THE CLASS OF STRONG PLACEMENT GAMES: COMPLEXES, VALUES, AND TEMPERATURE

by

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Abstract

Strong Placement (SP-) games are a class of combinatorial games in which pieces are placed on a board such that the order in which previously placed pieces have been played does not matter.

It is known that to each such game one can assign two square-free monomial ideals (the legal and illegal ideal) and two simplicial complexes (the legal and illegal complex). In this work we will show that reverse constructions also exist, in particular when restricting to invariant SP-games.

We then use this one-to-one correspondence between games, ideals, and simplicial complexes to study several properties of SP-games. This includes the structure of the game tree of an SP-game, and the set of possible game values.

The temperatures of SP-games are also considered. We prove a first general upper bound on the boiling point of a game, and will show through several games that this bound is particularly applicable for SP-games.

Motivated by the connection to commutative algebra, we then explore what it could mean for an SP-game to be Cohen-Macaulay, as well as several related properties.

List of Abbreviations and Symbols Used

Symbol	Page	Description
GAME	4	Text in small caps indicates a combinatorial game
(R, B)	5	The ruleset R played on the board B
$\mathbb{G}$	12	Group of all short, normal-play games
$\mathbb{G}_n$	68	Set of values of games born by day n
$\mathbb{G}^{\mathrm{I}}$	108	Set of impartial games
$\widetilde{\mathbb{G}}$	68	Set of literal forms of short games
$\widetilde{\mathbb{G}}_n$	68	Literal forms of games with formal birthday n
$\mathbb{V}$	68	Value set of SP-games
$\mathbb{V}_n$	69	Value set of SP-games with formal birthday n
$\mathbb{D}$	65	Set of dyadic rationals
$\mathscr{N}, \mathscr{P}, \mathscr{L}, \mathscr{R}$	10	Outcome classes
$o(G)$	10	Outcome class of G
G^L/G^R	4	Left/Right option of G
$G^{\mathcal{L}}/G^{\mathcal{R}}$	4	Set of Left/Right option of G
$G + H$	9	Disjunctive sum of G and H
$-G$	12	The negative of G
$G = H$	12	G is equal to H
$G > H$	12	G is greater than H
$G \cong H$	14	G and H are literally equal
$G \ngtrless H$	12	The games G and H are incomparable
SP-game	7	Strong placement game
iSP-game	33	Invariant strong placement game
$\Delta_{G,B}$	23	Legal complex of (G, B)
$\Gamma_{G,B}$	23	Illegal complex of (G, B)
$\Delta^{\mathrm{I}}_{G,B}$	108	Impartial legal complex of (G, B)
$\Gamma^{\mathrm{I}}_{G,B}$	108	Impartial illegal complex of (G, B)
$\mathcal{L}_{G,B}$	23	Legal ideal of (G, B)

Symbol	**Page**	**Description**
$\mathcal{ILL}_{G,B}$	24	Illegal ideal of (G, B)
$\mathcal{N}(\Delta)$	18	Stanley-Reisner ideal of the simplicial complex Δ
$\mathcal{F}(\Delta)$	18	Facet ideal of the simplicial complex Δ
$\mathcal{N}(I)$	18	Stanley-Reisner complex of the monomial ideal I
$\mathcal{F}(I)$	18	Facet complex of the monomial ideal I
$T_{G,B}$	13	Game tree of (G, B)
$\mathcal{G}_{G,B}$	49	Game graph of (G, B)
$\mathcal{P}_{G,B}$	57	Game poset of (G, B)
$P(\Delta)$	58	Face poset of Δ
$O(P)$	58	Order complex of P
$\mathrm{mex}(A)$	108	Minimal excluded value of the set A
$\mathrm{b}(G, B)$	68	The birthday of (G, B)
$\widetilde{\mathrm{b}}(G, B)$	68	The formal birthday of (G, B)
$*n$	66	Nimber: $\{0, *, *2, \ldots, *(n-1) \mid 0, *, *2, \ldots, *(n-1)\}$
$\uparrow$	66	Up: $\{0 \mid *\}$
$\downarrow$	66	Down: $\{* \mid 0\} = -\uparrow$
$+_G$	66	Tiny G: $\{0 \mid \{0 \mid -G\}\}$
$-_G$	66	Miny G: $\{\{G \mid 0\} \mid 0\} = -(+_G)$
$LS(G)/RS(G)$	80	Left stop/Right stop of G
$\mathcal{C}(G)$	85	Confusion interval of G
$\ell(G)$	85	Measure of the confusion interval of G
$m(G)$	82	Mean of G
G_t	82	G cooled by t
$t(G)$	82	Temperature of G
$BP(S)$	86	Boiling point of the set S
$\widetilde{G} = \{\widetilde{G}^L \mid \widetilde{G}^R\}$	87	Thermic version of G
$V(\Delta)$	16	Vertex set of Δ
$\dim \Delta$	16	Dimension of Δ
$\Delta^{[k]}$	17	kth skeleton of Δ
$\mathrm{link}_\Delta F$	17	Link of F in Δ

Symbol	**Page**	**Description**
$\Delta * \Gamma$	30	Join of Δ and Γ
Δ^V,Δ_M,Δ^c	17	Alexander dual/cover dual/complement of Δ
$\tilde{H}_i(\Delta; k)$	128	ith reduced homology of Δ over k
P_n	20	Path of n vertices
C_n	20	Cycle of n vertices
K_n	20	Complete graph on n vertices
$N_G(v)$	21	Neighbourhood of v in G

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Chapter 1

Introduction

Combinatorial games are two-player games of pure strategy such as CHESS or GO. Combinatorial game theory is an area of study in mathematics and computer science that develops new tools for determining who wins such a game and how.

Combinatorial commutative algebra is an area in which combinatorial concepts are used to study objects in commutative algebra and vice versa. One of the main roots of combinatorial algebra lies in the relationship between square-free monomial ideals and simplicial complexes.

Our main goal in this thesis is to give a one-to-one correspondence between a class of combinatorial games and simplicial complexes, making these games objects that can be studied using commutative algebra.

The majority of work in combinatorial game theory is to dissect one game at a time, whereas in this work we often look at an entire class together. This approach allows us to develop new tools applicable to all games in the class.

The combinatorial games SNORT, COL, DOMINEERING, and KAYLES have been studied for several decades. They are all examples of strong placement games, a class of combinatorial games in which the players place pieces on a finite graph. This class of games is the main focus of this thesis.

Faridi, Huntemann, and Nowakowski [26] showed that to each strong placement game one can assign two simplicial complexes, one representing legal positions called the legal complex, and the other minimal illegal positions called the illegal complex, and corresponding square-free monomial ideals. The main open question at the time was whether or not every simplicial complex in turn corresponds to a strong placement game. We will answer this question positively and will use the resulting one-to-one correspondence to study strong placement games further.

The remainder of this chapter is devoted to giving a background to the problems studied. We will introduce all required concepts from combinatorial game theory and commutative algebra, some graph theory concepts used, as well as demonstrate how to construct the legal and illegal complexes from a given strong placement game. We further show that two strong placement games with isomorphic legal complexes are literally equal, a very strong condition as they will be equal under any winning condition (see Proposition 1.52).

In Chapter 2, we will demonstrate the constructions which map a simplicial complex, considered as a legal complex or illegal complex, to a strong placement game (see Proposition 2.1). These constructions have the undesirable property that the ruleset highly depends on the board. We will thus look at invariant strong placement games, in which rulesets are independent of the board. In Theorems 2.16 and 2.17 we show that every simplicial complex is the legal complex of some invariant strong placement game, and a simplicial complex without isolated vertices is an illegal complex. As a consequence, we further show that any strong placement game is literally equal to an invariant strong placement game (see Theorem 2.20).

We then restrict to independence games, those strong placement games for which the illegal complex *always*, independent of the board, is a graph, or equivalently, the legal complex always is a flag complex. The invariant strong placement game we have constructed given a simplicial complex turns out to be an independence game when the simplicial complex is a graph. As a consequence, in Proposition 2.24 we show that given any graph, respectively flag complex, there exists an independence game for which this is the illegal complex, respectively legal complex.

Continuing, in Chapter 3, we will consider the structure of the game tree of a strong placement game. The game tree of a combinatorial game encodes a game completely, and is commonly used in computer game playing programs to direct the search for good positions, and is of interest in misère theory. We introduce the game graph, a simplified version of the game tree, and show in Proposition 3.4 that the two are equivalent if labellings of the positions are given. We then give a complete description of which game graphs come from strong placement games (see Proposition 3.11). Finally, we show in Proposition 3.18 that the game graph and legal complex are in one-to-one correspondence, thus can be used interchangeably as characterizations of a strong placement game.

With the one-to-one correspondence between strong placement games and simplicial complexes established, in Chapter 4 we will study the possible game values of strong placement games under normal play. Given a game or class of games, it is an important question as to what values occur, or more interestingly, which values do not occur, as this reflects the structure of the class. For the vast majority of games it is an open question what values they achieve. There is only one class, namely impartial games, for which the values are known. Despite ongoing research, very little is known for the class of strong placement games. We will show that many interesting values can be found in strong placement games, such as all numbers (Theorem 4.27), all nimbers (Proposition 4.32), up (Section 4.2), and several tinies (Proposition 4.31).

In Chapter 5, we then consider the temperature of strong placement games. The temperature of a game is a characteristic that can be used to choose a good move, but is difficult to calculate. Being able to bound temperature depending on features such as board size will make simpler algorithms choosing good positions possible, but apart from a few specific games very little is known. In Theorem 5.25 we prove an upper bound on the maximum temperature of a set of games, not necessarily strong placement games, based on the maximum length of the confusion intervals. This is the first known result which holds for all short games. Although this bound is still far from some conjectured bounds (see page 86) and computational evidence, it is optimal in the sense that there are examples of classes in which it is tight. We then apply this new bound to specific strong placement games, which seem particularly suitable for this approach. We further give a conjecture for the maximum temperature of SNORT.

For a general strong placement game, the associated simplicial complexes have a natural bipartition of the vertices depending on moves by the two players. Impartial games are combinatorial games in which both players have the same moves available. For impartial strong placement games, the simplicial complexes can be defined similarly without a bipartition. We will study this case further in Chapter 6.

Square-free monomial ideals that are Cohen-Macaulay usually have nice properties from the point of view of algebra or combinatorics. In Chapter 7 we will present some preliminary results on when the game ideal of the games SNORT and COL are Cohen-Macaulay and when they satisfy necessary or sufficient conditions for Cohen-Macaulayness. We also consider how a strong placement game changes if its legal complex is grafted, or what it means for the legal complex to be shellable. We then restrict to impartial games. This chapter can be considered a first step towards identifying what a Cohen-Macaulay strong placement game is.

As the approach of using simplicial complexes to study strong placement games is new, the work in this thesis opens up many new avenues of research. At the end of each chapter we will list some relevant problems and conclude the thesis with an overview of several overarching questions in Chapter 8.

The appendix has a summary of the rulesets used throughout the thesis for an easy reference, as well as the code used for various calculations.

All previously known results not proven by the author are stated as facts, and indicated as such. Unless otherwise stated, the results in this thesis are the author's work under guidance of their supervisors.

1.1 Combinatorial Game Theory

We will begin by introducing concepts from Combinatorial Game Theory required throughout. Good references for more information are the classic "Winning Ways" by Berlekamp, Conway, and Guy [6], the undergraduate text "Lessons in Play" by Albert, Nowakowski, and Wolfe [1], and the graduate text "Combinatorial Game Theory" by Siegel [57]. Most combinatorial game theory facts can be found in the latter.

1.1.1 Combinatorial Games

Definition 1.1. A **combinatorial game** is a 2-player game with perfect information and no chance devices, where the two players are **Left** and **Right** (denoted by $\boldsymbol{L}$ and $\boldsymbol{R}$ respectively) and they do not move simultaneously. A game is a set of **positions**. **Rules** determine which position the game starts with and from which position to which position the players are allowed to move. A **legal position** is a position that can be reached by playing the game according to the rules. A move from a position to another position is called a **legal move** if it is allowed according to the rules, and it is called an **illegal move** if it is not allowed. A **Left option** $\boldsymbol{P^L}$ (or similarly **Right option** $\boldsymbol{P^R}$) of a position P is a second position that Left (Right) can reach from P in one legal move. The set of Left options from a position P is denoted as $\boldsymbol{P^{\mathcal{L}}}$, the

Right options as $\boldsymbol{P^{\mathcal{R}}}$. A game is called **short** if it has finitely many positions, and no position can be reached from itself through a sequence of legal moves. A game G is often represented by its starting position P, and the game $G = P$ is denoted as $\boldsymbol{\{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}}$.

In this thesis, we only consider short games.

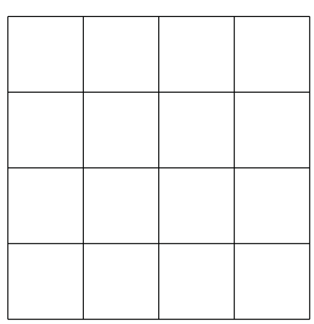
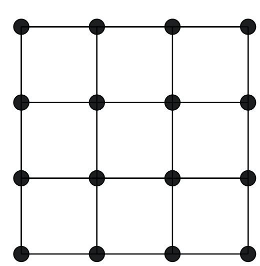
We denote the ruleset of a combinatorial game by its name in SMALL CAPS. Further, when the specific ruleset and board are of importance, we will denote the game consisting of the ruleset R played on the board B by (R, B). Note that we will use R to indicate a ruleset or in the context of options a single Right option (and later on pieces by Right as well). Which is intended will be clear from context.

Definition 1.2. A game G for which $P^{\mathcal{L}} = P^{\mathcal{R}}$ for all positions P, so both players have the same options at all times, is called **impartial**. A game G for which the set of Left options is unrelated to the set of Right options is called **partizan**.

1.1.2 Strong Placement Games

From now on, unless otherwise specified, a board will be a finite graph, and pieces can be thought of as tokens being placed on subgraphs of the board.

If a game is defined to be played on a board other than a graph, for example a checkerboard, we can represent this board by a graph: we assign to each space a vertex and two vertices are adjacent if and only if the two corresponding spaces are horizontally or vertically adjacent. For example the board on the left below is represented by the graph on the right.

We do not only consider graphs of this type for boards though, but *any* graph as long as the ruleset allows for it.

Brown et al. [12] introduced a subclass of combinatorial games, which they called "placement games", for which the question of counting the number of legal positions is particularly interesting. In their work, they used the concept of an "auxiliary board" for some specific games, which later on in Faridi et al. [26] was generalized to the "illegal complex" of a large class of games, which we call "strong placement games".

In this thesis we will call the games played by placing pieces on a board "placement games". We distinguish three different classes of placement games. The first is the broadest class, containing a large variety of combinatorial games.

Definition 1.3. A **weak placement game** is a combinatorial game which satisfies the following conditions:

(i) The board is empty at the beginning of the game.

(ii) Players place pieces on empty vertices of the board according to the rules.

(iii) Pieces are not moved or removed once placed.

In particular, note that pieces cannot overlap.

Well-known examples of (commercially published) weak placement games are BLOKUS and KULAMI.

Definition 1.4 (Brown et al. [12])**.** A **medium placement game** is a weak placement game which furthermore satisfies the following condition:

(iv) The rules are such that if it is legal to place piece X on subgraph Y on move i, then it was legal to place piece X on subgraph Y on move j for all $j \leq i$.

Note that condition (iv) implies that any subset of pieces placed forming a legal position also form a legal position. Further note that what we call medium placement games is the class of games called "placement games" by Brown et al. [12].

Both BLOKUS and KULAMI are not medium placement games as they fail condition (iv). In BLOKUS pieces need to be adjacent to previously placed pieces, while in KULAMI, which is played on subdivided grids, the piece has to be placed in the same row or column as the last piece placed and may not be in the same subgrid as the last and second-to-last piece. Other examples of weak placement games which are not medium include Chilled Domineering (see Kao, Wu, Shan, and Lin 2010 [36]) and the conjoin of two placement games (see Huggan and Nowakowski 2018 [32]).

An example of a medium placement game is TicTacToe. Note though that there exist positions P which end the game that also have subpositions P' that would end the game, i.e. moving from P' to P is illegal. Placement games which also satisfy that a legal position can be reached in any order of moves are called strong placement games.

Definition 1.5 (Faridi, Huntemann, Nowakowski [26])**.** A **strong placement game** (or **SP-game**) is a weak placement game which furthermore satisfies the following condition:

(iv') The rules are such that if it is possible to reach a position through a sequence of legal moves, then any sequence of moves leading to this position consists of legal moves.

Note that condition (iv') in the above definition implies that the order of moves does not matter, and that the last piece played could have been played at any previous point. Thus the class of SP-games is contained in the class of medium placement games. But the converse is not true, with TicTacToe being one example.

The property that positions are independent of the order of moves gives us commutativity, which will allow us to represent positions by monomials and faces of simplicial complexes (see Section 1.4).

The following rulesets together with a board are examples of SP-games and will be used throughout the document. For an easy reference, their rules are repeated in Appendix A.

Definition 1.6. In **Snort** (see [6]), players place a piece on a single vertex which is not adjacent to a vertex containing a piece from their opponent.

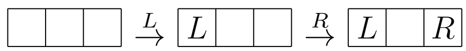
In **Col** (see [6]), players place a piece on a single vertex which is not adjacent to a vertex containing one of their own pieces.

In **NoGo** (see Chou, Teytaud, Yen 2011 [16]), players place a piece on a single vertex. At every point in the game, for each maximal group of connected vertices of the board that contain pieces placed by the same player, at least one of these needs to be adjacent to an empty vertex.

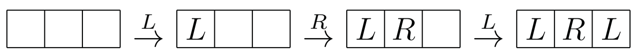
In **Domineering** (see Berlekamp 1988 [4] or Lachmann, Moore, Rapaport 2002 [39]), which is played on grids, both players place dominoes. Left may only place vertically, and Right only horizontally. The vertices of the board are the squares of the grid, and each piece occupies two vertices.

Example 1.7. Examples of legal alternating sequences of play for each of the first three of these games are the following:

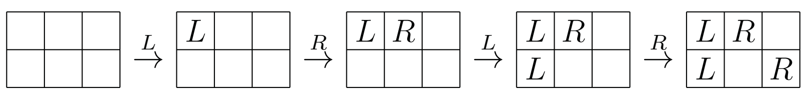
Snort:



Col:

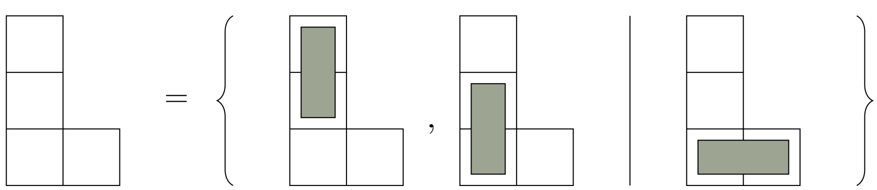


NoGo:



Note that other sequences of play are possible for each of these games, and we do not claim that the sequences given are optimal under any circumstances.

Example 1.8. Let G be Domineering played on an L-shaped board. The Left and Right options are listed in game notation below:

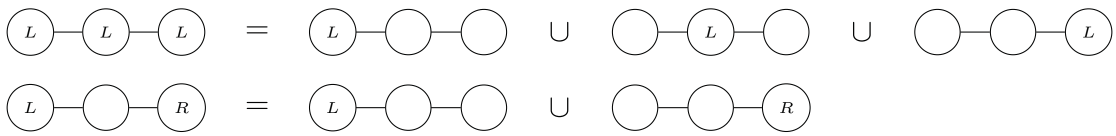


Other examples of SP-games are Node-Kayles and Arc-Kayles (see for example Schaefer 1978 [53], Bodlaender 1993 [9], Fleischer and Trippen 2006 [27], or Huggan and Stevens 2016 [33]) and some of the Partizan Octals (see Fraenkel and Kotzig 1987 [28] or Mesdal 2009 [43]). Their rules and how to interpret them as SP-games can be found in Appendix A.

For SP-games, since the order of moves taken does not matter, the positions with a single piece played become very important. We thus define a basic position:

Definition 1.9. A position with a single piece played, whether this is legal or not, is called a **basic position**.

Any position in an SP-game is the union of a finite number of basic positions. For example, positions in SNORT played on a path of three vertices break up as follows:



The following game is crucial to the study of impartial games. One of the earliest results in combinatorial game theory is that any position in an impartial game is equal to a NIM position and thus has as its value a nimber, see Section 6.1.

Definition 1.10 (Bouton 1902 [11]). The game of **NIM** is played using piles of tokens. On a turn, the player chooses a pile and removes any number of tokens from it. The game ends once no tokens remain.

Remark 1.11. Although it may not seem this way immediately, NIM is equivalent to an SP-game. Suppose the piles we are playing NIM on have sizes $a_1, \ldots, a_k$. Then let the board for the SP-game be the disjoint union of the complete graphs $K_{a_1}, \ldots, K_{a_k}$. Both players have as their possible pieces $K_1, \ldots, K_a$, where a is the maximum of the a_i, and may place anywhere on the board.

1.1.3 Tools from Combinatorial Game Theory

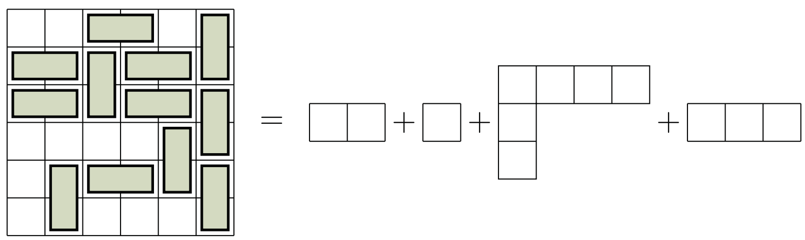
This subsection introduces several commonly used tools from combinatorial game theory. These concepts are mentioned throughout Chapter 2, and are used heavily in Chapters 4 and 5.

Many combinatorial games, especially SP-games, have a natural tendency to break up into smaller, independent components as play progresses. For example, after several moves the empty spaces could be split into many disconnected components and a player, on their move, then has to choose a component to move in. From this, we define a sum on games as follows:

Definition 1.12. The **disjunctive sum $G_1 + G_2$** of two games G_1 and G_2 is the game in which at each step the current player can decide to move in either game, but not both. Formally,

$$G_1 + G_2 = \{G_1^{\mathcal{L}} + G_2, G_1 + G_2^{\mathcal{L}} \mid G_1^{\mathcal{R}} + G_2, G_1 + G_2^{\mathcal{R}}\}.$$

Example 1.13. This property is especially apparent in DOMINEERING. Consider for example a 6×6 board. The position on the left below could occur during play. It is equal to the disjunctive sum of several smaller positions (empty boards) given on the right.



We will implicitly assume that all games we consider are a summand in a disjunctive sum. Due to this, the two players do not necessarily alternate their turns in any one component (in a disjunctive sum, the players can use different components). Only for some concepts will we consider playing in a single component, thus having alternating play (for example for Left and Right stops, see Definition 5.1).

Unless otherwise specified, the winning condition of a game does not matter to us. When considering game values (see Chapter 4) and temperature (see Chapter 5) we do require a fixed winning condition. The two most commonly considered winning conditions are **normal play**, in which the first player unable to move loses, and **misère play**, in which the first player unable to move wins.

Given a fixed winning condition, we can partition games into four outcome classes.

Definition 1.14 (Outcome classes)**.** The **outcome class $o(G)$** of a combinatorial game G indicates who will win the game when playing optimally. The outcome classes are:

- $\mathscr{N}$: the first (next) player can force a win;

- $\mathscr{P}$: the second (previous) player can force a win;

- $\mathscr{L}$: Left can force a win, no matter who plays first;
- $\mathscr{R}$: Right can force a win, no matter who plays first.

A game whose outcome is in $\mathscr{N}$, that is one in which the first player always has a good move, is also called a first-player win. Similarly, games whose outcomes are in the other classes are called second-player win, Left win, or Right win, respectively.

Convention in combinatorial game theory is to order games by how favourable they are to Left. Games in $\mathscr{L}$ are the most favourable as she can always force a win. There is no differentiation between games in $\mathscr{N}$ and $\mathscr{P}$ since in both cases she in some sense wins half the time, while games in $\mathscr{R}$ are the least favourable as she always loses. Thus we have the partial order on the outcome classes as in Figure 1.1.

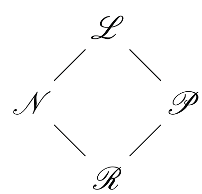


Figure 1.1: Partial Order of Outcome Classes

Example 1.15. As examples for the outcome classes, consider the DOMINEERING positions in Figure 1.2, assuming normal play.

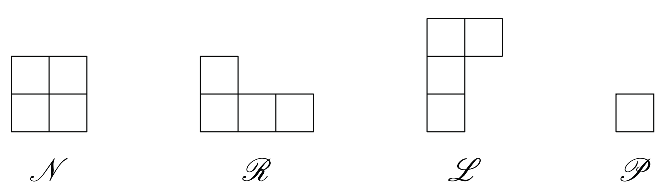


Figure 1.2: DOMINEERING Positions with their Outcome Classes under Normal Play

In the first position from the left, once either player has placed a domino, the other cannot place theirs, thus the first player to go wins. In the second position, Right going first can play in the two bottom left spaces, which leaves no moves for Left, while if Left goes first, she only has one move, leaving another move for Right. Thus Right wins this game, no matter if going first or second. The third position is similarly a Left win. In the fourth position, neither player can move. Thus no matter who goes first, they will lose, implying that the second player to go wins.

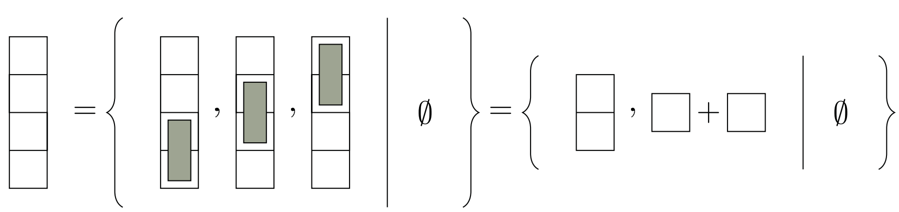
Given a fixed winning condition, we say two games G_1 and G_2 are **equal** and write $\boldsymbol{G_1 = G_2}$ if $o(G_1 + H) = o(G_2 + H)$ for all games H. This equivalence is an equivalence relation. The equivalence class of a game G under "=" is called its **game value**. The disjunctive sum of two game values is found by taking the disjunctive sum of any of the games in the two equivalence classes. The group of all possible game values of short games under normal play with disjunctive sum as operation is denoted as $\mathbb{G}$.

For a game G, we say that the **negative** $\boldsymbol{-G}$ is the game recursively defined as

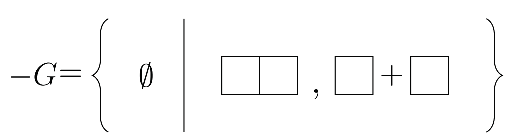
$$-G = \{-G^{\mathcal{R}} \mid -G^{\mathcal{L}}\},$$

i.e. the game in which the roles of Left and Right are reversed. For example, in DOMINEERING this is equivalent to rotating the board by 90°.

Example 1.16. For example, let G be DOMINEERING played on a 1×4 board. For this game we have



Now $-G$ consists of swapping the Left and Right options and taking the negative of them, i.e.



which is DOMINEERING played on a 4×1 board.

We use $\boldsymbol{G - H}$ as shorthand for $G + (-H)$.

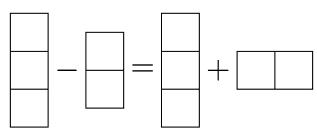
Similar to equality, we can also define inequalities: We say that $\boldsymbol{G_1 > G_2}$ if $o(G_1 + H) > o(G_2 + H)$ for all games H, with the partial order on the outcome classes as in Figure 1.1. Two games are incomparable, denoted $\boldsymbol{G_1 \not\gtrless G_2}$, if their outcome classes are incomparable, i.e. if one is a first-player win and the other a second-player win. Similarly defined are $G_1 \geq G_2$ and $G_1 \leq G_2$.

Under normal play, we are able to determine the relationship between two games using the following fact by simply determining the outcome class of their difference.

Fact 1.17 ([57, Section II.1])**.** *Under normal play, for two games G and H, we have:*

1. *$G = H$ if and only if $o(G - H) = \mathscr{P}$;*
2. *$G < H$ if and only if $o(G - H) = \mathscr{R}$;*
3. *$G > H$ if and only if $o(G - H) = \mathscr{L}$; and*
4. *$G \ngtrless H$ if and only if $o(G - H) = \mathscr{N}$.*

Example 1.18. We will show that the game G_1, which is DOMINEERING played on a 3×1 board, is equal under normal play to the game G_2, which is DOMINEERING played on a 2×1 board. We have



Now both players, no matter who starts, can make one move each, after which no moves remain. Thus under normal play, the second player wins this difference, showing that $G_1 = G_2$.

In combinatorial game theory, the game tree of a game is often used to study properties.

Definition 1.19. The **game tree T_G** of a combinatorial game G is a diagram constructed inductively as follows:

Step 0: Place a vertex representing the starting position of G.

Step k: For each vertex v representing a position P constructed in step $k - 1$ do the following: For each Left option of P place a vertex v_P below and to the left of v and connect v and v_P with an edge (thus with positive slope, or oriented to the left), and similarly for all Right options.

Note that we can think of the game tree also as an oriented, directed graph tree where the edges are labelled as L and R rather than being oriented to the left and right, respectively. Also note that many different vertices can represent the same legal position. Furthermore, since we only consider short games, in our case the game tree is finite.

Example 1.20. Consider SNORT played on P_3. The game tree is given in Figure 1.3. Empty spaces are indicated by "-".

In combinatorial game theory, two game trees T_1 and T_2 are called **isomorphic** if their structure is the same, that is if there exists a bijection ϕ of the vertices preserving the edges, i.e. if there is an edge from v_1 to v_2 pointing to the left, respectively to the right, in T_1 then there is an edge from $\phi(v_1)$ to $\phi(v_2)$ pointing to the left, respectively to the right, in T_2. Two games G_1 and G_2 with isomorphic game trees are called **literally equal**, written as $\boldsymbol{G_1 \cong G_2}$. Note that two games that are literally equal will be equal under any winning condition.

The games in Example 1.18 are equal under normal play (and also under misère play), but they are not literally equal as in G_1 Left has two basic positions and in G_2 she only has one. One can similarly show that DOMINEERING played on a 2×2 board under normal play is equal to DOMINEERING played on a 2×2 board with an extra space attached, but that they are not literally equal as the latter has an additional basic position for one of the players. In this case the games are further not equal under misère play as the former is a second-player win, while the latter is a first-player win.

Note that it is common when constructing the game tree of a game to only create a branch once for all symmetric options. Under normal play or misère play this does not change the game for calculating its game value. There are winning conditions though for which ignoring options does make a difference, thus when constructing the game tree we insist on all options being listed. When we calculate game values, we will often only list repeated options once.

Several other concepts will be introduced in the relevant chapters and sections.

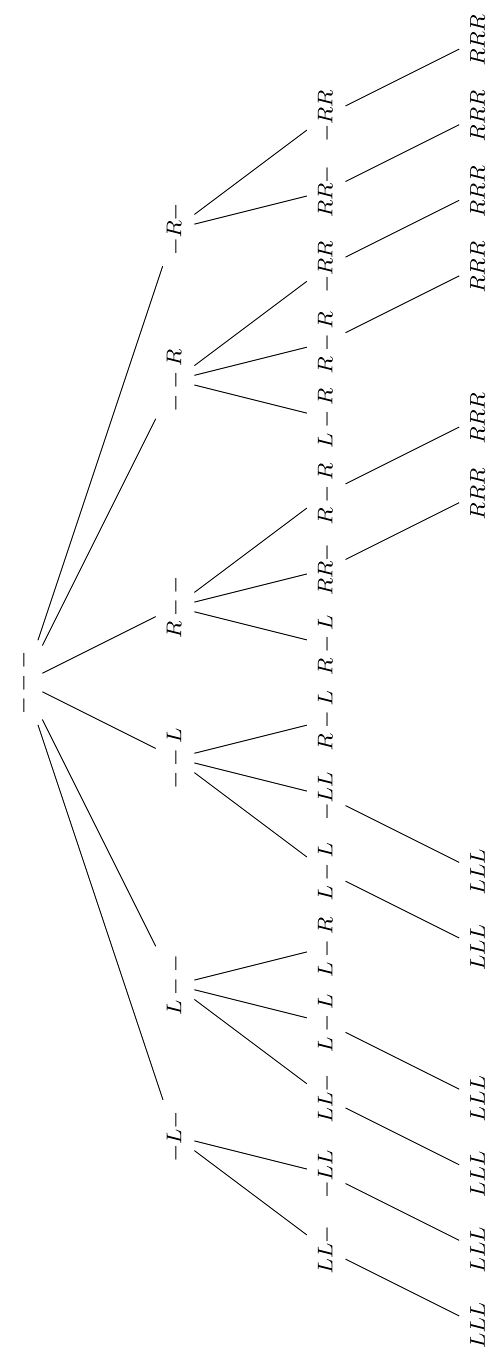


Figure 1.3: The Game Tree of SNORT Played on P_3

1.2 Combinatorial Commutative Algebra

The goal of this thesis is to translate strong placement games into combinatorial commutative algebra objects and vice versa. Commutative algebra is used to study properties of ideals, and properties of monomial ideals are the easiest to understand combinatorially. A major topic in combinatorial commutative algebra is the interplay between the algebra of monomial ideals and the combinatorics and topology of simplicial complexes. In this section we give the basic definitions.

We introduce two objects: simplicial complexes and square-free monomial ideals, and some of their properties. Good references are "Cohen-Macaulay Rings" by Bruns and Herzog [13] and "Monomial Ideals" by Herzog and Hibi [31].

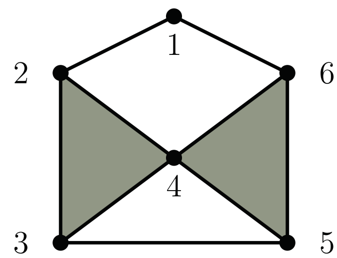
Simplicial complexes are one of the main constructions we use to study SP-games.

Definition 1.21. An (abstract) **simplicial complex $\boldsymbol{\Delta}$** on a finite vertex set V is a set of subsets of V, called **faces**, with the condition that if $A \in \Delta$ and $B \subseteq A$, then $B \in \Delta$. The **facets** of a simplicial complex Δ are the maximal faces of Δ with respect to inclusion. A **non-face** of Δ is a subset of its vertices that is not a face. The **dimension of a face** is one less than the number of vertices of that face. The **dimension $\mathbf{dim}(\boldsymbol{\Delta})$** of a simplicial complex Δ is the maximum dimension of any of its faces.

Note that a simplicial complex with a fixed vertex set is uniquely determined by its facets. Thus a simplicial complex Δ with facets $F_1, \ldots, F_k$ is denoted by $\Delta = \langle F_1, \ldots, F_k \rangle$. If we list all faces, the simplicial complex will be in set notation. The vertex set of Δ is also denoted as $\boldsymbol{V(\Delta)}$.

We will often represent a simplicial complex by its "geometric realization". Faces with single elements are drawn as vertices, with two elements as edges, with three elements as filled triangles and so on. These will overlap if the faces have non-empty intersection. Note that the geometric realization of a simplicial complex is a topological space (see for example [47]), but for our purposes diagrams such as in the next example are sufficient.

Example 1.22. Let $\Delta = \langle \{1,2\}, \{1,6\}, \{2,3,4\}, \{3,5\}, \{4,5,6\} \rangle$. Then the below is a geometric realization of Δ.



If a simplicial complex is of the form $\Delta = \langle\{x_1, x_2, \ldots, x_n\}\rangle$, where $\{x_1, x_2, \ldots, x_n\}$ is the vertex set of Δ, we call it a **simplex**.

The following properties and operations will be useful throughout.

Definition 1.23. A simplicial complex is called **pure** if all its facets are of the same size.

Definition 1.24. The ***k*-skeleton $\boldsymbol{\Delta^{[k]}}$** of a simplicial complex Δ is the simplicial complex whose facets are the k-dimensional faces of Δ.

Definition 1.25. A **minimal vertex cover** A of a simplicial complex Δ is a subset of the vertex set such that for every facet $F \in \Delta$ we have $F \cap A \neq \emptyset$ and no subset of A satisfies this. A simplicial complex is called **unmixed** if all its minimal vertex covers are of the same size.

Definition 1.26. Given a simplicial complex Δ, the **cover complex** or **cover dual $\boldsymbol{\Delta_M}$** is the simplicial complex whose facets are the minimal vertex covers of Δ, and the **Alexander dual $\boldsymbol{\Delta^\vee}$** is defined as $\{F \subseteq V \mid V \setminus F \notin \Delta\}$ where V is the vertex set of Δ. The **complement $\boldsymbol{\Delta^c}$** is defined as $\langle V \setminus F \mid F \text{ is a facet of } \Delta\rangle$.

Faridi showed in 2004 [24] that $(\Delta_M)_M = \Delta$, while a proof that $(\Delta^\vee)^\vee = \Delta$ can be found in [31]. It can be easily checked that $(\Delta^c)^c = \Delta$ as well.

Definition 1.27. Given a simplicial complex Δ and a face F of Δ, the **link of *F* in $\boldsymbol{\Delta}$**, denoted **$\mathrm{link}_{\boldsymbol{\Delta}}$ *F***, is defined as the subcomplex of Δ given by

$$\mathrm{link}_\Delta F = \{G \in \Delta \mid F \cap G = \emptyset, F \cup G \in \Delta\}.$$

In particular, if $F = \{v\}$ is a single vertex, then

$$\mathrm{link}_\Delta v = \{G \in \Delta \mid v \notin G, \{v\} \cup G \in \Delta\}.$$

Next we consider monomial ideals of polynomial rings associated to a simplicial complex.

Definition 1.28. Let k be a field and S the polynomial ring $k[x_1, \dots, x_n]$. A product $x_1^{a_1} \dots x_n^{a_n} \in S$, where the a_i are non-negative integers, is called a **monomial**. Such a monomial is called **square-free** if each a_i is either 0 or 1.

