COMPACT OPEN SUBSETS IN THE DUAL SPACE OF A WALLPAPER GROUP

by

Emma Carline

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Abstract

Knowledge of the compact open sets in the dual space of a locally compact group can be used to study projections in the L^1-algebra of the group. The wallpaper groups are a class of almost abelian groups which arise as the symmetry groups of wallpaper patterns. We characterize the compact open subsets in the dual space of a wallpaper group G. This is achieved by associating to G a graph that captures the stratified nature of the dual space. We show how this can by applied to the problem of finding projections in $L^1(G)$ by constructing a novel projection in the L^1-algebra of the wallpaper group, p2.

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Chapter 1

Introduction

A projection is a self-adjoint idempotent. That is, it is an element in a $*$-algebra over $\mathbb{C}$ that is equal to both its adjoint and its square. The study of projections, or more generally, idempotents, in algebras has been a fruitful area of research in both pure and applied mathematics. Projections can reveal structural information about an algebra. For instance, the central minimal projections in a finite-dimensional C^*-algebra allow a decomposition of the algebra into a direct sum of full matrix algebras. This is both elegant and useful. As as example, a version of this decomposition is used in coding theory for the construction of error-correcting codes [3]. More recently, projections in $L^1(G)$, for certain locally compact groups G acting as affine transformations of $\mathbb{R}^n$, were found to be closely connected to functions in $L^2(\mathbb{R}^n)$ that generate tight frames [21]. A tight frame in $L^2(\mathbb{R}^n)$ is a countable set of functions $\mathcal{T} \subseteq L^2(\mathbb{R}^n)$ such that for $f \in L^2(\mathbb{R}^n)$, $f = \sum_{w\in\mathcal{T}}\langle f, w\rangle f$. To form such a group, fix A a $n \times n$ matrix over $\mathbb{R}$ and let G_A be the group of affine transformations $G_A = \{[k, x] : k \in \mathbb{Z}, x \in \mathbb{R}^n\}$ defined by, for $z \in \mathbb{R}^n$, $[k, x]z = A^k(x + z)$. If all of the eigenvalues of A have absolute value greater than 1 then there exists a "projection generating function" which gives rise to both a projection in $L^1(G)$ and to a tight frame in $L^2(\mathbb{R}^n)$. Specifically, there is a function $\xi \in L^2(\widehat{\mathbb{R}^n}) \cap A(\widehat{\mathbb{R}^n})$ such that defining w as the inverse Fourier transform of ξ, the set $\{\rho[k, x]w : k \in \mathbb{Z}, x \in \mathbb{R}^n\}$ is a tight frame. Here, ρ is the natural representation of the group G_A on $L^2(\mathbb{R}^n)$. The space $L^2(\mathbb{R}^n)$ is important in applications as it corresponds to signals (functions on $\mathbb{R}^n$) with finite energy. These include acoustic and electric signals whose domain is continuous time or magnetic signals whose domain is 3-space. Tight frames in $L^2(\mathbb{R}^n)$ are useful for encoding such signals and are often employed in analysis, compression and other signal processing [17].

In finite-dimensional C^*-algebras, projections can be explicitly calculated using the decomposition theorem (discussed in Chapter 3). For the L^1 algebra of a finite group, the form of projections has also been completely described. This is usually not the case for the L^1 algebra of a general group. Here, finding projections is a non-trivial problem. In a slightly more general setting, the group ring over the complex field, Kaplansky conjectured that if the group is torsion-free then there are no non-trivial projections 4 [16]. While numerous supporting examples have been identified, the conjecture remains an open problem. Projections may be difficult to characterize in general, but one indicator of their presence or absence in the L^1 or C^* algebra of a group G is the collection of compact open sets in the dual space of G. The dual space of a locally compact group G is a, not-necessarily Hausdorff, topological space denoted by $\widehat{G}$. Dixmier [9] showed that every projection has an associated compact open set in $\widehat{G}$. For some groups, every compact open set in $\widehat{G}$ has an associated projection. For this reason, understanding the compact open sets in $\widehat{G}$ is a useful tool in understanding projections in $L^1(G)$.

We also will focus on algebras that come from a group. The ℓ^1-algebra of a discrete group G consists of complex-valued functions on the group that are absolutely summable. If G is countable, one can think of these functions as sequences indexed by the group. Projections in $\ell^1(G)$ have been characterized both when G is abelian and when G is finite. It leads one to wonder what projections look like in $\ell^1(G)$ when G is *almost* abelian. Consider the case where G is a group that has a normal abelian subgroup whose index in G is finite. Such a group decomposes into finitely many cosets of an abelian group. We will narrow in on a certain collection of almost abelian groups: the wallpaper groups. These are the 2-dimensional crystallographic groups, or, the symmetry groups of wallpaper patterns on the plane. Wallpaper groups are simple to describe and yet have diverse, topologically interesting dual spaces. They also have the advantage of possessing visual representations, both the groups themselves and their dual spaces.

This paper aims to describe the compact open subsets in the dual space of a wallpaper group G as a preliminary step to characterizing projections in $\ell^1(G)$. We will describe the topology of $\widehat{G}$ using explicit calculations of the group C^*-algebra and applying the representation theory of C^*-bundles. Finally, we obtain a complete characterization of the compact open subsets of $\widehat{G}$ by associating $\widehat{G}$ with a certain kind of graph. As an application of our description of the compact open subsets of the dual, we construct a novel projection in $\ell^1(p2)$, where $p2$ is one of the simplest wallpaper groups.

1.1 Background and Notation

This section provides background on projections in $\ell^1(G)$, including known results about the form of projections in certain algebras. We define the L^1 algebra and group C^*-algebras of a topological group, dual spaces, the hull-kernel topology on dual spaces, and revisit the definition of a projection.

1.1.1 Basic Definitions

Definition 1. A **-algebra* is an algebra $\mathcal{A}$ over $\mathbb{C}$ equipped with an operation * which satisfies the following conditions. For $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$:

(i) $A^{**} = A$

(ii) $(\lambda A + B)^* = \overline{\lambda} A^* + B^*$

(iii) $(AB)^* = B^* A^*$

The adjoint of an element $A \in \mathcal{A}$ is defined to be its image under the $*$ operation, that is, A^*. A *Banach *-algebra* is a normed *-algebra $\mathcal{A}$ that is complete and satisfies $\|AB\| \leq \|A\|\|B\|$ for all A, $B \in \mathcal{A}$.

Definition 2. A C^*-algebra is a Banach *-algebra with the additional property that for any $A \in \mathcal{A}$, $\|A^*A\| = \|A\|^2$. This is called the C^* condition.

Definition 3. A projection in a Banach *-algebra $\mathcal{A}$ is a self-adjoint idempotent. That is, $A \in \mathcal{A}$ is a projection if and only if $A = A^* = A^2$.

We will be dealing with the L^1-algebras of locally compact topological groups. To begin, a *topological group* is a group equipped with a topology such that the group operation and inversion are continuous. A topological group G is *locally compact* if every point in G has an open neighbourhood contained in a compact set. The L^1-algebra of a locally compact group, denoted by $L^1(G)$, is the set of functions $f : G \to \mathbb{C}$ such that $\|f\|_1 := \int_G |f| < \infty$. The integral here is the Lebesgue integral with respect to the left Haar measure μ. This is the unique (up to a scalar multiple) regular Borel measure on a locally compact group that is non-zero, left invariant, and finite on compact sets [18].

A locally compact group with the discrete topology is called discrete. Recall that the discrete topology is the topology where singleton subsets (and thus all the subsets of G) are open. When G is discrete $L^1(G)$ is denoted by $\ell^1(G)$. It is evident that the counting measure μ, defined by $\mu(S) = |S|$ when S has finitely many elements and $\mu(S) = \infty$ when S has infinitely many elements, is left invariant. This measure is clearly finite on compact sets, which are the finite sets in a discrete group. Thus the counting measure is the Haar measure for a discrete group. With the counting measure, Lebesgue integration becomes summation. For instance, $\|f\|_1 = \int_G |f(x)|\mu(x) = \sum_{x \in G} |f(x)|$. From here on, we assume that G is discrete.

On $\ell^1(G)$, $\|\cdot\|_1$ defines a norm which is referred to as the ℓ^1-norm. Addition and scalar multiplication can be defined pointwise on $\ell^1(G)$. Further structure is given to $\ell^1(G)$ by defining two additional operators:

(1) Convolution: $(f * g)(x) = \sum_{y \in G} f(y)g(y^{-1}x)$

(2) The *-operation: $f^*(x) = \overline{f(x^{-1})}$

Note that $\|f * g\|_1 < \infty$. In fact, $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. This shows that convolution is well defined on $\ell^1(G)$ and is submultiplicative with respect to the ℓ^1-norm. The $*$-operation is skew-linear and involutive. For $f, g \in \ell^1(G)$ and $\lambda \in \mathbb{C}$, $(\lambda f + g)^* = \overline{\lambda} f^* + g^*$. One can also check that $(f * g)^* = g^* * f^*$. These operations, together with pointwise addition and scalar multiplication, make $\ell^1(G)$ a *-algebra. Being complete with respect to the ℓ^1-norm, it is even a Banach algebra.

Consider projections in $\ell^1(G)$. Restating Definition 3, f is a projection if and only if, for all $x \in G$:

$$f(x) = \overline{f(x^{-1})} = \sum_{y \in G} f(y) f(y^{-1}x)$$

There are a couple of functions in $\ell^1(G)$ that are clearly projections. First is the function that is zero everywhere: $f(x) = 0$ for all $x \in G$. Second is the point mass at the identity of G, denoted by δ_{1_G}. This is defined on G by $\delta_{1_G}(x) = 0$ unless $x = 1_G$. Since these are projections in every discrete group, we consider them trivial.

Another important kind of topological space is a Hilbert space. This is not an algebra but a vector space with a strict geometric structure imposed by an inner product.

Definition 4. A Hilbert space is a complete inner product space.

Any locally compact topological group has an associated Hilbert space. Let $L^2(G)$ be the set of complex-valued functions on G such that $\|f\|_2 = \int_G |f|^2 < \infty$. Then $\|\cdot\|_2$ is a norm on $L^2(G)$ and comes from the inner product $\langle f, g \rangle = \int_G f\overline{g}$. When G is discrete $L^2(G)$ is denoted by $\ell^2(G)$ and the inner product becomes $\langle h_1, h_2 \rangle = \sum_{x \in G} h_1(x)\overline{h_2(x)}$. Note that $\ell^2(G)$ is indeed complete with respect to the resulting norm,

$$\|h\|_2 = \sum_{x \in G} |h(x)|^2$$

Let $\mathcal{B}(\mathcal{H})$ denote the set of bounded linear operators on the Hilbert space $\mathcal{H}$. An operator on $\mathcal{H}$ is a linear function $A : \mathcal{H} \to \mathcal{H}$ and is called bounded if $\sup\{\|A\xi\| : \xi \in \mathcal{H}, \|\xi\| < \infty\}$. Letting multiplication be composition of operators, $\mathcal{B}(\mathcal{H})$ forms an algebra. When $\mathcal{H}$ is finite-dimensional, $\mathcal{H}$ is isomorphic to $\mathbb{C}^n$ for some $n \in \mathbb{N}$ and $\mathcal{B}(\mathcal{H})$ is isomorphic to the set of $n \times n$ matrices over $\mathbb{C}$. The adjoint of an operator $A \in \mathcal{B}(\mathcal{H})$ is the unique operator A^* satisfying, for all $\xi, \eta \in \mathcal{H}$:

$$\langle A\xi, \eta \rangle = \langle \xi, A^*\eta \rangle$$

Taking adjoints defines a $*$-operation for $\mathcal{B}(\mathcal{H})$, that is, a skew linear involution. There is a norm on $\mathcal{B}(\mathcal{H})$ called the operator norm. The operator norm of an operator $A \in \mathcal{B}(\mathcal{H})$ is the supremum of the norm of A on the unit ball of $\mathcal{H}$. For $A \in \mathcal{A}$, $\|A\|_{op} = \sup\{\|A\xi\| : \xi \in \mathcal{H}, \|\xi\| \leq 1\}$. With this norm and *-operation, $\mathcal{B}(\mathcal{H})$ is a C^*-algebra.

1.1.2 Representations of G and $\ell^1(G)$

A *unitary representation* of a group G is a homomorphism $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$, where $\mathcal{U}(\mathcal{H}_\pi)$ denotes the group of unitary operators on a Hilbert space $\mathcal{H}_\pi$. Recall that a bounded linear operator $U : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is *unitary* if it is surjective and preserves the inner product. Equivalently, U is unitary if $U^*U = UU^* = I$, where U^* is the adjoint of U and I is the identity operator on $\mathcal{H}$. Similarly, a **-representation* of a *-algebra $\mathcal{A}$ is a homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ that preserves the *-operation: $\pi(a^*) = \pi(a)^*$. Usually, we will write $\mathcal{H}$ instead of $\mathcal{H}_\pi$, the Hilbert space of a representation π, when the context is clear. The dimension of a representation is the dimension of the Hilbert space on which it operates: $\dim \pi = \dim(\mathcal{H}_\pi)$.

Two unitary representations, $\pi_1 : G \to \mathcal{U}(\mathcal{H}_1)$ and $\pi_2 : G \to \mathcal{U}(\mathcal{H}_2)$, are *equivalent* if there exists a linear unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_1(x) = \pi_2(x)U$ for each $x \in G$. This does in fact form an equivalence relation on the collection of unitary representations of G. Similarly, two *-representations $\pi_1 : \ell^1(G) \to \mathcal{B}(\mathcal{H}_1)$ and $\pi_2 : \ell^1(G) \to \mathcal{B}(\mathcal{H}_2)$ are equivalent if there is a linear unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$

such that $U\pi_1(x) = \pi_2(x)U$ for all $x \in \ell^1(G)$. A unitary representation (or *-representation) π on $\mathcal{H}_\pi$ is *irreducible* if there are no non-trivial G-invariant ($\ell^1(G)$-invariant) closed subspaces in $\mathcal{H}$. That is, there is no closed subspace $W \subset \mathcal{H}$ (other than $\{0\}$ and $\mathcal{H}$) such that $\pi(x)w \in W$ for all $x \in G$ (or $x \in \ell^1(G)$ in the case of a *-representation) and $w \in W$. For a *-representation π, the *kernel* of π is the set of algebra elements that are mapped by π to zero in $\mathcal{B}(\mathcal{H})$. Any unitary representation π of G can be considered as a $*$-representation of $\ell^1(G)$, as discussed below.

The algebra $\ell^1(G)$ acts on $\ell^2(G)$ via convolution. One can show that for $f \in \ell^1(G)$ and $g \in \ell^2(G)$, $f * g$ is in $\ell^2(G)$. Consider the map $\lambda : \ell^1(G) \to \mathcal{B}(\ell^2(G))$ defined by $\lambda(f)g = f * g$. For $f \in \ell^1(G)$, $\lambda(f)$ is linear since convolution is distributive and is bounded since $\|f * g\|_2 \leq \|f\|_1\|g\|_2$. So $\lambda(f)$ is indeed a bounded linear operator on $\ell^2(G)$. We can thus view $\ell^1(G)$ as an algebra of bounded linear operators on a Hilbert space. In fact, λ also preserves the $*$ operation and convolution. Note that $\lambda(f^*) = \lambda(f)^*$, where $\lambda(f)^*$ is adjoint of $\lambda(f)$ in $\mathcal{B}(\ell^2(G))$, and that $\lambda(f * g) = \lambda(f)\lambda(g)$. We conclude that λ is a *-representation of $\ell^1(G)$. It is a special representation called the left regular representation.

Each group G also has an associated C^*-algebra. It is formed by completing $\ell^1(G)$ with respect to a new norm. For $f \in \ell^1(G)$, the C^*-norm of f is defined as the supremum of the operator norms of the non-degenerate *-representations of $\ell^1(G)$ evaluated at f: $\|f\|_{C^*} = \sup\{\|\pi(f)\|_{op} : \pi \text{ a non-degenerate *-representation of } \ell^1(G)\}$. A *-representation π is non-degenerate if the set $\{\pi(f)\xi : f \in \ell^1(G), \xi \in \mathcal{H}_\pi\}$ is dense in $\mathcal{H}_\pi$. The reduced group C^*-algebra of G is the completion of $\ell^1(G)$ with respect to another norm. Consider the operator norm on $\lambda(\ell^1(G))$: $\|\lambda(f)\|_op = \sup\{\|\lambda(f)g\| = \|f * g\| : g \in \ell^2(G), \|g\| \leq 1\}$. We identify $\ell^1(G)$ with the space of operators $\lambda(\ell^1(G))$ and write $\|f\|_{op} = \|\lambda(f)\|_{op}$. The reduced group C^*-algebra of G is then defined as:

$$C_r^*(G) = \overline{\lambda(\ell^1(G))}^{\|\cdot\|}$$

Lemma 5. $C^*(G)$ and $C_r^*(G)$ are C*-algebras.

When G is amenable, $C_r^*(G) = C^*(G)$ [1]. A locally compact Hausdorff group is amenable if it has a left or right invariant mean. We will not discuss amenability further as what is important is this equivalence of group C^*-algebras in the case of amenability.

A *-representation of a C^*-algebra is defined in the same way as a *-representation of a *-algebra except that its domain is, of course, a C^*-algebra. So a *-representation of a C^*-algebra $\mathcal{A}$ is a homomorphism of $\mathcal{A}$ with $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ that preserves the *-operation. The concepts of equivalence and irreducibility of representations is also the same for *-representations of a $C^*(G)$ as they are for *-representations of a *-algebra.

The dual space of a locally compact group G is the collection of equivalence classes of irreducible unitary representations of G, denoted by $\widehat{G}$. Likewise, the dual space of $C^*(G)$ is the collection of equivalence classes of irreducible *-representations of $C^*(G)$ and is denoted by $\widehat{C^*(G)}$. The dual space of a group and the dual space of the group's C^*-algebra are closely related. Given an irreducible unitary representation π of G, consider the map $L(\pi)$ defined by, for $f \in \ell^1(G)$,

$$L(\pi)(f) = \sum_{x \in G} \pi(x) f(x)$$

Note that $L(\pi)$ is well-defined on $\ell^1(G)$ since $\| \sum_{x \in G} \pi(x) f(x) \| \leq \sum_{x \in G} \|\pi(x)\| |f(x)|_1 = \|f\|_1 < \infty$, noting that $\pi(x)$ is unitary and so is an isometry. One can check that $L(\pi)$ is an irreducible *-representation of $\ell^1(G) \subseteq C^*(G)$. Furthermore, $L(\pi)$ may be uniquely extended to all of $C^*(G)$. If $\pi \sim \sigma \in \widehat{G}$ then $L(\pi) \sim L(\sigma)$ (the same unitary operator can be used to show both equivalences). At the same time, if $L(\pi) \sim L(\sigma)$ then $\pi \sim \sigma$. Thus L is a one-to-one map from $\widehat{G}$ to the collection of equivalence classes of *-representations of $\ell^1(G)$. Suppose $\Pi \in \widehat{C^*(G)}$. Then $L^{-1}(\Pi)(x) := \Pi(\delta_x)$ for $x \in G$ defines an irreducible unitary representation of G, where δ_x the point mass at $x \in G$, the function in $\ell^1(G)$ which takes the value 1 at x and 0 elsewhere. One can check that L^{-1} is indeed the inverse of L. Thus there is a bijection between the dual spaces of G and $C^*(G)$. Throughout we suppress explicit reference to the bijection, instead viewing a representation $\pi \in \widehat{G}$ as both a representation of G and of $C^*(G)$. For this reason, when we say $\widehat{G}$, we are referring to both the dual space of the group G and its C^*-algebra.

1.1.3 Topology on the Dual Space

The dual spaces of groups and of C^*-algebras are topological spaces. The most common topology on $\widehat{G}$ is the hull-kernel (or Jacobson) topology. It is given by first defining a topology on a certain space of ideals of $C^*(G)$. From here one can define a topology on $\widehat{C^*(G)}$. Using the bijection between $\widehat{G}$ and $\widehat{C^*(G)}$ discussed above, this gives a topology on $\widehat{G}$. To summarize, the hull-kernel topology is defined on three spaces in the following order:

$$\mathrm{Prim(C^*(G))} \to \widehat{\mathrm{C^*(G)}} \to \widehat{\mathrm{G}}$$

Definition 6. A primitive ideal is the kernel of an irreducible *-representation of $C^*(G)$. The space of primitive ideals of $C^*(G)$ is denoted by $\mathrm{Prim}(C^*(G))$

A primitive ideal is the preimage of the 0 ideal in $\mathcal{B}(\mathcal{H})$ under a *-homomorphism. Thus it is a closed two-sided *-ideal in $C^*(G)$. In fact, primitive ideals are prime [8]. A closed two-sided *-ideal is prime if whenever there are ideals $J_1,\ J_2 \in C^*(G)$ such that $J_1 J_2 \subseteq J$ then $J_1 \subseteq J$ or $J_2 \subseteq J$. Note that if π and σ are equivalent irreducible representations of $C^*(G)$ then $\ker \pi = \ker \sigma$. For $\pi \sim \sigma$ implies that there exists a unitary operator U such that $U\pi(F) = \sigma(F)U$ for all $F \in C^*(G)$. If $\pi(F) = 0$ then $0 = U\pi(F) = \sigma(F)U \implies \sigma(F) = 0$ since U is invertible. Thus for each equivalence class of irreducible representations there is a single associated kernel in $C^*(G)$. So there is a surjective map from $\widehat{C^*(G)}$ to $\mathrm{Prim}(C^*(G))$.

The *kernel* of a subset $\mathcal{J} \subseteq \mathrm{Prim}(C^*(G))$ is defined to be the intersection of its elements: $\ker(\mathcal{J}) = \cap_{J \in \mathcal{J}} J$. Note that $\ker(\mathcal{J})$ is an ideal. The hull of an ideal

$J \in \mathrm{Prim}(C^*(G))$ is the set of ideals that contain J, that is, $\mathrm{hull}(J) = \{A \in \mathrm{Prim}(C^*(G)) : J \subseteq A\}$. The *hull kernel topology* on $\mathrm{Prim}(C^*(G))$ is defined by, for a set $\mathcal{J} \subseteq \mathrm{Prim}(C^*(G))$, $\overline{\mathcal{J}} = \mathrm{hull}(\ker(\mathcal{J}))$. Since each primitive ideal is the kernel of an irreducible *-representation of $C^*(G)$, this defines a topology on $\widehat{C^*(G)}$. The closure of a set $S \subseteq \widehat{C^*(G)}$ is:

$$\overline{S} = \mathrm{hull}(\ker(\{\ker(\pi) : \pi \in S\})) = \{\sigma \in \widehat{C^*(G)} : \ker(\sigma) \supseteq \bigcap_{\pi \in S} \ker(\pi)\} \tag{1.1}$$

Lemma 7. The closure operation in Equation 1.1 satisfies the Kuratowski axioms and so defines a topology on $\widehat{C^*(G)}$.

A proof may be found in Davidson's book [8] on page 191. Recall that the Kuratowski closure axioms are a way to define a topology through a closure operation rather than a collection of open sets. We denote the closure operation by an overline (not to be confused with the complex conjugate of a number in $\mathbb{C}$). Let X be a topological space and $S \mapsto \overline{S}$ be the closure operation. The Kuratowski axioms are as follows:

(i) $\overline{\emptyset} = \emptyset$.

(ii) For $A \subseteq X$, A is a subset of $\overline{A}$.

(iii) For $A, B \subseteq X$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(iv) $\overline{\overline{A}} = \overline{A}$.

For the topology on $\widehat{G}$, we say that a set $\mathcal{S} \subseteq \widehat{G}$ is closed if the corresponding set in $\widehat{C^*(G)}$ is closed. Next is a well known result about the topology of the dual space of a locally compact group. A proof may be found in [9].

Theorem 8. Let G be a locally compact topological group. If G is compact then $\widehat{G}$ is discrete. If G is discrete then $\widehat{G}$ is compact.

1.2 Known Results

For a discrete finite abelian group G, the projections in $\ell^1(G)$ have been completely characterized. The irreducible unitary representations of an abelian group are called characters and are all one-dimensional. The Hilbert space of character is $\mathbb{C}$. In fact, the dual space of an abelian group is a group. It is not hard to show that each character $\chi \in \widehat{G}$ is a projection in $\ell^1(G)$. Furthermore, $\chi * \chi_0 = 0$ if $\chi \neq \chi_0$ in $\widehat{G}$. Rudin and Schneider [20] then showed that idempotents in $\ell^1(G)$ are exactly those functions of the form:

$$f(x) = \frac{1}{n} \sum_{\chi \in \widehat{G}} \chi(x) e_\chi$$

where $\chi \in \widehat{G}$ and $e_\chi \in \{0, 1\}$.

If a group is abelian but not necessarily finite, every idempotent has a finite support group [20]. The support group of a function f on G is the smallest group that contains the support of f. Rudin and Schneider also describe the idempotents in $\ell^1(G)$ of norm 1.

Theorem 9. Let G be any group and $f \in \ell^1(G)$ with $f * f = f$ and $\|f\|_1 = 1$. Then the support of f is a finite subgroup K of G. Furthermore,

$$|f(x)| = \tfrac{1}{|K|} \quad \text{for all } x \in K, \text{ and}$$

$$f(xy) = |K| f(x) f(y) \quad \text{for all } x, y \in G$$

This implies that idempotents of norm 1 are scaled characters on a finite subgroup. Being a scalar multiple of a homomorphism, such an idempotent automatically satisfies the * condition. So the theorem actually gives a complete description of the projections of norm 1. Conversely, every character on a finite subgroup K of G gives rise to a projection of norm 1 by scaling. Rudin and Schneider note that idempotents in $\ell^1(G)$ must have a norm that is at least 1. So this theorem, in a sense, describes the "smallest" idempotents in $\ell^1(G)$.

Functions in $\ell^1(G)$ have a kind of support in the dual space $\widehat{G}$. Consider the set of equivalence classes of unitary representations of G, evaluated at a fixed $f \in \ell^1(G)$. The *support set* of f is then the subset of representations in $\widehat{G}$ that are non-zero at f. We will either refer to this as the support set or as the support of f in $\widehat{G}$, to avoid confusion with the support of f in $\ell^1(G)$.

Definition 10. Let $f \in \ell^1(G)$ be a projection. The support set of f is defined as:

$$\operatorname{supp}(f) = \{\pi \in \widehat{G} : \pi(f) \neq 0\}$$

Next is a key result that will strongly direct our analysis of projections in $\ell^1(G)$.

Lemma 11. Suppose $f \in \ell^1(G)$ is a projection. Then $\operatorname{supp}(f)$ is compact and open in $\widehat{G}$ in the hull kernel topology.

This follows directly from 3.3.2 and 3.3.7 in [9]. The first lemma (3.3.2) actually shows that the map $\pi \mapsto \|\pi(f)\|$ is lower semi-continuous, that is, $\{\pi \in \widehat{G} : \|\pi(f)\| > \alpha\}$ is open in $\widehat{G}$ for each $\alpha > 0$. That $\operatorname{supp}(f) = \{\pi \in \widehat{G} : \pi(f) \neq 0\} = \{\pi \in \widehat{G} : \|\pi(f)\| > 0\}$ is open is an immediate consequence. The second lemma (3.3.7) shows compactness. There are always two compact open sets in $\widehat{G}$: the empty set and $\widehat{G}$. We consider these trivial. Lemma 11 tells us that if there are no non-trivial compact opens in $\widehat{G}$ then there are no non-trivial projections in $\ell^1(G)$.

It would be nice if the compact open subsets of $\widehat{G}$ were in perfect correspondence with the projections in $\ell^1(G)$. We will see that this is not the case. There are usually many compact open subsets that are not the support set of a projection and also many projections which share the same support set. The abelian case, however, is special [15]:

Theorem 12. If G is abelian then there is a bijection between the projections in $\ell^1(G)$ and the compact open sets in $\widehat{G}$.

Furthermore, when G is compact, Taylor and Kaniuth showed how to construct a projection whose support set is any chosen singleton set in $\widehat{G}$, as a simple consequence of the orthogonality relations for irreducible representations of a compact group. Recall that the dual space of a compact group is discrete [9] and so all singletons are both compact and open.

Theorem 13. Let G be compact and $\pi \in \widehat{G}$. Let $\xi \in \mathcal{H}_\pi$ such that $\|\xi\| = \sqrt{d_\pi}$. Define $f_\xi(x) = \langle \xi, \pi(x)\xi \rangle$ for $x \in G$. Then the following hold:

(i) f_ξ is a projection in $L^1(G)$

(ii) The support set of f_ξ is $\{\pi\}$

(iii) $\pi(f_\xi)$ is the rank one projection of $\mathcal{H}_\pi$ onto $\mathbb{C}\xi$.

As mentioned earlier, Kaniuth and Taylor also gave a method of constructing projections in the case that $G = A \rtimes H$ where A is abelian, H is σ-compact and there is an open free H orbit in $\widehat{A}$. With this set-up, H acts on $\widehat{A}$ through conjugation. They showed that there is an open point in $\widehat{G}$ and constructed a projection in $L^1(G)$ whose support set is the singleton set consisting of this point [15].

In this paper, we study projections in $\ell^1(G)$ when G is a wallpaper group. Roughly, a wallpaper group is the symmetry group of a wallpaper pattern (imagining that the pattern covers the entire plane). The wallpaper groups will be discussed in detail in Chapter 4. The known results just presented do not apply to the wallpaper groups. The wallpaper groups are not abelian (except for the trivial wallpaper p1, which consists purely of translations), they are not finite, they do not always split as a semi-direct product of an abelian group with a σ-compact group, and even if they do split in such a way, H does not have an open free orbit in $\widehat{A}$. Nonetheless, the wallpaper groups are very close to each of these types of groups. They have a normal abelian subgroup whose quotient in G is finite. Many wallpaper groups do split as the semi-direct product $\mathbb{Z}^2 \rtimes D$ where D is a finite point group. We thus expect to obtain somewhat analogous results on projections in $\ell^1(G)$ for the wallpaper groups.

One immediate result is that projections can be constructed from finite subgroups.

1.2.1 Projections from Finite Subgroups

In this section we present a simple method of constructing projections in $\ell^1(G)$ using finite subgroups of G.

Lemma 14. Let K be a finite subgroup of G. Then $f_K = \frac{1}{|K|}\sum_{g\in K}\delta_g$ is a projection in $\ell^1(G)$, where δ_g is the function on G that takes the value 1 at g and 0 elsewhere.

Proof. We must check that f_K is an idempotent, is self-adjoint and is an element of $\ell^1(G)$.

1. f_K is an idempotent.

For any $g, k \in G$ we have that $\delta_g * \delta_k = \delta_{gk}$. Then,

$$\begin{aligned} f_K * f_K &= \left(\frac{1}{|K|}\sum_{g\in K}\delta_g\right) * \left(\frac{1}{|K|}\sum_{g\in K}\delta_g\right) \\ &= \frac{1}{|K|^2}\sum_{g\in K}\sum_{k\in K}\delta_g * \delta_k \\ &= \frac{1}{|K|^2}\sum_{g\in K}\sum_{k\in K}\delta_{gk} \\ &= \frac{1}{|K|}\sum_{g\in K}\delta_g \end{aligned}$$

with the last equality occuring because for any $h \in K$ there are $|K|$ pairs (g, k) such that $gk = h$. If $h \notin K$ then $gk \neq h$ for all $g, k \in K$.

2. f_K is self-adjoint.

For any $g \in G$, $(\delta_g)^* = \delta_{g^{-1}}$. Then,

$$\begin{aligned}(f_K)^* &= \left(\frac{1}{|K|}\sum_{g\in K}\delta_g\right)^* \\ &= \frac{1}{|K|}\sum_{g\in K}(\delta_g)^* \\ &= \frac{1}{|K|}\sum_{g\in K}\delta_{g^{-1}} \\ &= \frac{1}{|K|}\sum_{g\in K}\delta_g\end{aligned} \tag{1.2}$$

with the last equality occurring because K is a group.

3. $f_K \in \ell^1(G)$.

$$\begin{aligned}\|f_K\|_1 &= \sum_{g\in G}|f_K(g)| \\ &= \sum_{g\in G}\frac{1}{|K|}\sum_{k\in K}\delta_k(g) \\ &= \frac{1}{|K|}\sum_{k\in K}\sum_{h\in K}\delta_k(h) \\ &= \frac{1}{|K|}\sum_{k\in K}1 \\ &= 1\end{aligned}$$

□

We next show that one can multiply f_K by a matrix coefficient of a representation of K to obtain another projection. We will need the Schur orthogonality relations for matrix coefficients of representations of finite groups.

Lemma 15. (Schur orthogonality relations) Let G be a finite group and $\pi, \eta \in \widehat{G}$. Let $u, v \in \mathcal{H}_\pi$ and $u_0, v_0 \in \mathcal{H}_\eta$. Then

$$\int_G \langle \pi(x)u, v\rangle \overline{\langle \eta(x)u_0, v_0\rangle}\, d\mu(x) = \begin{cases} \frac{1}{d_\pi}\langle u, u_0\rangle\overline{\langle v, v_0\rangle} & \text{if } \pi \cong \eta \\ 0 & \text{else}\end{cases}$$

where μ is the normalized Haar measure of G (i.e. $\mu(G) = 1$).

Lemma 16. Let K be a finite subgroup of a locally compact group G. Let $\pi \in \widehat{K}$ and $\xi \in \mathcal{H}_\pi$ with $\|\xi\|_2 = 1/\sqrt{d_\pi}$. Define $c_\xi^\pi : G \to \mathbb{C}$ by $c_\xi^\pi(x) = \langle \pi(x)\xi, \xi \rangle$ when $x \in K$ and 0 otherwise. Then $c_\xi^\pi f_K$ is a projection in $\ell^1(G)$.

Proof. To simplify the notation, let $c = c_\xi^\pi$. Note that:

$$\begin{aligned}(c\delta_x * c\delta_y)(z) &= \sum_{w\in G} c(w)\delta_x(w)c(w^{-1}z)\delta_y(w^{-1}z) \\ &= c(x)c(y)\delta_{xy}\end{aligned}$$

Then using this and linearity, we get that

$$\begin{aligned}cf_K * cf_K &= \frac{1}{|K|^2}(\sum_{k\in K} c\delta_k) * (\sum_{h\in K} c\delta_h) \\ &= \frac{1}{|K|^2}\sum_{k\in K}\sum_{h\in K} c\delta_k * c\delta_h \\ &= \frac{1}{|K|^2}\sum_{k\in K}\sum_{h\in K} c(k)c(h)\delta_{kh} \\ &= \frac{1}{|K|^2}\sum_{k\in K}\sum_{h\in K} c(k)c(k^{-1}h)\delta_h \\ &= \frac{1}{|K|}\sum_{h\in K}\delta_h\sum_{k\in K} c(k)c(k^{-1}h)\end{aligned}$$

Consider

$$\begin{aligned}\frac{1}{|K|^2}\sum_{k\in K} c(k)c(k^{-1}h) &= \frac{1}{|K|^2}\sum_{k\in K}\langle \pi(k)\xi, \xi\rangle\langle \pi(k^{-1}h)\xi, \xi\rangle \\ &= \frac{1}{|K|^2}\sum_{k\in K}\langle \pi(k)\xi, \xi\rangle\overline{\langle \pi(k)\xi, \pi(h)\xi\rangle}\end{aligned}$$

Applying the Schur orthogonality relations (recalling that they require that the Haar measure of the finite group be normalized),

$$\begin{aligned}\frac{1}{|K|^2}\sum_{k\in K}\langle \pi(k)\xi, \xi\rangle\overline{\langle \pi(k)\xi, \pi(h)\xi\rangle} &= \frac{1}{|K|}\langle \pi(h)\xi, \xi\rangle \\ &= \frac{1}{|K|}c(h)\end{aligned}$$

Thus $cf_K * cf_K = \frac{1}{|K|}\sum_{h\in K} c(h)\delta_h = cf_K$.

Note that when K is abelian, irreducible representations are one-dimensional and are thus homomorphisms $\chi : K \to \mathbb{T}$ (as $\mathbb{T}$ is the set of unitary operators on $\mathbb{C}$ through multiplication). Then for any $\xi \in \mathbb{C}$ with $|\xi| = 1$, we have $c_\chi^\xi(k) = \langle \chi(k)\xi, \xi \rangle = \chi(k)\langle \xi, \xi \rangle = \chi(k)$. So $c_\chi^\xi f_K = \chi f_K$. In this case, calculating $\chi f_K * \chi f_K$ does not actually require the Schur orthogonality relations, since χ jumps out of the inner product and is multiplicative:

$$\begin{aligned}
\chi f_K * \chi f_K &= \frac{1}{|K|^2}(\sum_{k \in K} \chi(k)\delta_k) * (\sum_{h \in K} \chi(h)\delta) \\
&= \frac{1}{|K|^2} \sum_{k \in K} \sum_{h \in K} \chi(k)\chi(h)\delta_k * \delta_h \\
&= \frac{1}{|K|^2} \sum_{k \in K} \sum_{h \in K} \chi(kh)\delta_{kh} \\
&= \frac{1}{|K|^2} \sum_{k \in K} \sum_{h \in K} \chi(h)\delta_h \\
&= \frac{1}{|K|} \sum_{h \in K} \chi(h)\delta_h \\
&= \chi f_K
\end{aligned}$$

Now check that cf_K is self-adjoint:

$$\begin{aligned}
(cf_K)^*(x) &= \overline{c(x^{-1} f_K(x^{-1})} \\
&= \overline{c(x^{-1}} f_K(x) \\
&= \overline{\langle \pi(x{-}1)\xi, \xi \rangle} f_K(x) \\
&= \overline{\langle \pi(x)^*\xi, \xi \rangle} f_K(x) \\
&= \overline{\langle \xi, \pi(x) \rangle} f_K(x) \\
&= \langle \pi(x)\xi, \xi \rangle f_K(x) \\
&= c(x) f_K(x)
\end{aligned}$$

(where we've used the fact that $f_K^* = f_K$.)

