GEOMETRIC EMBEDDING OF GRAPHS AND RANDOM GRAPHS

by

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To my mother

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Abstract

In a spatial graph model, vertices are embedded in a metric space, and the link probability between two vertices depends on this embedding in such a way that vertices that are closer together in the metric space are more likely to be linked. In this thesis we study spatial embedding of graphs and random graphs when the metric space is $(\mathbb{R}^n, \|.\|_\infty)$.

The first part of this thesis is devoted to the study of $(\mathbb{R}^2, \|.\|_\infty)$-*geometric graphs*, graphs whose vertices are points in $\mathbb{R}^2$ and two vertices are adjacent if and only if their distance is at most 1. Such graphs are called *square geometric graphs*. We present a polynomial-time algorithm for recognition of a subclass of square geometric graphs. Moreover if the input graph is a square geometric graph then the algorithm returns the orderings of the x and y coordinates that determine the embedding.

The second part of this thesis is devoted to the study of spatial embedding of random graphs when the metric space is $(\mathbb{R}, \|.\|_\infty)$. Let $w : [0,1]^2 \to [0,1]$ be a symmetric function, and consider the random process $G(n, w)$, where vertices are chosen from $[0,1]$ uniformly at random, and w governs the edge formation probability. Such a random graph is said to have a *linear embedding*, if the probability of linking to a particular vertex v decreases with distance. The rate of decrease, in general, depends on the particular vertex v. A linear embedding is called *uniform* if the probability of a link between two vertices depends only on the distance between them. In this thesis we give necessary and sufficient conditions for the existence of a uniform linear embedding for random graphs where w attains only a finite number of values.

List of Abbreviations and Symbols Used

$E(G)$ .... the edge set of graph G

$V(G)$ .... the vertex set of graph G

$G[S]$ .... the induced subgraph by S

$N(u)$ .... the open neighbourhood of vertex u

$N_S(u)$ .... the open neighbourhood of vertex u in S

BFS .... Breath-First Search

$L(G)$ .... the line graph of the graph G

$G(n, w)$ .... the w-random graph with vertex set $\{1, \ldots, n\}$

G^c .... the complement of graph G

P_n .... the path of order n

C_n .... the cycle of order n

K_n .... the complete graph of order n

$K_{m,n}$ .... the complete bipartite graph with parts of orders m and n

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Chapter 1

Introduction

Modeling real-life problems with graphs is a long-studied subject in many research areas such as social sciences, biology, and ecology. The graph model of a real-world problem consists of nodes or vertices which represent the users of a social network, the neurons of a neural network, or the habitats of an ecological network, and the links or the edges identifying the friendship relation between the members of a social network, the neural connections between the neurons, or the interactions between the habitats of an ecological network. These types of real-world networks usually share a common property that "the more similar the two entities are the higher the probability of being linked". An appropriate way to take this fact into account is to consider a metric space in which the nodes are embedded, so that the connections between nodes are influenced by their metric distance. So, the nodes are embedded in a metric space where similar nodes have smaller metric distance, and they are more likely to attach to each other if they are "close". Such graph models are called *spatial models*.

The metric space considered for a spatial graph model is usually the space $\mathbb{R}^k$ equipped with one of the metrics obtained from the L_p-norms. Consider the metric space $(\mathbb{R}^k, d)$, where d is a metric obtained from one of the L_p-norms. A natural question arising in the study of spatial graph models is as follows. Given a graph model G, whether the graph model is compatible with a notion of spatial graph model, when the metric space is $(\mathbb{R}^k, d)$? More precisely, is there an embedding of the vertices of the graph into the metric space $(\mathbb{R}^k, d)$ such that the link formation of the graph model G follows the spatial principal: "the probability of two vertices being linked is a decreasing function of their metric distances "?

A graph model may be obtained by a deterministic process or a random process. In the former case, the link formation is deterministic *i.e.* the probability of a link between two vertices is either 0 or 1. Then, a given graph G, has a spatial model if there is an embedding of the vertices of G into $(\mathbb{R}^k, d)$ and a threshold r such that two vertices x, y are adjacent if and only if their metric distance is at most r. Such graphs are called $(\mathbb{R}^k, d)$-*geometric graphs*. Note that, by scaling, we can always assume that $r = 1$. In a random graph model, the link formation is governed by a random process. So, a random graph model is compatible with a spatial model if there exists an embedding of the vertices of the graph into the metric space $(\mathbb{R}^k, d)$, and the function governing the edge formation is a decreasing function of the metric distances of the vertices.

In this thesis, our goal is to address the question raised earlier for special classes of graphs and random graphs. The metric space, we consider in this thesis, is $(\mathbb{R}^k, \|.\|_\infty)$, where $\|.\|_\infty$ is the metric derived from L_∞-norm. For $x, y \in \mathbb{R}^k$, the distance between $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ in the L_∞-metric is $\|x - y\|_\infty = \max_i |x_i - y_i|$. The reason for this choice of metric is that the L_∞-metric considers coordinates independently *i.e.* we take the maximum over the absolute value of differences of corresponding coordinates. In fact, in real-life networks, coordinates describe different attributes of the node, such as age or location of users of a social network, or temperature and eating habits of the species in an ecological network. So it makes sense to keep different attributes of a node independent while dealing with them as points of a metric space.

The class of $(\mathbb{R}^k, \|.\|_\infty)$-geometric graphs have a well-known representation as the intersection graph of k-dimensional cubes, the cartesian product of k intervals of unit length: each vertex corresponds to a k-dimensional cube and two vertices are adjacent if and only if their corresponding k-dimensional cubes intersect. The minimum dimension of the space $\mathbb{R}^k$ for which G has a k-dimensional cube presentation is a graph parameter called the *cubicity* of a graph (such a k always exists). The concept of cubicity of graphs has been studied extensively since its introduction in 1969 by Roberts (see [39]).

The problem of characterizing graphs with cubicity k, for any fixed $k \geq 2$, is an NP-hard problem [5, 43]. Equivalently, for any fixed $k \geq 2$, it is NP-hard to determine whether a graph has an $(\mathbb{R}^k, \|.\|_\infty)$-geometric representation. However recognizing $(\mathbb{R}, \|.\|_\infty)$-geometric graphs (the famous *unit interval graphs*), is a linear time problem (see [21]). For applications of unit interval graphs see [24]. The main direction of the study of $(\mathbb{R}^k, \|.\|_\infty)$-geometric graphs, in this thesis, is to investigate $(\mathbb{R}^2, \|.\|_\infty)$-geometric graphs for a class of graphs obtained by adding some edges between two unit interval graphs, *binate interval graphs*.

A basic subclass of binate interval graphs are cobipartite graphs, the union of two cliques with some edges added between the cliques. The cobipartite graphs are precisely the complements of bipartite graphs. In this thesis, first, we provide a polynomial-time algorithm for recognition of cobipartite graphs. This algorithm is based on a similar approach to a previous algorithm [38]. But our methods for the proofs are completely different, and the great advantage of the methods we use is that they are amenable to be adjusted for other classes of graphs. Then, applying the methods we developed for studying square geometric cobipartite graphs, we present a polynomial-time algorithm for characterizing square geometric graphs of another subclasses of binate interval graphs. We believe that the methods we developed to study $(\mathbb{R}^2, \|.\|_\infty)$-geometric cobipartite graphs are the backbone of studying $(\mathbb{R}^2, \|.\|_\infty)$-geometric binate interval graphs. More details on our methods can be find in Subsection 1.2.1. Our complete results on the study of $(\mathbb{R}^2, \|.\|_\infty)$-geometric graphs together with the algorithms are discussed in Chapters 2 and 3. A preliminary version of results from Chapter 3 was presented at conference 13th Cologne-Twente Workshop on Graphs and Combinatorial Optimization (CTW), [19].

In Chapter 4 we investigate random graphs that are compatible with a notion of spatial random graphs when the metric space is $([0, 1], |.|)$. Recall that in a spatial random graph, vertices are embedded in a metric space, and the link probability between two vertices depends on this embedding in such a way that vertices that are closer together in the metric space are more likely to be linked. So, let a random graph model be formed as follows: the vertices are chosen from the interval $[0, 1]$, and edges are chosen independently at random, but so that, for a given vertex x, the probability that there is an edge to a vertex y decreases as the distance between x and y increases. We call this a random graph with a *linear embedding*. The random graphs that have a linear embedding has been studied in a previous work [16].

The concept of a spatial random graph allows for the possibility that the link probability depends on the spatial position of the vertices, as well as their metric distance. Thus, in the graph we may have tightly linked clusters for two different reasons. On the one hand, such clusters may arise when vertices are situated in a region where the link probability is generally higher. On the other hand, clusters can still arise when the probability of a link between two vertices depends only on their distance, and not on their location. In this case, tightly linked clusters can arise if the distribution of vertices in the metric space is inhomogeneous. The central question addressed in this thesis is how to recognize spatial random graphs with a uniform link probability function. For our study we ask this question for a general edge-independent random graph model which generalizes the Erdős-Rényi random graph $G(n, p)$. In subsection 1.2.2, you can find the formal definition of a uniform link probability function and a brief description of our results on spatial random graphs. The results in Chapter 4 were presented at European Conference on Combinatorics, Graph Theory and Applications (EUROCOMB), and published in [18]. The complete results of Chapter 4 can be found in [17].

1.1 Preliminaries and Definitions

We devote this chapter to the definitions and results that will be used later in this thesis. Generally, we follow [42] for graph theory terminology, and [40] for real analysis terminology.

A *graph* G consists of a non-empty set of *vertex set* denoted by $V(G)$ and an *edge set* denoted by $E(G)$, where each edge e of G consists of a pair of vertices $\{u, v\}$(not necessarily distinct) called its *endpoints*. An edge $e \in E(G)$ with two endpoints $u, v \in V(G)$ is denoted by $e = uv$. An edge with only one endpoint is a *loop*. The edges with the same endpoints are *multiple edges*. A *simple graph* is a graph which contains no loops or multiple edges. Throughout the rest of this thesis, we assume that the graphs are simple graphs unless otherwise stated. The vertices u and v are *adjacent* or neighbors if they are endpoints of an edge $e = uv$, signified by $u \sim v$. The neighborhood of a vertex v, denoted by $N(v)$, consists of the vertices $u \in V(G) \setminus \{v\}$ such that $u \sim v$. For $S \subseteq V(G)$, $N_S(v)$ is the set $N(v) \cap S$. The *complement* of a graph G, denoted by G^c is a graph with vertex set $V(G)$ and the edge set $(E(G))^c$ which is formed as follows: Two vertices u and v are adjacent if and only if u and v are not adjacent in G. The *null graph* is the graph whose vertex set and edge set are empty. The *empty* graph G is a graph with edge set $E(G) = \emptyset$. The *order* of a graph is the number of vertices in G, and the *size* of a graph is the number of edges in G.

A vertex v is an *isolated vertex* if $N(v) = \emptyset$, and v is a *universal vertex* if $N(v) = V(G) \setminus \{v\}$. A *clique* in a graph is a subset of vertices of the graph that are pairwise adjacent. A *maximal clique* in a graph is a clique that is not contained in a larger clique. An *independent set* is a subset of vertices of the graph that are pairwise non-adjacent.

A graph H is an *induced subgraph* of a graph G if $V(H)$ is a subset of $V(G)$ and $E(H)$ consists of all the edges of $E(G)$ with both endpoints in $V(H)$. For a set $V \subseteq V(G)$ the notation $G[V]$ denotes the induced subgraph of G with vertex set V.

An *isomorphism* from a simple graph G to a simple graph H is a bijection $f : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say that G is *isomorphic to H*. An *H-free graph* is a graph with no induced subgraph isomorphic to H.

A graph G is *bipartite* if $V(G)$ is the union of two disjoint (possibly empty) independent sets called *partite sets* of G. The compliment of a bipartite graphs is called a *cobipartite graph*. A *complete bipartite graph* is a bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the partite sets have sizes r and s, the complete bipartite graph is denoted $K_{r,s}$. The complete bipartite graph $K_{1,3}$ is a *claw*.

The *line graph* of a graph G, denoted by $L(G)$, is a graph whose vertex set is $E(G)$ and two vertices of $L(G)$ are adjacent if their corresponding edges in G have a common endpoint.

A *clique-vertex matrix* of a graph G is a matrix which has a row for each maximal clique, and a column for each vertex, and the (i, j) entry of the matrix is 1 if the maximal clique corresponding to the i-th row contains the vertex corresponding to the j-th column. A clique-vertex matrix of a graph has the *consecutive ones property* if its rows and columns can be permuted so that the ones in each row and each column appear in consecutive positions.

A *threshold graph* is a graph which can be constructed from an empty graph by repeatedly adding either an isolated vertex or a universal vertex.

Let G be a graph with vertex set $V(G)$. Suppose there is a family of sets $\{S_v | v \in V(G)\}$ such that $u \sim v$ if and only if $S_u \cap S_v \neq \emptyset$. Then G is the *intersection graph* of the sets S_v.

A graph G is called a *chordal graph* if it has no induced cycle of size greater than 3. An *asteroidal triple* is a set of three independent vertices such that any two of them are connected by a path which has no intersection with the neighborhood of the third vertex.

An *interval representation* of a graph is a family of intervals assigned to the vertices so that vertices are adjacent if and only if their corresponding intervals intersect. A graph having such a representation is called an *interval graph*. A *unit interval* graph is an interval graph whose interval representation consists of intervals of the same length.

There are several characterizations of the families of interval graphs and unit interval graphs. Here we state two results which characterize interval and unit interval graphs based on their forbidden subgraphs. These results will be used later in this thesis.

Theorem 1.1.1. *[30] A graph G is an interval graph if and only if it is chordal and asteroidal triple-free.*

Theorem 1.1.2. *[24] A graph G is a unit interval graph if and only if G is a claw-free interval graph.*

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Suppose that $E_1(G), \ldots, E_k(G)$ are subsets of $E(G)$ such that, for all $1 \leq i \leq k$, the graph G_i with vertex set $V(G)$ and edge set $E_i(G)$ is a threshold graph. Moreover, let $E(G) = \cup_{i=1}^{k} E_i(G)$. Then $G_1, \ldots, G_k$ is called a *threshold cover* of G. The *threshold dimension* of G (or *threshold number* of G) is the minimum positive integer k for which a threshold cover exists. A graph G with threshold dimension k is called *k-threshold* graph.

A *partial order* is a binary relation, $<$, on a set S such that

- $<$ is reflexive: For all $u \in S$, $s < s$.

- $<$ is antisymmetric: For all $u, w \in S$, if $u < w$ and $w < u$ then $u = w$.

- $<$ is transitive: For all $u, w, z \in S$, if $u < w$ and $w < z$ then $u < z$.

A *linear order* is a partial order for which every pair is related.

A *metric* on a set S is a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$ the following conditions are satisfied:

- $d(x, y) \geq 0$.
- $d(x, y) = 0$ if and only if $x = y$.
- $d(x, y) = d(y, x)$.
- $d(x, z) \leq d(x, y) + d(y, z)$.

An ordered pair (S, d) is a *metric space* if the set S is equipped with metric d.

A *σ-algebra* $\mathcal{F}$ on a set S is a collection of subsets of S which satisfies the following conditions:

- $S \in \mathcal{F}$.
- If $A \in \mathcal{F}$ then the complement of A belongs to $\mathcal{F}$.
- If $\{A_i\}_{i=1}^{\infty}$ is a sequence of elements of $\mathcal{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Let S be a set with σ-algebra $\mathcal{F}$. Then the pair $(S, \mathcal{F})$ is called a *measurable space*, and the elements of $\mathcal{F}$ are called *measurable sets*. Let S and S' be sets with corresponding σ-algebras $\mathcal{F}$ and $\mathcal{F}'$, respectively. Moreover, suppose $(S, \mathcal{F})$ and $(S', \mathcal{F}')$ are measurable spaces. Then a function $f : S \to S'$ is a *measurable function* if for every $A' \in \mathcal{F}'$ we have that $f^{-1}(A') \in \mathcal{F}$. A function $m : \mathcal{F} \to [0, \infty)$ is called a *measure* if the following conditions are satisfied

- For all $A \in \mathcal{F}$ we have $m(A) \geq 0$.
- $m(\emptyset) = 0$.
- For all countable collections $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{F}$ we have $m(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} m(A_i)$.

The triple $(S, \mathcal{F}, m)$, where $m : \mathcal{F} \to [0, \infty)$ is a measure, is called a *measure space*. A *probability measure* is a measure m with $m(S) = 1$, and a *probability space* is a measure space with a probability measure. A property is said to hold *almost everywhere* if the set of points where it fails is a set of measure zero.

Suppose A is a subset of $\mathbb{R}$ and the length of any interval $I = (a,b)$ is given by $\ell(I) = b - a$. The *Lebesgue outer measure* $\mu^*(A)$ of A is

$$\mu^*(A) = \inf\{\sum_{i=1}^{\infty} \ell(I_i) \mid (I_i)_{i=1}^{\infty} \text{ is a sequence of open intervals with } A \subseteq \bigcup_{i=1}^{\infty} I_i\}.$$

A set $A \subseteq \mathbb{R}$ is *Lebesgue measurable* if for every $B \subseteq \mathbb{R}$ we have $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$, where B^c is the complement of the set B. Then the *Lebesgue measure* of A is defined to be its Lebesgue outer measure $\mu^*(A)$ and is denoted by $\mu(A)$.

A *norm* on a vector space V is a function $f : V \to \mathbb{R}$ which satisfies the following conditions:

- $f(\mathbf{x}) \geq \mathbf{0}$ for all $\mathbf{x} \in V$.
- $f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$.
- $f(\lambda \mathbf{x}) = |\lambda| f(\mathbf{x})$ for all $\lambda \in \mathbb{C}$ and $\mathbf{x} \in V$.
- $f(\mathbf{x}) = 0$ if and only if $\mathbf{x} = 0$.

For any $1 \leq p < \infty$, the L_p *norm* on $\mathbb{R}^n$ is defined as follows. Let $x = (x_1, x_2, \ldots, x_n)$ be a vector in $\mathbb{R}^n$, then

$$\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}.$$

The L_∞ norm or *maximum norm* is defined as $\|x\|_\infty = \max_{i=1}^{n} |x_i|$.

1.2 Background and Motivation

1.2.1 Square geometric graphs

A graph G is called *$(\mathbb{R}^k, \|.\|_\infty)$-geometric graph*, if the vertices of the graph can be embedded in $\mathbb{R}^k$ equipped with $\|.\|_\infty$ metric, the metric derived from the L_∞ norm, with two vertices being adjacent if and only if their metric distance is at most 1.

Another way to define $(\mathbb{R}^k, \|.\|_\infty)$-geometric graphs is to look at it as the problem of representing a graph as the intersection graph of k-dimensional cubes where a k-dimensional cube is the cartesian product of k closed intervals of unit length in the real line $\mathbb{R}$. More precisely, assume that a graph G is the intersection graph of k-dimensional cubes. Then two vertices are adjacent if and only if their corresponding cubes intersect if and only if the distance between the centers of their corresponding cubes is at most 1. Therefore, the vertices of the graph G can be embedded in $\mathbb{R}^k$ in such a way that each vertex maps to the center of its corresponding cube in the k-dimensional cube representation of G. Then two vertices are adjacent if and only if their distance is at most one, and thus G is a $(\mathbb{R}^k, \|.\|_\infty)$-geometric graph. Now let G be a $(\mathbb{R}^k, \|.\|_\infty)$-geometric graph. Then the vertices of the graph can be embedded in $\mathbb{R}^k$ with two vertices being adjacent if and only if their metric distance is at most 1. We can correspond with each vertex v a k-dimensional cube in such a way that the center of the cube is the image of v in $\mathbb{R}^k$ under the embedding of G into $(\mathbb{R}^k, \|.\|_\infty)$. Then two cubes intersect if and only if the distance between their centers is at most 1.

The minimum dimension of the space $\mathbb{R}^k$ for which G has a k-dimensional cube presentation is a graph parameter called the *cubicity* of a graph. The cubicity of a graph with n vertices is at most $\lfloor \frac{2n}{3} \rfloor$, see [39]. This implies that cubicity is a well-defined graph parameter.

A generalization of the cubicity of graphs defines another graph parameter called the *boxicity* of a graph. A k-box is the cartesian product of k closed intervals in the real line $\mathbb{R}$. The boxicity of a graph is the smallest k such that the graph G has a representation as the intersection graph of k-dimensional boxes. It is clear that boxicity of a graph is smaller than its cubicity.

Graphs with boxicity 1 are interval graphs, and graphs with cubicity 1 are unit interval graphs. Interval graphs have broad applications in real life problems, such as scheduling, genetics, transportation etc. See [24, 28, 41, 36] for more information on interval graphs and their application. The concepts of the boxicity and cubicity of a graph were first introduced by Roberts in [39]. Another well-studied geometric property of graphs is the *sphericity* of graphs. The concept of sphericity of graphs corresponds to embeddings of graphs into euclidean metric space $(\mathbb{R}^k, \|.\|_2)$. The sphericity of a graph G is the minimum dimension k for which G can be represented as a $(\mathbb{R}^k, \|.\|_2)$-geometric graph. It is known that the sphericity of a graph of order n is at most $n-1$, see [34]. The concept of sphericity of graphs was first introduced by Havel (see [25]). He discussed that characterizing graphs with sphericity at most 3 is very important in the calculation of molecular conformation. More results on the relation of cubicity and sphericity can be found in [22, 35, 37].

In his paper, [39], Roberts indicates that there is a tight connection between graphs with cubicity k and unit interval graphs.

Theorem 1.2.1 ([39]). *The cubicity of a graph G is k, where k is a positive integer, if and only if G is the intersection of k unit interval graphs.*

The same result is true for boxicity. We just need to replace unit interval graphs by interval graphs [39].

Earlier results on cubicity(boxicity) study the complexity of recognition of graphs with a certain cubicity(boxicity). In his paper, [43], Yannakakis shows that recognition of graphs with cubicity k is NP-hard for any $k \geq 3$. Later Brue in [5] proves that the problem of recognition of graphs with cubicity 2 in general is an NP-hard problem. As for $(\mathbb{R}, \|.\|_\infty)$-geometric graphs or unit interval graphs, there are several results presenting linear time algorithms for recognition of unit interval graphs. See [20, 21]. We also, know that the recognition of graphs with boxicity at least 2 is NP-hard, see [29, 43]. There are several linear time algorithms for recognition of interval graphs, [4, 26].

One of the main directions of the research on cubicity (boxicity) aims at finding bounds on the cubicity (boxicity) of graphs. Part of the results give lower and upper bounds for cubicity (boxicity) of graphs in general. In [39] Roberts showed that for any graph G, the cubicity of G is at most $\lfloor \frac{2n}{3} \rfloor$ and the boxicity of G is at most $\lfloor \frac{n}{2} \rfloor$. Chandran *et al*, in [2, 13], presented upper bounds for cubicity of a graph G in terms of the boxicity and the independence number of the graph. In [9], the authors introduced an upper bound for cubicity of a graph in terms of the minimum vertex cover and the number of vertices of the graph.

There are also results investigating the cubicity of specific families of graphs. In [14], for graphs with low chromatic number, an upper bound in terms of boxicity and the chromatic number is provided. The cubicity of interval graphs has been studied in [2, 10] and the authors give an upper bound for the cubicity of an interval graph in terms of the maximum degree of the graph. In [12, 15] the bounds on the cubicity of hypercube graphs have been derived. The cubicity of asteroidal-triple free graphs has been studied in [3]. The cubicity of threshold graphs and bipartite graphs has been investigated in [1, 9]. More results on cubicity of graphs can be found in [3, 11, 12, 22, 35].

In this thesis, we study the problem of recognition of a special class of $(\mathbb{R}^2, \|.\|_\infty)$-geometric graphs or graphs with cubicity 2. We refer to $(\mathbb{R}^2, \|.\|_\infty)$-geometric graphs as *square geometric graphs*. Our approach to study the recognition of square geometric graphs is inspired by the following characterization of unit interval graphs.

Theorem 1.2.2. *[31] A graph G is a unit interval graph if and only if there is an ordering $<$ on the vertex set of G such that for any $u, v, z \in V(G)$ we have*

$$u < z < v \quad \text{and} \quad u \sim v \Rightarrow u \sim z \quad \text{and} \quad v \sim z$$

A translation of the above property of unit interval graphs states: A graph G is a unit interval graph if and only if its vertex-clique matrix satisfies the consecutive ones property for both rows and columns. This implies that the cliques consists precisely of the consecutive vertices in the ordering $<$ of Equation 1.2.2. Therefore, a unit interval graph can be represented as a sequence of cliques. In Chapter 2, we will present a generalization of Theorem 1.2.2 for $(\mathbb{R}^k, \|.\|_\infty)$-graphs (Theorem 2.1.1). Then, using this theorem, we investigate the recognition of square geometric graphs for a subclass of *binate interval* graphs.

Definition 1.2.3. *A binate interval graph is a graph whose vertex set can be partitioned into two sets U and W such that the graphs induced by U and W are connected unit interval graphs.*

In this thesis, we study the problem of recognition of $(\mathbb{R}^2, \|.\|_\infty)$-geometric graphs for a subclass of binate interval graphs, called *$\mathcal{B}_{a,b}$-graphs*. We are interested in studying this class of graphs mainly because of its structure, that is two unit interval graphs and some edges between them. Therefore, the binate interval graphs can be seen as a model of interaction between two unit interval graphs. Since unit interval graphs have broad applications in practical problems studying binate interval graphs may find its application in future.

Definition 1.2.4. *A $\mathcal{B}_{a,b}$-graph is a binate interval graph whose vertex set can be partitioned into two sets $X_a \cup X_b$ and Y, where X_a, X_b, Y are cliques and $X_a \cap X_b \neq \emptyset$.*

Since unit interval graphs have a natural representation as a sequence of cliques the algorithm for this special class, $\mathcal{B}_{a,b}$-graph, will likely be a crucial component of the recognition of square geometric binate interval graphs. The following is an example of a $B_{a,b}$-graph which is not square geometric.

Example 1. *The graph G of Figure 1.1 is a $B_{a,b}$-graph with clique partitions $X_a = \{g, e\}$, $X_b = \{g, f\}$, and $Y = \{c, d\}$.*

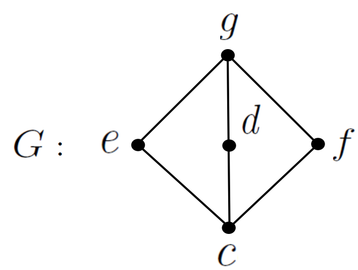


Figure 1.1: A $B_{a,b}$ graph G which is not square geometric.

The graph G of Figure 1.1 is not a square geometric graph. To show this, suppose to the contrary that G is square geometric. Then by Theorem 1.2.1 it is the intersection of two unit interval graphs, say U_1 and U_2, i.e. *$G = U_1 \cap U_2$ where U_1 and U_2 are unit interval graphs. By Theorem 1.1.2, we know that an induced 4-cycle is not a unit interval graph, as by Theorem 1.1.1 it is not an interval graph. Therefore, both U_1 and U_2 contains no induced 4-cycle. So let us first take a look at induced 4-cycles of G. If we add one chord of an induced 4-cycle then the obtained graph is chordal and claw-free, and thus it is a unit interval graph (by Theorem 1.1.2). Therefore, adding exactly one chord of an induced 4-cycle gives us a unit interval graph. See Figure 1.2. Each unit interval graph I_1 and I_2 contains exactly one chord of the induce 4-cycle. Because if one of I_1 or I_2 has both chords of the induced 4-cycle then the intersection of I_1 and I_2 contains at least one of the chords of the induced 4-cycle.*

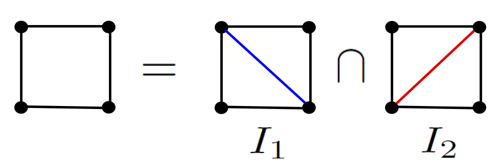


Figure 1.2: An induced 4-cycle is a square geometric graph.

As shown in Figure 1.1, the graph G has three induced 4-cycles: $gdcf$ with chords gc and df, $gdce$ with chords gc and de, and $gecf$ with chords gc and ef. See Figure 1.3.

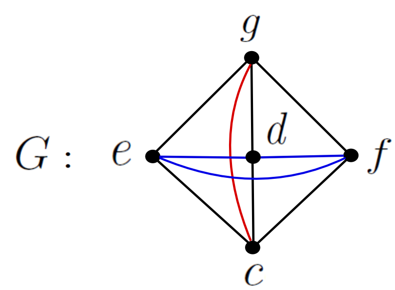


Figure 1.3: The $B_{a,b}$ graph G of Figure 1.3. The red and blue links denoted the non-edges of G that are chords of the induced 4-cycles of G.

Therefore, each unit interval graph U_1 and U_2 contains exactly one of the chords of each of the three induced 4-cycles of G. The non-edge gc is the chord of all of the three induced 4-cycles. Suppose without loss of generality that U_1 contains gc. Then U_1 does not contain the chords df, de, and ef. But this means that $G[\{c, d, e, f\}]$ (similarly $G[\{g, d, e, f\}]$) is a claw in U_1, and thus by Theorem 1.1.2 the graph U_1 is not a unit interval graph. Therefore, G is not the intersection of two unit interval graphs, and consequently, by Theorem 1.2.1, G is not square geometric.

In this thesis we take the first steps towards finding an algorithm for the class of $\mathcal{B}_{a,b}$-square geometric graphs. Let G be of one of the following forms. We present polynomial-time algorithms which recognize whether or not G is square geometric.

(i) Cobipartite graphs.

(ii) $\mathcal{B}_{a,b}$-graphs where $|X_a \setminus X_b| = |X_b \setminus X_a| = 1$.

(iii) $\mathcal{B}_{a,b}$-graphs where $|X_a \setminus X_b| = 2$ and $|X_b \setminus X_a| = 1$.

