# The Social Welfare Implications, Decomposability, and Geometry of the Sen Family of Poverty Indices

Kuan Xu and Lars Osberg*

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**Abstract**

This paper proposes a unified framework for the Sen indices of poverty intensity, which shows an explicit connection between the indices and the common underlying social evaluation function. This paper also identifies the common multiplicative decomposition of the indices that allows simple and similar geometric interpretations, easy numerical computation, and common subgroup decompositions. These results are useful to policy analysts.

**JEL Classfication**: C000, H000, O150

**Keywords**: poverty intensity, poverty rate, poverty gap, subgroup decomposition, equally-distributed-equivalent-income, social evaluation function, Gini index

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# 1 Introduction

Since Sen proposed an axiomatic approach to poverty research and an index of poverty intensity in 1976, poverty measurement has become an active research agenda. While a vast literature has developed over the years,[1] the Sen index (or S index[2]) and the modified Sen index (or SST index[3]) have been pursued actively for empirical poverty studies by Bishop, Formby and Zheng (1997), Myles and Picot (1999), Osberg and Xu (1997, 1999), and Rongve (1997) among others.

The Sen indices are based on a set of well-justified and commonly agreed axioms. From a policy point of view, it is desirable to understand the meaning of the Sen indices in terms of social welfare evaluation, but the common social evaluation function for the Sen indices has not yet been explicitly discussed in the literature on poverty measures.[4] Bourguignon and Fields (1997) pointed out that poverty measures can be interpreted as gauging the social welfare losses when persons have low income. Blackorby and Donaldson (1978, 1980) and Chakravarty (1983, 1997) laid a solid ground for interpreting the social welfare meaning of the Sen indices. This paper will examine the Sen indices and their common underlying social welfare function.

From a policy point of view, it is also desirable to understand the relationship between the Sen indices and their contributing components [see, for example, and Birdsall and Londono (1997), and Phipps (1999) among others]. Poverty indices that exhibit additive decomposability, such as the indices proposed by Theil (1967) and Foster, Greer and Thorbecke (1984), are often selected in economic studies.[5] Unfortunately, the Sen indices and

[1]See Zheng (1997), and the references therein, for a recent comprehensive survey.

[2]See Sen (1976). The index is called the S index in Sen (1997).

[3]The index is called the modified Sen index in Shorrocks (1995) and Sen (1997). Shorrocks (1995) proposed the index. Zheng (1997) noted that the modified Sen index is identical to the limit of Thon's modified Sen index [Thon (1979, 1983)]. Thus, we also call it the Sen-Shorrocks-Thon index [see Osberg and Xu (1997, 1999)].

[4]Dalton (1920) in his pioneering paper suggested that any measure of income inequality has an underlying social welfare function. This has been made precise by Kolm (1969), Atkinson (1970), and Sen (1973).

[5]See Chapter 7 of Chakravarty (1990) for a detailed survey of the additive decomposi-

their generalizations such as BD index[6] and C index[7] do not satisfy this axiom in general.[8] However, the Sen indices do have the property of multiplicative decomposability, as first briefly mentioned for the S index by Clark, Hemming, and Ulph (1981) for the S index and as examined for the SST index by Osberg and Xu (1997, 1999). Hence, the application of the Sen indices and their decompositions are no longer restricted by their lack of additive decomposability. In addition, as Bourguigon and Fields (1997) noted, some of additive poverty measures often lead to an antipoverty policy to pay attention to the richest of the poor. They noted that an appropriate poverty measure should facilitate a comprehensive evaluation of anti-poverty policy actions. Since the Sen indices have desirable properties and are multiplicatively decomposable, the Sen indices and their decomposed components can be readily used to measure the multidimensional impacts of anti-poverty policy actions.

In the literature on income inequality, the Gini index is perhaps the most-used index of inequality, partly because it has a useful and intuitive geometric interpretation. The S index has a useful but less intuitive geometric interpretation presented by Sen (1976). The SST index has a useful and intuitive geometric interpretation that has been discussed by Shorrocks (1995), Jenkins and Lambert (1997), Osberg and Xu (1997), Xu and Osberg (1998). The common multiplicative decomposability of the Sen indices suggests that the decompositions must have similar useful and intuitive geometric interpretations, which has been largely ignored in the literature. Given that the Sen indices are not additive decomposable, subgroup decomposition has not yet been considered in the current research agenda.

In this paper, we examine the common underlying social evaluation function, multiplicative decomposability, and geometric interpretations of the Sen indices, as well as further subgroup decompositions of their multiplicatively-decomposed components.[9]

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tion of the poverty intensity indices.

[6]See Blackorby and Donaldson (1980).

[7]See Chakravarty (1983).

[8]The Chakravarty index with the symmetric means of order $r$ ($r < 1$)is an exception.

[9]The Gini index also allows factor decomposition [see, for example, Fei, Ranis, and Kuo (1978) and Shorrocks (1982, 1983)]. Blackorby, Donaldson and Auersperg (1981) also considered subgroup decomposition of the indices of income inequality. Their approach requires that the social evaluation function satisfy a separability condition. Since the Gini

We show the following findings in this paper: (1) The Sen indices share a common Gini social evaluation function, in which the individual share of social welfare is based on the rankings of income rather than the sizes of incomes; (2) the Sen indices share a common multiplicative decomposition structure; each index can be expressed as the product of the poverty rate, the average poverty gap ratio of the poor and one plus the Gini index of the poverty gap ratios; (3) the SST index is a linear transformation of the S index and vice versa; (4) the common multiplicative decomposability of the Sen indices permits similar useful and intuitive geometric interpretations, renders them easy to understand and compute, and allows further subgroup decompositions; and (5) because of the common multiplicative decomposability, the Sen indices can be linearized so that they are additively decomposable —a useful result for empirical comparisons.

In section 2, the notation and some basic concepts are discussed. In section 3, via two general poverty indices—the BD and the C indices, we identify the common underlying social evaluation function for the Sen indices. We also analyze the common multiplicative decomposability, useful and intuitive geometric interpretations of the Sen indices. In section 3, we explore further subgroup decompositions of the multiplicative-decomposed components of the Sen indices. Section 4 concludes.

