FORMS AND VALUES OF NUMBER-LIKE AND NIMBER-LIKE GAMES

by

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To people and other living things of the past, present, and future whose lives benefit us all in ways we may not know.

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Abstract

We consider combinatorial games (positions) played by two players who move alternately. In a disjunctive sum of positions a player may play in any one summand. Plays in a particular summand may not alternate between players. Play ends in a finite number of moves when a player cannot move in any summand. The winner is determined by the last player to move. We primarily consider the case where the last player to move wins (the normal-play convention).

In a sum of nimbers both players have the same available moves (options). In a sum of numbers both players would rather it not be their turn. The canonical form theory partitions positions into equivalence classes which form a group with disjunctive sum. The unique (canonical) representative of an equivalence class is called a value. Nimber-valued and number-valued positions are closed under disjunctive sum. It has long been known how to identify positions that are nimbers or numbers. There are other methods to analyze positions that are not nimber-like and number-like, such as reduced canonical forms and atomic weight. Nimbers and numbers are both hereditarily transitive (where no player would benefit from moving twice in a row) and both are HACKENBUSH positions.

Dicotic positions (where both players have an option or neither does) are like nimbers and numbers. The dicotic hereditarily transitive positions are described using ordinal sum. We show how to recognize positions whose values are from the ruleset HACKENBUSH STALKS (whose summands are described by ordinal sums). We then consider outcomes of HACKENBUSH STALKS under the misère-play convention.

We end by considering the ruleset PARTIZAN EUCLID, a partizan dicotic ruleset, that is like nimbers and numbers in other ways.

List of Abbreviations and Symbols Used

$\tilde{\mathrm{b}}(G)$	formal birthday of G.
$\mathrm{b}(G)$	birthday of G.
$G^{\mathbf{L}}$	the Left options of G.
$G^{\mathbf{R}}$	the Right options of G.
$\overline{G}$	the negative of G, used in misère-play.
$G \to H$	the move from G to H.
$G : H$	the ordinal sum of G and H.
$\mathcal{L}$	the set of Left win positions.
$\mathcal{N}$	the set of first player win positions.
$\mathcal{P}$	the set of second player win positions.
$\mathcal{R}$	the set of Right win positions.
$\mathscr{L}$	the outcome $(\mathrm{Left}, \mathrm{Left})$.
$\mathscr{N}$	the outcome $(\mathrm{Left}, \mathrm{Right})$.
$\mathscr{P}$	the outcome $(\mathrm{Right}, \mathrm{Left})$.
$\mathscr{R}$	the outcome $(\mathrm{Right}, \mathrm{Right})$.
$\mathrm{rcf}(G)$	the reduced canonical form of G.
$\tilde{\mathbb{G}}$	the set of short positions.
$\mathbb{G}$	the set of short values.
$\mathrm{LS}(G)$	the Left stop of G.
$\mathrm{RS}(G)$	the Right stop of G.
$\mathrm{T}_{\mathrm{L}}(G)$	Left-transitive closure of G.
$\mathrm{T}_{\mathrm{R}}(G)$	Right-transitive closure of G.
$\mathrm{T}(G)$	transitive closure of G.
$\mathrm{HT}(G)$	hereditary transitive closure of G.
$\mathcal{D}$	the universe of dicotic positions.
0	$\{\cdot \mid \cdot\}$ "zero", "the empty position".
1	$\{0 \mid \cdot\}$ "one".
n	$\{n-1 \mid \cdot\}$ "n".

$*$	$\{0 \mid 0\}$ “star”.
$*^n$	$\{0, *, \ldots, *^{n-1} \mid 0, *, \ldots, *^{n-1}\}$ “star-n”.
$\bigstar$	“far star”.
$\boldsymbol{+}_x$	$\{0 \mid \{0 \mid -x\}\}$ “tiny-x”.
$\uparrow$	$\{0 \mid *\}$ “up”.
$\uparrow*$	$\{0, * \mid 0\}$ “up star”.
$\downarrow$	$\{* \mid 0\}$ “down”.
$\Uparrow$	$\{0 \mid \uparrow*\}$ “double-up”.
$⤊$	$\{0 \mid \Uparrow*\}$ “triple-up”.

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Chapter 1

Introduction

1.1 Introduction

This thesis is a work in Combinatorial Game Theory (CGT). We consider sums of games played concurrently. Our two players, called Left and Right, move alternately in the sum but not necessarily alternately in any particular summand. In this thesis, every summand (and every sum) ends in a bounded and finite number of moves made by either player, irrespective of alternation. A sum ends when the player whose turn it is has no allowed move in any summand. In the normal-play convention a player who cannot move loses. This thesis primarily considers normal-play. Another convention is that a player who is unable to move wins. This is the misère-play convention, which we consider in Chapter 5.

1.1.1 Background

Chapter 1 is an introduction to this thesis and a formal but incomplete introduction to combinatorial game theory. See *Lessons in Play* [1] for an engaging and approachable introduction to combinatorial game theory.

We expect a reader familiar with [6] has the background to read this thesis without consulting [1].

A reader familiar with [32] and in particular Sections I.1, I.2, Chapters II (excluding II.5) and III, and Section V.6, could (not recommended) skip Chapter 1 from Section 1.2 onward and start reading Chapter 2, referring back to this Chapter as necessary while reading the remainder of the thesis.

Chapters 15 and 16 of [14] are closely related to this thesis, but are not recommended to be read before this thesis.

In Section 1.1.2 we present the motivation for the work in this thesis presuming the reader is familiar with combinatorial game theory. For readers who are unfamiliar with the resources listed above, we recommend they read from Section 1.2 onward to the end of this Chapter and then read Section 1.1.2.

We present previously known statements (Propositions, Theorems, Lemmas, etc.) as Facts to help distinguish that work from work of the author (possibly including coauthors). We include some proofs of Facts because they are instructive and some proofs because the statement was originally given without proof. Some Facts are given without proof, but if the result is non-trivial we include a citation where the statement is demonstrated.

A ruleset describes the rules of a game. We identify a RULESET by varying the font as shown. For example, some familiar rulesets are NIM, GO, and HACKENBUSH.

In CGT, the word *game* is often ambiguous and we try to avoid it. We use the term *position* for an instance of a ruleset or as in Definition 1.2.1.

1.1.2 Purpose

In combinatorial games we have two main complementary considerations:

- Given a position, what can we say about its form, value, reduced value or relations to other positions?
- Given a sum of positions, how can we use what we know of the individual positions to say something about the sum?

Numbers and nimbers are reasonably well-understood from a CGT perspective. We understand how to play in a sum if the summands are all numbers, if the summands are all nimbers, or if the summands are both nimbers and numbers and canonical forms tell us when a position is a nimber or number. This is a great success of CGT.

However, there is a large gap between nimbers and numbers. We argue that the best positions to fill that gap are the dicotic positions. Dicotic positions include the nimbers. Another dicotic position is $\uparrow$, which is vaguely number-like in the context of infinitesimals (see Fact 1.13.8).

In the study of short normal-play partizan games, some other theoretical highlights are the temperature theory, the reduced canonical form, the theory of atomic weights, understanding the group-theoretic structure, and the order-theoretic structure. For dicotic positions, we get the best help from atomic weights, but there are position with atomic weight 0 (a very common atomic weight) in all outcome classes.

The form of a position is important. We will miss important concepts if we only study values. A good example is the value 0. This position, in addition to being the canonical form of every position in $\mathcal{P}$, is also dicotic, a nimber, and a number.

In this thesis we distinguish the value of a position and its form. We hope this helps the reader get a sense of when and why the distinction is important. To help comprehension we use "is" in the same sense as "$\frac{6}{3}$ is 2".

A reasonable approach to infinitesimals is to introduce $\uparrow$ and at some point later introduce $\uparrow\!* \cong \{0, * \mid 0\}$ as the value of $\uparrow + *$. However, when it comes to understanding dicotic positions we prefer to introduce $\{0, * \mid 0\}$ as a position that looks like $*$. We then observe that $\{0, * \mid 0\} + *$ is positive, which is maybe a little surprising, because neither of the summands look particularly strong for either player; call this position $\uparrow$. When we see $\{0, * \mid 0\}$ we note that Left has an option to $*$ in addition to the option to 0 both players have. It is cute that if the option to $*$ didn't exist the position would be $*$. This structure is the ordinal sum in hiding: we discovered that $\{0, * \mid 0\} \cong * : 1$. That is, $\uparrow\!*$ is literally $*$-based. Now we understand why $\uparrow$ is positive: because $\uparrow = * : 1 + *$ and whoever first plays in a $*$ loses because the opponent responds in the other; the winner of $\uparrow$ is the same as the winner of $1 + 0$, namely Left. In Chapter 2 we show that $* : 1$ is a value and thus there is no need to also name this position $\uparrow\!*$.

Many more positions than $\uparrow$ are disjunctive sums of $*$-based positions. Of the 67 dicotic values born by day 3, 50 are sum of $*$-based positions.

In Chapter 2 important distinctions between form and value lead us to extend the definition of ordinal sum. The "new" ordinal sum is still a binary operation and the old ordinal sum of G and H is the ordinal sum of their values.

In Chapter 3 we show that dicotic *hereditarily transitive* positions are $*$-based and it was actually this realization that helped reveal their structure. Despite that, $*$-based positions do not have a prominant role in the proofs that build to the main results, namely the existence and uniqueness of *stratified* forms of hereditarily transitive positions. The hereditarily transitive property of positions is very sensitive to minor changes in a game tree, even those that preserve equality. We discover a very strong relationship between positions with at most one option for each player and hereditarily transitive positions and thus it is not surprising that numbers are hereditarily transitive but it is noteworthy that nimbers are hereditarily transitive.

In Chapter 4 we consider a family of HACKENBUSH positions, called STALKS, where we can easily simplify positions from the HACKENBUSH form but using the canonical form is cumbersome. STALKS are hereditarily transitive and thus this work builds on the previous chapters. In particular, we learn something about the group structure of hereditarily transitive positions by investigating sums of STALKS.

In Chapter 5 we consider a subset of STALKS called SPRIGS, but under misère-play. There is a big payoff for our careful work distinguishing form and value when we change the play convention. Namely, by addressing ordinal sums literally in Chapter 2 and STALKS with care in Chapter 4 we can start analyzing MISÈRE STALKS without essentially repeating our introduction. However, we do not analyze as many sums under misère-play as we do under normal-play because we avoid $*^2$, which is notoriously more tricky under misère-play.

In Chapter 6 we consider a partizan dicotic ruleset, PARTIZAN EUCLID, where each player has at most one option. This investigation contrasts with the rest of the thesis which all relates directly to $*$-based positions. Of course, there are many values that are sums of $*$-based positions, but we determine the outcome of a position without resorting to values. When we look at values in PARTIZAN EUCLID we find many complex atomic weights which is surprising considering our positions have at most one option per player and that we are able to efficiently determine the outcome of a position.

1.1.3 Contents

In this Chapter we give a thorough formal introduction to the short normal-play theory of partizan combinatorial game theory.

In addition to distinguishing the form and value of positions, we present a new and more formal approach to the definition of outcome that codifies the normal-play convention in a definition. We also note that some properties of positions are hereditary: the property holds for all subpositions (including the position itself).

Chapter 2 is very important for the thesis as a whole and Chapters 3, 4 and

5 in particular. In Section 2.2 we present largely introductory ordinal sum results. We extend the definition of ordinal sum to include *bases* that are not necessarily in canonical form. Even with a base in canonical form, an ordinal sum is not necessarily in canonical form; Section 2.3 addresses canonical forms of extended ordinal sums. We see in Section 2.4 that under the extended definition, the ordinal sum of dicotic positions is dicotic. We also consider positions with a hereditary property that implies that positions are dicotic and an ordinal sum. We then discuss general methods of recognizing positions as ordinal sums.

In Chapter 3 we investigate the structure of a hereditarily transitive positions. We strengthen a result on reduced canonical forms. With regards to canonical forms, we classify the positions that are both in canonical form and hereditarily transitive. With regards to infinitesimals, we describe the values of number-ish positions, and tighten the bounds on the infinitesimals of non-numberish hereditarily transitive positions. We also given a new form for hereditarily transitive positions that is unique and analogous to canonical form for short positions under normal-play in general. Finally we investigate the partial order structure of values of hereditarily transitive positions where our work on the form of hereditarily transitive positions allows us to prove a result of [34].

In Chapter 4 we consider *stalks*. A disjunctive sum of *stalks* is equal to a STALKS position. We analyze outcomes for small numbers of stalks, and some families of stalks. We then consider values of some simpler families. We show that we can recognize positions (values) as disjunctive sums of stalks even for families where we do not know what the outcomes are. Finally, we consider $*$-based positions that are not stalks and we find that $*$-based numbers are special even in that more general context.

In Chapter 5 we analyze sums of SPRIGS (star-based dyadic rationals) under misère-play. Typically, rulesets are much harder to analyze under misère-play than under normal-play. We show that in some situations ordinal sums under misère-play exhibit very normal-play-like properties. In particular, outcomes of sums of SPRIGS under misère-play are directly relatable to sums of SPRIGS under normal-play.

In Chapter 6 we consider the ruleset PARTIZAN EUCLID which is inspired by the Euclidean algorithm. We use the structure of this ruleset to describe the form of the game tree. Each game tree is described by a particular path through the game tree called the *spine*. The *signature* of a position is the list of which player moves to stay on the spine. We show how to find the signature of a position and how to efficiently determine the outcome of a position from its signature. We also consider the values of PARTIZAN EUCLID POSITIONS.

1.2 Starting from empty

A game (position), say G, is an ordered pair of sets of positions and is denoted by $\left\{G^{\mathbf{L}} \,\middle|\, G^{\mathbf{R}}\right\}$ where $G^{\mathbf{L}}$ is the set of *Left options* and $G^{\mathbf{R}}$ is the set of *Right options*. We also write $G \cong \left\{G^{\mathbf{L}} \,\middle|\, G^{\mathbf{R}}\right\}$ to identify G with its options. Equality of positions is defined later, where we use '='. The bold superscripts of $G^{\mathbf{L}}$ and $G^{\mathbf{R}}$ represent that these objects are sets. We may refer to particular elements of the Left and Right options by G^L and G^R, respectively. In the case that a set of options is empty it is denoted by a '$\cdot$'.

A position with a superscript that is a string composed of letters $\mathbf{L}$, L, $\mathbf{R}$, and R is a position if the string ends with a non-bold letter (L or R) and is a set of positions if it ends with $\mathbf{L}$ or $\mathbf{R}$.

In representing a particular position we omit the braces of the set of options. At times, such as in Definition 1.4.1, we abuse notation and give sets instead of or along with options. The options include all elements of the sets and any positions.

The *empty* position is $\{\cdot \mid \cdot\}$ and is called 0.

Definition 1.2.1. *Let $\tilde{\mathbb{G}}_0 = \{0\}$ and for integers $n \geq 0$ let*

$$\tilde{\mathbb{G}}_{n+1} = \left\{\left\{G^{\mathbf{L}} \,\middle|\, G^{\mathbf{R}}\right\};\ G^{\mathbf{L}}, G^{\mathbf{R}} \subseteq \tilde{\mathbb{G}}_n\right\}.$$

A short *position is an element of*

$$\tilde{\mathbb{G}} = \bigcup_{n\geq 0} \tilde{\mathbb{G}}_n.$$

The positions we consider in this thesis are *short*. We depend on positions to be short especially when we use induction, but we do not always explicitly state that we rely on the positions to be short.

In Definition 1.2.1, $G^{\mathbf{L}}$ and $G^{\mathbf{R}}$ can be empty.

Definition 1.2.2. *A position, say G, is a* Left end *if G has no Left option; and G is a* Right end *if G has no Right option.*

The position $0 \cong \{\cdot \mid \cdot\}$ is both a Left end and a Right end. Also, 0 is an element of $\tilde{\mathbb{G}}_n$ for all $n \geq 0$.

We say an option of G can be *reached* in 1 *move*. A Left move in G gives a Left option of G; a Right move in G gives a Right option of G. An option of a position that can be reached in n moves can be reached in $n+1$ moves. A position is a *proper subposition* of G if it can be reached from G in 1 or more moves; the *subpositions* of G are G and its proper subpositions.

We write $G \xrightarrow{L} G^L$ in place of "the Left move from G to G^L", $G \xrightarrow{R} G^R$ in place of "the Right move from G to G^R". We also write $G \to H$ in place of "the move from G to H" and let context dictate which player(s) are meant.

Definition 1.2.3. *[32, Definition 1.4, p. 9]*
A finite run *of G of length k is a sequence of subpositions of G*

$$G_0, G_1, G_2, \ldots, G_k$$

such that $G_0 \cong G$ and G_{i+1} is an option of G_i for i from 0 to $k-1$.
A finite alternating run *of G of length k is a run where for $0 \leq i \leq k-2$:*
if $G_i \to G_{i+1}$ is a Left move then $G_{i+1} \to G_{i+2}$ is a Right move, and
if $G_i \to G_{i+1}$ is a Right move then $G_{i+1} \to G_{i+2}$ is a Left move.
A finite alternating run of G of length k is a play
if $G_{k-1} \to G_k$ is a Left move and G_k is a Right end; or
if $G_{k-1} \to G_k$ is a Right move and G_k is a Left end.

In CGT when we do not restrict ourselves to short positions there are non-finite runs and plays. A short position has finitely many options, and only finite runs. Short positions thus have only finitely many subpositions.

Definition 1.2.4. *The formal birthday of a position G, denoted $\tilde{\mathrm{b}}(G)$ is 0 if G is empty and $1 + \max(\tilde{\mathrm{b}}(G'))$ where G' ranges over all options of G.*

The length of a maximum run in G is $\tilde{\mathrm{b}}(G)$. If G is short, $\tilde{\mathrm{b}}(G)$ is finite.

A short position can be represented by a finite *game tree*; see Table 1.1. The position is identified with the root (the top vertex); the Left options are drawn to the left and the Right options are drawn to the right.

However, there are game trees that do not correspond directly with our definition of position as Left options and Right options are sets. For example, $\{0, 0 \mid\}$ is *not* technically a position (it would be $\{0 \mid\}$ instead) but could correspond to a game tree. We do not consider rulesets where this distinction is a concern.

Naming positions, such as $0 \cong \{\cdot \mid \cdot\}$ is essentially naming trees.

Definition 1.2.5. *The* negative *of a position G, denoted by $-G$ is*

$$-G \cong \left\{ -G^{\mathbf{R}} \,\middle|\, -G^{\mathbf{L}} \right\}$$

and the negative of a set is the set containing the negatives of all the elements of the set.

1.3 Outcomes

Our players are Left and Right. As shown in Table 1.2, we associate each of our players with a colour, a sign, and a pronoun.

For any position, we are interested in the result with Left moving first and also the result with Right moving first, although our sympathies are usually with Left.

A beautiful aspect of CGT is that we can analyze positions without knowing whose turn it is. In this section we develop the theory that allows us to do this.

We assume our players can play perfectly, winning if and only if it is possible. Hence, we are only concerned about whether a player can win and if so we say that player wins.

The Fundamental Theorem of CGT of [1, Theorem 2.1, p. 35] and [32, Theorem 1.5, p. 9] differ very slightly. Ours is very similar to the latter but we do not assume normal-play yet.

Theorem 1.3.1 (Fundamental Theorem of Combinatorial Games)**.** *Fix a short position G played between Left and Right, with Left moving first. If every play of G results in a Left win or a Right win, then either Left can force a win moving first, or Right can force a win moving second, but not both.*

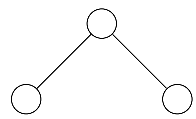
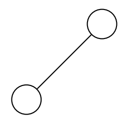
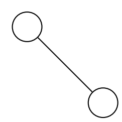
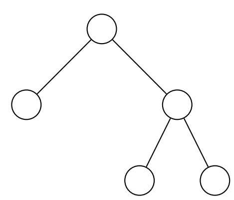
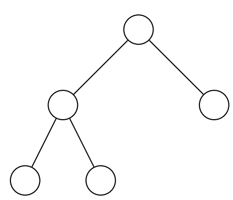
	Game tree	
0		zero
$*$		star
1		one
-1		negative one
$\uparrow$		up
$\downarrow$		down

Table 1.1: Some very short game trees and their names

Left	Right
bLue	Red
positive	negative
she	he

Table 1.2: Players

Proof. If G is a Left end, then G is already played. If G is a Left win then Left wins (going first); if G is a Right win then Right wins (going second).

Otherwise, consider a typical Left option of G, say G^L. As G is short, G^L has strictly fewer subpositions than G.

We assume by induction (and symmetry) that either Right can force a win playing first on G^L, or else Left can force a win playing second. If Right can win all such G^L playing first, then certainly he can win G playing second regardless of Left's opening move. Conversely, if Left can win any such G^L playing second, then he can win G by moving to it. Exactly one of these two possibilities must hold. $\square$

We have presented Theorem 1.3.1 supposing Left moves first. A similar statement holds if every instance of "Left" is replaced with "Right" and vice versa. In this case and in other such cases we usually omit the similar statement.

A *play convention* declares when a game finishes and the result at that time.

The function $o_L(G)$ is the result with Left going first and the function $o_R(G)$ is the result with Right going first.

Theorem 1.3.1 needs a play convention that results in exactly Left or Right being declared the winner when the player whose turn it is to move has no options. This condition is sufficient for short positions because a play of a position guarantees the occurrence of such a state. That is, o_L and o_R are well-defined on short positions.

If $o_L(G) = \text{Left}$, we say Left wins going first in G and if $o_L(G) = \text{Right}$, we say Right wins going second in G.

If $o_R(G) = \text{Left}$, we say Left wins going second in G and if $o_R(G) = \text{Right}$, we say Right wins going first in G.

There are exactly two possible starting players (Left and Right) and two possible results (Left wins or Right wins). Thus, each position corresponds to a function from $\{\text{Left}, \text{Right}\}$ to $\{\text{Left}, \text{Right}\}$ where the pre-image is the starting player and the image is the winner. There are $2^2 = 4$ such functions.

We represent such a function by a pair. The *outcome* of a position G, denoted $o(G)$, is the pair

$$o(G) = (o_L(G), o_R(G)).$$

The four outcomes are $(\text{Left}, \text{Left})$, $(\text{Right}, \text{Left})$, $(\text{Left}, \text{Right})$, and $(\text{Right}, \text{Right})$, which are denoted by $\mathscr{L}$, $\mathscr{P}$, $\mathscr{N}$, and $\mathscr{R}$, respectively.

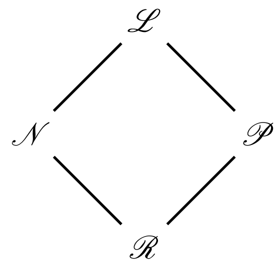


Figure 1.1: The partial order of outcomes

Our sympathies are usually with Left and we consider Left > Right. This induces a partial order on outcomes via the product order as shown in Figure 1.1.

Definition 1.3.2. *The* normal *play convention declares a game finished when the player to move has no options and declares the opponent as the winner. That is:*

if G is a Left end, then $o_L(G) = \text{Right}$; and

if G is a Right end, then $o_R(G) = \text{Left}$.

Example 1.3.3. *For the position 0, $o_L(0) = \text{Right}$ and $o_R(0) = \text{Left}$.*

As we consider short positions, every play finishes at an end for the player whose turn it is to move. By Theorem 1.3.1 every play results in Left or Right as the winner. Thus the normal-play convention gives a winner as follows:

For any position G,

$$o_L(G) = \begin{cases} \text{Left} & \text{if } o_R(G^L) = \text{Left for some } G^L \\ \text{Right} & \text{otherwise} \end{cases}$$

$$o_R(G) = \begin{cases} \text{Right} & \text{if } o_L(G^R) = \text{Right for some } G^R \\ \text{Left} & \text{otherwise.} \end{cases}$$

From here on (with the exception of Chapter 5) we assume the normal-play convention unless stated otherwise. However, formal Definitions (e.g those using $\cong$) apply without regard to play convention.

Short positions are thus partitioned into four ($\mathcal{L}$, $\mathcal{R}$, $\mathcal{N}$, $\mathcal{P}$) sets called *outcome classes* according to the outcome under normal-play:

- $\mathcal{L}$, (the set of positions where Left can win going first and second)

- $\mathcal{R}$, (the set of positions where Right can win going first and second)
- $\mathcal{N}$, (the set of positions where the next player to move wins regardless if this is Left or Right)
- $\mathcal{P}$, (the set of positions where the next player cannot win regardless if this is Left or Right).

Fact 1.3.4. *Let G be a position. The outcome of G is determined by the outcomes of its options. Specifically:*

- $o(G) = \mathscr{L}$ *iff* $\exists G^L$, $o(G^L) \in \{\mathscr{L}, \mathscr{P}\}$ *and* $\forall G^R$, $o(G^R) \in \{\mathscr{L}, \mathscr{N}\}$*;*
- $o(G) = \mathscr{P}$ *iff* $\forall G^L$, $o(G^L) \in \{\mathscr{N}, \mathscr{R}\}$ *and* $\forall G^R$, $o(G^R) \in \{\mathscr{L}, \mathscr{N}\}$*;*
- $o(G) = \mathscr{N}$ *iff* $\exists G^L$, $o(G^L) \in \{\mathscr{L}, \mathscr{P}\}$ *and* $\exists G^R$, $o(G^R) \in \{\mathscr{P}, \mathscr{R}\}$*;*
- $o(G) = \mathscr{R}$ *iff* $\forall G^L$, $o(G^L) \in \{\mathscr{N}, \mathscr{R}\}$ *and* $\exists G^R$, $o(G^R) \in \{\mathscr{P}, \mathscr{R}\}$*.*

Proof. This is a straightforward application of the definition of outcome. $\square$

The outcome class version of Fact 1.3.4 is as follows:

Corollary 1.3.5. *Let G be a position. The outcome class of G is determined by the outcome classes of its options. Specifically:*

- $G \in \mathcal{L}$ *iff* $\exists G^L, G^L \in \mathcal{L} \cup \mathcal{P}$ *and* $\forall G^R, G^R \in \mathcal{L} \cup \mathcal{N}$*;*
- $G \in \mathcal{P}$ *iff* $\forall G^L, G^L \in \mathcal{N} \cup \mathcal{R}$ *and* $\forall G^R, G^R \in \mathcal{L} \cup \mathcal{N}$*;*
- $G \in \mathcal{N}$ *iff* $\exists G^L, G^L \in \mathcal{L} \cup \mathcal{P}$ *and* $\exists G^R, G^R \in \mathcal{P} \cup \mathcal{R}$*;*
- $G \in \mathcal{R}$ *iff* $\forall G^L, G^L \in \mathcal{N} \cup \mathcal{R}$ *and* $\exists G^R, G^R \in \mathcal{P} \cup \mathcal{R}$*.*

Lemma 1.3.6. *If $H \in \mathcal{R} \cup \mathcal{N}$ then*

$$o(G) = o\left(\left\{H, G^{\mathbf{L}} \;\middle|\; G^{\mathbf{R}}\right\}\right).$$

Proof. No matter the outcome of G, adding a Left option in $\mathcal{R}$ or $\mathcal{N}$ does not change the existence of an option in $\mathcal{L} \cup \mathcal{P}$ and does not change whether all options are in $\mathcal{R} \cup \mathcal{N}$. $\square$

Proposition 1.3.7. *If G^L has exactly one Right option, namely to G^{LR}, then*

$$o(G) = o(\left\{G^{\mathbf{L}} \setminus \left\{G^L\right\}, G^{LR\mathbf{L}} \;\middle|\; G^{\mathbf{R}}\right\}).$$

Proof. We are proving a statement about outcome, which means we are playing a position in isolation (not as part of a sum). If Left plays to G^L, then Right has play to G^{LR} and thus Left will be choosing between the Left options of G^{LR}. Functionally, Left may have moved immediately to any of these. □

1.4 Equality and other relations

Definition 1.4.1. *The* disjunctive sum *of positions G and H is recursively defined as*

$$G + H \cong \left\{G^{\mathbf{L}} + H, G + H^{\mathbf{L}} \;\middle|\; G^{\mathbf{R}} + H, G + H^{\mathbf{R}}\right\}$$

where the sum of a set $\mathbf{A}$ with a position K is $\mathbf{A} + K = \{a + K; a \in A\}$.

Definition 1.4.2. *For positions G and H, we say G and H are* equal *and write $G = H$ if*

$$o(G + X) = o(H + X) \text{ for all short positions } X.$$

For positions G and H, $G \cong H$ implies $G = H$. If $G = H$ we also say G *is* H and if $G \cong H$ we say G *is literally* H.

The relation '=' is an equivalence relation on the set of all short positions [32, Proposition 1.8, p. 55].

Fact 1.4.3. *[32, Theorem 1.12, p. 56] The equivalence class of* 0 *is* $\mathcal{P}$*.*

We define subtraction as the addition of the negative (Definition 1.2.5), as seen in Fact 1.4.4, which expresses that negatives are inverses, at least in normal-play.

Fact 1.4.4. *[32, Theorem 1.13, p. 56] For a position G, $G - G = 0$.*

Definition 1.4.5 defines a partial order [32, Proposition 1.17, p. 57] on $\tilde{\mathbb{G}}$ that builds on the partial order of outcomes in Figure 1.1 (and our sympathy with Left).

Definition 1.4.5. *For positions G and H, $G \geq H$ if $o(G + X) \geq o(H + X)$ for all positions X.*

If $G \geq H$, then in a sum Left is at least as well off if G replaces H.

Fact 1.4.6. *[32, pp. 57–58]*

$G = H$ *if and only if* $o(G - H) = \mathscr{P}$*;*

$G \geq H$ *if and only if* $o(G - H) \geq \mathscr{P}$*.*

Definition 1.4.7. *If* G *is not equal to* H *we write* $G \neq H$*.*

If $G \geq H$ *and* $G \neq H$ *we write* $G > H$ *and say* G *is greater than* H*.*

If $G \leq H$ *and* $G \neq H$ *we write* $G < H$ *and say* G *is less than* H*.*

If $G \geq H$ *or* $G \leq H$ *we say* G *and* H *are comparable.*

If G *and* H *are not comparable we write* $G \parallel H$ *and say* G *is confused with* H*.*

Positions are not linearly ordered. For example, $* \parallel \uparrow$.

Example 1.4.8. *Real numbers are linearly ordered. Fix non-negative real numbers* x *and* y*. Exactly one of* $x = y$*,* $x > y$*, or* $x < y$ *holds.*

Here are two examples where we have partial information about the relation of x *and* y*: if* $x = 0$*, then* $x = y$ *or* $x < y$*, which we denote by* $x \leq y$*; if* x *and* y *are* distinct *then* $x > y$ *or* $x < y$*, which we denote by* $x \neq y$*.*

Because there are 3 *possibilities for the relation of* x *and* y*, exactly one of which holds, there are* $2^3 - 1$ *possible states of information on the relation of* x *and* y*:* $x \;\rho\; y$ *where* ρ *is one of* $=$, $>$, $<$, $\neq$, $\geq$, $\leq$, *or* ? *(no information).*

A position G has an outcome but sometimes we have partial information about its outcome. As there are 4 outcomes, there are $2^4 - 1 = 15$ possible states of information regarding the outcome of a position (having no outcome is impossible). Figure 1.2 shows these cases (and defines symbols for those cases where no symbols have been defined above). There are $4 = \binom{4}{1}$ cases where we know the outcome; $6 = \binom{4}{2}$ where we know it is one of two outcomes; $4 = \binom{4}{3}$ cases where we know one outcome that a position does not have; and $1 = \binom{4}{4}$ case where we know nothing about the outcome of a position.

Where ρ is any of the relations from Figure 1.2,

$$G \;\rho\; H \iff (G - H) \;\rho\; 0.$$

Fact 1.4.9. *[32, p. 62] If* G^L *is a Left option of* G*,* $G^L \lhd\!\mid G$*.*

ρ	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{L}$	$\mathcal{R}$	We say G is	Who wins G?
$=$	●	○	○	○	zero	Previous wins
$\parallel$	○	●	○	○	fuzzy	Next wins
$>$	○	○	●	○	positive	Left wins
$<$	○	○	○	●	negative	Right wins
$\neq$	○	●	●	●	non-zero	a player wins going first
$\gtreqless$	●	○	●	●	comparable with 0	a player loses going first
$\ngtr$	●	●	○	●	non-positive	Right wins for some start
$\nless$	●	●	●	○	non-negative	Left wins for some start
$\geq$	●	○	●	○	at least 0	Left wins playing second
$\leq$	●	○	○	●	at most 0	Right wins playing second
$\rhd\!\!\mid$	○	●	●	○	not ≤ 0	Left wins playing first
$\mid\!\!\lhd$	○	●	○	●	not ≥ 0	Right wins playing first
$\gtrless$	○	○	●	●	positive or negative	Left or Right wins
$\ngtrless$	●	●	○	○	zero or fuzzy	Previous or Next wins
?	●	●	●	●	no information	

Figure 1.2: Relations ρ and possible outcomes of G given $G \ \rho\ 0$.

1.5 Simplification and Canonical Form

In practice, Lemma 1.3.6 allows us to ignore options that are confused with 0 when we are proving statements about outcomes, but we can say something stronger.

Fact 1.5.1 (The Gift Horse Principle)**.** *[6, p. 72] [32, p. 63] If $H \lhd\!\mid G$ then*

$$G = \left\{ H, G^{\mathbf{L}} \;\middle|\; G^{\mathbf{R}} \right\}.$$

Given a position, a common desire is to find an equal position that is simpler. A stronger statement is Fact 1.5.8, the Canonical Form Theorem.

Definition 1.5.2. *Let G be a position. A Left option of G, say G^L, is* dominated *by some distinct Left option, say $G^{L'}$ if $G^{L'} \geq G^L$. We say G^L is* strictly dominated *if $G^{L'} > G^L$.*

Let G be a position. A Right option of G, say G^R, is dominated *by some distinct Right option, say $G^{R'}$, if $G^{R'} \leq G^R$. We say G^R is* strictly dominated *if $G^{R'} < G^R$.*

The following statement is easily proven. It provokes thought on the relations between equality, outcome, and form.

Lemma 1.5.3. *For all G, $G \in \mathcal{N} \iff G = \left\{ 0, G^{\mathbf{L}} \;\middle|\; 0, G^{\mathbf{R}} \right\}$.*

Proof. ($\Rightarrow$) If $G \in \mathcal{N}$ then (by Fact 1.3.4) there exists an $H \in G^{\mathbf{L}}$ such that $H \geq 0$, and there exists $K \in G^{\mathbf{R}}$ such that $K \leq 0$. The options to 0 are dominated and thus $G = \left\{0, G^{\mathbf{L}} \,\middle|\, 0, G^{\mathbf{R}}\right\}$.

($\Leftarrow$) Clearly, either player can win in $\left\{0, G^{\mathbf{L}} \,\middle|\, 0, G^{\mathbf{R}}\right\}$ moving first to 0. $\square$

Fact 1.5.4 (Remove dominated options)**.** *[32, Theorem 2.4, p. 65] If the Left option G^L is dominated by some $G^{L'}$,*

$$G = \left\{G^{\mathbf{L}} \setminus \left\{G^L\right\} \,\middle|\, G^{\mathbf{R}}\right\}.$$

Definition 1.5.5. *Let G be a position. A Left option of G, say G^L, is* reversible *through some Right option of G^L, say G^{LR}, if $G^{LR} \leq G$.*

Let G be a position. A Right option of G, say G^R, is reversible *through some Left option of G^R, say G^{RL}, if $G^{RL} \geq G$.*

Fact 1.5.6 (Bypass reversible options)**.** *[32, Theorem 2.5, p. 65] If G^L is reversible through G^{LR},*

$$G = \left\{G^{\mathbf{L}} \setminus \left\{G^L\right\}, G^{LR\mathbf{L}} \,\middle|\, G^{\mathbf{R}}\right\}.$$

As noted in [1, p. 80], bypassing reversible options is "not at all intuitive". With that in mind, we present a context in which reversibility is intuitive. In the sum $G+H$, a move in G, say for Left, is reversible by Right by playing in H when Right's incentive (Definition 1.8.1) in H exceeds Left's incentive in G. That is, domination is comparing two incentives for the same player and in some contexts reversibility is comparing incentives of opponents.

We often state that a particular option is dominated without stating the dominating option; or reversible without referencing the position though which the option is reversible. Usually this happens we when are focused on whether a particular position is in canonical form:

Definition 1.5.7. *A position is* in canonical form *if every subposition has no dominated option and no reversible option.*

Fact 1.5.8. *For every position G, there is a unique position in* canonical form*, say H, such that $G = H$.*

Generally $G \cong H$ is a stronger relationship than $G = H$, but when G and H are both in canonical form they are the same.

Fact 1.5.9. *[32, Theorem 2.9, p. 67] If G and H are in canonical form and $G = H$ then $G \cong H$.*

Positions that are not in canonical form may have dominated options, reversible options or both. A special case is the following.

Lemma 1.5.10. *If $G = 0$ but $G \not\cong 0$, then G has a reversible option.*

Proof. Suppose without loss of generality that Left has an option, G^L. As $G \in \mathcal{P}$, $G^L \in \mathcal{N} \cup \mathcal{R}$; that is $G^L \lhd\!\!| \; 0$ and so there is a $G^{LR} \in \mathcal{P} \cup \mathcal{R}$; that is, $G^{LR} \leq 0$ and G^L is reversible. □

If an option, say G^L, is reversible through G^{LR} but G^{LR} has no Left options, we say G^L *reverses out*. In the case of a zero position, eventually all options will reverse out.

The equivalence classes under $=$ form a group with the disjunctive sum. In [32] these equivalence classes are referred to as values. However, we use the term value more in the style of [14] and [1]: the *value* of a position G is the unique position in canonical form of the equivalence class of G. The set (or group) of values is denoted $\mathbb{G}$. The term value in this context refers to the practice of deliberate naming to inform the reader about outcome, group theoretic properties, order relations, or other properties.

The positions in Table 1.1 are all values; they are all in canonical form. Furthermore, the names invoke some familiar properties: $0 + G = G$ for all positions G, $1 + (-1) = 0$, and $\uparrow + \downarrow = 0$.

If $G = H$, then we say G and H have the same *value*. If we know the value of a position we know more about its subpositions.

Fact 1.5.11. *[32, Lemma 3.17, p. 175] Let K be a value and L a subposition of K. If $G = K$ then there is some subposition H of G with $H = L$.*

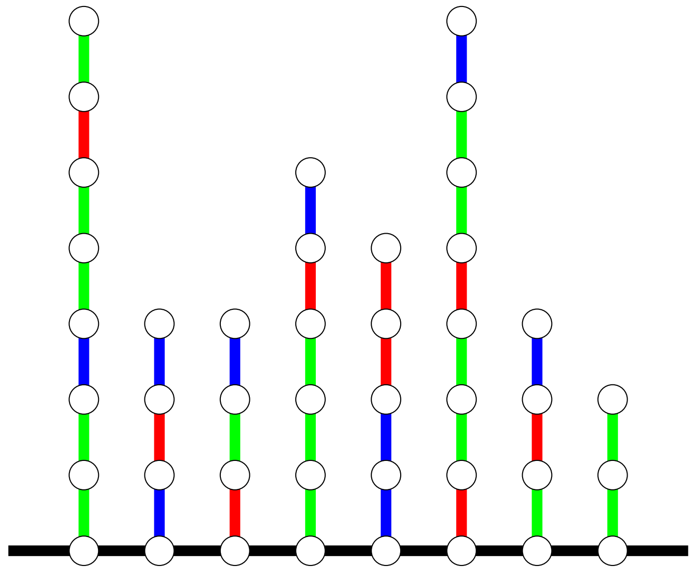


Figure 1.3: A STALKS position

1.6 Hackenbush

HACKENBUSH is a ruleset that is particularly important to this thesis. For a excellent introduction to HACKENBUSH there is no better place to look than [6] where HACKENBUSH is introduced on page 1.

We are concerned primarily with STALKS, such as in Figure 1.3.

In the study of normal-play games we almost always consider (either explicitly or implicitly) the value of a position. We look for positions with values we recognize and look for new families of positions to name. An open question is to classify the values of trees in HACKENBUSH.

When we name an instance of a ruleset we say for example: "Suppose G *is* a HACKENBUSH position" or "Let G be a HACKENBUSH position" to mean "consider a HACKENBUSH position which we call G".

form	$\{\cdot \mid \cdot\}$	$\{0 \mid 0\}$	$\{0 \mid \cdot\}$	$\{\cdot \mid 0\}$
value	0	$*$	1	-1
outcome	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{L}$	$\mathscr{R}$

Table 1.3: HACKENBUSH positions with at most one edge

If we are already discussing a position, say H, and wish to express that H has the same form as some HACKENBUSH POSITION we say "H is literally a HACKENBUSH position". For example, $*$ is literally a HACKENBUSH position.