Again, if K is abelian and χ is a character of K the calculation is simpler:

$$\begin{aligned}(\chi f_K) * (x) &= \overline{\chi(x^{-1}) f_K(x^{-1})} \\ &= \overline{\chi(x^{-1})} f_K(x) \\ &= \chi(x) f_K(x)\end{aligned}$$

Finally, it's clear that f_K, cf_K and χf_K are absolutely summable since their support in G is finite. Thus, each of these is a projection in $\ell^1(G)$. □

We call a projection whose support in G is a finite subgroup a *finite subgroup projection*. Usually, these are not the only kind of non-trivial projections in $\ell^1(G)$ (if any). We will show this for the wallpaper group $p2$ in Chapter 5. In fact, we will construct a projection whose support set is not the support set of a projection of the form cf_K.

This summarizes known results on projections in $\ell^1(G)$ that apply to the wallpaper groups. To our knowledge, compact open sets in the dual space of a group have not been extensively studied, besides in [14] where E. Kaniuth and K.F Taylor characterized the compact open subsets in the dual space of an $[FC]^-$ group. These are groups whose conjugacy classes are finite. They include the abelian groups and finite groups, but again, not the non-trivial wallpaper groups. You can see this even with the very uncomplicated wallpaper group, p2. In the next chapter we define the wallpaper groups and present some relevant results and examples.

Chapter 2

The Wallpaper Groups

"Symmetry, as wide or narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection."

–Hermann Weyl

2.1 Basic Concepts and Definitions

We begin with a quick overview of isometries of $\mathbb{R}^n$. In general, an isometry is a distance-preserving map between metric spaces. For $\mathbb{R}^n$ (and actually, any finite-dimensional inner product space) an isometry is a bijection. To see this, first note that an isometry T that fixes the origin preserves the inner product (expand $\langle Tx - Ty, Tx - Ty\rangle = \langle x - y, x - y\rangle$ on both sides and simplify). Using linearity of the inner product one can show that T is linear. Furthermore, since T preserves distance, it must be injective. Therefore T will map an orthonormal basis of $\mathbb{R}^n$ to an orthonormal basis. Being linear, this shows that T is bijective. Now any isometry can be translated so that it fixes the origin - and we're done. Because isometries map $\mathbb{R}^n$ onto itself without changing distances they are also called "rigid motions".

Denote the collection of isometries of $\mathbb{R}^n$ by $Iso(n)$. This space has an associative product, namely, composition. With respect to this product, the identity map is an identity. It is then easy to check that the inverse of an isometry in $Iso(n)$ is also in $Iso(n)$ and that the same goes for the composition of two isometries. Thus $Iso(n)$ is a group. As noted above, an isometry may be composed with a translation (also an isometry) such that the composition fixes the origin and is consequently linear. By definition, this distance-preserving, origin-fixing map is in the n-dimensional orthogonal group, $O(n)$. The matrix form of a member of the orthogonal group is an $n \times n$ orthogonal matrix. That is, each $U \in O(n)$ satisfies $UU^T = U^TU = I$, where U^T denotes the transpose of U. So every $T \in Iso(n)$ has the form $T = \tau_v \circ U$ where τ_v is translation by $v \in \mathbb{R}^n$ and $U \in O(n)$. We could switch the order of the composition by writing $T = U \circ (U^{-1} \circ \tau_v \circ U)$, noting that $U^{-1} \circ \tau_v \circ U$ is a translation. We will use the following notation for elements of $Iso(n)$:

Definition 17. For $M \in O(n)$ and $v \in \mathbb{R}^n$, let $[M, v]$ denoted the map defined by:

$$[M, v]x = M \circ \tau_v(x) = M(x + v)$$

Lemma 18. Each $[M, v]$ is an isometry. Thus $Iso(n) = \{[M, v] : M \in O(n), v \in \mathbb{R}^n\}$. We say M is the orthogonal part and v is the translational part of $[M, v]$.

The previous lemma shows that as a set, $Iso(n)$ is just $O(n) \times \mathbb{R}^n$. As a group, it is not the same, however. One can calculate that $[M, v][A, u] = [MA, u + A^{-1}v]$. So addition involves a twist in the translational part. Despite these differences, $Iso(n)$ is conventionally given the topology of $O(n) \times \mathbb{R}^n$. Recall that the topology on $O(n)$ is inherited by viewing it as a subset of $\mathbb{R}^{n^2}$ (simply reshape $n \times n$ matrices into vectors of length n^2).

A left group action of a group G on a set X is a map $\phi : G \times X \to X$ (we always write $g \cdot x = \phi(g, x)$) which satisfies:

(i) (Identity) $e \cdot x = x$ for all $x \in X$, where e is the identity of G.

(ii) (Compatibility) $g \cdot (h \cdot x) = (gh) \cdot x$.

The group $Iso(n)$ acts on $\mathbb{R}^n$ via point evaluation: $[M, v] \cdot x = [M, v]x = M(x + v)$. To check, note that $[I, 0]$ is the identity of $Iso(n)$ and $[I, 0]x = I(x + 0) = x$. For compatibility:

$$\begin{aligned}[M,v]\cdot([A,u]\cdot x) &= [M,v](A(x+u)) \\ &= M(Ax+Au+v) \\ &= MAu+MAx+MAA^{-1}v \\ &= MA(x+u+A^{-1}v) \\ &= [M,v][A,u]x\end{aligned}$$

A group action gives rise to an equivalence relation. For each $x \in X$, the orbit of x in G is the set $O_G(x) = \{g\cdot x : g \in G\}$. Then define $x \sim y$ if $y \in O_G(x)$. To show that this is an equivalence relation, use the following facts: $x = e \cdot x$, $y = g \cdot x \implies x = g^{-1} \cdot y$, $y = g \cdot x \wedge z = h \cdot y \implies z = hg \cdot x$. Clearly the equivalence classes are the $G-$orbits, i.e. $[x] = O_G(x)$. For $Iso(n)$ acting on $\mathbb{R}^n$ the orbit of $x \in \mathbb{R}^n$ is the image of x under the isometries of $\mathbb{R}^n$: $O_G(x) = \{[M,v]x : M \in O(n), v \in \mathbb{R}^n\}$. Since the group of translations is in $Iso(n)$, each $O_{Iso(n)}(x)$ is all of $\mathbb{R}^n$. Note however that the orbits of a subgroup of $Iso(n)$ may not be trivial. Take, for instance the subgroup of translations in $\mathbb{R}^2$ defined by $\mathcal{L} = \{[I,(a,b)] : (a,b) \in \mathbb{Z}^2\}$. The orbit of a point $x \in \mathbb{R}^2$ is $O_{\mathcal{L}}(x) = \{x+(a,b) : (a,b) \in \mathbb{Z}^2\}$, which is a discrete subset of $\mathbb{R}^2$. The quotient space corresponding to a group G acting on X is the set $X/G = \{O_G(x) : x \in X\}$, the set of orbits. The associated quotient map is $q : x \mapsto O_G(x)$. When X is a topological space, the quotient space carries the quotient topology. This is the topology on X/G where the open sets are exactly those sets whose preimage under the quotient map is open in X.

We are now ready to define the main item of interest, the crystallographic groups.

Definition 19. A *crystallographic group* of dimension n is a discrete subgroup of $Iso(n)$ with the property that $\mathbb{R}^n/G$ is compact.

We immediately restrict ourselves to the case that $n = 2$. The crystallographic groups of dimension 2 are called the *wallpaper groups*. The name comes from the fact that each such group is the symmetry group of a wallpaper pattern. That is, the group of isometries of $\mathbb{R}^2$ that each leave a wallpaper pattern on $\mathbb{R}^2$ looking the same. As an example, consider a pattern with only translational symmetry. A portion of such a pattern on $\mathbb{R}^2$ is shown in Figure 2.1.

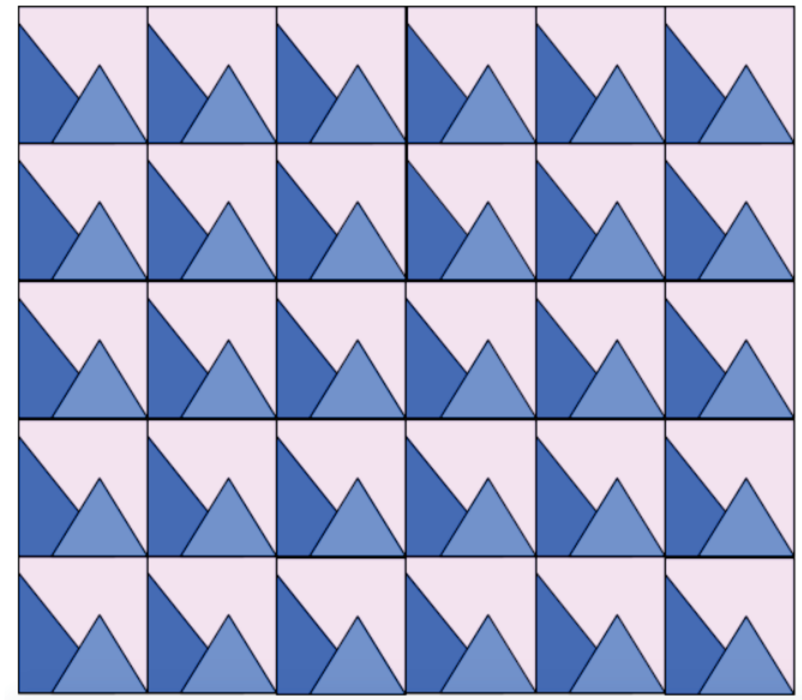

Figure 2.1: Portion of a wallpaper pattern with purely translational symmetry.

Recall that a discrete topological space is a space in which every subset is both open and closed. The discreteness condition for the wallpapers is actually very strong. For instance, a wallpaper group G being discrete forces each G-orbit in $\mathbb{R}^2$ to be discrete [10]:

Lemma 20. Let G be a crystallographic group. Then for each $a \in \mathbb{R}^n$, the orbit $O_G(a) = G \cdot a = \{x(a) : x \in G\}$ is discrete in $\mathbb{R}^n$.

This implies, for instance, that there must be a translation in G of minimal length, otherwise orbits would contain a cluster point. The collection of pure translations in G (elements of the form $[I, x]$ where $x \in \mathbb{R}^2$) forms a normal abelian subgroup. We call this the lattice $\mathcal{L}$ and think of it as a discrete subgroup of $\mathbb{R}^2$. Consider $\mathbb{R}^2/\mathbb{R}x$, where $y \sim z \iff y = z + \alpha x$ for some $\alpha \in \mathbb{R}$. This is isomorphic to $\mathbb{R}$. It's not hard to show that $\mathcal{L}/\mathbb{R}x \cap \mathbb{R}^2/\mathbb{R}x$ must also be discrete [10](p.16) and so is isomorphic to a discrete subgroup of $\mathbb{R}$. This leads to the following lemma:

Theorem 21. (Bieberbach) The lattice of a wallpaper group is isomorphic to $\mathbb{Z}^2$.

This is the first part of Bieberbach's theorem restricted to the case of the 2-dimensional crystallographic groups. A pair of vectors in $\mathbb{R}^2$ which generate the lattice is called a lattice basis. From now on we assume that group elements $[M, x]$ are written in terms of a lattice basis. The lattice is not only the largest abelian [10] subgroup of G but is also normal. To see this, first note that inverses are calculated as $[M, x]^{-1} = [M^{-1}, -Mx]$. Then $[M, x][I, a][M^{-1}, -Mx] = [M, a + x][M^{-1}, -Mx] = [I, M(a+x) - Mx] = [I, Ma]$. So in conjugating a translation by an affine transformation $[M, x]$, it is only the matrix part M that comes into play. Now $[M, x]$ and $[A, y]$ are in the same $\mathcal{L}$-coset if and only if $M = A$. The product of two equivalence classes in $G/\mathcal{L}$ corresponds to multiplication of their (unique) matrix parts. Representatives for the equivalence classes of $G/\mathcal{L}$ may be chosen as the matrix parts of the elements of G. This is called the point group for G, as each matrix part is in $O(2)$ and so fixes the origin. The second part of Bieberbach's theorem (in the case of the wallpaper groups) is the following:

Theorem 22. (Bieberbach) The point group $D = G/\mathcal{L}$ of a wallpaper group is finite.

The point group acts on $\mathcal{L}$ by conjugation. For $M \in D$ and $[I, x] \in \mathcal{L}$ choose $a \in \mathbb{Z}^2$ such that $[M, a] \in G$ and define $M \cdot [I, x] = [M, a][I, x][M^{-1}, -Ma] = [I, Mx]$. As noted above, it does not matter which $a \in \mathbb{Z}^2$ is used. Often we will view the lattice as $\mathbb{Z}^2$ via the map $[I, x] \mapsto x$. Then the action of the point group on $\mathbb{Z}^2$ becomes $M \cdot x = Mx$ for $M \in D$ and $x \in \mathbb{Z}^2$.

What is the point group of a wallpaper group like? Again, the discreteness of G forces strong conditions. One is the famous crystallographic restriction theorem: the rotations in D can only be of order $1, 2, 3, 4$ or 6. In fact there are only two kinds of finite groups of isometries of $\mathbb{R}^2$. This was known by Leonardo da Vinci, a while back.

Theorem 23. (Leonardo's Theorem) Let D be a finite group of isometries of the plane. Then D is either a cyclic group C_n of order $n \in \mathbb{Z}$ or a dihedral group D_{2n} of order $2n$, $n \in \mathbb{Z}$. No two of the groups $C_1, C_2, \ldots, D_1, D_2, \ldots$ are conjugate under an isometry.

Contrast groups being conjugate under an isometry with group equivalence. These are not the same. For instance, the groups C_2 and D_1 are isomorphic as groups but not conjugate under an isometry. The first consists of identity and rotation by 180$\circ$ while the latter contains a reflection. A proof of Leonardo's theorem may be found in [2]. Recall that C_n is the cyclic abelian group of order n, $\mathbb{Z}_n$. As a group of isometries, one can think of C_n as the group generated by a rotation by $360°/n$. The dihedral group D_{2n} is generated by C_n and a reflection. It has a crossed product structure $C_n \rtimes \{I, -I\}$ with the action of $\{I, -I\}$ defined by $-I \cdot R^m = R^{-m}$. The dihedral group D_{2n} is the symmetry group of a regular n-gon. Combining Leonardo's theorem with the crystallographic restriction, we see that the possible point groups of a wallpaper group are $C_1, C_2, C_3, C_4, C_6, D_2, D_4, D_6, D_8, D_{12}$. Each of these is the point group of at least one wallpaper group (see the wallpaper group descriptions in [19]). Note that it is possible for two wallpaper groups to have the same point group and yet be distinct groups. This happens because, in general, there are multiple ways a point group can act on the lattice.

The maximal normal abelian subgroup (the lattice) and the quotient of the group by this lattice (the point group) are actually defining features of a wallpaper group. Zassenhaus showed that the wallpaper groups are exactly the discrete groups that contain a finite index, normal, free abelian subgroup of rank 2, that is also maximal abelian [25]. In other words, a discrete group G is a wallpaper group if and only if there is a short exact sequence $0 \to \mathbb{Z}^2 \to G \to D \to 1$ where $\mathbb{Z}^2$ is maximal abelian and $[G : \mathbb{Z}^2]$ is finite.

Up to group isomorphism, there are exactly 17 distinct wallpaper groups. In fact, the number of n-dimensional crystallographic groups is finite for any n. This is the solution to part of Hilbert's 18^{th} problem, which was proved by Bieberbach in 1912. A proof that there are exactly 17 wallpaper groups may be found in [22]. Below is a table listing the wallpaper groups along with their point groups.

Wallpaper group	Point group
p1	C_1
p2	C_2
p3	C_3
p4	C_4
p6	C_6
cm, pm, pg	D_2
cmm, pmm, pmg, pgg	D_4
p31m, p3m1	D_6
p4mm, p4mg	D_8
p6mm	D_{12}

To get a sense of the connection between Definition 19 and the symmetry groups of crystal structures, consider the pattern in Figure 2.2 and imagine that it covers the entire plane. One can see that its symmetry group G is generated by a horizontal translation, a vertical translation, and a rotation by 180°. This wallpaper group is called p2 and will be revisited in Chapter 5.

Consider the square in Figure 2.3. Removing the top and right sides of the square gives a set of representatives for the equivalence classes in $\mathbb{R}^2/G$. For the topology of $\mathbb{R}^2/G$, note that the left side of the square is mapped to the right side by a horizontal translation. Thus, these two edges are identified. The rotation maps the left half side to the right half side of both the top and bottom edges. These identifications are shown in Figure 2.3. The vertical dotted line shows where one would fold in order to

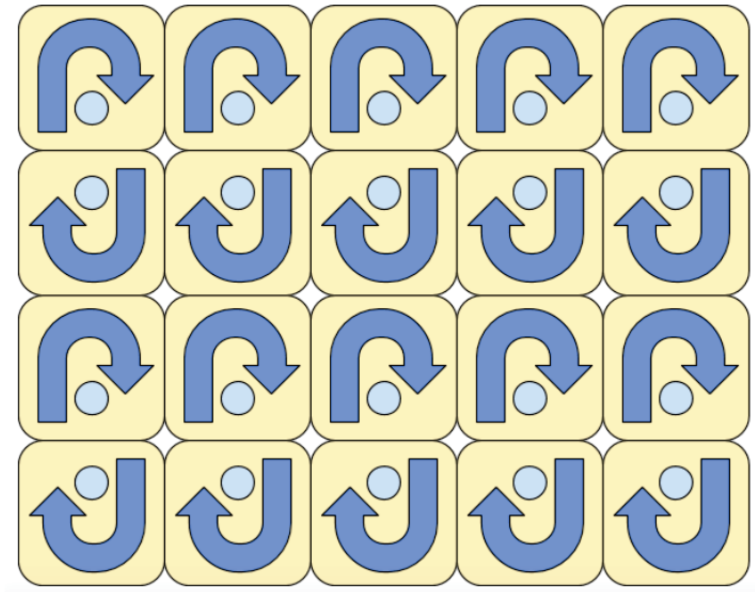

Figure 2.2: A wallpaper pattern with p2 symmetry.

glue the identified edges. Arrows mark these edges. Gluing these identified edges, we see that $\mathbb{R}^2/G$ is topologically isomorphic to a sphere. So $\mathbb{R}^2/G$ is indeed compact, as required in the definition of wallpaper group. You can also see in Figure 2.2 that each of the G-orbits in $\mathbb{R}^2$ is a discrete set.

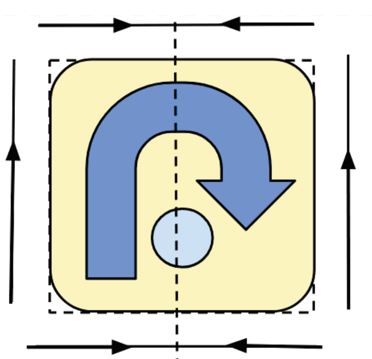

Figure 2.3: A picture of $\mathbb{R}^2/G$ with edge identifications.

A cross section of D in G is a map $\gamma : D \to G$ such that $q \circ \gamma = id_D$ where q is the quotient map $q : G \to D$. This allows us both to represent G by its $\mathbb{Z}^2$-coset decomposition and to express the action of D on $\mathbb{Z}^2$. With a cross section γ fixed, for any $x \in G$ there is a unique $d \in D$ and translation $a \in \mathbb{Z}^2$ such that $x = \gamma(d)a$. D acts on $\mathbb{Z}^2$ by conjugation:

$$d \cdot x = \gamma(d)x\gamma(d)^{-1}$$

Since γ is a cross section and the quotient map is a homomorphism, we know that $\gamma(c)\gamma(d)$ and $\gamma(cd)$ lie in the same $\mathbb{Z}^2$ coset. Thus there is some $\alpha(c,d) \in \mathbb{Z}^2$ such that $\gamma(c)\gamma(d) = \gamma(cd)\alpha(c,d)$. Doing this for each pair $c, d \in D$ we obtain a map $\alpha : D \times D \to \mathbb{Z}^2$. Using the identity $\alpha(b,c) = \gamma(bc)^{-1}\gamma(b)\gamma(c)$, one can see that α satisfies the following:

$$\alpha(b, cd)\alpha(c, d) = \alpha(bc, d)(d^{-1} \cdot \alpha(b, c))$$

This is called the 2-cocycle identity and α, a 2-cocycle.
Using the coset decomposition, we can express elements of G as $(d, x) = \gamma(d)x$ where $d \in D$ and $x \in \mathbb{Z}^2$. In this notation the group product becomes:

$$\begin{aligned}(c, x)(d, y) &= \gamma(c)x\gamma(d)y \\ &= \gamma(c)\gamma(d)(d^{-1} \cdot x)y \\ &= \gamma(cd)\alpha(c, d)(d^{-1} \cdot x)y \\ &= (cd, \alpha(c, d)(d^{-1} \cdot x)y)\end{aligned}$$

In order to work with explicit wallpaper groups we need to express the elements of G in terms of $x \in \mathcal{L}$ and $\gamma(d) \in G$. Explicit descriptions of the 17 wallpaper groups (along with their C^*-algebras and irreducible *-representations) were worked out in [19]. We will use these descriptions in our examples throughout.

A character of an abelian group is a homomorphism into the circle group $\mathbb{T}$. The characters of an abelian group form its dual space, which in this case is actually a group. Recall that the lattice of a wallpaper group is isomorphic to $\mathbb{Z}^2$. A character $\chi : \mathbb{Z}^2 \to \mathbb{T}$ is of the form $\chi_{z,w}$ where $(z, w) \in \mathbb{T}^2$ and:

$$\chi_{z,w}(x, y) = z^x w^y$$

Because of this, we will refer to $\widehat{\mathbb{Z}^2}$ as simply $\mathbb{T}^2$ and let $(z, w)^{(x,y)} = \chi_{z,w}(x, y) = z^x w^y$. The point group D also acts on $\widehat{\mathbb{Z}^2} \cong \mathbb{T}^2$. This action is defined by:

$$(d \cdot \chi_{z,w})(a, b) = \chi_{z,w}(d^{-1} \cdot (a, b))$$

We apply the discussion in this chapter thus far to two example wallpaper groups, p2 and pg.

Example 24. First consider **p2**, the symmetry group of the wallpaper pattern shown in Figure 2.2. Two linearly independent translations and a rotation by 180° generate this wallpaper group. Viewing $p2$ as a group of affine transformations, each element is of the form $[I, \underline{x}]$ or $[-I, \underline{x}]$ where $x \in \mathbb{Z}^2$. The point group D is isomorphic to $\mathbb{Z}_2 = \{1, -1\}$. One cross section is $\gamma(1) = [I, (0, 0)]$, $\gamma(-1) = [-I, (0, 0)]$ (in the future, we will always assume that $\gamma(id_D) = [I, (0, 0)]$). The D action on $\mathbb{Z}^2$ is then $-1 \cdot x = -x$. Note that the action of the identity of D is always trivial. For the chosen cross section γ, the 2-cocycle α is trivial (i.e. α takes the value 1 on all pairs of elements of D). This is because γ is a homomorphism: $\gamma(c)\gamma(d) = \gamma(cd)$ for all $c, d \in D$. In general, when γ is a homomorphism, D is isomorphic to a subgroup of G. When this occurs, G can be expressed as a semi-direct product $G = \mathbb{Z}^2 \rtimes D$. Note that the action of D on $\mathbb{Z}^2$ must also be defined in order for $\mathbb{Z}^2 \rtimes D$ to make sense as a group.

Example 25. Not all of the wallpaper groups split as a semi-direct product. Consider **pg**, the wallpaper generated by the lattice and one glide reflection. A portion of a pg wallpaper is shown in Figure 2.4. We can choose the glide reflection to be along the x-axis. A cross section for the action of D on pg is $\gamma(\sigma) = [\sigma, (\frac{1}{2}, 0)]$ where $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The D action is then $\sigma \cdot x = \sigma x$. Since $\gamma(\sigma)\gamma(\sigma) = [I, (1, 0)] \neq [I, (0, 0)]$, pg does not split as a semi-direct product $\mathbb{Z}^2 \rtimes D$. In fact, a wallpaper group splits if and only if it does not contain a glide reflection. Only four out of the 17 wallpapers do not split.

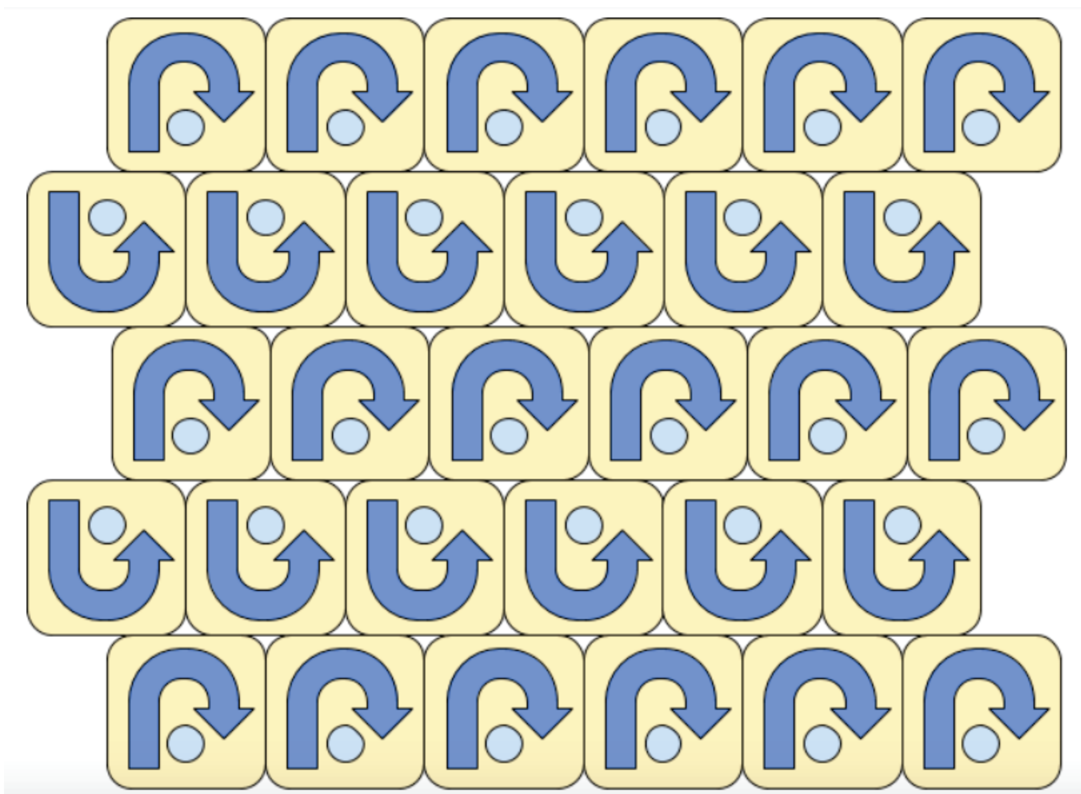

Figure 2.4: A wallpaper pattern with pg symmetry.

Chapter 3

Representation Theory

In the next section we discuss representation theory in greater depth and use this to work out an explicit description of the dual space of a wallpaper group. We begin by reviewing the representation theory of finite-dimensional C^*-algebras and introducing C^*-bundles and their representation theory. Next we present an explicit description of the C^*-algebra of a wallpaper group. Finally, we describe in detail the dual space of the C^*-algebra of a wallpaper group and its topology.

3.1 The Dual Space of a Finite-Dimensional C^*-algebra

A finite-dimensional C^*-algebra is a C^*-algebra which has a finite basis, when viewed as a normed vector space (i.e., when ignoring the product and * operations). Specifically, we will consider finite-dimensional C^*-subalgebras of $M_n(\mathbb{C})$, the algebra of $n \times n$ matrices with entries from $\mathbb{C}$ and we assume that the subalgebras contain the identity matrix. Note that $M_n(\mathbb{C})$ is itself a C^*-algebra. The $*$-operation here is defined by defining the adjoint of a matrix to be its conjugate transpose. Multiplication is the usual matrix multiplication. The norm is the operator norm, viewing matrices in $M_n(\mathbb{C})$ as linear operators on $\mathbb{C}^n$. Since $M_n(\mathbb{C})$ is finite-dimensional, so are all of its subalgebras.

Let $\mathcal{A}$ be a C^*-subalgebra of $M_n(\mathbb{C})$. Consider projections P, Q in $\mathcal{A}$. We say that $P \leq Q$ if $PQ = P$. With this relation, the projections in $\mathcal{A}$ form a lattice. That is, every pair of projections has a least upper bound and a greatest lower bound.

A projection P is called minimal if $P \geq Q$ implies $Q = 0$ or $Q = P$. Using the *-subalgebra generated by a non-zero operator $T \in \mathcal{A}$ and the spectral theorem of normal operators, one can show that $\mathcal{A}$ has at least one non-trivial projection, except in the degenerate case that $\mathcal{A} = \{0\}$. Furthermore, every projection dominates a minimal projection. Recall that the commutant of the sub-algebra $\mathcal{A}$ of $M_n(\mathbb{C})$ is the set of matrices of $M_n(\mathbb{C})$ that commute with every element of $\mathcal{A}$. The commutant of $\mathcal{A}$ is denoted by $\mathcal{A}'$. The centre of $\mathcal{A}$ is defined as the intersection of $\mathcal{A}$ and its commutant: $Z(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'$. These are the elements within $\mathcal{A}$ that commute with $\mathcal{A}$.

The first step is to characterize $\mathcal{A}$ when $\mathcal{A}$ is a factor, that is, if the centre of $\mathcal{A}$ is trivially $\mathbb{C}I = \{\alpha I : \alpha \in \mathbb{C}\}$, where I is the identity of $\mathcal{A}$. This is accomplished by finding a set of minimal projections $P_1, \ldots, P_k$ in $\mathcal{A}$ that sum to I. These give a system of matrix units V_{ij} (V_{ij} is a partial isometry with the property that $V_{ij}V_{ij}^* = P_i$ and $V_{ij}^*V_{ij} = P_j$) such that $P_i\mathcal{A}P_j = \alpha_{ij}V_{ij}$ for some $\alpha_{ij} \in \mathbb{C}$. The map $\mathcal{A} \mapsto \{\alpha_{ij}\}_{1 \leq i,j \leq k}$ defines an isomorphism of $\mathcal{A}$ with $M_k(\mathbb{C})$. Using a set of minimal projections $Q_1, \ldots, Q_m$ in the centre of $\mathcal{A}$, one can block diagonalize $\mathcal{A}$. Doing this, one finds that each element of $\mathcal{A}$ diagonalizes to a matrix consisting of a smaller matrix in $M_k(\mathbb{C})$ repeated m times along the diagonal and zeros elsewhere.

When $\mathcal{A}$ is not a factor, take a set of central minimal projections $C_1, \ldots, C_\ell$ that sum to I. Then each $C_i\mathcal{A}$ is a factor. We thus obtain a decomposition of $\mathcal{A}$. This will be especially useful in working out the dual space of $C^*(G)$, when we view it as the sections of a fibre bundle as the fibres are finite-dimensional matrix algebras. There are $\ell \in \mathbb{N}$, $k_i \in \mathbb{N}$ for $i = 1, \ldots, \ell$ such that for each element $A \in \mathcal{A}$, there are matrices $A_{k_i} \in M_{k_i}(\mathbb{C})$ for $i = 1 ... \ell$ such that:

$$A = \begin{bmatrix} M_1 & 0 & \dots & \dots & 0 \\ 0 & M_2 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \dots & \dots & 0 & M_\ell \end{bmatrix} \text{ where } M_i = \begin{bmatrix} A_{k_i} & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & A_{k_i} \end{bmatrix} \in M_{a_i k_i}(\mathbb{C}) \tag{3.1}$$

Specifically, k_i the number of minimal projections in the factor $C_i\mathcal{A}$, a_i the number of central minimal projections in $\mathcal{A}'$, and ℓ the number of minimal central projections in $\mathcal{A}$. For further details, see [8].

This decomposition determines the structure of the dual space of $\mathcal{A}$. The key insight is that the dual space of a full matrix algebra, i.e. $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, is trivial. There is only one equivalence class of irreducible representations: the class containing the representation $\mathcal{I}(M) = M$. Note that this is indeed a representation of M. The Hilbert space of $\mathcal{I}$ is $\mathbb{C}^n$ and M is the operator $M(x) = Mx$ (multiplication of x by M). For $\mathcal{A}$, fix k_i and consider the map $\pi_i : A \mapsto A_{k_i}$. Since A_{k_i} is a block on the diagonal of A (assume A has been block diagonalized), π_i is linear and preserves multiplication and the *-operation. Furthermore, A_{k_i} is an operator on the Hilbert space C^{k_i}. Thus π_i is a representation of $\mathcal{A}$. Since the set $\{\pi_i(A) : A \in \mathcal{A}\}$ is M_{k_i}, there are no π_i-invariant subspaces of $\mathbb{C}^{k_i}$. So π_i is irreducible. In fact, the representations π_i, $i = 1, \dots, \ell$ are a set of representatives for the elements of $\widehat{\mathcal{A}}$. Thus the dual space of $\mathcal{A}$ can be formed by restricting $\mathcal{A}$ to each (different) matrix block in its block diagonalization. This implies that $\widehat{\mathcal{A}}$ is a finite set with no more than n elements.

The topology of $\widehat{\mathcal{A}}$ is the discrete topology. The kernel of π_i is the set of elements in $\mathcal{A}$ for which the block in $\mathcal{A}$ corresponding to π_i is zero (assume $\mathcal{A}$ has been block diagonalized). Fix i and consider the subset $S_i = \{\pi_j : j = 1, \dots, i-1, i+1, \dots, \ell\}$. Then $\cap_{\pi_j \in S_i} \ker \pi_j$ is the set of elements of $\mathcal{A}$ that are zero except for the block corresponding to π_i. Since the kernel of π is the set of elements of $\mathcal{A}$ that are zero at this block, clearly π_i is not in the closure of S_i and so S_i is closed and $\pi_i = \widehat{\mathcal{A}} \setminus S_i$ is open. Thus each singleton subset of $\widehat{A}$ is open and so $\widehat{\mathcal{A}}$ is discrete.

3.2 C^*-bundle Theory

In working out the dual space of the C^*-algebra of a wallpaper group, it is convenient to view $C^*(G)$ as the sectional algebra of a C^*-bundle. A bundle is essentially a space consisting of fibres (each a C^*-algebra) which vary continuous over an underlying Hausdorff space. The isomorphism identifying $C^*(G)$ with a sectional algebra of a C^*-bundle is fairly involved and we will not go into too much depth. We already have an explicit description of the elements of $C^*(G)$ and are mainly interested in the dual space. This section is largely based on Fell and Doran's book [11].

Definition 26. A bundle over X is a triple (p, B, X) where B, X are Hausdorff spaces and $p : B \to X$ is an open and continuous surjection. X is referred to as the base space, B the bundle, and p the bundle projection. For each $x \in X$ the set $p^{-1}(x)$ is called the fibre over x and is denoted by B_x. Note that the fibres form a partition of B.

Definition 27. A C^*-bundle is a bundle (p, B, X) such that each fibre is a C^*-algebra with the "zero-limit" property: if $x \in X$ and $\{b_i\}_{i \in \mathcal{I}}$ is a net in B such that $\|b_i\| \to 0$ and $p(b_i) \to x$ in X then $b_i \to 0_x$, the zero element in the fibre B_x. Here, $\|b_i\|$ is the norm of b_i in the fibre B_{b_i}. This condition is equivalent to defining a neighbourhood basis for 0_x in B, for each $x \in X$, to be:

$$N(x : U, \varepsilon) := \{b \in B : p(B) \in U, \|b\| < \varepsilon\} \tag{3.2}$$

where U is a neighbourhood of x in X and $\varepsilon > 0$.

Addition, multiplication, and involution may be defined on B fibre-wise. Each of these operations is clearly continuous. It seems that B consists of seemingly very independent fibres. However, 3.2 shows that the fibres do fit together topologically. A neighbourhood of the zero element of a fibre B_x consists of elements in $p^{-1}(U)$

whose norm is less than some fixed ε, where U is a neighbourhood of x in X. So a neighbourhood of 0_x in B very well might contain elements from fibres other than B_x. A basis for the topology of B may be obtained by translating each neighbourhood $N(x : U, \varepsilon)$ along the fibre B_x, for each $x \in X$. So points b_1, b_2 in B whose projections in X are close and for which $\|b_1\|$ and $\|b_2\|$ are close (note: these are potentially different norms) are close in B. Thus the fibres of B vary continuously, in a sense.

Definition 28. A section of a C^*-bundle (p, B, X) is a continuous function $F : X \to B$ such that $p \circ F = id_X$, the identity on X. Thus F maps each $x \in X$ to something in the fibre B_x. Let $\Gamma(B)$ denote the space of sections on B.