## 2 Notation and Some Basic Concepts

Let $\mathbf{y} = [y_1, y_2, \ldots, y_n]^\top$ be the income vector of a population of size $n$ with (individual or family) incomes sorted in non-decreasing order. Let $\tilde{\mathbf{y}}$ be $\mathbf{y}$ with incomes sorted in non-increasing order where the notation " $\tilde{}$ " is used for sorting a vector $\mathbf{x}$ in opposite order. Let the poverty line be $z > 0$. Let the number of the poor be $q$. Hence the poverty rate $H$ is $\frac{q}{n}$. A censored income vector is obtained by setting $\overset{*}{y}_i = y_i$ if $y_i < z$ and $\overset{*}{y}_i = z$ otherwise,[10] that is $\overset{*}{\mathbf{y}} = [\overset{*}{y}_1, \overset{*}{y}_2, \ldots, \overset{*}{y}_n]^\top$. The income vector of the poor, $\mathbf{y}_p = [y_1, y_2, \ldots, y_q]^\top$, is a truncated income vector generated from $\overset{*}{\mathbf{y}}$ by deleting $z$'s. The average

social evaluation function used in the S and SST indices fails to satisfy the condition, this paper does not adopt their approach.

[10]We use the weak definition of the poor here—a poor person's income is less than the poverty line—as it is generally treated in the literature.

of an income vector $\mathbf{y}$ is given by $\overline{\mathbf{y}}=\frac{1}{n}\sum_{i=1}^{n} y_i$. Note that the order of the elements in $\mathbf{y}$ is not relevant in computing the mean.

Define $\mathbf{1}$ as a column vector of ones with an appropriate dimension. The poverty gap ratio vector of the population is defined as $\mathbf{x} = \frac{z\mathbf{1}-\overset{*}{\mathbf{y}}}{z}$ where the poor have poverty gap ratios $x_i = \frac{z-y_i}{z}$, $i = 1, 2, \ldots, q$, and the non-poor have zero poverty gap ratios. Similarly, the poverty gap ratio vector of the poor is given by $\mathbf{x}_p = \frac{z\mathbf{1}-\mathbf{y}_p}{z}$ where the poor have poverty gap ratios $x_i = \frac{z-y_i}{z}$, $i = 1, 2, \ldots, q$, and the non-poor's zero poverty gap ratios are excluded. Please note the elements in both $\mathbf{x}$ and $\mathbf{x}_p$ are in non-increasing order. The average poverty gap ratio of the population (the poor) is denoted by $\overline{\mathbf{x}}$ ($\overline{\mathbf{x}}_p$).

To analyze the social welfare implication of the Sen index and its extensions, we need to utilize the concept of the equally-distributed-equivalent-income (EDEI), or the representative income proposed by Atkinson (1970), Kolm (1969), and Sen (1973). For a particular social evaluation function (SEF), an EDEI given to every individual could be viewed as identical in terms of social welfare to an actual income distribution. Let $W(\mathbf{y}) = \phi(\overline{W}(\mathbf{y}))$ be a homothetic (ordinal) SEF of income with $\phi$ being an increasing function and $\overline{W}$ being a linearly homogeneous function. Let $\xi$ be the EDEI and $\mathbf{1}$ be a column vector of ones with an appropriate dimension. Then, $W(\xi \cdot \mathbf{1}) = W(\mathbf{y})$ or $\overline{W}(\xi \cdot \mathbf{1}) = \overline{W}(\mathbf{y})$. Given that $\overline{W}$ is positively linearly homogeneous, EDEI is computed by $\xi = \frac{\overline{W}(\mathbf{y})}{\overline{W}(\mathbf{1})} = \Xi(\mathbf{y})$. The SEF, $W$, and the EDEI, $\Xi$, have an one-to-one corresponding relationship.

For example, the Gini SEF is $\overline{W}_G(\mathbf{y}) = \frac{1}{n^2}\sum_{i=1}^{n}(2n-2i+1)y_i$.[11] Its corresponding EDEI function is

$$\Xi_G(\mathbf{y}) = \frac{1}{n^2}\sum_{i=1}^{n}(2n-2i+1)y_i \tag{1}$$

or

$$\Xi_{\widetilde{G}}(\tilde{\mathbf{y}}) = \frac{1}{n^2}\sum_{i=1}^{n}(2i-1)\tilde{y}_i \tag{2}$$

with $\Xi_G(\mathbf{y}) = \Xi_{\widetilde{G}}(\tilde{\mathbf{y}})$.[12] The Gini SEF attaches a higher weight to a lower

[11] This is because

$$\overline{W}(\mathbf{1}) = \frac{1}{n^2}\sum_{i=1}^{n}(2n-2i+1) = 1.$$

[12] This is because $y_i = \tilde{y}_{n-i+1}$ and $\tilde{y}_i = y_{n-i+1}$.

level of income and vice versa. The weight is determined by the rank of an income rather than the size of the income.[13]

The Gini index can be defined in terms of the Gini EDEI and the mean income as

$$G(\mathbf{y}) = 1 - \frac{\Xi_G(\mathbf{y})}{\overline{\mathbf{y}}} = 1 - \frac{1}{n^2\overline{\mathbf{y}}}\sum_{i=1}^{n}(2n-2i+1)y_i \tag{3}$$

or

$$\widetilde{G}(\tilde{\mathbf{y}}) = 1 - \frac{\Xi_{\widetilde{G}}(\tilde{\mathbf{y}})}{\overline{\mathbf{y}}} = 1 - \frac{1}{n^2\overline{\mathbf{y}}}\sum_{i=1}^{n}(2i-1)\tilde{y}_i, \tag{4}$$

where $\mathbf{y}$ ($\tilde{\mathbf{y}}$) has elements in non-decreasing (non-increasing) order.[14] Note that $G(\mathbf{y}) = \widetilde{G}(\tilde{\mathbf{y}})$ in equations (3) and (4) are identical but $G(\cdot)$ and $\widetilde{G}(\cdot)$ have different functional forms and the elements in $\mathbf{y}$ and $\tilde{\mathbf{y}}$ are sorted differently. Also note that[15]

$$G(\mathbf{y}) = -G(\tilde{\mathbf{y}}). \tag{5}$$

# 3 Common SEF and Multiplicative Decomposition

## 3.1 Common Gini SEF

The link between the S and SST indices can be better understood based on the BD and C indices introduced by Blackorby and Donaldson (1980) and Chakravarty (1983), respectively. Since these BD and C indices have a direct link to the Gini SEF, the analysis in this section permits an explicit interpretation of the S and SST indices in term of social welfare evaluation. The S index can be viewed as a special case of the BD index while the SST index can be considered as a special case of the C index.