Graphs of (i), (ii), and (iii) are all subgraphs of binate interval graphs. For the sake of simplicity, we call graphs of (ii), *type-1 $\mathcal{B}_{a,b}$-graphs*, and graphs of (iii), *type-2 $\mathcal{B}_{a,b}$-graphs*. It is known that the problem of recognizing square geometric cobipartite graphs is a polynomial-time problem. Indeed in [43], Yannakakis proves that the cubicity of cobipartite graphs is equal to the threshold number of split graphs. Ibraki and Peled, in [27], present a polynomial-time algorithm for the recognition of split graphs with threshold number 2. Later Raschle and Simon, [38], improved Ibraki and Peled's result and introduce an $\mathcal{O}(n^4)$ algorithm which decides whether or not a graph G has threshold number 2, where n is the order of the graph.

In this thesis, we represent an $\mathcal{O}(n^4)$ algorithm for recognizing square geometric cobipartite graphs. Our algorithm is based on a similar approach as in [27, 38]. But the methods we use in our proofs are amenable to be adjusted for recognition of other classes of square geometric graphs. In particular, we will apply our methods for recognition of square geometric cobipartite graphs to investigate the recognition of square geometric type-1 and type-2 $B_{a,b}$-graphs.

1.2.2 Uniform embedding of random graphs

Let $\mathcal{W}_0$ be the set of all symmetric measurable functions $w : [0,1]^2 \to [0,1]$. The *w-random graph* $G(n,w)$ is a graph with vertex set $\{1,\ldots,n\}$. The edge formation is as follow. Each vertex i is assigned a value x_i, where x_i is a real number drawn uniformly from $[0,1]$. Then vertices i and j, $i<j$, are connected with probability $w(x_i,x_j)$.

Alternatively, the random graph $G(n,w)$ may be seen as a one-dimensional spatial model, where the label x_i represents the coordinate of vertex i. In that case, the process $G(n,w)$ can be described as follows: a set P of n points is chosen uniformly from the metric space $[0,1]$. Any two points $x,y\in P$ are then linked with probability $w(x,y)$. For $G(n,w)$ to correspond to the notion of a spatial random graph, w must satisfy a certain type of monotonicity. This is captured by the following definition.

Definition 1.2.5. *[16] A function $w\in\mathcal{W}$ is* diagonally increasing *if for every $x,y,z\in[0,1]$, we have*

1. $x\le y\le z \Rightarrow w(x,z)\le w(x,y)$,

2. $y\le z\le x \Rightarrow w(x,y)\le w(x,z)$.

A function w in $\mathcal{W}$ is diagonally increasing almost everywhere if there exists a diagonally increasing function w' which is equal to w almost everywhere.

Next we formulate our central question: "which functions w are in fact uniform in disguise?"

Definition 1.2.6. *A diagonally increasing function $w \in \mathcal{W}_0$ has a uniform linear embedding if there exists a measurable injection $\pi : [0,1] \to \mathbb{R}$ and a decreasing function $f_{pr} : \mathbb{R}^{\geq 0} \to [0,1]$ such that for every $x, y \in [0,1]$,*

$$w(x,y) = f_{pr}(|\pi(x) - \pi(y)|)$$

The function f_{pr} is the *link probability function*, and the function π determines a probability distribution μ on $\mathbb{R}$, where for all $A \subseteq \mathbb{R}$, $\mu(A)$ equals the Lebesque measure of $\pi^{-1}(A)$. Indeed the function π replaces the uniform probability space $[0,1]$ by probability space $(\mathbb{R}, \mathcal{F}, \mu)$, where $\mathcal{F}$ is the set of all subsets of $\mathbb{R}$. An equivalent description of w-random graph $G(n,w)$ is as follows: a set of n points is chosen from $\mathbb{R}$ according to probability distribution μ. Two points are linked with probability given by f_{pr} of their distance. Note that in Definition 1.2.6, we restrict ourselves to injective embeddings π. Namely, we will see in Chapter 4 that any uniform embedding must be monotone. Therefore, the requirement that π is injective is equivalent to the requirement that the probability distribution μ has no points of positive measure.

The notion of diagonally increasing functions, and our interpretation of spatial random graphs, were first given in previous work, see [16]. In [16], a graph parameter Γ is given which aims to measure the similarity of a graph to an instance of a one-dimensional spatial random graph model. However, the parameter Γ fails to distinguish uniform spatial random graph models from the ones which are intrinsically nonuniform.

In this thesis, we give necessary and sufficient conditions for the existence of uniform linear embeddings for functions in $\mathcal{W}_0$. We consider only functions of finite range, for two reasons. Firstly, any function in $\mathcal{W}_0$ can be approximated by a function with finite range. Secondly, we will see that the necessary conditions become increasingly restrictive when the size of the range increases. This leads us to believe that for infinite valued functions, either the uniform linear embedding will be immediately obvious when considering w, or it does not exist.

1.3 Outline

In Chapter 2, we describe an $\mathcal{O}(n^4)$ algorithm for recognizing square geometric cobipartite graphs. Our algorithm returns a partial ordering of the sets of the bipartition if the graph is square geometric. This partial ordering provides us with the geometric embedding of the graph in $(\mathbb{R}^2, \|.\|_\infty)$. Indeed, if the input graph is square geometric then, using this partial order, our algorithm returns the ordering of the x and y coordinates that determine the embedding. The partial ordering obtained from our algorithm for a square geometric cobipartite graph is the key point in our approach to study square geometric $\mathcal{B}_{a,b}$ graphs.

In Chapter 3, applying methods of Chapter 2, we present necessary and sufficient conditions for type-1 and type-2 square geometric $B_{a,b}$-graphs (Theorem 3.1.9). Moreover, we give an $\mathcal{O}(n^4)$ algorithm that checks the necessary and sufficient conditions (Subsection 3.4.1).

In Chapter 4, we investigate the recognition of spatial random graphs which admit a uniform linear embedding. We present necessary and sufficient conditions for the existence of a uniform linear embedding for spatial random graphs when the metric space is $([0,1], |.|)$ (Theorem 4.1.9).

Chapter 2

Square Geometric Cobipartite Graphs

This chapter is devoted to the study of the problem of characterizing square geometric cobipartite graphs. We will represent a polynomial-time algorithm which characterizes square geometric cobipartite graphs in $O(n^4)$. The algorithm is efficient in the sense that it provides us with the exact reason, why or why not, a cobipartite graph G is embeddable in $(\mathbb{R}^2, \|.\|_\infty)$. More precisely, it either presents the embedding of the graph in $(\mathbb{R}^2, \|.\|_\infty)$ or finds a minimal non-embeddable induced subgraph of G. The basis of our algorithm is the same as the basis of an earlier algorithm for recognition of threshold graphs with dimension 2. However the methods we use for our proof are new and more importantly are amenable to be adjusted for other classes of graphs. We will apply these methods in Chapter 3 to study the problem of recognition of square geometric binate interval graphs.

2.1 A characterization of $(\mathbb{R}^k, \|.\|_\infty)$-geometric Graphs

As mentioned in the introduction, graphs with cubicity one or $(\mathbb{R}, \|.\|_\infty)$-geometric graphs, are the well-known unit interval graphs, and $(\mathbb{R}^k, \|.\|_\infty)$-geometric graphs with $k > 1$, can be seen as a generalization of unit interval graphs to higher dimensions.

Motivated by the characterization of unit interval graphs introduced in Theorem 1.1.2, we present the following characterization of $(\mathbb{R}^k, \|.\|_\infty)$-geometric graphs.

Theorem 2.1.1. *A graph G is an $(\mathbb{R}^k, \|.\|_\infty)$-geometric graph if and only if there exist k linear orderings $<_1, \ldots, <_k$ on the vertex set of G such that the following condition holds for all $a, b \in V(G)$: if for all $1 \leq i \leq k$ there exist vertices $u, v \in V(G)$ such that u and v are adjacent, and moreover $u <_i a <_i v$ and $u <_i b <_i v$, then a and b are adjacent.*

Proof. Suppose that G is an $(\mathbb{R}^k, \|.\|_\infty)$-geometric graph. By definition, there exists an embedding of G in $\mathbb{R}^k$ such that two vertices u, v of G are adjacent if and only if $\|u - v\|_\infty \leq 1$. Define $<_i$, $1 \leq i \leq k$, to be the ordering of vertices based on the increasing order of their coordinates in the i-th dimension, respectively. It is clear that $<_i$, $1 \leq i \leq k$, satisfy the condition mentioned in the statement of the theorem. More precisely, let $a, b \in V(G)$, and $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$. Suppose that we have the following: for all $1 \leq i \leq k$, there exist $u, v \in V(G)$ such that

$$\big(u <_i a <_i v \quad \text{and} \quad u <_i b <_i v\big), \quad \text{and} \quad u \sim v. \tag{2.1}$$

We prove that for all $1 \leq i \leq k$, we have that $|a_i - b_i| \leq 1$, which implies that $\|a - b\|_\infty \leq 1$, and thus $a \sim b$. Fix $j \in \{1, \ldots, k\}$, and suppose that u and v are the vertices corresponding to $<_j$ in Equation 2.1. Let $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$. Since $u \sim v$, we have that $\|u - v\|_\infty \leq 1$, and thus $|u_i - v_i| \leq 1$ for all $1 \leq i \leq k$. By definition of $<_j$, we have that $u_j < a_j < v_j$ and $u_j < b_j < v_j$ in the j-th dimension. This implies that $|a_j - b_j| \leq 1$. Therefore, for all $1 \leq i \leq k$, we have that $|a_i - b_i| \leq 1$, and thus $\|a - b\|_\infty \leq 1$. So $a \sim b$.

Now suppose that G is a graph with linear orderings $<_i$, $1 \leq i \leq k$, which satisfy the condition mentioned in the statement of the theorem. For all $<_i$, $1 \leq i \leq k$, we construct a corresponding set E_i as follows. If $v_r, v_s \in V(G)$ such that $v_r \sim v_s$ and $v_r <_i v_s$, then for any $v_t \in V(G)$ such that $v_r <_i v_t <_i v_s$, we add edges $v_t v_r$ and $v_t v_s$ to E_i.

Now define G_i, $1 \leq i \leq k$, to be the graph with vertex set $V(G_i) = V(G)$, and edge set $E(G_i) = E(G) \cup E_i$. For all $1 \leq i \leq k$, the linear order $<_i$ on vertices $V(G_i)$ satisfies Equation 1.2.2. Then, by Theorem 1.2.2, we have that, for all $1 \leq i \leq k$, the graph G_i is a unit interval graph. Now suppose that ab is an edge in $\bigcap_{i=1}^k E_i$. This implies that for all $1 \leq i \leq k$ there exist vertices $u, v \in V(G)$ such that u and v are adjacent, and moreover $u <_i a <_i v$ and $u <_i b <_i v$. Since linear orderings $<_i$, $1 \leq i \leq k$, satisfy the condition mentioned in the statement of the theorem then $ab \in E(G)$. This implies that $\bigcap_{i=1}^k E(G_i) = E(G)$. Therefore, $G = \bigcap_{i=1}^k G_i$. Since all G_i, $1 \leq i \leq k$, are unit interval graphs, then by Theorem 1.2.1 we have that G is square geometric. $\square$

The above theorem is a generalization of Theorem 1.2.2, which presents a characterization for unit interval graphs. We can also think of Theorem 2.1.1 as a geometric version of Theorem 1.2.1, which states that a graph G is a $(\mathbb{R}^k, \|.\|_\infty)$-geometric graph if and only if G is the intersection of k unit interval graphs.

2.2 Recognition of Square Geometric cobipartite Graphs

This section studies an $\mathcal{O}(n^4)$ algorithm for the recognition of square geometric cobipartite graphs. The methods and techniques we introduce in this section will be adjusted to study recognition of square geometric binate interval graphs in Chapter 3. Throughout this chapter, we assume that G is a cobipartite graph with clique bipartition X and Y. We start this section with the following theorem which presents a version of Theorem 2.1.1 for square geometric graphs.

Theorem 2.2.1. *A graph G is a square geometric graph if and only if there exist two linear orderings $<_1$ and $<_2$ on the vertex set of G such that for every $u, v, x, y, a, b \in V(G)$,*

$$\left.\begin{array}{lllll} u <_1 a <_1 v & \text{and} & u <_1 b <_1 v, & \text{and} & u \sim v \\ x <_2 a <_2 y & \text{and} & x <_2 b <_2 y, & \text{and} & x \sim y \end{array}\right. \Rightarrow a \sim b \tag{2.2}$$

Definition 2.2.2. *Let G be a square geometric graph with linear orders $<_1$ and $<_2$ as in Theorem 2.2.1. Define*

$$E_i = \{wz | \exists u, v \in V(G), u <_i w <_i v \quad \text{and} \quad u <_i z <_i v \quad \text{and} \quad u \sim v\}.$$

The completion *of $<_i$, denoted by $\mathcal{C}_i$, is $(E(G))^c \cap E_i$. Indeed $\mathcal{C}_i$ is the set of the non-edges of G whose ends are in between two adjacent vertices in $<_i$.*

Note that the completions $\mathcal{C}_1$ and $\mathcal{C}_2$ of Definition 2.2.2 are subsets of the set of non-edges of G. Let us now look at an example. The graph of Figure 2.1 is an induced 4-cycle. Consider the following linear orders $<_1$ and $<_2$ on the set of vertices of G.

$$x_1 <_1 x_2 <_1 y_1 <_1 y_2 \quad \text{and} \quad x_2 <_2 x_1 <_2 y_2 <_2 y_1.$$

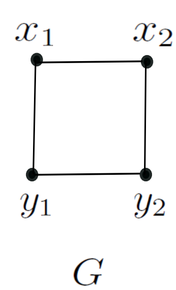


Figure 2.1: An induced 4-cycle.

The non-edges of G are x_1y_2 and x_2y_1. It is easy to see that x_1 and y_2 are not between two adjacent vertices in $<_1$, and x_2 and y_1 are not between two adjacent vertices in $<_2$. This implies that $<_1$ and $<_2$ satisfy Equation 2.2, and thus by Theorem 2.2.1 G is a square geometric graph. Therefore, we can find completions of each of linear orders $<_1$ and $<_2$ as in Definition 2.2.2.

We have $x_1 <_1 x_2 <_1 y_1 <_1 y_2$, and $x_1 \sim y_1$ and $x_2 \sim y_2$. Therefore, $E_1 = \{x_1x_2, x_1y_1, x_2y_1, x_2y_2, y_1y_2\}$. Similarly, $x_2 <_2 x_1 <_2 y_2 <_2 y_1$, and $x_1 \sim y_1$ and $x_2 \sim y_2$. So we have $E_2 = \{x_1x_2, x_1y_1, x_1y_2, x_2y_2, y_1y_2\}$. Since $E(G) = \{x_1x_2, x_1y_1, x_2y_2, y_1y_2\}$ and $(E(G))^c = \{x_1y_2, x_2y_1\}$, by Definition 2.2.2 we have $\mathcal{C}_1 = \{x_2y_1\}$ and $\mathcal{C}_2 = \{x_1y_2\}$. See Figure 2.2.

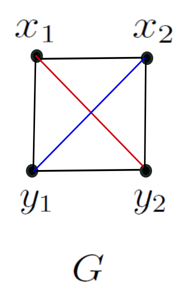


Figure 2.2: The red non-edge x_1y_2 belong to $\mathcal{C}_2$ and the blue non-edge x_2y_1 belongs to $\mathcal{C}_1$.

The following lemma shows the relation between linear orders of Theorem 2.2.1, $<_1$ and $<_2$, and the completions of Definition 2.2.2, $\mathcal{C}_1$ and $\mathcal{C}_2$.

Lemma 2.2.3. *Let G be a square geometric graph, and let $<_1$ and $<_2$ be linear orders on the vertex set of G. Then $<_1$ and $<_2$ satisfy Equation (2.2) if and only if $\mathcal{C}_1$ and $\mathcal{C}_2$, the completions of $<_1$ and $<_2$ respectively, have empty intersection.*

Proof. First suppose $<_1$ and $<_2$ satisfy Equation (2.2). By contradiction suppose $w, z \in V(G)$, and $wz \in \mathcal{C}_1 \cap \mathcal{C}_2$. Then by Definition 2.2.2, there are $u, v, x, y \in V(G)$ such that

$$\begin{array}{llll} u <_1 w <_1 v & \text{and} \quad u <_1 z <_1 v, & \text{and} & \quad u \sim v \\ x <_2 w <_2 y & \text{and} \quad x <_2 z <_2 y, & \text{and} & \quad x \sim y \end{array} \tag{2.3}$$

Since $<_1$ and $<_2$ satisfy Equation (2.2), we have $w \sim z$. This contradicts the fact that $\mathcal{C}_1$ and $\mathcal{C}_2$ are subsets of non-edges of G. Therefore, $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$.

Now suppose that $<_1$ and $<_2$ do not satisfy Equation (2.2). This implies that there are $u, v, x, y, w, z \in V(G)$ such that $w \nsim z$, and

$$\begin{array}{llll} u <_1 w <_1 v & \text{and} \quad u <_1 z <_1 v, & \text{and} & u \sim v \\ x <_2 w <_2 y & \text{and} \quad x <_2 z <_2 y, & \text{and} & x \sim y \end{array} \tag{2.4}$$

Then by the definition of completions (Definition 2.2.2) we have that $wz \in \mathcal{C}_1$ and $wz \in \mathcal{C}_2$. Therefore, $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$.

$\square$

Let G be a cobipartite graph. We define a graph associated with G.

Definition 2.2.4. *Let G be a cobipartite graph with clique bipartition (X, Y). The chord graph of G, denoted by $\tilde{G}$ is defined as follows.*

$$V(\tilde{G}) = \{xy | x \in X, y \in Y, xy \notin E\}.$$

Two vertices of $\tilde{G}$ are adjacent if and only if they are the missing chords of an induced 4-cycle of G, namely

$$E(\tilde{G}) = \{uv | u = xy, v = x'y', xy' \text{ and } x'y \text{ are in } E\}.$$

Below is an example of a cobiapartite graph and its corresponding chord graph.

Example 2. *The graph G, as shown in Figure 2.3, is a cobipartite graph. The graph G has three non-edges x_1y_2, x_1y_3, and x_2y_1. By definition of the chord graph, the vertex set of $\tilde{G}$ is $\{x_1y_2, x_1y_3, x_2y_1\}$. To find the edge set of $\tilde{G}$, we should look at the induced 4-cycles of G. As seen in Figure 2.3, the graph G has two induced 4-cycles $x_1x_2y_2y_1$ and $x_1x_2y_3y_1$. The missing chords of $x_1x_2y_2y_1$ are x_1y_2 and x_2y_1, which implies that x_1y_2 and x_2y_1 are adjacent in $\tilde{G}$. Also, the missing chords of $x_1x_2y_3y_1$ are x_1y_3 and x_2y_1, which implies that x_1y_3 and x_2y_1 are adjacent in $\tilde{G}$.*

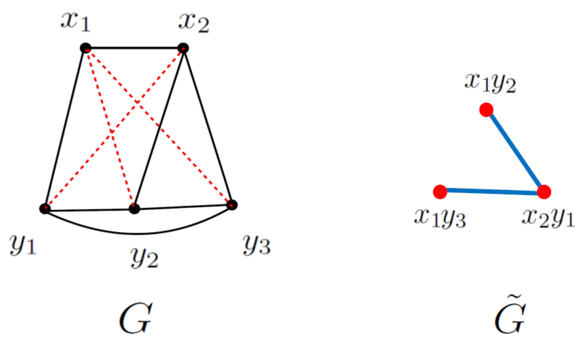


Figure 2.3: A cobipartite graph G with its corresponding chord graph $\tilde{G}$.

The vertex set of the chord graph, $\tilde{G}$, as in Definition 2.2.4 is the set of non-edges of G. From now on we may use a "vertex of $\tilde{G}$" and a "non-edge of G" interchangeably. For clarity, we denote the adjacency in graph $\tilde{G}$ by $\sim^*$. In Definition 2.2.5, for the sake of simplicity, we rename induced 4-cycles of a cobipartite graph.

Definition 2.2.5. *Let G be a cobipartite graph and x_1y_1, x_2y_2 are two edges of G with $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then $\{x_1y_1, x_2y_2\}$ is called a rigid pair of G if $x_1y_1y_2x_2$ is an induced 4-cycle of G. Moreover the non-edges x_1y_2 and x_2y_1 are called the chords of the rigid pair $\{x_1y_1, x_2y_2\}$. See Figure 2.4.*

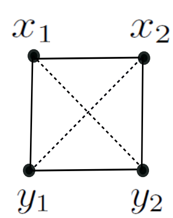


Figure 2.4: Rigid pair $\{x_1y_1, x_2y_2\}$ with chords x_1y_2 and x_2y_1.

Recall that completions $\mathcal{C}_1$ and $\mathcal{C}_2$ of Definition 2.2.2 are subsets of $V(\tilde{G})$. Proposition 2.2.6 shows us which non-edges are contained in the completions $\mathcal{C}_1$ and $\mathcal{C}_2$ for sure.

Proposition 2.2.6. *Let G be a square geometric cobipartite graph with linear orders $<_1, <_2$ as in Equation (2.2). Then every completion $\mathcal{C}_i$, $i \in \{1, 2\}$ contains exactly one chord of any rigid pair.*

Proof. Suppose G is a square geometric cobipartite graph with linear orders $<_1, <_2$ satisfying Equation (2.2). By Lemma 2.2.3, we know that $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. This implies that a chord of a rigid pair belongs to at most one of the completions $\mathcal{C}_i$, $i = 1, 2$. We now show that a chord of a rigid pair belongs to either $\mathcal{C}_1$ or $\mathcal{C}_2$. Let $\{x_1y_1, x_2y_2\}$ be a rigid pair of G. Without loss of generality let $x_1 <_1 x_2$. Using the fact that Equation (2.2) holds for $<_1$ and $<_2$, and $x_1 \sim y_1$ we have:

- If $x_1 <_1 y_1$, then either $x_1 <_1 y_1 <_1 x_2$ or $x_1 <_1 x_2 <_1 y_1$. Thus $x_2y_1 \in \mathcal{C}_1$.
- If $y_2 <_1 x_2$, then either $y_2 <_1 x_1 <_1 x_2$ or $x_1 <_1 y_2 <_1 x_2$. Thus $x_1y_2 \in \mathcal{C}_1$.
- If neither $x_1 <_1 y_1$ nor $y_2 <_1 x_2$, then we have $y_1 <_1 x_1 <_1 x_2 <_1 y_2$. This implies that $x_1y_2 \in \mathcal{C}_1$ and $x_2y_1 \in \mathcal{C}_1$.

Therefore $\mathcal{C}_1$ includes at least one chord of $\{x_1y_1, x_2y_2\}$. A similar discussion for $\mathcal{C}_2$ proves that $\mathcal{C}_2$ includes at least one chord of $\{x_1y_1, x_2y_2\}$. Note that since $\mathcal{C}_1$ and $\mathcal{C}_2$ contains at least one chord of $\{x_1y_1, x_2y_2\}$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ then the third case never occurs. $\square$

Note that by Definition 2.2.4, two vertices of $\tilde{G}$ are adjacent if and only if they are chords of a rigid pair. Therefore a non-edge of G is either a chord of a rigid pair or an isolated vertex of $\tilde{G}$. Proposition 2.2.6 shows that the set of non-isolated vertices of $\tilde{G}$ is a subset of $\mathcal{C}_1 \cup \mathcal{C}_2$, and two adjacent vertices of $\tilde{G}$ belong to different completions $\mathcal{C}_1$ and $\mathcal{C}_2$. This provides us with a bipartition for the set of non-isolated vertices of $\tilde{G}$. The following corollary is an immediate consequence of this bipartition.

Corollary 2.2.7. *Let G be cobipartite and square geometric. Then its chord graph $\tilde{G}$ is bipartite.*

Corollary 2.2.7 presents a necessary condition for a cobipartite graph to be square geometric. Indeed this necessary condition is also sufficient.

Theorem 2.2.8. *Let G be a cobipartite graph. Then G is square geometric if and only if its chord graph $\tilde{G}$ is bipartite.*

From Corollary 2.2.7, we have the necessity part of Theorem 2.2.8. The sufficiency will follow from a series of lemmas, presented in the rest of this chapter. We first give an informal discussion of the concepts involved in proving the sufficiency. We have that the chord graph of G is bipartite and we want to prove that G is square geometric. To prove that G is square geometric, by Theorem 2.2.1 and Lemma 2.2.3, we need to find linear orders $<_1$ and $<_2$ on $V(G)$ with corresponding completions $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$.

By Proposition 2.2.6 we know that if G is square geometric then non-isolated vertices of $\tilde{G}$ (chords of rigid pairs of G) belong to different completions $\mathcal{C}_1$ and $\mathcal{C}_2$. This is where the fact that $\tilde{G}$ is bipartite comes in. Indeed since $\tilde{G}$ is bipartite, $\tilde{G}$ is 2-colorable and the color classes form a bipartition of non-isolated vertices of $\tilde{G}$. So we may want to define linear orders $<_1$ and $<_2$ on $V(G)$ in such a way that each color class of a proper 2-coloring of $\tilde{G}$ is a subset of one of the completions. This makes sure that chords of a rigid pair belong to different completions.

In the following we give an example of the above discussion. In Figure 2.5, we have a simple example of a cobipartite graph G whose chord graph is bipartite.

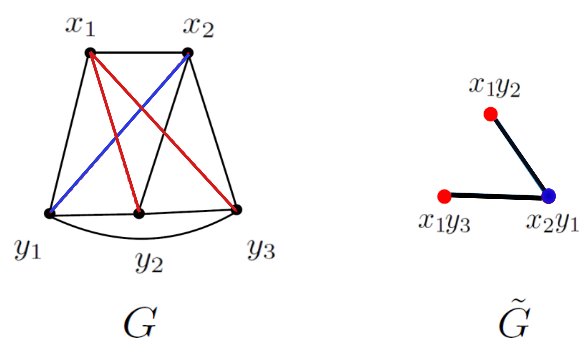


Figure 2.5: A cobipartite graph G with a bipartite chord graph $\tilde{G}$.

As we can see in the picture, a 2-coloring of G is indeed a bipartition of chords of rigid pairs of G. We define linear orders $<_1$ and $<_2$ such that $\mathcal{C}_1$ contains the blue color class and $\mathcal{C}_2$ contains the red color class:

$$x_1 <_1 x_2 <_1 y_1 <_1 y_2 <_1 y_3 \quad \text{and} \quad x_2 <_2 x_1 <_2 y_3 <_2 y_2 <_2 y_1.$$

By definition of completion it is easy to see that $\mathcal{C}_1 = \{x_2y_1\}$ and $\mathcal{C}_2 = \{x_1y_2, x_1y_3\}$. Therefore $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. This implies that the defined linear orders $<_1$ and $<_2$ satisfy Equation (2.2), and thus G is square geometric.

In the above example, the color classes of the 2-coloring of G and the completions $\mathcal{C}_1$ and $\mathcal{C}_2$ were the same. But this is not always the case. In Figure 2.6, we have a cobipartite graph G with a bipartite chord graph. The non-edges x_3y_1 and x_3y_2 are not chords of any rigid pair, and thus they corresponds to isolated vertices of $\tilde{G}$.

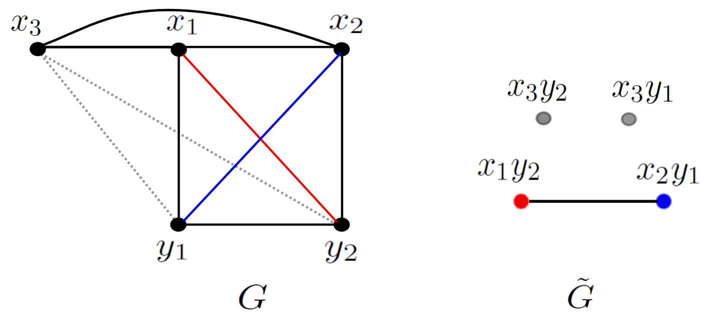


Figure 2.6: A cobipartite graph G with a bipartite chord graph $\tilde{G}$ which has isolated vertices.

In a proper 2-coloring of $\tilde{G}$ the vertices x_3y_1 and x_3y_2 may receive either red or blue, and even we can leave them uncolored. This corresponds to the fact that not every non-edge of G belongs to a completion. Indeed, isolated vertices of $\tilde{G}$, or equivalently, non-edges of G that are not chords of any rigid pair may, or may not belong to a completion. For example if we define linear orders $<_1$ and $<_2$ as

$$x_3 <_1 x_1 <_1 x_2 <_1 y_1 <_1 y_2 \quad \text{and} \quad x_3 <_2 x_2 <_2 x_1 <_2 y_2 <_2 y_1,$$

then $\mathcal{C}_1 = \{x_2y_1\}$ and $\mathcal{C}_2 = \{x_1y_2\}$, and thus $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. Now define

$$x_3 <_1 x_1 <_1 x_2 <_1 y_1 <_1 y_2 \quad \text{and} \quad x_2 <_2 x_3 <_2 x_1 <_2 y_2 <_2 y_1.$$

Here $\mathcal{C}_1 = \{x_2y_1\}$ and $\mathcal{C}_2 = \{x_1y_2, x_3y_2\}$, and still $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$.

Indeed, using a proper 2-coloring of $\tilde{G}$ we deal with the non-isolated vertices of $\tilde{G}$, the vertices which we know to be certainly part of the completions of the possible linear orders $<_1$ and $<_2$. The isolated vertices are dealt with at the very last step when we are defining the linear orders. We will see that the hard part is to deal with the non-isolated vertices. For the isolated ones it is easy to find an appropriate position in the orderings such that Equation (2.2) holds.

The rest of this chapter is devoted to the proof of the sufficiency part of Theorem 2.2.8. We observe, how from a 2-coloring of G, we obtain linear orders $<_1$ and $<_2$ which satisfy Equation (2.2).

We continue with the following example which consider possible embeddings of an induced 4-cycle into $(\mathbb{R}^2, \|.\|_\infty)$.