If X is compact then $\Gamma(B)$ can be given the supremum norm:

$$\|F\|_\infty = \sup_{x \in X} \|F(x)\|$$

This is finite because F is a continuous and X is compact. We can define addition, multiplication and the involution * pointwise. With respect to these, the sup norm is a C^*-norm, for it satisfies the C^* property: $\|F^*F\| = \|F^*\|\|F\|$ [11]. Indeed, $\Gamma(B)$ is a C^*-algebra. It is called the sectional C^*-algebra of B, or for short, the sectional algebra of B.

Let $C = (\Gamma(B), \|\cdot\|_\infty)$ be a sectional C^*-algebra of a C^*-bundle (p, B, X) with X compact. The dual space of such a C^*-algebra is particularly easy to describe. This was worked out by Fell and Doran [11]. For each $x \in X$ let $\widehat{B_x}$ be the dual space of the fibre B_x.

Lemma 29. If $\pi \in \widehat{B_x}$ and $F \in C$ then $\Pi(F) = \pi(F(x))$ defines an irreducible *-representation of C.

Fell and Doran also showed that if Π is an irreducible *-representation of C then there is a unique $x \in X$ such that $\Pi(F) = \pi(F(x))$. The action of forming Π from a representation $\pi \in B_x$ is called "lifting", as it extends a representation of a fibre to the sectional algebra of the entire bundle. This describes the dual space of C as the union of the dual spaces of its fibres. As for the topology on the dual of a C^*-bundle, Fell gave two very useful results.

Theorem 30. Let q be projection of $\widehat{C}$ onto X, i.e., $q : (\pi, x) \mapsto x$. Then q is continuous and open.

Theorem 31. For each fixed $\eta \in B_x$, the map $\pi \mapsto \Pi_{x,\pi}$ is a homeomorphism of $\widehat{B_x}$ into $\widehat{G}$.

Since q is continuous and open as well as surjective, it is a quotient map on $\widehat{C}$. Thus much of the topology of the dual space of the sectional algebra of a bundle is captured in the base space. The second result shows that dual space of each fibre is injected into $\widehat{G}$ with its topology completely preserved. As we will see, for the wallpaper groups this leads to a splitting up of $\widehat{G}$ into sheet-like pieces.

3.3 Representation Theory of the Wallpaper Groups

We defined the convolution algebra $\ell^1(G)$ for a discrete group G in Chapter 1. We review it quickly here as a refresher as well as the C^*-algebra, $C^*(G)$. After this we move to describing $C^*(G)$ for a wallpaper group G. This is a corollary of K.F. Taylor's description of the C^*-algebras of the crystallographic groups. For the wallpaper groups, $C^*(G)$ can be expressed as an algebra of matrices whose entries are continuous functions on the torus. Explicit calculations of the C^*-algebras of the wallpaper groups were done by E. Pohorecky in his Master's thesis [19]. We will use these in examples.

First is a quick review of the definitions of the algebras $\ell^1(G)$ and $C^*(G)$ from Chapter 1. For a wallpaper group G, $\ell^1(G)$ is the set of functions $f : G \to \mathbb{C}$ such that

$$\|f\|_1 = \sum_{x \in G} |f(x)|$$

Equipped with pointwise addition, scalar multiplication, convolution and the $*$ operation defined by $f^*(x) = \overline{f(x^{-1})}$, and the norm $\|\cdot\|_1$, $\ell^1(G)$ is a Banach *-algebra over $\mathbb{C}$.

Next, $\ell^2(G)$ is the space of square summable functions $f : G \to \mathbb{C}$. There is an inner product on $\ell^2(G)$ defined by $\langle f, g \rangle = \sum_{x \in G} f(x)\overline{g(x)}$. With this inner product and pointwise addition and scalar multiplication, $\ell^2(G)$ forms a Hilbert space, that is, it is a complete inner product space. It's not hard to see that $\ell^1(G)$ sits inside of $\ell^2(G)$. This comes from the fact that if a sequence is absolutely summable then it is also square summable.

Recall that a C^*-algebra is a Banach *-algebra with the additional norm condition [8]:

$$\|f^* * f\| = \|f\|^2$$

Now $\ell^1(G)$ is *not* a C^*-algebra. Nonetheless, we can extend it to a C^*-algebra by providing it with a new norm. This is the operator norm, for $f \in \ell^1(G)$ is an operator on $\ell^2(G)$ via convolution. That is, $\ell^1(G) \cong \lambda(G)$ where $\lambda(f)$ is the operator on $\ell^2(G)$ defined by $\lambda(f)g = f * g$. The norm of $\lambda(f)$ is then $\sup\{\|f * g\| : g \in \ell^2(G), \|g\|_2 \leq 1\}$. The wallpaper groups are amenable. This comes from the fact that locally compact abelian groups are amenable, as well as groups that have an amenable subgroup of finite index. For the wallpaper groups, $\mathbb{Z}^2$ is the locally compact abelian group of finite index. Thus the group C^*-algebra of a wallpaper group is the same as its reduced group C^*-algebra. This is denoted by $C^*(G)$ and can be defined to be the completion of $\ell^1(G)$ with respect to the operator norm. Note that $\ell^1(G)$ is a dense subset of $C^*(G)$ and that the C^*-norm of a function is never greater than its ℓ^1-norm for $\|f * g\| \leq \|f\|_1 \|g\|_2$.

3.3.1 Description of $C^*(G)$

K.F Taylor showed that the group C^*-algebra of a crystallographic group G is isomorphic to a subalgebra of matrices whose entries are continuous functions on the dual of the translational subgroup of the crystal [23]. We work out this identification in the specific case that G is a wallpaper group.

Since the lattice subgroup $\mathbb{Z}^2$ is abelian it carries a Fourier transform. This is defined by, for $f \in \ell^2(\mathbb{Z}^2)$,

$$\mathcal{F}(f)(\chi) = \sum_{(x,y)\in\mathbb{Z}^2} f(x,y)\chi(x,y)$$

where χ is a character in the dual group of $\mathbb{Z}^2$. As we've seen, the characters of $\mathbb{Z}^2$ are parametrized by $\mathbb{T}^2$ via $\chi_{z,w}(a,b) = z^a w^b$. Replacing $\chi_{z,w}$ with (z,w), the above equation may then be written as $\mathcal{F}(f)(z,w) = \sum_{x,y\in\mathbb{Z}^2} f(x,y)z^x w^y$. This is the usual Fourier series in two dimensions. Let $L^2(\mathbb{T}^2)$ be the set of square-integrable functions from $\mathbb{T}^2$ to $\mathbb{C}$. Given the inner product, $\langle f,g\rangle = \int_{\mathbb{T}^2} f(z,w)\overline{g(z,w)}d(z,w)$, $L^2(\mathbb{T}^2)$ is a Hilbert space. It is well know that $\mathcal{F}$ extends to a unitary map of $\ell^2(\mathbb{Z}^2)$ onto $L(\mathbb{T}^2)$, called the Plancherel transform. We denote this by $\mathcal{P}$.

Let $C(\mathbb{T}^2)$ be the set of continuous complex-valued functions on the torus. Equipped with the supremum norm, pointwise product and addition, and an involution defined by $f^*(z,w) = \overline{f(z,w)}$, $C(\mathbb{T}^2)$ is a C^*-algebra. Also let $\mathcal{B}(L^2(\mathbb{T}^2))$ denote the algebra of bounded linear operators on $L^2(\mathbb{T}^2)$, that is, linear functions $B : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ such that $\sup\{\|B\eta\|_2 : \eta \in L^2(\mathbb{T}^2), \|\eta\|_2 \leq 1\} < \infty$. This space carries pointwise additiion and scalar multiplication and composition defines a product. The operator norm is a natural norm on $\mathcal{B}(L^2(\mathbb{T}^2))$. It is well known that every bounded operator B on a Hilbert space has a unique operator, B^*, called the adjoint. This is the unique operator satisfying $\langle B\xi,\eta\rangle = \langle \xi, B^*\eta\rangle$ for every function ξ in the Hilbert space. The map that takes an operator to its adjoint is an involution. In fact, $\mathcal{B}(L^2(\mathbb{T}^2))$ forms a C^*-algebra.

As explained in Chapter 2, by choosing a cross section $\gamma : D \to G$ one can decompose G into cosets of the form $\gamma(d)\mathbb{Z}^2$. This also leads to a decomposition of $\ell^2(G)$. To see this, note that the set of functions in $\ell^2(G)$ whose support is in the coset $\gamma(d)\mathbb{Z}^2$ is a subspace of $\ell^2(G)$. This subspace is isometric with $\ell^2(\mathbb{Z}^2)$ via $f_d(x) = f(\gamma(d)x)$. As a set, $\ell^2(G)$ can then be written as $\oplus\ell^2(\mathbb{Z}^2)$ by mapping f to $(f_d)_{d\in D}$. Taking the Plancherel transform of each f_d we obtain an isometry $\Phi : \ell^2(G) \to \oplus L^2(\mathbb{T}^2)$ defined by $\Phi : f \mapsto (\mathcal{P}(f_d))$.

For $f \in \ell^1(G)$, consider $\Phi\lambda(f)\Phi^{-1}$. This is a bounded operator on $\oplus L^2(\mathbb{T}^2)$. In a sense $\Phi\lambda(F)\Phi^{-1}$ is a multiplication operator. Let $M_n(C(\mathbb{T}^2))$ denote the space of $n \times n$ matrices with entries in $C(\mathbb{T}^2)$. Define $\mathcal{M} : M_n(\mathbb{C}(\mathbb{T}^2)) \to \mathcal{B}(\oplus L^2(\mathbb{T}^2))$ by $(\mathcal{M}(F)(h))_d = \sum_{c \in D} F_{d,c}h_c$ for $h \in L^2(\mathbb{T}^2)$. It is not hard to show that $\mathcal{M}$ is a C^*-isomorphism of $M_n(C(\mathbb{T}^2))$ into $\mathcal{B}(\oplus L^2(\mathbb{T}^2))$. Then $\Phi\lambda(F)\Phi^{-1}$ can be represented by a matrix in $M_n(C(\mathbb{T}^2))$ in the sense that $\Phi\lambda(F)\Phi^{-1}$ is in the range of $\mathcal{M}$. The main result of [23] is the following:

Theorem 32. Let G be a wallpaper group with point group D. For $f \in \ell^1(G)$, let $\mathcal{F}(f) = \mathcal{M}^{-1}\Phi\lambda(f)\Phi^{-1}$. Then $\mathcal{F}$ extends to a C^*-isomorphism of $C^*(G)$ onto a C^*-subalgebra of $M_n(C(\mathbb{T}^2))$, where $n = |D|$.

Taylor used this to obtain a concrete description of the transform $\mathcal{F} : \ell^1(G) \to M_n(C(\mathbb{T}^2))$ and the form of the elements of $C^*(G)$:

Lemma 33. Let G be a wallpaper group with point group D, 2-cocycle α, and $n = |D|$. Let $f \in \ell^1(G)$. For each pair $(b, c) \in D \times D$,

$$(\mathcal{F}(f))_{b,c}(z, w) = (z, w)^{\alpha(bc^{-1}, c)}(\widehat{f}_{bc^{-1}})(c \cdot (z, w))$$

Furthermore, $\mathcal{F}(C^*(G))$ can be characterized in the following way. For $b, c \in D$, the (b, c)-entry of an element $F \in \mathcal{F}(C^*(G))$ has the form:

$$F_{b,c}(z, w) = (c \cdot (z, w))^{\alpha(b,c^{-1})\alpha(c,c^{-1})} F_{bc^{-1},1}(c \cdot (z, w)) \tag{3.3}$$

where $(z, w) \in \mathbb{T}^2$ and $F_{bc^{-1},1} \in C(\mathbb{T}^2)$. Thus the elements of $\mathcal{F}(C^*(G))$ are determined by a set of n continuous functions on $\mathbb{T}^2$.

As an example, consider the wallpaper group $G = \text{p2}$. We've seen that α is trivial. The action of $D = \mathbb{Z}_2$ on $\mathbb{T}^2$ can be worked out using our knowledge of the action of D on G:

$$-1 \cdot \chi_{z,w}(a, b) = \chi_{z,w}(-1^{-1} \cdot (a, b)) = \chi_{z,w}(-1 \cdot (a, b)) = \chi_{z,w}(-a, -b) = z^{-a}w^{-b} = \overline{z}^a\overline{w}^b$$

This shows that $-1 \cdot (z, w) = (\overline{z}, \overline{w})$.

Thus we have that, for $f \in \ell^1(p2)$,

$$\mathcal{F}(f) = \begin{bmatrix} \widehat{f}_1(z,w) & \widehat{f}_{-1}(\overline{z},\overline{w}) \\ \widehat{f}_{-1}(z,w) & \widehat{f}_1(\overline{z},\overline{w}) \end{bmatrix}.$$

Similarly, an element F of $C^*(p2)$ is of the form:

$$F = \begin{bmatrix} F_1(z,w) & F_{-1}(\overline{z},\overline{w}) \\ F_{-1}(z,w) & F_1(\overline{z},\overline{w}) \end{bmatrix}$$

where $F_1, F_{-1} \in C(\mathbb{T}^2)$.

3.4 Description of $\widehat{C^*(G)}$

To describe the dual space of $\widehat{G}$ it is helpful to view $C^*(G)$ as the sectional algebra of a C^*-bundle. The base space of $C^*(G)$ is $\mathbb{T}^2/D$. This is the quotient of $\mathbb{T}^2$ by the action of D. Note that $\mathbb{T}^2/D$ is a Hausdorff space. Let the D-orbit of an element (z, w) in $\mathbb{T}^2$ be denoted by $[z, w]$. This allows us a notation for the elements of $\mathbb{T}^2/D$. We define the C^*-bundle associated with $C^*(G)$ by defining its fibers. For $(z, w) \in \mathbb{T}^2$, let $\mathcal{A}_{[z,w]} = \{F(z, w) : F \in C^*(G)\}$. Since $C^*(G)$ is a C^*-algbra, it is not difficult to show that $\mathcal{A}_{[z,w]}$ is also a C^*-algebra. Let $B = \cup_{z,w\in\mathbb{T}^2/D}\mathcal{A}_{[z,w]}$ and p map elements of the fibre $\mathcal{A}_{[z,w]}$ to $[z, w] \in \mathbb{T}^2/D$. Then (somewhat trivially), (B, X, p) is a C^*-bundle. E. Pohorecky showed that the sectional algebra of this is isomorphic to $C^*(G)$ [19].

Consider the dual space of this sectional algebra. Recall that this is the union of the dual spaces of the fibres, lifted to the sectional algebra. Now $\mathcal{A}_{[z,w]}$ is finite-dimensional since it is a subalgebra of $M_n(\mathbb{C})$. Thus $\mathcal{A}_{[z,w]}$ may be decomposed into blocks, as shown in the first section of this chapter. The dual space of $\mathcal{A}_{[z,w]}$ is then finite and discrete, with one point for each central minimal projection in $\widehat{\mathcal{A}_{[z,w]}}$ in a list that sums to the identity of $\mathcal{A}_{[z,w]}$. Specifically, there is a change of basis matrix

U such that $U\mathcal{A}_{[z,w]}U^{-1}$ is block diagonal. For C_i, the projection corresponding to the i^{th} block of $U\mathcal{A}_{[z,w]}U^{-1}$, the map $M \mapsto C_iU\mathcal{A}_{[z,w]}U^{-1}$ is an irreducible representation of $\widehat{\mathcal{A}_{[z,w]}}$. Doing this for each block gives a complete set of representatives of the classes of irreducible representations of $\mathcal{A}_{[z,w]}$.

We now restate Theorems 30 and 31 for the case when G is a wallpaper group.

Theorem 34. Let q be the map that projects $\widehat{G}$ onto $\mathbb{T}^2/D$ via $(\pi, [z,w]) \mapsto [z,w]$. Then q is continuous and open.

Theorem 35. For each $[z,w] \in \mathbb{T}^2/D$, the map $\pi \mapsto \Pi_{[z,w],\pi}$ is a homeomorphism of $\widehat{\mathcal{A}_{[z,w]}}$ into $\widehat{G}$.

Thus at each point of the underlying space $\mathbb{T}^2/D$, the dual space of the fibre embeds into $\widehat{G}$. Since the dual space of a fibre is discrete and finite this implies that points in $\widehat{G}$ are closed. So $\widehat{G}$ is T_1. It is not Hausdorff, however.

Continuity of q leads to the following consequence. Let x_n be a sequence in $\mathbb{T}^2/D$. If $(\pi_n, x_n) \to (\pi, x)$ in $\widehat{G}$ then $q(\pi_n, x_n) \to q(\pi, x) \iff \pi_n \to \pi$. Thus a sequence converges in $\widehat{G}$ only if the underlying sequence in $\mathbb{T}^2/D$ converges. Next we will see that $\widehat{G}$, when G is a wallpaper group, is first countable - and so closure in $\widehat{G}$ is equivalent to sequential closure.

Lemma 36. Let G be a wallpaper group. Then $\widehat{G}$ is first countable.

This is clear as G is first countable as a set, being generated by a finite set (the point group) over $\mathbb{Z}^2$. As G is discrete, each singleton set in G is a neighbourhood basis for itself. Thus G is first countable. In a first countable space, a set is closed if and only if it is sequentially closed. That is, if $E \subseteq \widehat{G}$, a point $x \in \widehat{G}$ is in the closure of E if and only if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in E that converges to x. This allows us to describe the topology of $\widehat{G}$ in terms of convergence of sequences.

Let Ω be the set of elements of $\mathbb{T}^2/D$ that come from points that have a trivial D-stabilizer. From the formula for elements of $C^*(G)$ 3.3, it's clear that $\mathcal{A}_{[z,w]} = \{F(z,w) : F \in C^*(G)\}$ is the full matrix algebra $M_n(\mathbb{C})$ when $[z,w] \in \Omega$. Thus $\widehat{\mathcal{A}_{[z,w]}}$ consists of the identity representation of $M_N(\mathbb{C})$. The corresponding representation in $\widehat{G}$ is the evaluation of $C^*(G)$ at $[z,w]$:

$$\Pi_{[z,w]}(F) = \pi(F(z,w)) = F(z,w) \tag{3.4}$$

The kernel of $\Pi_{[z,w]}$ consists of all elements of $C^*(G)$ that are 0 at $[z,w]$. Suppose $[z_n,w_n] \to [z,w]$ in $\mathbb{T}^2/D$. For $F \in C^*(G)$, since F is continuous, $F(z_n,w_n) \to F(z,w)$. In particular, if $F(z_n,w_n) = 0$ for all $n \in \mathbb{N}$ then $F(z,w) = 0$. Recall the definition of closure in the hull-kernel topology. A point Π is in the closure of a set E if and only if $\ker \Pi \supseteq \cap_{\sigma \in E} \ker \sigma$. But this means that Equation 3.4 implies that $\Pi_{[z_n,w_n]} = (\pi_n, z_n) \to \Pi_{[z,w]} = (\pi, z)$. Thus we have that on Ω, $\Pi_{[z_n,w_n]} \to \Pi_{[z,w]} \iff [z_n,w_n] \to [z,w]$. This implies that $q^{-1}(\Omega)$ is homeomorphic to Ω (More succinctly, q is a homeomorphism of Ω and $q^{-1}(\Omega)$ since q is open, continuous, 1-1 and onto $q^{-1}(\Omega) = \{(\Pi, [z,w]) : [z,w] \in \Omega\}$) . In [19], you can see that Ω is a dense, connected subset of $\mathbb{T}^2/D$. In fact, Ω is second countable, Hausdorff and locally homeomorphic to 2-dimensional Euclidean space and so is a 2-manifold.

It remains then to describe the topology of $\widehat{G} \setminus q^{-1}(\Omega)$ and its boundary with $q^{-1}(\Omega)$. We do this in terms of sequences. Note that $\widehat{G}$ is compact since G is discrete and so every sequence in $\widehat{G}$ has a convergent subsequence. Recall that the dimension of representation is the dimension of its Hilbert space.

Lemma 37. Suppose $(\pi_n, [z_n,w_n])$ is a set of representations in $\widehat{G}$ of constant dimension with $[z_n,w_n] \to [z,w]$ in $\mathbb{T}^2/D$ and $(\pi_n, [z_n,w_n])(F) \to (\pi, [z,w])(F)$ for each $F \in C^*(G)$. Then $(\pi_n, [z_n,w_n]) \to (\pi, [z,w])$ in the hull kernel topology of $\widehat{G}$.

Proof. We must show that $\cap \ker(\pi_n, [z_n,w_n]) \subseteq \ker(\pi, [z,w])$. So let $F \in C^*(G)$ be such that $(\pi_n, [z_n,w_n])(F) = 0$ for all $n \in \mathbb{N}$. Then since $0 = (\pi_n, [z_n,w_n])(F) \to (\pi, [z,w])(F)$, we have that $(\pi, [z,w])(F) = 0$. □

Let K be a subgroup of D and let Ω_K be the set of elements of $\mathbb{T}^2/D$ with stabilizer K. We call Ω_K a strata of $\mathbb{T}^2/D$. Note that $\Omega_{1_D} = \Omega$ and that if $K_1 \neq K_2$ then $\Omega_{K_1} \neq \Omega_{K_2}$. Thus the strata form a partition of $\mathbb{T}^2/D$. From 3.3 it's clear that $\mathcal{A}_{[z,w]}$ block diagonalizes in the same way for each $(z, w) \in \Omega_K$. Thus over a strata there is a common number and dimension of representations of $\mathcal{A}_{[z,w]}$. So the previous lemma applies to sequences that converge within a strata. In practice, this is enough to efficiently determine the topology of Ω_K, as a subspace of $\widehat{G}$. If $(\pi_n, [z_n, w_n])$ does not converge to $(\pi, [z, w])$ pointwise we can use Appendix C to check that $\ker(\pi, [z, w])$ does not contain $\ker(\pi_n, [z_n, w_n])$. The Appendix uses the calculations in [19] of $\widehat{G}$ as a set to calculate the kernel of each representation. Note that there is one calculation for each Ω_K.

The strata of $\mathbb{T}^2/D$ come in only three different forms. There is Ω, which corresponds to a 2-dimensional subset of $\mathbb{T}^2/D$. Then there are strata that come from stabilizer subgroups that only contain a reflection, i.e., are of the form $\{1_D, \rho\}$ where ρ is a reflection. In Appendix A, you can see that these are 1-dimensional subsets of $\mathbb{T}^2/D$. Finally, there are strata whose stabilizer contains a rotation (note that if it contains more than one reflection, then it automatically contains a rotation). These strata consist of single points in $\mathbb{T}^2/D$.

We next show how to find the limits of sequences in $\widehat{G}$ above a strata that converge to a point $(\pi, [z, w])$, with $[z, w]$ lying in a strata of lower dimension. Note that the overlying sequence in $\widehat{G}$ must converge to some representation above (z, w), since $\widehat{G}$ is compact.

Lemma 38. Suppose π is a representation of $C^*(G)$ such that $\pi \sim \bigoplus_{i=1}^m \pi_i$ where each π_i is irreducible. Then $\ker \pi \subseteq \ker \pi_i$ for $i = 1, \ldots, m$.

Proof. Since $\pi \sim \bigoplus_{i=1}^m$, there are projections P_i, $i = 1, \ldots, m$ such that $\pi(F) = P_1\pi(F) + \ldots + P_m\pi(F)$. Thus if $\pi(F) = 0$ then $\pi_i(F) = P_i\pi(F) = 0$. □

Suppose $(\pi_n, [z_n, w_n])$ is a sequence in $\widehat{G}$ with $[z_n, w_n]$ in a strata of $\mathbb{T}^2/D$. Now

$(\pi_n, [z_n, w_n])(F)$ may be written as $PU_nF(z_n, w_n)U_n^{-1}$ where U_n is the change of basis matrix that block diagonalizes $\mathcal{A}_{[z,w]}$ and P is the projection corresponding to some block. Consider the function $F \mapsto PUF(z, w)U^{-1}$ where U is the limit of $\{U_n\}$. Claim: this is a (reducible) representation of $C^*(G)$. Call it $(\pi, [z, w])$. Note that $\ker(\pi, [z, w])$ contains $\ker(\pi_n, [z_n, w_n])$. Now $(\pi, [z, w])$ may be split into a sum of finitely many irreducible representations of $\widehat{G}$. By the previous lemma, $(\pi_n, [z_n, w_n])$ converges to each of these. This describes a method of finding limits at the boundaries of strata. In fact, it provides a complete set of limits. This can be checked using Appendix C.

Appendix A contains drawings of the dual spaces of the wallpaper groups which were made using the calculations of $\widehat{G}$ as a set in [19] and the results on the topology from this chapter. In the drawings, Ω is the large 2-dimensional piece in the centre. Small spheres are points and each point on a line is a representation. White lines and circles are empty - look nearby for the representations that live above them. Dotted lines connect 1D strata to the points in their closures.

Chapter 4

Compact Open Sets in $\widehat{G}$

In the last chapter we described the topology of the dual space of a wallpaper group. Now we delve further to arrive at a characterization of the compact open sets in $\widehat{G}$. This involves recognizing the preimage of a strata in $\widehat{G}$ as a fibre bundle, describing $\mathbb{T}^2/D$ as stratified space and mapping $\widehat{G}$ to a graph in which the compact open sets correspond to a certain type of subgraph.

4.1 More on the Topology of $\widehat{G}$

One can see from the drawings of the dual spaces (Appendix A) that the preimage of strata in $\widehat{G}$ has a fairly rigid structure. Locally, they look like a set of finite copies of the strata. When the wallpaper group splits, $q^{-1}(\Omega_K)$ is isomorphic to $\Omega_K \times \{1, \ldots, \ell\}$, for some $\ell \in \mathbb{N}$, where $\{1, \ldots, \ell\}$ is the discrete topological space with ℓ elements. When G does not split, this is still true locally: for each $x \in \mathbb{T}^2/D$ there is a neighbourhood $\mathcal{U}$ of x such that $q^{-1}(\mathcal{U})$ is isomorphic to $\mathcal{U} \times \{1, \ldots, \ell\}$ for some $\ell \in \mathbb{N}$. This leads to the notion of fibre bundles [6]. These are very similar to C^*-bundles. The main difference is that in a fibre bundle, the fibres are all one particular topological space.

Definition 39. Let E, B, F be topological spaces. A map $p : E \to B$ is a fibre bundle with fibre F if it satisfies the following:

(i) $p^{-1}(b) = F$ for all $b \in B$

(ii) $p : E \to B$ is surjective

(iii) For each $x \in B$ there is a neighbourhood $\mathcal{U}$ of x and a homeomorphism $\psi : p^{-1}(\mathcal{U}) \to \mathcal{U} \times F$ such that $proj \circ \psi = p$ where $proj$ is projection of $\mathcal{U} \times F$ onto the first coordinate.

Lemma 40. Let K be a subgroup of D. Let ℓ be the size of $\widehat{\mathcal{A}_{[z,w]}}$ for any $[z,w] \in \Omega_K$. Then q restricted to $q^{-1}(\Omega_K)$ is a fibre bundle with fibre $\{1, \ldots, \ell\}$.

This greatly simplifies our view of the topology of $\widehat{G}$, as we know what the topology of $\mathcal{U} \times \{1, \ldots, \ell\}$ looks like, where $\mathcal{U}$ is a subset of $\mathbb{T}^2/D$. Where strata intersect, again, we can make calculations using the definition of the hull-kernel topology.

Now consider a compact open subset C of $\widehat{G}$. We have already seen that $q^{-1}(\Omega)$ (where Ω is the set of elements of $\mathbb{T}^2/D$ with trivial stabilizer) is a dense 2-dimensional subset of $\mathbb{T}^2/D$. In particular, $q^{-1}(\Omega)$ is isomorphic to Ω, so we refer to $q^{-1}(\Omega)$ simply as Ω. From Appendix you can see that Ω is connected. $\mathbb{T}^2/D$ may be represented on a subset of square $[-1,1] \times [-1,1] \subseteq \mathbb{R}^2$, where certain edges are identified. Suppose C intersects Ω. Then $C \cap \Omega$ is an open subset of Ω. But since Ω is a subspace of $\mathbb{R}^2$, which is Hausdorff, compact subsets are closed. So $C \cap \Omega$ is an open and closed subset of Ω. But since Ω is a connected subset of $\mathbb{R}^2$ this means that $C \cap \Omega$ must be all of Ω or be empty.

Lemma 41. Let f be a non-zero projection in $\ell^1(G)$ with support set C in $\widehat{G}$. Then C contains Ω.

Proof. C is the support set of a projection in $\ell^1(G)$ so it is compact and open. Suppose C is empty. Now $C = \operatorname{supp}(f) = \{\pi \in \widehat{G} : \pi(f) \neq 0\}$. If this is empty then $\pi(f) = 0$ for all $\pi \in \widehat{G}$. This implies that $f = 0$. On the other hand, if $C \setminus \Omega$ is non-empty, then it contains some π from $q^{-1}(x)$ where x is not in Ω. But C is open, so it contains a neighbourhood of π. But every neighbourhood of π intersects Ω, since Ω is dense in $\widehat{G}$. Thus $C \cap \Omega$ is non-empty. This is a contradiction. ∎

Thus non-trivial projections in $\ell^1(G)$ have support sets that contain all of Ω. We will show that a similar result holds for the other strata of $\mathbb{T}^2/D$. Recall that a set is [sequentially] compact if every sequence has a convergent subsequence. Since the map $q : \widehat{G} \to \mathbb{T}^2/D$ is continuous, Cauchy sequences in $q^{-1}(\Omega)$ converge to each of the representations above the limit of this sequence in $\mathbb{T}^2/D$. Since Ω is dense in $\widehat{G}$ this means that, for each point $x \in \mathbb{T}^2/D \setminus \Omega$, C must contain at least one of the representations in the fibre $q^{-1}(x)$. In other words, for non-trivial projections, $q(C)$ is all of $\mathbb{T}^2/D$.

Definition 42. Let Ω_K be a strata of $\mathbb{T}^2/D$. Define a strata piece of Ω_K in $\widehat{G}$ to be a connected component of $q^{-1}(\Omega_K)$.

Note that for a wallpaper group that splits, a strata piece is isomorphic to the underlying strata. As we have seen, strata come in the form of line segments, points, and the 2-dimensional subspace Ω. Thus a strata piece in this case is one of these and so is a connected subset of $\mathbb{R}_0$ (a point), $\mathbb{R}$ or is Ω. For a strata piece S that is a point, it is clear that $C \cap S$ is all of S. For a strata piece S that is a line segment, we show that $C \cap S$ is closed in S. Consider a sequence $\{\pi_n\} \subseteq C \cap S$ that converges to some $\pi \in S$. Since $C \cap S$ is in C and C is compact in $\widehat{G}$, there is a limit of this sequence in C. But since S is Hausdorff and $\{\pi_n\}$ lies completely in S, the limit of S is unique. So $\pi \in C \cap S$. This shows that $\overline{C \cap S}$ is closed in S. Since C is open, $C \cap S$ is open in the subspace topology on S. But the only closed and open subsets of a line segment are the empty set and the whole line segment. Since $q(C) = \mathbb{T}^2/D$ if C is the support set of a non-trivial projection then C must contain an entire strata piece, for each strata.

For a non-splitting wallpaper group (of which there are 4), we can check on a case to case basis that strata are also points, circles, line segments and Ω. Strata pieces here also take this form, although they may be twisted in unexpected ways. For instance, pg has a strata that is a circle. The corresponding fibre is the set $\{1, 2\}$. However, instead of two circles as strata pieces, there is just one in the form of a doubly wound circle. Nevertheless, this is a Hausdorff space and so its intersection with C is closed and open and so must be empty or the whole strata piece. We have arrived at the following lemma.

Theorem 43. Let C be a compact open set in $\widehat{G}$ and S a strata piece of $\widehat{G}$. If $C \cap S$ is non-empty, then $C \cap S = S$.

Thus the problem of finding compact open sets becomes discretized in a sense, for there are finitely many strata and strata pieces. All that is left to figure out is which combinations of strata pieces yield compact open sets. What needs to be checked are the boundaries of strata. Note that at the boundary of Ω, we know that one piece from each strata must be contained in C. That leaves three other types of boundaries: point strata with point strata, point strata with line strata, line strata with line strata. The first case does not occur as points in $\mathbb{T}^2/D$ that are fixed by rotations are isolated. The last case also does not happen because in between strata that are fixed by two different reflections, there is always a point that is fixed by a rotation. This is because of the continuity of the D action on $\mathbb{T}^2$. A point on the boundary of these two strata is fixed by both reflections - and the composition of two reflections is a rotation. Thus the only case to consider is that of the boundary between a point strata and a line strata.

Suppose C contains a strata piece S that is a line segment and at its ends are pieces from a point strata. Then if C is compact, it must contain one of the points in the closure of S. If C contains a piece $\{\pi\}$ from a point strata, then openness of C requires that C contain the strata pieces which contain $\{\pi\}$ in their closures. Another way to express this is to define an order on strata. Say that $S_1 < S_2$ if $S_1 \subseteq \overline{S_2}$. If C contains a strata piece C_S from S and $S < S_1$ then C must contain all pieces from S_1 whose closure intersects C_S. If $S_0 < S$ then C must contain at least one piece from S_0 that intersects the closure of C_S. This leads to an algorithm for forming compact open sets in $\widehat{G}$:

(i) Let C contain Ω.

(ii) Choose a 1D strata, if any. Add to C at least one piece from this.

(iii) Move around the perimeter of Ω from here, adding at least one piece from each strata. If the previous strata was 1D, choose at least one piece from the closures of the 1D pieces in C. If the previous strata was a point, choose all 1D strata pieces that contain this point in their closures.

4.2 $\mathbb{T}^2/D$ as a Stratified Space

The problem of forming the compact open subsets of $\widehat{G}$ reduces to a sequence of choices of pieces from each strata. We really only need to know the strata pieces of G and how they connect to each other. This suggests that we map $\widehat{G}$ to a graph. This is what we will do in this section. To do this, we first introduce the concept of a stratified space and show that $\mathbb{T}^2/D$ is one of these, with its strata being the strata we've been talking about all along.

Definition 44. A filtered space is a Hausdorff space X endowed with a filtration by closed subsets:

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X_{n-1} \subset X_n = X$$

The formal dimension of X is defined to be n. The connected components of $X_i \setminus X_{i-1}$ are called *strata* and each have dimension i.

Definition 45. A stratified space is a filtered space with the following frontier condition:
For any two strata S, T such that $T \cap \overline{S} \neq \emptyset$, we have that $T \subseteq \overline{S}$.

Example 1. Let $X = X_2$ be a square, X_1 be the lines that form the perimeter of X, and X_0 the corner points of X. Then $X_0 \subseteq X_1 \subset X_2$ and each X_i is closed. The strata are $S_2 = X_2 \setminus X_1 =$ the open square, the connected components of $X_1 \setminus X_0 =$ the lines formed by removing the corner points from the perimeter, $X_0 =$ the corner

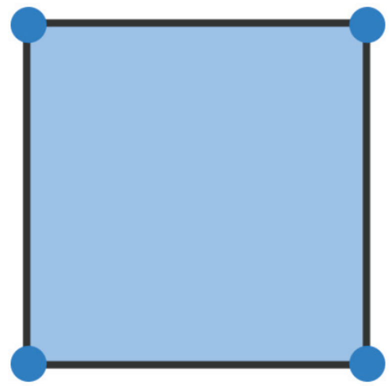

Figure 4.1: The square, decomposed into strata.

points. It's easy to see that if $S_i \cap \overline{S_j}$ is non-empty, then $S_i \subseteq S_j$.

As a simple non-example, consider extending one of the sides of the square (just the line), for instance the set $X = [0,1] \times [0,1] \cup \{1\} \times [0,2] \subseteq \mathbb{R}^2$. Use the same filtration as before except let X_1 include the line that extends past the square and remove the corner point there from X_0. Then this is still a filtration for each set is closed and $X_0 \subseteq X_1 \subset X_2 = X$. The corresponding strata are shown in Figure 4.2. This space is not stratified because the strata containing the extended line intersects the closure of the square without being completely contained in it.

Now consider $\mathbb{T}^2/D$. Let $X_2 = \mathbb{T}^2/D$. Let X_1 consist of the points that are stabilized by a non-trivial element of G. This corresponds to the union of the strata, without Ω. Finally let X_0 consist of the points in $\mathbb{T}^2/D$ that are stabilized by a rotation. This is the union of the point strata. Note that the formal dimensions of X_2, X_1 and X_0 are their dimensions as manifolds.

Lemma 46. X_2, X_1, and X_0 are closed subsets of $\mathbb{T}^2/D$ with $X_0 \subseteq X_1 \subset X_2 = \mathbb{T}^2$.