1.7 Form and heredity

To emphasize that we refer to the game tree structure of a position (and not its value) we may refer to the *literal form* of a position.

A property of a position is called *formal* if the property depends on the form of the position (not only the value).

For example, *formal birthday* (Definition 1.2.4) and Left end (Definition 1.2.2) are formal definitions.

Definition 1.7.1. *The* birthday *of a position G, denoted* $\mathrm{b}(G)$ *is the formal birthday of the canonical form of G. A position G is* born on day n *if* $\mathrm{b}(G) = n$ *and a position is* born by day n *if* $\mathrm{b}(G) \leq n$.

A property of a position is called *hereditary*, if when a position has a given property then all subpositions also have that property. That is, a hereditary property holds for every option of a position that has said property. Inductively, the property holds for every subposition. A set S is *hereditarily closed* if for every $G \in S$ and every option G' of G, $G' \in S$.

Example 1.7.2. *The property that a position is in canonical form is hereditary and formal. The set of values is hereditarily closed.*

The property that $\tilde{\mathrm{b}}(G) \leq n$ for some n is hereditary and formal. For $n \geq 0$, the set $\{G \in \tilde{\mathbb{G}}; \tilde{\mathrm{b}}(G) \leq n\}$ is hereditarily closed.

Rulesets also relate to heredity. If G is literally a HACKENBUSH position then every subposition is a literal HACKENBUSH position.

The property that G is literally a HACKENBUSH position is hereditary.

The formal birthday of a HACKENBUSH position is the number of edges.

1.8 Incentives

Definition 1.8.1. *For a position G, a* Left incentive *of G is a position*

$$\Delta_G^L = G^L - G, \text{ for some } G^L;$$

and a Right incentive *of G is a position*

$$\Delta_G^R = G - G^R, \text{ for some } G^R.$$

The subscript G is omitted when a particular G is implied. The order relations of the set of Left (Right) incentives of a position G are the same as those of the Left (Right) options of G. That is, if the incentive corresponding to a particular option is greater than the incentives other the other options, then that option dominates all others.

Fact 1.8.2. *[6, p. 197, without proof] In* HACKENBUSH *positions, moves corresponding to blue or red edges have negative incentives.*

Fact 1.8.3. *If Left wins playing first in G, then Right going first loses upon making a move of negative incentive.*

Proof. If Left wins playing first, then $G \mathrel{|\!\!\rhd} 0$. If Right makes a move with negative incentive from G then Right has moved to some G^R where $G^R > G$. That is, $G^R > G \mathrel{|\!\!\rhd} 0$; thus $G^R \mathrel{|\!\!\rhd} 0$ and Left wins moving first from G^R. □

Theorem 1.8.4 gives a method for deriving an expression for a position, G, when an incentive is known for each subpositions of G.

Theorem 1.8.4. *If G is a position and $(G^R, G^{RR}, \ldots, G^{R^n} = 0)$ is a sequence of n Right options from G to a zero position, then $G = \sum_{i=1}^n \Delta^{R^i}$, where Δ^{R^i} is the incentive of the move to G^{R^i}.*

Proof. If the sequence is (0) then $G - G^R = G - 0 = G$. Otherwise, by induction, $\sum_{i=1}^n \Delta^{R^i} = \Delta^R + \sum_{i=2}^n \Delta^{R^i} = (G - G^R) + G^R = G$. □

The strength of Theorem 1.8.4 is that we only need to know *one* Right incentive from every position to derive an expression for the position. A similar theorem also holds for Left options.

Theorem 1.8.5. *If $G_0, G_1, G_2, \ldots, G_k$ is a run of G, then*

$$G = G_k + \sum_{i=0}^{k} \Delta_i$$

where

$$\Delta_i = \begin{cases} -\Delta^L_{G_i} & \textit{if } G_i \xrightarrow{L} G_{i+1} \\ \Delta^R_{G_i} & \textit{if } G_i \xrightarrow{R} G_{i+1}. \end{cases}$$

Proof. This is a telescoping sum: $G = G_0$ and $\Delta_i = G_i - G_{i+1}$. □

While the proof of Theorem 1.8.5 is not itself interesting, the point is that when we consider a ruleset, if we understand the incentives, then to a great degree we understand the positions.

1.9 Numbers

This Section is about short numbers; for more about surreal numbers in general see [14].

Definition 1.9.1. *A position G is* cold *if every Left option is less than every Right option.*

Definition 1.9.2. *A* surreal number *is a hereditarily cold position.*

Definition 1.9.2 is formal. We say a position *is* a *number* if it is equal to a surreal number. Our surreal number is the same as *number* in [32, Section II.3] but we avoid the cumbersome "is equal to a number" phrasing in the sequel. When we do care about form the "surreal" alerts the reader to pay attention.

Fact 1.9.3. *[32, Proposition II.3.2, p. 69] Every incentive of a surreal number is negative.*

Fact 1.9.4. *[1, Theorem 6.19, p. 126] If every incentive of a position is negative then that position is a number.*

The dyadic rationals are the rationals whose denominators are 2^k for some $k \geq 0$ (2-*powers*).

$$
\begin{aligned}
0 &\cong \{\cdot \mid \cdot\} \\
1 &\cong \{0 \mid \cdot\} \\
-1 &\cong \{\cdot \mid 0\} \\
n &\cong \{n-1 \mid \cdot\} \\
-n &\cong \{\cdot \mid -(n-1)\} \\
\frac{m}{2^j} &\cong \left\{\frac{m-1}{2^j} \,\middle|\, \frac{m+1}{2^j}\right\} \text{ for odd } m \text{ and } j > 0 \text{ } (\tfrac{m-1}{2^j}, \tfrac{m+1}{2^j} \text{ in lowest terms})
\end{aligned}
$$

Figure 1.4: The dyadic rationals

Fact 1.9.5. *[32, p. 69] The (equivalence classes of) numbers form a subgroup of* $\mathbb{G}$ *that is isomorphic to the* dyadic rationals*.*

The values of (surreal) numbers, shown in Figure 1.4, are thus called *dyadic rationals*. The dyadic rational positions correspond to rationals in lowest terms, which is how the options of $\frac{m}{2^j}$ are to be interpreted. The subgroup of $\mathbb{G}$ of dyadic rationals is denoted $\mathbb{D}$. We say a position *is* an *integer* if it is a dyadic integer (denominator 2^0).

Fact 1.9.6. *[32, Theorem 3.6, p. 70] If m is odd and $j > 0$ then*

$$\frac{m}{2^j} = \left\{\frac{m-1}{2^j} \,\middle|\, \frac{m+1}{2^j}\right\}$$

is in canonical form.

Fact 1.9.7. *The canonical form of a number is a surreal number.*

Proof. By [32, Corollary 3.11, p. 72] a number is a dyadic rational.

The dyadic rationals are all surreal numbers: dyadic integers are vacuously cold and non-integers are cold by Fact 1.9.5; they are surreal by induction.

Dyadic integers are clearly in canonical form and dyadic non-integers are in canonical form by Fact 1.9.6.

A number that is in canonical form is a surreal number. □

Example 1.9.8. *The following positions all have value $\frac{3}{2}$:*

- $\{1 \mid 2\}$ *is a surreal number in canonical form*

- $\{0, 1 \mid 2\}$ *is a surreal number*
- $\left\{\frac{3}{2} + * \,\middle|\, \frac{3}{2} + *\right\}$ *is a number.*

Fact 1.9.9 (The Archimedean Principle). *[32, Theorem 3.7, p. 71] For any short position G, there is an integer n such that $n > G > -n$. In particular, any $n > \mathrm{b}(G)$ works.*

Fact 1.9.10. *[32] If G is a Left end, then G is an integer.*

Proof. Suppose that G is not an integer, then by [32, Theorem 3.27, p. 80] G has a Left incentive which contradicts the assumption that G is a Left end. □

Definition 1.9.11. *For positions G and an integer n. Let*

$$n \cdot G = \begin{cases} 0 & \text{if } n = 0; \\ G + (n-1) \cdot G & \text{if } n > 0; \\ -(-n \cdot G) & \text{if } n < 0. \end{cases}$$

The generalized Norton Product is defined using incentives and depends only on value.

Definition 1.9.12. *[32, p. 150] Let U be a positive position. The* generalized Norton product *of G by U, is*

$$G \cdot U = \begin{cases} n \cdot U & \text{if } G \text{ is the integer } n \\ \left\{G^{\mathbf{L}} \cdot U + U + \Delta \,\middle|\, G^{\mathbf{R}} \cdot U - U - \Delta\right\} & \text{otherwise} \end{cases}$$

where Δ ranges over all incentives of U.

Fact 1.9.13 (Number Avoidance Theorem [32, p. 72]). *Suppose that x is a number and G is not a number. If Left (resp. Right) has a winning move on $G + x$, then she has a winning move of the form $G^L + x$ (resp. $G^R + x$).*

Fact 1.9.14 (Number Translation Theorem [32, Theorem 3.21, p. 78]). *If x is a number and G is not a number, then*

$$G + x = \left\{G^{\mathbf{L}} + x \,\middle|\, G^{\mathbf{R}} + x\right\}.$$

1.10 Nimbers

Definition 1.10.1. *A position is* impartial *if Left and Right have exactly the same options for every subposition.*

The definition of impartial is formal and hereditary. The values of impartial games are *nimbers.*

Definition 1.10.2. *The* nimbers *are the positions*

$$*^n = \left\{ *^0, *^1, *^2, \ldots, *^{(n-1)} \;\middle|\; *^0, *^1, *^2, \ldots, *^{(n-1)} \right\}.$$

Where $*^0 = 0$ *and* $*^1 = *$.

We use the notation $*^n$ (instead of $*n$, for example) for enhanced readability, especially in ordinal sums. However, we also note the suitability of this notation in the context of Fact 2.2.1.

Fact 1.10.3 (Sprague-Grundy Theorem)**.** *[32, Theorem 1.3, p. 180] The canonical form of an impartial position is a nimber.*

Definition 1.10.4. *A position is* symmetric *if for every* G^L *there is a* G^R *such that* $G^L = -G^R$.

Lemma 1.10.5. *If* G *is symmetric,* $G + G = 0$.

Proof. For any Left move, Rights responds symmetrically: $G + G \xrightarrow{L} G^L + G \xrightarrow{R} G^L + (-G^L) = 0$. □

Lemma 1.10.6. *If* $G + G = 0$ *and* G *is in canonical form then* G *is symmetric.*

Proof. Suppose that $G + G = 0$ and G is in canonical form; an option of $G + G$ is in one G summand; a winning response must be in the other component, otherwise the first option was reversible, contradicting the assumption that G is in canonical form.

Suppose there exists a Left option G^{L_1} such that no $G^R = -G^{L_1}$.

Considering $G + G \xrightarrow{L} G^{L_1} + G$ there is a G^R such that $G^{L_1} + G^R \leq 0$. By supposition, $G^R \neq -G^{L_1}$ thus $G^{L_1} + G^R < 0$

Considering $G + G \xrightarrow{R} G + G^R$ there is a Left option G^{L_2} such that $G^{L_2} + G^R \geq 0$. Thus $G^{L_2} > G^{L_1}$, contradicting the claim that G is in canonical form. □

Fact 1.10.7. *[1, Exercise 7.1, p. 135] If G is impartial, $G + G = 0$.*

Proof. This holds for $G = 0$. If Left moves to $G^L = G'$, Rights responds to $G^R = G'$: $G+G \xrightarrow{L} G'+G \xrightarrow{R} G'+G'$ which is 0 by induction as impartiality is hereditary. $\square$

Corollary 1.10.8. *A position is impartial if and only if it is hereditarily symmetric.*

Proof. By Fact 1.10.7, an impartial position is symmetric. and impartiality is hereditary, so an impartial position is hereditarily symmetric.

Suppose G is hereditarily symmetric; every option of G is impartial by induction. For any G^L there exists a G^R such that $G^R = -G^L$ but as G^R and G^L are both impartial, $G^R = G^L$ and thus G is impartial. $\square$

Again, we clarify the issues relating to form and equality similar to those that arose with numbers. We say G *is* a nimber then G is equal to a nimber.

Example 1.10.9. *The following positions all have value $*^2$ and thus each is a nimber:*

- $\{0, * \mid 0, *\}$ *is in canonical form*
- $\{0, *, *^4 \mid 0, *, *^4\}$ *is impartial but is not in canonical form*
- $\{0, *, *^4 \mid 0, *, *^5\}$ *has only impartial options but it is not impartial*
- $\{0, \{0, * \mid 0, *^2\} \mid 0, \{0, *^2 \mid 0, *\}\}$ *is symmetric but it is not hereditarily symmetric*
- $\{0, \{1 \mid \{0, * \mid -1\}\} \mid 0, *\}$ *is not symmetric and has a subposition that is a number.*

Fact 1.10.10. *If G is impartial and H is impartial then $G + H$ is impartial.*

Proof. For both players, options of $G + H$ are of the form $G' + H$ or $G + H'$ where G' is an option of G or H' is an option of H and because G and H are impartial they have the same literal options. As impartiality is hereditary, $G + H$ is impartial by induction. $\square$

Corollary 1.10.11. *If G is a nimber and H is a nimber then $G + H$ is a nimber.*

Fact 1.10.12. *The (equivalence classes of) nimbers form a subgroup of $\mathbb{G}$.*

Proof. This follows from Facts 1.10.7 and 1.10.10. □

We present the definition of *nimsum*, denoted by $\oplus$, independently of NIM. For more context see [1, Chapter 7, p. 135] and [32, p. 2].

Definition 1.10.13. *The* 2*-adic valuation of a positive integer* n*, denoted* $\nu(n)$*, is the exponent of the largest* 2*-power dividing* n*.*

The exponent of the largest 2*-power less than or equal to a positive integer* n *is denoted by* $\mathrm{lb}(n) = \lfloor \log_2 n \rfloor$*.*

If $\nu(n) > \mathrm{lb}(m)$ then $n \oplus m = n + m$. If $\nu(n) = \mathrm{lb}(n)$ then n is a 2-power.

Definition 1.10.14. *The commutative binary operation* $\oplus$ *is defined as follows, where* $a' = a - 2^{\mathrm{lb}(a)}$ *and* $b' = b - 2^{\mathrm{lb}(b)}$*. If* $a \geq b$ *then*

- $a \oplus b = a$ *if* $b = 0$
- $a \oplus b = a' \oplus b'$ *if* $\mathrm{lb}(a) = \mathrm{lb}(b)$
- $a \oplus b = 2^{\mathrm{lb}(a)} + a' \oplus b'$ *if* $\mathrm{lb}(a) > \mathrm{lb}(b)$*.*

For an axiomatic definition of $\oplus$ see [1, Chapter 7, p. 135] or [32, p. 181].

We will see more nimbers in Chapter 4 where we prove Fact 4.5.9.

1.11 Dicots

Of particular interest in this thesis are *dicotic* positions.

Definition 1.11.1. *A position is* dicotic *if every subposition is a Left end if and only if it is a Right end.*

The dicotic property is formal and hereditary. A ruleset is called dicotic if every position of that ruleset is dicotic. Examples of dicotic rulesets include CLOBBER, SPRIGS, and NIM.

Proposition 1.11.2. *Impartial positions are dicotic.*

Example 1.11.3. *The following positions are all equal to the dicotic position* 0*:*

$\{\cdot \mid \cdot\}$ *is dicotic,*

$\{* \mid *\}$ *is dicotic,*

$\{-1 \mid 1\}$ *is not dicotic.*

We say a position is a *dicot* if it is equal to a dicotic position.

This is similar to issues relating to form and equality with surreal numbers (dyadic rationals) and impartial positions (nimbers) where we use *is* in the sense of equality.

This terminology is less satisfactory than in the cases of surreal numbers and impartial positions because unlike dyadic rationals and nimbers, we do not have unique names for values of dicots.

Proposition 1.11.4. *Let G and H be positions; $G + H$ is dicotic if and only if G and H are dicotic.*

Proof. Every subposition of G or H is a subposition of $G + H$ so if $G + H$ is dicotic so are G and H. If G and H are dicotic and $G + H$ is not an end then $G + H$ is dicotic by induction. Otherwise, without loss of generality suppose $G + H$ is a Left end, then both G and H are Left ends and because they are dicotic must both be Right ends; that is, $G + H$ is a Right end. $\square$

By Proposition 1.11.4, dicotic positions are closed under disjunctive sum. The dicotic values form a subgroup of $\mathbb{G}$.

Lemma 1.11.5. *Every non-empty dicotic short position involves $*$.*

Proof. The only dicotic position with formal birthday 0 is the empty position $\{\cdot \mid \cdot\}$. The only dicotic position with formal birthday 1 is $*$. Subsequent non-empty positions are formed by two non-empty sets of positions. If both are $\{0\}$ then the position is $*$ and thus involves $*$. Otherwise, one set contains a position that involves $*$ by induction. $\square$

The game trees of dicotic positions have many zero positions.

Theorem 1.11.6. *For any dicotic position G, strictly greater than half of the subpositions (vertices) of G are 0.*

Proof. We argue that there are more zero subpositions than non-zero subpositions. If $G \cong 0$ then there is 1 more zero subposition than there are zero subpositions. Now suppose $G \not\cong 0$; G is one subposition that is non-zero. As G is dicotic it has at least 2 options (at least one for both Left and Right) and by induction each option has more zero subpositions than non-zero subpositions so collectively there are more zero subpositions of G than non-zero subpositions of G. That is, more than half of the subpositions of G are 0. $\square$

Lemma 1.11.7. *For $n > 0$, $\frac{2}{3}$ of the subpositions (vertices) of $*^n$ are 0.*

Proof. There are 3 subpositions of $*^1$: one is $*^1$ and two are 0. For $n > 1$, $*^n$ is a subposition, two options are 0 and the other options are all nimbers for which exactly $\frac{2}{3}$ of the subpositions are 0 by induction. $\square$

1.12 Stops

The *Left stop* and *Right stop*, respectively given in Definition 1.12.1, give the *value* of the first number reached in alternating play on G starting with a given player.

Definition 1.12.1. *For a position G,*

$$\mathrm{LS}(G) = \begin{cases} x & \textit{if } G = x \textit{ and } x \textit{ is a number in canonical form} \\ \max_{G^L}(\mathrm{RS}(G^L)) & \textit{otherwise} \end{cases}$$

$$\mathrm{RS}(G) = \begin{cases} x & \textit{if } G = x \textit{ and } x \textit{ is a number in canonical form} \\ \max_{G^R}(\mathrm{LS}(G^R)) & \textit{otherwise.} \end{cases}$$

If we defined a stop to be a literal subposition of G instead of a value, then stops of non-canonical positions would not be well-defined, as in some cases there are equal options either of which could be the stop. The definition given is non-formal.

The following Facts are to remind the familiar reader of some relevant properties of stops. For proofs, see [32, Propositions 3.17–3.19, pp. 75–77]

Fact 1.12.2. *Let G and H be positions:*

$\mathrm{LS}(-G) = -\mathrm{RS}(G)$,

$\mathrm{LS}(G) + \mathrm{LS}(H) \geq \mathrm{LS}(G+H) \geq \mathrm{RS}(G+H) \geq \mathrm{RS}(G) + \mathrm{RS}(H)$, *and*

if $G \geq H$, then $\mathrm{LS}(G) \geq \mathrm{LS}(H)$ and $\mathrm{RS}(G) \geq \mathrm{RS}(H)$.

Fact 1.12.3. *Let G be a position and x be a number:*

If $\mathrm{LS}(G) < x$, *then* $G < x$.	*If* $\mathrm{RS}(G) > x$, *then* $G > x$.
If $\mathrm{LS}(G) > x$, *then* $G \,|\!\!\triangleright\, x$.	*If* $\mathrm{RS}(G) < x$, *then* $G \,\triangleleft\!\!|\, x$.
If $G \leq x$, *then* $\mathrm{LS}(G) \leq x$.	*If* $G \geq x$, *then* $\mathrm{RS}(G) \geq x$.
$\mathrm{LS}(G + x) = \mathrm{LS}(G) + x$.	$\mathrm{RS}(G + x) = \mathrm{RS}(G) + x$.

1.13 Infinitesimals

Definition 1.13.1. *The position G is* infinitesimal *if for all positive numbers x, $-x < G < x$.*

Some common infinitesimal positions are *tinies* (⧾) and *minies* (⧿).

Definition 1.13.2. *For $G \geq 0$,*

$⧾_G = \{0 \mid \{0 \mid -G\}\}$*;*

$⧿_G = \{\{G \mid 0\} \mid 0\}$*.*

Fact 1.13.3. *[32, Proposition 4.3, p. 84] A position G is infinitesimal if and only if* $\mathrm{LS}(G) = \mathrm{RS}(G) = 0$.

Theorem 1.13.4. *If a position is dicotic then it is (hereditarily) infinitesimal.*

Proof. By [32, Theorem 4.2, p. 83], if G is dicotic, then G is infinitesimal. As dicotic is a hereditary property, G is hereditarily infinitesimal. □

Proposition 1.13.5. *If G is dicotic, then the canonical form of G is dicotic.*

Proof. Let K be the canonical form of G. Suppose, by way of contradiction that a subposition of K, say L, is an end but is not empty. By Fact 1.9.10 it is an integer and because K is canonical is not 0 and thus by Fact 1.5.11 G was not hereditarily infinitesimal. □

Example 1.13.6. *A position is not necessarily dicotic even if it is hereditarily infinitesimal.*

For example, the position $G \cong \{* \mid \cdot\} = 0$ is infinitesimal and all options are infinitesimal, but G is not dicotic because G is a Right end but not a Left end.

The literature is confusing when it comes to the difference between hereditarily infinitesimal and dicotic. The first appearance of the term *all small* is [13, p. 101] and that same definition is unchanged in [14]. Conway's definition[1] is what we take as our definition for hereditarily infinitesimal. However, [6, p. 229] defines *all small* differently but it is essentially our dicotic (Definition 1.11.1). Example 1.13.6 shows that these two definitions of all small are not the same.

We can still show that the value of a hereditarily infinitesimal position is dicotic.

[1] “We call a game all small if all its positions are small games”

Fact 1.13.7. *If G is hereditarily infinitesimal, then G is a dicot.*

Proof. Let G be hereditarily infinitesimal. Suppose a subposition, say H, is an end. By Fact 1.9.10 every end is an integer. If H is non-zero, then G is not hereditarily infinitesimal, which is a contradiction; if H is zero, we may replace H by $\{\cdot \mid \cdot\}$ without changing the value. Upon replacing all zero ends, we will have a dicotic position equal to G. $\square$

In [1, Definition 9.1, p. 186] *all-small* is essentially our dicotic. Finally, in [32, p. 63], the term dicotic replaces *all-small*.

Dicotic positions are of interest in misère-play; see for example [4] and Chapter 5. However, the infinitesimal theory of normal-play does not transfer to misère-play so we are happy to adopt the new terminology.

The simplest positive dicot is $\uparrow = \{0 \mid *\}$. Despite $\uparrow$ being infinitesimal and having a small birthday, it is arguably a large infinitesimal.

Fact 1.13.8. *[32, Theorem 4.8, p. 85] If G is infinitesimal then $n \cdot \uparrow > G > n \cdot \downarrow$ for some n.*

In some sense many dicotic positions are large at least compared to tinies.

Lemma 1.13.9. *If H is dicotic and $H \in \mathcal{N}$ and x is a positive number, then*

$$H + \boldsymbol{+}_x \in \mathcal{N}.$$

Proof. Left wins by playing a winning move on H. Right wins playing first playing in the tiny, to which Left must respond locally to 0, allowing Right to win moving in H. $\square$

Fact 1.13.10 (Dicotic Translation Theorem [32, Exercise 4.17, p. 98])**.** *If* $\mathrm{LS}(G) > \mathrm{RS}(G)$ *and H is dicotic, then* $G + H = \left\{G^{\mathbf{L}} + H \;\middle|\; G^{\mathbf{R}} + H\right\}$.

We will see many more dicotic positions in the following Chapters, but a family we will see again soon is as follows.

Fact 1.13.11. *[1, Lemma 9.5, p. 187] For all $n \geq 1$, $\{0 \mid *^n\} = \uparrow + *^{n \oplus 1}$.*

1.14 Reduced Canonical Form

Whenever the stops of a sum are non-zero, infinitesimal summands are irrelevant to the outcome of the sum; the definitions in this section formalize this concept. A complete presentation of these concepts is [32, Chapter II.6]. This Section is a brief summary for the familiar reader.

To ignore infinitesimals, we declare G and H to be equivalent if their difference is infinitesimal.

Definition 1.14.1. *For positions G and H,*

$G =_I H$ *if* $G - H$ *is infinitesimal;*

$G \geq_I H$ *if* $G - H \geq -x$ *for every number* $x > 0$.

If $G =_I H$ we say G and H are *infinitesimally close* and also G is H-ish or H is G-ish. If G is x-ish for some number x, then we say G is *numberish.*

Proposition 1.14.2. *A position G is numberish if and only if* $\mathrm{LS}(G) = \mathrm{RS}(G)$.

Proof. Suppose $x = \mathrm{LS}(G) = \mathrm{RS}(G)$, then by Fact 1.12.3 $\mathrm{LS}(G-x) = \mathrm{LS}(G)-x = 0$ and $\mathrm{RS}(G-x) = \mathrm{RS}(G) - x = 0$. That is, $G - x$ is infinitesimal. Suppose that for some number x, $G - x$ is infinitesimal. That is, $\mathrm{LS}(G - x) = 0$ and $\mathrm{RS}(G - x) = 0$ so $\mathrm{LS}(G) = x = \mathrm{RS}(G)$. □

Fact 1.14.3. *[32, Lemma 6.4, p. 125] The following are equivalent:*

(i) $G \geq_I 0$.

(ii) $\mathrm{RS}(G) \geq 0$.

(iii) $G \geq \epsilon$ *for some infinitesimal* ϵ.

Corollary 1.14.4. *For any position H,* $\mathrm{LS}(H) \geq_I H \geq_I \mathrm{RS}(H)$.

Proof. As $\mathrm{LS}(H)$ is a number Fact 1.12.3 gives $\mathrm{RS}(\mathrm{LS}(H) - H) = \mathrm{LS}(H) + \mathrm{RS}(-H)$, which by Fact 1.12.2 is $\mathrm{LS}(H) - \mathrm{LS}(H) = 0$. By 1.14.3 where $G = \mathrm{LS}(H) - H$, $\mathrm{LS}(H) - H \geq_I 0$ and thus $\mathrm{LS}(H) \geq_I H$. The Right case is similar. □

Given a position, we can simplify it using a process similar to how we find the canonical form.

Definition 1.14.5. *[32, Definition II.6.6] For a position G:*

A Left option G^{L_1} is Inf-dominated (by G^{L_2}) if $G^{L_2} \geq_I G^{L_1}$ for some distinct G^{L_2}.

A Left option G^{L_1} is Inf-reversible (through $G^{L_1 R}$) if $G^{L_1 R} \leq_I G^{L_1}$ for some $G^{L_1 R}$. Definitions for Inf-dominated and Inf-reversible are similar for Right.

Definition 1.14.6. *A position G is in* reduced canonical form *if for every subposition K of G:*

K is a number in canonical form; or

$\mathrm{LS}(K) > \mathrm{RS}(K)$ *and no option of K is Inf-dominated or Inf-reversible.*

Removing Inf-dominated options and bypassing Inf-reversible options give a position that is infinitesimally close to the original. Removing Inf-dominated options is just that (they are removed as options) and we do not have to use Inf-reversibility directly.

Fact 1.14.7. *[32, p. 128] For a position G, there is a unique position $H =_I G$ in reduced canonical form.*

Definition 1.14.8. *The* reduced canonical form *of G, denoted* $\mathrm{rcf}(G)$*, is the unique simplest position infinitesimally close to G.*

As the reduced canonical form of a position is also a value we may refer to $\mathrm{rcf}(G)$ as the *reduced value* of G. For a position G we call $G - \mathrm{rcf}(G)$ the *infinitesimal part* of G.

Chapter 2

Ordinal sum

2.1 Introduction

The ordinal sum is a way of combining two games.

Definition 2.1.1. *The* ordinal sum *of positions G and H, denoted $G:H$ (and pronounced G* colon *H) is defined recursively as*

$$G:H \cong \left\{G^{\mathbf{L}}, G:H^{\mathbf{L}} \,\middle|\, G^{\mathbf{R}}, G:H^{\mathbf{R}}\right\}.$$

The position G is called the base *of $G:H$ and the position H is called the* subordinate *of $G:H$.*

The intuitive understanding of ordinal sum in play is that the subordinate is eliminated when a player plays in the base. Typically, the ordinal sum $G:H$ differs from the base, G, by an infinitesimal; if not (usually when G and H are numbers) the difference is likely small compared to G.

Ordinal sums arise naturally in HACKENBUSH [6], TOPPLING DOMINOES [16], LENRES [34], and more.

Ordinal sums and this Chapter are also very important for later chapters of this thesis.

More discussion of the definition of ordinal sum follows in Section 2.1.1. Section 2.2 is largely an introduction to the existing theory of ordinal sums (some work has been done to unify the presentation of remoteness used for both $\bigstar$ and Norton's Lemma).

We discuss domination, reversibility and canonical form in Section 2.3. This is the work that makes the extended definition of ordinal usable and valuable.

In Section 2.4 we see that ordinal sums of dicotic positions are now dicotic (Lemma 2.4.1). We then discover a class of positions, called star-like, that are readily described using ordinal sum. This further allows us to describe the atomic weights of star-like positions.

We generalized the techniques we used to identify star-like positions as ordinal sum in Section 2.5.

2.1.1 Form and Outcomes

Our definition of ordinal sum depends on the form of the base. In [14, p. 193] ordinal sum is defined using the canonical form of the base instead of the literal base. However, [14, p. 193] contains ordinal sums with non-canonical bases, which speaks volumes about their usefulness. Unfortunately some results such as [14, Theorem 94, p. 211] do not hold under our definition; our patched version of that result is Lemma 2.3.5. We also discuss this further in Section 2.3.1.

Example 2.1.2. *By Definition 2.1.1,*

$$\left\{*^2 \;\middle|\; *^2\right\} : *^2 \cong \left\{*^2, \left\{*^2 \;\middle|\; *^2\right\}, \left\{*^2 \;\middle|\; *^2\right\} : *^1 \;\middle|\; *^2, \left\{*^2 \;\middle|\; *^2\right\}, \left\{*^2 \;\middle|\; *^2\right\} : *^1\right\} = *^3.$$

Whereas in [14],

$$\left\{*^2 \;\middle|\; *^2\right\} : *^2 = 0 : *^2 = *^2.$$

Proposition 2.1.3. $\tilde{\mathrm{b}}(G : H) = \tilde{\mathrm{b}}(G) + \tilde{\mathrm{b}}(H)$

Proof. A longest run of $G : H$ is a longest run of H followed by a longest run of G. □

The idea of Lemma 2.1.4 is not new; see [34, Lemma 2.2.13, p. 70] and [32, Proposition 4.17, p. 90] and the proof is not interesting (essentially this proof is the same as the one in [34]) but what is noteworthy is that this result is literal and not just a result about equality.

Lemma 2.1.4. *If G and H are positions, $-(G : H) \cong (-G) : (-H)$.*

Proof. This follows from Definition 1.2.5 and induction:

$$\begin{aligned}
-(G:H) &\cong -G:H \\
&\cong -\left\{G^{\mathbf{L}}, G:H^{\mathbf{L}} \,\middle|\, G^{\mathbf{R}}, G:H^{\mathbf{R}}\right\} \\
&\cong \left\{-G^{\mathbf{R}}, -(G:H^{\mathbf{R}}) \,\middle|\, -G^{\mathbf{L}}, -(G:H^{\mathbf{L}})\right\} \\
&\cong \left\{-G^{\mathbf{R}}, (-G):(-H^{\mathbf{R}})) \,\middle|\, -G^{\mathbf{L}}, (-G):(-H^{\mathbf{L}})\right\} \\
&\cong \left\{(-G)^{\mathbf{L}}, (-G):(-H)^{\mathbf{L}}) \,\middle|\, (-G)^{\mathbf{R}}, (-G):(-H)^{\mathbf{R}}\right\} \\
&\cong (-G):(-H).
\end{aligned}$$

□

Theorem 2.1.5. *The outcome of $G:H$, $o(G:H)$, is given by Table 2.1.*

Proof. If a player wins going first in G then they can win going first in $G:H$. In such a case the opponent loses if they play in H because the first player still has the base option available. That is, $o(G:H) = o(G)$ if $o(G) \neq \mathscr{P}$.

If $o(G) = \mathscr{P}$, then $o(G:H) = o(H)$ as whomever can play last in the subordinate forces the opponent to make a losing move in the base. □

$o(G)$ \ $o(H)$	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{L}$	$\mathscr{R}$
$\mathscr{P}$	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{L}$	$\mathscr{R}$
$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$
$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$
$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$

Table 2.1: Outcome of the ordinal sum of G and H

In Table 2.1, the base and subordinate are not necessarily in canonical form.

2.2 Ordinal Sum Basics

Ordinal sums of nimbers are useful for an example in Section 2.2.1 and needed in Section 2.2.2 so we present them here.

It is easy to see that the ordinal sum of impartial positions is impartial. Thus the ordinal sum of a nimber with a nimber is a nimber; moreover it is literally a nimber.

Fact 2.2.1. *For all positive integers n and m, $*^n : *^m \cong *^{(n+m)}$.*

Proof. By induction, as clearly $* : *^n \cong *^{n+1}$ by Definition 1.10.2. □

2.2.1 Norton's Lemma

Determining the outcome of a lone ordinal sum is straightforward. Norton's Lemma will tell us something about the outcome of a disjunctive sum where (at least) one summand is an ordinal sum.

Definition 2.2.2. *We say G is* remote for *K if no subposition of K is equal to G.*

We emphasize that whether G is remote for K depends on the form of K but not on the form of G.

Example 2.2.3. *The nimber $*^2$ is remote for $\uparrow \cong \{0 \mid *\}$.*

Lemma 2.2.4. *If G is in canonical form, then G is remote for any proper subposition of G.*

Proof. If G is in canonical form then it is the simplest position with its value. If a proper subposition of G, say H, exists such that $G = H$ then H is simpler than G, contradicting the assumption that G is in canonical form. □

If G is not remote for K, then we say K *involves* G. That is, some subposition of K is equal to G. Furthermore, K involves all of its subpositions and all positions equal to its subpositions.

Both *remote for* and *involves* are formal as they depend on the form of K (but not G).

Example 2.2.5. *The position $*$ is remote for 0 but not for $\{*^2 \mid *^2\}$ (despite being equal to 0). The position $\{*^2 \mid *^2\}$ involves $*$.*

Fact 1.5.11 tells us that if the canonical form of G involves H, then so does G.

Lemma 2.2.6. *If G' is a subposition of G and G' is remote for K, then G is remote for K.*

If G is remote for K, then $G : H$ is remote for K.

Proof. If K involves G then K involves G'. If K involves $G{:}H$ then K involves G. □

Norton's Lemma. *[32, p. 91] Fix positions G, H, and X. If G is remote for X then*

$$o(G - X) = o(G : H \; - \; X).$$

Example 2.2.7. *For all H, $o(*^2 : H + \uparrow) = o(*^2 + \uparrow) = \mathscr{L}$.*

We reference Norton's Lemma often. In some contexts Norton's Lemma is tremendously powerful. In particular the Atomic Weight theory (Section 2.2.2) uses Norton's Lemma.

2.2.2 Atomic Weight

We now discuss *atomic weight* which is not closely associated with ordinal sums but does depend on Section 2.2.1. The theory of atomic weights is very important in the study of dicotic games. See [1, 6, 14, 32] for more intuition and a more complete presentation of atomic weights.

For every position G there is a nimber that is remote for G; in particular $*^{\tilde{\mathrm{b}}(G)+1}$ is remote for G. Furthermore, there are infinitely many remote nimbers for any position.

Corollary 2.2.8. *If $*^n$ is remote for G then $*^{(n+1)}$ is remote for G.*

Proof. This follows immediately from Lemma 2.2.6. $\square$

We use $\bigstar$ in a sum to represent a nimber that is remote for the given sum.

Example 2.2.9. *In the sum $\uparrow + *^2 + \bigstar$, $\bigstar$ represents $*^n$ for some $n \geq 4$. Despite $*^3$ being remote for $\uparrow$ and also for $*^2$ it is not remote for $\uparrow + *^2$ because $* + *^2 = *^3$ is a subposition (in particular a Right option) of $\uparrow + *^2$.*

Fact 2.2.10. *If $*^n$ and $*^m$ are remote for G then $o(G + *^n) = o(G + *^m)$.*

Proof. Without loss of generality, assume $n \geq m$. By Fact 2.2.1, $*^n \cong *^m : *^{n-m}$ and the result follows from Norton's Lemma. $\square$

Corollary 2.2.11. *For all n and positions G, $o(G + *^n + \bigstar) = o(G + \bigstar)$.*

Proof. There exists a sufficiently large k, such that $*^{2^k}$ is remote for $G + *^n$ and thus $*^{2^k} + *^n = *^{2^k+n}$ which is also remote and the result follows from Fact 2.2.10. $\square$

That is, in the presence of a remote nimber (also known as *far-star*), all nimber summands are equivalent (including 0). This equivalence is a particular case of a grander equivalence relation on $\mathbb{G}$.

Definition 2.2.12. *For all positions G and H,*

$$G \sim H \text{ if } o(G + X + \bigstar) = o(H + X + \bigstar) \text{ for all } X;$$
$$G \gtrsim H \text{ if } o(G + X + \bigstar) \geq o(H + X + \bigstar) \text{ for all } X.$$

Fact 2.2.13. *[32, p. 139] The relation $\sim$ is an equivalence relation.*

In a way similar to Fact 1.13.11, we use $\uparrow\bigstar$ to represent $\{0 \mid \bigstar\}$, where $\bigstar$ is remote for the other summand(s).

Fact 2.2.14. *[32, p. 140] For positions G and H, $G \sim H$ if and only if*

$$\uparrow\bigstar > G - H > \downarrow\bigstar.$$

Recall Definition 1.9.12.

Definition 2.2.15. *If $G \sim \uparrow \cdot H$ for some H we say G is* atomic *and write*

$$\mathrm{aw}(G) = H.$$

H is called the atomic weight *of G.*

Fact 2.2.16. *If G is atomic and $G = H$ then* $\mathrm{aw}(G) = \mathrm{aw}(H)$.

Proof. If $G = H$ (Definition 1.4.2) then $G \sim H$ (Definition 2.2.12). Thus $\uparrow \cdot \mathrm{aw}(G) \sim G$ and $G \sim H$ gives $\uparrow \cdot \mathrm{aw}(G) \sim H$ so H is atomic and $\mathrm{aw}(H) = \mathrm{aw}(G)$. $\square$

The atomic weight theory allows us to approximate positions on a *scale of* $\uparrow$*s*, which is reasonable to attempt considering Fact 1.13.8. As $\uparrow$ is the unit for our scale, $\mathrm{aw}(\uparrow) = 1$. The condensed version of the atomic weight theory, which is almost all the theory needed for integer atomic weights is given in Fact 2.2.17.

Fact 2.2.17. *[32, pp. 143–144]*

- *If G and H are atomic, then $G+H$ is atomic and* $\mathrm{aw}(G+H) = \mathrm{aw}(G)+\mathrm{aw}(H)$.
- *If* $\mathrm{aw}(G) \geq 2$ *then $G > 0$.*
- *If G is atomic,* $\mathrm{aw}(G) \geq 1$ *if and only if $G > \bigstar$.*

2.2.3 The Atomic Weight Calculus

Definition 2.2.18 essentially shows how to compute atomic weight for atomic positions by showing how to find $\upsilon(G)$ for hereditarily atomic positions.

Definition 2.2.18. *[32, p. 44, Definition 7.14]*

For a dicotic position G, $\upsilon(G)$ and $\tilde{\upsilon}(G)$ are defined recursively:

$$\tilde{\upsilon}(G) = \left\{ \upsilon(G^L) - 2 \;\middle|\; \upsilon(G^R) + 2 \right\}$$

If $\tilde{\upsilon}(G)$ is not an integer $\upsilon(G) = \tilde{\upsilon}$. Otherwise, $\upsilon(G)$ is an integer, namely:

$$\upsilon(G) = \begin{cases} \textit{the smallest element of } \mathcal{I} & \textit{if } G < \bigstar \\ \textit{the largest element of } \mathcal{I} & \textit{if } G > \bigstar \\ 0 & \textit{if } G \parallel \bigstar. \end{cases}$$

where

$$\mathcal{I} = \{n;\; \upsilon(G^L) - 2 \lhd\!\!| \; n \lhd\!\!| \; \upsilon(G^R) + 2 \textit{ for } G^L \in G^{\mathbf{L}} \textit{ and } G^R \in G^{\mathbf{R}}\}.$$

Fact 2.2.19. *[32, p. 44, Theorem 7.15] If G is dicotic then G is atomic and $\upsilon(G) =$ $\mathrm{aw}(G)$.*

Fact 2.2.19 shows that a dicotic position, say G, is atomic and thus also hereditarily atomic. In this thesis the only atomic weights we need are those of dicotic positions. However, by Fact 2.2.16, a dicot is also atomic.