Example 3. *As we discussed earlier an induced 4-cycle is a square geometric graph.*

In Figure 2.7 we have an embedding of an induced 4-cycle $xx'y'y$ *into the 2-dimensional space* $(\mathbb{R}^2, \|.\|_\infty)$*.*

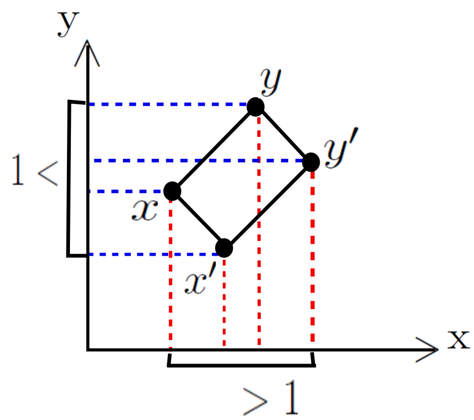


Figure 2.7: An embedding of an induced 4-cycle into $(\mathbb{R}^2, \|.\|_\infty)$.

The embedding of an induced 4-cycle as shown in Figure 2.8 is not possible.

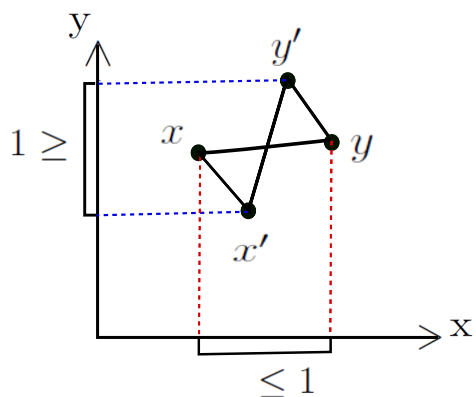


Figure 2.8: An impossible embedding of an induced 4-cycle into $(\mathbb{R}^2, \|.\|_\infty)$.

As we can see from the picture the distance between x *and* y' *in* x*-coordinate is less that the distance between* x_1 *and* y*. Since* x *and* y *are adjacent, their distance in* x*-coordinate is at most one. This implies that the distance between* x *and* y' *in* x*-coordinate is at most one. Similarly, since* x_2 *and* y_2 *are adjacent, their distance in* y*-coordinate is at most one. Thus the distance between* x *and* y' *in* y*-coordinate is at most one. Therefore,* $\|x - y'\|_\infty \leq 1$*, and thus* x *and* y' *must be adjacent in the embedding of Figure 2.8. But* $xx'y'y$ *is an induced 4-cycle.*

Indeed, in any embedding of an induced 4-cycle into $(\mathbb{R}^2, \|.\|_\infty)$*, the edges of the induced 4-cycle cannot cross each other. In Figure 2.8 the edges* xy *and* $x'y'$ *cross* *each other.*

From the above example, we can see that an embedding of an induced 4-cycle into $(\mathbb{R}^2, \|.\|_\infty)$ cannot contain an edge-crossing. Therefore, if a cobipartite graph G is square geometric then its embedding into metric space $(\mathbb{R}^2, \|.\|_\infty)$ contains no induced 4-cycle with edge-crossing. This observation inspires the following definition, The definition presents two partial orders on the clique bipartition of a cobipartite graph. These partial orders are the main tool for the proof of sufficiency.

Definition 2.2.9. *A* bi-ordering *of a cobipartite graph G with bipartition X and Y, denoted by $(<_X, <_Y)$, consists of two partial orders $<_X$ and $<_Y$ on the sets of vertices X and Y, respectively. An induced 4-cycle of G, $xyy'x'$ with $x, x' \in X$ and $y, y' \in Y$, is called a* crossed 4-cycle *with respect to a bi-ordering of G if the following condition holds*

$$\big(x <_X x' \quad \textit{and} \quad y' <_Y y\big) \qquad \textit{or} \qquad \big(x' <_X x \quad \textit{and} \quad y <_Y y'\big) \tag{2.5}$$

A bi-ordering of G with no crossed 4-cycle is called a proper bi-ordering.

For the cobipartite graph as shown in Figure 2.9 The relations $x_1 <_X x_2 <_X x_3$ and $y_1 <_Y y_2 <_Y y_3$ are partial orders on X and Y respectively. The relations $x_1 <_X x_3 <_X x_2$ and $y_1 <_Y y_2 <_Y y_3$ are partial orders on X and Y respectively. But the partial orders $<_X$ and $<_Y$ of (2) contains the crossed 4-cycle $x_2y_2y_3x_3$, so $(<_X, <_Y)$ is not a proper bi-ordering.

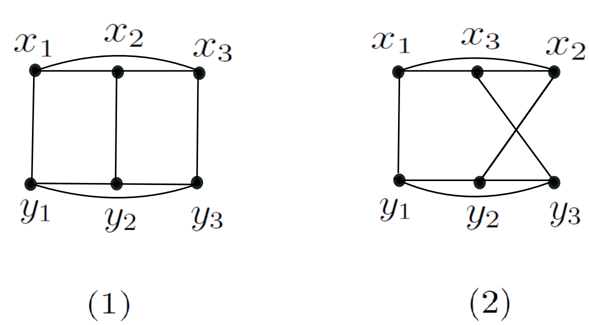


Figure 2.9: Structure (1) depicts partial orders $x_1 <_X x_2 <_X x_3$ and $y_1 <_Y y_2 <_Y y_3$, and structure (2) depicts partial orders $x_1 <_X x_3 <_X x_2$ and $y_1 <_Y y_2 <_Y y_3$.

Suppose now that G is a cobipartite graph with a bipartite chord graph $\tilde{G}$. We define two relations $<_X$ and $<_Y$ on X and Y, respectively, based on a 2-coloring of $\tilde{G}$.

Definition 2.2.10. *Let G be a cobipartite graph with a bipartite chord graph $\tilde{G}$. Consider a proper 2-coloring of $\tilde{G}$, $f: V(\tilde{G}) \to \{red, blue\}$. The relations associated with the coloring f, $<_X$ and $<_Y$, are defined as follows.*

- *$x <_X x'$ if there is a rigid pair $\{xy, x'y'\}$ such that $f(xy')$ is red and $f(x'y)$ is blue, or if $x = x'$.*
- *$y <_Y y'$ if there is a rigid pair $\{xy, x'y'\}$ such that $f(xy')$ is red and $f(x'y)$ is blue, or if $y = y'$.*

Note that by definition of the chord graph, we know that chords of a rigid pair are adjacent in the chord graph. Therefore, in Definition 2.2.10 "$f(xy')$ is red" if and only if "$f(x'y)$ is blue".

Example 4. *Consider the cobipartite graph G of Figure 2.10. The relations $<_X$ and $<_Y$ of Definition 2.2.10 associated with the proper 2-coloring of $\tilde{G}$ as shown in Figure 2.10 are as follow.*

$$<_X: x_1 <_X x_2, x_2 <_X x_3 \text{ and } <_Y: y_1 <_Y y_2, y_1 <_Y y_3.$$

Obviously, $<_X$ and $<_Y$ are partial orders. Moreover, $<_X$ and $<_Y$ create no crossed 4-cycle as defined in Definition 2.2.10. Therefore, $(<_X, <_Y)$ forms a proper bi-ordering.

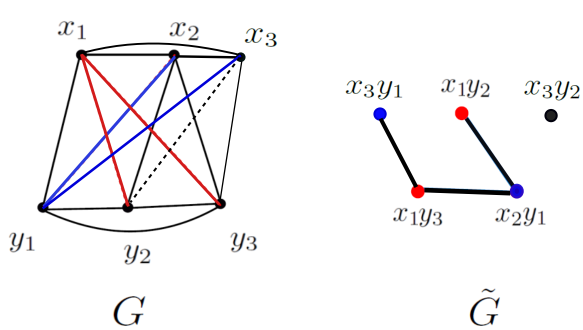


Figure 2.10: A cobipartite graph G and a proper 2-coloring of its chord graph $\tilde{G}$.

Remark 1. *The relations $<_X$ and $<_Y$ obtained from Definition 2.2.10 allow no crossed 4-cycle. Indeed, let $x_1y_1y_2x_2$ be an induced 4-cycle. Then by Definition 2.2.5 $\{x_1y_1, x_2y_2\}$ is a rigid pair, and by the definition of a chord graph, $\tilde{G}$, we know that $x_1y_2 \sim^* x_2y_1$. This implies that $f(x_1y_2) \neq f(x_2y_1)$. Without loss of generality let $f(x_1y_2) = red$, and so $f(x_2y_1) = blue$. Then by Definition 2.2.10 we have $x_1 <_X x_2$*

and $y_1 <_Y y_2$. Therefore, by Definition 2.2.9, $x_1y_1y_2x_2$ is not a crossed 4-cycle. Therefore, if the relations $<_X$ and $<_Y$ of Definition 2.2.10 are partial orders then $(<_X, <_Y)$ is a proper bi-ordering of G.

We now assume that there is a proper 2-coloring of $\tilde{G}$ such that its associated relations $<_X$ and $<_Y$ of Definition 2.2.10 form a proper bi-ordering. Then we prove that G is square geometric. So consider the following assumption.

Assumption 2.2.11. *Let G be a cobipartite graph with clique bipartition X and Y, and bipartite chord graph $\tilde{G}$. Suppose there is a coloring $f : V(\tilde{G}) \to \{red, blue\}$ such that relations $<_X$ and $<_Y$ associated with f, as given in Definition 2.2.10, form a proper bi-ordering for G.*

In what follows, we assume that G is a cobipartite graph, as given in Assumption 2.2.11. Then we use partial orders $<_X$ and $<_Y$ to obtain two linear orders $<_1$ and $<_2$ for graph G as in Equation 2.2 (Lemma 2.2.18). From now on, we assume that neither X nor Y has identical vertices *i.e.* there are no vertices $x_1, x_2 \in X$ such that $N_Y(x_1) = N_Y(x_2)$, and similarly there are no vertices $y_1, y_2 \in Y$ with $N_X(y_1) = N_X(y_2)$. This requirement is due to the fact that if such vertices exist then we can consider them as one vertex in linear orders $<_1$ and $<_2$. This means that in the geometric embedding of the graph we can place two identical vertices at the same position. So we can assume without loss of generality that there are no identical vertices. We now define two relations $<_1$ and $<_2$ on the set of vertices of a graph G, as in Assumption 2.2.11. We will show later that these relations are indeed linear orders.

Definition 2.2.12. *Let G and $(<_X, <_Y)$ be as in Assumption 2.2.11. Define two relations $<_1$ and $<_2$ on the set of vertices of G as follows.*

- *Relation $<_1$:*

  (1) $x <_1 x'$ if $x <_X x'$ or $N_Y(x) \subseteq N_Y(x')$

  (2) $y <_1 y'$ if $y <_Y y'$ or $N_X(y') \subseteq N_X(y)$

  (3) $x <_1 y$ for all $x \in X$ and all $y \in Y$

- *Relation $<_2$:*

(1) $x <_2 x'$ *if* $x' <_X x$ *or* $N_Y(x) \subseteq N_Y(x')$

(2) $y <_2 y'$ *if* $y' <_Y y$ *or* $N_X(y') \subseteq N_X(y)$

(3) $x <_2 y$ *for all* $x \in X$ *and for all* $y \in Y$

We now look at an example of Definition 2.2.12.

Example 5. *Consider the cobipartite graph G of Figure 2.11.*

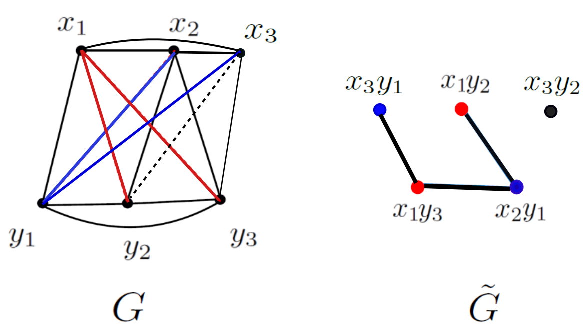


Figure 2.11: A cobipartite graph G and a proper 2-coloring of its chord graph $\tilde{G}$.

As we saw in Example 4, the relations $<_X$: $x_1 <_X x_2, x_1 <_X x_3$ *and* $<_Y$: $y_1 <_Y y_2, y_1 <_Y y_3$ *associated with the proper 2-coloring of* $\tilde{G}$ *as shown in Figure 2.11 form a proper bi-ordering. Therefore, we can define orderings* $<_1$ *and* $<_2$ *of Definition 2.2.12 as follows:*

- *Ordering* $<_1$: *Since* $x_1 <_X x_2$ *and* $x_1 <_X x_3$, *we have* $x_1 <_1 x_2$ *and* $x_1 <_1 x_3$, *respectively. Moreover, as seen in Figure 2.11,* $N_Y(x_3) \subseteq N_Y(x_2)$ *and thus* $x_3 <_1 x_2$. *Since* $y_1 <_Y y_2$ *and* $y_1 <_Y y_3$, *we have* $y_1 <_1 y_2$ *and* $y_1 <_1 y_3$, *respectively. Also, as seen in Figure 2.11,* $N_X(y_2) \subseteq N_X(y_3)$ *and thus* $y_3 <_1 y_2$. *Moreover,* $x_i <_1 y_j$ *for all* $i \in \{1,2,3\}$ *and* $j \in \{1,2,3\}$. *This gives us:* $x_1 <_1 x_3 <_1 x_2 <_1 y_1 <_1 y_3 <_1 y_2$.

- *Ordering* $<_2$: *Since* $x_1 <_X x_2$ *and* $x_1 <_X x_3$, *we have* $x_2 <_2 x_1$ *and* $x_3 <_2 x_1$, *respectively. Moreover, as seen in Figure 2.11,* $N_Y(x_3) \subseteq N_Y(x_2)$ *and thus* $x_3 <_2 x_2$. *Since* $y_1 <_Y y_2$ *and* $y_1 <_Y y_3$, *we have* $y_2 <_2 y_1$ *and* $y_3 <_2 y_1$, *respectively. Also, as seen in Figure 2.11,* $N_X(y_2) \subseteq N_X(y_3)$ *and thus* $y_3 <_2 y_2$. *Moreover,* $x_i <_2 y_j$ *for all* $i \in \{1,2,3\}$ *and* $j \in \{1,2,3\}$. *This gives us:* $x_3 <_2 x_2 <_2 x_1 <_2 y_3 <_2 y_2 <_2 y_1$.

The following remark states some facts regarding the relations $<_1$ and $<_2$ of Definition 2.2.12. These facts will be used later to prove Lemma 2.2.18.

Remark 2. *Let G and $(<_X, <_Y)$ be as in Assumption 2.2.11. Suppose $<_1$ and $<_2$ are as given in Definition 2.2.12.*

(1) If $x_1 <_1 x_2$ then, by Definition 2.2.12, either $N_Y(x_1) \subseteq N_Y(x_2)$ or $x_1 <_X x_2$. If $x_1 <_X x_2$ then by the definition of a proper bi-ordering we know that there is a rigid pair $\{x_1y_1, x_2y_2\}$. Moreover by Definition 2.2.10 we have that x_1y_2 is colored red and x_2y_1 is colored blue. Similarly, if $y_1 <_1 y_2$ then, by Definition 2.2.12, either $N_X(y_2) \subseteq N_X(y_1)$ or $y_1 <_Y y_2$. If $y_1 <_Y y_2$ then by the definition of a proper bi-ordering we know that there is a rigid pair $\{x_1y_1, x_2y_2\}$. Moreover by Definition 2.2.10 we have that x_1y_2 is colored red and x_2y_1 is colored blue. Similar results are true for $<_2$.

(2) Suppose $\{x_1y_1, x_2y_2\}$ is a rigid pair. By Definition 2.2.9 we know that a proper bi-ordering contains no crossed 4-cycle, and thus $x_1 <_X x_2$ if and only if $y_1 <_Y y_2$. Then by Definition 2.2.12 for all $i \in \{1, 2\}$ we have $x_1 <_i x_2$ if and only if $y_1 <_i y_2$.

(3) If $x_1, x_2 \in X$ are such that $N_Y(x_1) \subseteq N_Y(x_2)$ then there can be no rigid pair $\{x_1y_1, x_2y_2\}$. This implies that x_1 and x_2 are incomparable in $<_X$.

We now prove that relations $<_1$ and $<_2$ as given in Definition 2.2.12 are linear orders.

Lemma 2.2.13. *Let G and $(<_X, <_Y)$ be as in Assumption 2.2.11. Then relations $<_1$ and $<_2$ of Definition 2.2.12 are linear orders.*

Proof. We only prove the lemma for $<_1$. Similar arguments prove that $<_2$ is a linear order. It follows directly from the definition of $<_X$ and $<_Y$ (Definition 2.2.10) that $<_1$ is reflexive. We show that $<_1$ is antisymmetric, transitive, and total on the set of vertices of G, $X \cup Y$. By Definition 2.2.12, we know that $x <_1 y$ for all $x \in X$ and $y \in Y$. Therefore, if $<_1$ is antisymmetric, transitive, and total on X and Y, then $<_1$ is antisymmetric, transitive, and total on $X \cup Y$.

First note that for any two vertices $x_1, x_2 \in X$ if there is a rigid pair $\{x_1y_1, x_2y_2\}$ then $y_1 \in N_Y(x_1) \setminus N_Y(x_2)$ and $y_2 \in N_Y(x_2) \setminus N_Y(x_1)$. Therefore neighborhoods of

x_1 and x_2 are not nested. Moreover if neighborhoods of x_1 and x_2 are not nested then there is a rigid pair $\{x_1y_1, x_2y_2\}$. More precisely, if $N_Y(x_1) \not\subseteq N_Y(x_2)$ and $N_Y(x_2) \not\subseteq N_Y(x_1)$ then there are vertices $y_1, y_2 \in Y$ such that $y_1 \in N_Y(x_1) \setminus N_Y(x_2)$ and $y_2 \in N_Y(x_2) \setminus N_Y(x_1)$. Therefore, $\{x_1y_1, x_2y_2\}$ is a rigid pair. This proves that "the neighborhoods of x_1 and x_2 are not nested if and only if there is a rigid pair $\{x_1y_1, x_2y_2\}$. This implies that exactly one of the following can happen: x_1 and x_2 are related in $<_X$ or x_1 and x_2 have nested neighborhoods in Y. Similarly for $y_1, y_2 \in Y$, y_1 and y_2 are either related in $<_Y$ or have nested neighborhoods in X. Thus, by Definition 2.2.12, $<_1$ is total. Note that $x_1 <_X x_2$ implies that there is a rigid pair $\{x_1y_1, x_2y_2\}$ and thus neighborhoods of x_1 and x_2 in Y are not nested.

To prove that $<_1$ is antisymmetric, let $x_1, x_2 \in X$. Suppose $x_1 <_1 x_2$ and $x_2 <_1 x_1$. We know that if the neighborhoods of x_1 and x_2 are nested if and only if there is no rigid pair $\{x_1y_1, x_2y_2\}$. Then by Definition 2.2.12, either $x_1 <_X x_2$ and $x_2 <_X x_1$ or $N_Y(x_1) \subseteq N_Y(x_2)$ and $N_Y(x_2) \subseteq N_Y(x_1)$. We know $<_X$ is a partial order, and thus is antisymmetric. Now let $N_Y(x_1) \subseteq N_Y(x_2)$ and $N_Y(x_2) \subseteq N_Y(x_1)$. This implies that x_1 and x_2 are identical vertices. Since we assume X has no identical vertices $N_Y(x_1) \subseteq N_Y(x_2)$ and $N_Y(x_2) \subseteq N_Y(x_1)$ do not occur simultaneously. This implies that $<_1$ is antisymmetric on X. A similar argument shows that $<_1$ is antisymmetric on Y.

We now prove that $<_1$ is transitive on X. The proof of transitivity on Y is similar. Let $x_1, x_2, x_3 \in X$, such that $x_1 <_1 x_2$ and $x_2 <_1 x_3$. By Definition 2.2.12, there are a few possible cases. Suppose first $x_1 <_X x_2$ and $x_2 <_X x_3$. As $<_X$ is transitive, we have $x_1 <_X x_3$, and thus $x_1 <_1 x_3$. If $N_Y(x_1) \subseteq N_Y(x_2)$ and $N_Y(x_2) \subseteq N_Y(x_3)$, then by transitivity of subset relation, we have $N_Y(x_1) \subseteq N_Y(x_3)$. This implies that $x_1 <_1 x_3$. Now let $N_Y(x_1) \subseteq N_Y(x_2)$ and $x_2 <_X x_3$. Then there are $y_2, y_3 \in Y$ such that $\{x_2y_2, x_3y_3\}$ is a rigid pair. If $N_Y(x_1) \subseteq N_Y(x_3)$ then by definition $x_1 <_1 x_3$. So assume there is $y_1 \in N_Y(x_1)$ such that $y_1 \nsim x_3$. Since $N_Y(x_1) \subseteq N_Y(x_2)$, and $\{x_2y_2, x_3y_3\}$ is a rigid pair, we have $x_1 \nsim y_3$. Therefore, $\{x_1y_1, x_3y_3\}$ and $\{x_2y_1, x_3y_3\}$ are rigid pairs. We have that $\{x_2y_1, x_3y_3\}$ is a rigid pair and $x_2 <_X x_3$. Then by Remark 2, we have $y_1 <_Y y_3$. Now we know $\{x_1y_1, x_3y_3\}$ is a rigid pair and $y_1 <_Y y_3$. Again by Remark 2 we have $x_1 <_X x_3$.

A similar argument shows that $x_1 <_X x_3$, when $x_1 <_X x_2$ and $N_Y(x_2) \subseteq N_Y(x_3)$, and thus $x_1 <_1 x_3$. This finishes the proof of transitivity of $<_1$. Therefore $<_1$ and $<_2$ defined as in Definition 2.2.12 are linear orders. □

Now that we know that the $<_1$ and $<_2$ of Definition 2.2.12 are linear orders, the next step is to show that $<_1$ and $<_2$ satisfy Equation (2.2). Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are completions of $<_1$ and $<_2$, respectively. We first prove that the chords of a rigid pair belong to different completions $\mathcal{C}_1$ and $\mathcal{C}_2$. This will be a corollary of the following lemma.

Lemma 2.2.14. *Let G and $(<_X, <_Y)$ be as in Assumption 2.2.11. Suppose $<_1$ and $<_2$ are linear orders as in Definition 2.2.12. Suppose $\{x_1y_1, x_2y_2\}$ is a rigid pair. If $x_1 <_1 x_2$ then*

(i) For all $y \in N_Y(x_1)$, $y <_1 y_2$.

(ii) For all $x \in N_X(y_2)$, $x_1 <_1 x$.

(iii) For any $x \in X$ and $y \in Y$ with $x <_1 x_1$ and $y_2 <_1 y$, we have $x \nsim y$.

Similarly if $x_1 <_2 x_2$ and we replace $<_1$ by $<_2$ in the statements (i)-(iii), then statements (i), (ii), and (iii) hold.

Proof. We prove the lemma for $<_1$. The proof for $<_2$ follows similarly. Suppose that $\{x_1y_1, x_2y_2\}$ is a rigid pair with $x_1 <_1 x_2$. Therefore by Definition 2.2.12 we have $x_1 <_X x_2$, and by Remark 2 we have that $y_1 <_Y y_2$ and $y_1 <_1 y_2$.

First we prove (i) by contradiction. Let $w \in N_Y(x_1)$ and $y_2 <_1 w$. Since $\{x_1y_1, x_2y_2\}$ is a rigid pair, $x_1 \notin N_X(y_2)$. We also have $x_1 \in N_X(w)$, and thus $N_X(w) \nsubseteq N_X(y_2)$. But by assumption $y_2 <_1 w$, and thus by Remark 2 we must have $y_2 <_Y w$. Since y_2 and w are related in $<_Y$, by Definition 2.2.10, there are $u \in N_X(y_2) \setminus N_X(w)$ and $z \in N_X(w) \setminus N_X(y_2)$ such that $\{uy_2, zw\}$ is a rigid pair. Since $x_1 \in N_X(w) \setminus N_X(y_2)$ then $\{uy_2, x_1w\}$ is also a rigid pair. This together with the fact that $\{x_1y_1, x_2y_2\}$ is a rigid pair implies that $x_2y_1 \sim^* x_1y_2 \sim^* uw$ in $\tilde{G}$, as seen in Figure 2.12.

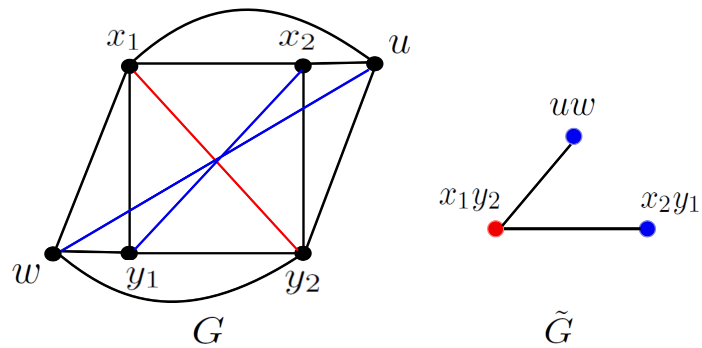


Figure 2.12: Rigid pairs $\{x_1y_1, x_2y_2\}$ and $\{uy_2, x_1w\}$ of G, and their corresponding structure in $\tilde{G}$.

Since uw and x_2y_1 receive the same color, then by Definition 2.2.10, we have that $x_1 <_X x_2$ if and only if $x_1 <_X u$. We know that $x_1 <_X x_2$. So $x_1 <_X u$, and thus $w <_Y y_2$. This together with Definition 2.2.12 implies that $w <_1 y_2$, which contradicts our assumption. A similar argument proves (ii).

We now prove (iii). Again by contradiction, assume that there are $v \in X$ and $z \in Y$ such that $v \sim z$ and $v <_1 x_1$ and $y_2 <_1 z$. By Part (i) we have $x_1 \nsim z$. But $v \sim z$, and so $N_Y(v) \nsubseteq N_Y(x_1)$. Therefore by Definition 2.2.12, $v <_X x_1$. This implies that there is rigid pair $\{vz, x_1w\}$ with $w \in N_Y(x_1)$. Since $v <_X x_1$ then $z <_Y w$. Then by Definition 2.2.12 we have $z <_1 w$. This is impossible since by Part (i) for all $y \in N_Y(x_1)$, we have $y <_1 y_2$. Therefore, $v \nsim z$. $\square$

Corollary 2.2.15. *Let G and $(<_X, <_Y)$ be as in Assumption 2.2.11, and $<_1$ and $<_2$ be as in Definition 2.2.12. Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are completions corresponding to $<_1$ and $<_2$, respectively. Then each chord of a rigid pair belongs to exactly one of $\mathcal{C}_1$ and $\mathcal{C}_2$.*

Proof. Suppose $\{x_1y_1, x_2y_2\}$ is a rigid pair. Without loss of generality, let $x_1 <_1 x_2$. Then by Remark 2 we have that $y_1 <_1 y_2$. We prove that $x_2y_1 \in \mathcal{C}_1 \setminus \mathcal{C}_2$ and $x_1y_2 \in \mathcal{C}_2 \setminus \mathcal{C}_1$. By Definition 2.2.12 we know that $x_1 <_X x_2$. Also by Definition 2.2.12, part (3) of the definition of $<_1$, we have that $x_1 <_1 x_2 <_1 y_1 <_1 y_2$. Since $x_1 \sim y_1$ then by the definition of completion we have that $x_2y_1 \in \mathcal{C}_1$. We now prove that $x_2y_1 \notin \mathcal{C}_2$. Since $x_1 <_X x_2$ then by Definition 2.2.12 for $<_2$ we have that $x_2 <_2 x_1$. Then by Remark 2 we have that $y_2 <_2 y_1$. Since $x_2 <_2 x_1$ then by Lemma 2.2.14 we know that x_2 and y_1 are not between two adjacent vertices, and thus by the definition of completion we have that $x_2y_1 \notin \mathcal{C}_2$. This implies that $x_2y_1 \in \mathcal{C}_1 \setminus \mathcal{C}_2$. A similar discussion shows that $x_1y_2 \in \mathcal{C}_2 \setminus \mathcal{C}_1$. $\square$

As we mentioned earlier, to prove that linear orders $<_1$ and $<_2$ satisfy Equation 2.2, we need to show that completions $\mathcal{C}_1$ and $\mathcal{C}_2$ have empty intersection. To show this, it is enough to prove that any vertex of $\tilde{G}$ (non-edge of G) belongs to at most one of the completions. Corollary 2.2.15 takes care of the non-isolated vertices of $\tilde{G}$, and the following lemma deals with the isolated ones.

Lemma 2.2.16. *Let G and $(<_X, <_Y)$ be as in Assumption 2.2.11. Suppose $<_1$ and $<_2$ are linear orders as in Definition 2.2.12. Suppose uw, $u \in X$ and $w \in Y$, is an isolated vertex of $\tilde{G}$. Then*

(i) For all $y \in N_Y(u)$ we have $y <_1 w$, and $y <_2 w$.

(ii) For all $x \in N_X(w)$ we have $u <_1 x$ or $u <_2 x$.

