and a short calculation on the right using the universal property of $\mathbb{X}_2$ shows that

$$\langle x_{1_A}, \delta_A(x_A)_1\rangle = \langle 1_{D(A)_0}, \delta_A\rangle\langle \pi_0 x_{1_A}, \pi_1(x_A)_1\rangle.$$

Recall that $\epsilon_A := \langle 1_D(A)_0, \delta_A\rangle$ to see that the bottom left square commutes by definition of ϵ and the bottom right square commutes by definition of θ_1. Together they show that $\epsilon\theta_1$ is uniquely induced by $\{\epsilon_A\langle \pi_0\iota_{1_A}, \pi_A(x_A)_1\rangle c\}_{A\in\mathcal{A}_0}$. Commutativity above shows

$$\epsilon_A\langle \pi_0\iota_{1_A}, \pi_A(x_A)_1\rangle c = e_A s\langle x_{1_A}, \delta_A(x_A)_1\rangle c = e_A(x_A)_1$$

and therefore

$$\epsilon\theta_1 = \theta_0 e$$

by uniqueness. ☐

Lemma 13. *The assignment* $(\theta_0, \theta_1) : \mathbb{D} \to \mathbb{X}$ *preserves composition.*

Proof. By definition of composition in $\mathbb{D}$ and the map θ_1, the composite $c\theta_1 : \mathbb{D}_2 \to \mathbb{X}_2$ is uniquely induced by the family of maps $c_{\varphi;\psi}\langle \pi_2 x_{\varphi\psi}, \pi_1(x_C)_1\rangle c$, where $\varphi : A \to B$ and $\psi : B \to C$ are composable morphisms in $\mathcal{A}$.

By the universal property of the coproduct $\mathbb{D}$ it suffices to show that

$$c_{\varphi;\psi}\langle \pi_2 x_{\varphi\psi}, \pi_1(x_C)_1\rangle c = \langle p_0\langle \pi_0 x_\varphi, \pi_1(x_B)_1\rangle \ , \ p_1\langle \pi_0 x_\psi, \pi_1(x_C)_1\rangle\rangle c$$

so that the middle pentagon (disguised as a triangle) in the diagram

$$\begin{tikzcd}
\mathbb{D}_2 \arrow[r, dotted, "c"] \arrow[r, dotted, bend left] & \mathbb{D}_1 \arrow[r, dotted, "\theta_1"] & \mathbb{X}_1 \\
D_{\varphi;\psi} \arrow[u, tail, "\iota_{\varphi;\psi}"] \arrow[r, "c_{\varphi;\psi}"'] \arrow[dr, "\langle p_0\langle \pi_0 x_\varphi{,}\pi_1(x_B)_1\rangle\ {,}\ p_1\langle \pi_0 x_\psi{,}\pi_1(x_C)_1\rangle\rangle"] \arrow[dd, tail, "\iota_{\varphi;\psi}"'] & D_{\varphi\psi} \arrow[u, "\iota_{\varphi\psi}"'] \arrow[r, "\langle \pi_0 x_{\varphi\psi}{,}\pi_1(x_C)_1\rangle"'] & \mathbb{X}_2 \arrow[u, "c"'] \arrow[dd, "c"] \\
& \mathbb{X}_2 \arrow[d, equal] \arrow[dr, "c"] & \\
\mathbb{D}_2 \arrow[r, dotted, "\langle \rho_0\theta_1{,}\rho_1\theta_1\rangle"'] & \mathbb{X}_2 \arrow[r, "c"'] & \mathbb{X}_1
\end{tikzcd}$$

commutes. The remaining squares and triangle in the diagram above commute by definition or by the identity law for composition in $\mathcal{E}$. Let $\delta^{(-1)}_{\varphi;\psi} : D(A)_0 \to D(C)_1$ denote the 'family of inverse coherence isomorphisms' associated to φ, ψ. In particular,

$$\langle \delta^{(-1)}_{\varphi;\psi}, \delta_{\varphi;\psi}\rangle c = D(\varphi)_0 D(\psi)_0 e_C$$

$$\begin{aligned}
&c_{\varphi;\psi}\langle \pi_0 x_{\varphi;\psi}, \pi_1(x_C)_1\rangle c \\
&= \langle p_0\pi_0 x_\varphi,\ p_0\pi_0 D(\varphi)_0 x_\psi,\ p_0\pi_0\delta^{(-1)}_{\varphi;\psi}(x_C)_1,\ p_0\pi_0\delta_{\varphi;\psi}(x_C)_1, \\
&\quad p_0\pi_1 D(\psi)_1(x_C)_1,\ p_1\pi_1(x_C)_1\rangle c \\
&= \langle p_0\pi_0 x_\varphi,\ p_0\pi_0 D(\varphi)_0 x_\psi,\ p_0\pi_0 D(\varphi)_0 D(\psi)_0 (x_C)_0 e, \\
&\quad p_0\pi_1 D(\psi)_1(x_C)_1,\ p_1\pi_1(x_C)_1\rangle c \\
&= \langle p_0\pi_0 x_\varphi,\ p_0\pi_0 D(\varphi)_0 x_\psi,\ p_0\pi_1 D(\psi)_1(x_C)_1,\ p_1\pi_1(x_C)_1\rangle c \\
&= \langle p_0\pi_0 x_\varphi,\ p_0\pi_1 s x_\psi\ ,\ p_0\pi_1 D(\psi)_1(x_C)_1,\ p_1\pi_1(x_C)_1\rangle c && \text{Id. Law} \\
&= \langle p_0\pi_0 x_\varphi,\ p_0\pi_1(x_B)_1\ ,\ p_0\pi_1 t x_\psi,\ p_1\pi_1(x_C)_1\rangle c && \text{Naturality } x_\psi \\
&= \langle p_0\pi_0 x_\varphi\ ,\ p_0\pi_1(x_B)_1,\ p_1\pi_0 x_\psi,\ p_1\pi_1(x_C)_1\rangle c && \text{Def. } D_{\varphi;\psi} \\
&= \langle\langle p_0\langle \pi_0 x_\varphi,\ \pi_1(x_B)_1\rangle c,\ p_1\langle \pi_0 x_\psi,\ \pi_1(x_C)_1\rangle c\rangle c && \text{Assoc. \& Factor}
\end{aligned}$$

□

Proposition 14. *The assignment* $\theta = (\theta_0, \theta_1) : \mathbb{D} \to \mathbb{X}$ *is an internal functor.*

Proof. Immediate from Lemmas 12 and 13. □

On the other hand, given an internal functor $\omega : \mathbb{D} \to \mathbb{X}$, for each $\varphi : A \to B$ in $\mathcal{A}$ define

$$\omega_A = \ell_A \omega \quad \text{and} \quad \omega_\varphi := \ell_\varphi \omega_1.$$

Notice that ω_A is an internal functor (by definition of internal functor composition) and we have that the source of ω_φ is ω_A

$$\omega_\varphi s = \ell_\varphi \omega_1 = \ell_\varphi s \omega_0 = (\ell_A)_0 \omega_0 = (\ell_A \omega)_0 = (\omega_A)_0,$$

its target is $D(\varphi)\omega_B$

$$\omega_\varphi t = \ell_\varphi \omega_1 t = \ell_\varphi t \omega_0 = (D(\varphi)\ell_B)_0 \omega_0 = D(\varphi)(\ell_B \omega)_0 = D(\varphi)(\omega_B)_0,$$

and the (family of) naturality square(s)

$$\begin{aligned}
\langle s\omega_\varphi, (D(\varphi)\omega_B)_1 \rangle c &= \langle s\ell_\varphi \omega_1, (D(\varphi)\ell_B \omega)_1 \rangle c \\
&= \langle s\ell_\varphi \omega_1, (D(\varphi)\ell_B)_1 \omega_1 \rangle c \\
&= \langle s\ell_\varphi, (D(\varphi)\ell_B)_1 \rangle c\omega_1 && \text{Functoriality } \omega \\
&= \langle (\ell_A)_1, t\ell_\varphi \rangle c\omega_1 \\
&= \langle (\ell_A)_1 \omega_1, t\ell_\varphi \omega_1 \rangle c \\
&= \langle \omega_A, t\omega_\varphi \rangle c
\end{aligned}$$

commutes so $\omega_\varphi : \omega_A \implies D(\varphi)\omega_B$ is an internal natural transformation. Putting these families of $\mathcal{A}_0$-indexed internal functors and $\mathcal{A}_1$-indexed internal natural transformations together precisely defines an internal lax transformation

$$\omega^* : D \implies \Delta\mathbb{X}$$

where $\Delta\mathbb{X} : \mathcal{A} \to \mathbf{Cat}(\mathcal{E})$ is the constant functor on $\mathbb{X}$.

Proposition 15. *For each $\mathbb{X} \in \mathbf{Cat}(\mathcal{E})$, lax transformations $D \implies \Delta\mathbb{X}$ correspond uniquely to internal functors $\mathbb{D} \to \mathbb{X}$.*

Proof. Suppose $x : D \implies \Delta\mathbb{X}$ is a lax transformation. Then for each object A in $\mathcal{A}$ there exists an internal functor $x_A : D(A) \to \mathbb{X}$ and for each morphism $\varphi : A \to B$ in $\mathcal{A}$ there exists an internal natural transformation $x_\varphi : x_A \implies D(\varphi)x_B$ that is coherent with respect to the composition and identity isomorphisms of D. Let $\theta : \mathbb{D} \to \mathbb{X}$ be the internal functor constructed above, and then consider the induced lax transformation θ^*. As seen above we have that

$$\theta_A = \ell_A \theta = x_A$$

for each $A \in \mathcal{A}_0$ and for each $\varphi \in \mathcal{A}_1$ we have

$$\theta_\varphi := \ell_\varphi \theta_1 = x_\varphi$$

so $\theta^* = x$.

On the other hand, let $\omega^* : D \implies \Delta\mathbb{X}$ denote the lax transformation constructed as above from an arbitrary internal functor $\omega : \mathbb{D} \to \mathbb{X}$. Let θ^* be the induced internal functor as above, then θ^*_0 is uniquely induced by the $\mathcal{A}_0$-indexed family of internal functors of ω^*. These are precisely the maps $(\omega_A)_0$ and so $\theta^*_0 = \omega_0$ by uniqueness. Using functoriality of ω and the definition of ω^* we can see that θ^*_1 is uniquely induced by the family of maps,$\iota_\varphi \omega_1$. We break the calculation up with terms on separate lines due to their length. We start with the functoriality of ω giving us

$$\langle \pi_0 \ell_\varphi \omega_1, \pi_1 (\ell_B \omega)_1 \rangle c = \langle \pi_0 \ell_\varphi,\ \pi_1 (\ell_B)_1 \rangle c \omega_1$$

and then by definition of ℓ_{φ, ℓ_B} the right-hand side is equal to

$$\langle \pi_0 \langle 1_{D(A)_0},\ D(\varphi)_0 e_B \rangle \iota_\varphi, \pi_1 \langle s,\ \langle s\delta_B,\ 1_{D(B)_1} \rangle c \rangle \iota_B \rangle c \omega_1.$$

The definition of $\iota_{\varphi;1_B}$ says the last term is equal to

$$\langle \pi_0 \langle 1_{D(A)_0},\ D(\varphi)_0 e_B \rangle, \pi_1 \langle s,\ \langle s\delta_B,\ 1_{D(B)_1} \rangle c \rangle \rangle \iota_{\varphi;1_B} c \omega_1$$

which is equal to

$$\langle \pi_0 \langle 1_{D(A)_0},\ D(\varphi)_0 e_B \rangle, \pi_1 \langle s,\ \langle s\delta_B,\ 1_{D(B)_1} \rangle c \rangle \rangle c_{\varphi;1_B} \iota_\varphi \omega_1$$

by definition of $c_{\varphi;1_B}$. The same definition gives that this is equal to

$$\langle \pi_0, \langle \pi_0 \delta_{\varphi;1_B},\ \pi_0 D(\varphi)_0 D(1_B)_0 e_B, \pi_1 \langle s\delta_B, 1_{D(B)_1}\rangle c\rangle c\rangle \iota_\varphi \omega_1$$

which is equal to

$$\langle \pi_0, \langle \pi_0 \delta_{\varphi;1_B}, \pi_1 \langle s\delta_B, 1_{D(B)_1}\rangle c\rangle c\rangle \iota_\varphi \omega_1$$

by the identity law in $D(B)$. Associativity of internal composition gives that this is equal to

$$\langle \pi_0,\ \langle \langle \pi_0 \delta_{\varphi;1_B}, \pi_1 s \delta_B\rangle c, \pi_1\rangle c\rangle \iota_\varphi \omega_1$$

which becomes

$$\langle \pi_0\ ,\ \langle \langle \pi_0 \delta_{\varphi;1_B}, \pi_0 D(\varphi)_0 \delta_B\rangle c, \pi_1\rangle c\rangle \iota_\varphi \omega_1$$

by definition of D_φ. Factoring pairing maps makes the last term equal to

$$\langle \pi_0\ ,\ \langle \pi_0 \langle \delta_{\varphi;1_B}, D(\varphi)_0 \delta_B\rangle c, \pi_1\rangle c\rangle \iota_\varphi \omega_1$$

which, by coherence of the structure isomorphisms for the pseudofunctor D, is equal to

$$\langle \pi_0\ ,\ \langle \pi_0 D(\varphi)_0 e_B, \pi_1\rangle c\rangle \iota_\varphi \omega_1.$$

The definition of D_φ and the identity law in $D(B)$ allows us to see the term above is really the left-hand side of the final equality:

$$\langle \pi_0,\ \pi_1\rangle \iota_\varphi \omega_1 = \iota_\varphi \omega_1$$

By the universal property of the coproduct $\mathbb{D}_1$ we have

$$\theta_1^* = \omega_1$$

and it follows that $\theta^* = \omega$. $\square$

3.3.4 Universal Property for 2-cells

Classical

When $\mathcal{E} = \mathbf{Set}$ and let X be a small category. Then any natural transformation $\alpha : \theta \implies \omega$ where $\theta, \omega : \mathbb{D} \to X$ induces a modification

$$\tilde{\alpha} : x \Rrightarrow y$$

where $x, y : D \implies \Delta\mathbb{X}$ are the lax natural transformations corresponding uniquely by Proposition 15 to θ and ω respectively. For each $A \in \mathcal{A}_0$ and $a \in D(A)_0$ we have

$$\tilde{\alpha}_{A,a} : x_A(a) \to y_A(a)$$

defined as the component

$$\alpha_{\ell_A(a)} : \theta(\ell_A(a)) \to \omega(\ell_A(a)).$$

For any $g : a \to a'$ in $D(A)_0$, the diagram

$$\begin{tikzcd}
x_A(a) \arrow[r, equal] \arrow[d, "x_A(g)"'] \arrow[rrr, bend left, "\tilde{\alpha}_{A,a}"] & \theta(\ell_A(a)) \arrow[r, "\alpha_{\ell_A(a)}"] \arrow[d, "\theta(\ell_A(g))"'] & \omega(\ell_A(a)) \arrow[r, equal] \arrow[d, "\omega(\ell_A(g))"] & y_A(a) \arrow[d, "y_A(g)"] \\
x_A(a') \arrow[r, equal] \arrow[rrr, bend right, "\tilde{\alpha}_{A,a'}"'] & \theta(\ell_A(a')) \arrow[r, "\alpha_{\ell_A(a')}"'] & \omega(\ell_A(a')) \arrow[r, equal] & y_A(a')
\end{tikzcd}$$

commutes by definition and naturality of α. Similarly, for any $\varphi : A \to B$ in $\mathcal{A}$ and any $a \in D(A)_0$ we have that the diagram

$$\begin{tikzcd}
x_A(a) \arrow[r, equal] \arrow[d, "x_{\varphi,a}"'] \arrow[rrr, bend left, "\tilde{\alpha}_{A,a}"] & (\ell_A\theta)(a) \arrow[r, "\alpha_{\ell_A(a)}"] \arrow[d, "\theta(\ell_{\varphi,a})"'] & (\ell_A\omega)(a) \arrow[r, equal] \arrow[d, "\omega(\ell_{\varphi,a})"] & y_A(a) \arrow[d, "y_{\varphi,a}"] \\
D(\varphi)x_B(a) \arrow[r, equal] \arrow[rrr, bend right, "\tilde{\alpha}_{A,a}"'] & D(\varphi)\ell_B\theta(a) \arrow[r, "\alpha_{(D(\varphi)\ell_B)(a)}"'] & D(\varphi)\ell_B\omega(a) \arrow[r, equal] & D(\varphi)y_B(a)
\end{tikzcd}$$

commutes. It follows that $\tilde{\alpha}$ is a modification $x \Rrightarrow y$.

Alternatively, given a modification $\gamma : x \Rrightarrow y$ between two lax natural transformations $x, y : D \implies \Delta\mathbb{X}$, let $\theta, \omega : \mathbb{D} \to X$ be the functors uniquely determined by x and y respectively. Then the middle two squares in the following diagram commute by definition of γ

$$\begin{tikzcd}
\theta((A,a)) \arrow[r, equal] \arrow[dd, "\theta((\varphi,f))"'] & x_A(a) \arrow[r, "\gamma_{A,a}"] \arrow[d, "x_{\varphi,a}"'] & y_A(a) \arrow[r, equal] \arrow[d, "y_{\varphi,a}"] & \omega((A,a)) \arrow[dd, "\omega((\varphi,f))"] \\
 & x_B(D(\varphi)(a)) \arrow[r, "\gamma_{B,D(\varphi)(a)}"] \arrow[d, "x_B(f)"'] & y_B(D(\varphi)(a)) \arrow[d, "y_B(f)"] & \\
\theta((B,b)) \arrow[r, equal] & x_B(b) \arrow[r, "\gamma_{B,b}"'] & y_B(b) \arrow[r, equal] & \omega((B,b))
\end{tikzcd}$$

and the left and right squares commute by definition of θ and ω respectively. This means

$$\overline{\gamma} := \{\gamma_{A,a} : (A,a) \in \mathbb{D}\}$$

is a natural transformation $\theta \implies \omega$. Notice that the lax natural transformation $\overline{\tilde{\alpha}}$ has components

$$\overline{\tilde{\alpha}}_{(A,a)} = \tilde{\alpha}_{A,a} = \alpha_{\ell_A(a)} = \alpha_{(A,a)}$$

so that $\overline{\tilde{\alpha}} = \alpha$. The modification $\tilde{\overline{\gamma}}$ has components

$$\tilde{\overline{\gamma}}_{A,a} = \overline{\gamma}_{\ell_A(a)} = \overline{\gamma}_{(A,a)} = \gamma_{A,a}$$

and so $\tilde{\overline{\gamma}} = \gamma$ by definition. The bijection follows and by uniqueness it suffices to see functoriality in one direction. Let $\gamma : x \Rrightarrow y$ and let $\eta : y \Rrightarrow z$ be modifications, then their composite $\gamma\eta$ has components

$$\begin{tikzcd}
x_A(a) \arrow[r, "\gamma_{A,a}"] \arrow[dr, "(\gamma\eta)_{A,a}"'] & y_A(a) \arrow[d, "\eta_{A,a}"] \\
 & z_A(a)
\end{tikzcd}$$

and so

$$\overline{\gamma\eta} := \{(\gamma\eta)_{A,a} : (A,a) \in \mathbb{D}\} = \{\gamma_{A,a}\eta_{A,a} : (A,a) \in \mathbb{D}\} =: (\overline{\gamma})(\overline{\eta})$$

where the right-hand side is the composite of lax natural transformations.

$$\theta \overset{\overline{\gamma}}{\Longrightarrow} \omega \overset{\overline{\eta}}{\Longrightarrow} \sigma$$

and $\theta, \omega, \sigma : \mathbb{D} \to \mathbb{X}$ are the internal functors uniquely determined by x, y, z respectively. It follows that composition of modifications is preserved by the bijection above and identities are trivially preserved because

$$(1_x)_{A,a} = 1_{x_A(a)}$$

for each $A \in \mathcal{A}_0$ and each $a \in D(A)_0$ by definition of the identity modification $1_x : x \Rrightarrow x$.

Internal

Let $\mathcal{E}$ be a category that admits an internal Grothendieck construction of $D : \mathcal{A} \to \mathbf{Cat}(\mathcal{E})$ and let $\mathbb{X}$ be an arbitrary internal category in $\mathcal{E}$. Let $\alpha : \theta \implies \omega$ be an internal natural transformation where $\theta, \omega : \mathbb{D} \to \mathbb{X}$ are internal functors and further let $x, y : D \implies \Delta\mathbb{X}$ denote the unique lax natural transformations induced by θ and ω respectively. For each A in $\mathcal{A}_0$ define

$$\begin{tikzcd}
D(A)_0 \arrow[r, "(\ell_A)_0"] \arrow[dr, "\tilde{\alpha}_A"'] & \mathbb{D}_0 \arrow[d, "\alpha\ ."] \\
& \mathbb{X}_1
\end{tikzcd}$$

Proposition 16. *The $\mathcal{A}_0$-indexed family of maps $\tilde{\alpha}_A : D(A)_0 \to \mathbb{X}_1$ defines an internal modification*

$$\tilde{\alpha} : x \Rrightarrow y.$$

Proof. First notice that

$$\tilde{\alpha}_A s = (\ell_A)_0 \alpha s = (\ell_A)_0 \theta_0 = (\ell_A \theta)_0 = (x_A)_0$$

and

$$\tilde{\alpha}_A t = (\ell_A)_0 \alpha t = (\ell_A)_0 \omega_0 = (\ell_A \omega)_0 = (y_A)_0.$$

Then

$$(s\tilde{\alpha}_A)t = s(y_A)_0 = (y_A)_1 s$$

and

$$(x_A)_1 t = t(x_A)_0 = t\tilde{\alpha}_A s$$

so there are two composable pairs given by the maps

$$\langle s\tilde{\alpha}, (y_A)_1\rangle, \langle (x_A)_1, t\tilde{\alpha}_A\rangle : D(A)_1 \to \mathbb{X}_2$$

which coincide after composition in $\mathbb{X}$.

$$\begin{aligned}
\langle (x_A)_1, t\tilde{\alpha}_A\rangle c &= \langle (\ell_A)_1\theta_1, t(\ell_A)_0\alpha\rangle c && \text{Def. } \tilde{\alpha}, x_A \\
&= \langle (\ell_A)_1\theta_1, (\ell_A)_1 t\alpha\rangle c && \text{Functoriality}\ell_A \\
&= (\ell_A)_1\langle \theta_1, t\alpha\rangle c && \text{Factor} \\
&= (\ell_A)_1\langle s\alpha, \omega_1\rangle c && \text{Naturality } \alpha \\
&= \langle (\ell_A)_1 s\alpha, (\ell_A)_1\omega_1\rangle c && \text{Factor} \\
&= \langle s(\ell_A)_0\alpha, (\ell_A)_1\omega_1\rangle c && \text{Functoriality} \\
&= \langle s\tilde{\alpha}_A, (y_A)_1\rangle c && \text{Def. } \tilde{\alpha}, y_A
\end{aligned}$$

This shows $\tilde{\alpha}_A : x_A \implies y_A$ is an internal natural transformation for each A in $\mathcal{A}_0$. Now since $\ell_\varphi s = (\ell_A)_0$ and $\ell_\varphi t = (D(\varphi)\ell_B)_0$ we have that

$$\tilde{\alpha} t = (\ell_A\omega)_0 = y_\varphi s \quad , \quad D(\varphi)_0\tilde{\alpha} s = (D(\varphi)\ell_A\theta)_0 = x_\varphi t$$

by definitions of the induced internal natural transformations $x, y : D \implies \Delta\mathbb{X}$. These give us the other two composable pairs which are equal in $\mathbb{X}$ after composition.

$$\begin{aligned}
\langle \tilde{\alpha}_A, y_\varphi \rangle c &= \langle (\ell_A)_0 \alpha, \ell_\varphi \omega_1 \rangle c && \text{Def.} \\
&= \langle \ell_\varphi s \alpha, \ell_\varphi \omega_1 \rangle c && \text{Def.} \ell_\varphi \\
&= \ell_\varphi \langle s\alpha, \omega_1 \rangle c && \text{Factor} \\
&= \ell_\varphi \langle \theta_1, t\alpha \rangle c && \text{Naturality } \alpha \\
&= \langle \ell_\varphi \theta_1, \ell_\varphi t \alpha \rangle c && \text{Factor} \\
&= \langle \ell_\varphi \theta_1, D(\varphi)_0 (\ell_B)_0 \alpha \rangle c && \text{Def. } \ell_\varphi \\
&= \langle x_\varphi, D(\varphi)_0 \tilde{\alpha}_B \rangle c && \text{Def.}
\end{aligned}$$

This last equality shows that the indexing of $\tilde{\alpha}$ is naturally compatible with the components of the natural transformations x_φ and y_φ, for each $\varphi : A \to B$ in $\mathcal{A}$. It follows that $\tilde{\alpha} : x \Rrightarrow y$ is a modification. ☐

On the other hand, suppose $\gamma : x \Rrightarrow y$ is a modification between two lax natural transformations $x, y : D \implies \Delta\mathbb{X}$. Let $\theta, \omega : \mathbb{D} \to \mathbb{X}$ be the unique internal functors corresponding to x and y respectively. Let $\overline{\gamma}$ be the map uniquely induced by the natural transformations of γ indexed by $\mathcal{A}_0$.

$$\begin{tikzcd}
\mathbb{D}_0 \arrow[r, dotted, "\overline{\gamma}"] & \mathbb{X}_1 \\
D(A)_0 \arrow[u, tail, "\iota_A"] \arrow[ur, "\gamma_A"'] &
\end{tikzcd}$$

Proposition 17. *The map $\overline{\gamma}$ is a natural transformation $\theta \implies \omega$.*

Proof. For each A in $\mathcal{A}_0$, the following diagrams commute by definition of θ and γ.

$$\begin{tikzcd}
\mathbb{D}_0 \arrow[r, dotted, "\overline{\gamma}"] \arrow[rr, dotted, bend left, "\theta_0"] & \mathbb{X}_1 \arrow[r, "s"] & \mathbb{X}_0 \\
D(A)_0 \arrow[u, tail, "\iota_A"] \arrow[ur, "\gamma_A"'] \arrow[urr, bend right, "(x_A)_0"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}_0 \arrow[r, dotted, "\overline{\gamma}"] \arrow[rr, dotted, bend left, "\omega"] & \mathbb{X}_1 \arrow[r, "t"] & \mathbb{X}_0 \\
D(A)_0 \arrow[u, tail, "\iota_A"] \arrow[ur, "\gamma_A"'] \arrow[urr, bend right, "(y_A)_0"'] & &
\end{tikzcd}$$

and so by the universal property of the coproduct $\mathbb{D}_0$ we have that

$$\theta_0 = \overline{\gamma} s \qquad , \qquad \omega_0 = \overline{\gamma} t.$$

Naturality is all that remains to show and this is done using the universal property of the coproduct $\mathbb{D}_1$. For each $\varphi : A \to B$ in $\mathcal{A}_1$ we can see

$$\iota_\varphi t\overline{\gamma} = \pi_1 t \iota_B \overline{\gamma} = \pi_1 t \gamma_B$$

and

$$\iota_\varphi s\overline{\gamma} = \pi_0 \iota_A \overline{\gamma} = \pi_0 \gamma_A$$

and therefore

$$\begin{aligned}
\iota_\varphi \langle \theta_1,\ t\overline{\gamma}\rangle c &= \langle \iota_\varphi \theta_1,\ \iota_\varphi t\overline{\gamma}\rangle c \\
&= \langle \langle \pi_0 x_\varphi, \pi_1 (x_B)_1\rangle c,\ \pi_1 t\gamma_B\rangle c \\
&= \langle \pi_0 x_\varphi,\ \pi_1 \langle (x_B)_1, t\gamma_B\rangle c\rangle c && \text{Assoc.} \\
&= \langle \pi_0 x_\varphi,\ \pi_1 \langle s\gamma_B, (y_B)_1\rangle c\rangle c && \text{Nat. } \gamma_B \\
&= \langle \langle \pi_0 x_\varphi, \pi_1 s\gamma_B\rangle c,\ \pi_1 (y_B)_1\rangle c && \text{Assoc.} \\
&= \langle \langle \pi_0 x_\varphi, \pi_0 D(\varphi)_0 \gamma_B\rangle c,\ \pi_1 (y_B)_1\rangle c && \text{Def. } D_\varphi \\
&= \langle \pi_0 \langle x_\varphi, D(\varphi)_0 \gamma_B\rangle c,\ \pi_1 (y_B)_1\rangle c && \text{Factor.} \\
&= \langle \pi_0 \langle \gamma_A, y_\varphi\rangle c,\ \pi_1 (y_B)_1\rangle c && \text{Def.}\gamma \\
&= \langle \pi_0 \gamma_A,\ \langle \pi_0 y_\varphi, \pi_1 (y_B)_1\rangle c\rangle c && \text{Assoc.} \\
&= \langle \iota_\varphi s\overline{\gamma},\ \iota_\varphi \omega\rangle c && \text{Def.} \\
&= \iota_\varphi \langle s\overline{\gamma}, \omega\rangle c && \text{Factor.}
\end{aligned}$$

By uniqueness we must have that

$$\langle \theta_1\ ,\ t\overline{\gamma}\rangle c = \langle s\overline{\gamma}, \omega\rangle c$$

and it follows that $\overline{\gamma} : \theta \implies \omega$ is an internal natural transformation. □

Proposition 18. *There is a one-to-one correspondence between modifications of lax natural transformations $D \implies \Delta\mathbb{X}$ and internal natural transformations between the corresponding internal functors of Proposition 15.*

Proof. The assignments $\overline{(-)}$ and $\tilde{(-)}$ are inverses. For any modification $\gamma : x \implies y$ we have that for each $A \in \mathcal{A}_0$,

$$\tilde{\overline{\gamma}}_A := (\ell_A)_0\overline{\gamma} = \iota_A\overline{\gamma} = \gamma_A$$

and so $\tilde{\overline{\gamma}} = \gamma$ by definition. On the other hand for any internal natural transformation $alpha : \theta \implies \omega$, and any $A \in \mathcal{A}$ we have that

$$\iota_A\overline{\tilde{\alpha}} = \tilde{\alpha}_A = (\ell_A)_0\alpha = \iota_A\alpha$$

and by the universal property of the coproduct $\mathbb{D}_0$, $\overline{\tilde{\alpha}} = \alpha$. $\square$

3.3.5 Internal Category of Elements as an OpLax Colimit

In the previous two subsections we've seen that internal functors $\mathbb{D} \to \mathbb{X}$ and internal natural transformations between them correspond uniquely to lax natural transformations $D \implies \Delta\mathbb{X}$ and modifications between them respectively. In this section we put this together as an equivalence of categories that establishes $\mathbb{D}$ as the oplax colimit of D in $\mathbf{Cat}(\mathcal{E})$.

Theorem 19 ($\mathbb{D}$ is the oplax colimit of D)**.** *Let $\mathcal{E}$ admit an internal Grothendieck construction of $D : \mathcal{A} \to \mathbf{Cat}(\mathcal{E})$, as in Definition 2. Then for every internal category $\mathbb{X} \in \mathbf{Cat}(\mathcal{E})$, the category of lax natural transformations $D \implies \Delta\mathbb{X}$ and their modifications is isomorphic to the category of internal functors $\mathbb{D} \to \mathbb{X}$ and their internal natural transformations.*

$$[D, \Delta\mathbb{X}]_\ell \cong \mathbf{Cat}(\mathcal{E})(\mathbb{D}, \mathbb{X})$$

Proof. The objects and morphisms are in bijection by Propositions 15 and 18 respectively. We only need to show composition and identities are preserved in one direction of the 2-cell correspondence. For any lax natural transformation $x : D \implies \Delta\mathbb{X}$, the identity modification 1_x consists by the family of identity internal natural transformations 1_{x_A} for each $A \in \mathcal{A}_0$. Let $\theta : \mathbb{D} \to \mathbb{X}$ and $\overline{1_\mathbb{X}} : \theta \implies \theta$ be the internal functor and internal natural transformation corresponding to x and $1_\mathbb{X}$ respectively. Then

$$\iota_A\overline{1_{\mathbb{X}}} := (1_{\mathbb{X}})_A = 1_{x_A} := e_A(x_A)_1 = (x_A)_0e = \iota_A\theta_0 e$$

shows $\overline{1_{\mathbb{X}}}$ is the identity natural transformation on θ.

Let $\gamma : x \Rrightarrow y$ and $\sigma : y \Rrightarrow z$ be modifications of lax natural transformations $x, y, z : D \implies \Delta\mathbb{X}$. For each $A \in \mathcal{A}_0$ let $\gamma_A : x_A \implies y_A$ and $\sigma_A : y_A \implies z_A$ be the internal natural transformations defining γ and σ so the composite

$$\gamma\sigma : x \Rrightarrow z$$

is defined by the composite of internal natural transformations

$$(\gamma\sigma)_A = \gamma_A\sigma_A$$

Internally this is given by the composite

$$\begin{tikzcd}
D(A)_0 \arrow[r, "\langle\gamma_A,\sigma_A\rangle"] \arrow[dr, "\gamma_A\sigma_A"'] & \mathbb{X}_2 \arrow[d, "c"] \\
& \mathbb{X}_1
\end{tikzcd}$$

and now we can see that

$$\iota_A\overline{\gamma\sigma} = (\gamma\sigma)_A = \gamma_A\sigma_A = \langle\gamma_A, \sigma_A\rangle c = \langle\iota_A\overline{\gamma}, \iota_A\overline{\sigma}\rangle c = \iota_A\langle\overline{\gamma}, \overline{\sigma}\rangle c.$$

By the universal property of $\mathbb{D}_0$ we have that

$$\overline{\gamma\sigma} = \langle\overline{\gamma}, \overline{\sigma}\rangle c$$

where the right-hand side defines the (horizontal) composition of natural transformations $\overline{\gamma} : \theta \implies \omega$, $\overline{\sigma}\omega \implies \nu$, where $\theta, \omega, \nu : \mathbb{D} \to \mathbb{X}$ correspond to $x, y,$ and z respectively. $\square$

Chapter 4

Internal Category of Fractions

In this chapter we give a suitable context, $\mathcal{E}$, and conditions on an internal category, $\mathbb{C}$, and a map $w : W \to \mathbb{C}_1$ that allow us to express a set of axioms for an internal category of (right) fractions. We show that such a pair $(\mathbb{C}, w)$ satisfying our Internal Fractions Axioms allows us to define an internal category, $\mathbb{C}[W^{-1}]$, which satisfies an analogous universal property expressed by Theorem 65. We write $(\mathbb{C}, W)$ for the pair from now on, as the map $w : W \to \mathbb{C}_1$ will be fixed and implied by W. The contextual conditions on $(\mathbb{C}, W)$ allow us to build the objects of diagrams in $\mathbb{C}$ we need in the Internal Fractions Axioms and also represent the arrows and paths of composable arrows in our internal category of fractions as equivalence classes of spans and paths of composable spans respectively. The contextual conditions on $\mathcal{E}$ then allow us formulate the Internal Fractions Axioms in terms of lifts of local witnesses to the axioms. This local data can be glued together to give globally defined structure maps provided a gluing condition is satisfied and we use this to define the internal category of fractions, $\mathbb{C}[W^{-1}]$, along with its structure maps and to prove it is an internal category. We will often give representations of our definitions and constructions as they would appear in $\mathcal{E} = \mathbf{Set}$ to help our readers and we will overload the symbols for structure maps of internal categories, namely s, t, c, and e. We will also abuse some notation and language by referring to arrows $W \to \mathbb{C}_1$ as representing 'arrows in W' when in general we mean it represents a family of arrows in an internal category $\mathbb{C}$ indexed by W. The symbols, π_i, will be overloaded and used for the i'th (pullback) projection of all pullbacks. Here i stands for 'number of components to the right of the left-most component,' naturally.

In Section 4.1 we describe the conditions we require for the pair $(\mathbb{C}, W)$ in order to state the Internal Fractions Axioms that culminate to Definition 28. In Section 4.2 we define the structure we need on the ambient category $\mathcal{E}$ in Definition 33 and present the Internal Fractions Axioms as part of Definition 34. We define the objects and structure maps for the internal category $\mathbb{C}[W^{-1}]$ in Section 4.3. We show that internal composition is assocaitive and satisfies the identity laws in $\mathbb{C}[W^{-1}]$, making $\mathbb{C}[W^{-1}]$ an internal category, in Section 4.4. In Section 4.5 we define the associated internal localization functor and prove that it inverts $w : W \to \mathbb{C}_1$ in a suitable sense. In the last section, Section 4.6, we prove the universal property of the internal category of fractions as Theorem 65.

4.1 The Context

For an internal category of fractions, we need to work in a suitably structured category $\mathcal{E}$ and with suitable internal categories $\mathbb{C}$ in $\mathcal{E}$. The conditions on $\mathcal{E}$ will allow us to define the category structure on our internal category of fractions, $\mathbb{C}[W^{-1}]$, and the conditions on the internal categories we consider, $\mathbb{C}$, allow us to construct objects necessary to describe the Internal Fractions Axioms and describe reflexive internal graphs of fractions. The following definition will be important for defining the context in this section and the Internal Fractions Axioms in the next section.

Definition 20. An *effective epimorphism* in a category $\mathcal{E}$ is the coequalizer of its kernel pair.

Effective epimorphisms appear in each of the Internal Fractions Axioms in Definition 34 and are used to define composition, source, and target structure maps for the internal category of fractions. For the rest of this chapter we assume $\mathcal{E}$ has a class of effective epimorphisms, $\mathcal{J}$, that is stable under pullback and composition. We call these epimorphisms *covers*. We will see these in the next section when we give the Internal Fractions Axioms in Section 4.2.

For the rest of this section we focus on the conditions we will impose on an internal category $\mathbb{C}$ and an arrow $w : W \to \mathbb{C}_1$ in $\mathcal{E}$ in order to construct the building blocks of our internal category of fractions, $\mathbb{C}[W^{-1}]$, as well as the objects of diagrams in $\mathbb{C}$ that we use to internalize the axioms for a category of fractions. For example, let spn denote the object of spans in $\mathbb{C}$ whose left leg is in W,

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "\circ" marking] \arrow[r] & \cdot \end{tikzcd},$$

let csp denote the object of cospans in $\mathbb{C}$ whose right leg is in W,

$$\begin{tikzcd}\cdot \arrow[r] & \cdot & \cdot \arrow[l, "\circ" marking] ,\end{tikzcd}$$

let $W_{\triangle}$ denote the object of pairs of arrows whose terminal arrow and composite is in W,

$$\begin{tikzcd} & & \cdot \arrow[d] \arrow[dll, "\circ" marking] \\ \cdot & & \cdot \arrow[ll, "\circ" marking] \end{tikzcd} ,$$

and let sb denote the object of the following commuting diagrams (in $\mathbb{C}$) that we will call sailboats.

$$\begin{tikzcd} & & \cdot \arrow[d] \arrow[dll, "\circ" marking] & \\ \cdot & & \cdot \arrow[ll, "\circ" marking] \arrow[r] & \cdot \end{tikzcd}$$

These are given by the following pullbacks in $\mathcal{E}$ respectively.

$$\begin{tikzcd}
\mathrm{spn} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}_1 \arrow[d, "s"] \\
W \arrow[r, "ws"'] & \mathbb{C}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{csp} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "wt"] \\
\mathbb{C}_1 \arrow[r, "t"'] & \mathbb{C}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
W_{\triangle} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "w"] \\
\mathbb{C}_1 {}_t\times_s W \arrow[r, "c"'] & \mathbb{C}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{sb} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}_1 \arrow[d, "s"] \\
W_{\triangle} \arrow[r, "\pi_0\pi_1 ws"'] & \mathbb{C}_0
\end{tikzcd}$$

The following pullback,

$$\begin{tikzcd}
W_{\circ} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "w"] \\
\mathbb{C}_1 {}_t\times_{ws} W {}_{wt}\times_{ws} W \arrow[r, "c"'] & \mathbb{C}_1
\end{tikzcd} ,$$

is the object of composable paths of length three where the last two arrows are in W and their composite is again in W. In $\mathcal{E} = \mathbf{Set}$, the elements of this set would be composable pairs of arrows in the image of $w : W \to \mathbb{C}_1$ along with a pre-composable arrow in $\mathbb{C}_1$ such that their composition (in $\mathbb{C}$) gives an element in the image of W.

We use this object to express a weak composition axiom for internal fractions. The pullback

$$\begin{tikzcd}
W_{\square} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}_1 {}_t\times_s W \arrow[d, "c"] \\
W {}_t\times_s \mathbb{C}_1 \arrow[r, "c"'] & \mathbb{C}_1
\end{tikzcd}$$

represents commuting diagrams in $\mathbb{C}$ that will be referred to as *Ore squares*:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "\circ" marking] & \cdot \arrow[d, "\circ" marking] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}$$

The arrows marked with $\circ$ denote arrows in the image of $w : W \to \mathbb{C}_1$. This object is used to express the internal (right) Ore condition. Let $P(\mathbb{C})$ denote the object of parallel arrows in $\mathcal{C}$ given by the pullback of pairing of source and target maps:

$$\begin{tikzcd}
P(\mathbb{C}) \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}_1 \arrow[d, "{(s,t)}"] \\
\mathbb{C}_1 \arrow[r, "{(s,t)}"'] & \mathbb{C}_0 \times \mathbb{C}_0
\end{tikzcd}$$

Let $\mathcal{P}_{eq}(\mathbb{C})$ and $\mathcal{P}_{cq}(\mathbb{C})$ be the objects of equalized and coequalized parallel arrows in $\mathcal{C}$ (that don't satisfy any kind of internal universal property) given by the equalizers

$$\begin{tikzcd}
\mathcal{P}_{eq}(\mathbb{C}) \arrow[r, tail, "\iota_{eq}"] & W \; {}_{wt}{\times}_s \, P(\mathbb{C}) \arrow[r, shift left, "{(\pi_0 w, \pi_1 \pi_0)c}"] \arrow[r, shift right, "{(\pi_0 w, \pi_1 \pi_1)c}"'] & \mathbb{C}_1
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{C}) \arrow[r, tail, "\iota_{eq}"] & P(\mathbb{C}) \; {}_t{\times}_{ws} \, W \arrow[r, shift left, "{(\pi_0 \pi_0, \pi_1 w)c}"] \arrow[r, shift right, "{(\pi_0 \pi_1, \pi_1 w)c}"'] & \mathbb{C}_1
\end{tikzcd}$$

in $\mathcal{E}$ (with the usual universal property). Let $\mathcal{P}(\mathbb{C})$ denote the following pullback

$$\begin{tikzcd}
\mathcal{P}(\mathbb{C}) \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{P}_{cq}(\mathbb{C}) \arrow[d, "\pi_0"] \\
\mathcal{P}_{eq}(\mathbb{C}) \arrow[r, "\pi_1"'] & P(\mathbb{C})
\end{tikzcd}$$

representing diagrams of the form:

$$\begin{tikzcd}
\cdot \arrow[r, "\circ" marking] & \cdot \arrow[r, shift left] \arrow[r, shift right] & \cdot \arrow[r, "\circ" marking] & \cdot
\end{tikzcd}$$

where the $\circ$ marked arrows represent arrows indexed by W. This will be used for internalizing what is sometimes called the 'right-cancellability' or 'lifting' condition for internal fractions. In this thesis we refer to this as 'zippering' in order to avoid confusion with the lifts in the Internal Fractions Axioms and because of the way it witnesses commutativity of parts of diagrams that don't commute prior to its application to make new ones that do commute. The best place to see this is in the process of defining composition and proving the associativity and identity laws. Note that these equalizers can be given as pullbacks of pairing maps

$$(1,(\pi_0\pi_0,\pi_1 w)c),(1,(\pi_0\pi_1,\pi_1 w)c) : (P(\mathbb{C})\ {}_{t}\!\times_{ws} W) \to (P(\mathbb{C})\ {}_{t}\!\times_{ws} W) \times \mathbb{C}_1$$

$$(1,(\pi 0 w,\pi_1\pi_0)c),(1,(\pi 0 w,\pi_1\pi_1)c) : (W\ {}_{wt}\!\times_{s} P(\mathbb{C})) \to (W\ {}_{wt}\!\times_{s} P(\mathbb{C})) \times \mathbb{C}_1$$

when they all exist in $\mathcal{E}$ with the pullback projections being made equal by the identity maps in the left-hand components of the pairing maps above and the equalizer condition being forced by the right-hand components respectively. The constructions above allow us to formalize the axioms for a category of (right) fractions and we now give a name to the collection of internal categories, $\mathbb{C}$, of $\mathcal{E}$ and maps, $w : W \to \mathbb{C}_1$, in $\mathcal{E}$ for which this happens. The next definition describes a setting in which we can internal the Internal Fractions Axioms. We'll be overloading notation for the structure maps of an internal category, and suppress $w : W \to \mathbb{C}_1$ when describing internal composition with arrows in $\mathbb{C}$ indexed by W.

Definition 21. Let $\mathbb{C}$ be an internal category in $\mathcal{E}$ and let $w : W \to \mathbb{C}_1$ be an arrow in $\mathcal{E}$. We say the pair $(\mathbb{C}, W)$ *is a pre-candidate for internal fractions* if the following pullbacks

$$\begin{tikzcd}
\mathrm{csp} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "wt"] \\
\mathbb{C}_1 \arrow[r, "t"'] & \mathbb{C}_0
\end{tikzcd}$$

$$\begin{tikzcd}
W \,{}_{wt}{\times}_{s}\, \mathbb{C}_1 \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "ws"] \\
\mathbb{C}_1 \arrow[r, "t"'] & \mathbb{C}_0
\end{tikzcd}$$

$$\begin{tikzcd}
W_{\square} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \,{}_{wt}{\times}_{s}\, \mathbb{C}_1 \arrow[d, "c"] \\
\mathbb{C}_1 \,{}_{t}{\times}_{ws}\, W \arrow[r, "c"'] & \mathbb{C}_1
\end{tikzcd}$$

$$\begin{tikzcd}
W_{\blacktriangle} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \,{}_{t}{\times}_{s}\, W \arrow[d, "\pi_0 ws"] \\
\mathbb{C}_1 \arrow[r, "t"'] & \mathbb{C}_0
\end{tikzcd}$$

$$\begin{tikzcd}
W_{\triangle} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "w"] \\
\mathbb{C}_1 \,{}_{t}{\times}_{s}\, W \arrow[r, "c"'] & \mathbb{C}_1
\end{tikzcd}$$

$$\begin{tikzcd}
P(\mathbb{C}) \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}_1 \arrow[d, "{(s,t)}"] \\
\mathbb{C}_1 \arrow[r, "{(s,t)}"'] & \mathbb{C}_0 \times \mathbb{C}_0
\end{tikzcd}$$

$$\begin{tikzcd}
P(\mathbb{C}) \,{}_{\pi_0 t}{\times}_{ws}\, W \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "ws"] \\
P(\mathbb{C}) \arrow[r, "\pi_0 t"'] & \mathbb{C}_0
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{spn} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}_1 \arrow[d, "s"] \\
W \arrow[r, "ws"'] & \mathbb{C}_0
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{C}_1 \,{}_{t}{\times}_{ws}\, W \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}_1 \arrow[d, "s"] \\
W \arrow[r, "wt"'] & \mathbb{C}_0
\end{tikzcd}$$

$$\begin{tikzcd}
W \,{}_{t}{\times}_{s}\, W \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "ws"] \\
W \arrow[r, "wt"'] & \mathbb{C}_0
\end{tikzcd}$$

$$\begin{tikzcd}
W_{\circ} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "w"] \\
W_{\blacktriangle} \arrow[r, "c"'] & \mathbb{C}_1
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{sb} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}_1 \arrow[d, "s"] \\
W_{\triangle} \arrow[r, "\pi_0\pi_1 ws"'] & \mathbb{C}_0
\end{tikzcd}$$

$$\begin{tikzcd}
W \,{}_{wt}{\times}_{\pi_0 s}\, P(\mathbb{C}) \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & P(\mathbb{C}) \arrow[d, "\pi_0 s"] \\
W \arrow[r, "wt"'] & \mathbb{C}_0
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{P}(\mathbb{C}) \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{P}_{cq}(\mathbb{C}) \arrow[d, "\pi_0"] \\
\mathcal{P}_{eq}(\mathbb{C}) \arrow[r, "\pi_1"'] & P(\mathbb{C})
\end{tikzcd}$$

exist in $\mathcal{E}$ along with the equalizers,

$$\begin{tikzcd}
\mathcal{P}_{eq}(\mathbb{C}) \arrow[r, tail, "\iota_{eq}"] & W \,{}_{wt}{\times}_{s}\, P(\mathbb{C}) \arrow[r, shift left, "{(\pi_0 w, \pi_1\pi_0)c}"] \arrow[r, shift right, "{(\pi_0 w, \pi_1\pi_1)c}"'] & \mathbb{C}_1
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{C}) \arrow[r, tail, "\iota_{eq}"] & P(\mathbb{C}) \,{}_{t}{\times}_{ws}\, W \arrow[r, shift left, "{(\pi_0\pi_0, \pi_1 w)c}"] \arrow[r, shift right, "{(\pi_0\pi_1, \pi_1 w)c}"'] & \mathbb{C}_1
\end{tikzcd} .$$

These pullbacks and equalizers give us the building blocks we need to construct an internal category of fractions so for the rest of this chapter we assume that $(\mathbb{C}, W)$ is a pre-candidate for internal fractions. Our construction requires a bit of 'scaffolding' however, in the form of a family of reflexive internal graphs encoding the data for an equivalence relation we hope to define on spans and paths of composable spans. At this point we can only define the first one given by the two maps $p_0, p_1 : \mathrm{sb} \to \mathrm{spn}$, defined explicitly as the pairing maps

$$p_0 = (\pi_0\pi_0\pi_1,\pi_1), \quad p_1 = (\pi_0\pi_1,(\pi_0\pi_0\pi_0,\pi_1)c)$$

by the universal property of the pullback spn. These maps represent projecting two different spans out of a commuting diagrams (in $\mathbb{C}$) which we call a sailboat. The idea is that coequalizing these will produce equivalence classes of spans that are related by being part of a common sailboat. For example, when $\mathcal{E} = \mathbf{Set}$, the maps p_0 and p_1 can be seen to project sailboats in sb to spans in spn like this:

$$\left[\begin{tikzcd} & & \cdot \arrow[dll, "\circ" marking] \arrow[d] & \\ \cdot & & \cdot \arrow[ll, "\circ" marking] \arrow[r] & \cdot \end{tikzcd}\right] \xmapsto{p_0=(\pi_0\pi_0\pi_1,\pi_1)} \left[\begin{tikzcd} \cdot & \cdot \arrow[l, "\circ" marking] \arrow[r] & \cdot \end{tikzcd}\right]$$

$$\left[\begin{tikzcd} & & \cdot \arrow[dll, "\circ" marking] \arrow[d] \arrow[dr, dotted] & \\ \cdot & & \cdot \arrow[ll, "\circ" marking] \arrow[r] & \cdot \end{tikzcd}\right] \xmapsto{p_1=(\pi_0\pi_1,(\pi_0\pi_0\pi_0,\pi_1)c)} \left[\begin{tikzcd} & \cdot \arrow[dl, "\circ" marking] \arrow[dr] & \\ \cdot & & \cdot \end{tikzcd}\right]$$

The dotted arrow on the left-hand side is just emphasizing that the pair is composable (in $\mathbb{C}$) in order to make the mapping more clear. The arrows in a category of fractions are equivalence classes of spans, where two distinct spans represent the same equivalence class whenever there exists an intermediate span and two sailboats such that the intermediate span forms the p_1 span projection of two different sailboats whose p_0 span projections are the original two spans. For $\mathcal{E} = \mathbf{Set}$ the coequalizer of p_0 and p_1 describes precisely this set of equivalence classes of spans. The following lemma shows p_0 and p_1 form a reflexive pair in general.

Lemma 22. *The parallel pair*

$$\begin{tikzcd} sb \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & spn \end{tikzcd}$$

is reflexive.

Proof. Define a map,

$$\varphi_s : \mathrm{spn} \to \mathrm{sb},$$

by the pairing map

$$\varphi_s = \big(((\pi_0 wse, \pi_0), \pi_0), \pi_1\big).$$

The component

$$\varphi_s \pi_0 = ((\pi_0 wse, \pi_0), \pi_0) : \mathrm{spn} \to W_\triangle$$

is well-defined by the identity law for composition in $\mathbb{C}$:

$$(\pi_0 wse, \pi_0 w)c = \pi_0 w(se, 1)c = \pi_0 w$$

The other component is well-defined because

$$\varphi_s \pi_1 s = \pi_1 s = \pi_0 ws = \varphi_s \pi_0 s.$$

To see that φ_s is a common section of p_0 and p_1 we get

$$\begin{aligned} \varphi_s p_0 \pi_0 &= \varphi_s \pi_0 \pi_0 \pi_1 \\ &= \pi_0 \end{aligned} \qquad \begin{aligned} \varphi_s p_0 \pi_1 &= \varphi_s \pi_1 \\ &= \pi_1 \end{aligned}$$

by definition and by the identity law in $\mathbb{C}$ we also get

$$\begin{aligned} \varphi_s p_1 \pi_0 &= \varphi_s \pi_0 \pi_1 \\ &= \pi_0 \end{aligned} \qquad \begin{aligned} \varphi_s p_1 \pi_1 &= \varphi_s (\pi_0 \pi_0 \pi_0, \pi_1)c \\ &= (\pi_0 wse, \pi_1)c \\ &(\pi_1 se, \pi_1)c \\ &\pi_1(se, 1)c \\ &\pi_1. \end{aligned}$$

By the universal property of the pullbacks sb and spn

$$\varphi_s p_0 = 1_{\mathrm{spn}} = \varphi_s p_1.$$

□

To define the source, target, and composition structure maps, and prove associativity and identity laws we need to reason about paths (or zig-zags more accurately) of composable spans and sailboats. In the following definition we give a sufficient condition for obtaining these as reflexive internal graphs.

Definition 23. We say that a pre-candidate for internal fractions, $(\mathbb{C}, W)$, in $\mathcal{E}$, *admits reflexive graphs of fractions* if the source and target maps on spn and sb,

$$\begin{tikzcd}
\mathrm{spn} \arrow[dr, "\pi_0 wt"'] & & \mathrm{spn} \arrow[dl, "\pi_1 t"] & & \mathrm{sb} \arrow[dr, "s", "\pi_0\pi_0\pi_1 wt"'] & & \mathrm{sb} \arrow[dl, "t"', "\pi_1 t"] \\
& \mathbb{C}_0 & & & & \mathbb{C}_0 &
\end{tikzcd}$$

admit pullbacks along one another.

The following lemma shows precisely which reflexive graphs are being referred to in Definition 23.

Lemma 24. *The pairs*

$$sb\ {}_t\times_s \dots\ {}_t\times_s\ sb \overset{p_0^n}{\underset{p_1^n}{\rightrightarrows}} spn\ {}_t\times_s \dots\ {}_t\times_s\ spn$$

are reflexive for each $n \geq 1$ where $n = 1$ is the case $p_0, p_1 : sb \to spn$ and $p_i^n = (\pi_0 p_i, \pi_1 p_i, ..., \pi_{n-1} p_i)$ is the unique map determined by the iterated pullback projections and the map p_i for $i = 0, 1$.

Proof. The proof is by induction on the number of pullbacks. The base case, $n = 1$, follows from Lemma 22. Assume p_0^n and p_1^n have a common section φ_s^n. Let sb^n denote the n-fold pullback of $t, s : \mathrm{sb} \to \mathbb{C}_0$, similarly for spn^n, for each natural number n. The induction step follows from the following commuting diagram:

$$\begin{tikzcd}
\mathrm{spn}^{n+1} \arrow[r, dotted] \arrow[d, "\cong"'] \arrow[rr, bend left, equal] & \mathrm{sb}^{n+1} \arrow[r, shift left, "p_0^{n+1}"] \arrow[r, shift right, "p_1^{n+1}"'] \arrow[d, "\cong"] & \mathrm{spn}^{n+1} \arrow[d, "\cong"] \\
\mathrm{spn}^n\ {}_t\times_s \mathrm{spn} \arrow[r, "\varphi_s^n \times \varphi_s"'] \arrow[rr, bend right, equal] & \mathrm{sb}^n\ {}_t\times_s \mathrm{sb} \arrow[r, shift left, "p_0^n \times p_0"] \arrow[r, shift right, "p_1^n \times p_1"'] & \mathrm{spn}^n\ {}_t\times_s \mathrm{spn}
\end{tikzcd}$$

The bottom commutes by the universal property of the pullbacks in the bottom row above:

$$\begin{aligned}
(\varphi_s^n \times \varphi_s)(p_0^n \times p_0) &= (\pi_0\varphi_s^n, \pi_1\varphi_s)(\pi_0 p_0^n, \pi_1 p_0)\\
&= (\pi_0\varphi_s^n p_0^n, p_1\varphi_s p_0)\\
&= (\pi_0, \pi_1)\\
&= 1\\
&= (\pi_0, \pi_1)\\
&= (\pi_0\varphi_s^n p_1^n, p_1\varphi_s p_1)\\
&= (\pi_0\varphi_s^n, \pi_1\varphi_s)(\pi_0 p_1^n, \pi_1 p_1)\\
&= (\varphi_s^n \times \varphi_s)(p_1^n \times p_1).
\end{aligned}$$

This implies the top commutes and we get a reflexive graph:

$$\begin{tikzcd}
\text{sb}\ {}_t\times_s \cdots\ {}_t\times_s \text{sb} \arrow[r, bend left, "p_0^{n+1}"] \arrow[r, bend right, "p_1^{n+1}"'] & \text{spn}\ {}_t\times_s \cdots\ {}_t\times_s \text{spn} \arrow[l, "\varphi_s^n"']
\end{tikzcd}$$

The result follows by induction. □

The arrows and composable paths in the internal category of fractions should be the coequalizers of the internal reflexive graphs in Lemmas 22 and 24 respectively. For this we need to require the existence of these coequalizers and the pullbacks of the induced source and target maps on the coequalizer of p_0 and p_1, and we need them to coincide. The following definition is used to restrict our focus to internal categories for which these coequalizers exist.

Definition 25. We say $(\mathbb{C}, W)$ *admits internal quotient graphs of fractions* if the coequalizer,

$$\begin{tikzcd}
\mathrm{sb}\ {}_t\times_s \cdots {}_t\times_s \mathrm{sb} \arrow[r, shift left, "p_0^n"] \arrow[r, shift right, "p_1^n"'] & \mathrm{spn}\ {}_t\times_s \cdots {}_t\times_s \mathrm{spn} \arrow[r, two heads, "q_n"] & \mathbb{C}[W^{-1}]_n \ ,
\end{tikzcd}$$

exists in $\mathcal{E}$ for each $n \geq 1$.

The coequalizers in Definition 25 are named suggestively. In particular $\mathbb{C}[W^{-1}]_1$ is how we will define the object of arrows for the internal category of fractions. Using its universal property we can define source and target structure maps by the following lemma.

Lemma 26 (Source and Target Structure for $\mathbb{C}[W^{-1}]$)**.** *The* ***source*** *and* ***target*** *maps for $\mathbb{C}[W^{-1}]$ are determined by the universal property of the coequalizer $\mathbb{C}[W^{-1}]_1$ and more precisely induced by $s' = \pi_0 wt$ and $t' = \pi_1 t$.*

$$\begin{tikzcd}
 & \mathbb{C}_1 \arrow[r, "t"] & \mathbb{C}_0 \\
sb \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & spn \arrow[u, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[r, two heads, "q"] \arrow[ur, "t'"] \arrow[dr, "s'"'] & \mathbb{C}[W^{-1}]_1 \arrow[u, dotted, "t"'] \arrow[d, dotted, "s"] \\
 & W \arrow[r, "wt"'] & \mathbb{C}_0
\end{tikzcd}$$

Proof. This is well-defined by the following calculations.

$$\begin{aligned}
p_0 s' &= (\pi_0\pi_0\pi_1, \pi_1)s' \\
&= (\pi_0\pi_0\pi_1, \pi_1)\pi_0 wt \\
&= \pi_0\pi_0\pi_1 wt \\
&= \pi_0\pi_0 ct \\
&= \pi_0\pi_1 wt \\
&= (\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c)\pi_0 wt \\
&= (\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c)s' \\
&= p_1 s'
\end{aligned}$$

$$\begin{aligned}
p_0 t &= (\pi_0\pi_0\pi_1, \pi_1)t' \\
&= (\pi_0\pi_0\pi_1, \pi_1)\pi_1 t \\
&= \pi_1 t \\
&= (\pi_0\pi_0\pi_0, \pi_1)ct \\
&= (\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c)\pi_1 t \\
&= (\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c)t' \\
&= p_1 t'
\end{aligned}$$

□

Now we define the pairs, $(\mathbb{C}, W)$, that admit internal quotient graphs of fractions in $\mathcal{E}$ for which the pullbacks of the induced source and target maps on the coequalizers

$\mathbb{C}[W^{-1}]_1$ exist. Notice this only requires the coequalizer of the pair $p_0, p_1 : \mathrm{sb} \to \mathrm{spn}$ so these could exist without the other reflexive graphs. Being able to construct proofs for coherences for associativity of composition for longer paths of arrows becomes unclear without the universal property of the coequalizers of the other internal reflexive graphs, p_0^n and p_1^n.

Definition 27. We say the pair $(\mathbb{C}, W)$ *admits paths of fractions* if the coequalizer $\mathbb{C}[W^{-1}]_1$ exists and $\mathcal{E}$ admits pullbacks of the induced source or target maps, $s, t : \mathbb{C}[W^{-1}]_1 \to \mathbb{C}_0$, along one another as well as the source and target maps of spn and sb.

The next definition is the last one in this section and describes all the structure we need for the internal categories we consider.

Definition 28. We say the pair $(\mathbb{C}, W)$ is a *candidate for internal fractions* if it satisfies Definitions 21, 25, and 27 and the induced left and right product functors on the slice category for each of the source and target maps for $\mathrm{sb}, \mathrm{spn}$, and $\mathbb{C}[W^{-1}]_1$

$$(-) \times s :, \quad t \times (-) : \mathcal{E}/\mathbb{C}_0 \to \mathcal{E}/\mathbb{C}_0$$

preserve reflexive coequalizers.

For the rest of this thesis we will assume the pair $(\mathbb{C}, W)$ is a candidate for internal fractions in $\mathcal{E}$. The remainder of this chapter consists of general lemmas which are combined to state that the coequalizers

$$\begin{tikzcd}
\mathrm{sb} \; {}_t\!\times_s \cdots {}_t\!\times_s \mathrm{sb} \arrow[r, shift left, "p_0^n"] \arrow[r, shift right, "p_1^n"'] & \mathrm{spn} \; {}_t\!\times_s \cdots {}_t\!\times_s \mathrm{spn} \arrow[r, two heads, "q_n"] & \mathbb{C}[W^{-1}]_n
\end{tikzcd}$$

exist in $\mathcal{E}$ for each $n \geq 1$. The following lemmas hold very generally and combine in Proposition 32 to show how the coequalizer, $\mathbb{C}[W^{-1}]_2$, of the reflexive pair, p_0^2 and p_1^2, above coincides with the pullback, $\mathbb{C}[W^{-1}]_1 \, {}_t\!\times_s \mathbb{C}[W^{-1}]_1$, of the induced source and target maps in Lemma 26.

Lemma 29. *For any category $\mathcal{E}$, the underlying-structure functor, $U : \mathcal{E}/\mathbb{C}_0 \to \mathcal{E}$, which maps objects $A \to \mathbb{C}_0$ in $\mathcal{E}/\mathbb{C}_0$ to objects A in $\mathcal{E}$ and is defined similarly on arrows, reflects coequalizers.*

Proof. Suppose we have a commuting diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, "a"'] & B \arrow[r, "h"] \arrow[d, "b"] & C \arrow[dl, "c"] \\
& \mathbb{C}_0 &
\end{tikzcd}$$

such that

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] & C
\end{tikzcd}$$

is a coequalizer in $\mathcal{E}$. Now suppose there exists an arrow $x : X \to \mathbb{C}_0$ and another arrow $\varphi : B \to X$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, "a"'] & B \arrow[r, "\varphi"] \arrow[d, "b"] & X \arrow[dl, "x"] \\
& \mathbb{C}_0 &
\end{tikzcd}$$

commutes in $\mathcal{E}$. In $\mathcal{E}$ we get a unique map $\theta : C \to X$ such that $\theta x = c$ by the universal property of the coequalizer:

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[ddrr, bend right, "a"'] & B \arrow[r, "h"] \arrow[dr, "\varphi"'] \arrow[ddr, "b"] & C \arrow[d, dotted, "\theta"] \arrow[dd, bend left, "c"] \\
& & X \arrow[d, "x"] \\
& & \mathbb{C}_0
\end{tikzcd}$$

This implies that for any $\varphi : b \to x$ in $\mathcal{E}/\mathbb{C}_0$, there exists a unique $\theta : c \to x$ such that the diagram

$$\begin{tikzcd}
a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \arrow[r, "h"] \arrow[dr, "\varphi"'] & c \arrow[d, dotted, "\theta"] \\
& & x
\end{tikzcd}$$

commutes. It follows that U reflects coequalizers. □

Notice that the proof above holds for reflexive coequalizers as well, since a section for a reflexive pair in $\mathcal{E}/\mathbb{C}_0$ is a section of the underlying reflexive pair in $\mathcal{E}$. Next we restate and prove Lemma 4.7 from [2]. It will be used in the proofs of Lemma 31 and Proposition 32 immediately after.

Lemma 30. *In any category, if the top row and right column are reflexive coequalizers and the middle column is a reflexive parallel pair, then the diagonal is a coequalizer.*

$$\begin{tikzcd}
A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "g"'] & B \arrow[l] \arrow[r, "h"] \arrow[d, shift right=2, "g'"'] \arrow[d, shift left=2, "f'"] & C \arrow[d, shift right=2, "g''"'] \arrow[d, shift left=2, "f''"] \\
& B' \arrow[u] \arrow[r, "h'"'] & C' \arrow[u] \arrow[d, "h''"] \\
& & C''
\end{tikzcd}$$

Proof. Let $x : B' \to X$ be any arrow in the category such that

$$ff'x = gg'x.$$

We claim the following diagram commutes where the notation for the sections, s, is suppressed:

$$\begin{tikzcd}
A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "g"'] & B \arrow[l] \arrow[r, "h"] \arrow[d, shift right=2, "g'"'] \arrow[d, shift left=2, "f'"] & C \arrow[d, shift right=2, "g''"'] \arrow[d, shift left=2, "f''"] \arrow[ddl, dotted, "\theta"'] \\
& B' \arrow[u] \arrow[r, "h'"'] \arrow[d, "x"'] & C' \arrow[u] \arrow[d, "h''"] \arrow[dl, dotted, "s\theta"] \\
& X & C'' \arrow[l, dotted, "\gamma"]
\end{tikzcd}$$

Pre-composing the common section of f and g gives

$$f'x = g'x$$

and induces the unique map $\theta : C \to X$ such that

$$h\theta = f'x = g'x$$

by the universal property of the coequalizer C. Then

$$hg''s\theta = g'h's\theta = g'sh\theta = g'sg'x = g'x$$

and similarly

$$hf''s\theta = f'x$$

which implies

$$hg''s\theta = hf''s\theta.$$

By the universal property of the coequalizer C, we have that

$$g''s\theta = f''s\theta$$

which induces the unique map $\gamma : C'' \to X$ such that

$$h''\gamma = s\theta.$$

Now we can also see that

$$h'h''\gamma = h's\theta = sh\theta = sf'x = x$$

and it is unique by the universal property of the coequalizer, C'', in the right-hand row. It follows that the diagonal is a coequalizer and it is reflexive with the section given by composing the common section of f' and g' and the common section of f and g.

□

We now apply Lemmas 29 and 30 to get the coequalizers we need to form the internal category of fractions.

Lemma 31. *The pullback*

$$\begin{tikzcd}
\mathbb{C}[W^{-1}]_1 \, {}_t\!\times_s \mathbb{C}[W^{-1}]_1 \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & \mathbb{C}[W^{-1}]_1 \arrow[d, "s"] \\
\mathbb{C}[W^{-1}]_1 \arrow[r, "t"'] & \mathbb{C}_0
\end{tikzcd}$$

is also a coequalizer

$$\begin{tikzcd}
sb \, {}_t\!\times_s sb \arrow[r, shift left, "p_0^2"] \arrow[r, shift right, "p_1^2"'] & spn \, {}_t\!\times_s spn \arrow[r, two heads, "q \times q"] & \mathbb{C}[W^{-1}]_1 \, {}_t\!\times_s \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

in $\mathcal{E}$.

Proof. Since $(\mathbb{C}, W)$ admits internal quotient graphs of fractions we know that the object $\mathbb{C}[W^{-1}]_1$ is a reflexive coequalizer of p_0 and p_1. By Lemma 29, the diagrams

$$\begin{tikzcd}
\mathrm{sb} \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] \arrow[dr, "s"'] & \mathrm{spn} \arrow[r, two heads, "q"] \arrow[d, "s"] & \mathbb{C}[W^{-1}]_1 \arrow[dl, "s"] \\
 & \mathbb{C}_0 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{sb} \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] \arrow[dr, "t"'] & \mathrm{spn} \arrow[r, two heads, "q"] \arrow[d, "t"] & \mathbb{C}[W^{-1}]_1 \arrow[dl, "t"] \\
 & \mathbb{C}_0 &
\end{tikzcd}$$

are coequalizers in $\mathcal{E}/\mathbb{C}_0$. These coequalizers are preserved by the left and right product functors on $\mathcal{E}/\mathbb{C}_0$ induced by the source and target maps for $\mathrm{sb}, \mathrm{spn}$, and $\mathbb{C}[W^{-1}]_1$ in $\mathcal{E}/\mathbb{C}_0$. This means the top row and right column in the following diagram are reflexive coequalizers in $\mathcal{E}/\mathbb{C}_0$

$$\begin{tikzcd}
\mathrm{sb}\ {}_t{\times}_s\ \mathrm{sb} \arrow[r, shift left=2, "1\times p_0"] \arrow[r, shift right=2, "1\times p_1"'] & \mathrm{sb}\ {}_t{\times}_s\ \mathrm{spn} \arrow[l] \arrow[r, "1\times q"] \arrow[d, shift right=2, "p_1\times 1"'] \arrow[d, shift left=2, "p_0\times 1"] & \mathrm{sb}\ {}_t{\times}_s\ \mathbb{C}[W^{-1}]_1 \arrow[d, shift right=2, "p_1\times 1"'] \arrow[d, shift left=2, "p_0\times 1"] \\
 & \mathrm{spn}\ {}_t{\times}_s\ \mathrm{spn} \arrow[u] \arrow[r, "1\times q"'] & \mathrm{spn}\ {}_t{\times}_s\ \mathbb{C}[W^{-1}]_1 \arrow[u] \arrow[d, "q\times 1"] \\
 & & \mathbb{C}[W^{-1}]_1\ {}_t{\times}_s\ \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

where we suppress the arrows into $\mathbb{C}_0$ given by the commuting pullback squares. The middle row is a reflexive pair whose coequalizer, $q \times 1$, is just not drawn in the diagram. Lemma 30 says the diagonal is a coequalizer in $\mathcal{E}/\mathbb{C}_0$:

$$\begin{tikzcd}
\mathrm{sb}\ {}_t{\times}_s\ \mathrm{sb} \arrow[r, shift left, "p_0^2"] \arrow[r, shift right, "p_1^2"'] \arrow[dr, "\pi_0 t"'] & \mathrm{spn}\ {}_t{\times}_s\ \mathrm{spn} \arrow[r, two heads, "q\times q"] \arrow[d, "\pi_0 t"] & \mathbb{C}[W^{-1}]_1\ {}_t{\times}_s\ \mathbb{C}[W^{-1}]_1 \arrow[dl, "\pi_0 t"] \\
 & \mathbb{C}_0 &
\end{tikzcd}$$

Let $q_2 : \mathrm{spn}\ {}_t{\times}_s\ \mathrm{spn} \to \mathbb{C}[W^{-1}]_2$ denote the coequalizer of p_0^2 and p_1^2 in $\mathcal{E}$. Notice the following diagram commutes in $\mathcal{E}$

$$\begin{tikzcd}
\mathrm{sb}\ {}_t\times_s \mathrm{sb} \arrow[r, shift left, "p_0^2"] \arrow[r, shift right, "p_1^2"'] \arrow[d, "\pi_i"'] & \mathrm{spn}\ {}_t\times_s \mathrm{spn} \arrow[r, two heads, "q_2"] \arrow[d, "\pi_i"] \arrow[dr, "q\times q"'] & \mathbb{C}[W^{-1}]_2 \arrow[d, dotted, "\overline{\pi_i q}"] \\
\mathrm{sb} \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & \mathrm{spn} \arrow[r, "q"'] & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

by the universal property of the coequalizer, $\mathbb{C}[W^{-1}]_2$, in $\mathcal{E}$. The same universal property induces the following unique map between the coequalizer and the pullback in the following commuting diagram:

$$\begin{tikzcd}
\mathrm{sb}\ {}_t\times_s \mathrm{sb} \arrow[r, shift left, "p_0^2"] \arrow[r, shift right, "p_1^2"'] \arrow[dr, "\pi_0 t"'] & \mathrm{spn}\ {}_t\times_s \mathrm{spn} \arrow[rr, two heads, "q_2"] \arrow[d, "\pi_0 t"] \arrow[drr, "q\times q"'] & & \mathbb{C}[W^{-1}]_2 \arrow[d, dotted, "\overline{\pi_0 q}\times\overline{\pi_1 q}"] \\
& \mathbb{C}_0 & & \mathbb{C}[W^{-1}]_1\ {}_t\times_s \mathbb{C}[W^{-1}]_1 \arrow[ll, "\pi_0 t"]
\end{tikzcd}$$

More precisely, since $\pi_0 t$ coequalizes p_0^2 and p_1^2 above, the universal property of $\mathbb{C}[W^{-1}]_2$ says there is a unique $\overline{\pi_0 t} : \mathbb{C}[W^{-1}]_2 \to \mathbb{C}_0$ such that $q_2\overline{\pi_0}t = \pi_0 t$. In particular

$$\overline{\pi_0 t} = (\overline{\pi_0 q} \times \overline{\pi_1 q})\pi_0 t = \overline{\pi_0 q}t$$

Now the diagram

$$\begin{tikzcd}
\mathrm{sb}\ {}_t\times_s \mathrm{sb} \arrow[r, shift left, "p_0^2"] \arrow[r, shift right, "p_1^2"'] \arrow[dr, "\pi_0 t"'] & \mathrm{spn}\ {}_t\times_s \mathrm{spn} \arrow[r, two heads, "q\times q"] \arrow[d, "\pi_0 t"] \arrow[ddr, "q_2"] & \mathbb{C}[W^{-1}]_1\ {}_t\times_s \mathbb{C}[W^{-1}]_1 \arrow[dl, "\pi_0 t"'] \arrow[dd, dotted, "\gamma"] \\
& \mathbb{C}_0 & \\
& & \mathbb{C}[W^{-1}]_2 \arrow[ul, "\overline{\pi_0 t}"]
\end{tikzcd}$$

commutes in $\mathcal{E}$ and induces the map γ on the right by the universal property of the coequalizer, $\pi_0 t : \mathbb{C}[W^{-1}]_1 \ {}_t\times_s \mathbb{C}[W^{-1}]_1 \to \mathbb{C}_0$, in $\mathcal{E}/\mathbb{C}_0$. In particular we have that $q_2 = q \times q\gamma$. Finally we can see

$$q_2(\overline{\pi_0 q} \times \overline{\pi_1 q})\gamma = (q \times q)\gamma = q_2$$

and

$$\gamma(\overline{\pi_0 q} \times \overline{\pi_1 q})\pi_0 t = \gamma\overline{\pi_0 q}t = \pi_0 t.$$

By the universal property of $\mathbb{C}[W^{-1}]_2$ we have that

$$(\overline{\pi_0 q}, \overline{\pi_1 q})\gamma = 1_{\mathbb{C}[W^{-1}]_2}$$

and by the universal property of the coequalizer, $\mathbb{C}[W^{-1}]_1 \ {}_t\times_s \mathbb{C}[W^{-1}]_1$, in $\mathcal{E}/\mathbb{C}_0$

$$\gamma(\overline{\pi_0 q}, \overline{\pi_1 q}) = 1_{\mathbb{C}[W^{-1}]_1 \, {}_t\times_s \mathbb{C}[W^{-1}]_1}.$$

It follows that

$$\mathbb{C}[W^{-1}]_2 \cong \mathbb{C}[W^{-1}]_1 \ {}_t\times_s \mathbb{C}[W^{-1}]_1.$$

□

Proposition 32. *The paths of composable arrows of length n in $\mathbb{C}[W^{-1}]$ given by pullbacks*

$$\mathbb{C}[W^{-1}]_1 \ {}_t\times_s \cdots \ {}_t\times_s \mathbb{C}[W^{-1}]_1$$

of n copies of $\mathbb{C}[W^{-1}]_1$ are coequalizers of the parallel pairs

$$\begin{tikzcd} sb \ {}_t\times_s \ldots \ {}_t\times_s sb \arrow[r, shift left, "p_0^n"] \arrow[r, shift right, "p_1^n"'] & spn \ {}_t\times_s \ldots \ {}_t\times_s spn \end{tikzcd} ,$$

for every $n \geq 2$.

Proof. This proof follows by induction on the length of path of composable arrows. Use Lemma 31 as the base case. Assume the result holds for paths of length n. Then the following diagram is a reflexive coequalizer,

$$\begin{tikzcd} \mathrm{sb}^n \arrow[r, shift left=2, "p_0^n"] \arrow[r, shift right=2, "p_1^n"'] & \mathrm{spn}^n \arrow[l] \arrow[r, "q\times\ldots\times q"] & \mathbb{C}[W^{-1}]_1 \ {}_t\times_s \cdots \ {}_t\times_s \mathbb{C}[W^{-1}]_1 \end{tikzcd} ,$$

where sb^n and spn^n are pullbacks defining paths of composable sailboats and spans of length n respectively. On the right we have iterated pullbacks of n copies of $\mathbb{C}[W^{-1}]_1$. By Lemma 29, we can view these as reflexive coequalizers in $\mathbb{C}_0$ using the induces source and target maps given by taking the left-most or right-most pullback projections and applying the source or target maps on $\mathrm{sb}, \mathrm{spn}$, and $\mathbb{C}[W^{-1}]_1$ respectively. Since $(\mathbb{C}, W)$ is a candidate for internal fractions, the top row and right column in the following diagram are reflexive coequalizers in $\mathcal{E}/\mathbb{C}_0$,

$$\begin{tikzcd}
\mathrm{sb}\ {}_t\times_s \mathrm{sb}^n \arrow[r, shift left, "1\times p_0"] \arrow[r, shift right, "1\times p_1"'] & \mathrm{sb}\ {}_t\times_s \mathrm{spn}^n \arrow[r, "1\times q"] \arrow[d, shift right=2, "p_1\times 1"'] \arrow[d, shift left=2, "p_0\times 1"] & \mathrm{sb}\ {}_t\times_s \mathbb{C}[W^{-1}]_n \arrow[d, shift right=2, "p_1\times 1"'] \arrow[d, shift left=2, "p_0\times 1"] \\
& \mathrm{spn}\ {}_t\times_s \mathrm{spn}^n \arrow[u] \arrow[r, "1\times q"'] & \mathrm{spn}\ {}_t\times_s \mathbb{C}[W^{-1}]_n \arrow[u] \arrow[d, "q\times 1"] & , \\
& & \mathbb{C}[W^{-1}]_1\ {}_t\times_s \mathbb{C}[W^{-1}]_n
\end{tikzcd}$$

and the middle column is a reflexive pair with common section $\varphi_s \times 1 : \mathrm{spn}\,{}_t\times_s \mathrm{spn}^n \to \mathrm{sb}\ {}_t\times_s \mathrm{spn}^n$. By Lemma 30, the diagonal is a (reflexive) coequalizer. Then

$$\mathbb{C}[W^{-1}]_{n+1} \cong \mathbb{C}[W^{-1}]_1\ {}_t\times_s \mathbb{C}[W^{-1}]_n \cong \mathbb{C}[W^{-1}]_1\ {}_t\times_s (\mathbb{C}[W^{-1}]_1\ {}_t\times_s \cdots\ {}_t\times_s \mathbb{C}[W^{-1}]_1)$$

and we can drop the brackets on the right-hand side due to a canonical isomorphism encoding associativity of taking pullbacks. □

4.2 The Axioms

Here we give an internal description of a weakened version of the axioms in [5] that allow for the construction of a category of fractions. In particular we internalize weaker conditions for the class of arrows, W, which we intend to invert by not assuming that W contains identities nor that it is closed under composition. Instead we assume that every object in $\mathbb{C}_0$ is the target of some map in W, and that every composable pair in W can be pre-composed by some arrow in $\mathbb{C}_1$ to give a composite in W. The purpose of this, as shown in [15], is to allow for a smaller class of arrows to be inverted when constructing the category of fractions. In particular, when applied to the Grothendieck construction, this allows us to invert a cleavage of the cartesian arrows in the category of elements rather than all of the cartesian arrows. This saves us some work in Section 5.1 because there is a convenient cleavage of the cartesian arrows that is easier to describe than all of the cartesian arrows.