Consistent with the S index, the BD index focuses on the incomes of the poor $\mathbf{y}_p$ or the truncated income distribution by excluding the non-poor population. It is defined as follows:

[13] The Gini SEF, as a rank dependent expected utility function, also draws some attention in economic theory; see, for example, Chew and Safra (1987), Quiggin (1982), Segal and Spivak (1990), and Yaari (1987).

[14] The two equations are identical because $\overline{\mathbf{y}}=\overline{\tilde{\mathbf{y}}}$, $y_i = \tilde{y}_{n-i+1}$, and $\tilde{y}_i = y_{n-i+1}$.

[15] See Fei, Ranis, and Kuo (1978).

**Definition 1** *The BD index is defined as*

$$I_{BD}(\mathbf{y}_p) = H\left[\frac{z - \Xi(\mathbf{y}_p)}{z}\right] \tag{6}$$

*where* $\Xi$ *is the EDEI function of* $\mathbf{y}_p$ *for some increasing and strict S-concave SEF.*

Please note that the EDEI function is generic. For the ease of comparison, the S index is defined as follows:

**Definition 2** *The S index is defined as*

$$I_S(\mathbf{y}_p) = H\left[\overline{\mathbf{x}}_p + (1 - \overline{\mathbf{x}}_p)\, G(\mathbf{y}_p)\right]. \tag{7}$$

**Lemma 1** *The BD index* $I_{BD}(\mathbf{y}_p)$ *with the Gini EDEI,* $\Xi_G(\mathbf{y}_p)$*, is the S index, that is*

$$I_S(\mathbf{y}_p) = I_{BD}^G(\mathbf{y}_p) = H\left[\frac{z - \Xi_G(\mathbf{y}_p)}{z}\right]. \tag{8}$$

**Proof**: See Blackorby and Donaldson (1980, pp. 1054–1055). □

Equation (8) provides a mathematical structure based on which one can see why the S index is explicitly related to the underlying Gini SEF.[16]

Following the idea of Thon (1979) and Takayama (1979), Chakravarty (1983) developed the C index for the censored income vector $\overset{*}{\mathbf{y}}$:

**Definition 3** *The C index is defined as*

$$I_C(\overset{*}{\mathbf{y}}) = \frac{z - \Xi(\overset{*}{\mathbf{y}})}{z}, \tag{9}$$

*where* $\Xi$ *is the EDEI function for some increasing and strict S-concave SEF.*

For the ease of comparison, the SST index is defined below:

[16] It should be noted that Sen (1976) started from Axioms R (Ordinal Rank Weights), M (Monotonic Welfare), and N (Normalized Poverty Value) which have Gini social welfare implications.

**Definition 4** *The SST index of poverty intensity is defined as either*

$$I_{SST}(\overset{*}{\mathbf{y}}) = \frac{1}{n^2}\sum_{i=1}^{n}(2n-2i+1)x_i \tag{10}$$

*or*

$$I_{SST}(\mathbf{y}_p) = \frac{1}{n^2}\sum_{i=1}^{q}(2n-2i+1)x_i. \tag{11}$$

**Lemma 2** *The C index, $I_C(\overset{*}{\mathbf{y}})$, with the Gini EDEI, $\Xi_G(\overset{*}{\mathbf{y}})$, is the SST index, $I_{SST}(\overset{*}{\mathbf{y}})$, that is*

$$I_{SST}(\overset{*}{\mathbf{y}}) = I_C^G(\overset{*}{\mathbf{y}}) = \frac{z-\Xi(\overset{*}{\mathbf{y}})}{z}. \tag{12}$$

**Proof**: See Chakravarty (1997). □

Equation (12) provides a mathematical structure based on which one can see why the SST index is explicitly related to the Gini SEF.

## 3.2 Common Multiplicative Decomposition

Both S index and SST index do not permit additive decomposition although they possess desirable properties. However, one can analyze the decomposition issue from the multiplicative point of view, which allows linearization in that the logarithm of the index is additive decomposable. We note that both the S index and SST index permit a common multiplicative decomposition into the poverty rate, the average poverty gap ratio and the Gini index of poverty gap ratios.

The following proposition states that the S index permits multiplicative decomposition.

**Proposition 1** *The S index has the following multiplicative decomposition:*

$$I_S(\mathbf{y}_p) = H\overline{\mathbf{x}}_p\left(1+G\left(\tilde{\mathbf{x}}_p\right)\right), \tag{13}$$

*where $\tilde{\mathbf{x}}_p$ has elements in non-decreasing order.*

**Proof**: From Lemma 1 and equation (1),

$$I_S(\mathbf{y}_p) = \frac{q}{n}\left(\frac{q^2 z - \sum_{i=1}^{q}(2q-2i+1)y_i}{q^2 z}\right). \tag{14}$$

The above equation can then be rewritten as[17]

$$I_S(\mathbf{y}_p) = \frac{q}{n}\left(\frac{2}{q}\sum_{i=1}^{q}\frac{z-y_i}{z} - \frac{1}{q^2}\sum_{i=1}^{q}(2i-1)\frac{z-y_i}{z}\right). \tag{15}$$

Further manipulation gives

$$I_S(\mathbf{y}_p) = \frac{q}{n}\left(2\overline{\mathbf{x}}_p - \frac{\overline{\mathbf{x}}_p}{q^2\overline{\mathbf{x}}_p}\sum_{i=1}^{q}(2i-1)x_i\right). \tag{16}$$

Based on equation (4) the above equation can be rewritten as

$$I_S(\mathbf{y}_p) = H\overline{\mathbf{x}}_p\left(1+\widetilde{G}\left(\mathbf{x}_p\right)\right) \tag{17}$$

or

$$I_S(\mathbf{y}_p) = H\overline{\mathbf{x}}_p\left(1+G\left(\tilde{\mathbf{x}}_p\right)\right), \tag{18}$$

where $\mathbf{x}_p$ ($\tilde{\mathbf{x}}_p$) has elements in non-increasing (non-decreasing) order. □

As can be seen from the above proposition, we can present the S index as the product of the poverty rate, the average poverty gap ratio, and the Gini index of poverty gap ratios of the *poor*.