Proof. We only prove the lemma for $<_1$. The proof for $<_2$ is identical. Let uw be an isolated vertex of $\tilde{G}$. By the definition of $\tilde{G}$ we know that uw is not a chord of any rigid pair of G. We first prove (i). Suppose by contradiction that $y \in N_Y(u)$ and $w <_1 y$. Since $w <_1 y$ then by Remark 2 we know either $N_X(y) \subseteq N_X(w)$ or $w <_Y y$. Since $u \in N_X(y) \setminus N_X(w)$ then $w <_Y y$. Therefore there is a rigid pair $\{xw, x'y\}$. Since $u \in N_X(y) \setminus N_X(w)$ then $\{xw, uy\}$ is also a rigid pair with chords xy and uw. This contradicts the fact that uw is an isolated vertex of $\tilde{G}$, and thus for all $y \in N_Y(u)$ we have $y <_1 w$. The proof of (ii) follows by a similar argument. $\square$

Corollary 2.2.17. *Let G and $(<_X, <_Y)$ be as in Assumption 2.2.11, and $<_1$ and $<_2$ be as in Definition 2.2.12. Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are completions corresponding to $<_1$ and $<_2$, respectively. Then the isolated vertices of $\tilde{G}$ do not belong to any of the completions $\mathcal{C}_1$ and $\mathcal{C}_2$.*

Proof. Let uw be an isolated vertex of $\tilde{G}$ with $u \in X$ and $w \in Y$. Then by Definition 2.2.12, $u <_1 w$. By (i) of Lemma 2.2.16 we know that if $y \in N_Y(u)$ then $y <_1 w$. Moreover by (ii) of Lemma 2.2.16, if $x \in N_X(w)$ then $N_Y(x) \nsubseteq N_Y(u)$ and thus $u <_1 x$. This implies that for all $y \in N_Y(u)$ and all $x \in N_X(w)$, we have $u <_1 x <_1 y <_1 w$. Therefore, u and w are not in between two adjacent vertices in linear orders $<_1$, and thus $uw \notin \mathcal{C}_1$. An analogous discussion for $<_2$ proves that $uw \notin \mathcal{C}_2$. $\square$

We are now ready to prove that linear orders $<_1$ and $<_2$ of Definition 2.2.12 satisfy Equation 2.2.

Lemma 2.2.18. *Suppose that G and $(<_X, <_Y)$ are as in Assumption 2.2.11. Then $<_1$ and $<_2$ as in Definition 2.2.12 are linear orders satisfying Equation 2.2.*

Proof. Let G and f be as in Assumption 2.2.11, and let linear orders $<_1$ and $<_2$ be as in Definition 2.2.12. By Lemma 2.2.13, we know that $<_1$ and $<_2$ are linear orders. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the completions of $<_1$ and $<_2$, respectively. By Corollary 2.2.15 we know that adjacent vertices of $\tilde{G}$ belong to different completions. Moreover $f : V(\tilde{G}) \to \{\text{red}, \text{blue}\}$ is a proper 2-coloring of G and adjacent vertices receive different colors. Suppose without loss of generality that the red color class is a subset of $\mathcal{C}_1$ and the blue color class is a subset of $\mathcal{C}_2$. By Corollary 2.2.17 we know that completions $\mathcal{C}_1$ and $\mathcal{C}_2$ consist of only non-isolated vertices of $\tilde{G}$. This implies that $\mathcal{C}_1$ is a subset of the red color class of f, and $\mathcal{C}_2$ is a subset of the blue color class of f. Therefore, $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, and we are done.

□

Corollary 2.2.19. *Let G be as in Assumption 2.2.11. Then G is square geometric.*

Proof. Let G be a cobipartite graph with proper bi-ordering $(<_X, <_Y)$ as in Assumption 2.2.11. We know by Lemma 2.2.18 that $<_1$ and $<_2$ as in Definition 2.2.12 are linear orders satisfying Equation 2.2. Then by Theorem 2.2.1 we have that G is square geometric. □

Recall that our aim is to prove Theorem 2.2.8, which states that a cobipartite graph G is square geometric if and only if its chord graph is bipartite. The proof of the necessity was easy (Corollary 2.2.7). For the proof of the sufficiency , our strategy is to use a 2-coloring of the chord graph of G to obtain two linear orders $<_1$ and $<_2$ for G which satisfy Equation 2.2. We defined proper bi-ordering $(<_X, <_Y)$ (Definition 2.2.9) which is the main tool to obtain linear orders $<_1$ and $<_2$ from a 2-coloring of $\tilde{G}$. Indeed, for any 2-coloring of $\tilde{G}$ we defined corresponding relations $<_X$ and $<_Y$. Then we proved in Lemma 2.2.18 that if $\tilde{G}$ is bipartite and there is a proper 2-coloring of $\tilde{G}$ such that relations $<_X$ and $<_Y$ as given in Definition 2.2.10 form a proper bi-ordering then G is square geometric. Therefore, to complete the proof of Theorem 2.2.8 we need to find a 2-coloring of $\tilde{G}$ such that the relations $<_X$ and $<_Y$ as given in Definition 2.2.10 form a proper bi-ordering.

2.3 Proper bi-ordering of a cobipartite graph with a bipartite chord graph

In this section we prove that if G is a cobipartite graph with a bipartite $\tilde{G}$, then $\tilde{G}$ has a 2-coloring whose corresponding relations $<_X$ and $<_Y$ as given in Definition 2.2.10 form a proper bi-ordering. Note that by Remark 1, we already know that $<_X$ and $<_Y$ of Definition 2.2.10 allow no crossed 4-cycle. Therefore, if relations $<_X$ and $<_Y$ are partial orders then $(<_X, <_Y)$ is a proper bi-ordering. The next remark presents some of the most useful properties of relations $<_X$ and $<_Y$ of Definition 2.2.10.

Remark 3. *Let $<_X$ and $<_Y$ be as in Definition 2.2.10. Then*

(1) Suppose $x_1, x_2 \in X$ are different vertices of G. Then x_1 and x_2 are related in $<_X$ if there is a rigid pair $\{x_1y_1, x_2y_2\}$ in G, otherwise x_1 and x_2 are incomparable in $<_X$. Similarly, for different vertices $y_1, y_2 \in Y$, y_1 and y_2 are related in $<_Y$ if there is a rigid pair $\{x_1y_1, x_2y_2\}$ in G, otherwise y_1 and y_2 are incomparable in $<_Y$.

(2) Let $\{x_1y_1, x_2y_2\}$ be a rigid pair of G. Then $x_1 <_X x_2$ if and only if $y_1 <_Y y_2$. This implies that $<_X$ is a partial order if and only if $<_Y$ is a partial order.

The aim is to obtain a proper 2-coloring of the chord graph $\tilde{G}$ such that its associated relations $<_X$ and $<_Y$ as in Definition 2.2.10 are reflexive, antisymmetric, and transitive. The case where $\tilde{G}$ is connected is simple to deal with as there is only one possible 2-coloring of $\tilde{G}$ (up to switching the colors). However for a disconnected $\tilde{G}$ the situation is more complicated. Here we have to make sure that the colorings of the components are chosen so that they are compatible in ways that will be explained later in this section.

By (2) of Remark 3 we know that $<_X$ is a partial order if and only if $<_Y$ is a partial order. Therefore, from now on, we state our results only for $<_X$. The same results hold for $<_Y$ accordingly.

We know from Definition 2.2.10 that the relation of vertices in $<_X$ is based on a 2-coloring of chords of rigid pairs of G or equivalently the vertices of the chord graph

$\tilde{G}$. Indeed a rigid pair $\{x_1y_1, x_2y_2\}$ of G corresponds to an edge of $\tilde{G}$ *i.e.* the chords of the rigid pair x_1y_2 and x_2y_1 are vertices of $\tilde{G}$ and $x_1y_2 \sim^* x_2y_1$. Suppose x_1y_2 receives red in a 2-coloring of $\tilde{G}$. Then by Definition 2.2.10 we have $x_1 <_X x_2$. Now let $\{x_3y_3, x_4y_4\}$ be another rigid pair of G. If there is a path in $\tilde{G}$ between x_1y_2 and x_3y_4 then the relation of x_3 and x_4 in $<_X$ is also determined by the color of x_1y_2. That is if the path between x_1y_2 and x_3y_4 is of even length then x_3y_4 is red and thus $x_3 <_X x_4$, and if the path is of odd length then x_3y_4 is blue and thus $x_4 <_X x_3$. This implies that the color of one vertex in a component of $\tilde{G}$ determines the relation between vertices of the rigid pairs whose chords belong to that component.

Therefore, to find our desired 2-coloring (the one which provides us with two partial orders $<_X$ and $<_Y$) we need to study the structure of a chord graph, and the paths between the chords of different rigid pairs in $\tilde{G}$.

2.3.1 The paths of the chord graph $\tilde{G}$, and their corresponding structure in G

In this subsection, we collect some information about the structure of the paths of chord graph $\tilde{G}$. We assume throughout that G is a cobipartite graph with a bipartite chord graph $\tilde{G}$.

Let $xy, x'y' \in V(\tilde{G})$. If there exists an even path between xy and $x'y'$ in $\tilde{G}$ then we denote this by $xy \approx_e x'y'$, and if there exists an odd path between xy and $x'y'$ in $\tilde{G}$ then we denote this by $xy \approx_o x'y'$. First we give an example of a path in $\tilde{G}$ and its corresponding structure in G.

Example 6. *Suppose that x_1y_2 and x_3y_4 are distinct vertices of $\tilde{G}$, and there is a path $\tilde{P}$ of length 4 in $\tilde{G}$ between x_1y_2 and x_3y_4. Then there are vertices $u_1v_1, u_2v_2, u_3v_3 \in V(\tilde{G})$ such that*

$$\tilde{P} : x_1y_2 \sim^* u_1v_1 \sim^* u_2v_2 \sim^* u_3v_3 \sim^* x_3y_4$$

Since $x_1y_2 \sim^* u_1v_1$ then, by definition of chord graph $\tilde{G}$, we know that $\{x_1v_1, u_1y_2\}$ is a rigid pair with chords x_1y_2 and u_1v_1. Similarly, since $u_1v_1 \sim^* u_2v_2$, $u_2v_2 \sim^* u_3v_3$, and $u_3v_3 \sim^* x_3y_4$ then $\{u_1v_2, u_2v_1\}$ is a rigid pair with chords u_1v_1, u_2v_2, $\{u_2v_3, u_3v_2\}$ is a rigid pair with chords u_2v_2, u_3v_3, and $\{u_3y_4, x_3v_3\}$ is a rigid pair with chords u_3, v_3, x_3y_4, respectively. See Figure 2.13. Note that in Figure 2.13, the edges between vertices of X, i.e. x_1, u_1, u_2, u_3, x_3, and the edges between vertices of Y, i.e.

y_2, v_1, v_2, v_3, y_4 are omitted to keep the figure simplified.

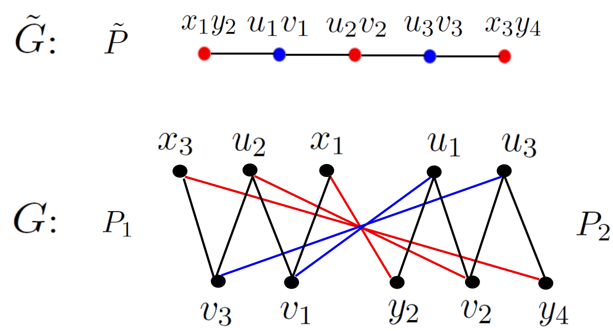


Figure 2.13: A path in $\tilde{G}$ and its corresponding structure in G.

Therefore, the path $\tilde{P}$ in $\tilde{G}$ corresponds to two paths $P_1 : x_1 \sim v_1 \sim u_2 \sim v_3 \sim x_3$, and $P_2 : y_2 \sim u_1 \sim v_2 \sim u_3 \sim y_4$ in G with the property that $\{x_1v_1, u_1y_2\}$, $\{u_1v_2, u_2v_1\}$, $\{u_2v_3, u_3v_2\}$, and $\{u_3y_4, x_3v_3\}$ are rigid pairs of G.

We now have a look at the structure of a path $\tilde{P}$ in $\tilde{G}$.] **Even paths of $\tilde{G}$:** Let k be a positive integer, and $\tilde{P}$ be a path of length $2k$ in $\tilde{G}$ between distinct vertices x_1y_2 and x_3y_4 of $\tilde{G}$. Then there are distinct vertices $u_iv_i \in V(\tilde{G})$, $1 \le i \le 2k-1$ such that

$$\tilde{P} : x_1y_2 \sim^* u_1v_1 \sim^* u_2v_2 \sim^* \ldots \sim^* u_{2k-1}v_{2k-1} \sim^* x_3y_4.$$

Note that the vertices u_iv_i of $\tilde{G}$ are distinct but this does not mean that all vertices u_i are distinct vertices of G or all v_i are distinct vertices of G. Indeed, we can have $u_i = u_j$ for $1 \le i < j \le 2k-1$. By definition of chord graph $\tilde{G}$ we know that

- $\{x_1v_1, u_1y_2\}$ is a rigid pair with chords x_1y_2 and u_1v_1.
- For $1 \le i \le 2k-2$, $\{u_iv_{i+1}, u_{i+1}v_i\}$ is a rigid pair with chords u_iv_i and $u_{i+1}v_{i+1}$.
- $\{x_3v_{2k-1}, u_{2k-1}y_4\}$ is a rigid pair with chords $u_{2k-1}v_{2k-1}$ and x_3y_4.

Moreover,

$$P_1 : x_1 \sim v_1 \sim u_2 \sim v_3 \sim \ldots \sim u_{2k-2} \sim v_{2k-1} \sim x_3,$$

is an even walk between x_1 and x_3 in G, and

$$P_2 : y_2 \sim u_1 \sim v_2 \sim u_3 \sim \ldots \sim v_{2k-2} \sim u_{2k-1} \sim y_4,$$

is an even walk between y_2 and y_4 in G.

Odd paths of $\tilde{G}$: Let k be a positive integer, and $\tilde{P}$ be a path of length $2k+1$ in $\tilde{G}$ between distinct vertices x_1y_2 and x_3y_4. Then there are vertices $u_iv_i \in V(\tilde{(G)})$, $1 \leq i \leq 2k$ such that

$$\tilde{P} : x_1y_2 \sim^* u_1v_1 \sim^* u_2v_2 \sim^* \ldots \sim^* u_{2k}v_{2k} \sim^* x_3y_4.$$

By definition of chord graph $\tilde{G}$ we know that

- $\{x_1v_1, u_1y_2\}$ is a rigid pair with chords x_1y_2 and u_1v_1.
- For $1 \leq i \leq 2k-1$, $\{u_iv_{i+1}, u_{i+1}v_i\}$ is a rigid pair with chords u_iv_i and $u_{i+1}v_{i+1}$.
- $\{x_3v_{2k}, u_{2k}y_4\}$ is a rigid pair with chords $u_{2k}v_{2k}$ and x_3y_4.

Moreover,

$$P_1 : x_1 \sim v_1 \sim u_2 \sim v_3 \sim \ldots \sim v_{2k-1} \sim u_{2k} \sim y_4,$$

is an odd walk between x_1 and y_4 in G, and

$$P_2 : y_2 \sim u_1 \sim v_2 \sim u_3 \sim \ldots \sim u_{2k-1} \sim v_{2k} \sim x_3,$$

is an odd walk between y_2 and x_3 in G.

2.3.2 A 2-coloring of $\tilde{G}$ whose associated relations $<_X$, $<_Y$ are partial orders

In this subsection we obtain a 2-coloring of G for which its corresponding relations $<_X$ and $<_Y$ as in Definition 2.2.10 are partial orders. We first prove that for any proper 2-coloring of G the associated relations $<_X$ and $<_Y$ are reflexive and antisymmetric.

Proposition 2.3.1. *Let G be a cobipartite graph with bipartite chord graph $\tilde{G}$.*

(1) Suppose $\{x_1y_1, x_2y_2\}$ and $\{x_1y_1', x_2y_2\}$ are rigid pairs of G with $y_1 \neq y_1'$. Then there is an even path between x_2y_1 and x_2y_1'.

(2) Suppose $\{x_1y_1, x_2y_2\}$ and $\{x_1y_1', x_2y_2'\}$ are rigid pairs of G with $y_1 \neq y_1'$ and $y_2 \neq y_2'$. Then there is an even path between x_2y_1 and x_2y_1', similarly there is an even path between x_1y_2 and $x_1y_2',$.

Proof. We first prove (1). Since $\{x_1y_1, x_2y_2\}$ is a rigid pair by definition of $\tilde{G}$ we know that $x_1y_2 \sim^* x_2y_1$. Similarly, since $\{x_1y_1', x_2y_2\}$ is a rigid pair we have that $x_1y_2 \sim^* x_2y_1'$. This implies that $x_2y_1 \sim^* x_1y_2 \sim^* x_2y_1'$, which is an even path in $\tilde{G}$ between x_2y_1 and x_2y_1'.

To prove (2) note that since $\{x_1y_1, x_2y_2\}$ and $\{x_1y_1', x_2y_2'\}$ are rigid pairs then $\{x_1y_1, x_2y_2\}$ and $\{x_1y_1', x_2y_2\}$ are rigid pairs, and thus by part (1) we have $x_2y_1 \approx_e x_2y_1'$. Similarly, $\{x_1y_1, x_2y_2\}$ and $\{x_1y_1, x_2y_2'\}$ are rigid pairs, and thus by part (1) we have $x_1y_2 \approx_e x_1y_2'$. $\square$

Lemma 2.3.2. *Let G be a cobipartite graph with bipartite chord graph $\tilde{G}$. Then for any proper 2-coloring of $\tilde{G}$ its associated relations $<_X$ and $<_Y$ as in Definition 2.2.10 are reflexive and antisymmetric.*

Proof. Let $f : V(\tilde{G}) \to \{\text{red}, \text{blue}\}$ be any proper 2-coloring of $\tilde{G}$. It is sufficient to show that $<_X$ is reflexive and antisymmetric (see (2) of Remark 3). We know by Definition 2.2.10 that for any $x \in X$ we have $x <_X x$. This gives us reflexivity of $<_X$. We now prove that $<_X$ is antisymmetric. Suppose that $x_1, x_2 \in X$ and x_1 and x_2 are related in $<_X$. Then by (1) of Remark 3 we know that there is a rigid pair $\{x_1y_1, x_2y_2\}$. By Proposition 2.3.1 we know that if there is another rigid pair $\{x_1y_1', x_2y_2'\}$ then $x_2y_1 \approx_e x_2y_1'$ (equivalently $x_1y_2 \approx_e x_1y_2'$). This implies that $f(x_2y_1) = f(x_2y_1')$, and thus for two distinct vertices x_1, x_2 always only one of $x_1 <_X x_2$ or $x_2 <_X x_1$ is true. This implies that if $x_1 <_X x_2$ and $x_2 <_X x_1$ then $x_1 = x_2$, and so $<_X$ is antisymmetric. $\square$

Let us now investigate the 2-colorings of $\tilde{G}$ whose associated relations $<_X$ and $<_Y$ are transitive. Suppose $f : V(\tilde{G}) \to \{\text{red}, \text{blue}\}$ is a proper 2-coloring of $\tilde{G}$ with associated relations $<_X$ and $<_Y$. Recall that the transitivity of $<_X$ gives us the transitivity of $<_Y$ by (2) of Remark 3 . To satisfy transitivity, $<_X$ must satisfy the following property. For any three vertices $x_1, x_2, x_3 \in X$ such that $x_1 <_X x_2$ and $x_2 <_X x_3$: (1) the vertices x_1 and x_3 must be related in $<_X$, and (2) $x_1 <_X x_3$. This implies that if $\{x_1y_1, x_2y_2\}$ and $\{x_2y_2', x_3y_3\}$ are rigid pairs of G such that $f(x_1y_2)$ and $f(x_2y_3)$ are red then: (1) $\{x_1y_1, x_3y_3\}$ is a rigid pair, and (2) $f(x_1y_3)$ is red.

The first requirement of the transitivity (the vertices x_1 and x_3 must be related in $<_X$) is a corollary of the following lemma.

Lemma 2.3.3. *Let G be a cobipartite graph with bipartite chord graph $\tilde{G}$. Suppose that x_1, x_2 and x_3 are different vertices of G. Let $\{x_1y_1, x_2y_2\}$ and $\{x_2y_2', x_3y_3\}$ be rigid pairs of G. If $\{x_1y_1, x_3y_3\}$ is not a rigid pair of G then there is an even path in $\tilde{G}$ between x_2y_1 and x_2y_3 i.e. $x_2y_1 \approx_e x_2y_3$.*

Proof. Suppose that G is cobipartite and $\tilde{G}$ is bipartite. Let $\{x_1y_1, x_2y_2\}$ and $\{x_2y_2', x_3y_3\}$ be rigid pairs of G. Suppose that $\{x_1y_1, x_3y_3\}$ is not a rigid pair. Then either $x_1 \sim y_3$ or $x_3 \sim y_1$.

If $x_1 \sim y_3$ then $\{x_1y_3, x_2y_2\}$ is a rigid pair. It follows that $x_1y_2 \sim^* x_2y_3$ in $\tilde{G}$. Since $\{x_1y_1, x_2y_2\}$ is a rigid pair in G we have $x_2y_1 \sim^* x_1y_2$ in $\tilde{G}$. Therefore $x_2y_1 \sim^* x_1y_2 \sim^* x_2y_3$ is an even path in $\tilde{G}$ between x_2y_1 and x_2y_3, see Figure 2.14.

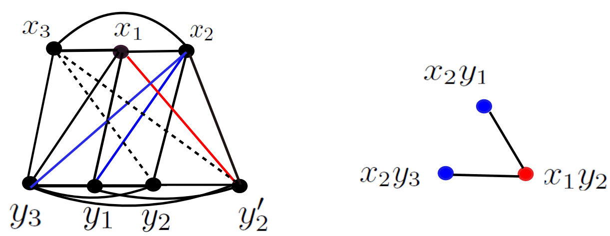


Figure 2.14: The even path $x_2y_1 \sim^* x_1y_2 \sim^* x_2y_3$ in $\tilde{G}$.

A similar discussion shows that if $x_3 \sim y_1$, then $x_2y_1 \approx_e x_2y_3$. □

Note that in Lemma 2.3.3 the vertices y_2 and y_2' are not necessarily different, that is we can have $y_2 = y_2'$.

Corollary 2.3.4. *Let G be a cobipartite graph with bipartite $\tilde{G}$. Suppose that x_1, x_2, and x_3 are different vertices of G. Let $\{x_1y_1, x_2y_2\}$ and $\{x_2y_2', x_3y_3\}$ be rigid pairs of G, and $f : V(\tilde{G}) \to \{red, blue\}$ be a proper 2-coloring of $\tilde{G}$. If $f(x_2y_1)$ is blue and $f(x_2y_3)$ is red, then $\{x_1y_1, x_3y_3\}$ is a rigid pair of G.*

Proof. We know by Lemma 2.3.3 that if $\{x_1y_1, x_3y_3\}$ is not a rigid pair then $x_2y_1 \approx_e x_2y_3$. This implies that in any proper 2-coloring of $\tilde{G}$ the vertices x_2y_1 and x_2y_3 receive the same color, which contradicts $f(x_2y_1) \neq f(x_2y_3)$. Therefore, $\{x_1y_1, x_3y_3\}$ is a rigid pair. □

By Corollary 2.3.4 we know that if $\{x_1y_1, x_2y_2\}$ and $\{x_2y_2', x_3y_3\}$ are rigid pairs of G such that x_1y_2 and x_2y_3 are red then $\{x_1y_1, x_3y_3\}$ is a rigid pair. We now discuss the second requirement of "$<_X$ being transitive" which is "$f(x_1y_3)$ is red". We will see in the next two lemmas that if $\tilde{G}$ has at most two components then any 2-coloring of G satisfies the second requirement. This concludes that if $\tilde{G}$ has at most two components then for any 2-coloring of G its associated ordering $<_X$ is transitive.

Lemma 2.3.5. *Suppose that G is cobipartite and $\tilde{G}$ is bipartite. Suppose x_1, x_2, and x_3 are different vertices of G. Let $\{x_1y_1, x_2y_2\}$ and $\{x_2y_2', x_3y_3\}$ be rigid pairs of G. If $x_2y_1 \approx_o x_2y_3$ then $\{x_1y_1, x_3y_3\}$ is a rigid pair and $x_2y_1 \approx_o x_1y_3$.*

Proof. Suppose that $\{x_1y_1, x_2y_2\}$ and $\{x_2y_2', x_3y_3\}$ are rigid pairs. Since $x_2y_1 \approx_o x_2y_3$ then by Lemma 2.3.3 we know that $\{x_1y_1, x_3y_3\}$ is a rigid pair. We now prove that $x_2y_1 \approx_o x_1y_3$. Let $\tilde{P}$ be an odd path connecting x_2y_1 and x_2y_3 in $\tilde{G}$.

$$\tilde{P} : x_2y_1 \sim^* u_1v_1 \sim^* u_2v_2 \sim^* \ldots \sim^* u_{2k}v_{2k} \sim^* x_2y_3,$$

where $u_iv_i \in V(\tilde{G})$ for $1 \leq i \leq 2k$.

Obviously we have the following :

$$\text{If } i \text{ is odd, then } u_iv_i \approx_o x_2y_1 \text{ and } u_iv_i \approx_e x_2y_3.$$

$$\text{If } i \text{ is even, then } u_iv_i \approx_e x_2y_1 \text{ and } u_iv_i \approx_o x_2y_3.$$

Claim. If there exists an integer i, $1 \leq i \leq 2k-1$, so that either $x_3 \sim v_i$ or $u_i \sim y_3$. then there exists an odd path between x_2y_1 and x_1y_3 in $\tilde{G}$.

Proof of Claim: To prove the claim, let m, $1 \leq m \leq 2k-1$ be the smallest positive integer for which either $x_3 \sim v_i$ or $u_i \sim y_3$. First assume that m is odd, so there is a non-negative integer t such that $m = 2t+1$.

Then $x_3 \sim v_{2t+1}$ and $x_3 \nsim v_{2t}$, and so $\{x_3v_{2t+1}, u_{2t+1}v_{2t}\}$ is a rigid pair. Hence $u_{2t+1}v_{2t+1} \sim^* x_3v_{2t}$. Also since m is the smallest positive integer for which either $x_3 \sim v_i$ or $u_i \sim y_3$, then for all $1 \leq i \leq m$ we have: $x_3 \nsim v_i$ and $u_i \nsim y_3$. Therefore

$$x_1y_3 \sim^* x_3y_1 \sim^* u_1y_3 \sim^* x_3v_2 \ldots \sim^* x_3v_{2t-2} \sim^* u_{2t-1}y_3 \sim^* x_3v_{2t},$$

is an odd path in $\tilde{G}$. Moreover $u_{2t+1}v_{2t+1} \approx_o x_2y_1$. Since $u_{2t+1}v_{2t+1} \sim^* x_3v_{2t}$ these create an odd path between x_2y_1 and x_1y_3 in $\tilde{G}$. A similar argument works for m even. This proves the claim. Next, we prove that a stronger claim holds.

Claim. If there exists an integer i, $1 \leq i \leq 2k-1$, so that $x_3 \sim v_i$ or $u_i \sim y_3$, then there exists an odd path between x_2y_1 and x_1y_3 in $\tilde{G}$.

Proof of Claim: Let m be the smallest integer so that $x_3 \sim v_m$ or $u_m \sim y_3$. We distinguish three cases.

- $m = 1$: First we prove $u_1 \nsim y_3$. If $u_1 \sim y_3$ then $\{u_1y_3, x_2v_1\}$ is a rigid pair. Hence $u_1v_1 \sim^* x_2y_3$. Since $u_1v_1 \approx_e x_2y_3$, this creates an odd walk in $\tilde{G}$ which contradicts the fact that $\tilde{G}$ is bipartite. Hence $u_1 \nsim y_3$. Thus $x_3 \sim v_1$, and it follows from the previous claim that there exists an odd path between x_2y_1 and x_1y_3 in $\tilde{G}$.

- m is even: There is a positive integer t such that $m = 2t$. If $x_3 \sim v_{2t+1}$ then, by the previous claim, $x_2y_1 \approx_o x_1y_3$ and we are done. Assume therefore that $x_3 \nsim v_{2t+1}$. If $v_{2t} \sim x_3$ then $\{x_3v_{2t}, u_{2t}v_{2t-1}\}$ is a rigid pair, and thus $u_{2t}v_{2t} \sim^* x_3v_{2t-1}$. Moreover $u_{2t}v_{2t} \approx_0 x_2y_3$ and $x_3v_{2t-1} \approx_0 x_2y_3$, this creates an odd closed walk in $\tilde{G}$ which contradicts the fact that $\tilde{G}$ is bipartite. Hence $v_{2t} \nsim x_3$. Therefore, $u_{2t} \sim y_3$, and it follows from the previous claim that $x_2y_1 \approx_o x_1y_3$.

- m is odd and $m > 1$: There is a non-negative integer t so that $m = 2t+1$. If $u_{2t+2} \sim y_3$ or $v_{2t+1} \sim x_3$ then we are done by the previous claim. Thus $u_{2t+1} \sim y_3$ and $u_{2t+2} \nsim y_3$. By a similar argument as in the previous case, this assumption leads to the existence of an odd closed walk in $\tilde{G}$, and thus a contradiction.

This concludes the proof of the claim. To complete the proof of the lemma, assume that $u_i \nsim y_3$ and $x_3 \nsim v_i$ for all $1 \leq i \leq 2k-1$. Moreover in path $\tilde{P}$, $u_{2k} \sim y_3$ which implies that $u_{2k}v_{2k} \sim^* u_{2k-1}y_3$. It follows that the following is a path in $\tilde{G}$:

$$x_1y_3 \sim^* x_3y_1 \sim^* u_1y_3 \sim^* \ldots \sim^* x_3v_{2k-2} \sim^* u_{2k-1}y_3 \sim^* u_{2k}v_{2k} \sim^* \ldots \sim^* u_1v_1 \sim^* x_2y_1$$

This is an odd path between x_2y_1 and x_1y_3, and we are done. $\square$

Corollary 2.3.6. . *Suppose that G is cobipartite and $\tilde{G}$ is bipartite and has at most two components. Suppose that $f : V(\tilde{G}) \to \{red, blue\}$ is a proper 2-coloring of $\tilde{G}$ with associated ordering $<_X$. If there are three vertices $x_1, x_2, x_3 \in V(\tilde{G})$ such that $x_1 <_X x_2$ and $x_2 <_X x_3$ then $x_1 <_X x_3$.*

Proof. Let $x_1, x_2, x_3 \in X$ such that $x_1 <_X x_2$ and $x_2 <_X x_3$. Then by Definition 2.2.10 there are rigid pairs $\{x_1y_1, x_2y_2\}, \{x_2y_2', x_3y_3\}$ for which $f(x_2y_1)$ is blue and $f(x_2y_3)$ is red. Then by Lemma 2.3.3 we have that $\{x_1y_1, x_3y_3\}$ is a rigid pair. First suppose that x_2y_1 and x_2y_3 are in the same component of $\tilde{G}$. Therefore, there is a path in $\tilde{G}$ between x_2y_1 and x_2y_3. Since $f(x_2y_1) \neq f(x_2y_3)$ then $x_2y_1 \approx_o x_2y_3$. Then by Lemma 2.3.5, we know that $x_2y_1 \approx_o x_1y_3$. This implies that $f(x_2y_1) \neq f(x_1y_3)$, and thus $f(x_1y_3)$ is red. Therefore, by Definition 2.2.10 we have that $x_1 <_X x_3$. Note that in this case all chords of rigid pairs $\{x_1y_1, x_2y_2\}$, $\{x_2y_2', x_3y_3\}$, and $\{x_1y_1, x_3y_3\}$ are in one component of $\tilde{G}$.