These axioms generally sound like, 'for any diagram of a certain shape, there exist some filler arrows that make a larger diagram commute.' Internalizing these statements in a category of spaces like **Top** becomes an issue because, while we can form the objects representing such diagrams in **Set** and give them topologies, picking out the arrows to fill in the larger diagrams can rarely be done globally and continuously. For topological spaces one might work with effective descent covers to witness local information on a space that, when it satisfies a certain gluing condition, can be pasted together to give global information. In general we ask that our category $\mathcal{E}$ has a class of effective epimorphisms, $\mathcal{J}$, that are stable under pullback and composition. These give a way of witnessing the fractions axioms in $\mathbb{C}$ locally and continuously and then construct global maps with them provided their coequalizer condition is satisfied. The coequalizer condition for these amounts to saying constructions we wish to define, such as composition of spans for example, are well-defined. Stability under pullback and composition is required in order to witness multiple applications of the Internal Fractions Axioms.

Definition 33. We say $(\mathcal{E}, \mathcal{J})$ is *a candidate context for internal fractions* if $\mathcal{J}$ is a class of effective epimorphisms that are stable under pullback and composition. We will refer to the elements of $\mathcal{J}$ as *covers.*

With a candidate context for internal fractions, $(\mathcal{E}, \mathcal{J})$, and a candidate for internal fractions, $(\mathbb{C}, W)$, we can formulate the Internal Fractions Axioms below and begin to ask whether $(\mathcal{E}, \mathcal{J})$ is a context for internal fractions, for a given candidate for internal fractions, $(\mathbb{C}, W)$ as defined in Definition 28 in the previous section.

Definition 34 (Internal Fractions Axioms)**.** Let $(\mathcal{E}, \mathcal{J})$ be a candidate context for internal fractions, as in Definition 33, and let $(\mathbb{C}, W)$ be a candidate for internal fractions in $\mathcal{E}$, as in Definition 28. We say $(\mathbb{C}, W)$ *satisfies the internal (right)* *fractions axioms* or *admits an internal category of fractions* (with respect to $w : W \to \mathbb{C}_1$) if the following conditions hold.

In.Frc(1) The identity map $1_{\mathbb{C}_0} : \mathbb{C}_0 \to \mathbb{C}_0$, admits a lift along $wt : W \to \mathbb{C}_0$.

$$\begin{tikzcd} & W \arrow[d, "wt"] \\ \mathbb{C}_0 \arrow[ur, dotted, "\tau"] \arrow[r, equal] & \mathbb{C}_0 \end{tikzcd}$$

In.Frc(2) There exists a cover $U \overset{u}{\not\longrightarrow} W \,{}_t{\times}_s\, W$ that admits a lift, $\omega : U \to W_\circ$, along $\pi_0\pi_{12} : W_\circ \to W \,{}_{wt}{\times}_{ws}\, W$.

$$\begin{tikzcd} & W_\circ \arrow[d, "\pi_0\pi_{12}"] \\ U \arrow[ur, dotted, "\omega"] \arrow[r, "/" marking, "u"'] & W \,{}_{wt}{\times}_{ws}\, W \end{tikzcd}$$

In.Frc(3) There exists a cover $U \overset{u}{\not\longrightarrow} \mathbb{C}_1 \,{}_t{\times}_{wt}\, W$ that admits a lift, $\theta : U \to W_\square$, along $(\pi_0\pi_1, \pi_1\pi_1) : W_\square \to \mathbb{C}_1 \,{}_t{\times}_{wt}\, W$.

$$\begin{tikzcd} & W_\square \arrow[d, "(\pi_0\pi_1{,}\pi_1\pi_1)"] \\ U \arrow[ur, dotted, "\theta"] \arrow[r, "/" marking, "u"'] & \mathbb{C}_1 \,{}_t{\times}_{wt}\, W \end{tikzcd}$$

In.Frc(4) There exists a cover $U \overset{u}{\not\longrightarrow} \mathcal{P}_{cq}$ that admits a lift, $\delta : U \to \mathcal{P}(\mathbb{C})$, along $\pi_1 : \mathcal{P}(\mathbb{C}) \to \mathcal{P}_{cq}(\mathbb{C})$.

$$\begin{tikzcd} & \mathcal{P}(\mathbb{C}) \arrow[d, "\pi_1"] \\ U \arrow[ur, dotted, "\delta"] \arrow[r, "/" marking, "u"'] & \mathcal{P}_{cq}(\mathbb{C}) \end{tikzcd}$$

The lifts in the axioms above represent the existence of fillers for diagrams in $\mathbb{C}$ represented by codomains of the covers. In order to prove our composition is well-defined, associative, and satisfies the identity laws, we want to have a notion of base change. The following lemma shows how this works with stability of covers under pullback.

Lemma 35. *If $u : U \twoheadrightarrow B$ is a cover that admits a lift along $f : A \to B$, then for any map $g : X \to B$, there exists a cover, $u' : U' \twoheadrightarrow X$, such that the diagram*

$$\begin{tikzcd}
 & & A \arrow[d, "f"] \\
U' \arrow[r, "/" marking, "u'"'] \arrow[urr, dotted, bend left, "\ell"] & X \arrow[r, "g"'] & B
\end{tikzcd}$$

commutes in $\mathcal{E}$.

Proof. Since covers are stable under pullback, taking the pullback of the cover $u : U \twoheadrightarrow B$ along the map $g : X \to B$ gives a cover $u' : U' \twoheadrightarrow X$. Then the desired lift $\ell : U' \to A_i$ is given by post-composing the pullback projection with the lift ℓ. This is seen in the following commuting diagram:

$$\begin{tikzcd}
 & & A_i \arrow[d, "f_i"] \\
 & U \arrow[r, "/" marking, "u"] \arrow[ur, dotted, bend left] & B_i \\
U' \arrow[r, "/" marking, "u'"'] \arrow[ur, "\pi"] \arrow[uurr, dotted, bend left, "\ell"] & X \arrow[ur, "g"'] &
\end{tikzcd}$$

$\square$

Lemma 35 allows us to apply the axioms **In.Frc(1) - In.Frc(4)**, in Definition 34 a little more broadly. For simplicity in notation in later proofs we will typically supress the pullbacks in Lemma 35 and just write $u : U \to X$ for the cover $u' : U' \to X$ with lift ℓ.

4.3 Defining the Internal Category of Fractions

In this section we define structure for an internal category of fractions, $\mathbb{C}[W^{-1}]$, for a pair $(\mathbb{C}, W)$ that admits an internal category in a context for internal fractions, $(\mathcal{E}, \mathcal{J})$, as in Definitions 28 and 34. Before we begin we should mention that the proofs in this section and Section 3.2, using axioms **In.Frc(1) - In.Frc(4)**, can be difficult to follow so we have labeled and colour coded diagrams in a particular way. The diagrams labeled with capital letters, $(A), (B), (C), ...$, are representing diagrams in $\mathbb{C}$ which contain the data of the usual proofs for the case $\mathcal{E} = \mathbf{Set}$. The 'cover diagrams' labeled with stars, $(\star), (\star\star), ...$, describe the corresponding applications of **In.Frc(1) - In.Frc(4)** whose covers and lifts allow us to witness the arrows represented by the diagrams $(A), (B), (C), ...$ Any reference to these diagrams or equations should be interpreted 'locally' within the scope of the proof in which the reference occurs. We use covers to define the composition structure for the internal category of fractions and we also use the fact that they are epimorphisms to show other maps out of spn ${}_t{\times}_s$ spn are equal by showing the can be equalized by covers or their composites with other epimorphisms (such as coequalizer maps).

The composition, source, target, and identity notation for internal categories is being overloaded, as well as notation for pullback and product projections. We have included colours in both kinds of diagrams mentioned above as well as the corresponding equations for the maps in $\mathcal{E}$ of the star-labeled cover diagrams. The reference scope between these diagrams is contained within respective lemmas and propositions so there should be no issue with re-using labeling and colour patterns for diagrams in different lemma and proposition representing these two things similarly.

The object of objects is that of $\mathbb{C}_0$, and the object of arrows is the coequalizer from Lemma 31:

$$\mathbb{C}[W^{-1}]_0 = \mathbb{C}_0, \qquad \left(\begin{tikzcd} \mathrm{sb} \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & \mathrm{spn} \arrow[r, "q"'] & \mathbb{C}[W^{-1}]_1 \end{tikzcd} \right)$$

The source and target maps $s, t : \mathbb{C}[W^{-1}]_1 \to \mathbb{C}[W^{-1}]_0$ are defined by the universal property of $\mathbb{C}[W^{-1}]_1$ as seen in Lemma 26:

$$\begin{tikzcd}
 & \mathbb{C}_1 \arrow[r, "t"] & \mathbb{C}_0 \\
\mathrm{sb} \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & \mathrm{spn} \arrow[u, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[r, two heads, "q"] & \mathbb{C}[W^{-1}]_1 \arrow[u, dotted, "t"'] \arrow[d, dotted, "s"] \\
 & W \arrow[r, "wt"'] & \mathbb{C}_0
\end{tikzcd}$$

To define the identity map, $e : \mathbb{C}[W^{-1}]_0 \to \mathbb{C}[W^{-1}]_1$, it helps to think about the case when $\mathcal{E} = \mathbf{Set}$ for a moment. In this case, the identity for an object, a, in a category of fractions is represented by any span with two of the same legs in W:

$$\begin{tikzcd} a & b \arrow[l, "u"', "\circ" marking] \arrow[r, "u", "\circ" marking] & a \end{tikzcd}$$

By **In.Frc(1)**, we have a section, α, of the target map $wt : W \to \mathbb{C}_0$ we can use to define the identity structure map. Take the unique span $\sigma_\alpha = (\alpha, \alpha w) : \mathbb{C}_0 \to \text{spn}$ induced by α and αw and post-compose it with the coequalizer map to define the identity map for $\mathbb{C}[W^{-1}]$.

$$\begin{tikzcd}
\mathbb{C}_0 \arrow[r, "\sigma_\alpha"] \arrow[d, equal] & \text{spn} \arrow[d, two heads, "q"] \\
\mathbb{C}[W^{-1}]_0 \arrow[r, dotted, "e"'] & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

Now we will prove that this definition does not depend on the choice of section, $\alpha : \mathbb{C}_0 \to W$, of $wt : W \to \mathbb{C}_0$.

Proposition 36. *The identity map, $e : \mathbb{C}[W^{-1}]_0 \to \mathbb{C}[W^{-1}]_1$ does not depend on the section, $\alpha : \mathbb{C}_0 \to W$.*

Proof. Let α and β be two sections of wt, and let

$$\sigma_\alpha = (\alpha, \alpha w) \qquad , \qquad \sigma_\beta = (\beta, \beta w)$$

be two spans $\mathbb{C}_0 \to \text{spn}$. We will show that $\sigma_\alpha q = \sigma_\beta q$ by finding a cover, $u : U \to \mathbb{C}_0$, to witness a family of intermediate spans $\sigma_{\alpha\beta} : U \to \text{spn}$ for which

$$u\sigma_\alpha q = \sigma_{\alpha\beta} q = u\sigma_\beta q.$$

Since u is an epimorphism, this will imply $\sigma_\alpha q = \sigma_\beta q$. Notice that $\alpha(wt) = 1_{\mathbb{C}_0} = \beta(wt)$ so there is an induced pairing map $(\alpha w, \beta) : \mathbb{C}_0 \to \text{csp}$ which is a section of both $\pi_0 t$ and $\pi_1 wt$. By **In.Frc(3)**, there exists a cover, $u_1 : U_0 \to \mathbb{C}_0$, and a lift, $\theta_{\alpha\beta}$ of $u_1(\alpha w, \beta)\mathbb{C}_0 \to \text{csp}$, along the cospan projection, $W_\Box \to \text{csp}$, in the bottom right of the following diagram:

$$\begin{tikzcd}
W_\circ \arrow[r, "(\pi_0\pi_1{,}\pi_0\pi_2)"] & W \ {}_t\!\times_s W & \\
U \arrow[u, dotted, "\omega_{\alpha\beta}"] \arrow[r, "u_0", "/" marking] & U_0 \arrow[u, "(\theta_{\alpha\beta}\pi_0\pi_0{,}u_1\alpha)"'] \arrow[r, "u_1", "/" marking] \arrow[d, dotted, "\theta_{\alpha\beta}"] & \mathbb{C}_0 \arrow[d, "(\alpha w{,}\beta)"] \\
& W_\Box \arrow[r, "(\pi_0\pi_1 w{,}\pi_1\pi_1)"'] & \text{csp}
\end{tikzcd} \tag{$\star$}$$

By definition of $W_\square$, we have that

$$\theta_{\alpha\beta}\pi_0\pi_0 wt = \theta_{\alpha\beta}\pi_0\pi_1 s = u_1(\alpha w, \beta)\pi_0 s = u_1\alpha w s$$

inducing the map $U_0 \to W\,{}_t{\times}_s\,W$ in the diagram abovel. By **In.Frc(2)**, there exists a cover, $u_0 : U \to U_0$, and a lift, $\omega_{\alpha\beta} : U \to W_\circ$ to make the square in the upper left of the diagram above commute. When $\mathcal{E} = \mathbf{Set}$ the process can be represented by the following picture with labels of arrows corresponding to the arrows in the diagram above that would be witnessing those below internally to $\mathbb{C}$.

$$\begin{tikzcd}
 & & \cdot \arrow[d, dotted, "\omega_{\alpha\beta}\pi_0\pi_0\pi_0"] \arrow[dddll, bend right, "\omega_{\alpha\beta}\pi_1"', "\odot" marking] \arrow[dddrr, bend left] & & \\
 & & \cdot \arrow[dl, dotted, "u_0\theta_{\alpha\beta}\pi_0\pi_0"', "\odot" marking] \arrow[dr, dotted, "u_0\theta_{\alpha\beta}\pi_1\pi_0"] & & \\
 & \cdot \arrow[dl, dotted, "u\alpha"', "\odot" marking] \arrow[dr, dotted, "u\alpha w"] & & \cdot \arrow[dl, dotted, "u\beta"', "\odot" marking] \arrow[dr, dotted, "u\beta w"] & \\
\cdot & & \cdot & & \cdot
\end{tikzcd} \tag{A}$$

where, since covers are stable under composition, we let $u = u_0 u_1$ denote the composite cover of u_0 and u_1. Note that by definition of $W_\square$ (ie. the Ore condition)

$$\begin{aligned}
\omega_{\alpha\beta}\pi_1 &= (\omega_{\alpha\beta}\pi_0\pi_0\pi_0,\ u_0\theta_{\alpha\beta}\pi_0\pi_0 w,\ u\alpha w)c \\
&= \bigl(\omega_{\alpha\beta}\pi_0\pi_0\pi_0,\ (u_0\theta_{\alpha\beta}\pi_0\pi_0 w,\ u\alpha w)c\bigr)c \\
&= \bigl(\omega_{\alpha\beta}\pi_0\pi_0\pi_0,\ (u_0\theta_{\alpha\beta}\pi_0\pi_0 w,\ u_0\theta_{\alpha\beta}\pi_0\pi_1)c\bigr)c \\
&= \bigl(\omega_{\alpha\beta}\pi_0\pi_0\pi_0,\ (u_0\theta_{\alpha\beta}\pi_1\pi_0,\ u_0\theta_{\alpha\beta}\pi_1\pi_1 w)c\bigr)c \\
&= \bigl(\omega_{\alpha\beta}\pi_0\pi_0\pi_0,\ (u_0\theta_{\alpha\beta}\pi_1\pi_0,\ u\beta\pi_1\pi_1 w)c\bigr)c \\
&= (\omega_{\alpha\beta}\pi_0\pi_0\pi_0,\ u_0\theta_{\alpha\beta}\pi_1\pi_0,\ u\beta\pi_1\pi_1 w)c.
\end{aligned}$$

So the outer span in Diagram (A) can be represented by the map, $\sigma_{\alpha\beta} : U \to \mathrm{spn}$, defined by the pairing map

$$\sigma_{\alpha\beta} = \bigl(\omega_{\alpha\beta}\pi_1,\ (\omega_{\alpha\beta}\pi_0\pi_0\pi_0,\ u_0\theta_{\alpha\beta}\pi_1\pi_0,\ u\beta\pi_1\pi_1 w)c\bigr)$$

whose right component is the composite

$$\begin{tikzcd}
U \arrow[r, "{(\omega_{\alpha\beta}\pi_0\pi_0\pi_0,\ u_0\theta_{\alpha\beta}\pi_1\pi_0,\ u\beta\pi_1\pi_1 w)c}"] \arrow[dr, "\sigma_{\alpha\beta}\pi_1"'] & \mathbb{C}_3 \arrow[d, "c"] & . \\
 & \mathbb{C}_1 &
\end{tikzcd}$$

This composite represents the internal triple composition in $\mathbb{C}$ of the arrows represented on the right side of Diagram (A) above. Now let

$$\mu_\alpha = (\omega_{\alpha\beta}\pi_0\pi_0\pi_0,\ u_0\theta_{\alpha\beta}\pi_0\pi_0)c \qquad , \qquad \mu_\beta = (\omega_{\alpha\beta}\pi_0\pi_0\pi_0,\ u_0\theta_{\alpha\beta}\pi_1\pi_0)c$$

and notice the map $\varphi_\alpha : U \to \mathrm{sb}$ given by

$$\varphi_\alpha = \big(((\mu_\alpha,\ u\alpha),\ \omega_{\alpha\beta}\pi_1),\ u\alpha w)$$

is well-defined by associativity of composition and the definitions above. Similarly we can see

$$\begin{aligned} \varphi_\alpha p_0 &= \varphi_\alpha(\pi_0\pi_0\pi_1, \pi_1) = (u\alpha, u\alpha w) \\ &= u\sigma_\alpha \end{aligned}$$

and

$$\begin{aligned} \varphi_\alpha p_1 &= \varphi_\alpha(\pi_0\pi_1,\ (\pi_0\pi_0\pi_0, \pi_1)c) \\ &= \big(\omega_{\alpha\beta}\pi_1,\ (\mu_\alpha, u\alpha w)c\big) \\ &= (\sigma_{\alpha\beta}\pi_0, \sigma_{\alpha\beta}\pi_1) \\ &= \sigma_{\alpha\beta}. \end{aligned}$$

On the other hand we have another map $\varphi_\beta : U \to \mathrm{sb}$ given by

$$\varphi_\beta = \big(((\mu_\beta, u\beta),\ \omega_{\alpha\beta}\pi_1),\ u\beta w)$$

for which

$$\varphi_\beta p_0 = \varphi_\beta(\pi_0\pi_0\pi_1, \pi_1) = (u\beta, u\beta w) = u\sigma_\beta$$

and

$$\varphi_\beta p_1 = \varphi_\beta(\pi_0\pi_1,\ (\pi_0\pi_0\pi_0, \pi_1)c) = \big(\omega_{\alpha\beta}\pi_1,\ (\mu_\beta, u\beta w)c\big) = (\sigma_{\alpha\beta}\pi_0, \sigma_{\alpha\beta}\pi_1) = \sigma_{\alpha\beta}.$$

From here we can conclude that

$$u\sigma_\alpha q = \varphi_\alpha p_0 q = \varphi_\alpha p_1 q = \sigma_{\alpha\beta} q = \varphi_\beta p_1 q = \varphi_\beta p_0 q = u\sigma_\beta q$$

and since u is epic

$$\sigma_\alpha q = \sigma_\beta q.$$

□

We can immediately see that the identity structure map, $\sigma_\alpha q : \mathbb{C}_0 \to \mathbb{C}[W^{-1}]_1$, is a section of both the source and target maps:

$$\begin{aligned} \sigma_\alpha q s &= (\alpha, \alpha w) q s \\ &= (\alpha, \alpha w)\hat{s} \\ &= (\alpha, \alpha w)\pi_0 w t \\ &= \alpha w t \\ &= 1_{\mathbb{C}_0} \end{aligned} \qquad\qquad \begin{aligned} \sigma_\alpha q t' &= (\alpha, \alpha w) q t \\ &= (\alpha, \alpha w)\hat{t} \\ &= (\alpha, \alpha w)\pi_1 t \\ &= \alpha w t \\ &= 1_{\mathbb{C}_0} \end{aligned}$$

The composition structure map needs to be defined out of the following pullback,

$$\begin{tikzcd} \mathbb{C}[W^{-1}]_1 \ {}_t\!\times_s \mathbb{C}[W^{-1}]_1 \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}[W^{-1}]_1 \arrow[d, "s"] \\ \mathbb{C}[W^{-1}]_1 \arrow[r, "t"'] & \mathbb{C}_0 \end{tikzcd},$$

in order for $\mathbb{C}[W^{-1}]$ to be an internal category. Since $(\mathbb{C}, W)$ is a candidate for internal fractions, by Lemma 31, this pullback is also the coequalizer of the parallel pair $p_0^2, p_1^2 : \mathrm{sb}\ {}_t\!\times_s \mathrm{sb} \to \mathrm{spn}\ {}_t\!\times_s \mathrm{spn}$. When $\mathcal{E} = \mathbf{Set}$ we can see how p_0^2 maps a pair of composable sailboats

$$\left[\begin{tikzcd} & \cdot \arrow[dl, "\circ" marking] \arrow[d] & & \cdot \arrow[dl, "\circ" marking] \arrow[d] & \\ \cdot & \cdot \arrow[l, "\circ" marking] \arrow[r] & \cdot & \cdot \arrow[l, "\circ" marking] \arrow[r] & \cdot \end{tikzcd}\right]$$

to the pair of composable spans

$$\left[\begin{tikzcd} \cdot & \cdot \arrow[l, "\circ" marking] \arrow[r] & \cdot & \cdot \arrow[l, "\circ" marking] \arrow[r] & \cdot \end{tikzcd}\right],$$

and how p_1^2 maps a pair of composable sailboats

$$\left[\begin{tikzcd} & \cdot \arrow[dl, "\circ" marking] \arrow[d] \arrow[dr, dotted] & & \cdot \arrow[dl, "\circ" marking] \arrow[d] \arrow[dr, dotted] & \\ \cdot & \cdot \arrow[l, "\circ" marking] \arrow[r] & \cdot & \cdot \arrow[l, "\circ" marking] \arrow[r] & \cdot \end{tikzcd}\right]$$

to the pair of composable spans

$$\left[\begin{tikzcd} & \cdot \arrow[dl, "\circ" marking] \arrow[dr] & & \cdot \arrow[dl, "\circ" marking] \arrow[dr] & \\ \cdot & & \cdot & & \cdot \end{tikzcd}\right]$$

where the dotted arrows are just used to point out the composite of the arrows in the sailboats respectively that make up the right legs of the spans being picked out by p_1^2.

The first thing to do is to use the Internal Fractions Axioms to obtain a cover, $u : U \to \text{spn}\ {}_t\times_s \text{spn}$, of composable spans which witnesses the span composition operation in the form of a map

$$U \xrightarrow{\sigma_\circ} \text{spn} \ .$$

When $\mathcal{E} = \mathbf{Set}$, span composition for fractions is defined by applying the right Ore condition followed by the weak-composition axiom to get a span whose left leg is in W, as shown in the following figure.

$$\left[\begin{tikzcd} & \cdot \arrow[dl, "\circ" marking] \arrow[dr] & & \cdot \arrow[dl, "\circ" marking] \arrow[dr] & \\ \cdot & & \cdot & & \cdot \end{tikzcd}\right]$$

$$\begin{tikzcd} {} \arrow[d, mapsto] \\ {} \end{tikzcd}$$

$$\left[\begin{tikzcd}[column sep=small]
 & & \cdot \arrow[d, dotted] \arrow[lldd, bend right, "\circ" marking] \arrow[rrdd, bend left] & & \\
 & & \cdot \arrow[ld, dotted, "\circ" marking] \arrow[rd, dotted] & & \\
 & \cdot \arrow[ld, dotted, "\circ" marking] \arrow[rd, dotted] & & \cdot \arrow[ld, dotted, "\circ" marking] \arrow[rd, dotted] & \\
\cdot & & \cdot & & \cdot
\end{tikzcd}\right].$$

To internalize this we construct a diagram of covers below, starting with a map, $\mathrm{spn}\ {}_t{\times}_s\ \mathrm{spn} \to \mathrm{csp}$, picking out a cospan whose right leg is in W from a pair of composable spans and apply **Int.Frc.(3)** along with Lemma 35 to get the cover $u_1 : U_0 \to \mathrm{spn}\ {}_{t'}{\times}_{s'}\ \mathrm{spn}$ that makes the bottom right square below commute. Next consider the map which picks out the composable pair in W from the Ore-square filler and the left leg of the first span in the original composable pair and apply **Int.Frc.(2)** along with Lemma 35 to get the cover $u_0 : U \to U_0$ that makes the top left square below commute.

$$\begin{tikzcd}
W_\circ \arrow[r, "(\pi_0\pi_1{,}\pi_0\pi_2)"] & W \times_{\mathbb{C}_0} W & \\
U \arrow[u, "\omega"] \arrow[r, "u_0", "/" marking] \arrow[d, "\sigma_\circ"'] & U_0 \arrow[u, "(\theta\pi_0\pi_0{,}u_1\pi_0\pi_0)"'] \arrow[r, "u_1", "/" marking] \arrow[d, "\theta"] & \mathrm{spn}\ {}_t{\times}_s\ \mathrm{spn} \arrow[d, "(\pi_0\pi_1{,}\pi_1\pi_0)"'] \\
\mathrm{spn} & W_\Box \arrow[r, "(\pi_0\pi_1{,}\pi_1\pi_1)"'] & \mathrm{csp}
\end{tikzcd}$$

Since covers are stable under composition we can take $u = u_0u_1 : U \to \mathrm{spn}\ {}_t{\times}_s\ \mathrm{spn}$ as our cover, and define $\sigma_\circ : U \to \mathrm{spn}$ by the pairing map

$$\sigma_\circ = \big(\omega\pi_1,\ (\omega\pi_0\pi_0\pi_0,\ u_0\theta\pi_1\pi_0,\ u\pi_1\pi_1)c\big).$$

We claim the construction represented by $\sigma_\circ$ is well-defined on equivalence classes in the sense that for any two choices of fillers for the Ore-square and weak-composition conditions above, there exists a sailboat relating them. Internally this is translated as independence of the choice of filler-arrows in the lifts, θ and σ, and is proven in Lemma 37 by finding a cover $\tilde{u} : \tilde{U} \to \ker(u)$ and two families of sailboats,

$$\varphi_0 : \tilde{U} \to \mathrm{sb} \qquad , \qquad \varphi_1 : \tilde{U} \to \mathrm{sb},$$

which witness commutativity of the square

$$\begin{tikzcd}
\ker u \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & U \arrow[d, "\sigma_\circ q"] \\
U \arrow[r, "\sigma_\circ q"'] & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

in $\mathcal{E}$. The proof is rather long and technical but full of colourful pictures. The cover u is an effective epimorphism so it is the coequalizer of its kernel pair and in Lemma 38 we use this universal property to induce a composition map on spans

$$c' : \mathrm{spn}\ {}_t\times_s \mathrm{spn} \to \mathbb{C}[W^{-1}]_1$$

such that the square

$$\begin{tikzcd}
U \arrow[r, "\sigma_\circ"] \arrow[d, "u"', "/" marking] & \mathrm{spn} \arrow[d, two heads, "q"] \\
\mathrm{spn}\ {}_t\times_s \mathrm{spn} \arrow[r, dotted, "c'"'] & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

commutes in $\mathcal{E}$. Finally, in the proof of Proposition 39 we show how to find an even finer cover $\hat{u} : \hat{U} \to \mathrm{sb}\,{}_t\times_s \mathrm{sb}$ witnessing that the map c' respects the sailboat relation. More precisely, the proof of Proposition 39, shows how to construct sailboats

$$\varphi_i : \hat{U} \to \mathrm{sb}$$

witnessing equivalences between the spans

$$\sigma_j : \hat{U} \to \mathrm{spn}$$

so that

$$\hat{u}p_0c' = \hat{\pi}_0\sigma_0 q = \varphi_0 p_0 q = \varphi_0 p_1 q = ... = \varphi_4 p_0 = \hat{\pi}_1 \sigma_0 q = \hat{u}p_1c'.$$

Then since $\hat{u}$ is an epimorphism, we can conlude that $p_0c' = p_1c'$ and induce the composition map, $c : \mathbb{C}[W^{-1}]_2 \to \mathbb{C}[W^{-1}]_1$. For the rest of this section we prove the lemmas and propositions we required to define composition.

Lemma 37. *There is a cover $\tilde{u} : \tilde{U} \to \ker(u)$, together with two maps*

$$\varphi_0 : \tilde{U} \to sb \qquad , \qquad \varphi_1 : \tilde{U} \to sb$$

, which witness that the composite $\sigma_\circ q$ coequalizes the kernel pair of $u : U \to spn\,{}_t\!\times_s spn$. That is, the diagram

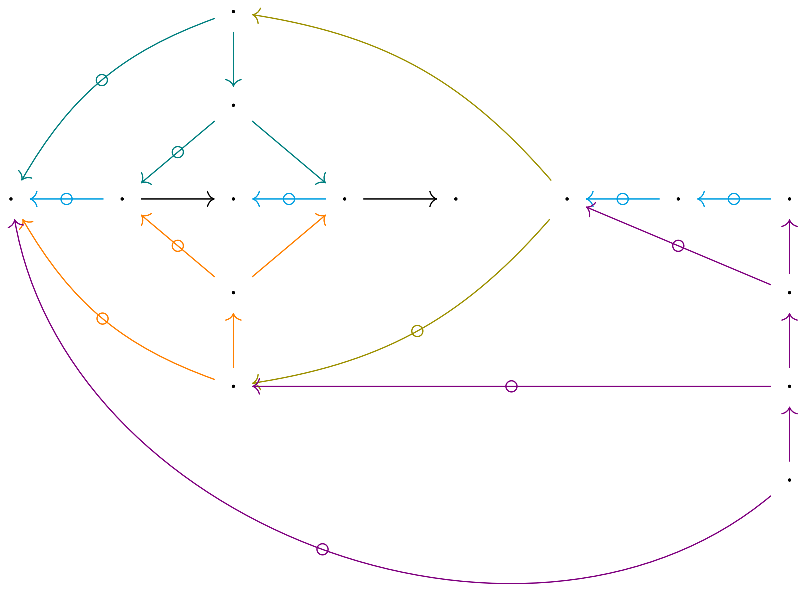
$$\begin{tikzcd}
\ker u \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & U \arrow[d, "\sigma_\circ q"] \\
U \arrow[r, "\sigma_\circ q"'] & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

commutes.

Proof.

We are essentially showing that any two choices of fillers above represent equivalent spans. Classically this can be done with the data in the following sketch.

- Take two composites (pictured in orange and teal below) of a single pair of composable spans
- Apply the right Ore condition (corresponding to **In.Frc(3)**) on the cospan determined by the left legs of the composites
- Apply the zippering axiom (corresponding to **In.Frc(4)**) to the parallel pair which, after post-composing with the left leg of the first span in the original composable pair, gives the two sides of the commuting Ore-square
- Apply zippering (corresponding to **In.Frc(4)**) to the parallel pair which is coequalized after post-composing with the left leg of the second span in the original composable pair
- Apply weak-composition (corresponding to **In.Frc(2)**) three times to obtain a span whose left leg is in W.



(A)

To translate this internally to $\mathcal{E}$, first note that the definition of $\sigma_\circ$ implies $\pi_0\sigma_\circ$ and $\pi_1\sigma_\circ$ have the same source.

$$\pi_0\sigma_\circ s' = \pi_0 u\pi_0\pi_0 wt = \pi_1 u\pi_0\pi_0 wt = \pi_1\sigma_\circ s'$$

Now take covers to witness the application of the axioms above in that order as follows. First take the Ore-square and zippering lifts given by **In.Frc(3)** and **In.Frc(4)** respectively:

$$\begin{tikzcd}
 & \mathcal{P}(\mathbb{C}) \arrow[r, "\pi_1"] & \mathcal{P}_{cq}(\mathbb{C}) & \\
\tilde{U}_2 \arrow[r, "\tilde{u}_3", "/" marking] \arrow[d, "\tilde{\delta}_1"'] & \tilde{U}_3 \arrow[r, "\tilde{u}_4", "/" marking] \arrow[u, "\tilde{\delta}_0"] \arrow[d, "\delta_1"] & \tilde{U}_4 \arrow[r, "\tilde{u}_5", "/" marking] \arrow[u, "\delta_0"'] \arrow[d, "\tilde{\theta}"'] & \ker u \arrow[d, "{(\pi_0\sigma_\circ\pi_0 w, \pi_1\sigma_\circ\pi_0)}"] \\
\mathcal{P}(\mathbb{C}) \arrow[r, "\pi_1"'] & \mathcal{P}_{cq}(\mathbb{C}) & W_\square \arrow[r, "{(\pi_0\pi_1, \pi_1\pi_1)}"'] & \mathrm{csp}
\end{tikzcd}$$

, $(\star)$

and then take the weak-composition lifts.

$$\begin{tikzcd}
W_\circ \arrow[r, "\pi_0\pi_{12}"] & W \ {}_{wt}\!\times_{ws} W & W_\circ \arrow[r, "\pi_0\pi_{12}"] & W \ {}_{wt}\!\times_{ws} W \\
\tilde{U} \arrow[u, "\tilde{\omega}_0"] \arrow[r, "\tilde{u}_0", "/" marking] & \tilde{U}_0 \arrow[u, "\omega_0"'] \arrow[d, "\tilde{\omega}_1"'] \arrow[r, "\tilde{u}_1", "/" marking] & \tilde{U}_1 \arrow[u, "\tilde{\omega}_2"] \arrow[d, "\omega_1"'] \arrow[r, "\tilde{u}_2", "/" marking] & \tilde{U}_2 \arrow[u, "\omega_2"'] \\
& W_\circ \arrow[r, "\pi_0\pi_{12}"'] & W \ {}_{wt}\!\times_{ws} W &
\end{tikzcd} \tag{$\star\star$}$$

The first vertical map representing cospans with right legs in W, seen on the right-hand side of Diagram ($\star$), is witnessing the following cospan of Diagram (A).

$$\begin{tikzcd}
& \cdot \arrow[dl, bend right, "\circ" marking] \\
\cdot & \\
& \cdot \arrow[ul, bend left, "\circ" marking]
\end{tikzcd} \tag{B}$$

Axiom **In.Frc(3)** along with Lemma 35 then give the cover and lift

$$\begin{tikzcd}
\tilde{U}_4 \arrow[r, "\tilde{u}_5", "/" marking] \arrow[d, "\tilde{\theta}"'] & \ker u \\
W_\square &
\end{tikzcd}$$

that make the bottom right square in Diagram ($\star$) commute. The map δ_0 is induced by a map, $\delta_0' : \tilde{U} \to P(\mathbb{C}) \ {}_{t}\!\times_{ws} W$, which can be found by expanding both sides of the commuting Ore square equation

$$\tilde{U}_4 \xrightarrow{(\tilde{\theta}\pi_0\pi_0 w,\ \tilde{u}_5\pi_0\sigma_\circ\pi_0 w)c=(\tilde{\theta}\pi_1\pi_0,\ \tilde{u}_5\pi_1\sigma_\circ\pi_0 w)c} \mathbb{C}_1$$

The arrows involved in this calculation are pictured:

$$\begin{tikzcd}
 & & \cdot \arrow[lldd, bend right, "\circ" marking] \arrow[d] & & \\
 & & \cdot \arrow[ld, "\circ" marking] & & \\
\cdot & \cdot \arrow[l, "\circ" marking] & & & \cdot \arrow[lluu, bend right] \arrow[lldd, bend left, "\circ" marking] \\
 & & \cdot \arrow[lu, "\circ" marking] & & \\
 & & \cdot \arrow[lluu, bend left, "\circ" marking] \arrow[u] & &
\end{tikzcd} \tag{C}$$

On the bottom we have

$$\begin{aligned}
&(\tilde{\theta}\pi_0\pi_0 w,\ \tilde{u}_5\pi_0\sigma_\circ\pi_0 w)c \\
&= (\tilde{\theta}\pi_0\pi_0 w,\ \tilde{u}_5\pi_0\omega\pi_1 w)c \\
&= \big(\tilde{\theta}\pi_0\pi_0 w,\ \tilde{u}_5\pi_0(\omega\pi_0\pi_0,\ \omega\pi_0\pi_1 w,\ \omega\pi_0\pi_2 w)c\big)c \\
&= \big(\tilde{\theta}\pi_0\pi_0 w,\ \tilde{u}_5\pi_0(\omega\pi_0\pi_0,\ u_0\theta\pi_0\pi_0 w,\ u\pi_0\pi_0 w)c\big)c \\
&= \big((\tilde{\theta}\pi_0\pi_0 w,\ \tilde{u}_5\pi_0\omega\pi_0\pi_0,\ \tilde{u}_5\pi_0 u_0\theta\pi_0\pi_0 w)c,\ \tilde{u}_5\pi_0 u\pi_0\pi_0 w\big)c
\end{aligned} \tag{4.1}$$

and on the top we have

$$\begin{aligned}
&(\tilde{\theta}\pi_1\pi_0,\ \tilde{u}_5\pi_1\sigma_\circ\pi_0 w)c \\
&= (\tilde{\theta}\pi_1\pi_0 w,\ \tilde{u}_5\pi_1\omega\pi_1 w)c \\
&= \big(\tilde{\theta}\pi_1\pi_0 w,\ \tilde{u}_5\pi_1(\omega\pi_0\pi_0,\ \omega\pi_0\pi_1 w,\ \omega\pi_0\pi_2 w)c\big)c \\
&= \big(\tilde{\theta}\pi_1\pi_0 w,\ \tilde{u}_5\pi_1(\omega\pi_0\pi_0,\ u_0\theta\pi_0\pi_0 w,\ u\pi_0\pi_0 w)c\big)c \\
&= \big((\tilde{\theta}\pi_1\pi_0 w,\ \tilde{u}_5\pi_1\omega\pi_0\pi_0,\ \tilde{u}_5\pi_1 u_0\theta\pi_0\pi_0 w)c,\ \tilde{u}_5\pi_1 u\pi_0\pi_0 w\big)c.
\end{aligned} \tag{4.2}$$

Since $\pi_0 u = \pi_1 u : \ker u \to \mathrm{spn}\ {}_t\!\times_s \mathrm{spn}$ by definition of $\ker u$, we have the equality

$$\tilde{U}_4 \xrightarrow{\tilde{u}_5\pi_0 u\pi_0\pi_0 w = \tilde{u}_5\pi_1 u\pi_0\pi_0 w} \mathbb{C}_1$$

between the final components in the bottom lines of calculations (1) and (2) which says there is an arrow in W coequalizing a parallel pair in $\mathbb{C}$. This determines a unique map, $\delta_0' : \tilde{U}_4 \to P(\mathbb{C})\ {}_t\!\times_{ws} W$, by the fact that

$$\delta_0'\pi_0\pi_1 = (\tilde{\theta}\pi_1\pi_0 w,\ \tilde{u}_5\pi_1\omega\pi_0\pi_0,\ \tilde{u}_5\pi_1 u_0\theta\pi_0\pi_0 w)c,$$

$$\delta_0'\pi_0\pi_0 = (\tilde{\theta}\pi_0\pi_0 w,\ \tilde{u}_5\pi_0\omega\pi_0\pi_0,\ \tilde{u}_5\pi_0 u_0\theta\pi_0\pi_0 w)c,$$

and

$$\delta_0'\pi_1 = \tilde{u}_5\pi_0 u\pi_0\pi_0;$$

and that the equality

$$\delta_0'(\pi_0\pi_0, \pi_1 w)c = \delta_0'(\pi_0\pi_1, \pi_1 w)c$$

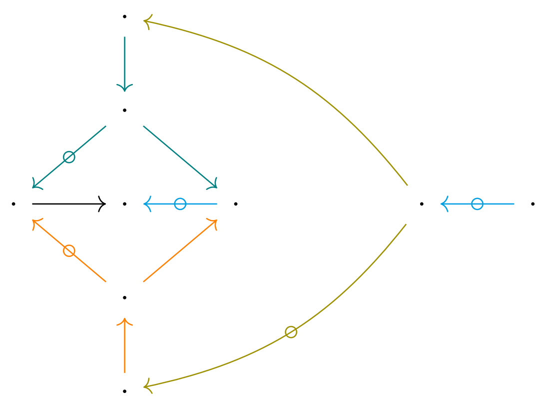
holds. The map δ_0' uniquely determines the map $\delta_0 : \tilde{U}_3 \to \mathcal{P}_{cq}(\mathbb{C})$ for which the equalizer diagram

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{C}) \arrow[r, tail, "\iota_{cq}"] & P(\mathbb{C})\ {}_t{\times}_{ws}\, W \arrow[r, shift left, "(\pi_0\pi_0{,}\pi_1 w)c"] \arrow[r, shift right, "(\pi_0\pi_1{,}\pi_1 w)c"'] & \mathbb{C}_1 \\
\tilde{U}_4 \arrow[u, dotted, "\delta_0"] \arrow[ur, "\delta_0'"'] & &
\end{tikzcd}$$

commutes in $\mathcal{E}$. By **In.Frc(4)** and Lemma 35 the cover and lift

$$\begin{tikzcd}
\mathcal{P}(\mathbb{C}) & \\
\tilde{U}_3 \arrow[u, "\tilde{\delta}_0"] \arrow[r, "\tilde{u}_4", "/"{marking}] & \tilde{U}_4
\end{tikzcd}$$

from Diagram ($\star$) exist. Similarly, the map δ_1 is induced by a map $\delta_1' : P(\mathbb{C})\ {}_t{\times}_{ws} W$. For readability purposes, let $\tilde{u}_{i;j} = \tilde{u}_i\tilde{u}_{i+1}...\tilde{u}_j$ for $0 \leq i < j \leq 5$ with $\tilde{u} = \tilde{u}_{0;5}$. Applying the zippering axiom and Ore conditions as we did above gives another equation, from the definitions of $\mathcal{P}$ and $W_\square$, which internally expresses the commutativity in the following picture:



(C)

$$\begin{aligned}
&\big((\tilde{\delta}_0\pi_0\iota_{eq}\pi_0,\ \tilde{u}_4\tilde{\theta}\pi_0\pi_0 w,\ \tilde{u}_{4;5}\pi_0\omega\pi_0\pi_0,\ \tilde{u}_{4;5}\pi_0u_0\theta\pi_1\pi_0)c,\ \tilde{u}_{4;5}\pi_0u\pi_1\pi_0w\big)c\\
&=\big(\tilde{\delta}_0\pi_0\iota_{eq}\pi_0,\ \tilde{u}_4\tilde{\theta}\pi_0\pi_0 w,\ \tilde{u}_{4;5}\pi_0\omega\pi_0\pi_0,\ (\tilde{u}_{4;5}\pi_0u_0\theta\pi_1\pi_0,\ \tilde{u}_{4;5}\pi_0u\pi_1\pi_0w)c\big)c\\
&=\big(\tilde{\delta}_0\pi_0\iota_{eq}\pi_0,\ \tilde{u}_4\tilde{\theta}\pi_0\pi_0 w,\ \tilde{u}_{4;5}\pi_0\omega\pi_0\pi_0,\ (\tilde{u}_{4;5}\pi_0u_0\theta\pi_0\pi_0w,\ \tilde{u}_{4;5}\pi_0u\pi_0\pi_1)c\big)c\\
&=\big((\tilde{\delta}_0\pi_0\iota_{eq}\pi_0,\ \tilde{u}_4\tilde{\theta}\pi_0\pi_0 w,\ \tilde{u}_{4;5}\pi_0\omega\pi_0\pi_0,\ \tilde{u}_{4;5}\pi_0u_0\theta\pi_0\pi_0w)c,\ \tilde{u}_{4;5}\pi_0u\pi_0\pi_1\big)c\\
&=\big((\tilde{\delta}_0\pi_0\iota_{eq}\pi_0,\ \tilde{u}_4\tilde{\theta}\pi_1\pi_0,\ \tilde{u}_{4;5}\pi_0\omega\pi_0\pi_0,\ \tilde{u}_{4;5}\pi_0u_0\theta\pi_0\pi_0w)c,\ \tilde{u}_{4;5}\pi_0u\pi_0\pi_1\big)c\\
&=\big(\tilde{\delta}_0\pi_0\iota_{eq}\pi_0,\ \tilde{u}_4\tilde{\theta}\pi_1\pi_0,\ \tilde{u}_{4;5}\pi_1\omega\pi_0\pi_0,\ (\tilde{u}_{4;5}\pi_1u_0\theta\pi_0\pi_0w,\ \tilde{u}_{4;5}\pi_0u\pi_0\pi_1)c\big)c\\
&=\big(\tilde{\delta}_0\pi_0\iota_{eq}\pi_0,\ \tilde{u}_4\tilde{\theta}\pi_1\pi_0,\ \tilde{u}_{4;5}\pi_1\omega\pi_0\pi_0,\ (\tilde{u}_{4;5}\pi_1u_0\theta\pi_1\pi_0,\ \tilde{u}_{4;5}\pi_0u\pi_1\pi_0w)c\big)c\\
&=\big((\tilde{\delta}_0\pi_0\iota_{eq}\pi_0,\ \tilde{u}_4\tilde{\theta}\pi_1\pi_0,\ \tilde{u}_{4;5}\pi_1\omega\pi_0\pi_0,\ \tilde{u}_{4;5}\pi_1u_0\theta\pi_1\pi_0)c,\ \tilde{u}_{4;5}\pi_0u\pi_1\pi_0w\big)c
\end{aligned}\tag{4.3}$$

The first and last lines in equation (4.3) correspond to the concatenations of the 'inside' paths in Diagram (C). They imply the existence of a map, $\delta_1' : \tilde{U}_3 \to P(\mathbb{C})\ {}_t\times_{ws} W$, uniquely determined by the projections

$$\begin{aligned}
\delta_1'\pi_0\pi_0 &= (\tilde{\delta}_0\pi_0\iota_{eq}\pi_0,\ \tilde{u}_4\tilde{\theta}\pi_0\pi_0w,\ \tilde{u}_{4;5}\pi_0\omega\pi_0\pi_0,\ \tilde{u}_{4;5}\pi_0u_0\theta\pi_1\pi_0)c,\\
\delta_1'\pi_0\pi_1 &= (\tilde{\delta}_0\pi_0\iota_{eq}\pi_0,\ \tilde{u}_4\tilde{\theta}\pi_1\pi_0,\ \tilde{u}_{4;5}\pi_1\omega\pi_0\pi_0,\ \tilde{u}_{4;5}\pi_1u_0\theta\pi_1\pi_0)c,\\
\delta_1'\pi_1 &= \tilde{u}_{4;5}\pi_0u\pi_1\pi_0w
\end{aligned}$$

for which

$$\delta_1'(\pi_0\pi_0,\pi_1w)c = \delta_1'(\pi_0\pi_1,\pi_1w)c$$

represents the inner cyan-colored arrow in Diagram C. The map δ_1' induces the unique map δ_1 that makes the following equalizer diagram

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{C}) \arrow[r, tail, "\iota_{cq}"] & P(\mathbb{C})\ {}_t\times_{ws} W \arrow[r, shift left, "(\pi_0\pi_0,\pi_1w)c"] \arrow[r, shift right, "(\pi_0\pi_1,\pi_1w)c"'] & \mathbb{C}_1 \\
\tilde{U}_3 \arrow[u, dotted, "\delta_1"] \arrow[ur, "\delta_1'"'] & &
\end{tikzcd}$$

commute in $\mathcal{E}$. By **In.Frc(4)** the cover lift

$$\begin{tikzcd}
\tilde{U}_2 \arrow[r, "\tilde{u}_3", "/" marking] \arrow[d, "\tilde{\delta}_1"'] & \tilde{U}_3 \\
\mathcal{P}(\mathbb{C}) &
\end{tikzcd}$$

in Diagram ($\star$) to make the bottom left square commute. The covers and lifts

$$\begin{tikzcd}
W_\circ & & W_\circ & \\
\tilde{U} \arrow[u, "\tilde{\omega}_0"] \arrow[r, "\tilde{u}_0", "|" marking] & \tilde{U}_0 \arrow[r, "\tilde{u}_1", "|" marking] \arrow[d, "\tilde{\omega}_1"'] & \tilde{U}_1 \arrow[u, "\tilde{\omega}_2"] \arrow[r, "\tilde{u}_2", "|" marking] & \tilde{U}_2 \\
& W_\circ & &
\end{tikzcd}$$

in Diagram ($\star\star$) are given by **In.Frc(2)** and Lemma 35. It suffices to define $\omega_i : \tilde{U}_i \to W \;_{wt}\!\times_{ws} W$, in Diagram ($\star\star$) that pick out composable pairs in W. The relevant representative diagram in $\mathbb{C}$ to keep in mind is:

$$\begin{tikzcd}
\cdot & & \cdot \arrow[dl, bend left, "\circ" marking] & \cdot \arrow[l, "\circ" marking] & \cdot \arrow[l, "\circ" marking] \\
& & & & \cdot \arrow[u] \arrow[ull, "\circ" marking] \\
& \cdot \arrow[ul, bend left, "\circ" marking] & & & \cdot \arrow[u] \arrow[lll, "\circ" marking] \\
& & & & \cdot \arrow[u] \arrow[uuullll, bend left=60, "\circ" marking]
\end{tikzcd} \tag{D}$$

The maps $\omega_i : \tilde{U}_i \to W_{wt}\!\times_{ws}$ are defined in sequence as follows. First, the pair of arrows obtained from the two diagram-extension conditions (colored in cyan in Diagram (D)) are composable by definition of $\mathcal{P}(\mathbb{C})$:

$$\begin{aligned}
\tilde{\delta}_1 \pi_0 \iota_{eq} \pi_0 wt &= \tilde{\delta}_1 \pi_0 \iota_{eq} \pi_1 \pi_0 s && \text{Def. } W\;_{wt}\!\times_s P(\mathbb{C}) \\
&= \tilde{\delta}_1 \pi_1 \iota_{cq} \pi_0 \pi_0 s && \text{Def. } \mathcal{P}(\mathbb{C}) \\
&= \tilde{u}_3 \delta_1 \iota_{cq} \pi_0 \pi_0 s && \text{Def. } \tilde{\delta}_1 \\
&= \tilde{u}_3 \delta_1' \pi_0 \pi_0 s && \text{Def. } \delta_1 \\
&= \tilde{u}_3 \tilde{\delta}_0 \pi_0 \iota_{eq} \pi_0 s && \text{Def. } \delta_1'
\end{aligned}$$

This uniquely determines the map

$$\tilde{U}_2 \xrightarrow{\omega_2=(\tilde{\delta}_1\pi_0\iota_{eq}\pi_0,\tilde{u}_3\tilde{\delta}_0\pi_0\iota_{eq}\pi_0)} W \ {}_{wt}{\times}_{ws} \, W$$

which gives the lift $\tilde{\omega}_2 : \tilde{U}_1 \to W_\circ$ in Diagram $(\star\star)$. The composite (in W) witnessed by $\tilde{\omega}_2$ can be composed with the arrow in W (colored olive in Diagram (D)) that filled the Ore square because

$$\begin{aligned}
\tilde{\omega}_2\pi_1 wt &= \tilde{\omega}_2\pi_0\pi_{12}\pi_1 wt && \text{Def. } W_\circ \\
&= \tilde{u}_2\omega_2\pi_1 wt && \text{Def. } \tilde{\omega}_2 \\
&= \tilde{u}_{2;3}\tilde{\delta}_0\pi_0\iota_{eq}\pi_0 wt && \text{Def. } \omega_2 \\
&= \tilde{u}_{2;3}\tilde{\delta}_0\pi_0\iota_{eq}\pi_1\pi_0 s && \text{Def. } W \ {}_{wt}{\times}_s \, P(\mathbb{C}) \\
&= \tilde{u}_{2;3}\tilde{\delta}_0\pi_1\iota_{cq}\pi_0\pi_0 s && \text{Def. } \mathcal{P}(\mathbb{C}) \\
&= \tilde{u}_{2;4}\delta_0\iota_{cq}\pi_0\pi_0 s && \text{Def.} \\
&= \tilde{u}_{2;4}\delta_0'\pi_0\pi_0 s && \text{Def. } \delta_0 \\
&= \tilde{u}_{2;4}\tilde{\theta}\pi_0\pi_0 ws && \text{Def. } \delta_0'
\end{aligned}$$

and it induces the pairing map

$$\tilde{U}_2 \xrightarrow{\omega_1=(\tilde{\omega}_2\pi_1,\tilde{u}_{2;4}\tilde{\theta}\pi_0\pi_0)} W \ {}_{wt}{\times}_{ws} \, W \ .$$

This gives the cover $\tilde{u}_1 : \tilde{U}_0 \to \tilde{U})1$ and lift $\tilde{\omega}_1 : \tilde{U}_0 \to W_\circ$ in Diagram $(\star\star)$. Finally, the composite witnessed by $\tilde{\omega}_1$ can be composed with the left leg of the original span (colored in orange in Diagram D and) witnessed by $\tilde{u}_{1;5}\pi_0\sigma_\circ : \tilde{U}_1 \to$ spn because

$$\begin{aligned}
\tilde{\omega}_1\pi_1 wt &= \tilde{\omega}_1\pi_0\pi_{12}\pi_1 wt && \text{Def. } W_\circ \\
&= \tilde{u}_1\omega_1\pi_1 wt && \text{Def. } \tilde{\omega}_1 \\
&= \tilde{u}_{1;4}\tilde{\theta}\pi_0\pi_0 wt && \text{Def. } \Omega_1 \\
&= \tilde{u}_{1;5}\pi_0\sigma_\circ\pi_0 ws && \text{Def. } \theta.
\end{aligned}$$

This gives the unique pairing map

$$\tilde{U}_2 \xrightarrow{\omega_0=(\tilde{\omega}_1\pi_1,\tilde{u}_{1;5}\pi_0\sigma_\circ\pi_0)} W\ {}_{wt}\!\times_{ws} W$$

which induces the cover $u_0 : \tilde{U} \to \tilde{U}_0$ and lift $\hat{\omega}_0 : \tilde{U} \to W_\circ$ in Diagram $(\star\star)$. Now let

$$\tilde{U} \xrightarrow{\omega=(\tilde{\omega}_0\pi_0\pi_0,\ \tilde{u}_0\tilde{\omega}_1\pi_0\pi_0,\tilde{u}_{0;1}\tilde{\omega}_2\pi_1 w)c} \mathbb{C}_1$$

witness the composite(s) of the three vertical violet-colored arrows and the two horizontal cyan-colored arrows in Diagram (D):

$$\begin{tikzcd}
\cdot & \cdot \arrow[l, dashed, "\circ" marking] & \cdot \arrow[l, dashed, "\circ" marking] \\
 & & \cdot \arrow[u, dashed] \arrow[ull, "\circ" marking] \\
 & & \cdot \arrow[u] \\
 & & \cdot \arrow[u] \arrow[uuull, dotted]
\end{tikzcd}$$

By zippering we get commutativity of the following piece of Diagram (A)

$$\begin{tikzcd}
\cdot \arrow[d] & & & & & \\
\cdot \arrow[dr] & & & & & \\
 & \cdot \arrow[r] & \cdot & \cdot \arrow[uulll, bend right] \arrow[ddlll, bend left, "\circ" marking] & \cdot \arrow[l, "\circ" marking] & \cdot \arrow[l, "\circ" marking] \\
\cdot \arrow[ur] & & & & & \\
\cdot \arrow[u] & & & & &
\end{tikzcd} \tag{E}$$

Internally we can use associativity of composition, definitions of the pairing maps involved, and the definition of $\mathcal{P}(\mathbb{C})$ to write this commutativity by the equation:

$$(\omega,\ \tilde{u}_{0;4}\tilde{\theta}\pi_0\pi_0 w,\ \tilde{u}\pi_0\sigma_\circ\pi_1)c = (\omega,\ \tilde{u}_{0;4}\tilde{\theta}\pi_1\pi_0,\ \tilde{u}\pi_1\sigma_\circ\pi_1)c. \tag{4.4}$$

or the commuting diagram

$$\begin{tikzcd}
\tilde{U} \arrow[r, "{(\omega,\ \tilde{u}_{0;4}\tilde{\theta}\pi_1\pi_0,\ \tilde{u}\pi_1\sigma_\circ\pi_1)}"] \arrow[d, "{(\omega,\ \tilde{u}_{0;4}\tilde{\theta}\pi_0\pi_0 w,\ \tilde{u}\pi_0\sigma_\circ\pi_1)}"'] & \mathbb{C}_3 \arrow[d, "c"] \\
\mathbb{C}_3 \arrow[r, "c"'] & \mathbb{C}_1
\end{tikzcd} .$$

For readability we define μ_0 and μ_1 by composition in $\mathbb{C}$

$$\begin{tikzcd}
\tilde{U} \arrow[r, "{(\omega,\ \tilde{u}_{0;4}\tilde{\theta}\pi_0\pi_0 w)}"] \arrow[dr, "\mu_0"'] & \mathbb{C}_2 \arrow[d, "c"] & \tilde{U} \arrow[r, "{(\omega,\ \tilde{u}_{0;4}\tilde{\theta}\pi_1\pi_0 w)}"] \arrow[dr, "\mu_1"'] & \mathbb{C}_2 \arrow[d, "c"] \\
& \mathbb{C}_1 & & \mathbb{C}_1
\end{tikzcd}$$

to internally represent the composites in the following piece of Diagram (A):

$$\begin{tikzcd}
\cdot & & & \\
& \cdot \arrow[ul, bend right] \arrow[ddl, bend left, "\circ" marking] & \cdot \arrow[l, "\circ" marking] & \cdot \arrow[l, "\circ" marking] \\
& & & \cdot \arrow[u] \\
\cdot & & & \cdot \arrow[u] \\
& & & \cdot \arrow[u]
\end{tikzcd} \tag{E}$$

Note that equation (4.4) above gives two descriptions of the right leg of an intermediate span $\sigma_{01} : \tilde{U} \to \mathrm{spn}$ given by the pairing

$$\sigma_{01} = (\tilde{\omega}_0\pi_1, \sigma_{01}\pi_1),$$

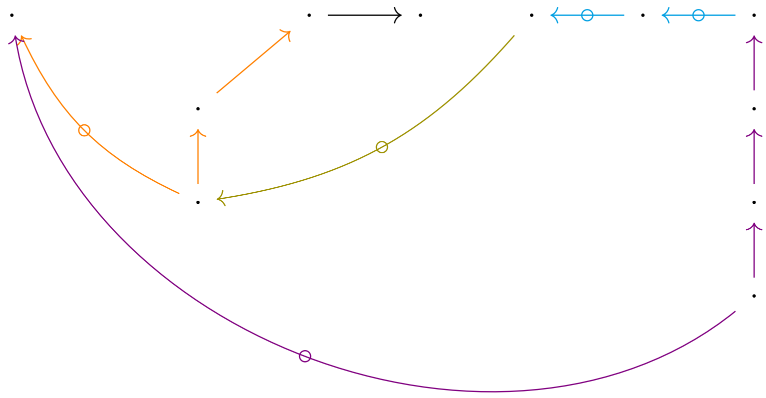
where the right-hand component can be rewritten as either one of the terms in the following equation:

$$(\mu_0, \tilde{u}\pi_0\sigma_\circ\pi_1)c = \sigma_{01}\pi_1 = (\mu_1, \tilde{u}\pi_1\sigma_\circ\pi_1)c$$

The different representations of the right leg of this intermediate span can be seen by the two paths in Diagram (A) given by combining Diagrams (C) and (E). Now by expanding internal composition in terms of pairing maps; by associativity of composition in $\mathbb{C}$; and by the definitions of $W_\circ, \mathcal{P}(\mathbb{C})$, and $W_\Box$ we can represent the left leg of the intermediate span σ_{01} by:

$$(\mu_0, \tilde{u}\pi_0\sigma_\circ\pi_0)c = \tilde{\omega}_0\pi_1 = (\mu_1, \tilde{u}\pi_1\sigma_\circ\pi_0)c.$$

Note that the sources of the composites (in $\mathbb{C}$) in the previous two equations are the sources of the maps $\mu_0, \mu_1 : \tilde{U} \to \mathbb{C}_1$, which have a common source in $\omega s : \tilde{U} \to \mathbb{C}_0$. We can now give well-defined explicit descriptions of the two sailboats, $\tilde{U} \to \mathrm{sb}$, using the universal property of sb. The first sailboat represents picking out the following piece of Diagram (A):



(F)

This is determined uniquely by the components in the pairing map:

$$\varphi_0 = \big(((\mu_0,\ \tilde{u}\pi_0\sigma_\circ\pi_0),\ \tilde{\omega}_0\pi_1),\ \tilde{u}\pi_0\sigma_\circ\pi_1\big).$$

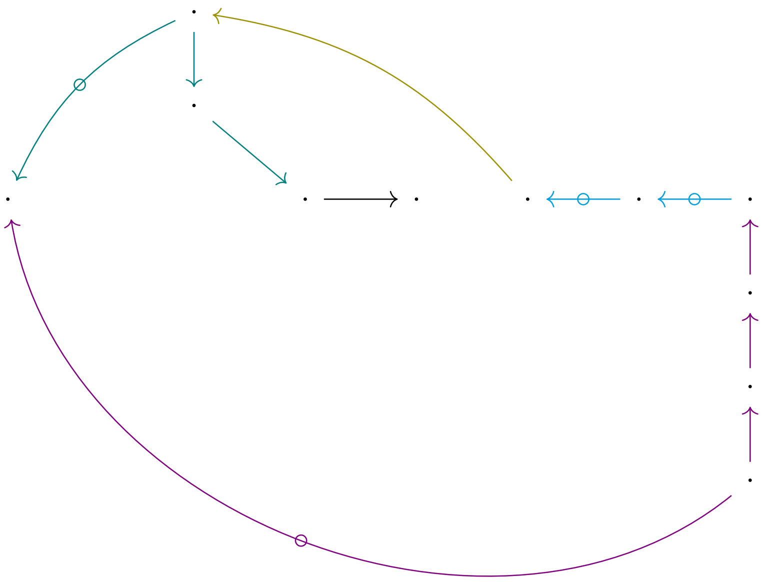
By definition of φ_0 we can compute

$$\varphi_0 p_0 = \varphi_0(\pi_0\pi_0\pi_1, \pi_1) = (\tilde{u}\pi_0\sigma_\circ\pi_0,\ \tilde{u}\pi_0\sigma_\circ\pi_1) = \tilde{u}\pi_0\sigma_\circ$$

and additionally with the definition of σ_{01} we can see

$$\begin{aligned}
\varphi_0 p_1 &= \varphi_0(\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c) \\
&= \big(\tilde{\omega}_0\pi_1,\ (\mu_0,\ \tilde{u}\pi_0\sigma_\circ\pi_1)c\big) \\
&= (\tilde{\omega}_0\pi_1,\ \sigma_{01}\pi_1) \\
&= \sigma_{01}.
\end{aligned}$$

The second sailboat represents picking out the following piece of Diagram (A):



(G)

This one is uniquely determined by the pairing map:

$$\varphi_1 = \big(((\mu_1,\ \tilde{u}\pi_1\sigma_\circ\pi_0),\ \tilde{\omega}_0\pi_1),\ \tilde{u}\pi_1\sigma_\circ\pi_1\big).$$

By definition of φ_1 we get

$$\begin{aligned}
\varphi_1 p_0 &= \varphi_1(\pi_0\pi_0\pi_1, \pi_1) \\
&= (\tilde{u}\pi_1\sigma_\circ\pi_0,\ \tilde{u}\pi_1\sigma_\circ\pi_1) \\
&= \tilde{u}\pi_1\sigma_\circ
\end{aligned}$$

and by definition of σ_{01}

$$\begin{aligned}
\varphi_0 p_1 &= \varphi_0(\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c) \\
&= \big(\tilde{\omega}_0\pi_1,\ (\mu_0,\ \tilde{u}\pi_0\sigma_\circ\pi_1)c\big) \\
&= (\tilde{\omega}_0\pi_1,\ \sigma_{01}\pi_1) \\
&= \sigma_{01}.
\end{aligned}$$

Putting the previous few computations together we can see

$$\begin{aligned}
\tilde{u}\pi_0\sigma_\circ q &= \varphi_0 p_0 q && \text{Def. } \varphi_0 \\
&= \varphi_0 p_1 q && \text{Def. } q \\
&= \sigma_{01} q && \text{Def. } \varphi_0 \\
&= \varphi_1 p_1 q && \text{Def. } \varphi_1 \\
&= \varphi_1 p_0 q && \text{Def. } q \\
&= \tilde{u}\pi_1\sigma_\circ q && \text{Def. } \varphi_1
\end{aligned}$$

and since $\tilde{u}$ is epic:

$$\pi_0\sigma_\circ q = \pi_1\sigma_\circ q$$

That is, the diagram

$$\begin{tikzcd}
\ker u \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & U \arrow[d, "\sigma_\circ q"] \\
U \arrow[r, "\sigma_\circ q"'] & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

commutes. $\square$

Lemma 38. *There exists a unique 'composition on representatives' map* $c' : spn\,{}_t\!\times_s spn \to \mathbb{C}[W^{-1}]_1$ *such that the diagram*

$$\begin{tikzcd}
U \arrow[r, "u", "/" marking] \arrow[d, "\sigma_\circ"'] & spn\,{}_t\!\times_s spn \arrow[d, dotted, "c'"] \\
spn \arrow[r, two heads, "q"'] & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

commutes in $\mathcal{E}$.

Proof. This follows by the universal property of u being the coequalizer of its kernel pair and Lemma 37 showing that $\sigma_\circ q : \mathrm{spn}\ {}_t\!\times_s \mathrm{spn} \to \mathbb{C}[W^{-1}]_1$ also coequalizes the kernel pair of u. □

Having defined composition on representative spans by a map $c' : \mathrm{spn}\ {}_t\!\times_s \mathrm{spn} \to \mathbb{C}[W^{-1}]_1$, the next thing to do is to check it is well-defined. This is translated internally by the following proposition.

Proposition 39. *The composition operation on spans,*

$$c' : spn\ {}_t\!\times_s spn \to \mathbb{C}[W^{-1}]_1,$$

is well-defined on equivalence classes in the sense that the square

$$\begin{tikzcd}
sb\ {}_t\!\times_s sb \arrow[r, "p_1^2"] \arrow[d, "p_0^2"'] & spn\ {}_t\!\times_s spn \arrow[d, "c'"] \\
spn\ {}_t\!\times_s spn \arrow[r, "c'"'] & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

commutes in $\mathcal{E}$.

Proof. By Lemma 40

$$\hat{u}p_0^2c' = \varphi_0 p_0 q = \varphi_3 p_0 q = \hat{u}p_1^2c'$$

and since $\hat{u}$ is epic

$$p_0^2c' = p_1^2c'$$

□

A direct consequence of Proposition 39 is it induces a unique composition map $c : \mathbb{C}[W^{-1}]_2 \to \mathbb{C}[W^{-1}]_1$ such that the diagram

$$\begin{tikzcd}
\mathrm{spn}\ {}_t\!\times_s \mathrm{spn} \arrow[r, two heads, "q_2"] \arrow[dr, "c'"'] & \mathbb{C}[W^{-1}]_2 \arrow[d, dotted, "c"] \\
 & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

commutes in $\mathcal{E}$, by the universal property of the coequalizer $\mathbb{C}[W^{-1}]_2$. Lemma 40 is doing all the heavy lifting for showing that c' is well-defined and subsequently defining the composition map $c : \mathbb{C}[W^{-1}]_2 \to \mathbb{C}[W^{-1}]_1$. We now prove this lengthy and technical lemma.

Lemma 40. *There exists a cover $\hat{U} \to sb\ {}_t{\times}_s\ sb$, and four families of sailboats, $\varphi_i : \hat{U} \to sb$ for $0 \leq i \leq 3$, such that the diagram*

$$\begin{tikzcd}
sb^2 \arrow[r, shift left, "p_0^2"] \arrow[r, shift right, "p_1^2"'] & spn^2 \arrow[ddr, "c'"] & \\
\hat{U} \arrow[u, "|" marking, "\hat{u}"] \arrow[r, shift left, "\hat{\pi}_0"] \arrow[r, shift right, "\hat{\pi}_1"'] \arrow[d, shift right, "\varphi_3"'] \arrow[d, shift left, "\varphi_0"] & U \arrow[u, "|" marking, "u"'] \arrow[d, "\sigma_\circ"] & \\
sb \arrow[r, "p_0"] & spn \arrow[r, two heads, "q"'] & \mathbb{C}[W^{-1}]
\end{tikzcd}$$

commutes in the sense that

$$\varphi_0 p_0 q = \hat{\pi}_0 \sigma_\circ q = \hat{\pi}_0 u c' = \hat{u} p_0^2 c',$$

$$\varphi_4 p_0 q = \hat{\pi}_1 \sigma_\circ q = \hat{\pi}_1 u c' = \hat{u} p_1^2 c',$$

and the sailboats glue together along comparison spans

$$\varphi_0 p_0 q = \varphi_0 p_1 q = \varphi_1 p_1 q = \varphi_1 p_0 q = \varphi_2 p_0 q = \varphi_2 p_1 q = \varphi_3 p_1 q = \varphi_3 p_0 q.$$

Proof. The main idea is to use the explicit definition

$$\sigma_\circ = \big(\omega\pi_1,\ (\omega\pi_0\pi_0\pi_0,\ u_0\theta\pi_1\pi_0,\ u\pi_1\pi_1)c\big)$$

and post-compose it with the maps p_0^2 and p_1^2 to get two different spans. To show these two spans are equivalent we construct a comparison span from the data involved in each of their constructions and show they're both equivalent to the comparison span. Each of these equivalences in turn requires constructing an additional comparison span and a witnessing sailboat. This accounts for the four sailboats.

To do this we need a common domain for the covers so take pullbacks of $u : U \to \mathrm{spn}\ {}_t{\times}_s\ \mathrm{spn}$ along p_0^2 and p_1^2 to get two covers of $\mathrm{sb}\ {}_t{\times}_s\ \mathrm{sb}$

$$\begin{tikzcd}
\bar{U}_0 \arrow[r, "\pi_1"] \arrow[d, "|" marking, "\bar{u}_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "|" marking, "u"] & \bar{U}_1 \arrow[l, "\pi_1"'] \arrow[d, "|" marking, "\bar{u}_1"] \arrow[dl, phantom, "\llcorner", very near start] \\
\mathrm{sb}\ {}_t{\times}_s\ \mathrm{sb} \arrow[r, "p_0^2"'] & \mathrm{spn}\ {}_t{\times}_s\ \mathrm{spn} & \mathrm{sb}\ {}_t{\times}_s\ \mathrm{sb} \arrow[l, "p_1^2"]
\end{tikzcd} \qquad (1)$$

Now take a refinement

$$\begin{tikzcd}
\bar{U} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, "\bar{u}", "/" marking] & \bar{U}_1 \arrow[d, "\bar{u}_1", "/" marking] \\
\bar{U}_0 \arrow[r, "\bar{u}_0"', "/" marking] & \mathrm{sb}\ {}_t\!\times_s \mathrm{sb}
\end{tikzcd} \tag{2}$$

by taking a pullback of $\bar{u}_0$ and $\bar{u}_1$ to get a cover of the pairs of composable sailboats. Note that

$$\bar{u}p_0^2 = \pi_0\pi_0p_0^2 = \pi_0\pi_1u$$

projects out the composable spans represented by

$$\left[\begin{tikzcd}
\cdot & \cdot \arrow[l, "\circ" marking] \arrow[r] & \cdot & \cdot \arrow[l, "\circ" marking] \arrow[r] & \cdot
\end{tikzcd}\right]$$

while

$$\bar{u}p_1^2 = \pi_1\pi_0p_1^2 = \pi_1\pi_1u$$

projects out the composable spans represented by

$$\left[\begin{tikzcd}
 & \cdot \arrow[dl, "\circ" marking] \arrow[d] & & \cdot \arrow[dl, "\circ" marking] \arrow[d] & \\
\cdot & \cdot \arrow[r] & \cdot & \cdot \arrow[r] & \cdot
\end{tikzcd}\right]$$

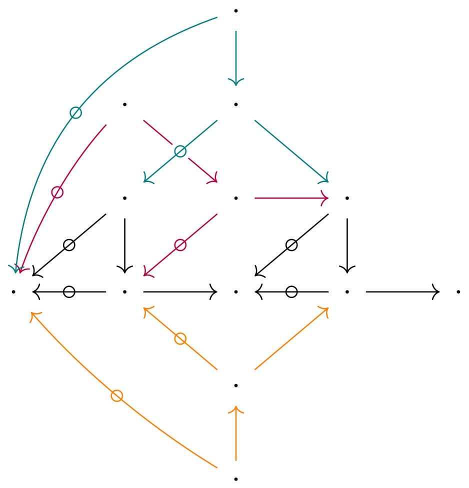
From this point the usual set-theoretic proof can be translated into a chain of covers and lifts. The outline is that for any pair of composable sailboats,

$$\left[\begin{tikzcd}
 & \cdot \arrow[dl, "\circ" marking] \arrow[d] & & \cdot \arrow[dl, "\circ" marking] \arrow[d] & \\
\cdot & \cdot \arrow[l, "\circ" marking] \arrow[r] & \cdot & \cdot \arrow[l, "\circ" marking] \arrow[r] & \cdot
\end{tikzcd}\right]$$

the composites of the spans represented by $\bar{u}p_0^2$ and $\bar{u}p_1^2$ are equivalent to the composite of a comparison pair of composable spans,

$$\left[\begin{tikzcd}
 & & & \cdot \arrow[dl, "\circ" marking] \arrow[d, dotted] \arrow[dr] & \\
\cdot & \cdot \arrow[l, "\circ" marking] \arrow[r] & \cdot & \cdot \arrow[r, dotted] & \cdot
\end{tikzcd}\right]$$

The following figure shows the construction of three different composites being constructed.



(A)

To internalize this we define the maps that pick out each of the three spans and their composites by finding a corresponding cover $\tilde{U} \overset{\tilde{u}}{\not\longrightarrow} \bar{U}$. Two of the spans can be given in terms of the composition, $\sigma_\circ$, on the cover U but the comparison span needs a finer covering to witness applying the Ore and weak composition conditions to arrows from both of the first two spans. Denote the comparison pair of composable spans by γ and define it by the universal property in the following pullback diagram.

$$\begin{tikzcd}
\bar{U} \arrow[r, "\bar{u}", "/" marking] \arrow[d, "\bar{u}"', "/" marking] \arrow[dr, dotted, "\gamma"] & \mathrm{sb}\ {}_t\times_s \mathrm{sb} \arrow[r, "p_1^2"] & \mathrm{spn}\ {}_t\times_s \mathrm{spn} \arrow[d, "\pi_1"] \\
\mathrm{sb}\ {}_t\times_s \mathrm{sb} \arrow[d, "p_0^2"'] & \mathrm{spn}\ {}_t\times_s \mathrm{spn} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & \mathrm{spn} \arrow[d, "s"] \\
\mathrm{spn}\ {}_t\times_s \mathrm{spn} \arrow[r, "\pi_0"'] & \mathrm{spn} \arrow[r, "t"'] & \mathbb{C}_0
\end{tikzcd} \tag{3}$$

The following diagram of covers shows how the intermediate span is constructed by a similar span-composition construction for γ. Note there is another way to do this by taking a pullback of the pairing map $(\pi_0 p_0, \pi_1 p_1) : \mathrm{sb}\ {}_t\times_s \mathrm{sb} \to \mathrm{spn}_t\times_s$ along $u : U \to \mathrm{spn}\ {}_t\times_s \mathrm{spn}$ and a refinement with the previous refinement of covers of $\mathrm{spn}\ {}_t\times_s \mathrm{spn}$ above, and then using the span composition $\sigma_\circ : U \to \mathrm{spn}$ to obtain the intermediate span σ_γ in Diagram ($\star$) below. Both approaches lead to the same result.

$$\begin{tikzcd}
W_\circ \arrow[r, "(\pi_0\pi_1{,}\pi_0\pi_2)"] & W \times_{\mathbb{C}_0} W & \\
\tilde{U} \arrow[u, "\omega_\gamma"] \arrow[r, "\tilde{u}_0", "/" marking] \arrow[d, "\sigma_\gamma"] \arrow[d, bend right, "\sigma_0"'] \arrow[d, bend left, "\sigma_1"] & \tilde{U}_0 \arrow[u, "(\theta_\gamma\pi_0\pi_0{,}\tilde{u}_1\gamma\pi_0\pi_0)"'] \arrow[r, "\tilde{u}_1", "/" marking] \arrow[d, "\theta_\gamma"'] & \bar{U} \arrow[d, "(\gamma\pi_0\pi_1{,}\gamma\pi_1\pi_0)"] \\
\mathrm{spn} & W_\square \arrow[r, "(\pi_0\pi_1{,}\pi_1\pi_1)"'] & \mathbb{C}_1 \, {}_t\!\times_{wt} W
\end{tikzcd} \tag{$\star$}$$

The left and right curved arrows, σ_0 and σ_1, into spn in the bottom left corner are defined by applying the composite of spans, $\sigma_\circ$, to the composable spans given by applying p_0^2 and p_1^2 to the pair of composable sailboats. Since $\sigma_\circ$ is only defined on U we need to pass through the appropriate cover. The colours in the previous diagram and following equations indicate which of the three different span compositions in Figure (A) the arrows in the following equations are witnessing.

$$\begin{aligned}\sigma_0 &= \tilde{u}\pi_0\pi_1\sigma_\circ \\ &= \tilde{u}\pi_0\pi_1\big(\omega\pi_1,(\omega\pi_0\pi_0,u_0\theta\pi_1\pi_0,u\pi_1\pi_1)c\big)\end{aligned} \tag{4.5}$$

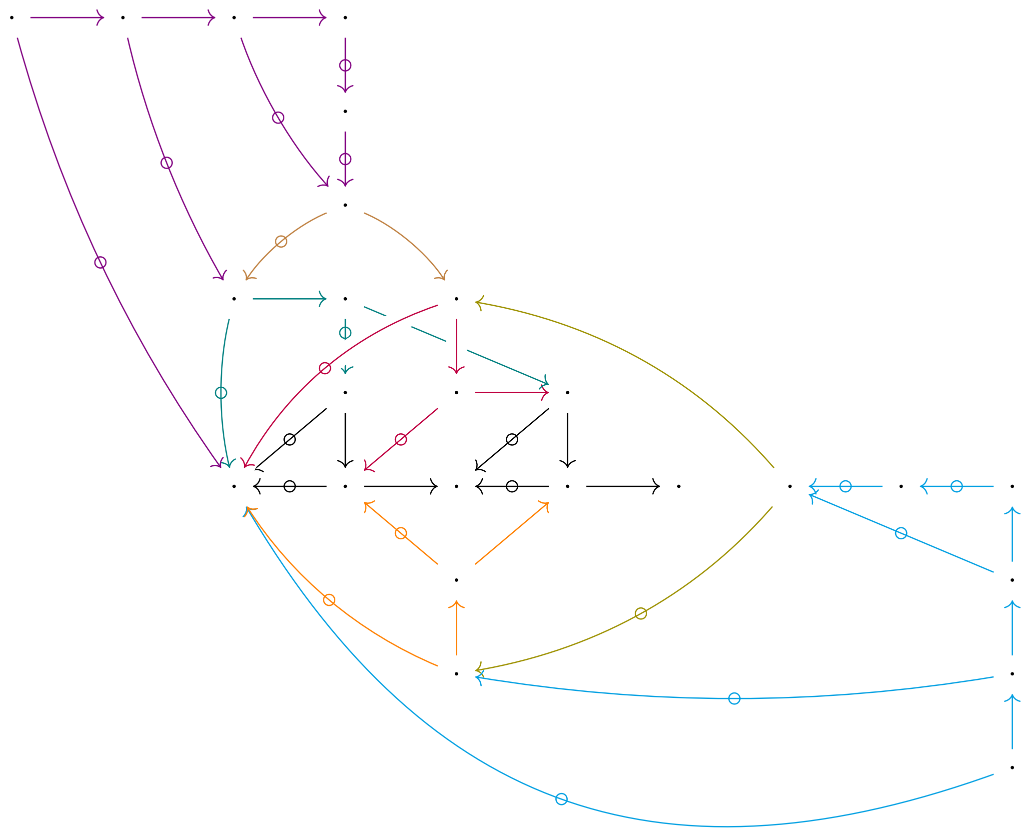
and

$$\begin{aligned}\sigma_1 &= \tilde{u}\pi_1\pi_1\sigma_\circ \\ &= \tilde{u}\pi_1\pi_1\big(\omega\pi_1,(\omega\pi_0\pi_0,u_0\theta\pi_1\pi_0,u\pi_1\pi_1)c\big).\end{aligned} \tag{4.6}$$

The arrow into spn on the bottom left side of the cover diagram is the universal map

$$\sigma_\gamma = \big(\omega_\gamma\pi_1,(\omega_\gamma\pi_0\pi_0,\tilde{u}'\theta_\gamma\pi_1\pi_0,\tilde{u}\bar{u}p_1^2\pi_1\pi_1)c\big).$$

The data necessary to construct witnessing sailboats for the equivalences between the pairs of spans σ_0, σ_1, and σ_γ can be obtained by applying the Ore condition, followed by the diagram-extension twice, and then weak composition three times. Internally this corresponds to a chain of six covers and lifts. All of this is color-coded below using olive and brown for the Ore condition and cyan and violet for the zippering and weak composition step(s) that follow. Note that in both cases the first zippering is done to parallel pairs of composites that can be post-composed by the left leg of the bottom left span. The second zipper is done to parallel pairs of composites that can be post-composed with the left leg of the bottom right span in the pair of composale sailboats. Weak composition is then applied three times in to get comparison spans, $\sigma_{0,\gamma}$ and $\sigma_{1,\gamma}$, whose left legs are in W.