Proof. If we show that the set of points stabilized by a subgroup of D is closed then this implies that X_2, X_1 and X_0 are closed. This is actually immediate from the fact that the action of a group element on $\mathbb{T}^2$ is continuous. For if $d \cdot x_n = x_n$ for all $n \in \mathbb{N}$ and $x_n \to x$, then $d \cdot x = x$. Thus the set of elements of $\mathbb{T}^2/D$ fixed by a particular

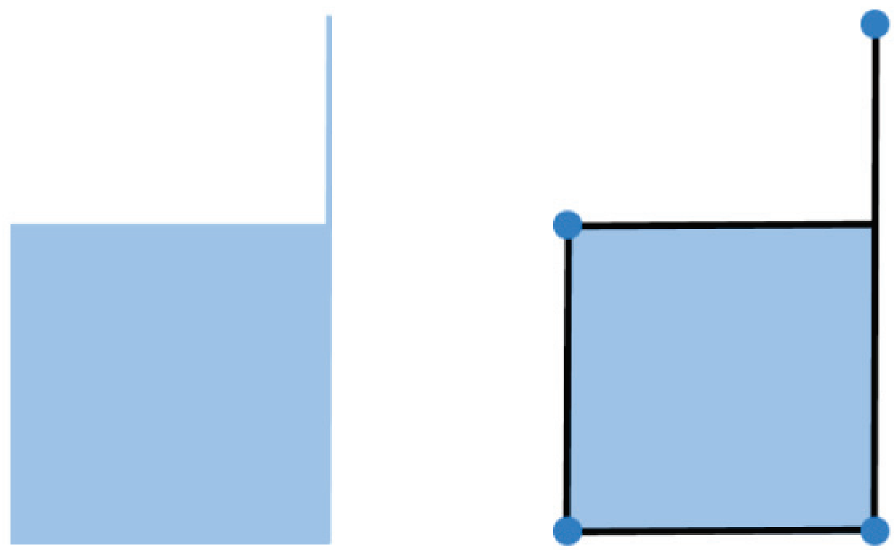

Figure 4.2: Square with extended side and decomposition into strata.

subgroup of D is closed. Clearly $X_0 \subseteq X_1 \subseteq X_2$. □

The previous lemma shows that $X_{-1} \subseteq X_0 \subseteq X_1 \subset X_2$ defines a filtration on $\mathbb{T}^2/D$. The associated strata are the strata we defined earlier. For $X_2 \setminus X_1$ is the set of points with trivial stabilizer $= \Omega$, $X_1 \setminus X_0$ are the points fixed by a reflection but not a rotation, and $X_0 \setminus X_{-1}$ is X_1, the points fixed by a rotation. You can see in [19] that $\mathbb{T}^2/D$ is either a sphere, disc, cylinder, or mobius band. To simplify notation, we will write $S_2 = X_2 \setminus X_1$, $S_1 = X_1 \setminus X_0$ and $S_0 = X_0$.

Proposition 47. With the filtration given above, $\mathbb{T}^2/D$ is a stratified space.

Proof. To prove this, we need to show that whenever a stratum intersects the closure of another, that stratum is entirely contained in the closure. Firstly, the closure of a 0-dimensional strata is itself since these strata are just isolated points. So it does not intersect any other stratum. The closure of a 1-dimensional stratum, however, may pick up a point at the end of the line segment, which is stabilized by a rotation. But such a point is a strata of its own and so is entirely contained in the closure. The closure of the 2-dimensional strata is all of $\mathbb{T}^2/D$, since the set of stabilized points in $\mathbb{T}^2/D$ form a set of measure 0. So all strata are contained in this closure. The only real case to check then is the case of a 1-dimensional stratum S intersecting the closure of another 1D strata T. But as we have seen, this does not happen as there is always a rotation-fixed point between reflection strata. □

4.3 The Graph of $\widehat{G}$

We use the notion of a stratified space to define a type of graph that we will associate to each wallpaper. Recall that a graph is a set of vertices and a set of pairs of vertices (edges). Two vertices are adjacent if there is an edge between them. A set of vertices is independent if no two vertices are adjacent. In a directed graph, the edges are ordered pairs. If (v_1, v_2) is in the edge set of a directed graph, we say there is an edge from v_1 to v_2 and denote this by an arrow in the drawing of the graph. When the vertices may be grouped into k disjoint sets for which no two vertices are adjacent, the graph is called k-partite or multipartite [24]:

Definition 48. A graph is multipartite if its vertex set may be partitioned into independent subsets.

Now construct a graph from $\widehat{G}$ using the following steps. We will call this the graph of $\widehat{G}$ and denote it by $\mathcal{G}$.

(i) Decompose $\widehat{G}$ into strata.

(ii) Within each strata, take strata pieces.

(iii) Assign each piece a vertex.

(iv) Draw a directed edge from piece A to piece B if $B \subseteq \overline{A}$.

Lemma 49. Let G be a wallpaper group. The graph of $\widehat{G}$ is a directed multipartite graph.

Proof. First, $\mathcal{G}$ is a directed graph by construction. To show that it is multipartite, we must define a partition of the vertex set and show that each subset in this is independent. Fix a strata S of $\mathbb{T}^2/D$ and let $S_{\mathcal{G}}$ be the vertices in $\mathcal{G}$ that correspond to strata pieces of S in $\widehat{G}$. We call this the set of vertices of the strata S. Since the strata pieces of S are the connected components of $q^{-1}(S)$, they are not contained in each others' closures as connected components are closed. Thus there are no edges between vertices in $S_{\mathcal{G}}$, showing that $S_{\mathcal{G}}$ is independent. Since the set $\{q^{-1}(S) : S, \text{ a strata of } \mathbb{T}^2/D\}$ is a partition of $\widehat{G}$, $\{S_{\mathcal{G}} : S, \text{ a strata of } \mathbb{T}^2/D\}$ is a partition of $\mathcal{G}$. □

Drawings of graphs of the dual spaces of the wallpaper groups are found in Appendix B. Vertices from the same strata are grouped together. Note that the direction of an edge is always from a higher dimensional strata to a strata of lower dimension. The strata of dimension 2 forms a distinguished vertex in the graph for it flows into every other vertex. Thus it is the source of $\mathcal{G}$. We will leave these edges out in the more complicated graph pictures in order to make the connectivity visually clear. The dimension of a strata can actually be gleaned from the graph; it is the length of the path from the source. The flow and connectivity of the partition are determined by the strata structure while the finer details are controlled by the dual space topology.

Consider what a non-trivial compact open subset C of $\widehat{G}$ looks like on the graph $\mathcal{G}$. Denote the image of C in $\mathcal{G}$ by $\mathcal{C}$. We know that C is a collection of strata pieces, one from each strata. Thus $\mathcal{C}$ corresponds to a collection of vertices of $\mathcal{G}$, one from each strata. Furthermore, C satisfies the property that if $C_S \subseteq C$ is a strata piece from a strata S then all strata pieces which contain C_S in their closure must be in C. On the graph, this means that the predecessors of each vertex of $\mathcal{C}$ are also in $\mathcal{C}$. The second property of C is that it contains at least one piece from each strata $S_0 < S$ that intersects the closure of C_S. For $\mathcal{C}$ this means for each vertex of C, at least one of its successors is also in C.

Theorem 50. Let G be a wallpaper group and $\mathcal{G}$ the associated graph. A compact open set in $\widehat{G}$ corresponds to a subgraph $\mathcal{S}$ of $\mathcal{G}$ with the property that for each vertex in $\mathcal{S}$, each of its predecessors is in $\mathcal{S}$ and at least one of its successors is in $\mathcal{S}$. Furthermore $\mathcal{S}$ must contain at least one vertex from each strata.

Chapter 5

Projections in $\ell^1(G)$: An Example

We have described the compact open subsets of $\widehat{G}$. Which of these are the support set of a projection? In this section we present a further condition for compact open sets in $\widehat{G}$ that are the support set of a projection in $\ell^1(G)$. Next do an example using $p2$ to show that the support sets of finite group projections are not always the only possible support sets in $\widehat{G}$. We construct a projection onto a compact open subset of $\widehat{p2}$ that is not the support set of a finite subgroup projection.

5.1 The Rank Condition

Lemma 51. Define a map, $\mathrm{trace} : \mathrm{M_n}(\mathbb{C}) \to \mathbb{C}$ by $\mathrm{trace(M)} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{M_{ii}}$. Suppose $M_n(\mathbb{C})$ is given the topology as bounded operators on $\mathbb{C}^n$. Then trace is continuous.

Proof. For any $1 \leq i \leq n$,

$$|M_{ii}| \leq \sup\{\|M\underline{x}\| : \|x\| \leq 1\} = \|M\|.$$

Thus $\sum_{i=1}^{n} |M_{ii}| \leq n\|M\|$. But $|\operatorname{trace}(M)| \leq \sum_{i=1}^{n} |M_{ii}|$ so that $|\operatorname{trace}(M)| \leq n\|M\|$. Since trace is linear this shows that it is continuous. □

Lemma 52. If $P \in M_n(\mathbb{C})$ is a projection, then $\operatorname{trace}(P) = \operatorname{rank}(P)$.

Proof. If P is a projection then there is a unitary matrix U and a matrix D with ones or zeros on the diagonal and zeros everywhere else so tha $P = UDU^{-1}$ (take U to be a basis of eigenvectors, and D the corresponding matrix of eigenvalues). Then:

$$\mathrm{trace(P)} = \mathrm{trace(UDU^{-1})} = \mathrm{trace(D)}$$

But $\mathrm{trace}(D) = \mathrm{rank}(D)$ and $\mathrm{rank}(D) = \mathrm{rank}(P)$. Thus $\mathrm{trace}(P) = \mathrm{rank}(P)$. $\square$

Theorem 53. Let $f \in \ell^1(G) \subseteq C^*(G)$ be a non-trivial projection. Then there is some $\alpha \in \mathbb{N} \cup \{0\}$ such that $\mathrm{rank}(\mathcal{F}(f)(\underline{a})) = \alpha$ for all $\underline{a} \in \mathbb{T}^2$. In this case we say that the rank of f is α.

Proof. Let $F = \mathcal{F}(f)$ and $F(\mathbb{T}^2) = \{F(\underline{a}) : \underline{a} \in \mathbb{T}^2\}$. Then $\mathrm{rank} : F(\mathbb{T}^2) \to \mathbb{C}$ is continuous. But for any $\underline{a} \in \mathbb{T}^2$, $\mathrm{rank}(F(\underline{a})) \in \mathbb{N} \cup \{0\}$. Since $\mathbb{T}^2$ is connected, this implies rank is constant on $F(\mathbb{T}^2)$. $\square$

Now $\mathrm{supp}(f) = \{\pi \in \widehat{G} : \pi(f) \neq 0\}$. From the previous chapter, defining $\Pi_{[z,w]}(F) = F(z, w)$, Π is a representation of $C^*(G)$ with each irreducible representation of $C^*(G)$ corresponding to the projection onto a block of $UF(z, w)U^{-1}$ where U is the matrix that block diagonalizes $F(z, w)$.

Lemma 54. Let $f \in \ell^1(G)$ be a projection. Then there exists $\alpha \in \mathbb{N} \cup \{0\}$ such that for each $[z, w] \in \mathbb{T}^2/D$, the elements in the set $\{\mathrm{rank}(\pi(f)) : \pi \in \mathrm{supp}(f), q(\pi) = [z, w]\}$ sum to α.

This puts a restriction on the number and size of representations at each point $[z, w] \in \mathbb{T}^2/D$ in a compact open set that is the support set of a projection. To reflect this in $\mathcal{G}$, we can add a weight to each vertex defined to be the dimension of the corresponding representation in $\widehat{G}$. We call a compact open set in $\widehat{G}$ *viable* if it satisfies the rank condition in lemma 54. In $\mathcal{G}$ a viable compact open set is a compact open set such that the sum of the weights in each strata is non-increasing as you move away from the source.

5.2 The Example: *p*2

Realize p2 as $p2 = \{[I, (a, b)] : a, b \in \mathbb{Z}\} \cup \{[-I, (a, b)] : a, b \in \mathbb{Z}\}$, a group of affine transformations.

The dual of $C^*(p2)$ consists of the representations given by point evaluation of $C^*(G)$ on $\mathbb{T}^2$, except at four special points: $(1,1),(-1,1),(1,-1),(-1,-1) \in \mathbb{T}^2/D$. Here the point evaluation representation splits into two 1-dimensional irreducible representations:

$$\pi^+_{z,w}(F) = F_1(z,w) + F_{-1}(z,w)$$

$$\pi^-_{z,w}(F) = F_1(z,w) - F_{-1}(z,w).$$

Figure 7 shows a picture of the dual space in $2D$ on the parametrized torus and in $3D$.

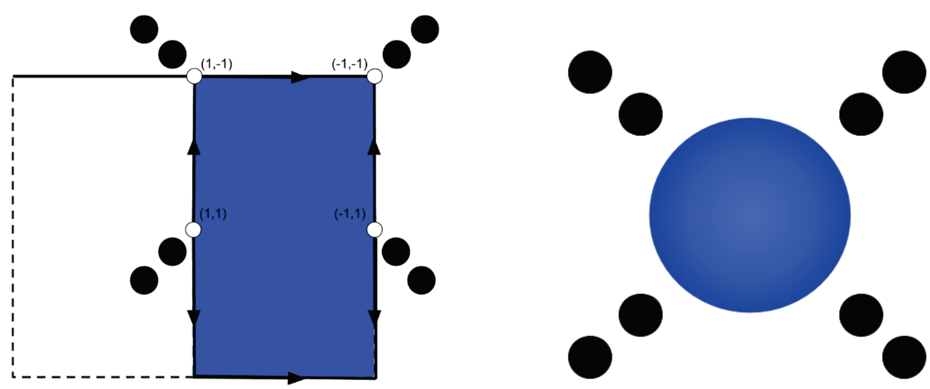

Figure 5.1: The dual space of p2 in 2D and 3D.

Then D is $\{I, -I\}$ and the finite subgroups of G are of the form $K_{a,b} = \{[-I,(a,b)],[I,(0,0)]\}$ where $(a,b) \in \mathbb{Z}^2$. Let $f_K = \frac{1}{2}(\delta_{[-I,(a,b)]} + \delta_{[I,(0,0)]})$. Each $K_{a,b}$ has two representations in its dual space: the identity character and the character which is 1 on $[1,(0,0)]$ and -1 on $[-I,(a,b)]$. Thus subgroup projections in $\ell^1(p2)$ have the following form:

$$f = \frac{1}{2}(\delta_{[1,(0,0)]} \pm \delta_{[\sigma,(n,m)]})$$

$$\mathcal{F}(f)(z,w) = \frac{1}{2}\begin{bmatrix} 1 & \pm\overline{z}^n\overline{w}^m \\ \pm z^n w^m & 1 \end{bmatrix}$$

The support sets of these subgroup projections are determined by the parity of n and m. This gives rise to 8 different sets. These can be seen in Figure 5.2. White dots are empty, while black dots are included in the support set. Of course the centre sphere is also included in the support set. To obtain the other four, make the white dots black and the black dots white. A light blue background is included in order to visually separate the pictures.

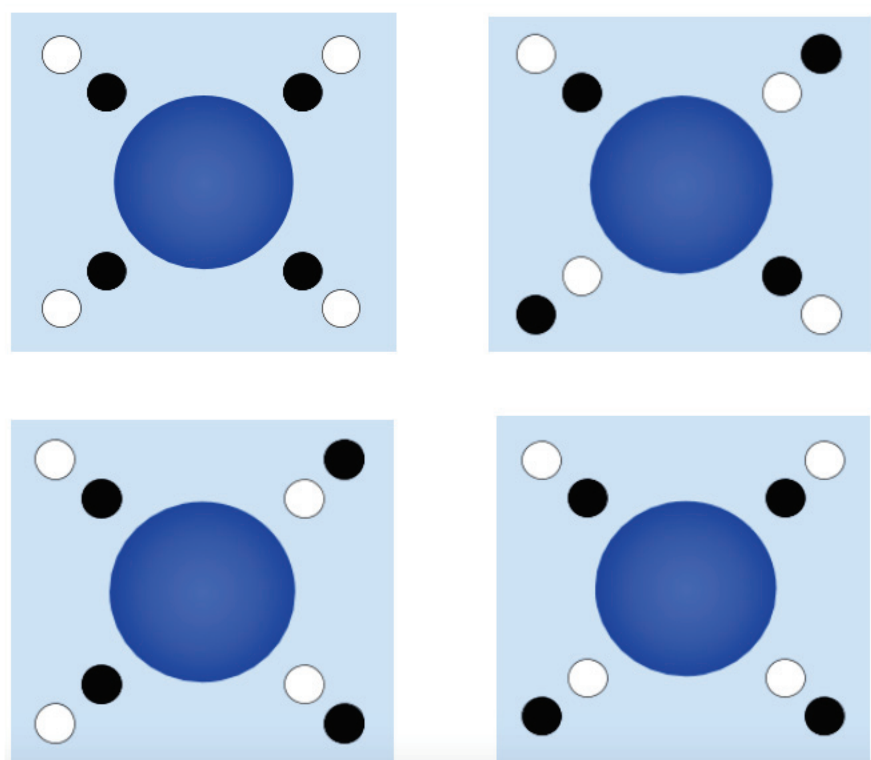

Figure 5.2: Support sets of the finite subgroup projections in $\ell^1(p2)$.

We will now construct a projection whose support set is not the support set of a finite subgroup projection. To do this, we derive a complete set of conditions that characterize projections in $\ell^1(p2)$. First, note that:

$$C^*(G) = \left\{ \begin{bmatrix} F_1(z,w) & F_{-1}(\overline{z},\overline{w}) \\ F_{-1}(z,w) & F_1(\overline{z},\overline{w}) \end{bmatrix} : F_1, F_{-1} \in C(\mathbb{T}^2)\} \right\}$$

$$\ell^1(p2) = \left\{ \begin{bmatrix} \widehat{f_1}(z,w) & \widehat{f_{-1}}(\overline{z},\overline{w}) \\ \widehat{f_{-1}}(zw) & \widehat{f_1}(\overline{z},\overline{w}) \end{bmatrix} : f_1, f_{-1} \in \ell^1(Z^2) \right\}$$

Since $\mathcal{F}$ is a C*-isomorphism, f is a projection in $C^*(p2)$ if and only if $\mathcal{F}(f)$ is a projection in $M_2(C(\mathbb{T}^2))^D$. So consider $F \in M_2(C(\mathbb{T}^2))^D$ such that $F = F^2 = F^*$. We can obtain an equivalent set of equations in terms of the functions F_1 and F_{-1}:

$F^2 = F$ is equivalent to:

$$\begin{aligned}
F_1(z,w) &= F_1(z,w)^2 + F_{-1}(z,w)F_{-1}(\overline{z},\overline{w}) \\
F_{-1}(z,w) &= F_1(z,w)F_{-1}(z,w) + F_{-1}(z,w)F_1(\overline{z},\overline{w}) \\
F_{-1}(\overline{z},\overline{w}) &= F_1(z,w)F_{-1}(\overline{z},\overline{w}) + F_{-1}(\overline{z},\overline{w})F_1(\overline{z},\overline{w}) \\
F_1(\overline{z},\overline{w}) &= F_{-1}(z,w)F_{-1}(\overline{z},\overline{w}) + F_1(\overline{z},\overline{w})^2
\end{aligned}$$

$F = F^*$ is equivalent to:

$$\begin{aligned}
F_1(z,w) &= \overline{F_1(z,w)} \\
F_{-1}(z,w) &= \overline{F_{-1}(\overline{z},\overline{w})} \\
F_{-1}(\overline{z},\overline{w}) &= \overline{F_{-1}(z,w)} \\
F_1(\overline{z},\overline{w}) &= \overline{F_1(\overline{z},\overline{w})}
\end{aligned}$$

Simplifying, these together become:

$$\begin{aligned}
&F_1(z,w) \in \mathbb{R} \\
&F_{-1}(\overline{z},\overline{w}) = \overline{F_{-1}(z,w)} \\
&F_{-1}(z,w)(1 - F_1(z,w) - F_1(\overline{z},\overline{w})) = 0 \\
&F_1(z,w) = \frac{1 \pm \sqrt{1 - 4|F_{-1}(z,w)|^2}}{2}
\end{aligned}$$

If F_1 is the positive root over all of $\mathbb{T}^2$ or the negative root over all of $\mathbb{T}^2$ then in particular $F_1(z,w) = F_1(\overline{z},\overline{w})$ so that the second equation implies that $F_1(z,w) = 1/2$ whenever $F_{-1}(z,w) \neq 0$. But if $F_{-1}(z,w) = 0$ then $F_1(z,w) = 0$. So since F_1 must be continuous, either F_{-1} is constantly 0 or F_1 is constantly $\frac{1}{2}$. The first case implies that F is the identity projection. The second gives:

$$F = \begin{bmatrix} \frac{1}{2} & \overline{F_{-1}(z,w)} \\ F_{-1}(z,w) & \frac{1}{2} \end{bmatrix}$$

with $|F_{-1}(z,w)| = \frac{1}{2}$, $\mathcal{F}^{-1}(F_{-1}) \in \ell^1(\mathbb{Z})$ and $F_{-1}(\overline{z},\overline{w}) = \overline{F_{-1}(z,w)}$ for all $(z,w) \in \mathbb{T}^2$.

Since F_{-1} is a continuous function onto the half unit circle, we know there exists a continuous $f : [-1,1] \times [-1,1] \to [-1,1]$ such that:

$$
\begin{aligned}
&F_{-1}(e^{\pi i x}, e^{\pi i y}) = \frac{1}{2} e^{\pi i f(x,y)} \\
&f(1,y) = f(-1,y) + 2k_y \\
&f(x,1) = f(x,-1) + 2k_x \\
&f(-x,-y) = -f(x,y) + 2k_{x,y}
\end{aligned}
$$

where k_x, k_y and $k_{x,y}$ are integers. The last condition shows that on each line through the origin f is an odd function shifted by an integer.

We could also force F_1 to be the positive root on a fundamental domain for the action of D on $\mathbb{T}$ and the negative root on the fundamental domain shifted by the action of D (i.e., $(z,w) \mapsto (\overline{z},\overline{w})$). We would need F_1 to be well-defined on the boundary of the domain. Note that with this definition of F_1, the second equation is automatically satisfied. An example would be to take the fundamental domain $\mathbb{T}^2_L = \{(z,w) : Im(z) > 0\}$ and define $F_{-1}(z,w) = z^2 + z$. Then $\overline{z}^2 + \overline{z} = \overline{z^2+z}$ and $\mathcal{F}^{-1}(F_{-1}) \in \ell^1(\mathbb{Z})$. On $\mathbb{T}^2_L$, $F_1(z,w) = \frac{1+\sqrt{1-4|z+1|^2}}{2}$ and $F_1(z,w) = \frac{1-\sqrt{1-4|z+1|^2}}{2}$ on the rest of the torus. Note that regardless of the choice of fundamental domain, $F_1(\pm 1, \pm 1) = 1/2$ and $|F_{-1}(\pm 1, \pm 1)| = 1/2$.

Lemma 55. If $F_1 : [-1,1] \times [-1,1] \to \mathbb{C}$ is twice continuously differentiable with $F(1,x) = F(-1,x)$ and $F(x,1) = F(x,-1)$ then $\mathcal{F}^{-1}(F) \in \ell^1(\mathbb{Z}^2)$.

Proof. First suppose $f : (-1,1] \to \mathbb{C}$ is differentiable. Then

$$
\begin{aligned}
\mathcal{F}^{-1}(\frac{df}{dx})(a) &= \int_{-1}^{1} \frac{df}{dx}(x) e^{\pi i a x} dx \\
&= f(x) \pi i a e^{\pi i a x}|_{-1}^{1} - \int_{-1}^{1} f(x) \pi i a e^{\pi i x a} dx \\
&= -\pi i a \mathcal{F}^{-1}(f)(a)
\end{aligned}
$$

so that $\mathcal{F}^{-1}(f)(a) = \frac{i}{\pi a}\mathcal{F}^{-1}(\frac{df}{dx})(a)$.

Now applying this once for each variable to $F : (-1, 1] \times (-1, 1] \to \mathbb{C}$ we get that

$$\mathcal{F}^{-1}(F)(a, b) = \frac{-1}{\pi^2 ab}\mathcal{F}^{-1}(F_{xy})$$

where $F_{xy} = (\frac{d^2 F}{dxdy})(a, b)$.

Now since the Plancherel transform $\mathcal{P} : \ell^2(\mathbb{Z}^2) \to L^2(\mathbb{T}^2)$ is an isometric isomorphism and since $C(\mathbb{T}^2) \subseteq L^2(\mathbb{T}^2)$, we have that $\mathcal{F}^{-1}(F)$ is square summable. The function $h(a, b) = \frac{-1}{\pi^2 ab}$ is also in $\ell^2(\mathbb{Z}^2)$. Thus the product $\mathcal{F}^{-1}(F)$ is in $\ell^1(\mathbb{Z}^2)$. $\square$

We will work on a parametrized torus. The conditions in Theorem 2 can be easily adjusted to reflect this. Parametrize by: $(x, y) \mapsto (e^{i\pi x}, e^{i\pi y})$. Next, let $\mathcal{L} = [-1, 0] \times [-1, 1]$ and $\mathcal{R} = [0, 1] \times [-1, 1]$, the left and right halves of the parametrized torus, respectively. Define:

$$F_{-1}(x, y) = \frac{1}{2}(\cos(\pi x)\cos(\pi xy) + ix^2 \sin(\pi y))$$

$$F_1(x, y) = \begin{cases} \frac{1}{2} + \frac{1}{2}\sqrt{1 - \cos^2(\pi x)\cos^2(\pi xy) + x^4 \sin^2(\pi y)} & \text{for } (x, y) \in \mathcal{L} \\ \\ \frac{1}{2} - \frac{1}{2}\sqrt{1 - \cos^2(\pi x)\cos^2(\pi xy) + x^4 \sin^2(\pi y)} & \text{for } (x, y) \in \mathcal{R} \end{cases}$$

Then F_1 and F_{-1} define a projection F in $C^*(p2)$. The support set of F is shown in Figure 5.3.

It is clear that F_{-1} has continuous first and second partial derivatives. For F_1, one must check that the partials are continuous at the points along the boundary of $\mathcal{L}$ and $\mathcal{R}$. This is indeed the case. Thus $\mathcal{F}^{-1}(F)$ is an element of $\ell^1(p2)$. We have constructed a projection in $\ell^1(p2)$ onto a certain viable compact open set that is not the support set of a finite subgroup projection. This suggests that knowledge of the form of the compact open subsets of $\widehat{G}$ can be helpful in finding "unusual" projections.

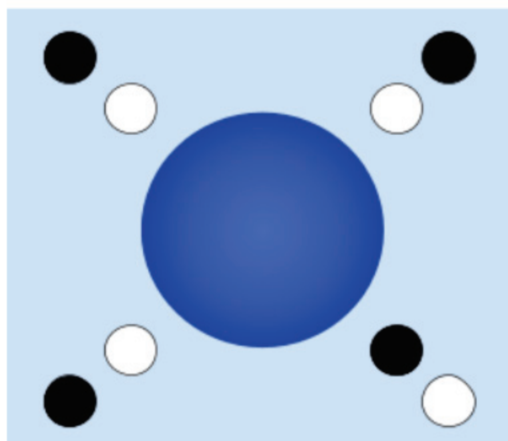

Figure 5.3: Support set of the constructed projection

Chapter 6

Conclusion

The dual space of a wallpaper group G consists of the union of the dual spaces of the fibres of a C^*-bundle whose underlying space is $\mathbb{T}^2/D$. We showed that $\mathbb{T}^2/D$ is a stratified space and how the topology of $\widehat{G}$ is reflected by this. Essentially, $\widehat{G}$ consists of a 2-manifold surrounded by "strata pieces" which locally are copies of the strata of $\mathbb{T}^2/D$. There is a graph associated with $\widehat{G}$ that encodes the topological information in $\widehat{G}$ that is relevant to compact open subsets in $\widehat{G}$. The compact open subsets of $\widehat{G}$ then correspond to a certain kind of subgraph. Using p2 as an example, we showed that it is possible to construct a projection onto a compact open set in $\widehat{p2}$ that is not a finite-subgroup projection. This compact open set is also not the support set of a finite-subgroup projection. Future work might aim to show how to construct a projection in $\ell^1(G)$ for a given viable compact open set in the dual space of a given wallpaper group G. A natural question is whether the compact open subsets in the dual space of a general crystallographic group G may be described in a similar way, i.e. as subgraphs of a graph associated with $\widehat{G}$.

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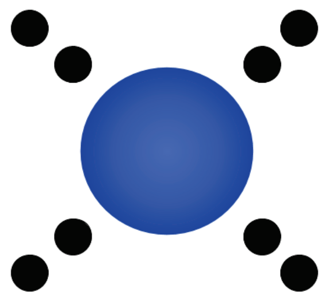
Appendix A

Pictures of the Dual Spaces

p2



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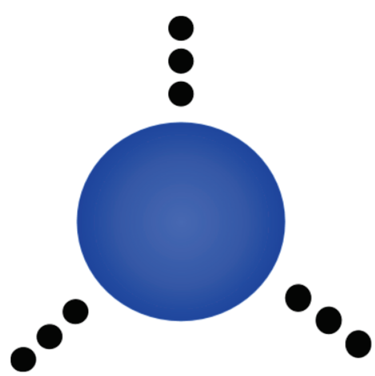

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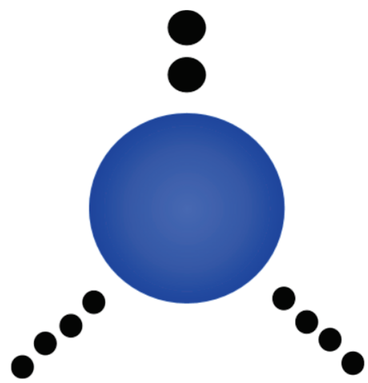

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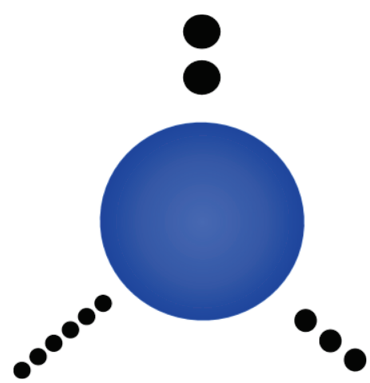

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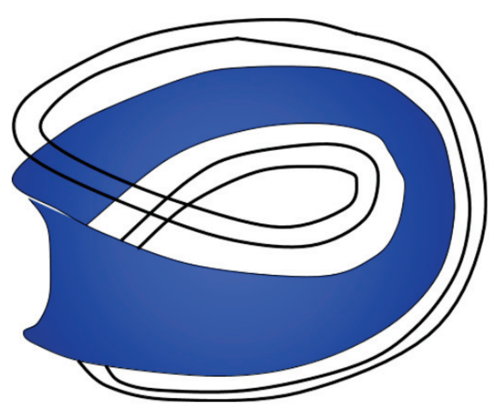

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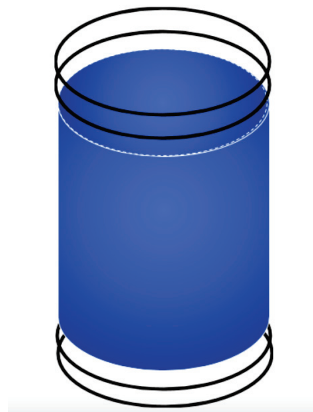

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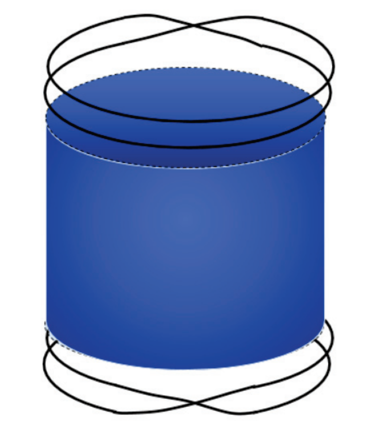

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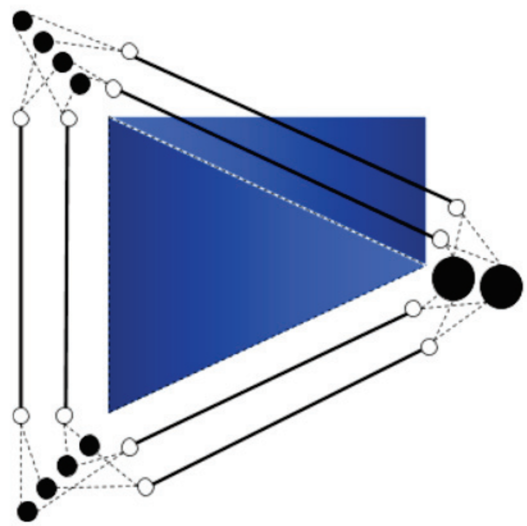

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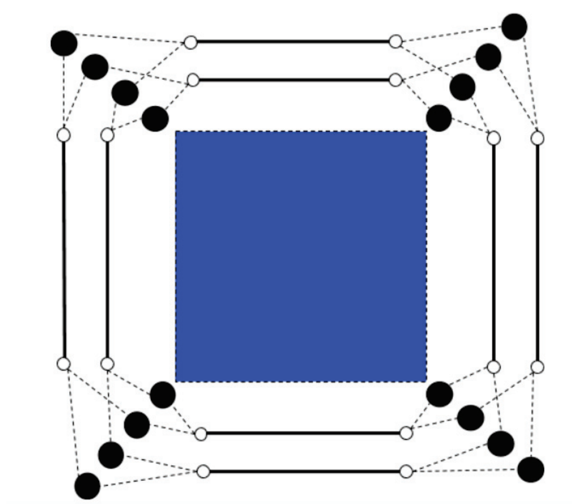
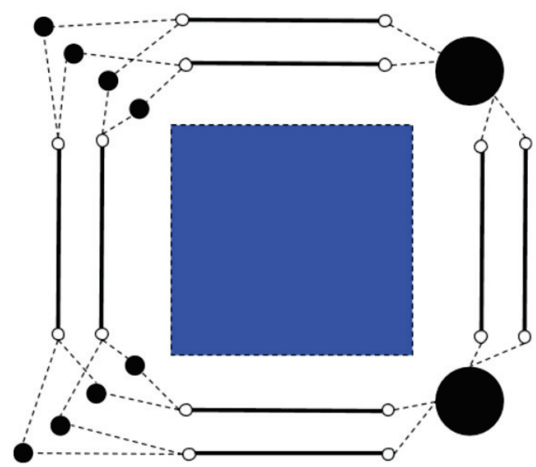
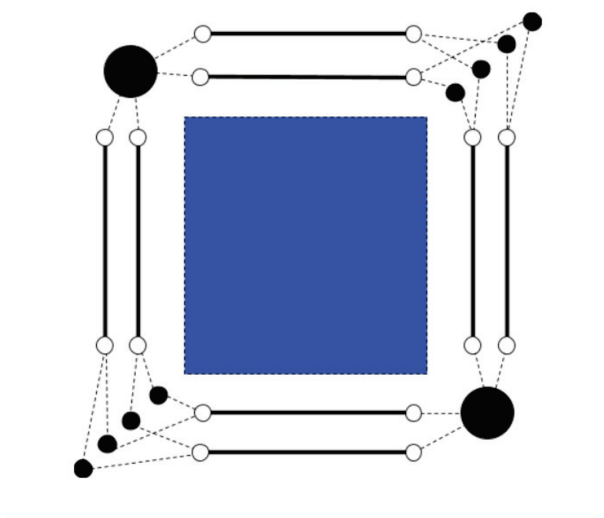


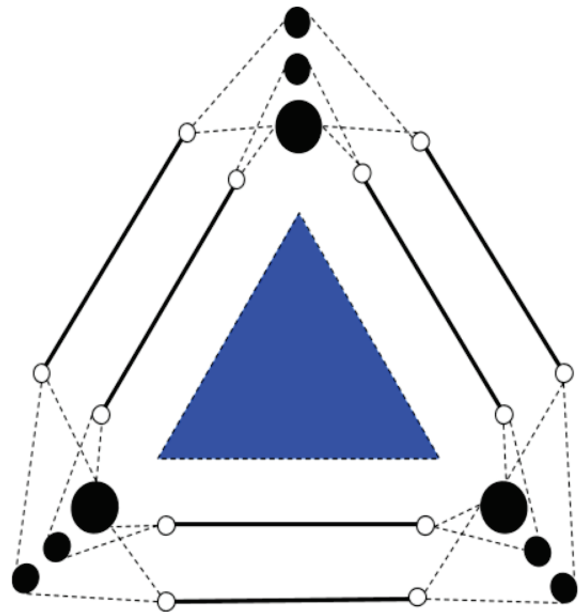
pmg



pgg

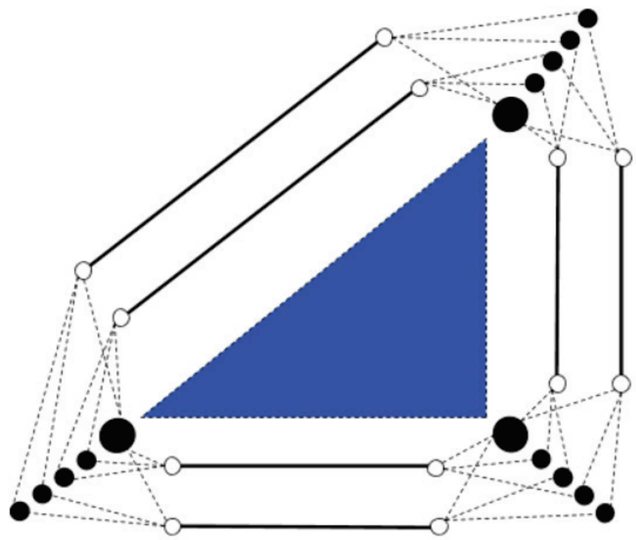


p3m1



p31m

p4mm



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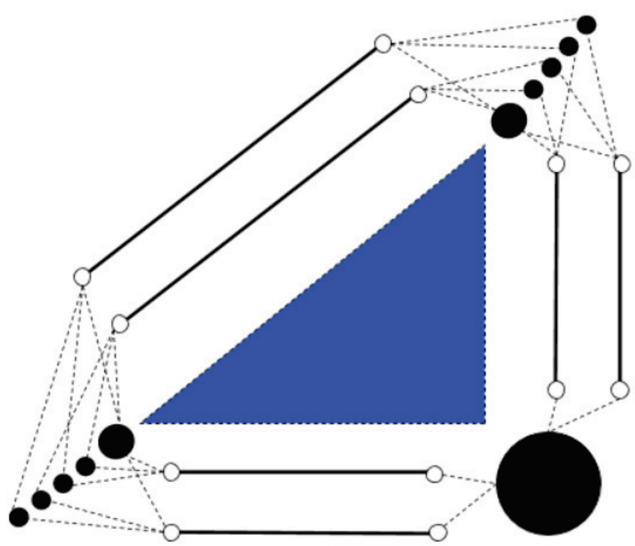