Definition 1.29. Let k be a field and S the polynomial ring $k[x_1, \dots, x_n]$. A **monomial ideal** of S is an ideal generated by monomials in S. A monomial ideal is called a **square-free monomial ideal** if it is generated by square-free monomials.

Note that every monomial ideal has a unique minimal monomial generating set (see for example [31, Proposition 1.1.6]).

Let k be a field and $S = k[x_1, \dots, x_n]$ a polynomial ring. Given a simplicial complex Δ on n vertices, we can label each vertex with an integer from 1 to n. Each subset $\{i_1, \dots, i_r\}$ of $\{1, \dots, n\}$ corresponds to a monomial $x_{i_1} \cdots x_{i_r}$ in S.

Definition 1.30. The **facet ideal** of a simplicial complex Δ, denoted by $\boldsymbol{\mathcal{F}(\Delta)}$, is the ideal

$$\mathcal{F}(\Delta) = (x_{i_1} \cdots x_{i_r} \mid \{i_1, \dots, i_r\} \text{ is a facet of } \Delta)\,.$$

The **Stanley-Reisner ideal** of Δ, denoted by $\boldsymbol{\mathcal{N}(\Delta)}$, is the ideal

$$\mathcal{N}(\Delta) = (x_{i_1} \cdots x_{i_r} \mid \{i_1, \dots, i_r\} \text{ is a minimal nonface of } \Delta)\,.$$

Definition 1.31. The **facet complex** of a square-free monomial ideal I, denoted by $\boldsymbol{\mathcal{F}(I)}$, is the simplicial complex

$$\mathcal{F}(I) = \langle \{i_1, \dots, i_r\} \mid x_{i_1} \cdots x_{i_r} \text{ is a minimal generator of } I \rangle.$$

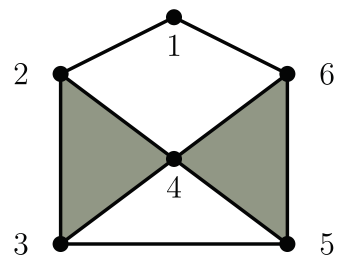
The **Stanley-Reisner complex** of I, denoted by $\boldsymbol{\mathcal{N}(I)}$, is the simplicial complex

$$\mathcal{N}(I) = \{\{i_1, \dots, i_r\} \mid x_{i_1} \cdots x_{i_r} \notin I\}.$$

Note that the Stanley-Reisner and facet operators respectively are inverses of each other, thus give one-to-one correspondences between square-free monomial ideals and simplicial complexes.

Since these concepts are heavily used in the remainder of this thesis, we now present some examples.

Example 1.32. Consider the simplicial complex Δ below with the labelling of the vertices as given.



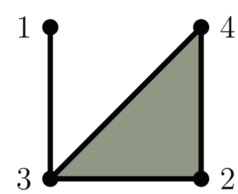
The facet ideal of Δ then is

$$\mathcal{F}(\Delta) = (x_1x_2, x_1x_6, x_2x_3x_4, x_3x_5, x_4x_5x_6),$$

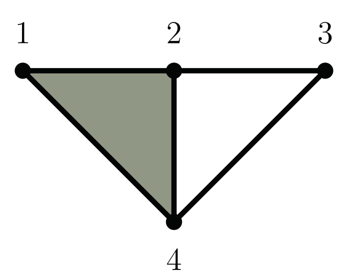
and the Stanley-Reisner ideal of Δ is

$$\mathcal{N}(\Delta) = (x_1x_3, x_1x_4, x_1x_5, x_2x_5, x_2x_6, x_3x_4x_5, x_3x_6).$$

Example 1.33. Consider the square-free monomial ideal $I = (x_1x_3, x_2x_3x_4)$. The facet complex $\mathcal{F}(I)$ is



and the Stanley-Reisner complex $\mathcal{N}(I)$ is



Note that in this thesis, we will occasionally see that not all variables of the underlying ring are vertices of a simplicial complex. These so-called "loops" will be excluded from the geometric realization of the simplicial complex (see Example 1.47).

Other concepts will again be introduced in the relevant chapters.

1.3 Graph Theory

To be able to discuss structures of the board, and for a few other concepts, we will need a few definitions from graph theory. See for example "Graph Theory" by Bondy and Murty [10].

Definition 1.34. A **graph** is a pair (V, E) where V is a set of objects called **vertices** and E a list of sets $\{v_i, v_j\}$, $v_i, v_j \in V$ called **edges**. A **simple graph** is a graph with no loops (an edge that has identical ends) and no parallel edges (two or more edges that share the same two ends). The **degree** of a vertex in a simple graph is the number of edges incident with it. The **size** of a graph is the size of its vertex set. Two vertices v and w are called **adjacent** if $\{v, w\} \in E$.

Note that all graphs we consider are simple graphs, even if not specified as such.

Definition 1.35. A **walk** between two vertices v and w in a graph G is a sequence of vertices $v = v_1, v_2, \ldots, v_k = w$ of G such that v_i and v_{i+1} are connected by an edge and no edge is repeated. A **path** between v and w is a walk in which no vertex is repeated. A graph G is called **connected** if there exists at least one path between any pair of vertices.

For many games we consider, sets of vertices which are not adjacent will be important.

Definition 1.36. Given a graph $G = (V, E)$, a subset of the vertices $A \subseteq V$ is called an **independent set** if no pair $v, w \in A$ is adjacent. The simplicial complex Δ with vertex set V and faces the independent sets of G is called the **independence complex** of G.

We will often consider playing a ruleset on specific classes of graphs. Four such classes are the following ones.

Definition 1.37. A **path** $\boldsymbol{P_n}$ is a connected simple graph on $n \geq 2$ vertices such that two vertices have degree 1 and $n - 2$ vertices have degree 2. A **cycle** $\boldsymbol{C_n}$ is a connected simple graph on $n \geq 3$ vertices such that all vertices have degree 2. A **complete graph** $\boldsymbol{K_n}$ is a simple graph on n vertices such that any two vertices are adjacent. A **bipartite graph** G is a simple graph whose vertices can be partitioned into two sets V_1 and V_2 such that every edge of G has one vertex in V_1 and the other in V_2; the sets V_1 and V_2 are the **parts** of G. A **tree** is a connected simple graph which does not contain any cycles.

Definition 1.38. The **line graph** of a graph B is the graph whose vertices are the edges of B, and two vertices are adjacent if the corresponding edges in E are incident (have an element in common).

We will occasionally also be talking about all vertices adjacent to a fixed one, and this set is called the neighbourhood.

Definition 1.39. Given a graph $G = (V, E)$ and a vertex $v \in V$, the **neighbourhood of v**, denoted $\boldsymbol{N_G(v)}$, is the set of vertices in V which are adjacent to v.

1.4 Game Complexes and Ideals

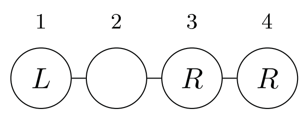
Brown et al. showed in [12] that when playing SNORT or COL on a fixed board, one can assign a second graph, which they called the "auxiliary board", whose independence sets are exactly the legal positions of the game. A similar assignment works for all SP-games, with the auxiliary board generalized to the "illegal complex", and the independence complex of the auxiliary board generalized to the "legal complex". Both of these simplicial complexes, and two associated ideals, are complete representations of the game. This allows us to translate problems from combinatorial game theory into commutative algebra.

In this section, we introduce the construction of simplicial complexes and square-free monomial ideals which are related to SP-games. Unless otherwise specified, let the underlying ring be $S = k[x_1, \ldots, x_m, y_1, \ldots, y_n]$, where k is a field, m is the number of basic positions with a Left piece, and n is the number of basic positions with a Right piece. Note that since we only consider finite games, both n and m are finite as well.

A square-free monomial z of S represents a position P in the game if it is the product over those x_i and y_j such that Left has played in the basic position i and Right has played in the basic position j in order to reach P. By condition (iv) in Definition 1.5, the order of moves to reach P does not matter, thus we have commutativity.

We begin with a few example of how to determine the underlying ring and assign the monomials to positions. The first demonstrates that the number of Left basic positions is not necessarily equal to the number of Right basic positions.

Example 1.40. Consider the game in which Left claims a single vertex, and Right two adjacent vertices, played on P_4. We number the vertices consecutively from one end as 1, 2, 3, 4. The Left basic position i is the position in which Left has played on vertex i, and the Right basic position j is the position in which Right has played on vertices j and $j+1$. Since Left has 4 basic positions, and Right has 3, the underlying ring is $S = k[x_1, x_2, x_3, x_4, y_1, y_2, y_3]$. The position



is represented by the monomial $x_1 y_3$.

In Domineering we face the issue that the pieces have an orientation. We get around this by letting the basic positions be in which either player places a domino, regardless of orientation, and then making some of them illegal, as demonstrated in the next example.

Example 1.41. Consider Domineering played on the board B given in Figure 1.4.

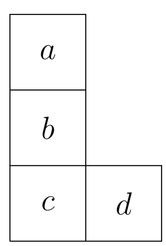


Figure 1.4: Example Board B for Domineering with Squares Labelled

Since Left and Right both play dominoes, the basic positions are to place a domino on vertices a, b (basic position 1), on b, c (basic position 2), or on c, d (basic position 3). Thus the underlying ring in this case is $S = k[x_1, x_2, x_3, y_1, y_2, y_3]$.

Since Left may only place a domino vertically, the basic position represented by x_1 and x_2 are legal, while x_3 is illegal. Similarly, for Right y_1 and y_2 are illegal, while y_3 is legal.

The monomial x_1y_3 represents the position in which Left has placed a domino on vertices a and b, and Right has played on c and d, which is a legal position. Similarly, x_2y_3 represents the position where Left has played on vertices b and c, while Right has played on c and d, which is illegal since the two dominoes overlap.

A legal position is called a **maximal legal position** if placing any further piece is illegal, i.e. it is not properly contained in any other legal position.

If we sort the monomials representing illegal positions by divisibility, the positions corresponding to the minimal elements are called **minimal illegal positions**. Equivalently, an illegal position is a minimal illegal position if any proper subset of the pieces placed forms a legal position.

With this terminology, the generalization of the "auxiliary board" in [12] is the illegal complex.

Definition 1.42 ([26])**.** The **illegal complex $\boldsymbol{\Gamma_{R,B}}$** of an SP-game (R, B) is the simplicial complex whose facets consist of those vertices labelled x_i and y_j such that Left has played in the basic position i and Right has played in the basic position j of the minimal illegal positions of (R, B).

The legal complex, the generalization of the independence complex of the auxiliary board, represents the legal positions.

Definition 1.43 ([26])**.** The **legal complex $\boldsymbol{\Delta_{R,B}}$** of an SP-game (R, B) is the simplicial complex whose faces consist of those vertices labelled x_i and y_j such that Left has played in the basic position i and Right has played in the basic position j of the legal positions of (R, B).

Some of the results we discuss in this thesis hold for both the legal and illegal complex of some game and board. For brevity, we will use the term **game complex** when discussing a simplicial complex which is a legal or illegal complex of some SP-game.

We can also assign two square-free monomial ideals to each game, which are the facet ideals of the game complexes (see Fact 1.46).

Definition 1.44 ([26])**.** The **legal ideal $\boldsymbol{\mathcal{L}_{R,B}}$** of an SP-game (R, B) is the ideal generated by the monomials representing maximal legal positions of (R, B).

Definition 1.45 ([26])**.** The **illegal ideal $\mathcal{ILL}_{R,B}$** of an SP-game (R,B) is the ideal generated by the monomials representing minimal illegal positions of (R,B).

For the construction of these simplicial complexes and ideals to work, we necessarily need strong placement games instead of just medium placement games. Since for medium placement games which are not strong placement games (such as TicTacToe) the order of the moves might matter, the underlying ring is non-commutative and the geometric structure would be a directed hypergraph instead of a simplicial complex. Since square-free monomial ideals of a commutative ring and simplicial complexes have more structure and are better understood, we prefer to work in this setting rather than the more general one.

Note that condition (iv) in Definition 1.5 implies that the order of moves does not matter, which gives us commutativity when representing positions by monomials. Thus the legal and illegal ideal are indeed commutative ideals. The condition also implies that given any legal position, any subset of the pieces played gives a legal position as well, and thus the hypergraphs representing the game are indeed simplicial complexes.

The following proposition is clear from the careful examination of the definitions of the facet and Stanley-Reisner operators and the game complexes and ideals. It is very useful for the study of placement games, and its consequences will be used throughout the thesis.

Fact 1.46 ([26, Proposition 3.4])**.** *For an SP-game (R,B) we have the following*

(1) $\mathcal{L}_{R,B} = \mathcal{F}(\Delta_{R,B})$,

(2) $\mathcal{ILL}_{R,B} = \mathcal{F}(\Gamma_{R,B}) = \mathcal{N}(\Delta_{R,B})$,

or equivalently

(1) $\Delta_{R,B} = \mathcal{F}(\mathcal{L}_{R,B}) = \mathcal{N}(\mathcal{ILL}_{R,B})$,

(2) $\Gamma_{R,B} = \mathcal{F}(\mathcal{ILL}_{R,B})$.

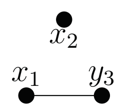
We will continue Example 1.41 to demonstrate these concepts. This also illustrates again that the vertices of the complexes are the basic positions, not the vertices of the graph/board.

Example 1.47. Consider DOMINEERING played on the board B given in Figure 1.4. Our underlying ring is $S = k[x_1, x_2, x_3, y_1, y_2, y_3]$.

The maximal legal positions are represented by the monomials x_1y_3 and x_2. Thus we have the legal ideal

$$\mathcal{L}_{\text{DOMINEERING},B} = (x_1y_3, x_2)$$

and the legal complex below



Note that although x_3, y_1, and y_2 are variables of R, they do not appear as vertices in the legal complex, and are thus not included in the above geometric realization.

The minimal illegal positions are represented by the monomials x_1x_2, x_2y_3, x_3, y_1, and y_2. Thus we have the illegal ideal

$$\mathcal{ILL}_{\text{DOMINEERING},B} = (x_1x_2, x_2y_3, x_3, y_1, y_2)$$

and the illegal complex below

It is important to note that the legal and illegal complexes and corresponding ideals have an extra layer of structure. The monomials have elements $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, y_2, \ldots, y_n\}$ and the complexes have their vertices partitioned into those corresponding to the Left and Right basic positions. In general, we call a simplicial complex whose vertex set is bipartioned into sets $\mathfrak{L}$ and $\mathfrak{R}$ an **$(\mathfrak{L}, \mathfrak{R})$-labelled simplicial complex**.

We now define homomorphisms between game complexes and ideals, which must preserve these partitions. We thus extend the definitions of homomorphisms of simplicial complexes and homomorphisms of ideals as follows.

Definition 1.48. Let Δ and Γ be two $(\mathfrak{L}, \mathfrak{R})$-labelled simplicial complexes. A map $\phi : \Delta \to \Gamma$ is a **simplicial complex $(\mathfrak{L}, \mathfrak{R})$-homomorphism** if

1. For all faces $F \in \Delta$ we have $\phi(F) = \bigcup_{z_i \in F} \phi(\{z_i\})$;

2. For all $F, G \in \Delta$, if $F \subseteq G$, then $\phi(F) \subseteq \phi(G)$; and

3. If $z \in \mathfrak{L}$ then $\phi(\{z\}) \in \mathfrak{L}$ and if $z \in \mathfrak{R}$ then $\phi(\{z\}) \in \mathfrak{R}$.

Definition 1.49. Let $S = k[x_1, \ldots, x_m, y_1, \ldots y_n]$ be a polynomial ring with variables partitioned into sets $\mathfrak{L} = \{x_1, \ldots, x_m\}$ and $\mathfrak{R} = \{y_1, \ldots, y_n\}$, and let I and J be ideals of R. A map $\phi : I \to J$ is an **ideal $(\mathfrak{L}, \mathfrak{R})$-homomorphism** if

1. $\phi(rx + sy) = r\phi(x) + s\phi(y)$ for all $x, y \in I$ and $r, s \in S$; and

2. For all $x \in \mathfrak{L}$ we have $\phi(x) \in \mathfrak{L}$ and for all $y \in \mathfrak{R}$ we have $\phi(y) \in \mathfrak{R}$.

If we say that two game complexes are isomorphic, we mean that there exists a simplicial complex $(\mathfrak{L}, \mathfrak{R})$-homomorphism which is a bijection (and similarly for the ideals).

We occasionally also talk about isomorphic boards, with which we mean the boards are isomorphic as graphs and contain the same pieces. For SP-games we formally define a board isomorphism as follows.

Definition 1.50. Let B_1 and B_2 be two boards, potentially not empty. A map $\phi : B_1 \to B_2$ is a **board isomorphism** if

1. ϕ is a graph isomorphism, that is a bijection of the vertex sets of B_1 and B_2 such that $\{v_1, v_2\}$ is an edge of B_1 if and only if $\{\phi(v_1), \phi(v_2)\}$ is an edge of B_2;

2. The vertex v of B_1 contains a Left piece if and only if the vertex $\phi(v)$ of B_2 contains a Left piece; and

3. The vertex w of B_1 contains a Right piece if and only if the vertex $\phi(w)$ of B_2 contains a Right piece.

We have the following relation between the Stanley-Reisner and facet operators together with the cover dual, Alexander dual, and complement.

Fact 1.51 (Faridi 2004 [24])**.** *Given two simplicial complexes Δ and Γ, if we have $\mathcal{N}(\Delta) = \mathcal{F}(\Gamma)$, then $\mathcal{N}(\Delta^\vee) = \mathcal{F}(\Gamma_M)$. Alternatively, given a simplicial complex Δ, we have*

- $\mathcal{N}(\mathcal{F}(\Delta)) = (\Delta_M)^c$

- $\mathcal{N}(\mathcal{F}(\Delta))^\vee = \mathcal{N}(\mathcal{F}(\Delta_M)) = \Delta^c$

By Fact 1.51 we have the diagram in Figure 1.5 for the simplicial complexes and ideals related to the SP-game (R, B).

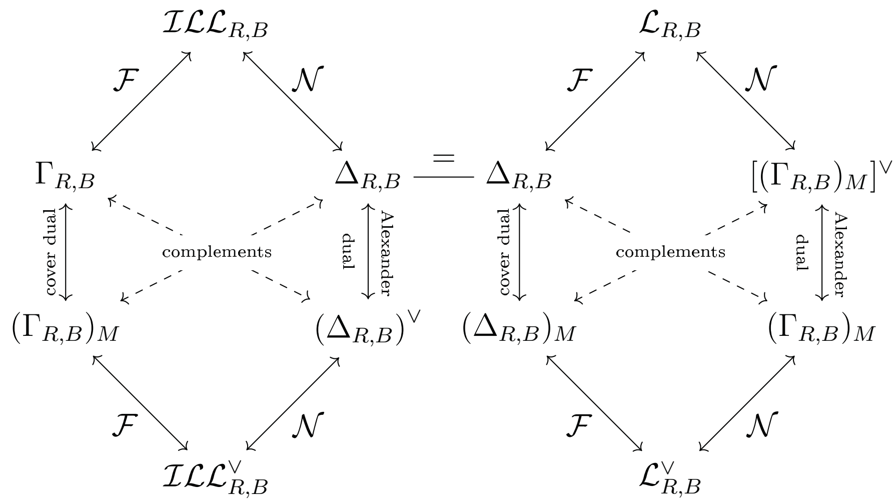


Figure 1.5: Relationship between Game Complexes, Ideals, and their Duals

The following proposition shows that two games with isomorphic legal complexes have isomorphic game trees, and as a consequence the same game value under most winning conditions (such as normal play and misère). Thus using simplicial complexes helps us to easily identify when two games are literally equal.

Proposition 1.52. *If two SP-games (R_1, B_1) and (R_2, B_2) have isomorphic legal complexes, then their game trees are isomorphic, i.e. they are literally equal.*

Proof. We prove that isomorphic legal complexes imply isomorphic game trees by induction on the size of the faces (i.e. the number of pieces in a position). Let $(\mathfrak{L}_1, \mathfrak{R}_1)$ be the labelling of Δ_{R_1,B_1} and $(\mathfrak{L}_2, \mathfrak{R}_2)$ be the labelling of Δ_{R_2,B_2}. Since the two legal complexes are isomorphic, the labelling are the same and we will thus write $\mathfrak{L}$ for $\mathfrak{L}_1$ and $\mathfrak{L}_2$ and similarly $\mathfrak{R}$ for $\mathfrak{R}_1$ and $\mathfrak{R}_2$.

The empty face (i.e. empty board) corresponds to the root of the game tree, thus the latter is trivially the same for both games.

Now assume that the game trees are isomorphic up to positions with k pieces played.

Consider a position P_1 in the game (R_1, B_1) with k pieces played. Let F_1 be the face of Δ_{R_1,B_1} (of dimension $k-1$) corresponding to P_1. Since Δ_{R_1,B_1} and Δ_{R_2,B_2} are isomorphic, there exists a face $F_2 \in \Delta_{R_2,B_2}$ (of dimension $k-1$) to which F_1 is mapped, corresponding to a position P_2 of (R_2, B_2), which also has k pieces placed.

Now let P_1' be any option of P_1 and F_1' be the corresponding face in Δ_{R_1,B_1}. Then there exists a vertex v such that $F_1' = F_1 \cup \{v\}$. Let F_2' be the face of Δ_{R_2,B_2} corresponding to F_1'. Then there exists a vertex w (corresponding to v) such that $F_2' = F_2 \cup \{w\}$. Thus the position P_2' corresponding to F_2' is an option of P_2.

Further, since the legal complexes have the same bipartition, we have that the following are equivalent:

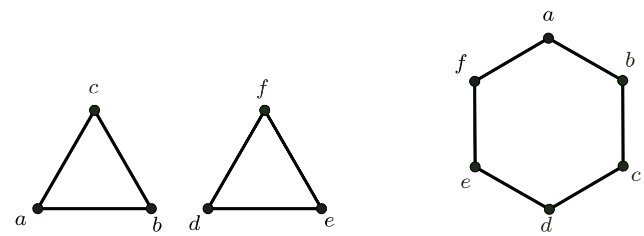
1. The position P_1' is a Left, respectively Right, option of P_1.
2. The vertex v belongs to $\mathfrak{L}$, respectively $\mathfrak{R}$.
3. The vertex w belongs to $\mathfrak{L}$, respectively $\mathfrak{R}$.
4. The position P_2' is a Left, respectively Right, option of P_2.

Thus for any option of P_1 there exists an option of P_2 and vice-versa, which shows that the game trees of (R_1, B_1) and (R_2, B_2) are isomorphic up to positions of $k+1$ pieces, and by induction they are entirely isomorphic. □

This is a very strong statement since isomorphic game trees imply that two games are in the same equivalence class, independent of the winning condition considered. Further, games in the same equivalence class can have different game trees.

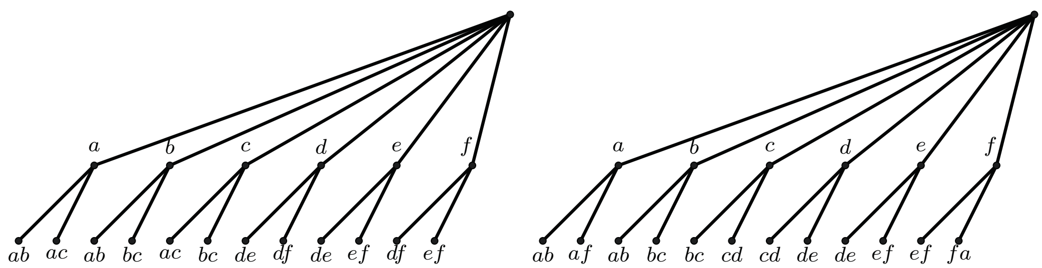
Note though that the converse is not true, as the following example demonstrates. We are grateful to Alex Fink for providing this example.

Example 1.53 (Alex Fink)**.** Consider a ruleset R in which all pieces occupy a single vertex, have to be adjacent to all previously placed ones, at most two pieces may be placed, and only Left may play. Then $\Delta_{R,B} \cong B$ for all boards B. In particular, consider B_1 being a disjoint union of two 3-cycles, and B_2 being a 6-cycle with labels for basic positions as below.



B_1 B_2

The game trees for (R, B_1) and (R, B_2) (shown below on the left and right respectively) are isomorphic even though the legal complexes are not.



Note that in the above example the game trees have isomorphic structure, but the labelling of positions is not isomorphic in the sense that the isomorphic structures of the trees do not preserve the labelling. We will show in Chapter 3 that if the labelling of the game tree is given, we are able to recover a unique SP-game corresponding to it.

This also gives an indication that the legal complex of an SP-game is a better representative for the SP-game than the tree as it conveys more structure.

On the other hand, if the illegal complexes are isomorphic, it is not always true that the game trees are isomorphic. For example, consider the ruleset R in which neither player can place on a vertex of degree 1. We then have

$$\Gamma_{R,P_2} = \langle x_1, x_2, y_1, y_2 \rangle \cong \Gamma_{R,P_3} = \langle x_1, x_3, y_1, y_3 \rangle.$$

The legal complexes $\Delta_{R,P_2} = \emptyset$ and $\Delta_{R,P_3} = \langle x_2, y_2 \rangle$ are not isomorphic. And since there are no legal moves in (R, P_2), but there are in (R, P_3), their game trees are not isomorphic either.

Another occurrence of isomorphic illegal complexes but non-isomorphic legal complexes is if there are moves that are always playable in one game, but these moves do not occur at all in the second game. This situation occurs in the proof of Theorem 2.17, where we construct a game with isomorphic illegal complex, but potentially a different underlying ring, and an adjustment has to be made to the game accordingly.

We are able to characterize what the legal and illegal complex of the disjunctive sum of two games looks like. For this, we require the join and disjoint union of two simplicial complexes.

Definition 1.54. Given two simplicial complexes $\Delta = \langle F_1, F_2, \ldots, F_i \rangle$ and $\Delta' = \langle F'_1, F'_2, \ldots, F'_j \rangle$ their **join** is defined as

$$\Delta * \Delta' = \langle F_1 \cup F'_1, F_1 \cup F'_2, \ldots, F_1 \cup F'_j, \ldots, F_i \cup F'_1, F_i \cup F'_2, \ldots, F_i \cup F'_j \rangle.$$

If the vertex set of Δ is $V = \{x_1, \ldots, x_m\}$ and the vertex set of Δ' is $V' = \{x'_1, \ldots, x'_n\}$, then their **disjoint union** is the simplicial complex with vertex set $V \cup V'$ given by

$$\Delta \sqcup \Delta' = \langle F_1, F_2, \ldots, F_i, F'_1, F'_2, \ldots, F'_j \rangle.$$

Theorem 1.55. *Let (R, B) and (R', B') be two SP-games with legal complexes $\Delta_{R,B}$ and $\Delta_{R',B'}$, and with illegal complexes $\Gamma_{R,B}$ and $\Gamma_{R',B'}$. Then*

$$\Delta_{(R,B)+(R',B')} = \Delta_{R,B} * \Delta_{R',B'}$$

is the legal complex and

$$\Gamma_{(R,B)+(R',B')} = \Gamma_{R,B} \sqcup \Gamma_{R',B'}$$

is the illegal complex of the disjunctive sum $(R, B) + (R', B')$.

Proof. A maximal legal position in the game $(R, B) + (R', B')$ is one where both the pieces placed in (R, B) and the ones placed in (R', B') form maximal legal positions. Thus a facet in the legal complex of $(R, B) + (R', B')$ is a union of a facet of $\Delta_{R,B}$ and a facet of $\Delta_{R',B'}$.

A minimal illegal position in the game $(R, B) + (R', B')$ is one where either the pieces placed in (R, B) or the ones placed in (R', B') form a minimal illegal position. Thus a facet in the illegal complex of $(R, B) + (R', B')$ is a facet of $\Gamma_{R,B}$ or a facet of $\Gamma_{R',B'}$. $\square$

Chapter 2

Simplicial Complexes are Games Complexes

In this chapter, we show that each simplicial complex is the legal complex of some invariant strong placement game (iSP-game). One implication is that in most situations when studying SP-games it is enough to consider those with invariance.

In Section 1.4, we demonstrated how to assign two simplicial complexes to each SP-game. One of the main questions is what complexes appear as game complexes. In Proposition 2.1 we show that every simplicial complex is both a legal and an illegal complex of some SP-game and board. The rulesets of these games can be quite complex, though, and depend highly on the board on which the game is being played.

Thus we introduce the concept of invariance for SP-games, which, in a sense, forces rulesets to be uniform and thus independent of the board. An example of a non-invariant SP-game is the game played on a complete graph in which the vertices of the board are partitioned and pieces on vertices in one part cannot be adjacent to other pieces, while there are no restrictions for the other vertices. Invariance is a concept that was introduced for subtraction games (see for example Dûchene and Rigo 2010 [21], Larsson, Hegarty, and Fraenkel 2011 [41], and Larsson 2012 [40]), where it is defined slightly differently due to the different class of games, but has the same intent, namely that the ruleset does not depend on the board. Similar to the previous question, we are interested in which simplicial complexes come from invariant SP-games (iSP games). We show that every simplicial complex without an isolated vertex is the illegal complex of some iSP-game, and also that every simplicial complex is a legal complex of an iSP-game. The constructions given in all cases prove the stronger result that such SP-games exist given *any* bipartition of the vertices of the simplicial complex (see Theorems 2.13 and 2.17) into Left and Right positions. This construction then allows us to show that for every SP-game there exists an iSP-game such that their game trees are isomorphic. This in turn implies that their game values are the same under both normal and misère winning conditions. Thus it is enough to only consider iSP-games in most situations, such as when calculating values.

Finally, we restrict to independence games, those games for which the ruleset played on any board gives an illegal complex which is a graph. This class includes many games actually played, such as SNORT, COL, and DOMINEERING, but not NOGO. We show that any SP-game whose illegal complex is a graph is literally equal to an invariant independence game.

2.1 Games from Simplicial Complexes

A natural and important question is whether any given simplicial complex Δ is the legal or illegal complex of some game. We will answer this question positively in both cases. This will allow us to view properties of SP-games as properties of simplicial complexes and vice-versa. We are able to show this for any bipartition of the vertices into Left $\mathfrak{L}$ and Right $\mathfrak{R}$, where $\mathfrak{L}$ or $\mathfrak{R}$ could even be the empty set.

Proposition 2.1 (**Games from Simplicial Complexes**)**.** *Given an* $(\mathfrak{L}, \mathfrak{R})$*-labelled simplicial complex* Δ*, there exist SP-games* (R_1, B) *and* (R_2, B) *such that*

(a) $\Delta = \Delta_{R_1,B}$ *and*

(b) $\Delta = \Gamma_{R_2,B}$

and the sets of Left (respectively Right) positions is $\mathfrak{L}$ *(respectively* $\mathfrak{R}$*).*

Proof. Let $m = |\mathfrak{L}|$ and $n = |\mathfrak{R}|$. Let B be the board consisting of m disjoint 3-cycles and n disjoint 4-cycles. In the games (R_1, B) and (R_2, B), Left will be playing 3-cycles, while Right will be playing 4-cycles.

In Δ, label the vertices belonging to $\mathfrak{L}$ as $1, \ldots, m$, and the vertices in $\mathfrak{R}$ as $m+1, \ldots, n+m$. Similarly, label the 3-cycles of B as $1, \ldots, m$, and the 4-cycles as $m+1, \ldots, n+m$.

(a) In R_1, playing on a set of cycles of B is legal if and only if the corresponding set of vertices in Δ forms a face.

(b) In R_2, playing on a set of cycles of B is legal if and only if the corresponding set of vertices in Δ does not contain a facet.

It is now easy to see that $\Delta = \Delta_{R_1,B}$ and $\Delta = \Gamma_{R_2,B}$. □

Note that we chose 3-cycles and 4-cycles for the pieces as these are the smallest graphs which allow us to guarantee that players can only place on "their" spaces.

Remark 2.2. There is a second construction that works for the legal complex, which can be thought of as playing on the simplicial complex:

Given an $(\mathfrak{L}, \mathfrak{R})$-labelled simplicial complex Δ, let the board be the 1-skeleton $\Delta^{[1]}$ of the given simplicial complex, i.e. the underlying graph. Left and Right will be claiming a single vertex, with Left only being allowed to place on vertices of B which belong to $\mathfrak{L}$ in Δ, and Right only on vertices in $\mathfrak{R}$. Playing on a set of vertices is legal if and only if they form a face in Δ.

As seen in the proof of Proposition 2.1, it is rather simple to construct games on fixed boards from simplicial complexes by restricting the legal moves to certain parts of the board. We now move on to look at games where such restrictions can be relaxed. We call these invariant games.

2.2 Invariant Games

As we have shown in Proposition 2.1, every $(\mathfrak{L}, \mathfrak{R})$-labelled simplicial complex is the legal or illegal complex of some SP-game and board. The rules created as part of this construction, however, depend heavily on the board. We now define the concept of invariance for SP-games, which in a sense forces the ruleset to be "uniform" across the board.

Definition 2.3. The ruleset of an SP-game is **invariant** if the following conditions hold:

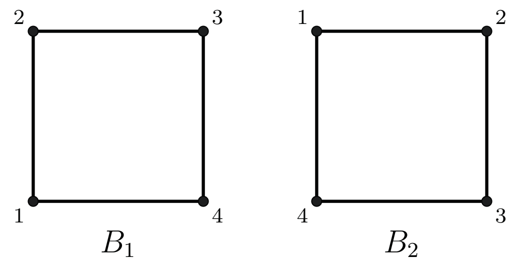
(1) Every basic position is legal.

(2) The ruleset does not depend on the board, i.e. if B_1 and B_2 are isomorphic boards, then a move in B_1 is legal if and only if its isomorphic image in B_2 is legal.

If the ruleset of an SP-game is invariant, we also say that the game is an **invariant strong placement game (iSP-game)**.

COL and SNORT are examples of rulesets that are invariant, while DOMINEERING and NOGO are not. In DOMINEERING half of the basic positions are illegal (Right cannot play vertically, while Left cannot play horizontally). That NoGo is not invariant is not as obvious. Indeed on most boards both conditions hold, but whenever the board has an isolated vertex, playing on it is illegal (thus the basic position corresponding to that vertex is illegal). An example of an SP-game which fails the second condition is the following.

Example 2.4. Consider playing on B_1 and B_2, both 4-cycles with labels as below, a game in which a Left piece on vertex 1 cannot be adjacent to another piece.



Now B_1 and B_2 are isomorphic graphs and, since neither contains pieces, also isomorphic boards. The position in which there is a Left piece in the top left corner and a Right piece in the top right corner is legal on B_1 but not on B_2. Thus this game is not invariant.

Similar to the question of the previous section, we are interested in which simplicial complexes appear as the legal or illegal complex of an iSP-game.

We will show below that the illegal complex of an iSP-game cannot contain an isolated vertex.

Proposition 2.5. *Let Γ be a simplicial complex. If Γ is the illegal complex of some iSP-game then Γ has no facets of dimension 0.*

Proof. Assume that Γ has a facet that has dimension 0, i.e. an isolated vertex, and label this vertex a. If Γ is the illegal complex of some SP-game (R, B), then since $\{a\}$ is a facet of Γ, there exists a basic position (corresponding to the vertex a) which is illegal. Thus G does not satisfy the first condition of invariance. $\square$

Other than the isolated vertex situation, there is no obstruction for a simplicial complex Γ being an illegal complex. We set out to prove this (see Theorem 2.13) by constructing a Γ-board and a Γ-ruleset from Γ.

Construction 2.6 (Γ-board). Given an $(\mathfrak{L}, \mathfrak{R})$-labelled simplicial complex Γ with no isolated vertices we can construct a graph B_Γ (called the **Γ-board**) as follows:

If Γ is empty, then let B_Γ be empty.

If Γ is non-empty, then let $H = \Gamma^{[1]}$, i.e. the underlying graph of Γ. Let n be the number of vertices in the graph H and (re)label the vertices of H as $1, \ldots, n$. Begin constructing the board B_Γ by using n cycles of sizes $n^4 + 4$ and $n^4 + 5$ and label these $1, \ldots, n$ so that cycle i will have size $n^4 + 4$ if the vertex i in H belongs to $\mathfrak{L}$, and size $n^4 + 5$ if the vertex i belongs to $\mathfrak{R}$. For each cycle, designate $n - 1$ consecutive vertices for joining, called **connection vertices** (see Figure 2.1).

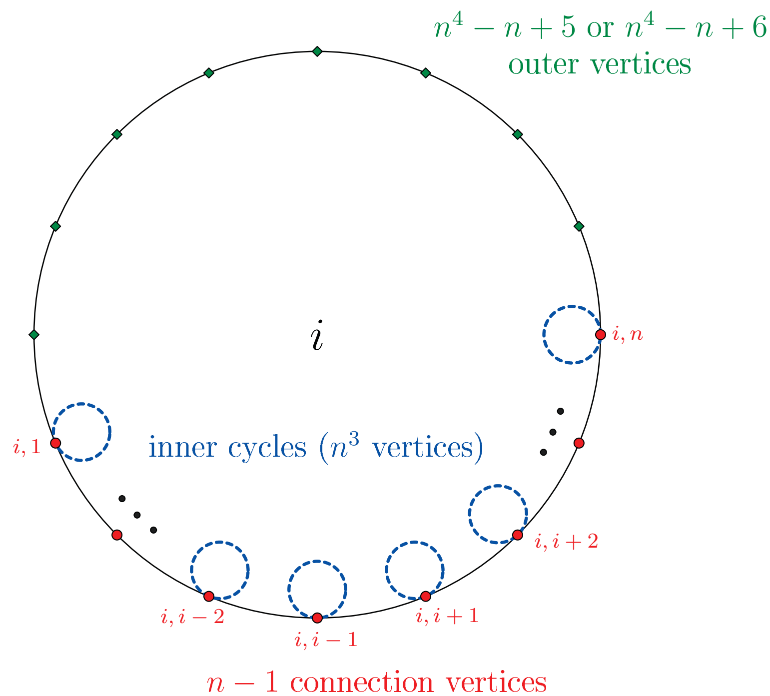


Figure 2.1: Cycle i in the Board B_Γ

Call the remaining vertices **outer vertices**. To each connection vertex, join a cycle of length n^3 (called **inner cycles**). In cycle i label the connection vertices as i, j where $j = 1, \ldots, n$ and $j \neq i$.