It is also interesting to compare the multiplicative decomposition of the original S index with the one presented here. As Sen (1976) pointed out, for a large $q$, the S index is defined as in equation (7) where the Gini index is for incomes of the poor. In this paper, we show that the S index can be written alternatively as in equation (13) where the Gini index is for poverty

[17]Here we use

$$\frac{1}{q^2}\sum_{i=1}^{q}(2i-1)\frac{z}{z} = 1$$

and

$$\frac{2}{q}\sum_{i=1}^{q}\frac{z}{z} = 2.$$

gap ratios of the poor. Equation (13) is a bit simpler than equation (7) and permits a simple geometric interpretation as shown later in this paper.

The following proposition states that the SST index permits similar multiplicative decomposition.

**Proposition 2** *The SST index has the following multiplicative decomposition:*

$$I_{SST}(\overset{*}{\mathbf{y}}) = H\overline{\mathbf{x}}_p \left(1 + G\left(\tilde{\mathbf{x}}\right)\right), \tag{19}$$

*where* $\tilde{\mathbf{x}}$ *has elements in non-decreasing order.*

**Proof**: Based on Lemma 2, the SST index can be written as

$$I_{SST}(\overset{*}{\mathbf{y}}) = \frac{z - \Xi_G(\overset{*}{\mathbf{y}})}{z} = \frac{1}{n^2}\sum_{i=1}^{n}(2n - 2i + 1)x_i. \tag{20}$$

By multiplying and dividing the right-hand-side of equation (20) by $\overline{\mathbf{x}}$, the SST index becomes

$$I_{SST}(\overset{*}{\mathbf{y}}) = \overline{\mathbf{x}} \left( \frac{2n}{n^2\overline{\mathbf{x}}} \sum_{i=1}^{n} x_i - \frac{1}{n^2\overline{\mathbf{x}}} \sum_{i=1}^{n}(2i - 1)x_i \right). \tag{21}$$

According to equation (4) and $\overline{\mathbf{x}} = H\overline{\mathbf{x}}_p$, the above equation can be simplified further into

$$I_{SST}(\overset{*}{\mathbf{y}}) = H\overline{\mathbf{x}}_p \left(1 + \tilde{G}\left(\mathbf{x}\right)\right) \tag{22}$$

or

$$I_{SST}(\overset{*}{\mathbf{y}}) = H\overline{\mathbf{x}}_p \left(1 + G\left(\tilde{\mathbf{x}}\right)\right), \tag{23}$$

where $\mathbf{x}$ ($\tilde{\mathbf{x}}$) has elements in non-increasing (non-decreasing) order. □

As can be seen from the above proposition, we can present the SST index as the product of the poverty rate, the average poverty gap ratio, and the Gini inequality measure of relative deprivations of the *population*.

From the multiplicative decomposition of the Sen indices, we note that the two indices differ only by the argument of $G(\cdot)$. The S index has a component $G\left(\tilde{\mathbf{x}}_p\right)$ while the SST index has a component $G\left(\tilde{\mathbf{x}}\right)$. Note that the poverty gap ratios of the non-poor subpopulation are zeros and the poor and the non-poor subgroups do not overlap in the censored income vector $\overset{*}{\mathbf{y}}$. Hence, as shown in the following lemma, the Gini index of poverty gap ratios of the *population* can be decomposed into two components: (1) the Gini index of subgroup average poverty gap ratios between the non-poor and poor subpopulations and (2) the product of the poverty ratio and the Gini index of poverty gap ratios of the *poor*.

**Lemma 3** *The Gini index of poverty gap ratios of the population, $G(\tilde{\mathbf{x}})$, is the sum of the Gini index of the average poverty gap ratios between the non-poor and the poor subpopulations,$(1-H)$, and the poverty-rate-weighted Gini index of poverty gap ratio of the poor, $HG(\tilde{\mathbf{x}}_p)$,as follows:*

$$G(\tilde{\mathbf{x}}) = (1-H) + HG(\tilde{\mathbf{x}}_p). \tag{24}$$

**Proof:** From equation (3), we have

$$G(\mathbf{x}) = 1 - \frac{1}{n^2\overline{\mathbf{x}}}\sum_{i=1}^{n}(2n-2i+1)\,x_i, \tag{25}$$

where the poverty gap ratio vector of the population, $\mathbf{x}$ has elements in non-increasing order (i.e., the poor subpopulation takes the top partition of the column vector while the non-poor subpopulation takes the bottom partition). Similarly, from equation (3), we have

$$G(\mathbf{x}_p) = 1 - \frac{1}{q^2\overline{\mathbf{x}}_p}\sum_{i=1}^{q}(2q-2i+1)\,x_i, \tag{26}$$

where the poverty gap ratio vector of the poor, $\mathbf{x}_p$, has elements in non-increasing order. It is known that $\overline{\mathbf{x}} = H\overline{\mathbf{x}}_p$. From equation (25) we get

$$G(\mathbf{x}) = 1 - \frac{q}{n}\frac{1}{q^2\overline{\mathbf{x}}_p}\sum_{i=1}^{q}(2q-2i+1)\,x_i - 2\left(1-\frac{q}{n}\right). \tag{27}$$

It can be further rewritten as

$$G(\mathbf{x}) = \frac{q}{n}\left\{1 - \frac{1}{q^2\overline{\mathbf{x}}_p}\sum_{i=1}^{q}(2q-2i+1)\,x_i\right\} - \left(1-\frac{q}{n}\right). \tag{28}$$

Thus,

$$G(\mathbf{x}) = (H-1) + HG(\mathbf{x}_p). \tag{29}$$

Applying equation (5) to $G$ on the left-hand-side, the above relation can be expressed alternatively as

$$G(\tilde{\mathbf{x}}) = -(H-1) - HG(\mathbf{x}_p). \tag{30}$$

Applying equation (5) to $G$ on the right-hand-side of equation (12), the equation becomes

$$G(\tilde{\mathbf{x}}) = (1-H) + HG(\tilde{\mathbf{x}}_p). \tag{31}$$

□

**Proposition 3** *The SST index and the S index are related in the following way:*

$$I_{SST}(\overset{*}{\mathbf{y}}) = HI_S(\mathbf{y}_p) + 2H(1-H)\overline{\mathbf{x}}_p. \tag{32}$$

**Proof:** Combining the results in Lemma 3 and Proposition 2 gives

$$I_{SST}(\overset{*}{\mathbf{y}}) = H\overline{\mathbf{x}}_p(2(1-H) + H(1+G(\tilde{\mathbf{x}}_p))). \tag{33}$$

Further manipulation of equation (33) gives equation (32). Zheng (1997) stated the same result [equation (3.9), p. 146] without giving the details of the proof. □

According to Chakravarty (1990, Theorem 6.9), if the SEF is completely strictly recursive, then

$$I_{BD}(\mathbf{y}_p) < I_C(\overset{*}{\mathbf{y}}). \tag{34}$$

In other words, the BD index is bounded above by the C index. It is known that the Gini SEF, which is the underlying SEF for the S and SST indices, is not completely strict recursive, the following proposition shows that the similar relationship holds.