Fact 2.2.20. *[6, p. 251] The value of every* HACKENBUSH *position with only green ground edges is dicotic with integer atomic weight.*

Proof. Let G be a simplest counter-example. Suppose G^L is obtained when Left hacks the edge α and G^L is the Left option of largest atomic weight, say a; and G^R is obtained when Right hacks the edge δ and G^R is the Right option of least atomic weight d.

Hacking both edges α and δ gives a position, say H. By induction H is atomic and $h = \mathrm{aw}(H)$ is an integer. Either $h = a$ or h is a Right option of a, thus

$$a - 2 \lhd\!\!| \; h, \text{ and similarly } h \lhd\!\!| \; d + 2$$

showing that $\upsilon(G)$ is an integer thus the atomic weight of G is an integer but it is not necessarily h. □

2.2.4 The Colon Principle

Whereas in Norton's Lemma the form of the base matters, The Colon Principle shows that for a subordinate only its value matters.

The Colon Principle. *[6, p. 219] If G and H are positions*

$$H \geq K \implies G : H \geq G : K.$$

In particular, $H = K \implies G : H = G : K$.

The Colon Principle comes from Winning Ways [6] where it is presented first in a HACKENBUSH setting and applies more generally than in just the case of ordinal sum.

A more general corollary of The Colon Principle is Corollary 2.2.21, which is similar to [34, Corollary 2.2.12, p. 70].

Corollary 2.2.21. *Where ρ is any of the relations from Figure 1.2,*

$$G : H \; \rho \; G : K \iff (H - K) \; \rho \; 0.$$

Proof. The Colon Principle can be extended to be if and only if (see [34, Theorem 2.2.10, p. 69] and [32, Proposition 4.17, p. 90]); that is $G : H \; \rho \; G : K$ is equivalent to $(H - K) \; \rho \; 0$ where ρ is $\geq$. By the same argument this works if ρ is $\leq$. Together these imply equivalence when ρ is $=$, which then implies equivalence when ρ is $>$ or $<$. Now that we have equivalence in three of the four cases (corresponding to outcome of $H - K$), equivalence holds if ρ is $\parallel$ too. Thus equivalence holds for any relation. □

If $G : H = G : J$ then by Corollary 2.2.21 $H = J$.

Definition 2.2.22. *If $K = G : H$ for some H, then K is called* G-based *and the* G-based-subordinate of *K is the value of H.*

If $K \cong G : H$ for some H, then K is called literally G-based *and H is the* literal G-based-subordinate of *K.*

2.3 Domination and Reversibility in Ordinal Sums

We can have an ordinal sum $G : H$ that has a dominated option despite G and H having no dominated options.

Example 2.3.1. *In the ordinal sum* $1 : 1 \cong \{0, 1 \mid \cdot\} = \{1 \mid \cdot\}$ *the Left option to* 0 *in the base is dominated, but* 1 *(the base and subordinate) has no dominated option.*

Proposition 2.3.2. *If* $G : H$ *has no dominated option then neither* G *nor* H *has a dominated option.*

Proof. If an option of G, say G_1, is dominated by the option G_2, then G_1 is again dominated by G_2 as an option of $G : H$. If an option of H, say H_1, is dominated by H_2, then $G : H_1$ is dominated by $G : H_2$ by Corollary 2.2.21. □

Corollary 2.3.3. *If* G *is a position and* K *is derived from* G *by removing dominated options then* $G : H = K : H$*.*

Now we consider reversibility; [14, Theorem 94, p. 211] states:

> If neither X nor Y has a reversible move, then neither does $X : Y$.

Example 2.3.4. *Note that* $\{\{0 \mid 3\} \mid \cdot\} = \{1 \mid \cdot\} = 2$ *and the only option is not reversible. Now consider* $\{\{0 \mid 3\} \mid \cdot\} : 1$*; where the Left option to* $\{0 \mid 3\}$ *is reversible.*

This example shows that the statement of [14, Theorem 94, p. 211] does not hold given our definitions. The proof relies on X^{LR} not involving X; in the example X is $\{\{0 \mid 3\} \mid \cdot\}$ and X^{LR} is 3, which does involve 2. If X would also be required to have its options in canonical form then the given example would not be a "counter-example".

Lemma 2.3.5. *[14, Theorem 94, p. 211] The ordinal sum* $G : H$ *has no reversible option if* G *is in canonical form and* H *has no reversible option.*

Proof. We give the argument for Left, the case for Right is similar.

By Corollary 2.2.21, $G : H^{LR} \leq G : H$ if and only if $H^{LR} \leq H$. That is, if some $G : H^L$ is reversible then some H^L is reversible.

It is sufficient to show $G^{LR} \leq G$ if and only if $G^{LR} \leq G : H$. This holds by Norton's Lemma as G is remote for G^{LR} by Lemma 2.2.4. □

Example 2.3.4 showed that a base without reversible options could produce an ordinal sum with a reversible option. Example 2.3.6 shows that a reversible Left option in a base does not guarantee a reversible Left option of the ordinal sum.

Example 2.3.6. *Consider* $G \cong \{\{1 \mid *\} \mid 0\}$, G^L *reverses through* $*$ *to* 0*; that is* $G = *$. *However,* $G : -1 \cong \{\{1 \mid *\} \mid 0, \{\{1 \mid *\} \mid 0\}\} = \{\{1 \mid *\} \mid 0, *\}$.

Because G *appears as an option,* $G : -1$ *is not in canonical form.*

Theorem 2.3.7. *If* $G : H$ *is in canonical form then* G *and* H *are in canonical form.*

Proof. If $G : H$ is in canonical form then G is in canonical form because G is a subposition of $G : H$.

If H is not in canonical form, let K be the canonical form of H. Then by Corollary 2.2.21, $G : K = G : H$ and $G : K$ is simpler than $G : H$ which contradicts $G : H$ being in canonical form. $\square$

The existence of a reversible option in a base may benefit a player especially in a case where the subordinate is good for the opponent of the player with the reversible option.

Example 2.3.8. *The following are three examples of ordinal sums with bases that are numbers that are not in canonical form.*

- $\{* \mid *\} = 0$ *but* $\{* \mid *\} : 1 = \{*, 0 \mid *\} = \uparrow$, *which is not a number and definitely not* $0 : 1 = 1$.
- $\{\{1 \mid 0\} \mid *\} = 0$ *but* $\{\{1 \mid 0\} \mid *\} : -1 = \boldsymbol{-}_1$*; the base is zero but the ordinal sum is better for Left than* $0 : -1$, *although both positions are negative.*
- *The base can also be non-zero:* $\{0 \mid \{1 \mid 2\}\} = 1$ *and* $1 : 1 = 2$ *but* $\{0 \mid \{1 \mid 2\}\} : 1 = 1 + \boldsymbol{+}_3$.

Despite Examples 2.3.6 and 2.3.8, if there is no reversible option in the base, the value of the ordinal sum is determined by the value of the base.

Lemma 2.3.9. *If* G *has no reversible options and* K *is the canonical form of* G*, then* $G : H = K : H$.

Proof. Because $G - K = 0$ any option from $G - K$ has a winning response. As neither G nor K has reversible options, the winning response is of the form $G^L - K^R$ or $G^R - K^L$ (where the response is in the other summand). These same winning responses are available from moves in a base in the sum $G : H - K : H$ and thus loses.

The result follows by induction as the subordinate options now all have obvious responses. □

Bases in $\mathcal{P}$ are always tricky, as every $\mathscr{P}$-position that is not in canonical form (literally 0) has at least one reversible option (Lemma 1.5.10). If the base is literally 0, the ordinal sum is literally the subordinate.

Corollary 2.3.10. *If x and y are numbers and x is free of reversible options then $x : y$ is a number.*

Proof. By Lemma 2.3.9 and The Colon Principle, $x : y$ is equal to the ordinal sum of the canonical form of the base with the canonical form of the subordinate, so we assume x and y are canonical. By Norton's Lemma, as the incentives in x are negative (see Fact 1.9.3) so are the incentives of options in the base of $x{:}y$. By Corollary 2.2.21, as the incentives in y are negative so are the incentives of options in the subordinate of $x : y$. By Fact 1.9.4, $x : y$ is a number. □

The value of $x : y$ was described by Berlekamp [6, p. 77] and is related to the *Hackenbush Number system.* This is also addressed in depth in Sections 2.2.1 and 2.2.2 of [34].

There are ordinal sums of positions in canonical form that are not canonical, as shown in Example 2.3.1.

Proposition 2.3.11. *If G is in canonical form and has a negative Left incentive and H has a Left option then $G : H$ is not in canonical form.*

Proof. Suppose $G^L - G < 0$. By Norton's Lemma and Lemma 2.2.4, $G^L < G : H^L$ thus the Left option to G^L is dominated. □

A similar result holds for Right.

Example 2.3.12. *A base with a negative incentive combined with a subordinate without a corresponding option may give an ordinal sum in canonical form such as*

$$1 : -1 \cong \{0 \mid 1\} \cong \frac{1}{2}.$$

Theorem 2.3.13. *Let G and H be positions in canonical form. If G has no negative incentives, then $G : H$ is in canonical form.*

Proof. The possibility of reversible options in $G:H$ is covered by Lemma 2.3.5.

The concern with regards to domination is that some G^L is comparable to some $G:H^L$ (or the equivalent for Right). As G has no negative incentives, every option G' of G is confused with G. By Norton's Lemma, and because G is in canonical form, if an option G' is confused with G, then it is also confused with $G:H'$ for all H', including where H' is an option of H. □

Theorem 2.3.13 allows us to declare that positions are in canonical form without much work which is great for results such as the following.

Corollary 2.3.14. *For any position H, $\mathrm{b}(*:H) = 1 + \mathrm{b}(H)$.*

Proof. Let K be the canonical form of H. By the definition of $\mathrm{b}(G)$ (Definition 1.7.1) and The Colon Principle, $\mathrm{b}(*:H) = \mathrm{b}(*:K)$. As $*$ is in canonical form and has no options with negative incentive, $*:K$ is in canonical form by Theorem 2.3.13 and thus by Proposition 2.1.3 we have

$$\mathrm{b}(*:H) = \mathrm{b}(*:K) = \tilde{\mathrm{b}}(*:K) = \tilde{\mathrm{b}}(*) + \tilde{\mathrm{b}}(K) = 1 + \mathrm{b}(H).$$

□

2.3.1 Associativity of ordinal sum

In [14], Conway defines the ordinal sum of G and H to be $K:H$ under Definition 2.1.1 where K is the canonical form of G. In the CGSuite software [30], ordinal sum is similar to that of Conway, but also the default behaviour is to return the canonical form of $K:H$; CGSuite is not a good tool for exploring the distinctions between these definitions of ordinal sum as discussed in this Chapter.

We compare the associativity of our definition of ordinal sum with Conway's [14, p. 192]. Our ordinal sum is associative in that $G:H:K$ is well-defined (unambiguous without using parentheses) which is what we show in Lemma 2.3.15. In [14, p. 212] it is shown using its definition that $G:H:K$ is well-defined and also that $G:H:K = G':H':K'$ if $G = G'$ and $H = H'$ and $K = K'$. This is not the case with our definition, but holds if none of G, G', H and H' have reversible options. When it comes to substitution of bases, we have Lemma 2.3.9 which is similar when used multiple times.

Composing ordinal sums highlights the importance of form because even an ordinal sum of canonical positions is not necessarily canonical. Thus when considering the ordinal sum of 3 or more positions, we should expect to confront non-canonical bases.

Lemma 2.3.15. *If G, H, and K are any positions then $(G : H) : K \cong G : (H : K)$.*

Proof. The Left options of the positions are:

$$\begin{aligned}
(G : (H : K))^{\mathbf{L}} &\cong G^{\mathbf{L}} \cup G : (H : K)^{\mathbf{L}} \\
&\cong G^{\mathbf{L}} \cup G : H^{\mathbf{L}} \cup G : (H : K^{\mathbf{L}}) \text{ and} \\
((G : H) : K)^{\mathbf{L}} &\cong (G : H)^{\mathbf{L}} \cup (G : H) : K^{\mathbf{L}} \\
&\cong G^{\mathbf{L}} \cup G : H^{\mathbf{L}} \cup (G : H) : K^{\mathbf{L}}.
\end{aligned}$$

By induction, $G : (H : K^L) \cong (G : H) : K^L$ which shows the Left options of the two positions are equal. The case is similar for the Right options. □

2.4 Dicots and ordinal sum

Lemma 2.4.1. *Let G be non-empty. The ordinal sum $G : H$ is dicotic if and only if G is dicotic.*

Proof. Suppose G is dicotic. Options of G are dicotic because G is dicotic (a hereditary property); options corresponding to moves in H are dicotic by induction and thus $G : H$ is dicotic.

If G is not dicotic, there is a subposition of G that is not dicotic; that subposition is also a subposition of $G : H$ and thus it is not dicotic. □

Conway's definition of ordinal sum [14] does not give Lemma 2.4.1. For example, $\{* \mid *\} : 1 \cong \{0, * \mid *\} = \uparrow$ and $\{* \mid *\} = 0$ but $0 : 1 = 1$ which is neither infinitesimal nor dicotic.

Theorem 2.4.2. *If x is a number with no reversible options and H is a dicot then*

$$x : H = x + H.$$

Proof. True if $H = 0$. Otherwise, by Lemma 2.3.9, it suffices to assume x is in canonical form. Then,

$$\begin{aligned}
x : H &= \left\{ x^L, x : H^{\mathbf{L}} \,\middle|\, x^R, x : H^{\mathbf{R}} \right\} && \text{(by definition)} \\
&= \left\{ x : H^{\mathbf{L}} \,\middle|\, x : H^{\mathbf{R}} \right\} && (x^L < x : H^{\mathbf{L}}) \text{ and } (x^R > x : H^{\mathbf{R}}) \\
&= \left\{ x + H^{\mathbf{L}} \,\middle|\, x + H^{\mathbf{R}} \right\} && \text{(by induction)} \\
&= x + H. && \text{(by Fact 1.9.14)}
\end{aligned}$$

□

2.4.1 Star-like positions

We consider positions for which every non-zero subposition has an option to 0 for both players; we call such positions *star-like*. We show these positions have a strong connection to ordinal sum.

Definition 2.4.3. *A position is star-like if it is in $\mathcal{N}$ and every option is star-like or zero.*

Example 2.4.4. *Every non-zero nimber is star-like.*

Lemma 2.4.5. *The position $* : G$ is star-like.*

Proof. Both players have options to 0 so the position is a next-player win. The non-zero options are equal to $* : G^L$ or $* : G^R$, which are star-like by induction. □

Theorem 2.4.6. *If G is star-like then G is $*$-based.*

Proof. By Lemma 1.5.3, $G = \left\{ 0, G^{\mathbf{L}} \,\middle|\, 0, G^{\mathbf{R}} \right\}$. Each non-zero option of G is star-like by definition and thus $*$-based by induction. That is,

$$G = \left\{ 0, * : K^{\mathbf{L}} \,\middle|\, 0, * : K^{\mathbf{R}} \right\} = * : \left\{ K^{\mathbf{L}} \,\middle|\, K^{\mathbf{R}} \right\}.$$

□

From Theorem 2.4.6 and Lemma 2.4.5, we see that star-like is equivalent to $*$-based; in the sequel we use the latter term.

Corollary 2.4.7. *If G is in canonical form and G is star-like, then $G \cong * : H$ for some H.*

Proof. As G is in canonical form a player has at most one option to zero from G. A player has a least one option to 0 because otherwise all options are to $\mathscr{N}$-positions which contradicts $G \in \mathcal{N}$ and because G is in canonical form the option is literally to 0. □

Corollary 2.4.8. *Let G be a position; G is in canonical form if and only if $* : G$ is in canonical form.*

Proof. This follows from Theorem 2.3.7 and Theorem 2.3.13. □

Example 2.4.9. *Any ruleset with multiple clearly differentiated components can be played with the option for either player on their turn to remove a component as their move and this new ruleset gives only values that are sums of $*$-based values.*

For example, one could play MAZE *(a ruleset played with multiple tokens on the same board where each is a summand) with the added stipulation that either player may use a turn to remove a token. Each token has value $* : G$, where G is the value of the token in that position under the original ruleset.*

Norton's Lemma does not help us compare $*$-based positions to other dicotic positions, as seen by Lemma 1.11.5.

However, we do use The Colon Principle to analyze dicots.

Fact 2.4.10. *$o(*^n : H + *^n : K) = o(H + K)$.*

Proof. This follows from Corollary 2.2.21 and Fact 1.10.7 (which shows that $*^n = -*^n$). □

2.4.2 Atomic weights of $*$-based positions

Theorem 2.4.11. *For a position H,*

$$\mathrm{aw}(* : H) \in \{1, 0, -1\}.$$

Proof. We use Fact 2.2.19 (and Definition 2.2.18). That is, $\mathrm{aw}(* : H) = \upsilon(* : H)$. By induction, $\mathrm{aw}(* : H') \in \{1, 0, -1\}$ for all options H' of H. For all $H^L \in H^{\mathbf{L}}$, $\mathrm{aw}(* : H^L) - 2 \in \{-1, -2, -3\}$ and for all $H^R \in H^{\mathbf{R}}$, $\mathrm{aw}(* : H^R) + 2 \in \{1, 2, 3\}$.

Let $G \cong * : H$, and recall that $\mathrm{aw}(0) = 0$. That is, $G^L - 2 < 0 < G^R + 2$ for all G^L and G^R, so $\tilde{\upsilon}(G)$ is 0. As $\tilde{\upsilon}(G)$ is an integer $\mathrm{aw}(* : H) \in \mathcal{I}$ where

$$\mathcal{I} = \{n;\ \upsilon(G^L) - 2 \lhd\!\mid n \lhd\!\mid \upsilon(G^R) + 2 \text{ for } G^L \in G^{\mathbf{L}} \text{ and } G^R \in G^{\mathbf{R}}\},$$

and because $G^{\mathbf{L}}$ and $G^{\mathbf{R}}$ contains 0, $\mathcal{I} \subseteq \{1, 0, -1\}$. That is, $\mathrm{aw}(* : H) \in \{1, 0, -1\}$. □

Theorem 2.4.12. *For a position H,*

$$\mathrm{aw}(* : H) = \begin{cases} 1 & \text{if } H > \bigstar \\ -1 & \text{if } H < \bigstar \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From Theorem 2.4.11 we know that $\mathrm{aw}(* : H) \in \{1, 0, -1\}$. Using Fact 2.2.17, if $* : H > \bigstar$, then $\mathrm{aw}(* : H) = 1$; if $* : H < \bigstar$, then $\mathrm{aw}(* : H) = -1$, and if $* : H$ is confused with $\bigstar$, then $\mathrm{aw}(* : H) = 0$ by process of elimination.

Let $\bigstar$ be remote for $* : H$; $\bigstar$ is also remote for H. By Corollary 2.2.8 $\bigstar : *$ is also remote for $* : H$. By Fact 2.2.1, $\bigstar : * = * : \bigstar$.

By Fact 2.4.10,

$$o(* : H + \bigstar) = o(* : H + * : \bigstar) = o(H + \bigstar).$$

Therefore if $H > \bigstar$ then $* : H > \bigstar$; and if $H < \bigstar$ then $* : H < \bigstar$. Finally, as $\bigstar$ is remote for H, $H \neq \bigstar$ so otherwise $H \parallel \bigstar$ and thus $* : H \parallel \bigstar$. □

2.5 Recognizing Ordinal Sums

In this Section we present methods for identifying positions as ordinal sums. Previously, ordinal sums were identified using ad hoc methods. We are interested to find, given a position K, positions G and H such that $K = G : H$.

2.5.1 Literal ordinal sums

We start by considering the literal case, that is, $K \cong G : H$. For every K, $K \cong 0 : K$ and $K \cong K : 0$. That is, if we allow 0 as the base or subordinate, it is trivial to represent positions as ordinal sums.

Lemma 2.5.1. *Let K be a short position. If $K \cong G : H$ then G is a subposition of K and $G^{\mathbf{L}} \subset K^{\mathbf{L}}$ and $G^{\mathbf{R}} \subset K^{\mathbf{R}}$.*

Proof. For every subposition J of H, $G{:}J$ is a subposition of $G{:}H$ and because 0 is a subposition of H, $G{:}0 \cong G$ is a subposition of K. The set relations $K^{\mathbf{L}} = G^{\mathbf{L}} \cup G{:}H^{\mathbf{L}}$ and $K^{\mathbf{R}} = G^{\mathbf{R}} \cup G : H^{\mathbf{R}}$ are from Definition 2.1.1. $\square$

In other words, possible literal bases of a position are subpositions whose options are subsets of the original.

For any position K, given a position G we can determine if K is literally G-based

Lemma 2.5.2. *Let K and G be positions; $K \cong G : H$ for some H if and only if $G^{\mathbf{L}} \subset K^{\mathbf{L}}$, $G^{\mathbf{R}} \subset K^{\mathbf{R}}$ and for each $K' \in (K^{\mathbf{L}} \setminus G^{\mathbf{L}}) \cup (K^{\mathbf{R}} \setminus G^{\mathbf{R}})$, $K' \cong G : H'$ for some H'.*

Proof. If $K \cong G : H$ for some H then $G^{\mathbf{L}} \subset K^{\mathbf{L}}$, $G^{\mathbf{R}} \subset K^{\mathbf{R}}$ and for each $K' \in (K^{\mathbf{L}} \setminus G^{\mathbf{L}}) \cup (K^{\mathbf{R}} \setminus G^{\mathbf{R}})$, $K' \cong G{:}H'$ for some option H' of H, as seen from Definition 2.1.1.

Let $H^{\mathbf{L}}$ be the set of literal G-based subordinates of $(K^{\mathbf{L}} \setminus G^{\mathbf{L}})$ and $H^{\mathbf{R}}$ be the set of literal G-based subordinates of $(K^{\mathbf{R}} \setminus G^{\mathbf{R}})$ then $K \cong G : \left\{ H^{\mathbf{L}} \,\middle|\, H^{\mathbf{R}} \right\}$. $\square$

Moreover, if there is an H such that $K \cong G : H$ we can find it.

Corollary 2.5.3. *If K is literally G-based then the options of the literal G-based subordinate are the literal G-based subordinates of the options of K.*

Lemma 2.5.1 says that the only positions G for which K may be G-based are the subpositions of K. That is, for any position K, we can write it as a literal ordinal sum in all possible ways: for each candidate G (i.e. subposition of K) test using Lemma 2.5.2 to see if K is G-based; if so, we can find the appropriate H such that $K \cong G{:}H$.

Remark 2.5.4. *The nimber $*^n$ is literally $*^m$-based for every $m \le n$. This shows that there exist positions with any finite number of distinct bases.*

Given positions K and H we may find a G such that $K \cong G : H$ using the same inefficient method as when we do not know the base or subordinate (trying every subposition as a base). Fortunately, we can improve this by restricting our candidates for G to those satisfying $\tilde{\mathrm{b}}(G) = \tilde{\mathrm{b}}(K) - \tilde{\mathrm{b}}(H)$. Furthermore, if $K \cong G{:}H$, then G is the only subposition of K with this formal birthday.

2.5.2 Positions equal to ordinal sums

Given a non-empty position K, is $K = G : H$ for some non-zero G and H? We have shown that if $K \cong G : H$ we can find G and H. Now we expand that search. As we are looking for G and H such that $G : H = K$ (emphasis on $=$), we may assume K is in canonical form. By The Colon Principle, it suffices to look for a position H in canonical form. We must not restrict our search to positions G is in canonical form.

We can always find $K = K : 0$ and $K = 0 : K$. We ignore the cases where the base or subordinate is empty when we ask if K is an ordinal sum. However, we do consider bases equal to the empty position (i.e. 0).

No position with formal birthday 1 is a literal ordinal sum because a non-trivial ordinal sum has formal birthday at least 2. There are also positions with formal birthday 2 that are not literal ordinal sums such as $\uparrow \cong \{0 \mid *\}$ which by Lemma 2.5.1 would have to be $*$-based which it is not (no Right option to 0).

However,

$$\{* \mid *\} : 1 \cong \{*, \{* \mid *\} \mid *\} = \{*, 0 \mid *\} = \{0 \mid *\},$$

so $\uparrow$ is an ordinal sum. Moreover, we could have used $\{\cdot \mid *\}$ as the base instead of $\{* \mid *\}$.

For many positions K, $K = G : H$ for some G and H. As seen in Example 2.3.8, $\text{⧿}_1 = \{\{1 \mid 0\} \mid *\} : -1$. However, given ⧿_1 it is not clear how to find pairs such as $\{\{1 \mid 0\} \mid *\}$ and -1.

Suppose we are given K and a candidate G. We can determine if there is an H such that $K = G : H$ and if so, determine H.

Theorem 2.5.5. *Let K and G be positions in canonical form; $K = G : H$ for some H in canonical form if and only if*

1. $K = \left\{G^{\mathbf{L}}, K^{\mathbf{L}} \,\middle|\, G^{\mathbf{R}}, K^{\mathbf{R}}\right\}$,
2. *for every $K^L \notin G^{\mathbf{L}}$, $K^L = G : H^L$ for some position H^L, and*
3. *for every $K^R \notin G^{\mathbf{R}}$, $K^R = G : H^R$ for some position H^R.*

Proof. Suppose the conditions hold. Let $H = \left\{H^{\mathbf{L}} \,\middle|\, H^{\mathbf{R}}\right\}$ where $H^{\mathbf{L}}$ and $H^{\mathbf{R}}$ respectively range over all the H^L and H^R from conditions 2 and 3. We give the argument that Left (it is similar for Right) loses going first in $G : H - K$.

If Left moves in $G : H$, Left loses moving to $G^L - K$ because it is a Left option of $\left\{G^{\mathbf{L}}, K^{\mathbf{L}} \,\middle|\, G^{\mathbf{R}}, K^{\mathbf{R}}\right\} - K$ (and thus Right has a winning response according to condition 1); Left loses moving to $G : H^L$ because H was constructed such that this option only exists if K has a response moving the sum to 0.

If Left moves in $-K$, Left either moves to $G:H - K^R$ where K^R is in $G^{\mathbf{R}}$ or moves to $G : H - K^R$ where $K^R = G : -H^R$; in either case Right responds in $G : H$ moving to 0 sum.

Now suppose $K = G : H$. That is, $K = \left\{G^{\mathbf{L}}, G : H^{\mathbf{L}} \,\middle|\, G^{\mathbf{R}}, G : H^{\mathbf{R}}\right\}$. Condition 1 holds as K is already the canonical form of a position with the options of G as its options. By Lemma 2.3.5, K is derived from $G : H$ by only removing dominated options so each option of K is a literal option of G or literally $G : H'$ for some option H' of H and thus Conditions 2 and 3 hold. □

Corollary 2.5.6. *If K is G-based then the options of the G-based subordinate are the G-based subordinates of the options of K.*

Theorem 2.5.7. *If K is the canonical form of $G : H$, then a subposition of K is the value of G.*

Proof. We may assume that H is in canonical form; let K be the canonical form of $G : H$. If $H = 0$ then $K = G$. If $H \neq 0$ then, there is some option $G : H'$ of which G is a subposition by induction (if it is not dominated or reversible).

Consider H^L a Left option of H. No $G^L \geq G : H^L$ because that would contradict Fact 1.4.9. The case for a Right option is similar. That is, an option of the form $G : H'$ where H' is an option of H is not dominated.

An option $G : H'$ is shown to not be reversible by Lemma 2.3.5. By induction, this shows that a subposition of K is equal to G; as all subpositions of K are in canonical form a subposition of K is the value of G. □

That is, if K is in canonical form and $K = G : H$ for some G in canonical form then G is literally a subposition of K. Therefore, given a canonical position we can find all bases G that are in canonical form by testing every subposition as a candidate base using Theorem 2.5.5.

If K is in canonical form and $K = G : H$ for some G that is not in canonical form then the value of G is a subposition of K. We mostly care about bases that are not in canonical form when no position in canonical form is a base.

If K is in canonical form and there is no G in canonical form such that $K = G\!:\!H$, but $K = J : H$, then J has a reversible option (by Lemma 2.3.9).

Given a value, say K', (which we are thinking of as a subposition of K) there are an infinite (though countable) number of positions $J = K'$ where J has a reversible option. It is not possible to check all non-canonical representations of a position to find those for which $K = J : H$ for some H.

Problem 2.5.8. *Given a position K in canonical form, find non-zero positions G and H such that $K = G : H$ if possible.*

2.5.3 Terminology

We end with some definitions that are important for Chapter 4 but apply to positions in general.

Definition 2.5.9. *The* nimber-base *of a position is the least simple nimber (possibly* 0*) that is a base for the position.*

Definition 2.5.10. *The* number-base *of a position is the least simple number (possibly* 0*) that is a base for the position.*

These definitions are well-defined because 0 is a nimber, a number, and a base for all positions. However, for the same reasons 0 is excluded from the following definition.

Definition 2.5.11. *A position, G, is* number-based *if there exists a* non-zero *number x, such that $G = x : H$ for some H.*

A position is number-based if it is 1-based or -1-based. We can use Theorem 2.5.5 with 1 and with -1 to determine if a position is number-based.

We may define *nimber-based* analogously to number-based. A position is nimber-based if and only if $*$-based and thus we usually use the term $*$-based. However, there is a difference between $*$-based and nimber-based: the set of $*$-based integers, $\{* : n;\ n \in \mathbb{Z}\}$ is a subset of the nimber-based integers, $\{*^k : n;\ n \in \mathbb{Z}, k \geq 0\}$.

Chapter 3

Transitive games

3.1 Introduction

Playing in a transitive position, a player can reach in one move any position that they can reach in two (or more). A player would derive no benefit from making multiple consecutive moves instead of one from a transitive position.

We adopt notation similar to [34, Definition 4.11].

Definition 3.1.1. *For a position G, $\mathbf{L}^n(G)$ denotes the set of subpositions of G that can be reached by Left from G in exactly n moves.*

For a set S,

$$\mathbf{L}^n(S) \cong \bigcup_{s \in S} \mathbf{L}^n(s).$$

In particular,

$$\begin{aligned} \mathbf{L}^0(G) &\cong \{G\} \\ \mathbf{L}^1(G) &\cong \mathbf{L}(G) \cong G^{\mathbf{L}} \\ \mathbf{L}^2(G) &\cong \mathbf{L}(\mathbf{L}(G)) \cong G^{\mathbf{LL}}. \end{aligned}$$

In general $\mathbf{L}^{n+1}(G) \cong \mathbf{L}(\mathbf{L}^n(G))$ and we also define (using Kleene-like notation):

$$\mathbf{L}^*(G) \cong \bigcup_{k \geq 0} \mathbf{L}^k(G)$$

$$\mathbf{L}^+(G) \cong \bigcup_{k \geq 1} \mathbf{L}^k(G).$$

For a position G, the set of purely Left subpositions *of G is $\mathbf{L}^+(G)$. For a set A, the set of* purely Left subpositions *of A is $\mathbf{L}^+(A) \cong \bigcup_{G \in A} \mathbf{L}^+(G)$.*

We have similar definitions for Right with $\mathbf{R}$ in place of $\mathbf{L}$.

Example 3.1.2. *The set of purely Right subpositions of $\left\{\frac{1}{2} \,\middle|\, *, \boldsymbol{+}_1\right\}$ is*

$$\mathbf{R}^+(\left\{\frac{1}{2} \,\middle|\, *, \boldsymbol{+}_1\right\}) \cong \{*, \boldsymbol{+}_1, 0, \{0 \mid -1\}, -1\}.$$

Definition 3.1.3. *A position is* Left-transitive *if* $\mathbf{L}^+(G) \subseteq \mathbf{L}(G)$*. A position is* Right-transitive *if* $\mathbf{R}^+(G) \subseteq \mathbf{R}(G)$*. A position is* transitive *if it is both Left- and Right-transitive.*

That is, a position is Left-transitive if all purely Left subpositions are Left options.

Proposition 3.1.4. *A position is Left-transitive if and only if* $\mathbf{L}^2(G) \subseteq \mathbf{L}(G)$*.*

Proof. Clearly, $\mathbf{L}^2(G) \subseteq \mathbf{L}^+(G)$. If G is Left-transitive $\mathbf{L}^+(G) \subseteq \mathbf{L}(G)$ so $\mathbf{L}^2(G) \subseteq \mathbf{L}(G)$.

If $\mathbf{L}^2(G) \subseteq \mathbf{L}(G)$ then $\mathbf{L}^k(G) \subseteq \mathbf{L}(G)$ for some $k > 1$. By induction, $\mathbf{L}^{k+1}(G) \cong \mathbf{L}(\mathbf{L}^k(G)) \subseteq \mathbf{L}(\mathbf{L}(G)) \cong \mathbf{L}^2(G) \subseteq \mathbf{L}(G)$. That is, $\mathbf{L}^+(G) \subseteq \mathbf{L}(G)$. □

Our definition of transitive is equivalent to [32, Definition 6.22]: a position is Left-transitive if $G^{\mathbf{LL}} \subseteq G^{\mathbf{L}}$. The definition of hereditarily transitive follows from the definition of transitive. Both definitions are formal. That is, a position is *hereditarily transitive* if all subpositions are transitive.

Example 3.1.5. *The positions* $0 \cong \{\cdot \mid \cdot\}$, $1 \cong \{0 \mid \cdot\}$, $-1 \cong \{\cdot \mid 0\}$*, and* $* \cong \{0 \mid 0\}$ *are all hereditarily transitive.*

However, $\{* \mid *\}$, and $\{\uparrow, 0 \mid 0, \downarrow\}$ are not Left- or Right-transitive despite equaling 0 and $*$, respectively. Form is very important in transitive positions. We show every position is equal to a position that is not transitive (Proposition 3.6.7) and some positions are not equal to transitive positions (Proposition 3.3.4).

The readers most familiar with the topic of this Chapter at the time of writing are likely to be familiar with them under the name *option-closed games* from [26] and [34]. In [32] the term hereditarily transitive replaces option-closed.

Transitive positions include the nimbers, numbers, some hot positions, and some dicotic positions. The theory of transitive positions is important because a good understanding of transitive positions helps to compare nimbers and numbers as families of positions.

Ordinal sums are important in describing the dicotic hereditarily transitive positions, which have a very similar structure to hereditarily transitive positions in general; in Section 3.2 we discuss ordinal sums and transitivity.

The reduced values of transitive positions are discussed in Section 3.4 and are easily found from the stops of positions which we discuss in Section 3.3. We show that the infinitesimal parts of numberish positions are $*$-based and improve the bound on the infinitesimal parts of non-numberish positions. In Section 3.5 we describe the positions that are in canonical form.

Most of the chapter builds to Theorems 3.10.7 and 3.10.5 which are analogous to Facts 1.5.8 and 1.5.9, respectively. With these results in hand, we get Theorem 3.11.1; this allows us to consider the values of hereditarily transitive positions without as much attention to form as we can go back and forth between values and stratified positions using the operations in Section 3.6.

In Section 3.12 we end by proving that the partial order of hereditarily transitive positions is a planar lattice (Theorem 3.12.9).

3.2 Transitivity and ordinal sum

Lemma 3.2.1. *Let G and H be positions; $G : H$ is transitive if and only if G and H are transitive.*

Proof. Suppose G and H are transitive. We want to show that $G : H$ is transitive. We consider the Left-Left options $\mathbf{L}^2(G : H) \cong \mathbf{L}(\{G^{\mathbf{L}} \cup G : H^{\mathbf{L}}\}) \cong G^{\mathbf{LL}} \cup G^{\mathbf{L}} \cup G : H^{\mathbf{LL}}$; as G is transitive $G^{\mathbf{LL}} \subseteq G^{\mathbf{L}}$; as H is transitive $G : H^{\mathbf{LL}} \subseteq G : H^{\mathbf{L}}$. The case for Right options is similar; we conclude that $G : H$ is transitive.

Suppose $G : H$ is transitive:

- $G^{\mathbf{LL}} \subseteq (G : H)^{\mathbf{L}} \cong G^{\mathbf{L}} \cup G : H^{\mathbf{L}}$. There is no $G^{LL} \cong G : H^{L}$ because $\tilde{\mathrm{b}}(G^{LL}) < \tilde{\mathrm{b}}(G) \leq \tilde{\mathrm{b}}(G : H^{L})$, so every G^{LL} is in $G^{\mathbf{L}}$; that is, $G^{\mathbf{LL}} \subseteq G^{\mathbf{L}}$.

- $G : \mathbf{L}^2(H) \subseteq (G : H)^{\mathbf{L}} \cong G^{\mathbf{L}} \cup G : H^{\mathbf{L}}$ but no $G : H^{LL} \cong G^{L}$ because $\tilde{\mathrm{b}}(G^{L}) < \tilde{\mathrm{b}}(G) \leq \tilde{\mathrm{b}}(G : H^{LL})$, so every $G : H^{LL}$ is in $G : H^{\mathbf{L}}$; that is, $H^{\mathbf{LL}} \subseteq H^{\mathbf{L}}$.

□

Theorem 3.2.2. *Let G and H be positions; $G : H$ is hereditarily transitive if and only if G and H are hereditarily transitive.*

Proof. If G and H are hereditarily transitive, then $G{:}H$ is transitive by Lemma 3.2.1. The hereditary transitivity follows by induction.

If $G : H$ is hereditarily transitive, then G is hereditarily transitive because it is a subposition. Every subposition of the form $G : H'$ where H' is a subposition of H is transitive; by Lemma 3.2.1 every H' is transitive and thus H is hereditarily transitive. □

Corollary 3.2.3. *Let G be a position;*

G is transitive if and only if $* : G$ is transitive,

G is hereditarily transitive if and only if $* : G$ is hereditarily transitive.

Proof. By Lemma 3.2.1 and Theorem 3.2.2, noting that $*$ is transitive and hereditarily transitive. □

Combined with Corollary 2.4.8, $* : G$ is canonical and transitive if and only if G is canonical and transitive.

Corollary 3.2.4. *Every nimber is hereditarily transitive.*

3.3 Stops of transitive positions

Proposition 3.3.1. *If G is Left-transitive and not a Left end, then Left has an option to a Left end.*

Proof. As G is not a Left end, Left has at least 1 option. If Left's option is not a Left end then it has a Left option and as G is Left-transitive we know G must have at least 2 options. If no Left-option is a Left end, then G has an unbounded number of Left options which contradicts G being a short position. □

Given any transitive position that is not an end, both players have an option to an end; by Fact 1.9.10 each player has an option to an integer. However, we can also say something stronger.

Lemma 3.3.2 is a generalization of [32, Lemma II.6.23, p.133]. We need only to assume transitive (instead of hereditarily transitive) and only for one player at a time. We give the case for Left.

Lemma 3.3.2. *If G is Left-transitive and G is not a number then* $\mathrm{LS}(G)$ *is a Left option of G.*

Proof. There is a G^L such that $\mathrm{RS}(G^L) = \mathrm{LS}(G)$ and $\mathrm{LS}(G^L) \geq \mathrm{RS}(G^L)$, so $\mathrm{LS}(G^L) \geq \mathrm{LS}(G)$. A sequence of purely Left options chosen to maximize the Right stop at each step contains only positions for which the Left stop is at least $\mathrm{LS}(G)$. As G is short this sequence eventually reaches a number x; in the extreme case, some purely Left option has no option and is thus an integer. As G is Left-transitive x is a Left option of G and thus $\mathrm{LS}(G) \geq x = \mathrm{LS}(x) \geq \mathrm{LS}(G)$. That is, some Left option of G is $\mathrm{LS}(G)$. $\square$

A similar result holds for Right and we use either as needed when considering transitive positions.

Example 3.3.3. *For all positions H, $\{0 \mid H\}$ is Left-transitive. In particular, $\Uparrow \cong \{0 \mid \uparrow *\}$ is Left-transitive.*

The position $\pm\Uparrow \cong \{\Uparrow \mid \Downarrow\}$ is not transitive, but is equal to a transitive position, namely $\{0, \Uparrow \mid 0, \Downarrow\}$.

We can also show that there are positions that are not equal to transitive positions.

Proposition 3.3.4. *There is no $G = \Downarrow$ such that G is Left-transitive.*

Proof. If $G = \Downarrow$ then $\mathrm{LS}(G) = 0$ and if G is Left-transitive then by Lemma 3.3.2 it has an option to 0 which contradicts $G = \Downarrow$ because $\Downarrow < 0$. $\square$

Lemma 3.3.5. *If G is Left-transitive, then for every Left option G^L of G,* $\mathrm{LS}(G) \geq \mathrm{LS}(G^L)$.