(B)

Internally this is given by finding a cover $\hat{U} \xrightarrow{\hat{u}} \tilde{U}$ that witnesses application of the Ore condition, zippering, and weak composition in that order. The first three covers, $\hat{u}_5, \hat{u}_4$, and $\hat{u}_3$, witness the Ore condition being to each of the two cospans and the two applications of zippering that follow from each Ore-square.

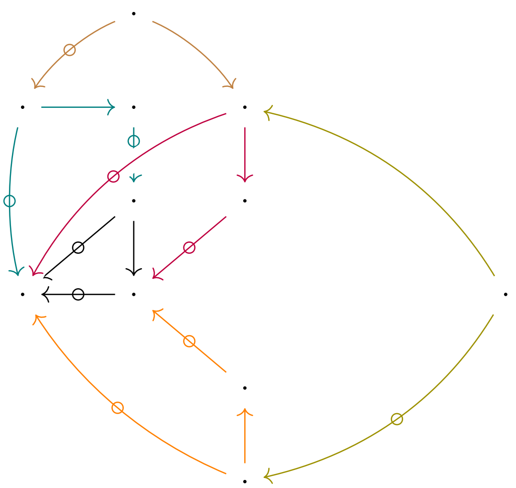
$$\begin{tikzcd}
 & \mathcal{P}(\mathbb{C}) \arrow[r, "\pi_1"] & \mathcal{P}_{cq}(\mathbb{C}) & \\
\hat{U}_3 \arrow[r, "\hat{u}_3", "/" marking] \arrow[d, shift right, "\delta_{\rho_1}"'] \arrow[d, shift left, "\delta_{\rho_0}"] & \hat{U}_4 \arrow[r, "\hat{u}_4", "/" marking] \arrow[u, shift left, "\delta_{\lambda_1}"] \arrow[u, shift right, "\delta_{\lambda_0}"'] \arrow[d, shift right, "\rho_1"'] \arrow[d, shift left, "\rho_0"] & \hat{U}_5 \arrow[r, "\hat{u}_5", "/" marking] \arrow[u, shift left, "\lambda_1"] \arrow[u, shift right, "\lambda_0"'] \arrow[d, shift right, "\theta_{\gamma_1}"'] \arrow[d, shift left, "\theta_{\gamma_0}"] & \tilde{U} \arrow[d, shift right, "(\sigma_1\pi_0 w{,}\sigma_\gamma\pi_0)"'] \arrow[d, shift left, "(\sigma_0\pi_0 w{,}\sigma_\gamma\pi_0)"] \\
\mathcal{P}(\mathbb{C}) \arrow[r, "\pi_1"'] & \mathcal{P}_{cq}(\mathbb{C}) & W_\square \arrow[r, "(\pi_0\pi_1{,}\pi_1\pi_1)"'] & \mathbb{C}_1 {}_t\times_{wt} W
\end{tikzcd} \quad (\star\star)$$

The covers, $\hat{u}_2, \hat{u}_1$, and $\hat{u}_0$, witness three applications of weak composition in each case as seen in the following continued sequence of covers:

$$\begin{tikzcd}
W_\circ \arrow[r, "(\pi_0\pi_1,\pi_0\pi_2)"] & W \times_{\mathbb{C}_0} W & W_\circ \arrow[r, "(\pi_0\pi_1,\pi_0\pi_2)"] & W \times_{\mathbb{C}_0} W \\
\hat{U} \arrow[u, shift right, "\omega_{1,0}"] \arrow[u, shift left, "\omega_{0,0}"'] \arrow[r, "\hat{u}_0", "/" marker] \arrow[d, shift left=3] \arrow[d, shift left=1] \arrow[d, shift right=1] \arrow[d, shift right=3] & \hat{U}_1 \arrow[u, shift right, "\omega'_{1,0}"] \arrow[u, shift left, "\omega'_{0,0}"'] \arrow[r, "\hat{u}_1", "/" marker] \arrow[d, shift right, "\omega_{1,1}"'] \arrow[d, shift left, "\omega_{0,1}"] & \hat{U}_2 \arrow[u, shift right, "\omega_{1,2}"] \arrow[u, shift left, "\omega_{0,2}"'] \arrow[r, "\hat{u}_2", "/" marker] \arrow[d, shift right, "\omega'_{1,1}"'] \arrow[d, shift left, "\omega'_{0,1}"] & \hat{U}_3 \arrow[u, shift right, "\omega'_{1,2}"] \arrow[u, shift left, "\omega'_{0,2}"'] \\
\mathrm{sb} & W_\circ \arrow[r, "(\pi_0\pi_1,\pi_0\pi_2)"'] & W \times_{\mathbb{C}_0} W &
\end{tikzcd}$$

$(\star\star\star)$

The diagrams above will allow us to extract four sailboats that relate the three spans above through two new intermediate spans, $\sigma_{0,\gamma}$ and $\sigma_{1,\gamma}$. All of these will be defined after we justify the maps in Diagrams $(\star\star)$ and $(\star\star\star)$ above. The maps, λ_0 and λ_1 are induced by maps λ'_0 and λ'_1 which pick out two parallel pairs of arrows in $\mathbb{C}$ along with a post-composable arrow in W that coequalizes them (in $\mathbb{C}$). The following figure serves as a guide to defining these.



(B)

The explicit definitions are obtained using the universal property of the equalizer $\mathcal{P}_{cq}(\mathbb{C})$. This is done similarly as in Lemma 37, by specifying two maps (on the left below) that also equalize the parallel pair on the right below.

$$\begin{tikzcd}
\hat{U}_5 \arrow[r, shift left, "\lambda'_1"] \arrow[r, shift right, "\lambda'_0"'] & P(\mathbb{C}) \times_{\mathbb{C}_0} W \arrow[r, shift left, "(\pi_0\pi_0,\pi_1 w)c"] \arrow[r, shift right, "(\pi_0\pi_1,\pi_1 w)c"'] & \mathbb{C}_1
\end{tikzcd}$$

The maps λ_0' and λ_1' are uniquely determined in a similar fashion to δ_0' in Lemma 37, namely by descending through the covers above and expanding both sides of the Ore-square equations witnessed. The calculations are lengthy and technical and can be found in Lemma 93 of Section B.1. The proof shows that the map λ_0' is uniquely determined by the projections

$$\begin{aligned}
\lambda_0'\pi_1 &= \hat{u}_5\tilde{u}\pi_0\pi_0p_0^2\pi_0\pi_0\\
\lambda_0'\pi_0\pi_0 &= (\theta_{\gamma_0}\pi_0\pi_0w\ ,\ \hat{u}_5\tilde{u}\pi_0\pi_1\omega\pi_0\pi_0\ ,\ \hat{u}_5\tilde{u}\pi_0\pi_1u_0\theta\pi_0\pi_0w)c\\
\lambda_0'\pi_0\pi_1 &= (\theta_{\gamma_0}\pi_1\pi_0w\ ,\ \hat{u}_5\omega_\gamma\pi_0\pi_0\ ,\ \hat{u}_5\tilde{u}'\theta_\gamma\pi_0\pi_0w)c
\end{aligned}$$

and that the equalizer diagram

$$\begin{tikzcd}
\mathcal{P}_{cq} \arrow[r, tail, "\iota_{cq}"] & P(\mathbb{C})\ {}_t\times_{ws} W \arrow[r, shift left, "(\pi_0\pi_0,\pi_1w)c"] \arrow[r, shift right, "(\pi_0\pi_1,\pi_1w)c"'] & \mathbb{C}_1 \\
\hat{U}_5 \arrow[u, "\lambda_0"] \arrow[ur, "\lambda_0'"'] & &
\end{tikzcd}$$

commutes in $\mathcal{E}$. The map λ_1', inducing λ_1 in Diagram ($\star\star$), such that the equalizer diagram

$$\begin{tikzcd}
\mathcal{P}_{cq} \arrow[r, tail, "\iota_{cq}"] & P(\mathbb{C})\ {}_t\times_{ws} W \arrow[r, shift left, "(\pi_0\pi_0,\pi_1w)c"] \arrow[r, shift right, "(\pi_0\pi_1,\pi_1w)c"'] & \mathbb{C}_1 \\
\hat{U}_5 \arrow[u, "\lambda_1"] \arrow[ur, "\lambda_1'"'] & &
\end{tikzcd}$$

commutes in $\mathcal{E}$ can be derived by a similar computation to the one in Lemma 93 in the appendix where one replaces θ_{γ_0} with θ_{γ_1} and factors through the cover $\bar{U}_1$ instead of $\bar{U}_0$ to access the arrows used to construct the span σ_1. The map λ_1' is uniquely determined by the following maps $\tilde{U}_4 \to \mathbb{C}_1$:

$$\begin{aligned}
\lambda_1'\pi_1 &= \hat{u}_5\tilde{u}\pi_0\pi_0p_0^2\pi_0\pi_0,\\
\lambda_1'\pi_0\pi_0 &= (\theta_{\gamma_1}\pi_0\pi_0w\ ,\ \hat{u}_5\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0\ ,\ \hat{u}_5\tilde{u}\pi_1\pi_1u_0\theta\pi_0\pi_0w\ ,\ \hat{u}_5\tilde{u}\bar{u}\pi_0\pi_0\pi_0\pi_0)c,\\
\lambda_1'\pi_0\pi_1 &= (\theta_{\gamma_1}\pi_1\pi_0w\ ,\ \hat{u}_5\omega_\gamma\pi_0\pi_0\ ,\ \hat{u}_5\tilde{u}'\theta_\gamma\pi_0\pi_0w)c
\end{aligned}$$

Applying **In.Frc(4)** along with Lemma 35 twice gives two covers and two lifts, one for each of λ_0 and λ_1. Since covers are stable under pullback and composition we can take a common refinement by pulling one cover back along the other and get the cover and two lifts

$$\begin{tikzcd}
\mathcal{P}(\mathbb{C}) & \\
\hat{U}_4 \arrow[u, shift right, color=violet, "\delta_{\lambda_1}"] \arrow[u, shift left, color=cyan, "\delta_{\lambda_0}"'] \arrow[r, "\hat{u}_4", "/" marking] & \hat{U}_5
\end{tikzcd}$$

in Diagram ($\star\star$).

The violet and cyan arrows in the figure below are witnessed by post-composing the maps, δ_{λ_1} and δ_{λ_0}, with the projection $\pi_0 : \mathcal{P} \to \mathcal{P}_{eq}(\mathbb{C})$. There are two pairs of parallel composites that begin at each of these arrows whose codomain is that of the vertical arrow in the second of the composable sailboats. These pairs are determined by the legs of the brown and olive spans respectively. As a consequence of commutativity of the teal and purple Ore squares along with the previous diagram extension, both parallel pairs are respectively coequalized after post-composing with the left leg of the bottom span in the second of the composable sailboats. The parallel pairs ρ_0 and ρ_1 can be seen beginning at each $\bullet$ and ending at $\blacksquare$ in the following figure with their common coequalizing arrow in W having domain $\blacksquare$:

$$\begin{tikzcd}
 & \bullet \arrow[d, color=violet, "\circ" marking] & & & & \\
 & \cdot \arrow[dl, bend right, color=brown, "\circ" marking] \arrow[dr, bend left, color=brown] & & & & \\
\cdot \arrow[r, color=teal] & \cdot \arrow[d, color=teal, "\circ" marking] \arrow[drr, color=teal] & \cdot \arrow[d, color=purple] & & & \\
 & \cdot \arrow[d] & \cdot \arrow[r, color=purple] \arrow[dl, color=purple, "\circ" marking] & \cdot \arrow[dl, "\circ" marking] \arrow[d] & & \\
 & \cdot \arrow[r] & \cdot & \blacksquare \arrow[l, "\circ" marking] & \cdot \arrow[uull, bend right, color=olive] \arrow[ddll, bend left, color=olive, "\circ" marking] & \bullet \arrow[l, color=cyan, "\circ" marking] \\
 & & \cdot \arrow[ul, color=orange, "\circ" marking] \arrow[ur, color=orange] & & & \\
 & & \cdot \arrow[u, color=orange] & & &
\end{tikzcd} \tag{C}$$

Let $\hat{u}_{i;j} = \hat{u}_i\hat{u}_{i+1}...\hat{u}_j$ for $0 \leq i < j \leq 5$ to make composition of covers a bit easier to read, where $\hat{u} = \hat{u}_{0;5}$. Internally, we use Figure (C) as a blueprint for defining the maps $\rho_0', \rho_1' : \hat{U}_4 \to P_{cq}(\mathbb{C})$ in termds of parallel pair of arrows in $\mathbb{C}$, $\hat{U}_4 \to P(\mathbb{C})$, which are coequalized (in $\mathbb{C}$) by an arrow $\hat{U}_4 \to W$:

$$\begin{tikzcd}
\hat{U}_4 \arrow[r, shift left, "\rho_1'"] \arrow[r, shift right, "\rho_0'"'] & P(\mathbb{C}) \times_{\mathbb{C}_0} W \arrow[r, shift left, "(\pi_0\pi_0{,}\pi_1 w)c"] \arrow[r, shift right, "(\pi_0\pi_1{,}\pi_1 w)c"'] & \mathbb{C}_1
\end{tikzcd}$$

completely determined by the following maps:

$$\rho_0'\pi_1 = \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_0,$$

$$\rho_0'\pi_0\pi_0 = (\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w, \hat{u}_4\omega_{\gamma_0}\pi_1\pi_0, \hat{u}_{4;5}\omega_{\gamma}\pi_0\pi_0, \hat{u}_{4;5}\tilde{u}'\theta_{\gamma}\pi_1\pi_0, \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0)c,$$

$$\rho_0'\pi_0\pi_1 = (\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w, \hat{u}_4\omega_{\gamma_0}\pi_0\pi_0, \hat{u}_{4;5}\tilde{u}\pi_0\pi_1\omega\pi_0\pi_0, \hat{u}_{4;5}\tilde{u}\pi_0\pi_1 u_0\theta\pi_1\pi_0)c$$

Another lengthy but straight forward computation that comes down to the definition of $\delta_{\lambda_0}\iota_{eq} : \hat{U}_4 \to \mathcal{P}_{eq}(\mathbb{C})$ and the Ore-square can be found in Lemma 94 of the appendix and shows that

$$(\rho_0'\pi_0\pi_0, \rho_0'\pi_1)c = (\rho_0'\pi_0\pi_1, \rho_0'\pi_1)c$$

implying the equalizer diagram

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{C}) \arrow[r, tail, "\iota_{cq}"] & P(\mathbb{C}) \ {}_t\times_{ws} W \arrow[r, shift left, "(\pi_0\pi_0{,}\pi_1 w)c"] \arrow[r, shift right, "(\pi_0\pi_1{,}\pi_1 w)c"'] & \mathbb{C}_1 \\
\tilde{U}_3 \arrow[u, dotted, "\rho_0"] \arrow[ur, "\rho_0'"'] & &
\end{tikzcd}$$

commutes in $\mathcal{E}$. The map inducing ρ_1 is a map $\rho_1' : \hat{U}'' \to P_{ceq}(\mathbb{C})$ similarly defined but replacing δ_{λ_0} with δ_{λ_1} and θ_{γ_0} with θ_{γ_1}. It is uniquely determined by

$$\rho_1'\pi_1 = \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_0,$$

$$\begin{aligned}\rho_1'\pi_0\pi_0 = (&\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\omega_{\gamma_1}\pi_1\pi_0,\\ &\hat{u}_{4;5}\omega_{\gamma}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_{\gamma}\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0)c,\end{aligned}$$

$$\rho_1'\pi_0\pi_1 = (\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_0\pi_0,$$
$$\hat{u}_{4;5}\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1 u_0\theta\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0)c$$

and a similar lengthy but straightforward computation found in Lemma 95 of the appendix shows that

$$(\rho_1'\pi_0\pi_0, \rho_1'\pi_1)c = (\rho_1'\pi_0\pi_1, \rho_1'\pi_1)c.$$

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{C}) \arrow[r, tail, "\iota_{cq}"] & P(\mathbb{C})\ {}_{t}\times_{ws} W \arrow[r, shift left, "(\pi_0\pi_0,\pi_1 w)c"] \arrow[r, shift right, "(\pi_0\pi_1,\pi_1 w)c"'] & \mathbb{C}_1 \\
\tilde{U}_3 \arrow[u, dotted, "\rho_1"] \arrow[ur, "\rho_1"'] & &
\end{tikzcd}$$

Applying **In.Frc(4)** along with Lemma 35 twice gives two covers and two lifts, one for each of ρ_0 and ρ_1. A common refinement given by pulling one cover back along the other gives the cover and two lifts

$$\begin{tikzcd}
\hat{U}_3 \arrow[r, "\hat{u}_3", "/" marking] \arrow[d, shift right, "\delta_{\rho_1}"'] \arrow[d, shift left, "\delta_{\rho_0}"] & \hat{U}_4 \\
\mathcal{P}(\mathbb{C}) &
\end{tikzcd}$$

in Diagram ($\star\star$). It remains to define the 'weakly-composable maps' being picked out by $\omega'_{i,j}$ for $i = 0, 1$ and $j = 0, 1, 2$ in Diagram ($\star\star\star$). First we have maps $\hat{U}_3 \to W\ {}_{wt}\times_{ws} W$ given by

$$\omega'_{0,2} = (\delta_{\rho_0}\pi_0\iota_{eq}\pi_0, \hat{u}_3\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0) \qquad \omega'_{1,2} = (\delta_{\rho_1}\pi_0\iota_{eq}\pi_0, \hat{u}_3\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0)$$

and by applying **In.Frc(2)** and Lemma 35 twice and taking a common refinement of covers we get the cover and lift

$$\begin{tikzcd}
W_\circ & \\
\hat{U}_2 \arrow[u, shift right, "\omega_{1,2}"] \arrow[u, shift left, "\omega_{0,2}"'] \arrow[r, "\hat{u}_2", "/" marking] & \hat{U}_3
\end{tikzcd}$$

in Diagram ($\star\star\star$). Next we have maps $\hat{U}_2 \to W\ {}_{wt}\times_{ws} W$ given by

$$
\begin{aligned}
\omega'_{0,1} &= (\omega_{0,2}\pi_1, \hat{u}_{2;4}\theta_{\gamma_0}\pi_0\pi_0) \\
&= \big((\omega_{0,2}\pi_0\pi_0, \hat{u}_2\omega'_{0,2}c)c, \hat{u}_{2;4}\theta_{\gamma_0}\pi_0\pi_0\big) \\
&= \big((\omega_{0,2}\pi_0\pi_0, \hat{u}_2\delta_{\rho_0}\pi_0\iota_{eq}\pi_0, \hat{u}_{2;3}\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0)c,\ \hat{u}_{2;4}\theta_{\gamma_0}\pi_0\pi_0\big)
\end{aligned}
$$

and

$$
\begin{aligned}
\omega'_{1,1} &= (\omega_{1,2}\pi_1, \hat{u}_{2;4}\theta_{\gamma_1}\pi_0\pi_0) \\
&= \big((\omega_{1,2}\pi_0\pi_0, \hat{u}_2\omega'_{1,2}c)c\ ,\ \hat{u}_{2;4}\theta_{\gamma_1}\pi_0\pi_0\big) \\
&= \big((\omega_{1,2}\pi_0\pi_0, \hat{u}_2\delta_{\rho_1}\pi_0\iota_{eq}\pi_0, \hat{u}_{2;3}\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0)c,\ \hat{u}_{2;4}\theta_{\gamma_1}\pi_0\pi_0\big)
\end{aligned}
$$

that, by applying **In.Frc(2)** and Lemma 35 twice and taking a common refinement of covers, gives the cover and lifts

$$\begin{tikzcd}
\hat{U}_1 \arrow[r, "\hat{u}_1", "/" marking] \arrow[d, shift right, "\omega_{1,1}"'] \arrow[d, shift left, "\omega_{0,1}"] & \hat{U}_2 \\
W_\circ &
\end{tikzcd}$$

in Diagram ($\star\star\star$). Finally, we have maps $\hat{U}_1 \to W\ {}_{wt}\!\times_{ws} W$ given by

$$
\begin{aligned}
\omega'_{0,0} &= (\omega_{0,1}\pi_1, \hat{u}_{1;5}\sigma_0\pi_0\pi_0) \\
&= \big((\omega_{0,1}\pi_0\pi_0, \hat{u}_1\omega'_{0,1}c)c,\ \hat{u}_{1;5}\sigma_0\pi_0\pi_0\big) \\
&= \big((\omega_{0,1}\pi_0\pi_0, \hat{u}_1\omega_{0,2}\pi_0\pi_0, \hat{u}_{1;2}\delta_{\rho_0}\pi_0\iota_{eq}\pi_0, \hat{u}_{1;3}\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0, \hat{u}_{1;4}\theta_{\gamma_0}\pi_0\pi_0)c, \\
&\quad \hat{u}_{1;5}\sigma_0\pi_0\pi_0\big)
\end{aligned}
$$

and

$$
\begin{aligned}
\omega'_{1,0} &= (\omega_{1,1}\pi_1, \hat{u}_{1;5}\sigma_1\pi_0) \\
&= \big((\omega_{1,1}\pi_0\pi_0, \hat{u}_1\omega'_{1,1}c)c,\ \hat{u}_{1;5}\sigma_1\pi_0\big) \\
&= \big((\omega_{1,1}\pi_0\pi_0, \hat{u}_1\omega_{1,2}\pi_0\pi_0, \hat{u}_{1;2}\delta_{\rho_1}\pi_0\iota_{eq}\pi_0, \hat{u}_{1;3}\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0, \hat{u}_{1;4}\theta_{\gamma_1}\pi_0\pi_0)c, \\
&\quad \hat{u}_{1;5}\sigma_1\pi_0\big)
\end{aligned}
$$

that, by applying **In.Frc(2)** and Lemma 35 twice and taking a common refinement of covers, gives the cover and lifts

$$\begin{tikzcd}
W_\circ & \\
\hat{U} \arrow[u, shift right, "\omega_{1,0}"] \arrow[u, shift left, "\omega_{0,0}"'] \arrow[r, "\hat{u}_0", "/" marking] & \hat{U}_1
\end{tikzcd}$$

in Diagram $(\star\star\star)$. The object $\hat{U}$ witnesses five spans, $\hat{U} \to \mathrm{spn}$ related by four sailboats, $\hat{U} \to \mathrm{sb}$, via the covers, $\hat{u}_{0;j} : \hat{U} \to \hat{U}_{j+1}$, and lifts in Diagrams $(\star)$, $(\star\star)$, and $(\star\star\star)$, and the covers and projections in Diagrams (1) and 2). The original three spans $\hat{u}\sigma_0, \hat{u}\sigma_\gamma$, and $\hat{u}\sigma_1$ are immediate; and two intermediate spans, $\sigma_{0,\gamma}$ and $\sigma_{1,\gamma}$, defined in technical Lemmas 96 and 97 in Section B.1 of the appendix. Lemma 98 in the same section shows that the sailboat $\varphi_0 : \hat{U} \to \mathrm{sb}$, defined by the pairing map

$$\varphi_0 = \big(((\mu_0,\ \hat{u}\sigma_0\pi_0),\ \omega_{0,0}\pi_1),\ \hat{u}\sigma_0\pi_1\big)$$

is well-defined, where

$$\mu_0 = (\omega_0,\ \hat{u}_{0;4}\theta_{\gamma_0}\pi_0\pi_0)c.$$

The same lemma shows that $\varphi_0 : \hat{U} \to \mathrm{sb}$ relates the spans $\hat{u}\sigma_0, \sigma_{0,\gamma} : \hat{U} \to \mathrm{spn}$ in the sense that

$$\begin{aligned} \varphi_0 p_0 &= \varphi_0(\pi_0\pi_0\pi_1, \pi_1) \\ &= (\hat{u}\sigma_0\pi_0, \hat{u}\sigma_0\pi_1)c \\ &= \hat{u}\sigma_0 \end{aligned} \qquad \begin{aligned} \varphi_0 p_1 &= \varphi_0(\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c) \\ &= (\omega_{0,0}\pi_1, \sigma_{0,\gamma}\pi_1) \\ &= \sigma_{0,\gamma}, \end{aligned}$$

and so

$$\hat{u}\sigma_0 q = \varphi_0 p_0 q = \varphi_0 p_1 q = \sigma_{0,\gamma} q. \tag{4.7}$$

Lemma 99 of Section B.1 in the appendix similarly shows that the sailboat, $\varphi_{0,\gamma} : \hat{U} \to \mathrm{sb}$, defined by

$$\varphi_{0,\gamma} = \big(((\mu_{0,\gamma},\ \hat{u}\sigma_\gamma\pi_0),\ \omega_{0,0}\pi_1),\ \hat{u}\sigma_\gamma\pi_1\big)$$

where

$$\mu_{0,\gamma} = (\omega_0,\ \hat{u}_{0;4}\theta_{\gamma_0}\pi_1\pi_0)c$$

is well-defined and relates σ_γ to $\sigma_{0,\gamma}$ in the sense that

$$\varphi_{0,\gamma}p_0 = \hat{u}\sigma_\gamma \qquad\qquad \varphi_{0,\gamma}p_1 = \sigma_{0,\gamma}.$$

This implies

$$\sigma_{0,\gamma}q = \varphi_{0,\gamma}p_1q = \varphi_{0,\gamma}p_0q = \hat{u}\sigma_\gamma q. \tag{4.8}$$

By Lemma 100, the sailboat, $\varphi_1 : \hat{U} \to \text{sb}$, defined by

$$\varphi_1 = \big(((\mu_1,\ \hat{u}\sigma_1\pi_0),\ \omega_{1,0}\pi_1),\ \hat{u}\sigma_1\pi_1\big)$$

where

$$\mu_1 = (\omega_1, \hat{u}_{0;4}\gamma_1\pi_0\pi_0)c$$

is well-defined and relates the spans $\sigma_1, \sigma_{1,\gamma} : \hat{U} \to \text{spn}$ in the sense that

$$\varphi_1p_0 = \hat{u}\sigma_1 \qquad\qquad \varphi_1p_1 = \sigma_{1,\gamma}$$

This implies

$$\hat{u}\sigma_1q = \varphi_1p_0q = \varphi_1p_1q = \sigma_{1,\gamma}q \tag{4.9}$$

By Lemma 101, the sailboat, $\varphi_{1,\gamma} : \hat{U} \to \text{sb}$, defined by

$$\varphi_{1,\gamma} = \big(((\mu_{1,\gamma},\ \hat{u}\sigma_\gamma\pi_0),\ \omega_{1,0}\pi_1),\ \hat{u}\sigma_\gamma\pi_1\big)$$

where

$$\mu_{1,\gamma} = (\omega_1,\ \hat{u}_{0;4}\theta_{\gamma_1}\pi_1\pi_0)c$$

relates the spans $\sigma_\gamma, \sigma_{1,\gamma} : \hat{U} \to \mathrm{spn}$ in the sense that

$$\varphi_{1,\gamma} p_0 = \hat{u}\sigma_\gamma \qquad\qquad \varphi_{1,\gamma} p_1 = \sigma_{1,\gamma}.$$

It follows that

$$\sigma_{1,\gamma} q = \varphi_{1,\gamma} q = \varphi_{1,\gamma} p_0 q = \sigma_\gamma q. \tag{4.10}$$

Equations (4.7), (4.8), (4.9), and (4.10) imply

$$\begin{aligned}
\hat{u}\sigma_0 q &= \varphi_0 p_0 q \\
&= \varphi_0 p_1 q \\
&= \sigma_{0,\gamma} q \\
&= \varphi_{0,\gamma} p_1 q \\
&= \varphi_{0,\gamma} p_0 q \\
&= \hat{u}\sigma_\gamma q \\
&= \varphi_{1,\gamma} p_0 q \\
&= \varphi_{1,\gamma} p_1 q \\
&= \sigma_{1,\gamma} q \\
&= \varphi_1 p_1 q \\
&= \varphi_1 p_0 q \\
&= \hat{u}\sigma_1 q.
\end{aligned}$$

By the definitions of $c' : \mathrm{spn}^2 \to \mathbb{C}[W^{-1}]_1$ in Lemma (37) and the spans $\sigma_1, \sigma_0 : \tilde{U} \to \mathrm{spn}$ in (4.5)and (4.6) above along with commutativity of Diagrams (1), (2), (3), and ($\star$) we can see

$$\begin{aligned}
\hat{u}\tilde{u}\bar{u}p_1^2c' &= \hat{u}\tilde{u}\pi_1\pi_0p_1^2c' \\
&= \hat{u}\tilde{u}\pi_1\pi_1uc' \\
&= \hat{u}\tilde{u}\pi_1\pi_1\sigma_\circ q \\
&= \hat{u}\sigma_1 q \\
&= \hat{u}\sigma_0 q \\
&= \hat{u}\tilde{u}\pi_0\pi_1\sigma_\circ q \\
&= \hat{u}\tilde{u}\pi_0\pi_1uc' \\
&= \hat{u}\tilde{u}\pi_0\pi_0p_0^2c' \\
&= \hat{u}\tilde{u}\bar{u}p_0^2c'.
\end{aligned}$$

where $\hat{u}\tilde{u}\bar{u} : \hat{U} \to \mathrm{sb}\ {}_t{\times}_s\ \mathrm{sb}$ is a cover because covers are closed under composition. The result follows by renaming the $\varphi_i : \hat{U} \to \mathrm{sb}$ for $0 \leq i \leq 3$ to match with $\varphi_0, \varphi_{0,\gamma}, \varphi_{1,\gamma}$ and φ_1 (in that order). The cover $\hat{u} : \hat{U} \to \mathrm{sb}\ {}_t{\times}_s\ \mathrm{sb}$ in the statement of the Lemma corresponds to the composite $\hat{u}\tilde{u}\bar{u}$ mentioned above and constructed in the proof. □

4.4 Associativity and Identity Laws

This section consists of technical proofs of associativity and identity laws for composition that are required to see that $\mathbb{C}[W^{-1}]$, as defined in Section 4.3, is an internal category in $\mathcal{E}$.

4.4.1 Associativity

The proof for associativity of composition in the internal category of fractions is rather involving so we have given it its own subsection. Before we prove associativity we give a remark about induced projection maps for the quotient objects of the reflexive graphs of fractions and prove a Lemma to give explicit descriptions of the two possible compositions

$$1 \times c, c \times 1 : \mathbb{C}[W^{-1}]_3 \to \mathbb{C}[W^{-1}]_2$$

in terms of the representative composition, $c' : \mathrm{spn} \to \mathbb{C}[W^{-1}]_1$, and the first quotient map $q : \mathrm{spn} \to \mathbb{C}[W^{-1}]_1$. We use these to differentiate between which maps are being composed first for a triple composite in $\mathbb{C}[W^{-1}]$ and then prove that they are equal using the universal property of the coequalizer $\mathbb{C}[W^{-1}]_3$.

Remark 41. By definition, the canonical pullback projections commute with the coequalizer diagram maps:

$$\begin{tikzcd}
\mathrm{sb}^k \arrow[r, shift left, "p_0^k"] \arrow[r, shift right, "p_1^k"'] \arrow[d, "\pi_{i_0,\dots,i_j}"'] & \mathrm{spn}^k \arrow[r, two heads, "q_k"] \arrow[d, "\pi_{i_0,\dots,i_j}"] & \mathbb{C}[W^{-1}]_k \arrow[d, "\pi_{i_0,\dots,i_j}"] \\
\mathrm{sb}^\ell \arrow[r, shift left, "p_0^\ell"] \arrow[r, shift right, "p_1^\ell"'] & \mathrm{spn}^\ell \arrow[r, two heads, "q_\ell"'] & \mathbb{C}[W^{-1}]_\ell
\end{tikzcd} .$$

Before we can prove associativity we need to define the maps that show up in the statement. We use Proposition 32 and the universal property of the coequalizer to do this.

Lemma 42. *Let* $c \times 1 = (\pi_{01}c, \pi_2)$ *and* $1 \times c = (\pi_0, \pi_{12}c)$ *denote the pairing maps* $\mathbb{C}[W^{-1}]_3 \to \mathbb{C}[W^{-1}]_2$, *and let*

$$q \times c' = (\pi_0 q,\pi_{12} c') \quad \textit{and} \quad c' \times q = (\pi_{01} c',\pi_2 q).$$

The diagram

$$\begin{tikzcd}
 & spn^3 \arrow[dl, "q\times c'"'] \arrow[d, two heads, "q_3"] \arrow[dr, "c'\times q"] & \\
\mathbb{C}[W^{-1}]_2 & \mathbb{C}[W^{-1}]_3 \arrow[l, dotted, "1\times c"] \arrow[r, dotted, "c\times 1"'] & \mathbb{C}[W^{-1}]_2
\end{tikzcd}$$

commutes in $\mathcal{E}$.

Proof. On the right we have

$$q_3(c \times 1) = (q_3\pi_{01}c, q_3\pi_2) = (\pi_{01}q_2c, \pi_2 q) = (\pi_{01}c', \pi_2 q) = c' \times q$$

and on the left we have

$$q_3(1 \times c) = (q_3\pi_0, q_3\pi_{12}c) = (\pi_0 q, \pi_{12}q_2c) = (\pi_0 q, \pi_{12}c') = q \times c'.$$

□

Now we can state and prove the associativity law. This proof is long and technical but follows a similar pattern to the proofs in Section 4.3. Recall that diagrams labeled with capital letters are guides for the usual proofs when $\mathcal{E} = \mathbf{Set}$ and represent diagrams in the internal category $\mathbb{C}$, with diagrams labeled with stars giving the internal translation involving covers and lifts from the Internal Fractions Axioms.

Proposition 43. *The diagram*

$$\begin{tikzcd}
\mathbb{C}[W^{-1}]_3 \arrow[r, "c\times 1"] \arrow[d, "1\times c"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}[W^{-1}]_2 \arrow[d, "c"] \\
\mathbb{C}[W^{-1}]_2 \arrow[r, "c"'] & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

commutes in $\mathcal{E}$.

Proof. The plan is to show there exists a cover $\hat{U} \to \mathrm{spn}^3$ with two sailboats, $\varphi_i : \hat{U} \to \mathrm{sb}$, with a common sail-projection, $\varphi_0 p_1 = \varphi_1 p_1$, so that

$$\hat{u}q_3(1\times c)c = \varphi_0 p_0 q = \varphi_0 p_1 q = \varphi_1 p_1 q = \varphi_1 p_0 q = \hat{u}q_3(c\times 1)c.$$

The result then follows from the fact that $\hat{u}$ and q_3 are epic. First we find representative spans for the equivalence classes of spans being picked out by $(c\times 1)c$ and $(1\times c)c$, then we build a comparison span and two sailboats witnessing their equivalence.

Begin by taking pullbacks of the projections, $\pi_{01}, \pi_{12} : \mathrm{spn}^3 \to \mathrm{spn}^2$, along the cover, $u : U \to \mathrm{spn}^2$, that witnesses the span composition construction

$$\begin{tikzcd}
\tilde{U}_0 \arrow[r, "\tilde{\pi}_{01}"] \arrow[d, "\tilde{u}_{0:0}"', "\shortmid" marking] \arrow[dd, bend right=60, "\tilde{u}_0"', "\shortmid" marking] & U \arrow[d, "u_0", "\shortmid" marking] & \tilde{U}_1 \arrow[l, "\tilde{\pi}_{12}"'] \arrow[d, "\tilde{u}_{0:1}", "\shortmid" marking] \arrow[dd, bend left=60, "\tilde{u}_1", "\shortmid" marking] \\
\tilde{U}_{0:0} \arrow[r, "\tilde{\pi}_{0:01}"] \arrow[d, "\tilde{u}_{1:0}"', "\shortmid" marking] & U_0 \arrow[d, "u_1", "\shortmid" marking] & \tilde{U}_{0:1} \arrow[l, "\tilde{\pi}_{0:12}"'] \arrow[d, "\tilde{u}_{1:1}", "\shortmid" marking] \\
\mathrm{spn}^3 \arrow[r, "\pi_{01}"'] & \mathrm{spn}^2 & \mathrm{spn}^3 \arrow[l, "\pi_{12}"]
\end{tikzcd} \tag{1}$$

and since the outer squares are pullbacks, as seen in [8] we also have

$$\begin{tikzcd}
\tilde{U}_0 \arrow[r, "\tilde{\pi}_{01}"] \arrow[d, "\tilde{u}_0"', "\shortmid" marking] & U \arrow[d, "u", "\shortmid" marking] & \tilde{U}_1 \arrow[l, "\tilde{\pi}_{12}"'] \arrow[d, "\tilde{u}_1", "\shortmid" marking] \\
\mathrm{spn}^3 \arrow[r, "\pi_{01}"'] & \mathrm{spn}^2 & \mathrm{spn}^3 \arrow[l, "\pi_{12}"]
\end{tikzcd} \tag{2}$$

and then taking a common refinement

$$\begin{tikzcd}
\tilde{U} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, "\tilde{u}", "/" marking] & \tilde{U}_1 \arrow[d, "\tilde{u}_1", "/" marking] \\
\tilde{U}_0 \arrow[r, "\tilde{u}_0"', "/" marking] & \mathrm{spn}^3
\end{tikzcd} . \tag{3}$$

This induces two maps, $\sigma_c \times 1$ and $1 \times \sigma_c$, $\tilde{U} \to \mathrm{spn}^2$ defined by

$$\sigma_c \times 1 = (\pi_0 \tilde{\pi}_{01} \sigma_c, \tilde{u}\pi_2) \quad \text{and} \quad 1 \times \sigma_c = (\tilde{u}\pi_0, \pi_1 \tilde{\pi}_{12} \sigma_c).$$

Taking pullbacks of these induced maps along the composites that make up $u : U \to \mathrm{spn}^2$ once again gives

$$\begin{tikzcd}
\bar{U}_0 \arrow[r, "\bar{\pi}_{01}"] \arrow[d, "\bar{u}_{0:0}"', "/" marking] \arrow[dd, bend right=60, "\bar{u}_0"', "/" marking] & U \arrow[d, "u_0", "/" marking] & \bar{U}_1 \arrow[l, "\bar{\pi}_{12}"'] \arrow[d, "\bar{u}_{0:1}", "/" marking] \arrow[dd, bend left=60, "\bar{u}_1", "/" marking] \\
\bar{U}_{0:0} \arrow[r, "\bar{\pi}_{0:01}"] \arrow[d, "\bar{u}_{1:0}"', "/" marking] & U_0 \arrow[d, "u_1", "/" marking] & \bar{U}_{0:1} \arrow[l, "\bar{\pi}_{0:12}"'] \arrow[d, "\bar{u}_{1:1}", "/" marking] \\
\tilde{U} \arrow[r, "\sigma_c \times 1"'] & \mathrm{spn}^2 & \tilde{U} \arrow[l, "1 \times \sigma_c"]
\end{tikzcd} \tag{4}$$

and taking a common refinement of the covers on the left and right

$$\begin{tikzcd}
\bar{U} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, "\bar{u}", "/" marking] & \bar{U}_1 \arrow[d, "\bar{u}_1", "/" marking] \\
\bar{U}_0 \arrow[r, "\bar{u}_0"', "/" marking] & \tilde{U}
\end{tikzcd} , \tag{5}$$

gives a cover $\bar{\tilde{u}} = \bar{u}\tilde{u} : \bar{U} \to \mathrm{spn}^3$ that witnesses representatives for the two ways to compose a composable triple of spans. Let

$$\sigma_0 : \bar{U} \to \mathrm{spn} \quad \text{and} \quad \sigma_1 : \bar{U} \to \mathrm{spn}$$

be defined by

$$\sigma_0 = \pi_0 \bar{\pi}_{01} \sigma_c \quad \text{and} \quad \sigma_1 = \pi_1 \bar{\pi}_{12} \sigma_c.$$

To see σ_0 represents the equivalence class of $\bar{\tilde{u}} q_3 (c \times 1) c : \bar{U} \to \mathbb{C}[W^{-1}]_1$ we use the left squares in the diagrams and definitions above along with the definitions of $c' : \mathrm{spn}^2 \to \mathbb{C}[W^{-1}]_1$ and $c : \mathbb{C}[W^{-1}]_2 \to \mathbb{C}[W^{-1}]_1$ to compute

$$\begin{aligned}
\sigma_0 q &= \pi_0 \bar{\pi}_{01} \sigma_c q \\
&= \pi_0 \bar{\pi}_{01} u c' \\
&= \pi_0 \bar{u}_0 (\sigma_c \times 1) c' \\
&= \pi_0 \bar{u}_0 (\sigma_c \times 1) q c \\
&= \pi_0 \bar{u}_0 (\pi_0 \tilde{\pi}_{01} \sigma_c, \tilde{u} \pi_2) q c \\
&= \pi_0 \bar{u}_0 (\pi_0 \tilde{\pi}_{01} \sigma_c q, \tilde{u} \pi_2 q) c \\
&= \pi_0 \bar{u}_0 (\pi_0 \tilde{\pi}_{01} u c', \tilde{u} \pi_2 q) c \\
&= \pi_0 \bar{u}_0 (\pi_0 \tilde{u}_0 \pi_{01} c', \tilde{u} \pi_2 q) c \\
&= \pi_0 \bar{u}_0 (\pi_0 \tilde{u}_0 \pi_{01} c', \tilde{u} \pi_2 q) c \\
&= \bar{u} (\tilde{u} \pi_{01} c', \tilde{u} \pi_2 q) c \\
&= \bar{u} \tilde{u} (\pi_{01} c', \pi_2 q) c \\
&= \bar{\tilde{u}} (c' \times q) c \\
&= \bar{\tilde{u}} q_3 (c \times 1) c.
\end{aligned}$$

A similar computation using the right squares in the diagrams above shows the spans $\sigma_1 : \bar{U} \to \text{spn}$ represent

$$\sigma_1 q = \bar{\tilde{u}} q_3 (1 \times c) c.$$

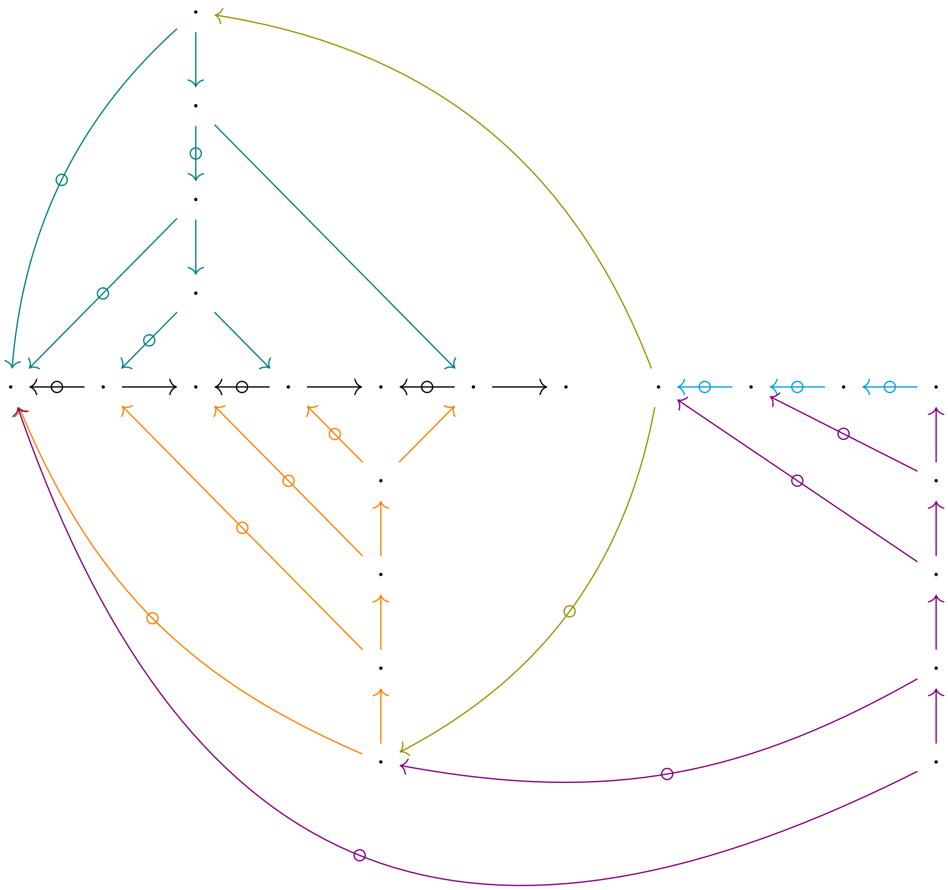
To see these representatives are equivalent we will show there exists a cover, $\hat{u} : \hat{U} \to \bar{U}$, along with two sailboats $\varphi_i : \hat{U} \to \text{sb}$ for $i = 0, 1$ such that

$$\varphi_0 p_0 = \hat{u} \sigma_0 \quad , \quad \varphi_0 p_1 = \varphi_1 p_1 \quad , \quad \text{and} \quad \varphi_1 p_0 = \hat{u} \sigma_1 .$$

.The following algorithm outlines the necessary steps for constructing the cover $\hat{u} : \hat{U} \to \bar{U}$

- Apply the Ore-condition to the cospan consisting of the left legs, $\sigma_0 \pi_0 : \bar{U} \to W$ and $\sigma_1 \pi_0 : \bar{U} \to W$
- Apply three zippers, one for each left leg of each span in the composable triple in order from initial to final.
- Apply weak-composition four times to get a span whose left leg is in W

The figure below illustrates it.



(A)

Internally taking the Ore-square and zippering three times corresponds to applying **In.Frc(3)** followed by **In.Frc(4)** four times to get the chain of covers and lifts:

$$\begin{tikzcd}
\mathcal{P}(\mathbb{C}) \arrow[r, "\pi_1"] & \mathcal{P}_{cq}(\mathbb{C}) & \mathcal{P}(\mathbb{C}) \arrow[r, "\pi_1"] & \mathcal{P}_{cq}(\mathbb{C}) & \\
\hat{U}_3 \arrow[u, dotted, "\delta_2"] \arrow[r, "\hat{u}_4", "/" marking] & \hat{U}_4 \arrow[u, "\delta_2'"'] \arrow[r, "\hat{u}_5", "/" marking] \arrow[d, dotted, "\delta_1"'] & \hat{U}_5 \arrow[u, dotted, "\delta_0"'] \arrow[r, "\hat{u}_6", "/" marking] \arrow[d, "\delta_1'"] & \hat{U}_6 \arrow[u, "\delta_0'"'] \arrow[r, "\hat{u}_7", "/" marking] \arrow[d, dotted, "\theta_a"'] & \bar{U} \arrow[d, "{(\sigma_0\pi_0 w,\sigma_1\pi_0)}"] \\
& \mathcal{P}(\mathbb{C}) \arrow[r, "\pi_1"'] & \mathcal{P}_{cq}(\mathbb{C}) & W_{\Box} \arrow[r, "{(\pi_0\pi_1,\pi_1\pi_1)}"'] & \text{csp}
\end{tikzcd} \tag{$\star$}$$

where δ_0' is induced by the map $\delta_0'' : \hat{U}_6 \to P(\mathbb{C})\,{}_{wt}\!\times_{ws} W$ that can be found by taking the equality

$$\theta_a(\pi_0\pi_0, \pi_0\pi_1)c = \theta_a(\pi_1\pi_0, \pi_1\pi_1)c$$

from the definition of $W_\square$; expanding the second components using the definitions of θ_a, $\sigma_c \times 1$, and $1 \times \sigma_c$ with the definition

$$\sigma_c = (\omega\pi_1, (\omega\pi_0\pi_0, u_0\theta\pi_1\pi_0, u\pi_1\pi_1)c) : U \to spn$$

and finding a common final arrow in W in this expansion process. This arrow is the left leg of the initial span in the original composable triple of spans and can be seen in Figure (A) above. The composite obtained from the teal upper half of the figure is longer than the other as it requires factoring through two weak-composition triangles. This is formalized by expanding

$$\begin{aligned}
\theta_a\pi_0\pi_1 &= \bar{u}_7\sigma_0\pi_0 w \\
&= \bar{u}_7\pi_0\bar{\pi}_{01}\sigma_c\pi_0 w \\
&= \bar{u}_7\pi_0\bar{\pi}_{01}\omega\pi_1 w \\
&= \bar{u}_7\pi_0\bar{\pi}_{01}(\omega\pi_0\pi_0, u_0\theta\pi_0\pi_0 w, u\pi_0\pi_0 w)c \\
&= \bar{u}_7\pi_0(\bar{\pi}_{01}\omega\pi_0\pi_0,\ \bar{\pi}_{01}u_0\theta\pi_0\pi_0 w,\ \bar{\pi}_{01}u\pi_0\pi_0 w)c \\
&= \bar{u}_7\pi_0(\bar{\pi}_{01}\omega\pi_0\pi_0,\ \bar{u}_{0:0}\bar{\pi}_{0:01}\theta\pi_0\pi_0 w,\ \bar{u}_0(\sigma_c \times 1)\pi_0\pi_0 w)c
\end{aligned} \tag{4.11}$$

and similarly

$$\begin{aligned}
\theta_a\pi_1\pi_1 &= \bar{u}_7\sigma_1\pi_0 \\
&\ \ \vdots \\
&= \bar{u}_7\pi_1\big(\bar{\pi}_{12}\omega\pi_0\pi_0,\ \bar{u}_{0:1}\bar{\pi}_{0:12}\theta\pi_0\pi_0 w,\ \bar{u}_1(1 \times \sigma_c)\pi_0\pi_0 w\big)c.
\end{aligned} \tag{4.12}$$

Now recall that

$$\sigma_c \times 1 = (\pi_0\tilde{\pi}_{01}\sigma_c, \tilde{u}\pi_2) \quad \text{and} \quad 1 \times \sigma_c = (\tilde{u}\pi_0, \pi_1\tilde{\pi}_{12}\sigma_c)$$

and we have

$$\begin{aligned}
(\sigma_c \times 1)\pi_0\pi_0 w &= \pi_0\tilde{\pi}_{01}\sigma_c\pi_0 w \\
&= \pi_0\tilde{\pi}_{01}\omega\pi_1 w \\
&= \pi_0\tilde{\pi}_{01}(\omega\pi_0\pi_0, u_0\theta\pi_0\pi_0 w, u\pi_0\pi_0 w)c
\end{aligned} \tag{4.13}$$

where the last map, $\tilde{U} \to \mathbb{C}_1$ in this composite is:

$$\begin{aligned}\pi_0\tilde{\pi}_{01}u\pi_0\pi_0 w &= \pi_0\tilde{u}_0\pi_{01}\pi_0\pi_0 w\\ &= \tilde{u}\pi_{01}\pi_0\pi_0 w\\ &= \tilde{u}\pi_0\pi_0 w\\ &= (1 \times \sigma_c)\pi_0\pi_0 w\end{aligned} \tag{4.14}$$

The definition of the cover $\bar{u}$ from Diagram (5) and equation (4.14) imply the final component of the internal composition defining $\theta_a\pi_1\pi_1$ in equation (4.12) is the final component of the internal composition defining $\theta_a\pi_0\pi_1$ in equation (4.11):

$$\begin{aligned}\hat{u}_7\pi_0\bar{u}_0\pi_0\tilde{\pi}_{01}u\pi_0\pi_0 w &= \hat{u}_7\pi_1\bar{u}_1\pi_0\tilde{\pi}_{01}u\pi_0\pi_0 w\\ &= \hat{u}_7\pi_1\bar{u}_1(1 \times \sigma_c)\pi_0\pi_0 w.\end{aligned} \tag{4.15}$$

With these calculations and the commuting diagrams defining the covers above we can see there exists a map $\delta_0'' : \hat{U}_6 \to P(\mathbb{C})\,{}_{wt}{\times}_{ws}W$, uniquely determined by the maps

$$\delta_0''\pi_1 = \hat{u}_7\bar{u}(1 \times \sigma_c)\pi_0\pi_0,$$

$$\delta_0''\pi_0\pi_0 = \big(\theta_a\pi_1\pi_0,\ \bar{u}_7\pi_1(\bar{\pi}_{12}\omega\pi_0\pi_0,\ \bar{u}_{0:1}\bar{\pi}_{0:12}\theta\pi_0\pi_0 w)c\big)c,$$

and

$$\begin{aligned}\delta_0''\pi_0\pi_1 = {}&\big(\theta_a\pi_0\pi_0,\\ &\quad\bar{u}_7\pi_0(\bar{\pi}_{01}\omega\pi_0\pi_0,\ \bar{u}_{0:0}\bar{\pi}_{0:01}\theta\pi_0\pi_0 w,\\ &\quad\bar{u}_0(\pi_0\tilde{\pi}_{01}(\omega\pi_0\pi_0, u_0\theta\pi_0\pi_0 w)c\ )c\ \big)c\\ = {}&\big(\theta_a\pi_0\pi_0,\\ &\quad\bar{u}_7\pi_0(\bar{\pi}_{01}\omega\pi_0\pi_0,\ \bar{u}_{0:0}\bar{\pi}_{0:01}\theta\pi_0\pi_0 w,\ \bar{u}_0(\sigma_c \times 1)\pi_0\pi_0 w)c\big).\end{aligned}$$

Moreover, the equations five equations above imply

$$\begin{aligned}
\delta_0''(\pi_0\pi_0,\pi_1)c = {} & \big(\theta_a\pi_1\pi_0, \\
& \quad \bar{u}_7\pi_1(\bar{\pi}_{12}\omega\pi_0\pi_0,\ \bar{u}_{0:1}\bar{\pi}_{0:12}\theta\pi_0\pi_0 w)c, \\
& \quad \hat{u}_7\bar{u}(1\times\sigma_c)\pi_0\pi_0\big)c \\
= {} & (\theta_a\pi_1\pi_0,\theta_a\pi_1\pi_1)c \\
= {} & (\theta_a\pi_0\pi_0,\theta_a\pi_0\pi_1)c \\
= {} & \big(\theta_a\pi_0\pi_0, \\
& \quad \bar{u}_7\pi_0(\bar{\pi}_{01}\omega\pi_0\pi_0,\ \bar{u}_{0:0}\bar{\pi}_{0:01}\theta\pi_0\pi_0 w,\ \bar{u}_0(\sigma_c\times 1)\pi_0\pi_0 w)c\big)c \\
= {} & \delta_0''(\pi_0\pi_1,\pi_1)c.
\end{aligned} \tag{4.16}$$

By the universal property of the equalizer,$\mathcal{P}_{cq}(\mathbb{C})$, equation (4.16) induces a unique map $\delta_0' : \hat{U}_6 \to \mathcal{P}_{cq}(\mathbb{C})$ such that the diagram

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{C}) \arrow[r, tail, "\iota_{cq}"] & P(\mathbb{C})\,{}_t\times_{ws}W \\
\hat{U}_6 \arrow[u, dotted, "\delta_0'"] \arrow[ur, "\delta_0''"']
\end{tikzcd}$$

commutes in $\mathcal{E}$. Next, to define the map $\delta_1' : \hat{U}_5 \to \mathcal{P}_{cq}(\mathbb{C})$, we start by considering the definition of the pullback, $\mathcal{P}(\mathbb{C})$, that says

$$\delta_0\pi_0\iota_{eq}(\pi_0,\pi_1\pi_0)c = \delta_0\pi_0\iota_{eq}(\pi_0,\pi_1\pi_1)c : \hat{U}_5 \to \mathbb{C}_1$$

are equal in $\mathcal{E}$ and represent that same family of arrows in $\mathbb{C}$. We can post-compose these internally to $\mathbb{C}$ with the family of arrows

$$\hat{u}_{6;7}\bar{u}\tilde{u}\pi_0\pi_1 : \hat{U}_5 \to \mathbb{C}_1$$

and after re-associating the internal composition to find each of the two Ore-squares arising at the two different covers in the following diagram.

$$\begin{tikzcd}
 & & \tilde{U}_0 \arrow[r, "\tilde{u}_{0:0}", "/" marking] & \tilde{U}_{0:0} \arrow[r, "\tilde{\pi}_{0:01}"] & U_0 \arrow[r, "\theta"] \arrow[d, "u_1", "/" marking] & W_\square \arrow[d, "\pi_0\pi_1"] \\
\hat{U}_5 \arrow[r, "\hat{u}_{6;7}", "/" marking] \arrow[urrrrr, bend left, "\square_{1,0}"] \arrow[drrrrr, bend right, "\square_{0,1}"'] & \bar{U} \arrow[r, "\bar{u}", "/" marking] \arrow[d, "\pi_1"'] & \tilde{U} \arrow[u, "\pi_0"] & & \mathrm{spn}^2 \arrow[r, "\pi_0\pi_1", "/" marking] & \mathbb{C}_1 \\
 & \bar{U}_1 \arrow[rr, "\bar{u}_{0:1}"', "/" marking] & & \bar{U}_{0:1} \arrow[r, "\bar{\pi}_{0:12}"'] & U_0 \arrow[r, "\theta"'] \arrow[u, "u_1"', "/" marking] & W_\square \arrow[u, "\pi_0\pi_1"']
\end{tikzcd} \qquad (\blacksquare)$$

This diagram commutes in the sense that the two maps on the outside, $\hat{U}_5 \to \mathbb{C}_1$, are equal. In the figure above this corresponds to the statement that the two left-most Ore-squares on the top and bottom agree on the projection, $\pi_0\pi_1 : W_\square \to \mathbb{C}_1$. Notice $\square_{1,0}$ represents one of the two Ore-squares in Diagram A of the triple composition construction on the top (in teal) while $\square_{0,1}$ represents the second of two Ore-squares in the triple composition construction on the bottom (in orange). Now compute the projection

$$\begin{aligned}
\square_{0,1}\pi_1\pi_1 &= \hat{u}_{6;7}\pi_1\bar{u}_{0:1}\bar{\pi}_{0:12}\theta\pi_1\pi_1 \\
&= \hat{u}_{6;7}\pi_1\bar{u}_{0:1}\bar{\pi}_{0:12}u_1\pi_1\pi_0 \\
&= \hat{u}_{6;7}\pi_1\bar{\pi}_{12}u_0u_1\pi_1 \\
&= \hat{u}_{6;7}\pi_1\bar{\pi}_{12}u\pi_1\pi_0 \\
&= \hat{u}_{6;7}\pi_1\bar{u}_1(1 \times \sigma_c)\pi_1\pi_0 \\
&= \hat{u}_{6;7}\bar{u}(1 \times \sigma_c)\pi_1\pi_0 \\
&= \hat{u}_{6;7}\bar{u}\pi_1\tilde{\pi}_{12}\sigma_c\pi_0 \\
&= \hat{u}_{6;7}\bar{u}\pi_1\tilde{\pi}_{12}\omega\pi_1.
\end{aligned}$$

Since $\omega\pi_1 = (\omega\pi_0\pi_0, \omega\pi_0\pi_1, \omega\pi_0\pi_2)c$ with $\omega\pi_0\pi_2 = u\pi_0\pi_0$ by definition of $\omega : U \to W_\circ$ we can expand and notice that the final map,

$$\hat{u}_{6;7}\bar{u}\pi_1\tilde{\pi}_{12}\omega\pi_0\pi_2 : \hat{U}_5 \to W$$

is equal to

$$\hat{u}_{6;7}\bar{u}\pi_1\tilde{\pi}_{12}u\pi_0\pi_0 = \square_{1,0}\pi_1\pi_1 : \hat{U}_5 \to W.$$

This induces a unique map $\delta_1'' : \hat{U}_5 \to P(\mathbb{C})\,{}_t{\times}_{ws}W$ determined by the map $\hat{U}_5 \to W$ given by

$$\delta_1''\pi_1 = \square_{1,0}\pi_1\pi_1,$$

and the map $\delta_1''\pi_0 : \hat{U}_5 \to P(\mathbb{C})$ whose left projection, $\delta_1''\pi_0\pi_0$, is the pairing map

$$\begin{aligned}
&\big(\delta_0\iota_{eq}\pi_0 w,\\
&\ \hat{u}_6\theta_a\pi_1\pi_0,\\
&\ \hat{u}_{6;7}\pi_1\bar{\pi}_{12}\omega\pi_0\pi_0,\\
&\ \hat{u}_{6;7}\pi_1\bar{\pi}_{12}u_0\theta\pi_1\pi_0,\\
&\ \hat{u}_{6;7}\bar{u}\pi_1\tilde{\pi}_{12}\omega\pi_0\pi_0,\\
&\ \hat{u}_{6;7}\bar{u}\pi_1\tilde{\pi}_{12}u_0\theta\pi_0\pi_0 w\big)c
\end{aligned}$$

and whose right projection, $\delta_1''\pi_0\pi_1$, is the pairing map

$$\begin{aligned}
&\big(\delta_0\iota_{eq}\pi_0 w,\\
&\ \hat{u}_6\theta_a\pi_0\pi_0 w,\\
&\ \hat{u}_{6;7}\pi_0\bar{\pi}_{01}\omega\pi_0\pi_0,\\
&\ \hat{u}_{6;7}\pi_0\bar{\pi}_{01}u_0\theta\pi_0\pi_0 w,\\
&\ \hat{u}_{6;7}\bar{u}\pi_1\tilde{\pi}_{01}\omega\pi_0\pi_0,\\
&\ \hat{u}_{6;7}\bar{u}\pi_0\tilde{\pi}_{01}u_0\theta\pi_1\pi_0\big)c.
\end{aligned}$$

The fact that δ_1'' satisfies the equalizer condition for $\mathcal{P}_{cq}(\mathbb{C})$, namely

$$\delta_1''(\pi_0\pi_0, \pi_1)c = \delta_1''(\pi_0\pi_1, \pi_1)c,$$

follows from the calculations above. This induces the unique map δ_1' that makes the following diagram commute:

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{C}) \arrow[r, "\iota_{cq}"] & P(\mathbb{C})\,{}_{t}{\times}_{ws}W \\
\hat{U}_5 \arrow[u, dotted, "\delta_1'"] \arrow[ur, "\delta_1''"'] &
\end{tikzcd}$$

Finally, the map $\delta_2' : \hat{U}_4 \to \mathcal{P}_{cq}(\mathbb{C})$ is similarly induced by a map $\delta_2'' : \hat{U}_4 \to P(\mathbb{C})\,{}_{t}{\times}_{ws}W$ which can be deduced expanding the right and left-hand sides of the equation

$$\delta_1 \iota_{eq}(\pi_0, \pi_1\pi_0)c = \delta_1 \iota_{eq}(\pi_0\pi_1\pi_1)c$$

whose common target is the middle object of the original composable triple of spans, post-composing with the arrow

$$\hat{u}_{5;6}\bar{u}\tilde{u}\pi_1\pi_1 : \hat{U}_4 \to \mathbb{C}_1,$$

which is final map given by applying the projection $\pi_0\pi_1 : W_\square \to \mathbb{C}_1$ of the Ore-squares

$$\begin{tikzcd}
& \tilde{U}_0 \arrow[rr, "\tilde{u}_{0:0}", "/" marking] & & \tilde{U}_{0:0} \arrow[r, "\tilde{\pi}_{0:01}"] & U_0 \arrow[r, "\theta"] \arrow[d, "u_1", "/" marking] & W_\square \arrow[d, "\pi_1\pi_1"] & \\
\hat{U}_4 \arrow[r, "\hat{u}_{4;7}", "/" marking] \arrow[urrrrr, teal, bend left=45, "\square_{1,1}"] \arrow[drrrrr, orange, bend right=45, "\square_{0,0}"'] & \bar{U} \arrow[r, "\bar{u}", "/" marking] \arrow[u, teal, "\pi_0"] & \tilde{U} \arrow[d, orange, "\pi_1"'] & & \mathrm{spn}^2 \arrow[r, "\pi_1\pi_0", "/" marking] & W & . \\
& & \bar{U}_1 \arrow[r, "\bar{u}_{0:1}"', "/" marking] & \bar{U}_{0:1} \arrow[r, "\bar{\pi}_{0:12}"'] & U_0 \arrow[r, "\theta"'] \arrow[u, "u_1", "/" marking] & W_\square \arrow[u, "\pi_1\pi_1"'] &
\end{tikzcd} \tag{$\blacksquare\blacksquare$}$$

Picking out the composable pairs in W to get a cover witnessing a family of spans whose left leg is in W is done in order from right to left in the diagram before. This is identical to how it was done in the proof of Lemma 40 from Section 4.3, except this time an extra zippering step leads to an extra composable pair in W. The chain of covers and lifts are given by applying **In.Frc(2)** four times as seen in diagrams

$$\begin{tikzcd}
W_{\circ} \arrow[r, "{(\pi_0\pi_1,\pi_0\pi_2)}"] & W\ {}_{wt}\times_{ws}W & \\
\hat{U}_1 \arrow[u, dotted, "\omega_1"] \arrow[r, "\hat{u}_2", "/" marking] & \hat{U}_2 \arrow[u, "{(\omega_0\pi_1,\hat{u}_{3;5}\delta_0\iota_{eq}\pi_0)}"'] \arrow[r, "\hat{u}_3", "/" marking] \arrow[d, dotted, "\omega_0"'] & \hat{U}_3 \arrow[d, "{(\delta_2\iota_{eq}\pi_0,\hat{u}_4\delta_1\iota_{eq}\pi_0)}"] \\
 & W_{\circ} \arrow[r, "{(\pi_0\pi_1,\pi_0\pi_2)}"'] & W\ {}_{wt}\times_{ws}W
\end{tikzcd} \tag{$\star\star$}$$

and

$$\begin{tikzcd}
W_{\circ} \arrow[r, "{(\pi_0\pi_1,\pi_0\pi_2)}"] & W\ {}_{wt}\times_{ws}W & \\
\hat{U} \arrow[u, dotted, "\omega_3"] \arrow[r, "\hat{u}_0", "/" marking] & \hat{U}_0 \arrow[u, "{(\omega_2\pi_1,\hat{u}_{1;7}\sigma_0\pi_0)}"'] \arrow[r, "\hat{u}_1", "/" marking] \arrow[d, dotted, "\omega_2"'] & \hat{U}_1 \arrow[d, "{(\omega_1\pi_1,\hat{u}_{2;6}\theta_a\pi_0\pi_0)}"] \\
 & W_{\circ} \arrow[r, "{(\pi_0\pi_1,\pi_0\pi_2)}"'] & W\ {}_{wt}\times_{ws}W
\end{tikzcd} . \tag{$\star\star\star$}$$

At this point we can define two sailboats, $\varphi_0, \varphi_1 : \hat{U} \to \mathrm{sb}$, whose deck-projections give the two composite representatives we care for,

$$\varphi_0 p_0 = \hat{u}\sigma_0 \quad , \quad \varphi_1 p_0 = \hat{u}\sigma_1$$

and whose sail-projections agree,

$$\varphi_0 p_1 = \varphi_1 p_1$$

by virtue of zippering. To define these explicitly we first expand both sides of the equation

$$\hat{u}_{0;3}\delta_2\iota_{eq}(\pi_0, \pi_1\pi_0)c = \hat{u}_{0;3}\delta_2\iota_{eq}(\pi_0, \pi_1\pi_1)c$$

into composites and post-compose both sides with the map represented by

$$\hat{u}\bar{u}\tilde{u}\pi_2\pi_1 : \hat{U} \to \mathbb{C}_1.$$

This gives two equal representations of the right leg of the intermediate span,

$$\varphi_0 p_1 \pi_1 = \varphi_1 p_1 \pi_1,$$

and by re-associating the composites in both representations we can get

$$\varphi_0 p_1 \pi_1 = (\mu_0, \hat{u}\sigma_0\pi_1)c$$

and

$$\varphi_1 p_1 \pi_1 = (\mu_1, \hat{u}\sigma_1\pi_1)c$$

for two maps, $\mu_0, \mu_1 : \hat{U} \to \mathbb{C}_1$, which represent the masts of the sailboats being picked out. This gives the maps $\hat{U} \to \mathrm{sb}$ defined by the pairing maps

$$\varphi_0 = \big(((\mu_0, \hat{u}\sigma_0),\ \omega_2\pi_1),\ \hat{u}\sigma_0\pi_1\big)$$

$$\varphi_1 = \big(((\mu_1, \hat{u}\sigma_1),\ \omega_2\pi_1),\ \hat{u}\sigma_1\pi_1\big).$$

It follows that

$$\hat{u}\sigma_0 q = \varphi_0 p_0 q = \varphi_0 p_1 q = \varphi_1 p_1 q = \varphi_1 p_0 q = \hat{u}\sigma_1 q$$

and since $\hat{u}$ is epic,

$$\sigma_0 q = \sigma_1 q.$$

□

4.4.2 Identity Laws

The only conditions left to check in order to see that $\mathbb{C}[W^{-1}]$ is an internal category are the left and right identity laws for composition. This rest of this section is dedicated precisely to this. The identity laws are typically proven, when $\mathcal{E} = \mathbf{Set}$, by looking at the composites

$$\begin{tikzcd}
 & & \cdot \arrow[d, "u"] \arrow[dddll, bend right, "y"', "\circ" marking] & & \\
 & & \cdot \arrow[dl, "v"', "\circ" marking] \arrow[dr, "h"] & & \\
 & \cdot \arrow[dl, "x"', "\circ" marking] \arrow[dr, "x"] & & \cdot \arrow[dl, "f"', "\circ" marking] \arrow[dr, "g"] & \\
\cdot & & \cdot & & \cdot
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & \cdot \arrow[d, "u"] \arrow[dddll, bend right, "y"', "\circ" marking] & & \\
 & & \cdot \arrow[dl, "v"', "\circ" marking] \arrow[dr, "h"] & & \\
 & \cdot \arrow[dl, "f"', "\circ" marking] \arrow[dr, "g"] & & \cdot \arrow[dl, "x"', "\circ" marking] \arrow[dr, "x"] & \\
\cdot & & \cdot & & \cdot
\end{tikzcd}$$

and producing the sailboats

$$\begin{tikzcd}
 & \cdot \arrow[dl, "y"', "\circ" marking] \arrow[d, "uh"] & & & & \cdot \arrow[dl, "y"', "\circ" marking] \arrow[d, "uv"] & \\
\cdot & \cdot \arrow[l, "f", "\circ" marking] \arrow[r, "g"'] & \cdot & & \cdot & \cdot \arrow[l, "f", "\circ" marking] \arrow[r, "g"'] & \cdot
\end{tikzcd}$$

that relate each composite to the span (f, g) respectively. Since proving the identity laws requires a lot of source and target maps for different objects and we have been overloading their notation, for the rest of this section we rename the source and target maps for spans and sailboats to keep our calculations somewhat more legible. Let $s', t' : \mathrm{spn} \to \mathbb{C}_0$ denote the source and target maps for spans, given by the pairing maps $s' = \pi_0 wt$ and $t' = \pi_1 t$ respectively. Also let $s'', t'' : \mathrm{sb} \to \mathbb{C}_0$ denote the source and target maps for sailboats given by the pairing maps $s'' = \pi_0\pi_1 t = \pi_0\pi_0\pi_1 t$ and $t'' = \pi_1 t$. Internalizing this will require covers that witness composition of spans along with the canonical left and right identity inclusions,

$$\mathrm{spn} \xrightarrow{(s'\sigma_\alpha,1)} \mathrm{spn} {}_{t'}\times_{s'} \mathrm{spn} \qquad\qquad \mathrm{spn} \xrightarrow{(1,t'\sigma_\alpha)} \mathrm{spn} {}_{t'}\times_{s'} \mathrm{spn}$$

and

$$\mathrm{sb} \xrightarrow{(s''\varphi_\alpha,1)} \mathrm{sb} {}_{t''}\times_{s''} \mathrm{sb} \qquad\qquad \mathrm{sb} \xrightarrow{(1,t''\varphi_\alpha)} \mathrm{sb} {}_{t''}\times_{s''} \mathrm{sb}\ ,$$

induced by $\sigma_\alpha = (\alpha, \alpha w)$, $\varphi_\alpha = \big(((\alpha swe, \alpha), \alpha), \alpha w\big)$, and the fact that α is a section of wt. The following lemma is used to define the identity inclusions $\mathbb{C}[W^{-1}]_1 \to \mathbb{C}[W^{-1}]_1 {}_t\times_s \mathbb{C}[W^{-1}]_1$ used in the identity law statement.

Lemma 44. *The diagrams*

$$\begin{tikzcd}
sb \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] \arrow[d, "{(s''\varphi_\alpha,1)}"'] & spn \arrow[r, two heads, "q"] \arrow[d, "{(s'\sigma_\alpha,1)}"] & \mathbb{C}[W^{-1}]_1 \arrow[d, dotted, "{(se,1)}"] \\
sb \;{}_{t''}\!\times_{s''} sb \arrow[r, shift left, "p_0^2"] \arrow[r, shift right, "p_1^2"'] & spn \;{}_{t'}\!\times_{s'} spn \arrow[r, "q^2"'] & \mathbb{C}[W^{-1}]_1 \;{}_{t}\!\times_{s} \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

and

$$\begin{tikzcd}
sb \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] \arrow[d, "{(1,t''\varphi_\alpha)}"'] & spn \arrow[r, two heads, "q"] \arrow[d, "{(1,t'\sigma_\alpha)}"] & \mathbb{C}[W^{-1}]_1 \arrow[d, dotted, "{(1,te)}"] \\
sb \;{}_{t''}\!\times_{s''} sb \arrow[r, shift left, "p_0^2"] \arrow[r, shift right, "p_1^2"'] & spn \;{}_{t'}\!\times_{s'} spn \arrow[r, "q^2"'] & \mathbb{C}[W^{-1}]_1 \;{}_{t}\!\times_{s} \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

commute in the sense that for $i = 0, 1$

$$(s''\varphi_\alpha, 1)p_i^2 = p_i(s'\sigma_\alpha, 1) \qquad , \qquad (t''\varphi_\alpha, 1)p_i^2 = p_i(t'\sigma_\alpha, 1),$$

which uniquely determines

$$(se, 1) \qquad \textit{and} \qquad (1, te)$$

respectively.

Proof. To see the squares on the left commute first notice that

$$p_0 s' = p_0\pi_0 wt = \pi_0\pi_0\pi_1 wt = s'' = \pi_0\pi_1 wt = p_1\pi_0 wt = p_1 s'$$

and

$$p_0 t' = p_0\pi_1 t = \pi_1 t = t'' = p_1\pi_1 t = p_1 t'.$$

Now for $i = 0, 1$ we have

$$\begin{aligned}
(s''\varphi_\alpha, 1)p_i^2 &= (s''\varphi_\alpha, 1)(\pi_0 p_i, \pi_1 p_i) \\
&= \big((s''\varphi_\alpha, 1)\pi_0 p_i,\ (s''\varphi_\alpha, 1)\pi_1 p_i\big) \\
&= \big(s''\varphi_\alpha p_i,\ p_i\big) \\
&= \big(s''(\alpha, \alpha w),\ p_i\big) \\
&= \big(s''\sigma_\alpha,\ p_i\big) \\
&= \big(p_i s'\sigma_\alpha,\ p_i\big) \\
&= p_i(s'\sigma_\alpha, 1)
\end{aligned}$$

and similarly

$$(t''\varphi_\alpha, 1)p_i^2 = p_i(t'\sigma_\alpha, 1)$$

showing that the squares on the left commute as described. Then since $p_0^2 q_2 = p_1^2 q_2$ we get

$$p_0(s'\sigma_\alpha, 1)q^2 = p_1(s'\sigma_\alpha, 1)q^2$$

and

$$p_0(1, t'\sigma_\alpha)q^2 = p_1(1, t'\sigma_\alpha)q^2$$

inducing the unique vertical maps on the right in the lemma's diagrams by the universal property of the coequalizer $\mathbb{C}[W^{-1}]_1$. Now we show these are precisely $(se, 1), (1, te) : \mathbb{C}[W^{-1}]_1 \to \mathbb{C}[W^{-1}]_1 \ {}_t\times_s \mathbb{C}[W^{-1}]_1$. Notice the outer squares of the following pullback diagrams

$$\begin{tikzcd}
\text{spn} \arrow[drr, bend left, "q"] \arrow[dr, dotted, "(s'\sigma_\alpha,1)q^2"] \arrow[ddr, bend right, "s'e"'] & & \\
& \mathbb{C}[W^{-1}]_1 \ {}_t\times_s \mathbb{C}[W^{-1}]_1 \arrow[r, "\pi_1^2"] \arrow[d, "\pi_0^2"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}[W^{-1}]_1 \arrow[d, "s"] \\
& \mathbb{C}[W^{-1}]_1 \arrow[r, "t"'] & \mathbb{C}[W^{-1}]_0
\end{tikzcd}$$

$$\begin{tikzcd}
\text{spn} \arrow[drr, bend left, "t'e"] \arrow[dr, dotted, "(1,t'\sigma_\alpha)q^2"] \arrow[ddr, bend right, "q"'] & & \\
& \mathbb{C}[W^{-1}]_1 \ {}_t\times_s \mathbb{C}[W^{-1}]_1 \arrow[r, "\pi_1^2"] \arrow[d, "\pi_0^2"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}[W^{-1}]_1 \arrow[d, "s"] \\
& \mathbb{C}[W^{-1}]_1 \arrow[r, "t"'] & \mathbb{C}[W^{-1}]_0
\end{tikzcd}$$

commute because $s' = qs$ implies

$$et = \sigma_\alpha qt = \sigma_\alpha t' = \sigma_\alpha \pi_1 t = (\alpha, \alpha w)\pi_1 t = \alpha wt = 1_{\mathbb{C}_0}$$

and similarly $t' = qt$ implies

$$es = \sigma_\alpha qs = \sigma_\alpha s' = \alpha wt = 1_{\mathbb{C}_0}.$$

The triangles in the left diagram commute because

$$(s'\sigma_\alpha,1)q^2\pi_0^2 = (s'\sigma_\alpha,1)\pi_0 q = s'\sigma_\alpha q = s'e$$

and

$$(s'\sigma_\alpha,1)q^2\pi_1^2 = (s'\sigma_\alpha,1)\pi_1 q = q$$

and a similar calculation shows the triangles on the right commute. Then we have

$$(s'\sigma_\alpha, 1)q^2 = (s'e, q) = (qse, q) = q(se, 1)$$

and

$$(1, t'\sigma_\alpha)q^2 = (q, t'e) = (q, qte) = q(1, te).$$

as required.

$\square$

Lemma 45. *The diagram*

$$\begin{tikzcd}
spn \arrow[r, "s'\sigma_\alpha{,}1)"] \arrow[dr, two heads, "q"'] & spn \, {}_{t'}\!\times_{s'} spn \arrow[d, "c'"] & spn \arrow[l, "(1{,}t'\sigma_\alpha)"'] \arrow[dl, two heads, "q"] \\
 & \mathbb{C}[W^{-1}]_1 &
\end{tikzcd}$$

commutes in $\mathcal{E}$.