Now suppose that x_2y_1 and x_2y_3 are in different components of $\tilde{G}$. Since $\tilde{G}$ has at most two components then x_1y_3 either belongs to the component of x_2y_3 or belongs to the component of x_2y_1. Suppose without loss of generality that x_1y_3 is in the same component as x_2y_3. If $x_1y_3 \approx_o x_2y_3$ then $x_3y_1 \approx_o x_3y_2$, and by Lemma 2.3.5 we have that $x_3y_2 \approx_o x_1y_2$. This together with $x_2y_1 \sim^* x_1y_2$ and $x_3y_2 \sim^* x_2y_3$ implies that there is a path between x_2y_3 and x_1y_2. This contradicts our assumption that x_2y_1 and x_2y_3 are in different components. Therefore $x_1y_3 \approx_e x_2y_3$, and thus $f(x_1y_3)$ is red. This implies that $x_1 <_X x_3$. □

The discussion for a disconnected $\tilde{G}$ with at least three components is more complicated. The difficulty arises since in this case possibly there are rigid pairs $\{x_1y_1, x_2y_2\}, \{x_2y_2', x_3y_3\}$, and $\{x_1y_1, x_3y_3\}$ of G with chords in 3 different components of $\tilde{G}$, and there are at least 2^3 possible proper 2-colorings of $\tilde{G}$. Indeed there are $2^{\tilde{c}}$ proper 2-colorings of $\tilde{G}$ where $\tilde{c}$ is the number of components of $\tilde{G}$. If the chords of rigid pairs $\{x_1y_1, x_2y_2\}, \{x_2y_2', x_3y_3\}$, and $\{x_1y_1, x_3y_3\}$ are in 3 different components of $\tilde{G}$, then each of these components must be colored properly so that transitivity holds for the relations $<_X$ and $<_Y$. The problem becomes even more complicated if there are more rigid pairs with chords in different components of $\tilde{G}$. Then it is essential to check if the coloring gives consistent relations for all these rigid pairs.

The following is a simple example of a cobipartite graph with a disconnected $\tilde{G}$.

Example 7. *Let G be the cobipartite graph of Figure 2.15*

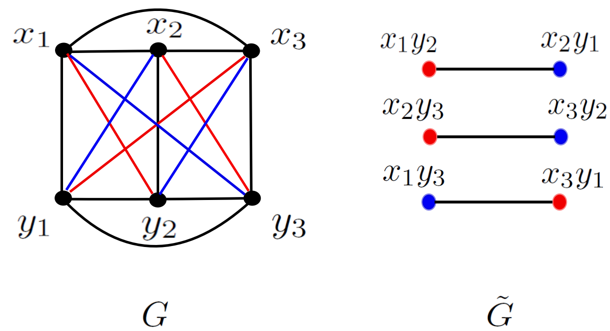


Figure 2.15: A cobipartite graph G with a disconnected chord graph $\tilde{G}$.

The ordering $<_X$ *associated to the given 2-coloring is* $x_1 <_X x_2$ *(since* x_1y_2 *is red),* $x_2 <_X x_3$ *(since* x_2y_3 *is red), and* $x_3 <_X x_1$ *(since* x_1y_2 *is red), which obviously is not transitive.*

This lead us to study the structures as in Example 7. We formulate these structures in the following definition.

Definition 2.3.7. *Let* G *be a cobipartite graph. The rigid pairs* $\{x_1y_1, x_2y_2\}$, $\{x_2y_2, x_3y_3\}$ *and* $\{x_1y_1, x_3y_3\}$ *form an* independent triple *in* G *if there is no path in* $\tilde{G}$ *between the chords of any two distinct pairs. We denote this independent triple by* $\{x_1y_1, x_2y_2, x_3y_3\}$*. The* chords of the independent triple $\{x_1y_1, x_2y_2, x_3y_3\}$ *are the chords of rigid pairs* $\{x_1y_1, x_2y_2\}$, $\{x_2y_2, x_3y_3\}$ *and* $\{x_1y_1, x_3y_3\}$.

As we noticed, despite the case where $\tilde{G}$ has at most two components, if $\tilde{G}$ has at least three components then not every proper 2-coloring of $\tilde{G}$ provides us with a transitive ordering $<_X$. In this case we need to obtain a 2-coloring which its associated ordering $<_X$ satisfies transitivity property. Recall that by (2) of Remark 3 transitivity of $<_X$ gives us the transitivity of $<_Y$.

2.3.3 Properties of independent triples

In this subsection we collect results about the properties of independent triples. These result will be used later to present a 2-coloring algorithm which provides us with a desired proper 2-coloring of $\tilde{G}$ (the one whose associated ordering $<_X$ is transitive).

Let G be a cobipartite graph with bipartition X and Y. For a path P in G we define X_P to be $V(P) \cap X$, and Y_P to be $V(P) \cap Y$.

Lemma 2.3.8. *Let G be a cobipartite graph and $x_3y_3 \in E(G)$. Let $x_1, x_2 \in X$ and P be a path in G connecting x_1 and x_2 such that y_3 has no neighbor in X_P and x_3 has no neighbor in Y_P. Then $x_1y_3 \approx_e x_2y_3$. Similarly, if $y_1, y_2 \in Y$ and P is a path in G connecting y_1 and y_2 such that y_3 has no neighbor in X_P and x_3 has no neighbor in Y_P then $x_3y_1 \approx_e x_3y_2$.*

Proof. We only prove the first statement of the lemma. The proof of the second statement is similar. Suppose that $P : x_1v_1u_1v_2 \ldots v_{2k+1}x_2$ is a path in G connecting x_1 and x_2, where $u_i \in X$, for $1 \leq i \leq 2k$, and $v_j \in Y$, for $1 \leq j \leq 2k+1$. If for all $1 \leq i \leq 2k$ and $1 \leq j \leq 2k+1$, we have $x_3 \nsim v_j$ and $y_3 \nsim u_i$ then

- $\{x_3y_3, x_1v_1\}$ is a rigid pair.
- $\{x_3y_3, u_tv_{t+1}\}$, $1 \leq t \leq 2k$, is a rigid pair.
- $\{x_3y_3, x_3v_{2k+1}\}$ is a rigid pair.

This creates the following path between x_1y_3 and x_2y_3 in $\tilde{G}$

$$\tilde{P} : x_1y_3 \sim^* x_3v_1 \sim^* y_3u_1 \sim^* x_3v_2 \sim^* \ldots \sim^* y_3u_{2k} \sim^* x_3v_{2k+1} \sim^* x_2y_3,$$

which is an even path. $\square$

Corollary 2.3.9. *Suppose that G is a cobipartite graph. Let $\{x_1y_1, x_2y_2, x_3y_3\}$ be an independent triple and P be a path in G between x_1 and x_2. Then either x_3 has a neighbor in Y_P or y_3 has a neighbor in X_P.*

Proof. If x_3 has no neighbor in Y_P and y_3 has no neighbor in X_P then by Lemma 2.3.8 we know that $x_1y_3 \approx_e x_2y_3$. But since $\{x_1y_1, x_2y_2, x_3y_3\}$ is an independent triple chords of $\{x_1y_1, x_3y_3\}$ and $\{x_2y_2, x_3y_3\}$ belong to different components. Therefore, either x_3 has a neighbor in Y_P or y_3 has a neighbor in X_P. $\square$

Lemma 2.3.10. *Let G be a cobipartite graph. Suppose that $\{x_1y_1, x_2y_2\}$ and $\{x_4y_4, x_5y_5\}$ are two rigid pairs of G, and $\{x_1y_1, x_2y_2, x_3y_3\}$ is an independent triple. Let $\tilde{P}$ be an even path in $\tilde{G}$ between x_2y_1 and x_5y_4.*

$$\tilde{P} : x_2y_1 \sim^* u_1v_1 \sim^* u_2v_2 \sim^* \ldots \sim^* u_{2k+1}v_{2k+1} \sim^* x_5y_4.$$

Then

(1) $P_1 : y_1u_1v_2u_3 \ldots v_{2k}u_{2k+1}y_4$, *and* $P_2 : x_2v_1u_2v_3 \ldots u_{2k}v_{2k+1}x_5$ *are paths in* G, *and* x_3 *has no neighbor in* Y_{P_1} *and* Y_{P_2}, *and* y_3 *has no neighbor in* X_{P_1} *and* X_{P_2}. *Moreover,* $x_4y_3 \approx_e x_1y_3$ *and* $x_2y_3 \approx_e x_5y_3$.

(2) $\{x_3y_3, x_4y_4, x_5y_5\}$ *is an independent triple. Moreover,* $x_4y_3 \approx_e x_1y_3$ *and* $x_2y_3 \approx_e x_5y_3$.

Proof. First we prove part (1). Consider the even path $\tilde{P}$.

$$\tilde{P} : x_2y_1 \sim^* u_1v_1 \sim^* u_2v_2 \sim^* \ldots \sim^* u_{2k+1}v_{2k+1} \sim^* x_5y_4,$$

From the definition of $\tilde{G}$, it follows that the u_iv_i's, $1 \le i \le 2k+1$, are non-edges of G. Also,

$$P_1 : y_1u_1v_2u_3 \ldots v_{2k}u_{2k+1}y_4,$$

$$P_2 : x_2v_1u_2v_3 \ldots u_{2k}v_{2k+1}x_5,$$

are paths in the graph G. Figure 2.16 is an example of paths $\tilde{P}$, P_1, and P_2 when $k = 1$.

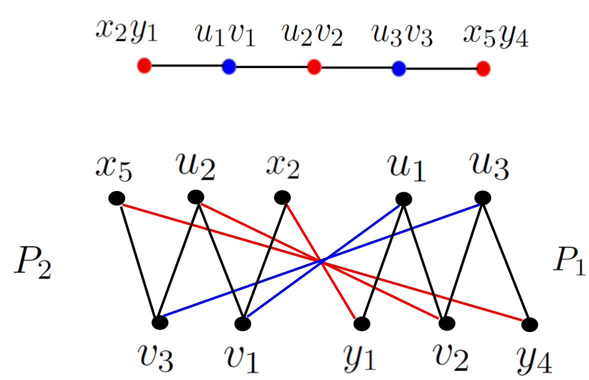


Figure 2.16: Example of a path $\tilde{P}$ in $\tilde{G}$ and its corresponding structure in G.

We now prove that for all $1 \le i \le 2k+1$ we have $x_3 \nsim v_i$ and $y_3 \nsim u_i$. Suppose $1 \le t \le 2k+1$ is the smallest integer such that either $x_3 \sim v_t$ or $y_3 \sim u_t$. Without loss of generality let $t = 2m$ be even and $x_3 \sim u_{2m}$. Then for any $1 \le i < 2m$ we have $x_3 \nsim v_i$ and $y_3 \nsim u_i$. Therefore

$$\tilde{Q} : x_3v_{2m-1} \sim^* u_{2m-2}y_3 \sim^* x_3v_{2m-3} \sim^* \ldots \sim^* x_3v_1 \sim^* x_2y_3,$$

is a path in $\tilde{G}$. Moreover $x_3v_{2m-1} \sim^* u_{2m}v_{2m}$ and there is a path in $\tilde{G}$ between $u_{2m}v_{2m}$ and x_2y_1. This together with path $\tilde{Q}$ forms a path in $\tilde{G}$ between x_2y_1 and x_2y_3. But

$\{x_1y_1, x_2y_2, x_3y_3\}$ is an independent triple. This implies that for all $1 \leq i \leq 2k+1$ we have $x_3 \nsim v_i$ and $y_3 \nsim u_i$. This proves the first statment of (1). To prove the second statement note that $P_1 : y_1u_1v_2u_3 \ldots v_{2k}u_{2k+1}y_4$ is a path between y_1 and y_4, and x_3 has no neighbor in Y_{P_1} and y_3 has no neighbor in X_{P_1}. Therefore, by (1) of Lemma 2.3.8 we have that $x_3y_1 \approx_e x_3y_4$. Moreover $P_2 : x_2v_1u_2v_3 \ldots u_{2k}v_{2k+1}x_5\ x_3$ is a path between x_2 and x_5, and x_3 has no neighbor in Y_{P_2}, and y_3 has no neighbor in X_{P_2}. Then by (1) of Lemma 2.3.8 we have that $x_2y_3 \approx_e x_5y_3$.

To prove (2), first we show that $\{x_3y_3, x_4y_4\}$ and $\{x_3y_3, x_5y_5\}$ are rigid pairs. Since $y_4 \in Y_{P_1}$ and $x_5 \in X_{P_2}$, by part (1), we know that $x_3 \nsim y_4$ and $y_3 \nsim x_5$. Therefore, it is enough to show that $y_3 \nsim x_4$ and $x_3 \nsim y_5$.

We first prove that $y_3 \nsim x_4$. If $u_1 = x_4$ then we know from part (1) that $y_3 \nsim x_4$. Suppose now that $u_1 \neq x_4$. If $y_3 \sim x_4$ then $\{x_5y_5, x_4y_3\}$ is a rigid pair and we have $x_4y_5 \sim^* x_5y_3$. By part (1), we have $x_5y_3 \approx_e x_2y_3$. Moreover we have $x_2y_1 \sim x_4y_5$. This creates a path between x_2y_1 and x_2y_3 which contradicts the fact that $\{x_1y_1, x_2y_2, x_3y_3\}$ is an independent triple. Therefore, $y_3 \nsim x_4$. A similar discussion proves that $x_3 \nsim y_5$. Therefore, we have that $\{x_3y_3, x_4y_4\}$, $\{x_3y_3, x_5y_5\}$, and $\{x_4y_4, x_5y_5\}$ are rigid pairs.

Now we prove that there is no path between chords of any two rigid pairs $\{x_3y_3, x_4y_4\}$, $\{x_3y_3, x_5y_5\}$, and $\{x_4y_4, x_5y_5\}$. We know that $x_2y_1 \approx_e x_5y_4$, $x_2y_3 \approx_e x_5y_3$, and $x_3y_1 \approx_e x_3y_4$. Moreover $\{x_1y_1, x_2y_2, x_3y_3\}$ is an independent triple, and thus there is no path between any two chords x_2y_1, x_2y_3, and x_3y_1. This implies that there is no path between any two chords x_5y_4, x_5y_3, and x_3y_4. Since $x_5y_4 \sim^* x_4y_5$, $x_5y_3 \sim^* x_3y_5$, and $x_3y_4 \sim^* x_4y_3$ then there is no path between chords of any two rigid pairs $\{x_3y_3, x_4y_4\}$, $\{x_3y_3, x_5y_5\}$, and $\{x_4y_4, x_5y_5\}$. This implies that $\{x_3y_3, x_4y_4, x_5y_5\}$ is an independent triple.

□

Note that the conclusion of Lemma 2.3.10 is true if $x_4y_4 = x_2y_2$.

Corollary 2.3.11. *Let $\{x_1y_1, x_2y_2\}$ and $\{x_2y_2, x_4y_4\}$ be rigid pairs and $\tilde{P}$ be a path in $\tilde{G}$ between x_2y_1 and x_2y_4. If there is $x_3y_3 \in E(G)$ such that $\{x_1y_1, x_2y_2, x_3y_3\}$ is an independent triple then $\tilde{P}$ is an even path.*

Proof. We first prove (1). Suppose to the contrary that $x_3y_3 \in E(G)$ such that $\{x_1y_1, x_2y_2, x_3y_3\}$ is an independent triple and $\tilde{P}$ is an odd path. Since $x_2y_4 \sim^* x_4y_2$

then there is an even path between x_2y_1 and x_4y_2. Then by (1) of Lemma 2.3.10 we have that $x_3y_1 \approx_e x_3y_2$ which is not possible since $\{x_1y_1, x_2y_2, x_3y_3\}$ is an independent triple. $\square$

Corollary 2.3.12. *Let G be a cobipartite graph with bipartite $\tilde{G}$. Suppose that C is a component of $\tilde{G}$. Then either each vertex of C is the chord of an independent triple, or no vertex of C is the chord of an independent triple.*

Proof. Suppose that x_1y_2 is a chord of an independent triple. Then there is an independent triple $\{x_1y_1, x_2y_2, x_3y_3\}$ in G. Let $x_4y_5 \in V(\tilde{G})$ and there is a path between x_4y_5 and x_1y_2. Since x_4y_5 is not an isolated vertex there is a rigid pair $\{x_4y_4, x_5y_5\}$ of G. We know that $x_4y_5 \sim^* x_5y_4$. Therefore, if $x_1y_2 \approx_o x_4y_5$ then $x_1y_2 \approx_e x_5y_4$. Since the role of x_4y_4 and x_5y_4 can be interchanged, then without loss of generality we assume that $x_1y_2 \approx_e x_4y_5$.

Now Since $x_1y_2 \approx_e x_4y_5$, then by (2) of Lemma 2.3.10 we have that $\{x_3y_3, x_4y_4, x_5y_5\}$ is an independent triple. Therefore, x_4y_5 is a chord of an independent triple as well. This finishes the proof of the corollary. $\square$

2.3.4 An algorithm to obtain a proper 2-coloring whose associated relations $<_X$ and $<_Y$ are partial orders

In this subsection we use the results of Subsection 2.3.3 to obtain a proper 2-coloring of $\tilde{G}$ whose associated relations $<_X$ and $<_Y$ are partial orders.

Definition 2.3.13. *Let G be a cobipartite graph with bipartite chord graph $\tilde{G}$. A partial 2-coloring of $\tilde{G}$ is a mapping $f : V(\tilde{H}) \to \{red, blue\}$ where $\tilde{H}$ is an induced subgraph of $\tilde{G}$. A partial 2-coloring f with associated relations $<_X$ and $<_Y$, as in Definition 2.2.10, is called a closed partial 2-coloring if the following condition is satisfied*

(1) The relations $<_X$ and $<_Y$ are partial orders.

(2) Every component of $\tilde{G}$ is either completely colored, or completely uncolored.

The following proposition is a direct consequence of Definition 2.3.13 and Definition 2.2.10.

Proposition 2.3.14. *Suppose G is a cobipartite graph with bipartite chord graph $\tilde{G}$. Let f be a closed partial 2-coloring with corresponding relations $<_X$ and $<_Y$. If $\{x_1y_1, x_2y_2\}$, $\{x_1y_1, x_3y_3\}$, and $\{x_2y_2, x_3y_3\}$ are rigid pairs with $f(x_1y_2)$ red and $f(x_2y_3)$ red then $f(x_1y_3)$ is red.*

We now define an algorithm which takes a closed partial 2-coloring of $\tilde{G}$ and extends it to another closed partial 2-coloring.

Algorithm 2.3.15. *[Extension of a closed partial 2-coloring]*
Input: A closed partial 2-coloring of $\tilde{G}$, $f_1 : V(\tilde{H}_1) \to \{red, blue\}$, with associated relations $<_X$ and $<_Y$ as in Definition 2.2.10, where $V(\tilde{H}_1)$ is a proper subset of $V(\tilde{G})$.
Output: A closed partial 2-coloring of $\tilde{G}$, $f_2 : V(\tilde{H}_2) \to \{red, blue\}$, where $V(\tilde{H}_1)$ is a proper subset of $V(\tilde{H}_2)$ and the restriction of f_2 to $\tilde{H}_1$ is equal to f_1.
Step 1. *Obtain the set $\overline{X}_{f_1}$ as follow.*

$$\overline{X}_{f_1} = \{x \in X | \exists\ xy \in V(\tilde{G}) \text{ such that } xy \text{ is non-isolated and is not colored in } f_1\}$$

Step 2. *Find a minimal element, x_1, of $\overline{X}_{f_1}$ under $<_X$.*
Step 3. *For all uncolored components of $\tilde{G}$ which contain a vertex of the form x_1y, do the following. Pick a vertex of the form x_1y of the component, and color it red. Then perform a BFS (Breath First Search) and extend the coloring to the whole component.*

Proposition 2.3.16. *Suppose that $f_1 : V(\tilde{H}_1) \to \{red, blue\}$ is a closed partial 2-coloring of $\tilde{G}$ with associated relations $<_X$ and $<_Y$, as in Definition 2.2.10. Let $\overline{X}_{f_1}$ be as in Algorithm 2.3.15, and let x_1 be a minimal element of $\overline{X}_{f_1}$ under $<_X$. If there is a rigid pair $\{x_1y_1, x_2y_2\}$ with $f_1(x_1y_2)$ blue then all the non-isolated vertices of $\tilde{G}$ of the form x_2y are colored in f_1.*

Proof. Suppose that there is a rigid pair $\{x_1y_1, x_2y_2\}$ with $f_1(x_1y_2)$ blue. Then, by Definition 2.2.10, we know that $x_2 <_X x_1$. We know that x_1 is the minimal element under $<_X$ among all the vertices of $x \in X$ such that there is an uncolored non-isolated vertex $xy \in V(\tilde{G})$. Since $x_2 <_X x_1$ then all the non-isolated vertices of $\tilde{G}$ of the form x_2y are colored in f_1. □

We now prove that the if the input of Algorithm 2.3.15 is a closed partial coloring then the output is a closed partial coloring as well.

Lemma 2.3.17. *Let G be a cobipartite graph with a bipartite chord graph $\tilde{G}$. If the input of Algorithm 2.3.15 is a closed partial 2-coloring then the output is a closed partial 2-coloring.*

Proof. Suppose that $\tilde{H}_1$ is a subgraph of $\tilde{G}$ and $f_1 : V(\tilde{H}_1) \to \{\text{red}, \text{blue}\}$ is a closed partial 2-coloring. Let f_2 be a partial 2-coloring of $\tilde{G}$ obtained from Algorithm 2.3.15 with input f_1. Let $<_X$ and $<_Y$ be the associated relations of f_2 as in Definition 2.2.10.

Suppose that x_1 is picked in step 2 of Algorithm 2.3.15 as a minimal element of the set $\overline{X}_{f_1}$. Assume that C is an uncolored component of $\tilde{G}$ which is colored in the extension process. Then, there is at least one vertex of the form $x_1y \in V(C)$. In step 3 of Algorithm 2.3.15, we choose a vertex of the form x_1y in C arbitrarily, we color it red, and then we extend the coloring to the whole component C. This implies that for all components of $\tilde{G}$ we have: either all vertices of the component are colored or no vertices of the component is colored. Therefore, Condition 2 of Definition 2.3.13 is satisfied. We now prove that Condition 1 of Definition 2.3.13 holds, *i.e.* the relations $<_X$ and $<_Y$ are partial orders.

By Lemma 2.3.2, we know that $<_X$ and $<_Y$ are reflexive and antisymmetric. Also by Corollary 2.3.6, we know that if there are rigid pairs $\{x_4y_4, x_5y_5\}$, $\{x_5y_5, x_6y_6\}$, and $\{x_4y_4, x_6y_6\}$ such that there is a path between x_4y_5 and x_5y_6 then the ordering $<_X$ is transitive among $x_4, x_5, x_6 \in X$. Therefore, to prove that $<_X$ is a partial order we need to prove that $<_X$ is transitive among $x_4, x_5, x_6 \in X$ such that there is an independent triple $\{x_4y_4, x_5y_5, x_6y_6\}$ in G. So suppose without loss of generality that $\{x_4y_4, x_5y_5, x_6y_6\}$ is an independent triple such that $x_4 <_X x_5$ and $x_5 <_X x_6$. We prove that $x_4 <_X x_6$. Since $x_4 <_X x_5$ and $x_5 <_X x_6$ then, by Definition 2.2.10, we have that $f_2(x_4y_5)$ and $f_2(x_5y_6)$ are red.

If $x_4y_5, x_5y_6 \in V(\tilde{H}_1)$ then we have that $f_1(x_4y_5)$ and $f_1(x_5y_6)$ are red. Since f_1 is a closed partial 2-coloring then $f_1(x_4y_6)$ is red, which implies that $f_2(x_4y_6)$ is red. Therefore, $x_4 <_X x_6$.

Now suppose that $x_4y_5 \in V(\tilde{H}_1)$ and $x_5y_6 \notin V(\tilde{H}_1)$. Then $f_1(x_4y_5)$ is red and x_5y_6 is not colored in f_1. Since $x_5y_6 \sim^* x_6y_5$ then x_6y_5 is not colored in f_1. This implies that $x_5, x_6 \in \overline{X}_{f_1}$. Since $x_5 <_X x_6$ then $f_2(x_5y_6)$ is red. This implies that there is a vertex of the form x_1y_2 such that there is an even path between x_1y_2 and x_5y_6. Then, by Lemma 2.3.10, we know that $\{x_1y_1, x_2y_2, x_4y_4\}$ is an independent triple and $x_1y_4 \approx_e x_5y_4$. This implies that $f_1(x_1y_4)$ is blue. Then, by Proposition 2.3.16, we have that x_4y_6 is colored in f_1. First let $f_1(x_4y_6)$ be blue. Then $f_1(x_6y_4)$ is red. Moreover, we know that $f_1(x_4y_5)$ is red. Then, by Proposition 2.3.14, $f_1(x_6y_5)$ is red. But we assumed that x_5y_6 is not colored in f_1. Therefore, $f_1(x_4y_6)$ must be red. Then $x_4 <_X x_6$, and we are done. □

Theorem 2.3.18. *Let G be a cobipartite graph with a bipartite $\tilde{G}$. Then there is a proper 2-coloring of $\tilde{G}$ such that its associated relations $<_X$ and $<_Y$ as in Definition 2.2.10 are partial orders.*

Proof. Let G be a cobipartite graph with a bipartite chord graph $\tilde{G}$. Let $\tilde{H}_0$ be the null subgraph of $\tilde{G}$, and $f_0 : V(\tilde{H}_0) \to \{\text{red}, \text{blue}\}$. Then f_0 is a closed partial 2-coloring of $\tilde{G}$. We apply Algorithm 2.3.15 with input f_0. By Lemma 2.3.17 we know that the output of Algorithm 2.3.15 is a closed partial 2-coloring f_1. Again we apply Algorithm 2.3.15 with input f_1, which gives us another closed partial 2-coloring f_3. We keep applying Algorithm 2.3.15 until the whole graph is colored. The 2-coloring we obtained at the end of this process is a closed partial 2-coloring. This implies that the associated ordering $<_X$ and $<_Y$ as in Definition 2.2.10 are partial orders, and we are done. □

2.3.5 The Algorithm and its Complexity

In this subsection, we present an algorithm for recognition of square geometric cobipartite graphs. Then we will prove that the complexity of the algorithm is $\mathcal{O}(n^4)$. The algorithm has three main steps. First, for a given graph G, it constructs the chord graph, $\tilde{G}$. Then it checks whether $\tilde{G}$ is bipartite or not, and at last it forms the two linear orders $<_1$ and $<_2$ of Equation 2.2. First we will see how the algorithm constructs graph $\tilde{G}$.

Lemma 2.3.19. *Let G be a cobipartite graph. Suppose that $L(G^c)$ is the line graph of the complement of G. Define the graph $[L(G^c)]_3$ to be a graph with the same vertex set of $L(G^c)$ (non-edges of G) and the edge set*

$$E([L(G^c)]_3) = \{uv \ \mid \ \textit{distance of } u \textit{ and } v \textit{ in graph } L(G^c) \textit{ is at least } 3\}.$$

The graphs $\tilde{G}$ and $[L(G^c)]_3$ are the same.

Proof of Lemma 2.3.19. The vertex set of both $\tilde{G}$ and $[L(G^c)]_3$ are the non-edges of G. First we show that any edge of $\tilde{G}$ is an edge of $[L(G^c)]_3$. Assume that u and v are two adjacent vertices of $\tilde{G}$. This means that their corresponding edges, e_u and e_v, in G^c are missing chords of a 4-cycle in G. It follows that there is no edge in G^c connecting the ends of e_u and e_v. Hence the distance of u and v in $L(G^c)$ is at least 3 and so they are adjacent in $[L(G^c)]_3$ as well. Now suppose that u and v are adjacent vertices of $[L(G^c)]_3$ with corresponding edges e_u and e_v in G^c. Hence their distance is at least 3 in graph $L(G^c)$. This means that in graph G^c there is no edge between the ends of e_u and e_v. This implies that there is an edge between their ends in graph G, which means that e_u and e_v are missing chords of a 4-cycle in G. Therefore u and v are adjacent in graph $\tilde{G}$. This proves that $\tilde{G}$ and $[L(G^c)]_3$ are the same graphs. □

Note that if $L(G^c)$ is a disconnected graph then two vertices of $L(G^c)$ which are in different components have distance ∞. These vertices are connected in $[L(G^c)]_3$. We are now ready to present the algorithm.

Algorithm 2.3.20. *Algorithm for recognition of cobipartite square geometric graphs:*

Part 1: Construct the chord graph $\tilde{G}$ and check if it is bipartite.

Input: A cobipartite graph G.

Output: Either a minimal subgraph of G that cannot be embedded or the chord graph $\tilde{G}$ with its components.