Proof. This has the same proof as Lemma II.6.24 of [32], but we only assume that G is Left-transitive instead of transitive. $\square$

3.4 Infinitesimals and transitivity

3.4.1 Reduced values of transitive positions

Theorem 3.4.1. *If G is transitive then* $\mathrm{rcf}(G)$ *is a number or a position of the form $\{a \mid b\}$ for numbers a and b with $a > b$.*

Proof. It suffices to use the same proof as Theorem II.6.25 of [32], but we only assume that G is transitive instead of hereditarily transitive and use 3.3.2 instead of Lemma II.6.23 of [32]. $\square$

This is stronger than the original theorem of [26], which only shows that hereditarily transitive positions have these simple reduced canonical forms.

In Lemma 3.4.3 for hereditarily transitive positions that are not numbers we bound the confusion interval of some of its options.

Definition 3.4.2. *A Left option of G, say G^L, is a* primary *Left option of G if it can be reached by Left in exactly one move only. That is, G^L is primary if $G^L \in \mathbf{L}(G) \setminus \mathbf{L}^+(\mathbf{L}(G))$.*

A Right option of G, say G^R, is a primary *Right option of G if it can be reached by Right in exactly one move only. That is, G^R is primary if $G^R \in \mathbf{R}(G) \setminus \mathbf{R}^+(\mathbf{R}(G))$.*

A primary *option is either a primary Left option or a primary Right option.*

The primary options are the key to understanding the structure of a hereditarily transitive position, which we discuss in Section 3.7.

Lemma 3.4.3. *Let G be a hereditarily transitive position that is not a number. If G^L is a non-dominated and non-reversible primary option of G, then*

$$\mathrm{LS}(G) \geq \mathrm{LS}(G^L) \geq \mathrm{RS}(G^L) \geq \mathrm{RS}(G).$$

Proof. If $G^L = \mathrm{LS}(G)$ then $\mathrm{LS}(G) = \mathrm{LS}(G^L) = \mathrm{RS}(G^L)$ and the result follows because $\mathrm{LS}(G) \geq \mathrm{RS}(G)$.

If $G^L \neq \mathrm{LS}(G)$ then G^L is not a number: by Lemma 3.3.5, $\mathrm{LS}(G^L) \leq \mathrm{LS}(G)$ and thus $G^L < \mathrm{LS}(G)$, contradicting our assumption that G^L is not dominated. As G is hereditarily transitive, Right has an option to $\mathrm{RS}(G^L)$; this option is reversible if $\mathrm{RS}(G^L) < \mathrm{RS}(G)$. Thus, $\mathrm{LS}(G^L) \geq \mathrm{RS}(G^L) \geq \mathrm{RS}(G)$ and $\mathrm{LS}(G^L) \leq \mathrm{LS}(G)$. $\square$

In the case that a hereditarily transitive position is numberish, then we can say a lot more.

Lemma 3.4.4. *If G is hereditarily transitive and x-ish for some number x, then $G = x$ or $G = x : * : K$ for some hereditarily transitive position K.*

Proof. If G is hereditarily transitive x-ish for some number x and $G \neq x$ then by Lemma 3.4.3, the primary options of G, say hereditarily transitive positions G^L and G^R, are both x-ish and thus $G = \mathrm{T}(\left\{G^L \mid G^R\right\})$ where $G^L = x : H^L$ and $G^R = x : H^R$ (H^L and H^R are possibly 0). By Theorem 2.5.5, we see that G is x-based.

By induction the x-based options are either $x = x : 0$, which is an option for both players by Lemma 3.3.2, or $x : * : H'$ for some hereditarily transitive H'. By Corollary 2.5.6, H is $*$-based and hereditarily transitive. □

Corollary 3.4.5. *If G is hereditarily transitive, infinitesimal, and non-zero then G is $*$-based.*

Proposition 3.4.6. *If G is hereditarily transitive, dicotic, and non-zero then G is literally $*$-based.*

Proof. As G is hereditarily transitive and non-zero, an option to an end exists; all subpositions of G that are ends are 0 because G is dicotic. That is, G has options to 0 for both players. All non-zero options are literally $*$-based by induction as they are dicotic, hereditarily transitive, and non-zero. □

We can also bound the infinitesimal parts of non-numberish hereditarily transitive positions. This is a tightening of the bound given by [26, Theorem 2.16, p. 146], which corresponds to the case $n = 1$ in the following:

Lemma 3.4.7. *If G is hereditarily transitive, then for all $n \geq 1$,*

$$G - \mathrm{rcf}(G) < 2(* : n) - * : (n-1).$$

Proof. If $m > n \geq 1$ then $2(* : m) - * : (m-1) < 2(* : n) - * : (n-1)$. We show that the result holds whenever $n > \mathrm{b}(G)$, and thus it holds for all $n \geq 1$.

We show that $2(*:n) - *:(n-1) > 0$ and then we simplify. Left can win by moving to $2(*:n)$ which is positive. Right's only non-dominated option is to $*:n - *:(n-1)$, which is positive by Fact 2.4.10. Left's option to $2(* : n)$ reverses to 0 and the others reverse out, giving $2(* : n) - * : (n-1) = \{0 \mid * : n - * : (n-1)\}$.

If G is a number, then $G - \mathrm{rcf}(G) = 0$ which is less than $\{0 \mid * : n - * : (n-1)\}$. If G is x-ish for some number x, then by Lemma 3.4.4 $G \cong * : H$ for some H and by Theorem 2.4.2 $G - \mathrm{rcf}(G) = x + * : H - x = * : H$. By the Archimedean Principle

(1.9.9) and our assumption that $n > \mathrm{b}(G)$, $n > H$ and thus by Fact 2.4.10 $*{:}n > *{:}H$. Also by Fact 2.4.10 $*{:}n - *{:}(n-1) > 0$, so $2(*{:}n) - *{:}(n-1) > *{:}H = G - \mathrm{rcf}(G)$.

Now assume that G is not numberish and let $\mathrm{rcf}(G) = \{a \mid b\}$. Consider the difference

$$G + \{-b \mid -a\} + \{* : (n-1) - * : n \mid 0\}.$$

We need to show that Right wins going first and Left loses going first.

If Right plays first, he can move to $b + \{-b \mid -a\} + \{* : (n-1) - * : n \mid 0\}$; by Facts 1.9.13 and 1.13.10, Left's best response is to $b - b + \{* : (n-1) - * : n \mid 0\}$ which is negative.

Left going first has to choose from moves in three summands. If Left plays to

$$G - b + \{* : (n-1) - * : n \mid 0\},$$

Right wins by responding to $b - b + \{* : (n-1) - * : n \mid 0\} = \{* : (n-1) - * : n \mid 0\}$ which is negative. If Left plays to

$$G + \{-b \mid -a\} + * : (n-1) - * : n,$$

Right can win by responding to $b + \{-b \mid -a\} + * : (n-1) - * : n$ as (by Facts 1.9.13 and 1.13.10) Left's best response is to $b - b + * : (n-1) - * : n = * : (n-1) - * : n$ which is negative.

Finally, we consider Left moves in G; let G^L be a Left option of G. If G^L is numberish, Right responds to $G^L - a + \{* : (n-1) - * : n \mid 0\}$ which is negative by induction if G^L is a-ish and more definitely negative if G^L is x-ish for $x < a$. If $G^L =_I \{a \mid b\}$ then $G^L + \{-b \mid -a\} + \{* : (n-1) - * : n \mid 0\}$ is negative by induction, and if $G^L =_I \{a' \mid b'\}$ for $a' < a$ or $b' < b$ then $G^L + \{-b \mid -a\} + \{* : (n-1) - * : n \mid 0\}$ is more definitely negative. However, if $a > b' > b$ then Right must respond to $G^L - a + \{* : (n-1) - * : n \mid 0\}$; by Facts 1.9.13 and 1.13.10 Left's response will be in G^L to some G^{LL} giving

$$G^{LL} - a + \{* : (n-1) - * : n \mid 0\};$$

if $G^{LL} =_I a$ Right wins responding to $a - a + \{* : (n-1) - * : n \mid 0\}$ or similarly if $G^{LL} < a$. Lastly, if $G^{LL} =_I \{a \mid b''\}$ with $b'' < a$ and Right thus wins by moving to $b'' - a + \{* : (n-1) - * : n \mid 0\}$.

$\square$

3.5 Hereditarily transitive values

There are some transitive positions that are in canonical form, but only a few.

Lemma 3.5.1. *If G is transitive and in canonical form then G has no positive Left option and no negative Right option.*

Proof. If some Left option, say G^L is greater than 0 then from it Left has an option to a position greater than or equal to 0; an option to 0 from G^L implies an option to 0 from G and this would be dominated. Otherwise, there is an option to another position, say G^{LL} that is greater than or equal to 0 and by induction this implies an non-finite run, which contradicts the fact that G is short. □

Lemma 3.5.2. *The numbers that are transitive and in canonical form are* 0, 1, -1, $\frac{1}{2}$, *and* $-\frac{1}{2}$.

Proof. By Lemma 3.5.1, the only numbers that can be Left options are non-positive integers and the only numbers that can be Right options are non-negative integers. The gap between the Left and Right options of a number in canonical form is at most 1 (otherwise there is an integer in the interval) so the above numbers are the only numbers that are transitive and in canonical form. □

This brings us to a good point to compare hereditarily transitive positions with *consecutive move ban games*; consecutive move ban games have more recently been called *alternating games* [25].

Definition 3.5.3. *[27, Definition 67, p. 46] A position G has a consecutive move ban if $G^{\mathbf{LL}} = G^{\mathbf{RR}} = \emptyset$ and all the subpositions of G have a consecutive move ban.*

Proposition 3.5.4. *If G has a consecutive move ban, then G is hereditarily transitive.*

Proof. If $G^{\mathbf{LL}} = G^{\mathbf{RR}} = \emptyset$, then by Proposition 3.1.4 G is transitive. The consecutive move ban property is hereditary and thus the subpositions of G are also transitive; that is, a position with a consecutive move is hereditarily transitive. □

Fact 3.5.5. *[27, Theorem 69, p. 46] The only possible values of positions with a consecutive move ban are* 0, $*$, 1, -1, $\frac{1}{2}$ *and* $-\frac{1}{2}$.

We return to the full set of hereditarily transitive positions to find more positions in canonical form.

Theorem 3.5.6. *If G is hereditarily transitive, in canonical form, and not a number, then G is literally $*$-based.*

Proof. By Lemma 3.3.2 as G is not a number, Lemma 3.5.1 constrains the possible stops. That is, we need $\mathrm{LS}(G) \leq 0$ and $\mathrm{RS}(G) \geq 0$ and by Fact 1.12.2, $\mathrm{LS}(G) \geq \mathrm{RS}(G)$ which implies $\mathrm{LS}(G) = \mathrm{RS}(G) = 0$. Because G is in canonical form, the option to 0 is a literal option to 0.

All non-zero Left options of G are confused with 0: for all G^L, $\mathrm{LS}(G^L) = 0$; if $\mathrm{RS}(G^L) < 0$ then G^L is reversible so $\mathrm{RS}(G^L) = 0$. That is, every non-zero option of G is hereditarily transitive, in canonical form, and infinitesimal. By induction and Corollary 3.4.5, G is literally $*$-based. $\square$

Corollary 3.5.7. *If G is hereditarily transitive, in canonical form and not a number, then $G \cong *^n : x$ for $n \geq 1$ and $x \in \{0, 1, -1, \frac{1}{2}, -\frac{1}{2}\}$.*

Proof. By Theorem 3.5.6, $G \cong *{:}H$; H is a hereditarily transitive position in canonical form by Corollaries 3.2.3 and 2.4.8. If H is a number then by Lemma 3.5.2 it is an x as given and $G \cong *^1 : x$. Otherwise, by induction, $G \cong * : *^m : x \cong *^n : x$ for some $n \geq 2$. $\square$

Only a small fraction of hereditarily transitive positions are in canonical form and thus it remains to be shown how useful canonical form is for hereditarily transitive positions. We do eventually make good use of values in Theorem 3.11.1.

3.6 Making transitive positions

Taking a position as input, we can generate a position that is (Left- or Right- or hereditarily) transitive. In particular, we do so in a way that if the input already satisfies the property it is the output.

With the introduction of $\mathbf{L}(G) = G^{\mathbf{L}}$, we remind the reader that

$$\left\{G^{\mathbf{L}} \,\middle|\, G^{\mathbf{R}}\right\} \cong \{\mathbf{L}(G) \mid \mathbf{R}(G)\}.$$

Definition 3.6.1. *The* Left-transitive closure *of a position G is*

$$\mathrm{T_L}(G) \cong \left\{ \mathbf{L}^+(G) \;\middle|\; \mathbf{R}(G) \right\}.$$

The Right-transitive closure *of a position G is*

$$\mathrm{T_R}(G) \cong \left\{ \mathbf{L}(G) \;\middle|\; \mathbf{R}^+(G) \right\}.$$

The transitive closure *of a position G is*

$$\mathrm{T}(G) \cong \left\{ \mathbf{L}^+(G) \;\middle|\; \mathbf{R}^+(G) \right\}.$$

The hereditary transitive closure *of a position G is*

$$\mathrm{HT}(G) \cong \mathrm{T}\left(\left\{ \mathrm{HT}(G^{\mathbf{L}}) \;\middle|\; \mathrm{HT}(G^{\mathbf{R}}) \right\} \right),$$

where the hereditary transitive closure *of a set S, is*

$$\mathrm{HT}(S) = \bigcup_{s \in S} \mathrm{HT}(s).$$

Proposition 3.6.2. *For any position G,*

(a) $\mathrm{T_L}(G)$ *is Left-transitive.*

(b) $\mathrm{T_R}(G)$ *is Right-transitive.*

(c) $\mathrm{T}(G)$ *is transitive.*

Proof. Transparent from the definitions. □

Proposition 3.6.3. *For any position G,*

(a) If G is Left-transitive, $G \cong \mathrm{T_L}(G)$.

(b) If G is Right-transitive, $G \cong \mathrm{T_R}(G)$.

(c) If G is transitive, $G \cong \mathrm{T}(G)$.

Proof. Transparent from the definitions. □

Example 3.6.4. *Consider the following position in canonical form*

$$G = \left\{1, \left\{\frac{3}{2}, \{2 \mid 0\} \mid 0\right\} \mid 0\right\}.$$

The purely Left options of G are

$$\mathbf{L}^{+}(G) \cong \{1, 0, \left\{\frac{3}{2}, \{2 \mid 0\} \mid 0\right\}, \frac{3}{2}, \{2 \mid 0\}, 2\}.$$

Thus

$$\mathrm{T_L}(G) \cong \left\{1, 0, \left\{\frac{3}{2}, \{2 \mid 0\} \mid 0\right\}, \frac{3}{2}, \{2 \mid 0\}, 2 \mid 0\right\}.$$

In this case, as G is Right-transitive, $\mathrm{T_L}(G)$ is also the transitive closure $\mathrm{T}(G)$.

The canonical form of $\mathrm{T}(G)$ is

$$\mathrm{T}(G) \cong \mathrm{T_L}(G) = \{2, \{2, \{2 \mid 0\} \mid 0\} \mid 0\} \cong \{2 \mid 0\} : 2.$$

That is, $\mathrm{T}(G) \neq G$.

In Example 3.6.4, $G \neq \mathrm{T_L}(G)$. In particular, 2 is a Left option of $\mathrm{T_L}(G)$ and $2 > G$. We can describe when $G = \mathrm{T_L}(G)$.

Lemma 3.6.5. *For all positions G, $G = \mathrm{T_L}(G)$ if and only if no purely Left subposition is at least G.*

Proof. Suppose, by way of contradiction, that some purely Left subposition, say H, is at least G. Then H is equal to a Left option of G which contradicts Fact 1.4.9.

Suppose, by way of contradiction, $G \neq \mathrm{T_L}(G)$. By Fact 1.5.1, no new Left option, say H, added to G changes the value unless $H \geq G$. □

Lemma 3.6.6. *For any position G, there exists a position $H = G$ with some $H^{LL} > H$.*

Proof. Let $H = \left\{G^{\mathbf{L}}, \pm(\mathrm{b}(G)+1) \,\middle|\, G^{\mathbf{R}}\right\}$; $H^{LL} \cong \mathrm{b}(G) + 1 > G$. We can see that $H = G$ because Left's option to $\pm(\mathrm{b}(G)+1)$ reverses out. □

Proposition 3.6.7. *For all G, there is an $H = G$ such that H is not transitive.*

Proof. By Lemma 3.6.5, taking H from Lemma 3.6.6 gives $H \neq \mathrm{T_L}(H)$. □

If we are given a position and asked to determine if it is equal to a Left-transitive position it is not sufficient to note that the given position is not equal to its Left-transitive closure.

3.7 Primary Options

Recall Definition 3.4.2.

Proposition 3.7.1. *If G is transitive, then G is literally the transitive closure of the position whose options are exactly the primary options of G.*

Proof. If G is transitive, $G^{\mathbf{L}}$ is the Left-transitive closure of its primary Left options, and $G^{\mathbf{R}}$ is the Right-transitive closure of its primary Right options. $\square$

Proposition 3.7.1 foreshadows the importance of primary options.

Example 3.7.2. *The position $\{2 \mid 0\} : 2$ is transitive and its only primary Left option is $\{2 \mid 0\} : 1$ and its only primary Right option is 0. We see that $\{2 \mid 0\} : 2 \cong \mathrm{T}(\{\{2 \mid 0\} : 1 \mid 0\})$.*

Proposition 3.7.3. *The transitive closure of a position with only hereditarily transitive options is hereditarily transitive.*

Proof. Suppose H is a position for which all its options are hereditarily transitive. Let $G = \mathrm{T}(H)$. All options of G were options of H and thus are hereditarily transitive. By Proposition 3.6.2, G is transitive and so G is hereditarily transitive. $\square$

Example 3.7.4. *The nimber $*^3$ is hereditarily transitive and $\mathrm{T}(\{*^2 \mid *^2\}) \cong *^3$.*

Proposition 3.7.5. *For any position G, $\mathrm{HT}(G)$ is hereditarily transitive.*

Proof. The hereditarily transitive closure of a position is the transitive closure of the hereditarily transitive closures of the options (Definition 3.6.1). Thus, $\mathrm{HT}(G)$ is hereditarily transitive by Proposition 3.7.3. $\square$

3.8 Transitive domination

Definition 3.8.1. *Let G and H be transitive positions. We write*

$$G \geq_L H$$

and say G Left-transitively-dominates H if for all $h \in \mathbf{L}^*(H)$, there is a $g \in \mathbf{L}^*(G)$ such that $g \geq h$. We write

$$G \leq_R H$$

and say G Right-transitively-dominates H if for all $h \in \mathbf{R}^*(H)$, there is a $g \in \mathbf{R}^*(G)$ such that $g \leq h$.

In a transitive position the existence of an option implies the existence of other options. If $G \geq_L H$, Left prefers G and the options it implies as a Left option to H and the options it implies. If $G \leq_R H$ Right prefers G and the options it implies as a Right option to H and the options it implies.

Transitive domination is closely related to the usual partial order.

Lemma 3.8.2. *If $G \geq_L H$ and $H \leq_R G$, then $G \geq H$.*

Proof. We need to show that Right loses $G - H$ going first. If Right moves in G (to some $G^R - H$) then Left wins because $H \leq_R G$; if Right moves in H (to some $G - H^L$) then Left wins because $G \geq_L H$. □

A position always Left- and Right- transitively-dominates its options.

Lemma 3.8.3. *If G is Left-transitive then $G \geq_L G^L$.*

Proof. Transparent from the definitions. □

Transitive domination is transitive.

Lemma 3.8.4. *If $G \geq_L H$ and $H \geq_L K$, then $G \geq_L K$.*

Proof. If $k \in \mathbf{L}^*(K)$ then because $H \geq_L K$, there is an $h \in \mathbf{L}^*(H)$ such that $h \geq k$; because $G \geq_L H$ there is a $g \in \mathbf{L}^*(G)$ such that $g \geq h$ and so $g \geq k$. □

Lemma 3.8.5. *Let G and H be hereditarily transitive. If $G \geq_L H^L$ for all primary Left options H^L of H and $G \geq H$ or there is a G^L such that $G^L \geq H$, then $G \geq_L H$.*

Proof. Let $h \in \mathbf{L}^*$. Note that $\mathbf{L}^*(H) = \{H\} \cup \mathbf{L}^1(H) \cup \mathbf{L}^2(H)$. As $G \geq H$ or there is a G^L such that $G^L \geq H$, there is a $g \in \{G\} \cup \mathbf{L}^1(G)$ such that $g \geq h$ if $h = H$. As $G \geq_L H^L$ for all primary Left options H^L, there is a $g \in \mathbf{L}^*(G)$ such that $g \geq h$ if h is a primary Left option of H. Lastly, the result holds for non-primary Left options of H because G Left-transitively-dominates the primary Left options. □

Theorem 3.8.6. *If G_1 and G_2 are hereditarily transitive positions, then one Left-transitively-dominates the other and one Right-transitively-dominates the other.*

Proof. We give the proof for Left-transitive domination. The proof for Right is similar. Suppose by way of contradiction that G_1 and G_2 are a simplest pair of hereditarily transitive positions where neither Left-transitively-dominates the other. By Lemmas 3.8.3 and 3.8.4, no $G_1^L \geq_L G_2$ and no $G_2^L \geq_L G_1$. As G_1 and G_2 are the simplest, we have $G_2 \geq_L G_1^L$ and $G_1 \geq_L G_2^L$ for all $G_1^L \in \mathbf{L}(G_1)$ and all $G_2 \in \mathbf{L}(G_2)$. That is, in the difference $G_1 + (-G_2)$, Left does not win playing first in G_1 and Right does not win playing first in $-G_2$.

If Left does not win playing first in $-G_2$ or Right does not win playing first in G_1 then a player loses going first in $G_1 + (-G_2)$ so G_1 and G_2 are comparable and thus by Lemma 3.8.5 one Left-transitively-dominates the other. Suppose that Left does win playing first in $-G_2$ (to the primary option option $-G_2^R$ or one of its options) and Right does win playing first in G_1 (to the primary option G_1^R or one of its options). But either $G_1^R \leq_R G_2^R$ or $G_2^R \leq_R G_1^R$ so they cannot both win. $\square$

3.9 Simplifying hereditarily transitive positions

In 3.5 we saw some positions that are both hereditarily transitive and in canonical form. However, the number of such positions increases linearly with birthday (on Day n for $n \geq 2$ there are $5(n-1)-1$ such positions.

This leaves us with questions about the canonical forms of hereditarily transitive positions and the hereditary transitive closure of positions in canonical forms.

Given a hereditarily transitive position, we wish to find an equal hereditarily transitive position that is simpler. We see that simplifying hereditarily transitive positions is like simplifying positions in general if we only work with primary options.

Lemma 3.9.1. *If G is transitive and H is formed from G by removing a primary option of G then H is transitive.*

Proof. Let H be formed from G by the removal of a primary Left option, G^L.

By definition, $G^L \notin \mathbf{L}^2(G)$. As $H^{\mathbf{L}} \subseteq G^{\mathbf{L}}$, $\mathbf{L}^2(H) \subseteq \mathbf{L}^2(G)$ but $G^L \notin \mathbf{L}^2(G)$ so $\mathbf{L}^2(G) = \mathbf{L}^2(H)$.

Also $\mathbf{L}^2(G) \subseteq G^{\mathbf{L}}$ but $G^L \notin \mathbf{L}^2(G)$, so $\mathbf{L}^2(G) \subseteq G^{\mathbf{L}} \setminus \{G^L\}$, that is, H is Left-transitive. $\square$

Corollary 3.9.2. *If G is hereditarily transitive and H is formed from G by removing a primary option of G then H is hereditarily transitive.*

Proof. Let H be formed from G by the removal of a primary Left option, G^L. By Lemma 3.9.1, H is transitive. As G is hereditarily transitive, the subpositions of H are hereditarily transitive and so H is hereditarily transitive. □

Lemma 3.9.3. *If G is hereditarily transitive and H is obtained from G by removing some (Left-transitively) dominated primary Left option G^{L_1}, then $H = G$ and H is hereditarily transitive.*

Proof. Suppose G^{L_1} is Left-transitively-dominated by G^{L_2}, then it is dominated by G^{L_2} or some G^{L_2L}. Removing a dominated option does not affect the value of a position, thus $H = G$. Lastly, H is hereditarily transitive by Corollary 3.9.2. □

Lemma 3.9.4. *If G is hereditarily transitive and G^L is a primary Left option reversible through G^{LR} then*

$$H \cong \left\{ \mathbf{L}(G^{LR}), \mathbf{L}(G) \setminus \left\{ G^L \right\} \,\middle|\, G^R \right\}$$

is hereditarily transitive and $H = G$.

Proof. Equality follows from normal reversibility (Fact 1.5.6). As G is hereditarily transitive all proper subpositions of H are also hereditarily transitive. To see that H is transitive note that $H^{\mathbf{L}}$ is formed by adding options and then removing G^{LR}. Whereas G is hereditarily transitive, adding all Left options of G^{LR} includes all purely Left subpositions of every position added thus maintaining hereditary transitivity. Finally, we remove G^L and H is hereditarily transitive by Corollary 3.9.2. □

Theorem 3.9.5. *If G is a hereditarily transitive position, then there is a hereditarily transitive position $H = G$ with at most one primary Left option and at most one primary Right option, neither of which is reversible.*

Proof. By Lemmas 3.9.3 and 3.9.4, removing dominated primary options and bypassing reversible primary options gives a new hereditarily transitive position. Bypassing a reversible option may result in more primary options that may be dominated or reversible. However, as G is short this process exhausts eventually.

If K has two primary or more Left options, then by Theorem 3.8.6 one is dominated. By Lemma 3.9.3, removing the dominated primary options gives a hereditarily transitive position $H = G$ as desired. □

Corollary 3.9.6. *A hereditarily transitive position with no dominated or reversible primary options has at most one primary option for each player.*

Corollary 3.9.7. *Every hereditarily transitive position is equal to a hereditarily transitive position that is literally the transitive closure of a position with at most one Left option and at most one Right option, neither of which are reversible.*

Proof. Let G be a hereditarily transitive position. If G is a number this is shown by Lemma 3.10.6.

Otherwise, by Theorem 3.9.5 G has exactly one primary Left option, say G^L and one primary Right option, say G^R. By Proposition 3.7.1 $G \cong \mathrm{T}(\left\{G^L \,\middle|\, G^R\right\}$ and that position is hereditarily transitive by Proposition 3.7.3. □

Example 3.9.8. *Consider the position* $G \cong \{0 \mid 0, -1\} : * : 1$*, which we consider because it is a non-trivial example that we can argue is hereditarily transitive: G is hereditarily transitive by Theorem 3.2.2, as* $\{0 \mid 0, -1\}$*,* $*$ *and* 1 *are hereditarily transitive.*

Without ordinal sum G would be this:

$$G \cong \{0, \{0 \mid 0, -1\}, \{0, \{0 \mid 0, -1\} \mid 0, -1, \{0 \mid 0, -1\}\} \mid 0, -1, \{0 \mid 0, -1\}\}.$$

None of the options of G are reversible and there is exactly one primary option for each player. The Left option $\{0, \{0 \mid 0, -1\} \mid 0, -1, \{0 \mid 0, -1\}\}$ *is primary and the Right option* $\{0 \mid 0, -1\}$ *is primary. We see that*

$$G = \mathrm{HT}(\{\{0, \{0 \mid 0, -1\} \mid 0, -1, \{0 \mid 0, -1\}\} \mid \{0 \mid 0, -1\}\}).$$

But if we are going to express G a closure, we can iteratively replace primary options with positions with at most one option for each player. Doing so, we find:

$$\begin{aligned} G &\cong \mathrm{HT}(\{\{0, \{0 \mid -1\} \mid 0, -1, \{0 \mid -1\}\} \mid \{0 \mid -1\}\}) \\ &\cong \mathrm{HT}(\{\{\{0 \mid -1\} \mid \{0 \mid -1\}\} \mid \{0 \mid -1\}\}) \\ &\cong \mathrm{HT}(\{\{0 \mid -1\} : * \mid \{0 \mid -1\}\}). \end{aligned}$$

As we simplified the presentation we were able to recognize $\{\{0 \mid -1\} \mid \{0 \mid -1\}\}$ *as* $\{0 \mid -1\} : *$. *At this point it is easy to recognize that* $G \cong \mathrm{HT}(\{0 \mid -1\} : * : 1)$.

Lemma 3.9.9. *Let G be hereditarily transitive. Fix some G^L and let G^{LL} be one of its Left options.*

1. *If $G^{LL} \leq_R G^L$ then $G^{LL} < G^L$.*
2. *If $G^L \leq_R G^{LL}$ and G^{LL} is reversible then G^L is reversible.*

Proof.

1. As given in Lemma 3.8.3, $G^L \geq_L G^{LL}$ and so by Lemma 3.8.2, $G^{LL} \leq G^L$. As G^{LL} is an option of G^L, they are not equal.
2. Suppose G^{LL} is reversible. That is, $G^{LLR} \leq G$ for some G^{LLR}. As $G^L \leq_R G^{LL}$, there is a $G^{LR} \leq G^{LLR}$ and so $G^{LR} \leq G$.

□

The parts of Lemma 3.9.9 combine using Theorem 3.8.6 to show that if a Left option of a hereditarily transitive position is reversible then it is dominated or it is a subposition of a Left option that is also reversible.

Lemma 3.9.10. *Let G be a hereditarily transitive position with at most one primary Left option and at most one primary Right option, neither of which are reversible. If a secondary option is reversible, then it is strictly dominated.*

Proof. Suppose G^{LL} is reversible and is a Left option of the primary option G^L. By Lemma 3.8.3, $G^L \geq_L G^{LL}$. By Theorem 3.8.6, either $G^L \leq_R G^{LL}$ or $G^{LL} \leq_R G^L$. If $G^L \leq_R G^{LL}$ then by Lemma 3.9.9 G^L is reversible which contradicts our assumption. If $G^{LL} \leq_R G^L$ then by Lemma 3.9.9 $G^{LL} < G^L$ and thus is dominated. □

Lemma 3.9.11. *If G and H are hereditarily transitive, $G = H$ and H has no dominated or reversible primary options, then $G \geq_L H$.*

Proof. By Corollary 3.9.6, H has at most one primary option for each player.

As $G \geq H$, it suffices to show that for all H^L, some $G^L \geq H^L$.

For each H^L where H^L is not strictly dominated consider $G-H \xrightarrow{R} G-H^L$; if H^L is strictly dominated by some H^{L_2} then (consider it first and) whichever $G^L \geq H^{L_2}$ is also at least H^L. We know $G - H^L \,|\!\!\!\!\triangleright\, 0$ because Right loses going first in $G - H$. We consider Left's responses to $G - H^{LR}$: if $G - H^{LR} \geq 0$ then $H \geq H^{LR}$ and thus H^{LR} is reversible: by Lemma 3.9.10 we have contradicted our assumption that H^L was not strictly dominated. Therefore, Left must have a response to some G^L such that $G^L - H^L \geq 0$. □

Lemma 3.9.12. *If $G \geq H$ and H has no dominated or reversible primary options, then $G \geq_L H$.*

Proof. If $H \geq G$ then $G = H$ and the result holds by Lemma 3.9.11; we may assume $H \neq G$. Now suppose by way of contradiction that $G \not\geq_L H$; by Theorem 3.8.6 $H \geq_L G$. But $H \not\geq G$, so there exists $H^L \geq G$; this contradicts $G \geq H$. □

Corollary 3.9.13. *If $G \leq H$ and H has no dominated or reversible primary options, then $G \leq_R H$.*

Lemma 3.9.14. *If $G^L \geq_L H^L$ and $H^R \leq_R G^R$ then $G \geq H$.*

Proof. We need to show Right loses going first in $G-H$. Right loses moving to G^R-H as $H^R \leq_R G^R$ implies the existence of some $H^R \leq G^R$ which gives $G^R - H^R \geq 0$. Right loses moving to $G - H^L$ as $G^L \geq_L H^L$ implies the existence of some $G^L \geq H^L$ which gives $G^L - H^L \geq 0$. □

An equivalent result is:

Corollary 3.9.15. *If $G^R \leq_R H^R$ and $H^L \geq_L G^L$ then $G \leq H$.*

Lemma 3.9.16. *If $G = \mathrm{T}(\left\{G^L \;\middle|\; G^R\right\})$ and $H = \mathrm{T}(\left\{H^L \;\middle|\; H^R\right\})$ where $G^L = H^L$ and $G^R = H^R$ and each of the positions G^L, G^R, H^L, and H^R are transitive, then $G = H$.*

Proof. Lemma 3.9.11 gives us transitive domination from equality which is use with Lemma 3.9.14 and Corollary 3.9.15 to give $G \geq H$ and $G \leq H$, respectively. Together, $G = H$. □

3.10 Stratified positions

Definition 3.10.1. *A hereditarily transitive position is* stratified *if no primary option is dominated or reversible, and all subpositions are stratified.*

The main result of this Section is Theorem 3.10.7, which shows that every hereditarily transitive position is equal to a stratified position. We also show that equal stratified positions are congruent and how to find the stratified position equal to a given value if it exists.

Example 3.10.2. *The position* $\{0, *, * : 1 \mid 0, -1\}$ *is stratified (and hereditarily transitive).*

The position $\{0, *, 1 \mid 0\}$ *is hereditarily transitive but not stratified (*$*$ *and* 1 *are both primary Left options); this position is equal to the stratified position* $\{0, 1 \mid 0\}$ *(obtained by removing the dominated* $*$*).*

The position $\{0, * \mid \cdot\}$ *is hereditarily transitive but not stratified as the primary option to* $*$ *is reversible;* $\{0, * \mid \cdot\} = \{0 \mid \cdot\} \cong 1$*, which is stratified.*

Example 3.10.3. *The position* $2 \cong \{1 \mid \cdot\}$ *has hereditarily transitive closure* $\{0, 1 \mid \cdot\}$*, which is stratified and equals* 2*;* 2 *equals a hereditarily transitive position, but is not hereditarily transitive.*

Theorem 3.10.4. *If* G *is stratified and* K *is the canonical form of* G*, then* $G \cong \mathrm{HT}(K)$*.*

Proof. Form a new position, say H, from G first by eliminating dominated options. By Definition 3.10.1 no primary options of G are reversible and thus the primary options of H are not reversible; by Lemma 3.9.10 no secondary options are reversible because they would have been dominated and we have already eliminated dominated options. That is, no remaining options are reversible.

We now form K by replacing the options of H with their canonical forms; this does not introduce any domination because we have not changed any values and this does not introduce reversibility because by induction we are only removing dominated options.

As G is stratified, by induction all the subpositions of G are literally the hereditarily transitive-closure of their canonical forms, so the primary Left and Right options of G are restored in $\mathrm{HT}(K^{\mathbf{L}})$ and $\mathrm{HT}(K^{\mathbf{R}})$ and the secondary options of G that were removed because they were dominated are restored by taking the transitive closure. $\square$

Theorem 3.10.5. *If G and H are stratified and $G = H$ then $G \cong H$.*

Proof. If $G = H$ then their value is the same, say K, and thus by Theorem 3.10.4, $G \cong \mathrm{HT}(K) \cong H$. $\square$

Lemma 3.10.6. *If x is a number in canonical form, then* $\mathrm{HT}(x)$ *is stratified and* $x = \mathrm{HT}(x)$*.*

Proof. By Proposition 3.7.5, $\mathrm{HT}(x)$ is hereditarily transitive. The only primary options are the options of x, and those options are not reversible. The subpositions are stratified by induction.

The incentives of options of numbers are negative, thus all non-primary options of $\mathrm{HT}(x)$ are dominated thus $x = \mathrm{HT}(x)$. $\square$

By Corollary 3.9.6 if G is stratified it has at most one primary Left option and at most one primary Right option. The coming Lemmas work towards showing that any hereditarily transitive position is equal to a hereditarily transitive position with at most one primary option for each player before showing that any hereditarily transitive position is equal to a stratified position.

Theorem 3.10.7. *If G is hereditarily transitive, then G is equal to a stratified position.*

Proof. If G is a number, this is Lemma 3.10.6.

Theorem 3.9.5 gives a hereditarily transitive position $H = G$ with one primary Left option and one primary Right option, both are hereditarily transitive and neither are reversible.

If the primary Left option of H is H^L and the primary Right option of H is H^R, then by Proposition 3.7.3, $H \cong \mathrm{T}(\left\{H^L \,\middle|\, H^R\right\})$ which is hereditarily transitive.

That is, $G = \mathrm{T}(\left\{H^L \,\middle|\, H^R\right\})$ for some hereditarily transitive H^L and H^R.

By induction, H^L and H^R are equal to stratified hereditarily transitive positions, say K^L and K^R. By Lemma 3.9.16, $G = \mathrm{T}(\left\{K^L \,\middle|\, K^R\right\})$.

If K^L or K^R is reversible, using Lemma 3.9.4 we see that we get another hereditarily transitive position equal to G for which all options are stratified. However, there may now be multiple primary options for each player and thus we may use Lemma 3.9.3 to give a stratified position. □

3.11 HT-values

Suppose we are given a position, say G, that is not hereditarily transitive but we wish to determine if there is some hereditarily transitive position equal to G. We can imagine that if K is the canonical form of G, $K \neq \mathrm{HT}(K)$ but that for some $H = G$ we have $H = \mathrm{HT}(H)$; how would we find such an H? Fortunately, we won't have to. Theorem 3.11.1 gives us a test to determine if a position is equal to a hereditarily transitive position:

Theorem 3.11.1. *Let G be a position and K its value; G is equal to a hereditarily transitive position if and only if $K = \mathrm{HT}(K)$.*

Proof. If $K = \mathrm{HT}(K)$ then clearly G is equal to a hereditarily transitive position.

Suppose $K \neq \mathrm{HT}(K)$ but $G = \mathrm{HT}(H)$. By Theorem 3.10.7, there is a stratified position $S = \mathrm{HT}(H) = G$ so K is the canonical form of S. By Theorem 3.10.4, $S \cong \mathrm{HT}(K)$ and so $K = \mathrm{HT}(K)$. That is, if $K \neq \mathrm{HT}(K)$, G is not equal to any hereditarily transitive position. □

We call a value that is equal to a hereditarily transitive position an HT-value. By Theorem 3.11.1, K is an HT-value if and only if $K = \mathrm{HT}(K)$.

Let $\mathcal{HT}_n$ denote the HT-values born by day n. In Table 3.1 we give the number of HT-values born by day n for $n \leq 5$; the number was previously given in [34] for $n \leq 3$.

Example 3.11.2.

$$\mathcal{HT}_0 = \{0\}$$

$$\mathcal{HT}_1 = \{0, *, 1, -1\}$$

$$\mathcal{HT}_2 = \Big\{ -2, -1, -1*, -\frac{1}{2}, 0, \{0 \mid -1\}, * : -1, *, \{0, * \mid -1\}, *^2, * : 1, \frac{1}{2}, 1, \pm 1, \{1 \mid 0, *\}, \{1 \mid 0\}, 1*, 2\Big\}$$

n	$\lvert\mathcal{HT}_n\rvert$
0	1
1	4
2	18
3	176
4	11363
5	397444242

Table 3.1: The number of HT-values born by day $n \leq 5$

By Corollary 3.4.5, the number of infinitesimal values born by day n is the number of $*$-based values plus 1 (for 0). By Corollaries 2.3.14, 2.4.8, and 3.2.3, the number of $*$-based values born on day $n+1$ is the number of HT-values born on day n.

3.12 Partial order of hereditarily transitive positions

Now we consider the partially order set (poset) of HT-values. The investigation into the structure of this poset is motivated by [8] where it is shown that the games born by day n form a distributive lattice and [31] where the partial order structure is used to count the number of dicotic games born by day 4. More recently, [2] and [9] have generated lattices from more general sets of positions.

We denote by $(\mathcal{HT}_n, <)$ the poset of HT-values born by day n. We will show that $(\mathcal{HT}_n, <)$ is a lattice and then in Theorem 3.12.9 that it is a planar lattice.

The poset $(\mathcal{HT}_n, <)$ was investigated in [34, Section 4.3, p. 158], where Theorem 3.12.9 is presented and argued, but not successfully[1].

For hereditarily transitive positions G and H we write $G >_L H$ and $G <_R H$ for $G \geq_L H$ but $G \neq H$ and $G \leq_R H$ but $G \neq H$, respectively. Theorem 3.8.6 shows that $G >_L H$ and $G <_R H$ are linear orders.

Theorem 3.12.1. *For hereditarily transitive positions G and H,*

$$G > H \iff G >_L H \text{ and } H <_R G.$$

Proof. ($\Rightarrow$) By Lemma 3.9.12 and Corollary 3.9.13, $G \geq_L H$ and $H \leq_R G$ but as $G > H$, $G >_L H$ and $H <_R G$.