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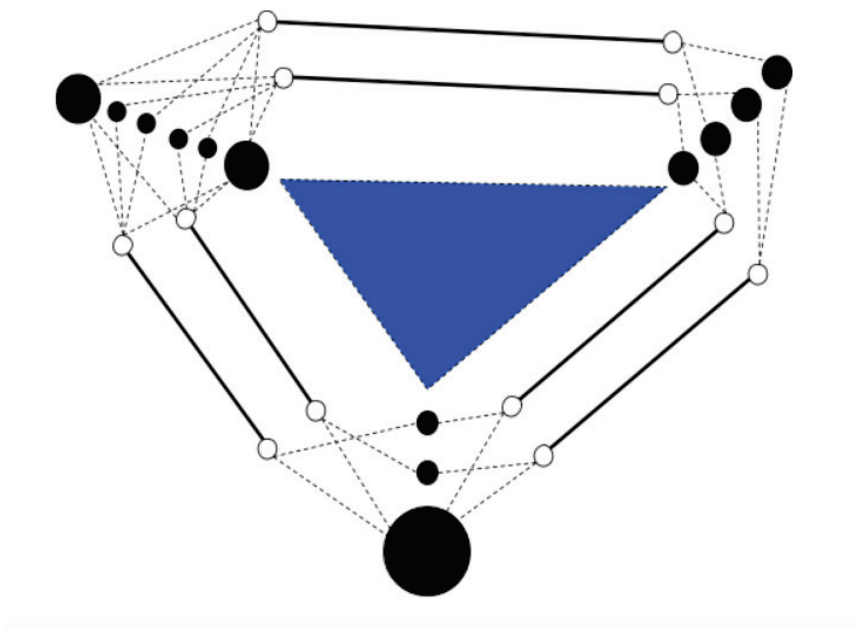
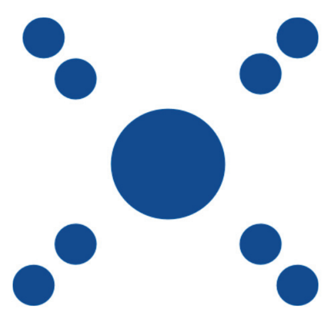


Appendix B

Dual Space Graphs

This sections contains the graphs of the dual spaces of the wallpaper groups. Vertices in the same strata are grouped together, spatially (along a line radiating from the centre vertex). There is an edge from the large centre vertex to each of the other vertices of the graph, but these edges are not shown in order to keep the pictures uncluttered. The graph of the dual space of p1 is not included as this group is abelian and so its graph consists of a single vertex.

p2



p3

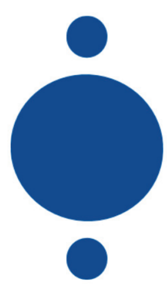
p4

p6

cm

pm

pg



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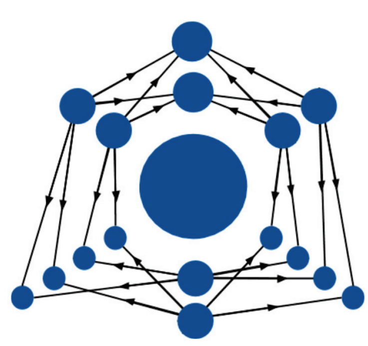

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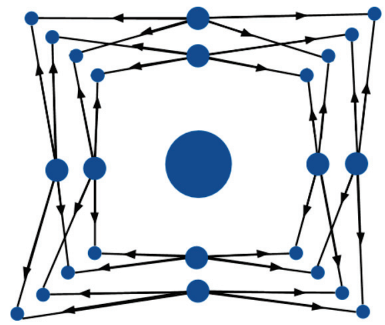
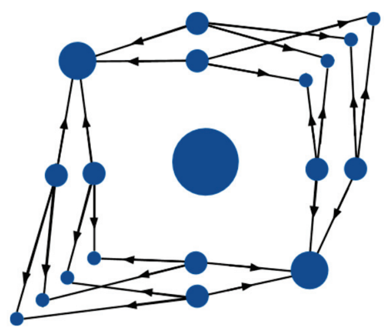
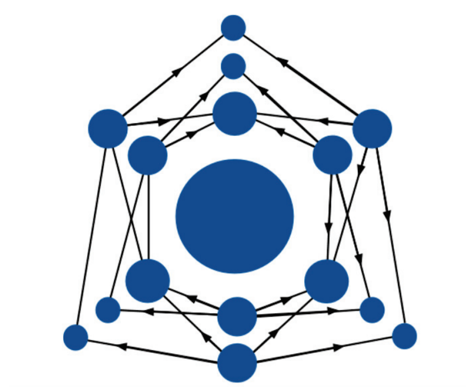


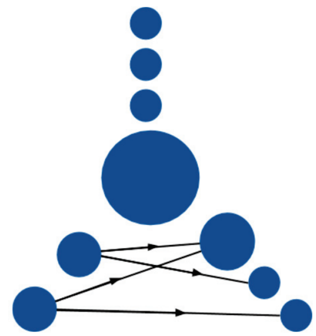
pgg



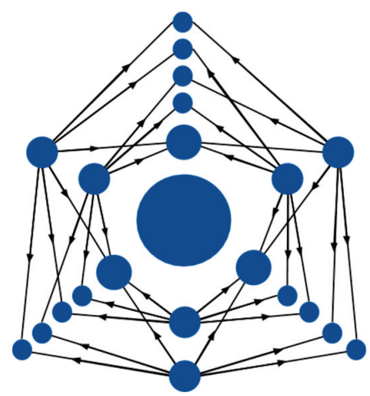
p3m1



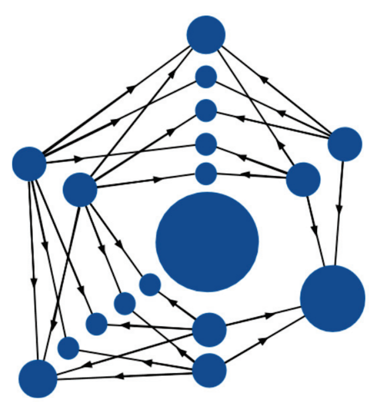
p31m



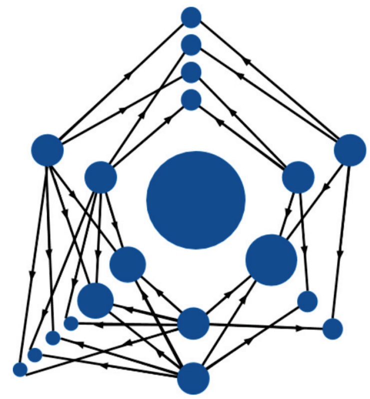
p4mm



p4mg



p6mm



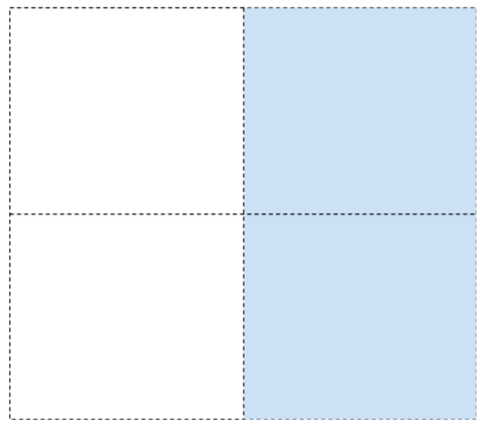
Appendix C

*-Representations of the Wallpaper Groups

This appendix gives a set of representatives for the equivalence classes of $\widehat{G}$, for each non-trivial wallpaper group. These calculations were worked out in [19]. For each wallpaper group, we also include a picture of a set of representatives for the equivalence classes of $\mathbb{T}^2/D$. This is shown on a torus which has been parametrized by $(x, y) \mapsto (e^{\pi i x}, e^{\pi i y})$, $(x, y) \in [-1, 1) \times [-1, 1)$. Be aware that some edges are identified (glued), according to the action of the point group D. Representations are organized by strata. The first strata listed is always Ω, which contains just one irreducible representation above each point. We leave the description of Ω blank, as it is the complement of the union of the rest of the strata in $\mathbb{T}^2/D$. The kernel of a $*-$representation π consists of functions F in $C^*(G)$ such that $\pi(F) = 0$. Note that $\pi(F)$ is a matrix in $M_n(\mathbb{C})$, for some $n \leq |D|$. Knowing the kernels of elements of $\widehat{G}$ allows us to explicitly check statements about the topology of $\widehat{G}$.

p2

Point group: $\mathbb{Z}^2 = \{1, \sigma\}$.



Strata

$$\pi(F) = \begin{bmatrix} F_1(z,w) & F_\sigma(\overline{z},\overline{w}) \\ F_\sigma(z,w) & F_1(\overline{z},\overline{w}) \end{bmatrix}$$

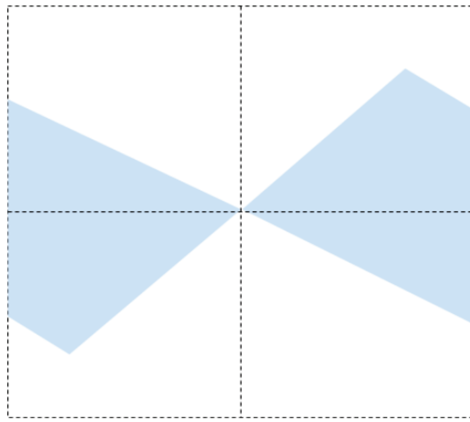
Strata $(1,1),(1,-1),(-1,1),(-1,-1)$

$$\pi_1(F) = [F_1(z,w) - F_\sigma(z,w)]$$

$$\pi_2(F) = [F_1(z,w) + F_\sigma(z,w)]$$

p3

Point group: $\mathbb{Z}_3 = \{1, \sigma, \sigma^2\}$



Strata

$$\pi(F) = \begin{bmatrix} F_1(z,w) & F_{\sigma^2}(\overline{zw}, z) & F_\sigma(w, \overline{zw}) \\ F_\sigma(z,w) & F_1(\overline{zw}, z) & F_{\sigma^2}(w, \overline{zw}) \\ F_{\sigma^2}(z,w) & F_\sigma(\overline{zw}, z) & F_1(w, \overline{zw}) \end{bmatrix}$$

Strata $(1,1), (e^{-i\pi/3}, e^{-i\pi/3}), e^{-2i\pi/3, -2i\pi/3}$

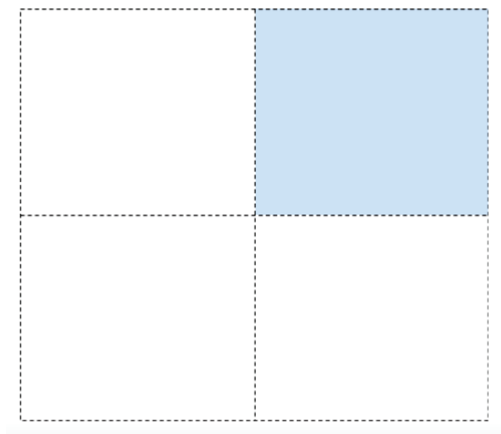
$$\pi_1(F) = F_1(z,w) + F_{\sigma^2}(z,w) + F_\sigma(z,w)$$

$$\pi_2(F) = F_1(z,w) + \tfrac{-1}{2}(1 - \sqrt{3}i)F_{\sigma^2} + \tfrac{-1}{2}(1 + \sqrt{3}i)F_\sigma(z,w)$$

$$\pi_3(F) = F_1(z,w) + \tfrac{-1}{2}(1 + \sqrt{3}i)F_{\sigma^2}(z,w) + \tfrac{-1}{2}(1 - \sqrt{3}i)F_\sigma(z,w)$$

p4

Point group: $\mathbb{Z}_4 = \{1, \sigma, \sigma^2, \sigma^3\}$



Strata

$$\pi(F) = \begin{bmatrix} F_1(z,w) & F_{\sigma^3}(\overline{w},z) & F_{\sigma^2}(\overline{z},\overline{w}) & F_\sigma(w,\overline{z}) \\ F_\sigma(z,w) & F_1(\overline{w},z) & F_{\sigma^3}(\overline{z},\overline{w}) & F_{\sigma^2}(w,\overline{z}) \\ F_{\sigma^2}(z,w) & F_\sigma(\overline{w},z) & F_1(\overline{z},\overline{w}) & F_{\sigma^3}(w,\overline{z}) \\ F_{\sigma^3}(z,w) & F_{\sigma^2}(\overline{w},z) & F_\sigma(\overline{z},\overline{w}) & F_1(w,\overline{z}) \end{bmatrix}$$

Strata $(1,1), (-1,1), (-1,-1)$

$$\pi_1(F) = [F_1(z,w) + F_{\sigma^3}(z,w) + F_{\sigma^2}(z,w) + F_\sigma(z,w)]$$

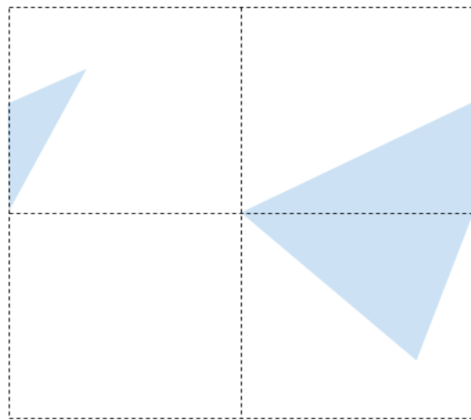
$$\pi_2(F) = [F_1(z,w) - F_{\sigma^3}(z,w) + F_{\sigma^2}(z,w) - F_\sigma(z,w)]$$

$$\pi_3(F) = [F_1(z,w) + iF_{\sigma^3}(z,w) - F_{\sigma^2}(z,w) - iF_\sigma(z,w)]$$

$$\pi_4(F) = [F_1(z,w) - iF_{\sigma^3}(z,w) - F_{\sigma^2}(z,w) + iF_{\sigma}(z,w)]$$

p6

Point group: $\mathbb{Z}_6 = \{1, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5\}$



Strata

$$\pi(F) = \begin{bmatrix} F_1(z,w) & F_{\sigma^5}(z\overline{w},z) & F_{\sigma^4}(\overline{w},z\overline{w}) & F_{\sigma^3}(\overline{z},\overline{w}) & F_{\sigma^2}(\overline{z}w,\overline{z}) & F_{\sigma}(w,\overline{z}w) \\ F_{\sigma}(z,w) & F_1(z\overline{w},z) & F_{\sigma^5}(\overline{w},z\overline{w}) & F_{\sigma^4}(\overline{z},\overline{w}) & F_{\sigma^3}(\overline{z}w,\overline{z}) & F_{\sigma^2}(w,\overline{z}w) \\ F_{\sigma^2}(z,w) & F_{\sigma}(z\overline{w},z) & F_1(\overline{w},z\overline{w}) & F_{\sigma^5}(\overline{z},\overline{w}) & F_{\sigma^4}(\overline{z}w,\overline{z}) & F_{\sigma^3}(w,\overline{z}w) \\ F_{\sigma^3}(z,w) & F_{\sigma^2}(z\overline{w},z) & F_{\sigma}(\overline{w},z\overline{w}) & F_1(\overline{z},\overline{w}) & F_{\sigma^5}(\overline{z}w,\overline{z}) & F_{\sigma^4}(w,\overline{z}w) \\ F_{\sigma^4}(z,w) & F_{\sigma^3}(z\overline{w},z) & F_{\sigma^2}(\overline{w},z\overline{w}) & F_{\sigma}(\overline{z},\overline{w}) & F_1(\overline{z}w,\overline{z}) & F_{\sigma^5}(w,\overline{z}w) \\ F_{\sigma^5}(z,w) & F_{\sigma^4}(z\overline{w},z) & F_{\sigma^3}(\overline{w},z\overline{w}) & F_{\sigma^2}(\overline{z},\overline{w}) & F_{\sigma}(\overline{z}w,\overline{z}) & F_1(w,\overline{z}w) \end{bmatrix}$$

Strata $(1,1)$

$$\pi_1(F) = [F_1 + F_{\sigma^5} + F_{\sigma^4} + F_{\sigma^3} + F_{\sigma^2} + F_{\sigma}](1,1)$$

$$\pi_2(F) = [F_1 - F_{\sigma^5} + F_{\sigma^4} - F_{\sigma^3} + F_{\sigma^2} - F_{\sigma}](1,1)$$

$$\pi_3(F) = [F_1 - F_{\sigma^3} + \tfrac{-1}{2}(1-\sqrt{3}i)F_{\sigma^2} + \tfrac{1}{2}(1+\sqrt{3}i)F_{\sigma} + \tfrac{-1}{2}(1+\sqrt{3}i)F_{\sigma^4} - \tfrac{1}{2}(1-\sqrt{3}i)F_{\sigma^5}](1,1)$$

$\pi_4(F) = [F_1 + F_{\sigma^3} + \frac{-1}{2}(1+\sqrt{3}i)F_{\sigma^2} + \frac{-1}{2}(1-\sqrt{3}i)F_\sigma + \frac{-1}{2}(1-\sqrt{3}i)F_{\sigma^4} + \frac{-1}{2}(1+\sqrt{3}i)F_{\sigma^5}](1,1)$

$\pi_5(F) = [F_1 + F_{\sigma^3} + \frac{-1}{2}(1-\sqrt{3}i)F_{\sigma^2} + \frac{1}{2}(1+\sqrt{3}i)F_\sigma + \frac{-1}{2}(1+\sqrt{3}i)F_{\sigma^4} + \frac{-1}{2}(1-\sqrt{3}i)F_{\sigma^5}](1,1)$

$\pi_6(F) = [F_1 - F_{\sigma^3} + \frac{-1}{2}(1+\sqrt{3}i)F_{\sigma^2} + \frac{1}{2}(1-\sqrt{3}i)F_\sigma + \frac{-1}{2}(1-\sqrt{3})F_{\sigma^4} + \frac{1}{2}(1+\sqrt{3}i)F_{\sigma^5}](1,1)$

Strata $(-1,1)$

$$\pi_1(F) = \begin{bmatrix} F_1 + F_{\sigma^3} & F_{\sigma^2} + F_{\sigma^3} & F_\sigma + F_{\sigma^4} \\ F_{\sigma^4} + F_\sigma & F_1 + F_{\sigma^3} & F_{\sigma^5} + F_{\sigma^2} \\ F_{\sigma^5} + F_{\sigma^2} & F_\sigma + F_{\sigma^4} & F_1 + F_{\sigma^3} \end{bmatrix}(-1,1)$$

$$\pi_2(F) = \begin{bmatrix} F_1 - F_{\sigma^3} & F_{\sigma^2} - F_{\sigma^3} & F_\sigma - F_{\sigma^4} \\ F_{\sigma^4} - F_\sigma & F_1 - F_{\sigma^3} & F_{\sigma^5} - F_{\sigma^2} \\ F_{\sigma^5} - F_{\sigma^2} & F_\sigma - F_{\sigma^4} & F_1 - F_{\sigma^3} \end{bmatrix}(-1,1)$$

Strata $\{(-e^{-\pi/3i}, e^{-2/3\pi i})\}$

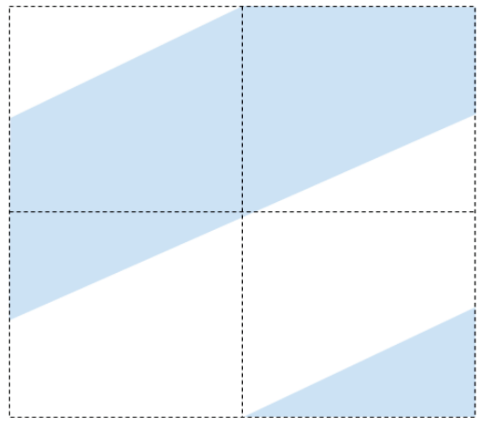
$$\pi_1(F) = \begin{bmatrix} F_1 - \frac{\sqrt{3}i+1}{2}F_{\sigma^4} + \frac{\sqrt{3}i-1}{2}F_{\sigma^2} & F_{\sigma^5} - \frac{\sqrt{3}i+1}{2}F_{\sigma^3} + \frac{\sqrt{3}i-1}{2}F_\sigma \\ F_\sigma - \frac{\sqrt{3}i+1}{2}F_\sigma^5 + \frac{\sqrt{3}i-1}{2}F_{\sigma^3} & F_1 - \frac{\sqrt{3}i+1}{2}F_{\sigma^4} + \frac{\sqrt{3}i-1}{2}F_{\sigma^2} \end{bmatrix}(-e^{-\pi/3i}, e^{-2/3\pi i})$$

$$\pi_2(F) = \begin{bmatrix} F_1 + F_{\sigma^4} + F_{\sigma^2} & F_\sigma + F_{\sigma^5} + F_{\sigma^3} \\ F_{\sigma^5} + F_{\sigma^3} + F_\sigma & F_1 + F_{\sigma^4} + F_{\sigma^2} \end{bmatrix}(-e^{-\pi/3i}, e^{-2/3\pi i})$$

$$\pi_3(F) = \begin{bmatrix} F_1 + \frac{\sqrt{3}i-1}{2}F_{\sigma^4} - \frac{\sqrt{3}i-1}{2}F_{\sigma^2} & F_{\sigma^5} + \frac{\sqrt{3}i-1}{2}F_{\sigma^3} - \frac{\sqrt{3}i+1}{2}F_{\sigma} \\ F_{\sigma} + \frac{\sqrt{3}i-1}{2}F_{\sigma^5} - \frac{\sqrt{3}i+1}{2}F_{\sigma^3} & F_1 + \frac{\sqrt{3}i-1}{2}F_{\sigma^4} - \frac{\sqrt{3}i+1}{2}F_{\sigma^2} \end{bmatrix} (-e^{-\pi/3i}, e^{-2/3\pi i})$$

cm

Point group: $\mathbb{Z}_2 = \{1, \rho\}$



Strata

$$\pi(F) = \begin{bmatrix} F_1(z,w) & F_\rho(z,z\overline{w}) \\ F_\rho(z,w) & F_1(z,z\overline{w}) \end{bmatrix}$$

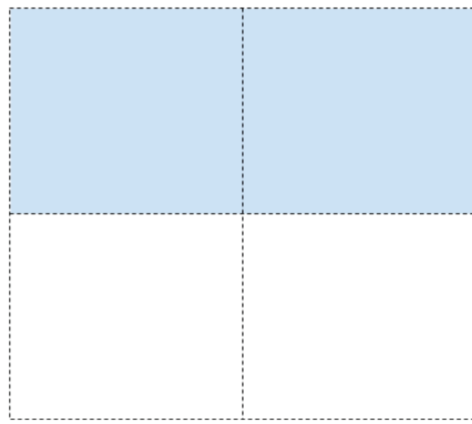
Strata $\{(w^2, w) : w \in \mathbb{T}^2\}$

$$\pi_1(F) = [F_1(w^2,w) + F_\rho(w^2,w)]$$

$$\pi_2(F) = [F_1(w^2,w) - F_\rho(w^2,w)]$$

pm

Point group: $\mathbb{Z}_2 = \{1, \rho\}$



Strata

$$\pi(F) = \begin{bmatrix} F_1(z,w) & F_\rho(z,\overline{w}) \\ F_\rho(z,w) & F_1(z,\overline{w}) \end{bmatrix}$$

Strata $\{(z,1) : z \in \mathbb{T}^2\}$

$$\pi_1(F) = [F_1(z,1) + F_\rho(z,1)]$$

$$\pi_2(F) = [F_1(z,1) - F_\rho(z,1)]$$

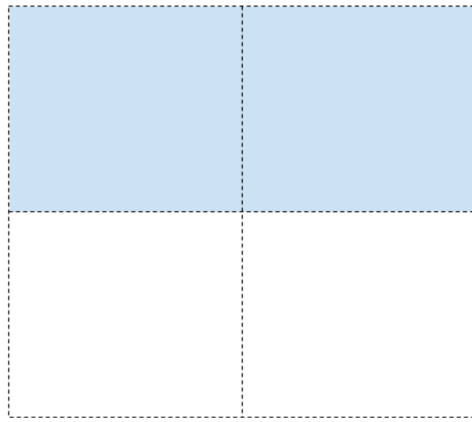
Strata $\{(z,-1) : z \in \mathbb{T}^2\}$

$$\pi_1(F) = [F_1(z, -1) + F_\rho(z, -1)]$$

$$\pi_2(F) = [F_1(z, -1) - F_\rho(z, -1)]$$

pg

Point group: $\mathbb{Z}_2 = \{1, \rho\}$



Strata

$$\pi(f) = \begin{bmatrix} F_1(z,w) & zF_\rho(z,\overline{w}) \\ F_\rho(z,w) & F_1(z,\overline{w}) \end{bmatrix}$$

Strata $\{(z,1) : z \in \mathbb{T}^2\}$

$$\pi_1(F) = [F_1(z,1) - z^{1/2}F_\rho(z,1)]$$

$$\pi_2(F) = [F_1(z,1) + z^{1/2}F_\rho(z,1)]$$

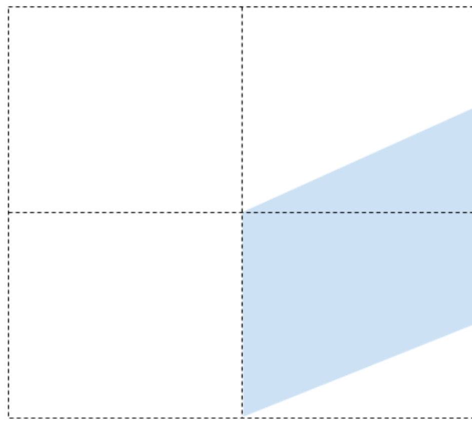
Strata $\{(z,-1) : z \in \mathbb{T}^2\}$

$$\pi_1(F) = [F_1(z,-1) + z^{1/2}F_\rho(z,-1)]$$

$$\pi_2(F) = [F_1(z,-1) - z^{1/2} F_\rho(z,-1)]$$

cmm

Point group: $D_4 = \{1, \rho_1, \rho_2, \sigma\}$



Strata

$$\pi(F) = \begin{bmatrix} F_1(z,w) & F_\sigma(\overline{z},\overline{w}) & F_{\rho_1}(z,z\overline{w}) & F_{\rho_2}(\overline{z},\overline{z}w) \\ F_\sigma(z,w) & F_1(\overline{z},\overline{w}) & F_{\rho_2}(z,z\overline{w}) & F_{\rho_1}(\overline{z},\overline{z}w) \\ F_{\rho_1}(z,w) & F_{\rho_2}(\overline{z},\overline{w}) & F_1(z,z\overline{w}) & F_\sigma(\overline{z},\overline{z}w) \\ F_{\rho_2}(z,w) & F_{\rho_1}(\overline{z},\overline{w}) & F_\sigma(z,z\overline{w}) & F_1(\overline{z},\overline{z}w) \end{bmatrix}$$

Strata $(1,1)$

$$\pi_1(F) = [F_1 + F_\sigma - F_{\rho_1} - F_{\rho_2}](1,1)$$

$$\pi_2(F) = [F_1 - F_\sigma - F_{\rho_1} + F_{\rho_2}](1,1)$$

$$\pi_3(F) = [F_1 + F_\sigma + F_{\rho_1} + F_{\rho_2}](1,1)$$

$\pi_4(F) = [F_1 - F_\sigma + F_{\rho_1} - F_{\rho_2}](1,1)$

Strata $\{(w^2, w) : w \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} F_1(w^2,w) - F_{\rho_1}(w^2,w) & F_\sigma(\overline{w}^2,\overline{w}) - F_{\rho_2}(\overline{w}^2,\overline{w}) \\ F_\sigma(w^2,w) - F_{\rho_2}(w^2,w) & F_1(\overline{w}^2,\overline{w}) - F_{\rho_1}(\overline{w}^2,\overline{w}) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} F_1(w^2,w) + F_{\rho_1}(w^2,w) & F_\sigma(\overline{w}^2,\overline{w}) + F_{\rho_2}(\overline{w}^2,\overline{w}) \\ F_\sigma(w^2,w) + F_{\rho_2}(w^2,w) & F_1(\overline{w}^2,\overline{w}) + F_{\rho_1}(\overline{w}^2,\overline{w}) \end{bmatrix}$$

Strata $(-1,1)$

$$\pi_1(F) = \begin{bmatrix} F_1(-1,1) - F_\sigma(-1,1) & F_{\rho_1}(-1,-1) - F_{\rho_2}(-1,-1) \\ F_{\rho_1}(-1,1) - F_{\rho_2}(-1,1) & F_1(-1,-1) - F_\sigma(-1,-1) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} F_1(-1,1) + F_\sigma(-1,1) & F_{\rho_1}(-1,-1) + F_{\rho_2}(-1,-1) \\ F_{\rho_1}(-1,1) + F_{\rho_2}(-1,1) & F_1(-1,-1) + F_\sigma(-1,-1) \end{bmatrix}$$

Strata $(1,-1)$

$\pi_1(F) = [F_1 + F_\sigma - F_{\rho_1} - F_{\rho_2}](1,-1)$

$\pi_2(F) = [F_1 - F_\sigma + F_{\rho_1} - F_{\rho_2}](1,-1)$

$\pi_3(F) = [F_1 - F_\sigma - F_{\rho_1} + F_{\rho_2}](1,-1)$

$\pi_4(F) = [F_1 + F_\sigma + F_{\rho_1} + F_{\rho_2}](1, -1)$

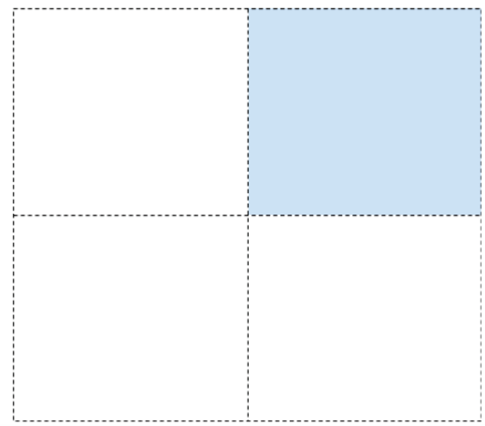
Strata $\{(1, w) : w \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} F_1(1,w) - F_{\rho_2}(1,w) & F_\sigma(1,\overline{w}) - F_{\rho_1}(1,\overline{w}) \\ F_\sigma(1,w) - F_{\rho_1}(1,w) & F_1(1,\overline{w}) - F_{\rho_2}(1,\overline{w}) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} F_1(1,w) + F_{\rho_2}(1,w) & F_\sigma(1,\overline{w}) + F_{\rho_1}(1,\overline{w}) \\ F_\sigma(1,w) + F_{\rho_1}(1,w) & F_1(1,\overline{w}) + F_{\rho_2}(1,\overline{w}) \end{bmatrix}$$

pmm

Point group: $D_4 = \{1, \rho_1, \rho_2, \sigma\}$



Strata

$$\pi(F) = \begin{bmatrix} F_1(z,w) & F_\sigma(\overline{z},\overline{w}) & F_{\rho_1}(z,\overline{w}) & F_{\rho_2}(\overline{z},w) \\ F_\sigma(z,w) & F_1(\overline{z},\overline{w}) & F_{\rho_2}(z,\overline{w}) & F_{\rho_1}(\overline{z},w) \\ F_{\rho_1}(z,w) & F_{\rho_2}(\overline{z},\overline{w}) & F_1(z,\overline{w}) & F_\sigma(\overline{z},w) \\ F_{\rho_2}(z,w) & F_{\rho_1}(\overline{z},\overline{w}) & F_\sigma(z,\overline{w}) & F_1(\overline{z},w) \end{bmatrix}$$

Strata $(1, 1)$

$$\pi_1(F) = [F_1(1,1) + F_\sigma(1,1) - F_{\rho_1}(1,1) - F_{\rho_2}(1,1)]$$

$$\pi_2(F) = [F_1(1,1) - F_\sigma(1,1) + F_{\rho_1}(1,1) - F_{\rho_2}(1,1)]$$

$$\pi_3(F) = [F_1(1,1) + F_\sigma(1,1) + F_{\rho_1}(1,1) + F_{\rho_2}(1,1)]$$

$\pi_4(F) = [F_1(1,1) - F_\sigma(1,1) - F_{\rho_1}(1,1) + F_{\rho_2}(1,1)]$

Strata $\{(1,w) : w \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} F_1(1,w) + F_{\rho_2}(1,w) & F_\sigma(1,\overline{w}) + F_{\rho_1}(1,\overline{w}) \\ F_\sigma(1,w) + F_{\rho_1}(1,w) & F_1(1,\overline{w}) + F_{\rho_2}(1,\overline{w}) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} F_1(1,w) - F_{\rho_2}(1,w) & F_\sigma(1,\overline{w}) - F_{\rho_1}(1,\overline{w}) \\ F_\sigma(1,w) - F_{\rho_1}(1,w) & F_1(1,\overline{w}) - F_{\rho_2}(1,\overline{w}) \end{bmatrix}$$

Strata $(1,-1)$

$\pi_1(F) = [F_1(1,-1) - F_\sigma(1,-1) + F_{\rho_1}(1,-1) - F_{\rho_2}(1,-1)]$

$\pi_2(F) = [F_1(1,-1) + F_\sigma(1,-1) - F_{\rho_1}(1,-1) - F_{\rho_2}(1,-1)]$

$\pi_3(F) = [F_1(1,-1) - F_\sigma(1,-1) - F_{\rho_1}(1,-1) + F_{\rho_2}(1,-1)]$

$\pi_4(F) = [F_1(1,-1) + F_\sigma(1,-1) + F_{\rho_1}(1,-1) + F_{\rho_2}(1,-1)]$

Strata $\{(z,-1) : z \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} F_1(z,-1) - F_{\rho_1}(z,-1) & F_\sigma(\overline{z},-1) - F_{\rho_2}(\overline{z},-1) \\ F_\sigma(z,-1) - F_{\rho_2}(z,-1) & F_1(\overline{z},-1) - F_{\rho_1}(\overline{z},-1) \end{bmatrix}$$

$$\pi_1(F) = \begin{bmatrix} F_1(z,-1) + F_{\rho_1}(z,-1) & F_\sigma(\overline{z},-1) + F_{\rho_2}(\overline{z},-1) \\ F_\sigma(z,-1) + F_{\rho_2}(z,-1) & F_1(\overline{z},-1) + F_{\rho_1}(\overline{z},-1) \end{bmatrix}$$

Strata $(-1,-1)$

$$\pi_1(F) = [F_1(-1,-1) - F_\sigma(-1,-1) + F_{\rho_1}(-1,-1) - F_{\rho_2}(-1,-1)]$$

$$\pi_2(F) = [F_1(-1,-1) + F_\sigma(-1,-1) - F_{\rho_1}(-1,-1) - F_{\rho_2}(-1,-1)]$$

$$\pi_3(F) = [F_1(-1,-1) + F_\sigma(-1,-1) + F_{\rho_1}(-1,-1) + F_{\rho_2}(-1,-1)]$$

$$\pi_4(F) = [F_1(-1,-1) - F_\sigma(-1,-1) - F_{\rho_1}(-1,-1) + F_{\rho_2}(-1,-1)]$$

Strata $\{(-1,w) : w \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} F_1(-1,w) + F_{\rho_2}(-1,w) & F_\sigma(-1,\overline{w}) + F_{\rho_1}(-1,\overline{w}) \\ F_\sigma(-1,w) + F_{\rho_1}(-1,w) & F_1(-1,\overline{w}) + F_{\rho_2}(-1,\overline{w}) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} F_1(-1,w) - F_{\rho_2}(-1,w) & F_\sigma(-1,\overline{w}) - F_{\rho_1}(-1,\overline{w}) \\ F_\sigma(-1,w) - F_{\rho_1}(-1,w) & F_1(-1,\overline{w}) - F_{\rho_2}(-1,\overline{w}) \end{bmatrix}$$

Strata $(-1,1)$

$$\pi_1(F) = [F_1(-1,1) - F_\sigma(-1,1) + F_{\rho_1}(-1,1) - F_{\rho_2}(-1,1)]$$

$$\pi_2(F) = [F_1(-1,1) + F_\sigma(-1,1) - F_{\rho_1}(-1,1) - F_{\rho_2}(-1,1)]$$

$$\pi_3(F) = [F_1(-1,1) - F_\sigma(-1,1) - F_{\rho_1}(-1,1) + F_{\rho_2}(-1,1)]$$

$$\pi_4(F) = [F_1(-1,1) + F_\sigma(-1,1) + F_{\rho_1}(-1,1) + F_{\rho_2}(-1,1)]$$