1.1 Form the graph G^c (complement of the graph G), and $L(G^c)$.

1.2 Build the graph $[L(G^c)]_3$ ($\tilde{G}$): Delete all the edges of $L(G^c)$. For all pairs of vertices $u, v \in V(L(G^c))$, add the edge uv if the distance of u and v in $L(G^c)$ is greater than 2.

1.3 Perform a BFS on $\tilde{G}$ obtained from step (2) to check if it is 2-colorable, and find its components. If $\tilde{G}$ is not 2-colorable find the odd cycle $\tilde{C}$. Let $V_G(\tilde{C})$ be the set of vertices of G that appear in the ends of edges of $\tilde{C}$. Form the induced graph of $V_G(\tilde{C})$, $G[V_G(\tilde{C})]$. Stop the algorithm and return "$G[V_G(\tilde{C})]$ is a minimal non-embeddable subgraph of G". If $\tilde{G}$ is 2-colorable then return all components of $\tilde{G}$.

The second part of the algorithm uses Algorithm 2.3.15 to find a closed partial 2-coloring of $\tilde{G}$ which gives us partial orders $<_X$ and $<_Y$.

Part 2: Obtain a proper bi-ordering of a square geometric cobipartite graph.

Input: Cobipartite graph G, chord graph $\tilde{G}$, and its components.

Output: A proper bi-ordering $<_X$ and $<_Y$.

2.1 Let $\tilde{H}_1$ be the null subgraph of $\tilde{G}$, and $f_1 : V(\tilde{H}_1) \to \{\text{red, blue}\}$. Perform Algorithm 2.3.15 with input f_1.

2.2 Repeat Algorithm 2.3.15 until all the non-isolated vertices of the graph $\tilde{G}$ are colored.

2.3 Find the ordering $<_X$ and $<_Y$ corresponding to the 2-coloring obtained in 2.2.

Part 3: Find the linear orders $<_1$ and $<_2$ of Equation (2.2) for a square geometric cobipartite graph. Use the partial orderings $<_X$ and $<_Y$ obtained in part 2, and construct the linear orders $<_1$ and $<_2$ as in Definition 2.2.12.

Note that the correctness of Algorithm 2.3.20 follows by Theorem 2.2.8, and Lemma 2.3.17.

Complexity of Algorithm 2.3.20 We assume that n is the order of G and m is the size of G.

Part 1: Construct the chord graph $\tilde{G}$, and check if it is bipartite: Suppose that n is the order of the graph G. Then the complexity of step 1.1 is $\mathcal{O}(n^2)$. The complexity of step 1.2 is $\mathcal{O}(n^4)$: Since the number of vertices of $L(G^c)$ is at most n^2 and to construct $[L(G^c)]_3$, for all the vertices of $L(G^c)$ we perform a BFS which its complexity is $\mathcal{O}(n^2)$. Then the complexity of step 1.2 is $\mathcal{O}(n^4)$. The step 1.3 is also a BFS on $\tilde{G}$. The order of $\tilde{G}$ is at most n^2. Therefore, the number of edges of $\tilde{G}$ is at most n^4. This implies that the complexity of step 1.3 is $\mathcal{O}(n^4)$, and thus the complexity of part 1 of the algorithm is $\mathcal{O}(n^4)$.

Part 2: Obtain a proper bi-ordering of a square geometric cobipartite graph: Part 2 is a repetition of Algorithm 2.3.15. We prove that the complexity of all iterations of Algorithm 2.3.15 is $\mathcal{O}(n^4)$.

First we discuss the complexity of all the iterations of step 1 of Algorithm 2.3.15: Let $C_1, \ldots, C_r$ be the components of $\tilde{G}$. We perform a BFS on each component, C_i, and make a corresponding list, L_i, of vertices of X that appear as endpoints of edges corresponding to vertices in C_i. For example, if $x_1y_2 \in V(C_i)$, then x_1 belongs to L_i. For each $x \in X$, define $c(x)$ to be the number of the lists, L_i, which contain x.

Now let f_k denote the coloring obtained in the k-th iteration of Algorithm 2.3.15. We use $L_1, \ldots, L_r$ to find the set $\overline{X}_{f_k}$ at each iteration as follows: For all lists L_i which its corresponding component C_i is colored in f_k and is not colored in f_{k-1} do the following: If $x \in L_i$, then let $c(x) = c(x) - 1$. A vertex $x \in X$ belongs to $\overline{X}_{f_k}$ if $c(x) > 0$. At each iteration we only check the lists of newly colored components and the cardinality of $\bigcup_{i=1}^{r} L_i$ is at most n^4. This implies that the complexity of all iterations of step 1 of Algorithm 2.3.15 is $\mathcal{O}(n^4)$.

Each vertex of G is contained in at most n relations of $<_X$. Then, the minimal element of $\overline{X}_{f_k}$ in the k-th iteration can be obtained in $\mathcal{O}(n^2)$ steps. Moreover, there are at most n iterations of Algorithm 2.3.15. This implies that the complexity of all iterations of step 2 of Algorithm 2.3.15 is $\mathcal{O}(n^3)$.

We now discuss the complexity of all iterations of step 3 of Algorithm 2.3.15. In the k-th iteration of step 3 of Algorithm 2.3.15, we perform a BFS on uncolored components of $\tilde{G}$ which contain a vertex of the form x_1y, where x_1 is the minimal element of $\overline{X}_{f_k}$. Once an uncolored component is colored in the k-th iteration, we never perform a BFS on it through any other iterations of Algorithm 2.3.15. Therefore, when step 2.2 of Algorithm 2.3.20 stops, we perform BFS on each component of $\tilde{G}$ once. Therefore, the complexity of all iterations of step 3 of Algorithm 2.3.15 is $\mathcal{O}(n^4)$. This implies that the complexity of part 2 of the algorithm is $\mathcal{O}(n^4)$.

Part 3: Find the linear orders $<_1$ and $<_2$ of Equation (2.2) for a square geometric cobipartite graph. In part 3 we need to compare the neighborhoods of the pairs of the vertices of G and their relations in the relations $<_X$ and $<_Y$ obtained in part 2 of the algorithm. Since for all $v \in V(G)$ we have that $|N(v)| < n$ then comparing the neighborhood of two vertices takes at most $(n-1)^2$ steps. Moreover, there are at most $n(n-1)/2$ pairs of vertices of X and Y. Therefore, the complexity of part 3 is $\mathcal{O}(n^4)$.

Chapter 3

Square Geometric $\mathcal{B}_{a,b}-$graphs

In this chapter, we apply the methods of Chapter 2 to investigate the problem of recognition of square geometric $\mathcal{B}_{a,b}$-graphs. Recall that G is a $\mathcal{B}_{a,b}$-graph if $V(G) = X_a \cup X_b \cup Y$ such that X_a, X_b, and Y are cliques and the induced subgraphs $G_a = G[X_a \cup Y]$ and $G_b = G[X_b \cup Y]$ are cobipartite graphs. If $|X_a \setminus X_b| = |X_b \setminus X_a| = 1$ then we call the graph a type-1 $\mathcal{B}_{a,b}$-graph, and if $|X_a \setminus X_b| = 2$ and $|X_b \setminus X_a| = 1$ then we call the graph a type-2 $\mathcal{B}_{a,b}$-graph. Below are examples of a type-1 and a type-2 $B_{a,b}$-graph.

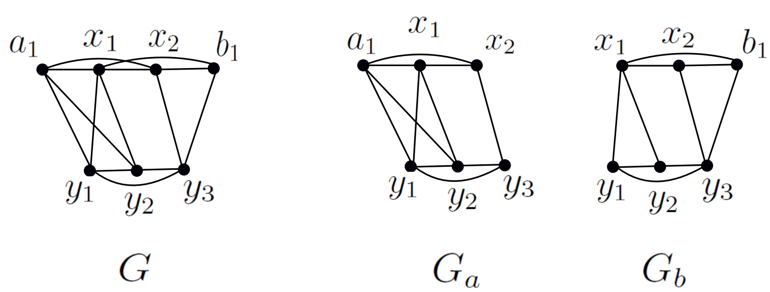


Figure 3.1: A type-1 $B_{a,b}$-graph with its corresponding cobipartite subgraphs G_a and G_b.

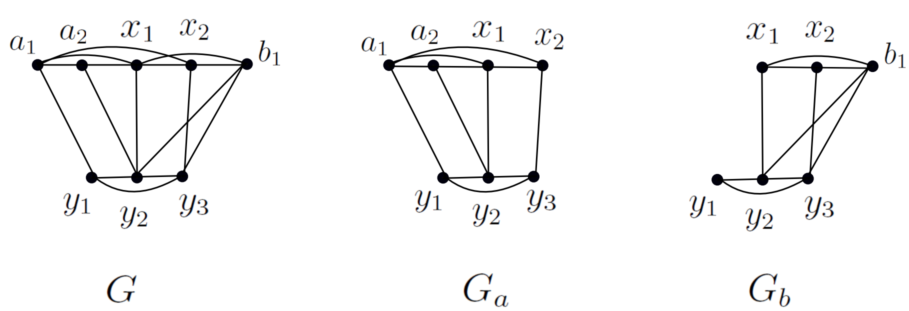


Figure 3.2: A type-2 $B_{a,b}$-graph with its corresponding cobipartite subgraphs G_a and G_b.

In Section 3.1, we present necessary and sufficient conditions for type-1 and type-2 $B_{a,b}$-graphs to be square geometric (Theorem 3.1.9). Part of the conditions are on specific forbidden structures of type-1 and type-2 $B_{a,b}$-square geometric graphs, and another part of the conditions require a specific 2-coloring of the graph $\tilde{G}_a \cup \tilde{G}_b$, where $\tilde{G}_a$ and $\tilde{G}_b$ are the chord graphs of G_a and G_b, respectively. The structural condition is formulated in Definition 3.1.4, and the 2-coloring condition is captured by Algorithm 3.4.7. The Algorithm 3.4.7 takes a $B_{a,b}$-graph as input and provides the desired 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$. If the algorithm fails then the graph is not square geometric.

In section 3.2, we prove the necessity of the conditions of Theorem 3.1.9, and then in Section 3.3, we prove the sufficiency of the conditions. Then in Subsection 3.4.2, we prove that the conditions of Theorem 3.1.9 can be checked in $\mathcal{O}(n^4)$ steps.

3.1 Preliminaries and Main Result

In this section, we first give an overview of the main themes of our approach for recognition of type-1, and type-2 $\mathcal{B}_{a,b}$-square geometric graphs. Then we state required definitions, and recall parts of results of Chapter 2 which will be used later in this chapter. At the end of the section we will state our main result which represents necessary and sufficient conditions for a type-1 and type-2 $B_{a,b}$-graph to be square geometric.

We know from Theorem 2.2.1 that a graph G is square geometric if and only if there are two linear orders $<_1$ and $<_2$ with corresponding completions $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. Recall that, for $i \in \{1,2\}$, $\mathcal{C}_i$ is the set of non-edges of G whose ends are in between two adjacent vertices in $<_i$. Consequently, to recognize if a graph G is square geometric or not, we need to study the structure of non-edges of G to see if such linear orders $<_1$ and $<_2$ exist.

Recall that, for a cobipartite graph H, the chord graph of H, $\tilde{H}$, is the graph whose vertices are non-edges of H, and two vertices are adjacent if they are chords of a rigid pair of H. We saw in Chapter 2 that for a cobipartite graph there are two types of non-edges: (1) the non-edges that are not chords of any rigid pair (the isolated vertices of $\tilde{H}$), and (2) the non-edges that are chords of some rigid pairs of G (the non-isolated vertices of $\tilde{H}$). In Chapter 2 we used the chord graph of a cobipartite graph H, $\tilde{H}$, to study the structure of the non-edges of H. We proved that if $\tilde{H}$ is bipartite then there exist linear orders $<_1$ and $<_2$ with $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$.

The isolated vertices of $\tilde{H}$ were easy to deal with. Indeed we located the ends of a non-edge corresponding to an isolated vertex of $\tilde{H}$ in linear orders $<_1$ and $<_2$ in such a way that the non-edge belongs to none of the completions (Corollary 2.2.17). For the non-isolated vertices of $\tilde{H}$ we obtained a 2-coloring of $\tilde{H}$ such that its associated relations $<_X$ and $<_Y$, as in Definition 2.2.10, are partial orders and form a proper bi-ordering. We used these partial orders to define linear orders $<_1$ and $<_2$ with $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ (Lemma 2.2.18).

In what follows we briefly discuss our approach. We first need the following definition.

Definition 3.1.1. *Let G be a $\mathcal{B}_{a,b}$-graph with clique bipartition U, Y, where $U = X_a \cup X_b$, and X_a, X_b and Y are cliques. A vertex $v \in \tilde{G}_a \cup \tilde{G}_b$ is called an a-vertex if there there exists $y \in Y$ such that $v = ay$ for $a \in X_a \setminus X_b$. For $b \in X_b \setminus X_a$, a b-vertex is defined similarly. Define $G_a = G[X_a \cup Y]$ and $G_b = G[X_b \cup Y]$.*

As we mentioned earlier, we can consider a $\mathcal{B}_{a,b}$-graph G as the union of its co-bipartite induced subgraphs G_a and G_b. This suggests the following partition of the set of non-edges of G:

(1) The isolated vertices of $\tilde{G}_a$ and $\tilde{G}_b$.

(2) The non-isolated vertices of $\tilde{G}_a$ and $\tilde{G}_b$.

(3) The non-edges with one end in $X_a \setminus X_b$ and the other end in $X_b \setminus X_a$.

Figure 3.3 is an example of a type-1 $B_{a,b}$-graph and its corresponding classes of non-edges.

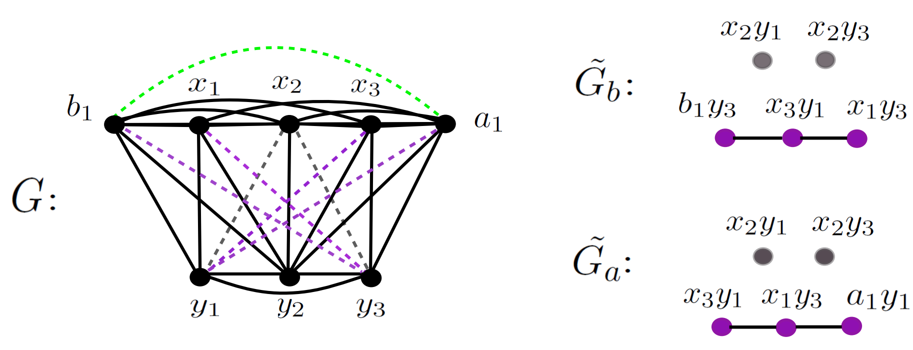


Figure 3.3: The grey non-edges are non-edges of class (1), the purple non-edges are non-edges of class (2), and the green non-edges are non-edges of class (3).

We want to determine under what conditions we can guarantee that there are linear orders $<_1$ and $<_2$ for a $B_{a,b}$-graph such that non-edges of classes (1)-(3) belong to at most one completion $\mathcal{C}_1$ and $\mathcal{C}_2$, *i.e.* $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. Similar to cobipartite graphs, it is easy to deal with the non-edges of class (1). We deal with them in the very last step when we are defining linear orders $<_1$ and $<_2$.

To deal with non-edges of class (2), similar to the method of Chapter 2 for cobipartite graphs, we use 2-colorings of $\tilde{G}_a$ and $\tilde{G}_b$, and their corresponding partial orders $<_X$ and $<_Y$. However, here the 2-colorings of $\tilde{G}_a$ and $\tilde{G}_b$ which give us the partial orders must be consistent on $\tilde{G}_a \cap \tilde{G}_b$.

The non-edges of class (3) are harder to deal with. This is due to the fact that we have no information about these non-edges, as they are not part of the non-edges of cobipartite graphs G_a and G_b. Therefore, we need to investigate their structure in a different way.

For the non-edges of class (3), the conditions which guarantee the existence of linear orders $<_1$ and $<_2$ with $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ are a combination of a condition on the 2-colorings of $\tilde{G}_a$ and $\tilde{G}_b$ that we mentioned above, and some conditions on the neighborhood of the ends of the non-edges of class (3) *i.e.* the neighborhood of a-vertices and b-vertices. Let us first have a look at the structural conditions *i.e.* the conditions on the neighborhoods of the ends of the non-edges of (3). There are some specific structures of the neighborhoods of a-vertices and b-vertices which force the non-edges of part (3) to belong to both completions $\mathcal{C}_1$ and $\mathcal{C}_2$ for any two linear orders $<_1$ and $<_2$. In what follows, we introduce such structures. Let G be a $B_{a,b}$-graph with cobipartite subgraphs G_a and G_b. A pair $\{x_1y_1, x_2y_2\}$ of edges of G is a *rigid pair* of G if $x_1y_1y_2x_2$ is an induced 4-cycle of G.

Definition 3.1.2. *Let G be a $\mathcal{B}_{a,b}$-graph with bipartition $X_a \cup X_b$ and Y. Assume $v \in X_a \cup X_b$ and $S \subseteq X_a \cup X_b$. Then v is called* rigid-free with respect to S *if there is no rigid pair $\{x_1y_1, x_2y_2\}$ of G with $\{x_1, x_2\} \subseteq S$ and $y_1, y_2 \in N_Y(v)$.*

In Figure 3.4, we have $y_1, y_2 \in N_Y(v)$ and $\{x_1y_1, x_2y_2\}$ is a rigid pair. Thus, by Definition 3.1.2, the vertex v is not rigid-free with respect to $\{x_1, x_2\}$. Indeed, for any $S \subseteq X_a \cup X_b$ such that $\{x_1, x_2\} \subseteq S$ the vertex v is not rigid-free with respect to S.

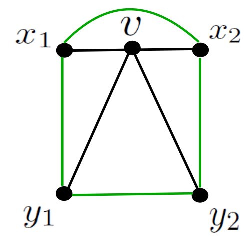


Figure 3.4: The vertex v is not rigid-free with respect to $\{x_1, x_2\}$.

Note that, in Definition 3.1.2, S is a subset of $X_a \cup X_b$. So the vertex v in Definition 3.1.2 is not necessarily adjacent to all vertices of S. For example, in Figure 3.5, the vertex a_1 is not rigid-free with respect to $\{x_1, b_1\}$, and any S with $\{x_1, b_1\} \subseteq S$. In particular, a_1 is not rigid-free with respect to X_b.

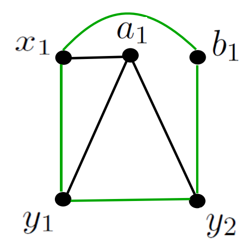


Figure 3.5: The vertex a_1 is not rigid-free with respect to $\{x_1, b_1\}$.

The set of non-edges of class (3) are the non-edges with one end in $X_a \setminus X_b$ and the other end in $X_b \setminus X_a$. For a type-1 $\mathcal{B}_{a,b}$-graph, we have $X_a \setminus X_b = \{a_1\}$ and $X_b \setminus X_a = \{b_1\}$. Therefore, the set of non-edges of class (3) for a type-1 graph is $\{a_1 b_1\}$. Similarly, for a type-2 $\mathcal{B}_{a,b}$-graph, we have $X_a \setminus X_b = \{a_1, a_2\}$ and $X_b \setminus X_a = \{b_1\}$. Therefore, the set of non-edges of class (3) for a type-2 graph is $\{a_1 b_1, a_2 b_1\}$. In what follows we will state the required conditions on the neighborhoods of a-vertices and b-vertices. Throughout this section, we assume that the graphs are as given in the following assumption, unless otherwise stated.

Assumption 3.1.3. *Let G be either a type-1 or a type-2 $\mathcal{B}_{a,b}$-graph with clique bipartition $X_a \cup X_b$ and Y. Suppose that G_a and G_b are cobipartite subgraphs of G as in Definition 3.1.1 with corresponding chord graphs $\tilde{G}_a$ and $\tilde{G}_b$, respectively. Let $X_b \setminus X_a = \{b_1\}$. If G is type-1, let $X_a \setminus X_b = \{a_1\}$, and if G is type-2, let $X_a \setminus X_b = \{a_1, a_2\}$. Suppose that for all $v \in \{a_1, a_2, b_1\}$ we have $0 < |N_Y(v)| < |Y|$. Also, suppose that there exists $u \in X_a \cup X_b$ such that $N_Y(b_1) \not\subseteq N_Y(u)$. Let $\tilde{G}_a \cap \tilde{G}_b \neq K_n^c$.*

In Assumption 3.1.3, we exclude some particular structures of a type-2 $\mathcal{B}_{a,b}$-graph G. The main challenge to characterize square geometric $B_{a,b}$-graphs is to find the conditions which make sure that in the intersection of G_a and G_b, the requirements for G_a being square geometric is consistent with the ones for G_b being square geometric. All the excluded cases of Assumption 3.1.3 have a simple-structured intersection $G_a \cap G_b$. This makes dealing with these cases easier. Our approach in this thesis is based on a well-built structure of $G_a \cap G_b$.

In Assumption 3.1.3, we do not include the case where one of a_1, a_2, and b_1 is adjacent to all vertices of Y. This case is either reducible to the study of cobipartite graphs or reducible to the cases which are already included in Assumption 3.1.3. For example, if $N_Y(b_1) = Y$ in G then $\{b_1\} \cup Y$ is a clique, and thus G is a cobipartite graph with bipartition X_a and $\{b_1\} \cup Y$. This reduces the problem to the study of square geometric cobipartite graphs. Also, we exclude the cases when $|N_Y(v)| = 0$ for $v \in \{a_1, a_2.b_1\}$, and $E(\tilde{G}_a \cap \tilde{G}_b) = \emptyset$. In the former case, the neighborhood of v is only one of the cliques X_a or X_b, and v has no neighbors in Y, and in the latter case the neighborhoods of vertices of $X_a \cap X_b$ in Y are nested.

Let G be a type-2 $B_{a,b}$-graph. If the neighborhood of a vertex in $\{a_1, a_2\}$ is a subset of the neighborhood of the other vertex, we always assume that $N_Y(a_1) \subseteq N_Y(a_2)$. Therefore, for a type-2 $B_{a,b}$-graph, either $N_Y(a_1) \subseteq N_Y(a_2)$ or there exists a rigid pair $\{a_1y_1, a_2y_2\}$ in G_a. The following definition formulates the forbidden structures of neighborhoods of a-vertices and b-vertices for type-1, and type-2 square geometric $B_{a,b}$-graphs.

Definition 3.1.4. *[Rigid-free conditions] Let G be as in Assumption 3.1.3. Then the rigid-free conditions are as follow.*

(i) If G is type-1, then either a_1 is rigid-free with respect to X_b or b_1 is rigid-free with respect to X_a.

(i) If G is type-2 and $N_Y(a_1) \subseteq N_Y(a_2)$ then either a_2 is rigid-free with respect to X_b or b_1 is rigid-free with respect to X_a.

(iii) If G is type-2 and there is a rigid pair $\{a_1y_1, a_2y_2\}$, then a_1 and a_2 are rigid-free with respect to X_b. Moreover, either there exists $a \in \{a_1, a_2\}$ such that $N_Y(a) \subseteq N_Y(b_1)$ or b_1 is rigid-free with respect to $\{a_1, a_2\}$.

We will prove, in Subsection 3.2.3, that if a graph G, as given in Assumption 3.1.3, is square geometric, then the rigid-free conditions of Definition 3.1.4 hold.

We will now give an insight into the coloring part of the necessary and sufficient conditions. As we mentioned earlier, an obvious necessary condition for a type-1 or type-2 $B_{a,b}$-graph G to be square geometric is that its induced cobipartite subgraphs G_a and G_b be square geometric. We saw, in Chapter 2, that for a square geometric cobipartite graph H, its chord graph $\tilde{H}$ has a 2-coloring for which its associated relations $<_X$ and $<_Y$, as in Definition 2.2.10, are partial orders (and thus by Remark 1 form a proper bi-ordering). Moreover, we saw that the partial orders $<_X$ and $<_Y$ provide us with the linear orders $<_1$ and $<_2$ as in Theorem 2.2.1 (Lemma 2.2.18). Therefore, a reasonable necessary condition for G to be square geometric is that: (1) $\tilde{G}_a$ has a 2-coloring f_a such that its associated relations $<_X$ and $<_Y$ as in Definition 2.2.10 are partial orders, and (2) $\tilde{G}_b$ has a 2-coloring f_b such that its associated relations $<_X$ and $<_Y$ as in Definition 2.2.10 are partial orders, and moreover (3) f_a and f_b agree on $V(\tilde{G}_a \cap \tilde{G}_b)$. The following definitions are versions of Definitions 2.2.9 and 2.2.10 for $B_{a,b}$-graphs.

Definition 3.1.5. *Let G be a $B_{a,b}$-graph. Suppose $<_X$ and $<_Y$ are relations on $X_a \cup X_b$ and Y, respectively. Then $(<_X, <_Y)$ is called a* proper bi-ordering *of G if*

(1) The restrictions of $<_X$ to X_a and X_b are partial orders on X_a and X_b, respectively. Moreover, the ordering $<_Y$ is a partial order on Y.

(2) For any rigid pair $\{xy, x'y'\}$ of G, we have that $x <_X x'$ if and only if $y <_Y y'$.

Definition 3.1.6. *Let G be a $B_{a,b}$-graph with cobipartite subgraphs G_a and G_b. Suppose $\tilde{G}_a$ and $\tilde{G}_b$ are bipartite, and $\tilde{G}_a \cup \tilde{G}_b$ is 2-colorable. Consider a proper 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$, $f : V(\tilde{G}) \rightarrow \{$red, blue$\}$. The relations associated with the coloring f, $<_X$ and $<_Y$, are defined as follows.*

- *$x <_X x'$ if there is a rigid pair $\{xy, x'y'\}$ such that $f(xy')$ is red and $f(x'y)$ is blue, or if $x = x'$.*

- *$y <_Y y'$ if there is a rigid pair $\{xy, x'y'\}$ such that $f(xy')$ is red and $f(x'y)$ is blue, or if $y = y'$.*

Remark 4. *Let $<_X$ and $<_Y$ be as in Definition 3.1.6.*

(1) *Let $\{x_1y_1, x_2y_2\}$ be a rigid pair. Then $x_1y_2 \sim^* x_2y_1$ in $\tilde{G}_a \cup \tilde{G}_b$. This implies that $f(x_1y_2) \neq f(x_2y_1)$. Without loss of generality, let $f(x_1y_2) =$ red, and so $f(x_2y_1) =$ blue. Then, by Definition 3.1.6, we have that $x_1 <_X x_2$ and $y_1 <_Y y_2$. This implies that if the restrictions of $<_X$ to X_a and X_b, and the relation $<_Y$ are partial orders, then $(<_X, <_Y)$ is a proper bi-ordering of G.*

(2) *For a rigid pair $\{xy, x'y'\}$, we know that $x <_X x'$ if and only if $y <_Y y'$. Therefore, the restrictions of $<_X$ to X_a and X_b are partial orders if and only if $<_Y$ is a partial order.*

(3) *Let $a \in X_a \backslash X_b$ and $b \in X_b \backslash X_a$. Since $a \nsim b$ then there is no rigid pair $\{ay, by'\}$ in G. This implies that a and b are not related in $<_X$. Moreover, if two vertices x_1 and x_2 are related in $<_X$ then either $x_1, x_2 \in X_a$ or $x_1, x_2 \in X_b$.*

The next proposition is a version of Lemma 2.3.2 for $B_{a,b}$-graphs.

Proposition 3.1.7. *Let G be a $B_{a,b}$-graph with cobipartite subgraphs G_a and G_b. Suppose $\tilde{G}_a \cup \tilde{G}_b$ is 2-colorable. Let f be an arbitrary proper 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$ with corresponding relations $<_X$ and $<_Y$ as in Definition 3.1.6. Then the restrictions of $<_X$ to X_a and X_b, and $<_Y$ are reflexive and antisymmetric.*

Proof. Let $f : V(\tilde{G}_a \cup \tilde{G}_b) \rightarrow \{\text{red}, \text{blue}\}$ be a proper 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$. By (2) of Remark 4, it is enough to show that the restrictions of $<_X$ to X_a and X_b are reflexive and antisymmetric. We know, by Definition 3.1.6, that for any $x \in X$ we have that $x <_X x$. This gives us reflexivity of the restrictions of $<_X$ to X_a and X_b. We now prove the antisymmetry. First suppose that $x_1, x_2 \in X_a$ and x_1 and x_2 are related in $<_X$. Then there is a rigid pair $\{x_1y_1, x_2y_2\}$ with $x_1, x_2 \in X_a$. Then consider the cobipartite graph G_a. By Proposition 2.3.1 we know that if there is another rigid pair $\{x_1y_1', x_2y_2'\}$ then $x_2y_1 \approx_e x_2y_1'$ (equivalently $x_1y_2 \approx_e x_1y_2'$). This implies that $f(x_2y_1) = f(x_2y_1')$, and thus for two distinct vertices x_1, x_2, only one of $x_1 <_X x_2$ or $x_2 <_X x_1$ is true. This implies that if $x_1 <_X x_2$ and $x_2 <_X x_1$ then $x_1 = x_2$, and so the restriction $<_X$ to X_a is antisymmetric. An analogous discussion proves that the restriction $<_X$ to X_b is antisymmetric. $\square$

Suppose now that G is a type-1 or type-2 $B_{a,b}$-graph such that $\tilde{G}_a \cup \tilde{G}_b$ has a 2-coloring so that its associated $<_X$ and $<_Y$, as in Definition 3.1.6, form a proper bi-ordering. Moreover, let the rigid-free conditions (Definition 3.1.4) hold for G. The following is an example of such a graph G which is not square geometric.