**Proposition 4** *The S index is bounded above by the SST index, i.e.,*

$$I_S(\mathbf{y}_p) < I_{SST}(\overset{*}{\mathbf{y}}). \tag{35}$$

**Proof**: It is obvious that $q < n$ and $\frac{q}{n} > 0$. These conditions imply that $\Xi_G(\overset{*}{\mathbf{y}}) < z$ and $\Xi_G(\mathbf{y}_p) < z$. From the strict S-concavity of $\Xi_G$, that is based on the strict S-concavity of $W$, we have

$$\Xi_G(\overset{*}{\mathbf{y}}) < \frac{q}{n}\Xi_G(\mathbf{y}_p) + \left(\frac{n-q}{n}\right)z. \tag{36}$$

Equation (36) implies

$$I_{SST}(\overset{*}{\mathbf{y}}) = \frac{z - \Xi_G(\overset{*}{\mathbf{y}})}{z} > \frac{q}{n}\left(\frac{z - \Xi_G(\mathbf{y}_p)}{z}\right) = I_S(\mathbf{y}_p). \tag{37}$$

Thus,

$$I_{SST}(\overset{*}{\mathbf{y}}) > I_S(\mathbf{y}_p). \tag{38}$$

□

The common multiplicative decomposition structure allows further analysis of poverty intensity in terms of the poverty rate, the average poverty gap ratio, and the Gini index of poverty gap ratios. The multiplicative decomposition of the S and SST indices can be changed to the additive decomposition of the indices through the logarithmic transformation. Because analysts are often concerned with percentage differences either over time or across jurisdictions, this is useful for empirical research of poverty intensity and its decomposition. In the following corollary, we use $I$ for either the S index or the SST index and $G$ for the Gini index of poverty gap ratios of either the poor or the population.

**Corollary 1** *Since the S and SST indices of poverty intensity take the form of*

$$I = H\overline{\mathbf{x}}_p(1 + G), \tag{39}$$

*then*

$$\Delta I = \Delta H + \Delta\overline{\mathbf{x}}_p + \Delta(1 + G), \tag{40}$$

*where* $\Delta x = \ln x_t - \ln x_{t-1} \approx \frac{x_t - x_{t-1}}{x_{t-1}}$ *approximates the percentage change in* $x$ *for a small change in* $x$.

Depending on the purpose of research, one may use the same poverty line $z_t = z_{t-1} = z$ for $I_t$ and $I_{t-1}$ and their components or different poverty lines $z_t$ and $z_{t-1}$, respectively, for $I_t$ and $I_{t-1}$ and their components.

Osberg and Xu (1997, 1999) used this multiplicative decomposition for the SST index to the international and regional comparative studies. Statistics Canada has also adopted this methodology to analyze low-income intensity among Canadian children [Myles and Picot (1999)].

The common multiplicative decomposition also allows policy makers to use three specific anti-poverty policy "targets" (rate, gap, and inequality) in reducing poverty intensity. These targets may be used to monitor the effectiveness of the anti-poverty policy.[18]

### 3.3 Similar Geometric Interpretations

The S index permits a simpler geometric interpretation that is somewhat different from that of Sen (1976) but is quite close to that of the SST index proposed by Shorrocks (1995). For comparison purpose, we also present and interpret the SST index geometrically.

Note again that the relative deprivation measure be $x_i = \frac{z-y_i}{z}$ if $z > y_i$, $x_i = 0$ otherwise, for $i = 1, 2, \ldots, n$. The $x_i$'s are in non-increasing order. The first $q$ $x_i$'s are positive for the poor who are deprived and the rest are zeros for the non-poor.

The deprivation profile can be graphed by plotting $\frac{1}{n}\sum_{i=1}^{r} x_i$ against $\frac{r}{n}$ for $r = 1, 2, \ldots, n$ in a unit square box. As shown in Figure 1, the poverty profile starts from the origin, reaches out concavely to the point $a$ and then becomes horizontal from $a$ to $H\overline{\mathbf{x}}_p$. The point $H$ is the poverty rate, and the point $H\overline{\mathbf{x}}_p$ represents the average poverty gap ratio of the population, $\overline{\mathbf{x}}$. Since the deprivation measures $\{x_i\}$ are in non-increasing order, the concave arc $0a$ is in fact an inverted Lorenz curve for the deprivation measures $\{x_i\}$, which represents the inequality of poverty gap ratios of the poor. The dotted straight line linking 0 and $a$ would be a segment of the poverty profile if the poor had identical incomes. Since the non-poor have zero deprivation, the horizontal segment $aH\overline{\mathbf{x}}_p$ of the deprivation profile has no significant information but shows the non-poor account for the $1 - H$ proportion of the population.

[Insert Figure 1 about here]

[18]See Bourguignon and Fields (1997) and Ravallion, van der Walle and Gautam (1995) discussed the relationship between poverty measures and anti-poverty policy actions. As Bourguignon and Fields (1997) noted, if there is a qualitative difference (e.g. in fuctions) between being poor or non-poor, the poverty rate is of specific interest. Similarly, if the aggregate deprivation level is of a major social concern, the average poverty gap ratio is of specific interest. If the dispersion of deprivations demands more social attention, inequality of deprivations is clearly of greater importance. But in practice the changes in inequality of deprivations over time or across jurisdictions, relative to those in the poverty rate or average poverty gap ratio, are of much smaller magnitude. See Osberg and Xu (1997,1999).