Proof. We show the left triangle commutes, the argument for the right triangle is similar. Pullback u_1 along $(s'\sigma_\alpha, 1)$ and then pullback along u_0 as shown in the diagram below to obtain a cover of spn that witnesses the entire composition process of an arbitary span and a pre-composable span representing the identity in $\mathbb{C}[W^{-1}]_1$. The following diagram commutes by definition.

$$\begin{tikzcd}
U^* \arrow[r, "\pi_1"] \arrow[d, "u_0^*", "\shortmid" marking] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, bend right=60, "u^*"', "\shortmid" marking] & U \arrow[r, "\sigma_\circ"] \arrow[d, "u_0"', "\shortmid" marking] \arrow[dd, bend left=60, "u", "\shortmid" marking] & \text{spn} \arrow[dd, two heads, "q"] \\
U_0^* \arrow[r, "\pi_1"] \arrow[d, "u_1^*", "\shortmid" marking] \arrow[dr, phantom, "\lrcorner", very near start] & U_0 \arrow[d, "u_1"', "\shortmid" marking] & \\
\text{spn} \arrow[r, "(s'\sigma_\alpha{,}1)"'] & \text{spn} \, {}_{t'}\!\times_{s'} \text{spn} \arrow[r, "c"'] & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

By commutativity of the outer square above and since u^* is epic, it suffices to show

$$u^*\pi_1\sigma_\circ q = u^*q.$$

This can be done by translating the usual proof of the left identity law for span composition and defining a sailboat $\varphi : U^* \to \mathrm{sb}$ such that

$$\varphi p_0 = u^* \quad \text{and} \quad \varphi p_1 = \pi_1\sigma_\circ$$

to give

$$u^*q = \varphi p_0 q = \varphi p_1 q = \pi_1\sigma_\circ q.$$

First compute

$$\begin{aligned}\pi_1\sigma_\circ &= \pi_1\big(\omega\pi_1,\ (\omega\pi_0\pi_1, u_0\theta\pi_1\pi_0,\ u\pi_1\pi_1)c\big)\\ &= \big(\pi_1\omega\pi_1,\ (\pi_1\omega\pi_0\pi_1, \pi_1u_0\theta\pi_1\pi_0,\ \pi_1u\pi_1\pi_1)c\big)\\ &= \big(\pi_1\omega\pi_1,\ (\pi_1\omega\pi_0\pi_1, u_0^*\pi_1\theta\pi_1\pi_0,\ u^*\pi_1)c\big).\end{aligned}$$

Now notice that

$$\begin{aligned}\pi_1u\pi_0\pi_0w &= u^*(s'\sigma_\alpha, 1)\pi_0\pi_0w\\ &= u^*(s'\sigma_\alpha, 1)\pi_0\pi_0w\\ &= u^*s'\sigma_\alpha\pi_0w\\ &= u^*s'\alpha w\\ &= u^*s'\sigma_\alpha\pi_1\\ &= u_0^*u_1^*(s'\sigma_\alpha, 1)\pi_0\pi_1\\ &= u_0^*\pi_1u_1\pi_0\pi_1\\ &= u_0^*\pi_1\theta\pi_0\pi_1\end{aligned}$$

and use this in the third line of the following calculation along with the definition of $W_\Box$ (the Ore-condition) in the fifth line.

$$\begin{aligned}
\pi_1\omega\pi_1 &= (\pi_1\omega\pi_0\pi_0,\ \pi_1\omega\pi_0\pi_1 w,\ \pi_1\omega\pi_0\pi_2 w)c \\
&= (\pi_1\omega\pi_0\pi_0,\ \pi_1 u_0\theta\pi_0\pi_0 w,\ \pi_1 u\pi_0\pi_0 w)c \\
&= (\pi_1\omega\pi_0\pi_0,\ u_0^*\pi_1\theta\pi_0\pi_0 w,\ u_0^*\pi_1\theta\pi_0\pi_1)c \\
&= \big(\pi_1\omega\pi_0\pi_0,\ u_0^*\pi_1(\theta\pi_0\pi_0 w,\ \theta\pi_0\pi_1)c\big)c \\
&= \big(\pi_1\omega\pi_0\pi_0,\ u_0^*\pi_1(\theta\pi_1\pi_0,\ \theta\pi_1\pi_1 w)c\big)c \\
&= (\pi_1\omega\pi_0\pi_0,\ u_0^*\pi_1\theta\pi_1\pi_0,\ u_0^*\pi_1\theta\pi_1\pi_1 w)c \\
&= (\pi_1\omega\pi_0\pi_0,\ u_0^*\pi_1\theta\pi_1\pi_0,\ u_0^*\pi_1 u_1\pi_1\pi_0 w)c \\
&= (\pi_1\omega\pi_0\pi_0,\ u_0^*\pi_1\theta\pi_1\pi_0,\ u^*(s'\sigma_\alpha,1)\pi_1\pi_0 w)c \\
&= (\pi_1\omega\pi_0\pi_0,\ u_0^*\pi_1\theta\pi_1\pi_0,\ u^*\pi_0 w)c
\end{aligned}$$

The last calculation shows that the sailboat $\varphi : U^* \to \text{sb}$ defined by

$$\varphi = \big((((\pi_1\omega\pi_0\pi_0, u_0^*\pi_1\theta\pi_1\pi_0 w)c,\ u^*\pi_0),\ \pi_1\omega\pi_1),\ u^*\pi_1\big)$$

is well-defined. Clearly we have

$$\varphi p_0 = \varphi(\pi_0\pi_0\pi_1, \pi_1) = (u^*\pi_o, u^*\pi_1) = u^*$$

and the first calculation along with associativity of composition in the last equality below shows us that

$$\begin{aligned}
\varphi p_1 &= \varphi(\pi_0\pi_1,\ (\pi_0\pi_0\pi_0, \pi_1)c) \\
&= \big(\pi_1\omega\pi_1,\ ((\pi_1\omega\pi_0\pi_0, u_0^*\pi_1\theta\pi_1\pi_0)c, u^*\pi_1\big) \\
&= \pi_1\sigma_\circ.
\end{aligned}$$

□

Proposition 46 (Identity Laws)**.** *The diagram*

$$\begin{tikzcd}
\mathbb{C}[W^{-1}]_1 \arrow[r, "(se{,}1)"] \arrow[dr, equal] & \mathbb{C}[W^{-1}]_1 {}_t\times_s \mathbb{C}[W^{-1}]_1 \arrow[d, "c"] & \mathbb{C}[W^{-1}]_1 \arrow[l, "(1{,}te)"'] \arrow[dl, equal] \\
& \mathbb{C}[W^{-1}]_1 &
\end{tikzcd}$$

commutes in $\mathcal{E}$.

Proof. By Lemma 44, the diagrams

$$\begin{tikzcd}
\mathrm{spn} \arrow[r, two heads, "q"] \arrow[d, "(s'\sigma_\alpha{,}1)"'] & \mathbb{C}[W^{-1}]_1 \arrow[d, dotted, "(se{,}1)"] \\
\mathrm{spn}\ {}_{t'}\times_{s'} \mathrm{spn} \arrow[r, "q^2"'] \arrow[dr, "c'"'] & \mathbb{C}[W^{-1}]_1\ {}_t\times_s \mathbb{C}[W^{-1}]_1 \arrow[d, dotted, "c"] \\
 & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{spn} \arrow[r, two heads, "q"] \arrow[d, "(1{,}t'\sigma_\alpha)"'] & \mathbb{C}[W^{-1}]_1 \arrow[d, dotted, "(1{,}te)"] \\
\mathrm{spn}\ {}_{t'}\times_{s'} \mathrm{spn} \arrow[r, "q^2"'] \arrow[dr, "c'"'] & \mathbb{C}[W^{-1}]_1\ {}_t\times_s \mathbb{C}[W^{-1}]_1 \arrow[d, dotted, "c"] \\
 & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

commute and by Lemma 45 the composites on the left sides are both equal to q. It follows that the right-hand sides are identities by uniqueness. ☐

4.5 The Internal Localization Funtor

In this section we define the (internal) localization functor, $L : \mathbb{C} \to \mathbb{C}[W^{-1}]$, prove it is an internal functor, define what it means for an internal functor to invert an arrow $w : W \to \mathbb{C}_1$, and then show that L inverts $w : W \to \mathbb{C}_1$.

Defining the Internal Functor

The localizing internal functor, $L : \mathbb{C} \to \mathbb{C}[W^{-1}]$, is defined on objects to be the identity map, $L_0 = 1_{\mathbb{C}_0}$, because $\mathbb{C}[W^{-1}]_0 = \mathbb{C}_0$. On arrows we use the section $\alpha : \mathbb{C}_0 \to W$ along with the source map and the identity to get a (family of) span(s) which can be mapped to $\mathbb{C}[W^{-1}]_1$ as follows.

$$\begin{tikzcd}
\mathbb{C}_1 \arrow[r, "\big(s\alpha{,}(s\alpha w{,}1)c\big)"] \arrow[dr, "L_1"'] & \mathrm{spn} \arrow[d, two heads, "q"] \\
 & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

When $\mathcal{E} = \mathbf{Set}$ this says L_1 maps an arrow $f : a \to b$ in $\mathbb{C}_1$ to the equivalence class of spans represented by the span

$$\begin{tikzcd}
a & a \arrow[l, "\alpha(a)"', "\circ" marking] \arrow[r, "\alpha(a)"', "\circ" marking] \arrow[rr, bend left, "\alpha(a)f"] & a \arrow[r, "f"'] & b
\end{tikzcd} .$$

Identities are preserved since $es = 1_{\mathbb{C}_0}$ and by the identity law, $(1, te)c = 1_{\mathbb{C}_1}$, in $\mathbb{C}$

$$\begin{aligned}
eL_1 &= e\big(s\alpha,(s\alpha w,1)c\big)q \\
&= \big(es\alpha,(es\alpha w,e)c\big)q \\
&= \big(\alpha,(\alpha w,e)c\big)q \\
&= \big(\alpha,\alpha w(1,te)c\big)q \\
&= (\alpha,\alpha w)q
\end{aligned}$$

where the last line is the identity structure map, $e = (\alpha,\alpha w)q : \mathbb{C}[W^{-1}]_0 \to \mathbb{C}[W^{-1}]_1$, for the internal category $\mathbb{C}[W^{-1}]$. This shows the diagram

$$\begin{tikzcd}
\mathbb{C}_0 \arrow[r, equal, "L_0"] \arrow[d, "e"'] & \mathbb{C}[W^{-1}]_0 \arrow[d, "e"] \\
\mathbb{C}_1 \arrow[r, "L_1"'] & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

commutes in $\mathcal{E}$ so $L = (L_0, L_1)$ preserves the identity structure. Composition is preserved in a less obvious way. We need Lemma 47 to see

$$\begin{aligned}
cL_1 &= c\big(s\alpha,(s\alpha w,1)c\big)q \\
&= \big(cs\alpha,(cs\alpha w,c)c\big)q \\
&= \big(\pi_0 s\alpha,(\pi_0 s\alpha w,c)c\big)q \\
&= \big(\pi_0(s\alpha,(s\alpha w,1)c), \pi_1(s\alpha,(s\alpha w,1)c)\big)c' \\
&= \big(\pi_0(s\alpha,(s\alpha w,1)c), \pi_1(s\alpha,(s\alpha w,1)c)\big)(q \times q)c \\
&= \big(\pi_0(s\alpha,(s\alpha w,1)c)q, \pi_1(s\alpha,(s\alpha w,1)c)q\big)c \\
&= (L_1 \times L_1)c
\end{aligned}$$

and conclude that the diagram

$$\begin{tikzcd}
\mathbb{C}_2 \arrow[r, "L_1\times L_1"] \arrow[d, "c"'] & \mathbb{C}[W^{-1}]_1^2 \arrow[d, "c"] \\
\mathbb{C}_1 \arrow[r, "L_1"'] & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

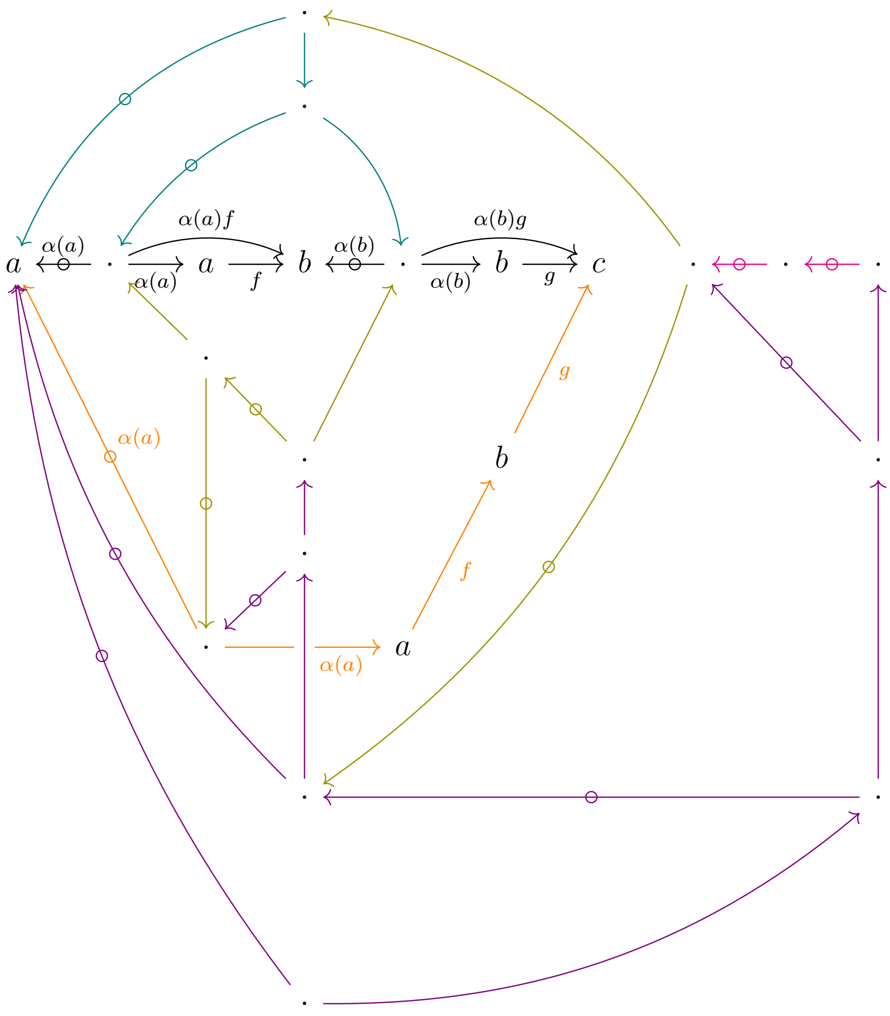
commutes in $\mathcal{E}$. It follows that $L = (L_0, L_1)$ is an internal functor.

Lemma 47. *The diagram*

$$\begin{tikzcd}
\mathbb{C}_2 \arrow[r, "\big(\pi_0(s\alpha,(s\alpha w,1)c){,}\pi_1(s\alpha,(s\alpha w,1)c)\big)"] \arrow[d, "\big(\pi_0 s\alpha{,}(\pi_0 s\alpha w{,}c)c\big)"'] & spn^2 \arrow[d, "c'"] \\
spn \arrow[r, two heads, "q"'] & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

commutes in $\mathcal{E}$.

Proof. Internalize the following figure



(A)

where olive coloured arrows are fillers of Ore-squares, violet coloured arrows W-composition fillers, and magenta coloured arrows are zippering fillers. To do this internally, start by taking the pullback

$$\begin{tikzcd}
\tilde{U} \arrow[rrr, "\pi"] \arrow[d, "\tilde{u}"', "|" marking] \arrow[drrr, phantom, very near start, "\lrcorner"] & & & U \arrow[d, "u", "|" marking] \\
\mathbb{C}_2 \arrow[rrr, "\big(\pi_0(s\alpha,(s\alpha w,1)c){,}\pi_1(s\alpha,(s\alpha w,1)c)\big)"'] & & & \mathrm{spn}^2
\end{tikzcd} \tag{0}$$

to give a cover witnessing span-composition along with the representative spans for L_1. In Figure (A), this gives access to the teal and black coloured arrows. We begin by internalizing the 'inner' part of Figure A

$$\begin{tikzcd}
a & \cdot \arrow[l, "\circ" marking, "\alpha(a)"'] \arrow[r, "\alpha(a)"'] \arrow[rr, bend left, "\alpha(a)f"] & a \arrow[r, "f"'] & b & \cdot \arrow[l, "\circ" marking, "\alpha(b)"'] \arrow[r, "\alpha(b)"'] \arrow[rr, bend left, "\alpha(b)g"] & b \arrow[r, "g"'] & c \\
& & \cdot \arrow[ul] \arrow[ddd, "\circ" marking] & & & & \\
& & & \cdot \arrow[ul, "\circ" marking] \arrow[uur] & & b \arrow[uur, "g"'] & \\
& & & \cdot \arrow[u] \arrow[dl, "\circ" marking] & & & \\
& & \cdot \arrow[uuuull, "\circ" marking, "\alpha(a)"'] \arrow[rr, "\alpha(a)"'] & & a \arrow[uur, "f"'] & & \\
& & & \cdot \arrow[uu] \arrow[uuuuulll, bend left, "\circ" marking] & & &
\end{tikzcd} \tag{B}$$

.

We can build diagram (1) below by noticing

$$\pi u\pi_0\pi_0 = \tilde{u}\pi_0 s\alpha \tag{a}$$

which means there is a unique map $(\tilde{u}\pi_0 s\alpha w, \pi u\pi_0\pi_0) : \tilde{U} \to \mathrm{csp}$. Applying the internal Ore condition, **Int.Frc.(3)**, witnesses the first (family of) Ore-square(s), $\theta_0 : \hat{U}_8 \to W_\square$, and using diagram (0) and equation (a) above we can rewrite

$$\hat{u}_9\pi u\pi_0\pi_1 = \hat{u}_9\tilde{u}\pi_0(s\alpha w, 1)c = (\hat{u}_9\tilde{u}\pi_0 s\alpha w, \hat{u}_9\tilde{u}\pi_0)c = (\theta_0\pi_1\pi_1 w, \hat{u}_9\tilde{u}\pi_0)c$$

and see its source coincides with the target of $\theta_0\pi_1\pi_0 : \hat{U}_8 \to W$ by definition of $W_\Box$. This induces the unique map

$$(\theta_0\pi_1\pi_0, \pi u\pi_0\pi_1) : \hat{U}_8 \to \mathbb{C}_2.$$

Now the target of $\pi u\pi_0\pi_1 : \tilde{U} \to \mathbb{C}_1$ is the target of $\pi u\pi_1\pi_0 : \tilde{U} \to W$ by definition of spn^2, and this gives rise to the map

$$\big((\theta_0\pi_1\pi_0, \pi u\pi_0\pi_1)c, \pi u\pi_1\pi_0\big) : \hat{U}_8 \to \mathrm{csp}$$

in diagram (1) below. Applying **Int.Frc.(3)** here witnesses the second (family of) Ore-square(s), $\theta_1 : \hat{U}_7 \to W_\Box$. The map representing pairs of composable arrows in W, that induce the cover $\tilde{u}_f$ and the lift $\omega_0 : \hat{U}_6 \to W_\circ$ by **Int.Frc.(2)** in diagram (1) below, are pretty self-explanatory. The one inducing $\omega_1 : \hat{U}_5 \to W_\circ$ can be justified by chasing through the already established parts of diagram (1) below. First notice that

$$\hat{u}_{7;8}\theta_0\pi_0\pi_1 = \hat{u}_{7;9}\tilde{u}\pi_0 s\alpha w$$

and

$$\omega_0\pi_1 = (\omega_0\pi_0\pi_0, \theta_1\pi_0\pi_0, \hat{u}_8\theta_0\pi_0\pi_0)c.$$

The target of this last composite is the target of the last arrow which is the source

$$\omega_0\pi_1 wt = \hat{u}_{7;8}\theta_0\pi_0\pi_0 wt = \hat{u}_{7;8}\theta_0\pi_0\pi_1 s = \hat{u}_{7;9}\tilde{u}\pi_0 s\alpha w s.$$

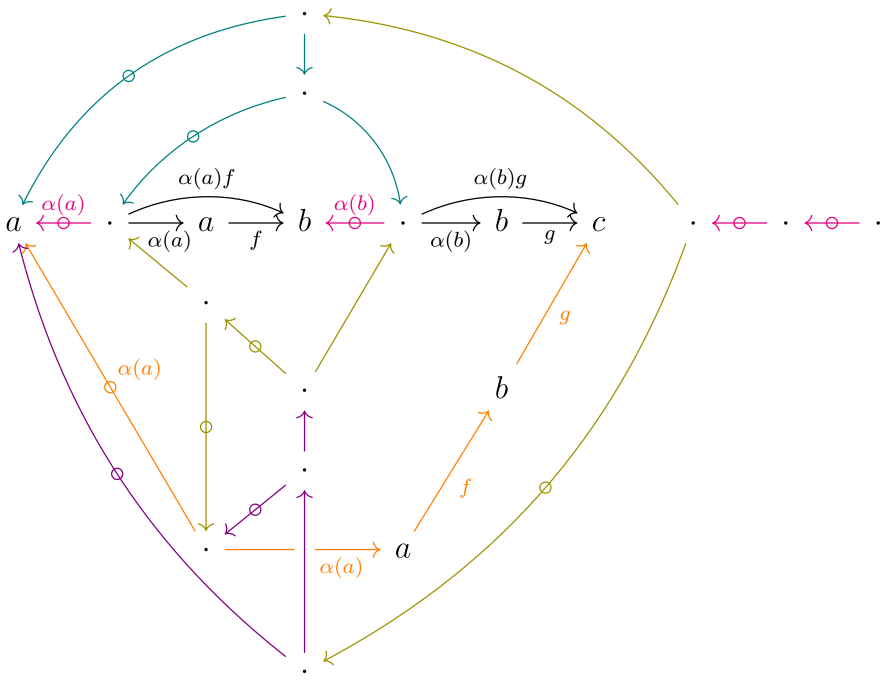
Applying **Int.Frc.(3)** induces the map $\omega_1 : \hat{u}_5 \to W_\circ$ and all together we get a commuting diagram of witnesses to the the inner part of Figure (A).

$$\begin{tikzcd}
W_{\square} \arrow[r, "{(\pi_0\pi_1,\pi_1\pi_1)}"] & \mathrm{csp} & \\
\hat{U}_7 \arrow[u, "\theta_1"] \arrow[r, "\hat{u}_8", "/" marking] & \hat{U}_8 \arrow[u, "{\big((\theta_0\pi_1\pi_0,\hat{u}_9\pi u\pi_0\pi_1)c,\hat{u}_9\pi u\pi_1\pi_0\big)}"'] \arrow[r, "\hat{u}_9", "/" marking] \arrow[d, "\theta_0"'] & \tilde{U} \arrow[d, "{(\tilde{u}\pi_0 s\alpha w,\pi u\pi_0\pi_0)}"] \\
& W_{\square} \arrow[r, "{(\pi_0\pi_1,\pi_1\pi_1)}"'] & \mathrm{csp}
\end{tikzcd}$$

(1)

$$\begin{tikzcd}
W_{\circ} \arrow[r, "{(\pi_0\pi_1,\pi_0\pi_2)}"] & W\ {}_{wt}\times_{ws} W & \\
\hat{U}_5 \arrow[u, "\omega_1"] \arrow[r, "\hat{u}_6", "/" marking] & \hat{U}_6 \arrow[u, "{(\omega_0\pi_1,\hat{u}_{7;9}\tilde{u}\pi_0 s\alpha)}"'] \arrow[r, "\hat{u}_7", "/" marking] \arrow[d, "\omega_0"'] & \hat{U}_7 \arrow[d, "{(\theta_1\pi_0\pi_0,\hat{u}_8\theta_0\pi_0\pi_0)}"'] \\
& W_{\circ} \arrow[r, "{(\pi_0\pi_1,\pi_0\pi_2)}"'] & W\ {}_{wt}\times_{ws} W
\end{tikzcd}$$

Applying **Int.Frc(3)** once followed by **Int.Frc(4)** twice gives local witnesses to the existence of the outer Ore-square and zippering arrows in magenta from Figure (A). Note that the first and second magenta arrows equalize the parallel pairs obtained by going around either side of the Ore-square and ending at the domains of $\alpha(a)$ and $\alpha(b)$ respectively.



(C)

For the additional Ore-square added in (C) recall the construction of the cover $u : u \to \mathrm{spn}^2$ and notice that target of the left leg of the composite coincides with the target of $\omega_1\pi_1 : \hat{U}_5 \to W$,

$$\begin{aligned}
\hat{u}_{6;9}\pi\sigma_\circ\pi_0 wt &= \hat{u}_{6;9}\pi\omega\pi_1 wt \\
&= \hat{u}_{6;9}\pi u\pi_0\pi_0 wt \\
&= \hat{u}_{6;9}s\alpha wt \\
&= \omega_1\pi_0\pi_2 wt \\
&= \omega_1\pi_1 wt.
\end{aligned}$$

This induces a unique map $(\omega_1\pi_1, \hat{u}_{6;9}\pi\sigma_\circ\pi_0) : \hat{U}_5 \to \mathrm{csp}$ which in turn gives a witnessing map $\theta_2 : \hat{U}_4 \to W_\Box$ in diagram (2) below. The map $\lambda' : \hat{U}_4 \to \mathcal{P}_{cq}(\mathbb{C})$ is induced by $\lambda'' : \hat{U} \to P(\mathbb{C})\,{}_{\pi_0 t}\!\times_s W$, which itself is induced by the universal property of the pullback $P(\mathbb{C})\,{}_{\pi_0 t}\!\times_s W$ and can be defined explicitly as a pairing map by expanding each side of the equality determined by commutativity of the last Ore-square. Internally this is captured by the definition of $W_\Box$ and the lift θ_2 from **Int.Frc.(3)**. On one side of the equality we have

$$\begin{aligned}
&(\theta_2\pi_0\pi_0, \theta_2\pi_0\pi_1)c \\
&= (\theta_2\pi_0\pi_0, \hat{u}_5\omega_1\pi_1)c \\
&= \big(\theta_2\pi_0\pi_0, \hat{u}_5(\omega_1\pi_0\pi_0, \hat{u}_6\omega_0\pi_1, \hat{u}_{6;9}\tilde{u}\pi_0 s\alpha w)c\big)c \\
&= \big(\theta_2\pi_0\pi_0, \hat{u}_5\omega_1\pi_0\pi_0, \hat{u}_{5;6}\omega_0\pi_1, \hat{u}_{5;9}\tilde{u}\pi_0 s\alpha w\big)c \qquad (4.17) \\
&= \big(\theta_2\pi_0\pi_0, \hat{u}_5\omega_1\pi_0\pi_0, \hat{u}_{5;6}(\omega_0\pi_0\pi_0, \hat{u}_7\theta_1\pi_0\pi_0, \hat{u}_{7;8}\theta_0\pi_0\pi_0)c, \hat{u}_{5;9}\tilde{u}\pi_0 s\alpha w\big)c \\
&= \big(\theta_2\pi_0\pi_0, \hat{u}_5\omega_1\pi_0\pi_0, \hat{u}_{5;6}\omega_0\pi_0\pi_0, \hat{u}_{5;7}\theta_1\pi_0\pi_0, \hat{u}_{5;8}(\theta_0\pi_0\pi_0 w, \hat{u}_9\tilde{u}\pi_0 s\alpha w)c\big)c \\
&= \big(\theta_2\pi_0\pi_0, \hat{u}_5\omega_1\pi_0\pi_0, \hat{u}_{5;6}\omega_0\pi_0\pi_0, \hat{u}_{5;7}\theta_1\pi_0\pi_0, \hat{u}_{5;8}\theta_0\pi_1\pi_0, \hat{u}_{5;9}\tilde{u}\pi_0 s\alpha w\big)c
\end{aligned}$$

and on the other we have

$$\begin{aligned} & (\theta_2\pi_1\pi_0, \theta_2\pi_1\pi_1 w)c \\ &= (\theta_2\pi_1\pi_0, \hat{u}_{5;9}\pi\sigma_\circ\pi_0 w)c \\ &= (\theta_2\pi_1\pi_0, \hat{u}_{5;9}\pi\omega\pi_1 w)c \\ &= \big(\theta_2\pi_1\pi_0, \hat{u}_{5;9}\pi(\omega\pi_0\pi_0, u_0\theta\pi_0\pi_0 w, u\pi_0 w)c\big)c \\ &= \big(\theta_2\pi_1\pi_0, \hat{u}_{5;9}\pi\omega\pi_0\pi_0, \hat{u}_{5;9}\pi u_0\theta\pi_0\pi_0 w, \hat{u}_{5;9}\pi u\pi_0 w\big)c \\ &= \big(\theta_2\pi_1\pi_0, \hat{u}_{5;9}\pi\omega\pi_0\pi_0, \hat{u}_{5;9}\pi u_0\theta\pi_0\pi_0 w, \hat{u}_{5;9}\tilde{u}\pi_0 s\alpha w\big)c \end{aligned} \quad . \tag{4.18}$$

Notice that the last coordinates of the internal compositions described in the last lines of equations (4.17) and (4.18) coincide. Then the map $\lambda'' : \hat{U}_4 \to P(\mathbb{C})\,{}_t{\times}_{ws}W$ is determined by the projections

$$\begin{aligned} \lambda''\pi_1 &= \hat{u}_{5;9}\tilde{u}\pi_0 s\alpha w \\ \lambda''\pi_0\pi_0 &= (\theta_2\pi_0\pi_0 w, \hat{u}_5\omega_1\pi_0\pi_0, \hat{u}_{5;6}\omega_0\pi_0\pi_0, \hat{u}_{5;7}\theta_1\pi_0\pi_0, \hat{u}_{5;8}\theta_0\pi_1\pi_0)c \\ \lambda''\pi_0\pi_1 &= (\theta_2\pi_1\pi_0, \hat{u}_{5;9}\pi\omega\pi_0\pi_0, \hat{u}_{5;9}\pi u_0\theta\pi_0\pi_0 w)c. \end{aligned}$$

The left-hand sides of equations (4.17) and (4.18) are equal by definition of $W_\Box$ and this induces the unique map $\lambda' : \hat{U}_4 \to \mathcal{P}_{cq}(\mathbb{C})$ such that the triangle

$$\begin{tikzcd} \mathcal{P}_{cq}(\mathbb{C}) \arrow[r, tail, "\iota_{cq}"] & P(\mathbb{C})\,{}_t{\times}_{ws}W \\ \hat{U}_4 \arrow[u, dotted, "\lambda'"] \arrow[ur, "\lambda"'] & \end{tikzcd}$$

commutes by the universal property of the equalizer $\mathcal{P}_{cq}(\mathbb{C})$. By definition of the pullback $\mathcal{P}(\mathbb{C})$ we have

$$\lambda\pi_0\iota_{eq}\pi_1 = \lambda\pi_1\iota_{cq}\pi_0 = \lambda'\iota_{cq}\pi_0 = \lambda''\pi_0,$$

so that

$$(\lambda\pi_0\iota_{eq}\pi_0 w, \lambda''\pi_0\pi_0)c = (\lambda\pi_0\iota_{eq}\pi_0 w, \lambda''\pi_0\pi_1)c. \tag{4.19}$$

Define

$$\eta = (\theta_2\pi_0\pi_0 w, \hat{u}_5\omega_1\pi_0\pi_0, \hat{u}_{5;6}\omega_0\pi_0\pi_0, \hat{u}_{5;7}\theta_1\pi_1\pi_0)c \tag{Def. η}$$

and then by definition of the first two Ore-square maps in diagram (1) we have

$$\begin{aligned}
&\big(\lambda''\pi_0\pi_0, \hat{u}_{5;9}\pi u\pi_0\pi_1\big)c \\
&= \big((\theta_2\pi_0\pi_0 w, \hat{u}_5\omega_1\pi_0\pi_0, \hat{u}_{5;6}\omega_0\pi_0\pi_0, \hat{u}_{5;7}\theta_1\pi_0\pi_0 w, \hat{u}_{5;8}\theta_0\pi_1\pi_0)c, \hat{u}_{5;9}\pi u\pi_0\pi_1\big)c \\
&= \big(\theta_2\pi_0\pi_0 w, \hat{u}_5\omega_1\pi_0\pi_0, \hat{u}_{5;6}\omega_0\pi_0\pi_0, \hat{u}_{5;7}\theta_1\pi_0\pi_0 w, \hat{u}_{5;8}(\theta_0\pi_1\pi_0, \hat{u}9\pi u\pi_0\pi_1)c\big)c \\
&= \big(\theta_2\pi_0\pi_0 w, \hat{u}_5\omega_1\pi_0\pi_0, \hat{u}_{5;6}\omega_0\pi_0\pi_0, \hat{u}_{5;7}\theta_1\pi_0\pi_0 w, \hat{u}_{5;7}\theta_1\pi_0\pi_1\big)c \\
&= \big(\theta_2\pi_0\pi_0 w, \hat{u}_5\omega_1\pi_0\pi_0, \hat{u}_{5;6}\omega_0\pi_0\pi_0, \hat{u}_{5;7}(\theta_1\pi_0\pi_0 w, \theta_1\pi_0\pi_1)c\big)c \\
&= \big(\theta_2\pi_0\pi_0 w, \hat{u}_5\omega_1\pi_0\pi_0, \hat{u}_{5;6}\omega_0\pi_0\pi_0, \hat{u}_{5;7}(\theta_1\pi_1\pi_0, \theta_1\pi_1\pi_1 w)c\big)c \\
&= \big(\theta_2\pi_0\pi_0 w, \hat{u}_5\omega_1\pi_0\pi_0, \hat{u}_{5;6}\omega_0\pi_0\pi_0, \hat{u}_{5;7}\theta_1\pi_1\pi_0, \hat{u}_{5;9}\pi u\pi_1\pi_0 w\big)c \\
&= (\eta, \hat{u}_{5;9}\pi u\pi_1\pi_0 w)c
\end{aligned} \quad . \tag{4.20}$$

It will help to define,

$$\nu = (\theta_2\pi_1\pi_0, \hat{u}_{5;9}\pi\omega\pi_0\pi_0, \hat{u}_{5;9}\pi u_0\theta\pi_1\pi_0 w)c \tag{Def. ν}$$

and by definition of the Ore-square in the definition of composition on representative spans, $\sigma_\circ : U \to \text{spn}$, we have

$$\begin{aligned}
&\big(\lambda''\pi_0\pi_1, \hat{u}_{5;9}\pi u\pi_0\pi_1\big)c \\
&= \big((\theta_2\pi_1\pi_0, \hat{u}_{5;9}\pi\omega\pi_0\pi_0, \hat{u}_{5;9}\pi u_0\theta\pi_0\pi_0 w)c, \hat{u}_{5;9}\pi u\pi_0\pi_1\big)c \\
&= \big(\theta_2\pi_1\pi_0, \hat{u}_{5;9}\pi\omega\pi_0\pi_0, \hat{u}_{5;9}\pi u_0(\theta\pi_0\pi_0 w, u_1\pi_0\pi_1)c\big)c \\
&= \big(\theta_2\pi_1\pi_0, \hat{u}_{5;9}\pi\omega\pi_0\pi_0, \hat{u}_{5;9}\pi u_0(\theta\pi_0\pi_0 w, \theta\pi_0\pi_1)c\big)c \\
&= \big(\theta_2\pi_1\pi_0, \hat{u}_{5;9}\pi\omega\pi_0\pi_0, \hat{u}_{5;9}\pi u_0(\theta\pi_1\pi_0 w, \theta\pi_1\pi_1)c\big)c \\
&= \big(\theta_2\pi_1\pi_0, \hat{u}_{5;9}\pi\omega\pi_0\pi_0, \hat{u}_{5;9}\pi u_0\theta\pi_1\pi_0 w, \hat{u}_{5;9}\pi u\pi_1\pi_0 w\big)c \\
&= (\nu, \hat{u}_{5;9}\pi u\pi_1\pi_0 w)c
\end{aligned} \tag{4.21}$$

Putting equations (4.19), (4.20), and (4.21) all together gives

$$\begin{aligned}
(\lambda\pi_0\iota_{eq}\pi_0 w, \hat{u}_4\eta, \hat{u}_{4;9}\pi u\pi_1\pi_0 w)c &= (\lambda\pi_0\iota_{eq}\pi_0 w, \hat{u}_4\lambda''\pi_0\pi_0, \hat{u}_{4;9}\pi u\pi_0\pi_1)c \\
&= (\lambda\pi_0\iota_{eq}\pi_0 w, \hat{u}_4\lambda''\pi_0\pi_1, \hat{u}_{4;9}\pi u\pi_0\pi_1)c \\
&= (\lambda\pi_0\iota_{eq}\pi_0 w, \hat{u}_4\nu, \hat{u}_{4;9}\pi u\pi_1\pi_0 w)c
\end{aligned} \tag{4.22}$$

and induces the unique map $\rho'' : \hat{U}_3 \to P(\mathbb{C})\,{}_t{\times}_{ws} W$ which is determined by the projections

$$\rho''\pi_1 = \hat{u}_{4;9}\pi u\pi_1\pi_0 w,$$

$$\rho''\pi_0\pi_0 = (\lambda\pi_0\iota_{eq}\pi_0 w, \hat{u}_4\eta)c,$$

$$\rho''\pi_0\pi_1 = (\lambda\pi_0\iota_{eq}\pi_0 w, \hat{u}_4\nu)c.$$

equation (4.22) induces the unique map $\rho' : \hat{U}_3 \to \mathcal{P}_{cq}(\mathbb{C})$ such that the triangle

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{C}) \arrow[r, tail, "\iota_{cq}"] & P(\mathbb{C})\,{}_t\!\times_{ws} W \\
\hat{U} \arrow[u, dotted, "\rho'"] \arrow[ur, "\rho"'] &
\end{tikzcd}$$

commutes by the universal property of the equalizer, $\mathcal{P}_{cq}(\mathbb{C})$.

$$\begin{tikzcd}
& \mathcal{P}(\mathbb{C}) \arrow[r, "\pi_1"'] & \mathcal{P}_{cq}(\mathbb{C}) & \\
\hat{U}_2 \arrow[r, "\hat{u}_3", "/" marking] \arrow[d, "\rho"'] & \hat{U}_3 \arrow[r, "\hat{u}_4", "/" marking] \arrow[u, "\lambda"] \arrow[d, "\rho'"] & \hat{U}_4 \arrow[r, "\hat{u}_5", "/" marking] \arrow[u, "\lambda'"'] \arrow[d, "\theta_2"'] & \hat{U}_5 \arrow[d, "(\omega_1\pi_1{,}\hat{u}_{6;9}\pi\sigma_\circ\pi_0)"] \\
\mathcal{P}(\mathbb{C}) \arrow[r, "\pi_1"'] & \mathcal{P}_{cq}(\mathbb{C}) & W_\Box \arrow[r, "(\pi_0\pi_1{,}\pi_1\pi_1)"'] & \text{csp}
\end{tikzcd} \quad . \tag{2}$$

Applying **Int.Frc(2)** three times gives a cover that witnesses everything in Diagram A, and from there we can find two sailboats, $\varphi, \psi : \hat{U} \to \text{sb}$, along with a comparison span, $\varphi p_1 = \psi p_1$, whose left leg is in W.

$$\begin{tikzcd}
W_\circ \arrow[r, "(\pi_0\pi_1{,}\pi_0\pi_2)"] & W\,{}_{wt}\!\times_{ws} W & W_\circ \arrow[r, "(\pi_0\pi_1{,}\pi_0\pi_2)"] & W\,{}_{wt}\!\times_{ws} W \\
\hat{U} \arrow[r, "\hat{u}_0", "/" marking] \arrow[u, "\omega_4"] \arrow[d, shift right, "\psi"'] \arrow[d, shift left, "\varphi"] & \hat{U}_0 \arrow[r, "\hat{u}_1", "/" marking] \arrow[u, "(\omega_3\pi_1{,}\hat{u}_{2;5}\omega_1\pi_1)"'] \arrow[d, "\omega_3"'] & \hat{U}_1 \arrow[r, "\hat{u}_2", "/" marking] \arrow[u, "\omega_2"] \arrow[d, "(\omega_2\pi_1{,}\hat{u}_{2;4}\theta_2\pi_0\pi_0)"] & \hat{U}_2 \arrow[u, "(\rho\pi_0\iota_{eq}\pi_0{,}\hat{u}_3\lambda\pi_0\iota_{eq}\pi_0)"] \\
\text{sb} & W_\circ \arrow[r, "(\pi_0\pi_1{,}\pi_0\pi_2)"'] & W\,{}_{wt}\!\times_{ws} W &
\end{tikzcd} \tag{3}$$

For defining the sailboats above we should notice that commutativity of the first two Ore-squares and weak-composition triangle along with the commuting forks given by zippering imply that the composites of solid arrows in Figure B below are equal.

$$\begin{tikzcd}
& & & \cdot \arrow[d] & & & & & & \\
& & & \cdot \arrow[dr, bend left] & & & & & & \\
a & \cdot \arrow[l, dotted, "\alpha(a)", "\circ" description] \arrow[r, dotted, "\alpha(a)"'] \arrow[rr, dotted, bend left, "\alpha(a)f"] & a \arrow[r, dotted, "f"'] & b & \cdot \arrow[l, dotted, "\alpha(b)", "\circ" description] \arrow[r, "\alpha(b)"'] \arrow[rr, bend left, "\alpha(b)g"] & b \arrow[r, "g"'] & c & \cdot \arrow[uullll, bend right] \arrow[dddddllll, bend left, "\circ" description] & \cdot \arrow[l, "\circ" description] & \cdot \arrow[l, "\circ" description] \\
& & \cdot \arrow[ul, dotted] \arrow[ddd, dotted, "\circ" description] & & & & & & & \\
& & & \cdot \arrow[ul, dotted, "\circ" description] \arrow[uur, dotted] & & b \arrow[uur, "g"] & & & & \\
& & & \cdot \arrow[u, dotted] \arrow[dl, "\circ" description] & & & & & & \\
& & \cdot \arrow[uuuull, dotted, "\alpha(a)", "\circ" description] \arrow[rr, "\alpha(a)"'] & & a \arrow[uur, "f"] & & & & & \\
& & & \cdot \arrow[uu] & & & & & &
\end{tikzcd} \tag{B}$$

This is seen internally by first taking the equation

$$(\rho\pi_0\iota_{eq}\pi_0 w, \rho\pi_0\iota_{eq}\pi_1\pi_0)c = (\rho\pi_0\iota_{eq}\pi_0 w, \rho\pi_0\iota_{eq}\pi_1\pi_1)c$$

from the definition of the equalizer $\mathcal{P}_{eq}(\mathbb{C})$, post-composing (in $\mathbb{C}$) on both sides with $\hat{u}_{3;9}\pi u\pi_1\pi_1 : \hat{U}_3 \to \mathbb{C}_1$ and using associativity to get the equation

$$(\rho\pi_0\iota_{eq}\pi_0 w, \rho\pi_0\iota_{eq}\pi_1\pi_0, \hat{u}_{3;9}\pi u\pi_1\pi_1)c = (\rho\pi_0\iota_{eq}\pi_0 w, \rho\pi_0\iota_{eq}\pi_1\pi_1, \hat{u}_{3;9}\pi u\pi_1\pi_1)c, \tag{4.23}$$

and then expanding the latter composites on both sides to get

$$\begin{aligned}
(\rho\pi_0\iota_{eq}\pi_1\pi_0, \hat{u}_{3;9}\pi u\pi_1\pi_1)c &= \big(\rho\pi_1\iota_{cq}\pi_0\pi_0, (\hat{u}_{3;9}\tilde{u}\pi_1(s\alpha w, 1)c\big)c \\
&= (\hat{u}_3\rho'\iota_{cq}\pi_0\pi_0, \hat{u}_{3;9}\tilde{u}\pi_1 s\alpha w, \hat{u}_{3;9}\tilde{u}\pi_1)c \\
&= (\hat{u}_3\rho''\pi_0\pi_0, \hat{u}_{3;9}\pi u\pi_1\pi_0 w, \hat{u}_{3;9}\tilde{u}\pi_1)c \\
&= \big(\hat{u}_3(\lambda\pi_0\iota_{eq}\pi_0 w, \hat{u}_4\eta)c, \hat{u}_{3;9}\pi u\pi_1\pi_0 w, \hat{u}_{3;9}\tilde{u}\pi_1\big)c \\
&= (\hat{u}_3(\lambda\pi_0\iota_{eq}\pi_0 w, \hat{u}_4\eta, \hat{u}_{4;9}\pi u\pi_1\pi_0 w)c, \hat{u}_{3;9}\tilde{u}\pi_1)c
\end{aligned}$$

from the left-hand side, and

$$\begin{aligned}(\rho\pi_0\iota_{eq}\pi_1\pi_1, \hat{u}_{3;9}\pi u\pi_1\pi_1)c &= (\rho\pi_1\iota_{cq}\pi_0\pi_1, \hat{u}_{3;9}\tilde{u}\pi_1(s\alpha w, 1)c)c\\ &= (\hat{u}_3\rho'\iota_{cq}\pi_0\pi_1, \hat{u}_{3;9}\tilde{u}\pi_1 s\alpha w, \hat{u}_{3;9}\tilde{u}\pi_1)c\\ &= (\hat{u}_3\rho''\pi_0\pi_1, \hat{u}_{3;9}\pi u\pi_1\pi_0 w, \hat{u}_{3;9}\tilde{u}\pi_1)c\\ &= (\hat{u}_3(\lambda\pi_0\iota_{eq}\pi_0 w, \hat{u}_4\nu)c, \hat{u}_{3;9}\pi u\pi_1\pi_0 w, \hat{u}_{3;9}\tilde{u}\pi_1)c\\ &= (\hat{u}_3(\lambda\pi_0\iota_{eq}\pi_0 w, \hat{u}_4\nu, \hat{u}_{4;9}\pi u\pi_1\pi_0 w)c, \hat{u}_{3;9}\tilde{u}\pi_1)c\end{aligned}$$

from the right-hand side. These expansions are used below in equation (4.24) where we start establishing how the middle arrows for the sailboats picked out by $\varphi, \psi : \hat{U} \to \text{sb}$ coincide. These middle arrows will be picked out by maps, $\mu_0, \mu_1 : \hat{U} \to \mathbb{C}_1$, which will be internal composites, $\mu_0 = (\omega', \hat{u}_{0;2}\mu'_0)c$ and $\mu_1 = (\omega', \hat{u}_{0;2}\mu'_1)c$, for the arrow $\omega' : \hat{U} \to \mathbb{C}_1$ that is defined after Figure (C). The map picking out the part of the middle arrows in the sailboats determined by $\varphi : \hat{U} \to \text{sb}$ is

$$\mu'_0 = (\rho\pi_0\iota_{eq}\pi_0 w, \hat{u}_3\lambda\pi_0\iota_{eq}\pi_0 w, \hat{u}_{3;4}\theta_2\pi_0\pi_0 w, \hat{u}_{3;5}\omega_1\pi_0\pi_0, \hat{u}_{3;6}\omega_0\pi_1 w)c$$

and by expanding the composites with the definitions of θ_0, θ_1, and ω_0 in 1 along with the fact that $\pi u\pi_0\pi_1 = \tilde{u}\pi_0(s\alpha w, 1)c$ from 0 we can see

$$(\rho\pi_0\iota_{eq}\pi_0 w, \hat{u}_3\lambda\iota_{eq}\pi_0 w, \hat{u}_{3;4}\eta, \hat{u}_{3;9}\pi u\pi_0\pi_1)c = \big(\mu'_0, \hat{u}_{3;9}\tilde{u}c(s\alpha w, 1)c\big)c.$$

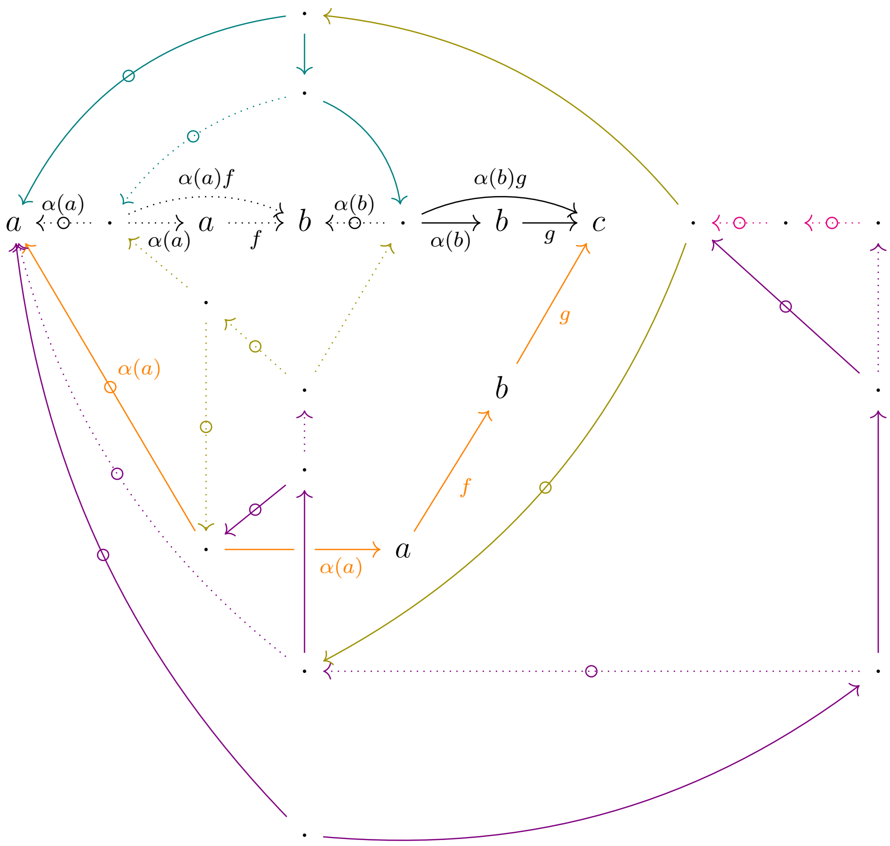
The middle of the sailboats being picked out by ψ are given explicitly by

$$\mu'_1 = (\rho\pi_0\iota_{eq}\pi_0 w, \hat{u}_3\lambda\pi_0\iota_{eq}\pi_0 w, \hat{u}_{3;4}\theta_2\pi_1\pi_0)c.$$

Putting together equation (4.23) with the expansions and definitions of μ'_0 and μ'_1 and the definitions (Def. η) and (Def. ν) and the definition of $\sigma_\circ : U \to \text{spn}$.

$$\begin{aligned}\big(\mu'_0, \hat{u}_{3;9}\tilde{u}c(s\alpha w, 1)c\big)c &= \big(\rho\pi_0\iota_{eq}\pi_0 w, \hat{u}_3\lambda\iota_{eq}\pi_0 w, \hat{u}_{3;4}\eta, \hat{u}_{3;9}\pi u\pi_1\pi_1)c\\ &= \big(\rho\pi_0\iota_{eq}\pi_0 w, \hat{u}_3\lambda\iota_{eq}\pi_0 w, \hat{u}_{3;4}\nu, \hat{u}_{3;9}\pi u\pi_1\pi_1)c. \qquad (4.24)\\ &= \big(\mu'_1, \hat{u}_{3;9}\pi\sigma_\circ\pi_1)c\end{aligned}$$

The maps φ and ψ picking out the sailboats can be seen in Figure (C) below as the appropriate composites of the solid arrows.



(C)

Explicitly define $\omega' : \hat{U} \to \mathbb{C}_1$ to be the composite of the weak-composition arrows in (3),

$$\omega' = (\omega_4\pi_0\pi_0, \hat{u}_0\omega_3\pi_0, \hat{u}_{0;1}\omega_2\pi_0)c, \tag{Def. ω'}$$

and then $\mu_0 : \hat{U} \to \mathbb{C}_1$ by

$$\mu_0 = (\omega', \hat{u}_{0;2}\mu_0')c. \tag{Def. μ_0}$$

By definition of $\omega_4 : \hat{U} \to W_\circ$ we have

$$\omega_4\pi_1 = (\mu_0, \hat{u}\tilde{u}\pi_0 s\alpha w)c$$

so the map $\varphi : \hat{U} \to \text{sb}$, that picks out the sailboats in the bottom of Figure (C) (consisting of orange and violet arrows and factoring through the bottom of the olive coloured Ore-square arrows), given by

$$\varphi = \big(((\mu_0, \hat{u}\tilde{u}\pi_0 s\alpha w), \omega_4\pi_1), \hat{u}\tilde{u}c(s\alpha w, 1)c\big) \tag{Def. φ}$$

is well-defined. Similarly define

$$\mu_1 = (\omega', \hat{u}_{0;2}\mu_1')c. \tag{ Def. μ_1}$$

By the first zippering, $\lambda : \hat{U}_3 \to \mathcal{P}(\mathbb{C})$, in (2), in particular by $\lambda\pi_0 : \hat{U}_3 \to \mathcal{P}_{eq}(\mathbb{C})$ and the equalizer in its codomain we have that

$$\begin{aligned}
&(\hat{u}_{0;2}\mu_1', \hat{u}_{0;9}\pi\omega\pi_0\pi_0, \hat{u}\pi u_0\theta\pi_0\pi_0 w)c \\
&= (\hat{u}_{0;2}\rho\pi_0\iota_{eq}\pi_0 w, \hat{u}_{0;3}\lambda\pi_0\iota_{eq}\pi_0 w, \hat{u}_{0;3}\lambda\pi_0\iota_{eq}\pi_1\pi_1)c \\
&= (\hat{u}_{0;2}\rho\pi_0\iota_{eq}\pi_0 w, \hat{u}_{0;3}\lambda\pi_0\iota_{eq}\pi_0 w, \hat{u}_{0;3}\lambda\pi_0\iota_{eq}\pi_1\pi_0)c
\end{aligned}$$

where the second to last line comes from the definition of μ_0' and the Ore-square picked out by $\theta_0 : \hat{U}_8 \to W_\Box$ and the last line is by definition of $\omega_4 : \hat{U} \to W_\circ$. This allows us to see

$$\begin{aligned}
&(\mu_1, \hat{u}\pi\sigma_\circ\pi_0 w)c \\
&= (\omega', \hat{u}_{0;2}\mu_1', \hat{u}\pi\omega\pi_0\pi_0, \hat{u}\pi u_0\theta\pi_0\pi_0 w, \hat{u}\pi u\pi_0\pi_0 w)c \\
&= (\omega', \hat{u}_{0;2}\mu_1', \hat{u}\pi\omega\pi_0\pi_0\hat{u}\pi u_0\theta\pi_0\pi_0 w, \hat{u}\pi u\pi_0\pi_0 w)c \\
&= (\omega', \hat{u}_{0;2}\rho\pi_0\iota_{eq}\pi_0 w, \hat{u}_{0;3}\lambda\pi_0\iota_{eq}\pi_0 w, \hat{u}_{0;3}\lambda\pi_0\iota_{eq}\pi_1\pi_0, \hat{u}\pi u\pi_0\pi_0 w)c \\
&= (\omega', \hat{u}_{0;2}\mu_0', \hat{u}\pi u\pi_0\pi_0 w) \\
&= (\mu_0, \hat{u}\tilde{u}\pi_0 s\alpha w) \\
&= \omega_4\pi_1
\end{aligned}$$

so that the map $\psi : \hat{U} \to \text{sb}$ determined by

$$\psi = \big(((\mu_1, \hat{u}\pi\sigma_\circ\pi_0), \omega_4\pi_1), \hat{u}\pi\sigma_\circ\pi_1\big) \tag{Def. ψ}$$

is well-defined. Notice the intermediate spans picked out by φ and ψ coincide due to the composites in Figure (B) being equal. Formally, by (Def. φ), (Def. ψ), and equation (4.24), we can see

$$\begin{aligned}
\varphi p_1 &= \varphi(\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c) \\
&= \bigl(\omega_4\pi_1, (\mu_0, \hat{u}_{3;9}\tilde{u}c(s\alpha w, 1)c\bigr)c \\
&= \bigl(\omega_4\pi_1, (\omega', \hat{u}_{0;2}\mu_0', \hat{u}_{3;9}\tilde{u}c(s\alpha w, 1)c\bigr)c \\
&= \bigl(\omega_4\pi_1, (\omega', \hat{u}_{0;2}\mu_1', \hat{u}_{3;9}\tilde{u}\pi\sigma_\circ\pi_1)c\bigr)c \\
&= \bigl(\omega_4\pi_1, (\mu_1, \hat{u}\tilde{u}\pi\sigma_\circ\pi_1)c\bigr)c \\
&= \psi(\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c) \\
&= \psi p_1
\end{aligned} \tag{4.25}$$

Also notice that by (Def. φ),

$$\begin{aligned}
\varphi p_0 &= \varphi(\pi_0\pi_0\pi_1, \pi_1) \\
&= \bigl(\hat{u}\tilde{u}\pi_0 s\alpha w, \hat{u}\tilde{u}c(s\alpha w, 1)c\bigr) \\
&= \hat{u}\tilde{u}\bigl(\pi_0 s\alpha w, c(s\alpha w, 1)c\bigr)
\end{aligned} \tag{4.26}$$

and by (Def. ψ),

$$\begin{aligned}
\psi p_0 &= \psi(\pi_0\pi_0\pi_1, \pi_1) \\
&= (\hat{u}\pi\sigma_\circ\pi_0, \hat{u}\pi\sigma_\circ\pi_1) \\
&= \hat{u}\pi\sigma_\circ.
\end{aligned} \tag{4.27}$$

Putting equations (4.25), (4.27), and (4.26) together gives

$$\begin{aligned}
\hat{u}\tilde{u}\bigl(\pi_0 s\alpha w, c(s\alpha w, 1)c\bigr)q &= \varphi p_0 q \\
&= \varphi p_1 q \\
&= \psi p_1 q \\
&= \psi p_0 q \\
&= \hat{u}\pi\sigma_\circ q \\
&= \hat{u}\pi u c' \\
&= \hat{u}\tilde{u}\bigl(\pi_0(s\alpha,(s\alpha w,1)c),\pi_1(s\alpha,(s\alpha w,1)c)\bigr)c'
\end{aligned}$$

and since $\hat{u}\tilde{u} : \hat{U} \to \mathbb{C}_2$ is epic, we get

$$\big(\pi_0 s\alpha w, c(s\alpha w, 1)c\big)q = \big(\pi_0(s\alpha,(s\alpha w,1)c),\pi_1(s\alpha,(s\alpha w,1)c)\big)c'$$

as promised. □

Inverting the Canonical Cartesian Cleavage

Now that we know $L : \mathbb{C} \to \mathbb{C}[W^{-1}]$ is an internal functor, we can show that it satisfies an important property. The rest of this section consists of lemmas leading to Proposition 55, which shows that the localization functor, $L : \mathbb{C} \to \mathbb{C}[W^{-1}]$, inverts $w : W \to \mathbb{C}_1$ in the sense of the following definition.

Definition 48. We say a map $x : X \to \mathbb{C}_1$ is *invertible* if there exists a map $x^{-1} : X \to \mathbb{C}_1$ such that the diagrams

$$\begin{tikzcd}
X \arrow[drr, bend left, "x^{-1}"] \arrow[dr, dotted, "{(x,x^{-1})}"] \arrow[ddr, bend right, "x"'] & & \\
& \mathbb{C}_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & \mathbb{C}_1 \arrow[d, "s"] \\
& \mathbb{C}_1 \arrow[r, "t"'] & \mathbb{C}_1
\end{tikzcd}, \qquad
\begin{tikzcd}
X \arrow[drr, bend left, "x"] \arrow[dr, dotted, "{(x^{-1},x)}"] \arrow[ddr, bend right, "x^{-1}"'] & & \\
& \mathbb{C}_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & \mathbb{C}_1 \arrow[d, "s"] \\
& \mathbb{C}_1 \arrow[r, "t"'] & \mathbb{C}_1
\end{tikzcd},$$

$$\begin{tikzcd}
X \arrow[r, "{(x,x^{-1})}"] \arrow[d, "x"'] & \mathbb{C}_2 \arrow[d, "c"] \\
\mathbb{C}_1 \arrow[r, "se"'] & \mathbb{C}_1
\end{tikzcd}, \qquad \text{and} \qquad
\begin{tikzcd}
X \arrow[r, "{(x^{-1},x)}"] \arrow[d, "x"'] & \mathbb{C}_2 \arrow[d, "c"] \\
\mathbb{C}_1 \arrow[r, "te"'] & \mathbb{C}_1
\end{tikzcd}$$

commute in $\mathcal{E}$. In this case we say x^{-1} *is an inverse for* x *in* $\mathbb{C}$.

The next definition describes what it means for an internal functor to invert a class of arrows in its domain.

Definition 49. We say an internal functor, $F : \mathbb{C} \to \mathbb{D}$ *inverts* $x : X \to \mathbb{C}_1$ if there exists a map $F(x)^{-1} : X \to \mathbb{D}_1$ such that $F(x)^{-1}$ is an inverse for the composite

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[dr, "F(x)"'] & \mathbb{C}_1 \arrow[d, "F_1"] \\
& \mathbb{D}_1
\end{tikzcd}$$

in $\mathbb{D}_1$.

One might expect that internal functors preserve inverses, and sure enough the following lemma states and proves this:

Lemma 50. *If* $x : X \to \mathbb{C}_1$ *is an arrow in* $\mathcal{E}$ *that has an inverse* $x^{-1} : X \to \mathbb{C}_1$ *and* $F : \mathbb{C} \to \mathbb{X}$ *is an internal functor, then the composite*

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[dr, "F(x)"'] & \mathbb{C}_1 \arrow[d, "F_1"] \\
 & \mathbb{D}_1
\end{tikzcd}$$

has an inverse given by

$$\begin{tikzcd}
X \arrow[r, "x^{-1}"] \arrow[dr, "F(x)^{-1}"'] & \mathbb{C}_1 \arrow[d, "F_1"] \\
 & \mathbb{D}_1
\end{tikzcd} .$$

Proof. By functoriality we can compute

$$\begin{aligned}
(F(X), F(X)^{-1})c &= (xF_1, x^{-1}F_1)c \\
&= (x, x^{-1})cF_1 \\
&= xseF_1 \\
&= xF_1se \\
&= F(X)se
\end{aligned}$$

and

$$\begin{aligned}
(F(X)^{-1}, F(X))c &= (x^{-1}F_1, xF_1)c \\
&= (x^{-1,x})cF_1 \\
&= xteF_1 \\
&= xF_1te \\
&= F(X)te
\end{aligned}$$

and the result follows from Definition 49. □

The following lemma shows how every span is equivalent to a canonical composite of spans and will be useful for proving our main result, Proposition 55. The idea is that every span, represented as

$$\begin{tikzcd} c & a \arrow[l, "v", "\circ" marking] \arrow[r, "f"] & b, \end{tikzcd}$$

is equivalent to a composite of the pair of composable spans, represented as

$$\begin{tikzcd} c & a \arrow[l, "v"', "\circ" marking] \arrow[r, equal] & a & d \arrow[l, "\alpha(b)", "\circ" marking] \arrow[r, dashed, "\alpha(a)", "\circ" marking] \arrow[rr, bend left, "\alpha(a)f"] & a \arrow[r, dashed, "f"] & b \end{tikzcd}$$

in particular. This is translated to the following statement about internal categories.

Lemma 51. *The triangle*

$$\begin{tikzcd} spn \arrow[rr, "\big((\pi_0{,}\pi_0 wse)\ ,\ (\pi_0 ws\alpha{,}(\pi_0 ws\alpha w{,}\pi_1)c)\big)"] \arrow[drr, two heads, "q"'] & & spn^2 \arrow[d, "c'"] \\ & & \mathbb{C}[W^{-1}]_1 \end{tikzcd}$$

commutes in $\mathcal{E}$.

Proof. Let $\gamma : \mathrm{spn} \to \mathrm{spn}^2$ be the unique pairing map

$$\gamma = \big((\pi_0{,}\pi_0 wse)\ ,\ (\pi_0 ws\alpha{,}(\pi_0 ws\alpha{,}\pi_1)c)\big).$$

Take the pullback of the cover $u : U \to \mathrm{spn}^2$, used to define $c' : \mathrm{spn}^2 \to \mathbb{C}[W^{-1}]_1$ in Lemmas 37 and 38, along γ to get a cover $\tilde{u} : \tilde{U} \to \mathrm{spn}$ that witnesses the span composition process for the family of composable spans represented by γ.

$$\begin{tikzcd} \tilde{U} \arrow[r, "\pi"] \arrow[d, "\tilde{u}", "|" marking] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "u", "|" marking] \\ \mathrm{spn} \arrow[r, "\gamma"'] & \mathrm{spn}^2 \end{tikzcd}$$

It suffices to construct a (family of) sailboat(s) $\varphi : \tilde{U} \to \mathrm{sb}$ such that

$$\varphi p_0 = \tilde{u} \quad \text{ and } \varphi p_1 = \pi\sigma_\circ$$

because that would give

$$\tilde{u}q = \varphi p_0 q = \varphi p_1 q = \pi\sigma_\circ q = \pi u c' = \tilde{u}\gamma c'$$

and since $\tilde{u}$ is epic we could conclude that

$$\gamma c' = q$$

as desired. This family of sailboats will be constructed using the definition of the span composition, but let us take a moment to consider how this works when $\mathcal{E} = \mathbf{Set}$. In this case, for each $f : a \to b$ in $\mathbb{C}_1$ and $u : a \to c$ in W we have a diagram that looks like

$$\begin{tikzcd}
 & & \cdot \arrow[d, "h"] \arrow[ddll, bend right, "v''"', "\circ" marking] & & & \\
 & & \cdot \arrow[dl, "v'"', "\circ" marking] \arrow[dr, "k"] & & & \\
c & a \arrow[l, "v"', "\circ" marking] \arrow[r, equal] & a \arrow[rrr, bend right, "f"'] & d \arrow[l, "\alpha(a)", "\circ" marking] \arrow[r, "\alpha(a)"', "\circ" marking] \arrow[rr, bend left, "\alpha(a)f"] & a \arrow[r, "f"] & b
\end{tikzcd}$$

which gives rise to the sailboat

$$\begin{tikzcd}
 & \cdot \arrow[dl, "v''"', "\circ" marking] \arrow[d, "hv'"] \arrow[dr, dotted, "hk\alpha(a)f"] & \\
c & a \arrow[l, "v", "\circ" marking] \arrow[r, "f"] & b
\end{tikzcd}$$

showing that the span (u, f) is equivalent to the composite $(u, 1) * (\alpha(a), \alpha(a)f)$. The map picking out such sailboats internally, $\varphi : \tilde{U} \to \mathrm{sb}$, can be constructed by picking out the corresponding arrows through the composition process witnessed by the cover $u : U \to \mathrm{spn}^2$. Explicitly, this is given by

$$\varphi = \big((((\pi\omega\pi_0\pi_0, \pi u_0\theta\pi_0\pi_0 w)c, \tilde{u}\pi_0), \pi\sigma_\circ\pi_0), \tilde{u}\pi_1\big)$$

and this is well-defined because $\tilde{u}\pi_0 = \tilde{u}\gamma\pi_0\pi_0$ and the definition of $W_\circ$ shows

$$\begin{aligned}
\big((\pi\omega\pi_0\pi_0, \pi u_0\theta\pi_0\pi_0 w)c, \tilde{u}\pi_0 w\big)c &= \big(\pi\omega\pi_0\pi_0, \pi u_0\theta\pi_1\pi_0, \tilde{u}\gamma\pi_0\pi_0 w\big)c \\
&= \big(\pi\omega\pi_0\pi_0, \pi u_0\theta\pi_0\pi_0 w, \pi u\pi_0\pi_0 w\big)c \\
&= \pi\omega\pi_1 w \\
&= \pi\sigma_\circ\pi_0 w.
\end{aligned}$$

We can immediately see that

$$\varphi p_0 = \varphi(\pi_0\pi_0\pi_1, \pi_1) = (\tilde{u}\pi_0, \tilde{u}\pi_1) = \tilde{u} \tag{4.28}$$

and by adding an identity map, $\tilde{U} \to \mathbb{C}_1$, given by

$$\tilde{u}\pi_0 wse = \tilde{u}\gamma\pi_0\pi_1 = \pi u\pi_0\pi_1$$

into the following computation we can use the definition of $W_\square$, the fact that

$$(\tilde{u}\pi_0 ws\alpha w, \tilde{u}\pi_1)c = \tilde{u}\gamma\pi_1 = \pi u\pi_1,$$

and the definition of $\sigma_\circ : U \to \text{spn}$ in Lemma 37.

$$\begin{aligned}
\varphi(\pi_0\pi_0\pi_0, \pi_1)c &= \big((\pi\omega\pi_0\pi_0, \pi u_0\theta\pi_0\pi_0 w)c, \tilde{u}\pi_1\big)c \\
&= (\pi\omega\pi_0\pi_0, \pi u_0\theta\pi_0\pi_0 w, \tilde{u}\pi_1)c \\
&= (\pi\omega\pi_0\pi_0, \pi u_0\theta\pi_0\pi_0 w, \tilde{u}\pi_0 wse, \tilde{u}\pi_1)c \\
&= (\pi\omega\pi_0\pi_0, \pi u_0\theta\pi_0\pi_0 w, \pi u\pi_0\pi_1, \tilde{u}\pi_1)c \\
&= (\pi\omega\pi_0\pi_0, (\pi u_0\theta\pi_0\pi_0 w, \pi u\pi_0\pi_1)c, \tilde{u}\pi_1)c \\
&= (\pi\omega\pi_0\pi_0, (\pi u_0\theta\pi_1\pi_0, \pi u\pi_1\pi_0 w)c, \tilde{u}\pi_1)c \\
&= (\pi\omega\pi_0\pi_0, \pi u_0\theta\pi_1\pi_0, \tilde{u}\pi_0 ws\alpha w, \tilde{u}\pi_1)c \\
&= \pi(\omega\pi_0\pi_0, u_0\theta\pi_1\pi_0, u\pi_1)c \\
&= \pi\sigma_\circ\pi_1.
\end{aligned}$$

Now we can easily see

$$\begin{aligned}
\varphi p_1 &= \varphi\big(\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c\big) \\
&= \big(\pi\sigma_\circ\pi_0, \pi\sigma_\circ\pi_1\big) \qquad (4.29) \\
&= \pi\sigma_\circ
\end{aligned}$$

The result follows from equations (4.28) and (4.29) as discussed at the beginning of this proof. □

The next lemma is used to give an equivalent representation of the identity spans in $\mathbb{C}[W^{-1}]$ which we use in the proof of Proposition 55.

Lemma 52. *The diagram*

$$\begin{tikzcd}
W \arrow[r, "{(1,w)}"] \arrow[d, "wt"'] & spn \arrow[d, two heads, "q"] \\
\mathbb{C}_0 \arrow[r, "{(\alpha,\alpha w)q}"'] & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

*commutes, where $\alpha : \mathbb{C}_0 \to W$ is a section of $wt : W \to \mathbb{C}_0$ from **Int.Frc.(1)**.*

Proof. By **Int.Frc.(2)** and **Int.Frc.(3)** there exist covers, $\tilde{u}_0$ and $\tilde{u}_1$, and lifts, $\tilde{\omega}$ and $\tilde{\theta}$, that make the squares in the following diagram commute respectively:

$$\begin{tikzcd}
W_\circ \arrow[r, "{(\pi_0\pi_1,\pi_0\pi_2)}"] & W\ {}_{wt}\times_{ws} W & \\
\tilde{U} \arrow[u, dotted, "\tilde{\omega}"] \arrow[r, "\tilde{u}_0", "/" marking] & \tilde{U}_0 \arrow[u] \arrow[r, "\tilde{u}_1", "/" marking] \arrow[d, dotted, "\tilde{\theta}"'] & W \arrow[d, "{(1,wt\alpha w)}"] \\
& W_\Box \arrow[r, "{(\pi_0\pi_1,\pi_1\pi_1)}"'] & \text{csp.}
\end{tikzcd}$$

The map $\tilde{\omega}\pi_1 : \tilde{U} \to W$ results in an intermediate (family of) span(s), $(\tilde{\omega}\pi_1, \tilde{\omega}\pi_1 w) : \tilde{U} \to \text{spn}$ and by definition of $W_\Box$ and the maps in the diagram above we have that

$$\tilde{\omega}\pi_1 = (\tilde{\omega}\pi_0\pi_0, \tilde{u}_0\tilde{\theta}\pi_0\pi_0 w, \tilde{u}w).$$

This gives a map $\tilde{U} \to W_\triangle$ and since $\text{sb} = W_\triangle\ {}_{\pi_0\pi_1 s}\times_s \mathbb{C}_1$ we can see this determines a sailboat, $\varphi : \tilde{U} \to \text{sb}$, given by the unique pairing map

$$\varphi = \big((((\tilde{\omega}\pi_0\pi_0, \tilde{u}_0\tilde{\theta}\pi_0\pi_0 w)c, \tilde{u}), \tilde{\omega}\pi_1), \tilde{u}w\big).$$

Similarly, we have

$$\tilde{\omega}\pi_1 = (\tilde{\omega}\pi_0\pi_0, \tilde{u}_0\tilde{\theta}\pi_1\pi_0, \tilde{u}wt\alpha w)$$

giving another unique map $\tilde{U} \to W_\triangle$ and determining a sailboat $\psi : \tilde{U} \to \text{sb}$, by the unique pairing map

$$\psi = \big((((\tilde{\omega}\pi_0\pi_0, \tilde{u}_0\tilde{\theta}\pi_1\pi_0)c, \tilde{u}wt\alpha), \tilde{\omega}\pi_1), \tilde{u}wt\alpha w\big).$$

First we can use the calculations and definitions above (along with the definition of the pullback projections and how they interact with pairing maps) to see

$$\begin{aligned}\varphi p_0 &= \varphi(\pi_0\pi_0\pi_1, \pi_1)\\ &= \tilde{u}(1, w)\end{aligned}$$

and

$$\begin{aligned}\varphi p_1 &= \varphi(\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c)\\ &= (\tilde{\omega}\pi_1, \tilde{\omega}\pi_1 w)\\ &= \tilde{\omega}(\pi_1, \pi_1 w)\end{aligned}$$

as well as

$$\begin{aligned}\psi p_0 &= \psi(\pi_0\pi_0\pi_1, \pi_1)\\ &= (\tilde{u}wt\alpha, \tilde{u}wt\alpha w)\\ &= (\tilde{u}wt(\alpha, \alpha w)\end{aligned}$$

and

$$\begin{aligned}\psi p_1 &= \psi(\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c)\\ &= (\tilde{\omega}\pi_1, \tilde{\omega}\pi_1 w)\\ &= \tilde{\omega}(\pi_1, \pi_1 w).\end{aligned}$$

Putting it all together shows

$$\tilde{u}(1, w)q = \varphi p_0 q = \varphi p_1 q = \psi p_1 q = \psi p_0 q = \tilde{u}wt(\alpha, \alpha w)q$$

and since $\tilde{u}$ is epic we get that

$$(1, w)q = wt(\alpha, \alpha w)q$$

as desired. □

An immediate corollary to Lemma 52 is that the internal localization functor, $L : \mathbb{C} \to \mathbb{C}[W^{-1}]$, maps the arrows from $w : W \to \mathbb{C}_1$ to arrows in $\mathbb{C}[W^{-1}]$ that have left inverses.

Corollary 53. *The map* $L_1 : \mathbb{C}_1 \to \mathbb{C}[W^{-1}]_1$ *has left inverses with respect to* $w : W \to \mathbb{C}_1$*, in the sense that the diagram*

$$\begin{tikzcd}
W \arrow[r, "\big((1{,}wse)q{,}wL_1\big)"] \arrow[d, "(1{,}w)"'] & \mathbb{C}[W^{-1}]_2 \arrow[d, "c"] \\
spn \arrow[r, "q"'] & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

commutes.

Proof. Consider the following diagram.

$$\begin{tikzcd}
& W \arrow[dl, bend right, "(1{,}w)"'] \arrow[d, "\big((1{,}wse)\ {,}\ (ws\alpha w{,}(ws\alpha{,}w)c)\big)"'] \arrow[dr, "\big((1{,}wse)q{,}wL_1\big)"] & \\
\text{spn} \arrow[r, "\big((\pi_0{,}\pi_0 wse)\ {,}\ (\pi_0 ws\alpha{,}(\pi_0 ws\alpha w{,}\pi_1)c)\big)"] \arrow[dr, two heads, "q"'] & \text{spn}^2 \arrow[r, two heads, "q\times q"] \arrow[d, "c'"] & \mathbb{C}[W^{-1}]_2 \arrow[dl, "c"] \\
& \mathbb{C}[W^{-1}]_1 &
\end{tikzcd}$$

The bottom left triangle commutes by Lemma 51; the bottom right commutes by definition of c; the top left triangle commutes by the universal property of the pullback spn^2; and the top right triangle commutes by the universal property of the pullback $\mathbb{C}[W^{-1}]_2$ along with the definitions of L_1 and $q \times q$. More precisely, post-composing the upper right triangle with the projection $\pi_1 : \mathbb{C}[W^{-1}]_2 \to \mathbb{C}[W^{-1}]_1$ gives precisely

$$(ws\alpha{,}(ws\alpha{,}w)c)q = w(s\alpha{,}(s\alpha{,}1)c)q = wL_1.$$

It follows that the diagram above commutes, in particular the outer square commutes.

□

Next we prove a lemma that shows the internal localization functor, $L : \mathbb{C} \to \mathbb{C}[W^{-1}]$, maps arrows coming from $w : W \to \mathbb{C}_1$ to arrows that have right inverses in $\mathbb{C}[W^{-1}]$.

Lemma 54. *The diagram*

$$\begin{tikzcd}
W \arrow[r, "\big(wL_1{,}(1{,}wse)q\big)"] \arrow[d, "ws(\alpha{,}\alpha w)"'] & \mathbb{C}[W^{-1}]_2 \arrow[d, "c"] \\
spn \arrow[r, "q"'] & \mathbb{C}[W^{-1}]_1
\end{tikzcd}$$

commutes.

Proof. Using the fact that

$$(ws\alpha,(ws\alpha,w)c)q = w(s\alpha,(s\alpha,1)c)q = wL_1$$

and the universal property of the pullback $\mathbb{C}[W^{-1}]_2$ we can see the top triangle in the diagram,

$$\begin{tikzcd}
W \arrow[r, "\big((ws\alpha,(ws\alpha w,w)c)\ ,\ (1,wse)\big)"] \arrow[d, "(ws\alpha,ws\alpha w)"'] \arrow[rr, bend left, "(wL_1,(1,wse)q\big)"] & \mathrm{spn}^2 \arrow[r, "q\times q"] \arrow[d, "c'"] & \mathbb{C}[W^{-1}]_2 \arrow[dl, "c"] \\
\mathrm{spn} \arrow[r, "q"'] & \mathbb{C}[W^{-1}]_1 &
\end{tikzcd},$$

commutes. The right triangle commutes by definition so it suffices to show the bottom left square commutes.

Let $\gamma = \big((ws\alpha,(ws\alpha w,w)c)\ ,\ (1,wse)\big)$ and take the pullback of the cover $u : U \to \mathrm{spn}^2$ along γ.

$$\begin{tikzcd}
\tilde{U} \arrow[r, "\pi"] \arrow[d, "\tilde{u}"', "/" marking] \arrow[dr, phantom, very near start, "\lrcorner"] & U \arrow[d, "u", "/" marking] \\
W \arrow[r, "\gamma"'] & \mathrm{spn}^2
\end{tikzcd}$$

The cover $\tilde{u} : \tilde{U} \to W$ witnesses the composition of the spans being picked out by γ. Thinking momentarily about the case when $\mathcal{E} = \mathbf{Set}$ for visualization purposes, this says that for every arrow $v : a \to b$ in W there exists a point in $\tilde{U}$ witnessing the commuting diagram:

$$\begin{tikzcd}
 & & \cdot \arrow[d, "h"] \arrow[ddll, bend right, "v''"', "\circ" marking] & & & \\
 & & \cdot \arrow[dl, "v'"', "\circ" marking] \arrow[drr, bend left, "k"] & & & \\
a & c \arrow[l, "\alpha(a)", "\circ" marking] \arrow[r, "\alpha(a)", "\circ" marking] \arrow[rr, bend right, "\alpha(a)v"'] & a \arrow[r, "v", "\circ" marking] & b & a \arrow[l, "v", "\circ" marking] \arrow[r, equal] & a
\end{tikzcd}$$

In this case the parallel pair of arrows, $hv'\alpha(a)$ and hk being coequalized by $v : a \to b$ allows us to zipper before applying weak composition for W to get a span whose left leg is in W, as pictured in the following diagram:

$$\begin{tikzcd}
 & \cdot \arrow[r, "g"] \arrow[dddl, bend right, "v'''"', "\circ" marking] & \cdot \arrow[d, "d", "\circ" marking] & & & \\
 & & \cdot \arrow[d, "h"] \arrow[ddll, bend right, "v''"', "\circ" marking] & & & \\
 & & \cdot \arrow[dl, "v'"', "\circ" marking] \arrow[drr, bend left=15, "k"] & & & \\
a & c \arrow[l, "\alpha(a)", "\circ" marking] \arrow[r, "\alpha(a)", "\circ" marking] \arrow[rr, bend right, "\alpha(a)v"'] & a \arrow[r, "v", "\circ" marking] & b & a \arrow[l, "v", "\circ" marking] \arrow[r, equal] & a
\end{tikzcd}$$

The zippering axiom says there exists a map d such that

$$dhv'\alpha(a) = dhk$$

and the weak-composition axiom says there exists a map g in the diagram above such that $gdv'' = v'''$ is in W. This data gives rise to two sailboats with a common projection,

$$\begin{tikzcd}
 & \cdot \arrow[dl, "v'''"'] \arrow[d, "gdhv'"'] \arrow[dr, dotted, "gdhk"] & \\
a & c \arrow[l, "\alpha(a)"] \arrow[r, "\alpha(a)"'] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \cdot \arrow[dl, "v'''"'] \arrow[d, "gd'"'] \arrow[dr, dotted, "gdhk"] & \\
a & \cdot \arrow[l, "v''"] \arrow[r, "hk"'] & a
\end{tikzcd}$$

implying that the composite of spans, $(\alpha(a), \alpha(a)v) * (v, 1_a) = (v'', hk)$, is equivalent to the span $(\alpha(a), \alpha(a))$ by transitivity. Translating this argument to the internal setting for $\mathcal{E}$ not necessarily equal to **Set** amounts to defining the map $\delta : \tilde{U} \to \mathcal{P}_{cq}(\mathbb{C})$ in the following diagram,

$$\begin{tikzcd}
W_\circ \arrow[r, "(\pi_0\pi_1{,}\pi_0\pi_2)"] & W\,{}_{wt}\times_{ws}W & \\
\hat{U} \arrow[u, "\hat{\omega}"] \arrow[r, "\hat{u}_0", "/" marking] & \hat{U}_0 \arrow[u, "(\hat{\delta}\pi_0\iota_{eq}\pi_0{,}\hat{u}_1\pi\sigma_\circ\pi_0)"'] \arrow[r, "\hat{u}", "/" marking] \arrow[d, "\hat{\delta}"'] & \tilde{U} \arrow[d, "\delta"] \\
 & \mathcal{P}(\mathbb{C}) \arrow[r, "\pi_1"'] & \mathcal{P}_{cq}(\mathbb{C})
\end{tikzcd}$$

applying **Int.Frc(4)** to get the cover $\hat{u}_1 : \hat{U}_0 \to \tilde{U}$ and the lift $\hat{\delta} : \hat{U}_1 \to \mathcal{P}(\mathbb{C})$, and then applying by **Int.Frc(2)** to get the cover $\hat{u}_0 : \hat{U} \to \hat{U}_0$ and the lift $\hat{\omega} : \hat{U} \to W_\circ$.