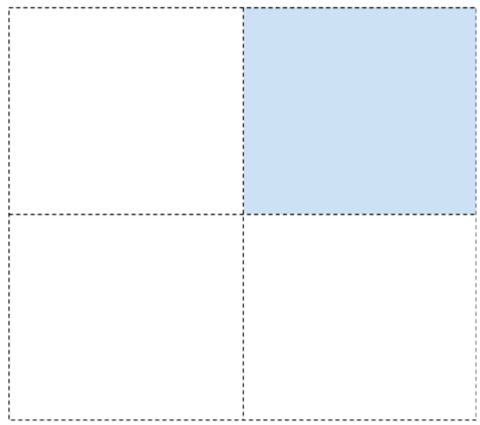
Strata $\{(z,1) : z \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} F_1(z,1) - F_{\rho_1}(z,1) & F_\sigma(\overline{z},1) - F_{\rho_2}(\overline{z},1) \\ F_\sigma(z,1) - F_{\rho_2}(z,1) & F_1(\overline{z},1) - F_{\rho_1}(\overline{z},1) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} F_1(z,1) + F_{\rho_1}(z,1) & F_\sigma(\overline{z},1) + F_{\rho_2}(\overline{z},1) \\ F_\sigma(z,1) + F_{\rho_2}(z,1) & F_1(\overline{z},1) + F_{\rho_1}(\overline{z},1) \end{bmatrix}$$

pmg

Point group: $D_4 = \{1, \rho_1, \rho_2, \sigma\}$



Strata

$$\pi(F) = \begin{bmatrix} F_1(z,w) & F_\sigma(\overline{z},\overline{w}) & zF_{\rho_1}(z,\overline{w}) & F_{\rho_2}(\overline{z},w) \\ F_\sigma(z,w) & F_1(\overline{z},\overline{w}) & F_{\rho_2}(z,\overline{w}) & F_{\rho_1}(\overline{z},w) \\ F_{\rho_1}(z,w) & F_{\rho_2}(\overline{z},\overline{w}) & F_1(z,\overline{w}) & F_\sigma(\overline{z},w) \\ F_{\rho_2}(z,w) & F_{\rho_1}(\overline{z},\overline{w}) & F_\sigma(z,\overline{w}) & F_1(\overline{z},w) \end{bmatrix}$$

Strata $(1,1)$

$$\pi_1(F) = [F_1(1,1) + F_\sigma(1,1) - F_{\rho_1}(1,1) - F_{\rho_2}(1,1)]$$

$$\pi_2(F) = [F_1(1,1) - F_\sigma(1,1) + F_{\rho_1}(1,1) - F_{\rho_2}(1,1)]$$

$$\pi_3(F) = [F_1(1,1) - F_\sigma(1,1) - F_{\rho_1}(1,1) + F_{\rho_2}(1,1)]$$

$$\pi_4(F) = [F_1(1,1) + F_\sigma(1,1) + F_{\rho_1}(1,1) + F_{\rho_2}(1,1)]$$

Strata $\{(1,w) : w \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} F_1(1,w) - F_{\rho_2}(1,w) & F_\sigma(1,\overline{w}) - F_{\rho_1}(1,\overline{w}) \\ F_\sigma(1,w) - F_{\rho_1}(1,w) & F_1(1,w) - F_{\rho_2}(1,\overline{w}) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} F_1(1,w) + F_{\rho_2}(1,w) & F_\sigma(1,\overline{w}) + F_{\rho_1}(1,\overline{w}) \\ F_\sigma(1,w) + F_{\rho_1}(1,w) & F_1(1,w) + F_{\rho_2}(1,\overline{w}) \end{bmatrix}$$

Strata $(1,-1)$

$$\pi_1(F) = [F_1(1,-1) - F_\sigma(1,-1) + F_{\rho_1}(1,-1) - F_{\rho_2}(1,-1)]$$

$$\pi_2(F) = [F_1(1,-1) + F_\sigma(1,-1) - F_{\rho_1}(1,-1) - F_{\rho_2}(1,-1)]$$

$$\pi_3(F) = [F_1(1,-1) - F_\sigma(1,-1) - F_{\rho_1}(1,-1) + F_{\rho_2}(1,-1)]$$

$$\pi_4(F) = [F_1(1,-1) + F_\sigma(1,-1) + F_{\rho_1}(1,-1) + F_{\rho_2}(1,-1)]$$

Strata $\{(z,-1) : z \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} F_1(z,-1) + z^{1/2}F_{\rho_1}(z,-1) & F_\sigma(\overline{z},-1) + z^{1/2}F_{\rho_2}(\overline{z},-1) \\ F_\sigma(z,-1) + \overline{z}^{1/2}F_{\rho_2}(z,-1) & F_1(\overline{z},-1) + \overline{z}^{1/2}F_{\rho_1}(\overline{z},-1) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} F_1(z,-1) + z^{1/2}F_{\rho_1}(z,-1) & F_\sigma(\overline{z},-1) + z^{1/2}F_{\rho_2}(\overline{z},-1) \\ F_\sigma(z,-1) + \overline{z}^{1/2}F_{\rho_2}(z,-1) & F_1(\overline{z},-1) + \overline{z}^{1/2}F_{\rho_1}(\overline{z},-1) \end{bmatrix}$$

Strata $(-1,-1)$

$$\pi(F) = \begin{bmatrix} F_1(-1,-1) + F_\sigma(-1,-1) & F_{\rho_1}(-1,-1) - F_{\rho_2}(-1,-1) \\ -F_{\rho_1}(-1,-1) - F_{\rho_2}(-1,-1) & F_1(-1,-1) - F_\sigma(-1,-1) \end{bmatrix}$$

Strata $\{(-1,w) : w \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} F_1(-1,w) + F_{\rho_2}(-1,w) & F_\sigma(-1,\overline{w}) + F_{\rho_1}(-1,\overline{w}) \\ F_\sigma(-1,w) - F_{\rho_1}(-1,w) & F_1(-1,\overline{w}) - F_{\rho_2}(-1,\overline{w}) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} F_1(-1,\overline{w}) + F_{\rho_2}(-1,\overline{w}) & F_\sigma(-1,w) + F_{\rho_1}(-1,w) \\ F_\sigma(-1,\overline{w}) - F_{\rho_1}(-1,\overline{w}) & F_1(-1,w) - F_{\rho_2}(-1,w) \end{bmatrix}$$

Strata $(-1,1)$

$$\pi(F) = \begin{bmatrix} F_1(-1,1) - F_\sigma(-1,1) & F_{\rho_2}(-1,1) + F_{\rho_1}(-1,1) \\ F_{\rho_2}(-1,1) - F_{\rho_1}(-1,1) & F_1(-1,1) + F_\sigma(-1,1) \end{bmatrix}$$

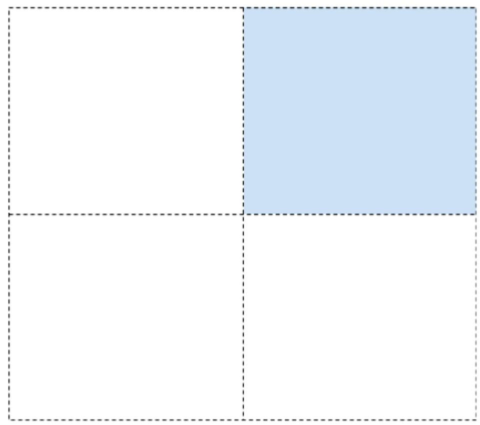
Strata $\{(z,1) : z \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} F_1(z,1) + z^{1/2}F_{\rho_1}(z,1) & F_\sigma(\overline{z},1) + z^{1/2}F_{\rho_2}(\overline{z},1) \\ F_\sigma(z,1) + \overline{z}^{1/2}F_{\rho_2} & F_1(\overline{z},1) + \overline{z}^{1/2}F_{\rho_1}(\overline{z},1) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} F_1(z,1) - z^{1/2}F_{\rho_1}(z,1) & F_\sigma(\overline{z},1) - z^{1/2}F_{\rho_2}(\overline{z},1) \\ F_\sigma(z,1) - \overline{z}^{1/2}F_{\rho_2} & F_1(\overline{z},1) - \overline{z}^{1/2}F_{\rho_1}(\overline{z},1) \end{bmatrix}$$

pgg

Point group: $D_4 = \{1, \rho_1, \rho_2, \sigma\}$



Strata

$$\pi(F) = \begin{bmatrix} F_1(z,w) & F_\sigma(\overline{z},\overline{w}) & zF_{\rho_1}(z,\overline{w}) & wF_{\rho_2}(\overline{z},w) \\ F_\sigma(z,w) & F_1(\overline{z},\overline{w}) & F_{\rho_2}(z,\overline{w}) & \overline{z}wF_{\rho_1}(\overline{z},w) \\ F_{\rho_1}(z,w) & \overline{w}F_{\rho_2}(\overline{z},\overline{w}) & F_1(z,\overline{w}) & \overline{z}F_\sigma(\overline{z},w) \\ F_{\rho_2}(z,w) & \overline{w}F_{\rho_1}(\overline{z},\overline{w}) & zF_\sigma(z,\overline{w}) & F_1(\overline{z},w) \end{bmatrix}$$

Strata $(1,1)$

$$\pi_1(F) = [F_1(1,1) - F_\sigma(1,1) + F_{\rho_1}(1,1) - F_{\rho_2}(1,1)]$$

$$\pi_2(F) = [F_1(1,1) + F_\sigma(1,1) - F_{\rho_1}(1,1) - F_{\rho_2}(1,1)]$$

$$\pi_3(F) = [F_1(1,1) - F_\sigma(1,1) - F_{\rho_1}(1,1) + F_{\rho_2}(1,1)]$$

$\pi_4(F) = [F_1(1,1) + F_\sigma(1,1) + F_{\rho_1}(1,1) + F_{\rho_2}(1,1)]$

Strata $\{(1,w) : w \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} F_1(1,w) + w^{1/2}F_{\rho_2}(1,w) & F_\sigma(1,\overline{w}) + \overline{w}^{1/2}F_{\rho_1}(1,\overline{w}) \\ F_\sigma(1,w) + w^{1/2}F_{\rho_1}(1,w) & F_1(1,\overline{w}) + \overline{w}^{1/2}F_{\rho_2}(1,\overline{w}) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} F_1(1,w) - w^{1/2}F_{\rho_2}(1,w) & F_\sigma(1,\overline{w}) - \overline{w}^{1/2}F_{\rho_1}(1,\overline{w}) \\ F_\sigma(1,w) - w^{1/2}F_{\rho_1}(1,w) & F_1(1,\overline{w}) - \overline{w}^{1/2}F_{\rho_2}(1,\overline{w}) \end{bmatrix}$$

Strata $(1,-1)$

$$\pi_1(F) = \begin{bmatrix} F_1(1,-1) - F_\sigma(1,-1) & F_{\rho_1}(1,-1) - F_{\rho_2}(1,-1) \\ F_{\rho_1}(1,-1) + F_{\rho_2}(1,-1) & F_1(1,-1) + F_\sigma(1,-1) \end{bmatrix}$$

Strata $\{(z,-1) : z \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} F_1(z,-1) - z^{1/2}F_{\rho_1}(z,-1) & F_\sigma(\overline{z},-1) + z^{1/2}F_{\rho_2}(\overline{z},-1) \\ F_\sigma(z,-1) - \overline{z}^{1/2}F_{\rho_2}(z,-1) & F_1(\overline{z},-1) + \overline{z}^{1/2}F_{\rho_1}(\overline{z},-1) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} F_1(z,-1) + z^{1/2}F_{\rho_1}(z,-1) & F_\sigma(\overline{z},-1) - z^{1/2}F_{\rho_2}(\overline{z},-1) \\ F_\sigma(z,-1) + \overline{z}^{1/2}F_{\rho_2}(z,-1) & F_1(\overline{z},-1) - \overline{z}^{1/2}F_{\rho_1}(\overline{z},-1) \end{bmatrix}$$

Strata $(-1,-1)$

$\pi_1(F) = [F_1(-1,-1) - F_\sigma(-1,-1) - iF_{\rho_1}(-1,-1) - iF_{\rho_2}(-1,-1)]$

$$\pi_2(F) = [F_1(-1,-1) + F_\sigma(-1,-1) + iF_{\rho_1}(-1,-1) - iF_{\rho_2}(-1,-1)]$$

$$\pi_3(F) = [F_1(-1,-1) - F_\sigma(-1,-1) + iF_{\rho_1}(-1,-1) + iF_{\rho_2}(-1,-1)]$$

$$\pi_4(F) = [F_1(-1,-1) + F_\sigma(-1,-1) - iF_{\rho_1}(-1,-1) + iF_{\rho_2}(-1,-1)]$$

Strata $\{(-1,w) : w \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} F_1(-1,w) - w^{1/2}F_{\rho_2}(-1,w) & F_\sigma(-1,\overline{w}) - \overline{w}^{1/2}F_{\rho_1}(-1,\overline{w}) \\ F_\sigma(-1,w) + w^{1/2}F_{\rho_1}(-1,w) & F_1(-1,\overline{w}) + \overline{w}^{1/2}F_{\rho_2}(-1,\overline{w}) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} F_1(-1,w) + w^{1/2}F_{\rho_2}(-1,w) & F_\sigma(-1,\overline{w}) + \overline{w}^{1/2}F_{\rho_1}(-1,\overline{w}) \\ F_\sigma(-1,w) - w^{1/2}F_{\rho_1}(-1,w) & F_1(-1,\overline{w}) - \overline{w}^{1/2}F_{\rho_2}(-1,\overline{w}) \end{bmatrix}$$

Strata $(-1,1)$

$$\pi(F) = \begin{bmatrix} F_1(-1,1) + F_\sigma(-1,1) & F_{\rho_1}(-1,1) - F_{\rho_2}(-1,1) \\ -F_{\rho_1}(-1,1) - F_{\rho_2}(-1,1) & F_1(-1,1) - F_\sigma(-1,1) \end{bmatrix}$$

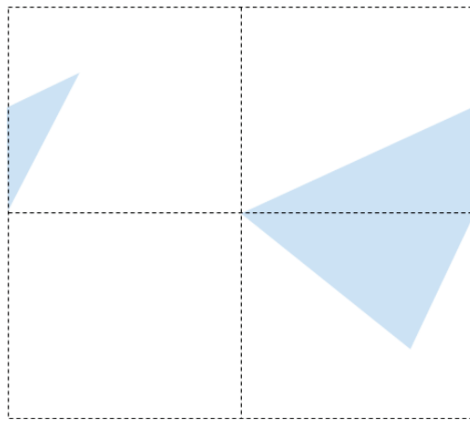
Strata $\{(z,1) : z \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} F_1(z,1) - z^{1/2}F_{\rho_1}(z,1) & F_\sigma(\overline{z},1) - z^{1/2}F_{\rho_2}(\overline{z},1) \\ F_\sigma(z,1) - \overline{z}^{1/2}F_{\rho_2}(z,1) & F_1(\overline{z},1) - z^{1/2}F_{\rho_1}(\overline{z},1) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} F_1(z,1) + z^{1/2}F_{\rho_1}(z,1) & F_\sigma(\overline{z},1) + z^{1/2}F_{\rho_2}(\overline{z},1) \\ F_\sigma(z,1) + \overline{z}^{1/2}F_{\rho_2}(z,1) & F_1(\overline{z},1) + z^{1/2}F_{\rho_1}(\overline{z},1) \end{bmatrix}$$

p3m1

Point group: $D_6 = \{1, \sigma, \sigma^2, \rho_1, \rho_2, \rho_3\}$



Strata

$$\pi(F) = \begin{bmatrix} F_1(z,w) & F_{\sigma^2}(\overline{w}, z\overline{w}) & F_\sigma(\overline{z}w, \overline{z}) & F_{\rho_1}(z, z\overline{w}) & F_{\rho_2}(\overline{z}w, w) & F_{\rho_3}(\overline{w}, \overline{z}) \\ F_\sigma(z,w) & F_1(\overline{w}, z\overline{w}) & F_{\sigma^2}(\overline{z}w, \overline{z}) & F_{\rho_2}(z, z\overline{w}) & F_{\rho_3}(\overline{z}w, w) & F_{\rho_1}(\overline{w}, \overline{z}) \\ F_{\sigma^2}(z,w) & F_\sigma(\overline{w}, z\overline{w}) & F_1(\overline{z}w, \overline{z}) & F_{\rho_3}(z, z\overline{w}) & F_{\rho_1}(\overline{z}w, w) & F_{\rho_2}(\overline{w}, \overline{z}) \\ F_{\rho_1}(z,w) & F_{\rho_2}(\overline{w}, z\overline{w}) & F_{\rho_3}(\overline{z}w, \overline{z}) & F_1(z, z\overline{w}) & F_{\sigma^2}(\overline{z}w, w) & F_\sigma(\overline{w}, \overline{z}) \\ F_{\rho^2}(z,w) & F_{\rho_3}(\overline{w}, z\overline{w}) & F_{\rho_1}(\overline{z}w, \overline{z}) & F_\sigma(z, z\overline{w}) & F_1(\overline{z}w, w) & F_{\sigma^2}(\overline{w}, \overline{z}) \\ F_{\rho_3}(z,w) & F_{\rho_1}(\overline{w}, z\overline{w}) & F_{\rho_2}(\overline{z}w, \overline{z}) & F_{\sigma^2}(z, z\overline{w}) & F_\sigma(\overline{z}w, w) & F_1(\overline{w}, \overline{z}) \end{bmatrix}$$

Strata $\{(w^2, w) : w \in \mathbb{T}^2\}$

$\pi_1(F) =$

$$\begin{bmatrix} F_1(\overline{w}, w) - F_{\rho_3}(\overline{w}, w) & F_{\sigma^2}(\overline{w}, \overline{w}^2) - F_{\rho_1}(\overline{w}, \overline{w}^2) & F_\sigma(w^2, w) - F_{\rho_2}(w^2, w) \\ F_\sigma(\overline{w}, w) - F_{\rho_1}(\overline{w}, w) & F_1(\overline{w}, \overline{w}^2) - F_{\rho_2}(\overline{w}, \overline{w}^2) & F_{\sigma^2}(w^2, w) - F_{\rho_3}(w^2, w) \\ F_{\sigma^2}(\overline{w}, w) - F_{\rho_2}(\overline{w}, w) & F_\sigma(\overline{w}, \overline{w}^2) - F_{\rho_3}(\overline{w}, \overline{w}^2) & F_1(w^2, w) - F_{\rho_1}(w^2, w) \end{bmatrix}$$

$\pi_2(F) =$

$$\begin{bmatrix} F_1(\overline{w},w)+F_{\rho_3}(\overline{w},w) & F_{\sigma^2}(\overline{w},\overline{w}^2)+F_{\rho_1}(\overline{w},\overline{w}^2) & F_\sigma(w^2,w)+F_{\rho_2}(w^2,w) \\ F_\sigma(\overline{w},w)+F_{\rho_1}(\overline{w},w) & F_1(\overline{w},\overline{w}^2)+F_{\rho_2}(\overline{w},\overline{w}^2) & F_{\sigma^2}(w^2,w)+F_{\rho_3}(w^2,w) \\ F_{\sigma^2}(\overline{w},w)+F_{\rho_2}(\overline{w},w) & F_\sigma(\overline{w},\overline{w}^2)+F_{\rho_3}(\overline{w},\overline{w}^2) & F_1(w^2,w)+F_{\rho_1}(w^2,w) \end{bmatrix}$$

Strata $\{(z,z^2) : z \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} F_1(\overline{z}^2,\overline{z})-F_{\rho_1}(\overline{z}^2,\overline{z}) & F_{\sigma^2}(z,\overline{z})-F_{\rho_2}(z,\overline{z}) & F_\sigma(z,z^2)-F_{\rho_3}(z,z^2) \\ F_\sigma(\overline{z}^2,\overline{z})-F_{\rho_2}(\overline{z}^2,\overline{z}) & F_1(z,\overline{z})-F_{\rho_3}(z,\overline{z}) & F_{\sigma^2}(z,z^2)-F_{\rho_1}(z,z^2) \\ F_{\sigma^2}(\overline{z}^2,\overline{z})-F_{\rho_3}(\overline{z}^2,\overline{z}) & F_\sigma(z,\overline{z})-F_{\rho_1}(z,\overline{z}) & F_1(z,z^2)-F_{\rho_2}(z,z^2) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} F_1(\overline{z}^2,\overline{z})+F_{\rho_1}(\overline{z}^2,\overline{z}) & F_{\sigma^2}(z,z^2)+F_{\rho_2}(z,z^2) & F_\sigma(z,\overline{z})+F_{\rho_3}(z,\overline{z}) \\ F_\sigma(\overline{z}^2,\overline{z})+F_{\rho_2}(\overline{z}^2,\overline{z}) & F_1(z,z^2)+F_{\rho_3}(z,z^2) & F_{\sigma^2}(z,\overline{z})+F_{\rho_1}(z,\overline{z}) \\ F_{\sigma^2}(\overline{z}^2,\overline{z})+F_{\rho_3}(\overline{z}^2,\overline{z}) & F_\sigma(z,z^2)+F_{\rho_1}(z,z^2) & F_1(z,\overline{z})+F_{\rho_2}(z,\overline{z}) \end{bmatrix}$$

Strata $\{(z,\overline{z}) : z \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} F_1(z,\overline{z})-F_{\rho_3}(z,\overline{z}) & F_{\rho_2}(z,z^2)-F_{\sigma^2}(z,z^2) & F_\sigma(\overline{z}^2,\overline{z})-F_{\rho_2}(\overline{z}^2,\overline{z}) \\ F_\sigma(z,\overline{z})-F_{\rho_1}(z,\overline{z}) & F_1(z,z^2)-F_{\rho_2}(z,z^2) & F_{\sigma^2}(\overline{z}^2,\overline{z})-F_{\rho_3}(\overline{z}^2,\overline{z}) \\ F_{\sigma^2}(z,\overline{z})-F_{\rho_2}(z,\overline{z}) & F_\sigma(z,z^2)-F_{\rho_3}(z,z^2) & F_1(\overline{z}^2,\overline{z})-F_{\rho_1}(\overline{z}^2,\overline{z}) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} F_1(\overline{z}^2,\overline{z})+F_{\rho_3}(\overline{z}^2,\overline{z}) & F_{\rho_2}(z,\overline{z})+F_{\sigma^2}(z,\overline{z}) & F_\sigma(z,z^2)+F_{\rho_2}(z,z^2) \\ F_\sigma(\overline{z}^2,\overline{z})+F_{\rho_1}(\overline{z}^2,\overline{z}) & F_1(z,\overline{z})+F_{\rho_2}(z,\overline{z}) & F_{\sigma^2}(\overline{z}^2,\overline{z})+F_{\rho_3}(z,z^2) \\ F_{\sigma^2}(\overline{z}^2,\overline{z})+F_{\rho_2}(\overline{z}^2,\overline{z}) & F_\sigma(z,\overline{z})+F_{\rho_3}(z,\overline{z}) & F_1(\overline{z}^2,\overline{z})+F_{\rho_1}(z,z^2) \end{bmatrix}$$

Strata $(-e^{\pi i/3}, e^{-2\pi i/3})$

$\pi_1(F) =$

$$\begin{bmatrix} F_1 - \frac{1}{2}F_{\sigma^2} + F_{\rho_2} - \frac{1}{2}F_{\rho_1} - \frac{1}{2}F_{\sigma} - \frac{1}{2}F_{\rho_3} & \frac{\sqrt{3}}{2}F_{\sigma} - \frac{\sqrt{3}}{2}F_{\sigma^2} - \frac{\sqrt{3}}{2}F_{\rho_3} + \frac{\sqrt{3}}{2}F_{\rho_1} \\ \frac{\sqrt{3}}{2}F_{\sigma^2} - \frac{\sqrt{3}}{2}F_{\sigma} + \frac{\sqrt{3}}{2}F_{\rho_1} - \frac{\sqrt{3}}{2}F_{\rho_3} & F_1 - \frac{1}{2}F_{\sigma} + \frac{1}{2}F_{\rho_1} - F_{\rho_2} + \frac{1}{2}F_{\rho_3} \end{bmatrix}$$

$\pi_2(F) = [F_1 + F_{\sigma^2} + F_{\sigma} - F_{\rho_1} - F_{\rho_2} - F_{\rho_3}]$

$\pi_3(F) = [F_1 + F_{\sigma^2} + F_{\sigma} + F_{\rho_1} + F_{\rho_2} + F_{\rho_3}]$

Strata $(e^{-2\pi i/3}, -e^{\pi i/3})$

$\pi_1(F) =$

$$\begin{bmatrix} F_1 - \frac{1}{2}F_{\sigma^2} + F_{\rho_2} - \frac{1}{2}F_{\rho_1} - \frac{1}{2}F_{\sigma} - \frac{1}{2}F_{\rho_3} & \frac{\sqrt{3}}{2}F_{\sigma} - \frac{\sqrt{3}}{2}F_{\sigma^2} - \frac{\sqrt{3}}{2}F_{\rho_3} + \frac{\sqrt{3}}{2}F_{\rho_1} \\ \frac{\sqrt{3}}{2}F_{\sigma^2} - \frac{\sqrt{3}}{2}F_{\sigma} + \frac{\sqrt{3}}{2}F_{\rho_1} - \frac{\sqrt{3}}{2}F_{\rho_3} & F_1 - \frac{1}{2}F_{\sigma} + \frac{1}{2}F_{\rho_1} - F_{\rho_2} + \frac{1}{2}F_{\rho_3} \end{bmatrix}$$

$\pi_2(F) = [F_1 + F_{\sigma^2} + F_{\sigma} - F_{\rho_1} - F_{\rho_2} - F_{\rho_3}]$

$\pi_3(F) = [F_1 + F_{\sigma^2} + F_{\sigma} + F_{\rho_1} + F_{\rho_2} + F_{\rho_3}]$

Strata $(1, 1)$

$\pi_1(F) =$

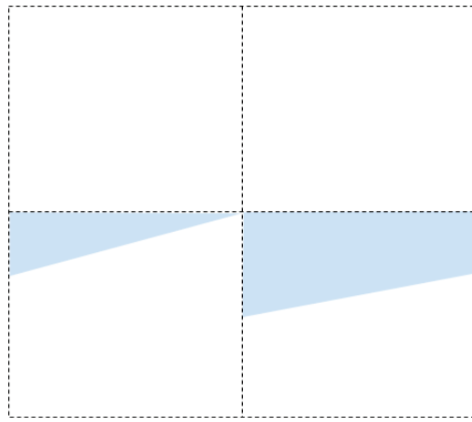
$$\begin{bmatrix} \frac{1}{2}(2F_1 - F_{\sigma^2} - F_{\rho_3} + 2F_{\rho_1} - F_{\sigma} - F_{\rho_2}) & \frac{\sqrt{3}}{2}(F_{\sigma} - F_{\sigma^2} - F_{\rho_2} + F_{\rho_3}) \\ \frac{\sqrt{3}}{2}(F_{\sigma^2} - F_{\sigma} - F_{\rho_2} + F_{\rho_3}) & \frac{1}{2}(2F_1 - F_{\sigma^2} - F_{\sigma} - 2F_{\rho_1} + F_{\rho_2} + F_{\rho_3}) \end{bmatrix}(1, 1)$$

$$\pi_2(F) = [F_1(1,1) + F_{\sigma^2}(1,1) + F_{\sigma}(1,1) - F_{\rho_1}(1,1) - F_{rho_2}(1,1) - F_{\rho_3}(1,1)]$$

$$\pi_3(F) = [F_1(1,1) + F_{\sigma^2}(1,1) + F_{\sigma}(1,1) + F_{\rho_1}(1,1) - F_{rho_2}(1,1) + F_{\rho_3}(1,1)]$$

p31m

Point group: $D_6 = \{1, \sigma, \sigma^2, \rho_1, \rho_2, \rho_3\}$



Strata

$\pi(F) =$

$$\begin{bmatrix} F_1(z,w) & F_{\sigma^2}(z\overline{w}^3, z\overline{w}^2) & F_\sigma(\overline{z}^2w^3, \overline{z}w) & F_{\rho_1}(z, z\overline{w}) & F_{\rho_2}(\overline{z}^2w^3, \overline{z}w^2) & F_{\rho_3}(z\overline{w}^3, \overline{w}) \\ F_\sigma(z,w) & F_1(z\overline{w}^3, z\overline{w}^2) & F_{\sigma^2}(\overline{z}^2w^3, \overline{z}w) & F_{\rho_2}(z, z\overline{w}) & F_{\rho_3}(\overline{z}^2w^3, \overline{z}w^2) & F_{\rho_1}(z\overline{w}^3, \overline{w}) \\ F_{\sigma^2}(z,w) & F_\sigma(z\overline{w}^3, z\overline{w}^2) & F_1(\overline{z}^2w^3, \overline{z}w) & F_{\rho_3}(z, z\overline{w}) & F_{\rho_1}(\overline{z}^2w^3, \overline{z}w^2) & F_{\rho_2}(z\overline{w}^3, \overline{w}) \\ F_{\rho_1}(z,w) & F_{\rho_2}(z\overline{w}^3, z\overline{w}^2) & F_{\rho_3}(\overline{z}^2w^3, \overline{z}w) & F_1(z, z\overline{w}) & F_{\sigma^2}(\overline{z}^2w, \overline{z}w^2) & F_\sigma(z\overline{w}^3, \overline{w}) \\ F_{\rho_2}(z,w) & F_{\rho_3}(z\overline{w}^3, z\emptyset^2) & F_{\rho_1}(\overline{z}^2w^3, \overline{z}w) & F_\sigma(z, z\overline{w}) & F_1(\overline{z}^2w^3, \overline{z}w^2) & F_{\sigma^2}(z\overline{w}^3, \overline{w}) \\ F_{\rho_3}(z,w) & F_{\rho_1}(z\overline{w}^3, z\overline{w}^2) & F_{\rho_2}(\overline{z}^2w^3, \overline{z}w) & F_{\sigma^2}(z, z\overline{w}) & F_\sigma(\overline{z}^2w^3, \overline{z}w^2) & F_1(z\overline{w}^3, \overline{w}) \end{bmatrix}$$

Strata $(1,1)$

$$\pi_1(F) = F_1(1,1) + F_{\sigma^2}(1,1) + F_\sigma(1,1) + F_{\rho_1}(1,1) + F_{\rho_2}(1,1) + F_{\rho_3}(1,1)]$$

$$\pi_2(F) = [F_1(1,1) + F_{\sigma^2}(1,1) + F_\sigma(1,1) - F_{\rho_1}(1,1) - F_{\rho_2}(1,1) - F_{\rho_3}(1,1)]$$

$$\pi_3(F) =$$

$$\begin{bmatrix} F_1 - \frac{1}{2}F_{\sigma^2} - \frac{1}{2}F_\sigma + \frac{1}{2}F_{\rho_1} - F_{\rho_2} + \frac{1}{2}F_{\rho_3} & \frac{\sqrt{3}}{2}F_{\sigma^2} - \frac{\sqrt{3}}{2}F_\sigma + \frac{\sqrt{3}}{2}F_{\rho_1} - \frac{\sqrt{3}}{2}F_{\rho_3} \\ -\frac{\sqrt{3}}{2}F_{\sigma^2} + \frac{\sqrt{3}}{2}F_\sigma + \frac{\sqrt{3}}{2}F_{\rho_1} - \frac{\sqrt{3}}{2}F_{\rho_3} & F_1 - \frac{1}{2}F_{\sigma^2} - \frac{1}{2}F_\sigma - \frac{1}{2}F_{\rho_1} + F_{\rho_2} - \frac{1}{2}F_{\rho_3} \end{bmatrix}(1,1)$$

Strata $\{(z,1) : z \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} F_1(\overline{z}^2,\overline{z}) - F_{\rho_1}(\overline{z}^2,\overline{z}) & F_{\sigma^2}(z,1) - F_{\rho_2}(z,1) & F_\sigma(z,z) - F_{\rho_3}(z,z) \\ F_\sigma(\overline{z}^2,\overline{z}) - F_{\rho_2}(\overline{z}^2,\overline{z}) & F_1(z,1) - F_{\rho_3}(z,1) & F_{\sigma^2}(z,z) - F_{\rho_1}(z,z) \\ F_{\sigma^2}(\overline{z}^2,\overline{z}) - F_{\rho_3}(\overline{z}^2,\overline{z}) & F_\sigma(z,1) - F_{\rho_1}(z,1) & F_1(z,z) - F_{\rho_2}(z,z) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} F_1(z,1) + F_{\rho_1}(z,1) & F_{\sigma^2}(z,z) + F_{\rho_2}(z,z) & F_\sigma(\overline{z}^2,\overline{z}) + F_{\rho_3}(\overline{z}^2,\overline{z}) \\ F_\sigma(z,1) + F_{\rho_2}(\overline{z}^2,\overline{z}) & F_1(z,z) + F_{\rho_3}(z,z) & F_{\sigma^2}(\overline{z}^2,\overline{z}) + F_{\rho_1}(\overline{z}^2,\overline{z}) \\ F_{\sigma^2}(z,1) + F_{\rho_3}(\overline{z}^2,\overline{z}) & F_\sigma(z,z) + F_{\rho_1}(z,z) & F_1(\overline{z}^2,\overline{z}) + F_{\rho_2}(\overline{z}^2,\overline{z}) \end{bmatrix}$$

Strata $(1, e^{-2/3\pi i})$

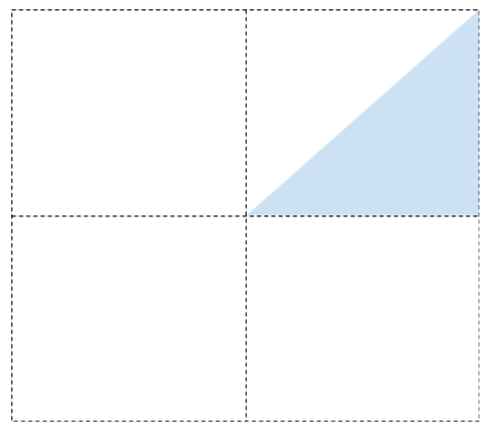
$$\pi_1(F) = \begin{bmatrix} F_1 + \frac{\sqrt{-3}-1}{2}F_{\sigma^2} - \frac{1+\sqrt{-3}}{2}F_\sigma & F_{\rho_1} - \frac{1+\sqrt{-3}}{2}F_{\rho_2} + \frac{\sqrt{-3}-1}{2}F_{\rho_3} \\ F_{\rho_1} + \frac{\sqrt{-3}-1}{2}F_{\rho_2} - \frac{\sqrt{-3}+1}{2}F_{\rho_3} & F_1 - \frac{\sqrt{-3}+1}{2}F_{\sigma^2} + \frac{\sqrt{-3}-1}{2}F_\sigma \end{bmatrix}(1, e^{-2/3\pi i})$$

$$\pi_2(F) = \begin{bmatrix} F_1 - \frac{\sqrt{-3}+1}{2}F_{\sigma^2} + \frac{-1+\sqrt{-3}}{2}F_\sigma & F_{\rho_1} + \frac{-1+\sqrt{-3}}{2}F_{\rho_2} - \frac{\sqrt{-3}+1}{2}F_{\rho_3} \\ F_{\rho_1} - \frac{\sqrt{-3}+1}{2}F_{\rho_2} + \frac{\sqrt{-3}-1}{2}F_{\rho_3} & F_1 + \frac{\sqrt{-3}-1}{2}F_{\sigma^2} - \frac{\sqrt{-3}+1}{2}F_\sigma \end{bmatrix}(1, e^{-2/3\pi i})$$

$$\pi_3(F) = \begin{bmatrix} F_1 + F_{\sigma^2} + F_\sigma & F_{\rho_1} + F_{\rho_2} + F_{\rho_3} \\ F_{\rho_1} + F_{\rho_2} + F_{\rho_3} & F_1 + F_{\sigma^2} + F_\sigma \end{bmatrix}(1, e^{-2/3\pi i})$$

p4mm

Point group: $D_8 = \{1, \sigma, \sigma^2, \sigma^3, \rho_1, \rho_2, \rho_3, \rho_4\}$



Strata

$\pi(F) =$

$$\begin{bmatrix}
F_1(z,w) & F_{\sigma^3}(\overline{w},z) & F_{\sigma^2}(\overline{z},\overline{w}) & F_{\sigma}(w,\overline{z}) & F_{\rho_1}(z,\overline{w}) & F_{\rho_2}(w,z) & F_{\rho_3}(\overline{z},w) & F_{\rho_4}(\overline{w},\overline{z}) \\
F_{\sigma}(z,w) & F_1(\overline{w},z) & F_{\sigma^3}(\overline{z},\overline{w}) & F_{\sigma^2}(w,\overline{z}) & F_{\rho_2}(z,\overline{w}) & F_{\rho_3}(w,z) & F_{\rho_4}(\overline{z},w) & F_{\rho_1}(\overline{w},\overline{z}) \\
F_{\sigma^2}(z,w) & F_{\sigma}(\overline{w},z) & F_1(\overline{z},\overline{w}) & F_{\sigma^3}(w,\overline{z}) & F_{\rho_3}(z,\overline{w}) & F_{\rho_4}(w,z) & F_{\rho_1}(\overline{z},w) & F_{\rho_2}(\overline{z},w) \\
F_{\sigma^3}(z,w) & F_{\sigma^2}(\overline{w},z) & F_{\sigma}(\overline{z},\overline{w}) & F_1(w,\overline{z}) & F_{\rho_4}(z,\overline{w}) & F_{\rho_1}(w,z) & F_{\rho_2}(\overline{z},w) & F_{\rho_3}(\overline{w},\overline{z}) \\
F_{\rho_1}(z,w) & F_{\rho_2}(\overline{w},z) & F_{\rho_3}(\overline{z},\overline{w}) & F_{\rho_4}(w,\overline{z}) & F_1(z,\overline{w}) & F_{\sigma^3}(w,z) & F_{\sigma^2}(\overline{z},w) & F_{\sigma}(\overline{w},\overline{z}) \\
F_{\rho_2}(z,w) & F_{\rho_3}(\overline{w},z) & F_{\rho_4}(\overline{z},\overline{w}) & F_{\rho_1}(w,\overline{z}) & F_{\sigma}(z,\overline{w}) & F_1(w,z) & F_{\sigma^3}(\overline{z},w) & F_{\sigma^2}(\overline{w},\overline{z}) \\
F_{\rho_3}(z,w) & F_{\rho_4}(\overline{w},z) & F_{\rho_1}(\overline{z},\overline{w}) & F_{\rho_2}(w,\overline{z}) & F_{\sigma^2}(z,\overline{w}) & F_{\sigma}(w,z) & F_1(\overline{z},w) & F_{\sigma^3}(\overline{w},\overline{z}) \\
F_{\rho_4}(z,w) & F_{\rho_1}(\overline{w},z) & F_{\rho_2}(\overline{z},\overline{w}) & F_{\rho_3}(w,\overline{z}) & F_{\sigma^3}(z,\overline{w}) & F_{\sigma^2}(w,z) & F_{\sigma}(\overline{z},w) & F_1(\overline{w},\overline{z})
\end{bmatrix}$$