[1] The approach to the proof fails to address to differing preferences of Left and Right. This is best seen in Definition 4.3.16, but a typo obscures the problem. The example following Definition 4.3.16 reveals a typo in the definition and we see that $\mathcal{B}_{n-1}$ is intended to be defined by $B_i = -A_{p-i}$. For small n Right's preference is the reverse of Left's but this does not continue.

($\Leftarrow$) As $G >_L H$ and $H <_R G$, $G \geq_L H$ and $H \leq_R G$ and $G \neq H$. By Lemma 3.8.2 $G \geq H$ and because $G \neq H$, we get $G > H$. $\square$

The forward direction of Theorem 3.12.1 shows that $>_L$ and $<_R$ are *linear extensions* of $>$. The backward direction shows that $>$ is *realized* by $>_L$ and $<_R$.

Definition 3.12.2. *[15, p. 601, Section 2] The dimension of a poset $P = (S, <)$ is the smallest number m such that P is realized by m linear orders on S.*

Lemma 3.12.3. *The poset of HT-values born by day n, $(\mathcal{HT}_n, <)$, has dimension exactly 2 for $n \geq 1$.*

Proof. Theorem 3.12.1 shows the poset $(\mathcal{HT}_n, <)$ is realized by the two linear extensions $>_L$ and $<_R$ [15, p. 600, Section 2]. Thus, the dimension is at most 2.

For $n \geq 1$, $\mathcal{HT}_n$ contains 0 and $*$; $\mathcal{HT}_1$ contains 0 and $*$ (see Example 3.11.2) and for $n \geq 0$, $\mathcal{HT}_n \subset \mathcal{HT}_{n+1}$. The positions 0 and $*$ are confused, so $(\mathcal{HT}_n, <)$ is not linear ordered for $n \geq 1$. Thus, the dimension of $(\mathcal{HT}_n, <)$ is exactly 2 for $n \geq 1$. $\square$

To show that $(\mathcal{HT}_n, <)$ is a lattice we define operations in Definition 3.12.4 that we soon show are the least upper bound and greatest lower bound. To accommodate ends, we extend the definition of $\geq_L$ and $\leq_R$ such that for all positions, G, $G \geq_L \cdot$ and $G \leq_R \cdot$ in Definition 3.12.4.

Definition 3.12.4. *Let G and H be stratified positions, and G^L, G^R, H^L, and H^R be the primary options of G and H as indicated, respectively, or $\cdot$ as appropriate, and let $K \cong \left\{K^L \,\middle|\, K^R\right\}$.*

$G \vee H \cong \mathrm{HT}(K)$ where

$$K^L = \begin{cases} G^L \text{ if } G^L \geq_L H^L \\ H^L \text{ if } H^L \geq_L G^L \end{cases} \qquad K^R = \begin{cases} G^R \text{ if } H^R \leq_R G^R \\ H^R \text{ if } G^R \leq_R H^R \end{cases}$$

and

$G \wedge H \cong \mathrm{HT}(K)$ where

$$K^L = \begin{cases} G^L \text{ if } H^L \geq_L G^L \\ H^L \text{ if } G^L \geq_L H^L \end{cases} \qquad K^R = \begin{cases} G^R \text{ if } G^R \leq_R H^R \\ H^R \text{ if } H^R \leq_R G^R. \end{cases}$$

For non-stratified positions, these operations are defined using the stratified position.

Lemma 3.12.5 shows that $G \vee H$ is an upper bound for G and H, and Lemma 3.12.6 shows that $G \vee H$ is the least upper bound.

Lemma 3.12.5. *For hereditarily transitive positions G and H,*

(i) $G \vee H \geq G$

(ii) $G \vee H \geq H$.

Proof. Let K be as in Definition 3.12.4. For (i) we play the difference $G \vee H - G$. Right has two sorts of options: Right loses moving to some $G \vee H - G^L$, because $K \geq_L G$; Right loses moving to some $K^R - G$ because $G^R \leq_R K^R$. The proof for (ii) is similar. □

Lemma 3.12.6. *For hereditarily transitive positions G, H and K, if $K \geq G$ and $K \geq H$ then $K \geq G \vee H$.*

Proof. We play the difference $K - G \vee H$. Right has two sorts of options: Right loses moving to some $K - G^L$ or $K - H^L$, because $K \geq_L G$ and $K \geq_L H$ by Lemma 3.9.12; Right loses moving to some $K^R - G \vee H$ because by Corollary 3.9.13 $G \leq_R K^R$. □

Results similar to Lemmas 3.12.5 and 3.12.6 show that $G \wedge H$ is a greatest lower bound for G and H.

Lemma 3.12.7. *The poset HT-values born by day n, $(\mathcal{HT}_n, <)$, is a lattice with join $\vee$ and meet $\wedge$ as given in Definition 3.12.4.*

Proof. This follows from Lemmas 3.12.5 and 3.12.6. □

Fact 3.12.8. *[5, p. 18] A finite lattice has dimension at most* 2 *if and only if it has a planar diagram.*

Theorem 3.12.9. *The lattice $(\mathcal{HT}_n, <)$ of HT-values born by day n is planar for $n \geq 0$.*

Proof. By Lemma 3.12.3, the poset $(\mathcal{HT}_n, <)$ of HT-values born by day n has dimension at most 2. By Lemma 3.12.7, $(\mathcal{HT}_n, <)$ is a lattice and thus the result thus follows from Fact 3.12.8. □

We have proved a strong result about the partial order of HT-values. A good next step would be to determine a formula for the number of HT-values born by day n or at least extend Table 3.1. Earlier in this chapter we described the structure of an individual hereditarily transitive position. In Chapter 4 we consider sums of positions whose values are HT-values — this addresses the group structure of HT-values.

Chapter 4

Stalks

In this chapter we consider disjunctive sums of *stalks* and some $*$-based positions in general. We introduce stalks and some families of stalks. We don't know all about sums of stalks. We know the canonical form when we have one stalk. As we consider sums of stalks there is a trade-off between how general the family is and what we can say. In Theorem 4.8.3 we consider only $*$-based integers but we can describe the canonical form. In Theorem 4.6.6 we consider nimber-based dyadic rationals and determine the outcome. In Section 4.7 we show that we can recognize positions (values) as disjunctive sums of stalks even for families where we do not know the outcomes.

4.1 Introduction

A STALK is a HACKENBUSH path with exactly one ground vertex where that vertex has degree at most one. The simplest STALKS are shown in Figure 4.1 and each is in canonical form. Other STALKS are literally ordinal sums of simpler STALKS. STALK values are ubiquitous; the most common are discussed in Section 4.2.

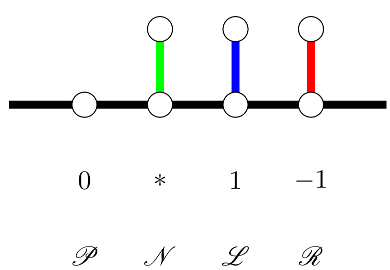


Figure 4.1: Values and outcomes of stalks with at most one edge

Proposition 4.1.1. STALKS *are hereditarily transitive.*

Proof. As seen in Proposition 3.1.5, the positions 1, -1, and $*$ are hereditarily transitive. By Theorem 3.2.2, all STALKS are hereditarily transitive. □

Moreover, STALKS are stratified: the primary option for a player if it exists is to chop the highest allowed.

Not every hereditarily transitive position is a STALK; for example, $\{0, 1 \mid -1, 0\}$. However, all the positions seen in Section 3.5 (those that are in canonical form and hereditarily transitive) are STALKS: either very short BLUE-RED STALKS or GENERALIZED FLOWERS. In particular, GREEN STALKS are literally nimbers. Most STALKS are not in canonical form. Unfortunately, the canonical form of a STALK in general does not have a tidy description.

In a STALK, every subposition is an option for some player. In particular, in playing a STALK, say G, any Left option after any Right move is a Left option of G and vice versa.

Proposition 4.1.2. *No option of a* STALK *is reversible.*

Proof. Let G be a STALK and suppose without loss of generality that $G^{RL} > G$. As G^{RL} is also a Left option of G, this contradicts Fact 1.4.9. □

Fact 4.1.3. *A* BLUE-RED HACKENBUSH *position is a number, and for every number x there is a* BLUE-RED HACKENBUSH *position G, such that $G = x$.*

Proof. That a BLUE-RED HACKENBUSH position is a number is shown by Corollary 2.3.10 and Proposition 4.1.2. The construction of a BLUE-RED HACKENBUSH position G, such that $G = x$ for every number x is found at [6, p. 77]. □

We will (in Definition 4.3.1) define a *stalk*, which is an intermediate form between a STALK and its canonical form. Many stalks are in canonical form.

4.2 Families of Stalks

In our analysis of STALKS we consider many families, shown in Table 4.1. However, not every STALK is in one of these particular families.

Numbers

A BLUE-RED STALK is a STALK with no green edges. A BLUE-RED STALK is a number by Fact 4.1.3. Some integer BLUE-RED STALKS are shown in Figure 4.2.

form	type	ruleset
n	integer	BLUE STALK or RED STALK
x	number	BLUE-RED STALK
$*^m$	nimber	GREEN STALK
$* : x$	$*$-based number	SPRIG
$*^m : n$	$*$-based integer	FLOWER
$*^m : x$	nimber-based number	GENERALIZED FLOWER

Table 4.1: Families of STALKS

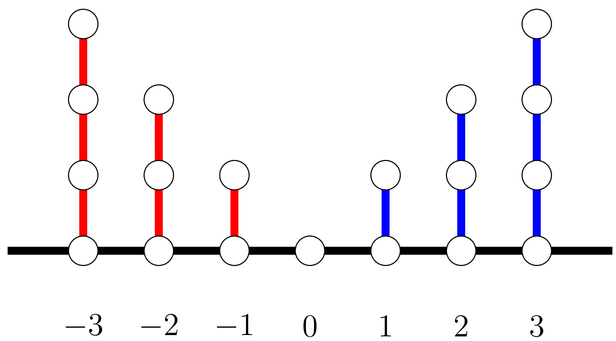


Figure 4.2: Non-green monochrome STALKS are integers

Some non-integer BLUE-RED STALKS are shown in Figure 4.3. For integers and non-integers in Figures 4.2 and 4.3, larger numbers are found by increasing the size of the (in these cases) blue path touching the ground; the negatives of these numbers are found by swapping red and blue.

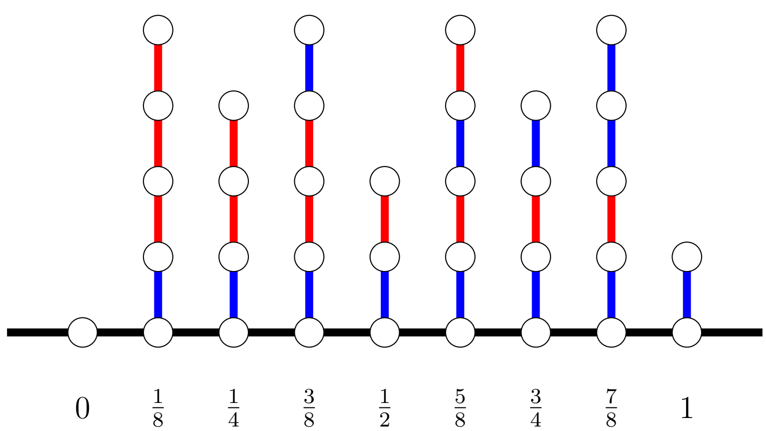


Figure 4.3: BLUE-RED STALKS are dyadic rationals

We find non-integers with larger denominators as the number of edges (excluding those in the monochrome edges nearest the ground) increases.

Nimbers

A GREEN STALK is a STALK with all edges green. GREEN STALKS are literally nimbers; some GREEN STALKS are shown in Figure 4.4.

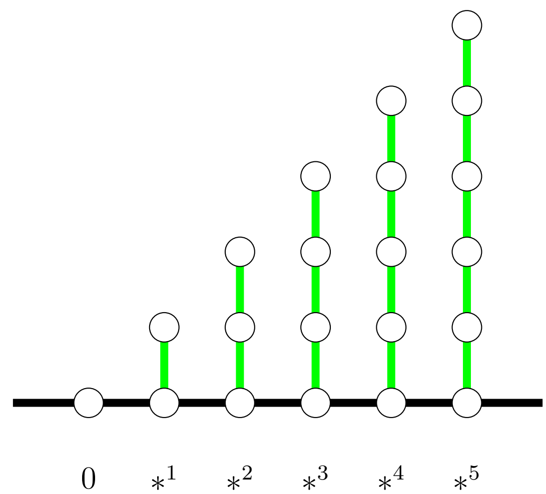


Figure 4.4: GREEN STALKS are nimbers

Sprigs

A SPRIG is a STALK with a green ground edge and all additional edges are not green (i.e. red or blue). A SPRIG is a $*$-based number. A SPRIGS position is shown in Figure 4.5.

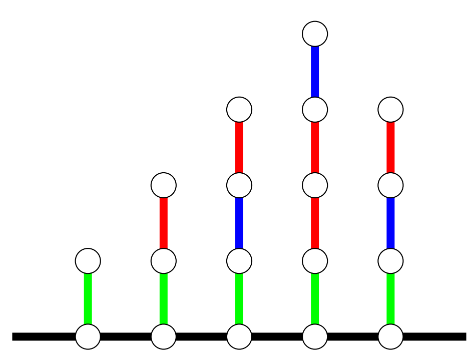


Figure 4.5: A SPRIGS position

Flowers

A FLOWER is a STALK with a green ground edge, possibly some more green edges and then all additional edges are red or all additional edges are blue. A GREEN STALK is also a FLOWER, and thus the set of FLOWERS positions is hereditarily closed. A FLOWER is a nimber-based integer.

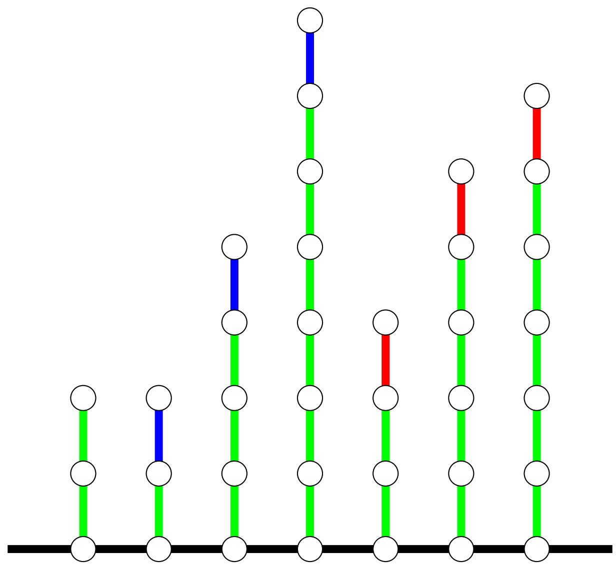


Figure 4.6: A FLOWERS position

There is no known good method for determining the outcome of a FLOWERS position. Furthermore, there is no known polynomial-time algorithm [32, p. 45] for determining the outcome of a sum of positions of the form $*^k : 1$ or $*^k : -1$, such as seen in Figure 4.6.

As the reader may observe, we have no rules to determine outcomes for general sums of stalks with arbitrary nimber-bases; to do so would imply a solution to FLOWERS. For more on the attempt to solve FLOWERS see [7].

Generalized Flowers

A GENERALIZED FLOWER is a STALK with a green ground edge, possibly some more green edges and then all additional edges are not green (i.e. red or blue). A GENERALIZED FLOWER is a nimber-based number.

GENERALIZED FLOWERS encompass GREEN STALKS, FLOWERS, and SPRIGS (all the rulesets of Table 4.1 except BLUE-RED STALKS).

4.3 A Simpler Stalk

We have seen from Propositions 4.1.1 and 4.1.2 that understanding the form of STALKS is useful. However, we are interested in outcomes of STALKS positions. It makes sense to consider these positions outside of the HACKENBUSH context.

Definition 4.3.1. *We define* stalks *recursively: the position S is a* stalk *if*

$$S \cong \begin{cases} *^k & \text{for some } k \geq 0 \\ x : D & \text{for some non-zero dyadic } x \text{ and dicotic stalk } D \\ *^k : x : D & \text{for some } k > 0, \text{ non-zero dyadic } x, \text{ and dicotic stalk } D. \end{cases}$$

A stalk, say D, is dicotic if and only if it is a nimber or is of the form $*^k : x : D'$ for a dicotic stalk D'.

We call the non-zero nimber and non-zero number ordinal summands of a stalk *segments*. The nimber and number segments of a stalk alternate. Every non-zero stalk also has a *highest non-zero segment* (which is either a number or a nimber).

Stalks are simpler than STALKS, as we have removed all but one option for each player from each number segment. In particular, nimber-based dyadic rationals are in canonical form.

Proposition 4.3.2. *Every* STALK *is equal to a stalk and vice versa.*

Proof. The red and blue edges by the ground (before any green edge) in a STALK can be replaced by a dyadic rational by Fact 4.1.3 and Lemma 2.3.9. Then by induction and The Colon Principle (2.2.4) the STALK from the first green edge on up can be replaced by a dicotic stalk. The same process works in reverse. □

Corollary 4.3.3. *The position S is a non-zero stalk if and only if for some stalk T,*

$$S = \begin{cases} 1 : T \\ * : T \\ -1 : T. \end{cases}$$

Lemma 4.3.4. *If K is a stalk, then $K = x : H$ where x is a number (possibly 0) and H is a dicotic stalk (possibly 0).*

If K is not a number then $K = x + * : H'$ for some stalk H'.

Proof. Let x be the number-base of K. The x-based-subordinate of K, say H, is a stalk. If H is not zero and not $*$-based, then H is 1-based or (-1)-based which contradicts x being the least-simple number-base of K.

If K is not a number then H is not zero and thus is $*$-based: $H = * : H'$ for some H'. By combining this with Theorem 2.4.2 we get $K = x : H = x + H = x + * : H'$. □

In practice, to identify a position as a stalk, we use the following:

Proposition 4.3.5. *If G is a stalk then:*

- *the reduced value of G, $\mathrm{rcf}(G)$, is a number; and*
- *the infinitesimal part of G, $G - \mathrm{rcf}(G)$, is a stalk.*

Proof. If G is dicotic then the reduced value is 0, a number; the result holds as G is the infinitesimal part.

Otherwise, $G \cong x : D$ for some non-zero dyadic x and dicotic stalk D. By Theorem 2.4.2, $G = x + D$. Thus, the reduced canonical form of G is a number, x, and the infinitesimal part, D, is a stalk. □

Theorem 4.3.6. *A sum of stalks is a number plus a sum of $*$-based stalks.*

Proof. By Lemma 4.3.4, each stalk is the sum of a number and its infinitesimal part (a $*$-based stalk or 0). As numbers are closed under disjunctive sum (Fact 1.9.5), our sum equals one number plus a sum of $*$-based stalks. □

Corollary 4.3.7. *Let G be a sum of stalks and x be the sum of the number-bases of those stalks.*

If $x \neq 0$, $o(G) = o(x)$. If $x = 0$, $o(G)$ is the outcome of the sum of $*$-based infinitesimal parts.

Proof. This is Theorem 4.3.6 combined with the observation that $*$-based stalks are infinitesimal, which they are by Theorem 1.13.4 because they are dicotic. □

Lemma 4.3.8. *A $*$-based stalk has atomic weight 1 if it is greater than its nimber-base, atomic weight 0 if it is a nimber, or atomic weight -1 if it is less than its nimber-base.*

Proof. By Theorem 2.4.12: for $k > 0$,

$$\operatorname{aw}(S) = \begin{cases} 0 & \text{if } S = *^k \\ 1 & \text{if } S = *^k : x : D \text{ for any positive number } x, \text{ and dicotic stalk } D. \\ -1 & \text{if } S = *^k : x : D \text{ for any negative number } x, \text{ and dicotic stalk } D. \end{cases}$$

□

Both nimbers and numbers form subgroups (Facts 1.9.5 and 1.10.12) but stalks do not:

Proposition 4.3.9. *The values of stalks are not closed under disjunctive sum.*

Proof. By Lemma 4.3.8 the atomic weight of a stalk is 1, 0, or −1. In particular the stalk $* : 1$ has atomic weight 1 and thus the sum of stalks $* : 1 + * : 1 = \Uparrow$ has atomic weight 2 and is thus not a stalk. □

The atomic weight theory considerably reduces the complexity of our arguments in cases where the outcome of a sum of three of more stalks is "clear".

In a sum of dicotic stalks, a summand with atomic weight 1 implies an incentive for Right of atomic weight 1; a summand with atomic weight −1 implies an incentive for Left of atomic weight 1. When a player exercises such an option (by moving to a nimber) we say a player moves the atomic weight in their favour. A move in a summand to 0 (but where the sum has not necessarily become 0) is called a *kill.* We associate summands of atomic weight 1 with Left and summands with atomic weight −1 with Right. A common move in a sum of stalks is to kill an opponent's stalk.

Lemma 4.3.10 in particular concerns the case where a stalk is ★ in disguise thus allowing us to use the atomic weight theory immediately.

Lemma 4.3.10. *Let* $S = \sum_{i \in I} *^{n_i} : s_i$ *be a sum of* $*$*-based stalks where no* s_i *is* $*$*-based. If for some summand, say* $*^{n_1} : s_1$, $\operatorname{lb}(n_1) > \operatorname{lb}(n_i)$ *for all* $i \neq 1$ *then:*

$$S \begin{cases} > 0 & \textit{if } \operatorname{aw}(S - *^{n_1} : s_1) > 0. \\ \parallel 0 & \textit{if } \operatorname{aw}(S - *^{n_1} : s_1) = 0. \\ < 0 & \textit{if } \operatorname{aw}(S - *^{n_1} : s_1) < 0. \end{cases}$$

Proof. If there is a summand, say $*^{n_1} : s_1$ such that $\mathrm{lb}(n_1) > \mathrm{lb}(n_i)$ for all $i \neq 1$ then $*^{n_1} : s_1$ is remote for $S - *^{n_1} : s_1$ and thus $o(S) = o(S - *^{n_1} : s_1 + \bigstar)$.

By Lemma 4.3.8 all atomic weights of stalks are integral, so if $\mathrm{aw}(S - *^{n_1} : s_1) \gtrless 0$ then $\mathrm{aw}(S - *^{n_1} : s_1) \geq 1$ or $\mathrm{aw}(S - *^{n_1} : s_1) \leq -1$. By Fact 2.2.17 we get: $o(S) = o(S - *^{n_1} : s_1 + \bigstar) = o(\mathrm{aw}(S - *^{n_1} : s_1))$.

Now suppose $\mathrm{aw}(S - *^{n_1} : s_1) = 0$. If $S - *^{n_1} : s_1$ is a nimber, $o(S) = o(S - *^{n_1} : s_1 + \bigstar) = \mathscr{N}$ because the presence of $\bigstar$ ensures $S \neq 0$; otherwise in $S - *^{n_1} : s_1$ there is at least one stalk with atomic weight 1 and at least one stalk with atomic weight -1 and both players thus have options to move the atomic weight in their favour which win by induction. □

4.4 Two Stalks

Our first result here is essentially Corollary 4.3.7 in the case of two stalks and with more notation. This result iterates well with Fact 2.4.10, which is why we especially include it.

Lemma 4.4.1. *Let*

$$S = x : S'$$

$$T = y : T'$$

be stalks where x and y are the number-bases of S and T, respectively, and S' and T' are dicots. Then

$$o(S - T) = \begin{cases} o(x - y) & \text{if } x \neq y \\ o(S' - T') & \text{if } x = y. \end{cases}$$

Proof. By Corollary 4.3.7, a numberish position has the same outcome as its reduced value unless the reduced value is 0.

The reduced value of $S - T$ is $x - y$. If $x - y \neq 0$, $o(S - T) = o(x - y)$.

If $x = y$ then $S = x : S'$ and $T = x : T'$. The result, $o(S - T) = o(S' - T')$, thus follows from The Colon Principle and Theorem 2.4.2. □

Theorem 4.4.2. *Let G and H be stalks; $G \parallel H$ if and only if for some $*$-based stalk K either $G = H : K$ or $H = G : K$.*

Proof. Let $K = * : K'$ and suppose without loss of generality that $H = G : K$, then $G = G : 0$ and $H = G : K$ are confused by Corollary 2.2.21 as $K \parallel 0$.

By Lemma 4.4.1, if $G \parallel H$ then the number-bases of G and H are equal. We give the proof for the case where both G and H have number-base 0 (i.e. G and H are dicots), the general situation is similar (but requires Theorem 2.4.2). Let $G = *^n : G'$ and $H = *^n : H'$ for some maximal n; at most one of G' and H' is $*$-based. Furthermore, $G' \parallel H'$ by Fact 2.4.10.

By induction, $G' = H' : K$ or $H' = G' : K$ giving $G = *^n : H' : K$ or $H = *^n : G : K$, where K is some $*$-based stalk. □

Lemma 4.4.1, Fact 2.4.10 and Theorem 4.4.2 work together and apply to STALKS as follows:

Corollary 4.4.3. *Suppose the* STALKS s_1 *and* s_2 *differ for some edge from the ground up. If* s_1 *has a blue edge where* s_2 *has red, green, or no edge, then* $s_1 > s_2$*; if* s_1 *has a red edge where* s_2 *has blue, green or no edge, then* $s_1 < s_2$*. Otherwise one* STALK *ends where the other has a green edge and in this case the* STALKS *are confused.*

Corollary 4.4.4. *Let* S *be a stalk with nimber-base* $*^n$*. If* S *is not a nimber, then* S *is confused with* $*^m$ *if and only if* $n \geq m + 1$*.*

Proof. Suppose S is confused with $*^m$. By Theorem, 4.4.2, S must have $*^m$ as a base (not the other way around because S is not a nimber) and have a dicotic subordinate, thus $*^{(m+1)}$ is also a base of S.

If $n \geq m + 1$, $*^m$ is both a Left option of s and a Right option of s, thus s is confused with $*^m$. □

We use Corollary 4.4.4 frequently. Winning moves from a sum of two stalks and a nimber are often to the sum of one stalk and a nimber. Corollary 4.4.4 shows that at times a player wants to make a summand to be a sufficiently large nimber.

4.4.1 Stalk Incentives

Recall that an incentive (Definition 1.8.1) of a position is a difference of a position and an option of that position. We can apply what we know about sums of two stalks to incentives.

Lemma 4.4.5. *A stalk incentive corresponding to an option in a number segment is negative. Moreover, any option in a number segment is dominated by every option in a nimber segment above it.*

Proof. Let $S = T : x : D$ be a stalk where x is a dyadic rational that is not a negative integer; T is 0 or its highest segment is a nimber; and D is dicotic.

Consider $S^L = T : x'$. As every incentive of x is negative then by Corollary 2.2.21 the incentive for $x : D \to x'$ is negative and $\Delta = T : x' - S = T : x' - T : x : D$ is negative.

If D is non-empty there is a nimber segment above x, let $R = T : x : D'$ be an option of S that corresponds to an option in a nimber segment higher than x. By the first part, $T : x'$ is an option of R with a negative incentive, thus R dominates $T : x'$. □

Corollary 4.4.6 also follows from Lemma 4.4.5 and also Fact 1.8.2, which is proved less formally.

Corollary 4.4.6. *An incentive corresponding to hacking a red or blue edge in a* STALK *is negative. Moreover, the option to hack a non-green edge below a green edge is dominated.*

Theorem 4.4.7. *If S is a stalk then $S : * : G = S + * : H$ for some H.*

Proof. Consider the difference $S : * : G - S$. Either player may move to 0 (i.e. $S - S$) by moving in the $*$. By Lemma 4.4.5, every move in a number segment of S in either summand is dominated. Every move in a nimber of S, say to S' is $*$-based by induction as $S = S' : * : T$ for some T. Every move in G is $*$-based by induction. Thus $S : * : G - S$ is $*$-based by Theorem 2.4.6. □

Example 4.4.8. *The position $G = * : 1 : *$ has three Left incentives, two of which are $*$-based: $-G$ and $* : 1 - G = * : \{1 \mid -1*\}$.*

Corollary 4.4.9. *An incentive corresponding to an option in a nimber segment in a stalk is confused with 0. Furthermore, any two such distinct incentives are confused.*

Proof. Let $S = T : *^k : S'$ be a stalk and consider $T : *^{k'}$ for some $k' < k$ which is an option for both players. By Theorem 4.4.7, $T : *^k : S' - T : *^{k'} = * : H$ for some H. That is, the incentive is confused with 0 as a Left or Right option.

If R is also an option of S corresponding to an option in a nimber segment, then either R is an option of T or vice versa; thus R and T are confused. □

We also get the HACKENBUSH version:

Corollary 4.4.10. *An incentive corresponding to hacking a green edge in a* STALK *is confused with* 0*. Furthermore, any two such distinct incentives are confused.*

Finally, Theorem 4.4.11 describes the canonical form of a stalk.

Theorem 4.4.11. *Every option in a nimber segment of a stalk corresponds to both a Left option and a Right option in the canonical form of the stalk. If the highest segment of the stalk is a number segment then each player with an option in that number segment has an additional corresponding option in the canonical form.*

Proof. By Proposition 4.1.2 there are no reversible options in a STALK and the options of a stalk are a subset of the options of the corresponding (and equal) STALK, so there are no reversible options in a stalk.

The rest of the proof is thus a clarification of which of the remaining options are dominated. Lemma 4.4.5 shows that any option in any number segment that may not be dominated is in the highest segment. Corollary 4.4.9 clarifies the rest: options in nimber segments are confused with options in higher number segments; and every option in a nimber segment is confused with every other option in a nimber segment. □

4.5 Three Stalks

Of course, we are naturally interested in the outcome of a sum of three stalks, having succeeded with two stalks. However, three stalks are also important because of incentives. An incentive of a kill is 1 stalk and other incentives are a difference (i.e. sum) of 2 stalks. So to compare incentives of different types, we must examine sums of 3 stalks.

Theorem 4.4.7 shows that hacking a green edge in a STALK gives a $*$-based incentive, but that incentive is not necessarily a stalk. If such an incentive is a stalk, then it is either explained by Theorem 2.4.2 or there will be a sum of three dicotic stalks equal to 0, which tells us about the group structure of stalks.

Play in a sum of three stalks eventually reaches a sum of two stalks and a nimber (possibly 0). To know about a sum of three stalks, we first need to know all about outcomes of two stalks and a nimber, which is what we discuss in this Section. This allows us to compare incentives of kills and incentives of other options in nimber-bases.

The most interesting result of this section is Theorem 4.5.11, which shows that two non-nimber stalks may sum to a nimber.

Theorem 4.5.1. *If G, H, and K are stalks and $*:G + *:H + *:K = 0$, then one (or three) of G, H, and K is a nimber.*

Proof. By Lemma 4.3.8, the parity of atomic weight of a non-nimber stalk is odd. The parity of the atomic weight of the right-hand side is even. That is, there must be an odd number of nimbers on the left-hand side. □

A closely related result is Lemma 4.5.2.

Lemma 4.5.2. *Let G be a sum of $*$-based positions. If $G \in \mathcal{P}$ then every position with one less summand is in $\mathcal{N}$.*

Proof. Every option resulting from playing in a $*$-base is available to both players. Thus, if such an option of G is in $\mathcal{P}$, $\mathcal{L}$, or $\mathcal{R}$ then $G \notin \mathcal{P}$. □

For the upcoming results, We remind the reader of the Definition 1.10.13.

Lemma 4.5.3. *If n is a positive integer and k is a non-negative integer, $\nu(n) > \mathrm{lb}(k)$ if and only if $n' \oplus k < n$ for all $n' < n$.*

Proof. ($\Rightarrow$) If $\nu(n) > \mathrm{lb}(k)$ then every 1-bit of n is more significant than any 1-bit of k. Any $n' < n$ lacks at least one 1-bit that n has. In particular the 1-bit 'missing' from n' is more significant than any 1-bit of k. Thus, $n' \oplus k$ is also lacking the same 1-bit.

($\Leftarrow$) Suppose that $n' \oplus k < n$ for all $n' < n$.

If $k = 0$ then $\nu(n) > \mathrm{lb}(k)$, so assume $k > 0$. If n and k share at least one 1-bit, choosing n' to be the largest such that it shares no 1-bits with k gives $n' \oplus k \geq n$; that is n and k share no 1-bits.

If n and k do not share any 1-bits, but some 1-bit of k is more significant than the least significant bit of n then n minus the smallest 1-bit of n, say j, gives $j \oplus k \geq n$, so every 1-bit of m is less significant than any 1-bit of n. □

Lemma 4.5.4. *If $\nu(n) > \mathrm{lb}(k)$ or $k = 0$ then $o(*^n : G + *^k + *^n : H) = o(G + *^k + H)$.*

Proof. If $k = 0$ then the result holds by Fact 2.4.10. By induction, those options of $*^n : G + *^k + *^n : H$ other than those in a $*^n$-base correspond to an option in $G + *^k + H$ and the corresponding option has the same outcome. Lemma 4.5.3 together with Corollary 4.4.4 show that that all the $*^n$-base options are in $\mathcal{N}$ and thus the existence of these options does not change the outcome of the sum by Lemma 1.3.6. □

Lemma 4.5.4 may look not particularly useful, but it is strong when the outcome is $\mathscr{P}$ because it then becomes a result about equality.

Corollary 4.5.5. *Let n and k be non-negative integers. If x is a number and $\nu(n) > \mathrm{lb}(k)$ then $*^n : x + *^k = *^n : x : *^k$.*

Proof. By Lemma 4.5.4, the difference $*^n : x + *^k + *^n : -x : *^k = x + *^k - x : *^k = *^k + *^k = 0$. □

It is worth noting that in Lemma 4.5.4 that G and H can be $*$-based. That is, $*^n$ is not necessarily the nimber-base of $*^n : G$.

Proposition 4.5.6. $*^n : 1 + * = \begin{cases} *^{n-1} : \uparrow & \text{if } n \text{ is odd} \\ *^n : (1 + *) & \text{if } n \text{ is even} \end{cases}$

Proof. The even cases follows easily from Corollary 4.5.5. The odd case uses Lemma 4.5.4 which using $* : 1 + * - \uparrow = 0$ implies $*^{n-1} : * : 1 + * - *^{n-1} : \uparrow = 0$ and the result follows as $*^n : 1 + * = *^{n-1} : * : 1 + *$. □

Example 4.5.7. *Consider the sum $*^5 : G + *^3 + *^5 : H$ and note that $\nu(5) < \mathrm{lb}(3)$ and thus we cannot apply Lemma 4.5.4. Despite this, we may rewrite the sum as $*^4 : * : G + *^3 + *^4 : * : H$. Now note that $\nu(4) > \mathrm{lb}(3)$. Thus, $o(*^5 : G + *^3 + *^5 : H) = o(* : G + *^3 + * : H)$ and the latter sum appears to be much simpler. Furthermore, tor some position G and H, $*^3$ is remote.*

Corollary 4.5.8. *If G is $*^n$-based and H is $*^m$-based where $\mathrm{lb}(n) > \mathrm{lb}(k)$ and $\mathrm{lb}(m) > \mathrm{lb}(k)$ then $o(G + *^k + H) = o(*^{n'} : G' + *^k + *^{m'} : H')$ where $n' = n - 2^{\mathrm{lb}(k)+1}$ and $G = *^n : G'$; $m' = m - 2^{\mathrm{lb}(k)+1}$ and $H = *^n : H'$.*

Furthermore, we can use Lemma 4.5.4 to prove the NIM-addition rule.

Fact 4.5.9. *[32, Theorem 4.7, p.85] For any a and b in $\mathbb{N}$, $*^a + *^b + *^c = 0$ if $a \oplus b \oplus c = 0$.*

Proof. If any two of a, b and c are equal then one of a, b and c is 0, and the result holds.

Without loss of generality, assume $a > b > c$. Consider the largest 2-power that is at most b, say 2^k. Then $a \geq 2^k$ and $c \not\geq 2^k$ because $a \oplus b \oplus c = 0$. By Lemma 4.5.4, $o(*^a + *^b + *^c) = o(*^{a'} + *^b + *^{c'})$ where $a' = a - 2^k$ and $b' = b - 2^k$. Note that $a' \oplus b' \oplus c = 0$. By induction, $*^{a'} + *^{b'} + *^c = 0$. □

Corollary 4.5.5 is neat, but we can use it multiple times to 'put' a nimber higher and higher in a stalk.

Example 4.5.10. *First by noting $*^2 = *^3 + *$ and then using Corollary 4.5.5 and Theorem 2.4.2 multiple times, we can get equations such as:*

$$*^{12} : \frac{3}{4} : *^8 : -2 : *^2 = *^{12} : \frac{3}{4} : *^8 : -2 : *^3 + *.$$

Finally theorem 4.5.11 answers a question suggested by Theorem 4.5.1, namely, *which stalks differ by a nimber?*

Theorem 4.5.11. *Let G and H be distinct non-nimber stalks:*

$$G = H + *^k$$

if and only if there exists a stalk J such that

- *the highest segment of J is a number,*
- *for every nimber segment $*^{n_i}$ of J, $\nu(n_i) > \mathrm{lb}(k)$,*
- *$G = J : *^g$,*
- *$H = J : *^h$, and*
- *$g \oplus h = k$.*

Proof. Suppose there exists a J such that the bullet points hold. By induction, if J is number-based, apply Lemma 4.3.4; if J is nimber-based apply Lemma 4.5.4. Eventually, $g \oplus h = k$ gives equality.

Now suppose $G = H + *^k$. By Theorem 4.4.2, either G or H is a base of the other. Without loss of generality, assume G is shorter than H; thus G is a base of H. Let J be the stalk that is the tallest base of G whose highest segment is a number; note that such a J exists because G is not a nimber. That is, $G = J : *^g$ and $H = J : *^g : * : H'$ for some stalk H'.

If J is number-based, G and H have the same number-base (the number-base of J). Using Lemma 4.3.4, the proof reduces to the case where J is a dicot. Assume J is a dicot and let $*^n$ be the nimber-base of J (and thus also G and H).

If there exists $n' < n$ such that $n' \oplus k > n$ then either player wins moving first in the difference; thus, $\nu(n) > \mathrm{lb}(k)$.

By induction using Lemma 4.5.4, the $*^n$-based subordinates of G and H also differ by $*^k$. Thus, for any nimber segment of J, say $*^{n_i}$, $\nu(n_i) > \mathrm{lb}(k)$. Induction gives that $*^g + *^g : * : H' = *^k$, implying that H' is a nimber and that $g \oplus h = k$. □

Theorem 4.5.11 implies that the outcome of a sum of two nimber-based numbers and a nimber is only in $\mathscr{P}$ if the nimber-based numbers are really nimbers. The other outcomes are given in Theorem 4.5.12.

Theorem 4.5.12. *Let*

$$G = *^n : x + *^m : y + *^a$$

where n and m are non-zero, and x and y are non-zero numbers.

If x and y have the same sign then $o(G) = o(x) = o(y)$.

Assume $x > 0$ and $y < 0$.

Assume $x + y \neq 0$ or $n \neq m$.

$$\text{If } m > n \text{ then: } o(G) = \begin{cases} \mathscr{N} & \text{if } n' \oplus a \geq m \text{ for some } n' \leq n \\ \mathscr{L} & \text{otherwise.} \end{cases}$$

$$\text{If } n = m \text{ then: } o(G) = \begin{cases} \mathscr{N} & \text{if } n' \oplus a \geq n \text{ for some } n' \leq n \\ o(x+y) & \text{otherwise.} \end{cases}$$

$$\text{If } m < n \text{ then: } o(G) = \begin{cases} \mathscr{N} & \text{if } m' \oplus a \geq n \text{ for some } m' \leq m \\ \mathscr{R} & \text{otherwise.} \end{cases}$$

Proof. If x and y are both positive, then G has atomic weight 2, thus G is positive. If x and y are both negative, then G has atomic weight -2, thus G is negative.

Consider $n = m$. If $x = -y$ then $G = *^a$.

Corollary 4.4.4 provides the outcome at the point where only one stalk is a non-nimber. If one player can win playing in a $*^n$-base, then so can the other; if not, neither can. The winner of $x+y$ can win by playing to $*^n : x + *^n : y$ and thus wins the sum; the loser of $x+y$ would lose (see Fact 1.8.3) by playing in a number subordinate, as the move would have a negative incentive.

Consider $n > m$. Right can win moving first to $*^n : x + *^m : y$ if $a > 0$, or to $*^m + *^m : y$ if $a = 0$. Left can win moving first if only if $m' \oplus a \geq n$ for some $m' \leq m$.

□

We now give a characterization of the conditions given in Theorem 4.5.12. For example, in the case where $a > n$, a has a 1-bit that n does not have which is more significant than any 1-bit of n that a does not have, so $m = 0$.