The map $\delta : \tilde{U} \to \mathcal{P}_{cq}(\mathbb{C})$ is induced by the universal property of the equalizer $\mathcal{P}_{cq}(\mathbb{C})$ and the map $\delta' : \tilde{U} \to P(\mathbb{C})\,{}_t\!\times_{ws} W$. The map δ' is induced by the universal property of the pullback $P(\mathbb{C})\,{}_t\!\times_{ws} W$ and to define it we start by using the definition of $W_\square$ to see

$$\pi(\omega\pi_0\pi_0, u_0\theta\pi_0\pi_0 w, u\pi_0\pi_1)c = \pi(\omega\pi_0\pi_0, u_0\theta\pi_1\pi_0, u\pi_1\pi_0 w)c. \tag{$\star$}$$

Since

$$\pi u\pi_0\pi_1 = \tilde{u}\gamma\pi_0\pi_1 = \tilde{u}(ws\alpha w, w)c$$

the left-hand side of equation ($\star$) becomes

$$\pi(\omega\pi_0\pi_0, u_0\theta\pi_0\pi_0 w, u\pi_0\pi_1) = (\pi\omega\pi_0\pi_0, \pi u_0\theta\pi_0\pi_0 w, \tilde{u}ws\alpha w, \tilde{u}w)c.$$

Rewriting equation ($\star$) while recalling that $\pi u = \tilde{u}\gamma$ and $\gamma\pi_1\pi_0 = 1_W$ gives

$$(\pi\omega\pi_0\pi_0, \pi u_0\theta\pi_0\pi_0 w, \tilde{u}ws\alpha w, \tilde{u}w)c = (\pi\omega\pi_0\pi_0, \pi u_0\theta\pi_1\pi_0, \tilde{u}w)c \tag{$\star\star$}$$

and induces a unique $\delta' : \tilde{U} \to P(\mathbb{C})\,{}_t\!\times_{ws} W$ such that

$$\begin{aligned} \delta'\pi_1 &= \tilde{u} \\ \delta'\pi_0\pi_0 &= (\pi\omega\pi_0\pi_0, \pi u_0\theta\pi_0\pi_0 w, \tilde{u}ws\alpha w)c \\ \delta'\pi_0\pi_0 &= (\pi\omega\pi_0\pi_0, \pi u_0\theta\pi_1\pi_0)c. \end{aligned}$$

equation ($\star\star$) can then be simplified as

$$\delta'(\pi_0\pi_0, \pi_1)c = \delta'(\pi_0\pi_1, \pi_1)c$$

which induces the unique map $\delta : \tilde{U} \to \mathcal{P}_{cq}(\mathbb{C})$ such that

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{C}) \arrow[r, tail, "\iota_{cq}"] & P(\mathbb{C})\,{}_t\!\times_{ws} W \\
\tilde{U}. \arrow[u, dotted, "\delta"] \arrow[ur, "\delta'"'] &
\end{tikzcd}$$

The two (families of) sailboats, $\varphi, \psi : \hat{U} \to \mathrm{sb}$, with a common projection, $\varphi\pi_1 = \psi\pi_1$, can now be defined. By definition of

$$W_\circ = (\mathbb{C}_1 \; {}_t{\times}_{ws} W \; {}_{wt}{\times}_{ws} W) \; {}_c{\times}_{ws} W$$

we have

$$\hat{\omega}\pi_1 = (\hat{\omega}\pi_0\pi_0, \hat{u}_0\hat{\delta}\pi_0\iota_{eq}\pi_0 w, \hat{u}\pi\sigma_\circ\pi_0 w)c$$

so let

$$\mu_0 = (\hat{\omega}\pi_0\pi_0, \hat{u}_0\hat{\delta}\pi_0\iota_{eq}\pi_0 w)c$$

to determine the unique pairing map

$$\psi = \big(((\mu_0, \hat{u}\pi\sigma_\circ\pi_0), \hat{\omega}\pi_1), \hat{u}\pi(\omega\pi_0\pi_0, u_0\theta\pi_1\pi_0)c\big).$$

Notice the last map in the composite

$$\hat{u}\pi u\pi_1\pi_1 = \hat{u}\tilde{u}wse$$

is the identity structure map for $\mathbb{C}$. The identity laws in $\mathbb{C}[W^{-1}]$ and $\mathbb{C}$ can then both be used in the final calculation we need to determine the span projections for $\psi : \hat{U} \to \text{sb}$.

$$\begin{aligned}
(\hat{u}\pi\omega\pi_0\pi_0, \hat{u}\pi u_0\theta\pi_1\pi_0)c &= \hat{u}\pi(\omega\pi_0\pi_0, u_0\theta\pi_1\pi_0)c \\
&= (\hat{u}\pi\omega\pi_0\pi_0, \hat{u}\pi u_0\theta\pi_1\pi_0)c \\
&= (\hat{u}\pi\omega\pi_0\pi_0, \hat{u}\pi u_0\theta\pi_1\pi_0(1, te)c)c \\
&= (\hat{u}\pi\omega\pi_0\pi_0, \hat{u}\pi u_0\theta\pi_1\pi_0, \hat{u}\pi u_0\theta\pi_1\pi_0 te)c \\
&= (\hat{u}\pi\omega\pi_0\pi_0, \hat{u}\pi u_0\theta\pi_1\pi_0, \hat{u}\pi u_0\theta\pi_1\pi_1 wse)c \\
&= (\hat{u}\pi\omega\pi_0\pi_0, \hat{u}\pi u_0\theta\pi_1\pi_0, \hat{u}\pi u\pi_1\pi_0 wse)c \\
&= (\hat{u}\pi\omega\pi_0\pi_0, \hat{u}\pi u_0\theta\pi_1\pi_0, \hat{u}\pi u\pi_1\pi_1 se)c \\
&= (\hat{u}\pi\omega\pi_0\pi_0, \hat{u}\pi u_0\theta\pi_1\pi_0, \hat{u}\tilde{u}wsese)c \\
&= (\hat{u}\pi\omega\pi_0\pi_0, \hat{u}\pi u_0\theta\pi_1\pi_0, \hat{u}\tilde{u}wse)c \\
&= (\hat{u}\pi\omega\pi_0\pi_0, \hat{u}\pi u_0\theta\pi_1\pi_0, \hat{u}\pi u\pi_1\pi_1)c \\
&= \hat{u}\pi(\omega\pi_0\pi_0, u_0\theta\pi_1\pi_0, u\pi_1\pi_1)c \\
&= \hat{u}\pi\sigma_\circ\pi_1
\end{aligned}$$

Now we can see

$$\psi p_0 = \psi(\pi_0\pi_0\pi_1, \pi_1) = (\hat{u}\pi\sigma_\circ\pi_0, \hat{u}\pi\sigma_\circ\pi_1) = \hat{u}\pi\sigma_\circ$$

and

$$\psi p_1 = \psi(\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c) = \big(\hat{\omega}\pi_1, (\mu_0, \hat{u}\pi\omega\pi_0\pi_0, \hat{u}\pi u_0\theta\pi_1\pi_0)c\big).$$

For $\varphi : \hat{U} \to \mathrm{sb}$ let

$$\mu_1 = (\mu_0, \hat{u}\pi\omega\pi_0\pi_0, \hat{u}\pi u_0\pi_0\pi_0 w)c$$

and notice on one hand that

$$(\mu_1, \hat{u}\tilde{u}ws\alpha w)c = (\mu_1, \hat{u}\pi u\pi_0\pi_0 w)c = (\mu_0, \hat{u}\pi\sigma_\circ\pi_0 w)c = \hat{\omega}\pi_1$$

and on the other hand that

$$\begin{aligned}(\mu_1, \hat{u}\tilde{u}ws\alpha w)c &= (\hat{\omega}\pi_0\pi_0, \delta\pi_0\iota_{eq}\pi_0, \delta\pi_0\iota_{eq}\pi_1\pi_0)c \\ &= (\hat{\omega}\pi_0\pi_0, \delta\pi_0\iota_{eq}\pi_0, \delta\pi_0\iota_{eq}\pi_1\pi_1)c \\ &= (\mu_0, \hat{u}\pi\omega\pi_0\pi_0, \hat{u}\pi u_0\theta\pi_1\pi_0)c\end{aligned}$$

Then define

$$\varphi = \big(((\mu_1, \hat{u}\tilde{u}ws\alpha), \hat{\omega}\pi_1), \hat{u}\tilde{u}ws\alpha w\big)$$

and we get

$$\begin{aligned}\varphi p_0 &= (\pi_0\pi_0\pi_1, \pi_1) \\ &= (\hat{u}\tilde{u}ws\alpha, \hat{u}\tilde{u}ws\alpha w)\end{aligned}$$

and

$$\begin{aligned}\varphi p_1 &= \varphi\big(\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c\big) \\ &= \big(\hat{\omega}\pi_1, \mu_1, \hat{u}\tilde{u}ws\alpha w)c\big) \\ &= (\hat{\omega}\pi_1, (\mu_0, \hat{u}\pi\omega\pi_0\pi_0, \hat{u}\pi u_0\theta\pi_1\pi_0)c \\ &= \psi p_1.\end{aligned}$$

Combining our computations gives us that

$$\begin{aligned}\hat{u}\tilde{u}(ws\alpha, ws\alpha w)q &= \varphi p_0 q = \varphi p_1 q\\ &= \psi p_1 q = \psi p_0 q\\ &= \hat{u}\pi\sigma_\circ q\\ &= \hat{u}\pi u c'\\ &= \hat{u}\tilde{u}\gamma c'\end{aligned}$$

and since the composite $\hat{u}\tilde{u}$ is epic we can conclude

$$(ws\alpha, ws\alpha w)q = \gamma c'.$$

□

Now we prove the second main result of this section.

Proposition 55. *The localization (internal) functor,* $L : \mathbb{C} \to \mathbb{C}[W^{-1}]$ *inverts* $w : W \to \mathbb{C}_1$.

Proof. Consider the composite

$$\begin{tikzcd} W \arrow[r, "(1{,}wse)"] \arrow[dr] & \mathrm{spn} \arrow[d, two heads, "q"] \\ & \mathbb{C}[W^{-1}]_1 \end{tikzcd}$$

In the proofs of Lemma 54 and Corollary 53 we have already seen that

$$(1, wse)qs = wL_1 t, \qquad (1, wse)qt = wL_1 s$$

so it suffices to show the last two diagrams from Definition 49 commute in this setting.

First note that $e = (\alpha, \alpha w)q : \mathbb{C}_0 \to \mathbb{C}[W^{-1}]_1$ is the identity structure map on $\mathbb{C}[W^{-1}]$, and that the source map $s : \mathbb{C}[W^{-1}]_1 \to \mathbb{C}_0$ is uniquely determined by the map $qs = \pi_0 wt : \mathrm{spn} \to \mathbb{C}_0$. Also recall that

$$wL_1 = (ws\alpha, (ws\alpha, w)c)q$$

and then compute

$$\begin{aligned} wL_1se &= wL_1s(\alpha, \alpha w)q \\ &= (ws\alpha, (ws\alpha, w)c)qs(\alpha, \alpha w)q \\ &= (ws\alpha, (ws\alpha, w)c)\pi_0 wt(\alpha, \alpha w)q \\ &= ws\alpha wt(\alpha, \alpha w)q \\ &= ws(\alpha, \alpha w)q. \end{aligned}$$

We can replace the left and bottom composite in the commuting square of Lemma 54 by the last equation to give the commuting square

$$\begin{tikzcd} W \arrow[r, "\left(wL_1, (1, wse)q\right)"] \arrow[d, "wL_1"'] & \mathbb{C}[W^{-1}]_2 \arrow[d, "c"] \\ \mathbb{C}[W^{-1}]_1 \arrow[r, "se"'] & \mathbb{C}[W^{-1}]_1 \end{tikzcd}$$

in $\mathcal{E}$ and shows $(1, wse)q : W \to \mathbb{C}[W^{-1}]_1$ satisfies half of Definition 49. For the rest of it we recall that $qt = \pi_1 t : \text{spn} \to \mathbb{C}_0$ uniquely determines the structure map $t : \mathbb{C}[W^{-1}]_1 \to \mathbb{C}_0$ and similarly compute

$$\begin{aligned} wL_1te &= wL_1t(\alpha, \alpha w)q \\ &= (ws\alpha, (ws\alpha, w)c)qt(\alpha, \alpha w)q \\ &= (ws\alpha, (ws\alpha, w)c)\pi_1 t(\alpha, \alpha w)q \\ &= (ws\alpha, w)ct(\alpha, \alpha w)q \\ &= wt(\alpha, \alpha w)q. \end{aligned}$$

Putting this together with Lemma 52 gives

$$wL_1te = wt(\alpha, \alpha w)q = (1, w)q$$

and allows us to rewrite the commuting square in Corollary 53 as

$$\begin{tikzcd} W \arrow[r, "\left((1, wse)q, wL_1\right)"] \arrow[d, "wL_1"'] & \mathbb{C}[W^{-1}]_2 \arrow[d, "c"] \\ \mathbb{C}[W^{-1}]_1 \arrow[r, "te"'] & \mathbb{C}[W^{-1}]_1 \end{tikzcd} .$$

This means $(1,wse)q$ inverts (wL_1) by Definition 49. □

4.6 Universal Property of Internal Fractions

The main result of this section is Theorem 65, the universal property of internal localization. It is an isomorphism of categories between the category of internal functors, $\mathbb{C} \to \mathbb{D}$, that invert $w : W \to \mathbb{C}_1$ and their natural transformations, and the category of internal functors $\mathbb{C}[W^{-1}] \to \mathbb{D}$ and their natural transformations. In Section 4.6.1 we prove that the objects in each category uniquely correspond to one another in Proposition 60, and then in Lemma 63 we show that the 2-cells in each category uniquely correspond to one another. In Lemma 64 we show that the correspondence between natural transformations is functorial, and Theorem 65 follows immediately.

4.6.1 Correspondence Between 1-cells

The results in this subsection come together to prove that for any internal functor $F : \mathbb{C} \to \mathbb{X}$ that inverts $w : W \to \mathbb{C}_1$, there exists a unique internal functor $[F] : \mathbb{C}[W^{-1}] \to \mathbb{X}$ such that the diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, "F"] \arrow[dr, "L"'] & & \mathbb{X} \\
& \mathbb{C}[W^{-1}] \arrow[ur, dotted, "{[F]}"'] &
\end{tikzcd}$$

commutes. First we define $[F]$ and prove it is an internal functor, then we notice how every internal functor $\mathbb{C}[W^{-1}] \to \mathbb{X}$ corresponds to an internal functor $\mathbb{C} \to \mathbb{X}$ that inverts W by pre-composition with $L : \mathbb{C} \to \mathbb{X}$. Finally we show that these assignments are inverses to one another to prove the main result of this subsection, Proposition 60.

It's clear how to define $[F]$ on objects:

$$\begin{tikzcd}
\mathbb{C}[W^{-1}]_0 \arrow[r, "{[F]_0}"] \arrow[d, equal] & \mathbb{X}_0 \arrow[d, equal] \\
\mathbb{C}_0 \arrow[r, "F_0"'] & \mathbb{X}_0
\end{tikzcd} .$$

On arrows we use the universal property of the coequalizer $\mathbb{C}[W^{-1}]_1$. By Definition 49, there exists a map $F(w)^{-1} : W \to \mathbb{X}_1$ that inverts $wF_1 : W \to \mathbb{X}_1$ in $\mathbb{X}$. That is, the diagrams

$$\begin{tikzcd}
W \arrow[r, "{(wF_1, F(w)^{-1})}"] \arrow[d, "wF_1"'] & \mathbb{X}_2 \arrow[d, "c"] \\
\mathbb{X}_1 \arrow[r, "se"'] & \mathbb{X}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, "{(F(w)^{-1}, wF_1)}"] \arrow[d, "wF_1"'] & \mathbb{X}_2 \arrow[d, "c"] \\
\mathbb{X}_1 \arrow[r, "te"'] & \mathbb{X}_1
\end{tikzcd}$$

commute in $\mathcal{E}$. In particular we have that

$$F(w)^{-1}t = wF_1 s$$

and this along with functoriality of F and the definition of $\mathrm{spn} = W\ {}_{ws}\!\times_s \mathbb{C}_1$ is enough to see that the outside of the diagram,

$$\begin{tikzcd}
\mathrm{spn} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, dotted, "{[F]'}"] & \mathbb{C}_1 \arrow[dr, bend left, "F_1"] & \\
W \arrow[dr, bend right, "F(w)^{-1}"'] & \mathbb{X}_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{X}_1 \arrow[d, "s"] \\
 & \mathbb{X}_1 \arrow[r, "t"'] & \mathbb{X}_0
\end{tikzcd},$$

commutes and induces the unique map $[F]' : \mathrm{spn} \to \mathbb{X}_2$. This map is used to define $[F]_1 : \mathbb{C}[W^{-1}]_1 \to \mathbb{X}_1$ in the following lemma by the universal property of the coequalizer $\mathbb{C}[W^{-1}]_1$.

Lemma 56. *The coequalizer diagram,*

$$\begin{tikzcd}
sb \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & spn \arrow[r, two heads, "q"] \arrow[d, "{[F]'}"'] & \mathbb{C}[W^{-1}]_1 \arrow[d, dotted, "{[F]_1}"] \\
 & \mathbb{X}_2 \arrow[r, "c"] & \mathbb{X}_1
\end{tikzcd}$$

commutes in $\mathcal{E}$ and uniquely determines the map $[F]_1 : \mathbb{C}[W^{-1}]_1 \to \mathbb{X}_1$.

Proof. The main idea here is that the left legs of the two spans inhabiting a sailboat, represented by $p_0, p_1 : \mathrm{sb} \to \mathrm{spn}$, are arrows coming from W. These are part of a commuting triangle represented by $\pi_0 : \mathrm{sb} \to W_\triangle$. More precisely, the left leg of the

p_1 projection factors through the left leg of the p_0 projection by the arrow represented by the map $\pi_0\pi_0\pi_0 : \mathrm{sb} \to \mathbb{C}_1$ in $\mathcal{E}$. This is shown in the following calculation:

$$\begin{aligned} p_1\pi_0 w &= \pi_0\pi_1 w \\ &= \pi_0(\pi_0\pi_0, \pi_0\pi_1 w)c \\ &= (\pi_0\pi_0\pi_0, \pi_0\pi_0\pi_1 w)c \\ &= (\pi_0\pi_0\pi_0, p_0\pi_0 w)c \end{aligned}$$

Functoriality of F then gives

$$p_1\pi_0 w F_1 = (\pi_0\pi_0\pi_0 F_1, p_0\pi_0 w F_1)c.$$

The internal functor F inverts the arrows coming from $w : W \to \mathbb{C}_1$ so we can internally post-compose with $p_0\pi_0 F(w)^{-1} : \mathrm{sb} \to \mathbb{X}_1$ to give the following calculation. This calculation uses associativity and the identity laws for internal composition in $\mathbb{X}$, along with the definitions of $F(w)^{-1}$ and sb and functoriality of F.

$$\begin{aligned} (p_1\pi_0 w F_1, p_0\pi_0 F(w)^{-1})c &= (\pi_0\pi_0\pi_0 F_1, p_0\pi_0 w F_1, p_0\pi_0 F(w)^{-1})c \\ &= (\pi_0\pi_0\pi_0 F_1, p_0\pi_0(w F_1, F(w)^{-1})c)c \\ &= (\pi_0\pi_0\pi_0 F_1, p_0\pi_0 w F_1 s e)c \\ &= (\pi_0\pi_0\pi_0 F_1, \pi_0\pi_0\pi_1 w F_1 s e)c \\ &= (\pi_0\pi_0\pi_0 F_1, \pi_0\pi_0\pi_1 w s F_0 e)c \\ &= (\pi_0\pi_0\pi_0 F_1, \pi_0\pi_0\pi_0 t F_0 e)c \\ &= (\pi_0\pi_0\pi_0 F_1, \pi_0\pi_0\pi_0 F_1 t e)c \\ &= \pi_0\pi_0\pi_0 F_1(1, te)c \\ &= \pi_0\pi_0\pi_0 F_1 \end{aligned}$$

A similar internal composition involving the first and last terms in the equation above with $p_1\pi_0 F(w)^{-1} : \mathrm{sb} \to \mathbb{X}_1$ gives

$$\begin{aligned}
(p_1\pi_0F(w)^{-1}, \pi_0\pi_0\pi_0F_1)c &= (p_1\pi_0F(w)^{-1}, p_1\pi_0wF_1, p_0\pi_0F(w)^{-1})c \\
&= (p_1\pi_0(F(w)^{-1}, wF_1)c, p_0\pi_0F(w)^{-1})c \\
&= (p_1\pi_0wF_1te, p_0\pi_0F(w)^{-1})c \\
&= (p_1\pi_0wtF_0e, p_0\pi_0F(w)^{-1})c \\
&= (p_0\pi_0wtF_0e, p_0\pi_0F(w)^{-1})c \\
&= (p_0\pi_0wF_1te, p_0\pi_0F(w)^{-1})c \\
&= (p_0\pi_0F(w)^{-1}se, p_0\pi_0F(w)^{-1})c \\
&= p_0\pi_0F(w)^{-1}(se, 1)c \\
&= p_0\pi_0F(w)^{-1}
\end{aligned}$$

Now we can substitute the last equation into the following calculation to see $[F]'c$ coequalizes the pair p_0 and p_1:

$$\begin{aligned}
p_0[F]'c &= p_0(\pi_0F(w)^{-1}, \pi_1F_1)c \\
&= (p_0\pi_0F(w)^{-1}, p_0\pi_1F_1)c \\
&= \big((p_1\pi_0F(w)^{-1}, \pi_0\pi_0\pi_0F_1)c, p_0\pi_1F_1\big)c \\
&= (p_1\pi_0F(w)^{-1}, \pi_0\pi_0\pi_0F_1, p_0\pi_1F_1)c \\
&= (p_1\pi_0F(w)^{-1}, (\pi_0\pi_0\pi_0F_1, p_0\pi_1F_1)c)c \\
&= (p_1\pi_0F(w)^{-1}, (\pi_0\pi_0\pi_0, p_0\pi_1)cF_1)c \\
&= (p_1\pi_0F(w)^{-1}, p_1\pi_1F_1)c \\
&= p_1(\pi_0F(w)^{-1}, \pi_1F_1)c \\
&= p_1[F]'c
\end{aligned}$$

The existence and uniqueness of the map $[F]_1 : \mathbb{C}[W^{-1}]_1 \to \mathbb{X}_1$ such that $q[F]_1 = [F]'c$ follows from the universal property of $\mathbb{C}[W^{-1}]_1$. □

The next step is to show that $[F] = ([F]_0, [F]_1)$ is an internal functor. First we show identities are preserved by proving the following lemma.

Lemma 57. *The diagram*

$$\begin{tikzcd}
\mathbb{C}_0 \arrow[r, "F_0"] \arrow[d, "(\alpha{,}\alpha w)q"'] & \mathbb{X}_0 \arrow[d, "e"] \\
\mathbb{C}[W^{-1}]_1 \arrow[r, "{[F]_1}"'] & \mathbb{X}_1
\end{tikzcd}$$

commutes in $\mathcal{E}$*.*

Proof. By the universal property of the pullback $\mathbb{X}_2$, the definition of $F(w)^{-1}$, functoriality of F, and the fact that α is a section of wt we have

$$\begin{aligned}
(\alpha, \alpha w)q[F]_1 &= (\alpha, \alpha w)[F]'c \\
&= (\alpha, \alpha w)(\pi_0 F(w)^{-1}, \pi_1 F_1)c \\
&= (\alpha F(w)^{-1}, \alpha w F_1)c \\
&= \alpha(F(w)^{-1}, wF_1)c \\
&= \alpha w F_1 te \\
&= \alpha wt F_0 e \\
&= F_0 e
\end{aligned}$$

□

The following lemma shows that $[F]$ preserves (internal) composition. When $\mathcal{E} = \mathbf{Set}$ this is saying that for any pair of composable spans

$$\begin{tikzcd}
a & b \arrow[l, "\circ" marking, "v"'] \arrow[r, "f"] & c & d \arrow[l, "\circ" marking, "v'"'] \arrow[r, "f'"] & e
\end{tikzcd}$$

with composite span,

$$\begin{tikzcd}
a & b' \arrow[l, "\circ" marking, "v''"'] \arrow[r, "f''"] & e
\end{tikzcd} ,$$

the diagram

$$\begin{tikzcd}
F(a) \arrow[r, "F(v)^{-1}"] \arrow[dd, "f(v'')^{-1}"'] & F(b) \arrow[r, "F(f)"] & F(c) \arrow[d, "f(v')^{-1}"] \\
 & & F(d) \arrow[d, "F(f')"] \\
F(b') \arrow[rr, "F(f'')"'] & & F(e)
\end{tikzcd}$$

commutes in $\mathbb{X}$. To see this we look at the composition data

$$\begin{tikzcd}
 & & e \arrow[ddll, bend right, "\circ" marking, "v''"'] \arrow[d, "h"] \arrow[ddrr, bend left, "f''"] & & \\
 & & d \arrow[dl, "\circ" marking, "v_0"'] \arrow[dr, "k"] & & \\
a & b \arrow[l, "\circ" marking, "v"] \arrow[r, "f"'] & c & b' \arrow[l, "\circ" marking, "v'"] \arrow[r, "f'"'] & a'
\end{tikzcd}$$

and apply the functor F to the weak-composition triangle on the left to get the equation

$$F(v'') = F(h)F(v_0)F(v).$$

Since F inverts W, we can pre-compose both sides by $F(v'')^{-1}$ and post-compose them both by $F(v)^{-1}$ to get the equation

$$F(v)^{-1} = F(v'')^{-1}F(h)F(v_0).$$

Similarly, applying F to the Ore-square gives the equation

$$F(v_0)F(f) = F(k)F(v')$$

and post-composing with $F(v')^{-1}$ gives

$$F(v_0)F(f)F(v')^{-1} = F(k).$$

Put it all together with functoriality of F to see the square commutes.

$$\begin{aligned}
F(v)^{-1}F(f)F(v')^{-1}F(f') &= F(v'')^{-1}F(h)F(v_0)F(f)F(v')^{-1}F(f') \\
&= F(v'')^{-1}F(h)F(k)F(f') \\
&= F(v'')^{-1}F(f'').
\end{aligned}$$

Lemma 58. *The diagram,*

$$\begin{tikzcd}
\mathbb{C}[W^{-1}]_2 \arrow[r, "{[F]_1 \times [F]_1}"] \arrow[d, "c"'] & \mathbb{X}_2 \arrow[d, "c"] \\
\mathbb{C}[W^{-1}]_1 \arrow[r, "{[F]_1}"'] & \mathbb{X}_1
\end{tikzcd}$$

where $[F]_1 \times [F]_1 = (\pi_0[F]_1, \pi_1[F]_1)$ *is the unique pairing map, commutes in* $\mathbb{X}$.

Proof. Recall that $u : U \twoheadrightarrow \text{spn}\,{}_t\!\times_s \text{spn}$ is the cover on which we defined composition, with $u = u_0u_1$ in the diagram constructed by the Internal Fractions Axioms:

$$\begin{tikzcd}
W_\circ \arrow[rr, "(\pi_0\pi_1{,}\pi_0\pi_2)"] & & W \times_{\mathbb{C}_0} W & \\
U \arrow[u, "\omega"] \arrow[rr, "u_0", "\shortmid" marking] \arrow[d, "\sigma_\circ"] & & U_0 \arrow[u, "(\theta\pi_0\pi_0{,}u_1\pi_0\pi_0)"'] \arrow[r, "u_1", "\shortmid" marking] \arrow[d, "\theta"'] & \text{spn}\,{}_t\!\times_s \text{spn} \arrow[d, "(\pi_0\pi_1{,}\pi_1\pi_0)"] \\
\text{spn} & & W_\Box \arrow[r, "(\pi_0\pi_1{,}\pi_1\pi_1)"'] & \text{csp}
\end{tikzcd}$$

We use this cover when we need to show certain maps out of $\text{spn}\,{}_t\!\times_s \text{spn}$ are equal. More precisely, by showing it (or its composition with other epimorphisms) equalizes two maps we are interested in proving are equal. We begin this proof with the weak-composition triangle, $\omega : U \to W_\circ$, and the equation encoding it is commutativity.

$$\omega_0\pi_1 = (\omega_0\pi_0\pi_0, u_0\theta\pi_0\pi_0 w, u\pi_0\pi_0 w)c$$

By functoriality of F we have

$$\omega\pi_1 wF_1 = (\omega_0\pi_0\pi_0F_1, u_0\theta\pi_0\pi_0wF_1, u\pi_0\pi_0wF_1)c.$$

We can internally pre-compose both sides of $\mathbb{X}$ with the map $\omega\pi_1F(w)^{-1} : U \to \mathbb{X}_1$ and internally post-compose with $u\pi_0\pi_0F(w)^{-1} : U \to \mathbb{X}_1$. Before writing down the new equation however we do the following intermediate calculation:

$$\begin{aligned}
(\omega\pi_1F(w)^{-1}, \omega\pi_1wF_1)c &= \omega\pi_1(F(w)^{-1}, wF_1)c \\
&= \omega\pi_1wF_1te \\
&= \omega\pi_1wteF_1 \\
&= u\pi_0\pi_0wteF_1 \\
&= u\pi_0\pi_0wF_1te \\
&= u\pi_0\pi_0F(w)^{-1}se
\end{aligned}$$

Using this along with the definitions of $W_\Box$, θ, and csp gives

$$\begin{aligned}(u\pi_0\pi_0 wF_1, u\pi_0\pi_0 F(w)^{-1})c &= u\pi_0\pi_0(wF_1, F(w)^{-1})c\\ &= u\pi_0\pi_0 wF_1 se\\ &= u\pi_0\pi_0 wseF_1\\ &= u_0u_1\pi_0\pi_1 wteF_1\\ &= u_0\theta\pi_0\pi_0 wteF_1\\ &= u_0\theta\pi_0\pi_0 wF_1 te.\end{aligned}$$

Now the pre/post-composed equation is

$$\begin{aligned}&(u\pi_0\pi_0 F(w)^{-1}se, u\pi_0\pi_0 F(w)^{-1})c\\ &== (\omega\pi_1 F(w)^{-1}, \omega_0\pi_0\pi_0 F_1, u_0\theta\pi_0\pi_0 wF_1, u_0\theta\pi_0\pi_0 wF_1 te)c\end{aligned}$$

where the left side simplifies via the identity law in $\mathbb{X}$ as

$$(u\pi_0\pi_0 F(w)^{-1}se, u\pi_0\pi_0 F(w)^{-1})c = u\pi_0\pi_0 F(w)^{-1}(se, 1)c = u\pi_0\pi_0 F(w)^{-1}$$

and the right side simplifies similarly as

$$\begin{aligned}&(\omega\pi_1 F(w)^{-1}, \omega_0\pi_0\pi_0 F_1, u_0\theta\pi_0\pi_0 wF_1, u_0\theta\pi_0\pi_0 wF_1 te)c\\ &= \big(\omega\pi_1 F(w)^{-1}, \omega_0\pi_0\pi_0 F_1, u_0\theta\pi_0\pi_0 wF_1(1, te)c\big)c\\ &= \big(\omega\pi_1 F(w)^{-1}, \omega_0\pi_0\pi_0 F_1, u_0\theta\pi_0\pi_0 wF_1\big)c\end{aligned}$$

giving the simplified equation:

$$u\pi_0\pi_0 F(w)^{-1} = \big(\omega\pi_1 F(w)^{-1}, \omega_0\pi_0\pi_0 F_1, u_0\theta\pi_0\pi_0 wF_1\big)c. \qquad (\star)$$

Now take the Ore-square, $u_0\theta : U \to W_\Box$, and the equation describing commutativity,

$$(u_0\theta\pi_0\pi_0 w, u\pi_0\pi_1)c = (u_0\theta\pi_1\pi_0, u\pi_1\pi_0 w)c,$$

and apply F to get the equation

$$(u_0\theta\pi_0\pi_0 wF_1, u\pi_0\pi_1 F_1)c = (u_0\theta\pi_1\pi_0 F_1, u\pi_1\pi_0 wF_1)c.$$

Since F inverts $w : W \to \mathbb{C}_1$ we can post-compose both sides with $u\pi_1\pi_0 F(w)^{-1} : U \to \mathbb{X}_1$ to get a new equation. The following computation showing how this is done follows more or less by the definitions of $F(w)^{-1}$ and θ and functoriality of F:

$$\begin{aligned}(u\pi_1\pi_0 wF_1, u\pi_1\pi_0 F(w)^{-1})c &= u\pi_1\pi_0(wF_1, F(w)^{-1})c \\ &= u\pi_1\pi_0 wF_1 se \\ &= u\pi_1\pi_0 wseF_1 \\ &= u_0\theta\pi_1\pi_0 teF_1 \\ &= u_0\theta\pi_1\pi_0 F_1 te\end{aligned}$$

Adding internal composition with $u_0\theta\pi_1\pi_0 F_1 : U \to \mathbb{X}_1$ on the right of both side of the internal compositions shown in the last equation and applying the identity law in $\mathbb{X}$ gives:

$$\begin{aligned}(u_0\theta\pi_1\pi_0 F_1, u\pi_1\pi_0 wF_1, u\pi_1\pi_0 F(w)^{-1})c &= (u_0\theta\pi_1\pi_0 F_1, u_0\theta\pi_1\pi_0 F_1 te)c \\ &= u_0\theta\pi_1\pi_0 F_1(1, te)c \\ &= u_0\theta\pi_1\pi_0 F_1.\end{aligned}$$

Then by the definition of $W_\square$ we get the equation

$$\begin{aligned}&(u_0\theta\pi_0\pi_0 wF_1, u\pi_0\pi_1 F_1, u\pi_1\pi_0 F(w)^{-1})c \\ &= ((u_0\theta\pi_0\pi_0 wF_1, u\pi_0\pi_1 F_1)c, u\pi_1\pi_0 F(w)^{-1})c \\ &= ((u_0\theta\pi_1\pi_0 F_1, u\pi_1\pi_0 wF_1)c, u\pi_1\pi_0 F(w)^{-1})c \\ &= (u_0\theta\pi_0\pi_0 wF_1, u\pi_0\pi_1 F_1, u\pi_1\pi_0 F(w)^{-1})c \\ &= u_0\theta\pi_1\pi_0 F_1. \qquad (\star\star)\end{aligned}$$

which simplifies to the equality

$$(u_0\theta\pi_0\pi_0 wF_1, u\pi_0\pi_1 F_1, u\pi_1\pi_0 F(w)^{-1})c = u_0\theta\pi_1\pi_0 F_1. \qquad (\star\star)$$

which we use in following calculation that shows $[F]$ preserves composition. By equations $(\star\star)$ and $(\star)$ along with functoriality of F and the identity law in $\mathbb{X}$ we have:

$$
\begin{aligned}
&u(q \times q)c[F]_1 \\
&= uc'[F]_1 \\
&= \sigma_c q[F]_1 \\
&= \sigma_c[F]' \\
&= (\sigma_c\pi_0 F(w)^{-1}, \sigma_c\pi_1 F_1)c \\
&= (\omega\pi_1 F(w)^{-1}, \omega\pi_0\pi_0 F_1, u_0\theta\pi_1\pi_0 F_1, u\pi_1\pi_1 F_1)c \\
&= (\omega\pi_1 F(w)^{-1}, \omega\pi_0\pi_0 F_1, u_0\theta\pi_0\pi_0 w F_1, u\pi_0\pi_1 F_1, u\pi_1\pi_0 F(w)^{-1}, u\pi_1\pi_1 F_1)c \\
&= ((\omega\pi_1 F(w)^{-1}, \omega\pi_0\pi_0 F_1, u_0\theta\pi_0\pi_0 w F_1)c, u\pi_0\pi_1 F_1, u\pi_1\pi_0 F(w)^{-1}, u\pi_1\pi_1 F_1)c \\
&= (u\pi_0\pi_0 F(w)^{-1}, u\pi_0\pi_1 F_1, u\pi_1\pi_0 F(w)^{-1}, u\pi_1\pi_1 F_1)c \\
&= \big(u\pi_0(\pi_0 F(w)^{-1}, \pi_1 F_1)c, u\pi_1(\pi_0 F(w)^{-1}, \pi_1 F_1)c\big)c \\
&= \big(u\pi_0[F]'c, u\pi_1[F]'c\big)c \\
&= u\big(\pi_0 q[F]_1, \pi_1 q[F]_1\big)c \\
&= u(q \times q)([F]_1 \times [F]_1)c.
\end{aligned}
$$

The composite $u(q \times q)$ is epic, so we get

$$c[F]_1 = ([F]_1 \times [F]_1)c$$

as desired. □

Proposition 59. *The maps* $[F]_0 = F_0 : \mathbb{C}_0 \to \mathbb{X}_0$ *and* $[F]_1 : \mathbb{C}[W^{-1}]_1 \to \mathbb{X}_1$ *determine an internal functor* $[F] : \mathbb{C}[W^{-1}] \to \mathbb{X}$ *such that the diagram*

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, "F"] \arrow[dr, "L"'] & & \mathbb{X} \\
& \mathbb{C}[W^{-1}] \arrow[ur, "{[F]}"'] &
\end{tikzcd}$$

commutes in $\mathcal{E}$.

Proof. Functoriality follows from Lemma 57 and Lemma 58. To see the diagram commutes we can immediately see

$$L_0[F]_0 = 1_{\mathbb{C}_0}F_0 = F_0$$

and then use the definitions of L_1, $[F]_1$, and $[F]'$, along with functoriality of F, the identity law, $(se, 1)c = 1$, in $\mathbb{X}$, and the fact that α is a section of wt to compute

$$\begin{aligned}
L_1[F]_1 &= (s\alpha, (s\alpha w, 1)c)q[F]_1 \\
&= (s\alpha, (s\alpha w, 1)c)[F]'c \\
&= (s\alpha, (s\alpha w, 1)c)(\pi_0 F(w)^{-1}, \pi_1 F_1)c \\
&= (s\alpha F(w)^{-1}, (s\alpha w, 1)cF_1)c \\
&= (s\alpha F(w)^{-1}, s\alpha w F_1, F_1)c \\
&= (s\alpha(F(w)^{-1}, wF_1)c, F_1)c \\
&= (s\alpha w F_1 te, F_1)c \\
&= (s\alpha wte F_1, F_1)c \\
&= (seF_1, F_1)c \\
&= (F_1 se, F_1)c \\
&= F_1(se, 1)c \\
&= F_1.
\end{aligned}$$

□

Proposition 60. *Every internal functor $F : \mathbb{C} \to \mathbb{X}$ that inverts $w : W \to \mathbb{C}_1$ corresponds uniquely to an internal functor $[F] : \mathbb{C}[W^{-1}] \to \mathbb{X}$.*

Proof. Lemma 59 implies the forward direction. Now notice that for any internal functor $G : \mathbb{C}[W^{-1}] \to \mathbb{X}$, there is an internal functor $LG : \mathbb{C} \to \mathbb{X}$ given by pre-composing with the localization functor $L : \mathbb{C} \to \mathbb{C}[W^{-1}]$. In Proposition 55 we saw that L inverts $w : W \to \mathbb{C}_1$, with $(wL)^{-1} = (1, wse)q : W \to \mathbb{C}[W^{-1}]_1$. Functoriality of G implies $(wL)^{-1}G_1 : W \to \mathbb{X}_1$ is an inverse of $wLG : W \to \mathbb{X}_1$ in $\mathbb{X}$ so this establishes the other direction of the correspondence.

For any $F : \mathbb{C} \to \mathbb{X}$ inverting $w : W \to \mathbb{C}_1$, let $[F] : \mathbb{C}[W^{-1}] \to \mathbb{X}$ be the corresponding internal functor. Pre-composing with $L : \mathbb{C} \to \mathbb{X}$ gives

$$L[F] = F$$

so composing the assignments in one direction gives an identity. On the other hand, for any $G : \mathbb{C}[W^{-1}] \to \mathbb{X}$, if we pre-compose with $L : \mathbb{C} \to \mathbb{C}[W^{-1}]$ and then find the corresponding internal functor $\mathbb{C}[W^{-1}] \to \mathbb{X}$ we get that

$$[LG] : \mathbb{C}[W^{-1}] \to \mathbb{X}$$

where $q[LG] = [LG]'c : \text{spn} \to \mathbb{X}_1$. Now expanding this with the explicit definition of

$$[LG]' = (\pi_0 (LG)(w)^{-1}, \pi_1 (LG)_1)$$

we can use the definition

$$(LG)(w)^{-1} = (wL)^{-1}G_1 = (1, wse)qG_1,$$

functoriality of L and G, the definition

$$L_1 = (s\alpha, (s\alpha w, 1)c)q,$$

a bit of factoring with pairing maps, the fact that $q_2 c = c' : \text{spn}\ {}_t{\times}_s\ \text{spn} \to \text{spn}$, and Lemma 51 in the last line to see:

$$\begin{aligned}
[LG]'c &= (\pi_0 (LG)(w)^{-1}, \pi_1 (LG)_1)c \\
&= (\pi_0 (wL)^{-1} G_1, \pi_1 L_1 G_1)c \\
&= (\pi_0 (wL)^{-1}, \pi_1 L_1)cG_1 \\
&= (\pi_0 (1, wse)q, \pi_1 (s\alpha, (s\alpha w, 1)c)q)cG_1 \\
&= ((\pi_0, \pi_0 wse)q, (\pi_1 s\alpha, (\pi_1 s\alpha w, \pi_1)c)q)cG_1 \\
&= ((\pi_0, \pi_0 wse), (\pi_0 s\alpha, (\pi_0 s\alpha w, \pi_1)c)q_2 cG_1 \\
&= ((\pi_0, \pi_0 wse), (\pi_0 s\alpha, (\pi_0 s\alpha w, \pi_1)c)c' G_1 \\
&= qG_1
\end{aligned}$$

This implies that $[LG] = G_1$ by the universal property of the coequalizer $\mathbb{C}[W^{-1}]_1$. We have shown that the assignments in either direction are inverses to one another, so this correspondence is unique. $\square$

4.6.2 Correspondence Between 2-Cells

Next we show the 2-cell correspondence between internal natural transformations for the internal functors in the 1-cell correspondence of Proposition 60.

In this subsection we see that internal natural transformations, $\alpha : F \implies G$, between internal functors, $F, G : \mathbb{C} \to \mathbb{X}$, that invert $w : W \to \mathbb{C}_1$ correspond uniquely to natural transformations, $[\alpha] : [F] \implies [G]$, between the uniquely corresponding internal functors $[F], [G] : \mathbb{C}[W^{-1}] \to \mathbb{X}$ from Section 4.6.1. The main result of Section 4.6 is the isomorphism of categories established in Theorem 65.

We begin with a lemma that shows one direction of the correspondence between the aforementioned natural transformations,

Lemma 61. *Every internal natural transformation,*

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathbb{X} \end{tikzcd} ,$$

induces a canonical natural transformation:

$$\begin{tikzcd} \mathbb{C}[W^{-1}] \arrow[r, bend left=50, "{[F]}"] \arrow[r, bend right=50, "{[G]}"'] \arrow[r, phantom, "\Downarrow {[\alpha]}"] & \mathbb{X} \end{tikzcd}$$

Proof. Since $\mathbb{C}[W^{-1}]_0 = \mathbb{C}_0$, define the components of $[\alpha]$ to be the components of α:

$$\begin{tikzcd} \mathbb{C}[W^{-1}]_0 \arrow[r, "{[\alpha]}"] \arrow[d, equal, "L_0"'] & \mathbb{X}_1 \\ \mathbb{C}_0 \arrow[ru, "\alpha"'] & \end{tikzcd}$$

To see this is well-defined we need to show the (naturality) square

$$\begin{tikzcd}
\mathbb{C}[W^{-1}]_1 \arrow[r, "{(s[\alpha],[g]_1)}"] \arrow[d, "{([f]_1,t[\alpha])}"'] & \mathbb{X}_2 \arrow[d, "c"] \\
\mathbb{X}_2 \arrow[r, "c"'] & \mathbb{X}_1
\end{tikzcd}$$

commutes in $\mathcal{E}$. Let $F(w)^{-1}, G(w)^{-1} : W \to \mathbb{X}_1$ denote the inverses of $wF_1, wG_1 : W \to \mathbb{X}_1$. Naturality of $\alpha : F \implies G$ implies the diagram

$$\begin{tikzcd}
W \arrow[r, "{(ws\alpha,wG_1)}"] \arrow[d, "{(wF_1,wt\alpha)}"'] & \mathbb{X}_2 \arrow[d, "c"] \\
\mathbb{X}_2 \arrow[r, "c"'] & \mathbb{X}_1
\end{tikzcd}$$

commutes in $\mathcal{E}$. Using internal composition in $\mathbb{X}$ to compose with $F(w)^{-1} : W \to \mathbb{X}_1$ on the left and $G(w)^{-1} : W \to \mathbb{X}_1$ on the right on both sides gives a new commuting diagram,

$$\begin{tikzcd}
W \arrow[r, "{(F(W)^{-1},ws\alpha)}"] \arrow[d, "{(wt\alpha,g(W)^{-1})}"'] & \mathbb{X}_2 \arrow[d, "c"] \\
\mathbb{X}_2 \arrow[r, "c"'] & \mathbb{X}_1
\end{tikzcd},$$

by cancellation using the identity law in $\mathbb{X}$. It will also be helpful to recall the following commuting diagrams from the definition of $\mathbb{C}[W^{-1}]$ and its universal property.

$$\begin{tikzcd}
\mathrm{spn} \arrow[r, two heads, "q"] \arrow[d, "{(\pi_0F(w)^{-1},\pi_1F_1)}"'] & \mathbb{C}[W^{-1}]_1 \arrow[d, dotted, "{[F]_1}"] \\
\mathbb{X}_2 \arrow[r, "c"'] & \mathbb{X}_1
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{spn} \arrow[r, two heads, "q"] \arrow[d, "\pi_0"'] & \mathbb{C}[W^{-1}]_1 \arrow[d, dotted, "s"] \\
W \arrow[r, "wt"'] & \mathbb{C}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{spn} \arrow[r, two heads, "q"] \arrow[d, "\pi_1"'] & \mathbb{C}[W^{-1}]_1 \arrow[d, dotted, "t"] \\
\mathbb{C}_1 \arrow[r, "t"'] & \mathbb{C}_0
\end{tikzcd}$$

Now consider the following diagram:

$$\begin{tikzcd}
\mathbb{C}[W^{-1}]_1 \arrow[d, "{([F]_1,t[\alpha])}"'] & & \mathrm{spn} \arrow[ll, two heads, "q"'] \arrow[rr, two heads, "q"] \arrow[dll, "{(q[F]_1,qt[\alpha])}"] \arrow[drr, "{(qs[\alpha],q[G]_1)}"'] & & \mathbb{C}[W^{-1}]_1 \arrow[d, "{(s[\alpha],[G]_1)}"] \\
\mathbb{X}_2 \arrow[drr, "c"] & & & & \mathbb{X}_2 \arrow[dll, "c"'] \\
& & \mathbb{X}_1 & &
\end{tikzcd}$$

The inside commutes by the following calculation which uses associativity of composition along with naturality of α and the definitions of $[F]_1$, $[G]_1$, and the pullback $\mathrm{spn} = W\ {}_{ws}{\times}_s\ \mathbb{C}_1$:

$$\begin{aligned}
(q[F]_1, qt\alpha)c &= \big((\pi_0 F(w)^{-1}, \pi_1 f_1)c, qt\alpha\big)c && \text{Def. } [F]_1\\
&= \big(\pi_0 F(w)^{-1}, \pi_1 f_1, \pi_1 t\alpha\big)c && \text{Assoc.}\\
&= \big(\pi_0 F(w)^{-1}, \pi_1 (F_1, t\alpha)c\big)c && \text{Assoc.}\\
&= \big(\pi_0 F(w)^{-1}, \pi_1 (s\alpha, G_1)c\big)c && \text{Nat. } \alpha\\
&= \big(\pi_0 F(w)^{-1}, \pi_1 s\alpha, \pi_1 G_1\big)c && \text{Assoc.}\\
&= \big(\pi_0 F(w)^{-1}, \pi_0 ws\alpha, \pi_1 G_1\big)c && \text{Def. spn}\\
&= \big(\pi_0 (F(w)^{-1}, ws\alpha)c, \pi_1 G_1\big)c && \text{Assoc.}\\
&= \big(\pi_0 (wt\alpha, G(w)^{-1})c, \pi_1 G_1\big)c && \text{Nat. } \alpha\\
&= \big(\pi_0 wt\alpha, \pi_0 G(w)^{-1}, \pi_1 G_1\big)c && \text{Assoc.}\\
&= \big(qs\alpha, (\pi_0 G(w)^{-1}, \pi_1 G_1)c\big)c && \text{Assoc.}\\
&= \big(qs\alpha, q[G]_1\big)c && \text{Def. } [g]_1.
\end{aligned}$$

Since q is an epi we can conclude that[α] satisfies the appropriate naturality condition:

$$([F]_1,t[\alpha])c = ([\alpha],[G]_1)c.$$

It's source and target can be computed component-wise by the following commuting diagrams

$$\begin{tikzcd}
\mathbb{C}[W^{-1}]_0 \arrow[r, "{[\alpha]}"] \arrow[d, equal] & \mathbb{X}_1 \arrow[d, "s"] \\
\mathbb{C}_0 \arrow[r, "F_0"'] \arrow[ur, "\alpha"'] \arrow[d, equal] & \mathbb{X}_0 \\
\mathbb{C}[W^{-1}]_0 \arrow[ur, "{[F]_0}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C}[W^{-1}]_0 \arrow[r, "{[\alpha]}"] \arrow[d, equal] & \mathbb{X}_1 \arrow[d, "t"] \\
\mathbb{C}_0 \arrow[r, "G_0"'] \arrow[ur, "\alpha"'] \arrow[d, equal] & \mathbb{X}_0 \\
\mathbb{C}[W^{-1}]_0 \arrow[ur, "{[G]_0}"'] &
\end{tikzcd} ,$$

It follows that $[\alpha] : [F] \implies [G]$ is an internal natural transformation. □

We continue with another lemma establishing the other direction of the correspondence between natural transformations.

Lemma 62. *Every internal natural tranformation,*

$$\begin{tikzcd}
\mathbb{C}[W^{-1}] \arrow[rr, bend left, "H", ""{name=U, below}] \arrow[rr, bend right, "K"', ""{name=D, above}] & & \mathbb{X}
\arrow[Rightarrow, from=U, to=D, "\beta"]
\end{tikzcd} ,$$

induces a canonical natural transformation:

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, bend left, "LH", ""{name=U, below}] \arrow[rr, bend right, "LK"', ""{name=D, above}] & & \mathbb{X}
\arrow[Rightarrow, from=U, to=D, "L\beta"]
\end{tikzcd}$$

Proof. The notation is suggestive of the fact that this these are given by composing with the internal functors $H, K : \mathbb{C}[W^{-1}] \to \mathbb{X}$ and whiskering the internal natural transformation $\beta : H \implies K$ with the internal functor $L : \mathbb{C} \to \mathbb{C}[W^{-1}]$. Note that the components of the whiskered transformation coincide with those of β because L is the identity on objects:

$$\begin{tikzcd}
\mathbb{C}_0 \arrow[r, "L\beta"] \arrow[d, equal, "L_0"'] & \mathbb{X}_1 \\
\mathbb{C}[W^{-1}]_0 \arrow[ur, "\beta"'] &
\end{tikzcd} .$$

□

Lemmas 61 and 62 show us the two directions of the correspondence we need to prove. Now we show the assignments described in the proofs of these lemmas are inverses to get the correspondence we need in the following lemma.

Lemma 63. *Let $F, G : \mathbb{C} \to \mathbb{X}$ be internal functors that invert $w : W \to \mathbb{C}_1$ in $\mathbb{X}$. Then the internal natural transformations*

$$\begin{tikzcd}
\mathbb{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathbb{X} \arrow[from=U, to=D, Rightarrow, "\alpha"]
\end{tikzcd}$$

(bijectively) correspond to the internal natural transformations

$$\begin{tikzcd}
\mathbb{C}[W^{-1}] \arrow[r, bend left=50, "{[F]}"{name=U}] \arrow[r, bend right=50, "{[G]}"'{name=D}] & \mathbb{X} \arrow[from=U, to=D, Rightarrow, "{[\alpha]}"]
\end{tikzcd} .$$

Proof. Let $\alpha : F \implies G$ be an internal natural transformation between internal functors $\mathbb{C} \to \mathbb{X}$ that invert $w : W \to \mathbb{C}_1$ in $\mathbb{X}$. We will show whiskering the internal natural transformation $[\alpha] : [F] \implies [G]$ with $L : \mathbb{C} \to \mathbb{C}[W^{-1}]$ recovers $\alpha : F \implies G$:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, bend left=50, "{L[F]}"{name=U}] \arrow[r, bend right=50, "{L[G]}"'{name=D}] & \mathbb{X} \arrow[from=U, to=D, Rightarrow, "{L[\alpha]}"]
\end{tikzcd} = \begin{tikzcd}
\mathbb{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathbb{X} \arrow[from=U, to=D, Rightarrow, "\alpha"]
\end{tikzcd}$$

By definition of $[F], [G] : \mathbb{C}[W^{-1}] \to \mathbb{X}$ we have

$$L[F] = F \qquad\qquad L[G] = G$$

and the following commuting diagram in $\mathcal{E}$ shows the components of $L[\alpha] : F \implies G$ are precisely those of α:

$$\begin{tikzcd}
\mathbb{C}_0 \arrow[r, "{L[\alpha]}"] \arrow[d, equal, "L_0"'] & \mathbb{X}_1 \\
\mathbb{C}[W^{-1}]_0 \arrow[r, equal] \arrow[ur, "{[\alpha]}"'] & \mathbb{C}_0 \arrow[u, "\alpha"']
\end{tikzcd}$$

On the other hand, for any natural transformation

$$\begin{tikzcd}
\mathbb{C}[W^{-1}] \arrow[rr, bend left, "H"] \arrow[rr, bend right, "K"'] \arrow[rr, phantom, "\Downarrow \beta"] & & \mathbb{X}
\end{tikzcd}$$

we can see that

$$\begin{tikzcd}
\mathbb{C}[W^{-1}] \arrow[rr, bend left, "{[Lh]}"] \arrow[rr, bend right, "{[LK]}"'] \arrow[rr, phantom, "\Downarrow {[L\beta]}"] & & \mathbb{X} & = & \mathbb{C}[W^{-1}] \arrow[rr, bend left, "H"] \arrow[rr, bend right, "K"'] \arrow[rr, phantom, "\Downarrow \beta"] & & \mathbb{X}
\end{tikzcd}$$

by first noticing that the triangles,

$$\begin{tikzcd}
\mathbb{X} & \mathbb{C} \arrow[l, "{L[LH]}"'] \arrow[r, "{L[LK]}"] \arrow[d, "L"] & \mathbb{X} \\
\mathbb{C} \arrow[u, "LH"] \arrow[r, "L"'] & \mathbb{C}[W^{-1}] \arrow[ul, dotted, "{[LH]}"] \arrow[ur, dotted, "{[LK]}"'] & \mathbb{C} \arrow[u, "LK"'] \arrow[l, "L"]
\end{tikzcd},$$

commute in $\mathcal{E}$ and imply that $[LH] = H$ and $[LK] = K$ by the 1-cell universal property of the internal localization in Proposition 60. The following commuting diagram shows that the components for the natural transformations agree too:

$$\begin{tikzcd}
\mathbb{C}[W^{-1}]_0 \arrow[r, "{[L\beta]}"] \arrow[d, equal] & \mathbb{X}_1 \\
\mathbb{C}_0 \arrow[r, equal, "L_0"'] \arrow[ur, "L\beta"] & \mathbb{C}[W^{-1}]_0 \arrow[u, "\beta"']
\end{tikzcd}$$

□

The only other piece we need to prove Theorem 65 is that the correspondence between 2-cells in Lemma 63 is functorial. It suffices to show functoriality in one direction. We show it in the direction of Lemma 61 and leave the other direction (involving whiskering) as an exercise to the reader who wants to take a break.

Lemma 64. *The assignment of natural transformations, $\alpha \mapsto [\alpha]$, in Lemma 61 is functorial.*

Proof. For any internal functor $f : \mathbb{C} \to \mathbb{X}$, we have

$$\begin{tikzcd}
\mathbb{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "F"'{name=D}] \arrow[from=U, to=D, Rightarrow, "1_F"] & \mathbb{X} \arrow[r, mapsto] & \mathbb{C}[W^{-1}] \arrow[r, bend left=50, "{[F]}"{name=U2}] \arrow[r, bend right=50, "{[F]}"'{name=D2}] \arrow[from=U2, to=D2, Rightarrow, "{[1_F]}"] & \mathbb{X}
\end{tikzcd}$$

where the commuting diagram

$$\begin{tikzcd}
\mathbb{C}[W^{-1}]_0 \arrow[r, "{[1_F]}"] \arrow[d, equal] & \mathbb{X}_1 \\
\mathbb{C}_0 \arrow[ur, bend right, "1_F"'] \arrow[d, equal] & \\
\mathbb{C}[W^{-1}]_0 \arrow[uur, bend right, "1_{[F]}"'] &
\end{tikzcd}$$

shows that the components of $[1_F]$ coincide with those of $1_{[f]}$. This means $[1_F] = 1_{[F]}$ are the same natural transformation and so identities are preserved.

To see composition is preserved suppose we have two vertically composable internal natural transformations:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{name=G}] \arrow[r, bend right=60, "H"'{name=H}] \arrow[from=F, to=G, Rightarrow, "\alpha"] \arrow[from=G, to=H, Rightarrow, "\beta"] & \mathbb{X} \arrow[r, mapsto] & \mathbb{C}[W^{-1}] \arrow[r, bend left=60, "{[F]}"{name=F2}] \arrow[r, "{[G]}"{name=G2}] \arrow[r, bend right=60, "{[H]}"'{name=H2}] \arrow[from=F2, to=G2, Rightarrow, "{[\alpha]}"] \arrow[from=G2, to=H2, Rightarrow, "{[\beta]}"] & \mathbb{X}
\end{tikzcd}$$

We can see

$$\begin{tikzcd}
\mathbb{C}[W^{-1}] \arrow[r, bend left=70, "{[F]}", ""{name=F, below}] \arrow[r, "{[G]}", ""{name=G}] \arrow[r, bend right=70, "{[H]}"', ""{name=H, above}] & \mathbb{X} & = & \mathbb{C}[W^{-1}] \arrow[r, bend left, "{[F]}", ""{name=F2, below}] \arrow[r, bend right, "{[H]}"', ""{name=H2, above}] & \mathbb{X}
\arrow[Rightarrow, from=F, to=G, "{[\alpha]}"]
\arrow[Rightarrow, from=G, to=H, "{[\beta]}"]
\arrow[Rightarrow, from=F2, to=H2, "{[\alpha\beta]}"]
\end{tikzcd}$$

by noticing that their components coincide via the following commuting diagram:

$$\begin{tikzcd}
\mathbb{C}[W^{-1}]_0 \arrow[r, "{([\alpha],[\beta])}"] \arrow[d, equal] & \mathbb{X}_2 \arrow[r, "c"] & \mathbb{X}_1 \\
\mathbb{C}_0 \arrow[ur, "{(\alpha,\beta)}"'] \arrow[urr, bend right, "\alpha\beta"'] \arrow[d, equal, "L_0"'] & & \\
\mathbb{C}[W^{-1}]_0 \arrow[uurr, bend right, "{[\alpha\beta]}"'] & &
\end{tikzcd}$$

□

The following theorem is a direct consequence of all the lemmas that came before in this section and formalizes the universal property of the internal localization, $\mathbb{C}[W^{-1}]$.

Theorem 65. *There is an isomorphism of categories*

$$[\mathbb{C}, \mathbb{X}]_W^{\mathcal{E}} \cong [\mathbb{C}[W^{-1}], \mathbb{X}]^{\mathcal{E}}$$

between internal functors $\mathbb{C} \to \mathbb{X}$ in $\mathcal{E}$ that invert $w : W \to \mathbb{C}_1$ and their internal natural transformations, and internal functors $\mathbb{C}[W^{-1}] \to \mathbb{X}$ in $\mathcal{E}$ and their internal natural transformations.

Proof. The objects are in bijection by Proposition 60, the arrows are in bijection by Lemma 63, and functoriality follows from Lemma 64. □

Chapter 5

Pseudocolimits of Small Filtered Diagrams of Internal Categories

5.1 Application to the Internal Grothendieck Construction

Exercise 6.6, of Exposé VI in [1] states that the pseudocolimit of a filtered diagram $\mathcal{A}^{op} \to \mathbf{Cat}$ can be obtained by localizing the Grothendieck construction with respect to the cartesian arrows. A current paper in progress, [15], by Bustillo-Vazquez, Pronk, and Szyld shows that with a weaker composition axiom for the category of fractions, the class of arrows one needs to invert to get the pseudocolimit can be reduced from all cartesian arrows to a convenient cleavage of them. For the rest of this chapter we consider an arbitrary but fixed filtered diagram, $D : \mathcal{A}^{op} \to \mathbf{Cat}(\mathcal{E})$ so that every finite diagram in $\mathcal{A}^{op}$ has a cone. The main theorem of this section states that, in a suitable context $\mathcal{E}$, the pseudocolimit of a filtered diagram of internal categories, $\mathcal{A}^{op} \to \mathbf{Cat}(\mathcal{E})$, can be computed by forming the internal category of (right) fractions of the internal Grothendieck construction with respect to the object representing the canonical cleavage of the cartesian arrows.

Note that the axioms we gave in Section 4.2 are for a category of *right* fractions, so we need to use the *contravariant* form of the internal Grothendieck construction for a functor $D : \mathcal{A}^{op} \to \mathbf{Cat}(\mathcal{E})$. In Section 5.1.1 we introduce the object representing the canonical cleavage and show that it satisfies the Internal Fractions Axioms in Definition 34. Section 5.2 is all about proving the main result of this thesis. Namely that, when it exists, the internal category of (right) fractions, $\mathbb{D}[W^{-1}]$, of the internal Grothendieck construction with respect to the canonical cleavage object, $(\mathbb{D}, W)$, is the pseudocolimit of the original filtered diagram $D : \mathcal{A}^{op} \to \mathbf{Cat}(\mathcal{E})$.

5.1.1 The Canonical Cleavage of the Internal Grothendieck Construction

The internal Grothendieck construction we need for a contravariant functor and a calculus of (right) fractions has an object of arrows defined by

$$\mathbb{D}_1 = \coprod_{\varphi \in \mathcal{A}_1} D_\varphi \qquad \text{where} \qquad \begin{tikzcd} D_\varphi \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & D(A)_1 \arrow[d, "t"] \\ D(B)_0 \arrow[r, "D(\varphi)_0"'] & D(A)_0 \end{tikzcd}$$

whenever $\varphi : A \to B$ is an arrow in $\mathcal{A}$. The subtle difference in this definition is that the vertical map on the right is a target rather than a source and that $D(\varphi) : D(B) \to D(A)$ for $\varphi : A \to B$ in $\mathcal{A}$. Another subtle but important difference is the definition of cofiber composition for this version of the Grothendieck construction. For $\varphi : A \to B$ and $\psi : B \to C$ in $\mathcal{A}$, the cofiber composition is given by

$$\begin{tikzcd} D_{\varphi;\psi} \arrow[r, "c'_{\varphi;\psi;\delta^{-1}}"] \arrow[d, "\pi_1"'] \arrow[dr, dotted, "c_{\varphi;\psi}"] & D(A)_3 \arrow[dr, bend left, "c"] & \\ D_\psi \arrow[dr, bend right, "\pi_0"'] & D_{\varphi\circ\psi} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & D(A)_1 \arrow[d, "t"] \\ & D(C)_0 \arrow[r, "D(\varphi\circ\psi)_0"'] & D(A)_0 \end{tikzcd}$$

where $c'_{\varphi;\psi;\delta^{-1}}$ is the universal map

$$\begin{tikzcd} D_{\varphi;\psi} \arrow[drr, bend left, "c'_{(\varphi;\psi);\delta^{-1}}"] \arrow[dr, dotted, "c'_{\varphi;\psi;\delta^{-1}}"] \arrow[ddr, bend right, "c'_{\varphi;\psi}"'] & & \\ & D(A)_3 \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & D(A)_2 \arrow[d, "\pi_0"] \\ & D(A)_2 \arrow[r, "\pi_1"'] & D(A)_1 \end{tikzcd} .$$

given explicitly by the triple:

$$c'_{\varphi;\psi;\delta^{-1}} = \left(\pi_0\pi_1, \pi_1\pi_1 D(\varphi)_1, \pi_1\pi_0\delta^{-1}_{\varphi;\psi}\right) \tag{$\star$}$$

in which $\delta^{-1}_{\varphi;\psi} : D(C)_0 \to D(A)_1$ represents the inverse components for the structure isomorphism of the pseudofunctor, $D : \mathcal{A}^{op} \to \mathbf{Cat}(\mathcal{E})$.

When $\mathcal{E} = \mathbf{Set}$, the arrows represented by $\mathbb{D}_1$ are pairs $(\varphi, f) : (A, a) \to (B, b)$ where $b \in D(B)_0$ and $f : a \to D(\varphi)(b)$ is in $D(A)_1$. The arrows being picked out by $w : W \to \mathbb{D}_1$ should correspond to pairs $(\varphi, 1_{D(\varphi)}(b)) : (A, D(\varphi)(b)) \to (B, b)$ where $b \in D(B)_0$ and $1_{D(\varphi)(b)} : D(\varphi)(b) \to D(\varphi)(b)$ is the identity map in $D(A)_1$. The following definition describes an extra condition on $\mathcal{E}$ that we need in order to work with an object of the canonical cleavage of cartesian arrows in $\mathbb{D}$.

Definition 66. Suppose $\mathcal{E}$ admits an internal Grothendieck construction of $D : \mathcal{A}^{op} \to \mathbf{Cat}(\mathcal{E})$. Then we say that $\mathbb{D}$ *admits a canonical cleavage of cartesian arrows* if for each $\varphi : A \to B$ in $\mathcal{A}$, the top pullback

$$\begin{tikzcd}
W_\varphi \arrow[r, "\pi_\varphi"] \arrow[d, tail, "w_\varphi"'] \arrow[dr, phantom, "\lrcorner", very near start] & D(A)_0 \arrow[d, tail, "e"] \arrow[dd, bend left=60, equal] \\
D_\varphi \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & D(A)_1 \arrow[d, "t"] \\
D(B)_0 \arrow[r, "D(\varphi)_0"'] & D(A)_0
\end{tikzcd}$$

exists and the coproduct

$$W = \coprod_{\varphi \in \mathcal{A}_1} W_\varphi$$

over all $\varphi \in \mathcal{A}_1$ exists in $\mathcal{E}$.

When $\mathbb{D}$ admits an object of the canonical cleavage of cartesian arrows as in Definition 66 we can use the universal property of the coproduct, W, to get the map $w : W \to \mathbb{D}_1$ in $\mathcal{E}$ as follows:

$$\begin{tikzcd}
W \arrow[r, dotted, "w"] & \mathbb{D}_1 \\
W_\varphi \arrow[u, "\iota_\varphi"] \arrow[ur, tail, bend right, "w_\varphi"']
\end{tikzcd}$$

This can be thought of as indexing the canonical cleavage of the cartesian arrows in the internal category $\mathbb{D}$. From this point on we assume that $\mathbb{D}$ admit a canonical cleavage of cartesian arrows. The first lemma we prove in this section shows that $(\mathbb{C}[W^{-1}], W)$ satisfies **Int.Frc.1**. In the case when $\mathcal{E} = \mathbf{Set}$, the sections of the target map are given by

$$(1_B, D(1_B)(b) : (B, D(1_B)(b)) \to (B, b)$$

which are completely determined by $B \in \mathcal{A}_0$ and the objects of $D(B)$.

Lemma 67 (**Int.Frc.1**)**.** *There exists a section of the target map* $wt : W \to \mathbb{D}_0$.

Proof. It suffices to show that the cofibers $w_\varphi \pi_0 : W_\varphi \to D(B)_0$ have sections. For each $B \in \mathcal{A}_0$, the cofiber section, $\alpha_B : D(B)_0 \to W_{1_B}$, of the target map on, $w_{1_B}\pi_0 : W_{1_B} \to D(B)_0$, is induced by the pair of maps $1_{D(B)_0}, D(1_B)_0 : D(B)_0 \to D(B)_0$. This is shown in the following commuting diagram, where the outer square clearly commutes and induces the dotted arrows on the left by the universal property of the two pullback squares on the inside.

$$\begin{tikzcd}
D(B)_0 \arrow[dr, dotted, "\alpha_B"] \arrow[ddr, dotted] \arrow[dddr, equal, bend right] \arrow[drr, bend left, "D(1_B)_0"] & & \\
& W_{1_B} \arrow[r, "\pi_{1_B}"] \arrow[d, tail, "w_{1_B}"'] \arrow[dr, phantom, "\lrcorner", very near start] & D(B)_0 \arrow[d, tail, "e"] \arrow[dd, equal, bend left] \\
& D_{1_B} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & D(B)_1 \arrow[d, "t"] \\
& D(B)_0 \arrow[r, "D(1_B)_0"'] & D(B)_0
\end{tikzcd}$$

Using the universal property of coproducts, the section $\alpha : \mathbb{D}_0 \to \mathbb{D}_1$ is induced by the family of maps $\{\alpha_B \iota_{1_B} : B \in \mathcal{A}_0\}$. Since the map $wt : W \to \mathbb{D}_0$ is induced by the family of maps $\{w_\varphi \pi_0 : \varphi \in \mathcal{A}_1\}$ we have that

$$\iota_B \alpha wt = \alpha_B \iota_{1_B} wt = \alpha_B w_{1_B} \pi_0 \iota_B.$$

This means the diagram

$$\begin{tikzcd}
\mathbb{D}_0 \arrow[r, "\alpha"] \arrow[dr, equal] & W \arrow[d, "wt"] \\
& \mathbb{D}_0
\end{tikzcd}$$

commutes by the universal property of the coproduct $\mathbb{D}_0$ and it follows that $\alpha : \mathbb{D}_0 \to W$ is a section of $wt : W \to \mathbb{D}_0$. □

Before we prove the second axiom, let us consider the case when $\mathcal{E} = \mathbf{Set}$. Here one typically shows that any composable arrows

$$(A, a) \xrightarrow{(\varphi,1)} (B, b) \xrightarrow{(\psi,1)} (C, c)$$

in $W \subseteq \mathbb{D}_1$ can be precomposed by an arrow,

$$\big(A, D(\varphi \circ \psi)(c)\big) \xrightarrow{\big(1_A, \delta_{1_A \circ (\varphi\circ\psi),c} D(1_A)(\delta_{\varphi\circ\psi,c})\big)} (A, a)$$

in $\mathbb{D}_1$ to make the diagram

$$\begin{tikzcd}
\big(A, D(\varphi \circ \psi)(c)\big) \arrow[d, "\big(1_A, \delta_{1_A \circ (\varphi\circ\psi),c} D(1_A)(\delta_{\varphi\circ\psi,c})\big)"'] \arrow[rrd, bend left, "(\varphi\circ\psi,1)"] & & \\
(A, a) \arrow[r, "(\varphi,1)"'] & (B, b) \arrow[r, "(\psi,1)"'] & (C, c)
\end{tikzcd}$$

commute in $\mathbb{D}$. A convenient way to show this is to first notice that $a = D(\varphi)(b)$ and $b = D(\psi)(c)$ by definition, $D(\varphi)(1_{D(\psi)(c)}) = 1_{D(\varphi)\circ D(\psi)(c)}$ by functoriality of $D(\varphi)$, and the composite of the two arrows in W is:

$$\begin{tikzcd}
(A, a) \arrow[r, "(\varphi,1)"] \arrow[rd, "(\varphi\circ\psi,\ \delta^{-1}_{\varphi\circ\psi,c})"'] & (B, b) \arrow[d, "(\psi,1)"] \\
& (C, c)
\end{tikzcd}$$

Now computing the composite

$$\begin{tikzcd}
\big(A, D(\varphi \circ \psi)(c)\big) \arrow[r, "\big(1_A, \delta_{1_A \circ (\varphi\circ\psi),c} D(1_A)(\delta_{\varphi\circ\psi,c})\big)"] \arrow[rd, "(\varphi\circ\psi\ ,\ 1)"'] & (A, a) \arrow[d, "(\varphi\circ\psi\ ,\ \delta^{-1}_{\varphi\circ\psi,c})"] \\
& (C, c)
\end{tikzcd}$$

in $\mathbb{D}$ is done by noting that $1_A \circ (\varphi \circ \psi) = \varphi \circ \psi$ in $\mathcal{A}^{op}$ and checking that

$$\begin{tikzcd}
D(\varphi \circ \psi)(c) \arrow[rr, equal] \arrow[d, "\delta_{1_A \circ (\varphi\circ\psi),c}"'] & & D(\varphi \circ \psi)(c) \\
D(1_A) \circ D(\varphi \circ \psi)(c) \arrow[rr, equal] \arrow[dr, "D(1_A)(\delta_{\varphi\circ\psi,c})"'] & & D(1_A) \circ D(\varphi \circ \psi)(c) \arrow[u, "\delta^{-1}_{1_A \circ (\varphi\circ\psi),c}"'] \\
& D(1_A) \circ D(\varphi) \circ D(\psi)(c) \arrow[ur, "D(1_A)(\delta^{-1}_{\varphi\circ\psi,c})"'] &
\end{tikzcd} \qquad (\star)$$

commutes in the category $D(A)$. The bottom left triangle commutes by functoriality of $D(1_A)$ and then the outer triangle commutes by definition of the natural isomorphism $\delta_{1_A \circ (\varphi\circ\psi)}$. We now give an internal version of this proof.

Lemma 68 (Int.Frc.2). *There exists a cover* $U \overset{u}{\nrightarrow} W\, {}_{wt}{\times}_{ws}\, W$ *and a lift* $\ell : U \to W_\circ$ *such that the diagram*

$$\begin{tikzcd}
& W_\circ \arrow[d, "\pi_0\pi_{12}"] \\
U \arrow[ur, dotted, bend left, "\ell"] \arrow[r, "u"', "/" marking] & W\, {}_{wt}{\times}_{ws} W
\end{tikzcd}$$

commutes in $\mathcal{E}$.