Label the edges in H as $1, \ldots, k$. If the endpoints of the edge l are the vertices i and j, then add a path of $2 + l$ vertices to B_Γ, whose end vertices are i, j and j, i (see Figure 2.2). The l vertices between i, j and j, i are called **centre vertices**.

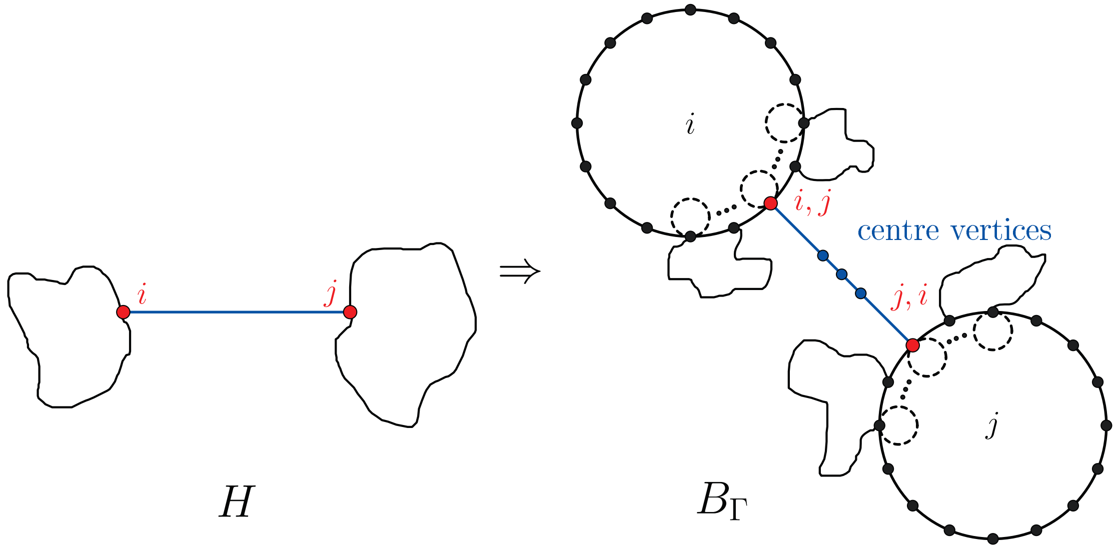


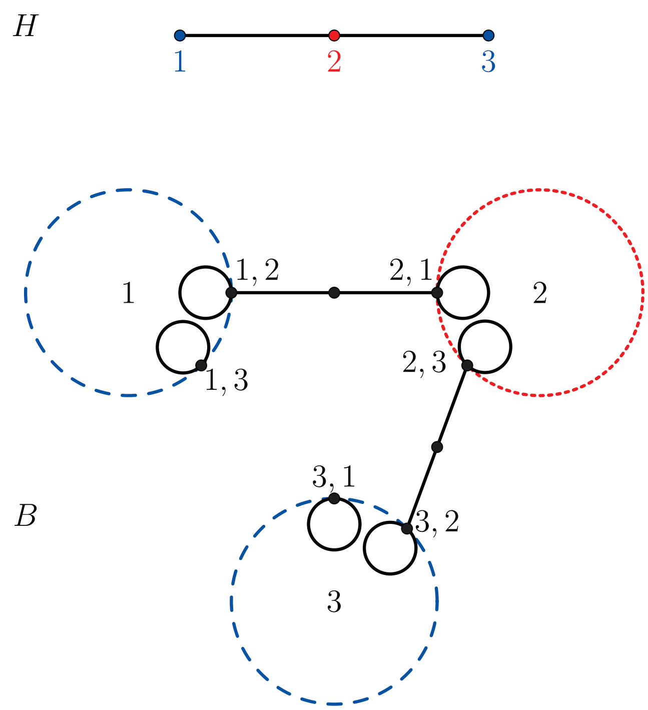
Figure 2.2: Effect of an Edge in H on the Board B_Γ

As an example for this construction, consider the following:

Example 2.7. Let Γ be a path of three vertices so that $H = \Gamma$. Let the two end vertices belong to $\mathfrak{L}$, and the centre vertex to $\mathfrak{R}$. Since Γ consists of three vertices, i.e. $n = 3$, the cycle i (where $i \in \mathfrak{L}$) is of length $3^4 + 4 = 85$ with two cycles of length $3^3 = 27$ joined to two adjacent vertices, and the cycle j (where $j \in \mathfrak{R}$) is of length 86 with two cycles of length 27 joined to two adjacent vertices.

Label the edge between vertex 1 (an end vertex) and vertex 2 (the centre vertex) as 1, and the edge between vertex 2 and vertex 3 (the other end vertex) as 2.

The board B_Γ is given below. Dashed, blue cycles consist of 85 vertices, and dotted, red cycles of 86 vertices, with the two labelled vertices adjacent in both cases. The smaller solid cycles consist of 27 vertices.



For the next construction, we will have to specify what is meant by distance between pieces.

Definition 2.8. Let two pieces P_1 and P_2 be placed on a board B and let V_1 and V_2 be the set of vertices on which P_1, respectively P_2, was placed. We then define the **distance** $d(P_1, P_2)$ between P_1 and P_2 by

$$d(P_1, P_2) = \min\{d(v_1, v_2) : v_1 \in V_1, v_2 \in V_2\},$$

where $d(v_1, v_2)$ is the graph theoretic distance between v_1 and v_2, i.e. the minimum number of edges of a path in B with endpoints v_1 and v_2.

Construction 2.9 (Γ-ruleset). Given an $(\mathfrak{L}, \mathfrak{R})$-labelled simplicial complex Γ with no isolated vertices we construct a ruleset R_Γ for an SP-game (called the Γ**-ruleset**).

If Γ is empty, then let R_Γ be the ruleset in which Left and Right place pieces on a single vertex with no restrictions.

If Γ is non-empty, then construct R_Γ as follows:

1. Let n be the number of vertices of Γ. Label the edges (the 1-dimensional faces) of Γ as $\{1, \ldots, k\}$.

2. Left plays cycles of length $n^4 + 4$ with cycles of length n^3 joined to $n - 1$ consecutive vertices,

3. Right plays cycles of length $n^4 + 5$ with cycles of length n^3 joined to $n - 1$ consecutive vertices (i.e. the pieces are as the structure given in Figure 2.1), and

4. Let F be a facet of Γ of dimension $f - 1$, whose 1-dimensional faces are labelled $k_1, \ldots, k_l$ (from 1.), where $l = \binom{f}{2}$. We call the set $\{k_1 + 1, \ldots, k_l + 1\}$ the **id-set** of F. Then no sets of f pieces are allowed such that the set of distances between any two pieces is exactly the id-set of F.

Example 2.10. Let Γ be a path of three vertices so that $n = 3$. Left's pieces are cycles of length $3^4 + 4 = 85$ with two cycles of length $3^3 = 27$ joined to two adjacent vertices, and Rights pieces are cycles of length 86 with two cycles of length 27 joined to two adjacent vertices.

Since the facets of Γ are the two edges (thus of size 2), the edge in one facet are labelled as 1, and in the other as 2. Thus the id-sets are $\{2\}$ and $\{3\}$, implying that in R_Γ no two pieces are allowed to have distance 2 or distance 3.

Example 2.11. Consider $\Gamma = \langle\{a, b, c\}, \{a, d\}\rangle$. Label the edge between a and b as 1, between b and c as 2, between c and a as 3, and between a and d as 4.

For the facet abc we have the id-set $\{1 + 1, 2 + 1, 3 + 1\} = \{2, 3, 4\}$. Thus in the Γ-ruleset R_Γ we cannot have three pieces where the distances between pairs are $\{2, 3, 4\}$, while two with any one of these distance are allowed.

For the facet ad we have the id-set $\{4 + 1\} = \{5\}$. Thus in R_Γ we cannot have any two pieces with distance 5.

Lemma 2.12. *Given an $(\mathfrak{L}, \mathfrak{R})$-labelled simplicial complex Γ with no isolated vertices, the Γ-ruleset R_Γ is invariant.*

Proof. If Γ is empty, then R_Γ played on any board has no illegal positions, thus is trivially invariant.

If Γ is non-empty, then since Γ has no isolated vertices, all facets have at least one edge and therefore all id-sets are non-empty. In particular, this means that every illegal position of R_Γ played on any board has at least two pieces, so there are no illegal basic positions.

Now suppose that we are playing R_Γ on isomorphic boards B_1 and B_2. A position P is legal on B_1 if and only if there is no id-set which is contained in the set of distances between pieces of P, which holds if and only if P is legal on B_2.

Thus R_Γ is invariant. □

The following statement will prove that every simplicial complex without isolated vertices can appear as the illegal complex of (many!) iSP-games.

Theorem 2.13 **(Invariant Game from Illegal Complex).** *Given an* $(\mathfrak{L},\mathfrak{R})$*-labelled simplicial complex* Γ *with no isolated vertices, fix labellings of the vertices and of the edges. Then* Γ *is the illegal complex of the* Γ*-ruleset* R_Γ *played on the* Γ*-board* B_Γ*, i.e.* $\Gamma_{R_\Gamma,B_\Gamma} = \Gamma$.

Proof. Let $G = (R, B)$ where $B = B_\Gamma$ and $R = R_\Gamma$ are the Γ-board and Γ-ruleset respectively, with the same labelling of the edges of Γ if Γ is nonempty.

If Γ is empty, then G has no illegal positions, thus $\Gamma_{R,B}$ is also empty.

To show that indeed $\Gamma_{R,B} = \Gamma$ for Γ nonempty, we will begin by showing that their vertex sets have the same size.

Let $H = \Gamma^{[1]}$. Clearly Left can place one of her pieces on the cycle labelled i in B if the vertex i of H belongs to $\mathfrak{L}$. Similarly Right can place on cycles labelled j where $j \in \mathfrak{R}$. Thus each vertex in H corresponds to a position in G.

We now need to show that there are no other ways for Left or Right to place pieces than what was previously mentioned, i.e. that the positions of G correspond exactly to the vertices of H.

Let n be the number of vertices of H and k be the number of edges. The cycles in B which only use connection and centre vertices have size at most $n(n-1) + \frac{k(k+1)}{2}$ (there are $n(n-1)$ connection vertices and $1 + \ldots + k$ centre vertices). Since there are at most $\binom{n}{2}$ edges in H, we have

$$\begin{aligned} n(n+1) + \frac{k(k+1)}{2} &\leq n(n+1) + \frac{\frac{n(n+1)}{2}\left(\frac{n(n+1)}{2}+1\right)}{2} \\ &= \frac{1}{8}n^4 + \frac{1}{4}n^3 + \frac{11}{8}n^2 + \frac{5}{4}n \end{aligned}$$

which is less than $n^4 + 4$ for all whole numbers.

Thus such cycles are shorter than $n^4 + 4$, and Left and Right will not be able to play on those.

Furthermore, any cycle of length $n^4 + 4$ or $n^4 + 5$ in B needs to include the outer vertices of some cycle i (since as above cycles using only connection and centre vertices are shorter, and the inner cycles are shorter). To then construct a cycle of that length without using all connection vertices of cycle i, the cycle would have to include at least one centre vertex. Since centre vertices do not have cycles of length n^3 added, this implies that neither Left or Right could play there.

Thus Left and Right are only able to play on the labelled cycles.

Further, since the pieces consist of cycles with a differing number of vertices, either player will only be able to play on the cycles of B that are designated to them. Thus there are n positions, in each of which only one player can play, all corresponding to vertices of Γ. The vertices of $\Gamma_{R,B}$ are thus a subset of the vertices of Γ and $\Gamma_{R,B}$ has less vertices than Γ if and only if there exists at least one position in which it is never illegal to play, which we will show cannot happen as part of the rest of the proof.

We next need to show that the facets of $\Gamma_{R,B}$ and Γ correspond.

Consider a facet consisting of the vertices $i_1, \ldots, i_k$ in Γ, thus any two vertices have an edge between them in H, and let these edges be $j_1, \ldots, j_l$. Then the positions i_a and i_b, $a, b \in \{1, \ldots, k\}$, in B have distance $j_c + 1$, where j_c is the edge between i_a and i_b in H, (since we joined a path of length $j_c + 2$ to their connection vertices). Thus it is illegal to play in all k positions (and this is a minimal illegal position), and thus there is a facet consisting of the vertices $i_1, \ldots, i_k$ in $\Gamma_{R,B}$.

Now let the vertices $i_1, \ldots, i_k$ form a facet in $\Gamma_{R,B}$. Assume that $i_1, \ldots, i_k$ do not form a facet in Γ. If some subset S of these vertices forms a facet, then by construction of R it would be illegal to play pieces on all of the cycles in B corresponding to vertices in S. Thus $i_1, \ldots, i_k$ is not a *minimal* illegal position, a contradiction to those vertices forming a facet in $\Gamma_{R,B}$. If on the other hand $i_1, \ldots, i_k$ is strictly contained in some facet F of Γ, then by construction of R it is legal to play on cycles $i_1, \ldots, i_k$ in B. Thus $i_1, \ldots, i_k$ is not an *illegal* position, a contradiction to those vertices forming a facet in $\Gamma_{R,B}$. Therefore $i_1, \ldots, i_k$ is a facet of Γ.

Finally, since H has no isolated vertices (by Γ not having such), the vertex set of

$\Gamma_{R,B}$ is a subset of the vertex set of H, i.e. the vertex set of Γ. Since furthermore the facets of $\Gamma_{R,B}$ and Γ correspond, we have that the vertex set of $\Gamma_{R,B}$ is equal to that of Γ.

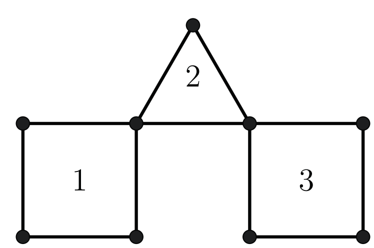
Consequently, the simplicial complexes Γ and $\Gamma_{R,B}$ have the same vertex and facet sets, which proves $\Gamma = \Gamma_{R,B}$. □

Example 2.14. Let Γ be a path of three vertices. Let $B = B_\Gamma$ (see Example 2.7) and $R = R_\Gamma$ (see Example 2.10).

Then $\Gamma_{R,B} = \Gamma$.

Note: Simpler constructions with smaller cycles and pieces are often possible (as shown in the next example), but the above construction is guaranteed to work.

Example 2.15. Let Γ be as in Example 2.14. Let Left play cycles of length 3, and Right cycles of length 4. For the board B' given below, it is easy to check that $\Gamma_{R',B'} = \Gamma$, where R' is the ruleset which forbids overlap between pieces.



The following theorem summarizes our results about illegal complexes of iSP-games.

Theorem 2.16. *A given simplicial complex* Γ *is the illegal complex of some iSP-game* (R, B) *if and only if* Γ *has no isolated vertices.*

Proof. By Proposition 2.5 we have that if Γ is the illegal complex of an iSP-game, then Γ has no isolated vertices.

Conversely, if Γ has no isolated vertices, then by Theorem 2.13, we have that Γ is the illegal complex of some iSP-game and board. □

We will now consider legal complexes. The first result shows that *every* simplicial complex is the legal complex of some iSP-game and board:

Theorem 2.17 (**Invariant Game from Legal Complex**)**.** *Given any* $(\mathfrak{L},\mathfrak{R})$*-labelled simplicial complex* Δ*, we can construct an iSP-game* (R,B) *such that* $\Delta = \Delta_{R,B}$ *and the sets of Left, respectively Right, positions is* $\mathfrak{L}$*, respectively* $\mathfrak{R}$*.*

Proof. Given Δ, let the underlying ring be $S = k[x_1,\dots,x_m,y_1,\dots,y_n]$, where the x_is are the 0-dimensional faces of Δ in $\mathfrak{L}$ and the y_js in $\mathfrak{R}$. With this ring, let $\Gamma = \mathcal{F}(\mathcal{N}(\Delta))$, i.e. the simplicial complex whose facets correspond to the minimal non-faces of Δ.

We will prove the statement separately for the case in which the simplicial complex Δ is not a simplex, i.e. when Γ has at least one 1-dimensional face, and when it is a simplex, i.e. when Γ is empty.

Case 1: If Δ is not a simplex, the construction is as follows:

Let i be a vertex in Δ. If Δ has at least one facet that does not contain i, then i will also be a vertex of Γ. Otherwise it is not a vertex of Γ.

Let the vertex set of Γ be bipartitioned into $\mathfrak{L}$ and $\mathfrak{R}$ the same way that the vertex set of Δ is. Let n be the number of vertices in Γ and let R be the Γ-ruleset and B_0 be the Γ-board, so that $\Gamma_{R,B_0} = \Gamma$. If Δ has a vertex v that is contained in every facet, then it is not a vertex of Γ, and there is no move in (R,B_0) corresponding to the vertex v. Thus the underlying rings of Γ_{R,B_0} and Γ are not the same, and we have to adjust the board as follows:

Without loss of generality, let $1,\dots,k$ be the vertices of Δ that are contained in every facet. Then for $l = 1,\dots,k$ let $B_l = B_{l-1} \cup C^l$ where C^l is a cycle of length n^4+4 (if the vertex l belongs to $\mathfrak{L}$) or length n^4+5 (if it belongs to $\mathfrak{R}$) with $n-1$ cycles of length n^3 joined to $n-1$ consecutive vertices. Let $B = B_k$. When playing the ruleset R on B, it is always legal to play on the disjoint C^l for either Left or Right, thus these positions are never part of a minimal illegal position, which shows that $\Gamma_{R,B_0} = \Gamma_{R,B}$. Furthermore, the underlying rings of Γ and $\Gamma_{R,B}$ are the same.

It immediately follows that

$$\Delta_{R,B} = \mathcal{N}(\mathcal{F}(\Gamma_{R,B})) = \mathcal{N}(\mathcal{F}(\Gamma)) = \Delta.$$

Case 2: If Δ is a simplex, we can construct R and B as follows:

Let n be the number of vertices in Δ and (re)label the vertices $1,\dots,n$. Let the board B be a disjoint union of n cycles of size 3 and 4 and label these $1,\dots,n$ so that cycle i will have size 3 if the vertex i in Δ belongs to $\mathfrak{L}$, and size 4 if the vertex i belongs to $\mathfrak{R}$.

Let R be the SP-ruleset in which Left plays cycles of length 3, and Right plays cycles of length 4. Note that R is invariant.

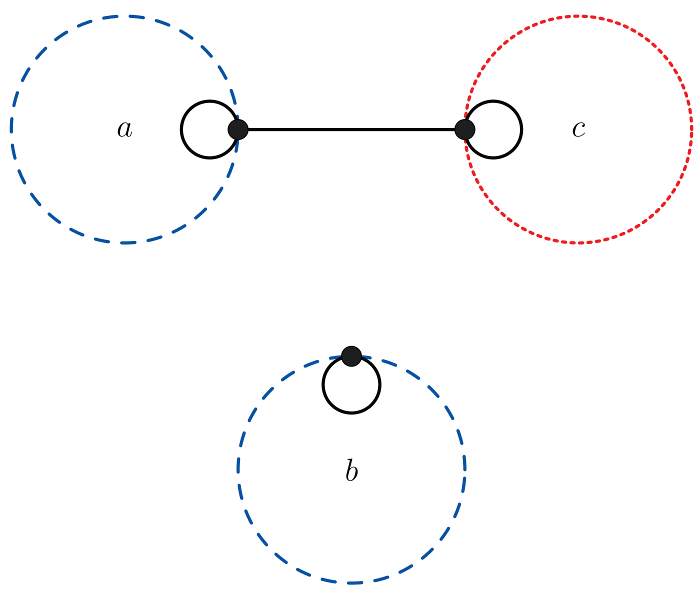
It is easy to see that $\Delta = \Delta_{R,B}$. $\square$

The following two examples demonstrate this construction in both the case where Δ is not a simplex and when it is.

Example 2.18. Consider the complex $\Delta = \langle\{a,b\},\{b,c\}\rangle$, where the vertices are partitioned as $\mathfrak{L} = \{a,b\}$ and $\mathfrak{R} = \{c\}$. Since Δ is not a simplex, we will follow the construction given in the first case of the proof of Theorem 2.17.

The only minimal nonface of Δ is ac, thus the graph H is P_2. Since $n = 2$, in the SP-ruleset R Left will play cycles of length $n^4 + 4 = 20$ with one cycle of length $n^3 = 8$ added to a vertex, while Right plays cycles of length $n^4 + 5 = 21$ with a cycle of length 8 added to a vertex.

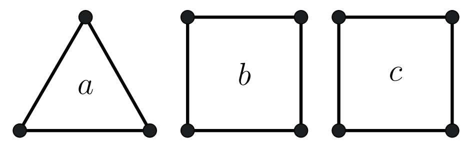
The board B is given below. Dashed, blue cycles consist of 20 vertices, and dotted, red cycles of 21 vertices. The smaller solid cycles consist of 8 vertices.



It is now easy to check that $\Delta_{R,B} = \Delta$.

Example 2.19. Consider the simplex $\Delta = \langle\{a,b,c\}\rangle$, where the vertices are partitioned as $\mathfrak{L} = \{a\}$ and $\mathfrak{R} = \{b,c\}$. Since Δ is a simplex, we will follow the second construction given in the proof of Theorem 2.17. Since $n = 3$, in the SP-ruleset R Left will play cycles of length 3, while Right plays cycles of length 4.

The board B is



It is now easy to check that $\Delta_{R,B} = \Delta$ and that $\Gamma_{R,B}$ is empty.

Concluding our discussion of iSP-games, we have the following result.

Theorem 2.20 (**Every SP-Game Tree Belongs To An iSP-Game**)**.** *Given an SP-game (R, B), there exists an iSP-game (R', B') so that their game trees are isomorphic, i.e. they are literally equal.*

Proof. Let $\Delta = \Delta_{R,B}$ with $\mathfrak{L}$ the vertices corresponding to Left basic positions, and similarly $\mathfrak{R}$. Then by Theorem 2.17 we know that there exists an iSP-game R' and a board B' such that $\Delta = \Delta_{R',B'}$ with the same bipartition. Since $\Delta_{R,B} = \Delta_{R',B'}$, we have by Proposition 1.52 that the game trees of R played on B and R' played on B' are isomorphic. $\square$

This in particular implies that under most winning conditions (such as normal play or misère play) the game values of R played on B and R' played on B' are the same, implying that we can replace one by the other.

2.3 Independence Games

Many of the games we have previously considered have illegal complexes that are graphs. This special class of SP-games is of further interest to us. This is also the class of SP-games whose legal complexes are flag complexes (see below for more).

When the illegal complex of an SP-game is a graph without isolated vertices, then its independence complex is the legal complex. Motivated by this we define the class of independence games.

Definition 2.21. An SP-ruleset R is called an **independence ruleset** if for *any* board B the illegal complex $\Gamma_{R,B}$ is a graph without isolated vertices (i.e. a pure one-dimensional simplicial complex). An SP-game (R, B) is called an **independence game** if R is an independence ruleset.

Many SP-games, such as Col and Snort, are independence games. NoGo is an example of an SP-game that is not an independence game. Even though $\Gamma_{\text{NoGo},B}$ is a graph for some boards (for example when B is P_2), there are many others for which this is not the case. For example, Γ_{NoGo,P_3}, given in Figure 2.3, has two-dimensional faces.

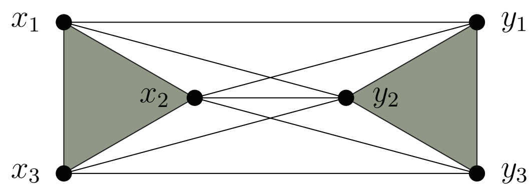


Figure 2.3: The Illegal Complex Γ_{NoGo,P_3}

Independence complexes are also called flag complexes in combinatorial commutative algebra (see for example Herzog and Hibi [31, p. 155]).

Definition 2.22. A simplicial complex is called **flag** if all the minimal non-faces are two element sets.

In the case of independence games, since $\Gamma_{R,B}$ is a graph without isolated vertices, we have that $\Delta_{R,B}$ is flag.

Lemma 2.23. *For an SP-game* (R, B)*,* $\Delta_{R,B}$ *is flag if and only if* $\Gamma_{R,B}$ *is a graph without isolated vertices.*

Proof. Since $\mathcal{N}(\Delta_{R,B}) = \mathcal{F}(\Gamma_{R,B})$ we have that the minimal non-faces of $\Delta_{R,B}$ are the facets of $\Gamma_{R,B}$. Thus if $\Gamma_{R,B}$ is a graph without isolated vertices, then $\Delta_{R,B}$ is flag, and conversely. $\square$

Consider the illegal complex $\Gamma_{R,B}$ of an independence ruleset R on a board B. Let $\Gamma'_{R,B}$ be the graph on the vertex set $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ (corresponding to the basic positions of R played on B) with edges those of $\Gamma_{R,B}$. Thus the difference between $\Gamma_{R,B}$ and $\Gamma'_{R,B}$ are isolated vertices corresponding to basic positions that are always legal. For many independence games we have $\Gamma'_{R,B} = \Gamma_{R,B}$.

Recall that the independence complex of a graph H is a simplicial complex with vertex set that of the graph and faces those sets of vertices that are independent in H, i.e. no two vertices are adjacent. The term 'independence game' was chosen for this class of games since the independent sets of $\Gamma'_{R,B}$ correspond to the legal positions of R played on B, i.e. the faces of $\Delta_{R,B}$. Thus in this case $\Delta_{R,B}$ is the independence complex of the graph $\Gamma'_{R,B}$.

One nice property of independence games is that playing an independence ruleset R on a board B is equivalent to forming independent sets of the graph $\Gamma'_{R,B}$ while Left picks vertices in $\mathfrak{L}$ and Right in $\mathfrak{R}$.

Further note that the Γ-ruleset in the case of Γ being a graph is always an independence ruleset (since minimal illegal positions are always pairs of pieces played). Using Theorem 2.17 this implies the following.

Proposition 2.24 (**iSP-Games of Flag Complexes**). *Given any SP-game (R, B) such that $\Gamma_{R,B}$ is a non-empty graph, there exists an invariant independence game (R', B') such that $\Delta_{R,B} = \Delta_{R',B'}$. In the case that $\Gamma_{R,B}$ has no isolated vertices, we also have $\Gamma_{R,B} = \Gamma_{R',B'}$.*

Proof. By Theorem 2.17 there exists an iSP-ruleset R' and board B' such that $\Delta_{R,B} = \Delta_{R',B'}$. In the case that $\Delta_{R,B}$ is not a simplex (if $\Gamma_{R,B}$ has at least one edge), the ruleset R' is the Γ-ruleset $R_{\Gamma_{R,B}}$. As mentioned above, this is an independence ruleset. In the case that $\Delta_{R,B}$ is a simplex, the ruleset R' has no illegal positions, and thus is an independence ruleset trivially.

If $\Gamma_{R,B}$ has no isolated vertices, then the underlying rings of $\Delta_{R,B}$ and $\Delta_{R',B'}$ are the same, thus

$$\Gamma_{R',B'} = \mathcal{F}(\mathcal{N}(\Delta_{R',B'})) = \mathcal{F}(\mathcal{N}(\Delta_{R,B})) = \Gamma_{R,B}. \qquad \square$$

Equivalently, this proposition states that given an SP-ruleset R and board B such that the minimal non-faces of $\Delta_{R,B}$ are all 1- and 2-element sets, there exists an SP-ruleset R' for which the legal complex is always flag and a board B' such that $\Delta_{R,B} = \Delta_{R',B'}$.

As a direct consequence of Proposition 2.24, applying Proposition 1.52, we have that these games also have isomorphic game trees.

Corollary 2.25. *Given any SP-game (R, B) such that $\Gamma_{R,B}$ is a non-empty graph, there exists an invariant independence game (R', B') such the game trees of (R, B) and (R', B') are isomorphic, i.e. they are literally equal.*

2.4 Further Work

The Γ-board and pieces of the Γ-ruleset have many more vertices than Γ itself. Thus we are interested in whether given any simplicial complex Γ simpler constructions of a ruleset R and board B are possible such that $\Gamma = \Gamma_{R,B}$. Ideally, we would like the pieces that Left and Right play to occupy only a single vertex. This seems unlikely though, thus an interesting question is for which class of simplicial complexes such a construction is possible.

Simplicial trees and forests, which are generalizations of graph trees and forests, are flag complexes (see Herzog and Hibi [31, Lemma 9.2.7]). Since many properties of simplicial trees are known (see for example Faridi 2004 [24] and Faridi 2005 [25]) it seems that this class of flag complexes provides a good start to studying whether simpler constructions are possible.

Chapter 3

Game Tree, Game Graph, and Game Poset of an SP-game

One step towards understanding properties of SP-games is to know the structure of the corresponding game trees. As the game tree for an SP-game has many repeated positions, we will introduce a simplified version called the game graph, and show that the game tree and game graph (both with positions labelled) are in one-to-one correspondence. We can thus equivalently ask what the structure of the game graph of an SP-game is, and we will answer this question in Proposition 3.11.

We then introduce the game poset and show that it is in one-to-one correspondence with the game graph. Due to the game poset's relationship with the legal complex – it is the face poset – we can then show that the legal complex and game graph are equivalent representations of an SP-game.

3.1 The Game Graph

Before introducing the game graph and for our later proof, we will have to fix some terminology regarding directed graphs.

Definition 3.1. A **directed graph** $D = (V, E)$ is a set of vertices V together with a set E of ordered pairs of vertices, called edges. Given an edge (v, w), v is called a **predecessor** of w and w is called a **successor** of v. The **in-degree** of a vertex v is the cardinality of the set $\{x \in V \mid (x, v) \in E\}$, and its **out-degree** the cardinality of $\{x \in V \mid (v, x) \in E\}$. A directed graph is called an **oriented graph** if whenever $(v, w) \in E$, then $(w, v) \notin E$. Two directed graphs are called **isomorphic** if there exists a bijection of the vertex sets preserving edges.

Recall from Definition 1.19 that the game tree of G can be thought of as an oriented directed graph with edges from each position to its options, and edges to Left options labelled L and similarly for Right options. As one can tell from Figure 1.3, even with a simple game, game trees tend to become very large even after a few moves, and the same position may be repeated many times. Motivated by this, we define the game graph below, in which each position corresponds to exactly one vertex.

Definition 3.2. The **game graph $\mathcal{G}_G$** of a combinatorial game G is a labelled, oriented graph where

(a) each legal position in G is represented by exactly one vertex; and

(b) if there is a Left move from position P to position Q, then there is an edge labelled L from v_P to v_Q, and correspondingly for Right moves.

Although the game tree is the most common representation of a game in combinatorial game theory, versions of the game graph are often used to model impartial or infinite games (see for example Nau 1983 [48], Cachat, Duparc, and Thomas 2002 [14]), or Berwanger and Serre 2012 [7]). For impartial games the labelling of the edges is removed (see Proposition 6.19) and for infinite games often the vertices, rather than the edges, represent moves and are labelled as L or R. We introduce the above version of a game tree for short, partizan games.

Note that the (unique) root of the game graph corresponds to the starting position, which in the case of SP-games is the empty position.

Example 3.3. The game graph of SNORT on P_3 is given in Figure 3.1, with vertices labelled with their corresponding legal positions. Compare this with the game tree in Figure 1.3.

We will show that game trees with positions labelled and game graphs with positions labelled of short combinatorial games are in a one-to-one correspondence. Note in particular that this is true for all games, not just SP-games.

Proposition 3.4. *Given a combinatorial game G, the game graph $\mathcal{G}_G$ and the game tree T_G (both with positions labelled) are in a one-to-one correspondence.*

Proof. Let A be the set of game trees with positions labelled and B the set of game graphs with positions labelled. We will describe maps $f : A \to B$ and $g : B \to A$ which preserve the game and are inverses of each other.

First, consider a game tree T_G with positions labelled of some game G. The map f sends this game tree to a game graph $\mathcal{G}$ by identifying vertices corresponding to

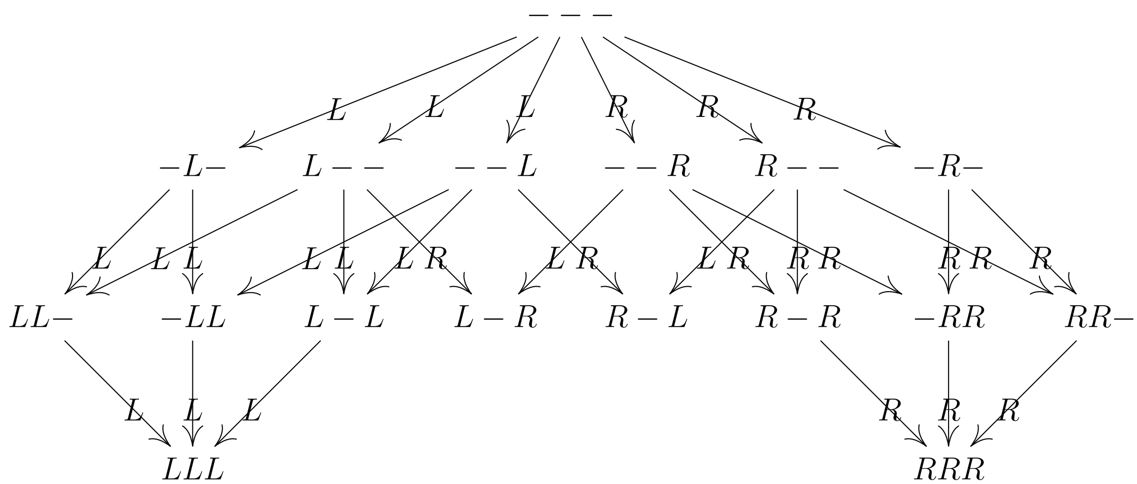


Figure 3.1: The Game Graph of SNORT on P_3

the same position and merge them into one. Edges that previously pointed to the left are labelled with an L, and similarly edges pointing to the right labelled with an R. This game graph is also the game graph of G.

On the other hand, consider the game graph $\mathcal{G}_G$ of some game G with positions labelled. This is mapped by g to a game tree T inductively by essentially splitting a vertex in the graph every time the in-degree is higher than 1. Formally the process is:

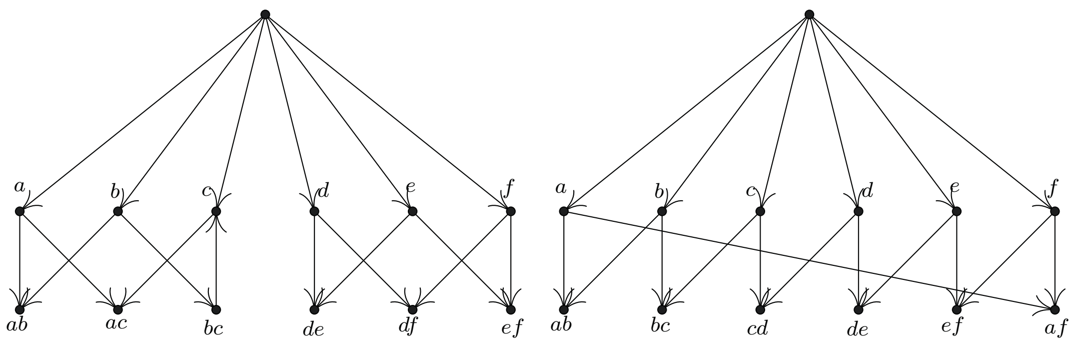
Step 1: Place a vertex in T for the starting position (the source of $\mathcal{G}_G$).

Step n: For every vertex v created in step $n-1$ do the following: Identify the vertex v' of $\mathcal{G}_G$ of which v is a copy. For each successor w' of v' place a vertex w in T such that if the edge from v' to w' is labelled with an L, the edge from v to w points to the left and similarly if the edge is labelled R.

This game tree is the game tree of G.

Since every game G has a unique game tree and game graph we have that f and g are inverses of each other. Thus the game graph and game tree are in one-to-one correspondence. □

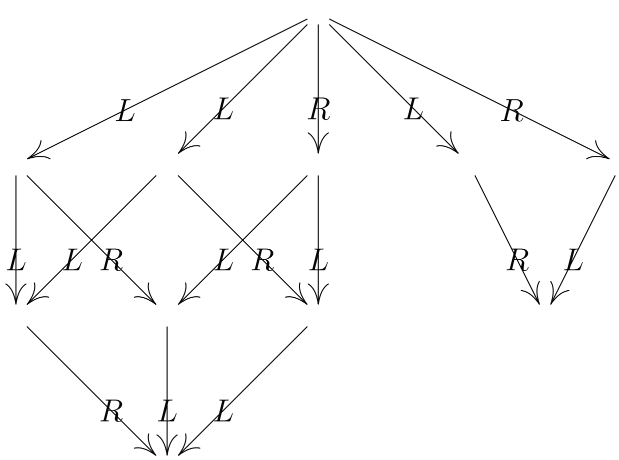
As in the above proof, the labelling of the game tree is needed to be able to identify the same positions when constructing the game graph from it. Given an unlabelled game graph, the structure of the game tree can be reconstructed using the same technique as in the proof, but the labelling cannot be reconstructed. Consider the two labelled game trees in Example 1.53. The two game graphs are given below, on the left for (R, B_1) and on the right for (R, B_2), and all edges will be labelled L.



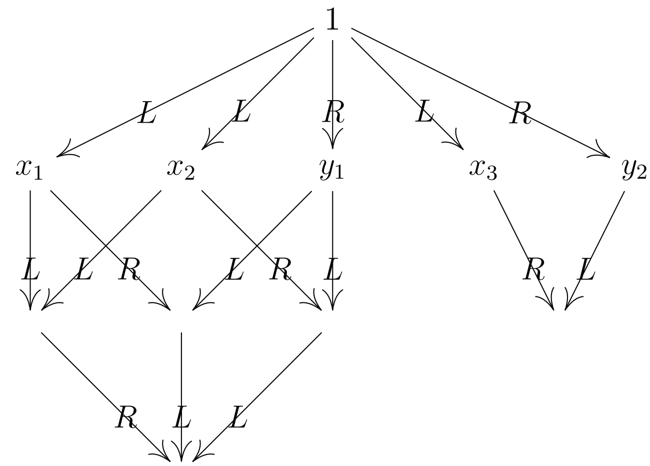
Notice in particular that the game graphs are not isomorphic due to the labellings of the trees being different, even though the trees are isomorphic. As with the legal complex, this indicates that the game graph is a better representative of a game than the (unlabelled) game tree.