Lemma 4.5.13. *If m is the smallest non-negative integer such that $m \oplus a \geq n$ then the set of 1-bits of m is the subset of 1-bits of n that are both not in a and more significant than any 1-bit of a in $a \oplus n$.*

Proof. Suppose that m is the smallest positive integer such that $m \oplus a \geq n$; m does not share any 1-bits with a; if m and a shared bits then the integer m' formed from the bits of m not shared with m is smaller and $m' \oplus a > m \oplus a$, a contradiction.

As $a \oplus (a \oplus n) = n \geq n$, it would be sufficient for m to be formed of all the 1-bits in n but not in a. Therefore m will not have any bits not in n: a bit that is less significant than a bit in n never decides that $m \oplus a \geq n$ and a bit more significant than those in n could be replaced by bits in n thus decreasing m.

If a only has 1-bits that n also has, then no 1-bit of a is in $a \oplus n$ and it is optimal for m to have exactly the 1-bits that n has that are not also in a. That is, $m = a \oplus n$.

Otherwise, consider the largest i such that the ith bit of a is 1 but the ith bit of n is 0. In this case it is sufficient (i.e. $m \oplus a \geq n$) if m is formed from all the 1-bits of n

that are not in a and more significant than the ith bit. The 1-bits in n that are less significant than the ith bit are unnecessary to ensure $a \oplus m \geq n$. The 1-bits of n more significant than the ith bit are necessary in m because i was chosen to be largest and for each 1-bit of n a bit at least as significant must be in either m or a. □

Theorems 4.5.11 and 4.5.12 give the outcome of a sum of two stalks and a nimber when the outcome is $\mathscr{P}$ and where the stalks are nimber-based numbers. Now we consider the remaining cases.

Consider a non-zero sum $*^n : x : S + *^m : y : T + *^k$ where $x > 0$ and $y < 0$.

If $\text{lb}(k) > \max(\text{lb}(m), \text{lb}(n))$ then $o(G) = \mathscr{N}$ because $*^k$ is remote. If $\text{lb}(k) < \max(\text{lb}(m), \text{lb}(n))$ then one of $*^m$ or $*^n$ is remote and the outcome is given by Lemma 4.3.10 or we apply Corollary 4.5.8 to find a simpler sum with the same outcome. Otherwise, $\text{lb}(n) = \text{lb}(m) = \text{lb}(k)$ or if $\text{lb}(n) \neq \text{lb}(m)$ then the larger of the two is $\text{lb}(k)$.

Theorem 4.5.14. *Let*

$$G = *^n : x : s + *^m : y : t + *^k$$

where s and t are dicotic stalks; x and y are numbers such that $x > 0$, $y < 0$; and $\text{lb}(k) = \max(\text{lb}(m), \text{lb}(n))$.

If $\text{lb}(n) = \text{lb}(m) = \text{lb}(k)$ *then* $G \parallel 0$.

If $\text{lb}(m) = \text{lb}(k) > \text{lb}(n)$ *then* $G \mathrel{|\!\!\triangleright} 0$ *with* $G \parallel 0$ *if and only if there exists* $n' < n$ *such that* $n' \oplus k \geq m$.

If $\text{lb}(n) = \text{lb}(k) > \text{lb}(m)$ *then* $G \mathrel{\triangleleft\!\!|} 0$ *with* $G \parallel 0$ *if and only if there exists* $m' < m$ *such that* $m' \oplus k \geq n$.

Proof. If $\text{lb}(n) = \text{lb}(m) = \text{lb}(k)$ then either player can win by hacking somewhere in the base of the opponents' stalk. Left can win by hacking in the $*^m$-base because $\text{lb}(m) = \text{lb}(k)$ there is some $m' < m$ such that $m' \oplus k = n$. The situation for Right is similar thus $G \parallel 0$.

Suppose $\text{lb}(m) = \text{lb}(k) > \text{lb}(n)$.

Going first, $*^k \xrightarrow{L} 0$ or $*^m : y : t \xrightarrow{L} 0$ wins. It is clear that Left can win in response to any Right move other than those moves in the nimber base of $*^n : x : s$. Right can win if $n' \oplus k \geq m$ for some $n' < n$ and otherwise $G > 0$. □

4.6 Outcomes of Sums of Nimber-based Numbers

A usable rule for determining outcomes of general sums of three or more stalks will require a more advanced approach. We suggest a two-step approach based on understanding outcomes of nimber-based numbers, and then determining when a sum of stalks has the same outcome as the sum of its nimber-based-number bases. In this Section we work on the first step and consider outcomes of sums of $*^n$-based numbers where n is fixed. The second step likely depends upon an understanding of the values of sums of nimber-based numbers, which we consider in Section 4.8.

The main result of this Section is Theorem 4.6.6 in which we generalize Conway's Theorem 88 [14, p. 194–195]. In Theorem 88 we have the positions $\uparrow x$ and $\downarrow y$; it suffices to know that $\uparrow x = * : x + *$ and $\downarrow y = * : -y + *$. From our perspective either is $* : x + * : 0$ for some x; that is, each is a sum of two $*$-based numbers.

Theorem 88 [1] from [14, p. 194–195]:

> Let X be a finite sum of terms $\uparrow x$ and $\downarrow y$, in which all the numbers x and y are positive, and no number x occurs also as a y (for then we could cancel). Then X is positive if and only if either the number of $\uparrow x$ terms exceeds the number of $\downarrow y$ terms, or these numbers are equal, and the least of the numbers x and y is a y. The game $X + *$ is positive if and only if $X + \downarrow \mathbf{On}$ is positive, where **On** temporarily denotes any number bigger than all the x and y. The game $X + *n$, with $n \geq 2$, is positive only if the number of $\uparrow x$ terms exceeds the number of $\downarrow y$ terms.

Of note is the $X + *$ case, which is apparently more difficult than the others. It becomes clear when we see the pattern again in Section 4.8.1 that in a sum of nimber-based numbers there are essentially two cases: an even number of summands and an odd number of summands.

Definition 4.6.1. *If M is a finite multiset of positions and G a position, let*

$$\langle G : M \rangle = \sum_{m \in M} G : m.$$

[1] Unfortunately, there is a minor mistake in the result as it was published. The corrected version is in the thesis. The only change is from $\uparrow$ **On** to $\downarrow$ **On** in the consideration of the outcome of $X + *$.

Similar to Conway's "no number x occurs also as a y (for then we could cancel)", we have Definition 4.6.2.

Definition 4.6.2. *A multiset of positions, C, is called a* collation *if it is* inverse-pair-free*; that is, for distinct elements a and b in C, $a + b \neq 0$. A collation contains at most 1 copy of the identity. Let C^+ denote the submultiset of positive elements, C^- denote the submultiset of negative elements, and C^0 denote the submultiset of C containing the elements equal to 0.*

If C is a collation of numbers then $|C| = |C^+| + |C^0| + |C^-|$.

Lemma 4.6.3. *If C is a multiset of numbers C and $n > 0$, then $\mathrm{aw}(\langle *^n : C \rangle) = |C^+| - |C^-|$.*

Proof. This follows from Lemma 4.3.8 and the additivity of atomic weight. $\square$

Corollary 4.6.4. *Let C be a multiset of numbers C and $n > 0$. If $\Big| |C^+| - |C^-| \Big| \geq 2$ then $o(\langle *^n : C \rangle) = o\left(\Big| |C^+| - |C^-| \Big|\right)$.*

Proof. By Fact 2.2.17 and Lemma 4.6.3. $\square$

A typical move in a sum of nimber-based numbers is to improve the atomic weight by playing in a nimber-base where the subordinate's sign is that of your opponent. When all the nimber-bases are the same it is clear that players will move in the base of the summand with the largest subordinate. When the outcome is not essentially decided by atomic weight it will be the smaller subordinates that remain after several moves and thus the player with the larger smallest subordinate has an advantage.

Definition 4.6.5. *For a collation C of numbers let*

$$\mathrm{A} = |C^+| - |C^-|$$

and

$$\epsilon(C) = \begin{cases} 0 & \text{\textit{if} } C^+ \text{ \textit{or} } C^- \text{ \textit{is empty}} \\ \min(C^+) - \min(C^-) & \text{\textit{otherwise}} \end{cases}$$

We now present a large result on sums of the form $\langle *^n : C\rangle + *^m$ where $|C| = |C^+| + |C^-|$, $|C| \geq 2$, and $n \geq 1$. That is, a sum of one nimber and some nimber-based numbers where the nimber-bases are all the same.

Theorem 4.6.6. *Let C be a collation of non-zero numbers with $|C| \geq 2$, and let $n \geq 1$. If*

$$G = \langle *^n : C\rangle + *^m$$

then $o(G)$ is given in Table 4.2 according to the following cases:

Case	*nimber condition*
a:	$m \geq n$
b:	$m = 0$, *or* $m < n$ *and* $\nu(n) > \mathrm{lb}(m)$
c:	$m \neq 0$, $m < n$, *and* $\nu(n) \leq \mathrm{lb}(m)$

A	$\epsilon > 0$			$0 > \epsilon$		
Case	a	b	c	a	b	c
≥ 2	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$
1	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{N}$	$\mathscr{N}$
0	$\mathscr{N}$	$\mathscr{L}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{R}$	$\mathscr{N}$
-1	$\mathscr{R}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
≤ -2	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$

Table 4.2: Outcomes of $*^n$-based integers

Proof. Corollary 4.6.4 covers the situations where $|A| \geq 2$. Otherwise, we prove this result by induction according to the Cases a, b, c given in the big table. For each there is a little table in the proof.

We proceed by induction on $|C^+|+|C^-|$. If $|C^+|+|C^-| < 3$ and $\left||C^+|-|C^-|\right| < 2$ the outcome is given by Lemma 4.4.1 if the nimber is 0 and otherwise by Theorem 4.5.12. For the remaining cases, $\left||C^+| - |C^-|\right| < 2$ and $|C^+| + |C^-| \geq 3$ combine to ensure $|C^+| \geq 1$ and $|C^-| \geq 1$.

For each little table there are thus two cases (0, and -1) where Left moves first and two cases ($1, 0$ and 0) where Right moves first; if the best move for a player is the same we may combine the cases.

By symmetry, we need only argue the case where $\epsilon > 0$. All the tables in the proof show outcomes for both $\epsilon > 0$ and $\epsilon < 0$ because it reveals when ϵ matters. Players choose to play in a summand with the opponents largest subordinate and thus ϵ does not change (now that we are past the inductive base cases).

Players do not play in subordinates (other than in the inductive base cases): if Left could win making a (negative incentive) subordinate move instead of improving the atomic weight the atomic weight would have to be 0 and $\epsilon > 0$, but in all Cases if where $\epsilon > 0$ and $A = 1$, we have $o(G) = \mathscr{L}$ so Left would have won by improving the atomic weight.

If $|C^+| - |C^-| = 1$ with Left moving first or if $|C^+| - |C^-| = -1$ with Right moving first have winning moves by Fact 2.2.17 because $|C^-| \geq 1$ and $|C^+| \geq 1$ guarantees the necessary options.

Case a: $m \geq n$

A	$\epsilon > 0$	$\epsilon < 0$
1	$\mathscr{L}$	$\mathscr{L}$
0	$\mathscr{N}$	$\mathscr{N}$
-1	$\mathscr{R}$	$\mathscr{R}$

In this case, $*^m$ could be remote in which case it would follow from Fact 2.2.17. Otherwise, the winner follows this strategy (argued here in favour of Left): moving first, move a summand with largest negative nimber-subordinate to 0, the Case is the same but Left increased A by 1 and wins by induction. Moving second, if Right moved a summand with positive largest nimber-subordinate to $*^k$, Left moves a summand with largest negative nimber-subordinate to $*^k$ and wins by induction.

Case b: $m = 0$, or $m < n$ and $\nu(n) > \text{lb}(m)$

A	$\epsilon > 0$	$\epsilon < 0$
1	$\mathscr{L}$	$\mathscr{N}$
0	$\mathscr{L}$	$\mathscr{R}$
-1	$\mathscr{N}$	$\mathscr{R}$

In case b, ϵ is very important, changing the sign of ϵ changes the outcome. However, this is covered in the inductive base cases.

Left wins moving first by moving Right's largest summand to $*^m$, which corresponds to a position in this Case but with A one greater.

If Right moves first he moves a Left summand to some $*^k$, but $*^m + *^k = *^{m'}$ where $m' < n$ because $\nu(n) > \mathrm{lb}(m)$, thus Right has moved to a position in this table but with A one less, which Left now wins as first player.

Case c: $m \neq 0$, $m < n$, and $\nu(n) \leq \mathrm{lb}(m)$

A	$\epsilon > 0$	$\epsilon < 0$
1	$\mathscr{L}$	$\mathscr{N}$
0	$\mathscr{N}$	$\mathscr{N}$
-1	$\mathscr{N}$	$\mathscr{R}$

In this Case, because $\mathrm{lb}(m) \geq \nu(n)$, the first player can move in a summand to (at least one) $*^k$ such that $k \oplus m \geq n$. This is at least achieved when k is as large as possible and shares no 1-bits with m.

If A $= 0$ then the winning move will be to a entry in Case a as the first player wins by choosing an appropriate nimber and changing A in their favour.

Left wins moving first if A $= -1$ by moving Right's largest summand to $*^m$, thus winning by Case b.

If Right moves first when A $= 1$, he will lose: ϵ does not change and the nimber is not relevant as in any Case, Left wins moving first when A $= 0$ and $\epsilon > 0$.

□

By thinking of $*^n$ as $*^n : 0$ (unlike in Theorem 4.6.6) we can spot nice patterns in the outcomes of sums of nimber-based numbers.

Lemma 4.6.7. *If C is a collation of numbers with $|C| \geq 4$ and C' is C without its greatest and least elements, then $o(\langle *^n : C\rangle) = o(\langle *^n : C'\rangle)$.*

Proof. If $\epsilon(C') \neq 0$ then $\epsilon(C) = \epsilon(C')$ because $\epsilon(C)$ is not 0 and the smallest positive and smallest negative are the same.

If neither the greatest nor least element of C is 0 then A also does not change and thus we are in the same case of Theorem 4.6.6 so the outcome is the same.

If the greatest or least element of C is 0 then $\mathrm{A}(C')$ does change but all the remaining elements of C' have the same sign and thus $|\,\mathrm{A}(C')| = |C'| \geq 4 - 2 = 2$; also $|\,\mathrm{A}(C)| \geq |C'| + 1 \geq 4 - 2 + 1 = 3$. That is, either both $\langle * : C\rangle$ and $\langle * : C'\rangle$ are in $\mathcal{L}$ or both are in $\mathcal{R}$ as they have sufficiently large (≥ 2) atomic weights of the same sign. $\square$

We demonstrate the niceness of sums of nimber-based numbers with common nimber-bases in Corollary 4.6.8.

Corollary 4.6.8. *If C is a non-empty collation of numbers and $|C| \equiv 0 \pmod 2$ then $\langle *^n : C\rangle \in \mathcal{L} \cup \mathcal{R}$.*

Proof. The outcomes of non-zero sums of nimber-based numbers are given in Theorem 4.6.6. There are only a few entries we have to check (those that can be a sum of an even number of $*^n$-based numbers. In particular, we are in Case a when $\mathrm{A} = 1$ or $\mathrm{A} = -1$ and $m = n$ and Case b when $\mathrm{A} = 0$ and $m = 0$. In these cases, the outcome is always $\mathscr{L}$ or $\mathscr{R}$. The outcomes in cases where $\mathrm{A} \notin \{-1, 0, 1\}$ are either $\mathscr{L}$ or $\mathscr{R}$ by Corollary 4.6.4. $\square$

4.7 Recognizing Sums of Stalks

In Section 2.5 we show how to identify positions as ordinal sums. We can apply those methods and also recognize stalks, as a stalk is an ordinal sum (or ordinal sums). Stalks are not closed under disjunctive sum, so identifying a position as a sum of stalks is much harder.

We are interested in determining if a given position is a sum of stalks. Of course, G is a sum of stalks if only if its value is a sum of stalks, so we restrict our attention to determining if a value is a sum of stalks.

If G is a sum of stalks then all its subpositions are sums of stalks and thus all the subpositions of the value of G are sums of stalk. If K is in canonical form and K' is a subposition of K that is not a sum of stalks, then K is not a sum of stalks.

To show that some position G is a sum of stalks, it is sufficient to show that some option, say the Left option G^L, is a sum of stalks and the corresponding incentive $G^L - G$ is a stalk, then $G = G^L - (G^L - G)$ is a sum of stalks.

4.7.1 Incentives of Sums of Stalks

Theorem 4.7.1. *Let G be a sum of dicotic stalks. If G is non-zero, then the canonical form of G has an incentive that is a $*$-based stalk.*

Proof. Let $G = \langle * : C \rangle$, where C is a non-empty collation of stalks, and C contains at most one nimber. Also we assume any stalk that is equal to a simpler stalk plus a nimber as in Theorem 4.5.11 has been replaced by said stalk and nimber.

If G is a nimber ($|C| = 1$ and the only element is a nimber) then the position is $*$-based and has itself as an incentive. If G is not a nimber, there is at least one non-nimber element of C.

For each player there is at least one kill that is not strictly dominated by another kill. Suppose otherwise, then for each summand there is a greater (lesser) summand; but there are a finite number of options and by transitivity some summand would have to be greater (less) than itself.

It is possible that each kill is dominated, as in the case where the maximally great and maximally least stalks are duplicates, such as the case $G = \langle * : 1, 1 \rangle$. However, those kills will not all be *strictly* dominated.

It remains to be shown that the kill option is reversible for at most one player.

Suppose the kill $* : c_1 \to 0$ is not dominated as a Left option but that it reverses through some non-dominated Right kill $* : c_2 \to 0$. (If $* : c_2 \to 0$ is dominated as a Right option by $* : c_3 \to 0$ then the Left kill $* : c_1 \to 0$ reverses through the Right kill $* : c_3 \to 0$; so we could have considered it instead.)

If $* : c_2 \to 0$ is reversible by some Left kill $* : c_4 \to 0$ then $* : c_1 \to 0$ was dominated by $* : c_4 \to 0$, contradicting the assumption that $* : c_1 \to 0$ was not dominated.

We must also consider the possibility that $* : c_1 \to 0$ is reversible by a non-kill. Suppose $* : c_2 \to 0$ is reversible by $* : c_4 \to *^k$. We argue that $* : c_2 \to 0$ is reversible by $* : c_4 \to 0$, which we already showed is a contradiction. We need to consider Theorems 4.5.11, 4.5.12, and 4.5.14 to argue that adding a nimber to two stalks does not change the outcome from $\mathscr{N}$ to something else (which would change our sum of incentives from confused to comparable with 0). Theorem 4.5.11 could have been a problem, but we have simplified these stalks and our sum is inverse-pair free. In Theorem 4.5.12 there is no case where the two stalks are confused. In Theorem 4.5.14

the case $\mathrm{lb}(m) = \mathrm{lb}(n)$ is where two stalks may be confused, but with the nimber the sum is confused with 0.

Finally, we note that a kill moving the atomic weight in a player's favour is not reversed by a move that does not improve the atomic weight in response and thus all remaining responses have been considered. □

Corollary 4.7.2. *The canonical form of a non-zero sum of stalks has an incentive that is a stalk.*

Proof. Let G be a non-zero sum of stalks. If G is a (non-zero) number it has an incentive that is a number which is also a stalk. If G is not a number, then by Theorem 4.3.6 G is a number plus a non-zero sum of dicotic stalks. By Theorem 4.7.1, the result holds. □

That is, we can (likely using a computer) determine if a position is a sum of stalks.

4.8 Values of Sums of $*$-based Numbers

Sums of $*$-based integers have previously been considered using the term *uptimal*.

4.8.1 Values of Uptimals

Definition 4.8.1. *For non-negative integers n let:*

$$\begin{aligned}
\uparrow^{[n]} &= * : n + * \\
\downarrow^{[n]} &= * : -n + * \\
\uparrow^{n+1} &= \{0 \mid * : -n\} \\
\downarrow^{n+1} &= \{* : n \mid 0\}.
\end{aligned}$$

Note that $\uparrow = \uparrow^{[1]} = \uparrow^{1}$, $\uparrow^{[n]} = -\downarrow^{[n]}$ and $\uparrow^{n} = -\downarrow^{n}$.

Fact 4.8.2. *[21, Lemma 4] For $n \geq 1$,*

$$\sum_{i=1}^{n} \uparrow^{i} = \uparrow^{[n]}.$$

The origin of Fact 4.8.2 is [14, p.195].

A sum of $\uparrow^n$s and $\downarrow^n$s may be represented in *uptimal notation*:

$$.d_1 \dots d_k = \sum_{i=1}^{k} d_i \cdot \uparrow^i.$$

Fact 4.8.2 shows that the group of values generated by sums of $\uparrow^{[n]}$ and $\downarrow^{[n]}$ is the same as that generated by sums of $\uparrow^n$ and $\downarrow^n$. Uptimals are prominent in [1], where uptimal notation is attributed to Conway and Ryba [1, p. 188].

There are at least 3 definitions of uptimal that define different subgroups:

1. In [32, p.95] uptimals are generated by $\{\uparrow^{n+1}; n \geq 0\}$.

2. In [21, p. 7124, Definition 7] uptimals are generated by $\{\uparrow^{n+1}; n \geq 0\} \cup \{*\}$.

3. In [30] uptimals are generated by $\{\uparrow^{n+1}; n \geq 0\} \cup \{*\} \cup \mathbb{D}$.

In Cases 2 and 3, uptimals are closed. We use the term *uptimal* as given in Case 2. In Case 1, uptimals are linearly ordered; we call a position that is an uptimal in this sense a *strict uptimal*. We do not consider positions in the sense of Case 3 as uptimals.

Theorem 4.8.3 gives the canonical form of uptimals.

Theorem 4.8.3. *[21, Theorem 11, p. 7126] Let $U = .d_1 \dots d_k$ be a strict uptimal and let*

$$\begin{aligned}
U_i &= \begin{cases} 0, & \textit{if } i = 0; \\ .d_1 \dots d_i, & \textit{otherwise,} \end{cases} \\
d_i^- &= d_i - 1, \\
d_i^+ &= d_i + 1, \\
U^+ &= .d_1^+ \dots d_k^+, \\
U^- &= .d_1^- \dots d_k^-, \\
U_i^+ &= .d_1^+ \dots d_i^+, \ \textit{and} \\
U_i^- &= .d_1^- \dots d_i^-.
\end{aligned}$$

Assume d_k is positive:
If $j > 0$, then $U = \left\{U_j^+ *, U_m \;\middle|\; U^- *\right\}$ and $U + * = \left\{U_j^+, U_m * \;\middle|\; U^-\right\}$,
If $j = 0$, then $U = \{U_m \mid U^- *\}$ and $U + *$ is as follows:

	$U > *$	$U \parallel *$
$m = 0$	$\{0 \mid U^-\}$	$\{0, * \mid U^-\}$
$m > 0$	$\{U_m * \mid U^-\}$	$\{0, U_m * \mid U^-\}$

where

$$m = \begin{cases} k-1 & \textit{if } d_k = 1, \\ 0 & \textit{if } d_i > 0 \textit{ for } 1 \le i \le k \textit{ and } d_k \neq 1, \\ \max\{i:\ d_i \le 0, 1 \le i \le k\} & \textit{otherwise;} \end{cases}$$

$$j = \begin{cases} 0 & \textit{if } d_i \ge 0 \textit{ for } 1 \le i \le k, \\ \max\{i:\ d_i < 0, 1 \le i \le k\} & \textit{otherwise.} \end{cases}$$

Uptimal notation is great for uptimals and some rulesets, such as CUTTHROAT STARS [19] which has only uptimal values [1]. For rulesets with uptimals and other $*$-based positions analysis is easier if we treat the position as a sum of $*$-based positions.

At first glance strict uptimals look nice but uptimals plus $*$ are tricky. When we think in terms of sums of $*$-based numbers we can see why this is misleading. For a strict uptimal U,

$$U = .d_1 \dots d_k = \sum_{i=1}^{k} d_i \cdot (* : (i+1) - * : i) = \langle * : C \rangle$$

where C is a collation of integers with $|C| \equiv 0 \pmod 2$. That is, a strict uptimal, say U, is a sum of an even number of $*$-based integers, but $U + *$ is a sum of an odd number of $*$-based integers.

In Theorem 4.8.3 we consider a strict uptimal $U = .d_1 \dots d_k$ where $d_k > 0$ and cases U and $U + *$. Instead we can consider some $G = \langle * : C \rangle$ for a collation C whose largest element is $k > 0$.

Now we interpret Theorem 4.8.3 from this perspective.

Corollary 4.8.4. *If U is a non-zero uptimal whose largest subordinate is $k \ge 0$ the canonical form of U has exactly one Right option, namely $U - * : k$.*

Corollary 4.8.4 is very close to a more general case we saw, Theorem 4.7.1.

The coefficients for uptimal notation can be derived from an uptimal presented as a sum of $*$-based integers using

$$d_u = \Big|\{a \in C^+; |a| \ge u\}\Big| - \Big|\{a \in C^-; |a| \ge u\}\Big|$$

for $u \geq 0$ where d_u is the coefficient of $*$.

4.9 Sums of $*$-based Numbers

Determining the canonical form of a number of $*$-based numbers requires significantly more attention than the integer case. We investigate the incentives of $*$-based numbers and then compare them to determine which options are dominated and which options are reversible.

Lemma 4.9.1. *Let x, y, and z be numbers. If $x \geq y \geq 0 \geq z$ then $o(\langle * : x, y, z\rangle) = \mathscr{L} \iff y > |z|$.*

Proof. ($\Rightarrow$) If $o(\langle * : x, y, z\rangle) = \mathscr{L}$, Right cannot win going first. In particular, the move to $\langle * : y, z\rangle$ must be winning for Left moving first and its outcome is $o(y + z)$; $y + z$ is a number and thus not confused with 0, hence $y > |z|$.

($\Leftarrow$) If $y > |z|$ then $\langle * : x, y\rangle$ is at least $\langle * : y, z\rangle$; the latter is positive: Left wins moving first.

Subordinate options have negative incentive and thus lose for Right. The move to $\langle * : y, z\rangle$ (which is positive) dominates the others. □

To find the canonical form we need to understand domination and reversibility. First, we tackle domination by way of incentives.

4.9.1 Incentives of $*$-based numbers

There are two types of options in $*$-based numbers: kills (moves in the base to 0) and non-kills (moves in the subordinate which is a number). The latter exists for both players if the subordinate is a non-integer, or exactly one player if the subordinate is a non-zero integer.

Comparing kills in $*$-based numbers

The incentive of killing $* : H$ is $-* : H$. As $*$-based numbers are ordered by their subordinates, so are the killing moves: in a sum of $*$-based numbers Left prefers killing the summand with the least subordinate and Right prefers killing the summand with the greatest subordinate.

Comparing non-kills in $*$-based numbers

We denote by Δ_x^L the non-killing Left incentive from $*:x$ where x is in canonical form, $*:x^L - *:x$; this incentive exists if x is a positive integer or a non-integer number. Denote by Δ_x^R the non-killing Right incentive from $*:x$ where x is in canonical form, $*:x - *:x^R$; this incentives exists if x is a negative integer or a non-integer number.

Numbers have only negative incentives. Thus, subordinate options of $*$-based numbers have negative incentives.

To compare non-killing incentives we examine the difference: $\Delta_x^L - \Delta_y^L$ (a sum of four $*$-based numbers). The following Lemma is good to keep in mind:

Lemma 4.9.2. *Let each of x and y be a positive integer or a non-integer number in canonical form. If $x \neq y$ then*

$$\Delta_x^L \gtrless \Delta_y^L$$

and $\Delta_x^L = \Delta_y^L$ if and only if $x = y$.

Proof. We consider the difference

$$\Delta_x^L - \Delta_y^L = *:x^L - *:x - (*:y^L - *:y) = \langle *: x^L, -x, -y^L, y\rangle.$$

First we consider equality. If $x = y$ then $x^L = y^L$ and $\langle *: x^L, -x, -y^L, y\rangle = 0$. Assume $x \neq y$. If the subordinates are inverse pairs it must be that $|-x| = |-y^L|$ because $|x| \neq |x^L|$ and $x \neq y$; thus also $|x^L| = |y|$. However, these cannot both be true as x^L is simpler than x and y^L is simpler than y.

The result thus follows from Corollary 4.6.8. □

In Lemma 4.9.3 we determine which of two non-killing Left incentives is greater.

Lemma 4.9.3. *Let each of x and y be a positive integer or a non-integer number in canonical form.*

1. *If $x > y > 0$ and $x^L > y^L$ then $\Delta_x^L > \Delta_y^L$.*

2. *If $x > y > 0$ and $x^L = y^L$ then $\Delta_y^L > \Delta_x^L$.*

3. *If $x > 0 > y$ and $x^L = |y|$ then $\Delta_x^L > \Delta_y^L$.*

4. *If $x > 0 > y$ and $x^L > |y|$ then $\Delta_x^L > \Delta_y^L$.*

5. *If* $x > 0 > y$ *and* $x^L < |y|$ *then* $\Delta^L_y > \Delta^L_x$.

6. *If* $0 > x > y$ *then* $\Delta^L_y > \Delta^L_x$.

Proof. For each case we play the difference

$$\Delta^L_x - \Delta^L_y = * : x^L - * : x - (* : y^L - * : y) = \langle * : x^L, -x, -y^L, y\rangle.$$

Each difference is a sum of 4 $*$-based numbers. Note that x^L and $-x$ do not have the same sign, and $-y^L$ and y do not have the same sign. (x^L and y^L are possibly 0 at times in some cases). By Lemma 4.6.7, if no pair of subordinates are inverses, the outcome of the sum is the same as the outcome of the sum where the largest positive and largest negative subordinates are excluded. Recall that $x^L < x$ and $y^L < y$ for all numbers.

1. $-x$ is the largest negative subordinate and both positive subordinates are larger than y^L, so $\langle * : x^L, -x, -y^L, y\rangle > 0$.

2. $\langle * : x^L, -x, -y^L, y\rangle = * : y - * : x < 0$.

3. as $x^L = |y|$, $\langle * : x^L, -x, -y^L, y\rangle = * : -x + * : -y^L$. Also y is simpler (has a smaller denominator) than x and the denominator determines the incentive so $x - x^L < y - y^L$; again noting that $x^L = -y$ we get $0 < -x - y^L$. Thus, $\langle * : x^L, -x, -y^L, y\rangle > 0$.

4. $-x$ is the largest negative subordinate and $-y^L$ is the largest positive subordinate so $o(\langle * : x^L, -x, -y^L, y\rangle) = o(x^L + y) = \mathscr{L}$.

5. $-x$ is the largest negative subordinate and $-y^L$ is the largest positive subordinate so $o(\langle * : x^L, -x, -y^L, y\rangle) = o(x^L + y) = \mathscr{R}$.

6. $-y^L$ is the largest positive subordinate and both negative subordinates are larger than x, so $\langle * : x^L, -x, -y^L, y\rangle < 0$.

□

Corollary 4.9.4. *If Left is choosing between two non-kill moves in* $*$*-based numbers with the same sign, Left prefers the move from the larger subordinate, unless they are both positive and the Left options are equal (in which case the smaller is preferred).*

If Left is choosing between playing in a positive subordinate (an x with largest possible Left option) and a negative subordinate (a y that is a largest non-integer), Left prefers the negative subordinate unless $x^L \geq |y|$, in which case Left prefers to play in a positive subordinate.

Intuition: If Left is to play in a summand with a positive subordinate, she tends to play in summands Right will kill soon (the greatest, but if Left's move would make another summand the greatest, she plays in it instead). If the positive summands are such that a move would change the ϵ for the sum, Left may seek to play in a negative (non-integer) summand if one exists, and a least summand would be best.

Definition 4.9.5. *Consider a sum of $*$-based numbers. A* Left target *is a summand for which Left's subordinate option is not strictly dominated by any other subordinate option in another summand. A* Right target *is a summand for which Right's subordinate option is not strictly dominated by any other subordinate option in another summand.*

Remark 4.9.6. *A sum of $*$-based numbers has a Left target unless all subordinates are non-positive integers. A sum of $*$-based numbers has a Right target unless all subordinates are non-negative integers.*

4.9.2 Comparing killing and non-killing moves

Theorem 4.9.7. *If G is a sum of $*$-based dyadic rationals, not all non-positive integers, then Left's kill and non-kill options are confused.*

Proof. Let the Left target of G be $*{:}t$. If $t < 0$, then t is a largest negative subordinate and thus Left's best kill move $(* : t \to 0)$ is in the same summand as Left's target move $(* : t \to * : t^L)$; these options are confused as the difference of their incentives is $* : t^L$.

If $t > 0$ and a summand, say $*{:}m$, has a negative subordinate, we compare $*{:}t^L - *{:}t$ to $-* : m$. That is, the difference is $\langle * : t^L, -t, m\rangle$. By Corollary 4.9.4 because $* : t$ is the target, $t^L \geq -m$. If $t^L = -m$ then $o(\langle * : t^L, -t, m\rangle) = o(\langle * : -t\rangle) = \mathscr{N}$. Otherwise, by Lemma 4.9.1 $o(\langle * : t^L, -t, m\rangle) = \mathscr{N}$ (two negative summands and one positive, but the largest is positive).

If $t > 0$ but no summand has a negative subordinate, we compare $* : t^L - * : t$ to $-* : s$ where s is the smallest subordinate. That is, the difference is $\langle * : t^L, -t, m \rangle$. By Lemma 4.9.1 $o(\langle * : t^L, -t, s \rangle) = \mathscr{N}$ (two positive summands and one negative, but the largest is negative). $\square$

Consider a sum of $*$-based numbers where a largest subordinate, s, is positive. Right's options are not reversible. Right's kill is clearly not reversible. Right's non-kill option is also in the summand with the largest subordinate, so if Left kills in response the difference from the orginal is still less than $* : s^R$, which is confused with 0. If Left's non-kill response would reverse Right's move then Right had a better non-kill to make.

Left's kill option is reversible except when there is only one summand, i.e. if the largest summand is positive.

Left's non-kill option with negative incentive will reverse sometimes but not always.

Now we describe which options are reversible, which we consider without describing which options will eventually remain in the canonical form.

Lemma 4.9.8. *Let S be a sum of $*$-based numbers where a largest subordinate, say s_0, is positive; and let $* : t$ be the Left target. Left's option in the Left target is reversible if and only if:*

- $t > 0$ *and there is another positive subordinate with the same Left option; or*
- $t < 0$

Proof. Consider Left's option in $* : t$ to $* : t^L$.

- If $t > 0$ and $t = s_0 = s_1$ (i.e. multiple largest subordinates) or $t < s_0$ ($t^L = s_0^L$) then Right can reverse by killing $* : s_0$. The difference from the original after Right's move is $* : s_0 + * : t - * : t^L$, which is positive by Lemma 4.9.1 as $t^L < t$ and $t^L < s_0$. If no other subordinate has the same option then either Right cannot reverse or a subordinate exceeds t in which case t is not the target.
- By hypothesis, $|s_0| > |t|$. As t is a negative number, $|t^L| > |t|$. The smallest in absolute value of $-t$, t^L and $-s_0$ is $-t$ which is also the lone positive subordinate.

If Right responds by killing $*: s_0$ the difference is $\langle * : -t, t^L, -s_0 \rangle$, which is less than 0 by the Right version of Lemma 4.9.1.

□

4.10 Group structure

From here until the end of the Chapter we consider stalks and other $*$-based positions. As shown by Theorem 4.4.7, understanding $*$-based positions is important for understanding stalks.

The order of stalks is reasonably easy to analyze. We also investigate finite even orders of $*$-based positions. There are no positions of odd order [32, p. 167], so this describes the possible orders.

4.10.1 On the Order of $*$-based Positions Including Stalks

We know from Fact1.10.7 that non-zero nimbers have order 2. Non-nimbers stalks have infinite order.

Lemma 4.10.1. *Non-nimber stalks have infinite order.*

Proof. A non-nimber stalk that is $*$-based has atomic weight 1 or -1 (Lemma 4.3.8) and thus has infinite order. Number-based stalks have non-zero mean and thus have infinite order. □

That nimbers have order 2 also follows by induction from Fact 2.4.10 with $n = 1$ using Proposition 4.10.2.

Proposition 4.10.2. *Let G be non-zero. The position $* : G$ has order 2 if and only if G has order 2.*

Proof. By Fact 2.4.10, $o(*:G + *:G) = o(G + G)$. That is $o(G+G) = \mathscr{P}$ if and only if $o(* : G + * : G) = \mathscr{P}$ and thus $* : G + * : G = 0$ if and only if $G + G = 0$, which is equivalent to G having order 2 if $G \neq 0$. □

Example 4.10.3. *For any position G, $\pm G = \{G \mid -G\}$ has order 2, so for example,*

$$* : \pm 1 + * : \pm 1 = 0.$$

Proposition 4.10.2 shows that there are lots of $*$-based positions of order 2, and Lemma 4.10.1 shows that we have lots of $*$-based stalks that have infinite order. The question of other finite orders is answered in Theorem 4.10.5, but first we need Lemma 4.10.4.

Lemma 4.10.4. *If* $4(* : G) = 0$ *then* $2(* : G) = 0$.

Proof. We may assume G is in canonical form. Suppose, by way of contradiction, that $2(* : G) \neq 0$.

If $2(* : G) \in \mathcal{L} \cup \mathcal{R}$ then $4(* : G) \in \mathcal{L} \cup \mathcal{R}$. Thus, we assume $2(* : G) \in \mathcal{N}$. By assumption, $4(* : G) \in \mathcal{P}$ and thus by Lemma 4.5.2, $3(* : G) \in \mathcal{N}$. As $3(* : G)$ is in $\mathcal{N}$ and its Left options are $2(* : G)$ and of the form $* : G^L + 2(* : G)$ there exists some G^{L_1} such that $* : G^{L_1} + 2(* : G) \geq 0$ (as $2(* : G) \in \mathcal{N}$).

Now consider Left's option from $4(* : G)$ to $* : G^{L_1} + 3(* : G)$, from which Right must have a good response as $4(* : G) = 0$.

Right's moves are to:

- $3(* : G)$ which is in $\mathcal{N}$;
- $* : G^{L_1} + 2(* : G)$, which is at least 0;
- $* : G^{L_1} + 2(* : G) + * : G^R$, to which Left has an option to $* : G^{L_1} + 2(* : G)$ (which is at least 0); and
- $* : G^{L_1 R} + 3(* : G)$.

Thus $* : G^{L_1 R} + 3(* : G)$ must be less than or equal to 0. By adding $* : G$ to both sides we see that $* : G^{L_1 R} \leq (* : G)$, which contradicts the assumption that G was canonical. $\square$

Theorem 4.10.5. *For* $n \geq 1$, *if* $2n(* : G) = 0$ *then* $2(* : G) = 0$.

Proof. This proof is very similar to the proof of Lemma 4.10.4. We may assume G is in canonical form. Suppose, by way of contradiction, that $2(* : G) \neq 0$. By induction, using Lemma 4.10.4 as a base case, we have a contradiction immediately unless $2k(* : G) \neq 0$ for $2 \leq k < n$.

If $2k(* : G) \in \mathcal{L} \cup \mathcal{R}$ then $2nk(* : G) \in \mathcal{L} \cup \mathcal{R}$, but $2n(* : G) = 0$ implies that $2nk(* : G) = 0$. Thus, we assume $2(* : G) \in \mathcal{N}$. By Lemma 4.5.2, $(2n - 1)(* : G) \in \mathcal{N}$.

As $(2n-1)(*:G)$ is in $\mathcal{N}$ and its Left options are $(2n-2)(*:G)$ and of the form $*:G^L+(2n-2)(*:G)$ there exists some G^{L_1} such that $*:G^{L_1}+(2n-2)(*:G)\geq 0$ (as $(2n-2)(*:G)\in\mathcal{N}$).

Now consider Left's option from $2n(*:G)$ to $*:G^{L_1}+(2n-1)(*:G)$, from which Right must have a good response as $2n(*:G)=0$.

Right's moves are to:

- $(2n-1)(*:G)$ which is in $\mathcal{N}$;
- $*:G^{L_1}+(2n-2)(*:G)$, which is at least 0;
- $*{:}G^{L_1}{+}(2n{-}2)(*{:}G){+}*{:}G^R$, to which Left has an option to $*{:}G^{L_1}{+}(2n{-}2)(*{:}G)$ (which is at least 0); and
- $*:G^{L_1R}+(2n-1)(*:G)$.

Thus $*:G^{L_1R}+(2n-1)(*:G)$ must be less than or equal to 0. By adding $*:G$ to both sides (and applying the assumption that $2n(*:G=0)$ we see that $*:G^{L_1R}\leq(*:G)$, which contradicts the assumption that G was canonical. □

4.10.2 On the Group Structure of Stalks

Non-zero nimbers have order 2, and we saw in Lemma 4.10.1 that non-nimber stalks have infinite order. However, we did see some interesting group structure concerning stalks: In Theorem 4.5.11 we saw three stalks add to 0. We can construct a sum of four stalks that add to 0 by matching two pairs of stalks (as in Theorem 4.5.11) that add to the same nimber. Similarly, we can construct sums of larger even numbers of non-nimber stalks that add to 0. It remains to be seen if there are 0 sums that are not explained using Theorem 4.5.11. We conjecture that all 0 sums of four stalks are explained using Theorem 4.5.11.