Proof. By extensivity we have that $W\, {}_{wt}{\times}_{ws} W \cong \coprod_{(\varphi,\psi)\in\mathcal{A}_2} W_{\varphi;\psi}$ where the cofibers are given by pullbacks

$$\begin{tikzcd}
W_{\varphi;\psi} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & W_\psi \arrow[r, tail, "w_\psi"] \arrow[d, "\pi_\psi"] \arrow[dr, "\pi_\psi e"] & D_\psi \arrow[d, "\pi_1"] \\
W_\varphi \arrow[r, "t_\varphi"'] \arrow[d, tail, "w_\varphi"'] & D(B)_0 & D(B)_1 \arrow[l, "s"] \\
D_\varphi \arrow[ur, bend right, "\pi_0"'] & &
\end{tikzcd}$$

Now using the component maps of the structure isomorphisms for the pseudofunctor $D : \mathcal{A}^{op} \to \mathbf{Cat}(\mathcal{E})$ we represent the composable vertical maps on the left-hand side in Diagram $(\star)$ by the internally composable pair, $D(C)_0 \to D(A)_2$, determined by the unviersal property of the following pullback:

$$\begin{tikzcd}
D(C)_0 \arrow[rr, bend left, "\delta_{\varphi\circ\psi}"] \arrow[dr, dotted, "\tilde{\delta}^w_{\varphi\circ\psi}"] \arrow[ddr, bend right, "\delta_{1_A\circ(\varphi\circ\psi)}"'] & & D(A)_1 \arrow[d, bend left, "D(1_A)_1"] \\
& D(A)_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & D(A)_1 \arrow[d, "s"] \\
& D(A)_1 \arrow[r, "t"'] & D(A)_0
\end{tikzcd}.$$

Specifying that such a pair comes from $W_{\varphi;\psi}$ and composing with the composition structure map of $D(A)$ gives the internal version of one component of pre-composable map in $\mathbb{D}$ that we need define the necessary lift:

$$\begin{tikzcd}
W_{\varphi;\psi} \arrow[r, "\pi_1 w_\psi \pi_0"] \arrow[drr, dotted, "\delta^w_{\varphi\circ\psi}"'] & D(C)_0 \arrow[r, "\tilde{\delta}^w_{\varphi\circ\psi}"] & D(A)_2 \arrow[d, "c"] \\
& & D(A)_1
\end{tikzcd}$$

To bring this together with the other component keeping track of the indexing, we need to map into the proper cofiber, D_{1_A}, which can be done using the universal property of the pullback:

$$\begin{tikzcd}
W_{\varphi;\psi} \arrow[drrr, bend left, "\delta^w_{\varphi\circ\psi}"] \arrow[drr, dotted, "\delta^w_{1_A;\varphi\circ\psi}"] \arrow[dd, bend right, "\pi_1 w_\psi \pi_0"'] & & & \\
& & D_{1_A} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & D(A)_1 \arrow[d, "t"] \\
D(C)_0 \arrow[r, "D(\psi)_0"'] & D(B)_0 \arrow[r, "D(\varphi)_0"'] & D(A)_0 \arrow[r, "D(1_A)_0"'] & D(A)_0
\end{tikzcd}$$

along with the fact that

$$\begin{aligned}\delta^w_{\varphi\circ\psi} t &= \pi_1 w_\psi \pi_0 \tilde{\delta}^w_{\varphi\circ\psi} c t \\ &= \pi_1 w_\psi \pi_0 \delta_{\varphi\circ\psi} D(1_A)_1 t \\ &= \pi_1 w_\psi \pi_0 \delta_{\varphi\circ\psi} t D(1_A)_0 \\ &= \pi_1 w_\psi \pi_0 D(\psi)_0 D(\varphi)_0 D(1_A)_0.\end{aligned}$$

Now we need to compose the cofiber triple and show that it factors through $W_{\varphi\circ\psi}$ via some map $c^w_{\varphi;\psi}$ in the following diagram.

$$\begin{tikzcd} W_{\varphi;\psi} \arrow[r, "\left(\delta^w_{1_A;\varphi\circ\psi}{,}\pi_0 w_\varphi{,}\pi_1 w_\psi\right)"] \arrow[d, "c^w_{\varphi;\psi}"'] & D_{1_A;\varphi;\psi} \arrow[d, "c_{1_A;\varphi;\psi}"] \\ W_{\varphi\circ\psi} \arrow[r, "w_{\varphi\circ\psi}"'] & D_{\varphi\circ\psi} \end{tikzcd} \qquad (\star\star)$$

We break this up into a couple steps using associativity of composition. First we compute the composite

$$\begin{tikzcd} W_{\varphi;\psi} \arrow[r, "(\pi_0 w_\varphi{,}\pi_1 \psi)"] \arrow[dr, "(\pi_1 w_\psi \pi_0{,}\pi_1 w_\psi \pi_0 \delta^{-1}_{\varphi;\psi})"'] & D_{\varphi;\psi} \arrow[d, "c_{\varphi;\psi}"] \\ & D_{\varphi\circ\psi} \end{tikzcd}$$

by calculating

$$\begin{aligned}(\pi_0 w_\varphi, \pi_1 w_\psi) c'_{\varphi;\psi;\delta^{-1}} &= (\pi_0 w_\varphi, \pi_1 w_\psi)(\pi_0 \pi_1, \pi_1 \pi_1 D(\varphi)_1, \pi_1 \pi_0 \delta^{-1}_{\varphi;\psi}) \\ &= (\pi_0 w_\varphi \pi_1, \pi_1 w_\psi \pi_1 D(\varphi)_1, \pi_1 w_\psi \pi_0 \delta^{-1}_{\varphi;\psi}) \\ &= (\pi_0 \pi_\varphi e, \pi_1 \pi_\psi e D(\varphi)_1, \pi_1 w_\psi \pi_0 \delta^{-1}_{\varphi;\psi}) \\ &= (\pi_0 \pi_\varphi e, \pi_0 w_\varphi \pi_0 e D(\varphi)_1, \pi_1 w_\psi \pi_0 \delta^{-1}_{\varphi;\psi}) \\ &= (\pi_0 \pi_\varphi e, \pi_0 w_\varphi \pi_0 D(\varphi)_0 e, \pi_1 w_\psi \pi_0 \delta^{-1}_{\varphi;\psi}) \\ &= (\pi_0 \pi_\varphi e, \pi_0 \pi_\varphi e, \pi_1 w_\psi \pi_0 \delta^{-1}_{\varphi;\psi})\end{aligned}$$

and then using the identity law in $D(A)$ twice in the last line below to see

$$\begin{aligned}
(\pi_0 w_\varphi,\pi_1 w_\psi)c_{\varphi;\psi} &= (\pi_0 w_\varphi,\pi_1 w_\psi)(\pi_1\pi_0, c'_{\varphi;\psi;\delta^{-1}}c) \\
&= \big(\pi_1 w_\psi \pi_0, (\pi_0 w_\varphi,\pi_1 w_\psi)c'_{\varphi;\psi;\delta^{-1}}c\big) \\
&= \big(\pi_1 w_\psi \pi_0, (\pi_0\pi_\varphi e, \pi_0\pi_\varphi e, \pi_1 w_\psi \pi_0 \delta^{-1}_{\varphi;\psi})c\big) \\
&= \big(\pi_1 w_\psi \pi_0, \pi_1 w_\psi \pi_0 \delta^{-1}_{\varphi;\psi}\big).
\end{aligned}$$

Now to see that we can pre-compose

$$(\pi_0 w_\varphi, \pi_1 w_\psi)c_{\varphi;\psi} : W_{\varphi;\psi} \to D_{\varphi\circ\psi}$$

with $\delta^w_{1_A;\varphi\circ\psi} : W_{\varphi;\psi} \to D_{1_A}$ at the cofiber $D_{1_A;(\varphi\circ\psi)}$, we check that

$$\begin{aligned}
\delta^w_{1_A;\varphi\circ\psi}\pi_0 &= \pi_1 w_\psi \pi_0 D(\psi)_0 D(\varphi)_0 \\
&= \pi_1 w_\psi \pi_0 \delta_{\varphi;\psi} t \\
&= \pi_1 w_\psi \pi_0 \delta^{-1}_{\varphi;\psi} s \\
&= (\pi_1 w_\psi \pi_0, \pi_1 w_\psi \pi_0 \delta^{-1}_{\varphi;\psi})\pi_1 s \\
&= (\pi_0 w_\varphi, \pi_1 w_\psi)c_{\varphi;\psi}\pi_1 s.
\end{aligned}$$

To see this cofiber composition factors through $W_{\varphi\circ\psi}$ we need to show that the arrow given by post-composing with the first projection, $\pi_1 : D_{\varphi\circ\psi} \to D(A)_1$ is an identity. To break this up a bit we first calculate

$$\begin{aligned}
&\big(\delta^w_{1_A;\varphi\circ\psi},\ (\pi_0 w_\varphi, \pi_1 w_\psi)c_{\varphi;\psi}\big)c'_{1_A;(\varphi\circ\psi);\delta^{-1}} \\
&= \big(\delta^w_{1_A;\varphi\circ\psi},\ (\pi_0 w_\varphi, \pi_1 w_\psi)c_{\varphi;\psi}\big)\big(\pi_0\pi_1,\ \pi_1\pi_1 D(1_A)_1,\ \pi_1\pi_0\delta^{-1}_{1_A;(\varphi\circ\psi)}\big) \\
&= \big(\delta^w_{1_A;\varphi\circ\psi}\pi_1,\ \pi_1 w_\psi \pi_0 \delta^{-1}_{\varphi;\psi} D(1_A)_1,\ \pi_1 w_\psi \pi_0 \delta^{-1}_{1_A;(\varphi\circ\psi)}\big) \\
&= \big(\delta^w_{\varphi\circ\psi},\ \pi_1 w_\psi \pi_0 \delta^{-1}_{\varphi;\psi} D(1_A)_1,\ \pi_1 w_\psi \pi_0 \delta^{-1}_{1_A;(\varphi\circ\psi)}\big) \\
&= \big(\pi_1 w_\psi \pi_0 \tilde{\delta}^w_{\varphi\circ\psi} c,\ \pi_1 w_\psi \pi_0 \delta^{-1}_{\varphi;\psi} D(1_A)_1,\ \pi_1 w_\psi \pi_0 \delta^{-1}_{1_A;(\varphi\circ\psi)}\big)
\end{aligned}$$

and then substituting it into the following calculation along with the definition

$$\tilde{\delta}^w_{\varphi\circ\psi} = (\delta_{1_A;\circ(\varphi\circ\psi}, \delta_{\varphi;\psi} D(1_A)_1)$$

gives:

$$\begin{aligned}
&\big(\delta^w_{1_A;\varphi\circ\psi}, (\pi_0 w_\varphi, \pi_1 w_\psi)c_{\varphi;\psi}\big)c_{1_A;(\varphi\circ\psi)}\pi_1\\
&= \big(\delta^w_{1_A;\varphi\circ\psi}, (\pi_0 w_\varphi, \pi_1 w_\psi)c_{\varphi;\psi}\big)c'_{1_A;(\varphi\circ\psi);\delta^{-1}}c\\
&= \big(\pi_1 w_\psi \pi_0 \tilde{\delta}^w_{\varphi\circ\psi}c,\ \pi_1 w_\psi \pi_0 \delta^{-1}_{\varphi;\psi}D(1_A)_1,\ \pi_1 w_\psi \pi_0 \delta^{-1}_{1_A;(\varphi\circ\psi)}\big)c\\
&= \big(\pi_1 w_\psi \pi_0 \tilde{\delta}^w_{\varphi\circ\psi}c,\ \pi_1 w_\psi \pi_0 (\delta^{-1}_{\varphi;\psi}D(1_A)_1, \delta^{-1}_{1_A;(\varphi\circ\psi)})c\big)c\\
&= \pi_1 w_\psi \pi_0 \big(\delta_{1_A;\circ(\varphi\circ\psi)},\ \delta_{\varphi;\psi}D(1_A)_1,\ \delta^{-1}_{\varphi;\psi}D(1_A)_1,\ \delta^{-1}_{1_A;(\varphi\circ\psi)}\big)c\\
&= \pi_1 w_\psi \pi_0 \big(\delta_{1_A;\circ(\varphi\circ\psi)},\ (\delta_{\varphi;\psi}D(1_A)_1,\ \delta^{-1}_{\varphi;\psi}D(1_A)_1)c,\ \delta^{-1}_{1_A;(\varphi\circ\psi)}\big)c\\
&= \pi_1 w_\psi \pi_0 \big(\delta_{1_A;\circ(\varphi\circ\psi)},\ (\delta_{\varphi;\psi},\ \delta^{-1}_{\varphi;\psi})cD(1_A)_1,\ \delta^{-1}_{1_A;(\varphi\circ\psi)}\big)c\\
&= \pi_1 w_\psi \pi_0 \big(\delta_{1_A;\circ(\varphi\circ\psi)},\ eD(\varphi\circ\psi)_1 D(1_A)_1,\ \delta^{-1}_{1_A;(\varphi\circ\psi)}\big)c\\
&= \pi_1 w_\psi \pi_0 \big(\delta_{1_A;\circ(\varphi\circ\psi)},\ D(\varphi\circ\psi)_0 D(1_A)_0 e,\ \delta^{-1}_{1_A;(\varphi\circ\psi)}\big)c\\
&= \pi_1 w_\psi \pi_0 \big(\delta_{1_A;\circ(\varphi\circ\psi)}, \delta_{1_A;\circ(\varphi\circ\psi)}te,\ \delta^{-1}_{1_A;(\varphi\circ\psi)}\big)c\\
&= \pi_1 w_\psi \pi_0 \big(\delta_{1_A;\circ(\varphi\circ\psi)},\ \delta^{-1}_{1_A;(\varphi\circ\psi)}\big)c\\
&= \pi_1 w_\psi \pi_0 D(1_A\circ(\varphi\circ\psi))_0 e\\
&= \pi_1 w_\psi \pi_0 D(\varphi\circ\psi)_0 e.
\end{aligned}$$

This internalizes the commutativity of Diagram ($\star$) and shows that for every composable $\varphi : A \to B$ and $\psi : B \to C$ in $\mathcal{A}$ there is a commuting diagram

$$\begin{tikzcd}
W_{\varphi;\psi} \arrow[ddd, bend right, "{\big(\delta^w_{1_A;\varphi\circ\psi},(\pi_0 w_\varphi,\pi_1 w_\psi)c_{\varphi;\psi}\big)}"'] \arrow[drr, bend left, "\pi_1 w_\psi \pi_0 D(\varphi\circ\psi)_0"] \arrow[dr, dotted, "c^w_{\varphi;\psi}"] & & \\
& W_{\varphi\circ\psi} \arrow[r, "\pi_{\varphi\circ\psi}"] \arrow[dd, tail, "w_{\varphi\circ\psi}"'] \arrow[ddr, phantom, very near start, "\lrcorner"] & D(A)_0 \arrow[dd, tail, "e"] \\
& & \\
D_{\varphi;\psi} \arrow[r, "c_{1_A;(\varphi\circ\psi)}"'] & D_{\varphi\circ\psi} \arrow[r, "\pi_1"] & D(A)_1
\end{tikzcd} .$$

The factorization we needed appears on the left of the diagram above and by associativity of composition in $\mathbb{D}$ we can conclude that the diagram we originally wanted ($\star\star$) involving composable triples commmutes. This allows us to compute

$$\begin{aligned}(\delta^w_{\varphi;\psi},1)(\pi_0\iota_{1_A},\iota^w_{\varphi;\psi})(\pi_0,\pi_1\pi_0 w,\pi_1\pi_1 w)c &= \big(\delta^w_{\varphi;\psi}\iota_{1_A}, \iota^w_{\varphi;\psi}\pi_0 w, \iota^w_{\varphi;\psi}\pi_1 w\big)c \\ &= \big(\delta^w_{\varphi;\psi}\iota_{1_A}, \pi_0\iota^w_{\varphi} w, \pi_1 iota^w_{\psi} w\big)c \\ &= \big(\delta^w_{\varphi;\psi}\iota_{1_A}, \pi_0 w_{\varphi}\iota_{\varphi}, \pi_1 w_{\psi}\iota_{\psi}\big)c \\ &= \big(\delta^w_{\varphi;\psi}, \pi_0 w_{\varphi}, \pi_1 w_{\psi}\big)\iota_{1_A;\varphi;\psi}c \\ &= \big(\delta^w_{\varphi;\psi}, \pi_0 w_{\varphi}, \pi_1 w_{\psi}\big)c_{1_A;\varphi;\psi}\iota_{\varphi\circ\psi} \\ &= c^w_{\varphi;\psi} w_{\varphi\circ\psi}\iota_{\varphi\circ\psi} \\ &= c^w_{\varphi;\psi}\iota^w_{\varphi\circ\psi} w\end{aligned}$$

and induce the unique cofiber lift, $\ell_{\varphi;\psi} : W_{\varphi;\psi} \to W_{\circ}$, by the universal property of the pullback, $W_{\circ}$, that makes the following diagram commute.

$$\begin{tikzcd}
W_{\varphi;\psi} \arrow[rr, "c^w_{\varphi;\psi}"] \arrow[dr, dotted, "\ell_{\varphi;\psi}"] \arrow[dd, "(\delta^w_{\varphi;\psi}{,}1)"'] \arrow[ddd, bend right, equal] & & W_{\varphi\circ\psi} \arrow[d, "\iota^w_{\varphi\circ\psi}"] \\
 & W_{\circ} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, very near start, "\lrcorner"] & W \arrow[d, tail, "w"] \\
D_{1_A}\,{}_t\times_s(W_{\varphi;\psi}) \arrow[r, "(\pi_0\iota_{1_A}{,}\iota^w_{\varphi;\psi})"'] \arrow[d, "\pi_1"'] & \mathbb{D}_1\,{}_t\times_{ws}(W\,{}_{wt}\times_{ws}W) \arrow[r, "(\pi_0{,}\pi_1\pi_0 w{,}\pi_1\pi_1 w)c"'] \arrow[d, "\pi_{12}"] & \mathbb{D}_1 \\
W_{\varphi;\psi} \arrow[r, "\iota^w_{\varphi l\psi}"'] & W\,{}_{wt}\times_{ws}W &
\end{tikzcd}$$

The universal property of coproducts then gives us the desired lift

$$\begin{tikzcd}
W\,{}_{wt}\times_{ws}W \arrow[r, dotted, "\ell"] \arrow[rr, bend left, equal] & W_{\circ} \arrow[r, "\pi_0\pi_{12}"] & W\,{}_{wt}\times_{ws}W \\
W_{\varphi;\psi} \arrow[u, "\iota^w_{\varphi;\psi}"] \arrow[ur, "\ell_{\varphi;\psi}"'] \arrow[urr, bend right, "\iota^w_{\varphi;\psi}"'] & &
\end{tikzcd}$$

where we take the identity map

$$W\,{}_{wt}\times_{ws}W \xlongequal{1_{W\,{}_{wt}\times_{ws}W}} W\,{}_{wt}\times_{ws}W$$

as our cover. □

The next thing we need to show is the right Ore condtition. Taking a look at the proof when $\mathcal{E} = \mathbf{Set}$ will be useful for guiding the reader through the internal version. Start by assuming there exists a cospan in $\mathbb{D}$ whose right leg is in W:

$$\begin{tikzcd}
 & \big(C, D(\psi)(b)\big) \arrow[d, "(\psi{,}1_{D(\psi)(b)})"] \\
\big(A, a\big) \arrow[r, "(\varphi{,}f)"'] & \big(B, b\big)
\end{tikzcd}$$

Since $\mathcal{A}$ is filtered, there exists an object $E \in \mathcal{A}_0$ and two maps $\varphi^* : E \to A$ and $\psi^* : E \to C$ such that the square

$$\begin{tikzcd}
E \arrow[r, "\psi^*"] \arrow[d, "\varphi^*"'] & C \arrow[d, "\psi"] \\
A \arrow[r, "\varphi"'] & B
\end{tikzcd}$$

commutes in $\mathcal{A}$. Now letting $\star$ denote the composition of arrows in the non-indexing compenent of the Grothendieck construction we can consider the commuting diagram:

$$\begin{tikzcd}
D(\varphi^*)(a) \arrow[r, equal] \arrow[ddd, equal] \arrow[drr, bend right, "1_{D(\varphi^*)(b)} \star f"] & D(\varphi^*)(a) \arrow[r, "D(\varphi^*)(f)"] & D(\varphi^*) \circ D(\varphi)(b) \arrow[d, "\delta^{-1}_{\varphi^*;\varphi,b}"] \\
 & & D(\varphi^* \circ \varphi)(b) \arrow[d, equal] \\
 & & D(\psi^* \circ \psi)(b) \arrow[d, "\delta_{\psi^*;\psi,b}"] \\
D(\varphi^*)(a) \arrow[r, "g"'] \arrow[urr, bend left, "g \star 1_{D(\psi)(b)}"] & D(\psi^*) \circ D(\psi)(b) \arrow[r, equal, "D(\psi^*)(1_{D(\psi)(b)}"'] & D(\psi^*) \circ D(\psi)(b)
\end{tikzcd}$$

where

$$g = D(\varphi^*)(f)\delta^{-1}_{\varphi^*;\varphi,b}\delta_{\psi^*;\psi,b}.$$

In particular we have

$$g \star 1_{D(\psi)(b)} = 1_{D(\varphi^*)(b)} \star f$$

and so the square

$$\begin{tikzcd}
\big(E, D(\varphi^*)(a)\big) \arrow[r, "{(\psi^*,g)}"] \arrow[d, "{(\varphi^*,1_{D(\varphi^*)(a)})}"', "\circ" marking] & \big(C, D(\psi)(b)\big) \arrow[d, "{(\psi,1_{D(\psi)(b)})}", "\circ" marking] \\
\big(A, a\big) \arrow[r, "{(\varphi,f)}"'] & \big(B, b\big)
\end{tikzcd}$$

commutes in $\mathbb{D}$. Now we show how to internalize this proof when $\mathcal{E}$ is not necessarily **Set**.

Lemma 69 (Int.Frc.3).

There exists a cover, $U \xrightarrow{u} \mathbb{D}_1 \,{}_t\!\times_{wt} W$ *and a lift* $U \xrightarrow{\ell} W_\Box$ *such that the following diagram commutes:*

$$\begin{tikzcd}
 & W_\Box \arrow[d, "{(\pi_0\pi_1,\pi_1\pi_1)}"] \\
U \arrow[ur, dotted, bend left, "\ell"] \arrow[r, "/" marking, "u"'] & \mathbb{D}_1 \,{}_t\!\times_{wt} W
\end{tikzcd}$$

where

$$W_\Box = (W \,{}_{wt}\!\times_s \mathbb{D}_1) \,{}_c\!\times_c (\mathbb{D}_1 \,{}_t\!\times_{ws} W).$$

Proof. Recall that $\mathrm{csp} = \mathbb{D}_1 \,{}_t\!\times_{wt} W$ and let $\mathrm{csp}(A)$ denote all the cospans in $\mathcal{A}$ for simpler notation. Since $\mathcal{E}$ is extensive we have the following isomorphisms:

$$\mathrm{csp} \cong \coprod_{(\varphi,\psi)\in \mathrm{csp}(A)} D_\varphi \,{}_{t_\varphi}\!\times_{w_\psi t_\psi} W_\psi$$

$$\mathbb{D}_1 \,{}_t\!\times_{ws} W \cong \coprod_{(\psi^*,\psi)\in \mathcal{A}_2} D_{\psi^*} \,{}_{t_{\psi^*}}\!\times_{w_\psi s_\psi} W_\psi$$

$$W \,{}_{wt}\!\times_s \mathbb{D}_1 \cong \coprod_{(\varphi^*,\varphi)\in \mathcal{A}_2} W_{\varphi^*} \,{}_{w_{\varphi^*} t_{\varphi^*}}\!\times_{s_\varphi} D_\varphi$$

Now we will define two families of maps

$$W_{\varphi^*} \,{}_{w_{\varphi^*} t_{\varphi^*}}\!\times_{s_\psi} D_\psi \xleftarrow{\ell_{\varphi;\psi,0}} D_\varphi \,{}_{t_\varphi}\!\times_{w_\psi t_\psi} W_\psi \xrightarrow{\ell_{\varphi;\psi,1}} D_{\psi^*} \,{}_{t_{\psi^*}}\!\times_{w_\psi s_\psi} W_\psi$$

before showing they agree after post-composing them each with the appropriate cofiber compositions. The left-hand side is simpler so we start there. Consider the following commuting pullback diagram:

$$\begin{tikzcd}
D_\varphi {}_{t_\varphi}\times_{w_\psi t_\psi} W_\psi \arrow[r, "\pi_0"] \arrow[ddd, "\pi_0"'] \arrow[drr, dotted, "\ell^w_{\varphi;\psi,0}"'] & W_\psi \arrow[r, "\pi_1"] & D(A)_1 \arrow[r, "s"] & D(A)_0 \arrow[d, "D(\varphi^*)_0"] \\
 & & W_{\varphi^*} \arrow[r, "\pi_{\varphi^*}"] \arrow[d, tail, "w_{\varphi^*}"'] \arrow[dr, phantom, "\lrcorner"] & D(E)_0 \arrow[d, tail, "e"] \arrow[dd, equal, bend left] \\
D_\varphi \arrow[dr, bend right, "\pi_1"'] & & D_{\varphi^*} \arrow[r, "\pi_1"] \arrow[d, tail, "\pi_0"'] \arrow[dr, phantom, "\lrcorner"] & D(E)_1 \arrow[d, "t"] \\
 & D(A)_1 \arrow[r, "s"'] & D(A)_0 \arrow[r, "D(\varphi^*)_0"'] & D(E)_0
\end{tikzcd}$$

The lower left commuting square above then induces the map we need, $\ell_{\varphi;\psi,0}$, by the following commuting pullback diagram:

$$\begin{tikzcd}
D_\varphi {}_{t_\varphi}\times_{w_\psi t_\psi} W_\psi \arrow[drr, bend left, "\pi_0"] \arrow[dr, dotted, "\ell_{\varphi;\psi,0}"] \arrow[ddr, bend right, "\ell^w_{\varphi;\psi,0}"'] & & & \\
 & W_{\varphi^*} {}_{w_{\varphi^*} t_{\varphi^*}}\times_{s_\varphi} D_\varphi \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner"] & D_\varphi \arrow[r, "\pi_1"] \arrow[d, "s_\varphi"'] & D(A)_1 \arrow[dl, "s"] \\
 & W_{\varphi^*} \arrow[r, "w_{\varphi^*} t_{\varphi^*}"] \arrow[d, tail, "w_{\varphi^*}"'] & D(A)_0 & \\
 & D_{\varphi^*} \arrow[ur, "\pi_0"'] & &
\end{tikzcd}$$

The map, $\ell_{\varphi;\psi,1}$, on the right-hand side is more involving to define, as we saw when $\mathcal{E} = \mathbf{Set}$, because it requires defining the map 'g' by composing with several other maps at hand. We first compute

$$\begin{aligned}
\pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi}s &= \pi_0\pi_0\delta_{\varphi^*;\varphi}t \\
&= \pi_0\pi_0 D(\varphi)_0 D(\varphi^*)_0 \\
&= \pi_0\pi_1 t D(\varphi^*)_0 \\
&= \pi_0\pi_1 D(\varphi^*)_1 t
\end{aligned}$$

to get one composable pair and then

$$\begin{aligned}
\pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi}t &= \pi_0\pi_0\delta_{\varphi^*;\varphi}s \\
&= \pi_0\pi_0 D(\varphi\circ\varphi^*)_0 \\
&= \pi_0\pi_0 D(\psi\circ\psi^*)_0 \\
&= \pi_0 t_\varphi D(\psi\circ\psi^*)_0 \\
&= \pi_1 w_\psi t_\psi D(\psi\circ\psi^*)_0 \\
&= \pi_1 w_\psi \pi_0 D(\psi\circ\psi^*)_0 \\
&= \pi_1 w_\psi \pi_0 \delta_{\psi^*;\psi}s
\end{aligned}$$

to get another with the same map in the middle. This gives a unique map

$$D_{\varphi\ t_\varphi}\times_{w_\psi t_\psi} W_\psi \xrightarrow{(\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi},\ \pi_1 w_\psi\pi_0\delta_{\psi;\psi^*})} D(E)_3$$

representing compsable triples in $D(E)$ whose composite we denote

$$\begin{tikzcd}
D_{\varphi\ t_\varphi}\times_{w_\psi t_\psi} W_\psi \arrow[r, "{(\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi},\ \pi_1 w_\psi\pi_0\delta_{\psi^*;\psi})}"] \arrow[dr, "\tilde{g}"'] & D(E)_3 \arrow[d, "c"] \\
& D(E)_1
\end{tikzcd} .$$

The target of this composite is

$$\tilde{g}t = \pi_1 w_\psi \pi_0 \delta_{\psi^*;\psi}t = \pi_1 w_\psi \pi_0 D(\psi)_0 D(\psi^*)_0$$

so there exists a unique map, g, in the commuting pullback diagram:

$$\begin{tikzcd}
D_{\varphi\ t_\varphi}\times_{w_\psi t_\psi} W_\psi \arrow[d, "\pi_1"'] \arrow[ddr, dotted, "g"] \arrow[ddrr, bend left, "\tilde{g}"] & & \\
W_\psi \arrow[d, tail, "w_\psi"'] \arrow[ddr, "\pi_\psi"] & & \\
D_\psi \arrow[d, "\pi_0"'] & D_{\psi^*} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & D(E)_1 \arrow[d, "t"] \\
D(B)_0 \arrow[r, "D(\psi)_0"'] & D(C)_0 \arrow[r, "D(\psi^*)_0"'] & D(E)_0
\end{tikzcd}$$

The left side of the diagram above, along with the fact that $s_\psi = \pi_0 s$, allows us to finally define the cofiber lift by the universal property of the pullback:

$$\begin{tikzcd}
D_\varphi\,{}_{t_\varphi}\times_{w_\psi t_\psi} W_\psi \arrow[drr, bend left, "\pi_1"] \arrow[dr, dotted, "\ell_{\varphi;\psi,1}"] \arrow[ddr, bend right, "g"'] & & & \\
& D_{\psi^*}\,{}_{t_{\psi^*}}\times_{w_\psi s_\psi} W_\psi \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & W_\psi \arrow[r, tail, "w_\psi"] \arrow[d, "\pi_\psi"'] \arrow[dr, "\pi_\psi e"] & D_\psi \arrow[d, "\pi_1"] \\
& D_{\psi^*} \arrow[r, "\pi_0"'] & D(C)_0 & D(C)_1 \arrow[l, "s"]
\end{tikzcd}$$

It only remains to show that the outside of the diagram,

$$\begin{tikzcd}
D_\varphi\,{}_{t_\varphi}\times_{w_\psi t_\psi} W_\psi \arrow[d, dotted, "\ell_{\varphi;\psi}"] \arrow[dd, bend right] \arrow[ddd, bend right=60, "\ell_{\varphi;\psi,0}"'] \arrow[dr, bend left] \arrow[drr, bend left, "\ell_{\varphi;\psi,1}"] & & \\
W_\square \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{D}_1\,{}_t\times_{ws} W \arrow[d, "c"] & D_{\psi^*}\,{}_{t_{\psi^*}}\times_{w_\psi s_\psi} W''_\psi \arrow[l, tail, "\iota_{\psi^*}\times\iota^w_\psi"'] \arrow[d, "1_{D_{\psi^*}}\times w_\psi"] \\
W\,{}_{wt}\times_s \mathbb{D}_1 \arrow[r, "c"'] & \mathbb{D}_1 & D_{\psi^*;\psi} \arrow[d, "c_{\psi^*;\psi}"] \\
W_{\varphi^*}\,{}_{w_{\varphi^*} t_{\varphi^*}}\times_{s_\varphi} D_\varphi \arrow[u, tail, "\iota^w_{\varphi^*}\times\iota_\varphi"'] \arrow[r, "w_{\varphi^*}\times 1_{D_\varphi}"'] & D_{\varphi^*;\varphi} \arrow[r, "c_{\varphi^*;\varphi}"'] & D_{\varphi^*\circ\varphi} \arrow[ul, tail, "\iota_{\varphi^*\circ\varphi}"]
\end{tikzcd},$$

commutes in $\mathcal{E}$ in order to induce the cofiber lift $\ell_{\varphi;\psi}$. Then the universal property of the coproduct, csp, will give the lift we need with the cover taken to be the identity on csp. All of the arrows involved have been defined by universal properties of pullbacks so we use pairing map notation to expand and manipulate them. Starting with the bottom composite, first we note, that by the universal properties of the pullbacks in the codomains of the following maps we have

$$w^*_\varphi \times 1_{D_\varphi} = (\pi_0 w_{\varphi^*}, \pi_1), \qquad c_{\varphi^*;\varphi} = (\pi_1\pi_0, c'_{\varphi^*;\varphi;\delta^{-1}} c)$$

where similarly, by equation $(\star)$ at the beginning of this section,

$$c'_{\varphi^*;\varphi;\delta^{-1}} = \big(\pi_0\pi_1, \pi_1\pi_1 D(\varphi^*)_1, \pi_1\pi_0\delta^{-1}_{\varphi^*;\varphi}\big).$$

Then in one component of the bottom composite we have

$$\begin{aligned}
\ell_{\varphi;\psi,0}(w_{\varphi^*} \times 1_{D_\varphi})c_{\varphi^*;\varphi}\pi_0 &= \ell_{\varphi;\psi,0}(\pi_0 w_{\varphi^*}, \pi_1)c_{\varphi^*;\varphi}\pi_0 \\
&= (\ell_{\varphi;\psi,0}\pi_0 w_{\varphi^*}, \ell_{\varphi;\psi,0}\pi_1)\pi_1\pi_0 \\
&= \ell_{\varphi;\psi,0}\pi_1\pi_0 \\
&= \pi_0\pi_0 \\
&= \pi_0 t_\varphi.
\end{aligned}$$

and in the other component we have

$$\begin{aligned}
\ell_{\varphi;\psi,0}(w_{\varphi^*} \times 1_{D_\psi})c_{\varphi^*;\varphi}\pi_1 &= \ell_{\varphi;\psi,0}(\pi_0 w_{\varphi^*}, \pi_1)c_{\varphi^*;\varphi}\pi_1 \\
&= (\ell_{\varphi;\psi,0}\pi_0 w_{\varphi^*}, \ell_{\varphi;\psi,0}\pi_1)c_{\varphi^*;\varphi}\pi_1 \\
&= (\ell_{\varphi;\psi,0}\pi_0 w_{\varphi^*}, \ell_{\varphi;\psi,0}\pi_1)c'_{\varphi^*;\varphi;\delta^{-1}}c \\
&= (\ell^w_{\varphi;\psi,0} w_{\varphi^*}, \pi_0)c'_{\varphi^*;\varphi;\delta^{-1}}c \\
&= (\ell^w_{\varphi;\psi,0} w_{\varphi^*}, \pi_0)\big(\pi_0\pi_1, \pi_1\pi_1 D(\varphi^*)_1, \pi_1\pi_0\delta^{-1}_{\varphi^*;\varphi}\big)c \\
&= \big(\ell^w_{\varphi;\psi,0} w_{\varphi^*}\pi_1, \pi_0\pi_1 D(\varphi^*)_1, \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi}\big)c.
\end{aligned}$$

Now the first component in the triple of the last line above can be rewritten using the definition of the pullback W_{φ^*} along with the definition of $\ell^w_{\varphi;\psi,0}$ and functoriality of $D(\varphi^*)$:

$$\begin{aligned}
\ell^w_{\varphi;\psi,0} w_{\varphi^*}\pi_1 &= \ell^w_{\varphi;\psi,0}\pi_{\varphi^*}e \\
&= \pi_0\pi_1 s D(\varphi^*)_0 e \\
&= \pi_0\pi_1 s e D(\varphi^*)_1.
\end{aligned}$$

Substituting this side calculation into the last line of the prior calculation and using associativity of composition, functoriality of $D(\varphi^*)$, and the identity law in $D(A)$ allows us to finally see that

$$\begin{aligned}
\ell_{\varphi;\psi,0}(w_{\varphi^*} \times 1_{D_\psi})c_{\varphi^*;\varphi}\pi_1 = \ldots &= \big(\ell^w_{\varphi;\psi,0} w_{\varphi^*}\pi_1, \pi_0\pi_1 D(\varphi^*)_1, \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi}\big)c \\
&= \big(\pi_0\pi_1 seD(\varphi^*)_1, \pi_0\pi_1 D(\varphi^*)_1, \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi}\big)c \\
&= \big((\pi_0\pi_1 seD(\varphi^*)_1, \pi_0\pi_1 D(\varphi^*)_1)c,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi}\big)c \\
&= \big((\pi_0\pi_1 se, \pi_0\pi_1)cD(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi}\big)c \\
&= \big(\pi_0\pi_1(se, 1_{D(A)_1})cD(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi}\big)c \\
&= \big(\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi}\big)c.
\end{aligned}$$

By the universal property of the pullback $D_{\varphi^*\circ\varphi} = D_{\psi^*\circ\psi}$, we can write the bottom composite as the following pairing map:

$$\ell_{\varphi;\psi,0}(w_{\varphi^*} \times 1_{D_\varphi})c_{\varphi^*;\varphi} = \big(\pi_0 t_\varphi, (\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi})c\big)$$

For the top composite, we begin similarly by noting that

$$1_{D^*_\psi} \times w_\psi = (\pi_0, \pi_1 w_\psi), \qquad c_{\psi^*;\psi} = (\pi_1\pi_0, c'_{\psi^*;\psi;\delta^{-1}}c)$$

where similarly, by equation ($\star$) at the beginning of this section,

$$c'_{\psi^*;\psi;\delta^{-1}} = \big(\pi_0\pi_1, \pi_1\pi_1 D(\psi^*)_1, \pi_1\pi_0\delta^{-1}_{\psi^*;\psi}\big).$$

Then in one component of the top composite we have

$$\begin{aligned}
\ell_{\varphi;\psi,1}(1_{D^*_\psi} \times w_\psi)c_{\psi^*;\psi}\pi_0 &= \ell_{\varphi;\psi,1}(\pi_0, \pi_1 w_\psi)c_{\psi^*;\psi}\pi_0 \\
&= (\ell_{\varphi;\psi,1}\pi_0, \ell_{\varphi;\psi,1}\pi_1 w_\psi)c_{\psi^*;\psi}\pi_0 \\
&= (\ell_{\varphi;\psi,1}\pi_0, \ell_{\varphi;\psi,1}\pi_1 w_\psi)\pi_1\pi_0 \\
&= \ell_{\varphi;\psi,1}\pi_1 w_\psi \pi_0 \\
&= \pi_1 w_\psi \pi_0 \\
&= \pi_1 w_\psi t_\psi.
\end{aligned}$$

In the other component we get

$$\begin{aligned}
\ell_{\varphi;\psi,1}(1_{D^*_\psi} \times w_\psi)c_{\psi^*;\psi}\pi_1 &= \ell_{\varphi;\psi,1}(\pi_0, \pi_1 w_\psi)c_{\psi^*;\psi}\pi_1 \\
&= (\ell_{\varphi;\psi,1}\pi_0, \ell_{\varphi;\psi,1}\pi_1 w_\psi)c'_{\psi^*;\psi;\delta^{-1}}c \\
&= (\ell_{\varphi;\psi,1}\pi_0, \ell_{\varphi;\psi,1}\pi_1 w_\psi)\big(\pi_0\pi_1, \pi_1\pi_1 D(\psi^*)_1, \pi_1\pi_0\delta^{-1}_{\psi^*;\psi}\big)c \\
&= \big(\ell_{\varphi;\psi,1}\pi_0\pi_1, \ell_{\varphi;\psi,1}\pi_1 w_\psi \pi_1 D(\psi^*)_1, \ell_{\varphi;\psi,1}\pi_1 w_\psi \pi_0 \delta^{-1}_{\psi^*;\psi}\big)c \\
&= \big(g\pi_1, \pi_1\pi_\psi e D(\psi^*)_1, \pi_1 w_\psi \pi_0 \delta^{-1}_{\psi^*;\psi}\big)c \\
&= \big(\tilde{g}, \pi_1\pi_\psi e D(\psi^*)_1, \pi_1 w_\psi \pi_0 \delta^{-1}_{\psi^*;\psi}\big)c \\
&= \big(\tilde{g},\ (\pi_1\pi_\psi e D(\psi^*)_1, \pi_1 w_\psi \pi_0 \delta^{-1}_{\psi^*;\psi})c\big)c.
\end{aligned}$$

Now looking at the last line above recall the definition of $\tilde{g}$:

$$\tilde{g} = (\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi},\ \pi_1 w_\psi \pi_0 \delta_{\psi^*;\psi})c$$

By definition of the pullback W_ψ, functoriality of $D(\psi^*)$, the definition of the structure isomorphism components $\delta^{-1}_{\psi^*;\psi}$, and the identity law for internal composition in $D(E)$ we get

$$\begin{aligned}
(\pi_1\pi_\psi e D(\psi^*)_1, \pi_1 w_\psi \pi_0 \delta^{-1}_{\psi^*;\psi})c &= (\pi_1\pi_\psi 1_{D(C)_1} e D(\psi^*)_1, \pi_1 w_\psi \pi_0 \delta^{-1}_{\psi^*;\psi})c \\
&= (\pi_1\pi_\psi 1_{D(C)_1} e D(\psi^*)_1, \pi_1 w_\psi \pi_0 \delta^{-1}_{\psi^*;\psi})c \\
&= (\pi_1 w_\psi \pi_0 D(\psi)_0 e D(\psi^*)_1, \pi_1 w_\psi \pi_0 \delta^{-1}_{\psi^*;\psi})c \\
&= (\pi_1 w_\psi \pi_0 D(\psi)_0 D(\psi^*)_0 e, \pi_1 w_\psi \pi_0 \delta^{-1}_{\psi^*;\psi})c \\
&= (\pi_1 w_\psi \pi_0 \delta^{-1}_{\psi^*;\psi} se, \pi_1 w_\psi \pi_0 \delta^{-1}_{\psi^*;\psi})c \\
&= \pi_1 w_\psi \pi_0 \delta^{-1}_{\psi^*;\psi}(se, 1_{D(E)_1})c \\
&= \pi_1 w_\psi \pi_0 \delta^{-1}_{\psi^*;\psi}.
\end{aligned}$$

Taking these side calculations into account and applying associativity of composition; the definition of the structure isomorphism components $\delta_{\psi^*;\psi}$ and $\delta^{-1}_{\psi^*;\psi}$; the definitions of t_ψ, t_φ, and the pullback $D_\varphi {}_{t_\varphi}\!\times_{w_\psi t_\psi} W_\psi$; the assumption that $\varphi^*\varphi = \psi^*\psi$ in $\mathcal{A}$ which means $\varphi \circ \varphi^* = \psi \circ \psi^*$ in $\mathcal{A}^{op}$; and the identity law for internal composition in $D(E)$ gives:

$$\begin{aligned}
&\big(\tilde{g},\ (\pi_1\pi_\psi e D(\psi^*)_1, \pi_1 w_\psi \pi_0 \delta^{-1}_{\psi^*;\psi})c\big)c \\
&= \big(\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi},\ \pi_1 w_\psi \pi_0 \delta_{\psi^*;\psi},\ \pi_1 w_\psi \pi_0 \delta^{-1}_{\psi^*;\psi}\big)c \\
&= \big(\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi},\ (\pi_1 w_\psi \pi_0 \delta_{\psi^*;\psi}, \pi_1 w_\psi \pi_0 \delta^{-1}_{\psi^*;\psi})c\big)c \\
&= \big(\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi},\ \pi_1 w_\psi \pi_0 (\delta_{\psi^*;\psi}, \delta^{-1}_{\psi^*;\psi})c\big)c \\
&= \big(\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi},\ \pi_1 w_\psi \pi_0 \delta_{\psi^*;\psi} s e\big)c \\
&= \big(\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi},\ \pi_1 w_\psi \pi_0 \delta_{\psi^*;\psi} s e\big)c \\
&= \big(\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi},\ \pi_1 w_\psi \pi_0 D(\psi \circ \psi^*)_0 e\big)c \\
&= \big(\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi},\ \pi_1 w_\psi \pi_0 D(\varphi \circ \varphi^*)_0 e\big)c \\
&= \big(\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi},\ \pi_1 w_\psi t_\psi D(\varphi \circ \varphi^*)_0 e\big)c \\
&= \big(\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi},\ \pi_0 t_\varphi D(\varphi \circ \varphi^*)_0 e\big)c \\
&= \big(\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi},\ \pi_0 \pi_0 D(\varphi \circ \varphi^*)_0 e\big)c \\
&= \big(\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi},\ \pi_0 \pi_0 \delta^{-1}_{\varphi^*;\varphi} t e\big)c \\
&= \big(\pi_0\pi_1 D(\varphi^*)_1,\ (\pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi},\ \pi_0 \pi_0 \delta^{-1}_{\varphi^*;\varphi} t e)c\big)c \\
&= \big(\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi}(1_{D(E)_1},\ te)c\big)c \\
&= \big(\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi}\big)c.
\end{aligned}$$

Putting all these calculations together along with the universal property of the pullback $D_{\varphi^*\circ\varphi} = D_{\psi^*\circ\psi}$ allows us to write the top composite as the following pairing map:

$$\ell_{\varphi;\psi,1}(1_{D^*_\psi} \times w_\psi)c_{\psi^*;\psi} = \big(\pi_1 w_\psi t_\psi, (\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi})c\big).$$

By definition of the pullback $D_\varphi\ {}_{t_\varphi}\!\times_{w_\psi t_\psi} W_\psi$ we know that $\pi_0 t_\varphi = \pi_1 w_\psi t_\psi$ and so both components in the following pairing maps agree:

$$\begin{aligned}
\ell_{\varphi;\psi,0}(w_{\varphi^*} \times 1_{D_\varphi})c_{\varphi^*;\varphi} &= \big(\pi_0 t_\varphi, (\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi})c\big) \\
&= \big(\pi_1 w_\psi t_\psi, (\pi_0\pi_1 D(\varphi^*)_1,\ \pi_0\pi_0\delta^{-1}_{\varphi^*;\varphi})c\big) \\
&= \ell_{\varphi;\psi,1}(1_{D^*_\psi} \times w_\psi)c_{\psi^*;\psi}.
\end{aligned}$$

This finally shows that the outside of the last diagram commutes and induces the cofiber lift

$$D_\varphi {}_{t_\varphi}\times_{w_\psi t_\psi} W_\psi \xrightarrow{\ell_{\varphi;\psi}} W_\Box \ .$$

The universal property of the coproduct csp gives a candidate for the lift we need:

$$\begin{tikzcd}
\mathrm{csp} \arrow[r, dotted, "\ell"] & W_\Box \\
D_\varphi {}_{t_\varphi}\times_{w_\psi t_\psi} W_\psi \arrow[u, "\iota_\varphi \times \iota_\psi^w"] \arrow[ur, "\ell_{\varphi;\psi}"'] &
\end{tikzcd} .$$

To see this is in fact the lift we need we need to see that the diagram

$$\begin{tikzcd}
W_\Box \arrow[r, "(\pi_0\pi_1, \pi_1\pi_1)"] & \mathrm{csp} \\
D_\varphi {}_{t_\varphi}\times_{w_\psi t_\psi} W_\psi \arrow[u, "\ell_{\varphi;\psi}"] \arrow[ur, "\iota_\varphi \times \iota_\psi^w"'] &
\end{tikzcd}$$

also commutes. This can be done by considering the commuting diagram

$$\begin{tikzcd}
D_\varphi {}_{t_\varphi}\times_{w_\psi t_\psi} W_\psi \arrow[d, dotted, "\ell_{\varphi;\psi}"] \arrow[dr, bend left] \arrow[drr, bend left, "\ell_{\varphi;\psi,1}"] \arrow[dd, bend right] \arrow[ddd, bend right=60, "\ell_{\varphi;\psi,0}"'] & & \\
W_\Box \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{D}_1 {}_t\times_{ws} W \arrow[d, "c"] \arrow[dr, "\pi_1"] & D_{\psi^*} {}_{t_{\psi^*}}\times_{w_\psi s_\psi} W_\psi \arrow[l, tail, "\iota_{\psi^*} \times \iota_\psi^w"'] \arrow[d, "\pi_1 \iota_\psi^w"] \\
W {}_{wt}\times_s \mathbb{D}_1 \arrow[r, "c"'] \arrow[dr, "\pi_1"] & \mathbb{D}_1 & W \\
W_{\varphi^*} {}_{w_{\varphi^*} t_{\varphi^*}}\times_{s_\varphi} D_\varphi \arrow[u, tail, "\iota_{\varphi^*}^w \times \iota_\varphi"'] \arrow[r, "\pi_1 \iota_\varphi"'] & \mathbb{D}_1 &
\end{tikzcd}$$

and recalling that

$$\ell_{\varphi;\psi,0}\pi_0 = \pi_0 \quad \text{and} \quad \ell_{\varphi;\psi,1}\pi_1 = \pi_1.$$

This allows us to see

$$\begin{aligned}
\ell_{\varphi;\psi}(\pi_0\pi_1, \pi_1\pi_1) &= (\ell_{\varphi;\psi}\pi_0\pi_1, \ell_{\varphi;\psi}\pi_1\pi_1) \\
&= (\ell_{\varphi;\psi,0}\pi_1\iota_\varphi, \ell_{\varphi;\psi,1}\pi_1\iota^w_\psi) \\
&= (\pi_0\iota_\varphi, \pi_1\iota^w_\psi) \\
&= \iota_\varphi \times \iota^w_\psi
\end{aligned}$$

and by the universal property of the coproduct csp we get the commuting diagram:

$$\begin{tikzcd}
\text{csp} \arrow[r, dotted, "\ell"] \arrow[rr, bend left, equal] & W_\Box \arrow[r, "{(\pi_0\pi_1,\pi_1\pi_1)}"] & \text{csp} \\
D_\varphi {}_{t_\varphi}\times_{w_\psi t_\psi} W_\psi \arrow[u, "\iota_\varphi\times\iota^w_\psi"] \arrow[ur, "\ell_{\varphi;\psi}"'] \arrow[urr, bend right, "\iota_\varphi\times\iota^w_\psi"'] & &
\end{tikzcd} .$$

The top triangle in the previous diagram shows that when taking the identity map $1_{\text{csp}} : \text{csp} \to \text{csp}$ as our cover, the map $\ell : \text{csp} \to W_\Box$ is precisely the lift we need. $\Box$

The last condition we need to check is the internal right-cancellation property which we have referred to as 'zippering.' The objects in $\mathcal{E}$ representing diagrams in $\mathbb{D}$ that are important to recall for this part are those of parallel pairs, $P(\mathbb{D})$, parallel pairs that are coequalized by an arrow in W, $\mathcal{P}_{cq}(\mathbb{D})$, parallel pairs that are equalized by an arrow in W, $\mathcal{P}_{eq}(\mathbb{D})$, and parallel pairs that are simultaneously equalized and coequalized by arrows in W respectively, $\mathcal{P}(\mathbb{D})$. The explicit constructions of these can be reviewed in Section 4.1.

As is our tradition by now, we first review the usual proof for when $\mathcal{E} = \mathbf{Set}$ before translating it internally to a more general category $\mathcal{E}$. Consider the following commuting diagram in $\mathbb{D}$:

$$\begin{tikzcd}
(A,a) \arrow[r, shift left, "{(\varphi,f)}"] \arrow[r, shift right, "{(\psi,g)}"'] & (B, D(\gamma)(c)) \arrow[r, "\circ" marking, "{(\gamma,1_{D(\gamma)(c)})}"] & (C,c)
\end{tikzcd} .$$

By definition of composition in $\mathbb{D}$, this means $\varphi \circ \gamma = \psi \circ \gamma$ in $\mathcal{A}^{op}$ and the diagram

$$\begin{tikzcd}
D(\psi)\circ D(\gamma)(c) \arrow[d, equal, "D(\psi)(1_{D(\gamma)(c)})"'] & a \arrow[l, "g"'] \arrow[r, "f"] & D(\varphi)\circ D(\gamma)(c) \arrow[d, equal, "D(\varphi)(1_{D(\gamma)(c)})"] \\
D(\psi)\circ D(\gamma)(c) \arrow[d, "\delta^{-1}_{\psi;\gamma,c}"'] & & D(\varphi)\circ D(\gamma)(c) \arrow[d, "\delta^{-1}_{\varphi;\gamma,c}"] \\
D(\psi\circ\gamma)(c) \arrow[rr, equal] & & D(\varphi\circ\gamma)(c)
\end{tikzcd} \tag{$\star$}$$

commutes in the category $D(A)$. Since $\mathcal{A}$ is filtered, there exists a map $\mu : E \to A$ such that the square

$$\begin{tikzcd}
E \arrow[r, "\mu"] \arrow[d, "\mu"'] & A \arrow[d, "\psi"] \\
A \arrow[r, "\varphi"'] & B
\end{tikzcd}$$

commutes in $\mathcal{A}$ and so $\mu\circ\psi = \mu\circ\varphi$ in $\mathcal{A}^{op}$. There is an obvious candidate equalizing arrow in W for the parallel pair, (f,φ) and (g,ψ), seen in the following diagram:

$$\begin{tikzcd}
(E, D(\mu)(a)) \arrow[r, "{(\mu,1_{D(\mu)(a)})}", "\circ" marking] & (A,a) \arrow[r, shift left, "{(\varphi,f)}"] \arrow[r, shift right, "{(\psi,g)}"'] & (B, D(\gamma)(c))
\end{tikzcd} \tag{$\star\star$}$$

To see this diagram commutes in $\mathbb{D}$ first notice that the diagram

$$\begin{tikzcd}
D(\mu)\circ D(\psi)\circ D(\gamma)(c) \arrow[d, "\delta^{-1}_{\mu;\psi,D(\gamma)(c)}"'] \arrow[ddr, "D(\mu)(\delta^{-1}_{\psi;\gamma,c})"'] & D(\mu)(a) \arrow[l, "D(\mu)(g)"'] \arrow[r, "D(\mu)(f)"] & D(\mu)\circ D(\varphi)\circ D(\gamma)(c) \arrow[dl, "D(\mu)(\delta^{-1}_{\varphi;\gamma,c})"'] \arrow[d, "\delta^{-1}_{\mu;\varphi,D(\gamma)(c)}"] \\
D(\mu\circ\psi)\circ D(\gamma)(c) \arrow[d, "\delta^{-1}_{\mu\circ\psi;\gamma,c}"'] & D(\mu)\circ D(\varphi\circ\gamma)(c) \arrow[d, equal] \arrow[ddr, "\delta^{-1}_{\mu;\varphi\circ\gamma,c}"] & D(\mu\circ\varphi)\circ D(\gamma)(c) \arrow[d, "\delta^{-1}_{\mu\circ\varphi;\gamma,c}"] \\
D((\mu\circ\psi)\circ\gamma) \arrow[d, equal] & D(\mu)\circ D(\psi\circ\gamma)(c) \arrow[dl, "\delta^{-1}_{\mu;\psi\circ\gamma,c}"] & D((\mu\circ\varphi)\circ\gamma) \arrow[d, equal] \\
D(\mu\circ(\psi\circ\gamma))(c) \arrow[rr, equal] & & D(\mu\circ(\varphi\circ\gamma))(c)
\end{tikzcd} \tag{$\star^3$}$$

commutes in $D(E)$. The top square commutes by functoriality of $D(\mu)$ and commutativity of diagram $(\star)$ above, the left and right squares commute by coherence of the structure isomorphisms for the pseudofunctor, D, and the bottom square commutes trivially because $\psi\circ\gamma = \varphi\circ\gamma$ in $\mathcal{A}$. Then the outside of the previous diagram commutes and implies that the following diagram commutes as well:

$$\begin{tikzcd}
D(\mu)(a) \arrow[r, equal] \arrow[d, equal, "1_{D(\mu)(a)}"] \arrow[ddd, bend right=60, "1_{D(\mu)(a)} * g"'] & D(\mu)(a) \arrow[d, equal, "1_{D(\mu)(a)}"'] \arrow[ddd, bend left=60, "1_{D(\mu)(a)} * f"] \\
D(\mu)(a) \arrow[d, "D(\mu)(g)"] & D(\mu)(a) \arrow[d, "D(\mu)(f)"'] \\
D(\mu) \circ D(\psi) \circ D(\gamma)(c) \arrow[d, "\delta^{-1}_{\mu;\psi,D(\gamma)(c)}"] & D(\mu) \circ D(\varphi) \circ D(\gamma)(c) \arrow[d, "\delta^{-1}_{\mu;\varphi,D(\gamma)(c)}"'] \\
D(\mu \circ \psi) \circ D(\gamma)(c) \arrow[r, equal] \arrow[d, "\delta^{-1}_{\mu\circ\psi;\gamma,c}"] & D(\mu \circ \varphi) \circ D(\gamma)(c) \arrow[d, "\delta^{-1}_{\mu\circ\psi;\gamma,c}"'] \\
D((\mu \circ \psi) \circ \gamma)(c) \arrow[r, equal] & D((\mu \circ \varphi) \circ \gamma)(c)
\end{tikzcd} \qquad (\star^4)$$

This shows that the original diagram ($\star\star$) commutes in $\mathbb{D}$ and proves the desired property in the case $\mathcal{E} = \mathbf{Set}$.

Lemma 70 (Int.Frc.4). *There exists a cover* $U \xrightarrow{u} \mathcal{P}_{cq}(\mathbb{D})$ *and a lift*

$$\begin{tikzcd}
 & \mathcal{P}(\mathbb{D}) \arrow[d, "\pi_1"] \\
U \arrow[ur, dotted, bend left, "\ell"] \arrow[r, "u", "/" marking] & \mathcal{P}_{cq}(\mathbb{D})
\end{tikzcd} .$$

Proof. For a more general extensive category $\mathcal{E}$ with a terminal object, products can be written as pullbacks over the terminal object and equalizers can then be written as pullbacks over products. In particular we have the pullback diagram

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{D}) \arrow[r, tail, "\iota_{cq}"] \arrow[d, tail, "\iota_{cq}"'] \arrow[dr, phantom, "\lrcorner", very near start] & P(\mathbb{D})\,{}_t{\times}_{ws}W \arrow[d, "\rho_1"] \\
P(\mathbb{D})\,{}_t{\times}_{ws}W \arrow[r, "\rho_0"'] & \big(P(\mathbb{D})\,{}_t{\times}_{ws}W\big) \times \mathbb{D}_1
\end{tikzcd} ,$$

where

$$\rho_0 = (1_{P(\mathbb{D})}, (\pi_0\pi_0, \pi_1 w)c)$$

and

$$\rho_1 = (1_{P(\mathbb{D})}, (\pi_0\pi_1, \pi_1 w)c).$$

Note that each object is a pullback of coproducts, and since $\mathcal{E}$ is extensive each of these pullbacks can be expressed as a coproduct of pullbacks of their corresponding cofibers. The cofibers for the parallel pairs objects are denoted

$$P(\mathbb{D})_{(\varphi,\psi)} = D_{\varphi\ (s,t)}\!\times_{(s,t)} D_{\psi}$$

The corresponding pullback diagram of the cofiber corresponding to the maps φ, ψ, and γ in $\mathcal{A}$ such that $\varphi\gamma = \psi\gamma$ is

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{D})_{(\varphi,\psi);\gamma} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & P(\mathbb{D})_{(\varphi,\psi)\ t}\!\times_{w_\gamma s_\gamma} W_\gamma \arrow[d, "\rho_{1,(\varphi;\psi);\gamma}"] \\
P(\mathbb{D})_{(\varphi,\psi)\ t}\!\times_{w_\gamma s_\gamma} W_\gamma \arrow[r, "\rho_{0,(\varphi;\psi);\gamma}"'] & \big(P(\mathbb{D})_{(\varphi,\psi)\ t}\!\times_{w_\gamma s_\gamma} W_\gamma\big) \times D_{\varphi\circ\gamma}
\end{tikzcd} \quad , \qquad (\star)$$

where

$$\rho_{0,(\varphi;\psi);\gamma} = \big(1_{P(\mathbb{D})_{(\varphi,\psi)} \times W_\gamma}, (\pi_0\pi_0, \pi_1 w_\gamma) c_{\varphi;\gamma}\big)$$

and

$$\rho_{1,(\varphi;\psi);\gamma} = \big(1_{P(\mathbb{D})_{(\varphi,\psi)} \times W_\gamma}, (\pi_0\pi_1, \pi_1 w_\gamma) c_{\psi;\gamma}\big).$$

Similarly we have the pullback diagram for the object of parallel pairs that are equalized by an arrow in W

$$\begin{tikzcd}
\mathcal{P}_{eq}(\mathbb{D}) \arrow[r, tail, "\iota_{eq}"] \arrow[d, tail, "\iota_{eq}"'] \arrow[dr, phantom, "\lrcorner", very near start] & W\ {}_{wt}\!\times_s P(\mathbb{D}) \arrow[d, "\lambda_1"] \\
W\ {}_{wt}\!\times_s P(\mathbb{D}) \arrow[r, "\lambda_0"'] & \big(W\ {}_{wt}\!\times_s P(\mathbb{D})\big) \times \mathbb{D}_1
\end{tikzcd} \quad ,$$

where

$$\lambda_0 = (1_{P(\mathbb{D})}, (\pi_1 w, \pi_0\pi_0) c)$$

and

$$\lambda_1 = (1_{P(\mathbb{D})}, (\pi_1 w, \pi_0\pi_1) c).$$

The corresponding pullback of a cofiber indexed by a maps μ, φ, and ψ such that $\mu\varphi = \mu\psi$ in $\mathcal{A}$ is

$$\begin{tikzcd}
\mathcal{P}_{eq}(\mathbb{D})_{\mu;(\varphi,\psi)} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & W_{\mu\ w_\mu t_\mu}\!\times_s (P(\mathbb{D})_{(\varphi,\psi)}) \arrow[d, "\lambda_{1,\mu;(\varphi,\psi)}"] \\
W_{\mu\ w_\mu t_\mu}\!\times_s (P(\mathbb{D})_{(\varphi,\psi)}) \arrow[r, "\lambda_{0,\mu;(\varphi,\psi)}"'] & \big(W_{\mu\ w_\mu t_\mu}\!\times_s P(\mathbb{D})_{(\varphi,\psi)}\big) \times D_{\mu\circ\varphi}
\end{tikzcd} \quad , \qquad (\star\star)$$

where

$$\lambda_{0,\mu;(\varphi,\psi)} = \big(1_{W_\mu \times P(\mathbb{D})_{(\varphi,\psi)}}, (\pi_0 w_\mu, \pi_1\pi_0)c_{\mu;\varphi}\big)$$

and

$$\lambda_{1,\mu;(\varphi,\psi)} = \big(1_{W_\mu \times P(\mathbb{D})_{(\varphi,\psi)}}, (\pi_0 w_\mu, \pi_1\pi_1)c_{\mu;\psi}\big).$$

We use the cofibers in Diagrams $(\star)$ and $(\star\star)$ to translate the usual proof for when $\mathcal{E} = \mathbf{Set}$ and then the universal property of coproducts will give us the result we want. Since $\mathcal{A}$ is filtered, there exists a map $\mu : E \to A$ in $\mathcal{A}$ such that the diagram

$$\begin{tikzcd}
E \arrow[r, "\mu"] \arrow[d, "\mu"'] & A \arrow[d, "\psi"] \\
A \arrow[r, "\varphi"'] & B
\end{tikzcd}$$

commutes in $\mathcal{A}$. Picking out the arrow we need to precompose was done by taking the source of the parallel pair and applying $D(\mu)$ to it. Internally this is done at the level of cofibers by first considering the following commuting diagram,

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{D})_{(\varphi,\psi);\gamma} \arrow[r, "\pi_0\pi_0"] \arrow[dd, "\pi_0\pi_0"'] \arrow[ddrr, dotted, "\ell^w_\mu"] \arrow[dddrr, dotted] & D_\varphi {}_{(s,t)}\times_{(s,t)} D_\psi \arrow[r, "\pi_0"] & D_\varphi \arrow[r, "\pi_1"] & D(A)_1 \arrow[d, "s"] \\
& & & D(A)_0 \arrow[d, "D(\mu)_0"] \\
D_\varphi {}_{(s,t)}\times_{(s,t)} D_\psi \arrow[d, "\pi_0"'] & & W_\mu \arrow[r, "\pi_\mu"] \arrow[d, "w_\mu"'] \arrow[dr, phantom, very near start, "\lrcorner"] & D(E)_0 \arrow[d, "e"] \arrow[dd, bend left, equal] \\
D_\varphi \arrow[d, "\pi_1"'] & & D_\mu \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, very near start, "\lrcorner"] & D(E)_1 \arrow[d, "t"] \\
D(A)_1 \arrow[rr, "s"'] & & D(A)_0 \arrow[r, "D(\mu)_0"'] & D(A)_0
\end{tikzcd}$$

The left side of the previous diagram then makes up the outside of the following pullback diagram:

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{D})_{(\varphi,\psi);\gamma} \arrow[d, dotted, "\tilde{\ell}_{\mu,(\varphi,\psi)}"] \arrow[dr, bend left, "\pi_0\pi_0"] \arrow[dd, bend right=60, "\ell^w_\mu"'] & & \\
W_\mu {}_{w_\mu t_\mu}\!\times_s P(\mathbb{D})_{(\varphi,\psi)} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & D_\varphi {}_{(s,t)}\!\times_{(s,t)} D_\psi \arrow[r, "\pi_0"] \arrow[d, "s"'] & D_\varphi \arrow[d, "\pi_1"] \\
W_\mu \arrow[r, "w_\mu t_\mu"] \arrow[d, tail, "w_\mu"'] & D(A)_0 & D(A)_1 \arrow[l, "s"] \\
D_\mu \arrow[ur, "\pi_0"'] & &
\end{tikzcd}$$

There are two ways to compose the arrows in the diagrams being represented by the previous universal map. To show they agree we show the the following diagram commutes

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{D})_{(\varphi,\psi);\gamma} \arrow[r, "\tilde{\ell}_{\mu,(\varphi,\psi)}"] \arrow[d, "\tilde{\ell}_{\mu,(\varphi,\psi)}"'] & W_\mu {}_{w_\mu t_\mu}\!\times_s P(\mathbb{D})_{(\varphi,\psi)} \arrow[dr, bend left, "(\pi_0 w_\mu{,}\pi_1\pi_1)"] & \\
W_\mu {}_{w_\mu t_\mu}\!\times_s P(\mathbb{D})_{(\varphi,\psi)} \arrow[dr, bend right, "(\pi_0 w_\mu{,}\pi_1\pi_0)"'] & & D_{\mu;\psi} \arrow[d, "c_{\mu;\psi}"] \\
& D_{\mu;\varphi} \arrow[r, "c_{\mu;\varphi}"'] & D_{\mu\circ\varphi}
\end{tikzcd}.$$

By definition of $\tilde{\ell}_{\mu,(\varphi,\psi)}$ the first two maps on either both sides can be composed to give the top and left arrows in the following square:

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{D})_{(\varphi,\psi);\gamma} \arrow[r, "(\ell^w_\mu w_\mu{,}\pi_0\pi_0\pi_1)"] \arrow[d, "(\ell^w_\mu w_\mu{,}\pi_0\pi_0\pi_0)"'] & D_{\mu;\psi} \arrow[d, "c_{\mu;\psi}"] \\
D_{\mu;\varphi} \arrow[r, "c_{\mu;\varphi}"'] & D_{\mu\circ\varphi}
\end{tikzcd}$$

To see that this square commutes we use the universal property of the pullback $D_{\mu\circ\varphi}$. It will help to recall the definition of cofiber composition in $\mathbb{D}$. In particular

$$c_{\mu;\varphi} = (\pi_1\pi_0, c'_{\mu;\varphi;\delta^{-1}}c) \quad \text{and} \quad c_{\mu;\psi} = (\pi_1\pi_0, c'_{\mu;\psi;\delta^{-1}}c)$$

where

$$c'_{\mu;\varphi;\delta^{-1}} = (\pi_0\pi_1, \pi_1\pi_1 D(\mu)_1, \pi_1\pi_0\delta^{-1}_{\mu;\varphi})$$

and similarly

$$c'_{\mu;\psi;\delta^{-1}} = (\pi_0\pi_1, \pi_1\pi_1 D(\mu)_1, \pi_1\pi_0\delta^{-1}_{\mu;\psi}).$$

To see both sides agree on the projection $\pi_0 : D_\mu \circ \varphi \to D(E)_0$ we can compute

$$\begin{aligned}(\ell^w_\mu w_\mu, \pi_0\pi_0\pi_0) c_{\mu\circ\varphi}\pi_0 &= (\ell^w_\mu w_\mu, \pi_0\pi_0\pi_0)\pi_1\pi_0 \\ &= \pi_0\pi_0\pi_0\pi_0\end{aligned}$$

and

$$\begin{aligned}(\ell^w_\mu w_\mu, \pi_0\pi_0\pi_1) c_{\mu\circ\psi}\pi_0 &= (\ell^w_\mu w_\mu, \pi_0\pi_0\pi_1)\pi_1\pi_0 \\ &= \pi_0\pi_0\pi_1\pi_0\end{aligned}$$

and notice that that the last lines are equal by definition of the pullback $\mathcal{P}_{cq}(\mathbb{D})_{(\varphi,\psi);\gamma)}$. To see that the other projection $\pi_1 : D_\mu \circ \varphi \to D(E)_0$ also coequalizes both sides of the square is more involving. First notice that the diagram

$$\begin{tikzcd}
P(\mathbb{D})_{(\varphi,\psi)} {}_t\times_{w_\gamma s_\gamma} W_\gamma \arrow[d, "\pi_0"] \arrow[drr, bend left, "\pi_0\pi_1"] \arrow[dd, bend right=60, "\pi_0\pi_0"'] & & \\
D_\varphi {}_{(s,t)}\times_{(s,t)} D_\psi \arrow[rr, "\pi_1"] \arrow[d, "\pi_0"'] & & D_\psi \arrow[d, "{(\pi_1 s, \pi_0)}"'] \arrow[dd, bend left=60, "\pi_0"] \\
D_\varphi \arrow[rr, "{(\pi_1 s, \pi_0)}"'] \arrow[drr, bend right, "\pi_0"'] & & D(B)_0 \times D(A)_0 \arrow[d, "\pi_1"] \\
& & D(A)_0
\end{tikzcd} \tag{A}$$

commutes and precomposing with the projection

$$\mathcal{P}_{cq}(\mathbb{D})_{(\varphi,\psi);\gamma} \xrightarrow{\pi_0} (D_\varphi {}_{(s,t)}\times_{(s,t)} D_\psi) {}_t\times_{w_\gamma s_\gamma} W_\gamma$$

gives the last line in the following side-calculation:

$$
\begin{aligned}
\ell_\mu^w w_\mu \pi_1 &= \ell_\mu^w \pi_\mu e \\
&= \pi_0\pi_0\pi_0\pi_1 s D(\mu)_0 e \\
&= \pi_0\pi_0\pi_0\pi_1 D(\mu)_1 s e.
\end{aligned}
$$

We use this side calculation in the fourth equality of the following calculation:

$$
\begin{aligned}
&(\ell_\mu^w w_\mu, \pi_0\pi_0\pi_0) c_{\mu\circ\varphi}\pi_1 \\
&= (\ell_\mu^w w_\mu, \pi_0\pi_0\pi_0) c'_{\mu;\varphi;\delta^{-1}} c \\
&= (\ell_\mu^w w_\mu, \pi_0\pi_0\pi_0)(\pi_0\pi_1, \pi_1\pi_1 D(\mu)_1, \pi_1\pi_0\delta^{-1}_{\mu;\varphi})c \\
&= \big(\ell_\mu^w w_\mu\pi_1,\ \pi_0\pi_0\pi_0\pi_1 D(\mu)_1,\ \pi_0\pi_0\pi_0\pi_0\delta^{-1}_{\mu;\varphi}\big)c \\
&= \big(\pi_0\pi_0\pi_0\pi_1 D(\mu)_1 s e,\ \pi_0\pi_0\pi_0\pi_1 D(\mu)_1,\ \pi_0\pi_0\pi_0\pi_0\delta^{-1}_{\mu;\varphi}\big)c \\
&= \big(\pi_0\pi_0\pi_0\pi_1 D(\mu)_1 (se, 1)c,\ \pi_0\pi_0\pi_0\pi_0\delta^{-1}_{\mu;\varphi}\big)c \\
&= \big(\pi_0\pi_0\pi_0\pi_1 D(\mu)_1,\ \pi_0\pi_0\pi_0\pi_0\delta^{-1}_{\mu;\varphi}\big)c
\end{aligned}
$$

The last calculation we need requires a side calculation along with the coherences for the structure isomorphisms of the pseudo functor $D : \mathcal{A}^{op} \to \mathbf{Cat}(\mathcal{E})$. We start similarly by noticing that the diagram

$$
\begin{tikzcd}
P(\mathbb{D})_{(\varphi,\psi)}\, {}_t\times_{w_\gamma s_\gamma} W_\gamma \arrow[d, "\pi_0"] \arrow[dr, bend left, "\pi_0\pi_1"] \arrow[dd, bend right, "\pi_0\pi_0"'] & \\
D_\varphi\, {}_{(s,t)}\times_{(s,t)} D_\psi \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & D_\psi \arrow[d, "{(\pi_1 s, \pi_0)}"'] \arrow[dd, bend left, "\pi_1 s"] \\
D_\varphi \arrow[r, "{(\pi_1 s, \pi_0)}"'] \arrow[dr, bend right, "\pi_1 s"'] & D(B)_0 \times D(A)_0 \arrow[d, "\pi_0"] \\
& D(A)_0
\end{tikzcd}
$$

(B)

commutes in $\mathcal{E}$ and noticing that pre-composing with the projection

$$
\mathcal{P}_{cq}(\mathbb{D})_{(\varphi,\psi);\gamma} \xrightarrow{\pi_0} (D_\varphi\, {}_{(s,t)}\times_{(s,t)} D_\psi)\, {}_t\times_{w_\gamma s_\gamma} W_\gamma
$$

gives the last line in the following side-calculation:

$$\begin{aligned} \ell^w_\mu \pi_\mu e &= \pi_0\pi_0\pi_0\pi_1 s D(\mu)_0 e \\ &= \pi_0\pi_0\pi_1\pi_1 s D(\mu)_0 e \\ &= \pi_0\pi_0\pi_1\pi_1 D(\mu)_1 s e. \end{aligned}$$

Another side-calculation we will need can be seen in the following commuting diagram:

$$\begin{tikzcd}
 & D(C)_0 \arrow[r, "D(\gamma)_0"] & D(B)_0 \\
 & D_\gamma \arrow[u, "\pi_0"] \arrow[r, "\pi_1"] \arrow[r, phantom, "\urcorner", very near start, shift left=2ex] & D(B)_1 \arrow[u, "t"'] \\
P(\mathbb{D})_{(\varphi,\psi)}\, {}_t\!\times_{w_\gamma s_\gamma} W_\gamma \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & W_\gamma \arrow[u, tail, "w_\gamma"] \arrow[d, tail, "w_\gamma"'] \arrow[r, "\pi_\gamma"'] \arrow[r, phantom, "\urcorner", very near start, shift left=2ex] \arrow[r, phantom, "\lrcorner", very near start, shift right=2ex] & D(B)_0 \arrow[u, tail, "e"] \arrow[d, tail, "e"] \arrow[uu, bend right, equal] \arrow[dd, bend left, equal] \\
D_\varphi\, {}_{(s,t)}\!\times_{(s,t)} D_\psi \arrow[d, "\pi_1"'] \arrow[dr, "t"'] & D_\gamma \arrow[r, "\pi_1"] \arrow[d, "s_\gamma"'] & D(B)_1 \arrow[d, "s"] \\
D_\varphi \arrow[r, "\pi_0"'] & D(B)_0 \arrow[r, equal] & D(E)_0
\end{tikzcd} \tag{C}$$

In particular we will use commutativity of the outside several times which says:

$$\pi_0\pi_1\pi_0 = \pi_1 w_\gamma \pi_0 D(\gamma)_0$$

Coherence of the structure isomorphisms of the pseudofunctor D says that the diagrams

$$\begin{tikzcd}
D(C)_0 \arrow[r, "(\delta_{\mu\circ\psi;\gamma}{,}D(\gamma)_0\delta_{\psi;\gamma})"] \arrow[d, "(\delta_{\mu;\psi\circ\gamma}{,}\delta_{\psi;\gamma}D(\mu)_1)"'] & D(E)_2 \arrow[d, "c"] \\
D(C)_2 \arrow[r, "c"'] & D(E)_1
\end{tikzcd}$$

and

$$\begin{tikzcd}
D(C)_0 \arrow[r, "(\delta_{\mu\circ\varphi;\gamma}{,}D(\gamma)_0\delta_{\varphi;\gamma})"] \arrow[d, "(\delta_{\mu;\varphi\circ\gamma}{,}\delta_{\varphi;\gamma}D(\mu)_1)"'] & D(E)_2 \arrow[d, "c"] \\
D(C)_2 \arrow[r, "c"'] & D(E)_1
\end{tikzcd}$$

commute in $\mathcal{E}$. Using the internal composition in $D(E)$ to pre-compose with the inverse structure isomorphism components $\delta^{-1}_{\mu\circ\psi;\gamma} : D(C)_0 \to D(E)_1$ and $\delta^{-1}_{\mu\circ\varphi;\gamma} : D(C)_0 \to D(E)_1$ respectively and then applying the identity law in $D(E)$ gives new commuting diagrams:

$$\begin{tikzcd}
D(C)_0 \arrow[r, "D(\gamma)_0"] \arrow[d, "(\delta^{-1}_{\mu\circ\psi;\gamma}{,}\delta_{\mu;\psi\circ\gamma}{,}\delta_{\psi;\gamma}D(\mu)_1)"'] & D(E)_0 \arrow[d, "\delta_{\psi;\gamma}"] \\
D(C)_3 \arrow[r, "c"'] & D(E)_1
\end{tikzcd}$$

and

$$\begin{tikzcd}
D(C)_0 \arrow[r, "D(\gamma)_0"] \arrow[d, "(\delta^{-1}_{\mu\circ\varphi;\gamma}{,}\delta_{\mu;\varphi\circ\gamma}{,}\delta_{\varphi;\gamma}D(\mu)_1)"'] & D(E)_0 \arrow[d, "\delta_{\varphi;\gamma}"] \\
D(C)_3 \arrow[r, "c"'] & D(E)_1
\end{tikzcd}$$

Taking inverses in $D(E)$ then gives the commuting diagrams,

$$\begin{tikzcd}
D(C)_0 \arrow[r, "D(\gamma)_0"] \arrow[d, "(\delta^{-1}_{\psi;\gamma}D(\mu)_1{,}\delta^{-1}_{\mu;\psi\circ\gamma}{,}\delta_{\mu\circ\psi;\gamma})"'] & D(B)_0 \arrow[d, "\delta^{-1}_{\mu;\psi}"] \\
D(C)_3 \arrow[r, "c"'] & D(E)_1
\end{tikzcd}$$

$$\begin{tikzcd}
D(C)_0 \arrow[r, "D(\gamma)_0"] \arrow[d, "(\delta^{-1}_{\varphi;\gamma}D(\mu)_1{,}\delta^{-1}_{\mu;\varphi\circ\gamma}{,}\delta_{\mu\circ\varphi;\gamma})"'] & D(B)_0 \arrow[d, "\delta^{-1}_{\mu;\varphi}"] \\
D(C)_3 \arrow[r, "c"'] & D(E)_1
\end{tikzcd}$$

which we will use in the calculation(s) below. The first of the latest side-calculations along with associativity of composition and the identity law for composition in $D(E)$

allows us to see

$$\begin{aligned}
&(\ell^w_\mu w_\mu,\pi_0\pi_0\pi_1)c_{\mu\circ\psi}\pi_1\\
&= (\ell^w_\mu w_\mu,\pi_0\pi_0\pi_1)c'_{\mu;\psi;\delta^{-1}}c\\
&= (\ell^w_\mu w_\mu,\pi_0\pi_0\pi_1)(\pi_0\pi_1, \pi_1\pi_1 D(\mu)_1, \pi_1\pi_0\delta^{-1}_{\mu;\psi})c\\
&= \big(\ell^w_\mu w_\mu\pi_1, \pi_0\pi_0\pi_1\pi_1 D(\mu)_1, \pi_0\pi_0\pi_1\pi_0\delta^{-1}_{\mu;\psi}\big)c\\
&= \big(\pi_0\pi_0\pi_1\pi_1 D(\mu)_1 se, \pi_0\pi_0\pi_1\pi_1 D(\mu)_1, \pi_0\pi_0\pi_1\pi_0\delta^{-1}_{\mu;\psi}\big)c\\
&= \big((\pi_0\pi_0\pi_1\pi_1 D(\mu)_1 se, \pi_0\pi_0\pi_1\pi_1 D(\mu)_1)c, \pi_0\pi_0\pi_1\pi_0\delta^{-1}_{\mu;\psi}\big)c\\
&= \big(\pi_0\pi_0\pi_1\pi_1 D(\mu)_1(se, 1_{D(E)_1})c, \pi_0\pi_0\pi_1\pi_0\delta^{-1}_{\mu;\psi}\big)c\\
&= \big(\pi_0\pi_0\pi_1\pi_1 D(\mu)_1, \pi_0\pi_0\pi_1\pi_0\delta^{-1}_{\mu;\psi}\big)c
\end{aligned}$$

The second of the latest side-calculations along with the left square deduced from the coherence diagrams allow us to see

$$\begin{aligned}
&\big(\pi_0\pi_0\pi_1\pi_1 D(\mu)_1, \pi_0\pi_0\pi_1\pi_0\delta^{-1}_{\mu;\psi}\big)c\\
&= \big(\pi_0\pi_0\pi_1\pi_1 D(\mu)_1, \pi_0\pi_1 w_\gamma\pi_0 D(\gamma)_0\delta^{-1}_{\mu;\psi}\big)c\\
&= \big(\pi_0\pi_0\pi_1\pi_1 D(\mu)_1, \pi_0\pi_1 w_\gamma\pi_0(\delta_{\psi;\gamma}D(\mu)_1,\delta^{-1}_{\mu;\psi\circ\gamma},\delta_{\mu\circ\psi;\gamma})c\big)c
\end{aligned}$$

The next side calculation shows how we can replace the last line above. The first line below comes from the definition of $\mathcal{P}_{cq}(\mathbb{D})_{(\varphi;\psi);\gamma}$. The second and third lines follow from the definition of cofiber composition and the fourth line is a standard computation using the calculus of pairing maps by the universal property of pullbacks. The fifth, sixth, and seventh lines are consequences of the definition of W_γ and the eighth line follows by definition of $\delta^{-1}_{\psi;\gamma}$. The ninth line comes from associativity of composition, and in the tenth line we apply the identity law for composition in $D(A)$.

$$
\begin{aligned}
&\pi_0(\pi_0\pi_0, \pi_1 w_\gamma)c_{\varphi;\gamma}\pi_1 D(\mu)_1\\
&= \pi_1(\pi_0\pi_1, \pi_1 w_\gamma)c_{\psi;\gamma}\pi_1 D(\mu)_1\\
&= \pi_1(\pi_0\pi_1, \pi_1 w_\gamma)c'_{\psi;\gamma;\delta^{-1}} c D(\mu)_1\\
&= \pi_1(\pi_0\pi_1, \pi_1 w_\gamma)(\pi_0\pi_1, \pi_1\pi_1 D(\psi)_1, \pi_1\pi_0\delta^{-1}_{\psi;\gamma})cD(\mu)_1\\
&= \big(\pi_1\pi_0\pi_1\pi_1, \pi_1\pi_1 w_\gamma\pi_1 D(\psi)_1, \pi_1\pi_1 w_\gamma\pi_0\delta^{-1}_{\psi;\gamma}\big)cD(\mu)_1\\
&= \big(\pi_1\pi_0\pi_1\pi_1, \pi_1\pi_1\pi_\gamma e D(\psi)_1, \pi_1\pi_1 w_\gamma\pi_0\delta^{-1}_{\psi;\gamma}\big)cD(\mu)_1\\
&= \big(\pi_1\pi_0\pi_1\pi_1, \pi_1\pi_1\pi_\gamma D(\psi)_0 e, \pi_1\pi_1 w_\gamma\pi 0\delta^{-1}_{\psi;\gamma}\big)cD(\mu)_1\\
&= \big(\pi_1\pi_0\pi_1\pi_1, \pi_1\pi_1 w_\gamma\pi_0 D(\gamma)_0 D(\psi)_0 e, \pi_1\pi_1 w_\gamma\pi_0\delta^{-1}_{\psi;\gamma}\big)cD(\mu)_1\\
&= \big(\pi_1\pi_0\pi_1\pi_1, \pi_1\pi_1 w_\gamma\pi_0\delta^{-1}_{\psi;\gamma} se, \pi_1\pi_1 w_\gamma\pi_0\delta^{-1}_{\psi;\gamma}\big)cD(\mu)_1\\
&= \big(\pi_1\pi_0\pi_1\pi_1, \pi_1\pi_1 w_\gamma\pi_0\delta^{-1}_{\psi;\gamma}(se, 1_{D(A)_1})c\big)cD(\mu)_1\\
&= \big(\pi_1\pi_0\pi_1\pi_1, \pi_1\pi_1 w_\gamma\pi_0\delta^{-1}_{\psi;\gamma}\big)cD(\mu)_1\\
&= \big(\pi_1\pi_0\pi_1\pi_1 D(\mu)_1, \pi_1\pi_1 w_\gamma\pi_0\delta^{-1}_{\psi;\gamma} D(\mu)_1\big)c
\end{aligned}
$$

Functoriality of $D(\mu)$ gives the final line in the computation above. Expanding the composition in the last line of the previous calculation and recalling that that the diagram

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{D})_{(\varphi,\psi);\gamma} \arrow[r, shift left, "\pi_0\pi_0"] \arrow[r, shift right, "\pi_1\pi_0"'] & \big(P(\mathbb{D})_{(\varphi,\psi)}\ {}_t\times_{w_\gamma s_\gamma} W_\gamma\big) \times D_{\varphi\circ\gamma}
\end{tikzcd}$$

commutes by definition of the pullback $\mathcal{P}_{cq}(\mathbb{D})_{(\varphi,\psi);\gamma}$ allows us to see that up to this point we have:

$$
\begin{aligned}
(\ell^w_\mu w_\mu, \pi_0\pi_0\pi_1)c_{\mu\circ\psi}\pi_1 &= \big(\pi_0\pi_0\pi_1\pi_1 D(\mu)_1, \pi_0\pi_0\pi_1\pi_0\delta^{-1}_{\mu;\psi}\big)c\\
&= \big(\pi_0(\pi_0\pi_0, \pi_1 w_\gamma)c_{\varphi;\gamma}\pi_1 D(\mu)_1, \pi_1\pi_1 w_\gamma\pi_0\delta^{-1}_{\mu;\psi\circ\gamma}\delta_{\mu\circ\psi;\gamma}\big)c
\end{aligned}
$$

A similar side calculation to the last one, where we can cancel composition with an identity map in the middle, shows

$$\begin{aligned}
&\pi_0(\pi_0\pi_0, \pi_1 w_\gamma)c_{\varphi;\gamma}\pi_1 D(\mu)_1\\
&= \pi_0(\pi_0\pi_0, \pi_1 w_\gamma)c'_{\varphi;\gamma;\delta^{-1}}cD(\mu)_1\\
&= \pi_0(\pi_0\pi_0, \pi_1 w_\gamma)(\pi_0\pi_1, \pi_1\pi_1 D(\varphi)_1, \pi_1\pi_0\delta^{-1}_{\varphi;\gamma})cD(\mu)_1\\
&= \big(\pi_0\pi_0\pi_0\pi_1, \pi_0\pi_1 w_\gamma \pi_1 D(\varphi)_1, \pi_0\pi_1 w_\gamma \pi_0 \delta^{-1}_{\varphi;\gamma}\big)cD(\mu)_1\\
&= \big(\pi_0\pi_0\pi_0\pi_1, \pi_0\pi_1\pi_\gamma e D(\varphi)_1, \pi_0\pi_1 w_\gamma \pi_0 \delta^{-1}_{\varphi;\gamma}\big)cD(\mu)_1\\
&\vdots\\
&= \big(\pi_0\pi_0\pi_0\pi_1 D(\mu)_1, \pi_0\pi_1 w_\gamma \pi_0 \delta^{-1}_{\varphi;\gamma} D(\mu)_1\big)c.
\end{aligned}$$

Note that the parallel arrows

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{D})_{(\varphi,\psi);\gamma} \arrow[r, shift left, "\pi_1\pi_1"] \arrow[r, shift right, "\pi_0\pi_1"'] & W_\gamma
\end{tikzcd}$$

are equal by definition of the pullback $\mathcal{P}_{cq}(\mathbb{D})_{(\varphi,\psi);\gamma}$. Since $\psi \circ \gamma = \varphi \circ \gamma$ and $\mu \circ \psi = \mu \circ \varphi$, composition in $D(E)$ is associative, the commuting square(s) deduced from the coherence of the structure isomorphisms for D, and by diagrams (C) and (A), we have that

$$\begin{aligned}
&(\ell^w_\mu w_\mu, \pi_0\pi_0\pi_1)c_{\mu\circ\psi}\pi_1\\
&= \big(\pi_0\pi_0\pi_0\pi_1 D(\mu)_1, \pi_0\pi_1 w_\gamma \pi_0 \delta^{-1}_{\varphi;\gamma} D(\mu)_1, \pi_1\pi_1 w_\gamma \pi_0 \delta^{-1}_{\mu;\varphi\circ\gamma}\delta_{\mu\circ\varphi;\gamma}\big)c\\
&= \big(\pi_0\pi_0\pi_0\pi_1 D(\mu)_1, \pi_0\pi_1 w_\gamma \pi_0 \delta^{-1}_{\varphi;\gamma} D(\mu)_1, \pi_0\pi_1 w_\gamma \pi_0 \delta^{-1}_{\mu;\varphi\circ\gamma}\delta_{\mu\circ\varphi;\gamma}\big)c\\
&= \big(\pi_0\pi_0\pi_0\pi_1 D(\mu)_1, \pi_0\pi_1 w_\gamma \pi_0 (\delta^{-1}_{\varphi;\gamma} D(\mu)_1, \delta^{-1}_{\mu;\varphi\circ\gamma}\delta_{\mu\circ\varphi;\gamma})c\big)c\\
&= \big(\pi_0\pi_0\pi_0\pi_1 D(\mu)_1, \pi_0\pi_1 w_\gamma \pi_0 D(\gamma)_0 \delta^{-1}_{\mu;\varphi}\big)c\\
&= \big(\pi_0\pi_0\pi_0\pi_1 D(\mu)_1, \pi_0\pi_0\pi_1\pi_0 \delta^{-1}_{\mu;\varphi}\big)c\\
&= \big(\pi_0\pi_0\pi_0\pi_1 D(\mu)_1, \pi_0\pi_0\pi_0\pi_0 \delta^{-1}_{\mu;\varphi}\big)c\\
&= (\ell^w_\mu w_\mu, \pi_0\pi_0\pi_0)c_{\mu\circ\psi}\pi_1
\end{aligned}$$

It follows that

$$(\ell^w_\mu w_\mu, \pi_0\pi_0\pi_0)c_{\mu\circ\psi} = (\ell^w_\mu w_\mu, \pi_0\pi_0\pi_1)c_{\mu\circ\psi}$$

so there exists a unique map $\ell_{\mu;(\varphi,\psi)} : \mathcal{P}_{cq}(\mathbb{D})_{(\varphi,\psi);\gamma} \to \mathcal{P}_{eq}(\mathbb{D})_{\mu;(\varphi,\psi)}$ at the cofiber level:

$$\begin{tikzcd}
& \mathcal{P}_{cq}(\mathbb{D})_{(\varphi,\psi);\gamma} \arrow[ddl, bend right, "\tilde{\ell}_{\mu;(\varphi,\psi)}"'] \arrow[ddr, bend left, "\tilde{\ell}_{\mu;(\varphi,\psi)}"] \arrow[d, dotted, "\ell_{\mu;(\varphi;\psi)}"] & \\
& \mathcal{P}_{eq}(\mathbb{D})_{\mu;(\varphi,\psi)} \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] \arrow[d, phantom, "\vee"] & \\
W_{\mu}\,{}_{w_\mu t_\mu}\!\times_s P(\mathbb{D})_{(\varphi,\psi)} \arrow[dr, "\left(1_{P(\mathbb{D})_{(\varphi,\psi)}},(\pi_0 w_\mu,\pi_1\pi_0)c_{\mu;\varphi}\right)"'] & & W_{\mu}\,{}_{w_\mu t_\mu}\!\times_s P(\mathbb{D})_{(\varphi,\psi)} \arrow[dl, "\left(1_{P(\mathbb{D})_{(\varphi,\psi)}},(\pi_0 w_\mu,\pi_1\pi_1)c_{\mu;\psi}\right)"] \\
& \left(W_{\mu}\,{}_{w_\mu t_\mu}\!\times_s P(\mathbb{D})_{(\varphi,\psi)}\right) \times D_{\mu\circ\varphi} &
\end{tikzcd}$$

Since this is true for arbitrary parallel pairs (φ,ψ) in $\mathcal{A}$ and an arbitrary γ in $\mathcal{A}$ which coequalizes them, the universal property of coproducts induces unique maps $\tilde{\ell}_0 : \mathcal{P}_{cq}(\mathbb{D}) \to W\,{}_{wt}\!\times_s P(\mathbb{D})$ and $\ell_0 : \mathcal{P}_{cq}(\mathbb{D}) \to \mathcal{P}_{eq}(\mathbb{D})$ such that the diagram

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{D}) \arrow[rd, bend left, "\tilde{\ell}_0"] \arrow[d, dotted, "\ell_0"] \arrow[dd, bend right=60, "\tilde{\ell}_0"'] & \\
\mathcal{P}_{eq}(\mathbb{D}) \arrow[r, tail, "\iota_{eq}"] \arrow[d, tail, "\iota_{eq}"'] \arrow[rd, phantom, "\lrcorner", very near start] & W\,{}_{wt}\!\times_s P(\mathbb{D}) \arrow[d, "(1_{P(\mathbb{D})},(\pi_1 w,\pi_0\pi_1)c)"] \\
W\,{}_{wt}\!\times_s P(\mathbb{D}) \arrow[r, "(1_{P(\mathbb{D})},(\pi_1 w,\pi_0\pi_0)c)"'] & \left(W\,{}_{wt}\!\times_s P(\mathbb{D})\right) \times \mathbb{D}_1
\end{tikzcd}$$

commutes in $\mathcal{E}$. The lift we need is then given by the following pullback diagram

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{D}) \arrow[rrd, bend left, equal] \arrow[rd, dotted, "\ell"] \arrow[rdd, bend right, "\ell_0"'] & & \\
& \mathcal{P}(\mathbb{D}) \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[rd, phantom, "\lrcorner", very near start] & \mathcal{P}_{cq}(\mathbb{D}) \arrow[d, "\iota_{cq}\pi_0"] \\
& \mathcal{P}_{eq}(\mathbb{D}) \arrow[r, "\iota_{eq}\pi_1"'] & P(\mathbb{D})
\end{tikzcd}$$

where the upper triangle shows we can assume the cover to be the identity map $1_{\mathcal{P}_{cq}(\mathbb{D})} : \mathcal{P}_{cq}(\mathbb{D}) \to \mathcal{P}_{cq}(\mathbb{D})$. The outside of the pullback diagram commutes because by definition of ℓ_0 we have

$$\ell_0 \iota_{eq} \pi_1 = \ell_0^w \pi_1$$

and then the following diagram

$$\begin{tikzcd}
\mathcal{P}_{cq}(\mathbb{D}) \arrow[r, dotted, "\tilde{\ell}_0"] \arrow[rr, dotted, bend left, "\iota_{cq}\pi_0"] & W {}_{wt}\times_s P(\mathbb{D}) \arrow[r, dotted, "\pi_1"] & P(\mathbb{D}) \\
\mathcal{P}_{cq}(\mathbb{D})_{(\varphi,\psi);\gamma} \arrow[u, "\iota_{(\varphi,\psi);\gamma}"] \arrow[r, "\tilde{\ell}_{\mu;(\varphi,\psi)}"'] \arrow[rr, bend right, "\pi_0\pi_0"'] & W_\mu {}_{w_\mu t_\mu}\times_s P(\mathbb{D})_{(\varphi,\psi)} \arrow[u, "\iota_\mu^w \times \iota_{(\varphi,\psi)}"'] \arrow[r, "\pi_1"'] & D_\varphi {}_{(s,t)}\times_{(s,t)} D_\psi \arrow[u, "\iota_{(\varphi,\psi)}"']
\end{tikzcd}$$

commutes in $\mathcal{E}$. ☐

The preceding lemmas in this section come together to show that the object of the convenient cleavage of cartesian arrows we consider for the internal Grothendieck construction can be formally inverted to give an internal category of (right) fractions, $\mathbb{D}[W^{-1}]$.