For an SP-game, we can take advantage of its structure, namely that any order of moves is possible, and recover a (unique) labelling of its game graph. Before proving this result in general, we will demonstrate it using an example.

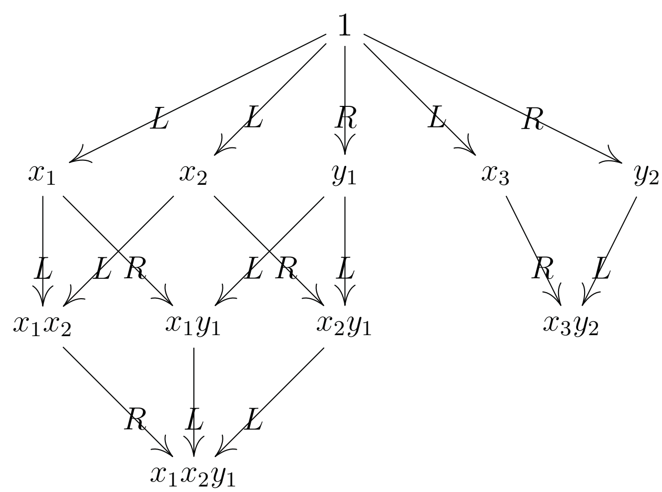
Example 3.5. Consider the game graph below whose positions are unlabelled and which we claim is the game graph of some SP-game (we show this is indeed the case in Proposition 3.11).



We begin by identifying the unique source, and label it 1 (representing the starting position). We then give a label to every basic positions (the successors of the source). Ones with the edge from the starting position labelled L will be labelled with an x_i, and ones with the edge labelled R with a y_j.



We then inductively label the rest of the vertices with the least common multiples of the monomials of the predecessors since every position is the union of previous positions. This gives us the labelled game graph.



Note that this labelling is not the only possible one — we labelled with monomials representing positions, not specific positions of a game — but any other labelling of this game graph, if coming from an SP-game, is equivalent. Thus such a labelling is unique.

This construction works for all game graphs of SP-games, so that we get the following general result:

Lemma 3.6. *Given the game graph of an SP-game with positions unlabelled, we can reconstruct the (unique) labelling and thus construct the game tree.*

Proof. To recover the labelling, we will take advantage of the order of moves not mattering in an SP-game.

Given the game graph, begin by identifying the unique root, which will be labelled as the starting position. Then label the basic positions which can be identified as being the followers of the root/starting position. For the positions at the end of edges labelled L, use variables $x_1, \ldots, x_m$, and of edges labelled R the variables $y_1, \ldots, y_n$.

Now inductively label every other vertex v with the monomial which is the least common multiple of the monomial labels of all predecessors of v.

Given the labelled game graph, we then get the unique game tree from Proposition 3.4. □

To be able to use Lemma 3.6, we first need to be able to identify when a game graph comes from an SP-game. To this end, we will investigate the structure of the game graph of an SP-game next.

3.1.1 Structure of the Game Graph

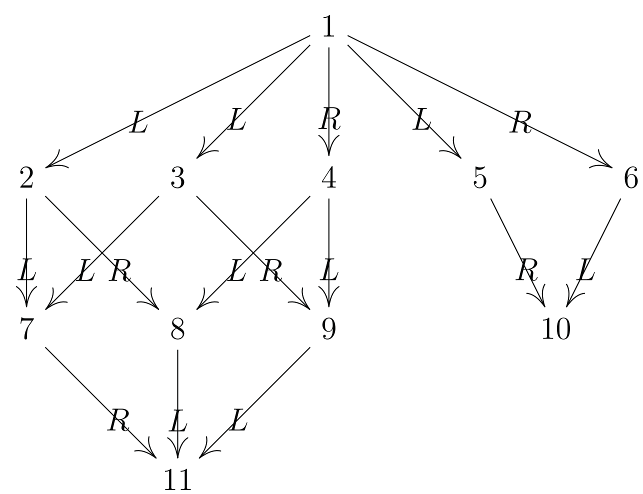
Consider the game graph $\mathcal{G}_G$ of an SP-game G. Consider positions P and P' of G where P' can be reached with a sequence of moves $M_1, M_2, \ldots, M_k$ from P, with each M_i in $\mathfrak{L}$ or $\mathfrak{R}$. Further, let v be the vertex in $\mathcal{G}_G$ for position P, and w the vertex for P'.

Since the order of moves in an SP-game does not matter, we have that each permutation of $M_1, \ldots, M_k$ is also a sequence of moves from P to P'. This gives exactly $k!$ sequences of moves from P to P', i.e. $k!$ paths between v and w in the game graph.

Further, in the game graph moves are only identified by whether they are Left moves or Right moves. Without loss of generality, assume that $M_1, \ldots, M_a$ are Left moves and $M_{a+1}, \ldots, M_k$ are Right moves (of which there are $b = k - a$). There are

$\binom{k}{a}$ ways to order the Ls and Rs. And for each of these, permuting $M_1, \ldots, M_a$ and $M_{a+1}, \ldots, M_k$, we have $a! \cdot b!$ different paths.

Example 3.7. The game graph given in Example 3.5, and repeated below, is known to be one of an SP-game.



Notice for example that between the starting position labelled 1 and the position labelled 11 there are a total of $3! = 6$ paths. There are two Ls and one R, and of each of the $\binom{3}{2}$ ways to order them there are $2! \cdot 1! = 2$ paths.

We can similarly observe the above properties between any two other positions between which there exists at least one path.

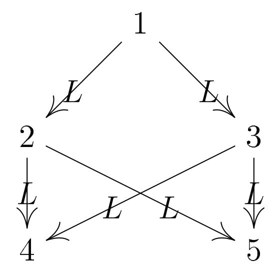
A directed graph is called **graded** if the vertex set can be partitioned into sets $V_0, \ldots, V_k$ such that every edge points from a vertex in V_i to a vertex in V_{i+1} for some i.

Observe that the game graph of an SP-game is graded, with the set V_i being those vertices whose corresponding positions break into i basic positions, i.e. can be reached in i moves, or equivalently the degree of the monomial representing the position is i.

Example 3.8. Consider again the game graph in Example 3.7. This graph is graded with vertex partitions $V_0 = \{1\}$ (the starting position), $V_1 = \{2, 3, 4, 5, 6\}$ (the basic positions), $V_2 = \{7, 8, 9, 10\}$ (positions reached in 2 moves), and $V_3 = \{11\}$ (the position reached in 3 moves).

Note that the above two properties about the number of paths between vertices and being graded are necessary conditions for a game graph to be of an SP-game, but not sufficient. The next example, due to Alex Fink, demonstrates this.

Example 3.9 (Alex Fink)**.** Consider the following game graph:



Although this game graph satisfies the properties described above for the number of paths between any two vertices and being graded, it is not the game graph of an SP-game. The vertices labelled 2 and 3 have two common successors (namely 4 and 5). In an SP-game this is not possible since a common successor of two positions would consist of the union of the basic positions, which is unique.

Thus for the game graph of an SP-game we additionally have that given any two vertices v and w in the same graded part V_i, there exists at most one successor in common, which corresponds to the unique position which is the union of the basic positions making up v and w.

Based on this discussion, we define the SP-property of a game graph as follows.

Definition 3.10. A game graph is said to satisfy the **SP-property** if it is graded and if, whenever there exists a path from a vertex v to a vertex w consisting of a edges belonging to $\mathfrak{L}$ and b belonging to $\mathfrak{R}$, there exist exactly $a! \cdot b!$ paths between v and w of each of the $\binom{a+b}{a}$ orderings of edges labelled L and R. Furthermore, any two vertices have at most one common successor.

Proposition 3.11. *Let $\mathcal{G}_G$ be the game graph of the combinatorial game G. Then G is an SP-game if and only if $\mathcal{G}_G$ satisfies the SP-property.*

Proof. Given G being an SP-game, we will show that the game graph satisfies the SP-property using that the order of the moves does not matter. Consider a vertex v and a vertex w in G such that there is a path from v to w. If we consider the a

Ls and b Rs as all different in this path (since they all represent different moves), then any order of these moves (i.e. edges) gives a path between v and w. Then considering these moves to be the same again there are $\binom{a+b}{a}$ different orders, and of each there are $a! \cdot b!$. Furthermore, as discussed above the game graph is graded with the grading coming from the number of basic positions, and two vertices have at most one common successor. Thus $\mathcal{G}_G$ statisfies the SP-property.

For the reverse, we show that there exists an SP-game G' with $\mathcal{G}_G$ as a game graph. Any other game with $\mathcal{G}_G$ as a game graph is then literally G' since the unlabelled game graph of an SP-game maps to exactly one game tree (Lemma 3.6) and thus G and G' are literally equal.

Begin by labelling the vertices adjacent to the root (that is its direct successors) as $\{x_1\}, \ldots, \{x_m\}$ (if the edge is labelled with L) and $\{y_1\}, \ldots, \{y_n\}$ (if the edge is labelled with R). Inductively, now label all other vertices with the union of the label of their direct predecessors. Due to the SP-property, each vertex at distance k to the root has in-degree k and its label has size k, and each label appears at most once. Let the labels of the vertices with out-degree 0 (those representing maximal/final positions) give the facets of a simplicial complex Δ and let G' be an SP-game such that the legal complex of G' is Δ. This definition implies that the vertices with out-degree 0 have the same distance to the root in both $\mathcal{G}_G$ and $\mathcal{G}_{G'}$. Since $\mathcal{G}_{G'}$ also satisfies the SP-property as we have shown above, it is isomorphic to $\mathcal{G}_G$ as a directed graph. Furthermore, the labelling of the edges will be identical as well since having the same labels for the vertices with out-degree 0 implies having the same number of Left and Right moves. □

3.2 The Game Poset

The positions in an SP-game can be given a partial order through sequences of moves between them. For more background on partially ordered sets (posets) and their correspondence to simplicial complexes see "Ordered Sets" by Schröder [54]. Recall in particular that for an element x we say that y is a **cover** of x if $x < y$ and for all z such that $x \leq z \leq y$ we have $z = x$ or $z = y$, and two posets are called **isomorphic** if there exists a bijection of the elements preserving the ordering.

Given the partial order of positions in an SP-game, we define the game poset. Note that this is a new structure not previously studied.

Definition 3.12. The **game poset $\mathcal{P}_G$** of an SP-game G is the poset where

- the elements are the legal positions of G;
- for two elements P_1 and P_2 we have $P_1 \leq P_2$ if there exists a sequence of legal moves starting at P_1 and ending at P_2; and
- given a cover P_2 of P_1 we say that P_2 is an L-cover (R-cover) of P_1 if the move from P_1 to P_2 is a Left (Right) move.

Note that the game poset is a poset where we additionally specify that each element is either an L-cover or R-cover of elements directly below. We will call such a poset an **$(\mathfrak{L}, \mathfrak{R})$-labelled poset**.

We define the game poset for SP-games only. For non-SP-games, when ordering positions by sequences of moves, some options might not be covers, which implies that a poset is not a good representation of such a game. For SP-games though this cannot happen, and the game poset is a useful tool which allows us to move from the game graph to the legal complex (see Propositions 3.14 and 3.18).

On the other hand, for each finite $(\mathfrak{L}, \mathfrak{R})$-labelled poset there exists a short game for which we can define the game poset without losing information about the game. Such a game for example is the following one: start at any minimal element. At any point, the Left options are the L-covers of the element and the Right options the R-covers. The game ends when a maximal element is reached.

In the Hasse diagram of an $(\mathfrak{L}, \mathfrak{R})$-labelled poset we will indicate L- and R-covers by labelling the edges between an element and its cover(s).

Example 3.13. The Hasse diagram of the game poset of SNORT on P_3 is given in Figure 3.2.

We can quite easily move between the game poset and the game graph:

Proposition 3.14. *Given an SP-game G, its game poset $\mathcal{P}_G$ and game graph $\mathcal{G}_G$ are in a one-to-one correspondence.*

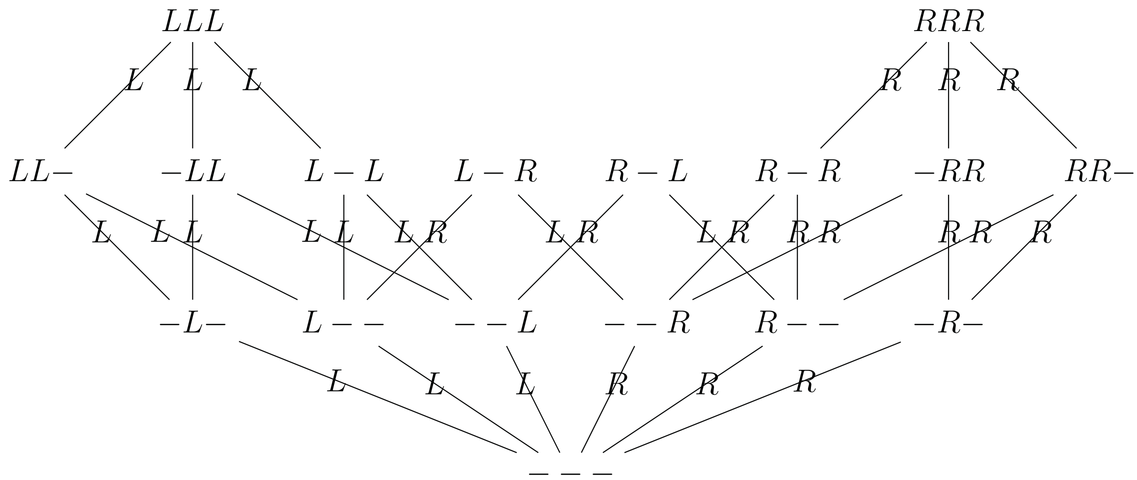


Figure 3.2: The Game Poset of Snort on P_3

Proof. Let A be the set of game posets of SP-games and B the set of game graphs of SP-games. We will give a map $f : A \to B$ which is a bijection preserving the game.

The map f will take the game poset $\mathcal{P}_G$ of a game G to the game graph $\mathcal{G}_G$ as follows: create a vertex for each element in the poset. Whenever x covers y in the poset, create an edge from the vertex representing y to the one for x. If x is an L-cover of y, then label the edge with L and similarly if it is an R-cover.

The function f is surjective. Given a game graph of an SP-game, the game poset mapping to it can be found by doing the above construction in reverse.

Furthermore, f is injective. Suppose that two game posets map to the same game graph. Then by construction their elements are the same, as are the covers of each element, and thus the two posets are isomorphic. □

We can further move between the legal complex and the game poset. One direction uses the face poset.

Definition 3.15. Given a simplicial complex Δ, the **face poset** $\boldsymbol{P(\Delta)}$ is the poset whose elements are the faces of Δ with $F_1 \leq F_2$ whenever $F_1 \subseteq F_2$.

The definition of a face poset can be generalized to $(\mathfrak{L}, \mathfrak{R})$-labelled simplicial complexes with the labelling preserved and giving an $(\mathfrak{L}, \mathfrak{R})$-labelled poset:

Definition 3.16. Given an $(\mathfrak{L}, \mathfrak{R})$-labelled simplicial complex Δ, the **face poset** $\boldsymbol{P(\Delta)}$ is the $(\mathfrak{L}, \mathfrak{R})$-labelled poset whose elements are the faces of Δ with $F_1 \leq F_2$

whenever $F_1 \subseteq F_2$. Furthermore, a cover F_2 of F_1 is an L-cover if the vertex in $F_2 \setminus F_1$ is in $\mathfrak{L}$, and correspondingly for R-covers.

Since the game poset has positions ordered by containment, the following result can be easily seen.

Lemma 3.17. *Given an SP-game (R, B), the game poset $\mathcal{P}_{R,B}$ is the face poset of the legal complex $\Delta_{R,B}$.*

As a consequence, we are able to move from the legal complex to the game graph and vice versa, which demonstrates that they are equivalent characterizations of a game:

Proposition 3.18. *Given an SP-game (R, B), the legal complex $\Delta_{R,B}$ and the game graph $\mathcal{G}_{R,B}$ are in a one-to-one correspondence.*

Proof. By Proposition 3.14 we have that the game graph $\mathcal{G}_{R,B}$ and the game poset $\mathcal{P}_{R,B}$ are in a one-to-one correspondence. Thus it remains to show that there is a bijective map from the game poset to the legal complex which preserves the game. By Lemma 3.17 the legal complex $\Delta_{R,B}$ can be mapped onto the game poset using the face poset construction.

The inverse, mapping the game poset $\mathcal{P}_{R,B}$ onto a simplicial complex Δ which is the legal complex, is as follows: Let the vertices of Δ be the covers of the bottom element in $\mathcal{P}_{R,B}$, i.e. those corresponding to the basic positions, with an $(\mathfrak{L}, \mathfrak{R})$-labelling depending on if the cover is an L-cover or an R-cover. Now inductively add faces to Δ by taking the union of faces corresponding to elements in the poset with the same cover. The simplicial complex Δ is then the legal complex of (R, B). $\square$

In this thesis, we will not be using game posets beyond this chapter. Due to the easy correspondence between the game graph and the game poset in the case of SP-games, the structure of the poset is interesting in itself though. In the remainder of this section we will make some further observation about the game poset given the below concepts from poset theory. Since the game poset is the face poset of the legal complex, results independent of the $(\mathfrak{L}, \mathfrak{R})$-labelling are likely known. We will give proofs from a game theoretic point of view.

Definition 3.19. A **bottom element** of a poset P is an element b such that $b \leq x$ for all $x \in P$. A **lower bound** of two elements $a, b \in P$ is an element c such that $c \leq a$ and $c \leq b$. The **greatest lower bound** (or **meet**) $\boldsymbol{a \wedge b}$ of two elements $a, b \in P$ is a lower bound c of a, b such that for every other lower bound d of a, b we have $d \leq c$. A **meet semilattice** is a poset in which any two elements have a meet.

A top element and the join (least upper bound) $a \vee b$ are similarly defined.

In the game poset, the bottom element is the starting position, which for SP-games is the empty position.

Ignoring the $(\mathfrak{L}, \mathfrak{R})$-labelling of a game poset, we have the following.

Lemma 3.20. *Every game poset of an SP-game is a meet semilattice.*

Proof. The meet of two elements P_1 and P_2 is the largest position contained in both P_1 and P_2 (in the worst case scenario, this is the empty position). Equivalently, it is the position which is the greatest common divisor of the monomials representing P_1 and P_2 or the intersection of the sets of basic positions making up each of the positions. □

Definition 3.21. A poset is called a **lattice** if any two elements have a meet and a join. A **distributive lattice** is a lattice in which $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. A lattice is called **complemented** if it has a bottom and top element and for any element a there exists an element b such that $a \vee b$ is the top element and $a \wedge b$ is the bottom element. A **Boolean lattice** is a complemented, distributive lattice.

Note that the join of two positions, if it exists, is the smallest position containing both. Equivalently, it is the position which is the least common multiple of the monomials representing the positions or the union of the sets of basic positions making up the positions. A game poset of an SP-game has a top element if and only if it has a unique maximal legal position, or equivalently the legal complex is a simplex. We then have the following:

Lemma 3.22. *If an SP-game G has a unique maximal legal position, then its game poset is a Boolean lattice.*

Proof. Since G has a maximal legal position, the join of any two elements in the game poset exists. Thus $\mathcal{P}_G$ is a lattice. Further, since the meet and join correspond to intersection and union of sets in this case, it also is a distributive lattice.

Finally, we will show that $\mathcal{P}_G$ is complemented. Let V be the set of basic positions in G. Now let a be some element of $\mathcal{P}_G$, and let V_1 be the set of basic positions which make up a. Let b be the element of $\mathcal{P}_G$ whose basic positions are $V \setminus V_1$. Then the join of a and b is the position with basic positions $V_1 \cup (V \setminus V_1) = V$, i.e. the top element. And the meet of a and b is the position with basic positions $V_1 \cap (V \setminus V_1) = \emptyset$, i.e. the bottom element. Thus $\mathcal{P}_G$ is also complemented, and thus a Boolean lattice. □

In general, the subposet between any two points in the game poset of an SP-game forms a Boolean lattice.

3.3 Further Work

Describing the structure of the game tree might be useful in learning more about the class of SP-games, especially which values are possible. Although we are able to describe this through the structure of the game graph and the one-to-one correspondence between the two when labelled, it would be useful to have a more direct and succinct description.

Although in this thesis we have only used the game poset to show the correspondence between game tree and legal complex, the poset gives another complete characterization of the game beyond the game tree, game graph, and legal complex. Another avenue of research would be to study the connection between combinatorial games and poset theory via SP-games, or a larger class of games for which the game poset is well-defined, for example weak placement games.

Chapter 4

Game Values under Normal Play

Recall that the value of a game G is the equivalence class it belongs to.

A problem of interest in combinatorial game theory is the range of values that occur in a game. One of the most celebrated results in combinatorial game theory is the Sprague-Grundy Theorem (Fact 6.1) which in essence states that the impartial game NIM takes on all game values possible for impartial games (see Section 6.1 for more information). Motivated by this is the search for a nontrivial short game which takes on all games values possible, even for partizan games.

A game taking on all possible values is called **universal**, and Carvalho and Santos [15] recently constructed the first known nontrivial universal game. This is not an SP-game though, so our question is what values SP-games can take on under normal play.

This problem has received attention for some specific SP-games. The only complete result for partizan SP-games is that COL only takes on numbers and numbers plus star as shown by Berlekamp, Conway, and Guy in 1982 [5] (also found in [6, p.47]). Some partial results are known for DOMINEERING (see for example Kim 1996 [38] or Uiterwijk and Barton 2015 [61]) and for SNORT (see Berlekamp et al. [6, pp.181–183]).

Since SP-games are much easier to understand than many other combinatorial games, if we are able to show that SP-games take on all possible game values, the class of SP-games would provide an excellent new tool for studying combinatorial games. But even if SP-games are not universal, being able to restrict the possible values would simplify game value calculations.

Knowing that each simplicial complex is the legal complex of some SP-game will be extremely useful in the exploration of the universality of SP-games. Although we are not able to determine their universality either way, we are able to show that many interesting values are possible.

In the remainder of this chapter, we will assume normal play winning condition.

4.1 Introduction to Game Values

In this section, we will introduce a few more concepts from combinatorial game theory needed specifically for the study of values, as well as give useful results. We will also discuss some specifics regarding SP-games.

Remark 4.1. In SP-games, each move corresponds to a basic position, thus making a move is essentially the same as claiming a vertex in the legal complex. Furthermore, since a position is only legal if all corresponding vertices form a face, only those faces containing it are relevant from now on. This means that if we consider a game with legal complex $\Delta = \langle \{v\} \cup F_1, \{v\} \cup F_2, \ldots, \{v\} \cup F_k, F_{k+1}, \ldots, F_j \rangle$ where $F_{k+1}, \ldots, F_j$ do not contain v, then making the move corresponding to the vertex v is to a position equivalent to the game with legal complex $\Delta' = \langle F_1, F_2, \ldots, F_k \rangle$, which is the link of v in Δ. From here on, we will often say that a move is to Δ' when we mean the move equivalent to claiming the vertex v.

Remark 4.2. Since the negative of a game switches Left and Right options, the legal complex of the negative of an SP-game is obtained by switching the vertices belonging to $\mathfrak{L}$ and $\mathfrak{R}$. Due to this, we will in this chapter not demonstrate the existence of negative games, but rather assume their existence once the existence of their positive counterpart has been shown.

Remark 4.3. As shown in Theorem 1.55, given any two SP-games their disjunctive sum has as its legal complex the join of the legal complexes of the individual games. Thus if we show that two game values are taken on by SP-games, their disjunctive sum is also taken on.

We define 0 to be the game $\{\emptyset \mid \emptyset\}$, so the game in which neither player has any available moves. Adding 0 to any other games does not change it. Thus the following result is a consequence of Fact 1.17, and we will use it throughout the thesis to demonstrate when a game is 0.

Fact 4.4 ([57, Theorem II.1.12])**.** *For all games G, $o(G) = \mathscr{P}$ if and only if $G = 0$.*

When either of the set of options is empty, we will often leave that side of the braces empty. Thus we can also write $0 = \{\,|\,\}$.

There are two simplifications we can use on games while still remaining in the same equivalence class. The first is to remove so-called dominated options, i.e. the ones where another option is clearly preferred.

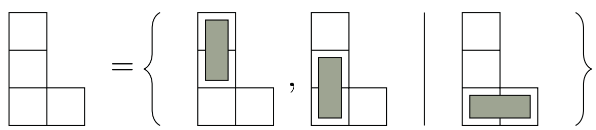
Definition 4.5. Given a game G, a Left option G^{L_1} is **dominated** by the Left option G^{L_2} if $G^{L_2} \geq G^{L_1}$. Similarly, a Right option G^{R_1} is dominated by the Right option G^{R_2} if $G^{R_2} \leq G^{R_1}$.

Fact 4.6 ([57, Theorem II.2.4])**.** *If for a given game G the Left option G^{L_1} is dominated by some other Left option, then*

$$G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\} = \{G^{\mathcal{L}} \setminus G^{L_1} \mid G^{\mathcal{R}}\}.$$

Similarly for Right dominated options.

For example, when playing DOMINEERING on an L-shaped board, the game is



The Left option in the upper two squares is a Right win (only Right has a move), while the Left option in the lower two squares, having no remaining moves, is a second-player win. Thus the latter option is greater than the former, and the option in the upper two squares is dominated and can thus be removed without changing the game value. This is also apparent as Left would never make this move which gives her opponent an advantage.

The second simplification is to replace reversible options, which are in some sense those options which have a guaranteed response.

Definition 4.7. Given a game G, a Left option G^{L_1} is **reversible** through $G^{L_1R_1}$ if $G^{L_1R_1} \leq G$. Similarly, a Right option G^{R_1} is reversible through $G^{R_1L_1}$ if $G^{R_1L_1} \geq G$.

Fact 4.8 ([57, Theorem II.2.5])**.** *If for a given game G the Left option G^{L_1} is reversible through $G^{L_1R_1}$, then*

$$G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\} = \{G^{\mathcal{L}} \setminus G^{L_1}, G^{L_1R_1\mathcal{L}} \mid G^{\mathcal{R}}\}.$$

Similarly for Right options that are reversible.

Note, this is still true if $G^{L_1 R_1 \mathcal{L}}$ is empty.

Reversibility is unfortunately not as easy to see as domination as it requires comparing options of an option with the game itself. There will be a few cases later on where reversibility is used as a simplification. In those cases, we have used the combinatorial game theory program CGSuite [56] to find the reversible options.

We say that a game is in **canonical form** if it has no dominated or reversible options. A game in canonical form in some sense is the simplest game in its equivalence class. Even more, there is only one game in canonical form in each equivalence class, so that we can talk about the canonical form of a game:

Fact 4.9 ([57, Theorems II.2.7 and II.2.9])**.** *For each game G there exists exactly one game H in canonical form such that $G = H$.*

For example, the game of DOMINEERING on the L-shaped board above, after removing the dominated option, has the canonical form $\{0 \mid \{0 \mid \}\}$. There are no reversible options in this case.

Note that the canonical form of an SP-game is not necessarily an SP-game itself. We give an example demonstrating this in Section 4.8.

When talking about game values, we will often represent a game value by its unique representative that is in canonical form.

As before, the game value with canonical form $\{\,\mid\,\}$ is called 0 as neither player has a move. The game $\{0 \mid \}$ is called 1 as Left has a guaranteed move, while $\{\mid 0\}$ is called -1. The game $\{1 \mid \}$ is then called 2 (two guaranteed moves for Left), and we can continue to recursively define integers.

The game $\{0 \mid 1\}$ is a slight advantage to Left, but not quite as much as 1 since Right actually does have a move. It turns out though that $\{0 \mid 1\} + \{0 \mid 1\} = 1$. Thus we call this game $\frac{1}{2}$. Recursively, we then set $\{0 \mid \frac{1}{2^{n-1}}\} = \frac{1}{2^n}$.

Addition of the integers and fractions as above turn out to work as in the rationals. For example $1 + 2 = \{0 \mid \} + \{1 \mid \} = \{2 \mid \} = 3$.

Remark 4.10. Although all fractions can be found in combinatorial games, a short game can only take on a fraction whose denominator is a power of 2 (a dyadic rational) (see [57, Corollary II.3.11]). The set of dyadic rationals is indicated by $\mathbb{D}$.

We will formally define the integers and dyadic rationals below, as well as several other game values that often appear in combinatorial game theory and are therefore given shorthand notation.

Definition 4.11. We define the following values by their canonical forms:

- **Integers**: For zero we have $0 = \{\,|\,\}$ and the other integers are recursively defined as $n = \{n-1 \mid \}$ for $n > 0$ and $n = \{ \mid n+1\}$ for $n < 0$.
- **Fractions**: Unit fractions are recursively defined as $\frac{1}{2^n} = \left\{0 \,\middle|\, \frac{1}{2^{n-1}}\right\}$. Other fractions are sums of these games.
- **Numbers**: A game whose value is either an integer or a fraction is called a number.
- **Switches**: A game with canonical form $\{a \mid b\}$, where $a \geq b$ are numbers, is called a switch and is written $\dfrac{a+b}{2} \pm \dfrac{a-b}{2}$.
- **Nimbers**: Nimbers are recursively defined as $*1 = \{0 \mid 0\}$ (shorthand $*$) and $*n = \{0, *, *2, \ldots, *(n-1) \mid 0, *, *2, \ldots, *(n-1)\}$. Note that for recursive purposes we often also set $*0 = 0$.
- **Up** and **down**: We have up as $\uparrow = \{0 \mid *\}$ and down as $\downarrow = -\uparrow$.
- **Tiny** and **miny**: For $G \geq 0$ a game, we have tiny-G as $+_G = \{0 \mid \{0 \mid -G\}\}$ and miny-G as $-_G = -(+_G)$.

Finally, note that disjunctive sums of numbers, nimbers, ups, and tinies are often shortened and the '+' omitted. To avoid confusion between the sum $2 + *$ and the nimber $*2$ for example, we will observe the order number, then up (or down), then nimbers and tinies. For example $2 + * + \frac{1}{2} + \downarrow$ will be written as $2\frac{1}{2} \downarrow *$. If we are writing a product, such as $\uparrow + \uparrow + \uparrow$, we will use a centre dot, i.e. $3 \cdot \uparrow$.

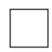
We will use the following fact at times when showing that a game is a number.

Fact 4.12 ([57, Proposition II.3.12]). *If $G = \{a \mid b\}$ where a and b are numbers and $a < b$, then G is a number, and we have*

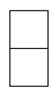
$$G = \begin{cases} n & \text{if } a - b > 1 \text{ and } n \text{ is the integer closest to zero such that } a < n < b; \\ \frac{p}{2^q} & \text{if } a - b \leq 1 \text{ and } q \text{ is the smallest positive integer such that} \\ & \qquad \text{there exists a } p \text{ such that } a < \frac{p}{2^q} < b. \end{cases}$$

Example 4.13. As examples for several of these values, we will consider DOMINEERING positions under normal play, including the ones in Figure 1.2.

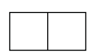
(a)

 $= \{\ \mid\ \} = 0$

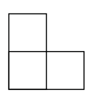
(b)

 $= \{0 \mid\ \} = 1$

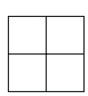
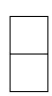
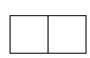
and to get the negative we rotate the board:

 $= \{\ \mid 0\} = -1$

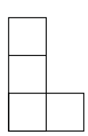
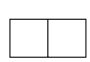
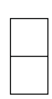
(c)

 $= \{0 \mid 0\} = *$

(d)

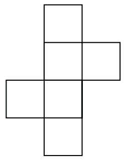
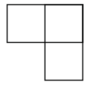
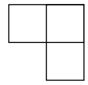
 $= \{$  $\mid$  $\} = \{1 \mid -1\} = \pm 1$

(e)

 $= \{$  $, 0 \mid$  $\} = \{0 \mid 1\} = \frac{1}{2}$

Note that the Left option to -1 is dominated by the option to 0, so can be ignored.

(f)

 $= \{$  $, 0 \mid$  $\} = \{*, 0 \mid *\} = \{0 \mid *\} = \uparrow$

Note that we have only listed one of each of Left and Right's options to $*$, and that Left's option to $*$ is reversible, thus gets replaced with the Left options of 0, the empty set.

Definition 4.14. The **value set** $\mathbb{V}$ of SP-games is the set of all possible values SP-games can exhibit under normal play, i.e. all equivalence classes that contain an SP-game. It is the set $\mathbb{G}$ restricted to SP-games only.

The question of interest is what the set $\mathbb{V}$ looks like. In the remainder, smaller examples have been calculated by hand. For larger examples, we have used the computer algebra program Macaulay2 [29] to construct the game in bracket and slash notation from its legal complex (for the code, see Appendix B), which has then been put into the combinatorial game theory program CGSuite [56] to obtain the canonical form.

We will show that all numbers, all nimbers, many switches, and many tinies are possible game values of SP-games, as well as that all games with small game tree are equal to some SP-game. The universality of SP-games still remains open though.

We will begin by looking at small dimensional legal complexes.

4.2 Small Birthdays

In this section we will consider game values whose canonical forms have small game trees. For this, we will take advantage of the recursive construction of games:

Definition 4.15. The set of all short games $\widetilde{\mathbb{G}}$ can be defined as

$$\widetilde{\mathbb{G}} = \bigcup_{n \geq 0} \widetilde{\mathbb{G}}_n,$$

where $\widetilde{\mathbb{G}}_0 = \{0\}$ and for $n \geq 0$

$$\widetilde{\mathbb{G}}_{n+1} = \{\{A \mid B\} : A, B \subseteq \widetilde{\mathbb{G}}_n\}.$$

If we let $\mathbb{G}_n$ be the set of values of elements of $\widetilde{\mathbb{G}}_n$, then the **birthday $\mathrm{b}(G)$** of a game G is the least n such that the game value of G is in $\mathbb{G}_n$. Similarly, the **formal birthday $\widetilde{\mathrm{b}}(G)$** is the least n such that the literal form of G is in $\widetilde{\mathbb{G}}_n$.

Note in particular that the birthday of a game is related to its game value, while the formal birthday relates to the first appearance of its literal form in the recursive construction of games.

The height of a game tree is the maximum number of moves from the starting position to an ending position. The elements in $\widetilde{\mathbb{G}}_n$ are precisely those games whose game trees have height n. We thus have the following fact.

Fact 4.16 ([57, p.61])**.** *Given a game G, its formal birthday is equal to the height of its game tree.*

Given this and that the height of the game tree of an SP-game (R, B) is equal to the size of the largest facet in the legal complex, we have the following proposition.

Proposition 4.17. *Given an SP-game (R, B) we have*

$$\widetilde{\mathrm{b}}(R, B) = \dim(\Delta_{R,B}) + 1.$$

To illustrate the difference between the birthday and formal birthday, we will give a short example.

Example 4.18. Consider Domineering played on a 2×5 grid. The maximal legal positions contain up to 5 pieces. Thus the legal complex has dimension 4, giving that the formal birthday is 5. This game has canonical form $\frac{1}{2}$ though, which is contained in $\mathbb{G}_2$, giving a birthday of 2.

Motivated by the relationship between the formal birthday of an SP-game and the dimension of its legal complex we define the following sets for SP-games:

Definition 4.19. We set $\mathbb{V}_n$ to be the set of game values of SP-games whose legal complexes have dimension $n - 1$.

Thus if

$$\widetilde{\mathbb{V}}_n = \{(R, B) \mid (R, B) \text{ is an SP-game and } \dim(\Delta_{R,B}) = n - 1\},$$

where the elements of $\widetilde{\mathbb{V}}_n$ again are in literal forms, then $\mathbb{V}_n$ is the set of values in $\widetilde{\mathbb{V}}_n$.

Note that with the above notation we have

$$\mathbb{V} = \bigcup_{n\geq 0} \mathbb{V}_n,$$

as well as $\mathbb{V}_n \subseteq \mathbb{G}_n$.

When studying the structure of $\mathbb{V}$, a natural start is to consider whether $\mathbb{V}\cap\mathbb{G}_n = \mathbb{G}_n$ for small n. We will do so for $n = 0, 1, 2$. We will in addition also consider $\mathbb{G}_n \setminus \mathbb{V}_n$.

Fact 4.20 ([57, Section III.1])**.** *The sets $\mathbb{G}_n$ for $n = 0, 1, 2$ are*

$$\begin{aligned}
\mathbb{G}_0 &= \{0\} \\
\mathbb{G}_1 &= \{0, *, 1, -1\} \\
\mathbb{G}_2 &= \Big\{0, *, *2, \pm 1, \uparrow, \downarrow, \uparrow *, \downarrow *, \{1 \mid 0, *\}, \{0, * \mid -1\}, \frac{1}{2}, -\frac{1}{2}, \{1 \mid *\}, \{* \mid -1\}, \\
&\qquad \frac{1}{2} \pm \frac{1}{2}, -\frac{1}{2} \pm \frac{1}{2}, 1, -1, 1*, -1*, 2, -2\Big\}
\end{aligned}$$

4.2.1 Formal Birthday 0

Consider an SP-game (R, B) with $\widetilde{\mathrm{b}}(R, B) = 0$. Then the legal complex has dimension -1, i.e. $\Delta_{R,B} = \emptyset$. Thus in (R, B) neither Left nor Right have moves, so that $(R, B) = \{\,\mid\,\} = 0$, which gives $\mathbb{V}_0 = \{0\}$. Thus $\mathbb{V} \cap \mathbb{G}_0 = \mathbb{V}_0 \cap \mathbb{G}_0 = \mathbb{G}_0$.

4.2.2 Formal Birthday 1

Now consider an SP-game (R, B) with formal birthday 1, so that the legal complex $\Delta_{R,B}$ has dimension 0, i.e. only consists of isolated vertices.