Conjecture 4.10.6. *A sum of four non-nimber stalks is equal to* 0 *only if the four summands form two pairs each summing to the same nimber.*

4.11 On the relation of $*$-based positions to $*$-based integers

The main result of this section is Theorem 4.11.5, which is a generalization of Fact 4.8.2 and Fact 1.13.11. It shows that a position of the form $\{* : G \mid 0\}$ is a sum of $*$-based positions, in particular a $*$-based position and a $*$-based integer. This suggests that $*$-based integers are indeed special among the $*$-based numbers.

We start with a pair of curious results that show the relevance of the size of a $*$-based subordinate in a sum of $*$-based positions.

Lemma 4.11.1. *If $G + H \in \mathcal{N}$ then $* : G + * : H + * : n \in \mathcal{L}$ for all sufficiently large positive integers n.*

Proof. By Fact 1.9.9, there exists some integer n that is greater than G and H, in particular any $n > \mathrm{b}(G + H)$. The only Right option of the sum from which Left cannot respond to either $* : G + * : n$ or $* : H + * : n$ is $* : G + * : H$, from which Left can win by the hypothesis. Left going first is easy, as she kills either $* : G$ or $* : H$. $\square$

Corollary 4.11.2. *If $G + H \in \mathcal{N}$ then $* : G + * : H + * : -n \in \mathcal{R}$ for all sufficiently large positive integers n.*

We reformat Fact 4.8.2 to be analogous to Theorem 4.11.5.

Lemma 4.11.3. *If n is an integer then $* : n + \{0 \mid * : -n\} = * : n + 1$.*

Proof. Play the difference: $* : n + \{0 \mid * : -n\} + * : -n - 1$. $\square$

In [21] $\uparrow^{x}$ and $\uparrow^{[x]}$ are defined for numbers x. The following result was originally presented in the form implied by Definition 4.8.1.

Fact 4.11.4. *[20, Theorem 3.14, p. 41] If x is a number then $* : x + \{0 \mid * : -x\} = * : n$ where n is the least positive integer larger than x.*

Theorem 4.11.5 generalizes Fact 4.11.4, and exposes a new invariant for short positions.

Theorem 4.11.5 (Main Theorem of this Section)**.** *For all positions G, there is an integer $n > 0$ such that*

$$* : G + \{0 \mid * : -G\} = * : n.$$

The proof of Theorem 4.11.5 comes later in this section, after some Lemmas needed in the proof.

Lemma 4.11.6. *For all positions G, $\{0 \mid * : G\} \in \mathcal{L}$.*

For all positions G and H, $\{0 \mid * : G\} + * : H \mathrel{|\!\!\rhd} 0$.

Proof. As $* : G \in \mathcal{N}$ for all G, $\{0 \mid * : G\} \in \mathcal{L}$. For all H, Left wins $\{0 \mid * : G\} + * : H$ by moving to $\{0 \mid * : G\}$. $\square$

Lemma 4.11.7. *For all G and H,*

$$o(\{0 \mid * : G\} + \{* : H \mid 0\}) = o(G + H).$$

Proof. Either player moving first loses by moving a summand to 0, as the opponent can respond to 0 in the other summand. Without loss of generality, suppose Left moves first (to $\{0 \mid * : G\} + * : H$). Right loses if he plays in $* : H$ (see Lemma 4.11.6. Right's only reasonable response is to $* : G + * : H$. That is, $o(\{0 \mid * : G\} + \{* : H \mid 0\}) = o(* : G + * : H)$ and by Fact 2.4.10 this is the same as $o(G + H)$. $\square$

In this section we consider $* : G + \{0 \mid * : -G\} + * : -n$ for various integers n. For clarity in proofs, we note here the following bad moves for all integers n:

If Left moves to:

- $* : G + \{0 \mid * : -G\}$, Right responds to $* : G + * : -G$.

If Right moves to:

- $\{0 \mid * : -G\} + * : -n$, Left responds to $\{0 \mid * : -G\}$.
- $* : G + \{0 \mid * : -G\}$, Left responds to $\{0 \mid * : -G\}$.
- $* : G + * : -G + * : -n$, Left responds to $* : G + * : -G$.
- $* : G^R + \{0 \mid * : -G\} + * : -n$, Left responds to $* : G^R + \{0 \mid * : -G\}$. From here, Right's move to $\{0 \mid * : -G\}$ is losing, and Right's move to $* : G^{RR} + \{0 \mid * : -G\}$ is losing because Left responds to $\{0 \mid * : -G\}$. Lastly, Right's move to $* : G^R + * : -G$ loses as Left responds to $* : G^R + * : -(G^R)$.

Lemma 4.11.8. *For all positions G, $* : G + \{0 \mid * : -G\} + * : 0 > 0$.*

Proof. If $G \geq 0$ then the result holds as $* : G + * \geq 0$ and $\{0 \mid * : -G\} > 0$.

If $G \ngeq 0$ then $-G \mathrel{|\!\triangleright} 0$. In this case, Left wins moving first to $\{0 \mid * : -G\} + *$; Right clearly loses moving to $\{0 \mid * : -G\}$; Right loses moving to $* : -G + *$ because $-G \mathrel{|\!\triangleright} 0$. Right's other moves are bad as previously demonstrated. □

Proof of Main Theorem. The result is new only for non-integers so we give the proof only for non-integers (so as to ignore the edge cases of $n = G$ and $n = -G$). Let G be a non-integer position and consider the difference $* : G + \{0 \mid * : -G\} + * : -n$. This difference becomes more favourable to Right as n increases and we show that it is 0 for some sufficiently large n.

If $n \ngtr G$ then $* : G + * : -n \nless 0$, and thus $* : G + \{0 \mid * : -G\} + * : -n \mathrel{|\!\triangleright} 0$.

We have previously considered the case where G is an integer so if $n \ngtr -G$ then we may assume $-G \mathrel{|\!\triangleright} n$; Left wins $* : G + \{0 \mid * : -G\} + * : -n$ by moving first to $\{0 \mid * : -G\} + * : -n$. Hence, we assume $n > G$ and $n > -G$.

Left loses moving to

- $* : G + 0 + * : -n$, which is negative; and
- $\{0 \mid * : -G\} + * : -n$; because Right can respond to $* : -G + * : -n$ which is in $\mathcal{R}$, as $n > -G$.

The only Left moves not previously demonstrated to be losing are to $* : G^L + \{0 \mid * : -G\} + * : -n$ for some G^L, to which Right can respond to $* : G^L + * : -G + * : -n$.

After Right's response if Left moves to:

- $* : G^L + * : -n$, Right wins if $n > G^L$.
- $* : -G + * : -n$, Right wins if $n > G$.
- $* : G^L + * : -G$, Right wins by moving to $* : G^L + * : -(G^L)$.
- $* : G^{LL} + * : -G + * : -n$, Right wins by moving to $* : -G + * : -n$ if $n > G$.
- $* : G^L + * : -(G^R) + * : -n$, Right wins if $n > G^L$ or $n > G^R$ by moving to $* : G^L + * : -n$ or $* : -G^R + * : -n$, respectively.

That is, Left loses moving first from $*:G+\{0 \mid *:-G\}+*:-n$ for some $n=N$ and also for $n \geq N$. Right's only move not previously demonstrated to be losing is Right moving to $*:G+\{0 \mid *:-G\}+*:-n+1$.

Recall that Left wins if $n=0$. Consider the largest $n=k$ such that Left wins moving first. By construction, Left loses moving first from the position with $n=k+1$. As Right has only one possibly winning move, Right also loses moving first when $n=k+1$, by construction. □

From Theorem 4.11.5 we have a new invariant (n) for short positions (G). We do not have a rule for determining n in general. In the case where G is a number, Fact 4.11.4, n is the simplest integer larger than G. We can also determine the invariant for dicotic positions, as we do in Theorem 4.11.9.

Theorem 4.11.9. *If G is dicotic, $*:G+*:1=\{0 \mid *:G\}$.*

Proof. We show that both players lose going first in the difference:

$$*:G+*:1+\{*:-G \mid 0\}.$$

If Right plays to $*:G+*:1$ he loses as $1>G$.

If Right plays to $*:G+\{*:-G \mid 0\}$, Left wins by responding to $*:G+*:-G=0$.

If Right plays to $*:1+\{*:-G \mid 0\}$, Left wins by responding to $*:1+*:-G$ as $1>-G$.

If Right plays to $*:G^R+*:1+\{*:-G \mid 0\}$, Left responds to $*:G^R+*:1+*:-G$; if Right moves to $*:G^R+*:-G$ Left responds to $*:G^R+*:-G^R=0$; a Right kill of $*:G^R$ or $*:-G$ moves to a positive position because 1 is greater than any dicotic position. Similarly, if Right had played in a subordinate Left could respond by killing the summand leaving a positive position.

If Left plays to $*:G+*+\{*:-G \mid 0\}$, Right will win but response depends on G. If $G>0$, Right responds to $*+\{*:-G \mid 0\}$. Otherwise, Right has an option in G to some $G^R \leq 0$ and thus Right responds to $*:G^R+*+\{*:-G \mid 0\}$ which is the sum of $*:G^R+*$ which is less than or equal to 0, and $\{*:-G \mid 0\}$ which is negative.

The other options have obvious winning responses. □

It is an open problem to determine a more general rule for the invariant. Theorem 4.11.5 may also generalize to loopy games (not otherwise discussed in this thesis) as the result is related to an example of Siegel [32, p.323].

Chapter 5

Misère-play and $*$-based Dyadic Rationals

Dicotic games occupy a middle-ground between the generality of partizan games and the relative simplicity of impartial games. This is particularly true under misère-play. As shown in [24], ends cause trouble; no end (Definition 1.2.2) is comparable to any non-end. Every end in a dicotic position is an end for both players, which alleviates some misery.

In particular, ordinal sum behaves nicely with dicotic positions (2.4.1). In this Chapter, we determine the outcome for SPRIGS, a dicotic ruleset; we do so by showing that the canonical form under misère-play of a SPRIG is a $*$-based dyadic rational. If g is a BLUE-RED STALK with canonical form x, then by Corollary 2.4.8 $* : x$ is the normal-play canonical form of $* : g$. Culminating in Theorem 5.2.8, we show that the misère-play canonical form of a SPRIG $* : g$ is also $* : x$ where x is the normal-play canonical form of g.

Given a multiset of dyadic rationals, X, what is the outcome under misère-play of $\langle * : X \rangle$? We answer our question in Theorems 5.3.2 and 5.3.4, depending on the members of the multiset.

We find the outcome of a disjunctive sum of $*$-based dyadic rationals under misère-play and show that it is the same as the outcome of that sum plus $*$ but under normal-play. Along the way we show that the sum of a SPRIG and its negative is equivalent to 0 under misère-play in the universe of dicotic positions, answering a question of Allen.

This Chapter is largely joint work that appeared in [22].

5.1 Introduction to Misère-play

The positions we consider under misère-play are the same as those we consider under normal-play. Many positions (especially normal-play canonical positions) have acquired names. Despite having names picked based on the normal-play theory, we use those same names when we consider misère-play. For example, $0 \cong \{\cdot \mid \cdot\}$; $* \cong \{0 \mid 0\}$; for a positive integer n, $n \cong \{n-1 \mid \cdot\}$; and non-integer dyadic rationals are $\frac{m}{2^q} \cong \left\{\frac{m-1}{2^q} \,\middle|\, \frac{m+1}{2^q}\right\}$ for odd $m \in \mathbb{Z}$ and $q \in \mathbb{N}$.

In Section 5.2, we present the general results needed for analyzing misère-play SPRIGs, in particular proving that $*:x - *:x \equiv 0$ in some universes (Lemma 5.2.4 and Corollary 5.2.5). This is an important result since, in the universe of all positions under misère-play, $G - G \not\equiv 0$ unless $G = \{\cdot \mid \cdot\}$.

The *negative* of a position is $-G \cong \left\{-G^{\mathbf{R}} \,\middle|\, -G^{\mathbf{L}}\right\}$. In normal-play, the negative is the additive inverse (Fact 1.4.4). However, in misère-play whether the negative is also the additive inverse depends on the universe being considered. To avoid confusion and inappropriate cancellation, in misère-play we represent the negative by $\overline{G}$ instead of $-G$.

Recall that an outcome is a pair: $(o_L(G), o_R(G))$. For a position G, we now use $o^+(G)$ for the outcome of G under normal-play. We will use $o^-(G)$ for the outcome under misère-play. Analogously, we use o_L^+, o_R^+, o_L^-, and o_R^-.

Definition 5.1.1. *The* misère *play convention declares a game finished when the player to move has no options and declares said player the winner. That is:*

if G is a Left end, then $o_L^-(G) = \text{Left}$; and

if G is a Right end, then $o_R^-(G) = \text{Right}$.

Example 5.1.2. *For the position 0, $o_L^-(0) = \text{Left}$ and $o_R^-(0) = \text{Right}$.*

Under the misère-play convention, every play finishes at an end for the player whose turn it is to move. By Theorem 1.3.1 every play results in Left or Right as the winner. Thus the misère-play convention gives a winner as follows:

For any position G,

$$o_L^-(G) = \begin{cases} \text{Left} & \text{if } G^{\mathbf{L}} \text{ is empty or } o_R^-(G^L) = \text{Left for some } G^L \\ \text{Right} & \text{otherwise} \end{cases}$$

$$o_R^-(G) = \begin{cases} \text{Right} & \text{if } G^{\mathbf{R}} \text{ is empty or } o_L^-(G^R) = \text{Right for some } G^R \\ \text{Left} & \text{otherwise.} \end{cases}$$

Short positions are thus partitioned into four ($\mathcal{L}^-$, $\mathcal{R}^-$, $\mathcal{N}^-$, $\mathcal{P}^-$) sets, the outcome classes under misère-play, according to their outcome:

- $\mathcal{L}$eft (Left can win going first and second under misère-play)
- $\mathcal{R}$ight (Right can win going first and second under misère-play)
- $\mathcal{N}$ext (the next player to move wins regardless if this Left or Right under misère-play)
- $\mathcal{P}$revious (the next player cannot win regardless if this Left or Right under misère-play).

Outcomes of negatives under misère-play work just as we hope and expect.

Lemma 5.1.3. *For a position G,*

$$\begin{array}{c|c} o^-(G) & o^-(\overline{G}) \\ \hline \mathscr{L} & \mathscr{R} \\ \mathscr{R} & \mathscr{L} \\ \mathscr{N} & \mathscr{N} \\ \mathscr{P} & \mathscr{P}. \end{array}$$

Despite positions having a outcome under normal-play and an outcome under misère-play there are still only four outcomes, as seen in Figure 1.1. However, there are four outcome classes under normal-play and four outcome classes under misère-play. For example, $o^+(0) = \mathscr{P}$ and $o^-(0) = \mathscr{N}$; we write $0 \in \mathcal{P}^+$ and $0 \in \mathcal{N}^-$, where the superscript denotes the play convention. We use $\between$ to represent either $+$, denoting normal-play, or $-$, denoting misère-play; for example, $o^{\between}(*^2) = \mathscr{N}$. If $\between$ appears multiple times in a line, it represents the same symbol each time.

In normal-play and misère-play *equality* and *inequality* are as follows:

Definition 5.1.4. $G =^{\between} H$ *if* $o^{\between}(G+X) = o^{\between}(H+X)$ *for all positions* X*;*

$G \geq^{\between} H$ *if* $o^{\between}(G+X) \geq o^{\between}(H+X)$ *for all positions* X.

In normal-play, there are tests for equality and inequality of positions as seen in Figure 1.2. In misère-play, showing equality involves comparing with all (i.e. infinitely many) positions.

Definition 5.1.5. *A* universe *is an additively closed and hereditarily closed set of positions.*

It was shown in [29] that if we restrict the X in Definition 5.1.4 to a universe we can show some positions are equivalent, in particular with impartial positions. Subsequently, [3] explored partizan misère games, using this idea.

Given a universe $\mathcal{X}$,

$$G \equiv H \pmod{\mathcal{X}} \text{ if } o^-(G+X) = o^-(H+X) \text{ for all positions } X \in \mathcal{X};$$

$$G \geq^- H \pmod{\mathcal{X}} \text{ if } o^-(G+X) \geq o^-(H+X) \text{ for all positions } X \in \mathcal{X}.$$

If $G \equiv H \pmod{\mathcal{X}}$ then G and H are said to be *indistinguishable* modulo $\mathcal{X}$. Otherwise, there exists a position $X \in \mathcal{X}$ such that $o^-(G+X) \neq o^-(H+X)$ and we say G and H are *distinguishable* in $\mathcal{X}$.

To fight the proliferation of superscripts, we will distinguish between normal-play and misère-play relations by reserving $=, >, <, \geq, \leq$ for the order relations in normal-play and $\equiv, \gtrdot, \lessdot, \geqq, \leqq$ for the corresponding relations in misère-play, which should also be accompanied by a reference to a universe.

The universes of [4] and [29] are defined by starting with a single position G and adding all the subpositions and all disjunctive sums that can be formed. We consider $\mathcal{D}$, the universe of all dicotic positions, and $\mathcal{S}$, the set of all positions that are a finite sum of sprigs.

Distinguishable elements in a given universe will also be distinguishable in a larger universe.

Fact 5.1.6. *[3, Proposition 1.3.19, p. 21] Let $\mathcal{X}$ and $\mathcal{Y}$ be sets of positions with $\mathcal{X} \subset \mathcal{Y}$. If $G \not\equiv H \pmod{\mathcal{X}}$ then $G \not\equiv H \pmod{\mathcal{Y}}$.*

Proof. If there is some position $X \in \mathcal{X}$ with $o^-(G+X) \neq o^-(H+X)$ for $X \in \mathcal{X}$, then X also distinguishes G and H in the universe $\mathcal{Y}$. □ □

As $\mathcal{S} \subset \mathcal{D}$, if two elements of $\mathcal{S}$ are distinguishable in $\mathcal{S}$ then they are distinguishable in $\mathcal{D}$.

As mentioned earlier, in the entire misère universe, $G+\overline{G} \not\equiv 0$ unless $G = \{\cdot \mid \cdot\} = 0$. It is shown in [3, Section 4.2, p. 99], that $* + * \equiv 0 \pmod{\mathcal{D}}$ and Allen [4, p. 11, Section 6, Question 3] asked which G satisfy $G + \overline{G} \equiv 0 \pmod{\mathcal{D}}$. We shall see that every sprig has this desirable property.

A position has a unique normal-play canonical form. Similarly, a position has a unique *misère-play canonical form* [24, 33], found by removing dominated options and bypassing reversible options. The normal-play canonical form and the misère-play canonical form of a position are rarely the same, because the definition of equality depends on the play convention and hence domination and reversibility are dependent on the play convention. In Section 5.2 we show the two canonical forms are the same if the position is a SPRIG.

5.1.1 Ordinal Sums Under Misère-play

Because we were careful about form in Chapter 2, we can start considering ordinal sum under misère-play without much extra work. Ordinal sum is *formally* associative as shown in Lemma 2.3.15, so combining multiple ordinal sums, such as in the construction of a STALK is unambiguous even under misère-play.

Lemma 5.1.7. *If G and H are any positions then $\overline{G : H} \cong \overline{G} : \overline{H}$.*

Proof. This is Lemma 2.1.4 with the misère notation. □

Theorem 5.1.8. *If $G \in \mathcal{P}^{\between}$ then $o^{\between}(G : H) = o^{+}(H)$.*

Proof. Playing in G loses; players play in H until forced to play in G, the player who can play last in H is given by $o^{+}(H)$. □

Theorem 5.1.9. *If G is not an end, then the outcome of $G : H$ under misère-play is given by Table 2.1.*

Proof. If a player wins going first in the base then they can win going first in the ordinal sum. In such a case the opponent loses if they play in the subordinate because the first player still has the base option available. That is, the outcome is the outcome of the base unless possibly if the base is a second-player win.

If $o^{-}(G) = \mathscr{P}$ this is Theorem 5.1.8. □

Tables 5.1 and 2.1 are essentially the same, but with different play conventions for G and $G : H$. The biggest difference is that the misère-play version excludes ends; no end G has $o^{-}(G) \neq \mathscr{P}$ so that case is not really missing.

Lemma 5.1.10 and Corollary 5.1.11 show that if $o^{-}(G) \in \{\mathscr{L}, \mathscr{R}\}$ and G is an end, then we would get the correct outcome if we used Table 5.1 anyway.

$o^+(H)$ → / $o^-(G)$ ↓	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{L}$	$\mathscr{R}$
$\mathscr{P}$	$\mathscr{P}$	$\mathscr{N}$	$\mathscr{L}$	$\mathscr{R}$
$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$	$\mathscr{N}$
$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$
$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$

Table 5.1: Misère-play outcome of ordinal sums with non-end bases

Lemma 5.1.10. *If G is a Left end and $o^-(G) = \mathscr{L}$, then $o^-(G:H) = \mathscr{L}$.*

Proof. As $o^-(G) = \mathscr{L}$, Right loses going first in G. Right going first in $G:H$ loses unless he plays to $G:H^R$ for some H^R, but by induction $o^-(G:H^R) = \mathscr{L}$ so he loses.

If H is also Left end, Left wins $G:H$ going first. Otherwise, Left can move to some $G:H^L$, which, by induction, is a winning move. □

Corollary 5.1.11. *If G is a Right end and $o^-(G) = \mathscr{R}$, then $o^-(G:H) = \mathscr{R}$.*

The only cases, which we call *exceptions*, where Table 5.1 may not give the correct outcome if one insisted on using it even when G is an end are when $o^-(G) = \mathscr{N}$.

Let $o^-(G) = \mathscr{N}$. If G and H are both Left ends or both Right ends then $o^-(G:H) = \mathscr{N}$ and it is not an exception. The exceptions are as follows:

- the base is empty: $o^-(G:H) = o^-(\{\cdot \mid \cdot\}:H) = o^-(H)$ (technically not an exception if $o^-(H) = \mathscr{N}$).
- G is a Left end but H is not, and Right wins going first in G: $o^-(G:H) = \mathscr{R}$.
- G is a Right end but H is not, and Left wins going first in G: $o^-(G:H) = \mathscr{L}$.

Theorem 5.1.12. *If G is not an end and $H \geq 0$ then in any universe $\mathcal{X}$, $G:H \geqq G$* (mod $\mathcal{X}$).

Proof. Given a strategy in $G + X$, where $X \in \mathcal{X}$, Left can do at least as well in $G:H + X$ by following this amended strategy: if Right plays in H then respond in H, otherwise follow the original strategy for $G + X$. As $H \geq 0$, Left can always respond to Right's moves in H. Also, G has options for both players so the addition of H is of no benefit to Right. □

Corollary 5.1.13. *If G is not a Left end and $H \in \mathcal{L}^+$, then $G:H \geq^{\between} G$.*

Proof. This is a unification of Theorem 5.1.12 and The Colon Principle (2.2.4). □

If g is a BLUE-RED STALK, its lowest edge is blue if and only if $g > 0$; its lowest edge is red if and only if $g < 0$.

In some cases the subordinate can be replaced with its normal-play canonical form, such as when the subordinate is a BLUE-RED STALK. Recall that the normal-play canonical form of a BLUE-RED STALK is a dyadic rational (by Fact 4.1.3).

5.2 On $*$-based Dyadic Rationals under Misère-play

In this section we present general results useful for analyzing SPRIGS under misère-play.

Corollary 5.2.1. *For any position G, $o^-(*:G) = o^+(G)$.*

Proof. This is a corollary of Theorem 5.1.8 as $* \in \mathcal{P}^-$. □

Theorem 5.2.2. *If $H + \overline{H} \in \mathcal{N}^-$ for all subpositions H of G, then*

$$G + \overline{G} \equiv 0 \pmod{\mathcal{D}}.$$

Proof. Let X be an finite sum of positions in $\mathcal{D}$ and suppose Left wins X. We give a strategy for Left to win $G + \overline{G} + X$. Left follows her original strategy for X unless no move is available in X for Left (or Right) in which case Left plays her winning move in $G + \overline{G}$. If Right at some point plays in $G + \overline{G}$, Left mirrors Right's move leaving a position of the form $G^R + \overline{G^R}$ or $G^L + \overline{G^L}$, which is equivalent to 0 by induction. Right must resume play in X and thus loses. □

Fact 5.2.3. *[4, Corollary 3.4, p. 3] In the universe of dicotic positions, $* + * \equiv 0 \pmod{\mathcal{D}}$.*

Proof. This is a corollary of Theorem 5.2.2. □

Lemma 5.2.4. *If g is a* BLUE-RED STALK *then $*:g + *:\overline{g} \equiv 0 \pmod{\mathcal{D}}$.*

Proof. Any non-zero subposition of $*:g$ is also of the form $*:g'$ for a STALK g' (possibly $g' \cong 0$). Since $\overline{*:g} = *:\overline{g}$, it suffices by Theorem 5.2.2 to show that $*:g + *:\overline{g} \in \mathcal{N}^-$ for any BLUE-RED STALK g.

If $g \cong 0$ then this is just Fact 5.2.3. Otherwise, g is either Left-win or Right-win under normal-play, so assume without loss of generality that $g \in \mathcal{L}^+$. Left playing first on $* : g + * : \overline{g}$ moves $* : \overline{g}$ to 0 and wins playing by Corollary 5.2.1, since $g \in \mathcal{L}^+$. By the same argument, since $\overline{g} \in \mathcal{R}^+$, Right can similarly win this sum playing first, and so $* : g + * : \overline{g} \in \mathcal{N}^-$. □

Corollary 5.2.5. *If x is a dyadic rational then $* : x + * : \overline{x} \equiv 0 \pmod{\mathcal{D}}$.*

We omit the proof of Corollary 5.2.5. It has a proof very similar to that of Lemma 5.2.4. It also follows from Lemma 5.2.4 using Theorem 5.2.8.

Recall that $\mathcal{S}$ denotes the set of SPRIGS positions. In the following results, note the distinction between $G > H$ or $G \geq H$ (normal-play inequality) and $G \gtrdot H$ or $G \geqq H$ (misère-play inequality).

Lemma 5.2.6. *Let g and h be* BLUE-RED STALKS*. If h is blue-based then*

$$* : g : h \gtrdot * : g \pmod{\mathcal{S}}.$$

Proof. By Theorem 5.1.12, $* : g : h \geqq * : g$. It remains to find a position in $\mathcal{S}$ that can distinguish $* : g : h$ and $* : g$.

Consider the position $* : \overline{g} + *$ which is in $\mathcal{S}$. We note $* : g + * : \overline{g} + * \in \mathcal{P}^-$ as $* : g + * : \overline{g} \equiv 0 \pmod{\mathcal{D}}$ by Lemma 5.2.4 and $* \in \mathcal{P}^-$. Going first in $* : g : h + * : \overline{g} + *$, Left wins by moving to $* : g + * : \overline{g} + *$. □

Theorem 5.2.7. *Let g and h be* BLUE-RED STALKS*. If $g > h$ then*

$$* : g \gtrdot * : h \pmod{\mathcal{S}}.$$

Proof. Let k be the longest STALK such that $g = k{:}g'$ and $h = k{:}h'$, for some STALKS g' and h'. Note that k could be empty, but at most one of g' and h' is empty, since $g \neq h$. The STALKS g' and h' cannot begin with the same colour edge, or else k is not maximal, and since $g' > h'$, we know that g' may only start with a blue edge and h' may only start with a red edge. Thus, by Lemma 5.2.6, $* : g \equiv * : k : g' \geqq * : k \geqq * : k : h' \equiv * : h \pmod{\mathcal{S}}$, and equivalence does not hold throughout because not both g and h can be equal to k so at least one inequality is strict. □

In normal-play, a SPRIG with x as the value of the BLUE-RED STALK subordinate has the value $*:x$ since all moves are dominated except for $\left\{0, *: x^L \mid 0, *: x^R\right\}$. We now show the same for misère-play.

Theorem 5.2.8. *The canonical form of a* SPRIG *under misère-play is* $*:x$ *where* x *is the normal-play value of the* BLUE-RED STALK *subordinate of the* SPRIG*.*

Proof. In the BLUE-RED STALK corresponding to x, removing the highest blue (red) edge is the best move under normal-play for Left (Right); any other move is strictly dominated. By Theorem 5.2.7, the same holds true under misère-play SPRIG for red or blue moves. Removing the green edge leaves 0 which is incomparable with $*:g$, namely, $* + 0 \in \mathcal{P}^-$ and $* + *:g \in \mathcal{N}^-$.

For reversibility, assume without loss of generality that Left moves first. If Right hacks a red edge in response, the result is not less than the original position; if Right responds by hacking the green edge, we are at 0 which is incomparable with the original. Therefore the canonical form of the SPRIG is $\left\{0, *: x^L \mid 0, *: x^R\right\} = *{:}x$. □

5.3 Outcomes of Sums

In this section, we give the outcomes under misère-play for disjunctive sums of $*$-based dyadic rationals. If X and Y are multisets of positive dyadic rationals then let (X, Y) denote the disjunctive sum $\sum_{x \in X} *: x + \sum_{y \in Y} *: -y$. A SPRIGS position has the same outcome as a sum of the form (X, Y) or $(X, Y) + *$.

We call $G = (X, Y)$ *reduced* if $X \cap Y = \emptyset$. Under normal-play $*: H + *: \overline{H} = 0$; Lemma 5.2.4 shows this is also true under misère-play in the universe of dicotic positions. Given a position $G = (X, Y)$, there is a *unique reduced* position equal to G, namely, $G' = (X', Y')$, where $X' = X \setminus Y$ and $Y' = Y \setminus X$.

We define the *advantage* of $G = (X, Y)$ to be $\mathrm{A}(G) = |X| - |Y|$ (which is the same as $|X'| - |Y'|$), where $|X|$ denotes the cardinality of the multiset X. We define the *tilt* of G to be $\tau(G) = \min(X') - \min(Y')$. If X' or Y' is empty then we take $\tau(G) = 0$. The tilt cannot be zero in any other way: X' and Y' have no elements in common, so $\min(X') \neq \min(Y')$.

Lemma 5.3.1. *If* $G = (X, Y)$ *with* $\mathrm{A}(G) = 0$ *and* $\tau(G) = 0$*, then* $X = Y$ *and* $G \equiv 0$ $(\mathrm{mod}\ \mathcal{D})$*.*

Proof. As $\mathrm{A}(G) = 0$, $|X| = |Y|$ and $|X'| = |Y'|$. As $\tau(G) = 0$, at least one of X' or Y' must be empty, so they both must be as they are the same size. This means that all terms cancel in $G = (X, Y)$, and so $G \equiv 0 \pmod{\mathcal{D}}$. □

The advantage of a position is sufficient to determine the outcome of the position.

Theorem 5.3.2. *If* $G = (X, Y)$ *then*

$$o^-(G) = \begin{cases} \mathscr{L} \textit{ if } \mathrm{A}(G) > 0; \\ \mathscr{R} \textit{ if } \mathrm{A}(G) < 0; \\ \mathscr{N} \textit{ if } \mathrm{A}(G) = 0. \end{cases}$$

Proof. Let $G = (X, Y)$. We proceed by induction on $|X| + |Y|$. If $|X| + |Y| = 0$ then $G = 0$ which is a next-player win. If $|X| = 1$ and $|Y| = 0$ then $G \in \mathcal{L}^-$ by Corollary 5.2.1.

Suppose $|X| = |Y| > 0$. As $|Y| > 0$, Left going first can move some $* : \overline{y}$ to 0; the resulting position, G^L, has $\mathrm{A}(G^L) > 0$ and so Left wins by induction. Similarly, Right can win moving first. Thus, if $\mathrm{A}(G) = 0$, then $G \in \mathscr{N}^-$.

Suppose $|X| > |Y|$. If $|Y| > 0$, then Left wins going first as above. If $|Y| = 0$ then we need only consider $|X| > 1$. Left wins going first by moving $* : x$ to 0 for some $x \in X$, which is a winning move by induction. If Right moves first to G^R such that $\mathrm{A}(G^R) \geq 0$, then Left wins because $G^R \in \mathscr{N}^- \cup \mathcal{L}^-$; if Right's move does not change A, then Left wins using the first-player argument previously given.

Omitted arguments for $|X| < |Y|$ are similar. □

Lemma 5.3.3. *Let* $G = (X, Y)$ *and consider* $G + *$. *If* $\mathrm{A}(G) > 0$ *then*

- *Left can win by playing first;*
- *if Right can win playing first, he can do so moving* $*{:}x$ *to* 0 *where* $x = \max(X')$.

Proof. Playing in $G + *$, if $\mathrm{A}(G) > 0$ then Left can win by playing first by moving $*$ to 0, leaving G which is winning by Theorem 5.3.2.

If Right does not move a SPRIG to 0, then he must play in some $* : x$ to $* : x^R$. As $x^R > x$, Theorem 5.2.7 then says that $* : x^R \gtrdot * : x$. By the definition of $\gtrdot$, the new position is better for Left and so Left wins playing first from this position.

Right loses if he does not eliminate a SPRIG. By Theorem 5.2.7 Right's options, such as $G+*-*:x_1$, are ordered and so Right eliminates the SPRIG that is best for the opponent. $\square$

The advantage and tilt of a position are sufficient to determine the outcome of the sum of a position with $*$.

Theorem 5.3.4. *If $G=(X,Y)$ then*

$$o^-(G+*)=\begin{cases}\mathscr{L} \textit{ if } \mathrm{A}(G)>1 \textit{ or } \mathrm{A}(G)=0,1 \textit{ and } \tau(G)>0;\\ \mathscr{R} \textit{ if } \mathrm{A}(G)<-1 \textit{ or } \mathrm{A}(G)=0,-1 \textit{ and } \tau(G)<0;\\ \mathscr{N} \textit{ if } \mathrm{A}(G)=1 \textit{ and } \tau(G)\leq 0 \textit{ or } \mathrm{A}(G)=-1 \textit{ and } \tau(G)\geq 0;\\ \mathscr{P} \textit{ if } \mathrm{A}(G)=0 \textit{ and } \tau(G)=0.\end{cases}$$

Proof. The outcome is the same as the outcome of the reduced position, so assume $G=(X,Y)$ is reduced and let $H=G+*$. We proceed by induction on $|X|+|Y|$. We focus on Left; omitted arguments for Right are similar.

- If $|X|+|Y|=0$ then $H=*$ which is a previous-player win.

- If $|X|+|Y|=1$ then $H\in\mathscr{N}^-$ as either player wins by moving to $*$.

- If $|X|=|Y|=1$, then $H=*:x+*:\overline{y}+*$. The first player to move any summand to 0 loses; play proceeds in x and $\overline{y}$ until someone is forced to do so, at which point the opponent responds by eliminating a second SPRIG, leaving a SPRIG that is at least as good for them as $*$. In particular, $o^-(H)=o^+(x-y)=o^+(\tau(G))$.

- If $|X|=|Y|>1$ and $\tau(G)>0$, then Left wins going first in H to some H^L by moving one of Right's SPRIGs to 0, changing Left's advantage to 1. By Lemma 5.3.3, Right must respond by moving one of Right's SPRIGs to 0, leaving H^{LR}. This position is in $\mathcal{L}^-$ by induction as both Left and Right will have removed the opponent's best sprig and the tilt has thus not changed. As H is a win for Left going first, when Right goes first he must move one of Left's SPRIGs to 0, to which Left responds by moving to $G^{RL}=G^{LR}$, a winning move.

- If $|X| > |Y|$ then Left can win going first by Lemma 5.3.3. Right's best move going first is to move a Sprig to 0. If $|X|-|Y| > 2$, Right's move loses. However, if $|X| - |Y| = 1$ and $\tau(G) < 0$, then Right wins by induction.

□

5.4 Distinguishability

In normal-play, any given position G is equivalent to many other positions (the other positions in its equivalence class). It is generally true that under misère-play there are few positions equivalent to (indistinguishable from) G. Under normal-play 'modding out' by equivalence leads to a group. Under misère-play, 'modding out' by equivalence leads to a monoid. This was part of Plambeck's breakthrough [29].

Lemma 5.4.1. *Let $G = (X_1, Y_1)$ and $H = (X_2, Y_2)$ be reduced positions. If $\beta \in \{0, 1\}$, then $G + \beta \cdot * \equiv H + \beta \cdot * \pmod{\mathcal{D}}$ if and only if $X_1 = X_2$ and $Y_1 = Y_2$.*

Proof. The sufficiency is clear. Conversely, suppose without loss of generality that $X_1 \neq X_2$. In this case, the reduced form of $G + \overline{H}$ is not 0: without loss of generality there is at least one $x \in X_1$ that is not in X_2, so there is no $\overline{* : x} = * : -x$ in $\overline{H}$, and since G is reduced, there is no $* : -x$ in G. So $G + \overline{H}$ has at least the term $* : x$. The positions $G + *$ and $H + *$ are now distinguished by $\overline{H}$, since by Lemma 5.3.4, $G + \overline{H} + * \notin \mathcal{P}^-$ while $H + \overline{H} + * \equiv * \pmod{\mathcal{D}}$ and $* \in \mathcal{P}^-$. Similarly G and H are distinguished by $\overline{H} + *$. □

Lemma 5.4.2. *If $G = (X_1, Y_1)$ and $H = (X_2, Y_2)$, then $G \not\equiv H + * \pmod{\mathcal{S}}$.*

Proof. The positions are distinguished by $\overline{H}$, since $H + \overline{H} + * \in \mathcal{P}^-$ while $G + \overline{H} \notin \mathcal{P}^-$. □

Corollary 5.4.3. *Let $G = (X_1, Y_1)$ and $H = (X_2, Y_2)$, $G \equiv H \pmod{\mathcal{D}}$ if and only if $X_1 = X_2$ and $Y_1 = Y_2$.*

Our only reductions are $* + * \equiv 0 \pmod{\mathcal{D}}$ and $* : x + * : -x \equiv 0 \pmod{\mathcal{D}}$, which follow from Fact 5.2.3 and Lemma 5.2.4.

For any sum of $*$-based dyadic rational positions (or any SPRIGS position), the outcome is unchanged by adding $*$ and toggling between normal and misère play conventions. This is based on comparing Theorem 5.3.2 to Theorem 4.6.6 Case a and Theorem 5.3.4 to Theorem 4.6.6 Case b, the normal-play equivalents.

Corollary 5.4.4. *If G is a collection of $*$-based dyadic rationals, then $o^+(G) = o^-(G+*)$ and $o^+(G+*) = o^-(G)$.*

A result analogous to Corollary 5.4.4 but where G is a GENERALIZED FLOWER is investigated in [18], where Lo manages to reduce the outcome question for FLOWERS under misère-play to the normal-play case, which is open (see 4.2).

5.5 Discussion

Generalizing results from $*$ to $*^2$ (and other nimbers) is troublesome because $*^2+*^2 \not\equiv 0 \pmod{\mathcal{D}}$. However, the relative simplicity of the partizan component makes this appear tractable.

Theorems in Section 5.2 and Corollary 5.2.5 give great hope for further progress in the study of misère-play games, especially dicotic positions and ordinal sums.

We focused on SPRIGS and the dicotic universe, which gives a very dicotic feel to the whole Chapter. Theorem 5.2.2 is critical for us but only requires a dicotic universe, we can build on it with non-dicotic positions.

Theorem 5.5.1. *If all subpositions of G are in $\mathcal{L}^-$ and have a Right option to 0 or are empty, then $G + \overline{G} \equiv 0 \pmod{\mathcal{D}}$.*

Proof. If G is empty so is $\overline{G}$ and thus $G+\overline{G} \in \mathcal{N}^-$. Otherwise, from $\overline{G}$ Left has an option to 0, so $G+\overline{G}$ has a Left option to G which is in $\mathcal{L}^-$. By Lemma 5.1.3, Right can similarly win by playing to $\overline{G}$; thus $G+\overline{G} \in \mathcal{N}^-$ and the result holds by Theorem 5.2.2. □

Furthermore, combining this with our work on ordinal sums we find another family of positions that satisfy $G + \overline{G} \equiv 0 \pmod{\mathcal{D}}$.