Example 8. *As we discussed in the introduction, Example 1, we know that the graph of Figure 3.6 is not square geometric.*

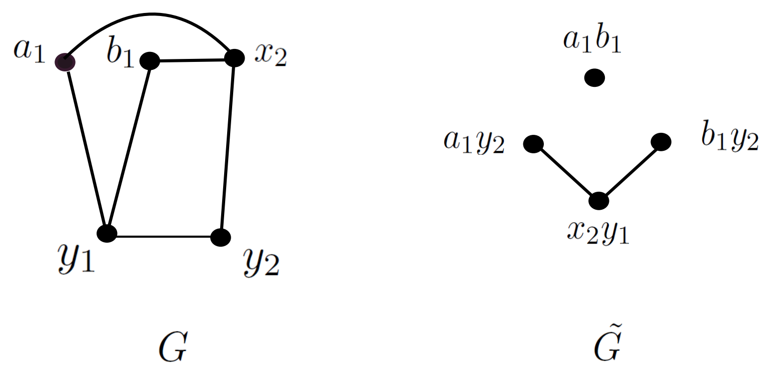


Figure 3.6: An example of a type-1 $B_{a,b}$-graph which is not square geometric, but satisfies rigid-free conditions.

As we can see in the figure, there is only one possible 2-coloring (up to switching the colors) of the non-isolated vertices of $\tilde{G}$. Suppose a_1y_2 is red, then x_2y_1 is blue and b_1y_2 is red. Then the relations $<_X$ and $<_Y$ of Definition 3.1.6 are as follow: $<_X$: $a_1 <_X x_2$, $b_1 <_X x_2$, and $<_Y$: $y_1 <_Y y_2$. The relations $<_X$ and $<_Y$, obviously, form a proper bi-ordering of G.

We will now briefly explain the necessary 2-coloring condition which is not satisfied in the above example. Let $\mathcal{A}$ be the set of all a-vertices which have a neighbor in $V(\tilde{G}_a \cap \tilde{G}_b)$, and $\mathcal{B}$ be the set of all b-vertices which have a neighbor in $V(\tilde{G}_a \cap \tilde{G}_b)$. Indeed, $\mathcal{A}$ and $\mathcal{B}$ belong to the non-edges of class (2) *i.e.* the non-edges which correspond to non-isolated vertices of the chord graphs $\tilde{G}_a$ and $\tilde{G}_b$.

In the above example, we have that $a_1y_2 \sim x_2y_1$ and $b_1y_2 \sim x_2y_1$, and thus $a_1y_2 \in \mathcal{A}$ and $b_1y_2 \in \mathcal{B}$.

Indeed, a necessary condition for a $B_{a,b}$-graph G to be square geometric is that there exists a 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$ such that its associated relations $<_X$ and $<_Y$ form a proper bi-ordering of G, and moreover $\mathcal{A}$ is a subset of one color class, and $\mathcal{B}$ is the subset of the other color class. But as we saw in the above example, in any 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$, the vertices a_1y_2 and b_1y_2 belong to the same color class. This does not allow G to be a square geometric graph.

We are now ready to state the main result on recognition of square geometric $B_{a,b}$-graphs of Assumption 3.1.3. First we need the following definition which is a version of Definition 2.3.13 for $B_{a,b}$-graphs.

Definition 3.1.8. *Let G be a $B_{a,b}$-graph with cobipartite subgraphs G_a and G_b. Suppose $\tilde{G}_a \cup \tilde{G}_b$ is 2-colorable. A* partial 2-coloring *of $\tilde{G}_a \cup \tilde{G}_b$ is a mapping $f : V(\tilde{H}) \to \{red, blue\}$ where $\tilde{H}$ is an induced subgraph of $\tilde{G}_a \cup \tilde{G}_b$. A partial 2-coloring f of $\tilde{G}_a \cup \tilde{G}_b$ with associated $<_X$ and $<_Y$, as in Definition 3.1.6, is called a* closed partial 2-coloring *if the following conditions are satisfied.*

(1) The relations $<_X$ and $<_Y$ are partial orders.

(2) Every component of $\tilde{G}_a \cup \tilde{G}_b$ is either completely colored, or completely uncolored.

The following theorem presents necessary and sufficient conditions for a graph G as in Assumption 3.1.3 to be square geometric.

Theorem 3.1.9. *Let G be a $B_{a,b}$-graph as given in Assumption 3.1.3. Then G is square geometric if and only if the following conditions are satisfied:*

(i) There is a closed partial 2-coloring $f : V(\tilde{G}_a \cup \tilde{G}_b) \to \{red, blue\}$ such that colors all vertices of $\mathcal{A}$ red and all vertices of $\mathcal{B}$ blue.

(ii) The rigid-free conditions of Definition 3.1.4 hold.

We will see, in Subsection 3.4.1, that the conditions of Theorem 3.1.9 can be checked in polynomial-time. We prove the necessity of the conditions of Theorem 3.1.9 in Section 3.2, and then in Section 3.3 we prove the sufficiency.

3.2 Necessity of Theorem 3.1.9

In this section, we assume that a graph G, as in Assumption 3.1.3, is square geometric We then prove the necessity of the conditions of Theorem 3.1.9. We assume throughout this section that we are in the settings of Assumption 3.2.1, unless otherwise stated.

Assumption 3.2.1. *Let G be a $B_{a,b}$-graph as given in Assumption 3.1.3. Suppose G is square geometric with linear orders $<_1$ and $<_2$ as in Equation (2.2). Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be completions of $<_1$ and $<_2$, respectively. Suppose G_a and G_b are cobipartite subgraphs of G as in Definition 3.1.1 with corresponding chord graphs $\tilde{G}_a$ and $\tilde{G}_b$, respectively.*

First we gather some important properties of completions $\mathcal{C}_1$ and $\mathcal{C}_2$, as in Definition 2.2.2, of a square geometric graph. Then we use these properties to prove the "only if" direction of Theorem 3.1.9. In Subsection 3.2.2, we prove that if G is a square geometric graph, then $\tilde{G}_a \cup \tilde{G}_b$ has a closed partial 2-coloring such that all vertices of $\mathcal{A}$ are red and all vertices of $\mathcal{B}$ are blue. Then, in Subsection 3.2.3, we prove that if G is a square geometric graph then the rigid-free conditions of Definition 3.1.4 hold.

3.2.1 Properties of completions $\mathcal{C}_1$ and $\mathcal{C}_2$

In this subsection, we collect some properties of completions $\mathcal{C}_1$ and $\mathcal{C}_2$ of Assumption 3.2.1. These properties will be used to prove the necessity part of Theorem 3.1.9. The following lemma is an easy consequence of Definition 2.2.2 (definition of completions). However the lemma is very useful, as the results of the lemma will be used to a great extent in future proofs.

Lemma 3.2.2. *Let G be a square geometric graph with linear orders $<_1$ and $<_2$ and corresponding completions $\mathcal{C}_1$ and $\mathcal{C}_2$. Then the following statements hold for all $i \in \{1, 2\}$.*

(1) Let $u, v, w, z \in V(G)$ be such that $w \notin N(z)$ and $u, v \in N(z)$. If $u <_i w <_i v$ then $zw \in \mathcal{C}_i$.

(2) Let $u, v, w \in V(G)$ be such that $w \in N(v) \setminus N(u)$. If $u <_i v$ and $wu \notin \mathcal{C}_i$ then $u <_i w$. Similarly if $v <_i u$ and $wu \notin \mathcal{C}_i$ then $w <_i u$.

(3) Let G be a $B_{a,b}$-graph. Suppose $x \in X_a \cup X_b$ and $y_1, y_2 \in Y$. If $y_1 <_i x <_i y_2$ then for all $y \in Y \setminus N_Y(x)$ we have $xy \in \mathcal{C}_i$. Similarly, let $y \in Y$ and $x_1, x_2 \in X_a \cap X_b$. If $x_1 <_i y <_i x_2$ then for all $x \in (X_a \cap X_b) \setminus N_X(y)$ we have $xy \in \mathcal{C}_i$.

(4) *Let* $u_1, u_2, w, z \in V(G)$ *be such that* $w \notin N(z)$*,* $u_1 \in N(z)$ *and* $u_2 \in N(w)$*. If* $u_1 <_i w$ *and* $u_2 <_i z$ *then* $zw \in \mathcal{C}_i$*. Similarly, if* $w <_i u_1$ *and* $z <_i u_2$ *then* $zw \in \mathcal{C}_i$*.*

(5) *Let* $u_1, u_2, v_1, v_2, w, z \in V(G)$ *be such that* $w \notin N(z)$*,* $u_1 \in N(u_2)$ *and* $v_1 \in N(v_2)$*. If* $u_1 <_i w <_i v_2$ *and* $v_1 <_i z <_i u_2$ *then* $zw \in \mathcal{C}_i$*.*

Proof. We first prove (1). Suppose $u <_i w <_i v$. We know either $z <_i w$ or $w <_i z$. If $z <_i w$ then we have $z <_i w <_i v$, and since $v \in N(z)$ by definition of completion we have $zw \in \mathcal{C}_i$. If $w <_i z$ then we have $u <_i w <_i z$, and since $u \in N(z)$ we have $zw \in \mathcal{C}_i$. We now prove (2). First let $u <_i v$ and $wu \notin \mathcal{C}_i$. If $w <_i u$ then we have $w <_i u <_i v$, and $w \in N(v)$. This implies that $wu \in \mathcal{C}_i$. But $wu \notin \mathcal{C}_i$, and thus $u <_i w$.

To prove (3), suppose G is a $B_{a,b}$-graph. Let $x \in X_a \cup X_b$ and $y_1, y_2 \in Y$ be such that $y_1 <_i x <_i y_2$. Suppose $y \in Y \setminus N_Y(x)$. Without loss of generality let $y <_i x$. Then we have $y <_i x <_i y_2$. Since $y_2 \in N(y)$ then $xy \in \mathcal{C}_i$. Therefore, for all $y \in Y \setminus N_Y(x)$ we have $xy \in \mathcal{C}_i$. The proof of the other statement of (3) follows similarly, by symmetry of $X_a \cap X_b$ and Y.

We now prove (4). First let $w <_i z$. Then we have $u_1 <_i w <_i z$. Since $u_1 \in N(z)$ we have $wz \in \mathcal{C}_i$. If $z <_i w$ then we have $u_2 <_i z <_i w$. Since $u_2 \in N(w)$ we have $wz \in \mathcal{C}_i$. To prove (5) without loss of generality assume $w <_i z$. Then we have $u_1 <_i w <_i z <_i u_2$. Since $u_1 \in N(u_2)$ then $wz \in \mathcal{C}_i$.

$\square$

We saw in Chapter 2 that for a square geometric cobipartite graph each chord of a rigid pair belongs to exactly one completion (Proposition 2.2.6). In what follows, we will see that the same result holds for square geometric $B_{a,b}$-graphs. Let G be a $B_{a,b}$-square geometric graph with linear orders $<_1$ and $<_2$, and corresponding completions $\mathcal{C}_1$ and $\mathcal{C}_2$. Suppose G_a and G_b are cobipartite subgraphs of G. Define $<_1^a$ to be the restriction of $<_1$ to $V(G_a)$, and $<_2^a$ to be the restriction of $<_2$ to $V(G_a)$. Moreover, let $\mathcal{C}_1^a$ and $\mathcal{C}_2^a$ be the completions of $<_1^a$ and $<_2^a$, respectively. Similarly, for G_b let $<_1^b$ and $<_2^b$ be the restriction of $<_1$ to $V(G_b)$, and the restriction of $<_2$ to $V(G_b)$, respectively. Suppose $\mathcal{C}_1^b$ and $\mathcal{C}_2^b$ are the completions of $<_1^b$ and $<_2^b$, respectively.

Remark 5. *The linear orders* $<_1^a$ *and* $<_2^a$ *for cobipartite graph* G_a *satisfy Equation 2.2. Indeed, the graph* G_a *is an induced subgraph of* G*, and thus, by Definition 2.2.2, we have that* $\mathcal{C}_1^a \subseteq \mathcal{C}_1$ *and* $\mathcal{C}_2^a \subseteq \mathcal{C}_2$*. This implies that* $(\mathcal{C}_1^a \cap \mathcal{C}_2^a) \subseteq (\mathcal{C}_1 \cap \mathcal{C}_2) = \emptyset$*, and thus, by Proposition 2.2.6,* $<_1^a$ *and* $<_2^a$ *satisfy Equation 2.2. Similarly, linear orders* $<_1^b$ *and* $<_2^b$*, for cobipartite graph* G_b*, satisfy Equation 2.2.*

Recall that a $B_{a,b}$-graph G with cobipartite subgraphs G_a and G_b has three classes of non-edges: (1) the isolated vertices of $\tilde{G}_a$ and $\tilde{G}_b$, (2) the non-isolated vertices of $\tilde{G}_a$ and $\tilde{G}_b$ (the chords of a rigid pair of G), and (3) the non-edges with one end in $X_a \setminus X_b$ and the other end in $X_b \setminus X_a$. Note that the non-edges of class (3) are not vertices of $\tilde{G}_a$ or $\tilde{G}_b$. Indeed, for a type-1 and type-2 $B_{a,b}$-graph the sets of the non-edges of class (3) are $\{a_1b_1\}$ and $\{a_1b_1, a_2b_2\}$, respectively. In the next few results, we collect some properties of the non-edges of class (2).

Lemma 3.2.3. *Let* G *be a* $B_{a,b}$*-square geometric graph with linear orders* $<_1$ *and* $<_2$*, and corresponding completions* $\mathcal{C}_1$ *and* $\mathcal{C}_2$*. Then*

(1) Let $u \in V(\tilde{G}_a)$ *be a non-isolated vertex. Then* $u \in \mathcal{C}_1$ *if and only if* $u \in \mathcal{C}_1^a$*, and* $u \in \mathcal{C}_2$ *if and only if* $u \in \mathcal{C}_2^a$*.*

(2) Let $u \in V(\tilde{G}_b)$ *be a non-isolated vertex. Then* $u \in \mathcal{C}_1$ *if and only if* $u \in \mathcal{C}_1^b$*, and* $u \in \mathcal{C}_2$ *if and only if* $u \in \mathcal{C}_2^b$*.*

Proof. We only prove (1). The proof of (2) follows by symmetry between X_a and X_b. Let $u \in V(\tilde{G}_a)$ be a non-isolated vertex. Since u is not an isolated vertex of $\tilde{G}_a$ then, by Definition 2.2.4, u is a chord of a rigid pair in G_a. We know, by Remark 5, that $<_1^a$ and $<_2^b$ satisfy Equation 2.2. Therefore, by Proposition 2.2.6, either $u \in \mathcal{C}_1^a \setminus \mathcal{C}_2^a$ or $u \in \mathcal{C}_2^a \setminus \mathcal{C}_1^a$. We first prove the "if" direction of the first statement of (1). Let $u \in \mathcal{C}_1^a$. Since $\mathcal{C}_1^a \subseteq \mathcal{C}_1$ then $u \in \mathcal{C}_1$. Now suppose $u \in \mathcal{C}_1$. Since $\mathcal{C}_2^a \subseteq \mathcal{C}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ then $u \notin \mathcal{C}_2^a$. This implies that $u \in \mathcal{C}_1^a$.

To prove the "if" direction of the second statement of (1), suppose that $u \in \mathcal{C}_2^a$. Since $\mathcal{C}_2^a \subseteq \mathcal{C}_2$ then $u \in \mathcal{C}_2$. Now suppose $u \in \mathcal{C}_2$. Since $\mathcal{C}_1^a \subseteq \mathcal{C}_1$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ then $u \notin \mathcal{C}_1^a$. This implies that $u \in \mathcal{C}_2^a$. □

We now state a version of Proposition 2.2.6 for square geometric $\mathcal{B}_{a,b}$-graphs.

Proposition 3.2.4. *Let G be a $B_{a,b}$-square geometric graph with linear orders $<_1$ and $<_2$, and corresponding completions $\mathcal{C}_1$ and $\mathcal{C}_2$. Then every completion $\mathcal{C}_i$, $i \in \{1, 2\}$, contains exactly one chord of any rigid pair.*

Proof. Suppose $\{x_1y_1, x_2y_2\}$ is a rigid pair of G. Then $\{x_1y_1, x_2y_2\}$ is either a rigid pair of G_a or a rigid pair of G_b. Without loss of generality, suppose that $\{x_1y_1, x_2y_2\}$ is a rigid pair of G_a. Then x_1y_2 and x_2y_1 are non-isolated vertices of $\tilde{G}_a$. By Proposition 2.2.6, we know that chords of a rigid pair belong to different completions $\mathcal{C}_1^a$ and $\mathcal{C}_2^a$. Without loss of generality, let $x_1y_2 \in \mathcal{C}_1^a \setminus \mathcal{C}_2^a$ and $x_2y_1 \in \mathcal{C}_2^a \setminus \mathcal{C}_1^a$. Then by (i) of Lemma 3.2.3, we have that $x_1y_2 \in \mathcal{C}_1 \setminus \mathcal{C}_2$ and $x_2y_1 \in \mathcal{C}_2 \setminus \mathcal{C}_1$. □

Recall that $u \approx_e v$ denotes "there is an even path in $\tilde{G}_a \cup \tilde{G}_b$ between u and v", and $u \approx_o v$ denotes "there is an odd path in $\tilde{G}_a \cup \tilde{G}_b$ between u and v". The following corollary follows from Proposition 3.2.4.

Corollary 3.2.5. *Let G be as in Assumption 3.2.1. Suppose $u, v \in V(\tilde{G}_a \cup \tilde{G}_b)$ and there is a path in $\tilde{G}_a \cup \tilde{G}_b$ between u and v. Then*

(i) If $u \approx_o v$ then u and v belong to different completions.

(ii) If $u \approx_e v$ then u and v belong to the same completion.

Proof. We only prove (i). The proof of part (ii) is analogous. Assume $z_1z_2 \ldots z_{2k-1}z_{2k}$ is an odd path between u and v, where $z_1 = u$ and $z_{2k} = v$. By the definition of chord graph, $\tilde{G}_a$ and $\tilde{G}_b$, for every pair of consecutive vertices of the path z_j and z_{j+1} there is a rigid pair of G with chords z_j and z_{j+1}. Therefore by Proposition 2.2.6, z_j and z_{j+1} belong to different completions $\mathcal{C}_i$, $i \in \{1, 2\}$. It follows that $u = z_1$ and $v = z_{2k}$ belong to different completions, which finishes the proof. □

Let $<_1$ and $<_2$ be linear orders of a square geometric graph. Suppose $S \subseteq V(G)$ and $v \in V(G)$. For any $i \in \{1, 2\}$, we denote the statement "for all $s \in S$ we have $v <_i s$ " by $v <_i S$.

Lemma 3.2.6. *Let G be a $B_{a,b}$-square geometric graph with linear orders $<_1$ and $<_2$, and corresponding completions $\mathcal{C}_1$ and $\mathcal{C}_2$. Let $\{xy, x'y'\}$ be a rigid pair of G and $x'y \in \mathcal{C}_1$.*

(1) If $x <_1 x'$ then either $x <_1 x' <_1 y <_1 y'$ or $x <_1 y <_1 x' <_1 y'$. Moreover for any $u \in N(y')$ we have $x <_1 u$, and for any $v \in N(x)$ we have $v <_1 y'$. In particular, $x <_1 Y$ and $(X_a \cap X_b) <_1 y'$.

(2) If $x' <_1 x$ then either $y' <_1 y <_1 x' <_1 x$ or $y' <_1 x' <_1 y <_1 x$. Moreover for any $u \in N(y')$ we have $u <_1 x$, and for any $v \in N(x)$ we have $y' <_1 v$. In particular, $Y <_1 x$ and $y' <_1 (X_a \cap X_b)$.

(3) $x <_1 x'$ if and only if $y <_1 y'$.

If $x'y \in \mathcal{C}_2$ then statements (1)-(3) hold if we replace $<_1$ by $<_2$.

Proof. Let $<_1$ and $<_2$ be linear orders on $V(G)$ as in Equation 2.2, with corresponding completions $\mathcal{C}_1$ and $\mathcal{C}_2$. By Proposition 3.2.4 we know that every chord of a rigid pair belongs to exactly one completion. Since $x'y \in \mathcal{C}_1$ then $x'y \notin \mathcal{C}_2$, and moreover $xy' \in \mathcal{C}_2 \setminus \mathcal{C}_1$. We now prove (1). Assume $x <_1 x'$. We know $xy' \notin \mathcal{C}_1$, and $y' \in N_Y(x') \setminus N_Y(x)$. Then by part (2) of Lemma 3.2.2 we have $x <_1 x' <_1 y$. If $y <_1 x <_1 x' <_1 y'$ then by definition of completion and the fact that $y \sim y'$ we have $xy' \in \mathcal{C}_1$ which is not true. Similarly if $x <_1 x' <_1 y' <_1 y$ then by definition of completion and the fact that $x \sim y$ we have $xy' \in \mathcal{C}_1$ which is not true. This implies that either $x <_1 x' <_1 y <_1 y'$ or $x <_1 y <_1 x' <_1 y'$.

Now suppose $u \in N(y')$. Then $y' \in N(u) \setminus N(x)$. If $u <_1 x <_1 y'$ then by part (2) of Lemma 3.2.2 we have $xy' \in \mathcal{C}_1$ which contradicts our assumption ($xy' \in \mathcal{C}_2$). Therefore $x <_1 u$. Since for all $u \in Y$ we have $u \in N(y')$ then $x <_1 Y$. Now let $v \in N(x)$. Then $x \in N(v) \setminus N(y')$. If $x <_1 y' <_1 v$ then by part (2) of Lemma 3.2.2 we have $xy' \in \mathcal{C}_1$ which contradicts our assumption ($xy' \in \mathcal{C}_2$). Therefore $v <_1 y'$. Since for all $v \in X_a \cap X_b$ we have $v \in N(x)$ then $(X_a \cap X_b) <_1 y'$.

The proof of (2) is analogous. To prove part (3), suppose that $x <_1 x'$. Then, by part (1), we have that $y <_1 y'$. Moreover, if $y <_1 y'$ then we know that part (2) does not occur, and thus we have $x <_1 x'$. This finishes the proof of the lemma. The proof for the case $x'y \in \mathcal{C}_2$ is analogous. Part (3) is an immediate result of (1) and (2). $\square$

Lemma 3.2.6 says that the embedding of a rigid pair $\{xy, x'y'\}$ in $(\mathbb{R}^2, \|.\|_\infty)$ has a general form as shown in Figure 3.2.1. More precisely, for a rigid pair $\{xy, x'y'\}$ and for all $i \in \{1, 2\}$, we have that $x <_i x'$ if and only if $y <_i y'$. This embedding property of rigid pairs will be used greatly in future proofs. In Figure 3.2.1, we assumed that the x-coordinates of the vertices give us the relation $<_1$ and the y-coordinates give the ordering $<_2$.

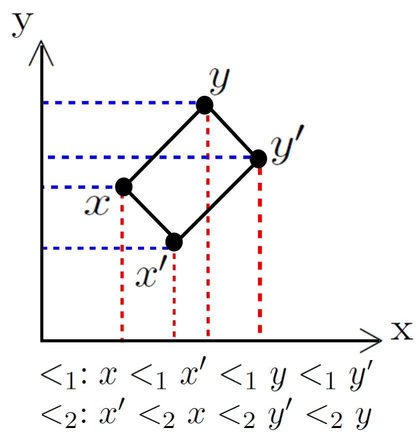


Figure 3.7: The embedding of a rigid pair $\{xy, x'y'\}$ in $(\mathbb{R}^2, \|.\|_\infty)$.

Lemma 3.2.7. *Let G be a $B_{a,b}$-square geometric graph with linear orders $<_1$ and $<_2$, and corresponding completions $\mathcal{C}_1$ and $\mathcal{C}_2$. Suppose $\{x_1y_1, x_2y_2\}$ and $\{x'_1y'_1, x'_2y'_2\}$ are rigid pairs of G with $x_1, x'_1 \in X_a$ or $x_1, x'_1 \in X_b$.*

(1) If x_1y_2 and $x'_1y'_2$ belong to the same completion then, for all $i \in \{1, 2\}$, we have $x_1 <_i x_2$ if and only if $x'_1 <_i x'_2$.

(2) If x_1y_2 and $x'_1y'_2$ belong to different completions then, for all $i \in \{1, 2\}$, we have $x_1 <_i x_2$ if and only if $x'_2 <_i x'_1$.

Proof. We only prove the lemma for X_a. The proof for X_b follows by symmetry of X_a and X_b. We prove (1) for $i = 1$. The proof for $i = 2$ is analogous. Assume $\{x_1y_1, x_2y_2\}$ and $\{x'_1y'_1, x'_2y'_2\}$ are rigid pairs of G. First note that if x_1y_2 and $x'_1y'_2$ belong to the same completion then x_2y_1 and $x'_2y'_1$ belong to the same completion as well.

Suppose without loss of generality that $x_2y_1 \in \mathcal{C}_1$ and $x'_2y'_1 \in \mathcal{C}_1$. Consequently, $x_1y_2 \in \mathcal{C}_2$ and $x'_1y'_2 \in \mathcal{C}_2$. Let $x_1 <_1 x_2$. Since $x_2y_1 \in \mathcal{C}_1$ by part (1) of Lemma 3.2.6 we have $x_1 <_1 y_2$. Also since $y'_2 \in N(y_2)$ then by part (1) of Lemma 3.2.6 we have $x_1 <_1 y'_2$. Similarly, since $x_1, x'_1 \in X_a$ we know that, $x'_1 \in N(x_1)$, and thus $x'_1 <_1 y_2$. Consequently, if $y'_2 <_1 x'_1$ then $y'_2 <_1 x'_1 <_1 y_2$ and by part (2) of Lemma 3.2.2 we have that $x_1'y_2' \in \mathcal{C}_1$, which contradicts $x_1'y_2' \in \mathcal{C}_2$. Therefore, $x_1' <_1 y_2'$, and thus by Lemma 3.2.6 we know that $x_1' <_1 x_2'$. If $x_1' <_1 x_2'$ then an analogous discussion proves that $x_1 <_1 x_2$.

We now prove (2). Suppose that x_1y_2 and $x_1'y_2'$ belong to different completions. Without loss of generality let $x_1'y_2' \in \mathcal{C}_1$ and $x_1y_2 \in \mathcal{C}_2$. This implies that $x_2'y_1' \in \mathcal{C}_2$ and $x_2y_1 \in \mathcal{C}_1$. Then we have that $x_1'y_2'$ and x_2y_1 belong to the same completion. Therefore by part (1) we have that $x_1' <_i x_2'$ if and only if $x_2 <_i x_1$. □

In the next few lemmas, we assume that the $B_{a,b}$-graph G is as in Assumption 3.2.1. We collect some properties of the non-edges of class (3) *i.e.* the non-edges a_1b_1, a_2b_1, if G is type-2, and a_1b_1, if G is type-1. The next two auxiliary lemmas (Lemmas 3.2.8 and 3.2.9) give us some information about the relation of the vertices of G in the linear orders $<_1$ and $<_2$. This information helps us to discuss the necessity of conditions of Theorem 3.1.9 in Subsections 3.2.2 and 3.2.3.

Lemma 3.2.8. *Let G be a square geometric $\mathcal{B}_{a,b}$-graph as in Assumption 3.2.1. Suppose $a_1b_1 \notin \mathcal{C}_1$.*

(1) Let $b_1 <_1 a_1$. Then we have $b_1 <_1 (X_a \cap X_b) <_1 a_1$. Moreover, $N_Y(b_1) <_1 a_1$ and $b_1 <_1 N_Y(a_1)$.

(2) Let $a_1 <_1 b_1$. Then we have $a_1 <_1 (X_a \cap X_b) <_1 b_1$. Moreover, $N_Y(a_1) <_1 b_1$ and $a_1 <_1 N_Y(b_1)$.

If $a_1b_1 \notin \mathcal{C}_2$ then (1) and (2) hold if we replace $<_1$ by $<_2$. Moreover, the lemma holds if we replace a_1 by a_2.

Proof. We only prove (1). The proof of (2) follows by exchanging a_1 and b_1 in the discussion for the proof of (1). Let $b_1 <_1 a_1$ and $x \in X_a \cap X_b$. Assume by contradiction that $b_1 <_1 a_1 <_1 x$ or $x <_1 b_1 <_1 a_1$. Since $x \in N(b_1)$ and $x \in N(a_1)$, by the definition of completion, for both cases we have $a_1b_1 \in \mathcal{C}_1$, which contradicts our assumption. Therefore, for all $x \in X_a \cap X_b$ we have $b_1 <_1 x <_1 a_1$. Now let $y \in N_Y(b_1)$. Then if $b_1 <_1 a_1 <_1 y$ then since $y \in N(b_1)$ we have that $a_1b_1 \in \mathcal{C}_1$ which contradicts $a_1b_1 \notin \mathcal{C}_1$. Therefore, for all $y \in N(b_1)$ we have $y <_1 a_1$. Similarly for all $y \in N(a_1)$ we have $b_1 <_1 y$. □

Lemma 3.2.9. *Let G be a square geometric $\mathcal{B}_{a,b}$-graph as in Assumption 3.2.1. Suppose $a_1b_1 \notin \mathcal{C}_1$.*

(1) Let $b_1 <_1 a_1$. Then either $b_1 <_1 Y$ or $Y <_1 a_1$.

(2) Let $a_1 <_1 b_1$. Then either $a_1 <_1 Y$ or $Y <_1 b_1$.

If $a_1b_1 \notin \mathcal{C}_2$ then (1) and (2) hold if we replace $<_1$ by $<_2$. Moreover, (1) and (2) hold if we replace a_1 by a_2.

Proof. We only prove (1). The proof of (2) follows by exchanging a_1 and b_1 in the proof of (1). Let $b_1 <_1 a_1$. Suppose to the contrary that there are $y_1, y_2 \in Y$ such that $y_1 <_1 b_1 <_1 a_1 <_1 y_2$. Then, since $y_1 \in N(y_2)$ by definition of completion we have $a_1b_1 \in \mathcal{C}_1$, which is a contradiction. □

The next two lemmas investigate the properties of the vertices of the sets $\mathcal{A}$, and $\mathcal{B}$. Recall that $\mathcal{A}$ is the set of a-vertices which have a neighbor in $\tilde{G}_a \cap \tilde{G}_b$, and $\mathcal{B}$ is the set of b-vertices which have a neighbor in $\tilde{G}_a \cap \tilde{G}_b$. These results will be used, in Subsection 3.2.2 (Lemma 3.2.12), to prove the necessity of Condition (1) of Theorem 3.1.9 (there is a closed partial coloring of $\tilde{G}_a \cup \tilde{G}_b$ which colors all the vertices of $\mathcal{A}$ red, and all the vertices of $\mathcal{B}$ blue).