- If all vertices belong to $\mathfrak{L}$, then Right has no moves while Left can move to the empty game. Thus $(R, B) = \{0 \mid\,\} = 1$.

- Similarly, if all vertices belong to $\mathfrak{R}$, then $(R, B) = \{\,\mid 0\} = -1$.

- If both $\mathfrak{L}$ and $\mathfrak{R}$ are non-empty, then both Left and Right have moves to the empty game. Thus $(R, B) = \{0 \mid 0\} = *$.

Thus $\mathbb{V}_1 = \{*, 1, -1\}$, giving $\mathbb{G}_1 \setminus \mathbb{V}_1 = \{0\}$, but $\mathbb{V} \cap \mathbb{G}_1 = \mathbb{G}_1$.

4.2.3 Formal Birthday 2

Now consider an SP-game (R, B) whose legal complex has dimension 1, i.e. it is a graph. Thus (R, B) is born by day 2. We will show below that $\mathbb{G}_2 \setminus \mathbb{V}_2 = \{1, -1\}$. Note that as previously mentioned as soon as we have shown a positive value exists, we assume to have shown the existence of the negative as well (through switching the bipartition).

- If all vertices belong to $\mathfrak{L}$, then Left can move to a single vertex belonging to $\mathfrak{L}$, i.e. to the game 1. Thus $(R, B) = \{1 \mid \} = 2$.
- If $\Delta_{R,B} = \langle\{x_1, y_1\}\rangle$, then Left can move to $\langle\{y_1\}\rangle$, i.e. -1. Similarly for Right, thus $(R, B) = \{-1 \mid 1\} = 0$.
- If $\Delta_{R,B} = \langle\{x_1, x_2\}, \{x_1, y_1\}\rangle$, then $(R, B) = \{1, * \mid 1\}$. The Left option to $*$ is reversible and gets replaced with the empty set, thus $(R, B) = \{1 \mid 1\} = 1*$.
- If $\Delta_{R,B} = \langle\{x_1, x_2\}, \{y_1, y_2\}\rangle$, then $(R, B) = \{1 \mid -1\} = \pm 1$.
- If $\Delta_{R,B} = \langle\{x_1, x_2\}, \{x_2, y_1\}, \{y_1, y_2\}, \{y_2, x_3\}\rangle$, then $(R, B) = \{1, *, -1 \mid *\} = \{1 \mid *\}$.
- If $\Delta_{R,B} = \langle\{x_1, x_2\}, \{y_1\}\rangle$, then $(R, B) = \{1 \mid 0\} = \dfrac{1}{2} \pm \dfrac{1}{2}$.
- If $\Delta_{R,B} = \langle\{x_1, y_1\}, \{x_2\}\rangle$, then $(R, B) = \{-1, 0 \mid 1\} = \{0 \mid 1\} = \dfrac{1}{2}$.
- If $\Delta_{R,B} = \langle\{x_1, y_1\}, \{x_2\}, \{y_2\}\rangle$, then $(R, B) = \{-1, 0 \mid 1, 0\} = \{0 \mid 0\} = *$.
- If $\Delta_{R,B} = \langle\{x_1, y_1\}, \{y_1, y_2\}, \{y_2, x_2\}, \{x_2, x_3\}, \{x_3, y_3\}, \{x_4\}, \{y_4\}\rangle$, then we have $(R, B) = \{-1, *, 0 \mid *, 1, 0\} = \{0, * \mid 0, *\} = *2$.
- If $\Delta_{R,B} = \langle\{x_1, y_1\}, \{y_1, y_2\}, \{y_2, x_2\}, \{x_2, x_3\}, \{x_3, y_3\}, \{x_4\}\rangle$, then $(R, B) = \{-1, *, 0 \mid *, 1\} = \{0 \mid *\} = \uparrow$.
- If $\Delta_{R,B} = \langle\{x_1, x_2\}, \{x_2, y_1\}, \{y_1, y_2\}, \{y_2, x_3\}, \{y_3\}\rangle$, then $(R, B) = \{1, *, -1 \mid *, 0\} = \{1 \mid 0, *\}$.
- If $\Delta_{R,B} = \langle\{x_1, y_1\}, \{x_1, x_2\}, \{x_2, y_1\}, \{y_2\}, \{x_3\}\rangle$, then $(R, B) = \{*, 0 \mid 1, 0\} = \{0, * \mid 0\} = \uparrow *$.

The values 1 and -1 are not possible if the legal complex has dimension 1 (see Proposition 4.23). Thus we get all values born by day 2, except 1 and -1 (which appeared in dimension 0 already though), implying $\mathbb{V} \cap \mathbb{G}_2 = \mathbb{G}_2$.

Although it seems reasonable to next look at whether all values of other birthdays are possible, the size of $\mathbb{G}_3$ alone is 1474 [57]. We will thus turn to more general existence results independent of the birthday.

4.3 Integers

We will begin by showing that all positive integers (and thus also the negatives) are possible values of SP-games.

Proposition 4.21. *Let (R, B) be an SP-game with legal complex the simplex $\Delta_{R,B} = \langle\{x_1, \ldots, x_n\}\rangle$ with $n \geq 0$. Then $(R, B) = n$.*

Proof. We will prove this by induction on n.

If $n = 0$, then $\Delta_{R,B} = \langle\emptyset\rangle$. We have shown previously that $(R, B) = 0$ in this case.

Now assume without loss of generality that the SP-game with legal complex $\langle\{x_1, \ldots, x_{n-1}\}\rangle$ has value $n - 1$.

If $\Delta_{R,B} = \langle\{x_1, \ldots, x_n\}\rangle$, then Right has no moves, while Left, without loss of generality, can move to $\langle\{x_1, \ldots, x_{n-1}\}\rangle$. By the induction hypothesis, we then have $(R, B) = \{n - 1 \mid \} = n$. □

From this and our knowledge about disjunctive sums, we get an immediate corollary on the value of a game whose legal complex is a simplex.

Corollary 4.22. *Let (R, B) be an SP-game such that $\Delta_{R,B}$ is the simplex*

$$\langle\{x_1, \ldots, x_m, y_1, \ldots, y_n\}\rangle.$$

Then (R, B) has value $m - n$.

Proof. We can write $\Delta_{R,B}$ as a join:

$$\begin{aligned}\Delta_{R,B} &= \langle\{x_1, \ldots, x_m, y_1, \ldots, y_n\}\rangle \\ &= \langle\{x_1, \ldots, x_m\}\rangle * \langle\{y_1, \ldots, y_n\}\rangle.\end{aligned}$$

If we let (R', B') and (R'', B'') be SP-games such that $\Delta_{R',B'} = \langle\{x_1, \ldots, x_m\}\rangle$ and $\Delta_{R'',B''} = \langle\{y_1, \ldots, y_n\}\rangle$, then by Theorem 1.55 we have $(R, B) = (R', B') + (R'', B'')$. By our previous result we further have $(R', B') = m$ and $(R'', B'') = -n$, so that $(R, B) = m - n$ as desired. □

With these two results on the existence of integers, we now turn to looking at the existence of integers in specific dimensions, thus checking if $n \in \mathbb{V}_k$ where we let k vary.

Note that since $\widetilde{\mathrm{b}}(R, B) = \dim \Delta_{R,B} + 1$, and the integer n has birthday $|n|$, we cannot get n as a value at dimension less than $|n| - 1$.

4.3.1 Integer n in Dimension $n - 1$

We have already shown that if $\Delta_{R,B} = \langle\{x_1, x_2, \ldots, x_n\}\rangle$, then $(R, B) = n$.

4.3.2 Integer n in Dimension n

In dimension n the integer n is not possible.

Proposition 4.23. *An SP-game (R, B) with $\dim \Delta_{R,B} = n$ cannot take on the value n (or $-n$) under normal play.*

Proof. We will show by induction that the value n is not possible. That $-n$ is not possible follows immediately since it is the negative, i.e. could be achieved by switching the bipartition of vertices.

Base case: As shown in Section 4.2, we cannot get 0 with dimension 0.

Induction hypothesis: Assume that an SP-game with legal complex of dimension $n - 1$ cannot take on value $n - 1$.

Induction step: Assume that (R, B) has the value n, i.e. $(R, B) = \{n - 1| \}$. Since (R, B) is born by day $n + 1$ (since $\dim \Delta = n$), we have that all Left options of (R, B) have to be born by day n. Thus the Left option to $n - 1$ in the canonical form of (R, B) cannot have come through reversing (reversing an option born by day k results in options born by day $k - 2$). Thus in Δ there exists a facet F of dimension n such that $F = \{x_i\} \cup F'$ where the game equivalent to $\langle F' \rangle$ has value $n - 1$. But $\langle F' \rangle$ has dimension $n - 1$, a contradiction to the induction hypothesis. □

4.3.3 Integer n in Dimension $n+1$

From Corollary 4.22, we immediately have the following.

Proposition 4.24. *If $\Delta_{R,B} = \langle\{x_1, x_2, \ldots, x_{n+1}, y_1\}\rangle$, then (R, B) under normal play has the value n.*

4.3.4 Integer n in Dimension $\geq n+1$

Proposition 4.25. *Let*

$$U = \big\{\{x_1, x_2, \ldots, x_{k+1}\}\big\} \cup \big\{\{x_{i_0}, x_{i_1}, \ldots, x_{i_n}, y\} \mid 1 \leq i_0, \ldots, i_n \leq k+1\big\}$$

where $k \geq n+1$ and let the facets of $\Delta_{R,B}$ be the elements of U. Then $\Delta_{R,B}$ has dimension k and (R, B) has value n.

Proof. Since $k \geq n+1$, we have that $k+1 \geq n+1+1$, i.e. there are at least as many elements in $\{x_1, \ldots, x_{k+1}\}$ as in $\{x_{i_0}, x_{i_1}, \ldots, x_{i_n}, y\}$. Thus $\{x_1, \ldots, x_{k+1}\}$ is a facet of maximal dimension, which shows that $\dim(\Delta_{R,B}) = k$.

In (R, B), Right's only move is to $\langle\{x_{i_0}, x_{i_1}, \ldots, x_{i_n}\} \mid 1 \leq i_0, \ldots, i_n \leq k+1\rangle$, which has value $n+1$. Left's moves are symmetric, so assume without loss of generality she moves in x_{k+1}. This is then to Δ' which has the facets $\{x_1, x_2, \ldots, x_k\}$ and $\{x_{i_0}, x_{i_1}, \ldots, x_{i_{n-1}}, y\}$ where $1 \leq i_0, \ldots, i_n \leq k$. By induction, it can now be easily seen that (R, B) has value

$$(R, B) = \{\ldots\{\{k-1-n|0\}|1\}\ldots|n+1\}.$$

To prove that (R, B) has value n, we will use induction on n.

Base case: If $n = 0$, then $\Delta_{R,B} = \langle\{x_1, \ldots, x_{k+1}\}, \{x_1, y\}, \ldots, \{x_{k+1}, y\}\rangle$. Thus $(R, B) = \{\{k-1|0\}|1\}$ which is 0 for all $k \geq 1$ since it is a second player win.

Induction hypothesis: Assume that for a fixed j and for all $k \geq j+1$, we have $\{\ldots\{\{k-1-j|0\}|1\}\ldots|j+1\} = j$.

Induction step: Suppose that $(R, B) = \{\{\ldots\{\{k-1-(j+1)|0\}|1\}\ldots|j+1\}|(j+1)+1\}$ for $k \geq j+2$. By the induction hypothesis we have that $\{\ldots\{\{(k-1)-1-j|0\}|1\}\ldots|j+1\} = j$ since $k-1 \geq j+1$. Thus $(R, B) = \{j|j+2\} = j+1$. $\square$

Example 4.26. We will construct a simplicial complex Δ of dimension 2 that is the legal complex of an SP-game (R, B) with value 1. In this case $U = \{\{x_1, x_2, x_3\}\} \cup \{\{x_1, x_2, y\}, \{x_1, x_3, y\}, \{x_2, x_3, y\}\}$ and the facets of Δ are the sets of U.

Right's only move is on y, after which Left has two remaining moves, i.e. this option has value 2.

All of Left's options are symmetric, so we will assume without loss of generality that she moves on x_1.

- After Right moved on y, Left still has one move, thus this Right option has value 1.

- After Left moved on x_2, both Left and Right have options to 0. Thus this Left option has value $\{0 \mid 0\} = *$.

Thus the Left option of x_1 has value $\{* \mid 1\}$.

In total, (R, B) has value $\{\{* \mid 1\} \mid 2\} = 1$.

To summarize, the integer n is a possible value of an SP-game if the legal complex has dimension $n-1$ or greater than n, but not dimensions n or less than $n-1$. In particular, this also shows that $n - 1 \in \mathbb{G}_n \setminus \mathbb{V}_n$ while all other integers are contained in $\mathbb{V}_n$ as we have previously seen for small n.

4.4 Fractions

The following construction shows that all fractions of the form $\frac{1}{2^n}$ are possible values of SP-games. This construction is also minimal in the sense that the dimension of the legal complex is one lower than the birthday of the fraction. All other fractions can be obtained through disjunctive sums.

Theorem 4.27. *Let $S_1, S_2, \ldots, S_{2^n}$ be the subsets of $\{y_1, y_2, \ldots, y_n\}$. Let*

$$\Delta_{R,B} = \langle \{x_1\} \cup S_1, \{x_2\} \cup S_2, \ldots, \{x_{2^n}\} \cup S_{2^n} \rangle.$$

Then (R, B) has value $\dfrac{1}{2^n}$.

Proof. We will prove this by induction on n.

Base case: For $n = 0$ we have $\Delta_{R,B} = \langle \{x_1\} \rangle$. We have shown in the previous section that $(R, B) = 1$ in that case.

Induction hypothesis: Let $S'_1, S'_2, \ldots, S'_{2^{n-1}}$ be the subsets of $\{y_1, y_2, \ldots, y_{n-1}\}$. Assume that the game with legal complex $\Delta' = \langle \{x'_1\} \cup S'_1, \ldots, \{x'_{2^{n-1}}\} \cup S'_{2^{n-1}} \rangle$ has value $\dfrac{1}{2^{n-1}}$.

Induction step: Without loss of generality, assume that $S_1, S_2, \ldots, S_{2^n}$ are ordered such that $S_{2^n} = \emptyset$ and the sets $S_1, S_2, \ldots, S_{2^{n-1}}$ are those containing y_n.

Left has the options to move to the games with legal complexes $\langle S_1 \rangle$, $\langle S_2 \rangle$, …, $\langle S_{2^n} \rangle$. All of those options, except for the one corresponding to $\langle S_{2^n} \rangle = \langle \emptyset \rangle$, will be negative. The option corresponding to $\langle \emptyset \rangle$ is 0, and thus dominates all other options.

All of Right's moves are symmetric. We will assume without loss of generality that he moves in y_n. This option leaves us with the game with legal complex $\langle \{x_1\} \cup S_1 \setminus \{y_n\}, \{x_2\} \cup S_2 \setminus \{y_n\}, \ldots, \{x_{2^{n-1}}\} \cup S_{2^{n-1}} \setminus \{y_n\} \rangle$. This game has value $\dfrac{1}{2^{n-1}}$ by the induction hypothesis.

Thus $(R, B) = \left\{ 0 \,\middle|\, \dfrac{1}{2^{n-1}} \right\} = \dfrac{1}{2^n}$. □

The following is then an immediate consequence using disjunctive sums.

Corollary 4.28. *Given any dyadic rational* $\dfrac{a}{2^n}$ *there exists an SP-game* (R, B) *such that* $(R, B) = \dfrac{a}{2^n}$.

4.5 Switches

We will show that all switches $\{a \mid b\}$ with $a \geq b$ being integers are possible as game values of SP-games. We will begin with a non-negative and b non-positive.

Proposition 4.29. *If* $a, b \geq 0$ *are integers, then the SP-game* (R, B) *with* $\Delta_{R,B} = \langle \{x_0, \ldots, x_a\}, \{y_0, \ldots, y_b\} \rangle$ *has value* $\{a \mid -b\}$.

Proof. Left's moves are all to a simplex consisting of a Left vertices, thus has value a. Similarly Right going first will move to $-b$. Thus (R, B) has value $\{a \mid -b\} = \frac{a-b}{2} \pm \frac{a+b}{2}$. □

If a connected legal complex is desired and $a, b \geq 1$, then we can also add in the face $\{x_0, y_0\}$, and a move in this face will be dominated, thus giving the same value.

Next we consider the case in which a is positive and b non-negative. The case of $0 \geq a > b$ is the negative of this.

Proposition 4.30. *If $a > b \geq 0$ are integers, then the SP-game (R, B) with $\Delta_{R,B} = \langle \{x_1, \ldots, x_{a+1}\}, \{x_1, \ldots, x_b, y\}\rangle$ has value $\{a \mid b\}$.*

Proof. We will prove this by induction on a.

If $a = 1$, then necessarily $b = 0$, so that $\Delta_{R,B} = \langle \{x_1, x_2\}, \{y\}\rangle$, which we have shown in the previous result has value $\{1 \mid 0\}$

Now assume that if $a > k > 0$, $j \geq 0$ and $\Delta_{R',B'} = \langle \{x_1, \ldots, x_{k+1}\}, \{x_1, \ldots, x_j, y\}\rangle$ then $(R', B') = \{k \mid j\}$.

If Left moves in any of $x_1, \ldots, x_b$, say without loss of generality in x_1, this is to $\langle \{x_2, \ldots, x_a\}, \{x_2, \ldots, x_b, y\}\rangle$. By induction, this has value $\{a - 1 \mid b - 1\}$.

If Left moves in any of $x_{b+1}, \ldots, x_a$, say without loss of generality in x_a, then it is to $\langle \{x_0, \ldots, x_{a-1}\}\rangle$. This has value a.

Right's only move is to $\langle \{x_1, \ldots, x_b\}\rangle$, which has value b. Thus (R, B) has value $\{\{a - 1 \mid b - 1\}, a \mid b\} = \{a \mid b\}$. □

Note that the above can also be shown by using that disjunctive sum of games corresponds to the join of simplicial complexes (see Theorem 1.55). The simplicial complex can be written as the following join:

$$\langle \{x_1, \ldots, x_{a+1}\}, \{x_1, \ldots, x_b, y\}\rangle = \langle \{x_1, \ldots, x_b\}\rangle * \langle \{x_{b+1}, \ldots, x_{a+1}\}, \{y\}\rangle$$

The first gives a game with value b by Proposition 4.21, and the second a game with value $\{a - b \mid 0\}$ by Proposition 4.29. And we indeed have

$$b + \{a - b \mid 0\} = \{a \mid b\}.$$

4.6 Tiny and Miny

We will show that all $+_n$, where n is a positive integer, are possible game values of SP-games. Since $+_0 = \uparrow$, we have already shown the existence of this value (see Subsection 4.2.3).

Proposition 4.31. *If n is a positive integer, then the SP-game (R, B) with $\Delta_{R,B} = \langle \{y_1, \ldots, y_{n+1}\}, \{x_1, y_1\}, \ldots, \{x_1, y_{n+1}\}, \{x_2\}\rangle$ has value $+_n$.*

Proof. Left's move in x_1 is to $\langle\{y_1\},\ldots,\{y_{n+1}\}\rangle$, which has value -1. The move in x_2 is to 0, and this also dominates the move to -1.

Right's moves are all symmetric, so assume without loss of generality that he makes the move corresponding to y_1. Then this is to $\langle\{y_2,\ldots,y_{n+1}\},\{x_1\}\rangle$, and it can be easily seen that this has value $\{0 \mid -n\}$.

Thus (R,B) has value $\{0 \mid \{0 \mid -n\}\} = +_n$. $\square$

We know from Uiterwijk and Barton 2015 [61] that several other tinies are also possible values of DOMINEERING, thus are elements of $\mathbb{V}$, for example $+_{1/2}$, $+_{1/4}$, and $+_{(1/2)*}$.

4.7 Nimbers

Contrary to other values, we will show the existence of nimbers as game values of SP-games by constructing the ruleset and board directly, rather than through the legal complex.

Proposition 4.32. *For every nimber $*n$ there exists an SP-game (R,B) that has value $*n$.*

Proof. Let R be the ruleset in which both Left and Right have as their pieces $K_1,K_2,\ldots,K_n$, played on $B=K_n$. We will show by induction that this has value $*n$. Note that $*0=0$ and $*1=*$.

Base case: For $n=0$, the board is empty, and Left and Right have no options, thus $(R,B)=0$.

Induction hypothesis: Assume that for all $j<n$, the game in which Left and Right can play pieces $K_1,\ldots,K_j$ on the board K_j has value $*j$.

Induction step: We now have as our board $B=K_n$ and pieces $K_1,\ldots,K_n$. Suppose that either player places K_l as their first piece. The game from this point is now equivalent to playing $K_1,\ldots,K_n$ on K_{n-l}. Since the pieces $K_{l+1},\ldots,K_n$ cannot be placed, this has value $*l$ by induction hypothesis. Thus we have $(R,B) = \{0,*,\ldots,*(n-1) \mid 0,*,\ldots,*(n-1)\} = *n$. $\square$

Remark 4.33. Note that each game defined above is equivalent to the game NIM on a single pile. Playing NIM on several piles is equivalent to playing a disjunctive sum of this game, showing that NIM can be thought of as an SP-game (see also Remark 1.11).

4.8 Further Work

The study of whether specific game values are elements of $\mathbb{V}$ can be continued by looking at values such as switches $\{a \mid b\}$ where a and/or b are not integers, tiny $+_G$ where G is a fraction or non-number, or other values we have not yet discussed at all, for example $\uparrow^n$ and $\uparrow^{[n]}$.

Ideally, we would like to have a recursive construction that works similarly to $\mathbb{G}_n$. This seems difficult though as simply combining two simplicial complexes, such as joining at a vertex or a face, often creates unwanted options besides the ones needed.

In case that $\mathbb{V}$ is not equal to $\mathbb{G}$, the question of course is which values are not possible. It is generally more difficult to show non-existence of a value though, and here again the legal complexes, and the structure of the game graph as shown in Chapter 3 should be of value.

Related to this, we can also ask when the canonical form of an equivalence class containing an SP-game is itself literally equal to an SP-game. Using that the game graph has to have the SP-property, we know that this is not always the case. Consider the following example:

Example 4.34. We have shown that $-\frac{1}{2}$ is the game value of some SP-game. Now consider the canonical form of $-\frac{1}{2}$, which is $\{\{ \mid 0\} \mid 0\}$. The game tree of this canonical form is

$$\begin{tikzcd}[column sep=small] & \{\{ \mid 0\} \mid 0\} \arrow[dl, dash] \arrow[dr, dash] & \\ \{ \mid 0\} \arrow[dr, dash] & & 0 \\ & 0 & \end{tikzcd}$$

Since there exists a Left move followed by a Right, but no Right followed by a Left the game graph for this game cannot have the SP-property.

It would also be interesting to further study which elements are in $\mathbb{G}_n \setminus \mathbb{V}_n$ besides the integer $n-1$.

In more general terms, one can also ask which values are possible under misère play, and all related questions above.

Chapter 5

Temperature

The temperature of a combinatorial game in essence indicates how urgent it is for a player to make a certain move. In practice, it is very hard to calculate, or even bound, the temperature of a game. In this chapter, we will be looking at the temperature of partizan SP-games in particular. After a brief introduction to temperature, we will give an upper bound on the temperatures of a set of games in Theorem 5.25—the first such known bound. We then give several examples of how to apply this bound to SP-games, which seem particularly suitable for this approach. Next, we will discuss SNORT in particular, giving a conjecture of a bound on temperature based on computational results. Finally, we discuss some further work.

5.1 Introduction to Temperature

We will begin by introducing further concepts from combinatorial game theory which are needed before defining temperature. Examples for all of these and proofs of statements can be found in "Combinatorial Game Theory" by Siegel [57].

Definition 5.1. The **Left stop** and **Right stop** of a combinatorial game G, denoted by $\boldsymbol{LS(G)}$ and $\boldsymbol{RS(G)}$ respectively, are recursively defined as

$$LS(G) = \begin{cases} x & \text{if } G = x \text{ is a number,} \\ \max\limits_{G^L \in G^{\mathcal{L}}} \left\{ RS\left(G^L\right) \right\} & \text{otherwise;} \end{cases}$$

$$RS(G) = \begin{cases} x & \text{if } G = x \text{ is a number,} \\ \min\limits_{G^R \in G^{\mathcal{R}}} \left\{ LS\left(G^R\right) \right\} & \text{otherwise.} \end{cases}$$

The Left stop is the first number reached when playing the game with Left going first and alternating play. Once a game is equal to a number, play becomes uncompelling as it is clear which player will win and what their advantage is.

We will use the following properties of Left and Right stops:

Fact 5.2 ([57, Section II.3]). *Let G and H be any two games and x any number. Then*

1. $LS(G) \geq RS(G)$;
2. $LS(-G) = -RS(G)$;
3. $RS(G) + LS(H) \leq LS(G+H) \leq LS(G) + LS(H)$ *and* $RS(G) + LS(H) \geq RS(G+H) \geq RS(G) + RS(H)$;
4. $RS(G^L) \leq LS(G)$ *for every* G^L *and* $LS(G^R) \leq RS(G)$ *for every* G^R;
5. $G \leq x$ *implies* $LS(G) \leq x$; *and*
6. $LS(G+x) = LS(G) + x$.

We know that two games G and H are equal to each other when $G - H = 0$. There are cases in which games are not equal, but their difference is almost insignificant, especially when replacing one by another in a sum. Such games are called infinitesimally close.

Definition 5.3. Two combinatorial games G and H are called **infinitesimally close** if $LS(G-H) = 0$ and $RS(G-H) = 0$.

Example 5.4. Consider the games $G = 1$ and $H = \{1 \mid 1\}$. These two games are not equal since $G - H = 1 + \{-1 \mid -1\} = *$, but they are very similar to each other since after a single move in H by either player they are equal. In fact, $LS(*) = RS(0) = 0$ and $RS(*) = LS(0) = 0$, so that G and H are infinitesimally close.

The temperature of a game intuitively indicates the advantage one receives by playing in it. When a game is equal to a number, there is no advantage gained, but rather lost, and thus the temperature is negative. The largest advantage that can be lost is 1 – when the game is an integer the player reduces their number of moves by 1, while not affecting the other player's (nonexisting) option – and thus temperature will be greater or equal to -1. There is no upper bound as an advantage gained can be arbitrarily large.

We now define formally what it means to cool a combinatorial game, and what its temperature is.

Definition 5.5. Fix a combinatorial game G and $t \geq -1$. Then ***G* cooled by *t***, denoted $\boldsymbol{G_t}$, is defined to be

- n if G is the integer n,
- $\tilde{G}_t = \{G_t^{\mathcal{L}} - t \mid G_t^{\mathcal{R}} + t\}$ if G is not an integer and there is no $t' < t$ such that $\tilde{G}_{t'}$ is infinitesimally close to a number x;
- x if G is not a number and there exists a $t' < t$ such that $\tilde{G}_{t'}$ is infinitesimally close to the number x and t' is the smallest such.

Note that in the last point, there is indeed a unique smallest t' such that $\tilde{G}_{t'}$ is infinitesimally close to a number, but this is not immediate (see [57, Section II.5] for more information).

Definition 5.6. The **temperature** of a game G played on a board B, denoted by $\boldsymbol{t(G)}$ is the smallest $t \geq -1$ such that G_t is infinitesimally close to a number.

A game is called **cold** if $t(G) < 0$, **tepid** if $t(G) = 0$, and **hot** if $t(G) > 0$. The number to which $G_{t(G)}$ is infinitesimally close to is the **mean** of G and indicated by $m(G)$.

Fact 5.7 ([57, Proposition II.5.20])**.** *Let G be a game.*

1. *G is cold if and only if G is equal to a number;*
2. *G is tepid if and only if G is infinitesimally close, but not equal, to a number; and*
3. *G is hot if and only if $LS(G) > RS(G)$.*

In the case that G is a switch or a number we have formulas for the temperature and the mean. We demonstrate both in the next example.

Example 5.8. Given $a \geq b$ integers, the switch $G = \{a \mid b\} = \frac{a+b}{2} \pm \frac{a-b}{2}$ has $t(G) = \frac{a-b}{2}$ and $m(G) = \frac{a+b}{2}$. For example if $G = \{2 \mid -1\} = \frac{1}{2} \pm \frac{3}{2}$, then $G_t = \{2-t \mid -1+t\}$ for $t \leq \frac{3}{2}$. For $t = \frac{3}{2}$ we have $G_t = \frac{1}{2} + *$, which is infinitesimally close to $\frac{1}{2}$. Thus the temperature is $\frac{3}{2}$ and the mean is $\frac{1}{2}$.

Given a number $G = \frac{m}{2^n}$ in simplest terms, we have $t(G) = -\frac{1}{2^n}$ and $m(G) = G$. For example, if $G = \frac{1}{2} = \{0 \mid 1\}$, then $G_t = \{-t \mid 1+t\}$ for $t \leq -\frac{1}{2}$. At $t = -\frac{1}{2}$ we have $G_t = \frac{1}{2} + *$, which is infinitesimally close to $\frac{1}{2}$. Thus the temperature is $-\frac{1}{2}$ and the mean is $\frac{1}{2}$.

The game G_t for $t \leq t(G)$ can be thought of as playing G but having to pay a penalty of t when making a move. The temperature $t(G)$ is then the point after which the penalty is too high for either player to be interested in the game. Temperature thus gives a sense of how valuable a component in a disjunctive sum is to the players, or which component is the most urgent to move in.[1] Thus a goal of computer scientists in combinatorial game theory is finding heuristics to evaluate the temperature [46], as it points to a good, hopefully the best, move.

Given a disjunctive sum, we can bound the temperature based on the temperature of the components as in the next fact. This result will be used frequently in our calculations later on.

Fact 5.9 ([57, Theorem II.5.18])**.** *For all games G and H we have*

$$t(G + H) \leq \max\{t(G), t(H)\}.$$

When cooling a game, the Left stop and Right stop become closer, until they are equal to the mean. The behaviour of the stops can be seen visually by the graphical representation of the thermograph.

Definition 5.10. Given a game G, the ordered pair $(LS(G_t), RS(G_t))$, as a function of t, is called the **thermograph** of G.

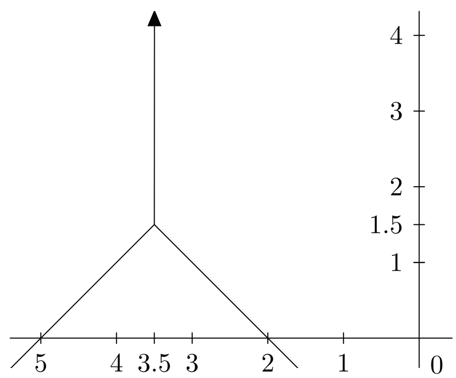
The graphical representation of a thermograph uses the following conventions: $LS(G_t)$ and $RS(G_t)$ are simultaneously plotted along the horizontal axis, with positive values on the left and negative on the right, while t is plotted along the vertical axis. Note that $LS(G)$ and $RS(G)$ are the points at which the thermograph crosses the horizontal axis. The two sides meet at vertical value $t(G)$ and horizontal value $m(G)$, and are then topped by an infinite vertical mast.

By definition of the Left and Right stops we can construct the thermograph of a game inductively from the thermographs of the options. The left wall is the leftmost

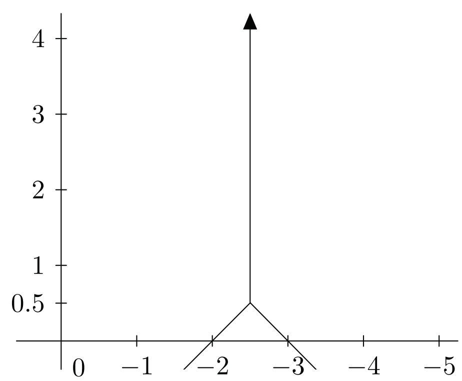
[1] Temperature can sometimes be misleading though. There are rare examples in which the component of highest temperature is not actually the most desirable one to move in, but it is still a good move.

right wall of the thermographs of the options, sheared by subtracting t. The right wall is similarly the rightmost left wall of the options sheared by adding t. The mast begins where these two intersect.

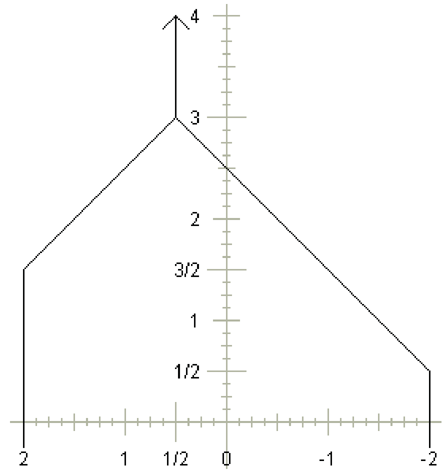
Example 5.11. Consider the game $G = \{\{5 \mid 2\} \mid \{-2 \mid -3\}\}$. We will construct the thermograph of G inductively from its options. The thermograph of a number n is simply the vertical line at n. Thus the thermograph of $\{5 \mid 2\}$ is as below, with the line at 5 sheared clockwise, the line at 2 counterclockwise, and the mast starting at their intersection.



The thermograph of $\{-2 \mid -3\}$ is similarly given in the diagram below.



To construct the thermograph of G, we take the right wall of the thermograph of $\{5 \mid 2\}$ (as it gives the Right stop of the Left option) and shear it clockwise by subtracting t, and the left wall of the thermograph of $\{-2 \mid -3\}$ and shear it counterclockwise by adding t, until their intersection, which is then topped by the mast. The thermograph of G is then given below.



Note that the temperature of a game G, being the vertical value at which the mast starts, is the length of all vertical segments plus the length of the oblique segments of either the right wall or the left wall above the horizontal axis and below the mast. We will use this property to bound the temperature of G. To do so, we will need the confusion interval of a game.

Definition 5.12. The **confusion interval** of G is $\boldsymbol{\mathcal{C}(G)}= \{x \in \mathbb{D} : G \not\gtrless x\}$. The endpoints of the confusion interval are the Left stop and Right stop. The measure of the confusion interval, $LS(G) - RS(G)$, is indicated by $\boldsymbol{\ell(G)}$.

Similarly to temperature, we can bound the measure of the confusion interval of a disjunctive sum of games based on those of the components:

Lemma 5.13. *For any two games G and H we have*

$$\ell(G + H) \leq \ell(G) + \ell(H).$$

Proof. We have

$$\begin{aligned}
\ell(G + H) &= LS(G + H) - RS(G + H) \\
&\leq LS(G) + LS(H) - RS(G) - RS(H) \\
&= \ell(G) + \ell(H).
\end{aligned}$$

□

Computer game playing programs often use temperature to find potentially good moves. As temperature is difficult to calculate though, it would be helpful to be able to bound the temperature of classes of games.

Definition 5.14. Given a class of games S, the **boiling point** of S, denoted $\boldsymbol{BP(S)}$, is the supremum of the temperatures of all games in S, thus

$$BP(S) = \sup_{G \in S} (t(G)) .$$

Of particular interest are the boiling points of classes of games which are the same ruleset played on different boards, often even the entire (infinite) set of boards.

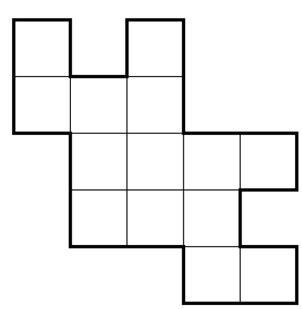
Historically, there has been much interest in temperatures of specific games. For some SP-games already mentioned the background is as follows:

- Any impartial game, including NIM, has a boiling point of 0 as the only possible values are the nimbers, which are infinitesimally close to 0.

- In Winning Ways, Berlekamp, Conway, and Guy show in 1982 [5] (also found in [6, p.47]) that COL played on any board is equal n or $n + *$ where n is a number. This implies that COL is always cold or tepid, giving a boiling point of 0.

- For SNORT values of some positions are known, and thus their temperatures, but there are no general results. We will further discuss SNORT in Section 5.3.

- For DOMINEERING, there have been five papers since 1995 (see Kim 1995 [37] and 1996 [38], Shankar and Sridharan 2005 [55], and Drummond-Cole 2004 [19] and 2005 [20]) which have demonstrated DOMINEERING positions which in turn each had new, higher, temperatures. Berlekamp conjectured in the late 1980's that the boiling point of DOMINEERING is 2 (see [30, 55]). However, there has been no theorem which states an upper bound for the temperature of DOMINEERING. The position below, found by Drummond-Cole in 2004 [19] has a game value of $\{2* \mid -2*\}$, which is infinitesimally close to ± 2 and thus has temperature 2.



In particular, there has been no bound proven to hold for games in general.

5.2 An Upper Bound on the Boiling Point of a Game

In this section, we will demonstrate an upper bound on the boiling point of a class of games dependent solely on the maximum difference between Left and Right stops. This is joint work with Carlos Pereira dos Santos.

Note that in this section we will assume that G is a general short game, not necessarily an SP-game.

We will being by showing that for any game G there exists a game $\widetilde{G}$ which has a single Left option and a single Right option such that $t(G) = t(\widetilde{G})$.

Theorem 5.15. *Let G be a hot game. Then, there are options G^L and G^R such that $t(G) = t(\{G^L \mid G^R\})$.*

Proof. Consider $G_{t(G)}$, i.e. G cooled by its temperature. Since G is hot we know that $G_{t(G)}$ is tepid, being equal to $m(G)$ plus some infinitesimal.

We have that $LS(G_{t(G)}) = RS(G_{t(G)}) = m(G)$ and those stops are achieved with Left and Right options of $G_{t(G)}$. These options are obtained from some G^L and G^R, being $G^L_{t(G)} - t(G)$ and $G^R_{t(G)} + t(G)$. Therefore, $t(\{G^L \mid G^R\}) = t(G)$. □

Definition 5.16. Let G be a hot game. Then (G^L, G^R) is called a pair of **thermic options** of G if $t(G) = t(\{G^L \mid G^R\})$. We set $\widetilde{G} = \{\widetilde{G}^L \mid \widetilde{G}^R\}$ where $\widetilde{G}^L = G^L$ and $\widetilde{G}^R = G^R$ are thermic options. We say $\boldsymbol{\widetilde{G}}$ is a **thermic version** of G.