Theorem 5.5.2. *If $G \cong 1 : H$, then $G + \overline{G} \equiv 0 \pmod{\mathcal{D}}$.*

Proof. By Lemma 5.1.10 every non-empty subposition of $\overline{G} \cong \overline{1} : \overline{H}$ is in $\mathcal{L}^-$. As every non-empty subposition of $\overline{1} : \overline{H}$ has a Right option to 0, Theorem 5.5.1 gives $\overline{G} + \overline{\overline{G}} \cong \overline{G} + G \equiv 0 \pmod{\mathcal{D}}$. □

Chapter 6

Partizan Euclid

PARTIZAN EUCLID is a game based on the Euclidean Algorithm. The outcome of any position (p, q) is determined by a single path of the game tree, this path has connections to the furthest integer continued fraction of p/q. We convert the question of 'Who wins?' to a word problem, then give a list of reductions that reduces the word/position to one of 9 positions. Work in this Chapter has been published in [23].

Suggested by EUCLID [10] and Richard K. Guy, the game of PARTIZAN EUCLID is played on a pair of positive integers (p, q) with $p > q$. Let $p = kq + t$ where $0 \leq t < q$. If $q \mid p$ (i.e. $t = 0$) then the game is finished, otherwise, Left moves to (q, t) and Right moves to $(q, q - t)$.

The game may seem trivial as there is only one move available for each player. However, as we shall show, answering the question 'Who wins?' reveals some of the interesting structure of the game. We would like to answer the question of who wins in the *disjunctive sum* of this game, but this appears to be difficult. See the last section for a discussion of that problem.

In the (impartial) game EUCLID, which is also played with (p, q), a pair of positive integers, a player is allowed to remove any multiple of the smaller from the larger provided the remainder is positive. Lengyel [17] reports that Schwartz first found that EUCLID is the sequential sum [35] of nim-heaps: given (p, q), suppose the normal continued fraction of $\frac{p}{q}$ is $[a_1, a_2, \ldots, a_n]$ ($a_n > 1$ except if Fibonacci numbers are involved) then the EUCLID position (p, q) corresponds to playing the sequential sum of NIM with nim-heaps $a_1, a_2, \ldots, a_n$. EUCLID has attracted much attention and has been generalized, see [11,12] for example. PARTIZAN EUCLID is related to *nearest* and *farthest* integer continued fractions (NICF and FICF) (see [28]).

In the case of both NICFs and FICFs we write rational numbers as a sum or difference of an integer and a rational less than 1. For example, the FICF for $\frac{11}{8}$ is obtained by rewriting, noting that

- $\frac{11}{8} = 2 - \frac{1}{8/5}$ since 2 is further away from $\frac{11}{8}$ than 1;
- $\frac{8}{5} = 1 + \frac{1}{5/3}$ since 1 is further away than 2;
- $\frac{5}{3} = 1 + \frac{1}{3/2}$ since 1 is further away than 2;
- and $\frac{3}{2} = 2 - \frac{1}{2/1} = 1 + \frac{1}{2/1}$ since 1 and 2 are equally distant.

We are not interested in the continued fraction itself but in noting that during the calculation (i) 'integer subtract fraction' corresponds to a move by Right and (ii) 'fraction subtract integer' corresponds to a Left move. We'll use the word *rlle* to represent this where e is the common move to $(2, 1)$. Section 6.1 reports on the structure of the game tree and shows there is one path, the path obtained from the FICF algorithm, that determines the whole game tree.

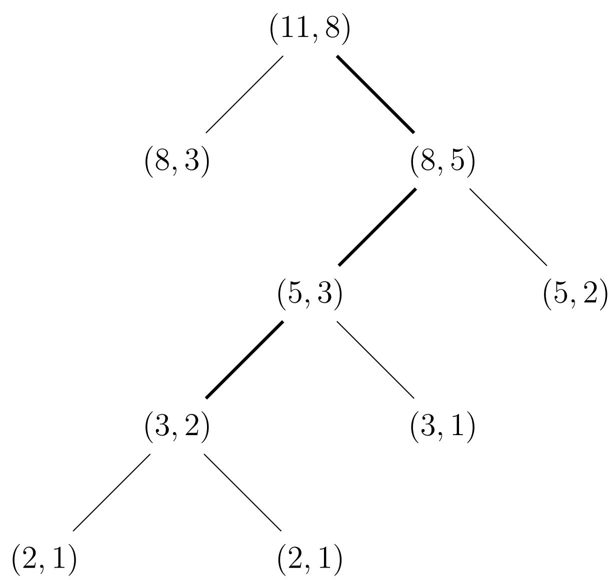


Figure 6.1: Some of the game tree of $(11, 8)$

For example, in Figure 6.1 the path formed by the moves

$$(11, 8) \xrightarrow{R} (8, 5) \xrightarrow{L} (5, 3) \xrightarrow{L} (3, 2)$$

contains positions isomorphic to all non-trivial parts of the tree that are not on that path; '$(8, 3)$' is isomorphic to '$(5, 3)$', '$(5, 2)$' is isomorphic to '$(3, 2)$', and '$(3, 1)$' is isomorphic to '$(2, 1)$'. All the information needed to determine the outcome and value of $(11, 8)$ is found on this path.

A game tree (position) is represented as a word from the alphabet r, l ending in e. In Lemma 6.2.3, we find reduction rules that preserve the outcome of the word, moreover, any word reduces to one of just 9 words each with length at most 4. This can be accomplished in time linear in the length of the corresponding FICF. Unfortunately, these reductions most of the time do not preserve the value.

We will denote a game position as $E(p, q)$. Also, we will use $p\%q$ for $p \bmod q$. If $t = p\%q$, then

$$E(p,q) \cong \begin{cases} 0 & \text{if t = 0} \\ \{E(q,t) \mid E(q,q-t)\} & \text{otherwise.} \end{cases}$$

Let $g \cong E(p, q)$; as there is at most one option for each player we will abuse notation and in place of $o(g) = o\left(\left\{g^L \,\middle|\, g^R\right\}\right)$ we write $\left\{o(g^L) \,\middle|\, o(g^R)\right\}$. For example,

$$\{\mathscr{N} \mid \mathscr{P}\} = \mathscr{R}.$$

The reader may review Fact 1.3.4 or reference Table 6.1.

		$o(G^R)$	
		$\mathscr{P}$ or $\mathscr{R}$	$\mathscr{L}$ or $\mathscr{N}$
$o(G^L)$	$\mathscr{L}$ or $\mathscr{P}$	$\mathscr{N}$	$\mathscr{L}$
	$\mathscr{N}$ or $\mathscr{R}$	$\mathscr{R}$	$\mathscr{P}$

Table 6.1: Outcome of a position with exactly one option for each player

6.1 Game Tree Structure

Lemmas 6.1.1 and 6.1.2 each show that for every position there are infinitely many congruent positions. Positions G and H are congruent if they have the same game tree and we may write $G \cong H$.

Lemma 6.1.1. *For all k, $E(kp, kq) \cong E(p, q)$.*

Proof. Recall that $q \mid p$ if and only if $kq \mid kp$. Thus, $E(p, q) \cong \{\cdot \mid \cdot\}$ if and only if $E(kp, kq) \cong \{\cdot \mid \cdot\}$. Let $t = p\%q$, then by induction $E(p, q) \cong \{E(q, t) \mid E(q, q - t)\} \cong \{E(kq, kt) \mid E(kq, kq - kt)\} \cong E(kp, kq)$. □

Note that if $h \cong E(n, m)$ is a subposition of a position $g \cong E(p, q)$ with $\gcd(p, q) = 1$, then $\gcd(n, m) = 1$. In the rest of this Chapter we will assume that every position has $\gcd(p, q) = 1$ and thus $p\%q = 0$ if and only if $q = 1$.

Lemma 6.1.2. *If $p > 2q$ then $E(p, q) \cong E(p - q, q)$.*

Proof. Let $p = kq + t$, $0 \leq t < q$ and $k \geq 2$. Then $p - q = (k-1)q + t$, $0 \leq t < q$ and $k - 1 \geq 1$. Consider the options of both positions:

$$E(p, q) \cong \{E(q, t) \mid E(q, q - t)\}$$
$$E(p - q, q) \cong \{E(q, t) \mid E(q, q - t)\}.$$

Since they have identical options the two positions are congruent. □

A position $E(p, q)$ will be called *standard* if $q < p < 2q$. All positions $E(p, q)$ with $q > 2$ are congruent to some standard position (which we derive with repeated applications of Lemma 6.1.2). Notably $E(2, 1)$ is not standard and positions of the form $E(k, 1)$ are neither standard nor congruent to some standard position, but are all congruent to each other. A subposition of a standard position may not be standard. For example, $E(3, 2)$ has exactly one proper subposition, $E(2, 1)$, which is not standard.

Lemma 6.1.3. *Let $g \cong E(p, q)$ and $t = p\%q$. If $t \neq 0$ then g has exactly one standard option except when $q = 2t$ (i.e. $E(3, 2)$). Moreover;*

- *if $2t > q$ then exactly g^L is standard,*
- *if $0 < 2t < q$ then exactly g^R is standard.*

Proof. As $t > 0$, $g^L \cong E(q, t)$ and $g^R \cong E(q, q - t)$. Recall by the definition of standard; that $t > \frac{q}{2}$ if and only if g^L is standard and $t < \frac{q}{2}$ if and only if g^R is standard. Otherwise, $2t = q$, implying that $p = 3t$ and subsequently that $g \cong E(3, 2)$; from $E(3, 2)$ both players have the option $E(2, 1)$, which is not standard. □

Let h be a PARTIZAN EUCLID position. There is a unique standard position with Left option h and a unique standard position with Right option h.

Lemma 6.1.4. *Let $g \cong E(p, q)$ be standard and let $0 < t < q$.*

- *If $g^L \cong E(q, t)$ then $g \cong E(q + t, q)$.*
- *If $g^R \cong E(q, q - t)$ then $g \cong E(q + t, q)$.*

Proof. If $g^L \cong E(q,t)$ then $p = kq + t$; as g is standard, $k = 1$. If $g^R \cong E(q, q-t)$ then $p = kq + (t-q) = (k-1)q + t$; as g is standard, $k = 2$. $\square$

The essential game tree structure is given by Theorem 6.1.5.

Theorem 6.1.5. *Let $g \cong E(p,q)$ and $t = p\%q$.*

- *If $t = 0$ then $g \cong 0$.*
- *If $2t = q$ then $g^L \cong g^R$.*
- *If $2t > q$ then $g^{LL} \cong g^R$.*
- *If $0 < 2t < q$ then $g^L \cong g^{RL}$.*

Proof. If $t = 0$ then the game is finished and $g \cong 0$. Thus we suppose $t > 0$ and hence $g^L \cong E(q,t)$ and $g^R \cong E(q, q-t)$.

Suppose $2t = q$, then $g^L \cong E(q,t) \cong E(q, q-2t+t) \cong E(q, q-t) \cong g^R$. Moreover, $q - t \mid q$, so $E(q,t) \cong E(q, q-t) \cong 0$. That is, $g \cong \{0 \mid 0\} \cong *$.

Suppose $2t > q$, then $q > 2(q-t)$ and $t > q - t$ so $g^R \cong E(q, q-t) \cong E(q-(q-t), q-t) \cong E(t, q-t)$ by Lemma 6.1.2, and $g^{LL} \cong E(q,t)^L \cong E(t, q-t)$, giving

$$g^R \cong E(t, q-t) \cong g^{LL}.$$

Suppose $2t < q$, then $g^L \cong E(q,t) \cong E(q-t, t)$ by Lemma 6.1.2. Since $g^R \cong E(q, q-t)$ we have that $g^{RL} \cong E(q-t, t)$, giving $g^{RL} \cong g^L$. $\square$

Corollary 6.1.6. *Let $p > q$ with $p = kq + t$, $0 < t < q$ and let $g \cong E(p,q)$.*

- *If $2t > q$ then $g \cong \left\{g^L \,\middle|\, g^{LL}\right\}$.*
- *If $2t < q$ then $g \cong \left\{g^{RL} \,\middle|\, g^R\right\}$.*

Proof. This is a simplification of Theorem 6.1.5. $\square$

We note the similarity of Lemma 6.1.3 and Corollary 6.1.6. In Corollary 6.1.6 we see that the two options of a position are the standard option and the standard option's Left option. The case $2t > q$ is when the lower integer is the 'farthest' integer when calculating the FICF and $2t < q$ is when the higher integer is the 'farthest'.

This motivates our next definition, the signature of a position, in which we highlight the standard option at each stage. Recall that when we refer to $E(p,q)$ we are assuming that $\gcd(p,q) = 1$.

Definition 6.1.7. *Let $g \cong E(p,q)$. The* signature *of g, denoted S_g, is defined as follows. If $q = 1$ then $S_g \cong \lambda$, the empty word. If $q = 2$ then $S_g \cong e$. Otherwise, let the standard option from g be h and let S_h be the signature of h. If $g^L \cong h$ then $S_g \cong lS_h$. If $g^R \cong h$ then $S_g \cong rS_h$.*

The position g and the standard positions that are successively the standard option (as per Lemma 6.1.3) starting from g are the spine *of g.*

For example, if $g \cong E(12, 7)$ then the signature of g is $lrle$ and the spine of g is $\{E(12, 7), E(7, 5), E(5, 3), E(3, 2)\}$. Often, we will write the signature with superscripts; for example, $lllrrllre$ is the same as $l^3r^2l^3e$.

If two positions have the same signature then they are congruent (by Lemma 6.1.4 and induction). We use signatures liberally to represent positions. Furthermore, we use αf to denote the position g where $S_g \cong \alpha S_f$.

The position $E(3, 2)$ is the unique standard position with signature e. This position is at the bottom of every spine for every position other than $E(k, 1)$ and $E(2k + 1, 2)$ for $k \geq 2$.

Theorem 6.1.8. *Let g be a* PARTIZAN EUCLID *position. Every subposition of g not of the form $E(k, 1)$ is congruent to some position on the spine of g.*

Proof. A position is on its spine, so we only need consider *proper* subpositions. If the length of the signature is 0 then there are no proper subpositions. If the length of the signature is 1 then $S_g \cong e$ and $g \cong E(3, 2)$ where the only proper subposition is $E(2, 1)$. We proceed by induction on the length of signature. If the length of the signature is at least 2 then the standard option is on the spine; the non-standard option is (by Theorem 6.1.5) either $g^L \cong g^{RL}$ or $g^R \cong g^{LL}$, which is the Left option of the standard option and is by induction on the spine of the standard option or of the form $E(k, 1)$. As the spine of the standard option is part of the spine of g, this completes the proof. □

Corollary 6.1.9. *Consider the position g, and let k represent an unfixed non-negative integer.*

We can write S_g as either $r^k l\alpha e$ or $r^k e$; Left's move has signature αe or λ, respectively.

We can also write S_g as either $r\alpha e$, $lr^k l\alpha e$, or $lr^k e$; Right's move has signature αe, αe, or λ, respectively.

Proof. If S_g is $r^k l\alpha e$ or $r^k e$, then S_{g^L} is αe or λ, respectively, because $g^L \cong g^{RL} \cong g^{RRL} \cong g^{RRRL} \cong \cdots \cong g^{R^kL}$.

If $S_g \cong r\alpha e$ then $S_{g^R} \cong \alpha e$. Otherwise, $g^R \cong g^{LL}$ so S_{g^R} is the signature of the position resulting from two Left moves namely αe or λ, as seen by the first part. $\square$

In the examples below, we repeatedly use Theorem 6.1.5, but using Corollary 6.1.9 one can easily jump from the leftmost term in a line of congruences to the rightmost. Let $g \cong llrle$ then

$$\begin{aligned}
e &\cong \left\{e^L \,\middle|\, e^R\right\} \cong \{\lambda \mid \lambda\},\\
le &\cong \left\{le^L \,\middle|\, le^R\right\} \cong \left\{e \,\middle|\, le^{LL}\right\} \cong \left\{e \,\middle|\, e^L\right\} \cong \{e \mid \lambda\},\\
rle &\cong \left\{rle^L \,\middle|\, rle^R\right\} \cong \left\{rle^{RL} \,\middle|\, le\right\} \cong \left\{le^L \,\middle|\, le\right\} \cong \{e \mid le\},\\
lrle &\cong \left\{lrle^L \,\middle|\, lrle^R\right\} \cong \left\{rle \,\middle|\, lrle^{LL}\right\} \cong \left\{rle \,\middle|\, rle^L\right\} \cong \left\{rle \,\middle|\, rle^{RL}\right\} \cong \{rle \mid e\},\\
llrle &\cong \left\{llrle^L \,\middle|\, llrle^R\right\} \cong \left\{lrle \,\middle|\, llrle^{LL}\right\} \cong \left\{lrle \,\middle|\, lrle^L\right\} \cong \{lrle \mid rle\}.
\end{aligned}$$

6.2 Reducing the Signature

The *paired outcome* of a position g (or signature S_g) is the pair $(o(g^L), o(g))$, denoted by $po(g)$ or $po(S_g)$. For example, if $S_g \cong e$, then $po(g) = po(e) = (\mathscr{P}, \mathscr{N})$. Note that $po(\lambda)$ is not defined.

The definition of paired outcome shows the inherent asymmetry of PARTIZAN EUCLID; this is a particular object that helps with the analysis of outcomes for this ruleset.

Lemma 6.2.1. *Let g be a* PARTIZAN EUCLID *position with exactly one standard option, h. If $S_g \cong lS_h$ then $po(g) = (o(h), \left\{o(h) \,\middle|\, o(h^L)\right\})$. If $S_g \cong rS_h$ then $po(g) = (o(h^L), \left\{o(h^L) \,\middle|\, o(h)\right\})$.*

Proof. Follows immediately from Theorem 6.1.5. $\square$

That is, the paired outcome of a position, g, with a standard option, h, is determined by the paired outcome of h. We use l and r to denote functions on paired outcomes; we write $l \circ po(S_g)$ to mean $po(l \circ S_g)$ and $r \circ po(S_g)$ to mean $po(r \circ S_g)$.

There are $4 \times 4 = 16$ ordered pairs of outcomes. However, as stated in Lemma 1.3.4, there are relationships between the outcome of a position and the outcome of its options; there are only 8 ordered pairs that are *paired outcomes* of positions.

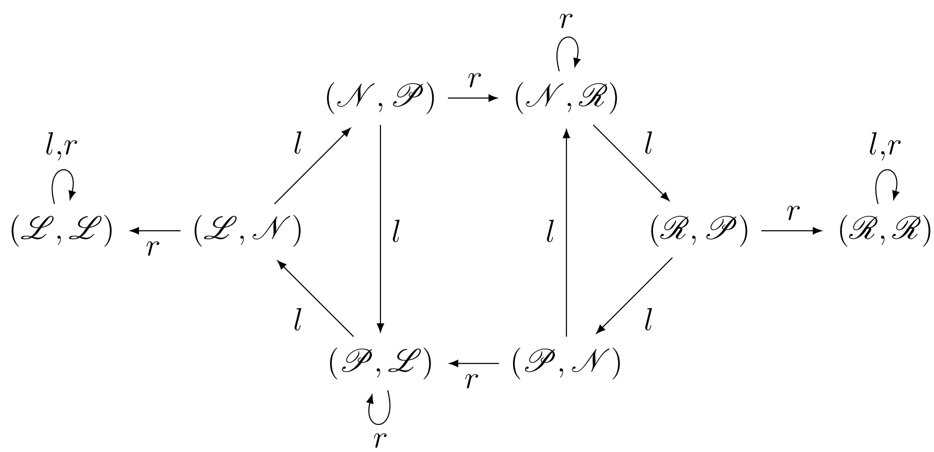


Figure 6.2: Paired outcome of signatures.

Figure 6.2 has a vertex for each of the 8 paired outcomes. The directed edges labelled l and r from a paired outcome, say $\mathbf{x}$, lead to $l \circ \mathbf{x}$ and $r \circ \mathbf{x}$, respectively. The correctness of Figure 6.2 can be verified using Lemma 6.2.1.

The outcome of a position is given by the paired outcome of the signature. Provided we know the paired outcome of some suffix of the signature, we can find the paired outcome of the next larger suffix using Figure 6.2 and eventually the desired paired outcome. As e is the suffix of every non-empty signature, we only need to know that $po(e) = po(E(3,2)) = (\mathscr{P},\mathscr{N})$ is where we start and to read the signature from the right starting after e.

We have now described a relatively efficient method to determine the outcome of a position given its signature, but we give a better way to determine the outcome than a walk through the graph for each letter in the signature.

Just as l and r are functions on paired outcomes, we use words (denoted by Greek letters) from $\{l,r\}^*$ such as $\alpha = lrr$ as functions on paired outcomes in the natural way where $\alpha \circ \mathbf{x} = l \circ r \circ r \circ \mathbf{x}$. For any such α, $\alpha \circ po(S_h) = po(\alpha S_h)$.

We give reduction rules by which we can simplify a word (signature) that preserves outcome, in the sense that if two positions have the same reduced signature then they have the same outcome. The main goal of this section is to prove Theorem 6.2.5, in which we give a short list of words to which any signature will reduce.

Lemma 6.2.2. *If* $\alpha \in \{l, r\}^*$ *then* $\alpha \circ (\mathscr{L}, \mathscr{L}) = (\mathscr{L}, \mathscr{L})$ *and* $\alpha \circ (\mathscr{R}, \mathscr{R}) = (\mathscr{R}, \mathscr{R})$.

Proof. Immediate from Figure 6.2. □

Lemma 6.2.3. *Let* $\mathbf{x}$ *be a paired outcome.*

1. $l^3 \circ \mathbf{x} = \mathbf{x}$;
2. $r^2 \circ \mathbf{x} = r \circ \mathbf{x}$;
3. $\alpha rlr \circ \mathbf{x} = rlr \circ \mathbf{x}$;
4. $r(llr)^2 \circ \mathbf{x} = r \circ \mathbf{x}$;
5. $rllr \circ (\mathscr{P}, \mathscr{N}) = l \circ (\mathscr{P}, \mathscr{N})$;
6. $\alpha rll \circ (\mathscr{P}, \mathscr{N}) = rll \circ (\mathscr{P}, \mathscr{N})$;
7. $rl \circ (\mathscr{P}, \mathscr{N}) = l \circ (\mathscr{P}, \mathscr{N})$;

Proof. In this proof we make extensive use of Figure 6.2. The most common use is to find a vertex with the desired paired outcome then look up the paired outcomes reached by the directed edges. For the first 4 rules, we need to show that these equation holds for all paired outcomes $\mathbf{x}$.

Rule 1: we apply l to each $\mathbf{x}$ three times to observe that we are still at $\mathbf{x}$ or have returned to where we started.

Rule 2: following two edges marked r and, regardless, the second edge is a loop.

Rule 3: following three edges marked r, l and r results in $(\mathscr{L}, \mathscr{L})$ or $(\mathscr{R}, \mathscr{R})$ so the first part of the signature it is irrelevant by Lemma 6.2.2.

Rule 4: if we start at $(\mathscr{L}, \mathscr{L})$ or $(\mathscr{R}, \mathscr{R})$ then we remain at $(\mathscr{L}, \mathscr{L})$ or $(\mathscr{R}, \mathscr{R})$ respectively; if we start at $(\mathscr{L}, \mathscr{N})$ or $(\mathscr{R}, \mathscr{P})$ then the first r-edge goes to $(\mathscr{L}, \mathscr{L})$ or $(\mathscr{R}, \mathscr{R})$, respectively, and we remain there. Otherwise, following an r puts us at either $(\mathscr{P}, \mathscr{L})$ or $(\mathscr{N}, \mathscr{R})$; following llr puts us at the other and following llr again returns us to where we were after following one r.

Rule 5: obvious from the figure.

Rule 6: from $(\mathscr{P}, \mathscr{N})$ the walk l, l, r ends at $(\mathscr{R}, \mathscr{R})$ and any further edges does not change the paired outcome.

S_g	$o(g)$	g
λ	$\mathscr{P}$	$E(2,1)$
e	$\mathscr{N}$	$E(3,2)$
re	$\mathscr{L}$	$E(4,3)$
le	$\mathscr{R}$	$E(5,3)$
lle	$\mathscr{P}$	$E(8,5)$
lre	$\mathscr{N}$	$E(7,4)$
$rlre$	$\mathscr{L}$	$E(10,7)$
$rlle$	$\mathscr{R}$	$E(11,8)$
$llre$	$\mathscr{P}$	$E(11,7)$

Table 6.2: Irreducible signatures with corresponding positions and outcomes

Rule 7: obvious from the figure. □

Lemma 6.2.4. *If* $\mathbf{x}$ *is a paired outcome and* $\gamma \circ \mathbf{x} = \delta \circ \mathbf{x}$*, then* $po(\alpha\gamma\beta e) = po(\alpha\delta\beta e)$*.*

Proof. $po(\alpha\gamma\beta e) = \alpha\gamma \circ po(\beta e) = \alpha\delta \circ po(\beta e) = po(\alpha\delta\beta e)$. □

In applying Lemma 6.2.4 to reduce a word, we make reference to particular rules from 6.2.3.

We call a word *irreducible* if none of the reduction rules are applicable. Reduction rules may be applied in any order to the signature of a position, say g, to derive an irreducible word; the irreducible word corresponds to some other position, say f. The outcome of g is the same as the outcome of f, as they have the same paired outcome, which is a stronger condition. As there are a finite number of irreducible words, we will be able to compute and store the outcomes of those positions.

Theorem 6.2.5. *There are 9 irreducible words:* λ*,* e*,* re*,* le*,* lle*,* lre*,* $rlre$*,* $llre$*, and* $rlle$*.*

For the proof of Theorem 6.2.5 we need the following Lemma:

Lemma 6.2.6. *A word containing 4* r*s is reducible.*

Proof. Suppose α is an irreducible word containing 4 rs. By Rule 2, each pair of consecutive rs is separated by at least one l. By Rule 3, each pair of rs except possibly the leftmost, is separated by more than one l. By Rule 1, each pair of rs is separated by at most 2 ls. That is, the rightmost 3 rs form the pattern $rllrllr$, which contradicts the assumption that α is irreducible by Rule 4. □

Proof of Theorem 6.2.5. The irreducible words that do not end in e are easy to list and count with the help of Lemma 6.2.6; such words have at most 3 rs and by Rule 1 at most 2 consecutive ls. In this proof, α and β are one of either λ, l, or ll.

Words with 3 rs are of the form $rlrllr\beta$, of which there are 3.

Words with 2 rs are of the form $\alpha rllr\beta$ or $rlr\beta$, of which there are 12.

Words with 1 r are of the form $\alpha r\beta$, of which there are 9.

Words with no r are of the form α, of which there are 3.

There are a total of 27 such words. The only irreducible signatures are among the set containing these strings but with a trailing e appended, and the empty word.

We show that 19 of the 27 strings reduce to the remaining 8: e, le, lle, re, lre, $llre$, $rlle$ and $rlre$.

- The 7 strings of the form $\gamma rlle$ where γ is non-empty reduce by Rule 6 to $rlle$.

- The 4 words of the form $\gamma rllrle$ reduce to $\gamma rllle$ and then to γre by Rules 7 and 1, respectively, leaving re, lre, $llre$, and $rlre$.

- The 4 words of the form $\gamma rllre$ reduce to γle by Rule 5, leaving the 3 strings with no r (note $llle = e$) and $rlle$.

- The 4 words of the form γrle reduce to γle by Rule 7, leaving the 3 strings with no r (note $llle = e$) and $rlle$.

None of the reductions apply to the 9 claimed irreducible words (8 from above and λ). $\square$

6.2.1 Algorithm

We present an algorithm that efficiently determines the outcome of a PARTIZAN EUCLID position.

Step 0: Let S be the signature of $E(p,q)$. Let S' be the empty string.

Step 1: If S is non-empty, remove the first letter of S and add it to the end of S'; go to Step 2. Otherwise, go to Step 3.

Step 2:

- If you added l to S', use Rule 1 on the suffix of S' if applicable. Go to Step 1.
- If you added r to S', use Rule 2, 3 or 4 on the suffix of S' if applicable, at most one will apply. Go to Step 1.
- If you added e to S', use Rule 5, 6 or 7 on the suffix of S' if applicable, at most one will apply. If you applied Rule 5 or 7, then use Rule 1 if applicable. Go to Step 3.

Step 3: The outcome of $E(p,q)$ is the outcome of S' given in Table 6.2.

Reductions occur at the end of the word and the application of a reduction does not cause another reduction, except possibly in Step 2: part 3, with an lll reduction. As such, each Step 2 finishes in constant time (as do Steps 1 and 3). Step 1 takes about as long as the Euclidean algorithm. Steps 1 and 2 have to be performed at most p times; Steps 0 and 3 are each performed once.

By Lemma 6.2.6, S' is of length of at most 8, as demonstrated by $llrllrll$. That is, if at any point the length of S' is 9, then it will be irreducible in the next step. In the algorithm as given above, S is computed in full at the beginning, for ease of description. However, we can easily modify our algorithm to be an on-line algorithm by computing the next letter of S as we need it to add to S'. In that case, to run the algorithm we store at most 3 integers no larger than p and a string of length at most 9.

6.3 Outcome Observations

There are several interesting observations that can be made about the outcomes which may be useful in actual play.

Observation 6.3.1. *If $S_g = r\beta e$ then $o(g) \in \{\mathscr{L}, \mathscr{R}\}$.*

All signatures in Table 6.2 starting with r are in $\mathcal{L}$ or $\mathcal{R}$. The reductions (from Lemma 6.2.3) change signatures starting with r to shorter signatures starting with r or to le, which is in $\mathcal{R}$.

Observation 6.3.2. *Let $g \cong E(p,q)$ be a standard position. If $o(g) \in \{\mathscr{N}, \mathscr{P}\}$, then $\frac{2p}{3} \geq q > \frac{p}{2}$.*

If $o(g) \in \{\mathscr{N}, \mathscr{P}\}$, then S_g is λ (in which case g is not standard), e (in which case $\frac{2p}{3} = q$), or starts with l. As g is standard, if $t = p\%q$, then $t = p - q$ and $q > \frac{p}{2}$. For the signature to start with l, we need $2t > q$, which is $2(p-q) > q$ or $\frac{2p}{3} > q$; combining this with $q > \frac{p}{2}$ gives the result.

6.4 Open Questions

Our main work is describing the structure of positions of PARTIZAN EUCLID and giving an efficient algorithm for determining the outcome. Thus we arrive at one main open question.

Question 6.4.1. *Is there an efficient method to play disjunctive sums of* PARTIZAN EUCLID *positions?*

For some families of positions (signatures) we can give the value easily. Observations 6.4.2 and 6.4.3 happen to correspond to the extreme cases of the Euclidean algorithm.

Observation 6.4.2. *Positions of the form $E(k+1, k)$ have value $* + (k-2)(* : 1)$ for $k \geq 2$.*

The signature $r^k e$ corresponds to $E(k+1, k)$. When $k \geq 2$ the Left option is to $E(k, 1)$ which is equal to 0, and the Right option is to $E(k, k-1)$.

Observation 6.4.3. *Let f_n be the nth Fibonacci number where $f_0 = 0$ and $f_1 = 1$. The position $E(f_k, f_{k-1})$ has the signature $l^k e$ and the value is periodic in k; $E(f_{3k}, f_{3k-1}) = \uparrow$, $E(f_{3k+1}, f_{3k}) = *$, and $E(f_{3k+2}, f_{3k+1}) = 0$. Starting with $E(2, 1) \cong 0$, $E(3, 2) \cong *$, and $E(5, 3) \cong \uparrow$, an easy induction gives the result.*

We present a general rule that we have found that gives values. Note from Corollary 6.1.9 that a Left move from a position whose signature has at least one l in it, removes exactly one l; and a Right move from a position whose signature has at least two ls in it, removes exactly one r or exactly two ls.

Observation 6.4.4. *Let two positions g and h have signatures $\alpha l r^a l \beta e$ and $\alpha l r^b l \gamma e$ respectively. If $r^a l \beta e = r^b l \gamma e$ and $\beta e = \gamma e$ then $g = h$.*

To see this, if $\alpha = \lambda$, then $g^L = r^a l\beta e = r^b l\gamma e = h^L$ and $g^R = g^{LL} = \beta e = \gamma e = h^{LL} = h^R$. If $\alpha = l$, then $g^L = lr^a l\beta e = lr^b l\gamma e = h^L$ and $g^R = \beta e = \gamma e = h^R$. If $\alpha = r$, then $g^L = r^a l\beta e = r^b l\gamma e = h^L$ and $g^R = lr^a l\beta e = lr^b l\gamma e = h^R$. For longer α, corresponding options from g and h are either equal as shown above or by induction.

As there are many non-trivial $\mathscr{P}$-positions in PARTIZAN EUCLID, and all $\mathscr{P}$-positions have value 0, we think it is reasonable to expect many other values to occur repeatedly, perhaps with similar patterns to that of the $\mathscr{P}$-positions.

Question 6.4.5. *Which signature reductions preserve value in which instances?*

Values are messy even for small p and q.

We expect a solution for sums of PARTIZAN EUCLID would include an analysis using the theory of atomic weights. Atomic weight is harder to calculate than outcome and there are few easy reductions of the signature that allow short cuts.

Many interesting atomic weights seem to appear, such as in Conjecture 6.4.6, when p and q follow arithmetic sequences.

Conjecture 6.4.6. *For $i \geq 7$,*

$$\mathrm{aw}(E(p,q)) = \{i+3 \mid 5+*\}$$

where $p = 19 + 22i$ and $q = 19 + 22(i+1)$.

The atomic weights of Tables 6.3 and 6.4 were generated by CGSuite [30]. (In versions 1.0 and 1.1 of CGSuite PARTIZAN EUCLID is used as an example and tables of values and atomic weights can be generated easily.)

Table 6.3 shows the variance of the means of the atomic weights of some positions for small p and q.

p	$q = 11$	12	13	14	15	16	17	18	19	20
10	$\{6 \mid 2\}$	$3/2$	2	0	-1	0	2	0	0	0
11		$\{7 \mid 2\}$	0	-3	0	4	3	0	-4	0
12			$\{8 \mid 2\}$	$\{2 \mid 2\}$	1	0	-3	-1	-1	0
13				$\{9 \mid 2\}$	0	2	3	0	5	4
14					$\{10 \mid 2\}$	$\{3 \mid 2\}$	-3	0	0	2
15						$\{11 \mid 2\}$	0	$3/2$	0	0

Table 6.3: $\mathrm{aw}(E(q,p))$, for $q = 11, \ldots, 20$, $p = 10, \ldots, 15$.

In addition to the means of atomic weights of positions varying greatly, there is also great complexity among the atomic weights. Some complex atomic weights are shown in Table 6.4.

(p, q)	$\mathrm{aw}(E(p, q))$
$(233, 318)$	$7\Uparrow *$
$(316, 181)$	$13\boldsymbol{+}_{8}$
$(869, 1187)$	$9⤊$
$(907, 1146)$	$13 \mid \{12 \mid \{11 \mid 10\}\}$
$(926, 1263)$	$28 \mid 28 + \boldsymbol{+}_{21}$
$(992, 1355)$	$10 \mid \{10 \mid 10 + \boldsymbol{+}_{1}\}$

Table 6.4: Some atomic weights from PARTIZAN EUCLID

Chapter 7

Conclusion

Form is important and positions with small formal birthdays are a great place to look at form. We note that the simplest position in $\mathcal{N}$ is $*$ and thus some interesting structure is exposed in dicotic positions (and especially $*$-based positions) at smaller birthdays and can subsequently be found in non-dicotic positions.

A focus of this thesis was properties of positions understood by considering their presentation as disjunctive sums or ordinal sums. For example, the following representations of $*^3$ expose different properties.

$$\begin{aligned}
*^3 &\cong * : * : * \\
*^3 &\cong *^2 : * \\
*^3 &\cong *^2 + * \\
*^3 &= *^2 + * \\
*^3 &= *^7 + *^4 \\
*^3 &\cong \mathrm{HT}(\left\{ *^2 \;\middle|\; *^2 \right\}).
\end{aligned}$$

Similarly, we see something different from $\uparrow + * = \uparrow*$ than we do from $\uparrow = * : 1 + *$ and $\uparrow \cong \{\cdot \mid *\} : 1$. In practice, we found it easier to find $* + * : 1$ in positions than to find $\uparrow$.

We note that our ordinal sums often have a base with birthday 1 and in this case ordinal sum is the same as a sum where the "subordinate" disappears only when a player moves the "base" to zero; this is a binary operation worth studying.

7.1 Open Problems

We end with a small selection of open problems inspired by results in the thesis and open problems that might be easier having read the thesis.

7.1.1 Ordinal Sum

The values born on day 1 are not the canonical form of any ordinal sum of positions born on or before that days unless we consider trivial ordinal sums where either the base or subordinate is empty. Counting these is somewhat difficult in practice when we allow non-canonical bases.

Problem 7.1.1. *For small n, count the number of values born on day n that are not non-trivial ordinal sums. On day 0 there is 1, on day 1 there are 3, and on day 2 there are 7 (the hot values).*

Numbers can have negative incentive options for both players in canonical form. Sums of star-based numbers can too. These families are both numberish with infinitesimal parts that have integer atomic weights.

Problem 7.1.2. *Find and investigate families of positions that have negative incentive options for both players (other than families based on ordinal sums with number subordinates).*

7.1.2 Hereditarily Transitive

We have computed $|\mathcal{HT}_n|$ for $n \leq 5$ (see Table 3.1). Now that we understand the structure of hereditarily transitive positions so well there is hope that we can determine the number for larger n.

Problem 7.1.3. *Find a formula for $|\mathcal{HT}_n|$.*

We can recognize a position as having the value of a sum of STALKS. We note that a STALK is hereditarily transitive.

Problem 7.1.4. *Find an efficient algorithm to determine if a position is a sum of hereditarily transitive positions (and return suitable summands).*

7.1.3 Hackenbush

There are many more problems that we are interested in that we cannot efficiently solve if we cannot efficiently determine the outcome of a FLOWERS position.

Problem 7.1.5. *Determine the outcome of an arbitrary sum of* FLOWERS.

As a lone HACKENBUSH TREE may be, for example, a $*$-based HACKENBUSH FOREST, recognizing a position as a HACKENBUSH TREE is more difficult than recognizing a position as a STALKS position.

Problem 7.1.6. *Develop an efficient test to decide if a position is a* HACKENBUSH TREE.

7.1.4 Infinitesimals

Theorem 4.11.5 is a result about incentives in disguise, as are the next three conjectures, which we list with increasing confidence.

Conjecture 7.1.7. *If G is infinitesimal then $\{0 \mid G\} - G$ is $*$-based only if G is dicotic.*

Note: $\{0 \mid G\} - G$ is $*$-based for many $*$-based sums, but not all, for example when $G = \Downarrow *$.

Conjecture 7.1.8. *If G is a $*$-based sum and $G \mathrel{|\!\!\!>} 0$ then $\{0 \mid G\} - G$ is a $*$-based sum.*

Conjecture 7.1.9. *If G is atomic and $G \mathrel{|\!\!\!>} 0$ then $\operatorname{aw}(\{0 \mid G\} - G) = 1$.*

The next conjecture has been verified by computer for small n.

Conjecture 7.1.10. *Let $\mathcal{U}_n$ be the set of uptimal values born by day n, If $n \geq 2$ then $|\mathcal{U}_n| = 3|\mathcal{U}_{n-1}|$.*

7.1.5 Partizan Euclid

We suspect an efficient method of determining the outcome of a sum of PARTIZAN EUCLID positions would consider the atomic weights of the summands. Ideally we would like to determine the atomic weight efficiently from the signature.

Question 7.1.11. *Is there an efficient way to determine the atomic weight of a position from its signature?*

In practice, this may be easier if we know what the atomic weights may be.

Question 7.1.12. *Which values are atomic weights of* PARTIZAN EUCLID *positions?*

The strongest conjecture we have regarding the atomic weights is the following. If true, does it describe the atomic weights completely?

Conjecture 7.1.13. *The Left option of the atomic weight of a* PARTIZAN EUCLID *position is an integer.*

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Rulesets

HACKENBUSH A game played on a graph where some vertices are part of the *ground*. Edges are coloured Red, bLue, or grEen. A legal move for a player is to *hack* (i.e. remove) an edge:

- Either player can hack a grEen edge,
- only Left can hack a bLue edge,
- only Right can hack a Red edge.

After a player hacks an edge, edges that are disconnected from the ground (no path to the ground) are removed.

The *summands* of a HACKENBUSH position are the connected components.

In GREEN HACKENBUSH ALL EDGES ARE GREEN.

In BLUE-RED HACKENBUSH ALL EDGES ARE EITHER BLUE OR RED sort.

STALK A HACKENBUSH position whose graph is a path and where exactly one vertex is part of the ground. A HACKENBUSH STALKS position is a HACKENBUSH position where each summand is a stalk. Plurals are used similarly for other HACKENBUSH variants that are stalks.

FLOWER	A GENERALIZED FLOWER is a STALK where all green edges in the stalk are closer to the ground than any non-green edge. A FLOWER is a GENERALIZED FLOWER where every non-green edge is the same colour.
SPRIG	A GENERALIZED FLOWER with at most one green edge.
NIM	An impartial game played on piles of tokens. A legal move for a player is to remove any number of tokens from exactly one pile.