Proposition 71. *Let $D : \mathcal{A}^{op} \to \mathbf{Cat}(\mathcal{E})$ be a pseudofunctor such that $\mathcal{E}$ is a candidate context for internal fractions and admits an internal Grothendieck construction, $\mathbb{D}$, which is a candidate for internal fractions. Let $W = \coprod_{\varphi \in \mathcal{A}_1} W_\varphi$ be the object of the canonical cleavage of the cartesian arrows we defined at the beginning of this section. Then $(\mathbb{D}, W)$ admits an internal category of fractions.*

Proof. Lemmas 67, 68, 69, and 70 come together to show that the Internal Fractions Axioms of Definition 34 are satisfied and the result follows by Definition 27. ☐

5.2 Pseudocolimits of Certain Small Filtered Diagrams of Internal Categories

The crux of our main theorem is an observation that in the correspondence between oplax natural transformations $D \implies \Delta\mathbb{X}$ and internal functors $\mathbb{D} \to \mathbb{X}$ for an arbitrary internal category $\mathbb{X}$ established in Theorem 19, the components of the natural transformations factor through the family of arrows $w : W \to \mathbb{D}_1$ that get inverted by the internal localization $L : \mathbb{D} \to \mathbb{D}[W^{-1}]$. The oplax natural transformations $D \implies \Delta$ are required here since we are dealing with a contravariant pseudofunctor and constructing a category of right fractions.

Recall that there is a canonical oplax natural transformation $D \implies \Delta\mathbb{D}$ whose components are internal functors $\ell_B : D(B) \to \mathbb{D}$ defined by:

$$\begin{tikzcd}
D(B)_0 \arrow[r, tail, "(\ell_B)_0 = \iota_B"] & \mathbb{D}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
D(B)_1 \arrow[r, "\left(t{,}(1_{D(B)_1}{,}t\delta_B^{-1})c\right)"] \arrow[dr, "(\ell_B)_1"'] & D_{1_A} \arrow[d, tail, "\iota_{1_A}"] \\
 & \mathbb{D}_1
\end{tikzcd} .$$

For each $\varphi : A \to B$ in $\mathcal{A}$, the internal transformation , $\ell_\varphi : D(\varphi)\ell_A \implies \ell_B$ is defined by its components, $\ell_\varphi \iota_\varphi : D(B)_0 \to \mathbb{D}_1$, which factor through W_φ as a consequence of the commuting diagram:

$$\begin{tikzcd}
D(B)_0 \arrow[drr, bend left, "D(\varphi)_0"] \arrow[dr, dotted, "\ell_\varphi^w"] \arrow[ddr, dotted, "\ell_\varphi"] \arrow[dddr, equals] & & \\
 & W_\varphi \arrow[r, "\pi_\varphi"] \arrow[d, tail, "w_\varphi"'] \arrow[dr, phantom, "\lrcorner", very near start] & D(A)_0 \arrow[d, tail, "e"] \arrow[dd, equals, bend left] \\
 & D_\varphi \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & D(A)_1 \arrow[d, "t"] \\
 & D(B)_0 \arrow[r, "D(\varphi)_0"'] & D(A)_0
\end{tikzcd}$$

More precisely, the components of the natural transformation, ℓ_φ, are picked out by the composite $\ell_\varphi \iota\varphi : D(B)_0 \to \mathbb{D}_1$, which represents the arrows in the component D_φ which are given by applying the identity structure map, $e : D(A)_0 \to D(A)_1$, after applying $D(\varphi)_0 : D(B)_0 \to D(A)_0$.

Definition 72. For an arbitrary oplax natural transformation $x : D \implies \Delta\mathbb{X}$, the induced internal functor $\theta_x : \mathbb{D} \to \mathbb{X}$ is defined on components and then induced by the universal property of the coproduct as follows:

$$\begin{tikzcd}
\mathbb{D}_0 \arrow[r, dotted, "(\theta_x)_0"] & \mathbb{X}_0 \\
D(B)_0 \arrow[u, tail, "\iota_B"] \arrow[ur, "(x_B)_0"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}_1 \arrow[r, dotted, "(\theta_x)_1"] & \mathbb{X}_1 \\
D_\varphi \arrow[u, tail, "\iota_\varphi"] \arrow[r, "\left(\pi_1(x_A)_1{,}\pi_0 x_\varphi\right)"'] & \mathbb{X}_2 \arrow[u, "c"']
\end{tikzcd}$$

The subtle difference here from the induced internal functor in Section 3.3.3 is the map

$$D_\varphi \xrightarrow{\left(\pi_1(x_A)_1,\pi_0 x_\varphi\right)} \mathbb{X}_2$$

which twists the order of composition in $\mathbb{X}$ to account for working with a contravariant functor the covariant used in Lemma 19. Next we review how every internal functor $\mathbb{D} \to \mathbb{X}$ induces an oplax natural transformation by whiskering. This is the same as in Section 3.3.4 but we restate it for our reader's convenience and the fact that we're working with a contravariant functor and oplax transformations.

Definition 73. For an arbitrary internal functor $F : \mathbb{D} \to \mathbb{X}$, the induced oplax natural transformation $F^* : D \implies \Delta\mathbb{X}$ has components that are internal functors $F^*_B : D(B) \to \mathbb{X}$ defined by post-composition

$$\begin{tikzcd}
D(B) \arrow[r, "\ell_B"] \arrow[dr, "F^*"'] & \mathbb{D} \arrow[d, "F"] \\
& \mathbb{X}
\end{tikzcd}$$

For each $\varphi : A \to B$ in $\mathcal{A}$, the induced transformation $F^*_\varphi : D(\varphi)F^* \implies F^*$ is defined by whiskering. More precisely, the components are given by post-composing the components of ℓ_φ with F_1:

$$\begin{tikzcd}
D(B)_0 \arrow[r, "\ell_\varphi"] \arrow[dr, "F^*_\varphi"'] & \mathbb{D}_1 \arrow[d, "F_1"] \\
& \mathbb{X}_1
\end{tikzcd}$$

A similar proof to the one in Proposition 15 shows the assignments in Definitions 72 and 73 are inverses. The following Lemma shows that the induced internal functor in Definition 72 inverts the cartesian arrows, $w : W \to \mathbb{D}_1$.

Lemma 74. *If $x : D \implies \Delta\mathbb{X}$ is a natural isomorphism, then the induced internal functor $\theta_x : \mathbb{D} \to \mathbb{X}$ inverts the family of cartesian arrows, $w : W \to \mathbb{D}$, as in Definition 49.*

Proof. By Definition 49 we need to show that the composite

$$\begin{tikzcd}
W \arrow[r, "w"] \arrow[rd, "\theta_x(W)"'] & \mathbb{D}_1 \arrow[d, "(\theta_x)_1"] \\
 & \mathbb{X}_1
\end{tikzcd}$$

is invertible in $\mathbb{X}$ as in Definition 48. It suffices to produce a map $\theta_x(w)^{-1} : W \to \mathbb{X}_1$ such that such that

$$(\theta_x(w)^{-1}, \theta_x(w))c = \theta_x(w)te, \quad (\theta_x(w), \theta_x(w)^{-1})c = \theta_x(w)se.$$

Since $x : D \implies \Delta\mathbb{X}$ is a natural isomorphism, for each $B \in \mathcal{A}_0$ the components $x_\varphi : D(B)_0 \to \mathbb{X}_1$ are invertible in $\mathbb{X}$. By Definition 48 there exists $x_\varphi^{-1} : D(B)_0 \to \mathbb{X}_1$ such that

$$x_\varphi^{-1}s = x_\varphi t, \quad x_\varphi^{-1}t = x_\varphi s$$

and

$$(x_\varphi^{-1}, x_\varphi)c = x_\varphi te, \quad (x_\varphi, x_\varphi^{-1})c = x_\varphi se.$$

commutes in $\mathcal{E}$. Define a candidate inverse for $w\theta_x : W \to X_1$ (with respect to composition in $\mathbb{X}$) to be the universal map induced by the family of composites,

$$\begin{tikzcd}
W_\varphi \arrow[r, tail, "w_\varphi"] \arrow[rrd, "\theta_x(W_\varphi)^{-1}"'] & D_\varphi \arrow[r, "\pi_0"] & D(B)_0 \arrow[d, "x_\varphi^{-1}"] \\
 & & \mathbb{X}_1
\end{tikzcd},$$

in $\mathcal{E}$ for each arrow $\varphi : A \to B$ in $\mathcal{A}$. Note that by definition of x_φ and $\theta_x(W)^{-1}$

$$\begin{aligned}
\theta_x(W_\varphi)^{-1}s &= w_\varphi\pi_0 x_\varphi^{-1}s \\
&= w_\varphi\pi_0 x_\varphi t \\
&= w_\varphi(\pi_1(x_A)_1, \pi_0 x_\varphi)ct \\
&= w_\varphi(\theta_x)_1 t \\
&= \iota_\varphi^w w(\theta_x)_1 \\
&= \iota_\varphi^w \theta_x(W).
\end{aligned}$$

For the rest of our argument we need a nicer characterization of $\theta_x(W) = w(\theta)_1 :$ $W \to$ the following commuting diagram

$$\begin{tikzcd}
W_\varphi \arrow[r, "\pi_\varphi"] \arrow[d, tail, "w_\varphi"'] \arrow[dr, phantom, very near start, "\lrcorner"] & D(A)_0 \arrow[d, tail, "e"] & \\
D_\varphi \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, very near start, "\lrcorner"] & D(A)_1 \arrow[r, "(x_A)_1"] \arrow[d, "t"] & \mathbb{X}_1 \arrow[d, "t"] \\
D(B)_0 \arrow[r, "D(\varphi)_0"'] \arrow[d, "x_\varphi"'] & D(A)_0 \arrow[r, "(x_A)_0"] & \mathbb{X}_0 \\
\mathbb{X}_1 \arrow[urr, bend right, "s"'] & &
\end{tikzcd}$$

in $\mathcal{E}$, where the pullback squares commute by definition, the bottom right square commutes by functoriality, and the bottom part of the diagram commutes by definition of the natural transformation $x_\varphi : D(\varphi)x_A \implies x_B$. Use the previous diagram's commutativity along with the identity law for internal composition in $\mathbb{X}$ to compute the composite

$$\begin{aligned} w_\varphi\big(\pi_1(x_A)_1,\pi_0 x_\varphi\big)c &= \big(w_\varphi\pi_1(x_A)_1,w_\varphi\pi_0 x_\varphi\big)c \\ &= \big(\pi_\varphi e(x_A)_1,w_\varphi\pi_0 x_\varphi\big)c \\ &= \big(\pi_\varphi ete(x_A)_1,w_\varphi\pi_0 x_\varphi\big)c \\ &= \big(w_\varphi\pi_0 D(\varphi)_0 e(x_A)_1,w_\varphi\pi_0 x_\varphi\big)c \\ &= \big(w_\varphi\pi_0 D(\varphi)_0 (x_A)_0 e,w_\varphi\pi_0 x_\varphi\big)c \\ &= \big(w_\varphi\pi_0 x_\varphi se,w_\varphi\pi_0 x_\varphi\big)c \\ &= w_\varphi\pi_0 x_\varphi(se,1_{\mathbb{X}_1})c \\ &= w_\varphi\pi_0 x_\varphi. \end{aligned}$$

Now we can see the target of $\theta_x(W_\varphi)^{-1}$ is the source of $\iota^w_\varphi\theta_x(W_\varphi) : W_\varphi \to \mathbb{X}_0$,

$$\begin{aligned} \theta_x(W_\varphi)^{-1}t &= w_\varphi\pi_0 x_\varphi^{-1}t \\ &= w_\varphi\pi_0 x_\varphi s \\ &= w_\varphi(\pi_1(x_A)_1,\pi_0 x_\varphi)cs \\ &= w_\varphi\iota_\varphi(\theta_x)_1 s \\ &= \iota^w_\varphi w(\theta_x)_1 s \\ &= \iota^w_\varphi\theta_x(W)s, \end{aligned}$$

and get a convenient description of the cofibers of the map $w(\theta_x)_1 : W \to \mathbb{X}_1$ as shown in the commuting diagram:

$$\begin{tikzcd}
W \arrow[r, dotted, "w"] & \mathbb{D}_1 \arrow[r, dotted, "(\theta_x)_1"] & \mathbb{X}_1 \arrow[dr, equal] & \\
W_\varphi \arrow[u, tail, "\iota^w_\varphi"] \arrow[r, tail, "w_\varphi"'] \arrow[dr, tail, "w_\varphi"'] & D_\varphi \arrow[u, tail, "\iota_\varphi"] \arrow[r, "\big(\pi_1(x_A)_1{,}\pi_0 x_\varphi\big)"'] & \mathbb{X}_2 \arrow[u, "c"'] \arrow[r, "c"'] & \mathbb{X}_1 \, . \\
& D_\varphi \arrow[r, "\pi_0"'] & D(B)_0 \arrow[ur, bend right, "x_\varphi"'] &
\end{tikzcd}$$

Now we can compute

$$\begin{aligned}\iota^w_\varphi(\theta_x(W)^{-1}, \theta_x(W))c &= (\iota^w_\varphi\theta_x(W)^{-1}, \iota^w_\varphi\theta_x(W))c \\ &= (w_\varphi\pi_0 x_\varphi^{-1}, \iota^w_\varphi w(\theta_x)_1)c \\ &= (w_\varphi\pi_0 x_\varphi^{-1}, w_\varphi\iota_\varphi(\theta_x)_1)c \\ &= (w_\varphi\pi_0 x_\varphi^{-1}, w_\varphi(\pi_1(x_A)_1, \pi_0 x_\varphi)c)c \\ &= (w_\varphi\pi_0 x_\varphi^{-1}, w_\varphi(\pi_1(x_A)_1, \pi_0 x_\varphi)c)c \\ &= (w_\varphi\pi_0 x_\varphi^{-1}, w_\varphi\pi_0 x_\varphi)c \\ &= w_\varphi\pi_0(x_\varphi^{-1}, x_\varphi)c \\ &= w_\varphi\pi_0 x_\varphi te \\ &= \theta_x(W_\varphi)^{-1}se \\ &= \iota^w_\varphi\theta_x(W)te\end{aligned}$$

as well as

$$\begin{aligned}\iota^w_\varphi(\theta_x(W), \theta_x(W)^{-1})c &= (\iota^w_\varphi\theta_x(W), \iota^w_\varphi\theta_x(W)^{-1})c \\ &= (\iota^w_\varphi w(\theta_x)_1, w_\varphi\pi_0 x_\varphi^{-1})c \\ &= (w_\varphi\iota_\varphi(\theta_x)_1, w_\varphi\pi_0 x_\varphi^{-1})c \\ &= (w_\varphi(\pi_1(x_A)_1, \pi_0 x_\varphi)c, w_\varphi\pi_0 x_\varphi^{-1})c \\ &= (w_\varphi(\pi_1(x_A)_1, \pi_0 x_\varphi)c, w_\varphi\pi_0 x_\varphi^{-1})c \\ &= (w_\varphi\pi_0 x_\varphi, w_\varphi\pi_0 x_\varphi^{-1})c \\ &= w_\varphi\pi_0(x_\varphi, x_\varphi^{-1})c \\ &= w_\varphi\pi_0 x_\varphi se \\ &= \theta_x(W_\varphi)^{-1}te \\ &= \iota^w_\varphi\theta_x(W)^{-1}te \\ &= \iota^w_\varphi\theta_x(W)se\end{aligned}$$

and by the universal property of the coproduct W we get

$$(\theta_x(W)^{-1}, \theta_x(W))c = \theta_x(W)te$$

and

$$(\theta_x(W), \theta_x(W)^{-1})c = \theta_x(W)se.$$

It follows that $\theta_x(W)$ has an inverse in X and that θ_x inverts $w : W \to \mathbb{D}_1$. $\square$

The next lemma we will need in our main result will help us establish that every natural transformation induced by an internal functor that inverts $w : W \to \mathbb{D}_1$ under the equivalence in Theorem 19 is a pseudonatural transformation. This is done by seeing that for each $\varphi : A \to B$ in $\mathcal{A}$ the internal natural transformation obtained by whiskering ℓ_φ with L,

$$\begin{tikzcd}
D(B) \arrow[d, "D(\varphi)"'] \arrow[dr, bend left, "\ell_B"] & & \\
D(A) \arrow[r, "\ell_A"'] \arrow[ur, phantom, "\overset{\ell_\varphi}{\Nearrow}"] & \mathbb{D} \arrow[r, "L"'] & \mathbb{D}[W^{-1}]
\end{tikzcd},$$

gives a natural isomorphism. The key observation to make here is that the 2-cells from the canonical oplax natural transformation $\ell : D \implies \mathbb{D}$ have components that factor through $w : W \to \mathbb{D}_1$ so that whiskering with the internal localization functor $L : \mathbb{D} \to \mathbb{D}[W^{-1}]$ inverts them to give a natural isomorphism after whiskering.

Lemma 75. *For each* $\varphi : A \to B$ *in* $\mathcal{A}$*, the internal natural transformation*

$$\ell_\varphi L : D(\varphi)\ell_A L \implies \ell_B L$$

given by whiskering,

$$\begin{tikzcd}
D(B) \arrow[r, "\ell_B"] \arrow[d, "D(\varphi)"'] & \mathbb{D} \arrow[r, "L"] \arrow[d, equal] & \mathbb{D}[W^{-1}] \arrow[d, equal] & D(B) \arrow[r, "\ell_B L"] \arrow[d, "D(\varphi)"'] & \mathbb{D}[W^{-1}] \arrow[d, equal] \\
D(A) \arrow[r, "\ell_A"'] \arrow[ur, phantom, "\overset{\ell_\varphi}{\Nearrow}"] & \mathbb{D} \arrow[r, "L"'] \arrow[ur, phantom, "\overset{1_L}{\Nearrow}"] & \mathbb{D}[W^{-1}] \arrow[ur, phantom, "="] & D(A) \arrow[r, "\ell_A L"'] \arrow[ur, phantom, "\overset{\ell_\varphi L}{\Nearrow}"] & \mathbb{D}[W^{-1}]
\end{tikzcd},$$

is an isomorphism.

Proof. Recall that the internal localization functor, L, inverts $w : W \to \mathbb{D}_1$. In particular $(wL_1)^{-1} = (1, wse)q : W \to \mathbb{D}[W^{-1}]_1$ is an inverse of $wL1 : W \to \mathbb{D}[W^{-1}]_1$ by Proposition 55. Also recall from Definition 3.3.1 that the components of the natural transformation ℓ_φ are given by $\ell_\varphi \iota_\varphi : D(B)_0 \to \mathbb{D}_1$ where $\ell_\varphi = (1_{D(B)_0}, D(\varphi)_0 e)$ is the unique pairing map induced by the universal property of the pullback D_φ. Notice this map factors through $w : W \to \mathbb{D}_1$ since $\ell_\varphi : D(B)_0 \to D_\varphi$ factors through W_φ via the composite:

$$\begin{tikzcd}
D(B)_0 \arrow[r, "\ell^w_\varphi"] \arrow[dr, "\ell_\varphi"'] & W_\varphi \arrow[d, tail, "w_\varphi"] \\
& D_\varphi
\end{tikzcd} .$$

Then since $w_\varphi \iota_\varphi = \iota^w_\varphi w : W_\varphi \to \mathbb{D}_1$, we have the following commuting diagram:

$$\begin{tikzcd}
D(B)_0 \arrow[r, "\ell^w_\varphi w_\varphi"] \arrow[d, dotted, "\ell^w_\varphi \iota^w_\varphi"'] \arrow[dr, "\ell_\varphi \iota_\varphi"] & D_\varphi \arrow[d, "\iota_\varphi"] \\
W \arrow[r, "w"] & \mathbb{D}_1 .
\end{tikzcd} \tag{$\star$}$$

Abusing notation by reusing the label $\ell_\varphi L$ we can see that the composite

$$\begin{tikzcd}
D(B)_0 \arrow[r, "\ell^w_\varphi w_\varphi"] \arrow[drr, "\ell_\varphi L"'] & D_\varphi \arrow[r, "\iota_\varphi"] & \mathbb{D}_1 \arrow[d, "L_1"] \\
& & \mathbb{D}[W^{-1}]_1
\end{tikzcd}$$

represents the components of the transformation $\ell_\varphi L : D(\varphi)\ell_A L \implies \ell_B L$. Diagram ($\star$) implies these components are all invertible in $\mathbb{D}[W^{-1}]$ via an inverse given by the composite

$$\begin{tikzcd}
D(B)_0 \arrow[r, "\ell^w_\varphi"] \arrow[drr, "(\ell_\varphi L)^{-1}"'] & W_\varphi \arrow[r, "\iota^w_\varphi"] & W \arrow[d, "(wL_1)^{-1}"] \\
& & \mathbb{D}[W^{-1}]_1
\end{tikzcd}$$

in $\mathcal{E}$. To see this is really an inverse we can use the definitions and diagrams above in the proof of this lemma along with the fact that $(wL_1)^{-1}$ is an inverse of wL_1 to see

$$\begin{aligned}(\ell_\varphi L, (\ell_\varphi L)^{-1})c &= (\ell_\varphi^w w_\varphi \iota_\varphi L_1, \ell_\varphi^w \iota_\varphi^w (wL_1)^{-1})c \\ &= (\ell_\varphi^w \iota_\varphi^w L_1, \ell_\varphi^w \iota_\varphi^w (wL_1)^{-1})c \\ &= \ell_\varphi^w \iota_\varphi^w (L_1, (wL_1)^{-1})c \\ &= \ell_\varphi^w \iota_\varphi^w L_1 se \\ &= \ell_\varphi^w w_\varphi \iota_\varphi L_1 se \\ &= (\ell_\varphi L)se\end{aligned}$$

and a similar proof shows

$$((\ell_\varphi L)^{-1}, \ell_\varphi L)c = (\ell_\varphi L)te.$$

It follows that the whiskered transformation

$$\ell_\varphi L : D(\varphi)\ell_A L \implies \ell_B L$$

has an inverse, $(\ell_\varphi L)^{-1}$ and is an internal natural isomorphism between internal functors for each $\varphi : A \to B$ in $\mathcal{A}$. □

Lemma 76. *Every internal functor $F : \mathbb{D} \to \mathbb{X}$ that inverts $w : W \to \mathbb{D}_1$ corresponds to a pseudonatural transformation $D \implies \Delta\mathbb{X}$ via the oplax version of the isomorphism of categories in Theorem 19*

Proof. Suppose $F : \mathbb{D} \to \mathbb{X}$ is an internal functor that inverts $w : W \to \mathbb{D}_1$. Then by the universal property of the internal localization, in Proposition 60, there exists a unique $[F] : \mathbb{D}[W^{-1}] \to \mathbb{X}$ such that $L[F] = F$. The natural transformation corresponding to F under the contravariant version of the isomorphism of categories in Theorem 19 is obtained by whiskering

$$\begin{tikzcd}
D(B) \arrow[d, "D(\varphi)"'] \arrow[dr, bend left, "\ell_B"] & & \\
D(A) \arrow[r, "\ell_A"'] \arrow[ur, Rightarrow, shorten=4mm, "\ell_\varphi"'] & \mathbb{D} \arrow[r, "F"] & \mathbb{X}
\end{tikzcd}$$

for each $\varphi : A \to B$ in $\mathcal{A}$. Since $F = L[F]$ this whiskering can be done in two steps. Starting with

$$\begin{tikzcd}
D(B) \arrow[d, "D(\varphi)"'] \arrow[dr, bend left, "\ell_B"] & & & \\
D(A) \arrow[r, "\ell_A"'] \arrow[ur, Rightarrow, phantom, "\Nearrow\ \ell_\varphi"] & \mathbb{D} \arrow[r, "L"] & \mathbb{D}[W^{-1}] \arrow[r, "{[F]}"] & \mathbb{X}
\end{tikzcd},$$

we can use Lemma 75 to get a natural isomomorphism

$$\begin{tikzcd}
D(B) \arrow[d, "D(\varphi)"'] \arrow[dr, bend left, "\ell_B L"] & & \\
D(A) \arrow[r, "\ell_A L"'] \arrow[ur, Rightarrow, phantom, "\Nearrow\ \ell_\varphi L"] & \mathbb{D}[W^{-1}] \arrow[r, "{[F]}"] & \mathbb{X}
\end{tikzcd}$$

and then we can whisker once more to get

$$\begin{tikzcd}
D(B) \arrow[d, "D(\varphi)"'] \arrow[dr, bend left, "{\ell_B L[F]}"] & \\
D(A) \arrow[r, "{\ell_A L[F]}"'] \arrow[ur, Rightarrow, phantom, "{\Nearrow\ \ell_\varphi L[F]}"] & \mathbb{X}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
D(B) \arrow[d, "D(\varphi)"'] \arrow[dr, bend left, "\ell_B F"] & \\
D(A) \arrow[r, "\ell_A F"'] \arrow[ur, Rightarrow, phantom, "\Nearrow\ \ell_\varphi F"] & \mathbb{X}
\end{tikzcd}.$$

Recall that the components of ℓ_φ are $\ell_\varphi \iota_\varphi : D(B)_0 \to \mathbb{D}_1$ and notice that the components of $\ell_\varphi F$ are precisely

$$\begin{tikzcd}
D(B)_0 \arrow[r, "\ell_\varphi \iota_\varphi"] \arrow[rr, bend left, "\ell_\varphi L"] \arrow[dr, "\ell_\varphi F"'] & \mathbb{D}_1 \arrow[r, "L_1"] \arrow[d, "F_1"] & \mathbb{D}[W^{-1}]_1 \arrow[dl, "{[F]_1}"] \\
& \mathbb{X}_1 &
\end{tikzcd}.$$

Lemma 75 shows that $\ell_\varphi L : D(B)_0 \to \mathbb{D}[W^{-1}]_1$ is invertible in $\mathbb{D}[W^{-1}]$ and since $[F] : \mathbb{D}[W^{-1}] \to \mathbb{X}$ is an internal functor, the total composite in the diagram above is invertible in $\mathbb{X}$ by Lemma 50. $\square$

The next two lemmas allow us to contextualize the previous two lemmas more precisely in terms of the oplax version of the isomorphism of categories in Theorem 19. This helps us avoid many explicit but unnecessary details in the proof of the isomorphism of categories in our main result.

Lemma 77. *The underlying-structure functor*

$$[D, \Delta\mathbb{X}]_{ps} \xrightarrow{U} [D, \Delta\mathbb{X}]_{op\ell}$$

is fully faithful.

Proof. If two modifications, $\mu, \nu : \alpha \to \beta$ between pseudonatural transformations $\alpha, \beta : D \implies \Delta\mathbb{X}$ are equal after forgetting the additional pseudonaturality structure then they are the same modification by definition. This implies U is faithful. It is clearly full because any modification between pseudonatural transformations is what it is. $\square$

Lemma 78. *The underlying-structure functor*

$$[\mathbb{D}, \mathbb{X}]^{\mathcal{E}}_{W} \xrightarrow{U'} [\mathbb{D}, \mathbb{X}]^{\mathcal{E}}$$

is fully faithful.

Proof. Any two natural transformations $\alpha, \beta : f \implies g$ between internal functors $f, g : \mathbb{D} \to \mathbb{X}$ that invert $w : W \to \mathbb{D}_1$ which become equal after forgetting that f and g invert $w : W \to \mathbb{D}_1$ must be the same natural transformations by definition. This implies U' is faithful. Any natural transformation between internal functors $\mathbb{D} \to \mathbb{X}$ that invert $w : W \to \mathbb{D}_1$ is precisely that, so U' is also clearly full. $\square$

The previous four lemmas come together in the following lemma which does most of the work for the proof of our main theorem which follows immediately after.

Lemma 79. *There is an isomorphism of categories*

$$[D, \Delta\mathbb{X}]_{ps} \cong [\mathbb{D}, \mathbb{X}]^{\mathcal{E}}_{W}$$

between the category of pseudonatural transformations $D \implies \Delta\mathbb{X}$ and their modifications; and internal functors, $\mathbb{D} \to \mathbb{X}$, that invert the cartesian arrows, $w : W \to \mathbb{D}$, and their natural transformations.

Proof. By Lemma 74, every pseudonatural transformation $D \implies \Delta\mathbb{X}$ induces an internal functor $\mathbb{D} \to \mathbb{X}$ that inverts $w : W \to \mathbb{D}$ by the following composition of functors

$$\begin{tikzcd}
{[D, \Delta\mathbb{X}]_{ps}} \arrow[d, "U"'] \arrow[dr, dotted] & \\
{[D, \Delta\mathbb{X}]_{\mathrm{op}\ell}} \arrow[r, "\cong"'] & {[\mathbb{D}, \mathbb{X}]^{\mathcal{E}}}
\end{tikzcd}$$

where the bottom isomorphism of categories is the oplax version of Theorem 19. By Lemma 75, the composite

$$\begin{tikzcd}
& {[\mathbb{D}, \mathbb{X}]^{\mathcal{E}}_W} \arrow[d, "U'"] \arrow[dl, dotted] \\
{[D, \Delta\mathbb{X}]_{\mathrm{op}\ell}} & {[\mathbb{D}, \mathbb{X}]^{\mathcal{E}}} \arrow[l, "\cong"]
\end{tikzcd}$$

factors through $[D, \Delta\mathbb{X}]_{ps}$. By Lemmas 77 and 78, we know $[D, \Delta\mathbb{X}]_{ps}$ and $[\mathbb{D}, \mathbb{X}]^{\mathcal{E}}_W$ are both fully faithful subcategories of $[D, \Delta\mathbb{X}]_{\mathrm{op}\ell}$ and $[\mathbb{D}, \mathbb{X}]^{\mathcal{E}}$ respectively so the isomorphism of categories in Theorem 19 restricts to an isomorphism between these subcategories. $\square$

We can finally state and prove the main theorem of this paper.

Theorem 80. *Let $\mathcal{A}$ be a cofiltered category and let $\mathcal{E}$ admit an internal Grothendieck constructoin, $\mathbb{D}$, for the pseudofunctor $D : \mathcal{A}^{op} \to \mathbf{Cat}(\mathcal{E})$. If $(\mathbb{D}, W)$ admits an internal category of fractions, $\mathbb{D}[W^{-1}]$, then $\mathbb{D}[W^{-1}]$ is the pseudocolimit of $D : \mathcal{A}^{op} \to \mathbf{Cat}(\mathcal{E})$.*

Proof. Under the given assumptions, we can apply Lemmas 79 and Theorem 65 to get a chain of isomorphisms (of categories) which can be composed to prove the result.

$$[D, \Delta\mathbb{X}]_{ps} \cong [\mathbb{D}, \mathbb{X}]^{\mathcal{E}}_W \cong [\mathbb{D}[W^{-1}], \mathbb{X}]^{\mathcal{E}}$$

$\square$

Chapter 6

Conclusion

Having given contexts for an internal Grothendieck construction and an internal category of (right) fractions, we have implicitly described a context for computing (op)lax colimits of certain diagrams of internal categories and another for computing pseudocolimits of certain filtered diagrams of internal categories. The purpose of doing this was to isolate and better understand the categorical constructions that are used when working in the context of **Set**, with diagrams of small categories, and to give a new formalism for gluing constructions in categories of internal categories.

For the internal Grothendieck construction of a pseudofunctor we required specific pullbacks along source (or target) maps of our internal categories, and certain disjoint coproducts that commute with these pullbacks. Any extensive category that has these pullbacks will satisfy these conditions, for example $\mathbf{Set}, \mathbf{Cat}, \mathbf{Top}$, and the category of smooth manifolds all admit internal Grothendieck constructions for small diagrams of their internal categories this way. We state these conditions so carefully in order to include other possible examples of larger categories which may not be extensive all around, or which may not contain all pullbacks, but which have these coproducts and pullbacks that interact well with one another.

The internal category of fractions construction requires a collection of pullbacks and equalizers in order to define the objects involved in the internal description of (a weakened version) of the (right) fractions axioms, as well as the relations and quotient objects which were required to define objects of paths of arrows with an appropriate universal property.

An interesting part of our main result, Theorem 80, is that some of the internal fractions structure from the definition of the Internal Fractions Axioms (Definition 34) becomes trivial when the internal category being considered is an internal Grothendieck construction. In this case we proved a formal gluing construction for these internal categories by showing that the resulting internal category of fractions is the pseudocolimit of the original diagram.

Future work in this area includes exploring more examples of diagrams of internal categories arising in contexts that satisfy our conditions. We have plenty extensive categories that allow for an internal Grothendieck construction, and it would be interesting to find an example where the entire category is not extensive, but the pullbacks and coproducts we have interact nicely for other reasons.

We hope to use our construction to eventually study the homotopy theory of certain generalized spaces called stacks. These are special pseudofunctors that can be represented by internal groupoids and a proposed topic of PhD research in the coming years. Another direction for future work is translating this result into the language of a proof assistant, like Lean or Agda, in order to make the lengthy calculations easier to verify and accept. Having this framework would make it more feasible to consider diagrams of internal higher categories and try to replicate higher categorical colimit constructions such as in [15] by extending the notion of internal fractions appropriately.

Appendix A

Internal Grothendieck Construction

This section of the appendix contains technical lemmas used in Chapter 3.

A.1 Associativity of Composition

This first lemma we need states that the source and target of a composite coincides with the source and target of the first and second map in the composite respectively.

Lemma 81. *For any composable pair* $(\varphi, \psi) \in \mathcal{A}(W, X) \times \mathcal{A}(X, Y)$ *in* $\mathcal{A}$ *we have that 'the source (target) of the composite is the source (target) of the first (second) map (respectively).'*

$$c_{\varphi;\psi} t_{\varphi\psi} = p_1 t_\psi \qquad , \qquad c_{\varphi;\psi} s_{\varphi\psi} = p_0 s_\varphi .$$

Proof. By definition of $t_{\varphi\psi}$, $s_{\varphi\psi}$, $c'_{\delta;\varphi;\psi}$, $c'_{\varphi;\psi}$, and $c_{\varphi;\psi}$.

$$\begin{aligned} c_{\varphi;\psi} t_{\varphi\psi} &= c_{\varphi;\psi} \pi_1 t \iota_Y \\ &= c_{\varphi;\psi} \pi_1 t \iota_Y \\ &= c'_{\delta;\varphi;\psi} c t \iota_Y \\ &= c'_{\delta;\varphi;\psi} q_{12} q_1 t \iota_Y \\ &= c'_{\varphi;\psi} q_1 t \iota_Y \\ &= p_1 \pi_1 t \iota_Y \\ &= p_1 t_\psi \end{aligned} \qquad\qquad \begin{aligned} c_{\varphi;\psi} s_{\varphi\psi} &= c_{\varphi;\psi} \pi_0 \iota_W \\ &= p_0 \pi_0 \iota_W \\ &= p_0 s_\varphi \end{aligned}$$

□

This next two lemmas contain calculations that show how to compute cofiber composition of the first and last two maps of a composable triple in the internal category of fractions, $\mathbb{D}$. These results are used to prove associativity of composition in $\mathbb{D}$ in Proposition 85.

Lemma 82. *For any φ, ψ, γ composable in $\mathcal{A}$*

$$c'_{01}c'_{\delta;\varphi\psi;\gamma} = (p_{01}p_0\pi_0\delta_{\varphi\psi;\gamma}, p_{01}c'_{\delta;\varphi;\psi}cD(\gamma)_1, p_{12}p_1\pi_1)$$

where

$$c'_{\delta;\varphi;\psi}cD(\gamma)_1 = (p_0\pi_0\delta_{\varphi;\psi}D(\gamma)_1, p_0\pi_1D(\psi)_1D(\gamma)_1, p_1\pi_1D(\gamma)_1)c$$

Proof. By the universal property of the relevant pullback of 'composable-triples,' it suffices to check that

$$\begin{aligned} c'_{01}c'_{\delta;\varphi\psi;\gamma}q_{01}q_0 &= c'_{01}c'_{\delta;(\varphi\psi;\gamma)}q_0 \\ &= c'_{01}p_0\pi_0\delta_{\varphi\psi;\gamma} \\ &= p_{01}c_{\varphi;\psi}\pi_0\delta_{\varphi\psi;\gamma} \\ &= p_{01}p_0\pi_0\delta_{\varphi\psi;\gamma}, \end{aligned}$$

$$\begin{aligned} c'_{01}c'_{\delta;\varphi\psi;\gamma}q_{01}q_1 &= c'_{01}c'_{\delta;(\varphi\psi;\gamma)}q_1 \\ &= c'_{01}p_0\pi_1D(\gamma)_1 \\ &= p_{01}c_{\varphi;\psi}\pi_1D(\gamma)_1 \\ &= p_{01}c'_{\delta;\varphi;\psi}cD(\gamma)_1, \end{aligned}$$

and

$$\begin{aligned} c'_{01}c'_{\delta;\varphi\psi;\gamma}q_{12}q_1 &= c'_{01}c'_{\delta;\varphi\psi;\gamma}q_{12}q_1 \\ &= c'_{01}c'_{\varphi\psi;\gamma}q_1 \\ &= c'_{01}p_1\pi_1 \\ &= p_{12}p_1\pi_1 \end{aligned}$$

respectively. By functoriality of $D(\gamma)$ and associativity of composition the middle component in that triple composite factors

$$\begin{aligned} c'_{\delta;\varphi;\psi} c D(\gamma)_1 &= c'_{\delta;\varphi;\psi}(q_{01}q_1 D(\gamma)_1, q_{01}q_1 D(\gamma)_1, q_{12}q_1 D(\gamma)_1)c \\ &= (p_0\pi_0\delta_{\varphi;\psi} D(\gamma)_1, p_0\pi_1 D(\psi)_1 D(\gamma)_1, p_1\pi_1 D(\gamma)_1)c. \end{aligned}$$

□

Lemma 83. *For any φ, ψ, γ composable in $\mathcal{A}$*

$$c'_{12}c'_{\delta;\varphi;\psi\gamma} = (p_{01}p_0\pi_0\delta_{\varphi;\psi\gamma}, p_{01}p_0\pi_1 D(\psi\gamma)_1, p_{12}c'_{\delta;\varphi;\psi\gamma}c).$$

Proof. By the universal property of the relvant 'composable-triples' pullback, it suffices to check that

$$\begin{aligned} c'_{12}c'_{\delta;\varphi;\psi\gamma}q_{01}q_0 &= c'_{12}c'_{\delta;\varphi;\psi\gamma}q_{01}q_0 \\ &= c'_{12}c'_{\delta;(\varphi;\psi\gamma)}q_0 \\ &= c'_{12}p_0\pi_0\delta_{\varphi;\psi\gamma} \\ &= c'_{12}p_0\pi_0\delta_{\varphi;\psi\gamma} \\ &= p_{01}p_0\pi_0\delta_{\varphi;\psi\gamma}, \end{aligned}$$

$$\begin{aligned} c'_{12}c'_{\delta;\varphi;\psi\gamma}q_{01}q_1 &= c'_{12}c'_{\delta;\varphi;\psi\gamma}q_{01}q_1 \\ &= c'_{12}c'_{\delta;(\varphi;\psi\gamma)}q_1 \\ &= c'_{12}p_0\pi_1 D(\psi\gamma)_1 \\ &= p_{01}p_0\pi_1 D(\psi\gamma)_1, \end{aligned}$$

and

$$\begin{aligned} c'_{12}c'_{\delta;\varphi;\psi\gamma}q_{12}q_1 &= c'_{12}c'_{\varphi;\psi\gamma}q_1 \\ &= c'_{12}p_1\pi_1 \\ &= p_{12}c_{\psi;\gamma}\pi_1 \\ &= p_{12}c'_{\delta;\psi;\gamma}c. \end{aligned}$$

□

The last lemma provides some calculations using the internal coherence for the composition natural isomorphisms associated to the pseudofunctor along with naturality and functoriality. In the classical Grothendieck construction (when $\mathcal{E} = \mathbf{Set}$), Lemma 84 internally encodes the intermediate step

$$\delta_{\varphi\psi;\gamma,a}D(\gamma)(\delta_{\varphi;\psi,a})D(\gamma)(D(\psi)(D(\varphi)(f))) = \delta_{\varphi;\psi\gamma,a}D(\psi\gamma)(D(\varphi)(f))\delta_{\psi;\gamma}$$

for each $a \in D(A)_0$ when proving associativity of composition.

Lemma 84. *For any φ, ψ, γ composable in $\mathcal{A}$*

$$(\pi_0\delta_{\varphi\psi;\gamma}, \pi_0\delta_{\varphi;\psi}D(\gamma)_1, \pi_1(D(\psi)_1D(\gamma)_1))c = (\pi_0\delta_{\varphi;\psi\gamma}, \pi_1D(\psi\gamma)_1, \pi_1t\delta_{\psi;\gamma})c$$

Proof. By coherence of composition isomorphisms for the original pseudofunctor, D, we have that

$$(\delta_{\varphi;\psi\gamma}, D(\varphi)_0\delta_{\psi;\gamma})c = (\delta_{\varphi\psi;\gamma}, \delta_{\varphi;\psi}D(\gamma)_1)c,$$

and by definition of the natural isomorphism $\delta_{\psi;\gamma} : D(\psi\gamma) \implies D(\psi)D(\gamma)$

$$(D(\psi\gamma)_1, t\delta_{\psi;\gamma})c = (s\delta_{\psi;\gamma}, D(\psi)_1D(\gamma)_1)c.$$

Putting coherence and naturality together with associativity we get the following equality of triple composites

$$
\begin{aligned}
&(\pi_0\delta_{\varphi\psi;\gamma}, \pi_0\delta_{\varphi;\psi}D(\gamma)_1, \pi_1(D(\psi)_1D(\gamma)_1))c\\
&= (\pi_0(\delta_{\varphi\psi;\gamma}, \delta_{\varphi;\psi}D(\gamma)_1)c, (D(\psi)_1D(\gamma)_1))c\\
&= (\pi_0(\delta_{\varphi;\psi\gamma}, D(\varphi)_0\delta_{\psi;\gamma})c, \pi_1D(\psi)_1D(\gamma)_1)c\\
&= (\pi_0\delta_{\varphi;\psi\gamma}, (\pi_0D(\varphi)_0\delta_{\psi;\gamma}, \pi_1D(\psi)_1D(\gamma)_1)c)c\\
&= (\pi_0\delta_{\varphi;\psi\gamma}, (\pi_1s\delta_{\psi;\gamma}, \pi_1D(\psi)_1D(\gamma)_1)c)c\\
&= (\pi_0\delta_{\varphi;\psi\gamma}, \pi_1(s\delta_{\psi;\gamma}, D(\psi)_1D(\gamma)_1)c)c\\
&= (\pi_0\delta_{\varphi;\psi\gamma}, \pi_1(D(\psi\gamma)_1, t\delta_{\psi;\gamma})c)c\\
&= (\pi_0\delta_{\varphi;\psi\gamma}, \pi_1D(\psi\gamma)_1, \pi_1t\delta_{\psi;\gamma})c
\end{aligned}
$$

□

We're now ready to prove associativity of composition in $\mathbb{D}$.

Proposition 85. *Composition in $\mathbb{D}$ is associative.*

Proof. The object of composable triples is given by pulling back the pullback projections $\rho_0, \rho_1 : \mathbb{D}_2 \to \mathbb{D}_1$. Denote its canonical maps by ρ_0' and ρ_1' respectively. By Definition 2 we have

$$\mathbb{D}_3 \cong \coprod_{(\varphi,\psi,\gamma)\in\mathcal{A}_3} D_{\varphi;\psi;\gamma}$$

where $D_{\varphi;\psi;\gamma}$ is given by pulling back the projections $p_1 : D_{\varphi;\psi} \to D_\psi$ and $p_0 : D_{\psi;\gamma} \to D_\psi$. More precisely, for any composable triple

$$W \xrightarrow{\varphi} X \xrightarrow{\psi} Y \xrightarrow{\gamma} Z$$

we have the following commuting diagram where the squares on the front and back are all pullbacks.

$$\begin{tikzcd}
 & & & \mathbb{D}_3 \arrow[r, "\rho_{12}"] \arrow[d, "\rho_{01}"'] & \mathbb{D}_2 \arrow[r, "\rho_1"] \arrow[d, "\rho_0"] & \mathbb{D}_1 \arrow[d, "s"] \\
D_{\varphi;\psi;\gamma} \arrow[r, "p_{12}"] \arrow[d, "p_{01}"'] \arrow[urrr, dotted, "\iota_{\varphi;\psi;\gamma}"] & D_{\psi;\gamma} \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[urrr, dotted, "\iota_{\psi;\gamma}"] & D_\gamma \arrow[d, "s_\gamma"'] \arrow[urrr, "\iota_\gamma"] & \mathbb{D}_2 \arrow[r, "\rho_1"] \arrow[d, "\rho_0"'] & \mathbb{D}_1 \arrow[r, "t"] \arrow[d, "s"] & \mathbb{D}_0 \\
D_{\varphi;\psi} \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[urrr, dotted, "\iota_{\varphi;\psi}"] & D_\psi \arrow[r, "t_\psi"'] \arrow[d, "s_\psi"'] \arrow[urrr, "\iota_\psi"] & \mathbb{D}_0 \arrow[urrr, equal] & \mathbb{D}_1 \arrow[r, "t"] & \mathbb{D}_0 & \\
D_\varphi \arrow[r, "t_\varphi"'] \arrow[urrr, "\iota_\varphi"] & \mathbb{D}_0 \arrow[urrr, equal] & & & &
\end{tikzcd}$$

By the universal property of the coproduct $\mathbb{D}_3$, we have maps c_{01} and c_{12} which represent composing the first two and last two maps in a composable triple respectively. These are uniquely determined on cofibers by the maps c'_{01} and c'_{12} respectively. The following diagrams are pastings of commuting cubes that show how c'_{01} and c_{01} are related. The coproduct inclusions from left to right are suppressed for readability but are indicated with the bent dotted arrows.

$$\begin{tikzcd}
 & & & \mathbb{D}_3 \arrow[r, "\rho_{12}"] \arrow[d, "\rho_{01}"'] \arrow[dr, dotted, "c_{01}"] & \mathbb{D}_2 \arrow[dr, bend left, "\rho_1"] & \\
D_{\varphi;\psi;\gamma} \arrow[r, "p_{12}"] \arrow[d, "p_{01}"'] \arrow[dr, dotted, "c'_{01}"] \arrow[urrr, dotted, bend left, "\iota_{\varphi;\psi;\gamma}"] & D_{\psi;\gamma} \arrow[dr, bend left, "p_1"] & & \mathbb{D}_2 \arrow[dr, bend right, "c"'] & \mathbb{D}_2 \arrow[r, "\rho_1"] \arrow[d, "\rho_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{D}_1 \arrow[d, "s"] \\
D_{\varphi;\psi} \arrow[dr, bend right, "c_{\varphi;\psi}"'] & D_{(\varphi\psi);\gamma} \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[urr, dotted, "\iota_{(\varphi\psi);\gamma}"] & D_\gamma \arrow[d, "s_\gamma"] & & \mathbb{D}_1 \arrow[r, "t"'] & \mathbb{D}_0 \\
 & D_{\varphi\psi} \arrow[r, "t_{\varphi\psi}"'] & \mathbb{D}_0 \arrow[urrr, equal, dotted, bend right] & & &
\end{tikzcd} \tag{c'_{01}}$$

A similar diagram shows the relation between c'_{12} and c_{12} and in particular the following squares commute by the universal property of $\mathbb{D}_2$.

$$\begin{tikzcd} \mathbb{D}_3 \arrow[r, "c_{01}"] & \mathbb{D}_2 \\ D_{\varphi;\psi;\gamma} \arrow[u, tail, "\iota_{\varphi;\psi;\gamma}"] \arrow[r, "c'_{01}"'] & D_{\varphi\psi;\gamma} \arrow[u, tail, "\iota_{\varphi\psi;\gamma}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}_3 \arrow[r, "c_{12}"] & \mathbb{D}_2 \\ D_{\varphi;\psi;\gamma} \arrow[u, tail, "\iota_{\varphi;\psi;\gamma}"] \arrow[r, "c'_{12}"'] & D_{\varphi;\psi\gamma} \arrow[u, tail, "\iota_{\varphi;\psi\gamma}"'] \end{tikzcd}$$

To show that composition is associative, we need to show that the front of the commuting cube below commutes.

$$\begin{tikzcd} & \mathbb{D}_3 \arrow[rr, "c_{12}"] \arrow[dd, "c_{01}"'] & & \mathbb{D}_2 \arrow[dd, "c"] \\ D_{\varphi;\psi;\gamma} \arrow[ur, tail, "\iota_{\varphi;\psi;\gamma}"] \arrow[rr, "c'_{12}"] \arrow[dd, "c'_{01}"'] & & D_{\varphi;\psi\gamma} \arrow[ur, tail, "\iota_{\varphi;\psi\gamma}"] \arrow[dd, "c_{\varphi;\psi\gamma}"] & \\ & \mathbb{D}_2 \arrow[rr, "c"'] & & \mathbb{D}_1 \\ D_{\varphi\psi;\gamma} \arrow[ur, tail, "\iota_{\varphi\psi;\gamma}"'] \arrow[rr, "c_{\varphi\psi;\gamma}"'] & & D_{\varphi\psi\gamma} \arrow[ur, tail, "\iota_{\varphi\psi\gamma}"'] & \end{tikzcd}$$

We'll use the universal property of the pullback $D_{\varphi\psi\gamma}$. First notice that

$$D_{(\varphi\psi)\gamma} = D_{\varphi\psi\gamma} = D_{\varphi(\psi\gamma)}$$

because of associativity in $\mathcal{A}$. That is,

$$(\varphi\psi)\gamma = \varphi(\psi\gamma)$$

so we drop the parentheses and just write $\varphi\psi\gamma$ for the triple composite in $\mathcal{A}$ without loss of generality. On one hand by Lemma 81 we have

$$p_{01}c\pi_0\iota_A = p_{01}cs_{\varphi\psi} = p_{01}p_0s_\varphi = p_{01}p_0\pi_0\iota_A$$

and since ι_A is monic,

$$p_{01}c\pi_0 = p_{01}p_0\pi_0. \tag{*}$$

Now recall by the definition of cofiber composition we have

$$c_{\varphi\psi;\gamma} = (p_0\pi_0, c'_{\delta;\varphi\psi;\gamma}c) \qquad\qquad c_{\varphi;\psi\gamma} = (p_0\pi_0, c'_{\delta;\varphi;\psi\gamma}c)$$

and so for the π_0 projection we get:

$$\begin{aligned}
c'_{01}c_{\varphi\psi;\gamma}\pi_0 &= c'_{01}p_0\pi_0 && \text{Def. } c_{\varphi\psi;\gamma}\\
&= p_{01}c\pi_0 && \text{Dgm. } (c'_{01})\\
&= p_{01}p_0\pi_0 && \text{Eq. } *\\
&= c'_{12}p_0\pi_0 && \text{Dgm. } (c'_{01})\\
&= c'_{12}c_{\varphi;\psi\gamma}\pi_0 && \text{Def. } c_{\varphi;\psi\gamma}
\end{aligned}$$

For the π_1 projection we have the following calculation split up on separate lines for readability. By definition of $c_{\varphi\psi;\gamma}$:

$$c'_{01}c_{\varphi\psi;\gamma}\pi_1 = c'_{01}c'_{\delta;\varphi\psi;\gamma}c$$

then by Lemma 82 the right-hand side is:

$$(p_{01}p_0\pi_0\delta_{\varphi\psi;\gamma},\ p_{01}c'_{\delta;\varphi;\psi}cD(\gamma)_1,\ p_{12}p_1\pi_1)c$$

The definition of $c'_{\delta;\varphi;\psi}$ says this is equal to

$$(p_{01}p_0\pi_0\delta_{\varphi\psi;\gamma}, p_{01}(p_0\pi_0\delta_{\varphi;\psi}D(\gamma)_1,\ p_0\pi_1D(\psi)_1D(\gamma)_1,\ p_1\pi_1D(\gamma)_1)c, p_{12}p_1\pi_1)c$$

which, by associativity of internal composition (and factoring out a p_0 from the pairing map into the object of composable paths of length 4, $\mathbb{C}_4$) is equal to

$$((p_{01}p_0\pi_0\delta_{\varphi\psi;\gamma}, p_{01}p_0(\pi_0\delta_{\varphi;\psi}D(\gamma)_1, \pi_1D(\psi)_1D(\gamma)_1)c)c, p_{01}p_1\pi_1D(\gamma)_1,\ p_{12}p_1\pi_1)c.$$

More associativity of internal composition and factoring $p_{01}p_0$ from the pairing map being post-composing with internal composition gives

$$(p_{01}p_0(\pi_0\delta_{\varphi\psi;\gamma}, (\pi_0\delta_{\varphi;\psi}D(\gamma)_1, \pi_1D(\psi)_1D(\gamma)_1)c)c, p_{01}p_1\pi_1D(\gamma)_1, p_{12}p_1\pi_1)c$$

By associativity of internal composition and the definition of $D_{\varphi;\psi;\gamma}$ this becomes:

$$(p_{01}p_0(\pi_0\delta_{\varphi\psi;\gamma}, \pi_0\delta_{\varphi;\psi}D(\gamma)_1, \pi_1D(\psi)_1D(\gamma)_1)c, p_{12}p_0\pi_1D(\gamma)_1, p_{12}p_1\pi_1)c$$

By Lemma 84 this is equal to:

$$(p_{01}p_0(\pi_0\delta_{\varphi;\psi\gamma}, \pi_1 D(\psi\gamma)_1, \pi_1 t\delta_{\psi;\gamma})c, p_{12}p_1\pi_1)c$$

By associativity of internal composition we get

$$(p_{01}p_0\pi_0\delta_{\varphi;\psi\gamma},\ p_{01}p_0\pi_1 D(\psi\gamma)_1,\ p_{01}p_0\pi_1 t\delta_{\psi;\gamma}, p_{12}p_0\pi_1 D(\gamma)_1,\ p_{12}p_1\pi_1)c$$

and then by more associativity

$$(p_{01}p_0\pi_0\delta_{\varphi;\psi\gamma}, p_{01}p_0\pi_1 D(\psi\gamma)_1, (p_{01}p_0\pi_1 t\delta_{\psi;\gamma}, p_{12}p_0\pi_1 D(\gamma)_1, p_{12}p_1\pi_1)c)c$$

By definition of $D_{\varphi;\psi}$ this becomes

$$(p_{01}p_0\pi_0\delta_{\varphi;\psi\gamma}, p_{01}p_0\pi_1 D(\psi\gamma)_1, (p_{01}p_1\pi_0\delta_{\psi;\gamma}, p_{12}p_0\pi_1 D(\gamma)_1, p_{12}p_1\pi_1)c)c$$

and by definition of $D_{\varphi;\psi;\gamma}$ we get

$$(p_{01}p_0\pi_0\delta_{\varphi;\psi\gamma}, p_{01}p_0\pi_1 D(\psi\gamma)_1, (p_{12}p_0\pi_0\delta_{\psi;\gamma}, p_{12}p_0\pi_1 D(\gamma)_1, p_{12}p_1\pi_1)c)c$$

Factoring gives

$$(p_{01}p_0\pi_0\delta_{\varphi;\psi\gamma}, p_{01}p_0\pi_1 D(\psi\gamma)_1, p_{12}(p_0\pi_0\delta_{\psi;\gamma}, p_0\pi_1 D(\gamma)_1, p_1\pi_1)c)cp_{12}$$

and the definition of $c'_{\psi;\gamma}$ says this is equal to

$$(p_{01}p_0\pi_0\delta_{\varphi;\psi\gamma}, p_{12}p_0\pi_1 D(\gamma)_1, p_{01}p_0\pi_1 D(\psi\gamma)_1, p_{12}(c'_{\delta;(\psi;\gamma)}q_0, c'_{\delta;(\psi;\gamma}q_1, c'_{\psi;\gamma}q_1)c)c$$

The definitions of Def. $c'_{\delta;\psi;\gamma}$ and $c'_{\delta;(\psi;\gamma)}$ imply the last term is equal to

$$(p_{01}p_0\pi_0\delta_{\varphi;\psi\gamma}, p_{01}p_0\pi_1 D(\psi\gamma)_1, p_{12}(c'_{\delta;\psi;\gamma}q_{01}q_0, c'_{\delta;\psi;\gamma}q_{01}q_1, c'_{\delta;\psi;\gamma}q_{12}q_1)c)c$$

and the definition of $c'_{\delta;\psi;\gamma}$ makes it

$$(p_{01}p_0\pi_0\delta_{\varphi;\psi\gamma}, p_{01}p_0\pi_1 D(\psi\gamma)_1, p_{12}c'_{\delta;\psi;\gamma}c)c$$

By Lemma 83 this is equal to the left-hand side of the final equation

$$c'_{12}c'_{\delta;\varphi;\psi\gamma}c = c'_{12}c_{\varphi;\psi\gamma}\pi_1$$

which follows from the definition of $c_{\varphi;\psi\gamma}$. Then the universal property of pullbacks says

$$c'_{01}c_{\varphi\psi;\gamma} = c'_{12}c_{\varphi;\psi\gamma}.$$

This shows composition is associative on cofibers/components of the coproduct. Associativity of composition in $\mathbb{D}$ now follows by the universal property of the coproduct $\mathbb{D}_3$. □

A.2 Lemmas for 1-cells of the Canonical Lax Transformation

The following are technical lemmas used in Section 3.3 of Chapter 3.

Lemma 86. *For any $A \in \mathcal{A}_0$:*

$$(q_0(\ell_A)'_1,q_1(\ell_A)'_1)c_{1_A;1_A} = c(\ell_A)'_1$$

Proof. First compute the 0'th projection:

$$\begin{aligned}
(q_0(\ell_A)'_1,q_1(\ell_A)'_1)c_{1_A;1_A}\pi_0 &= (q_0(\ell_A)'_1,q_1(\ell_A)'_1)p_0\pi_0 && \text{Def.}\\
&= q_0(\ell_A)'_1\pi_0 && \text{Def.}\\
&= q_0 s && \text{Def. } (\ell_A)'_1\\
&= cs && \text{Def. } c\\
&= c(\ell_A)'_1\pi_0 && \text{Def. } (\ell_A)'_1
\end{aligned}$$

For the first projection we break up equalities on separate lines and provide justification for each step in between once again for readability. Starting with the equation,

$$(q_0(\ell_A)'_1,q_1(\ell_A)'_1)c_{1_A;1_A}\pi_1 = (q_0(\ell_A)'_1,q_1(\ell_A)'_1)c'_{\delta;1_A;1_A}c,$$

the right-hand side is equal to

$$(q_0 s\delta_{1_A;1_A}, q_0(s\delta_A, 1_{D(A)_1})cD(1_A)_1\ , q_1(s\delta_A, 1_{D(A)_1})c)c$$

by definition of $c'_{1_A;1_A}$. By functoriality of $D(1_A)$ the last term is equal to

$$(q_0 s\delta_{1_A;1_A}, q_0(s\delta_A D(1_A)_1, D(1_A)_1)c, q_1(s\delta_A, 1_{D(A)_1})c)c$$

which, by associativity of internal composition, is equal to

$$(q_0 s\delta_{1_A;1_A},\ q_0 s\delta_A D(1_A)_1, (q_0 D(1_A)_1, q_1 s\delta_A)c,\ q_1)c.$$

The definition of $D(A)_2$ makes this equal to

$$(q_0 s\delta_{1_A;1_A},\ q_0 s\delta_A D(1_A)_1, (q_0 D(1_A)_1, q_0 t\delta_A)c,\ q_1)c$$

which, by factoring maps with respect to pairing maps, is equal to

$$(q_0 s\delta_{1_A;1_A},\ q_0 s\delta_A D(1_A)_1, q_0(D(1_A)_1, t\delta_A)c,\ q_1)c.$$

Naturality of δ_A makes this equal to

$$(q_0 s\delta_{1_A;1_A},\ q_0 s\delta_A D(1_A)_1, q_0(s\delta_A, 1_{D(A)_1})c,\ q_1)c$$

and by associativity we get

$$(q_0 s\delta_{1_A;1_A}, (q_0 s\delta_A D(1_A)_1, q_0 s\delta_A)c,\ q_0,\ q_1)c.$$

Factoring with respect to pairing maps gives

$$(q_0 s\delta_{1_A;1_A}, q_0 s(\delta_A D(1_A)_1, \delta_A)c,\ q_0,\ q_1)c$$

and associativity then gives

$$((q_0 s\delta_{1_A;1_A} q_0 s\delta_A D(1_A)_1)c, q_0 s\delta_A,\ q_0,\ q_1)c.$$

By factoring again we get

$$(q_0 s(\delta_{1_A;1_A}, \delta_A D(1_A)_1)c, q_0 s\delta_A,\ q_0,\ q_1)c$$

and by coherence of the structure isomorphisms for the pseudofunctor D this becomes

$$(q_0 se_A,\ q_0 s\delta_A,\ q_0,\ q_1)c.$$

By associativity of internal composition we have equality with

$$(q_0 s(e_A, \delta_A)c,\ q_0,\ q_1)c$$

and by factoring with pairing maps we get equality with

$$(q_0 s(1_{D(A)_0}, \delta_A)(e_A, 1_{D(A)_1})c, q_0,\ q_1)c]$$

The identity law in $D(A)$ makes the last term equal to

$$(q_0 s(1_{D(A)_0}, \delta_A)p_1,\ q_0,\ q_1)c$$

and by definition of the pullback projections we get

$$(q_0 s\delta_A,\ q_0\ ,\ q_1)c$$

Definition of internal composition gives equality with

$$(cs\delta_A,\ q_0\ ,\ q_1)c$$

and associativity gives

$$(cs\delta_A,\ (q_0, q_1)c)c.$$

The universal property of the pullbacks $D(A)_2$ make this equal to

$$(cs\delta_A,\ 1_{D(A)_2}c)c$$

which becomes the left-hand side of the final equality:

$$c(s\delta_A,\ 1_{D(A)_1})c = c(\ell_A)'_1\pi_1$$

The result follows by the universal property of the pullback D_φ. □

A.3 Lemmas for 2-cells of the Canonical Lax Transformation

Lemma 87.

$$((\ell_A)'_1\iota_{1_A}\ ,\ t(1_{D(A)_0}, D(\varphi)_0 e_B)\iota_\varphi) = ((\ell_A)'_1\ ,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))\iota_{1_A;\varphi}$$

Proof. By the universal property of $\mathbb{D}_2$ it suffices to compute

$$\begin{aligned}((\ell_A)'_1\ ,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))\iota_{1_A;\varphi}\rho_0 &= ((\ell_A)'_1\ ,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))p_0\iota_{1_A}\\ &= (\ell_A)'_1\iota_{1_A}\end{aligned}$$

and

$$\begin{aligned}((\ell_A)'_1\ ,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))\iota_{1_A;\varphi}\rho_1 &= ((\ell_A)'_1\ ,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))p_1\iota_\varphi\\ &= t(1_{D(A)_0}, D(\varphi)_0 e_B)\iota_\varphi.\end{aligned}$$

□

Notice the following computation contains the first functoriality argument for the naturality proof above in the case $\mathcal{E} = \mathbf{Set}$.

Lemma 88.

$$((\ell_A)'_1,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))c'_{\delta;1_A;\varphi} = (s\delta_{1_A;\varphi},\ (s\delta_A D(\varphi)_1, D(\varphi)_1)c,\ te_A D(\varphi)_1)$$

Proof. By the universal property of $D(B)_3$ it suffices to check three equalities. First we have

$$\begin{aligned}((\ell_A)'_1,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))c'_{\delta;1_A;\varphi}q_{01}q_0 &= ((\ell_A)'_1,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))p_0\pi_0\delta_{1_A;\varphi}\\ &= (\ell_A)'_1\pi_0\delta_{1_A;\varphi}\\ &= s\delta_{1_A;\varphi}\end{aligned}$$

where the first equality is by definition of $c'_{\delta;1_A;\varphi}$, the second line is by definition of the pullback projection, p_0, and the third line is by definition of $(\ell_A)'_1$. Second,

$$\begin{aligned}((\ell_A)'_1,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))c'_{\delta;1_A;\varphi}q_{01}q_1 &= ((\ell_A)'_1,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))p_0\pi_1 D(\varphi)_1 \\ &= (\ell_A)'_1\pi_1 D(\varphi)_1 \\ &= (s\delta_A, 1_{D(A)_1})cD(\varphi)_1 \\ &= (s\delta_A D(\varphi)_1, D(\varphi)_1)c\end{aligned}$$

where the first line is by definition of $c'_{\delta;1_A}$, the second line is by definition of the pullback projection p_0 and the pairing map it is precomposed with, the third line is be definition of $(\ell_A)'_1$, and the last line is by functoriality of $D(\varphi)$. Finally we can see

$$\begin{aligned}((\ell_A)'_1,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))c'_{\delta;1_A;\varphi}q_{12}q_1 &= ((\ell_A)'_1,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))c'_{1_A;\varphi}q_1 \\ &= ((\ell_A)'_1,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))p_1\pi_1 \\ &= t(1_{D(A)_0}, D(\varphi)_0 e_B)\pi_1 \\ &= tD(\varphi)_0 e_B \\ &= te_A D(\varphi)_1 \\ &= D(\varphi)_1\end{aligned}$$

where the first line is by definition of $c'_{\delta;1_A;\varphi}$, the second line is by definition of $c'_{1_A;\varphi}$, the third line is by definition of the pullback projection p_1, the third line is by definition of the pullback projection p_1, the fourth line is by functoriality of $D(\varphi)$ and the last line is by definition of the identity structure map, e_A, of $D(A)$. □

The previous calculation is an intermediate step for the following lemma which we use in our naturality computation at the end of this subsection.

Lemma 89.

$$((\ell_A)'_1\ ,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))c_{1_A;\varphi} = (s,\ (s\delta_{1_A;\varphi}, s\delta_A D(\varphi)_1, D(\varphi)_1)c)$$

Proof. By the universal property of D_φ it suffices to compute the pullback projections and check that they're equal. First we can see

$$\begin{aligned}((\ell_A)'_1\ ,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))c_{1_A;\varphi}\pi_0 &= ((\ell_A)'_1\ ,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))p_0\pi_0 \\ &= (\ell_A)'_1\pi_0 \\ &= s\end{aligned}$$

by definition of the pullback projections, p_0 and π_0, and the map $(\ell_A)'_1$. Next we can see

$$\begin{aligned}&((\ell_A)'_1\ ,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))c_{1_A;\varphi}\pi_1 \\ &= ((\ell_A)'_1\ ,\ t(1_{D(A)_0}, D(\varphi)_0 e_B))c'_{\delta' 1_A;\varphi}c \\ &= (s\delta_{1_A;\varphi},\ (s\delta_A D(\varphi)_1, D(\varphi)_1)c,\ te_A D(\varphi)_1)c \\ &= (s\delta_{1_A;\varphi},\ s\delta_A D(\varphi)_1,\ (D(\varphi)_1, te_A D(\varphi)_1)c)c \\ &= (s\delta_{1_A;\varphi},\ s\delta_A D(\varphi)_1,\ (1_{D(A)_1}, te_A)cD(\varphi)_1)c \\ &= (s\delta_{1_A;\varphi},\ s\delta_A D(\varphi)_1,\ (1_{D(A)_1}, t)(p_0, p_1 e_A)cD(\varphi)_1)c \\ &= (s\delta_{1_A;\varphi},\ s\delta_A D(\varphi)_1,\ (1_{D(A)_1}, t)p_0 D(\varphi)_1)c \\ &= (s\delta_{1_A;\varphi},\ s\delta_A D(\varphi)_1,\ D(\varphi)_1)c.\end{aligned}$$

where the first line is by definition of $c_{1_A;\varphi}$, the second line is by Lemma 88, the third line is by associativity of composition, the fourth line is by functoriality of $D(\varphi)$, the fifth line is given by factoring a pairing map, the sixth line is coming from the identity law in $D(A)$, and the last line is by definition of the pullback projection p_0.

□

The remaining lemmas are side calculations that show different ways of representing internal compositions involving certain pairing maps. We used them to prove results about the 1-cells of the canonical lax natural transformation, ℓ.