When bounding temperatures, instead of working with G we can work with a thermic version $\widetilde{G}$. This has the advantage that when we are constructing the thermograph, we only need to consider the thermographs of the unique options, rather than having to find which option has the leftmost right wall at every point.

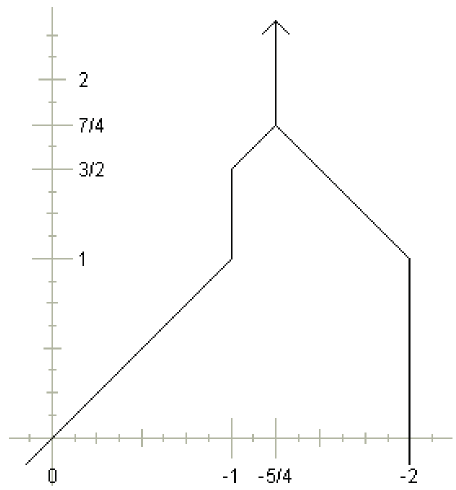
In general, it is tedious to find a thermic version of a game G as, in a worst-case scenario, all temperatures of combinations of Left and Right options would have to be checked. We will be using the thermic version for theoretical purposes only though, and for calculations will return to the original game.

Note that a thermic version of a game is not necessarily unique, as demonstrated in the following example:

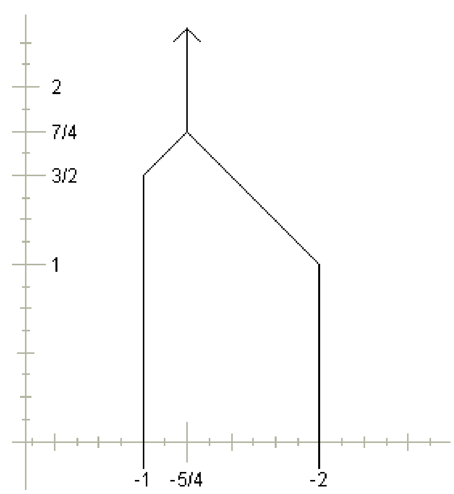
Example 5.17. Consider $G = \{\{\{3 \mid 1\} \mid 0\}, \{2 \mid 0\} \mid \{-1 \mid -2\}\}$. Both $\widetilde{G}_1 = \{\{\{3 \mid 1\} \mid 0\} \mid \{-1 \mid -2\}\}$ and $\widetilde{G}_2 = \{\{2 \mid 0\} \mid \{-1 \mid -2\}\}$ are thermic versions of G.

Further, in many cases the thermographs of G and a thermic version $\widetilde{G}$ are not identical, as shown in the following example.

Example 5.18. Let $G = \{\{2 \mid -1\}, 0 \mid \{-2 \mid -4\}\}$. The thermograph of G is



A thermic version of G is given by $\widetilde{G} = \{\{2 \mid -1\} \mid \{-2 \mid -4\}\}$. The thermograph of $\widetilde{G}$ is



Although the thermograph of a thermic version might not be identical to the one of the original game, the following proposition shows that it always lies to the inside:

Proposition 5.19. *Given a hot game G with a thermic version $\widetilde{G}$, we have for all $t \geq -1$ that $LS(\widetilde{G}_t) \leq LS(G_t)$ and $RS(\widetilde{G}_t) \geq RS(G_t)$. In particular $LS(\widetilde{G}) \leq LS(G)$ and $RS(\widetilde{G}) \geq RS(G)$.*

Proof. Note first that G_t and $\widetilde{G}_t$ are numbers only if $t \geq t(G)$, at which point the thermographs are identical.

For $t < t(G)$, we then have by definition $LS(G_t) = \max_{G^L}\left(RS(G^L_t) - t\right)$ and $LS(\widetilde{G}_t) = RS(\widetilde{G}^L_t) - t$ and similarly for the Right stops. Now since $\widetilde{G}^L$ is also a Left option of G, we have $LS(\widetilde{G}_t) \leq LS(G_t)$ and $RS(\widetilde{G}_t) \geq RS(G_t)$.

The special case of $t = 0$ then follows immediately. □

Corollary 5.20. *Given a hot game G with a thermic version $\widetilde{G}$, we have*

$$\ell(\widetilde{G}) \leq \ell(G).$$

Proof. From the previous proposition we have

$$\ell(\widetilde{G}) = LS(\widetilde{G}) - RS(\widetilde{G}) \leq LS(G) - RS(G) = \ell(G). \qquad \square$$

In this section, we will often consider segments of the thermograph. By length of such a segment we mean the change in t.

As we will be bounding the temperature of a game from the length of the vertical and oblique segments of the left and right walls of its thermograph, the turning points of the thermograph will be very important. We will be concentrating on the left wall throughout, but the same definitions and results apply to the right wall.

Definition 5.21. Let G be a hot game and $\widetilde{G}$ a thermic version of G. Let $t_0 = 0, t_1, t_2, \ldots, t_k = t(G)$ be the sequence of the vertical coordinates of the turning points of the left boundary of the thermograph of $\widetilde{G}$. The Right stops of the sequence of $G^L(i) = \widetilde{G}^L_{t_i} - t_i$ define the segments of the boundary. If $RS(G^L(i+1)) = RS(G^L(i))$, we have a vertical segment; on the other hand, if $RS(G^L(i+1)) < RS(G^L(i))$, we have an oblique segment. Define

- t_i to be **left vertical** if $RS(G^L(i+1)) = RS(G^L(i))$;
- t_i to be **left oblique** if $RS(G^L(i+1)) < RS(G^L(i))$.

We further define

$$T^L_{\text{vert}} = \sum_{\substack{t_i \text{ is left} \\ \text{vertical}}} (t_{i+1} - t_i)$$

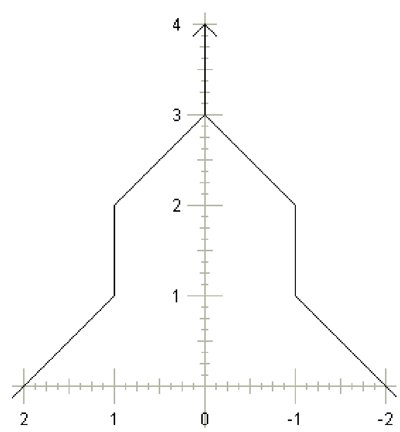
and

$$T^L_{\text{obl}} = \sum_{\substack{t_i \text{ is left} \\ \text{oblique}}} (t_{i+1} - t_i).$$

Right vertical, right oblique, T^R_{vert}, and T^R_{obl} are defined similarly.

Essentially, T^L_{vert} measures the length of the vertical segments of the left boundary between 0 and $t(G)$, while T^L_{obl} measures the oblique segments.

Example 5.22. Consider $G = \{\{\{6 \mid 4\} \mid \{2 \mid 0\}\} \mid \{\{0 \mid -2\} \mid \{-4 \mid -6\}\}\}$. Note that G is its own thermic version as there are only a single Left option and a single Right option. The thermograph is given below:



The turning points of the left boundary are $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, and $t_3 = 3$. We have

$$RS(G^L(0)) = RS(\{\{6 \mid 4\} \mid \{2 \mid 0\}\}) = 2$$
$$RS(G^L(1)) = RS(\{3* \mid 1*\}) = 1$$
$$RS(G^L(2)) = RS(\{1 \mid 1\}) = 1$$
$$RS(G^L(3)) = RS(*) = 0$$

We have that $t_0 = 0$ and $t_2 = 2$ are left oblique and $t_1 = 1$ is left vertical. Thus

$$T^L_{\text{vert}} = 2 - 1 = 1 \quad \text{and} \quad T^L_{\text{obl}} = (1 - 0) + (3 - 2) = 2.$$

Since the thermograph is symmetric, $T^R_{\text{vert}} = 1$ and $T^R_{\text{obl}} = 2$.

The following is an immediate consequence of the previous definition and that oblique segments have slope ± 1.

Lemma 5.23. *Given a hot game G and a thermic version $\widetilde{G}$ we have*

$$T^L_{\text{vert}} + T^L_{\text{obl}} = T^R_{\text{vert}} + T^R_{\text{obl}} = t(G)$$

and

$$\ell(\widetilde{G}) = T^L_{\text{obl}} + T^R_{\text{obl}}.$$

We will now demonstrate the bound on the temperature of a game from the measure of its confusion interval and those of its options using the vertical and oblique segments of the thermograph.

Theorem 5.24. *Let G be a hot game and $\widetilde{G}$ a thermic version of G. Then*

$$t(G) \leq \ell(H) + \frac{\ell(G)}{2}$$

where $H = \widetilde{G}^L$ if $T^L_{\text{vert}} \geq T^R_{\text{vert}}$ and $H = \widetilde{G}^R$ otherwise.

Proof. We will demonstrate this bound in the case of $T^L_{\text{vert}} \geq T^R_{\text{vert}}$. The second case follows similarly.

First consider T^L_{vert}. Since the left boundary of the thermograph of $\widetilde{G}$ comes from the right boundary of the thermograph of $\widetilde{G}^L$, we know that T^L_{vert} is at most the length of the oblique segments of the latter. Further, we know that the length of these oblique segments are at most the distance between the Left and Right stops. Thus $T^L_{\text{vert}} \leq \ell(\widetilde{G}^L)$.

On the other hand, $T^L_{\text{vert}} + T^L_{\text{obl}} = T^R_{\text{vert}} + T^R_{\text{obl}}$ and, by assumption, $T^L_{\text{vert}} \geq T^R_{\text{vert}}$. Hence, $T^L_{\text{obl}} \leq T^R_{\text{obl}}$. We have

$$2 \times T^L_{\text{obl}} \leq T^L_{\text{obl}} + T^R_{\text{obl}} = \ell(\widetilde{G}) \leq \ell(G)$$

so that

$$T^L_{\text{obl}} = \frac{\ell(\widetilde{G})}{2} \leq \frac{\ell(G)}{2}.$$

Therefore, $t(G) = T^L_{\text{vert}} + T^L_{\text{obl}} \leq \ell(\widetilde{G}^L) + \frac{\ell(G)}{2}$. $\square$

Working with the thermic options above had the advantage that we only had to consider a single option for each player. Note that the thermic version of a game G is difficult to determine in general though. We will instead bound the measure of the confusion interval for *all* options in our applications in the next section.

The following theorem is a direct result of the previous one. It is the first known theorem giving an upper bound on the boiling point of a class of games.

Theorem 5.25. *Let S be a class of short games and J, K be two non-negative numbers. If for all $G \in S$, we have $\ell(G) \leq K$ and for all G^L and G^R that $\ell(G^L), \ell(G^R) \leq J$, then*

$$BP(S) \leq \frac{K}{2} + J.$$

The next example will demonstrate that this bound is tight in some cases.

Example 5.26. Consider $S = \{G : \ell(G), \ell(G^L), \ell(G^R) \leq 6\}$ the set of short games G for which $\ell(G) \leq 6$, closed under options.

By the previous theorem we know that $BP(S) \leq 9$. Consider the following sequence of games, all of which belong to S:

$$\begin{aligned}
G_0 &= \pm\{9 \mid 3\} & t(G_0) &= 6 \\
G_1 &= \pm\{\{15 \,|\, 9\} \,|\, 3\} & t(G_1) &= \frac{15}{2} \\
G_2 &= \pm\{\{\{21 \,|\, 15\} \,|\, 9\} \,|\, 3\} & t(G_2) &= \frac{33}{4} \\
G_3 &= \pm\{\{\{\{27 \,|\, 21\} \,|\, 15\} \,|\, 9\} \,|\, 3\} & t(G_3) &= \frac{69}{8} \\
&(\ldots)
\end{aligned}$$

For this sequence, we have $t(G_n) = 9 - \dfrac{3}{2^n}$, and as n increases the temperature approaches 9. Therefore, $BP(S) = 9$.

We will use Theorem 5.25 in the remainder of this section to give upper bounds on the boiling point for specific SP-games. The following proposition will be used to bound $\ell(G)$.

Proposition 5.27. *Given a hot game G, if we know that $G^L - G - K + \epsilon \leq 0$ for all Left options G^L, a fixed number K, and infinitesimal ϵ, then $\ell(G) \leq K$.*

Proof. Since $G^L - G + \epsilon \leq K$, we have $LS(G^L - G) \leq K$. Thus (remember that $LS(G) = RS(G^L)$ for some G^L):

$$\begin{aligned}
\ell(G) &= LS(G) - RS(G) \\
&= RS(G^L) - RS(G) \\
&= RS(G^L) + LS(-G) \\
&\leq LS(G^L - G) \\
&\leq K
\end{aligned}$$

□

The game $G^L - G - K + \epsilon$ corresponds, in some sense, to letting Left play twice in G, balancing that with $K - \epsilon$. If we can bound the effect of that second move, we bound the confusion intervals of a class of short games.

Our goal thus is to find the minimal number K and an infinitesimal ϵ for which Right has a winning strategy going second in $G^L - G - K + \epsilon$.

As examples, we will show how to apply this to some Partizan Octals, to Partizan Subtraction, and some Domineering snakes. Note that since $t(G + H) \leq \max\{t(G), t(H)\}$, it is sufficient to bound the temperature when playing on a connected board.

5.2.1 Partizan Octals

We will consider the partizan octal games Oab where both octal codes are of the form **0.00...07**, with the 7 in the ath position for the Left code and in the bth position for the Right code. These games are also called partizan Splittles (see [43]). They further are equivalent to Left placing dominoes of length a and Right of length b onto a strip. Thus these are strong placement games.

The games Oaa are impartial. Thus the temperature is either -1 (for example when the board is empty, thus the value is 0) or 0 (for example when the strip has length a, thus the value is $*$), giving a boiling point of 0.

For the remainder of the section we will assume $a \neq b$. Also note that Oab = $-$Oba, thus Oab and Oba have the same boiling point.

The following result shows that it is sufficient to bound $\ell(G)$ for playing on empty strips.

Corollary 5.28. *If $\ell(G) \le k$ for all games G which are* Oab *played on an empty strip then*

$$\ell(G^L), \ell(G^R) \le 2k.$$

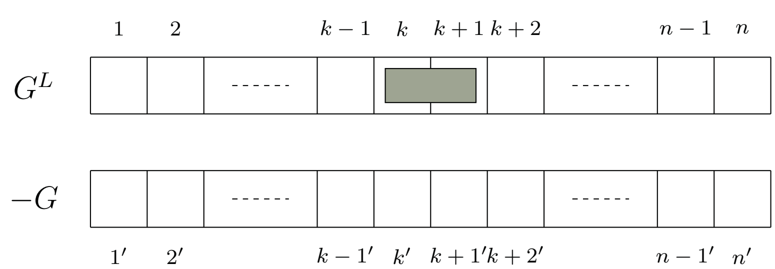
Proof. Any move, whether by Left or Right, results in a disjunctive sum of two smaller boards. Thus if we bound $\ell(G)$, we also bound $\ell(G^L)$ and $\ell(G^R)$ by Lemma 5.13. □

As a first non-trivial example, we begin with O12. The strategy we will employ is slightly different from the general case, but it will illustrate well some properties we take advantage of.

Proposition 5.29. *The boiling point of* O12 *and* O21 *is at most* 5*.*

Proof. Let G be the game of O21 played on a strip of length n. If $n = 1$, then $G = -1$ and we have $t(G) = -1$. If $n = 2$, then $G = \{0 \mid -1\}$ and $t(G) = \frac{1}{2}$. For $n \ge 3$ we will show that $G^L - G - 2$ is a Right win if Left goes first. By Proposition 5.27 we then have $\ell(G) \le 2$, and therefore $\ell(G^L), \ell(G^R) \le 4$. The result then follows by Theorem 5.25.

Our convention is that the spaces in G^L are labelled as $1, \ldots, n$, with the move already made in G^L being in spaces k and $k+1$ and called move 0. The spaces in $-G$ are labelled $1', \ldots, n'$.



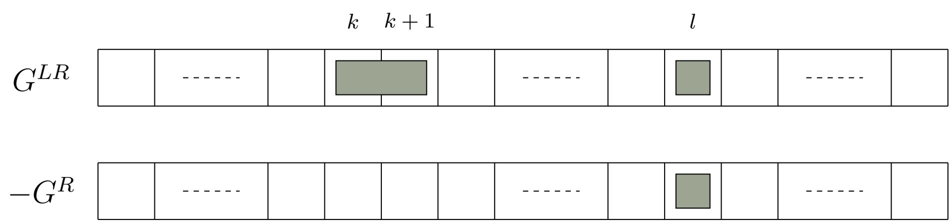
If Left plays her first move so that it is not in space k' or $k+1'$, then Right can mimic the move in the other game. This results in each component breaking into a disjunctive sum at the same spot, two of which are the negative of each other, the other two have the difference of the 0th Left move. In the case of Left having made her move in $-G$, we then have (see the diagram below)

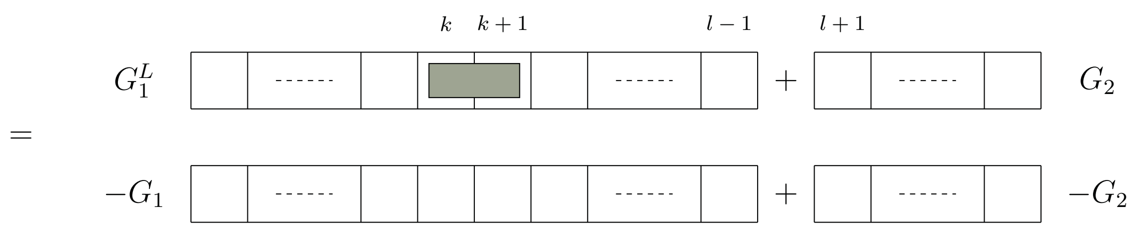
$$(-G)^L = -G^R = -G_1 - G_2 \qquad \text{and} \qquad G^{LR} = G_1^L + G_2.$$

This gives

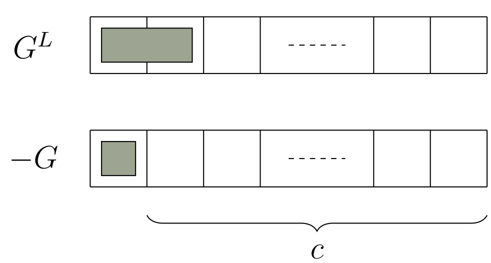
$$G^{LR} - G^{R} = G_1^{L} + G_2 - G_1 - G_2 = G_1^{L} - G_1,$$

so that we reduce the game to being played on a smaller board. Left having moved in G^L is similar. We can thus assume by induction that Left's first move is either in space k' or $k+1'$. Similarly, we can assume without loss of generality that any future Left move overlaps a previously made move by the same argument.

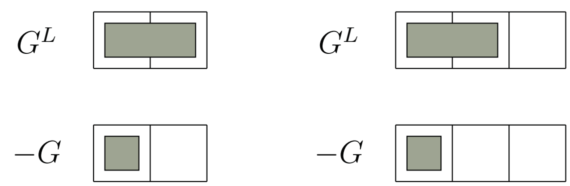




We will assume without loss of generality that Left's move is in k'. The game is now equivalent to the disjunctive sum of the strips 1 through $k-1$ and $1'$ through $k-1'$, which add to 0, and the strips k to n and k' to n' with spaces k, $k+1$, and k' all occupied. Let $c = n - k$. Our game has been reduced to the situation below.



We let Right respond by playing in $k+1'$ and $k+2'$. Again, the only Left move we have to consider is in $k+2$ and $k+3$, which reduces the game to the previous situation with c decreased by 2. We continue this strategy until $c=1$ or $c=2$.



If $c = 1$, Right has no responding move in $-G$, and instead moves -2 to -1. Left has one more move, to which Right responds by moving -1 to 0, thus wins. If $c = 2$, Right moves in $-G$ and wins. $\square$

The group of authors under the pseudonym G.A. Mesdal showed in 2009 [43] the following:

Fact 5.30 (Mesdal 2009 [43])**.** *The canonical form of* O12 *on a path of length n is*

$$\begin{cases} k & \textit{if } n = 4k-3 \\ \{k \mid k-1\} & \textit{if } n = 4k-2 \\ \{k+1 \mid k\} & \textit{if } n = 4k-1 \\ k & \textit{if } n = 4k \end{cases}$$

Thus the boiling points of O12 *and* O21 *are* $\frac{1}{2}$.

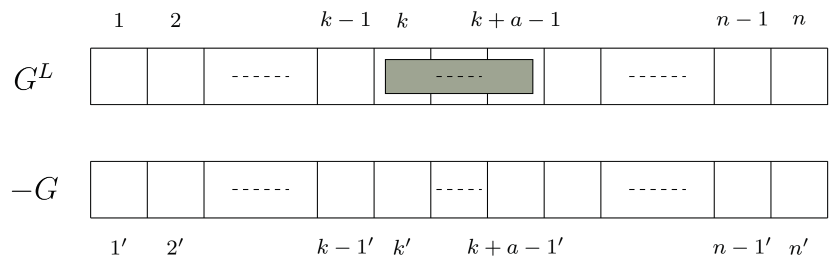
Thus our bound given is only a constant away from the actual bound.

Having demonstrated this simplest case, we will turn to Oab in general. The proof is similar to the case of $a = 2$, $b = 1$, but requires a slightly more involved strategy.

Proposition 5.31. *The boiling point of Oab with $a > b$ is at most* $\frac{5}{2}\left(\left\lfloor \frac{a-2}{b} \right\rfloor + 2\right)$.

Proof. Let n be the length of the strip. For $n < b$ neither player can move and the game is 0, thus the temperature trivially satisfies the bound. For $b \le n < a$, only Right can move, meaning the game is an integer and thus has temperature -1, again satisfying the bound. For $n \ge a$ we will show that $G^L - G - \left(\left\lfloor \frac{a-2}{b} \right\rfloor + 2\right)$ is a Right win if Left goes first. By Proposition 5.27 we then have $\ell(G) \le \left\lfloor \frac{a-2}{b} \right\rfloor + 2$, and therefore $\ell(G^L), \ell(G^R) \le 2\left(\left\lfloor \frac{a-2}{b} \right\rfloor + 2\right)$. The result then follows by Theorem 5.25.

Our convention similarly to O12 is that the spaces in G^L are labelled as $1, \ldots, n$, with the move already made in G^L being in spaces k to $k + a - 1$ and called move 0. The spaces in $-G$ are labelled $1', \ldots, n'$. See below.



If Left plays her first move so that it does not cover any of spaces k' through $k+a-1'$, then Right can mimic the move in the other game. As in the special case of O12, this results in the game breaking into a disjunctive sum, and we are able to reduce to a smaller board. By induction we may thus assume that Left's move does overlap one or more of these spaces (or any other previous move in the later game).

For Left's moves overlapping move 0, there are three cases to consider:

1. Neither $k-1'$ nor $k+a'$ is covered: In this case there are at most $\left\lfloor \frac{a}{b} \right\rfloor \leq \left\lfloor \frac{a-2}{b} \right\rfloor + 2$ moves of this type. Note that this is automatically the case if $b = 1$.

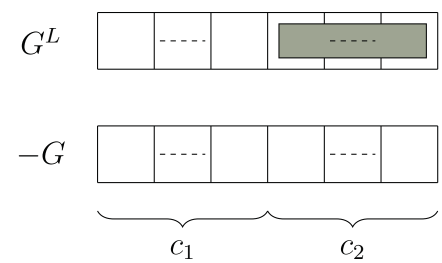
2. Exactly one of $k-1'$ and $k+a'$ is covered: In this case there are at most $\left\lfloor \frac{a-1}{b} \right\rfloor \leq \left\lfloor \frac{a-2}{b} \right\rfloor + 1$ moves strictly overlapping move 0.

3. Both of $k-1'$ and $k+a'$ are covered: In this case there are at most $\left\lfloor \frac{a-2}{b} \right\rfloor$ moves strictly overlapping move 0.

To each move strictly overlapping move 0, that does not cover either $k-1'$ or $k+a'$, Right responds by moving in the integer summand, reducing it by 1.

Suppose now that Left's move also covers one of $k-1'$ and $k+a'$. Without loss of generality we let that be $k-1'$. Right responds in G^L, playing in spaces $k-b$ through $k-1$ (as close to move 0 as possible). We continue this strategy whenever Left makes a move in $-G$ overlapping a previous one and not strictly overlapping move 0, in either direction. If Left ever moves non-overlapping, Right can again mimic, giving a smaller sum and the result is true by induction. Thus we may assume that all moves by Left overlap a previous one, and thus are all in $-G$.

If Left leaves empty spaces, these cannot be played by either player (if more than b spaces there would be no overlap with a previous piece). Thus at each step we again reduce the game by induction. At the end, Left might make one move to which Right cannot respond due to insufficient space. This can happen at one end (case 2) or both ends, with the two ends being a disjunctive sum (case 3). The end is then of the form seen below considered just before Left's move to which Right cannot respond.



Note that $c_1 < b$ as we assume Right has no response to Left's move. We may also assume that $c_2 < b$. If the occupied spaces on the top right are the end of move 0, then we can simply set c_2 to be the amount that Left's move overlaps with move 0 without changing the strategy. If these spaces are not part of move 0, then they were part of a Right responding move and $c_2 < b$ since otherwise there would not have been an overlap with Left's previous move.

In the case that $c_1+c_2 < b$, then Left has no moves and Right wins. If $b \leq c_1+c_2 < 2b$, then Left will move in $-G$ such that there are less than b spaces on either end and neither player can move in G^L or $-G$. Right then responds by reducing the integer by 1. $\square$

Using the Oab code for CGSuite in Appendix B, we have the following maximum temperatures on paths up to length 50 for various values of a and b.

	1	2	3	4	5	6	7	8	9	10
1	0	1/2	1	3/2	2	5/2	3	7/2	4	9/2
2	1/2	0	1	3/4	7/4	3/2	5/2	9/4	13/4	3
3	1	1	0	1	1	1	7/4	7/4	3/2	5/2
4	3/2	3/4	1	0	1	1	1	1	7/4	7/4
5	2	7/4	1	1	0	1	1	1	1	1
6	5/2	3/2	1	1	1	0	1	1	1	1
7	3	5/2	7/4	1	1	1	0	1	1	1
8	7/2	9/4	7/4	1	1	1	1	0	1	1
9	4	13/4	3/2	7/4	1	1	1	1	0	1
10	9/2	3	5/2	7/4	1	1	1	1	1	0

Based on this data it appears that given $a > b$ the boiling point would at most be $\left\lfloor \dfrac{a-1}{b} \right\rfloor$, which is approximately 2/5 of the bound we proved in Proposition 5.31.

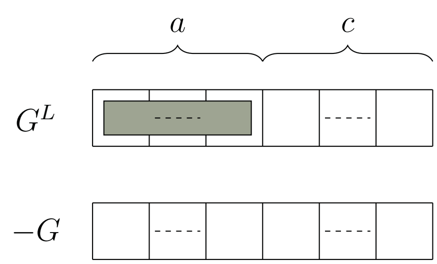
5.2.2 Subtraction Games

Partizan subtraction games are, similarly to Nim, equivalent to SP-games. For the purposes of this section though, we will think of partizan subtraction games as playing on strips similarly to Oab, and with all pieces having to be placed adjacent to the Left end. We let the Left subtraction set be $\{a\}$ and the Right set $\{b\}$. If $a = b$, the game is impartial and the temperature is at most 0. We show the bound for $a \neq b$ below.

Proposition 5.32. *The boiling point of the partizan subtraction game with Left subtraction set $\{a\}$ and Right subtraction set $\{b\}$ with $b > a$ is at most $\dfrac{5}{2}\left(\left\lfloor \dfrac{b-1}{a} \right\rfloor + 1\right)$.*

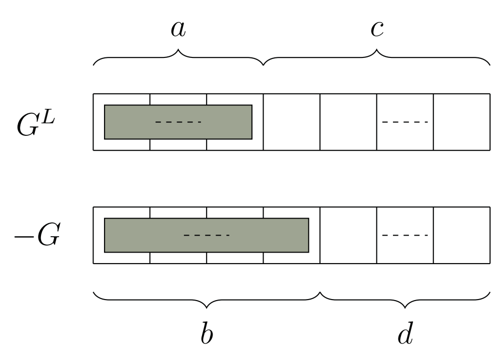
Proof. Suppose without loss of generality that $a < b$ ($a > b$ is simply the negative of this game, thus has the same boiling point). Let n be the length of the strip. For $n < a$ neither player can move and the game is 0, thus the temperature trivially satisfies the bound. For $n \geq a$ we will show that $G^L - G - \left(\lfloor \frac{b-1}{a} \rfloor + 1\right)$ is a Right win if Left goes first. By Proposition 5.27 we then have $\ell(G) \leq \lfloor \frac{b-1}{a} \rfloor + 1$, and therefore $\ell(G^L), \ell(G^R) \leq \left(\lfloor \frac{b-1}{a} \rfloor + 1\right)$. The result then follows by Theorem 5.25

The strategy for Right is to copy Left's move in the other game, so that both strips get shortened by the same amount (a if Left had moved in G^L and b if Left had moved in $-G$). This continues until either Left has no moves remaining or Right cannot respond, thus the situation is as below, again considered just before Left's move.



If $a + c < b$, then Left cannot move in either game and Right wins.

If $a + c \geq b$, Left will move in $-G$. Since we are assuming that Right can no longer respond in G^L, we have $c < b$. We are now in the situation below:



If $d \geq a$, then Right can move in $-G$, to which Left responds in G^L. Thus we may assume that $d < a$. Right has no remaining moves in G^L or $-G$, and thus moves the number to $-\lfloor \frac{b-1}{a} \rfloor$. Left still has $\lfloor c/a \rfloor$ moves remaining in G^L and no moves in $-G$. To each of these moves Right responds by reducing the number. Since $b - 1 \geq c$, Right has the last move and thus wins. $\square$

Using CGSuite, we have the following maximum temperatures on paths up to length 50 for various values of a and b.

	1	2	3	4	5	6	7	8	9	10
1	0	1/2	1	3/2	2	5/2	3	7/2	4	9/2
2	1/2	0	1/2	1/2	1	1	3/2	3/2	2	2
3	1	1/2	0	1/2	1/2	1/2	1	1	1	3/2
4	3/2	1/2	1/2	0	1/2	1/2	1/2	1/2	1	1
5	2	1	1/2	1/2	0	1/2	1/2	1/2	1/2	1/2
6	5/2	1	1/2	1/2	1/2	0	1/2	1/2	1/2	1/2
7	3	3/2	1	1/2	1/2	1/2	0	1/2	1/2	1/2
8	7/2	3/2	1	1/2	1/2	1/2	1/2	0	1/2	1/2
9	4	2	1	1	1/2	1/2	1/2	1/2	0	1/2
10	9/2	2	3/2	1	1/2	1/2	1/2	1/2	1/2	0

Based on this data it appears that given $b > a$ the boiling point would be $\frac{1}{2}\left(\left\lceil \frac{b}{a} \right\rceil - 1\right)$, which is approximately 1/5 of the bound we have proven above.

5.2.3 Domineering Snakes

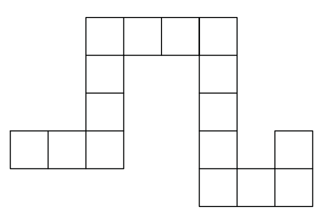
We will consider Domineering snakes that fit within a $2 \times n$ grid.

A Domineering snake is a Domineering position in which the board in some sense has 'width' 1. They can be inductively constructed as follows:

Step 1: Place a single square.

Step n: Attach a new square at the top, right, or bottom edge of the square placed in step $n-1$. When doing so, no 2×2 subgrid may be formed.

An example of a snake is below.



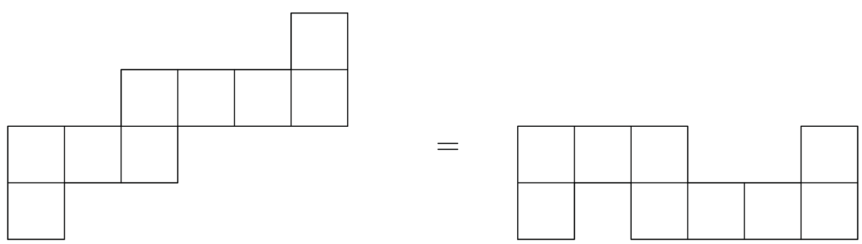
Domineering snakes are interesting as any move, whether by Left or by Right, results in a disjunctive sum of two smaller snakes. Thus they are amenable to a recursive study. Further, they often naturally occur during play on larger grids.

Conway in 1976 [17] (also in [18, pp.114-121]) gives a characterization of snakes in which a square is added to the right alternating with a square up. His results show that such snakes have temperature at most 1.

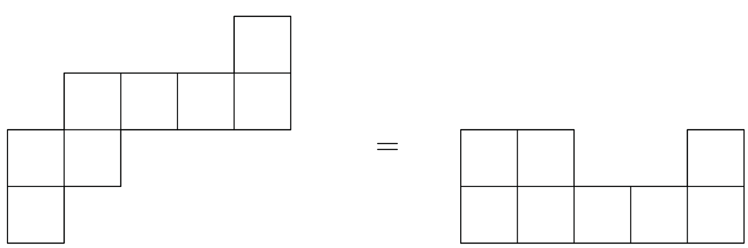
Wolfe in 1993 [64] gives reductions that show it is sufficient to consider snakes in which at most 4 squares are added in either direction. He proceeds to give values of all periodic snakes - those snakes in which, after an initial chain, the number of squares added vertically is always the same, as is the number of squares added horizontally. And finally, values are given for many repeating snakes fitting within a $3 \times n$ grid, so snakes in which always two squares are added horizontally.

The snakes we consider fit into a $2 \times n$ grid, thus they are snakes where always at least two squares are added to the right and exactly one square horizontally, alternating between up and down. We do not make any assumptions on repeating patterns though, so that many of the cases considered here are not covered by the results by Wolfe in 1993 [64].

As an example, the snake below on the left is considered as fitting into a $2 \times n$ grid as it can be folded by alternating vertical addition up and down into the snake on the right without changing the game.



On the other hand, the snake below does not fit into a $2 \times n$ grid since folding it the same way results in a 2×2 subgrid, thus changing the game.

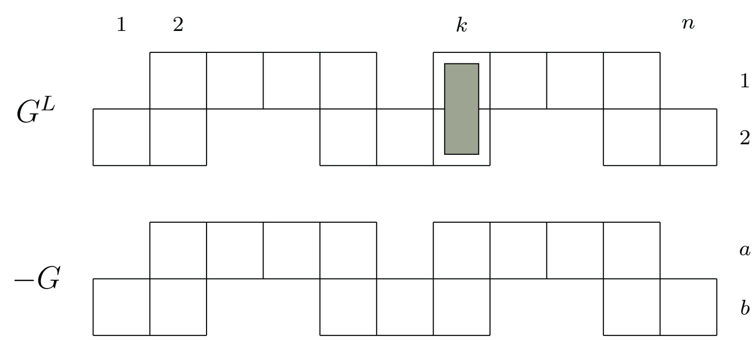


We again are in a situation where any move results in a disjunctive sum of two smaller games, which are still DOMINEERING snakes fitting within a $2 \times n$ grid. Thus bounding $\ell(G)$ for empty boards is sufficient.

Proposition 5.33. *The boiling point of* DOMINEERING *played on a snake fitting within a $2 \times n$ grid is at most 5.*

Proof. Let G be DOMINEERING played on a snake fitting within a $2 \times n$ grid. We will show that $G^L - G - 2$ is a Right win if Left goes first. By Proposition 5.27 we then have $\ell(G) \leq 2$, and therefore $\ell(G^L), \ell(G^R) \leq 4$. The result then follows by Theorem 5.25.

We will label the columns as $1, \ldots, n$ and use the convention that the move already made in G^L in column k and called move 0. The rows in G^L are labelled 1 and 2 and those in $-G$ are labelled a and b. We will denote a move occupying the two squares (x, y) and (u, v) by $\{(x, y), (u, v)\}$. For example move 0 would be $\{(k, 1), (k, 2)\}$. Without loss of generality we will also assume that the square adjacent to move 0 on the left is in row 2 as below.



If Left plays her first move in G^L or in $-G$ without overlapping column k, then Right can mimic the move in the other game, resulting in a disjunctive sum and reducing the board to a smaller size. We may thus assume that Left's move is $(a\,k, a\,k+1)$ or in $(b\,k-1, b\,k)$. When Left makes this move, Right will respond by moving in -2, and then in -1 when Left makes the second of these moves. Note that due to the form of snakes we have chosen, Left cannot overlap either of these moves again, so that Right can continue a mimicking strategy until the end of the game and wins. $\square$

Note that for DOMINEERING snakes in general any move results in a disjunctive sum, but for the above strategy to work we do require the snake fits into a $2 \times n$ grid. If this were not the case, then Left could potentially overlap a move overlapping move 0 again.

5.3 Snort

Winning Ways [6] contains a list of values for various Snort positions. From this list, the temperature of Snort played on a path up to 6 vertices is at most 2. Using the CGSuite code for Snort in Appendix B, we have the following temperatures t for Snort on a path of n vertices:

n	2	3	4	5	6	7	8	9	10	11	12	13	14
t	1	2	3/2	1	0	1	2	2	3/2	3/2	1	2	2

For Snort on a $2 \times n$ grid, the following temperatures were found.

n	2	3	4	5	6	7
t	-1	9/4	-1	5/2	-1	1

More generally, we make the following conjecture:

Conjecture 5.34. *The temperature of* Snort *on a board B is at most the degree of B.*

Intuitively, this conjectures comes from the degree of B being the maximum number of spaces one can 'reserve' for themselves with a single move. There are cases in which the temperature is equal to the degree of the board. Here we need the idea of a **universal vertex**, which is a vertex adjacent to all other vertices in the graph, i.e. one for which the degree is $|V| - 1$.