In Figure 2, we show the S index has a simple geometric interpretation that is similar to that of the Gini index. Note that triangle $0H'H$ is area $E$. Triangle $0Ha$ is area $C$. The space between arc $0a$ and the dotted straight line linking 0 and $a$ is area $D$. Thus,

$$\text{Area } E = \frac{1}{2}H. \tag{41}$$

$$\text{Area } C = \frac{1}{2}H^2\overline{\mathbf{x}}_p. \tag{42}$$

Area $D$ can be computed from the fact that the Gini index of poverty gap ratios of the poor is given by[19]

$$G(\tilde{\mathbf{x}}_p) = \frac{\text{Area } D}{\text{Area } C'} = \frac{\text{Area } D}{\text{Area } C}. \tag{43}$$

Using equations (42) and (43) yields

$$\text{Area } D = \text{Area } C \times G(\tilde{\mathbf{x}}_p) = \frac{1}{2}H^2\overline{\mathbf{x}}_p G(\tilde{\mathbf{x}}_p)$$

The S index is simply the ratio of the sum of areas $C$ and $D$ to area $E$, i.e.,

$$\begin{aligned} I_S(\mathbf{y}_p) &= \frac{\text{Area } C + \text{ Area } D}{\text{Area } E} \\ &= \frac{\frac{1}{2}H^2\overline{\mathbf{x}}_p + \frac{1}{2}H^2\overline{\mathbf{x}}_p G(\tilde{\mathbf{x}}_p)}{\frac{1}{2}H} \\ &= H\overline{\mathbf{x}}_p(1 + G(\tilde{\mathbf{x}}_p)). \end{aligned} \tag{44}$$

[Insert Figure 2 about here]

For a better understanding of the common multiplicative decomposition and similar geometric interpretations, we also analyze the geometric interpretation of the SST index in the similar fashion in Figure 3. Let the lower triangle of the unit box in Figure 3 be area $A$ and the rectangle at the lower right-hand corner of the unit box be area $B$. Thus

$$\text{Area } A = \frac{1}{2} \tag{45}$$

[19] Note that area $C'$, the triangle formed by two dotted straight lines and the vertical axis, is identical to area $C$.

and

$$\text{Area } B = (1-H)H\overline{\mathbf{x}}_p = H\overline{\mathbf{x}}_p - H^2\overline{\mathbf{x}}_p. \tag{46}$$

According to equation (19), the SST index can be expressed as

$$I_{SST}(\overset{*}{\mathbf{y}}) = H\overline{\mathbf{x}}_p(1 + G(\tilde{\mathbf{x}})). \tag{47}$$

Further, using equations (24), equation (47) becomes

$$I_{SST}(\overset{*}{\mathbf{y}}) = H\overline{\mathbf{x}}_p(2 - H + HG(\tilde{\mathbf{x}}_p)). \tag{48}$$

Now compute the ratio of the sum of areas $B$, $C$, and $D$ to area $A$, i.e.,

$$\begin{aligned} I_{SST}(\overset{*}{\mathbf{y}}) &= \frac{\text{Area } B + \text{Area } C + \text{Area } D}{\text{Area } A} \\ &= \frac{H\overline{\mathbf{x}}_p\left[(1-H)+\frac{1}{2}H+\frac{1}{2}HG(\tilde{\mathbf{x}}_p)\right]}{\frac{1}{2}} \\ &= H\overline{\mathbf{x}}_p(2 - H + HG(\tilde{\mathbf{x}}_p)) \\ &= H\overline{\mathbf{x}}_p(1 + G(\tilde{\mathbf{x}})). \end{aligned} \tag{49}$$

For the last two equalities, we have used equations (48) and (49). Thus, the SST index is the ratio of the sum of areas $B$, $C$ and $D$ to area $A$.

[Insert Figure 3 about here]

The similar geometric interpretation puts both S and SST indices in a Gini-like framework which shows clearly that $H$, $\overline{\mathbf{x}}_p$, and $G$ are three key components determining the poverty intensity. For applied economists and policy analysts, this graphical approach can convey the information about the poverty effectively.

## 3.4 Further Subgroup Decompositions

The Sen indices can be decomposed multiplicatively into three familiar and commonly used components ($H$, $\overline{\mathbf{x}}_p$, and $G$), which permit further subgroup decomposition.

The income vector for subgroup $k$ is $\mathbf{y}_{(k)} = [y_{1(k)}, y_{2(k)}, \ldots, y_{n_k(k)}]^\top$, where $y_{i(k)}$'s represent the income of individual $i$ ($i = 1, 2, \ldots, n$) in subgroup $k$ ($k = 1, 2, \ldots, l$) and are in non-decreasing order. The number of the elements in $\mathbf{y}_{(k)}$, $n_k$, depends on the number of individuals belonging to subgroup

$k$. Note that the sum of the group sizes gives the size of the population; that is $n = \sum_{k=1}^{l} n_k$. The vector of income of the poor in subgroup $k$ is $\mathbf{y}_{p(k)} = [y_{1(k)}, y_{2(k)}, \ldots, y_{q_k(k)}]^\top$, where the elements are in non-decreasing order. The number of the elements in $\mathbf{y}_{p(k)}$, $q_k$, depends on the number of poor individuals belonging to subgroup $k$. Note that the sum of the group poor gives the total number of the population poor; that is $q = \sum_{k=1}^{l} q_k$.

To discuss the subgroup decomposition of the Gini index of income later, the subgroup income vectors need to be stacked into a vector $\mathbf{y}_{(\cdot)}$ as $\mathbf{y}_{(\cdot)} = [\mathbf{y}_{(1)}^\top, \mathbf{y}_{(2)}^\top, \ldots, \mathbf{y}_{(l)}^\top]^\top$ so that the subgroup income vectors are sorted by the subgroup average incomes in non-decreasing order. Note that $\mathbf{y}$ differs from $\mathbf{y}_{(\cdot)}$ because the incomes in $\mathbf{y}$ are sorted by the individual incomes. $\mathbf{y}_p$ and $\mathbf{y}_{p(\cdot)}$ can be defined similarly as $\mathbf{y}$ and $\mathbf{y}_{(\cdot)}$, respectively. The average income of subgroup $k$ is denoted as $\overline{\mathbf{y}}_{(k)}$.