Lemma 90. *The cofiber composition, $D(A)_0 \to D_\varphi$, given by the term*

$$(s(1_{D(A)_0}, e_A D(\varphi)_1),\ (sD(\varphi)_0, D(\varphi)_1(\ell_B)'_1\pi_1))c'_{\delta;\varphi;1_B}$$

is equal to

$$(s\delta_{\varphi;1_B}, se_A D(\varphi)_1 D(1_B)_1, D(\varphi)_1(\ell_B)'_1\pi_1)$$

Proof. By the universal property of $D(B)_3$, it suffices to check the three projections $D(B)_3 \to D(B)_1$. First we have

$$\begin{aligned}
&(s(1_{D(A)_0}, e_A D(\varphi)_1),\ (sD(\varphi)_0, D(\varphi)_1(\ell_B)'_1\pi_1))c'_{\delta;\varphi;1_B}q_{01}q_0 \\
&= (s(1_{D(A)_0}, e_A D(\varphi)_1),\ (sD(\varphi)_0, D(\varphi)_1(\ell_B)'_1\pi_1))p_0\pi_0\delta_{\varphi;1_B} \\
&= s(1_{D(A)_0}, e_A D(\varphi)_1)\pi_0\delta_{\varphi;1_B} \\
&= s\delta_{\varphi;1_B} \qquad ,
\end{aligned}$$

Second we have

$$\begin{aligned}
&(s(1_{D(A)_0}, e_A D(\varphi)_1),\ (sD(\varphi)_0, D(\varphi)_1(\ell_B)'_1\pi_1))c'_{\delta;\varphi;1_B}q_{12}q_0 \\
&= (s(1_{D(A)_0}, e_A D(\varphi)_1),\ (sD(\varphi)_0, D(\varphi)_1(\ell_B)'_1\pi_1))p_0\pi_1 D(1_B)_1 \\
&= s(1_{D(A)_0}, e_A D(\varphi)_1)\pi_1 D(1_B)_1 \\
&= se_A D(\varphi)_1 D(1_B)_1 \qquad ,
\end{aligned}$$

and finally

$$\begin{aligned}
&(s(1_{D(A)_0}, e_A D(\varphi)_1),\ (sD(\varphi)_0, D(\varphi)_1(\ell_B)'_1\pi_1))c'_{\delta;\varphi;1_B}q_{12}q_1 \\
&= (s(1_{D(A)_0}, e_A D(\varphi)_1),\ (sD(\varphi)_0, D(\varphi)_1(\ell_B)'_1\pi_1))p_1\pi_1 \\
&= (sD(\varphi)_0, D(\varphi)_1(\ell_B)'_1\pi_1)\pi_1 \\
&= D(\varphi)_1(\ell_B)'_1\pi_1
\end{aligned}$$

. □

Lemma 91. *The pairing map*

$$(s,\ (s\delta_{\varphi;1_B}, sD(\varphi)_0 e_B D(1_B)_1, sD(\varphi)_0\delta_B, D(\varphi)_1(\ell_B)'_1\pi_1)c)$$

is equal to the cofiber composition

$$(s(1_{D(A)_0}, e_A D(\varphi)_1),\ (sD(\varphi)_0, D(\varphi)_1(\ell_B)'_1\pi_1))c_{\varphi;1_B}$$

Proof. By the universal property of D_φ, it suffices to check that

$$\begin{aligned}
&(s(1_{D(A)_0}, e_A D(\varphi)_1),\ (sD(\varphi)_0, D(\varphi)_1(\ell_B)'_1\pi_1))c_{\varphi;1_B}\pi_0 \\
&= (s(1_{D(A)_0}, e_A D(\varphi)_1),\ (sD(\varphi)_0, D(\varphi)_1(\ell_B)'_1\pi_1))p_0\pi_0 \\
&= s(1_{D(A)_0}, e_A D(\varphi)_1)\pi_0 \\
&= s1_{D(A)_0} \\
&= s
\end{aligned}$$

and by Lemma 90 and functoriality of $D(\varphi)$ we have

$$\begin{aligned}
&(s(1_{D(A)_0}, e_A D(\varphi)_1),\ (sD(\varphi)_0, D(\varphi)_1(\ell_B)'_1\pi_1))c_{\varphi;1_B}\pi_1 \\
&= (s(1_{D(A)_0}, e_A D(\varphi)_1),\ (sD(\varphi)_0, D(\varphi)_1(\ell_B)'_1\pi_1))c'_{\delta;\varphi;1_B}c \\
&= (s\delta_{\varphi;1_B}, se_A D(\varphi)_1 D(1_B)_1, D(\varphi)_1(\ell_B)'_1\pi_1)c \\
&= (s\delta_{\varphi;1_B}, sD(\varphi)_0 e_B D(1_B)_1, D(\varphi)_1(\ell_B)'_1\pi_1)c
\end{aligned}$$

. □

Lemma 92.

$$(s(1_{D(A)_0}, e_A D(\varphi)_1),\ D(\varphi)_1(\ell_B)'_1)\iota_{\varphi;1_B} = (s(1_{D(A)_0}, e_A D(\varphi)_1)\iota_\varphi,\ D(\varphi)_1(\ell_B)'_1\iota_{1_B})$$

Proof. By the universal property of $\mathbb{D}_2$, it suffices to check

$$\begin{aligned}
(s(1_{D(A)_0}, e_A D(\varphi)_1),\ D(\varphi)_1(\ell_B)'_1)\iota_{\varphi;1_B}\rho_0 &= (s(1_{D(A)_0}, e_A D(\varphi)_1),\ D(\varphi)_1(\ell_B)'_1)p_0\iota_\varphi \\
&= s(1_{D(A)_0}, e_A D(\varphi)_1)\iota_\varphi
\end{aligned}$$

and

$$\begin{aligned}
(s(1_{D(A)_0}, e_A D(\varphi)_1),\ D(\varphi)_1(\ell_B)'_1)\iota_{\varphi;1_B}\rho_1 &= (s(1_{D(A)_0}, e_A D(\varphi)_1),\ D(\varphi)_1(\ell_B)'_1)p_1\iota_{1_B} \\
&= D(\varphi)_1(\ell_B)'_1\iota_{1_B}
\end{aligned}$$

□

Appendix B

Internal Category of Fractions

B.1 Defining Span Composition on Representatives

This appendix consists of technical lemmas which are really just computations used in the proof of Lemma 40 in Chapter 4. We use these to define the composition structure of the internal category of fractions and prove it forms an internal category. They are heavily dependant on their context in that lemma so we restate the beginning of that proof and include the diagrams of covers that define the lifts from the fractions axioms being referred to in the lemmas.

First, pullbacks of $u : U \to \mathrm{spn}\ {}_t{\times}_s\ \mathrm{spn}$ are taken along p_0^2 and p_1^2 to get two covers of $\mathrm{sb}\ {}_t{\times}_s\ \mathrm{sb}$ that witness composition of the sailboat projections, p_0^2 and p_1^2:

$$\begin{tikzcd}
\bar{U}_0 \arrow[r, "\pi_1"] \arrow[d, "\bar{u}_0"', "/" marking] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "u", "/" marking] & \bar{U}_1 \arrow[l, "\pi_1"'] \arrow[d, "\bar{u}_1", "/" marking] \arrow[dl, phantom, "\llcorner", very near start] \\
\mathrm{sb}\ {}_t{\times}_s\ \mathrm{sb} \arrow[r, "p_0^2"'] & \mathrm{spn}\ {}_t{\times}_s\ \mathrm{spn} & \mathrm{sb}\ {}_t{\times}_s\ \mathrm{sb} \arrow[l, "p_1^2"]
\end{tikzcd} \tag{1}$$

A refinement

$$\begin{tikzcd}
\bar{U} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, "\bar{u}", "/" marking] & \bar{U}_1 \arrow[d, "\bar{u}_1", "/" marking] \\
\bar{U}_0 \arrow[r, "\bar{u}_0"', "/" marking] & \mathrm{sb}\ {}_t{\times}_s\ \mathrm{sb}
\end{tikzcd} \tag{2}$$

is given by a pullback of $\bar{u}_0$ and $\bar{u}_1$ and provides us with a common cover domain for the cover. Next we need to describe composition for the intermediate pair of composable spans:

$$\left[\begin{tikzcd}
 & & & & \cdot \arrow[dll, "\circ" marking] \arrow[d, dotted] \arrow[dr] & \\
\cdot & \cdot \arrow[l, "\circ" marking] \arrow[r] & \cdot & & \cdot \arrow[r, dotted] & \cdot
\end{tikzcd}\right]$$

The following figure shows the construction of three different composites being constructed.

$$\begin{tikzcd}
 & & \cdot \arrow[d, teal] \arrow[dddll, teal, bend right, "\circ" description] & & \\
 & \cdot \arrow[ddl, purple, bend right, "\circ" description] \arrow[dr, purple] & \cdot \arrow[dl, teal, "\circ" description] \arrow[dr, teal] & & \\
 & \cdot \arrow[dl, "\circ" description] \arrow[d] & \cdot \arrow[r, purple] \arrow[dl, purple, "\circ" description] & \cdot \arrow[dl, "\circ" description] \arrow[d] & \\
\cdot & \cdot \arrow[l, "\circ" description] \arrow[r] & \cdot & \cdot \arrow[l, "\circ" description] \arrow[r] & \cdot \\
 & & \cdot \arrow[ul, orange, "\circ" description] \arrow[ur, orange] & & \\
 & & \cdot \arrow[u, orange] \arrow[uull, orange, bend left, "\circ" description] & &
\end{tikzcd} \tag{A}$$

We define composition for this intermediate span similarly to how we defined $\sigma_\circ$. This could actually have been done by taking a pullback of the cover, $u : U \to \mathrm{spn}\,{}_t{\times}_s\,\mathrm{spn}$, witnessing span composition in general and finding a common refinement for this with the previous refinement. The same result holds either way. Denote the comparison pair of composable spans by γ and define it by the universal property in the following pullback diagram.

$$\begin{tikzcd}
\bar{U} \arrow[r, "\bar{u}", "/" marking] \arrow[d, "\bar{u}"', "/" marking] \arrow[dr, dotted, "\gamma"] & \mathrm{sb}\,{}_t{\times}_s\,\mathrm{sb} \arrow[r, "p_1^2"] & \mathrm{spn}\,{}_t{\times}_s\,\mathrm{spn} \arrow[d, "\pi_1"] \\
\mathrm{sb}\,{}_t{\times}_s\,\mathrm{sb} \arrow[d, "p_0^2"'] & \mathrm{spn}\,{}_t{\times}_s\,\mathrm{spn} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & \mathrm{spn} \arrow[d, "s"] \\
\mathrm{spn}\,{}_t{\times}_s\,\mathrm{spn} \arrow[r, "\pi_0"'] & \mathrm{spn} \arrow[r, "t"'] & \mathbb{C}_0
\end{tikzcd} \tag{3}$$

The following diagram of covers shows how the intermediate composite span is constructed for γ.

$$\begin{tikzcd}
W_\circ \arrow[r, "{(\pi_0\pi_1,\pi_0\pi_2)}"] & W \times_{\mathbb{C}_0} W & \\
\tilde{U} \arrow[u, purple, "\omega_\gamma"] \arrow[r, "\tilde{u}_0", "/" marking] \arrow[d, purple, "\sigma_\gamma"] \arrow[d, orange, bend right=60, "\sigma_0"'] \arrow[d, teal, bend left=60, "\sigma_1"] & \tilde{U}_0 \arrow[u, purple, "{(\theta_\gamma\pi_0\pi_0,\tilde{u}_1\gamma\pi_0\pi_0)}"'] \arrow[r, "\tilde{u}_1", "/" marking] \arrow[d, purple, "\theta_\gamma"'] & \bar{U} \arrow[d, purple, "{(\gamma\pi_0\pi_1,\gamma\pi_1\pi_0)}"] \\
\mathrm{spn} & W_\Box \arrow[r, "{(\pi_0\pi_1,\pi_1\pi_1)}"'] & \mathbb{C}_1\,{}_t{\times}_{wt}\,W
\end{tikzcd} \tag{$\star$}$$

The left and right curved arrows, σ_0 and σ_1, into spn in the bottom left corner are defined by applying the composite of spans, $\sigma_\circ$, to the composable spans given by applying p_0^2 and p_1^2 to the pair of composable sailboats. Since $\sigma_\circ$ is only defined on U we need to pass through the appropriate cover. The colours in the previous diagram and following equations indicate which of the three different span compositions in Figure (A) the arrows in the following equations are witnessing.

$$\begin{aligned}\sigma_0 &= \tilde{u}\pi_0\pi_1\sigma_\circ \\ &= \tilde{u}\pi_0\pi_1\big(\omega\pi_1, (\omega\pi_0\pi_0, u_0\theta\pi_1\pi_0, u\pi_1\pi_1)c\big)\end{aligned} \tag{B.1}$$

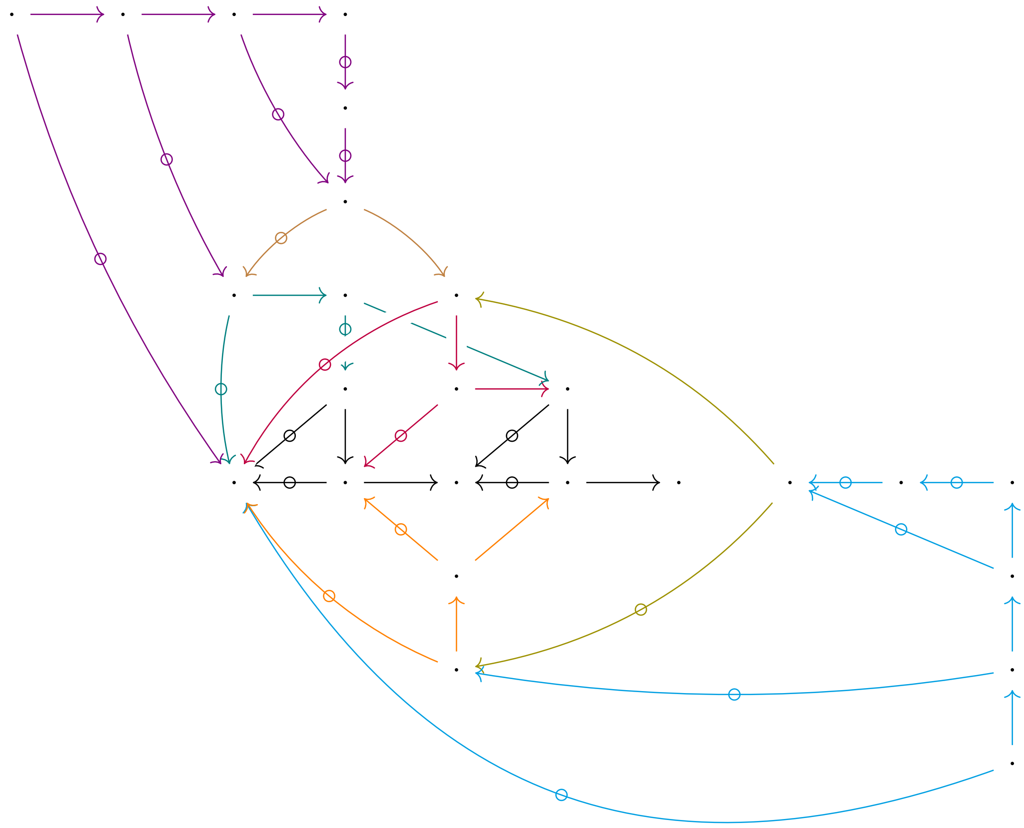
and

$$\begin{aligned}\sigma_1 &= \tilde{u}\pi_1\pi_1\sigma_\circ \\ &= \tilde{u}\pi_1\pi_1\big(\omega\pi_1, (\omega\pi_0\pi_0, u_0\theta\pi_1\pi_0, u\pi_1\pi_1)c\big).\end{aligned} \tag{B.2}$$

The arrow into spn on the bottom left side of the cover diagram is the universal map

$$\sigma_\gamma = \big(\omega_\gamma\pi_1, (\omega_\gamma\pi_0\pi_0, \tilde{u}'\theta_\gamma\pi_1\pi_0, \tilde{u}\bar{u}p_1^2\pi_1\pi_1)c\big).$$

The data necessary to construct witnessing sailboats for the equivalences between the pairs of spans σ_0, σ_1, and σ_γ can be obtained by applying the Ore condition, followed by the diagram-extension twice, and then weak composition three times. Internally this corresponds to a chain of six covers and lifts. All of this is color-coded below using olive and brown for the Ore condition and cyan and violet for the zippering and weak composition step(s) that follow. Note that in both cases the first zippering is done to parallel pairs of composites that can be post-composed by the left leg of the bottom left span. The second zipper is done to parallel pairs of composites that can be post-composed with the left leg of the bottom right span in the pair of composale sailboats. Weak composition is then applied three times in to get comparison spans, $\sigma_{0,\gamma}$ and $\sigma_{1,\gamma}$, whose left legs are in W.



(B)

The corresponding diagrams of covers and lifts which witness the arrows in the Ore squares and zippering in Diagram B are:

$$\begin{tikzcd}
 & \mathcal{P}(\mathbb{C}) \arrow[r, "\pi_1"] & \mathcal{P}_{cq}(\mathbb{C}) & \\
\hat{U}_3 \arrow[r, "\hat{u}_3", "/" marking] \arrow[d, shift right, "\delta_{\rho_1}"'] \arrow[d, shift left, "\delta_{\rho_0}"] & \hat{U}_4 \arrow[r, "\hat{u}_4", "/" marking] \arrow[u, shift right, "\delta_{\lambda_1}"] \arrow[u, shift left, "\delta_{\lambda_0}"'] \arrow[d, shift right, "\rho_1"'] \arrow[d, shift left, "\rho_0"] & \hat{U}_5 \arrow[r, "\hat{u}_5", "/" marking] \arrow[u, shift right, "\lambda_1"] \arrow[u, shift left, "\lambda_0"'] \arrow[d, shift right, "\theta_{\gamma_1}"'] \arrow[d, shift left, "\theta_{\gamma_0}"] & \tilde{U} \arrow[d, shift right, "(\sigma_1\pi_0 w{,}\sigma_\gamma\pi_0)"'] \arrow[d, shift left, "(\sigma_0\pi_0 w{,}\sigma_\gamma\pi_0)"] \\
\mathcal{P}(\mathbb{C}) \arrow[r, "\pi_1"'] & \mathcal{P}_{cq}(\mathbb{C}) & W_\square \arrow[r, "(\pi_0\pi_1{,}\pi_1\pi_1)"'] & \mathbb{C}_1 \, {}_t\!\times_{wt} W
\end{tikzcd}$$

(★★)

The covers, $\hat{u}_2, \hat{u}_1$, and $\hat{u}_0$, witness three applications of weak composition in each case as seen in the following continued sequence of covers:

$$\begin{tikzcd}
W_\circ \arrow[r, "{(\pi_0\pi_1,\pi_0\pi_2)}"] & W \times_{\mathbb{C}_0} W & W_\circ \arrow[r, "{(\pi_0\pi_1,\pi_0\pi_2)}"] & W \times_{\mathbb{C}_0} W \\
\hat{U} \arrow[u, shift right, "\omega_{1,0}"'] \arrow[u, shift left, "\omega_{0,0}"] \arrow[r, "\hat{u}_0", "/" marking] \arrow[d, shift right=3] \arrow[d, shift right=1] \arrow[d, shift left=1] \arrow[d, shift left=3] & \hat{U}_1 \arrow[u, shift right, "\omega'_{1,0}"'] \arrow[u, shift left, "\omega'_{0,0}"] \arrow[r, "\hat{u}_1", "/" marking] \arrow[d, shift right, "\omega_{1,1}"'] \arrow[d, shift left, "\omega_{0,1}"] & \hat{U}_2 \arrow[u, shift right, "\omega_{1,2}"'] \arrow[u, shift left, "\omega_{0,2}"] \arrow[r, "\hat{u}_2", "/" marking] \arrow[d, shift right, "\omega'_{1,1}"'] \arrow[d, shift left, "\omega'_{0,1}"] & \hat{U}_3 \arrow[u, shift right, "\omega'_{1,2}"'] \arrow[u, shift left, "\omega'_{0,2}"] \\
\mathrm{sb} & W_\circ \arrow[r, "{(\pi_0\pi_1,\pi_0\pi_2)}"'] & W \times_{\mathbb{C}_0} W &
\end{tikzcd} \tag{$\star\star\star$}$$

The following lemmas refer to the labeled diagrams and equations above.

Lemma 93. *The maps*

$$\begin{tikzcd}
\hat{U} \arrow[r, shift left, "\lambda'_1"] \arrow[r, shift right, "\lambda'_0"'] & P(\mathbb{C}) \, {}_t{\times}_{ws} \, W
\end{tikzcd}$$

are defined in a similar fashion to δ'_0 in Lemma 37, namely by descending through the preceeding covers and expanding both sides of the Ore-square equations witnessed.

Proof. To define λ'_0 we expand both sides of the Ore-square equation

$$\big(\theta_{\gamma_0}\pi_0\pi_0 w \ , \ \sigma_{\gamma_0}\pi_0\pi_1\big)c = \big(\theta_{\gamma_0}\pi_1\pi_0 w \ , \ \sigma_{\gamma_0}\pi_1\pi_1 w\big)c$$

On the left-hand side we have

$$\begin{aligned}
&\big(\theta_{\gamma_0}\pi_0\pi_0 w \ , \ \sigma_{\gamma_0}\pi_0\pi_1\big)c \\
=\ &\big(\theta_{\gamma_0}\pi_0\pi_0 w \ , \ \hat{u}_5\sigma_0\pi_0 w\big)c \\
=\ &\big(\theta_{\gamma_0}\pi_0\pi_0 w \ , \ \hat{u}_5\tilde{u}\pi_0\pi_1\sigma\pi_0 w\big)c \\
=\ &\big(\theta_{\gamma_0}\pi_0\pi_0 w \ , \ \hat{u}_5\tilde{u}\pi_0\pi_1\omega\pi_1 w\big)c \\
=\ &\big(\theta_{\gamma_0}\pi_0\pi_0 w \ , \ \hat{u}_5\tilde{u}\pi_0\pi_1\omega\pi_1 w\big)c \\
=\ &\big(\theta_{\gamma_0}\pi_0\pi_0 w \ , \ \hat{u}_5\tilde{u}\pi_0\pi_1(\omega\pi_0\pi_0 \ , \ u_0\theta\pi_0\pi_0 w \ , \ u\pi_0\pi_0 w)c\big)c \\
=\ &\big(\theta_{\gamma_0}\pi_0\pi_0 w \ , \ (\hat{u}_5\tilde{u}\pi_0\pi_1\omega\pi_0\pi_0 \ , \ \hat{u}_5\tilde{u}\pi_0\pi_1 u_0\theta\pi_0\pi_0 w \ , \ \hat{u}_5\tilde{u}\pi_0\pi_1 u\pi_0\pi_0 w)c\big)c \\
=\ &\big(\theta_{\gamma_0}\pi_0\pi_0 w \ , \ \hat{u}_5\tilde{u}\pi_0\pi_1\omega\pi_0\pi_0 \ , \ \hat{u}_5\tilde{u}\pi_0\pi_1 u_0\theta\pi_0\pi_0 w \ , \ \hat{u}_5\tilde{u}\pi_0\pi_1 u\pi_0\pi_0 w\big)c \\
=\ &\big(\theta_{\gamma_0}\pi_0\pi_0 w \ , \ \hat{u}_5\tilde{u}\pi_0\pi_1\omega\pi_0\pi_0 \ , \ \hat{u}_5\tilde{u}\pi_0\pi_1 u_0\theta\pi_0\pi_0 w \ , \ \hat{u}_5\tilde{u}\pi_0\pi_0 p_0^2\pi_0\pi_0 w\big)c \\
=\ &\big((\theta_{\gamma_0}\pi_0\pi_0 w \ , \ \hat{u}_5\tilde{u}\pi_0\pi_1\omega\pi_0\pi_0 \ , \ \hat{u}_5\tilde{u}\pi_0\pi_1 u_0\theta\pi_0\pi_0 w)c \ , \ \hat{u}_5\tilde{u}\pi_0\pi_0 p_0^2\pi_0\pi_0 w\big)c
\end{aligned} \tag{B.3}$$

and on the right we have

$$\begin{aligned}
&\big(\theta_{\gamma_0}\pi_1\pi_0 w\ ,\ \sigma_{\gamma_0}\pi_1\pi_1 w\big)c\\
&=\big(\theta_{\gamma_0}\pi_1\pi_0 w\ ,\ \hat{u}_5\sigma_{\gamma}\pi_0 w\big)c\\
&=\big(\theta_{\gamma_0}\pi_1\pi_0 w\ ,\ \hat{u}_5\omega_{\gamma}\pi_1 w\big)c\\
&=\big(\theta_{\gamma_0}\pi_1\pi_0 w\ ,\ \hat{u}_5(\omega_{\gamma}\pi_0\pi_0\ ,\ \omega_{\gamma}\pi_0\pi_1 w\ ,\ \omega_{\gamma}\pi_0\pi_2 w)c\big)c\\
&=\big(\theta_{\gamma_0}\pi_1\pi_0 w\ ,\ \hat{u}_5(\omega_{\gamma}\pi_0\pi_0\ ,\ \tilde{u}'\theta_{\gamma}\pi_0\pi_0 w\ ,\ \tilde{u}\gamma\pi_0\pi_0 w)c\big)c\\
&=\big(\theta_{\gamma_0}\pi_1\pi_0 w\ ,\ \hat{u}_5(\omega_{\gamma}\pi_0\pi_0\ ,\ \tilde{u}'\theta_{\gamma}\pi_0\pi_0 w\ ,\ \tilde{u}\pi_0\pi_1 u\pi_0\pi_0 w)c\big)c\\
&=\big(\theta_{\gamma_0}\pi_1\pi_0 w\ ,\ \hat{u}_5(\omega_{\gamma}\pi_0\pi_0\ ,\ \tilde{u}'\theta_{\gamma}\pi_0\pi_0 w\ ,\ \tilde{u}\pi_0\pi_0 p_0^2\pi_0\pi_0 w)c\big)c\\
&=\theta_{\gamma_0}\pi_1\pi_0 w\ ,\ \hat{u}_5(\omega_{\gamma}\pi_0\pi_0\ ,\ \tilde{u}'\theta_{\gamma}\pi_0\pi_0 w\ ,\ \tilde{u}\pi_0\pi_0 p_0^2\pi_0\pi_0 w)c\big)c\\
&=\theta_{\gamma_0}\pi_1\pi_0 w\ ,\ \hat{u}_5\omega_{\gamma}\pi_0\pi_0\ ,\ \hat{u}_5\tilde{u}'\theta_{\gamma}\pi_0\pi_0 w\ ,\ \hat{u}_5\tilde{u}\pi_0\pi_0 p_0^2\pi_0\pi_0 w\big)c\\
&=\big((\theta_{\gamma_0}\pi_1\pi_0 w\ ,\ \hat{u}_5\omega_{\gamma}\pi_0\pi_0\ ,\ \hat{u}_5\tilde{u}'\theta_{\gamma}\pi_0\pi_0 w)c\ ,\ \hat{u}_5\tilde{u}\pi_0\pi_0 p_0^2\pi_0\pi_0 w\big)c
\end{aligned}
\quad . \qquad \text{(B.4)}$$

The last lines in equations (1) and (2) uniquely determine λ_0' by

$$\begin{aligned}
\lambda_0'\pi_1 &= \hat{u}_5\tilde{u}\pi_0\pi_0 p_0^2\pi_0\pi_0 w\\
\lambda_0'\pi_0\pi_0 &= (\theta_{\gamma_0}\pi_0\pi_0 w\ ,\ \hat{u}_5\tilde{u}\pi_0\pi_1\omega\pi_0\pi_0\ ,\ \hat{u}_5\tilde{u}\pi_0\pi_1 u_0\theta\pi_0\pi_0 w)c\\
\lambda_0'\pi_0\pi_1 &= (\theta_{\gamma_0}\pi_1\pi_0 w\ ,\ \hat{u}_5\omega_{\gamma}\pi_0\pi_0\ ,\ \hat{u}_5\tilde{u}'\theta_{\gamma}\pi_0\pi_0 w)c
\end{aligned}$$

The map λ_1' is similarly determined by expanding both sides of the Ore-square equation:

$$\big(\theta_{\gamma_1}\pi_0\pi_0 w\ ,\ \sigma_{\gamma_1}\pi_0\pi_1\big)c = \big(\theta_{\gamma_1}\pi_1\pi_0 w\ ,\ \sigma_{\gamma_1}\pi_1\pi_1 w\big)c$$

On the left-hand side we get

$$
\begin{aligned}
&\bigl(\theta_{\gamma_1}\pi_0\pi_0 w,\ \sigma_{\gamma_1}\pi_0\pi_1\bigr)c\\
&= \bigl(\theta_{\gamma_1}\pi_0\pi_0 w,\ \hat{u}_5\sigma_1\pi_0 w\bigr)c\\
&= \bigl(\theta_{\gamma_1}\pi_0\pi_0 w,\ \hat{u}_5\tilde{u}\pi_1\pi_1\sigma\pi_0 w\bigr)c\\
&= \bigl(\theta_{\gamma_1}\pi_0\pi_0 w,\ \hat{u}5\tilde{u}\pi_1\pi_1\omega\pi_1 w\bigr)c\\
&= \bigl(\theta_{\gamma_1}\pi_0\pi_0 w,\ \hat{u}_5\tilde{u}\pi_1\pi_1\omega\pi_1 w\bigr)c\\
&= \bigl(\theta_{\gamma_1}\pi_0\pi_0 w,\\
&\quad \hat{u}_5\tilde{u}\pi_1\pi_1(\omega\pi_0\pi_0,\ u_0\theta\pi_0\pi_0 w,\ u\pi_0\pi_0 w)c\bigr)c\\
&= \bigl(\theta_{\gamma_1}\pi_0\pi_0 w,\\
&\quad (\hat{u}_5\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_5\tilde{u}\pi_1\pi_1 u_0\theta\pi_0\pi_0 w,\ \hat{u}_5\tilde{u}\pi_1\pi_1 u\pi_0\pi_0 w)c\bigr)c\\
&= \bigl(\theta_{\gamma_0}\pi_0\pi_0 w,\ \hat{u}_5\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_5\tilde{u}\pi_1\pi_1 u_0\theta\pi_0\pi_0 w,\ \hat{u}_5\tilde{u}\pi_1\pi_1 u\pi_0\pi_0 w\bigr)c\\
&= \bigl(\theta_{\gamma_1}\pi_0\pi_0 w,\ \hat{u}_5\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_5\tilde{u}\pi_1\pi_1 u_0\theta\pi_0\pi_0 w,\ \hat{u}_5\tilde{u}\pi_1\pi_0 p_1^2\pi_0\pi_0 w\bigr)c\\
&= \bigl((\theta_{\gamma_1}\pi_0\pi_0 w,\ \hat{u}_5\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_5\tilde{u}\pi_1\pi_1 u_0\theta\pi_0\pi_0 w)c,\\
&\quad \hat{u}_5\tilde{u}\pi_1\pi_0 p_1^2\pi_0\pi_0 w\bigr)c \qquad\qquad \text{(B.5)}\\
&= \bigl((\theta_{\gamma_1}\pi_0\pi_0 w,\ \hat{u}_5\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_5\tilde{u}\pi_1\pi_1 u_0\theta\pi_0\pi_0 w)c,\\
&\quad \hat{u}_5\tilde{u}\pi_1\pi_0\pi_0\pi_0\pi_1 w\bigr)c\\
&= \bigl((\theta_{\gamma_1}\pi_0\pi_0 w,\ \hat{u}_5\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_5\tilde{u}\pi_1\pi_1 u_0\theta\pi_0\pi_0 w)c,\\
&\quad \hat{u}_5\tilde{u}\bar{u}\pi_0\pi_0\pi_1 w\bigr)c\\
&= \bigl((\theta_{\gamma_1}\pi_0\pi_0 w,\ \hat{u}_5\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_5\tilde{u}\pi_1\pi_1 u_0\theta\pi_0\pi_0 w)c,\\
&\quad (\hat{u}_5\tilde{u}\bar{u}\pi_0\pi_0\pi_0\pi_0,\ \hat{u}_5\tilde{u}\bar{u}\pi_0\pi_0\pi_0\pi_1 w)c\bigr)c\\
&= \bigl((\theta_{\gamma_1}\pi_0\pi_0 w,\ \hat{u}_5\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_5\tilde{u}\pi_1\pi_1 u_0\theta\pi_0\pi_0 w,\ \hat{u}_5\tilde{u}\bar{u}\pi_0\pi_0\pi_0\pi_0)c,\\
&\quad \hat{u}_5\tilde{u}\bar{u}\pi_0\pi_0\pi_0\pi_1 w\bigr)c\\
&= \bigl((\theta_{\gamma_1}\pi_0\pi_0 w,\ \hat{u}_5\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_5\tilde{u}\pi_1\pi_1 u_0\theta\pi_0\pi_0 w,\ \hat{u}_5\tilde{u}\bar{u}\pi_0\pi_0\pi_0\pi_0)c,\\
&\quad \hat{u}_5\tilde{u}\pi_0\pi_0\pi_0\pi_0\pi_0\pi_1 w\bigr)c\\
&= \bigl((\theta_{\gamma_1}\pi_0\pi_0 w,\ \hat{u}_5\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_5\tilde{u}\pi_1\pi_1 u_0\theta\pi_0\pi_0 w,\ \hat{u}_5\tilde{u}\bar{u}\pi_0\pi_0\pi_0\pi_0)c,\\
&\quad \hat{u}_5\tilde{u}\pi_0\pi_0 p_0^2\pi_0\pi_0 w\bigr)c
\end{aligned}
$$

and on the right-hand side we have

$$\begin{aligned}
&\big(\theta_{\gamma_1}\pi_1\pi_0 w \ , \ \sigma_{\gamma_1}\pi_1\pi_1 w\big)c \\
&= \big(\theta_{\gamma_1}\pi_1\pi_0 w \ , \ \hat{u}_5\sigma_{\gamma}\pi_0 w\big)c \\
&= \big(\theta_{\gamma_1}\pi_1\pi_0 w \ , \ \hat{u}_5\omega_{\gamma}\pi_1 w\big)c \\
&= \big(\theta_{\gamma_1}\pi_1\pi_0 w \ , \ \hat{u}_5(\omega_{\gamma}\pi_0\pi_0 \ , \ \omega_{\gamma}\pi_0\pi_1 w \ , \ \omega_{\gamma}\pi_0\pi_2 w)c\big)c \\
&= \big(\theta_{\gamma_1}\pi_1\pi_0 w \ , \ \hat{u}_5(\omega_{\gamma}\pi_0\pi_0 \ , \ \tilde{u}'\theta_{\gamma}\pi_0\pi_0 w \ , \ \tilde{u}\gamma\pi_0\pi_0 w)c\big)c \\
&= \big(\theta_{\gamma_1}\pi_1\pi_0 w \ , \ \hat{u}_5(\omega_{\gamma}\pi_0\pi_0 \ , \ \tilde{u}'\theta_{\gamma}\pi_0\pi_0 w \ , \ \tilde{u}\pi_0\pi_1 u\pi_0\pi_0 w)c\big)c \\
&= \big(\theta_{\gamma_1}\pi_1\pi_0 w \ , \ \hat{u}_5(\omega_{\gamma}\pi_0\pi_0 \ , \ \tilde{u}'\theta_{\gamma}\pi_0\pi_0 w \ , \ \tilde{u}\pi_0\pi_0 p_0^2\pi_0\pi_0 w)c\big)c \\
&= \big(\theta_{\gamma_1}\pi_1\pi_0 w \ , \ \hat{u}_5(\omega_{\gamma}\pi_0\pi_0 \ , \ \tilde{u}'\theta_{\gamma}\pi_0\pi_0 w \ , \ \tilde{u}\pi_0\pi_0 p_0^2\pi_0\pi_0 w)c\big)c \\
&= \big(\theta_{\gamma_1}\pi_1\pi_0 w \ , \ \hat{u}_5\omega_{\gamma}\pi_0\pi_0 \ , \ \hat{u}_5\tilde{u}'\theta_{\gamma}\pi_0\pi_0 w \ , \ \hat{u}_5\tilde{u}\pi_0\pi_0 p_0^2\pi_0\pi_0 w\big)c \\
&= \big((\theta_{\gamma_1}\pi_1\pi_0 w \ , \ \hat{u}_5\omega_{\gamma}\pi_0\pi_0 \ , \ \hat{u}_5\tilde{u}'\theta_{\gamma}\pi_0\pi_0 w)c \ , \ \hat{u}_5\tilde{u}\pi_0\pi_0 p_0^2\pi_0\pi_0 w\big)c
\end{aligned} \tag{B.6}$$

The last lines of equations (3) and (4) uniquely determine the λ_1' by

$$\begin{aligned}
\lambda_1'\pi_1 &= \hat{u}_5\tilde{u}\pi_0\pi_0 p_0^2\pi_0\pi_0, \\
\lambda_1'\pi_0\pi_0 &= (\theta_{\gamma_1}\pi_0\pi_0 w \ , \ \hat{u}_5\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0 \ , \ \hat{u}_5\tilde{u}\pi_1\pi_1 u_0\theta\pi_0\pi_0 w \ , \ \hat{u}_5\tilde{u}\bar{u}\pi_0\pi_0\pi_0\pi_0)c, \\
\lambda_1'\pi_0\pi_1 &= (\theta_{\gamma_1}\pi_1\pi_0 w \ , \ \hat{u}_5\omega_{\gamma}\pi_0\pi_0 \ , \ \hat{u}_5\tilde{u}'\theta_{\gamma}\pi_0\pi_0 w)c
\end{aligned}$$

□

Lemma 94. *The equation*

$$(\rho_0'\pi_0\pi_0, \rho_0'\pi_1)c = (\rho_0'\pi_0\pi_1, \rho_0'\pi_1)c$$

holds.

Proof. This follows from equality between the first and last lines in the following straightforward but tedious calculation. We repeatedly use associativity for internal composition in $\mathbb{C}$ along with the definitions of the arrows and objects in Diagrams $(\star)$, $(\star\star)$, and $\star\star\star)$ of Lemma 40 in this calculation.

$$
\begin{aligned}
&\big((\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0)c,\\
&\hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_0 w\big)c\\
={}&\big((\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0)c,\\
&(\hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_0 w)c\big)c\\
={}&\big((\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0)c,\\
&(\hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_1 w)c\big)c\\
={}&\big((\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0)c,\\
&\hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_1 w\big)c\\
={}&\big((\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0)c,\\
&(\hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_1 w)c\big)c\\
={}&\big((\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0)c,\\
&(\hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_1^2\pi_1\pi_0 w)c\big)c\\
={}&\big((\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0)c,\\
&(\hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_0\pi_0 w,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_0\pi_1)c\big)c\\
={}&\big((\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0)c,\\
&(\hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_0\pi_0 w,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_0\pi_1)c\big)c\\
={}&\big((\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_0\pi_0 w)c,\\
&\hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_0\pi_1\big)c\\
={}&\big((\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_0\pi_1\omega\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_0\pi_1 u_0\theta\pi_0\pi_0 w)c,\\
&\hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_0\pi_1\big)c\\
={}&\big((\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_0\pi_1\omega\pi_0\pi_0)c,\\
&(\hat{u}_{4;5}\tilde{u}\pi_0\pi_1 u_0\theta\pi_0\pi_0 w,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_0\pi_0\pi_1)c\big)c\\
={}&\big((\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_0\pi_1\omega\pi_0\pi_0)c,\\
&(\hat{u}_{4;5}\tilde{u}\pi_0\pi_1 u_0\theta\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_0 w)c\big)c\\
={}&\big((\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_0\pi_1\omega\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_0\pi_1 u_0\theta\pi_1\pi_0)c,\\
&\hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_0 w\big)c.
\end{aligned}
$$

uniquely determine the map ρ'_0, for which

$$\rho'_0\pi_1 = \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_0$$

and $\rho'_0\pi_0$ is the parallel pair with components

$$\rho'_0\pi_0\pi_0 = (\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w, \hat{u}_4\omega_{\gamma_0}\pi_1\pi_0, \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0, \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0, \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0)c$$

and

$$\rho'_0\pi_0\pi_1 = (\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w, \hat{u}_4\omega_{\gamma_0}\pi_0\pi_0, \hat{u}_{4;5}\tilde{u}\pi_0\pi_1\omega\pi_0\pi_0, \hat{u}_{4;5}\tilde{u}\pi_0\pi_1u_0\theta\pi_1\pi_0)c.$$

This means that the first and last terms of the calculation above precisely says

$$(\rho'_0\pi_0\pi_0, \rho'_0\pi_1)c = (\rho'_0\pi_0\pi_1, \rho'_0\pi_1)c.$$

□

Lemma 95. *The equation*

$$(\rho'_1\pi_0\pi_0, \rho'_1\pi_1)c = (\rho'_1\pi_0\pi_1, \rho'_1\pi_1)c$$

holds.

Proof. This follows from the first and last lines of the following computation which is saimilar to the one in Lemma 94:

$$
\begin{aligned}
&\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0)c,\\
&\hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_0 w\big)c\\
=&\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0)c,\\
&(\hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_0 w)c\big)c\\
=&\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0)c,\\
&(\hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_1 w)c\big)c\\
=&\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0)c,\\
&\hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_1 w\big)c\\
=&\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0)c,\\
&(\hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_1 w)c\big)c\\
=&\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0)c,\\
&(\hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_1^2\pi_1\pi_0 w)c\big)c\\
=&\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0)c,\\
&(\hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_0\pi_0 w,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_0\pi_1)c\big)c\\
=&\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0)c,\\
&(\hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_0\pi_0 w,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_0\pi_1)c\big)c
\end{aligned}
$$

This calculation continues below, we just had to separate because it wouldn't fit on one page.

$$\begin{aligned}
&\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0)c,\\
&(\hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_0\pi_0 w,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_0\pi_1)c\big)c\\
=\ &\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_0\pi_0 w)c,\\
&\hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_0\pi_1\big)c\\
=\ &\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\\
&\hat{u}_{4;5}\tilde{u}\pi_1\pi_1u_0\theta\pi_0\pi_0 w,\ \hat{u}_{4;5}\tilde{u}\bar{u}\pi_0\pi_0\pi_0\pi_0)c,\\
&\hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_0\pi_1\big)c\\
=\ &\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1u_0\theta\pi_0\pi_0 w)c,\\
&(\hat{u}_{4;5}\tilde{u}\bar{u}\pi_0\pi_0\pi_0\pi_0, \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_0\pi_1)c\big)c\\
=\ &\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1u_0\theta\pi_0\pi_0 w)c,\\
&\hat{u}_{4;5}\tilde{u}\bar{u}p_1^2\pi_0\pi_1\big)c\\
=\ &\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0)c,\\
&(\hat{u}_{4;5}\tilde{u}\pi_1\pi_1u_0\theta\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_1^2\pi_1\pi_0 w)c\big)c\\
=\ &\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1u_0\theta\pi_1\pi_0)c,\\
&\hat{u}_{4;5}\tilde{u}\bar{u}p_1^2\pi_1\pi_0 w\big)c\\
=\ &\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1u_0\theta\pi_1\pi_0)c,\\
&(\hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0, \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_1 w)c\big)c\\
=\ &\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\\
&\hat{u}_{4;5}\tilde{u}\pi_1\pi_1u_0\theta\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0)c,\\
&\hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_1 w\big)c\\
=\ &\big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\\
&\hat{u}_{4;5}\tilde{u}\pi_1\pi_1u_0\theta\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0)c,\\
&\hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_0 w\big)c.
\end{aligned}$$

Unsurprisingly we get the same coequalizing arrow in W for ρ_1' as for ρ_0'

$$\rho_1'\pi_1 = \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_0$$

and the parallel pair $\rho_1'\pi_0$ is given by the pair of components

$$\rho'_1\pi_0\pi_0 = (\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\omega_{\gamma_1}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_{\gamma}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_{\gamma}\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0)c$$

and

$$\begin{aligned}\rho'_0\pi_0\pi_1 = (\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1u_0\theta\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0)c\end{aligned}$$

The first and last terms of the big equation above being equal then reduces to

$$(\rho'_1\pi_0\pi_0, \rho'_1\pi_1)c = (\rho'_1\pi_0\pi_1, \rho'_1\pi_1)c.$$

□

Lemma 96. *There is a unique map* $\sigma_{0,\gamma} : \hat{U} \to spn$ *determined by*

$$\begin{aligned}\sigma_{0,\gamma}\pi_0 &= \omega_{0,0}\pi_1\\ \sigma_{0,\gamma}\pi_1 &= (\omega_0,\ \hat{u}_{0;4}\theta_{\gamma_0}\pi_0\pi_0,\ \hat{u}\sigma_0\pi_1)c\\ &= (\omega_0,\ \hat{u}_{0;4}\theta_{\gamma_0}\pi_1\pi_0\ ,\ \hat{u}\sigma_{\gamma}\pi_1)c\end{aligned}$$

where $\omega_0 : \hat{U} \to W_\circ$ *is defined by*

$$\omega_0 = (\omega_{0,0}\pi_0\pi_0,\ \hat{u}_0\omega_{0,1}\pi_0\pi_0,\ \hat{u}_{0;1}\omega_{0,2}\pi_1)c$$

Proof. First, by definition of $W_\circ$ we have

$$\omega_0 s = \omega_{0,0}\pi_0\pi_0 s = \omega_{0,0}\pi_1 s =$$

showing that $\omega_{0,\gamma} : \hat{U} \to$ spn is well-defined. Now let $\omega'_0 : \hat{U} \to \mathbb{C}_1$ be defined by

$$\omega'_0 = (\omega_{0,0}\pi_0\pi_0,\ \hat{u}_0\omega_{0,1}\pi_0\pi_0,\ \hat{u}_{0;1}\omega_{0,2}\pi_0\pi_0)c.$$

By definition of $W_\circ$, ω_0, and ω'_0 we have

$$\begin{aligned}
\omega_0 &= (\omega_{0,0}\pi_0\pi_0,\ \hat{u}_0\omega_{0,1}\pi_0\pi_0,\ \hat{u}_{0;1}\omega_{0,2}\pi_1)c \\
&= (\omega_{0,0}\pi_0\pi_0,\ \hat{u}_0\omega_{0,1}\pi_0\pi_0,\ \hat{u}_{0;1}\omega_{0,2}\pi_0\pi_0,\ \hat{u}_{0;2}\omega'_{0,2}c)c \\
&= (\omega_{0,0}\pi_0\pi_0,\ \hat{u}_0\omega_{0,1}\pi_0\pi_0,\ \hat{u}_{0;1}\omega_{0,2}\pi_0\pi_0,\ \hat{u}_{0;2}\delta_{\rho_0}\pi_0\iota_{eq}\pi_0,\ \hat{u}_{0;3}\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0)c \\
&= \big((\omega_{0,0}\pi_0\pi_0,\ \hat{u}_0\omega_{0,1}\pi_0\pi_0,\ \hat{u}_{0;1}\omega_{0,2}\pi_0\pi_0,\ \hat{u}_{0;2}\delta_{\rho_0}\pi_0\iota_{eq}\pi_0)c,\ \hat{u}_{0;3}\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0\big)c \\
&= \big(\omega'_0,\ \hat{u}_{0;2}(\delta_{\rho_0}\pi_0\iota_{eq}\pi_0,\ \hat{u}_3\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0)c\big)c
\end{aligned} \tag{B.7}$$

Notice that by definition of σ_0 and $\rho'_0\pi_0\pi_1$ and the refinement of covers $\bar{u} : \bar{U} \to \mathrm{sb}\ {}_t{\times}_s\, \mathrm{sb}$, we have

$$\begin{aligned}
&(\rho'_0\pi_0\pi_1, \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_1)c \\
&= \big((\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_0\pi_1\omega\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_0\pi_1 u_0\theta\pi_1\pi_0)c, \\
&\quad \hat{u}_{4;5}\tilde{u}\pi_0\pi_1 u\pi_1\pi_1\big)c \\
&= \big(\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_0\pi_0, \\
&\quad (\hat{u}_{4;5}\tilde{u}\pi_0\pi_1\omega\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_0\pi_1 u_0\theta\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_0\pi_1 u\pi_1\pi_1)c\big)c \\
&= \big(\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_0\pi_0,\ \hat{u}_{4;5}\sigma_0\pi_1\big)c
\end{aligned} \tag{B.8}$$

and similarly by definition of σ_γ and $\rho'_0\pi_0\pi_0$

$$\begin{aligned}
&(\rho'_0\pi_0\pi_0, \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0)c \\
&= \big((\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_1\pi_0\ ,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0)c, \\
&\quad \hat{u}_{4;5}\tilde{u}\pi_0\pi_1 u\pi_1\pi_1\big)c \\
&= \big((\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_1\pi_0\ ,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0)c, \\
&\quad (\hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_0\pi_1 u\pi_1\pi_1)c\big)c \\
&= \big((\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_1\pi_0\ ,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0)c, \\
&\quad \hat{u}_{4;5}\tilde{u}\bar{u}p_1\pi_1\pi_1\big)c \\
&= \big(\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_1\pi_0\ , \\
&\quad (\hat{u}_{4;5}\omega_\gamma\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_1\pi_1\pi_1)c\big)c \\
&= \big(\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_1\pi_0\ ,\ \hat{u}_{4;5}\sigma_\gamma\pi_1\big)c.
\end{aligned} \tag{B.9}$$

Also notice since $\mathcal{P}(\mathbb{C})$ is a pullback of $\mathcal{P}_{eq}(\mathbb{C})$ and $\mathcal{P}_{cq}(\mathbb{C})$ over the object of parallel pairs in $\mathbb{C}$, $P(\mathbb{C})$, we have

$$\begin{aligned}\delta_{\rho_0}\pi_0\iota_{eq}\pi_1 &= \delta_{\rho_0}\pi_1\iota_{ceq}\pi_0\\ &= \hat{u}_3\rho_0\iota_{ceq}\pi_0\\ &= \hat{u}_3\rho_0'\pi_0\end{aligned}$$

and then by definition of $\mathcal{P}_{eq}(\mathbb{C})$ the composable pairs

$$\delta_{\rho_0}\pi_0\iota_{eq}(\pi_0, \pi_1\pi_0) = (\delta_{\rho_0}\pi_0\iota_{eq}\pi_0, \delta_{\rho_0}\pi_0\iota_{eq}\pi_1\pi_0) = (\delta_{\rho_0}\pi_0\iota_{eq}\pi_0, \hat{u}_3\rho_0'\pi_0\pi_0)$$

and

$$\delta_{\rho_0}\pi_0\iota_{eq}(\pi_0, \pi_1\pi_1) = (\delta_{\rho_0}\pi_0\iota_{eq}\pi_0, \delta_{\rho_0}\pi_0\iota_{eq}\pi_1\pi_1) = (\delta_{\rho_0}\pi_0\iota_{eq}\pi_0, \hat{u}_3\rho_0'\pi_0\pi_1)$$

are equal after post-composing with the composition structure map in $\mathbb{C}$, $c : mC_2 \to \mathbb{C}_1$:

$$(\delta_{\rho_0}\pi_0\iota_{eq}\pi_0, \hat{u}_3\rho_0'\pi_0\pi_0)c = (\delta_{\rho_0}\pi_0\iota_{eq}\pi_0, \hat{u}_3\rho_0'\pi_0\pi_1)c \tag{B.10}$$

Associativity of composition and equations $(B.7)$; (B.8); (B.9); and (B.10), allow us to see

$$\begin{aligned}(\omega_0,\ \hat{u}_{0;4}\theta_{\gamma_0}\pi_0\pi_0,\ \hat{u}\sigma_0\pi_1)c &= \big(\omega_0',\ \hat{u}_{0;2}\delta_{\rho_0}\pi_0\iota_{eq}\pi_0,\\ &\quad \hat{u}_{0;3}(\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_0\pi_0,\ \hat{u}_{4;5}\sigma_0\pi_1)c\big)c\\ &= \big(\omega_0',\ \hat{u}_{0;2}\delta_{\rho_0}\pi_0\iota_{eq}\pi_0,\\ &\quad \hat{u}_{0;3}(\rho_0'\pi_0\pi_1,\ \hat{u}_{4;5}\tilde{u}\pi_0\pi_1 u\pi_1\pi_1)c)c\\ &= \big(\omega_0',\ \hat{u}_{0;2}(\delta_{\rho_0}\pi_0\iota_{eq}\pi_0,\ \hat{u}_3\rho_0'\pi_0\pi_1)c,\\ &\quad \hat{u}_{0;5}\tilde{u}\pi_0\pi_1 u\pi_1\pi_1\big)c\\ &= \big(\omega_0',\ \hat{u}_{0;2}(\delta_{\rho_0}\pi_0\iota_{eq}\pi_0,\ \hat{u}_3\rho_0'\pi_0\pi_0)c,\\ &\quad \hat{u}_{0;5}\tilde{u}\pi_0\pi_1 u\pi_1\pi_1\big)c\\ &= \big((\omega_0',\ \hat{u}_{0;2}\delta_{\rho_0}\pi_0\iota_{eq}\pi_0)c,\\ &\quad \hat{u}_{0;3}(\rho_0'\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_0\pi_1 u\pi_1\pi_1)c\big)c\\ &= \big((\omega_0',\ \hat{u}_{0;2}\delta_{\rho_0}\pi_0\iota_{eq}\pi_0)c,\\ &\quad \hat{u}_{0;3}(\delta_{\lambda_0}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_0}\pi_1\pi_0,\ \hat{u}_{4;5}\sigma_\gamma\pi_1)c\big)c\\ &= (\omega_0,\ \hat{u}_{0;4}\theta_{\gamma_0}\pi_1\pi_0\ ,\ \hat{u}\sigma_\gamma\pi_1)c\end{aligned}$$

□

Lemma 97. *There is a unique map* $\sigma_{1,\gamma} : \hat{U} \to spn$ *determined by*

$$\begin{aligned} \sigma_{1,\gamma}\pi_0 &= \omega_{1,0}\pi_1 \\ \sigma_{1,\gamma}\pi_1 &= (\omega_1,\ \hat{u}_{0;4}\omega_{\gamma_1}\pi_0\pi_0,\ \hat{u}\sigma_1\pi_1)c \\ &= \big(\omega_1,\ \hat{u}_{0;4}\omega_{\gamma_1}\pi_1\pi_0,\ \hat{u}\sigma_\gamma\pi_1\big)c \end{aligned}$$

where $\omega_1 : \hat{U} \to W_\circ$ *is defined by*

$$\omega_1 = (\omega_{1,0}\pi_0\pi_0,\ \hat{u}_0\omega_{1,1}\pi_0\pi_0,\ \hat{u}_{0;1}\omega_{1,2}\pi_1)c$$

Proof. Similarly define

$$\omega_1' = (\omega_{1,0}\pi_0\pi_0,\ \hat{u}_0\omega_{1,1}\pi_0\pi_0,\ \hat{u}_{0;1}\omega_{1,2}\pi_0\pi_0)c$$

and we have:

$$\begin{aligned} \omega_1 &= (\omega_{1,0}\pi_0\pi_0,\ \hat{u}_0\omega_{1,1}\pi_0\pi_0,\ \hat{u}_{0;1}\omega_{1,2}\pi_1)c \\ &= (\omega_{1,0}\pi_0\pi_0,\ \hat{u}_0\omega_{1,1}\pi_0\pi_0,\ \hat{u}_{0;1}\omega_{1,2}\pi_0\pi_0,\ \hat{u}_{0;2}\delta_{\rho_1}\pi_0\iota_{eq}\pi_0,\ \hat{u}_{0;3}\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0)c \\ &= \big(\omega_1',\ \hat{u}_{0;2}(\delta_{\rho_1}\pi_0\iota_{eq}\pi_0,\ \hat{u}_3\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0)c\big)c. \end{aligned}$$

By definition of σ_1 and $\rho_1'\pi_0\pi_1$:

$$
\begin{aligned}
&(\rho'_1\pi_0\pi_1, \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_1)c \\
&= \big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1 u_0\theta\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0)c, \\
&\quad \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_1\big)c \\
&= \big(\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1 u_0\theta\pi_1\pi_0, \\
&\quad (\hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_1)c\big)c \\
&= \big(\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\pi_1\pi_1 u_0\theta\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_1^2\pi_1\pi_1\big)c \\
&= \big(\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_0\pi_0, \\
&\quad \hat{u}_{4;5}(\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \tilde{u}\pi_1\pi_1 u_0\theta\pi_1\pi_0,\ \tilde{u}\pi_1\pi_0 p_1^2\pi_1\pi_1)c\big)c \\
&= \big(\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_0\pi_0, \\
&\quad \hat{u}_{4;5}(\tilde{u}\pi_1\pi_1\omega\pi_0\pi_0,\ \tilde{u}\pi_1\pi_1 u_0\theta\pi_1\pi_0,\ \tilde{u}\pi_1\pi_1 u\pi_1\pi_1)c\big)c \\
&= \big(\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\theta_{\gamma_1}\pi_0\pi_0,\ \hat{u}_{4;5}\sigma_1\pi_1\big)c
\end{aligned}
\tag{B.11}
$$

Similarly by definition of σ_γ and $\rho'_1\pi_0\pi_0$:

$$
\begin{aligned}
&(\rho'_1\pi_0\pi_0, \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_1)c \\
&= \big((\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\omega_{\gamma_1}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0)c, \\
&\quad \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_1\big)c \\
&= \big(\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\omega_{\gamma_1}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0, \\
&\quad (\hat{u}_{4;5}\tilde{u}\bar{u}\pi_1\pi_0\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_1)c\big)c \\
&= \big(\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\omega_{\gamma_1}\pi_1\pi_0,\ \hat{u}_{4;5}\omega_\gamma\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}'\theta_\gamma\pi_1\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_1^2\pi_1\pi_1\big)c \\
&= \big(\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\omega_{\gamma_1}\pi_1\pi_0, \\
&\quad \hat{u}_{4;5}(\omega_\gamma\pi_0\pi_0,\ \tilde{u}'\theta_\gamma\pi_1\pi_0,\ \tilde{u}\bar{u}p_1^2\pi_1\pi_1)c\big)c \\
&= \big(\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\omega_{\gamma_1}\pi_1\pi_0,\ \hat{u}_{4;5}\sigma_\gamma\pi_1\big)c
\end{aligned}
\tag{B.12}
$$

Since $\mathcal{P}(\mathbb{C})$ is a pullback of $\mathcal{P}_{eq}(\mathbb{C})$ and $\mathcal{P}_{cq}(\mathbb{C})$ over the object of parallel pairs in $\mathbb{C}$, $P(\mathbb{C})$,

$$\begin{aligned}\delta_{\rho_1}\pi_0\iota_{eq}\pi_0 &= \delta_{\rho_1}\pi_1\iota_{ceq}\pi_0\\ &= \hat{u}_3\rho_1\iota_{ceq}\pi_0\\ &= \hat{u}_3\rho'_1\pi_0\end{aligned}$$

By definition of $\mathcal{P}_{eq}(\mathbb{C})$ the composable pairs

$$\delta_{\rho_1}\pi_0\iota_{eq}(\pi_0,\pi_1\pi_0) = (\delta_{\rho_1}\pi_0\iota_{eq}\pi_0, \delta_{\rho_1}\pi_0\iota_{eq}\pi_1\pi_0) = (\delta_{\rho_1}\pi_0\iota_{eq}\pi_0, \hat{u}_3\rho'_1\pi_0\pi_0)$$

and

$$\delta_{\rho_1}\pi_0\iota_{eq}(\pi_0,\pi_1\pi_1) = (\delta_{\rho_1}\pi_0\iota_{eq}\pi_0, \delta_{\rho_1}\pi_0\iota_{eq}\pi_1\pi_1) = (\delta_{\rho_1}\pi_0\iota_{eq}\pi_0, \hat{u}_3\rho'_1\pi_0\pi_1).$$

are coequalized (in $\mathcal{E}$) by the composition structure map of $\mathbb{C}$. This implies

$$(\delta_{\rho_1}\pi_0\iota_{eq}\pi_0, \hat{u}_3\rho'_1\pi_0\pi_0)c = (\delta_{\rho_1}\pi_0\iota_{eq}\pi_0, \hat{u}_3\rho'_1\pi_0\pi_1)c \tag{B.13}$$

Now the span

$$\sigma_{1,\gamma} = (\omega_{0,1}\pi_1,\ \sigma_{1,\gamma}\pi_1)$$

is well-defined because

$$\omega_{1,0}\pi_1 ws = \omega_{1,0}\pi_0\pi_0 s$$

where the right leg, $\sigma_{1,\gamma}\pi_1$, is given by the composite

$$\begin{aligned}
(\omega_1,\ \hat{u}_{0;4}\omega_{\gamma_1}\pi_0\pi_0,\ \hat{u}\sigma_1\pi_1)c &= \big(\omega_1',\ \hat{u}_{0;2}\delta_{\rho_1}\pi_0\iota_{eq}\pi_0,\\
&\quad \hat{u}_{0;3}(\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0,\ \hat{u}_4\omega_{\gamma_1}\pi_0\pi_0,\ \hat{u}_{4;5}\sigma_1\pi_1)c\big)c\\
&= \big(\omega_1',\ \hat{u}_{0;2}\delta_{\rho_1}\pi_0\iota_{eq}\pi_0,\\
&\quad \hat{u}_{0;3}(\rho_1'\pi_0\pi_1,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_1)c\big)c\\
&= \big(\omega_1',\\
&\quad \hat{u}_{0;2}(\delta_{\rho_1}\pi_0\iota_{eq}\pi_0,\ \hat{u}_3\rho_1'\pi_0\pi_1)c,\\
&\quad \hat{u}_{0;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_1\big)c\\
&= \big(\omega_1',\\
&\quad \hat{u}_{0;2}(\delta_{\rho_1}\pi_0\iota_{eq}\pi_0,\ \hat{u}_3\rho_1'\pi_0\pi_0)c,\\
&\quad \hat{u}_{0;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_1\big)c\\
&= \big(\omega_1',\ \hat{u}_{0;2}\delta_{\rho_1}\pi_0\iota_{eq}\pi_0,\\
&\quad \hat{u}_{0;3}(\rho_1'\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_1)c\big)c\\
&= \big(\omega_1',\ \hat{u}_{0;2}\delta_{\rho_1}\pi_0\iota_{eq}\pi_0,\\
&\quad \hat{u}_{0;3}(\rho_1'\pi_0\pi_0,\ \hat{u}_{4;5}\tilde{u}\bar{u}p_0^2\pi_1\pi_1)c\big)c\\
&= \big(\omega_1',\ \hat{u}_{0;2}\delta_{\rho_1}\pi_0\iota_{eq}\pi_0,\\
&\quad \hat{u}_{0;3}(\delta_{\lambda_1}\pi_0\iota_{eq}\pi_0 w,\ \hat{u}_4\omega_{\gamma_1}\pi_1\pi_0,\ \hat{u}_{4;5}\sigma_\gamma\pi_1)c\big)c\\
&= \big(\omega_1,\ \hat{u}_{0;4}\omega_{\gamma_1}\pi_1\pi_0,\ \hat{u}\sigma_\gamma\pi_1\big)c
\end{aligned}$$

□

Lemma 98. *There exists a sailboat* $\varphi_0 : \hat{U} \to sb$*, uniquely determined by the pairing map*

$$\varphi_0 = \big(((\mu_0,\ \hat{u}\sigma_0\pi_0),\ \omega_{0,0}\pi_1),\ \hat{u}\sigma_0\pi_1\big)$$

where

$$\mu_0 = (\omega_0,\ \hat{u}_{0;4}\theta_{\gamma_0}\pi_0\pi_0)c$$

such that

$$\varphi_0 p_0 = \hat{u}\sigma_0$$
$$\varphi_0 p_1 = \sigma_{0,\gamma}.$$

Proof. Recall that $\mathrm{sb} = W_{\triangle} \, {}_{\pi_0\pi_0\pi_1 s}{\times}_s \, \mathbb{C}_1$, where $W_{\triangle} = (\mathbb{C}_1 \, {}_t{\times}_{ws} \, W) \, {}_c{\times}_w \, W$ so to see $\varphi_0 : \hat{U} \to \mathrm{sb}$ is well-defined we need to show that

$$((\mu_0,\ \hat{u}\sigma_0\pi_0),\ \omega_{0,0}\pi_1) : \hat{U} \to W_{\triangle}$$

is well-defined and that

$$\hat{u}\sigma_0\pi_1 s = \mu_0 t = \hat{u}\sigma_0\pi_0 s.$$

By definition of μ_0 and the lift $\theta_{\gamma_0} : \hat{U}_5 \to W_{\square}$ we have

$$\mu_0 t = \hat{u}_{0;4}\theta_{\gamma_0}\pi_0\pi_0 w t = \hat{u}\sigma_0\pi_0 w s = \hat{u}\sigma_0\pi_1 s$$

showing that $(\mu_0, \hat{u}\sigma_0\pi_0) : \hat{U} \to \mathbb{C}_1 \, {}_t{\times}_{ws} \, W$ are composable with respect to the internal composition structure of $\mathbb{C}$ (after appliying $w : W \to \mathbb{C}_1$ in the right-hand component) and that $\hat{u}\sigma_0\pi_1$ is well-defined in the right-most component. It remains to see that the $\varphi_0 p_0 : \hat{U} \to W_{\triangle}$ is well-defined. For this we use the definitions of μ_0 and the lifts in Diagrams ($\star\star$) and ($\star\star$) along with associativity of composition in $\mathbb{C}$ to compute

$$\begin{aligned}
(\mu_0,\ \hat{u}\sigma_0\pi_0),\ \omega_{0,0}\pi_1)\pi_1 &= \omega_{0,0}\pi_1\\
&= (\omega_{0,0}\pi_0\pi_0, \omega_{0,0}\pi_0\pi_1, \omega_{0,0}\pi_0\pi_2)c\\
&= (\omega_{0,0}\pi_0\pi_0, \hat{u}_0\omega_{0,1}\pi_1, \hat{u}\sigma_0\pi_0)c\\
&= \big(\omega_{0,0}\pi_0\pi_0,\\
&\quad \hat{u}_0(\omega_{0,1}\pi_0\pi_0, \omega_{0,1}\pi_0\pi_1, \omega_{0,1}\pi_0\pi_2)c,\\
&\quad \hat{u}\sigma_0\pi_0\big)c\\
&= \big(\omega_{0,0}\pi_0\pi_0,\ \hat{u}_0\omega_{0,1}\pi_0\pi_0,\ \hat{u}_0\omega_{0,1}\pi_0\pi_1,\ \hat{u}_0\omega_{0,1}\pi_0\pi_2,\ \hat{u}\sigma_0\pi_0\big)c\\
&= \big(\omega_{0,0}\pi_0\pi_0,\ \hat{u}_0\omega_{0,1}\pi_0\pi_0,\ \hat{u}_{0;1}\omega_{0,2}\pi_1,\ \hat{u}_{0;4}\theta_{\gamma_0}\pi_0\pi_0,\ \hat{u}\sigma_0\pi_0\big)c\\
&= \big((\omega_{0,0}\pi_0\pi_0,\ \hat{u}_0\omega_{0,1}\pi_0\pi_0,\ \hat{u}_{0;1}\omega_{0,2}\pi_1)c,\\
&\quad \hat{u}_{0;4}\theta_{\gamma_0}\pi_0\pi_0,\ \hat{u}\sigma_0\pi_0\big)c\\
&= \big(\omega_0,\ \hat{u}_{0;4}\theta_{\gamma_0}\pi_0\pi_0,\ \hat{u}\sigma_0\pi_0\big)c\\
&= \big((\omega_0,\ \hat{u}_{0;4}\theta_{\gamma_0}\pi_0\pi_0)c,\ \hat{u}\sigma_0\pi_0\big)c\\
&= (\mu_0, \hat{u}\sigma_0\pi_0)c\\
&= (\mu_0,\ \hat{u}\sigma_0\pi_0),\ \omega_{0,0}\pi_1)(\pi_0\pi_0, \pi_0\pi_1)c.
\end{aligned}$$

This gives that

$$\varphi_0 p_0 = (\mu_0,\ \hat{u}\sigma_0\pi_0),\ \omega_{0,0}\pi_1) : \hat{U} \to W_\triangle$$

is well-defined. Similar techniques allow us to see

$$\begin{aligned}
(\mu_0, \hat{u}\sigma_0\pi_1)c &= \big((\omega_0,\ \hat{u}_{0;4}\theta_{\gamma_0}\pi_0\pi_0)c,\ \hat{u}\sigma_0\pi_1\big)c\\
&= (\omega_0,\ \hat{u}_{0;4}\theta_{\gamma_0}\pi_0\pi_0,\ \hat{u}\sigma_0\pi_1)c\\
&= \sigma_{0,\gamma}\pi_1.
\end{aligned}$$

which shows

$$\begin{aligned}\varphi_0 p_0 &= \varphi_0(\pi_0\pi_0\pi_1, \pi_1) \\ &= (\hat{u}\sigma_0\pi_0, \hat{u}\sigma_0\pi_1) \\ &= \hat{u}\sigma_0\end{aligned} \qquad \begin{aligned}\varphi_0 p_1 &= \varphi_0\big(\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c\big) \\ &= \big(\omega_{0,0}\pi_1, (\mu_0, \hat{u}\sigma_0\pi_1)c\big) \\ &= (\omega_{0,0}\pi_1, \sigma_{0,\gamma}\pi_1) \\ &= \sigma_{0,\gamma}.\end{aligned}$$

□

Lemma 99. *There exists a sailboat* $\varphi_{0,\gamma} : \hat{U} \to sb$*, uniquely determined by the pairing map*

$$\varphi_{0,\gamma} = \big(((\mu_{0,\gamma},\ \hat{u}\sigma_\gamma\pi_0),\ \omega_{0,0}\pi_1),\ \hat{u}\sigma_\gamma\pi_1\big)$$

where

$$\mu_{0,\gamma} = (\omega_0,\ \hat{u}_{0;4}\theta_{\gamma_0}\pi_1\pi_0)c$$

such that

$$\varphi_{0,\gamma}p_0 = \hat{u}\sigma_\gamma$$
$$\varphi_{0,\gamma}p_1 = \sigma_{0,\gamma}$$

Proof. First note that the components of the pairing map defining $\varphi_{0,\gamma} : \hat{U} \to \mathrm{sb}$ are appropriately composable with respect to the internal composition structure of $\mathbb{C}$:

$$\mu_{0,\gamma}t = \hat{u}_{0;4}\theta_{\gamma_0}\pi_1\pi_0 t = \hat{u}\sigma_\gamma\pi_0 ws = \hat{u}\sigma_\gamma\pi_1 s.$$

By Definition of $\mu_{0,\gamma}$ and the lifts, $\omega_{0,0} : \hat{U} \to W_\circ$ and $\theta_{\gamma_0} : \hat{U}_5 \to W_\Box$, we have

$$\begin{aligned}\omega_{0,0}\pi_1 &= \big(\omega_0,\ \hat{u}_{0;4}\theta_{\gamma_0}\pi_0\pi_0 w,\ \hat{u}\sigma_0\pi_0 w\big)c \\ &= \big(\omega_0,\ (\hat{u}_{0;4}\theta_{\gamma_0}\pi_0\pi_0 w,\ \hat{u}\sigma_0\pi_0 w)c\big)c \\ &= \big(\omega_0,\ (\hat{u}_{0;4}\theta_{\gamma_0}\pi_1\pi_0,\ \hat{u}\sigma_\gamma\pi_0 w)c\big)c \\ &= \big((\omega_0,\ \hat{u}_{0;4}\theta_{\gamma_0}\pi_1\pi_0)c,\ \hat{u}\sigma_\gamma\pi_0 w\big)c \\ &= (\mu_{0,\gamma}, \hat{u}\sigma_\gamma\pi_0 w)c.\end{aligned}$$

showing that the map

$$\varphi_{0,\gamma}\pi_0 = ((\mu_{0,\gamma},\ \hat{u}\sigma_\gamma\pi_0),\ \omega_{0,0}\pi_1) : \hat{U} \to W_\triangle$$

is well-defined. Then by associativity of composition in $\mathbb{C}$ and the definitions of $\mu_{0,\gamma}$ and $\sigma_{0,\gamma}$ we get

$$\begin{aligned}(\mu_{0,\gamma}, \hat{u}\sigma_\gamma\pi_1)c &= \big((\omega_0,\ \hat{u}_{0;4}\theta_{\gamma_0}\pi_1\pi_0)c,\ \hat{u}\sigma_\gamma\pi_1\big)c \\ &= \big(\omega_0,\ \hat{u}_{0;4}\theta_{\gamma_0}\pi_1\pi_0,\ \hat{u}\sigma_\gamma\pi_1\big)c \\ &= \sigma_{0,\gamma}\pi_1.\end{aligned}$$

The previous equation implies the unique pairing, $\hat{U} \to \mathrm{sb}$, given by

$$\varphi_{0,\gamma} = \big(((\mu_{0,\gamma},\ \hat{u}\sigma_\gamma\pi_0),\ \omega_{0,0}\pi_1),\ \hat{u}\sigma_\gamma\pi_1\big)$$

is well-defined. From here it is straightforward to calculate

$$\begin{aligned}\varphi_{0,\gamma}p_0 &= \varphi_{0,\gamma}(\pi_0\pi_0\pi_1, \pi_1) \\ &= (\hat{u}\sigma_\gamma\pi_0, \hat{u}\sigma_\gamma\pi_1) \\ &= \hat{u}\sigma_\gamma\end{aligned} \qquad \begin{aligned}\varphi_{0,\gamma}p_1 &= \varphi_{0,\gamma}\big(\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c\big) \\ &= \big(\omega_{0,0}\pi_1, (\mu_{0,\gamma}, \hat{u}\sigma_\gamma\pi_1)c\big) \\ &= (\omega_{0,0}\pi_1, \sigma_{0,\gamma}\pi_1) \\ &= \sigma_{0,\gamma}.\end{aligned}$$

□

Lemma 100. *The sailboat, $\varphi_1 : \hat{U} \to sb$, defined by*

$$\varphi_1 = \big(((\mu_1,\ \hat{u}\sigma_1\pi_0),\ \omega_{1,0}\pi_1),\ \hat{u}\sigma_1\pi_1\big)$$

where

$$\mu_1 = (\omega_1, \hat{u}_{0;4}\gamma_1\pi_0\pi_0)c$$

is well-defined and relates the spans $\sigma_1, \sigma_{1,\gamma} : \hat{U} \to spn$ in the sense that

$$\varphi_1 p_0 = \hat{u}\sigma_1 \qquad\qquad \varphi_1 p_1 = \sigma_{1,\gamma}$$

Proof. First notice that the components defining φ_1 are apporpriately composable by checking

$$\mu_1 t = \hat{u}_{0;4}\gamma_1\pi_0\pi_0 wt = \hat{u}\sigma_1\pi_1 s = \hat{u}\sigma_1\pi_0 ws.$$

Now to see that the component

$$\varphi_1 p_0 = ((\mu_1,\ \hat{u}\sigma_1\pi_0),\ \omega_{1,0}\pi_1) : \hat{U} \to W_\triangle$$

is well-defined we use the definitions of the lifts $\omega_{1,0} : \hat{U} \to W_\circ$ and $\theta_{\gamma_1} : \hat{U}_5 \to W_\square$ to see

$$\begin{aligned}
\omega_{1,0}\pi_1 &= (\omega_{1,0}\pi_0\pi_0, \omega_{1,0}\pi_0\pi_1, \omega_{1,0}\pi_0\pi_2)c \\
&= (\omega_{1,0}\pi_0\pi_0, \hat{u}_0\omega_{1,1}\pi_1, \hat{u}\sigma_1\pi_0)c \\
&= \big(\omega_{1,0}\pi_0\pi_0, \\
&\quad \hat{u}_0(\omega_{1,1}\pi_0\pi_0, \omega_{1,1}\pi_0\pi_1, \omega_{1,1}\pi_0\pi_2)c, \\
&\quad \hat{u}\sigma_1\pi_0\big)c \\
&= \big(\omega_{1,0}\pi_0\pi_0,\ \hat{u}_0\omega_{1,1}\pi_0\pi_0,\ \hat{u}_0\omega_{1,1}\pi_0\pi_1,\ \hat{u}_0\omega_{1,1}\pi_0\pi_2,\ \hat{u}\sigma_1\pi_0\big)c \\
&= \big(\omega_{1,0}\pi_0\pi_0,\ \hat{u}_0\omega_{1,1}\pi_0\pi_0,\ \hat{u}_{0;1}\omega_{1,2}\pi_1,\ \hat{u}_{0;4}\theta_{\gamma_1}\pi_0\pi_0,\ \hat{u}\sigma_1\pi_0\big)c \\
&= \big((\omega_{1,0}\pi_0\pi_0,\ \hat{u}_0\omega_{1,1}\pi_0\pi_0,\ \hat{u}_{0;1}\omega_{1,2}\pi_1)c, \\
&\quad \hat{u}_{0;4}\theta_{\gamma_1}\pi_0\pi_0,\ \hat{u}\sigma_1\pi_0\big)c \\
&= \big(\omega_1,\ \hat{u}_{0;4}\theta_{\gamma_1}\pi_0\pi_0,\ \hat{u}\sigma_1\pi_0\big)c \\
&= \big((\omega_1,\ \hat{u}_{0;4}\theta_{\gamma_1}\pi_0\pi_0)c,\ \hat{u}\sigma_1\pi_0\big)c \\
&= (\mu_1, \hat{u}\sigma_1\pi_0)c.
\end{aligned}$$

This shows $\varphi_1 : \hat{U} \to \text{sb}$ is well-defined. By definition of μ_1 and $\sigma_{1,\gamma}$ in Lemma 97 we have

$$\begin{aligned}
(\mu_1, \hat{u}\sigma_1\pi_1)c &= \big((\omega_1,\ \hat{u}_{0;4}\theta_{\gamma_1}\pi_0\pi_0)c,\ \hat{u}\sigma_1\pi_1\big)c \\
&= (\omega_1,\ \hat{u}_{0;4}\theta_{\gamma_1}\pi_0\pi_0,\ \hat{u}\sigma_1\pi_1)c \\
&= \sigma_{1,\gamma}\pi_1
\end{aligned}$$

which implies

$$
\begin{aligned}
\varphi_1 p_0 &= \varphi_1(\pi_0\pi_0\pi_1, \pi_1) \\
&= (\hat{u}\sigma_1\pi_0, \hat{u}\sigma_1\pi_1)c \\
&= \hat{u}\sigma_1
\end{aligned}
\qquad
\begin{aligned}
\varphi_1 p_1 &= \varphi_1(\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c) \\
&= (\mu_1, \hat{u}\sigma_1\pi_1)c \\
&= (\omega_{1,0}\pi_1, \sigma_{1,\gamma}\pi_1) \\
&= \sigma_{1,\gamma}
\end{aligned}
$$

□

Lemma 101. *The sailboat, $\varphi_{1,\gamma} : \hat{U} \to sb$, defined by*

$$\varphi_{1,\gamma} = \big(((\mu_{1,\gamma},\ \hat{u}\sigma_\gamma\pi_0),\ \omega_{1,0}\pi_1),\ \hat{u}\sigma_\gamma\pi_1\big)$$

where

$$\mu_{1,\gamma} = (\omega_1,\ \hat{u}_{0;4}\theta_{\gamma_1}\pi_1\pi_0)c$$

relates the spans $\sigma_\gamma, \sigma_{1,\gamma} : \hat{U} \to spn$ in the sense that

$$\varphi_{1,\gamma} p_0 = \hat{u}\sigma_\gamma \qquad\qquad \varphi_{1,\gamma} p_1 = \sigma_{1,\gamma}.$$

Proof. First use the definition of μ_{01}, the lift $\theta_{\gamma_1} : \hat{U}_5 \to W_\square$, and the span σ_γ to see that the components of $\varphi_{1,\gamma}$ are appropriately composable in $\mathbb{C}$:

$$\mu_{1,\gamma} t = \hat{u}_{0;4}\theta_{\gamma_1}\pi_1\pi_0 t = \hat{u}\sigma_\gamma\pi_0 w s = \hat{u}\sigma_\gamma\pi_1 s.$$

Now use those definitions to compute

$$
\begin{aligned}
\omega_{1,0}\pi_1 &= \big(\omega_1,\ \hat{u}_{0;4}\theta_{\gamma_1}\pi_0\pi_0 w,\ \hat{u}\sigma_1\pi_0 w\big)c \\
&= \big(\omega_1,\ (\hat{u}_{0;4}\theta_{\gamma_1}\pi_0\pi_0 w,\ \hat{u}\sigma_1\pi_0 w)c\big)c \\
&= \big(\omega_1,\ (\hat{u}_{0;4}\theta_{\gamma_1}\pi_1\pi_0,\ \hat{u}\sigma_\gamma\pi_0 w)c\big)c \\
&= \big((\omega_1,\ \hat{u}_{0;4}\theta_{\gamma_1}\pi_1\pi_0)c,\ \hat{u}\sigma_\gamma\pi_0 w\big)c \\
&= (\mu_{1,\gamma}, \hat{u}\sigma_\gamma\pi_0 w)c.
\end{aligned}
$$

This shows that the component

$$\varphi_{1,\gamma}\pi_0 = ((\mu_{1,\gamma},\ \hat{u}\sigma_\gamma\pi_0),\ \omega_{1,0}\pi_1) : \hat{U} \to W_\triangle$$

is well-defined. Similarly,

$$\begin{aligned}(\mu_{1,\gamma}, \hat{u}\sigma_\gamma\pi_1)c &= \big((\omega_1,\ \hat{u}_{0;4}\theta_{\gamma_1}\pi_1\pi_0)c,\ \hat{u}\sigma_\gamma\pi_1\big)c \\ &= \big(\omega_1,\ \hat{u}_{0;4}\theta_{\gamma_1}\pi_1\pi_0,\ \hat{u}\sigma_\gamma\pi_1\big)c \\ &= \sigma_{1,\gamma}\pi_1.\end{aligned}$$

implies

$$\begin{aligned}\varphi_{1,\gamma}p_0 &= \varphi_{1,\gamma}(\pi_0\pi_0\pi_1, \pi_1) \\ &= (\hat{u}\sigma_\gamma\pi_0, \hat{u}\sigma_\gamma\pi_1) \\ &= \hat{u}\sigma_\gamma\end{aligned} \qquad \begin{aligned}\varphi_{1,\gamma}p_1 &= \varphi_{1,\gamma}\big(\pi_0\pi_1, (\pi_0\pi_0\pi_0, \pi_1)c\big) \\ &= \big(\omega_{1,0}\pi_1, (\mu_{1,\gamma}, \hat{u}\sigma_\gamma\pi_1)c\big) \\ &= (\omega_{1,0}\pi_1, \sigma_{1,\gamma}\pi_1) \\ &= \sigma_{1,\gamma}.\end{aligned}$$

□

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