Proposition 5.35. *Suppose B is a graph with a universal vertex and $|V| = n + 1$. For $G = (\text{Snort}, B)$ we have $G = \pm n$, thus $t(G) = n$.*

Proof. Let v be a universal vertex of B. The good move for either player is to play on v, thus reserving all other vertices for themselves. All other possible moves will be dominated by this move. Thus $G = \{n \mid -n\}$. □

Using the techniques from the last section, it should be possible to at least bound the boiling point for Snort on a path—this would require bounding the length of the confusion interval for all empty paths and paths with one end already having been played on by Left or Right. The strategy would again mostly involve mirroring moves, except for some influence zone around move 0 (simply excluding overlap is not sufficient). Preliminary work indicates that this would result in the boiling point being bounded by 10.

5.4 Further Work

In Theorem 5.25 and all our applications we have bounded the confusion interval for all options. To improve these bounds, we will look at what the thermic version specifically is, thus only having to bound the length of the confusion interval for the thermic options. We have also specifically used that each option is a disjunctive sum in the examples considered and bounded the confusion intervals through this. In reality, the confusion intervals of the options are much smaller though, and looking at these measures specifically would improve the bounds on the boiling point significantly.

Further, all bounds on the length of the vertical and oblique segments in the proof of Theorem 5.24 are tight in certain cases, but are often much larger than the actual length of these segments. When restricting to specific classes of games it should be possible to improve these bounds and therefore the bound on the temperature itself.

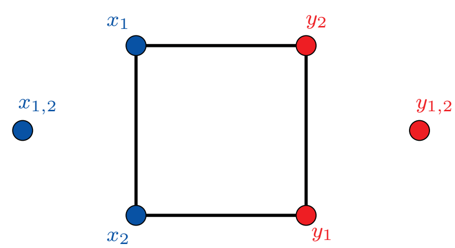
Another approach is to bound the Left and Right stops, and thus the measure of the confusion interval, based on the structure of the legal or illegal complex. We know that moves correspond to the link of the vertex, and that simplices have as their values integers. If we are further able to identify when they have as their values a number, we can inductively identify the stops, and through this process hopefully bound them.

Chapter 6

Impartial Games

Much of the theory in combinatorial game theory, especially during its early days, is about impartial games. An impartial game is one in which the options for both players are the same and thus we do not distinguish Left from Right. This chapter will deal exclusively with impartial SP-games.

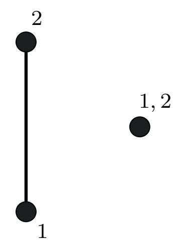
Due to the moves for both players being the same in an impartial game, the legal complex is in some sense symmetric. For example, the legal complex for NIM played on a pile of 2 tokens, or equivalently on K_2, is given by



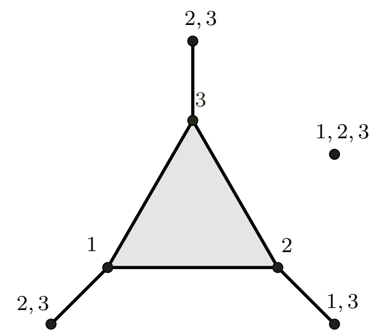
where the variables are indexed by which vertices are occupied by the basic position. For NIM played on K_3, the legal complex becomes

$$\begin{aligned}
&\langle \{x_1, x_2, x_3\}, \{y_1, y_2, y_3\},\\
&\{x_1, x_2, y_3\}, \{y_1, y_2, x_3\}, \{x_1, y_2, x_3\}, \{y_1, x_2, y_3\}, \{y_1, x_2, x_3\}, \{x_1, y_2, y_3\},\\
&\{x_{1,2}, x_3\}, \{x_{1,3}, x_2\}, \{x_{2,3}, x_1\}, \{y_{1,2}, y_3\}, \{y_{1,3}, y_2\}, \{y_{2,3}, y_1\},\\
&\{x_{1,2}, y_3\}, \{x_{1,3}, y_2\}, \{x_{2,3}, y_1\}, \{y_{1,2}, x_3\}, \{y_{1,3}, x_2\}, \{y_{2,3}, x_1\},\\
&\{x_{1,2,3}\}, \{y_{1,2,3}\}\rangle
\end{aligned}$$

Unfortunately, very quickly it becomes difficult to see how the game progresses just from the simplicial complex. A natural step forward, due to the symmetry, is to identify vertices with the same indices. The legal complex of NIM on K_2 then becomes



and for K_3 it becomes



This is the general idea behind this chapter.

We will give some more background on impartial games, then formally define the simplified game complexes. With the impartial game complexes introduced, we will give equivalent statements to our major results on partizan SP-games including in particular that every simplicial complex is the legal complex of some impartial iSP-game.

We then study what game values might be possible given certain structures of the impartial legal complex. We show for example that when the impartial legal complex is pure, the only possible values are 0 and $*$. We also consider an impartial version of the game graph, and its structure when it comes from an impartial SP-game.

If a winning condition is needed, we will again assume normal play.

6.1 Introduction to Impartial Games

For an impartial combinatorial game, since there is no differentiation between Left and Right, game options are just listed in curly brackets, without a divider. For example, the value $*$ can be written as $\{0\}$. A general option of the position P is indicated as P', so $P = \{P'_1, P'_2, \ldots, P'_k\}$.

Fact 6.1 (Sprague-Grundy Theorem, [57, Theorem IV.1.3])**.** *Every short impartial game under normal play is equal to the nimber* $*n$ *for some* n*.*

To find which nimber an impartial game G is equal to, we use the **minimal excluded value** (mex). The $\mathbf{mex}(\boldsymbol{A})$ of a finite set of non-negative integers A is the least non-negative integer not contained in A.

Fact 6.2 ([57, Theorem IV.1.2])**.** *If the impartial game* G *inductively is equal to* $\{*n_1, \ldots, *n_k\}$*, then* $G = *n$ *where* $n = \text{mex}\{n_1, \ldots, n_k\}$*.*

Example 6.3. Consider the impartial game $G = \{0, *1, *4\}$. Since $\text{mex}\{0, 1, 4\} = 2$, we have $G = *2$.

The values of short impartial games under normal play with disjunctive sum as operation form a group (see [57, Section IV.1]). This group is indicated by $\mathbb{G}^{\text{I}}$.

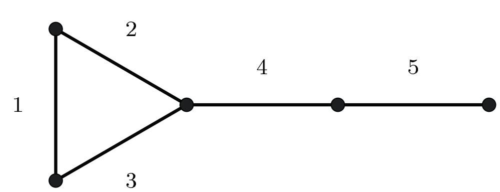
6.2 Impartial Game Complexes

For impartial games, since the options for Left and Right are the same, the legal and illegal complex can be simplified. A **basic position** is, as before, a position in which a single piece has been placed. Our underlying ring is $R = k[x_1, \ldots, x_n]$ where k is any field and n is the number of basic positions.

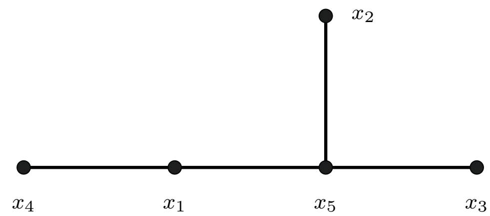
Definition 6.4. For an impartial SP-game (R, B) the **impartial legal complex** $\boldsymbol{\Delta}^{\text{I}}_{\boldsymbol{R,B}}$ is the simplicial complex with vertex set $\{x_1, \ldots, x_n\}$ and whose faces consist of vertices corresponding to the basic positions forming a legal position. The **impartial illegal complex** $\boldsymbol{\Gamma}^{\text{I}}_{\boldsymbol{R,B}}$ is the simplicial complex whose facets correspond to the minimal illegal positions of (R, B).

Example 6.5. The game Arc-Kayles (see Schaefer 1978 [53] or Huggan and Stevens 2016 [33]) is an impartial SP-game. The board can be any graph, and play consists of claiming two adjacent vertices.

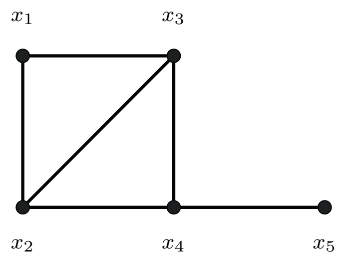
We will look at Arc-Kayles played on the board given below.



We will use the convention that basic position i is the move to claim the two endpoints of edge i. The impartial legal complex then is



And the impartial illegal complex is given by



Remark 6.6. The impartial illegal complex of ARC-KAYLES played on a board B is the line graph of B since the basic positions correspond to the edges, and two moves are illegal together if the two edges have a vertex in common.

Just as previously, we can again get the impartial legal complex of a disjunctive sum from the impartial legal complexes of the summands using the join. And the impartial illegal complex is the disjoint union. The proofs are the same.

Proposition 6.7 (Impartial version of Theorem 1.55)**.** *Let (R, B) and (R', B') be two impartial SP-games with impartial legal complexes $\Delta^I_{R,B}$ and $\Delta^I_{R',B'}$, and with impartial illegal complexes $\Gamma^I_{R,B}$ and $\Gamma^I_{R',B'}$. Then*

$$\Delta^I_{(R,B)+(R',B')} = \Delta^I_{R,B} * \Delta^I_{R',B'}$$

is the impartial legal complex and

$$\Gamma^I_{(R,B)+(R',B')} = \Gamma^I_{R,B} \sqcup \Gamma^I_{R',B'}$$

is the impartial illegal complex of the disjunctive sum $(R, B) + (R', B')$.

Many of the results we have discussed in Chapter 2 also hold when we change to the impartial setting. We will give the equivalent statements here. The proofs are similar by letting $\mathfrak{L}$ be simply the entire vertex set, and thus representing moves by both players, and $\mathfrak{R}$ be empty.

Proposition 6.8 (Impartial version of Proposition 2.1)**.** *Given a simplicial complex Δ, there exist impartial SP-games (R_1, B) and (R_2, B) such that*

(a) $\Delta = \Delta^I_{R_1,B}$ *and*

(b) $\Delta = \Gamma^I_{R_2,B}$.

Theorem 6.9 (Impartial version of Theorem 2.16)**.** *A given simplicial complex Γ is the impartial illegal complex of some impartial iSP-game (R, B) if and only if Γ has no isolated vertices.*

Theorem 6.10 (Impartial version of Theorem 2.17)**.** *Given any simplicial complex Δ, we can construct an impartial iSP-game (R, B) such that $\Delta = \Delta^I_{R,B}$.*

Theorem 6.11 (Impartial version of Theorem 2.20)**.** *Given an impartial SP-game (R, B), there exists an impartial iSP-game (R', B') so that their game graphs are isomorphic.*

With these results established, we will look at possible game values next.

6.3 Game Values and the Game Graph

By the Sprague-Grundy Theorem, impartial games can only take on nimbers as values, and we know that all nimbers are possible since Nim is equivalent to an impartial SP-game (see Remark 1.11). Although this answers the question of whether or not all values of impartial games are possible, it is still interesting to see if certain structures of the legal or illegal complex allow for only certain nimbers.

Using that a game has value 0 if and only if it is a second-player win, i.e. the second player always has a good responding move, we have the following two results.

Proposition 6.12. *Let (R, B) be an impartial SP-game. If all facets of $\Delta^I_{R,B}$ have odd dimension, thus even size, then (R, B) has value 0.*

Proof. Since all facets of $\Delta^{\mathrm{I}}_{R,B}$ have even size, all maximal legal positions of (R, B) also have an even number of pieces. Thus whenever the game ends, the second player will have made the last move, thus wins. □

Proposition 6.13. *If an impartial SP-game* (R, B) *has value* 0*, then the impartial legal complex has at least one facet of odd dimension, thus even size.*

Proof. An impartial SP-game can only be a second-player win if there is at least one maximal legal position with an even number of moves, thus there is a facet of even size. □

We are also able to determine the value of (R, B) immediately if its impartial legal complex is pure.

Proposition 6.14. *If* $\Delta^I_{R,B}$ *is a pure* $(n-1)$*-dimensional simplicial complex, then the corresponding impartial SP-game* (R, B) *has value* 0 *if* n *is even and* $*$ *if* n *is odd.*

Proof. The case in which n is even was already proven in Proposition 6.12.

We will now consider the case in which n is odd. In this case, any move will be to a pure $(n-2)$-dimensional simplicial complex Δ'. We already know that the impartial SP-game corresponding to Δ' has value 0. Thus $(R, B) = \{0\} = *$. □

The above result should be particularly useful in applications since proving the impartial complex is pure immediately results in restricting the possible game values to 0 and $*$. For example, we can use it to give an alternative proof to the following fact shown by Huggan and Stevens in 2016. Note that an **equimatchable graph** is a graph in which all maximal matchings (sets of edges, no two incident) have the same size. Also recall that a simplicial complex is called unmixed if all its minimal vertex covers have the same size.

Fact 6.15 (Huggan and Stevens 2016 [33, Theorem 1])**.** *The value of a game of* Arc-Kayles *played on an equimatchable graph* B*, where* m *is the size of all maximal matchings, is 0 if* m *is even and* $*$ *if* m *is odd.*

Proof. Let $R =$ Arc-Kayles. The complements of the maximal matchings of B correspond to the minimal vertex covers of its line graph $\Gamma^{\mathrm{I}}_{R,B}$. Since B is equimatchable, we then have that $\Gamma^{\mathrm{I}}_{R,B}$ is unmixed, implying that the impartial illegal complex is pure. Furthermore, the maximal independence sets in any graph are the complements of its minimal vertex covers. Thus the size of the maximal matchings of B is equal to the size of the maximal independence sets of $\Gamma^{\mathrm{I}}_{R,B}$. Thus if m is even, then

$\Delta^{\mathrm{I}}_{R,B}$ has odd dimension, i.e. (R, B) has value 0, and if m is odd, then (R, B) has value $*$. □

We also have

Corollary 6.16. *If Δ is the disjoint union of pure simplicial complexes $\Delta_1, \ldots, \Delta_k$ with dimensions $d_1, \ldots, d_k$, then the corresponding impartial SP-game (R, B) has value 0 if all d_i are odd, $*$ if all d_i are even, and $*2$ otherwise.*

Proof. In this case, a move always forces the game into a single component. If all d_i are odd, then any move is in a component with value 0, which is thus the value of the entire complex. Similarly if all d_i are even. If there is a mix though, there are moves to 0 (moving in an even-dimensional complex to an odd dimensional one, all pure) and to $*$ (moving in odd dimensional complex), thus the value is $*2$. □

These preliminary results on the relationship between the structure of the legal complex and possible game values can potentially be expanded upon by considering the structure of the game graph. The game graph is defined similar as in the partizan case, but without labelling the edges:

Definition 6.17. The **game graph** $\mathcal{G}^{\mathrm{I}}_{G}$ of an impartial combinatorial game G is a directed graph where

(a) the vertices represent the legal position in G; and

(b) if there is a move from position P to position Q, then there is an edge from v_P to v_Q.

Similarly as in the partizan case, the game graph of an impartial SP-game needs to reflect that all possible orders of moves are legal between two legal positions. We thus define an equivalent SP-property, which the game graph has to satisfy to come from an impartial SP-game.

Definition 6.18. A game graph of an impartial game is said to satisfy the **impartial SP-property** if it is graded and if, whenever there exists a path from a vertex v to a vertex w consisting of n edges, there exist exactly $n!$ paths between v and w. Furthermore, any two vertices have at most one common successor.

Proposition 6.19 (Impartial version of Proposition 3.11)**.** *Let $\mathcal{G}_G^I$ be the game graph of the impartial combinatorial game G. Then G is an SP-game if and only if $\mathcal{G}_G^I$ satisfies the impartial SP-property.*

Recall that the face poset of a simplicial complex Δ is the the poset whose elements are the faces of Δ ordered by containment. Given an impartial SP-game (R, B) we will call the face poset of $\Delta^{\mathrm{I}}_{R,B}$ the **impartial game poset**, denoted $\boldsymbol{\mathcal{P}}^{\mathrm{I}}_{\boldsymbol{R,B}}$.

Again, the impartial versions of our results regarding the game poset hold:

Proposition 6.20 (Impartial version of Proposition 3.14)**.** *Given an impartial SP-game G, its impartial game poset $\mathcal{P}_G^I$ and impartial game graph $\mathcal{G}_G^I$ are in a one-to-one correspondence.*

Proposition 6.21 (Impartial version of Proposition 3.18)**.** *Given an impartial SP-game (R, B), the impartial legal complex $\Delta^I_{R,B}$ and the game graph $\mathcal{G}^I_{R,B}$ are in one-to-one correspondence.*

Lemma 6.22 (Impartial version of Lemma 3.20)**.** *Every impartial game poset of an impartial SP-game is a meet semilattice.*

6.4 Further Work

Similar to questions in the general case, we are interested in whether we are able to completely describe how the structure of the impartial legal or illegal complex determines the normal play value of the corresponding impartial game.

Of course, we are again also interested in considering all these questions under misère play.

Chapter 7

Experimentation Towards Cohen-Macaulayness

Combinatorial structures, whose equivalent algebraic structures are Cohen-Macaulay, usually turn out to have best possible behaviour (see [13, Chapter 5] and [31, Chapters 8 and 9]). We are interested in the effects of Cohen-Macaulayness on strong placement games. This chapter will focus on investigating when the ideals of SP-games are Cohen-Macaulay and understanding the implications for the games. Some possible "good behaviour" for SP-games related to Cohen-Macaulayness, for example, could be that only certain game values are possible or that the game tree needs to have a specific structure.

Although we are not able to give a good characterization of Cohen-Macaulay games, this chapter will give some indications of what the possibilities are, in particular that game value is not a good characterizing measure.

We begin with the definition of Cohen-Macaulay ideals and combinatorial properties of their facet and Stanley-Reisner complexes. We then look at the Cohen-Macaulayness of the games Snort and Col. Some common combinatorial structures that guarantee Cohen-Macaulayness are shellability and graftedness. We examine each of these and their implications on the SP-games. We then slightly change direction and consider the Cohen-Macaulayness of impartial games since no $(\mathfrak{L}, \mathfrak{R})$-labellings need to be considered. A result of particular interest is that the Cohen-Macaulayness of the illegal ideal, whether in the impartial or partizan case, is "hereditary", i.e. if a game has a Cohen-Macaulay illegal ideal, then all its options will also have a Cohen-Macaulay illegal ideal (see Subsection 7.5.1).

This chapter should be considered as a first step towards studying Cohen-Macaulay games.

7.1 Background on Cohen-Macaulay Ideals

We begin by defining when a square-free monomial ideal is Cohen-Macaulay before going into computational and partial results. Note that the definition of Cohen-Macaulayness we give below can be generalized to other algebraic structures, but we can restrict to the polynomial ring and monomial ideal case for our purposes.

Definition 7.1. Given a polynomial ring $S = k[x_1, \dots, x_n]$ and an ideal I, an element $a \in S$ is said to be ***S/I*-regular** if for all $m \in S/I$ the equality $am = 0$ implies $m = 0$. A sequence $a_1, \dots, a_n$ of S is said to be an ***S/I*-regular sequence** if (i) a_j is $(S/I)/[(a_1, \dots, a_{j-1})(S/I)]$-regular and (ii) $(S/I)/[(a_1, \dots, a_j)(S/I)] \neq 0$ for all $j = 1, \dots, n$. The **depth** of S/I is the maximum length of an S/I-regular sequence with elements from the maximum ideal $(x_1, \dots, x_n)$.

The **Krull dimension** of a commutative ring S is the supremum of the lengths of chains of prime ideals of S.

Definition 7.2. The quotient S/I is called **Cohen-Macaulay** if its depth is equal to its Krull dimension. An ideal I over a polynomial ring S is called Cohen-Macaulay if S/I is Cohen-Macaulay.

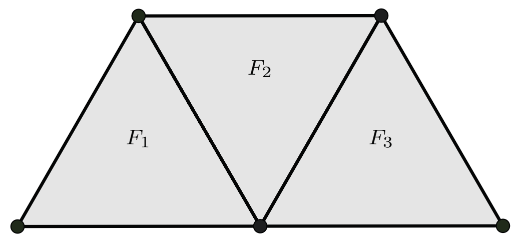
Cohen-Macaulayness is a very algebraic property, which results in nice combinatorial properties though. We will now consider some useful necessary and sufficient conditions for an ideal to be Cohen-Macaulay.

Fact 7.3 ([13])**.** *If an ideal I is Cohen-Macaulay, then $\mathcal{F}(I)$ is unmixed and $\mathcal{N}(I)$ is pure.*

Shellability, usually considered for pure simplicial complexes, was generalized by Björner and Wachs in 1996 [8] for nonpure shellable complexes. We will use their definition:

Definition 7.4. A simplicial complex is called **shellable** if its facets can be ordered as $F_1, \dots, F_k$ such that $\langle F_1, F_2, \dots, F_{j-1}\rangle \cap \langle F_j \rangle$ is pure and $(\dim F_j - 1)$-dimensional for all $j = 2, \dots, k$.

Example 7.5. The simplicial complex below is shellable with facets ordered as F_1, F_2, F_3.



The following statement is easy to prove, but does not seem to appear in literature. We will use that every connected graph contains a spanning tree, thus a subgraph which is a tree and contains all vertices.

Proposition 7.6. *A graph has a shelling if and only if it is connected.*

Proof. First, suppose that the graph is not connected. When ordering the facets, thus the edges, at some point one is forced to list an edge which is not incident with any of the previously listed ones (when switching from one connected component to another), thus the intersection of this edge with all previous ones is empty and one dimension lower than required for a shelling.

On the other hand, assume that the graph G is connected. Fix a spanning tree T. We will order the vertices and edges as follows: Pick a vertex v_1 and an edge $E_1 = \{v_1, v_2\}$ contained in T. Next, pick an edge $E_2 = \{v_2, v_3\}$ in T. Continue this process until a leaf has been reached, at which point one picks an edge of T incident with a previous vertex. Repeat this procedure until all edges of T have been ordered. Then list any remaining edges of G in any order.

For all $j = 2, \ldots, k$, the simplicial complex $\langle E_1, E_2, \ldots, E_{j-1}\rangle \cap \langle E_j\rangle$ consists of only one or two vertices, i.e. is 0-dimensional and pure, as the process above ensures that after the first edge every other edge intersects with a previous one. Thus this is a shelling of the connected graph. $\square$

Shellability is connected with Cohen-Macaulayness via the following fact.

Fact 7.7 (Stanley 1996 [59]). *If a simplicial complex Δ is pure and shellable, then $\mathcal{N}(\Delta)$ is Cohen-Macaulay.*

Recall that the legal complex $\Delta_{R,B}$ is the facet complex of the legal ideal $\mathcal{L}_{R,B}$ and the Stanley-Reisner complex of the illegal ideal $\mathcal{ILL}_{R,B}$, while the illegal complex $\Gamma_{R,B}$ is the facet complex of the illegal ideal (Proposition 2.1). Given Fact 7.3 we then know that for any placement game (R, B) we have the following corollaries:

Corollary 7.8. *If $\mathcal{L}_{R,B}$ is Cohen-Macaulay, then $\Delta_{R,B}$ is unmixed.*

Corollary 7.9. *If $\mathcal{ILL}_{R,B}$ is Cohen-Macaulay, then $\Delta_{R,B}$ is pure and $\Gamma_{R,B}$ is unmixed.*

Note that $\Delta_{R,B}$ being pure and $\Gamma_{R,B}$ being unmixed are equivalent since $\Delta_{R,B}$ is pure if and only if $(\Delta_{R,B})^c = (\Gamma_{R,B})_M$ is pure (by definition of the complement and Fact 1.51) if and only if $\Gamma_{R,B}$ is unmixed (by definition of unmixedness).

As a consequence of Corollaries 7.8 and 7.9 two interesting questions are the following:

Question 7.10. When is $\Delta_{R,B}$ pure and $\Gamma_{R,B}$ unmixed?

Question 7.11. When is $\Delta_{R,B}$ unmixed?

In particular, we would like to understand whether there are necessary conditions for the ruleset R or the board B such that the game complexes are pure/unmixed, as this would give necessary conditions for an SP-game to be considered Cohen-Macaulay.

Furthermore, by Fact 7.7 we have

Corollary 7.12. *If $\Delta_{R,B}$ is pure and shellable, then $\mathcal{ILL}_{R,B}$ is Cohen-Macaulay.*

Thus we are interested in

Question 7.13. When is $\Delta_{R,B}$ shellable?

We are hoping that shellability of the legal complex will reveal a good underlying structure of the game.

7.2 Results for Specific SP-Games

We will begin with computational and partial results exploring Questions 7.10, 7.11, and 7.13 for specific SP-games, namely SNORT and COL.

Board	Δ unmixed	Δ pure/ Γ unmixed	Δ shellable	$\mathcal{ILL}$ CM
K_3	yes	yes	no	no
K_4	yes	yes	no	no
K_5	yes	yes	no	no
K_6	yes	yes	no	no
C_4	yes	no	no	no
C_5	yes	no	no	no
C_6	no	no	no	no
C_7	no	no	no	no
C_8	no	no	no	no
P_3	no	no	no	no
P_4	no	no	no	no
P_5	no	no	no	no

Table 7.1: Properties of the Game Complexes and Ideals of SNORT on Different Boards

7.2.1 SNORT

Table 7.1 gives some answers to the above questions for SNORT played on a variety of boards. The calculations for the table have been done using the computer algebra software Macaulay2 [29]. Here, K_n indicates the complete graph on n vertices, C_n the cycle on n vertices, and P_n the path on n vertices.

We will now prove some of these results in greater generality.

Proposition 7.14. *The legal complex $\Delta_{\text{SNORT},B}$ is pure if and only if B is a disjoint union of complete graphs.*

Proof. When the board B is a disjoint union of several connected graphs $B_1, \ldots, B_k$, playing SNORT on B corresponds to the disjunctive sum of playing SNORT on each of the boards. Thus the legal complex $\Delta_{\text{SNORT},B}$ will be the join $\Delta_{\text{SNORT},B_1} * \cdots * \Delta_{\text{SNORT},B_k}$ by Theorem 1.55. Since in addition the join of several simplicial complexes is pure if and only if each one is pure, it is sufficient for us to show the statement for B being a connected graph.

When playing SNORT on the complete graph K_n, since all vertices are connected, as soon as one player has placed a piece, the other cannot play in any other vertex. Thus the maximal legal positions are $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$, giving that $\Delta_{\text{SNORT},B}$ is pure.

Conversely, we will show that if B is not a complete graph, then $\Delta_{\textsc{Snort},B}$ is not pure. Assume that B is a connected graph of size n which is not a complete graph. We know that two facets of $\Delta_{\textsc{Snort},B}$ are $\{x_1,\ldots,x_n\}$ and $\{y_1,\ldots,y_n\}$, both of size n. Since B is not complete, there exist two vertices i and j which are not connected, but have a common neighbour k. It is possible for Left to play in position i and for Right to play in position j, implying that neither player can play in position k. Thus any face including x_i and y_j cannot include x_k or y_k. We know that there has to be a facet which does include x_i and y_j, and therefore has to be of size smaller than n. This shows that $\Delta_{\textsc{Snort},B}$ is not pure. □

Proposition 7.15. *The illegal ideal $\mathcal{ILL}_{\textsc{Snort},B}$ is not Cohen-Macaulay for any board B.*

Proof. From Proposition 7.14 and Corollary 7.9, we immediately have that if B is a graph which is not a disjoint union of complete graphs, then $\mathcal{ILL}_{\textsc{Snort},B}$ is not Cohen-Macaulay.

Now if B is a complete graph or a disjoint union of complete graphs, then the legal complex $\Delta_{\textsc{Snort},B}$ has at least two disconnected components, which implies that $\mathcal{ILL}_{\textsc{Snort},B} = \mathcal{N}(\Delta_{\textsc{Snort},B})$ is not Cohen-Macaulay. □

This is a first indication that an SP-games whose illegal ideal is Cohen-Macaulay needs to have a high degree of "connectivity" in some sense. In SNORT there is not enough intersection between all legal positions.

Proposition 7.16. *The legal complex $\Delta_{\textsc{Snort},K_n}$ is unmixed.*

Proof. We know that the facets of $\Delta_{\textsc{Snort},K_n}$ are $\{x_1,\ldots,x_n\}$ and $\{y_1,\ldots,y_n\}$, which are disjoint simplices. Thus any minimal vertex cover is of the form $\{x_i,y_j\}$, i.e. all have size 2. □

Given Table 7.1, we speculate the following might be true:

- For $n \geq 6$ $\Delta_{\textsc{Snort},C_n}$ is not unmixed. Thus $\mathcal{L}_{\textsc{Snort},C_n}$ is not Cohen-Macaulay for $n \geq 6$.
- The legal complex $\Delta_{\textsc{Snort},P_n}$ is not unmixed. Thus $\mathcal{L}_{\textsc{Snort},P_n}$ is not Cohen-Macaulay.

Note that we will show in Corollary 7.30 that $\Delta_{\text{SNORT},B}$ is not shellable if B is connected and has at least two vertices.

7.2.2 COL

Similar to SNORT, we also consider Questions 7.10, 7.11, and 7.13 for the SP-game COL. Table 7.2 summarizes the calculations done in Macaulay2 [29].

Board	Δ unmixed	Δ pure/ Γ unmixed	Δ shellable	$\mathcal{ILL}$ CM
K_3	no	yes	yes	yes
K_4	no	yes	yes	yes
K_5	no	yes	yes	yes
K_6	no	yes	yes	yes
C_4	yes	no	no	no
C_5	no	yes	no	no
C_6	no	no	no	no
C_7	no	no	no	no
C_8	no	no	no	no
P_3	no	no	no	no
P_4	no	no	no	no
P_5	no	no	no	no

Table 7.2: Properties of the Game Complexes and Ideals of COL on Different Boards

We have the following more general results when playing COL on a complete graph or a path.

Proposition 7.17. *For $n \geq 3$, Δ_{COL,K_n} is pure and shellable, but not unmixed. Thus $\mathcal{ILL}_{\text{COL},K_n}$ is Cohen-Macaulay and $\mathcal{L}_{\text{COL},K_n}$ is not Cohen-Macaulay.*

Proof. When playing on the complete graph K_n, at most two pieces can be placed (one from each player) since all vertices are connected. Thus Δ_{COL,K_n} is a graph and therefore pure. The edges are all of the form $\{x_i, y_j\}$ where $i \neq j$.

We will first show that this graph is connected. Given any two vertices, we can find a path between them as follows:

Case 1: If the vertices are y_i and y_j, then a path would be (y_i, x_k, y_j), where k is chosen such that $i, j \neq k$. Similar if the vertices are x_i and x_j.

Case 2: If the vertices are x_i and y_j with $i \neq j$, then the two vertices have an edge in common. If $i = j$, then a path would be (x_i, y_k, x_l, y_i), where k, l are chosen such that $k, l \neq i$ and $k \neq l$.

Thus $\Delta_{\textsc{Col},K_n}$ is connected, implying that $\mathcal{ILL}_{\textsc{Col},K_n}$ is Cohen-Macaulay by Proposition 7.6 and Fact 7.7.

Now consider the sets $A = \{x_1, \dots, x_n\}$ and $B = \{x_2, \dots, x_n, y_2, \dots, y_n\}$. We will show that both A and B are minimal vertex covers of $\Delta_{\textsc{Col},K_n}$. Clearly, for any facet F (i.e. edge) of $\Delta_{\textsc{Col},K_n}$ we have $F \cap A \neq \emptyset$ and $F \cap B \neq \emptyset$, thus both A and B are vertex covers. Now A is a minimal vertex cover since when removing x_j the facet $\{x_j, y_k\}$ with $k \neq j$ is no longer covered. Similarly, when removing x_i from B the edge $\{x_i, y_1\}$ will not be covered anymore, and when removing y_j the edge $\{x_1, y_j\}$ is no longer covered. Thus B also is a minimal vertex cover. Since $|B| = 2n - 2 > |A| = n$ for $n \geq 3$, we have that $\Delta_{\textsc{Col},K_n}$ is not unmixed. $\square$

Note that $\Delta_{\textsc{Col},K_1} = \langle \{x_1\}, \{y_1\} \rangle$ and $\Delta_{\textsc{Col},K_2} = \langle \{x_1, y_2\}, \{x_2, y_1\} \rangle$ are both pure and unmixed, but not shellable as they are disconnected.

Proposition 7.18. *For $n \geq 3$, $\Gamma_{\textsc{Col},P_n}$ is not unmixed, thus $\mathcal{ILL}_{\textsc{Col},P_n}$ is not Cohen-Macaulay.*

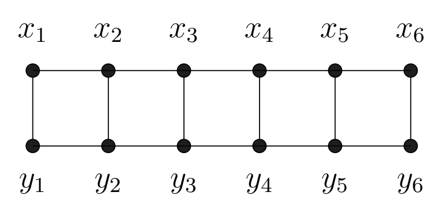
Proof. The illegal complex $\Gamma_{\textsc{Col},P_n}$ consists of the edges $\{x_i, x_{i+1}\}, \{y_i, y_{i+1}\}$ $(i = 1, \dots, n-1)$ and $\{x_j, y_j\}$ $(j = 1, \dots, n)$, thus it is the Cartesian product of P_n with P_2.

If n is even, two minimal vertex covers are $\{x_1, y_2, x_3, \dots, x_{n-1}, y_n\}$ (of size n) and $\{x_1, x_2, y_2, y_3, x_4, y_5, x_6, \dots, y_{n-1}, x_n\}$ (of size $n+1$).

If n is odd, two minimal vertex covers are $\{x_1, y_2, x_3, \dots, y_{n-1}, x_n\}$ (of size n) and $\{x_1, x_2, y_2, y_3, x_4, y_5, x_6, \dots, x_{n-1}, y_n\}$ (of size $n+1$). $\square$

Example 7.19. For $\textsc{Col}$ played on P_6 we have the illegal complex

$\Gamma_{\textsc{Col},P_6} =$



Two minimal vertex covers of differing size then are

$$\{x_1, y_2, x_3, y_4, x_5, y_6\} \text{ and } \{x_1, x_2, y_2, y_3, x_4, y_5, x_6\}.$$

In Huntemann's Masters thesis [34] (Note after Lemma 3.16) it was shown that $\Delta_{\text{SNORT},B}$ and $\Delta_{\text{COL},B}$ are isomorphic for bipartite graphs B when ignoring the $(\mathfrak{L}, \mathfrak{R})$-labelling. A bipartite graph, except for a disjoint union of K_2s, is not a disjoint union of complete graphs. In particular paths are not disjoint unions of complete graphs, and as a result we can also deduce Proposition 7.18 from Proposition 7.15.

More generally, we have

Corollary 7.20. *If B is a bipartite graph which is not a disjoint union of copies of K_2, then $\Delta_{\text{COL},B}$ is not pure, and thus $\mathcal{ILL}_{\text{COL},B}$ is not Cohen-Macaulay.*

Note that Proposition 7.18 is also a corollary to this, but an alternative, independent proof is given above.

Given Table 7.2, the following might furthermore be true:

- The legal complex Δ_{COL,P_n} is not unmixed and not shellable. Thus $\mathcal{L}_{\text{COL},P_n}$ is not Cohen-Macaulay.
- For $n \geq 6$, Δ_{COL,C_n} is not pure, not unmixed, and not shellable. Thus neither $\mathcal{L}_{\text{COL},C_n}$ nor $\mathcal{ILL}_{\text{COL},C_n}$ are Cohen-Macaulay for $n \geq 6$.

7.3 Whiskering and Grafting

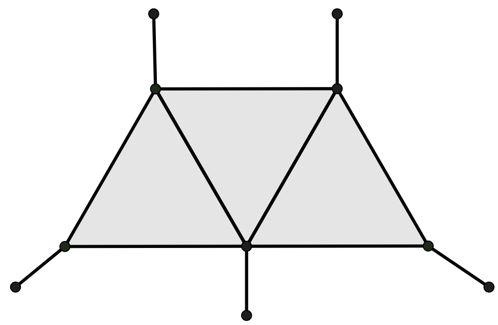
Whiskering and grafting are two operations that turn any simplicial complex into one with a Cohen-Macaulay facet ideal (Fact 7.26 below). We will look at these two operations in this section from a game theoretical point of view.

In 1990, Villarreal [63] showed that the facet ideal of a "whiskered" graph, also known as a "suspended" graph, is Cohen-Macaulay. Whiskering of a graph can be generalized to all simplicial complexes as follows:

Definition 7.21 (Faridi 2005 [25])**.** Let Δ be a simplicial complex with vertex set $V = \{v_1, \ldots, v_n\}$. The simplicial complex Δ' with the vertex set $V \cup \{w_1, \ldots, w_n\}$ is the **whiskering** of Δ if

$$\Delta' = \Delta \cup \langle \{v_i, w_i\} \mid v_i \in V \rangle.$$

Example 7.22. The below simplicial complex is the whiskering of the simplicial complex in Example 7.5.



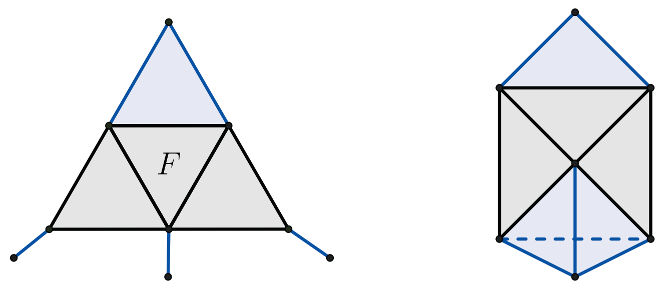
A generalization of whiskering is grafting, for which we require the definition of a leaf.

Definition 7.23 (Faridi 2004 [24])**.** Given a simplicial complex $\Delta = \langle F, F_1, \ldots, F_k \rangle$, a facet F is called a **leaf** of Δ if either it is the only facet or there exists another facet F_i such that $F \cap F_j \subseteq F \cap F_i$ for all $F_j \in \langle F_1, \ldots, F_k \rangle$ with $i \neq j$. If in addition $F \cap F_i \neq \emptyset$, then F_i is called a **joint** of F.

Definition 7.24 (Faridi 2005 [25])**.** Let $\Delta = \langle F_1, \ldots, F_k \rangle$ be a simplicial complex with vertex set $V = \{v_1, \ldots, v_n\}$. The simplicial complex $\Delta' = \Delta \cup \langle G_1, \ldots, G_\ell \rangle$ on the vertex set $V \cup \{w_1, \ldots, w_k\}$ is a **grafting** of Δ if

1. Every vertex of Δ is contained in some G_i;
2. $G_1, \ldots, G_\ell$ are exactly the leaves of Δ';
3. $F_i \neq G_j$ for any i, j;
4. $G_i \cap G_j = \emptyset$ for any $i \neq j$; and
5. If F_i is a joint of Δ', then $\langle F_1, \ldots, F_{i-1}, F_{i+1}, \ldots, F_k \rangle \cup \langle G_1, \ldots, G_\ell \rangle$ is also grafted.