Strata $(1, 1)$

$$\pi_1(F) = [F_1 + F_{\sigma^3} + F_{\sigma^2} + F_\sigma - F_{\rho_1} - F_{\rho_2} - F_{\rho_3} - F_{\rho_4}](1,1)$$

$$\pi_2(F) = [F_1 - F_{\sigma^3} + F_{\sigma^2} + F_\sigma + F_{\rho_1} + F_{\rho_2} + F_{\rho_3} - F_{\rho_4}](1,1)$$

$$\pi_3(F) = [F_1 + F_{\sigma^3} + F_{\sigma^2} + F_\sigma + F_{\rho_1} - F_{\rho_2} + F_{\rho_3} + F_{\rho_4}](1,1)$$

$$\pi_4(F) = \begin{bmatrix} F_1 - F_{\sigma^2} - F_{\rho_1} + F_{\rho_3} & F_\sigma - F_{\sigma^3} + F_{\rho_2} - F_{\rho_4} \\ F_{\sigma^3} - F_\sigma + F_{\rho_2} - F_{\rho_4} & F_1 - F_{\sigma^2} + F_{\rho_1} - F_{\rho_3} \end{bmatrix}(1,1)$$

$$\pi_5(F) = [F_1 - F_{\sigma^3} + F_{\sigma^2} - F_\sigma - F_{\rho_1} + F_{\rho_2} - F_{\rho_3} + F_{\rho_4}](1,1)$$

Strata $\{(z,z) : z \in \mathbb{T}^2\}$

$\pi_1(F) =$

$$\begin{bmatrix} F_1(\overline{z},z) + F_{\rho_4}(\overline{z},z) & F_{\sigma^3}(\overline{z},\overline{z}) + F_{\rho_1}(\overline{z},\overline{z}) & F_{\sigma^2}(z,\overline{z}) + F_{\rho_2}(z,\overline{z}) & F_\sigma(z,z) + F_{\rho_3}(z,z) \\ F_\sigma(\overline{z},z) + F_{\rho_1}(\overline{z},z) & F_1(\overline{z},\overline{z}) + F_{\rho_2}(\overline{z},\overline{z}) & F_{\sigma^3}(z,\overline{z}) + F_{\rho_3}(z,\overline{z}) & F_{\sigma^2}(z,z) + F_{\rho_4}(z,z) \\ F_{\sigma^2}(\overline{z},z) + F_{\rho_2}(\overline{z},z) & F_\sigma(\overline{z},\overline{z}) + F_{\rho_3}(\overline{z},\overline{z}) & F_1(z,\overline{z}) + F_{\rho_4}(z,\overline{z}) & F_{\sigma^3}(z,z) + F_{\rho_1}(z,z) \\ F_{\sigma^3}(\overline{z},z) + F_{\rho_3}(\overline{z},z) & F_{\sigma^2}(\overline{z},\overline{z}) + F_{\rho_4}(\overline{z},\overline{z}) & F_\sigma(z,\overline{z}) + F_{\rho_1}(z,\overline{z}) & F_1(z,z) + F_{\rho_2}(z,z) \end{bmatrix}$$

$\pi_2(F) =$

$$\begin{bmatrix} F_1(z,z) - F_{\rho_4}(z,z) & F_{\sigma^3}(\overline{z},\overline{z}) - F_{\rho_1}(\overline{z},z) & F_{\sigma^2}(\overline{z},\overline{z}) - F_{\rho_2}(\overline{z},\overline{z}) & F_\sigma(z,\overline{z}) - F_{\rho_3}(z,\overline{z}) \\ F_\sigma(z,z) - F_{\rho_1}(z,z) & F_1(\overline{z},z) - F_{\rho_2}(\overline{z},z) & F_{\sigma^3}(\overline{z},\overline{z}) - F_{\rho_3}(\overline{z},\overline{z}) & F_{\sigma^2}(z,\overline{z}) - F_{\rho_4}(z,\overline{z}) \\ F_{\sigma^2}(z,z) - F_{\rho_2}(z,z) & F_\sigma(\overline{z},z) - F_{\rho_3}(\overline{z},z) & F_1(\overline{z},\overline{z}) - F_{\rho_4}(\overline{z},\overline{z}) & F_{\sigma^3}(z,\overline{z}) - F_{\rho_1}(z,\overline{z}) \\ F_{\sigma^3}(z,z) - F_{\rho_3}(z,z) & F_{\sigma^2}(\overline{z},z) - F_{\rho_4}(\overline{z},z) & F_\sigma(\overline{z},\overline{z}) - F_{\rho_1}(\overline{z},\overline{z}) & F_1(z,\overline{z}) - F_{\rho_2}(z,\overline{z}) \end{bmatrix}$$

Strata $(-1,-1)$

$$\pi_1(F) = [F_1 + F_{\sigma^3} + F_{\sigma^2} + F_\sigma + F_{\rho_1} + F_{\rho_2} + F_{\rho_3} + F_{\rho_4}](-1,-1)$$

$$\pi_2(F) = \begin{bmatrix} F_1 - F_{\sigma^2} + F_{\rho_2} - F_{\rho_4} & F_\sigma - F_{\sigma^3} - F_{\rho_3} + F_{\rho_1} \\ F_{\sigma^3} - F_\sigma + F_{\rho_1} - F_{\rho_3} & F_1 - F_{\sigma^2} - F_{\rho_2} + F_{\rho_4} \end{bmatrix}$$

$$\pi_3(F) = [F_1 - F_{\sigma^3} + F_{\sigma^2} - F_\sigma - F_{\rho_1} + F_{\rho_2} - F_{\rho_3} + F_{\rho_4}]$$

$$\pi_4(F) = [F_1 + F_{\sigma^3} + F_{\sigma^2} + F_\sigma - F_{\rho_1} - F_{\rho_2} - F_{\rho_3} - F_{\rho_4}]$$

$$\pi_5(F) = [F_1 - F_{\sigma^3} + F_{\sigma^2} - F_\sigma + F_{\rho_1} - F_{\rho_2} + F_{\rho_3} - F_{\rho_4}]$$

Strata $\{(-1,w) : w \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} A & B \end{bmatrix}$$

where

$$A = \begin{bmatrix} F_1(\overline{w},-1) - F_{\rho_1}(\overline{w},-1) & F_{\sigma^3}(-1,\overline{w}) - F_{\rho_2}(-1,\overline{w}) \\ F_\sigma(\overline{w},-1) - F_{\rho_2}(\overline{w},-1) & F_1(-1,\overline{w}) - F_{\rho_3}(-1,\overline{w}) \\ F_{\sigma^2}(\overline{w},-1) - F_{\rho_3}(\overline{w},-1) & F_\sigma(-1,\overline{w}) - F_{\rho_4}(-1,\overline{w}) \\ F_{\sigma^3}(\overline{w},-1) - F_{\rho_4}(\overline{w},-1) & F_{\sigma^2}(-1,\overline{w}) - F_{\rho_1}(-1,\overline{w}) \end{bmatrix}$$

$$B = \begin{bmatrix} F_{\sigma^2}(w,-1) - F_{\rho_3}(w,-1) & F_\sigma(-1,w) - F_{\rho_4}(-1,w) \\ F_{\sigma^3}(w,-1) - F_{\rho_4}(w,-1) & F_{\sigma^2}(-1,w) - F_{\rho_1}(-1,w) \\ F_1(w,-1) - F_{\rho_1}(w,-1) & F_{\sigma^3}(-1,w) - F_{\rho_2}(-1,w) \\ F_\sigma(w,-1) - F_{\rho_2}(w,-1) & F_1(-1,w) - F_{\rho_3}(-1,w) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} A & B \end{bmatrix}$$

where

$$A = \begin{bmatrix} F_1(\overline{w},-1) + F_{\rho_1}(\overline{w},-1) & F_{\sigma^3}(-1,\overline{w}) + F_{\rho_2}(-1,\overline{w}) \\ F_\sigma(\overline{w},-1) + F_{\rho_2}(\overline{w},-1) & F_1(-1,\overline{w}) + F_{\rho_3}(-1,\overline{w}) \\ F_{\sigma^2}(\overline{w},-1) + F_{\rho_3}(\overline{w},-1) & F_\sigma(-1,\overline{w}) + F_{\rho_4}(-1,\overline{w}) \\ F_{\sigma^3}(\overline{w},-1) + F_{\rho_4}(\overline{w},-1) & F_{\sigma^2}(-1,\overline{w}) + F_{\rho_1}(-1,\overline{w}) \end{bmatrix}$$

$$B = \begin{bmatrix} F_{\sigma^2}(w,-1) + F_{\rho_3}(w,-1) & F_\sigma(-1,w) + F_{\rho_4}(-1,w) \\ F_{\sigma^3}(w,-1) + F_{\rho_4}(w,-1) & F_{\sigma^2}(-1,w) + F_{\rho_1}(-1,w) \\ F_1(w,-1) + F_{\rho_1}(w,-1) & F_{\sigma^3}(-1,w) + F_{\rho_2}(-1,w) \\ F_\sigma(w,-1) + F_{\rho_2}(w,-1) & F_1(-1,w) + F_{\rho_3}(-1,w) \end{bmatrix}$$

Strata $(-1,1)$

$$\pi_1(F) = \begin{bmatrix} F_1 + F_{\sigma^2} - F_{\rho_3} - F_{\rho_1} & F_\sigma + F_{\sigma^3} - F_{\rho_2} - F_{\rho_4} \\ F_{\sigma^3} + F_\sigma - F_{\rho_2} - F_{\rho_4} & F_1 + F_{\sigma^2} - F_{\rho_1} - F_{\rho_3} \end{bmatrix}(-1,1)$$

$$\pi_2(F) = \begin{bmatrix} F_1 - F_{\sigma^2} - F_{\rho_3} + F_{\rho_1} & -F_\sigma + F_{\sigma^3} + F_{\rho_2} - F_{\rho_4} \\ -F_{\sigma^3} + F_\sigma + F_{\rho_2} - F_{\rho_4} & F_1 - F_{\sigma^2} - F_{\rho_1} + F_{\rho_3} \end{bmatrix}(-1,1)$$

$$\pi_1(F) = \begin{bmatrix} F_1 - F_{\sigma^2} - F_{\rho_3} + F_{\rho_1} & F_\sigma + F_{\sigma^3} - F_{\rho_2} + F_{\rho_4} \\ F_{\sigma^3} - F_\sigma - F_{\rho_2} + F_{\rho_4} & F_1 - F_{\sigma^2} - F_{\rho_1} + F_{\rho_3} \end{bmatrix}(-1,1)$$

$$\pi_4(F) = \begin{bmatrix} F_1 + F_{\sigma^2} + F_{\rho_3} + F_{\rho_1} & F_\sigma + F_{\sigma^3} - F_{\rho_2} + F_{\rho_4} \\ F_{\sigma^3} + F_\sigma + F_{\rho_2} + F_{\rho_4} & F_1 + F_{\sigma^2} + F_{\rho_1} + F_{\rho_3} \end{bmatrix}(-1,1)$$

Strata $\{(z,1) : z \in \mathbb{T}^2\}$

$\pi_1(F) =$

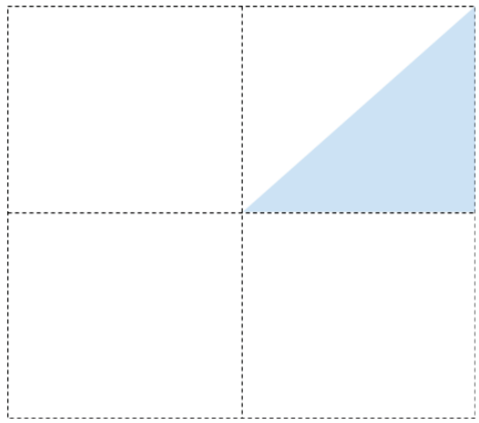
$$\begin{bmatrix} F_1(1,z)+F_{\rho_3}(1,z) & F_{\sigma^3}(\overline{z},1)+F_{\rho_4}(\overline{z},1) & F_{\sigma^2}(1,\overline{z})+F_{\rho_1}(1,\overline{z}) & F_{\sigma}(z,1)+F_{\rho_2}(z,1) \\ F_{\sigma}(1,z)+F_{\rho_4}(1,z) & F_1(\overline{z},1)+F_{\rho_1}(\overline{z},1) & F_{\sigma^3}(1,\overline{z})+F_{\rho_2}(1,\overline{z}) & F_{\sigma^2}(z,1)+F_{\rho_3}(z,1) \\ F_{\sigma^2}(1,z)+F_{\rho_1}(1,z) & F_{\sigma}(\overline{z},1)+F_{\rho_2}(\overline{z},1) & F_1(1,\overline{z})+F_{\rho_3}(1,\overline{z}) & F_{\sigma^3}(z,1)+F_{\rho_4}(z,1) \\ F_{\sigma^3}(1,z)+F_{\rho_2}(1,z) & F_{\sigma^2}(\overline{z},1)+F_{\rho_3}(\overline{z},1) & F_{\sigma}(1,\overline{z})+F_{\rho_4}(1,\overline{z}) & F_1(z,1)+F_{\rho_1}(z,1) \end{bmatrix}$$

$\pi_2(F) =$

$$\begin{bmatrix} F_1(1,z)-F_{\rho_3}(1,z) & F_{\sigma^3}(\overline{z},1)-F_{\rho_4}(\overline{z},1) & F_{\sigma^2}(1,\overline{z})-F_{\rho_1}(1,\overline{z}) & F_{\sigma}(z,1)-F_{\rho_2}(z,1) \\ F_{\sigma}(1,z)-F_{\rho_4}(1,z) & F_1(\overline{z},1)-F_{\rho_1}(\overline{z},1) & F_{\sigma^3}(1,\overline{z})-F_{\rho_2}(1,\overline{z}) & F_{\sigma^2}(z,1)-F_{\rho_3}(z,1) \\ F_{\sigma^2}(1,z)-F_{\rho_1}(1,z) & F_{\sigma}(\overline{z},1)-F_{\rho_2}(\overline{z},1) & F_1(1,\overline{z})-F_{\rho_3}(1,\overline{z}) & F_{\sigma^3}(z,1)-F_{\rho_4}(z,1) \\ F_{\sigma^3}(1,z)-F_{\rho_2}(1,z) & F_{\sigma^2}(\overline{z},1)-F_{\rho_3}(\overline{z},1) & F_{\sigma}(1,\overline{z})-F_{\rho_4}(1,\overline{z}) & F_1(z,1)-F_{\rho_1}(z,1) \end{bmatrix}$$

p4mg

Point group: $D_8 = \{1, \sigma, \sigma^2, \sigma^3, \rho_1, \rho_2, \rho_3, \rho_4\}$



Strata

$$\pi(F) = \begin{bmatrix} A & B \end{bmatrix}$$

where

$$A = \begin{bmatrix} F_1(z,w) & zF_{\sigma^3}(\overline{w},z) & F_{\sigma^2}(\overline{z},\overline{w}) & wF_\sigma(w,\overline{z}) \\ F_\sigma(z,w) & F_1(\overline{w},z) & \overline{z}F_{\sigma^3}(\overline{z},\overline{w}) & \overline{z}F_{\sigma^2}(w,\overline{z}) \\ F_{\sigma^2}(z,w) & z\overline{w}F_\sigma(\overline{w},z) & F_1(\overline{z},\overline{w}) & F_{\sigma^3}(w,\overline{z}) \\ F_{\sigma^3}(z,w) & zF_{\sigma^2}(\overline{w},z) & \overline{z}F_\sigma(\overline{z},\overline{w}) & F_1(w,\overline{z}) \\ F_{\rho_1}(z,w) & F_{\rho_2}(\overline{w},z) & \overline{z}F_{\rho_3}(\overline{z},\overline{w}) & wF_{\rho_4}(w,\overline{z}) \\ F_{\rho_2}(z,w) & zF_{\rho_3}(\overline{w},z) & \overline{z}wF_{\rho_4}(\overline{z},\overline{w}) & wF_{\rho_1}(w,\overline{z}) \\ F_{\rho_3}(z,w) & \overline{w}F_{\rho_4}(\overline{w},z) & \overline{z}F_{\rho_1}(\overline{z},\overline{w}) & F_{\rho_2}(w,\overline{z}) \\ F_{\rho_4}(z,w) & F_{\rho_1}(\overline{w},z) & \overline{z}wF_{\rho_2}(\overline{z},\overline{w}) & \overline{z}wF_{\rho_3}(w,\overline{z}) \end{bmatrix}$$

$$B = \begin{bmatrix} zF_{\rho_1}(z,\overline{w}) & F_{\rho_2}(w,z) & wF_{\rho_3}(\overline{z},w) & z\overline{w}F_{\rho_4}(\overline{w},\overline{z}) \\ F_{\rho_2}(z,\overline{w}) & F_{\rho_3}(w,z) & \overline{z}F_{\rho_4}(\overline{z},w) & \overline{w}F_{\rho_1}(\overline{w},\overline{z}) \\ z\overline{w}F_{\rho_3}(z,\overline{w}) & F_{\rho_4}(w,z) & F_{\rho_1}(\overline{z},w) & z\overline{w}F_{\rho_2}(\overline{w},\overline{z}) \\ zF_{\rho_4}(z,\overline{w}) & F_{\rho_1}(w,z) & F_{\rho_2}(\overline{z},w) & \overline{w}F_{\rho_3}(\overline{w},\overline{z}) \\ F_1(z,\overline{w}) & F_{\sigma^3}(w,z) & wF_{\sigma^2}(\overline{z},w) & \overline{w}F_{\sigma}(\overline{w},\overline{z}) \\ zF_{\sigma}(z,\overline{w}) & F_1(w,z) & wF_{\sigma^3}(\overline{z},w) & F_{\sigma^2}(\overline{w},\overline{z}) \\ \overline{w}F_{\sigma^2}(z,\overline{w}) & F_{\sigma}(w,z) & F_1(\overline{z},w) & \overline{w}F_{\sigma^3}(\overline{w},\overline{z}) \\ F_{\sigma^3}(z,\overline{w}) & F_{\sigma^2}(w,z) & \overline{z}wF_{\sigma}(\overline{z},w) & F_1(\overline{w},\overline{z}) \end{bmatrix}$$

Strata $(1,1)$

$$\pi_1(F) = [F_1 + F_{\sigma^3} + F_{\sigma^2} + F_{\sigma} - F_{\rho_1} - F_{\rho_2} - F_{\rho_3} - F_{\rho_4}](1,1)$$

$$\pi_2(F) = [F_1 - F_{\sigma^3} + F_{\sigma^2} - F_{\sigma} + F_{\rho_1} - F_{\rho_2} + F_{\rho_3} - F_{\rho_4}](1,1)$$

$$\pi_3(F) = [F_1 + F_{\sigma^3} + F_{\sigma^2} + F_{\sigma} + F_{\rho_1} - F_{\rho_2} + F_{\rho_3} + F_{\rho_4}](1,1)$$

$$\pi_4(F) = \begin{bmatrix} F_1 - F_{\sigma^2} - F_{\rho_1} + F_{\rho_3} & F_{\sigma} - F_{\sigma^3} + F_{\rho_2} - F_{\rho_4} \\ F_{\sigma^3} - F_{\sigma} + F_{\rho_2} - F_{\rho_4} & F_1 - F_{\sigma^2} + F_{\rho_1} - F_{\rho_3} \end{bmatrix}$$

$$\pi_5(F) = [F_1 - F_{\sigma^3} + F_{\sigma^2} - F_{\sigma} - F_{\rho_1} + F_{\rho_2} - F_{\rho_3} + F_{\rho_4}](1,1)$$

Strata $\{(z,z) : z \in \mathbb{T}^2\}$

$$\pi_1(F) =$$

$$\begin{bmatrix} F_1(z,z)-F_{\rho_2}(z,z) & zF_{\sigma^3}(\overline{z},z)-zF_{\rho_3}(\overline{z},z) & F_{\sigma^2}(\overline{z},\overline{z})-F_{\rho_4}(\overline{z},\overline{z}) & zF_{\sigma}(z,\overline{z})+zF_{\rho_1}(z,\overline{z}) \\ F_{\sigma}(z,z)-F_{\rho_3}(z,z) & F_1(\overline{z},z)-\overline{z}F_{\rho_4}(\overline{z},z) & \overline{z}F_{\sigma^3}(\overline{z},\overline{z})-\overline{z}F_{\rho_1}(\overline{z},\overline{z}) & \overline{z}F_{\sigma^2}(z,\overline{z})-F_{\rho_2}(z,\overline{z}) \\ F_{\sigma^2}(z,z)-F_{\rho_4}(z,z) & F_{\sigma}(\overline{z},z)-F_{\rho_1}(\overline{z},z) & F_1(\overline{z},\overline{z})-F_{\rho_2}(\overline{z},\overline{z}) & F_{\sigma^3}(z,\overline{z})-F_{\rho_3}(z,\overline{z}) \\ F_{\sigma^3}(z,z)-F_{\rho_1}(z,z) & zF_{\sigma^2}(\overline{z},z)-F_{\rho_2}(\overline{z},z) & \overline{z}F_{\sigma}(\overline{z},\overline{z})-\overline{z}F_{\rho_3}(\overline{z},\overline{z}) & F_1(z,\overline{z})-zF_{\rho_4}(z,\overline{z}) \end{bmatrix}$$

$\pi_2(F) =$

$$\begin{bmatrix} F_1(z,\overline{z})+F_{\rho_2}(z,\overline{z}) & zF_{\sigma^3}(\overline{z},z)+zF_{\rho_3}(\overline{z},z) & F_{\sigma^2}(\overline{z},\overline{z})+F_{\rho_4}(\overline{z},\overline{z}) & zF_{\sigma}(z,z)+zF_{\rho_1}(z,z) \\ F_{\sigma}(z,\overline{z})+F_{\rho_3}(z,\overline{z}) & F_1(\overline{z},z)+\overline{z}F_{\rho_4}(\overline{z},z) & \overline{z}F_{\sigma^3}(\overline{z},\overline{z})+\overline{z}F_{\rho_1}(\overline{z},\overline{z}) & \overline{z}F_{\sigma^2}(z,z)+F_{\rho_2}(z,z) \\ F_{\sigma^2}(z,\overline{z})+F_{\rho_4}(z,\overline{z}) & F_{\sigma}(\overline{z},z)+F_{\rho_1}(\overline{z},z) & F_1(\overline{z},\overline{z})+F_{\rho_2}(\overline{z},\overline{z}) & F_{\sigma^3}(z,z)+F_{\rho_3}(z,z) \\ F_{\sigma^3}(z,\overline{z})+F_{\rho_1}(z,\overline{z}) & zF_{\sigma^2}(\overline{z},z)+F_{\rho_2}(\overline{z},z) & \overline{z}F_{\sigma}(\overline{z},\overline{z})+\overline{z}F_{\rho_3}(\overline{z},\overline{z}) & F_1(z,z)+zF_{\rho_4}(z,z) \end{bmatrix}$$

Strata $(-1,-1)$

$$\pi_1(F) = [F_{\rho_4} - F_{\sigma^2} + F_1 - F_{\rho_2} + i(F_{\sigma^3} + F_\sigma - F_{\rho_1} - F_{\rho_3})](-1,-1)$$

$$\pi_2(F) = [F_1 - F_{\sigma^2} + F_{\rho^2} - F_{\rho_4} + i(-F_{\sigma^3} - F_\sigma - F_{\rho_1} - F_{\rho_3})](-1,-1)$$

$$\pi_3(F) = [F_1 - F_{\sigma^2} + F_{\rho_2} - F_{\rho_4} + i(F_{\sigma^3} + F_\sigma + F_{\rho_1} + F_{\rho_3})](-1,-1)$$

$$\pi_4(F) = \begin{bmatrix} F_1 + F_{\sigma^2} + i(-F_{\rho_1} + F_{\rho_3}) & -F_{\sigma^3} + F_\sigma + i(-F_{\rho_2} - F_{\rho_4}) \\ -F_{\sigma^3} + F_\sigma + i(F_{\rho_2} + F_{\rho_4}) & F_1 + F_{\sigma^2} + i(F_{\rho_1} - F_{\rho_3}) \end{bmatrix}(-1,-1)$$

$$\pi_5(F) = [F_{\rho_4} - F_{\sigma^2} + F_1 - F_{\rho_2} + i(-F_{\sigma^3} - F_\sigma + F_{\rho_1} + F_{\rho_3})](-1,-1)$$

Strata $(-1,w) : w \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} A & B \end{bmatrix}$$

where

$$A = \begin{bmatrix} F_1(-1,w) + w^{1/2}F_{\rho_3}(-1,w) & -F_{\sigma^3}(\overline{w},-1) + \overline{w}^{1/2}F_{\rho_4}(\overline{z},-1) \\ F_\sigma(-1,w) - \overline{w}^{1/2}F_{\rho_4}(-1,w) & F_1(\overline{w},-1) - \overline{w}^{1/2}F_{\rho_1}(\overline{w},-1) \\ F_{\sigma^2}(-1,w) + \overline{w}^{1/2}F_{\rho_1}(-1,w) & -wF_\sigma(\overline{w},-1) + \overline{w}^{1/2}F_{\rho_2}(\overline{w},-1) \\ F_{\sigma^3}(-1,w) + \overline{w}^{1/2}F_{\rho_2}(-1,w) & -F_{\sigma^2}(\overline{w},-1) - \overline{w}^{1/2}F_{\rho_3}(\overline{w},-1) \end{bmatrix}$$

$$B = \begin{bmatrix} F_{\sigma^2}(-1,\overline{w}) - w^{1/2}F_{\rho_1}(-1,\overline{w}) & wF_\sigma(w,-1) + w^{1/2}F_{\rho_2}(w,-1) \\ -F_{\sigma^3}(-1,\overline{w}) + w^{1/2}F_{\rho_2}(-1,\overline{w}) & -F_{\sigma^2}(w,-1) + w^{1/2}F_{\rho_3}(w,-1) \\ F_1(-1,\overline{w}) - \overline{w}^{1/2}F_{\rho_3}(-1,\overline{w}) & F_{\sigma^3}(w,-1) + w^{1/2}F_{\rho_4}(w,-1) \\ -F_\sigma(-1,\overline{w}) - w^{1/2}F_{\rho_4}(-1,\overline{w}) & F_1(w,-1) + w^{1/2}F_{\rho_1}(w,-1) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} A & B \end{bmatrix}$$

where

$$A = \begin{bmatrix} F_1(-1,w) - w^{1/2}F_{\rho_3}(-1,w) & -F_{\sigma^3}(\overline{w},-1) - \overline{w}^{1/2}F_{\rho_4}(\overline{z},-1) \\ F_\sigma(-1,w) + \overline{w}^{1/2}F_{\rho_4}(-1,w) & F_1(\overline{w},-1) + \overline{w}^{1/2}F_{\rho_1}(\overline{w},-1) \\ F_{\sigma^2}(-1,w) - \overline{w}^{1/2}F_{\rho_1}(-1,w) & -wF_\sigma(\overline{w},-1) - \overline{w}^{1/2}F_{\rho_2}(\overline{w},-1) \\ F_{\sigma^3}(-1,w) - \overline{w}^{1/2}F_{\rho_2}(-1,w) & -F_{\sigma^2}(\overline{w},-1) + \overline{w}^{1/2}F_{\rho_3}(\overline{w},-1) \end{bmatrix}$$

$$B = \begin{bmatrix} F_{\sigma^2}(-1,\overline{w}) + w^{1/2}F_{\rho_1}(-1,\overline{w}) & wF_\sigma(w,-1) - w^{1/2}F_{\rho_2}(w,-1) \\ -F_{\sigma^3}(-1,\overline{w}) - w^{1/2}F_{\rho_2}(-1,\overline{w}) & -F_{\sigma^2}(w,-1) - w^{1/2}F_{\rho_3}(w,-1) \\ F_1(-1,\overline{w}) + \overline{w}^{1/2}F_{\rho_3}(-1,\overline{w}) & F_{\sigma^3}(w,-1) - w^{1/2}F_{\rho_4}(w,-1) \\ -F_\sigma(-1,\overline{w}) + w^{1/2}F_{\rho_4}(-1,\overline{w}) & F_1(w,-1) - w^{1/2}F_{\rho_1}(w,-1) \end{bmatrix}$$

Strata $(-1, 1)$

$$\pi(F) = \begin{bmatrix} F_1 + F_{\sigma^2} & F_{\sigma^3} + F_\sigma & F_{\rho_2} + F_{\rho_4} & F_{\rho_1} + F_{\rho_3} \\ F_{\sigma^3} - F_\sigma & F_1 + F_{\sigma^2} & F_{\rho_1} - F_{\rho_3} & F_{\rho_2} + F_{\rho_4} \\ F_{\rho_2} - F_{\rho_4} & F_{\rho_3} + F_{\rho_1} & F_1 - F_{\sigma^2} & F_\sigma + F_{\sigma^3} \\ -F_{\rho_1} + F_{\rho_3} & F_{\rho_2} - F_{\rho_4} & -F_{\sigma^3} + F_\sigma & F_1 - F_{\sigma^2} \end{bmatrix}$$

Strata $\{(z,1) : z \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} A & B \end{bmatrix}$$

where

$$A = \begin{bmatrix} F_1(z,1) + z^{1/2}F_{\rho_1}(z,1) & zF_{\sigma^3}(1,z) + z^{1/2}F_{\rho_2} \\ F_\sigma(z,1) + \overline{z}^{1/2}F_{\rho_2}(z,1) & F_1(1,z) + z^{1/2}F_{\rho_3} \\ F_{\sigma^2}(z,1) + z^{1/2}F_{\rho_4}(z,1) & zF_{\sigma^2}(1,z) + z^{1/2}F_{\rho_1}(1,z) \\ F_{\sigma^3}(z,1) + z^{1/2}F_{\rho_4}(z,1) & zF_{\sigma^2}(1,z) + z^{1/2}F_{\rho_1}(1,z) \end{bmatrix}$$

$$B = \begin{bmatrix} F_{\sigma^2}(\overline{z},1) + \overline{z}^{1/2}F_{\rho_3}(\overline{z},1) & F_\sigma(1,\overline{z}) + z^{1/2}F_{\rho_4}(1,\overline{z}) \\ \overline{z}F_{\sigma^3}(\overline{z},1) + \overline{z}^{3/2}F_{\rho_4}(\overline{z},1) & \overline{z}F_{\sigma^2}(1,\overline{z}) + \overline{z}^{1/2}F_{\rho_1}(1,\overline{z}) \\ \overline{z}F_\sigma(\overline{z},1) + \overline{z}^{1/2}F_{\rho_2}(\overline{z},1) & F_{\sigma^3}(1,\overline{z}) + z^{1/2}F_{\rho_2}(1,\overline{z}) \\ \overline{z}F_\sigma(\overline{z},1) + \overline{z}^{1/2}F_{\rho_2}(\overline{z},1) & F_1(1,\overline{z}) + \overline{z}^{1/2}F_{\rho_3}(1,\overline{z}) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} A & B \end{bmatrix}$$

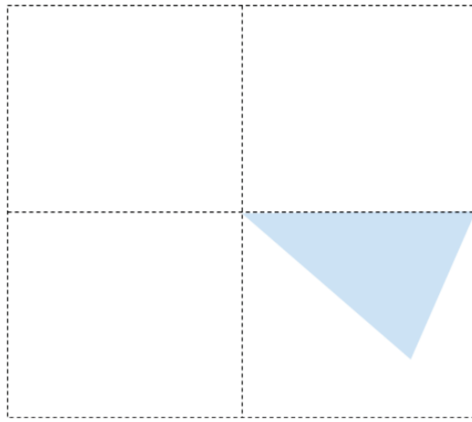
where

$$A = \begin{bmatrix} F_1(z,1) - z^{1/2}F_{\rho_1}(z,1) & zF_{\sigma^3}(1,z) - z^{1/2}F_{\rho_2} \\ F_\sigma(z,1) - \overline{z}^{1/2}F_{\rho_2}(z,1) & F_1(1,z) - z^{1/2}F_{\rho_3} \\ F_{\sigma^2}(z,1) - z^{1/2}F_{\rho_4}(z,1) & zF_{\sigma^2}(1,z) - z^{1/2}F_{\rho_1}(1,z) \\ F_{\sigma^3}(z,1) - z^{1/2}F_{\rho_4}(z,1) & zF_{\sigma^2}(1,z) - z^{1/2}F_{\rho_1}(1,z) \end{bmatrix}$$

$$B = \begin{bmatrix} F_{\sigma^2}(\overline{z},1) - \overline{z}^{1/2}F_{\rho_3}(\overline{z},1) & F_{\sigma}(1,\overline{z}) - z^{1/2}F_{\rho_4}(1,\overline{z}) \\ \overline{z}F_{\sigma^3}(\overline{z},1) - \overline{z}^{3/2}F_{\rho_4}(\overline{z},1) & \overline{z}F_{\sigma^2}(1,\overline{z}) - \overline{z}^{1/2}F_{\rho_1}(1,\overline{z}) \\ \overline{z}F_{\sigma}(\overline{z},1) - \overline{z}^{1/2}F_{\rho_2}(\overline{z},1) & F_{\sigma^3}(1,\overline{z}) - z^{1/2}F_{\rho_2}(1,\overline{z}) \\ \overline{z}F_{\sigma}(\overline{z},1) - \overline{z}^{1/2}F_{\rho_2}(\overline{z},1) & F_{1}(1,\overline{z}) - \overline{z}^{1/2}F_{\rho_3}(1,\overline{z}) \end{bmatrix}$$

p6mm

Point group: $D_12 = \{1, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6\}$