The vector of poverty gap ratios may be also defined for subgroup $k$ as $\mathbf{x}_{p(k)} = \frac{z\mathbf{1}-\mathbf{y}_{p(k)}}{z}$ where $\frac{z-y_{i(k)}}{z}$, $k = 1, 2, \ldots, l$ and $i = 1, 2, \ldots, q_k$. To discuss the subgroup decomposition of the Gini index of poverty gap ratios later, the vectors of subgroup poverty gap ratios need to be stacked into a vector as $\mathbf{x}_{p(\cdot)} = \left[\mathbf{x}_{p(1)}^\top, \mathbf{x}_{p(2)}^\top, \ldots, \mathbf{x}_{p(l)}^\top\right]^\top$, where the vectors of subgroup poverty gap ratios are sorted by the subgroup average poverty gap ratios in non-decreasing order. Note that $\mathbf{x}_p$ may differ from $\mathbf{x}_{p(\cdot)}$ because the poverty gap ratios in $\mathbf{x}_p$ are sorted by the individual poverty gap ratios. The average poverty gap ratio of the poor in subgroup $k$ is denoted by $\overline{\mathbf{x}}_{p(k)}$.

**Definition 5** *The poverty rate and the average poverty gap ratio for subgroup or region $k$ is defined and computed as $H_k = \frac{q_k}{n_k}$ and $\overline{\mathbf{x}}_{p(k)}$, respectively.*

**Proposition 5** *The poverty rate and the average poverty gap ratio are related to their subgroup counterparts, respectively, as follows:*

$$H = \sum_{k=1}^{l} w_k H_k, \tag{50}$$

*and*

$$\overline{\mathbf{x}}_p = \sum_{k=1}^{l} w_k \overline{\mathbf{x}}_{p(k)}, \tag{51}$$

*where $w_k = \frac{n_k}{n}$.*

**Proof**: The decompositions based on Foster, Greer and Thorbecke (1984) are quite intuitive. □

It is also possible to decompose the Gini index by subgroup following Yitzhaki and Lerman (1991) and Lambert and Aronson (1993) as follows:

**Proposition 6** *The Gini index $G(\mathbf{x}_p)$ can be decomposed into the between-group Gini index $G_B$,*[20] *the within-group Gini indices $G_k$'s, and overlapping component $R$ as follows:*

$$G(\mathbf{x}_p) = G_B(\mathbf{x}_p) + \sum_{k=1}^{l} b_k G_k(\mathbf{x}_{p(k)}) + R(\mathbf{x}_p, \mathbf{x}_{p(\cdot)}), \tag{52}$$

*where* $b_k = \frac{q_k^2}{q^2} \frac{\overline{\mathbf{x}}_{p(k)}}{\overline{\mathbf{x}}_p}$.

**Proof**: See Yitzhaki and Lerman (1991) and Lambert and Aronson (1993). □

We present the multiplicative decomposition of the S index and its components' subgroup decompositions in the following corollary.

**Corollary 2** *The S index can be expressed as*

$$\begin{gathered} I_S(\mathbf{y}_p) = \\ \left(\sum_{k=1}^{l} w_k H_k\right)\left(\sum_{k=1}^{l} w_k \overline{\mathbf{x}}_{p(k)}\right) \\ \times \left[1 - \left(G_B(\mathbf{x}_p) + \sum_{k=1}^{l} b_k G_k(\mathbf{x}_{p(k)}) + R(\mathbf{x}_p, \mathbf{x}_{p(\cdot)})\right)\right]. \end{gathered} \tag{53}$$

**Proof**: Combining the results in equation (5) and Propositions 5 and 6 gives Corollary 2.□

The Sen indices permit further subgroup decomposition of their decomposed components. The further subgroup decomposition of the S index is more straightforward and attractive than the SST index is since the $R$ term in the S index, which depends only on $\mathbf{x}_p$ and $\mathbf{x}_{p(\cdot)}$ . However, according to Proposition 3, the SST index can be expressed as a linear function of the S index. Hence, we have the following corollary:

[20] $G_B$ is computed as the Gini index but with the actual values of $\mathbf{x}$ being replaced by the corresponding subgroup averages.

**Corollary 3** *The SST index can be expressed as*

$$\begin{aligned} I_{SST}(\overset{*}{\mathbf{y}}) = \left(\sum_{k=1}^{l} w_k H_k\right)\left(\sum_{k=1}^{l} w_k \overline{\mathbf{x}}_{p(k)}\right) \\ \times\{2\left[1-\left(\sum_{k=1}^{l} w_k H_k\right)\right]+\left(\sum_{k=1}^{l} w_k H_k\right) \\ \times\left[1-\left(G_B(\mathbf{x}_p)+\sum_{k=1}^{l} b_k G_k(\mathbf{x}_{p(k)})+R(\mathbf{x}_p,\mathbf{x}_{p(\cdot)})\right)\right]\}. \end{aligned} \tag{54}$$

**Proof**: Combining the results in Lemma 3, equation (5) and Propositions 2 and 5 gives Corollary 3. □

The mathematical presentation of the further subgroup decomposition may appear to be somewhat complex. But it is inevitable because both the S and SST indices are the products of the most-used poverty and inequality measures (the poverty rate, the average poverty gap ratio and the Gini index of poverty gap ratios) which enable subgroup decomposability and similar geometric interpretations.

# 4 Concluding Remarks

This paper analyzes, from a theoretical point of view, the common underlying social evaluation function for the Sen indices of poverty intensity and presents a unified multiplicative decomposition framework for the Sen indices. The underlying social evaluation function gives these indices clear social welfare interpretation. The common multiplicative decomposition allows simple and intuitive geometric interpretations, easy numerical computation, and further subgroup decompositions.

The paper first shows that the BD index is a generalization of the S index while the S index is a special BD index with Gini social evaluation function. The C index is a generalization of the SST index while the SST index is a special C index with Gini social evaluation function. This paper then demonstrates that the S and SST indices have a common multiplicative decomposition structure which indicates that the indices can be viewed as the product of the poverty rate, the average poverty gap ratio of the poor and one plus the Gini index of the poverty gap ratios as follows:

$$\begin{pmatrix}\text{The S}\\ \text{index}\end{pmatrix} = \begin{pmatrix}\text{poverty}\\ \text{rate}\end{pmatrix}\begin{pmatrix}\text{poverty}\\ \text{gap}\end{pmatrix}\begin{pmatrix}1+\text{Gini index}\\ \text{of poverty gaps}\\ \text{of the poor}\end{pmatrix}$$

and

$$\begin{pmatrix} \text{The SST} \\ \text{index} \end{pmatrix} = \begin{pmatrix} \text{poverty} \\ \text{rate} \end{pmatrix} \begin{pmatrix} \text{poverty} \\ \text{gap} \end{pmatrix} \begin{pmatrix} 1 + \text{Gini index} \\ \text{of poverty gaps} \\ \text{of the population} \end{pmatrix}.$$

This common multiplicative decomposition structure (1) gives the two indices a much more straightforward interpretation of poverty intensity, (2) allows the indices to be computed much more easily via the commonly known poverty measures (the poverty rate and the average poverty gap ratio) and inequality measures (the Gini index of the poverty gap ratios), and (3) permits the indices to have the Gini-index-like geometric interpretations.