Example 7.25. The whiskered simplicial complex in Example 7.22 is also a grafted simplicial complex. Both of the simplicial complexes below are not graftings of the simplicial complex in Example 7.5. The one on the left fails the last condition since removing the facet labelled F, which is the joint of the 2-dimensional leaf, results in a simplicial complex which is not grafted. The simplicial complex on the right is not a grafting as the tetrahedron is not a leaf.



Fact 7.26 (Faridi 2005 [25])**.** *If Δ is a grafted simplicial complex, then $\mathcal{F}(\Delta)$ is Cohen-Macaulay.*

Since whiskering and grafting turn any simplicial complex into one whose facet ideal is Cohen-Macaulay, we are interested in how a game changes if either of its game complexes are whiskered or grafted. Below we have a partial answer to this in the case of the legal complex.

Theorem 7.27. *Given an $(\mathfrak{L}, \mathfrak{R})$-labelled simplicial complex Δ, there exists a grafting Δ' such that the game (R, B) with legal complex Δ' has game value 0.*

Proof. Let the vertex set of Δ be $V = \{v_1, \dots, v_n\}$. Let Δ_1 be any grafting of Δ with vertex set $V \cup \{w_1, \dots, w_m\}$. Then fix an $(\mathfrak{L}, \mathfrak{R})$-labelling for Δ_1 such that the labels for vertices in V are preserved.

Let the leaves of Δ_1 be $H_1, \dots, H_\ell$. To each H_i add sufficiently many new vertices $z_{i_1}, \dots, z_{i_j}$ labelled such that $|V(H_i) \cap \mathfrak{L}| = |V(H_i) \cap \mathfrak{R}|$ where $V(H_i)$ is the vertex set of H_i and such that at least one vertex z_{i_a} is labelled $\mathfrak{L}$, and at least one labelled $\mathfrak{R}$. Setting $G_i = H_i \cup \{z_{i_1}, \dots, z_{i_j}\}$, we then define $\Delta' = \Delta \cup \langle G_1, \dots, G_\ell \rangle$. It is easy to see that this still is a grafting of Δ.

Next, we will show that the SP-game (R, B) with legal complex Δ' has value 0 by showing that it is a second-player win (Fact 4.4).

Assume without loss of generality that Left moves first. Let the vertex corresponding to her first move be x_k, and let G_i be a leaf of Δ' containing x_k.

Further, by construction, there exists at least one vertex $y_j \in \mathfrak{R}$ which is only contained in G_i and no other facet of Δ'. A good move for Right is to make the one corresponding to y_j. We now have $\mathrm{link}_{\Delta'}(\{x_k, y_j\}) = \langle G_i \setminus \{x_k, y_j\} \rangle$, which has an equal number of vertices belonging to $\mathfrak{L}$ and to $\mathfrak{R}$. We have shown in Corollary 4.22 that the game with legal complex $\mathrm{link}_{\Delta'}(\{x_k, y_j\})$ has value 0, thus is a second player win. In turn, this also implies that the second player has a good move in (R, B) and wins. $\square$

Note though that not every whiskering or grafting turns a game into a second player win—how the game changes depends greatly on how the $(\mathfrak{L}, \mathfrak{R})$-labelling is extended to the new vertices.

Remark 7.28. Consider an SP-game G with legal complex Δ and a second SP-game G' with legal complex Δ' where Δ is a subcomplex of Δ'. Theorem 7.27 implies that the value of G' does not give us any information about the possible values of G.

Further, not every SP-game with Cohen-Macaulay legal ideal has value 0, and an SP-game having value 0 does not imply a Cohen-Macaulay legal ideal. For example, we have shown in Section 4.2 that the game (R, B) with legal complex $\Delta_{R,B} = \langle\{x_1, x_2\}, \{x_1, y_1\}\rangle$ has game value $1*$, but as this is a whiskered graph, its legal ideal is Cohen-Macaulay. On the other hand, Col played on the complete graph K_n is a second player win, thus has value 0, but we have shown in Proposition 7.17 that the legal ideal is not Cohen-Macaulay.

This implies that game value is not a good characteristic to define what a Cohen-Macaulay game is.

7.4 Shellability

In the case that the illegal complex is a graph, for example for all independence games, we know that the legal complex is the independence complex of a related graph (see Section 2.3). There are many known results on the shellability of an independence complex based on properties of the graph.

For example Van Tuyl and Villarreal [62] show the following:

Fact 7.29 (Van Tuyl and Villarreal 2008 [62])**.** *Let G be a bipartite graph. Then the independence complex of G is shellable if and only if G has two adjacent vertices x and y with $\deg(x) = 1$ such that the bipartite graphs $G \backslash (\{x\} \cup N_G(x))$ and $G \backslash (\{y\} \cup N_G(y))$ have shellable independence complexes.*

A consequence of this is the following:

Corollary 7.30. *For* SNORT*, we have* $\Delta_{\text{SNORT},B}$ *is not shellable if* B *is connected and has at least two vertices.*

Proof. For SNORT, the legal complex is the independence complex of the illegal complex since all basic positions are contained in some minimal illegal position. The illegal complex is a bipartite, connected graph when B is connected, with the two parts being $\mathfrak{L}$ and $\mathfrak{R}$. Furthermore, there are no vertices of degree 1 if B has at least two vertices. Thus by Fact 7.29 we have that the legal complex is not shellable. □

The following result, first proven by Van Tuyl and Villarreal in [62] for chordal graphs, then generalized by Woodroofe in 2009 [65] to include graphs with induced cycles of length 5, will be useful.

Fact 7.31 (Woodroofe 2009 [65])**.** *If* G *is a graph with no chordless cycles of length other than 3 or 5, then its independence complex is shellable.*

We will give an application of this result for the game of NODE-KAYLES in the next section.

7.5 Impartial Games

A major issue with understanding Cohen-Macaulayness and related properties for partizan games is that for the games themselves the bipartition of the vertices in the legal complexes are vital, while from the algebraic side this is not being considered.

We will now change direction and instead consider the same questions for impartial games, where no bipartition has to be considered. The idea is that understanding implications of Cohen-Macaulayness for impartial games will give a direction for defining Cohen-Macaulay $(\mathfrak{L}, \mathfrak{R})$-labelled simplicial complexes.

We begin with two results related to possible game values. Unlike for partizan games, Cohen-Macaulayness, through purity, narrows down the value set to finitely many possible values.

Proposition 7.32. *If the impartial illegal ideal of* (R, B) *is Cohen-Macaulay, then* (R, B) *has value* 0 *or* $*$.

Proof. If the impartial illegal ideal is Cohen-Macaulay, then the legal complex has to be pure, and by Proposition 6.14 we then get the result. □

For the partizan case on the other hand, we have infinitely many possible values. We showed in Corollary 4.22 that all integers are values of simplices alone. In Section 4.2 we also showed that many other values come from pure simplicial complexes, such as $1*$, ± 1, and $\{1 \mid *\}$. Nonetheless, computational evidence seems to indicate that the possible values given a pure simplicial complex are restricted.

Restricting again to impartial games, the next result implies that a game with whiskered impartial legal complex has value 0.

Proposition 7.33. *For an impartial SP-game (R, B), if the impartial legal complex is grafted with all leaves having even size, then (R, B) has value 0.*

Proof. We will show that (R, B) is a second player win. First, let Δ' be the simplicial complex with vertex set $\{v_1, \ldots, v_m\}$ from which we obtain $\Delta^{\mathrm{I}}_{R,B}$ by grafting, and let $\{v_1, \ldots, v_m, w_1, \ldots, w_n\}$ be the vertex set of $\Delta^{\mathrm{I}}_{R,B}$.

Suppose the first player chooses the vertex v_i as their move, and let F_j be a leaf containing v_i. The second player can respond by playing some w_k contained in F_j. Now the game is forced to continue in $F_j \setminus \{v_i, w_k\}$, which has even size, and is thus a second player win.

If the first player moves on a vertex w_k though, play is already forced to continue in the leaf containing w_k, thus is again a second player win. □

In the game of Node-Kayles, players place tokens not adjacent to any previously placed tokens. Thus the impartial illegal complex is equal to the board.

Proposition 7.34. *If B is a graph with no chordless cycles of length other than 3 or 5, then $\Delta^{I}_{\text{Node-Kayles},B}$ is shellable. If in addition all maximal independent sets of B are the same size, then the impartial illegal ideal is Cohen-Macaulay.*

Proof. If B, and thus $\Gamma^{\mathrm{I}}_{\text{Node-Kayles},B}$, has no chordless cycles of length other than 3 or 5, then by Fact 7.31 we have that $\Delta^{\mathrm{I}}_{\text{Node-Kayles},B}$ is shellable.

If all maximal independence sets are the same size, then $\Delta^{\mathrm{I}}_{\text{Node-Kayles},B}$, being the independence complex of B, is pure. □

7.5.1 Reisner's Criterion

Reisner's criterion gives both a necessary and sufficient condition for the Cohen-Macaulayness of the Stanley-Reisner ideal of a simplicial complex. Before stating this criterion, we will define the reduced homology of a simplicial complex.

Given a simplicial complex Δ with a fixed order on the vertices, and a field k, let C_i be the vector space over k whose basis elements correspond to the faces of Δ with $i+1$ vertices. We can then construct a chain complex

$$\cdots \to C_{i+1} \xrightarrow{\delta_{i+1}} C_i \xrightarrow{\delta_i} C_{i-1} \to \ldots \to C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \to 0$$

where the boundary map δ_i for $i \geq 0$ is defined by $\delta_i(\sigma) = \sum_{k=0}^{i}(-1)^k\,(\sigma \setminus \{x_{j_k}\})$ for a face $\sigma = \{x_{j_0}, \ldots, x_{j_i}\}$ of Δ. Then:

Definition 7.35. The ***i*th reduced homology** of a simplicial complex Δ over a field k is defined as $\tilde{H}_i(\Delta; k) = \ker \delta_i / \operatorname{im} \delta_{i+1}$ where δ_i is the boundary map as given above.

The special case in which the reduced homologies are all 0 is called acyclicity.

Definition 7.36. A simplicial complex Δ is called **acyclic** if $\tilde{H}_i(\Delta; k) = 0$ for all i.

The following theorem is known as Reisner's Criterion.

Fact 7.37 (Reisner 1976 [52])**.** *Fix a simplicial complex Δ. Then $\mathcal{N}(\Delta)$ is Cohen-Macaulay if and only if for all $i < \dim(\operatorname{link}_\Delta F)$ and all faces F of Δ (including the empty face) the following holds:*

$$\tilde{H}_i(\operatorname{link}_\Delta F; k) = 0.$$

An equivalent statement to Reisner's Criterion is the following:

Fact 7.38 (Reisner 1976 [52])**.** *Fix a simplicial complex Δ. Then $\mathcal{N}(\Delta)$ is Cohen-Macaulay if and only if $\tilde{H}_i(\Delta; k) = 0$ for all $i < \dim(\Delta)$ and the links of all vertices of Δ are Cohen-Macaulay.*

Given Δ being the (impartial) legal complex of an (impartial) SP-game G, we know that the links of the vertices correspond to the options of G (see Remark 4.1). This indicates that if the illegal ideal of G is Cohen-Macaulay, then the illegal ideals of all its options are as well.

Game properties that are closed under options are particularly nice for inductive arguments, especially structure of a game tree. Such properties are called hereditarily closed:

Definition 7.39. Let A be a set of short games H. Then A is called **hereditarily closed** if for all games $G \in A$, whenever H is an option of G, then $H \in A$.

So the set of (impartial) SP-games with Cohen-Macaulay illegal ideal is hereditarily closed.

Given this information, to understand Cohen-Macaulayness for a game, we need to understand what it means for the legal complex to be acyclic, and acyclicity to be hereditary.

7.6 Further Work

Building onto the preliminary results in this chapter, the following give some ideas for future work on the Cohen-Macaulayness of SP-games.

Meaning for Placement Games

The main reason why we study properties of game complexes is that we want to learn more about their corresponding SP-games. Thus we are very interested in what it means for an SP-game if its legal or illegal complex is pure, unmixed, or shellable, and if its legal or illegal ideal is Cohen-Macaulay, amongst others. Except for the legal complex being pure meaning that all maximal legal positions are of the same size, there does not seem to be an intuitive answer to this.

The Simplicial Complexes $\mathcal{N}(\mathcal{L}_{R,B})$ and Δ^c

Furthermore, since $\mathcal{L}_{G,B}$ being Cohen-Macaulay implies that its Stanley-Reisner complex is pure, we are interested in what the Stanley-Reisner complex of the legal ideal looks like and how to construct it easily from a given game and board. Related to this is the question about the complement of the legal complex (which by Figure 1.5 we know to be the Alexander dual of $\mathcal{N}(\mathcal{L}_{G,B})$).

Cohen-Macaulay Bigraded Ideals

When considering a partizan game, or an $(\mathfrak{L}, \mathfrak{R})$-labelled simplicial complex, the underlying ring has a natural bigrading assigned to it via $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$.

As we have previously seen, the $(\mathfrak{L}, \mathfrak{R})$-labelling influences the SP-game corresponding to this simplicial complex significantly, and understanding Cohen-Macaulayness and related properties is difficult due to not respecting this partition.

Once these concepts and the related games properties are understood in the impartial case, a next step would be to define generalizations within the bigraded ring such that the relationships still exist. In particular we would want an ideal being Cohen-Macaulay to imply a pure Stanley-Reisner complex and unmixed facet complex, and a shellable and pure $(\mathfrak{L}, \mathfrak{R})$-labelled complex implying a Cohen-Macaulay Stanley-Reisner ideal.

One example of a generalization, based on our previous results, is that a whiskering in a bigraded case should be so that the new vertices added belong to the opposite part from the vertex adjacent to it, as this would preserve a whiskered complex always being a second player win.

Note that a definition for a Cohen-Macaulay bigraded module already exists in literature (see for example [51]), but this definition might not be ideal for SP-games.

Chapter 8

Conclusion

8.1 Summary

In this thesis, we have studied strong placement games as a class, demonstrating several new tools applicable to these games.

The first major result of this thesis is the existence of the two one-to-one correspondences between SP-games and simplicial complexes (Theorems 2.16 and 2.17), which in particular allows us to use the legal complex as a representation of the SP-game. This result has the potential to be a powerful new tool for the study of SP-games. As an example, we apply it in the investigation of which game values are possible.

We further show that the game graph of an SP-game holds the same amount of information as its legal complex, and give a full characterization of when a game graph comes from an SP-game (Proposition 3.11).

The second major result of the thesis is the upper bound on the temperature of a game using measures of confusion intervals (Theorem 5.24), the first such known bound to hold in general. We then show that this theorem is particularly useful for SP-games and give upper bounds on the boiling point of several SP-games (Propositions 5.31 to 5.33).

Finally, we study how algebraic properties of the game complexes, especially Cohen-Macaulayness, may be reflected in the game itself. Our preliminary results indicate that a characterization of the simpler impartial case is required before the partizan case will be understood.

8.2 Further Questions

In this section, we will conclude this thesis by discussing some potential further questions and avenues to explore beyond applying the new tools to the research directions mentioned at the end of each chapter. This includes looking further at how specific properties of SP-games translate into the game complexes and vice versa, extending to scoring SP-games, and even more generally looking at games from other combinatorial objects.

8.2.1 Properties of Placement Games and the Game Complexes

The one-to-one correspondences between SP-games and simplicial complexes discussed in this thesis opens many questions about these relationships. We have already seen how being a disjunctive sum or an impartial game are reflected in the game complexes, and have begun to study how Cohen-Macaulayness might appear in the game. There are many more properties of and operations on either object that will be interesting to study and see how they translate to the other. Here we focus on restricted universes and sums on the game side and partitionability and forests on the simplicial complex side.

Restricted Universes

Recall that two games G_1 and G_2 are defined as equal if $o(G_1 + H) = o(G_2 + H)$ for all games H. In the study of misère games, it has proven very useful to restrict the universe in the definition of equality by letting H only belong to a smaller class of games (see Plambeck 2005 [49] and Plambeck and Siegel 2008 [50]). Two examples of classes that turned out particularly nice are dicotic games (if either player has an option, then so does the other, and this is also true for all options) and dead-ending games (if a player at one point has no move in a component, then they will not have a move later on). Dicotic games have been studied by Allen in 2009 [2] and 2015 [3] and dead-ending games were introduced and studied by Milley and Renault in 2013 [44].

All SP-games are by definition dead-ending, but we are interested in how an SP-game belonging to other classes is reflected in the structure of the legal complex. This in particular includes understanding the structure when fixing a ruleset, such as taking all DOMINEERING games as a restricted universe.

Although understanding the structure of the legal complex of classes of games will likely be difficult in general, results have the potential for many applications, including understanding value sets and improving our temperature bounds.

In the reverse direction, we can also ask whether SP-games with specific structures in their legal complex make for nice restricted universes.

Operations on Games

We have seen in Theorem 1.55 how the disjunctive sum is reflected both in the legal and the illegal complex. Although disjunctive sum is the most commonly considered sum on games, there are also other operations (see [57, Figure 4.9]), such as the conjunctive, ordinal, selective, and sequential sums and Norton product, which might be of interest to SP-games. Not all of these operations will result in an SP-game. For example the ordinal sum of two SP-games is a weak placement game, but not strong. It will be of interest to study how these operations are reflected in the game complexes as long as the resulting game is still an SP-game.

In turn, it will also be interesting to see if known operations on simplicial complexes or ideals, besides join and disjoint union, lead to interesting operations on games.

Games Corresponding to Partitionable Complexes

Partitionable simplicial complexes are those that can be written as a disjoint union of intervals, i.e. sets of nested faces.

It is well-known that any shellable simplicial complex is also partitionable. In fact, Stanley conjectured in 1979 [58] that for a not necessarily pure simplicial complex Δ if $\mathcal{N}(\Delta)$ is Cohen-Macaulay, then Δ is partionable. The conjecture was only recently disproven by Duval, Goeckner, Klivans, and Martin in 2016 [22], but for many classes of simplicial complexes the implication holds.

Beyond the relationship to Cohen-Macaulayness, partitionable complexes are very interesting due to their structure. If the legal complex of an SP-game is partitionable, the partitioning provides us with a grouping of positions, and it would be interesting to see if this would aid in calculations of values and temperatures. Thus a natural consequence for us is to be interested in when $\Delta_{R,B}$ is partitionable.

In addition, we are also interested in when $\Gamma_{R,B}$ is partitionable, and what it means for the underlying SP-game if either of its game complexes is partitionable.

Game Complexes that are Simplicial Forests

When we considered grafted simplicial complexes, we considered the notion of a leaf (see Definition 7.23). Faridi introduced leaves in 2004 to generalize the concept of a tree from graphs to simplicial complexes:

Definition 8.1 (Faridi 2004 [24])**.** A connected simplicial complex Δ is called a **tree** if every non-empty subcomplex of Δ which is generated by facets of Δ has a leaf. If Δ is not necessarily connected, but every non-empty subcomplex generated by facets of Δ has a leaf, then we call Δ a **forest**. Equivalently, a simplicial complex is a forest if each of its connected components is a tree.

If Δ is a graph, these definitions are equivalent to the graph theoretic definitions of trees and forests.

Trees and forests (and their facet and Stanley-Reisner ideals) have many nice properties. For example, Faridi showed in 2004 [24] that the localization of a tree is a forest and in 2005 [25] that a tree is unmixed if and only if it is grafted. A consequence of this, given in the latter paper, is that if $\mathcal{F}(I)$ is a forest, then I is Cohen-Macaulay if and only if $\mathcal{F}(I)$ is unmixed (or equivalently $\mathcal{N}(I)$ is pure). In our case that means that if $\Delta_{R,B}$ is a forest and unmixed, then $\mathcal{L}_{R,B}$ is Cohen-Macaulay, and if $\Gamma_{R,B}$ is a forest and unmixed, then $\mathcal{IL}\mathcal{L}_{R,B}$ is Cohen-Macaulay.

Similarly to the partitionability, the additional structure of the legal or illegal complex being a tree might again make calculations easier as well. Thus we are interested in when the legal or illegal complex form a tree or forest.

8.2.2 Scoring Variants

Scoring games have recently received renewed attention. A scoring game is one in which at the end of the game a score is assigned depending on the current position. A positive score indicates a Left win, while a negative one indicates a Right win, and zero a tie. The exact assignment of these scores varies from author to author, with the goal always being though to generalize as much of the normal play theory as possible without being too restrictive (see for example Milnor 1953 [45], Ettinger 2000 [23], Stewart 2011 [60], and Johnson 2014 [35]). Larsson, Nowakowski, Neto, and Santos [42] introduced the class of Guaranteed Scoring Games in 2016, which contains most of the previously explored classes and has normal-play games embedded, while still having unique canonical form and negative of every game, thus having the strongest tools.

We are interested in how SP-games can be generalized to scoring SP-games, ideally while falling into the class of guaranteed scoring games. A representative simplicial complex of a scoring variant of an SP-game would then have weights assigned to the facets. Many questions studied in this thesis are also of interest for scoring SP-games, especially which values are possible.

8.2.3 Games from Other Objects

Given an SP-game (R, B), the game is equivalent to playing on $\Delta_{R,B}$ by letting Left only play on vertices belonging to $\mathfrak{L}$ and Right belonging to $\mathfrak{R}$, and claimed vertices have to form a face. Similarly, an impartial SP-game is equivalent to playing on the impartial legal complex.

Motivated by this, we are interested in classes of combinatorial games coming from other mathematical objects. Two examples would be combinatorial designs and posets:

Combinatorial Designs: We can think about playing on combinatorial designs, such as finite projective planes, triple systems, and more generally block designs, by letting players claim points. Play can then be defined in many different ways, from claimed points having to be contained in a block, to any three points of a player forming an independence set.

Posets: Similarly, one can play on a poset by having players alternatingly picking covers of the previous element until a maximal element has been reached. Effectively, this means that play is to form a chain. Other methods of play could include pieces claimed having to form antichains.

For each of these classes of games, we are interested in many of the same questions as for SP-games:

Game Values: First and foremost we are interested in which game values can be achieved by each of these classes of games under normal play, and in particular if a class takes on all possible values, i.e. is universal. It seems likely that the structure of the design or poset would give an indication of which game values are possible.

Misère Play: Under misère play the situation is generally much more complicated than under normal play. One of the advantages of our approach of representing positions through other objects is that they are independent of the winning condition, and likely these tools will be very useful in the harder case as well. Two challenges often found in misère play are determining the disjunctive sum of two games or the inverse of a game, both relatively easy under normal play. For the former question at least we have a representation of the operation in the legal simplicial complexes for SP-games, and likely also for the other objects. Similarly as under normal play, we would also like to study the possible game values and how certain structures influence which values are possible.

Temperature: Similar to our study of temperature of SP-games, for the other classes of games the related combinatorial structure should also give an indication of possible temperatures.

Appendix A

Rulesets

This appendix is a summary of the rulesets of games considered. Unless specified otherwise, play ends once no more moves are available.

Arc-Kayles

An impartial SP-game.

The board can be any graph. Play consists of claiming two adjacent vertices.

Note: Arc-Kayles is usually defined as deleting an edge, its two incident vertices, and any adjacent edges. The above definition is equivalent and shows it is an SP-game.

Col

A partizan SP-game.

The board can be any graph. Players place a piece on a single vertex which is not adjacent to a vertex containing one of their own pieces.

Col was introduced by Colin Vout as a map-colouring game in which each player has a fixed colour and no two adjacent regions may be coloured the same.

Domineering

A partizan SP-game.

The board is a grid. Both players place dominoes—Left may only place vertically, and Right only horizontally.

Domineering is also known as Crosscram or Dominoes. Traditionally it is played on a checkerboard.

HEX

Equivalent to a partizan SP-game.

The board is a hexagonal grid forming a parallelogram. Players take turn claiming spaces. Play ends once Left has connected the left and right edge or Right has connected the top and bottom edge with their pieces.

Note: HEX is an SP-game if play continues until all spaces are filled. As only one set of parallel edges can be connected, the outcome remains the same.

NIM

Equivalent to an impartial SP-game.

The board is a collection of piles of tokens. On a turn, the player chooses a pile and removes any number of tokens from it.

NODE-KAYLES

An impartial SP-game.

The board can be any graph. Players place pieces on a single vertex such that no two pieces are adjacent.

NOGO

A partizan SP-game.

The board can be any graph. Players place a piece on a single vertex. At every point in the game, for each maximal group of connected vertices of the board that contain pieces placed by the same player, one of these needs to be adjacent to an empty vertex.

PARTIZAN OCTAL

PARTIZAN OCTALS are a class of games, some of which are equivalent to SP-games.

The board is a collection of stacks of tokens. Each player has assigned an octal code $.\mathbf{d_1 d_2 d_3}\ldots$, with $0 \leq \mathbf{d_i} \leq 7$ for all i. If one writes $\mathbf{d_i} = a \cdot 2^0 + b \cdot 2^1 + c \cdot 2^2$ with $a, b, c \in \{0, 1\}$ then:

- If $a = 1$, then i tokens may be removed to empty the pile.
- If $b = 1$, then i tokens may be removed leaving a nonempty pile.
- If $c = 1$, then i tokens may be removed and the remaining tokens are split into two nonempty piles.

For example, NIM is the game .**33333**... and O12 is the game with Left octal code .**7** and Right octal code .**07**. A PARTIZAN OCTAL with only 3's in the octal code is PARTIZAN SUBTRACTION.

SNORT

A partizan SP-game.

The board can be any graph. Players place a piece on a single vertex which is not adjacent to a vertex containing a piece from their opponent.

SNORT was introduced by Simon Norton. It is also known as CATS AND DOGS, with one player placing cats, the other dogs, and they may never be adjacent.

PARTIZAN SUBTRACTION

PARTIZAN SUBTRACTION is a class of games equivalent to SP-games.

Left and Right each have a subtraction set specified. The board is a stack of tokens. A legal move is to remove a number of tokens from the stack that is part of the subtraction set, leaving a nonnegative number of tokens.

NIM is the special case in which both subtraction sets are the positive integers.

Appendix B

Code

B.1 CGSuite Oab

```
/*
 * Oab.cgs
 *
 * Partizan Octals in which both subtraction sets have
 *      cardinality 1.
 *
 * Examples:
 * g := Oab(2,3,Strip(10));
 * g.CanonicalForm
 * P := n -> Oab(2,3,Strip(n));
 * P(10).CanonicalForm
 * tableof([n,P(n).CanonicalForm] for n from 1 to 10)
 */

class Oab extends StripGame

  var a;    // The element of the Left subtraction set.
  var b;    // The element of the Right subtraction set.

  /*
   * constructor Oab
   *
   * Input: Number a - Left subtraction set
   *        Number b - Right subtraction set
   *        grid     - Board
   *
```

```
 * Output: None
 *
 */
method Oab(Number a, Number b, grid)
  this.StripGame(grid);
end

/*
 * method Options
 *
 * Input: Player player - Player to evaluate for
 *
 * Output: Set - Options for player
 *
 * Checks for each location in grid whether player can
 *     place a piece.
 * If so, the piece is placed and the remaining spaces
 *     to left and right are added as a disjunctive sum
 *     to the output Set.
 *
 */
override method Options(Player player)

  options := {};

  if player == Player.Left then
    nDelta := a-1;
  else
    nDelta := b-1;
  end

  L := grid.ColumnCount;
```

```
  for n from 1 to L-nDelta do
    f := 0;
    for m from n to n+nDelta do
      if grid[m] == 1 then
        f := 1;
        break;
      end;
    end;
    if f == 0 then
      leftCopy := Strip(n-1);
      for m from 1 to n-1 do
        leftCopy[m] := grid[m];
      end;
      rightCopy := Strip(L-n-nDelta);
      for m from 1 to L-n-nDelta do
        rightCopy[m] := grid[m+n+nDelta];
      end;
      options.Add(Oab(a,b,copy1)+Oab(a,b,copy2));
    end;
  end;

  return options;

end

override property CharMap.get
  return ".#";
end

override property Icons.get
  return
  [
    GridIcon.Blank,
```

```
        GridIcon.GraySquare
      ];
  end

end
```

B.2 CGSuite Snort

```
/*
 * Snort.cgs
 *
 * In Snort players place pieces on the vertices of a graph.
 * Pieces by opposing players may not be adjacent.
 * This implementation works for any grid.
 *
 * Examples:
 * g := Snort(Grid(2,3));
 * g.CanonicalForm
 * P := n -> Snort(Grid(2,n));
 * P(10).CanonicalForm
 * tableof([n,P(n).CanonicalForm] for n from 1 to 10)
 */

class Snort extends GridGame

  /*
   * constructor Snort
   *
   * Input: grid - Board
   *
   * Output: None
   *
   */
  method Snort(grid)
    this.GridGame(grid);
  end

  /*
   * method Options
   *
   * Input: Player player - Player to evaluate for
```

```
 *
 * Output: Set - Options for player
 *
 * Checks for each location in grid whether player can
 * place a piece. If so, the piece is placed and the new
 * grid is added to the output Set.
 *
 */
override method Options(Player player)

  options := {};

  us := player.Ordinal;
  them := player.Opponent.Ordinal;

  for m from 1 to grid.RowCount do
    for n from 1 to grid.ColumnCount do
      moveOK:= true;
      if grid[m,n] == 0 then
        for d in Direction.Orthogonal do
          if grid[m+d.RowShift,n+d.ColumnShift] == them then
            moveOK:=false;
          end
        end
        if moveOK == true then
          copy := grid;
          copy[m,n] := us;
          options.Add(Snort(copy));
        end
      end
    end
  end

  return options;
```

```
  end

  override property CharMap.get
    return ".xo";
  end

  override property Icons.get
    return
    [
      GridIcon.Blank,
      GridIcon.BlackStone,
      GridIcon.WhiteStone
    ];
  end

end
```

B.3 Macaulay2 Combinatorial Games Package

```
-- Copyright 2015: Svenja Huntemann, Gwyn Whieldon
-- You may redistribute this file under the terms of the GNU
-- General Public License as published by the Free Software
-- Foundation, either version 2 of the License, or any later
-- version.

------------------------------------------------------------------
------------------------------------------------------------------
-- Header
------------------------------------------------------------------
------------------------------------------------------------------


if version#"VERSION" <= "1.4" then (
  needsPackage "SimplicialComplexes", needsPackage "Graphs"
  )

newPackage select((
  "CombinatorialGames",
  Version => "0.0.1",
  Date => "29. May 2015",
  Authors => {
     { Name => "Svenja Huntemann",
       Email => "svenja.huntemann@dal.ca",
       HomePage => "http://mathstat.dal.ca/~svenjah/" },
     { Name => "Gwyn Whieldon",
       Email => "whieldon@hood.edu",
       HomePage => "http://cs.hood.edu/~whieldon" }
  },
  Headline => "Package for computing combinatorial game
               representations of bipartitioned simplicial
               complexes.",
  Configuration => {},
  DebuggingMode => true,
```

```
  if version#"VERSION" > "1.4" then PackageExports => {
    "SimplicialComplexes", "Graphs"
  }
), x -> x =!= null)

if version#"VERSION" <= "1.4" then (
  needsPackage "SimplicialComplexes", needsPackage "Graphs"
)

export {
  "gameRepresentation"
}

------------------------------------------------------------
------------------------------------------------------------
-- Methods
------------------------------------------------------------
------------------------------------------------------------


------------------------------------------------------------
--Internal method to concatenate strings
--from Left/Right move sets.
------------------------------------------------------------
cgtJoin = method()
------------------------------------------------------------
cgtJoin(List,HashTable) := String => (L,dummyH) -> (
  if #L === 0 then ("")
  else (
    fold(
      concatenate,
      apply(#L, i-> if i=!=#L-1 then
                      concatenate(dummyH#(L_i),",")
                    else
```

```
                              dummyH#(L_i)))
            )
    )
```

```
gameRepresentation = method()
gameRepresentation(SimplicialComplex,List,List) :=
     String => (Delta,L,R) -> (
   S := ring Delta;
   L = apply(L,v-> sub(v,S));
   R = apply(R,v-> sub(v,S));
   V := S_*;
   if ((#L+#R) === #V) and (all(join(L,R),v->member(v,V))) then (
     S = QQ[L, R,
           Degrees => join(apply(L,i->{1,0}),apply(R,i->{0,1}))];
     L = (S_*)_(toList(0..#L-1));
     R = (S_*)_(toList(#L..#V-1));
     Delta = sub(Delta,S);
     d := dim(Delta);
     Fvec := apply(toList(0..d+1),
                  i->flatten entries faces(d-i,Delta));
     H := hashTable apply(Fvec_0, f-> f=>"{|}");
     E := hashTable {{}=>""};
     dummyH := merge(H,E,join);
     tempStringL := {};
     tempStringR := {};
     Hnew :={};
     for F in drop(Fvec,1) do (
       tempStringL =
           apply(
             apply(F,
                  m->select(keys H,
                           k->(degree(k)-degree(m) == {1,0}) and
                              (k % m == 0))),
             i->if i=={} then {} else i);
```

```
        tempStringR =
            apply(
              apply(F,
                    m-> select(keys H,
                                k->(degree(k)-degree(m) == {0,1}) and
                                   (k % m == 0))),
              i-> if i=={} then {} else i);
        Hnew = hashTable apply(#F, i-> F_i =>
         concatenate("{",
                    cgtJoin(tempStringL_i,dummyH),
                    "|",
                    cgtJoin(tempStringR_i,dummyH),
                    "}"));
        H = merge(H,Hnew,join);
        --print H;
        dummyH = merge(dummyH,Hnew,join);
      );
      --print H;
      H#(sub(1,ring Delta))
    )
    else "Variables of Delta not bi-partitioned."
    )

----------------------------------------------
----------------------------------------------
-- Documentation
----------------------------------------------
----------------------------------------------


beginDocumentation()

-- Front Page
doc ///
  Key
```

```
    CombinatorialGames
  Headline
    A package for outputting combinatorial game suite formatted
    games
  Description
    Text
      @SUBSECTION "Definitions"@
          Let $\Delta$ be a simplicial complex with vertices
          labelled by a partition of variables
          $L = \{x_1,x_2,...,x_n\}$ and $R = \{y_1,y_2,...,y_m\}$
          where $L\cup R = V$.
        Text

        Text
          Put description of how this constructor works in here.
    Text
      @SUBSECTION "Other acknowledgements"@
      --
      This package started at Macaulay2 Workshop in Boise,
      supported by NSF grant and organized by Zach Teitler,
      Hirotachi Abo and Frank Moore.
///

-------------------------------------------------------
-- Data type & constructor
-------------------------------------------------------


-- gameRepresentation
doc ///
    Key
        gameRepresentation
    (gameRepresentation,SimplicialComplex,List,List)
    Headline
        compute the game representation to export to cgsuite
```

```
␣␣␣␣Usage
␣␣␣␣␣␣␣␣G␣=␣gameRepresentation(Delta,L,R)
␣␣␣␣Inputs
␣␣␣␣␣␣␣␣Delta:SimplicialComplex
␣␣␣␣␣␣␣␣simplicial␣complex␣in␣bipartitioned␣variables
␣␣␣␣L:List
␣␣␣␣␣␣␣␣list␣of␣left-labeled␣vertices␣in␣Delta
␣␣␣␣R:List
␣␣␣␣␣␣␣␣list␣of␣right-labeled␣vertices␣in␣Delta
␣␣␣␣Outputs
␣␣␣␣␣␣␣␣G:String
␣␣␣␣␣␣␣␣string␣to␣export␣to␣cgsuite
␣␣␣␣Description
␣␣␣␣␣␣␣␣Text
␣␣␣␣␣␣␣␣This␣takes␣a␣simplicial␣complex␣and␣a␣partition␣of␣its
␣␣␣␣␣␣␣␣vertices␣into␣two␣subsets␣and␣outputs␣the␣game
␣␣␣␣␣␣␣␣representation␣of␣Delta␣in␣the␣format␣understood␣by
␣␣␣␣␣␣␣␣open-source␣software
␣␣␣␣␣␣␣␣@HREF("http://cgsuite.sourceforge.net/",␣"cgsuite")@.
␣␣␣␣Example
␣␣␣␣␣␣␣␣--␣create␣ring␣for␣simplicial␣complex␣Delta
␣␣␣␣␣␣␣␣S␣=␣QQ[x_1,x_2,x_3,y_1];
␣␣␣␣␣␣␣␣--␣create␣Delta,␣here␣input␣by␣list␣of␣facet␣vertices
␣␣␣␣␣␣␣␣Delta␣=␣simplicialComplex(
␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣apply({{x_1,x_2,x_3},
␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣{x_2,x_3,y_1}},product));
␣␣␣␣␣␣␣␣L␣=␣{x_1,x_2,x_3}␣--␣list␣of␣left␣player␣(L)␣vertex␣labels
␣␣␣␣␣␣␣␣R␣=␣{y_1}␣--␣list␣of␣right␣player␣(R)␣vertex␣labels
␣␣␣␣␣␣␣␣␣␣␣␣gameRepresentation(Delta,L,R)
␣␣␣␣␣␣␣␣Text
␣␣␣␣␣␣␣␣␣␣␣␣Here␣calling␣the␣variables␣$x_1,...,x_n$␣and
␣␣␣␣␣␣␣␣␣␣␣␣$y_1,...,y_m$␣is␣for␣convenience,␣and␣is␣not␣necessary
␣␣␣␣␣␣␣␣␣␣␣␣in␣inputting␣the␣game.
␣␣␣␣Example
```

```
        S = QQ[a,b,c,d,e,f]
        F = {{a,b,c},{a,c,d},{a,d,f},{c,d,e},
             {b,c,e},{d,e,f},{b,e,f}}
        Delta = simplicialComplex(apply(F,product))
        L = {a,b,d,f}
        R = {c,e}
        gameRepresentation(Delta,L,R)

///
```

end

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