Strata

$$\pi(F) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where

$$A = \begin{bmatrix} F_1(z,w) & F_{\sigma^5}(z\overline{w},z) & F_{\sigma^4}(\overline{w},z\overline{w}) & F_{\sigma^3}(\overline{z},\overline{w}) & F_{\sigma^2}(\overline{z}w,\overline{z}) & F_{\sigma}(w,\overline{z}w) \\ F_{\sigma}(z,w) & F_1(z\overline{w},z) & F_{\sigma^5}(\overline{w},z\o) & F_{\sigma^4}(\overline{z},\overline{w}) & F_{\sigma^3}(\overline{z}w,\overline{z}) & F_{\sigma^2}(w,\overline{z}w) \\ F_{\sigma^2}(z,w) & F_{\sigma}(z\overline{w},z) & F_1(\overline{w},z\overline{w}) & F_{\sigma^5}(\overline{z},\overline{w}) & F_{\sigma^4}(\overline{z}w,\overline{z}) & F_{\sigma^3}(w,\overline{z}w) \\ F_{\sigma^3}(z,w) & F_{\sigma^2}(z\overline{w},z) & F_{\sigma}(\overline{w},z\overline{w}) & F_1(\overline{z},\overline{w}) & F_{\sigma^5}(\overline{z}w,\overline{z}) & F_{\sigma^4}(w,\overline{z}w) \\ F_{\sigma^4}(z,w) & F_{\sigma^3}(z\overline{w},z) & F_{\sigma^2}(\overline{w},z\overline{w}) & F_{\sigma}(\overline{z},\overline{w}) & F_1(\overline{z}w,\overline{z}) & F_5(w,\overline{z}w) \\ F_{\sigma_5}(z,w) & F_{\sigma_4}(z\overline{w},z) & F_{\sigma^3}(\overline{w},z\overline{w}) & F_{\sigma^2}(\overline{z},\overline{w}) & F_{\sigma}(\overline{z}w,\overline{z}) & F_1(w,\overline{z}w) \end{bmatrix}$$

$$C = \begin{bmatrix} F_{\rho_1}(z,w) & F_{\rho_2}(z\overline{w},z) & F_{\rho_3}(\overline{w},z\overline{w}) & F_{\rho_4}(\overline{z},\overline{w}) & F_{\rho_5}(\overline{z}w,\overline{z}) & F_{\rho_6}(w,\overline{z}w) \\ F_{\rho_2}(z,w) & F_{\rho_3}(z\overline{w},z) & F_{\rho_4}(\overline{w},z\overline{w}) & F_{\rho_5}(\overline{z},\overline{w}) & F_{\rho_6}(\overline{z}w,\overline{z}) & F_{\rho_1}(w,\overline{z}w) \\ F_{\rho_3}(z,w) & F_{\rho_4}(z\overline{w},z) & F_{\rho_5}(\overline{w},z\overline{w}) & F_{\rho_6}(\overline{z},\overline{w}) & F_{\rho_1}(\overline{z}w,\overline{z}) & F_{\rho_2}(w,\overline{z}w) \\ F_{\rho_4}(z,w) & F_{\rho_5}(z\overline{w},z) & F_{\rho_6}(\overline{w},z\overline{w}) & F_{\rho_1}(\overline{z},\overline{w}) & F_{\rho_2}(\overline{z}w,\overline{z}) & F_{\rho_3}(w,\overline{z}w) \\ F_{\rho_5}(z,w) & F_{\rho_6}(z\overline{w},z) & F_{\rho_1}(\overline{w},z\overline{w}) & F_{\rho_2}(\overline{z},\overline{w}) & F_{\rho_3}(\overline{z}w,\overline{z}) & F_{\rho_4}(w,\overline{z}w) \\ F_{\rho_6}(z,w) & F_{\rho_1}(z\overline{w},z) & F_{\rho_2}(\overline{w},z\overline{w}) & F_{\rho_3}(\overline{z},\overline{w}) & F_{\rho_4}(\overline{z}w,\overline{z}) & F_{\rho_5}(w,\overline{z}w) \end{bmatrix}$$

$$B = \begin{bmatrix} F_{\rho_1}(z, z\overline{w}) & F_{\rho_2}(w, z) & F_{\rho_3}(\overline{z}w, w) & F_{\rho_4}(\overline{z}, \overline{z}w) & F_{\rho_5}(\overline{w}, \overline{z}) & F_{\rho_6}(z\overline{w}, \overline{w}) \\ F_{\rho_2}(z, z\overline{w}) & F_{\rho_3}(w, z) & F_{\rho_4}(\overline{z}w, w) & F_{\rho_5}(\overline{z}, \overline{z}w) & F_{\rho_6}(\overline{w}, \overline{z}) & F_{\rho_1}(z\overline{w}, \overline{w}) \\ F_{\rho_3}(z, z\overline{w}) & F_{\rho_4}(w, z) & F_{\rho_5}(\overline{z}w, w) & F_{\rho_6}(\overline{z}, \overline{z}w) & F_{\rho_1}(\overline{w}, \overline{z}) & F_{\rho_2}(z\overline{w}, \overline{w}) \\ F_{\rho_4}(z, z\overline{w}) & F_{\rho_5}(w, z) & F_{\rho_6}(\overline{z}w, w) & F_{\rho_1}(\overline{z}, \overline{z}w) & F_{\rho_2}(\overline{w}, \overline{z}) & F_{\rho_3}(z\overline{w}, \overline{w}) \\ F_{\rho_5}(z, z\overline{w}) & F_{\rho_6}(w, z) & F_{\rho_1}(\overline{z}w, w) & F_{\rho_2}(\overline{z}, \overline{z}w) & F_{\rho_3}(\overline{w}, \overline{z}) & F_{\rho_4}(z\overline{w}, \overline{w}) \\ F_{\rho_6}(z, z\overline{w}) & F_{\rho_1}(w, z) & F_{\rho_2}(\overline{z}w, w) & F_{\rho_3}(\overline{z}, \overline{z}w) & F_{\rho_4}(\overline{w}, \overline{z}) & F_{\rho_5}(z\overline{w}, \overline{w}) \end{bmatrix}$$

$$D = \begin{bmatrix} F_1(z, z\overline{w}) & F_{\sigma^5}(w, z) & F_{\sigma^4}(\overline{z}w, w) & F_{\sigma^3}(\overline{z}, \overline{z}w) & F_{\sigma^2}(\overline{w}, \overline{z}) & F_{\sigma}(z\overline{w}, \overline{w}) \\ F_{\sigma}(z, z\overline{w}) & F_1(w, z) & F_{\sigma^5}(\overline{z}w, w) & F_{\sigma^4}(\overline{z}, \overline{z}w) & F_{\sigma^3}(\overline{w}, \overline{z}) & F_{\sigma^2}(z\overline{w}, \overline{w}) \\ F_{\sigma^2}(z, z\overline{w}) & F_{\sigma}(w, z) & F_1(\overline{z}w, w) & F_{\sigma^5}(\overline{z}, \overline{z}w) & F_{\sigma^4}(\overline{w}, \overline{z}) & F_{\sigma^3}(z\overline{w}, \overline{w}) \\ F_{\sigma^3}(z, z\overline{w}) & F_{\sigma^2}(w, z) & F_{\sigma}(\overline{z}w, w) & F_1(\overline{z}, \overline{z}w) & F_{\sigma^5}(\overline{w}, \overline{z}) & F_{\sigma^4}(z\overline{w}, \overline{w}) \\ F_{\sigma^4}(z, z\overline{w}) & F_{\sigma^3}(w, z) & F_{\sigma^2}(\overline{z}w, w) & F_{\sigma}(\overline{z}, \overline{z}w) & F_1(\overline{w}, \overline{z}) & F_5(z\overline{w}, \overline{w}) \\ F_{\sigma_5}(z, z\overline{w}) & F_{\sigma_4}(w, z) & F_{\sigma^3}(\overline{z}w, w) & F_{\sigma^2}(\overline{z}, \overline{z}w) & F_{\sigma}(\overline{w}, \overline{z}) & F_1(z\overline{w}, \overline{w}) \end{bmatrix}$$

Strata $\{(z, 1) : z \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} A & B \end{bmatrix}$$

where

$$A = \begin{bmatrix} (F_1 + F_{\rho_2})(z, z) & (F_{\sigma^5} + F_{\rho_3})(1, z) & (F_{\sigma^4} + F_{\rho_4})(\overline{z}, 1) \\ (F_{\sigma} + F_{\rho_3})(z, z) & (F_1 + F_{\rho_4})(1, z) & (F_{\sigma^5} + F_{\rho_5})(\overline{z}, 1) \\ (F_{\sigma^2} + F_{\rho_4})(z, z) & (F_{\sigma} + F_{\rho_5})(1, z) & (F_1 + F_{\rho_6})(\overline{z}, 1) \\ (F_{\sigma^3} + F_{\rho_5})(z, z) & (F_{\sigma^2} + F_{\rho_6})(1, z) & (F_{\sigma} + F_{\rho_1})(\overline{z}, 1) \\ (F_{\sigma^4} + F_{\rho_6})(z, z) & (F_{\sigma^3} + F_{\rho_1})(1, z) & (F_{\sigma^2} + F_{\rho_2})(\overline{z}, 1) \\ (F_{\sigma^5} + F_{\rho_1})(z, z) & (F_{\sigma^4} + F_{\rho_2})(1, z) & (F_{\sigma^3} + F_{\rho_3})(\overline{z}, 1) \end{bmatrix}$$

$$B = \begin{bmatrix} (F_{\sigma^3}+F_{\rho_5})(\overline{z},\overline{z}) & (F_{\sigma^2}+F_{\rho_6})(1,\overline{z}) & (F_{\sigma}+F_{\rho_1})(z,1) \\ (F_{\sigma^4}+F_{\rho_6})(\overline{z},\overline{z}) & (F_{\sigma^3}+F_{\rho_1})(1,\overline{z}) & (F_{\sigma^2}+F_{\rho_2})(z,1) \\ (F_{\sigma^5}+F_{\rho_1})(\overline{z},\overline{z}) & (F_{\sigma^4}+F_{\rho_2})(1,\overline{z}) & (F_{\sigma^3}+F_{\rho_3})(z,1) \\ (F_{1}+F_{\rho_2})(\overline{z},\overline{z}) & (F_{\sigma^5}+F_{\rho_3})(1,\overline{z}) & (F_{\sigma^4}+F_{\rho_4})(z,1) \\ (F_{\sigma}+F_{\rho_3})(\overline{z},\overline{z}) & (F_{1}+F_{\rho_4})(1,\overline{z}) & (F_{\sigma^5}+F_{\rho_5})(z,1) \\ (F_{\sigma^2}+F_{\rho_4})(\overline{z},\overline{z}) & (F_{\sigma}+F_{\rho_5})(1,\overline{z}) & (F_{1}+F_{\rho_6})(z,1) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} A & B \end{bmatrix}$$

where

$$A = \begin{bmatrix} (F_{1}-F_{\rho_2})(\overline{z},\overline{z}) & (F_{\sigma^5}-F_{\rho_3})(1,z) & (F_{\sigma^4}-F_{\rho_4})(z,z) \\ (F_{\sigma}-F_{\rho_3})(\overline{z},\overline{z}) & (F_{1}-F_{\rho_4})(1,z) & (F_{\sigma^5}-F_{\rho_5})(z,z) \\ (F_{\sigma^2}-F_{\rho_4})(\overline{z},\overline{z}) & (F_{\sigma}-F_{\rho_5})(1,z) & (F_{1}-F_{\rho_6})(z,z) \\ (F_{\sigma^3}-F_{\rho_5})(\overline{z},\overline{z}) & (F_{\sigma^2}-F_{\rho_6})(1,z) & (F_{\sigma}-F_{\rho_1})(z,z) \\ (F_{\sigma^4}-F_{\rho_6})(\overline{z},\overline{z}) & (F_{\sigma^3}-F_{\rho_1})(1,z) & (F_{\sigma^2}-F_{\rho_2})(z,z) \\ (F_{\sigma^5}-F_{\rho_1})(\overline{z},\overline{z}) & (F_{\sigma^4}-F_{\rho_2})(1,z) & (F_{\sigma^3}-F_{\rho_3})(z,z) \end{bmatrix}$$

$$B = \begin{bmatrix} (F_{\sigma^3}-F_{\rho_5})(1,\overline{z}) & (F_{\sigma^2}-F_{\rho_6})(\overline{z},1) & (F_{\sigma}-F_{\rho_1})(z,1) \\ (F_{\sigma^4}-F_{\rho_6})(1,\overline{z}) & (F_{\sigma^3}-F_{\rho_1})(\overline{z},1) & (F_{\sigma^2}-F_{\rho_2})(z,1) \\ (F_{\sigma^5}-F_{\rho_1})(1,\overline{z}) & (F_{\sigma^4}-F_{\rho_2})(\overline{z},1) & (F_{\sigma^3}-F_{\rho_3})(z,1) \\ (F_{1}-F_{\rho_2})(1,\overline{z}) & (F_{\sigma^5}-F_{\rho_3})(\overline{z},1) & (F_{\sigma^4}-F_{\rho_4})(z,1) \\ (F_{\sigma}-F_{\rho_3})(1,\overline{z}) & (F_{1}-F_{\rho_4})(\overline{z},1) & (F_{\sigma^5}-F_{\rho_5})(z,1) \\ (F_{\sigma^2}-F_{\rho_4})(1,\overline{z}) & (F_{\sigma}-F_{\rho_5})(\overline{z},1) & (F_{1}-F_{\rho_6})(z,1) \end{bmatrix}$$

Strata $\{(z,\overline{z}) : z \in \mathbb{T}^2\}$

$$\pi_1(F) = \begin{bmatrix} A & B \end{bmatrix}$$

where

$$A = \begin{bmatrix} (F_1 + F_{\rho_3})(\overline{z}, \overline{z}^2) & (F_{\sigma^4} + F_{\rho_5})(z^2, z) & (F_{\sigma^3} + F_{\rho_6})(z^2, z) \\ (F_{\sigma^2} + F_{\rho_5})(\overline{z}, \overline{z}^2) & (F_1 + F_{\rho_1})(z^2, z) & (F_{\sigma^5} + F_{\rho_2})(z, z^2) \\ (F_{\sigma^3} + F_{\rho_6})(\overline{z}, \overline{z}^2) & (F_{\sigma} + F_{\rho_2})(z^2, z) & (F_1 + F_{\rho_3})(z, z^2) \\ (F_{\sigma^5} + F_{\rho_2})(\overline{z}, \overline{z}^2) & (F_{\sigma^3} + F_{\rho_4})(z^2, z) & (F_{\sigma^2} + F_{\rho_5})(z, z^2) \\ (F_{\sigma^4} + F_{\rho_1})(\overline{z}, \overline{z}^2) & (F_{\sigma^2} + F_{\rho_3})(z^2, z) & (F_{\sigma} + F_{\rho_4})(z, z^2) \\ (F_{\sigma} + F_{\rho_4})(\overline{z}, \overline{z}^2) & (F_{\sigma^5} + F_{\rho_6})(z^2, z) & (F_{\sigma^4} + F_{\rho_1})(z, z^2) \end{bmatrix}$$

$$B = \begin{bmatrix} (F_{\sigma} + F_{\rho_2})(\overline{z}^2, \overline{z}) & (F_{\sigma^2} + F_{\rho_1})(\overline{z}, z) & (F_{\sigma^5} - F_{\rho_4})(z, \overline{z}) \\ (F_{\sigma^3} + F_{\rho_4})(\overline{z}^2, \overline{z}) & (F_{\sigma^4} + F_{\rho_3})(\overline{z}, z) & (F_{\sigma} + F_{\rho_6})(z, \overline{z}) \\ (F_{\sigma^4} + F_{\rho_5})(\overline{z}^2, \overline{z}) & (F_{\sigma^5} + F_{\rho_4})(\overline{z}, z) & (F_{\sigma^2} + F_{\rho_1})(z, \overline{z}) \\ (F_1 + F_{\rho_1})(\overline{z}^2, \overline{z}) & (F_{\sigma} + F_{\rho_6})(\overline{z}, z) & (F_{\sigma^4} + F_{\rho_3})(z, \overline{z}) \\ (F_{\sigma^5} + F_{\rho_6})(\overline{z}^2, \overline{z}) & (F_1 + F_{\rho_5})(\overline{z}, z) & (F_{\sigma^3} + F_{\rho_2})(z, \overline{z}) \\ (F_{\sigma^2} + F_{\rho_3})(\overline{z}^2, \overline{z}) & (F_{\sigma^3} + F_{\rho_2})(\overline{z}, z) & (F_1 + F_{\rho_3})(z, \overline{z}) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} A & B \end{bmatrix}$$

where

$$A = \begin{bmatrix} (F_1 - F_{\rho_3})(\overline{z}, \overline{z}^2) & (F_{\sigma^5} - F_{\rho_4})(z, \overline{z}) & (F_{\sigma^4} - F_{\rho_5})(z^2, z) \\ (F_{\sigma} - F_{\rho_4})(\overline{z}, \overline{z}^2) & (F_1 - F_{\rho_5})(z, \overline{z}) & (F_{\sigma^5} - F_{\rho_6})(z^2, z) \\ (F_{\sigma^2} - F_{\rho_5})(\overline{z}, \overline{z}^2) & (F_{\sigma} - F_{\rho_6})(z, \overline{z}) & (F_1 - F_{\rho_1})(z^2, z) \\ (F_{\sigma^3} - F_{\rho_6})(\overline{z}, \overline{z}^2) & (F_{\sigma^2} - F_{\rho_1})(z, \overline{z}) & (F_{\sigma} - F_{\rho_2})(z^2, z) \\ (F_{\sigma^4} - F_{\rho_1})(\overline{z}, \overline{z}^2) & (F_{\sigma^3} - F_{\rho_2})(z, \overline{z}) & (F_{\sigma^2} - F_{\rho_3})(z^2, z) \\ (F_{\sigma^5} - F_{\rho_2})(\overline{z}, \overline{z}^2) & (F_{\sigma^4} - F_{\rho_3})(z, \overline{z}) & (F_{\sigma^3} - F_{\rho_4})(z^2, z) \end{bmatrix}$$

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$$B = \begin{bmatrix} (F_{\sigma^3} - F_{\rho_6})(\overline{z}, z) & (F_{\sigma^2} - F_{\rho_1})(\overline{z}, z) & (F_{\sigma} - F_{\rho_2})(\overline{z}^2, \overline{z}) \\ (F_{\sigma^4} - F_{\rho_1})(z, z^2) & (F_{\sigma_3} - F_{\rho_2})(\overline{z}, z) & (F_{\sigma_2} - F_{\rho_3})(\overline{z}^2, \overline{z}) \\ (F_{\sigma^5} - F_{\rho_2})(z, z^2) & (F_{\sigma^4} - F_{\rho_3})(\overline{z}, z) & (F_{\sigma^3} - F_{\rho_4})(\overline{z}^2, \overline{z}) \\ (F_1 - F_{\rho_3})(z, z^2) & (F_{\sigma^5} - F_{\rho_4})(\overline{z}, z) & (F_{\sigma^4} - F_{\rho_5})(\overline{z}^2, \overline{z}) \\ (F_{\sigma} - F_{\rho_4})(z, z^2) & (F_1 - F_{\rho_3})(\overline{z}, z) & (F_{\sigma^5} - F_{\rho_6})(\overline{z}^2, \overline{z}) \\ (F_{\sigma^2} - F_{\rho_5})(z, z^2) & (F_{\sigma} - F_{\rho_6})(\overline{z}, z) & (F_1 - F_{\rho_1})(\overline{z}^2, \overline{z}) \end{bmatrix}$$

Strata $\{(z, z^2) : z \in \mathbb{T}^2/D\}$

$$\pi_1(F) = \begin{bmatrix} A & B \end{bmatrix}$$

where

$$A = \begin{bmatrix} (F_1 - F_{\rho_5})(\overline{z}, z) & (F_{\sigma^5} - F_{\rho_6})(\overline{z}^2, \overline{z}) & (F_{\sigma^2} - F_{\rho_3})(z^2, z) \\ (F_{\sigma} - F_{\rho_6})(\overline{z}, z) & (F_1 - F_{\rho_1})(\overline{z}^2, \overline{z}) & (F_{\sigma^3} - F_{\rho_4})(z^2, z) \\ (F_{\sigma^4} - F_{\rho_3})(\overline{z}, z) & (F_{\sigma^2} - F_{\rho_4})(\overline{z}^2, \overline{z}) & (F_{\sigma^5} - F_{\rho_6})(z^2, z) \\ (F_{\sigma^3} - F_{\rho_2})(\overline{z}, z) & (F_{\sigma^2} - F_{\rho_3})(\overline{z}^2, \overline{z}) & (F_{\sigma^5} - F_{\rho_6})(z^2, z) \\ (F_{\sigma^2} - F_{\rho_1})(\overline{z}, z) & (F_{\sigma} - F_{\rho_2})(\overline{z}^2, \overline{z}) & (F_{\sigma^4} - F_{\rho_5})(z^2, z) \\ (F_{\sigma^5} - F_{\rho_4})(\overline{z}, z) & (F_{\sigma^4} - F_{\rho_5})(\overline{z}^2, \overline{z}) & (F_{\sigma} - F_{\rho_2})(z^2, z) \end{bmatrix}$$

$$B = \begin{bmatrix} (F_{\sigma^3} - F_{\rho_2})(z, \overline{z}) & (F_{\sigma^4} - F_{\rho_1})(\overline{z}, \overline{z}^2) & (F_{\sigma} - F_{\rho_4})(z, z^2) \\ (F_{\sigma^4} - F_{\rho_3})(z, \overline{z}) & (F_{\sigma^5} - F_{\rho_2})(\overline{z}, \overline{z}^2) & (F_{\sigma^2} - F_{\rho_5})(z, z^2) \\ (F_{\sigma} - F_{\rho_6})(z, \overline{z}) & (F_{\sigma^2} - F_{\rho_5})(\overline{z}, \overline{z}^2) & (F_{\sigma^5} - F_{\rho_2})(z, z^2) \\ (F_1 - F_{\rho_5})(z, \overline{z}) & (F_{\sigma} - F_{\rho_4})(\overline{z}, \overline{z}^2) & (F_{\sigma^4} - F_{\rho_1})(z, z^2) \\ (F_{\sigma^5} - F_{\rho_4})(z, \overline{z}) & (F_1 - F_{\rho_3})(\overline{z}, \overline{z}^2) & (F_{\sigma^3} - F_{\rho_6})(z, z^2) \\ (F_{\sigma^2} - F_{\rho_1})(z, \overline{z}) & (F_{\sigma^3} - F_{\rho_6})(\overline{z}, \overline{z}^2) & (F_1 - F_{\rho_3})(z, z^2) \end{bmatrix}$$

$$\pi_2(F) = \begin{bmatrix} A & B \end{bmatrix}$$

where

$$A = \begin{bmatrix} (F_1 + F_{\rho_3})(z, z^2) & (F_{\sigma^5} + F_{\rho_4})(\overline{z}, z) & (F_{\sigma^4} + F_{\rho_5})(\overline{z}^2, \overline{z}) \\ (F_\sigma + F_{\rho_4})(z, z^2) & (F_1 + F_{\rho_5})(\overline{z}, z) & (F_{\sigma^5} + F_{\rho_6})(\overline{z}^2, \overline{z}) \\ (F_{\sigma^2} + F_{\rho_5})(z, z^2) & (F_\sigma + F_{\rho_6})(\overline{z}, z) & (F_1 + F_{\rho_1})(\overline{z}^2, \overline{z}) \\ (F_{\sigma^3} + F_{\rho_6})(z, z^2) & (F_{\sigma^2} + F_{\rho_1})(\overline{z}, z) & (F_\sigma + F_{\rho_2})(\overline{z}^2, \overline{z}) \\ (F_{\sigma^4} + F_{\rho_1})(z, z^2) & (F_{\sigma^3} + F_{\rho_2})(\overline{z}, z) & (F_{\sigma^2} + F_{\rho_3})(\overline{z}^2, \overline{z}) \\ (F_{\sigma^5} + F_{\rho_2})(z, z^2) & (F_{\sigma^4} + F_{\rho_3})(\overline{z}, z) & (F_{\sigma^3} + F_{\rho_4})(\overline{z}^2, \overline{z}) \end{bmatrix}$$

$$B = \begin{bmatrix} (F_{\sigma^3} + F_{\rho_6})(\overline{z}, \overline{z}^2) & (F_{\sigma^2} + F_{\rho_1})(\overline{z}, \overline{z}) & (F_\sigma + F_{\rho_2})(z^2, z) \\ (F_{\sigma^4} + F_{\rho_1})(\overline{z}, \overline{z}^2) & (F_{\sigma^3} + F_{\rho_2})(z, \overline{z}) & (F_{\sigma^2} + F_{\rho_3})(z^2, z) \\ (F_{\sigma^5} + F_{\rho_2})(\overline{z}, \overline{z}^2) & (F_{\sigma^4} + F_{\rho_3})(z, \overline{z}) & (F_{\sigma^3} + F_{\rho_4})(z^2, z) \\ (F_1 + F_{\rho_3})(\overline{z}, \overline{z}^2) & (F_{\sigma^5} + F_{\rho_4})(z, \overline{z}) & (F_{\sigma^4} + F_{\rho_5})(z^2, z) \\ (F_\sigma + F_{\rho_4})(\overline{z}, \overline{z}^2) & (F_1 + F_{\rho_5})(z, \overline{z}) & (F_{\sigma^5} + F_{\rho_6})(z^2, z) \\ (F_{\sigma^2} + F_{\rho_5})(\overline{z}, \overline{z}^2) & (F_\sigma + F_{\rho_6})(z, \overline{z}) & (F_1 + F_{\rho_1})(z^2, z) \end{bmatrix}$$

Strata $(1, 1)$

$$\pi_1(F) = [F_1 + F_{\sigma^5} + F_{\sigma^4} + F_{\sigma^3} + F_{\sigma^2} + F_\sigma + F_{\rho_1} + F_{\rho_2} + F_{\rho_3} + F_{\rho_4} + F_{\rho_5} + F_{\rho_6}](1, 1)$$

$$\pi_2(F) = [F_1 + F_{\sigma^5} + F_{\sigma^4} + F_{\sigma^3} + F_{\sigma^2} + F_\sigma - F_{\rho_1} - F_{\rho_2} - F_{\rho_3} - F_{\rho_4} - F_{\rho_5} - F_{\rho_6}](1, 1)$$

$$\pi_3(F) = [F_1 - F_{\sigma^5} + F_{\sigma^4} - F_{\sigma^3} + F_{\sigma^2} - F_\sigma - F_{\rho_1} + F_{\rho_2} - F_{\rho_3} + F_{\rho_4} - F_{\rho_5} + F_{\rho_6}](1, 1)$$

$$\pi_4(F) = [F_1 - F_{\sigma^5} + F_{\sigma^4} - F_{\sigma^3} + F_{\sigma^2} - F_\sigma + F_{\rho_1} - F_{\rho_2} + F_{\rho_3} - F_{\rho_4} + F_{\rho_5} - F_{\rho_6}](1, 1)$$

$$\pi_5(F) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where

$$a = \frac{1}{2}\left((1-\sqrt{3}i)F_{\sigma} + (-1+\sqrt{3}i)F_{\sigma^4}(1+\sqrt{3})F_{\sigma^5} + (-1-\sqrt{3}i)F_{\sigma_2} + 2F_1 - 2F_{\sigma^3}\right)$$

$$b = \frac{1}{2}\left((1-\sqrt{3}i)F_{\rho_2} + (-1-\sqrt{3}i)F_{\rho_3} + (-1+\sqrt{3}i)F_{\rho_5} + (1+\sqrt{3}i)F_{\rho_6} + 2F_{\rho_1} - 2F_{\rho_4}\right)$$

$$c = \frac{1}{2}\left((1+\sqrt{3}i)F_{\rho_2} + (-1+\sqrt{3}i)F_{\rho_3} + (-1-\sqrt{3}i)F_{\rho_5} + (1-\sqrt{3}i)F_{\rho_6} + 2F_{\rho_1} - 2F_{\rho_4}\right)$$

$$d = \frac{1}{2}\left((1+\sqrt{3}i)F_{\sigma} + (-1\sqrt{3}i)F_{\sigma^4}(1-\sqrt{3})F_{\sigma^5} + (-1+\sqrt{3}i)F_{\sigma_2} + 2F_1 - 2F_{\sigma^3}\right)$$

$$\pi_6(F) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where

$$a = \frac{1}{2}\left((-1+\sqrt{3}i)F_{\sigma} + (-1-\sqrt{3}i)F_{\sigma^4}(-1+\sqrt{3})F_{\sigma^5} + (-1+\sqrt{3}i)F_{\sigma_2} + 2F_1 + 2F_{\sigma^3}\right)$$

$$b = \frac{1}{2}\left((-1-\sqrt{3}i)F_{\rho_2} + (-1+\sqrt{3}i)F_{\rho_3} + (-1-\sqrt{3}i)F_{\rho_5} + (-1+\sqrt{3}i)F_{\rho_6} + 2F_{\rho_1} + 2F_{\rho_4}\right)$$

$$c = \frac{1}{2}\left((-1+\sqrt{3}i)F_{\rho_2} + (1-\sqrt{3}i)F_{\rho_3} + (-1+\sqrt{3}i)F_{\rho_5} + (-1-\sqrt{3}i)F_{\rho_6} + 2F_{\rho_1} + 2F_{\rho_4}\right)$$

$$d = \frac{1}{2}\left((-1+\sqrt{3}i)F_{\sigma} + (-1+\sqrt{3}i)F_{\sigma^4}(-1-\sqrt{3})F_{\sigma^5} + (-1-\sqrt{3}i)F_{\sigma_2} + 2F_1 + 2F_{\sigma^3}\right)$$

Strata $(-1, 1)$

$$\pi_1(F) = \begin{bmatrix} [A \quad B \quad C] \end{bmatrix}$$

where

$$A = \begin{bmatrix} F_1 - F_{\sigma^3} + F_{\rho_3} - F_{\rho_6} \\ F_\sigma - F_{\sigma^4} + F_{\rho_4} - F_{\rho_1} \\ F_{\sigma^2} - F_{\sigma^5} + F_{\rho_5} - F_{\rho_2} \end{bmatrix} (-1, 1)$$

$$B = \begin{bmatrix} F_{\sigma^5} - F_{\sigma^2} - F_{\rho_1} + F_{\rho_4} \\ F_1 - F_{\sigma^3} - F_{\rho_2} + F_{\rho_5} \\ F_\sigma - F_{\sigma^4} - F_{\rho_3} + F_{\rho_6} \end{bmatrix} (-1, -1)$$

$$C = \begin{bmatrix} F_{\sigma^4} - F_\sigma - F_{\rho_2} + F_{\rho_3} \\ F_{\sigma^5} - F_{\sigma^2} - F_{\rho_3} + F_{\rho_6} \\ F_1 - F_{\sigma^3} - F_{\rho_4} + F_{\rho_1} \end{bmatrix} (1, -1)$$

$$\pi_2(F) = \begin{bmatrix} A & B & C \end{bmatrix}$$

where

$$A = \begin{bmatrix} F_1 - F_{\sigma^3} + F_{\rho_2} - F_{\rho_6} \\ F_\sigma + F_{\sigma^4} + F_{\rho_3} + F_{\rho_6} \\ F_{\sigma^2} + F_{\sigma^5} + F_{\rho_1} + F_{\rho_4} \end{bmatrix} (-1, 1)$$

$$B = \begin{bmatrix} F_{\sigma^5} - F_{\sigma^2} + F_{\rho_3} - F_{\rho_6} \\ F_1 - F_{\sigma^3} + F_{\rho_4} - F_{\rho_1} \\ F_{\sigma^4} - F_\sigma + F_{\rho_2} - F_{\rho_5} \end{bmatrix} (1, -1)$$

$$C = \begin{bmatrix} F_\sigma - F_{\sigma^4} - F_{\rho_4} + F_{\rho_1} \\ F_{\sigma^2} - F_{\sigma^5} - F_{\rho_5} + F_{\rho_2} \\ F_1 - F_{\sigma^3} - F_{\rho_3} + F_{\rho_6} \end{bmatrix} (-1, 1)$$

$$\pi_3(F) = \begin{bmatrix} A & B & C \end{bmatrix}$$

where

$$A = \begin{bmatrix} F_1 - F_{\sigma^3} + F_{\rho_2} - F_{\rho_3} \\ F_\sigma - F_{\sigma^4} + F_{\rho_3} - F_{\rho_6} \\ F_{\sigma^5} - F_{\sigma^2} + F_{\rho_1} - F_{\rho_4} \end{bmatrix} (-1,-1)$$

$$B = \begin{bmatrix} F_{\sigma^5} - F_{\sigma^2} + F_{\rho_3} - F_{\rho_6} \\ F_1 - F_{\sigma^3} + F_{\rho_4} - F_{\rho_1} \\ F_{\sigma^4} - F_\sigma + F_{\rho_2} - F_{\rho_5} \end{bmatrix} (1,-1)$$

$$C = \begin{bmatrix} F_\sigma - F_{\sigma^4} - F_{\rho_4} + F_{\rho_1} \\ F_{\sigma^2} - F_{\sigma^5} - F_{\rho_5} + F_{\rho_2} \\ F_1 - F_{\sigma^3} - F_{\rho_3} + F_{\rho_6} \end{bmatrix} (-1,1)$$

$$\pi_4(F) = \begin{bmatrix} A & B & C \end{bmatrix}$$

where

$$A = \begin{bmatrix} F_1 + F_{\sigma^3} - F_{\rho_3} - F_{\rho_6} \\ F_\sigma + F_{\sigma^4} - F_{\rho_4} - F_{\rho_1} F_{\sigma^2} + F_{\sigma^5} - F_{\rho_5} - F_{\rho_2} \end{bmatrix} (-1,1)$$

$$B = \begin{bmatrix} F_{\sigma^5} + F_{\sigma^2} - F_{\rho_1} - F_{\rho_4} \\ F_1 + F_{\sigma^3} - F_{\rho_2} - F_{\rho_5} \\ F_\sigma + F_{\sigma^4} - F_{\rho_3} - F_{\rho_6} \end{bmatrix} (-1,-1)$$

$$C = \begin{bmatrix} F_{\sigma^4} + F_\sigma - F_{\rho_2} - F_{\rho_5} \\ F_{\sigma^2} + F_{\sigma^2} - F_{\rho_3} - F_{\rho_6} \\ F_1 + F_{\sigma^3} - F_{\rho_4} - F_{\rho_1} \end{bmatrix} (1,-1)$$

Strata $(e^{2/3\pi i}, e^{-2/3\pi i})$

$$\pi_1(F) = \begin{bmatrix} a_{11} & a_{12} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

where

$$a_{11} = \tfrac{1}{2}(2F_1 - F_{\sigma^4} - F_{\sigma^2} + F_{\rho_1} - 2F_{\rho_3} + F_{\rho_5})(e^{2/3\pi i}, e^{-2/3\pi i})$$

$$a_{12} = \tfrac{1}{2}(2F_{\sigma^5} - F_{\sigma^3} - F_{\sigma} + F_{\rho_2} - 2F_{\rho_4} + F_{\rho_6})(e^{-2/3\pi i}, e^{2/3\pi i})$$

$$a_{13} = \tfrac{\sqrt{3}}{2}(F_{\sigma^4} - F_{\sigma^2} + F_{\rho_1} - F_{\rho_5})(e^{2/3\pi i}, e^{-2/3\pi i})$$

$$a_{14} = \tfrac{\sqrt{3}}{2}(F_{\sigma^3} - F_{\sigma} + F_{\rho_2} - F_{\rho_6})(e^{-2/3\pi i}, e^{2/3\pi i})$$

$$a_{21} = \tfrac{1}{2}(2F_{\sigma} - F_{\sigma^5} - F_{\sigma^3} + F_{\rho_2} - 2F_{\rho_4} + F_{\rho_6})(e^{2/3\pi i}, e^{-2/3\pi i})$$

$$a_{22} = \tfrac{1}{2}(2F_1 - F_{\sigma^4} - f_{\sigma^2} + F_{\rho_3} - 2F_{\rho_5} + F_{\rho_1})(e^{-2/3\pi i}, e^{2/3\pi i})$$

$$a_{23} = \tfrac{\sqrt{3}}{2}(F_{\sigma^5} - F_{\sigma^3} + F_{\rho_2} - F_{\rho_6})(e^{2/3\pi i}, e^{-2/3\pi i})$$

$$a_{24} = \tfrac{\sqrt{3}}{2}(F_{\sigma^4} - F_{\sigma^2} + F_{\rho_3} - F_{\rho_1})(e^{-2/3\pi i}, e^{2/3\pi i})$$

$$a_{31} = \tfrac{\sqrt{3}}{2}(F_{\sigma^2} - F_{\sigma^4} - F_{\rho_5} + F_{\rho_1})(e^{2/3\pi i}, e^{-2/3\pi i})$$

$$a_{32} = \tfrac{\sqrt{3}}{2}(F_{\sigma} - F_{\sigma^3} - F_{\rho_6} + F_{\rho_2})(e^{-2/3\pi i}, e^{2/3\pi i})$$

$$a_{33} = \tfrac{1}{2}(2F_1 - F_{\sigma^4} + 2F_{\rho_3} - F_{\rho_1} - F_{\sigma^2} - F_{\rho_5})(e^{2/3\pi i}, e^{-2/3\pi i})(e^{2/3\pi i}, e^{-2/3\pi i})$$

$a_{34} = \frac{1}{2}(2F_{\sigma_5} - F_{\sigma^3} + 2F_{\rho_4} - F_{\rho_2} - F_{\sigma} - F_{\rho_6})(e^{-2/3\pi i}, e^{2/3\pi i})$

$a_{41} = \frac{\sqrt{3}}{2}(F_{\sigma^3} - F_{\sigma^5} - F_{\rho_6} + F_{\rho_2})(e^{2/3\pi i}, e^{-2/3\pi i})$

$a_{42} = \frac{\sqrt{3}}{2}(F_{\sigma^2} - F_{\sigma^4} - F_{\rho_1} + F_{\rho_3})(e^{-2/3\pi i}, e^{2/3\pi i})(e^{-2/3\pi i}, e^{2/3\pi i})$

$a_{43} = \frac{1}{2}(2F_{\sigma} - F_{\sigma^5} + 2F_{\rho_4} - F_{\rho_2} - F_{\sigma^3} - F_{\rho_6})(e^{2/3\pi i}, e^{-2/3\pi i})$

$a_{44} = \frac{1}{2}(2F_1 - F_{\sigma^4} + 2F_{\rho_5} - F_{\rho_3} - F_{\sigma^2} - F_{\rho_1})(e^{-2/3\pi i}, e^{2/3\pi i})$

$$\pi_2(F) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where

$a = (F_1 + F_{\sigma^4} + F_{\sigma^2} - F_{\rho_3} - F_{\rho_5} - F_{\rho_1})(e^{-2/3\pi i}, e^{2/3\pi i})$

$b = (F_{\sigma} + F_{\sigma^5} + F_{\sigma^3} - F_{\rho_2} - F_{\rho_4} - F_{\rho_6})(e^{2/3\pi i}, e^{-2/3\pi i})$

$c = (F_{\sigma} + F_{\sigma^5} + F_{\sigma^3} - F_{\rho_2} - F_{\rho_4} - F_{\rho_6})(e^{-2/3\pi i}, e^{2/3\pi i})$

$d = (F_1 + F_{\sigma^4} + F_{\sigma^2} - F_{\rho_3} - F_{\rho_5} - F_{\rho_1})(e^{2/3\pi i}, e^{-2/3\pi i})$

$$\pi_3(F) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where

$a = (F_1 + F_{\sigma^4} + F_{\sigma^2} + F_{\rho_3} + F_{\rho_5} + F_{\rho_1})(e^{2/3\pi i}, e^{-2/3\pi i})$

$$b = (F_\sigma + F_{\sigma^5} + F_{\sigma^3} + F_{\rho_2} + F_{\rho_4} - F_{\rho_6})(e^{-2/3\pi i}, e^{2/3\pi i})$$

$$c = (F_\sigma + F_{\sigma^5} + F_{\sigma^3} + F_{\rho_2} + F_{\rho_4} - F_{\rho_6})(e^{2/3\pi i}, e^{-2/3\pi i})$$

$$a = (F_1 + F_{\sigma^4} + F_{\sigma^2} + F_{\rho_3} + F_{\rho_5} + F_{\rho_1})(e^{-2/3\pi i}, e^{2/3\pi i})$$