The practical implication of the multiplicative decomposition is that the Sen indices can be linearized so that the percentage change in these indices are additively decomposable. The three decomposed components of the S and SST indices have further subgroup decompositions. Since the underlying causes of changes over time or differences across jurisdictions in poverty intensity are of great interest to policy analysts, this class of poverty measures is indeed useful.

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Figure 1: Deprivation Profile

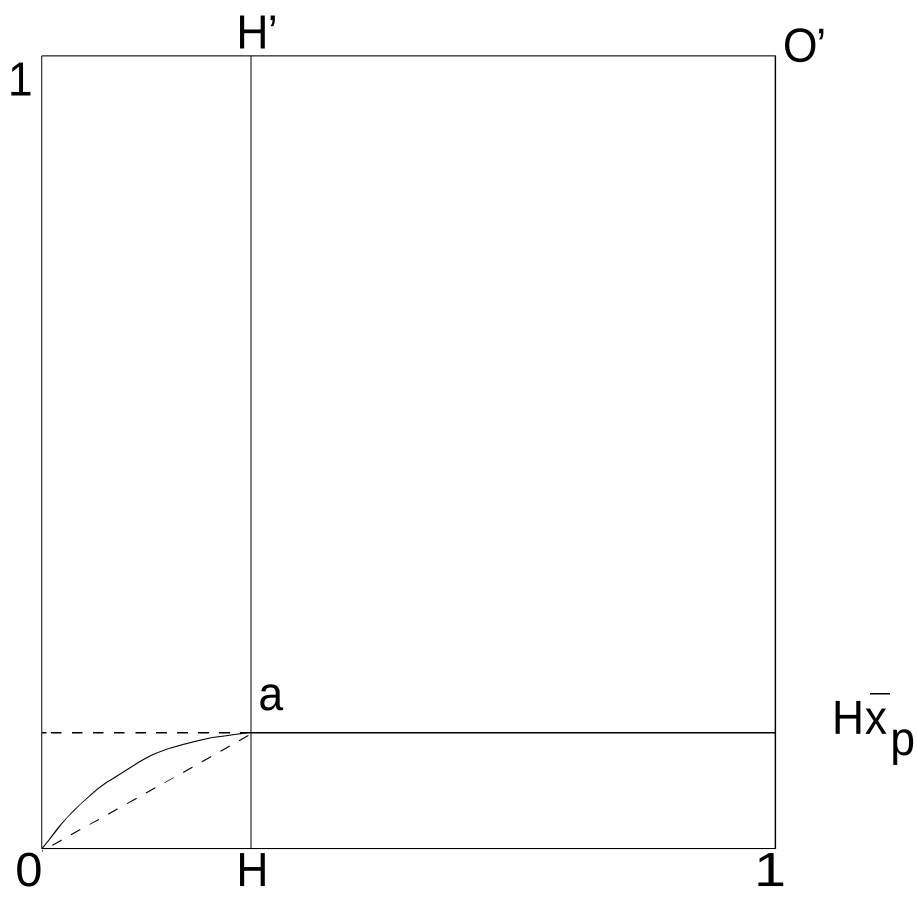

Figure 2: Geometric Interpretation of the S Index

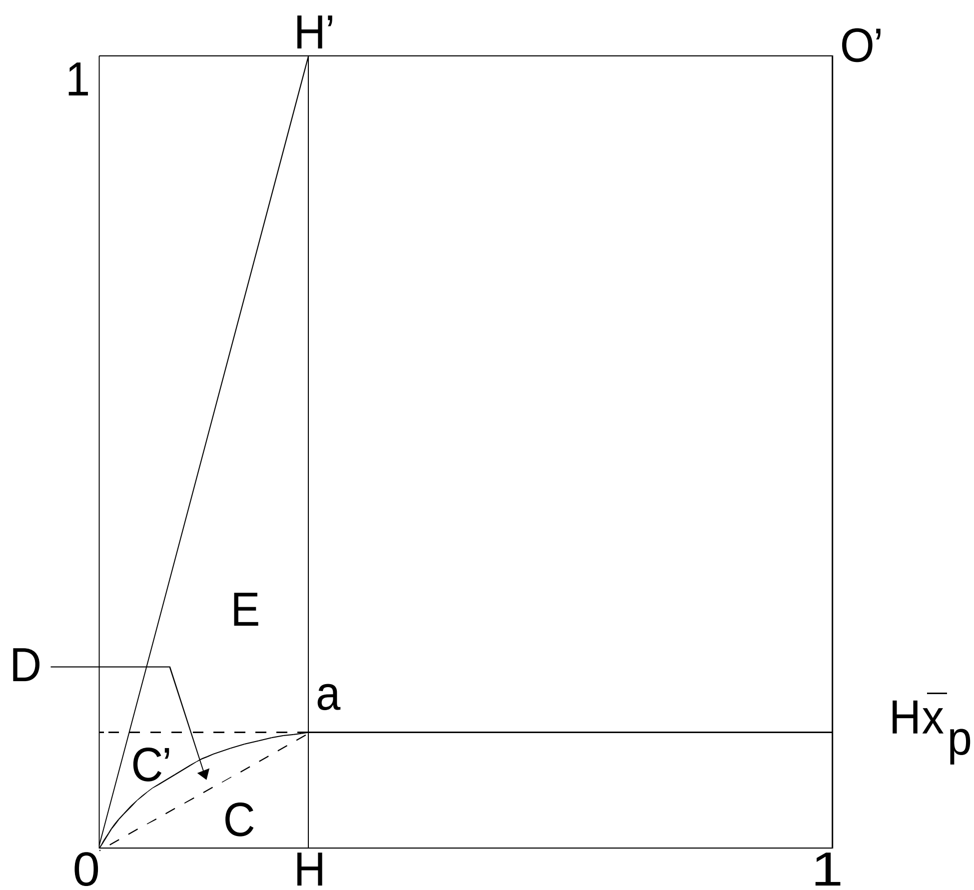

Figure 3: Geometric Interpretation of the SST Index