Lemma 3.2.10. *Let G be a square geometric $B_{a,b}$-graph as in Assumption 3.2.1.*

(1) Let $\{a_1y_1, x_2y_2\}$ and $\{b_1y_1', x_2'y_2'\}$ be rigid pairs. If a_1y_2 and b_1y_2' belong to the same completion $\mathcal{C}_i$, $1 \le i \le 2$, then $a_1b_1 \in \mathcal{C}_i$.

(2) Let $\{a_1y_1, x_2y_2\}$ and $\{a_1y_1', x_2'y_2'\}$ be rigid pairs. Then a_1y_2 and a_1y_2' belong to the same completion $\mathcal{C}_1$ or $\mathcal{C}_2$. Similarly, if $\{b_1y_1, x_2y_2\}$ and $\{b_1y_1', x_2'y_2'\}$ are rigid pairs then b_1y_2 and b_1y_2' belong to the same completion $\mathcal{C}_1$ or $\mathcal{C}_2$.

(3) Fix $i \in \{1, 2\}$. If $a_1 <_i X_a \cap X_b <_i b_1$ and $a_1 <_i Y <_i b_1$ then $E(\tilde{G}_a \cap \tilde{G}_b) = \emptyset$.

Moreover, if G is a type-2 $B_{a,b}$-graph then (1)-(3) hold if we replace a_1 by a_2.

Proof. To prove (1), suppose without loss of generality that $a_1y_2, b_1y_2' \in \mathcal{C}_1$. Consider the rigid pair $\{a_1y_1, x_2y_2\}$. Suppose without loss of generality $a_1 <_1 x_2$. Since $a_1y_2 \in \mathcal{C}_1$ then by (1) of Lemma 3.2.6 we have that $y_1 <_1 a_1 <_1 x_2$, $y_1 <_1 X$ and $Y <_1 x_2$.

Now consider the rigid pair $\{b_1y_1', x_2'y_2'\}$. First let $b_1 <_1 x_2'$. Since $a_1 <_1 x_2$ then by (4) of Lemma 3.2.2 we have that $a_1b_1 \in \mathcal{C}_1$. Now let $x_2' <_1 b_1$. Since $b_1y_2' \in \mathcal{C}_1$ then by Lemma 3.2.6 we have that $x_2' <_1 b_1 <_1 y_1'$. This together with $y_1 <_1 a <_1 x_2$, $y_1 \in N(y_1')$, $x_2 \in N(x_2')$, and (5) of Lemma 3.2.2 implies that $a_1b_1 \in \mathcal{C}_1$.

We now prove (2). Let $\{a_1y_1, x_2y_2\}$ and $\{a_1y_1', x_2'y_2'\}$ be rigid pairs. Suppose to the contrary that $a_1y_2 \in \mathcal{C}_1$ and $a_1y_2' \in \mathcal{C}_2$. Then, by (2) of Lemma 3.2.7, for all $i \in \{1, 2\}$ we have that $a_1 <_i x_2$ if and only if $x_2' <_i a_1$. First let $i = 1$, and without loss of generality assume that $a_1 <_1 x_2$. Then $x_2' <_1 a_1 <_1 x_2$. Since $x_2, x_2' \in N(b_1)$ then by (1) of Lemma 3.2.2 we have that $a_1b_1 \in \mathcal{C}_1$. An analogous discussion for $i = 2$ shows that $a_1b_1 \in \mathcal{C}_2$. This implies that $a_1b_1 \in \mathcal{C}_1 \cap \mathcal{C}_2$, which contradicts $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. This proves the first statement of (2). The proof of the second statement follows from an analogous discussion.

Without loss of generality, we prove (3) for $i = 1$. Let $a <_1 X_a \cap X_b <_1 b_1$ and $a <_1 Y <_2 b_1$. Then, for all $y \in Y$, and all $x \in X_a \cap X_b$ we have that either $a_1 <_1 x <_1 y <_1 b_1$, or $a_1 <_1 y <_1 x <_1 b_1$. Moreover, $a_1, b_1 \in N(x)$, and thus for all $y \in Y$ and all $x \in X_a \cap X_b$ we have $xy \in \mathcal{C}_1$. Now suppose to the contrary that $E(\tilde{G}_a \cap \tilde{G}_b) \neq \emptyset$. Then there is a rigid pair $\{x_1y_1, x_2y_2\}$ in G and $x_1y_2 \sim^* x_2y_1$. By Proposition 3.2.4, we know that x_1y_2 and x_2y_1 belong to different completions. But we know that $x_1y_2, x_2y_1 \in \mathcal{C}_1$. This implies that $E(\tilde{G}_a \cap \tilde{G}_b) = \emptyset$.

$\square$

Lemma 3.2.11. *Let G be a square geometric $B_{a,b}$-graph as in Assumption 3.2.1. Let $\{a_1y_1, x_2y_2\}$ and $\{a_2y_1', x_2'y_2'\}$ be rigid pairs. If a_1y_2 and a_2y_2' belong to different completions $\mathcal{C}_1$ and $\mathcal{C}_2$, then*

(1) Each completion $\mathcal{C}_1$ and $\mathcal{C}_2$ contains exactly one of the non-edges a_1b_1 and a_2b_1.

(2) Fix $i \in \{1, 2\}$, and suppose $a_1b_1 \notin \mathcal{C}_i$

(2.1) If $a_1y_2 \notin \mathcal{C}_i$ then for all $x \in X_a \cup X_b$ and all $y \in N_Y(b_1) \setminus N_Y(x)$, we have $xy \in \mathcal{C}_i$.

(2.2) If $a_1y_2 \in \mathcal{C}_i$ then for all $y \in Y \setminus N_Y(b_1)$ we have $b_1y \in \mathcal{C}_i$.

Moreover, if G is a type-2 $B_{a,b}$-graph then the result of part (2) holds if we replace a_1y_2 by a_2y_2' and a_1b_1 by a_2b_1.

Proof. Suppose $\{a_1y_1, x_2y_2\}$ and $\{a_2y_1', x_2'y_2'\}$ are rigid pairs and a_1y_2 and a_2y_2' belong to different completions $\mathcal{C}_1$ and $\mathcal{C}_2$.

We first prove (1). By (2) of Lemma 3.2.7, for all $i \in \{1, 2\}$ we have that $a_1 <_i x_2$ if and only if $x_2' <_i a_2$. Suppose without loss of generality that $a_1 <_i x_2$ and $x_2' <_i a_2$. Then either $a_1 <_i a_2$ or $a_2 <_i a_1$.

(i) Let $a_2 <_i a_1$. Then $x_2' <_i a_2 <_i a_1 <_i x_2$. Since $x_2, x_2' \in N(b_1)$ then by (1) of Lemma 3.2.2 we have $a_1b_1, a_2b_1 \in \mathcal{C}_i$.

(ii) Let $a_1 <_i a_2$. If $b_1 <_i a_1$ then $b_1 <_i a_1 <_i x_2$. Since $x_2 \in N(b_1)$ then $a_1b_1 \in \mathcal{C}_i$. If $a_2 <_i b_1$ then $x_2' <_i a_2 <_i b_1$, and thus $a_2b_1 \in \mathcal{C}_i$. Moreover, if $a_1 <_i b_1 <_i a_2$ then $a_1b_1, a_2b_1 \in \mathcal{C}_i$.

This implies that each completion $\mathcal{C}_1$ and $\mathcal{C}_2$ contains at least one of a_1b_1 and a_2b_1. Since $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ then either $a_1b_1 \in \mathcal{C}_2 \setminus \mathcal{C}_1$ and $a_2b_1 \in \mathcal{C}_1 \setminus \mathcal{C}_2$ or $a_1b_1 \in \mathcal{C}_1 \setminus \mathcal{C}_2$ and $a_2b_1 \in \mathcal{C}_2 \setminus \mathcal{C}_1$. Therefore, case (i) and case (ii) when $a_1 <_i b_1 <_i a_2$ cannot occur.

We now prove (2) for $i = 1$. The proof for $i = 2$ is analogous. First let $a_1y_2 \in \mathcal{C}_1$ and $a_1b_1 \notin \mathcal{C}_1$. Consider the rigid pair $\{a_1y_1, x_2y_2\}$. Suppose without loss of generality that $a_1 <_1 x_2$. Since $a_1y_2 \in \mathcal{C}_1$ then by (2) of Lemma 3.2.6 we know that $y_1 <_1 a_1 <_1 x_2$. Since $x_2 \in N(b_1)$ and $a_1b_1 \notin \mathcal{C}_1$ then by (2) of Lemma 3.2.2 we have $a_1 <_1 b_1$. Then by (2) of Lemma 3.2.8 we have $a_1 <_1 X_a \cap X_b <_1 b_1$. Moreover, since $y_1 <_1 a_1$ then by (2) of Lemma 3.2.9, we know that $Y <_1 b_1$, and thus $y_1 <_1 a_1 <_1 N_Y(b_1) <_1 b_1$.

For all $x \in X_a \cap X_b$ and all $y \in N_Y(b_1) \setminus N_Y(x)$, we have that $a_1 <_1 x <_1 b_1$ and $a_1 <_1 y <_1 b_1$. Therefore, either $x <_1 y <_1 b_1$ or $a_1 <_1 y <_1 x$. Since $a_1, b_1 \in N(x)$, we have that $xy \in \mathcal{C}_1$.

We know that $y_1 <_1 a_1 <_1 N_Y(b_1)$, and thus by (3) of Lemma 3.2.2, for all $y \in Y \setminus N_Y(a_1)$, we have that $a_1y \in \mathcal{C}_1$. In particular, for all $y \in N_Y(b) \setminus N_Y(a_1)$, $a_1y \in \mathcal{C}_1$.

Consider the rigid pairs $\{a_1y_1, x_2y_2\}$ and $\{a_2y_1', x_2'y_2'\}$. Since a_1y_2 and a_2y_2' belong to different completions and $a_1 <_1 x_2$ then by (2) of Lemma 3.2.7 we have that $x_2' <_1 a_2$. This together with $a_1 <_1 X_a \cap X_b <_1 b_1$ implies that $a_1 <_1 x_2' <_1 a_2$. Since $a_2 \in N(a_1)$ and $a_1b_1 \notin \mathcal{C}_1$ then $a_1 <_1 x_2' <_1 a_2 <_1 b_1$. Moreover, $a_1 <_1 N_Y(b_1) <_1 a_2$. Therefore, for all $y \in N_Y(b_1) \setminus N_Y(a_2)$ we have that either $x_2' <_1 y <_1 b_1$ or $a_1 <_1$

$y <_1 x_2' <_1 a_2$. If the latter occurs then $a_2 y \in \mathcal{C}_1$. If the former occurs then either $x_2' <_1 y <_1 a_2 <_1 b_1$ or $x_2' <_1 a_2 <_1 y <_1 b_1$. Since $x_2' \in N(b_1)$ then $a_2 y \in \mathcal{C}_1$. This implies that, for all $x \in X_a \cup X_b$ and all $y \in N_Y(b_1) \setminus N_Y(x)$, we have that $xy \in \mathcal{C}_1$.

We now prove (2.2). If $a_1 y_2 \notin \mathcal{C}_i$ and $a_1 b_1 \notin \mathcal{C}_i$ then for all $x \in Xa \cup X_b$ and all $y \in N_Y(b_1) \setminus N_Y(x)$ we have $xy \in \mathcal{C}_i$.

Let $a_1 y_2 \notin \mathcal{C}_1$ and $a_1 b_1 \notin \mathcal{C}_1$. Consider the rigid pair $\{a_1 y_1, x_2 y_2\}$. Suppose without loss of generality that $a_1 <_1 x_2$. Since $a_1 y_2 \notin \mathcal{C}_1$ then by (1) of Lemma 3.2.6 we know that $a_1 <_1 x_2 <_1 y_2$ and $a_1 <_1 Y$. This together with (1) of Lemma 3.2.8 implies that $N_Y(a_1) <_1 b_1$.

Now consider the rigid pairs $\{a_1 y_1, x_2 y_2\}$ and $\{a_2 y_1', x_2' y_2'\}$. Since $a_1 y_2$ and $a_2 y_2'$ belong to different completions and $a_1 <_1 x_2$ then by (2) of Lemma 3.2.7 we have that $x_2' <_1 a_2$. This together with $a_1 <_1 X_a \cap X_b <_1 b_1$ implies that $a_1 <_1 x_2' <_1 a_2$. Moreover, since $a_1 y_2 \notin \mathcal{C}_1$ then $a_2 y_2' \in \mathcal{C}_1$. Therefore, by (1) of Lemma 3.2.7 we have that either $x_2' <_1 y_2' <_1 a_2 <_1 y_1'$ or $x_2' <_1 a_2 <_1 y_2' <_1 y_1'$. If $y_1' <_1 b_1$ then since $x_2' \in N(b_1)$ we have $x_2' y_1, a_2 y_2' \in \mathcal{C}_1$ which contradicts Proposition 3.2.4. This implies that $b_1 <_1 y_1'$, and thus $N_Y(a_1) <_1 b_1 <_1 y_1'$. Therefore, by (3) of Lemma 3.2.2 for all $y \in Y \setminus N_Y(b_1)$ we have that $b_1 y \in \mathcal{C}_1$. $\square$

3.2.2 Necessity of Condition (1) of Theorem 3.1.9

In this subsection, we will prove that if G is a square geometric graph, as given in Assumption 3.1.3, then there exists a closed partial 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$ such that all vertices of $\mathcal{A}$ are red and all vertices of $\mathcal{B}$ are blue. Recall that $\mathcal{A}$ is the set of all a-vertices with a neighbor in $V(\tilde{G}_a \cap \tilde{G}_b)$ and $\mathcal{B}$ is the set of all b-vertices with a neighbor in $V(\tilde{G}_a \cap \tilde{G}_b)$.

Throughout the rest of this section, we assume that G is a square geometric graph as in Assumption 3.2.1.

Lemma 3.2.12. *Let G be a square geometric $B_{a,b}$-graph as in Assumption 3.2.1. Then*

(1) Either $\mathcal{A} \subseteq \mathcal{C}_1$ and $\mathcal{B} \subseteq \mathcal{C}_2$, or $\mathcal{A} \subseteq \mathcal{C}_2$ and $\mathcal{B} \subseteq \mathcal{C}_1$.

(2) Let $\{x_1 y_1, x_2 y_2\}, \{x_2 y_2', x_3 y_3\}$, and $\{x_1 y_1, x_3 y_3\}$ be rigid pairs of G such that $x_1 y_2 \in \mathcal{C}_1$ and $x_2 y_3 \in \mathcal{C}_1$. Then $x_1 y_3 \in \mathcal{C}_1$.

Proof. Let G be as in Assumption 3.2.1. We first prove (1). Let $a_1y, a_1y' \in \mathcal{A}$. Then there are rigid pairs $\{a_1y_1, x_2y_2\}$ and $\{a_1y'_1, x'_2y'_2\}$. By (2) of Lemma 3.2.10, we know that the a_1-vertices, a_1y_2 and $a_1y'_2$, belong to the same completions. This implies that all the a_1-vertices of $\mathcal{A}$ belong to the same completion. An analogous discussion and (2) of Lemma 3.2.10 prove that all the a_2-vertices of $\mathcal{A}$ belong to the same completion, and all the b_1-vertices of $\mathcal{B}$ belong to the same completion. Now consider the following cases:

Case 1. $\mathcal{A} = \emptyset$ and $\mathcal{B} \neq \emptyset$: As we discussed above, we know that all the b_1-vertices in $\mathcal{B}$ belong to the same completion $\mathcal{C}_1$ and $\mathcal{C}_2$.

Case 2. $\mathcal{A} \neq \emptyset$ and $\mathcal{B} = \emptyset$: Again, by the above discussion, we know that all the a_1-vertices in $\mathcal{A}$ belong to the same completion, and all the a_2-vertices in $\mathcal{A}$ belong to the same completion. We now prove that all the a_2-vertices and a_1-vertices of $\mathcal{A}$ belong to the same completion. So suppose that $a_1y_2, a_2y'_2 \in \mathcal{A}$. Then there are rigid pairs $\{a_1y_1, x_2y_2\}$ and $\{a_2y'_1, x'_2y'_2\}$. Suppose to the contrary that a_1y_2 and $a_2y'_2$ belong to different completions. Without loss of generality let $a_1y_2 \in \mathcal{C}_1$ and $a_2y'_2 \in \mathcal{C}_2$. Then by (1) of Lemma 3.2.11 we know that either $a_1b_1 \in \mathcal{C}_1 \setminus \mathcal{C}_2$ and $a_2b_1 \in \mathcal{C}_2 \setminus \mathcal{C}_1$ or $a_1b_1 \in \mathcal{C}_2 \setminus \mathcal{C}_1$ and $a_2b_1 \in \mathcal{C}_1 \setminus \mathcal{C}_2$.

First let $a_1b_1 \in \mathcal{C}_1 \setminus \mathcal{C}_2$ and $a_2b_1 \in \mathcal{C}_2 \setminus \mathcal{C}_1$. Since $a_1b_1 \in \mathcal{C}_1$ and $a_1y_2 \in \mathcal{C}_1$ then, by (2.1) of Lemma 3.2.11, for all $y \in Y \setminus N_Y(b_1)$, we have that $b_1y \in \mathcal{C}_1$. Moreover, $a_2b_1 \in \mathcal{C}_2$ and $a_2y'_2 \in \mathcal{C}_2$, and thus by (2.1) of Lemma 3.2.11 for all $y \in Y \setminus N_Y(b_1)$ we have that $b_1y \in \mathcal{C}_2$. This implies that, for all $y \in Y \setminus N_Y(b_1)$, we have that $b_1y \in \mathcal{C}_1 \cap \mathcal{C}_2$. Since $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ then $Y \setminus N_Y(b_1) = \emptyset$, which implies that $N_Y(b_1) = Y$. This case is not part of Assumption 3.2.1.

Now let $a_1b_1 \in \mathcal{C}_2 \setminus \mathcal{C}_1$ and $a_2b_1 \in \mathcal{C}_1 \setminus \mathcal{C}_2$. Since $a_1b_1 \in \mathcal{C}_2$ and $a_1y_2 \notin \mathcal{C}_2$ then by (2.2) of Lemma 3.2.11 for all $x \in X_a \cup X_b$ and all $y \in N_Y(b_1) \setminus N_Y(x)$ we have that $xy \in \mathcal{C}_2$. Moreover, $a_2b_1 \in \mathcal{C}_1$ and $a_2y'_2 \notin \mathcal{C}_1$, and thus by (2.2) of Lemma 3.2.11 for all $x \in X_a \cup X_b$ and all $y \in N_Y(b_1) \setminus N_Y(x)$ we have that $xy \in \mathcal{C}_2$. This implies that for all $x \in X_a \cup X_b$ and all $y \in N_Y(b_1) \setminus N_Y(x)$ we have that $xy \in \mathcal{C}_1 \cap \mathcal{C}_2$. Since $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ then, for all $x \in X_a \cup X_b$, we have that $N_Y(b_1) \setminus N_Y(x) = \emptyset$, which implies that for all $x \in X_a \cup X_b$, $N_Y(b_1) \subseteq N_Y(x)$. This case is not part of Assumption 3.2.1. Therefore, for all $a \in \{a_1, a_2\}$, we have that all a-vertices of $\mathcal{A}$ belong to the same completion.

Case 3. $\mathcal{A} \neq \emptyset$ and $\mathcal{B} \neq \emptyset$: First recall that, as we discussed in the very beginning of the proof, for all $a \in \{a_1, a_2\}$, all the a-vertices of $\mathcal{A}$ belong to the same completion, and all the b_1-vertices of $\mathcal{B}$ belong to the same completion. We now prove that an a-vertex of $\mathcal{A}$, and a b_1-vertex of $\mathcal{B}$ belong to different completions.

Suppose that $ay \in \mathcal{A}$, where $a \in \{a_1, a_2\}$, and $b_1y' \in \mathcal{B}$ the vertices ay and b_1y' belong to different completions. Since vertices ay and b_1y' have neighbors in $V(\tilde{G}_a \cap \tilde{G}_b)$ then there are rigid pairs $\{ay_1, xy\}$ and $\{b_1y_2, x'y'\}$ in G with $x, x' \in X_a \cap X_b$. By contradiction let $ay \in \mathcal{C}_1$ and $b_1y' \in \mathcal{C}_1$. Then by (1) of Lemma 3.2.10 we know that $ab_1 \in \mathcal{C}_1$, and thus by Proposition 3.2.4 we have that $ab_1 \notin \mathcal{C}_2$. Suppose without loss of generality that $a <_2 b_1$. Then by (2) of Lemma 3.2.8 we have that $a <_2 X_a \cap X_b <_2 b_1$, $a <_2 N_Y(b_1)$, and $N_Y(a) <_2 b_1$. Moreover, by (2) of Lemma 3.2.9 either $a_1 <_2 Y$ or $Y <_2 b_1$. Without loss of generality let $a <_2 Y$. First suppose that we also have $Y <_2 b_1$. This implies that $a <_2 X_a \cap X_b <_2 b_1$ and $a <_2 Y <_2 b_1$. Then by (3) of Lemma 3.2.10 we know that $E(\tilde{G}_a \cap \tilde{G}_b) \neq \emptyset$. This case is excluded in Assumption 3.1.3.

Therefore, $Y <_2 b_1$ cannot occur, and thus there is a vertex $y_3 \in Y$ such that $b_1 <_2 y_3$. We also have $N_Y(a) <_2 b_1$. Then $N_Y(a) <_2 b_1 <_2 y_3$. Since $y' \in Y \setminus N_Y(b_1)$ then, by (3) of Lemma 3.2.2, we have that $b_1y' \in \mathcal{C}_2$, which contradicts our assumption ($b_1y' \in \mathcal{C}_1$). This proves that either $\mathcal{A} \subseteq \mathcal{C}_1$ and $\mathcal{B} \subseteq \mathcal{C}_2$, or $\mathcal{A} \subseteq \mathcal{C}_2$ and $\mathcal{B} \subseteq \mathcal{C}_1$.

Now we prove (2). Suppose there are rigid pairs $\{x_1y_1, x_2y_2\}$ and $\{x_2y_2', x_3y_3\}$ with $x_1y_2 \in \mathcal{C}_1$ and $x_2y_3 \in \mathcal{C}_1$. Then by Lemma 3.2.7 we have $x_1 <_1 x_2$ if and only if $x_2 <_1 x_3$. Without loss of generality let $x_1 <_1 x_2 <_1 x_3$. Moreover by part (3) of Lemma 3.2.6 we have $y_1 <_1 y_2$ and $y_1 <_1 y_3$. Since $x_1y_2 \in \mathcal{C}_1$ by part (2) of Lemma 3.2.6 we have $y_1 <_1 x_1$, and thus either $y_1 <_1 x_1 <_1 y_3$ or $y_1 <_1 y_3 <_1 x_1$. Since $y_3, x_1 \in N(y_1)$ then, by definition of completion (Definition 2.2.2), we have that $x_1y_3 \in \mathcal{C}_1$. □

The necessity of Condition (1) of Theorem 3.1.9 is a result of Lemma 3.2.12.

Proposition 3.2.13. *Let G be a $B_{a,b}$-graph as in Assumption 3.2.1. Then there is a closed partial 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$ such that all vertices of $\mathcal{A}$ are red and all vertices of $\mathcal{B}$ are blue.*

Proof. Let G be a $B_{a,b}$-graph as in Assumption 3.2.1. Let $f : V(\tilde{G}_a \cup \tilde{G}_b) \to$

$\{$red, blue$\}$ be a 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$ as follows: for $\tilde{u} \in V(\tilde{G}_a \cup \tilde{G}_b)$ define $f(\tilde{u}) =$ red if $\tilde{u} \in \mathcal{C}_1$, and $f(\tilde{u}) =$ blue if $\tilde{u} \in \mathcal{C}_2$. Now let $<_X$ and $<_Y$ be the relations of Definition 3.1.6 corresponding to f. By (1) of Lemma 3.2.12, we know that $\mathcal{A} \subseteq \mathcal{C}_1$ and $\mathcal{B} \subseteq \mathcal{C}_2$. Therefore, in the 2-coloring f, all vertices of $\mathcal{A}$ are colored red and all vertices of $\mathcal{B}$ are colored blue. We now prove that $<_X$ and $<_Y$ form a proper bi-ordering of G. By (1) of Remark 4, we know that if the restrictions of $<_X$ to X_a and X_b, and $<_Y$ are partial orders then $(<_X, <_Y)$ form a proper bi-ordering for G. Moreover, by Proposition 3.1.7, we know that the restrictions of $<_X$ to X_a and X_b, and $<_Y$ are reflexive and antisymmetric. We now prove that the restrictions of $<_X$ to X_a and X_b are transitive. First let $x_1, x_2, x_3 \in X_a$ such that $x_1 <_X x_2$ and $x_2 <_X x_3$. Then, by Definition 3.1.6, there are rigid pairs $\{x_1y_1, x_2y_2\}$ and $\{x_2y_2', x_3y_3\}$ of G such that $f(x_1y_2) = f(x_2y_3) =$ red and $f(x_2y_1) = f(x_3y_2') =$ blue. Since $x_1, x_2, x_3 \in X_a$ then $\{x_1y_1, x_2y_2\}$ and $\{x_2y_2', x_3y_3\}$ are rigid pairs of G_a, which is a cobipartite graph. Moreover, $f(x_2y_1)$ is blue and $f(x_2y_3)$ is red. This together with Corollary 2.3.4, imply that $\{x_1y_1, x_3y_3\}$ is a rigid pair. Since $f(x_1y_2) = f(x_2y_3) =$ red then $x_1y_2 \in \mathcal{C}_1$ and $x_2y_3 \in \mathcal{C}_1$. Therefore, by (2) of Lemma 3.2.12 we know that $x_1y_3 \in \mathcal{C}_1$, and thus $f(x_1y_3) =$ red. It follows that $x_1 <_X x_3$. This proves that the restriction of $<_X$ to X_a is a partial order. An analogous discussion proves that the restriction of $<_X$ to X_b is a partial order. Then by (2) of Remark 4, we have that $<_Y$ is a partial order. This proves that $(<_X, <_Y)$ form a proper bi-ordering of G, and we are done. $\square$

3.2.3 Necessity of Condition (2) of Theorem 3.1.9

We devote this subsection to the proof of necessity of Condition (2) of Theorem 3.1.9. As we mentioned in Section 3.1, the rigid-free conditions exclude specific structures of neighborhoods of vertices a_1, a_2, b_1. We prove in this subsection that the occurrence of these structures does not allow the existence of linear orders $<_1$ and $<_2$ as in Equation 2.2. Indeed, if such structures occur then for every pair of linear orders $<_1$ and $<_2$ at least one of the non-edges a_1b_1 and a_2b_1 belongs to $\mathcal{C}_1 \cap \mathcal{C}_2$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are completions corresponding to $<_1$ and $<_2$, respectively.

We prove the necessity of Condition (2) of Theorem 3.1.9 in stages through a number of lemmas. These lemmas collect information on non-edges a_1b_1 and a_2b_1, and the location of a_1, a_2, b_1 in linear orders $<_1$ and $<_2$. The main tools of all proofs of this subsection are Lemma 3.2.2 and Definition 2.2.2 (definition of completion).

Type-1 $B_{a,b}$-graphs

We first present the results for type-1 $B_{a,b}$-graphs. The next lemma gives us the required information to prove that structures (1) and (2), as shown in Figure 3.8, do not occur simultaneously.

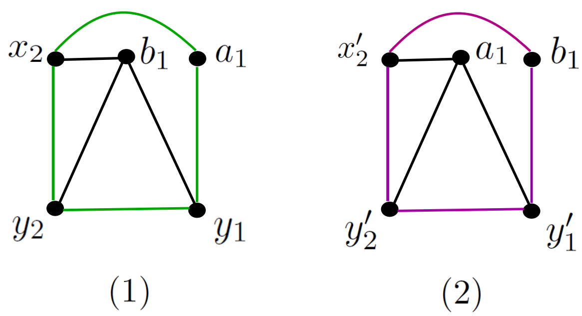


Figure 3.8: In subgraph (1), the vertex b_1 is not rigid-free with respect to $\{a_1, x_2\}$, and in subgraph (2), the vertex a_1 is not rigid-free with respect to $\{b_1, x_2'\}$.

Lemma 3.2.14. *Suppose that G is a square geometric $B_{a,b}$-graph.*

(1) Let $\{a_1y_1, x_2y_2\}$ be a rigid pair with $y_1, y_2 \in N_Y(b_1)$, as in Figure 3.8 (1). Then a_1y_2 and a_1b_1 belong to the same completion.

(2) Let $\{b_1y_1', x_2'y_2'\}$ be a rigid pair with $y_1', y_2' \in N_Y(a_1)$, as in Figure 3.8 (2). Then b_1y_2' and a_1b_1 belong to the same completion.

Proof. We only prove (1). The proof of (2) is analogous. Suppose $\{a_1y_1, x_2y_2\}$ is a rigid pair with $y_1, y_2 \in N_Y(b_1)$ and $a_1y_2 \in \mathcal{C}_1$. First suppose $a_1 <_1 x_2$. Then by (2) of Lemma 3.2.6 we have either $y_1 <_1 y_2 <_1 a_1 <_1 x_2$ or $y_1 <_1 a_1 <_1 y_2 <_1 x_2$. In both cases, $y_1 <_1 a_1 <_1 x_2$. Since $x_2, y_1 \in N(b_1)$, by part (1) of Lemma 3.2.2 we have $a_1b_1 \in \mathcal{C}_1$.

Now suppose that $x_2 <_1 a_1$. Then, by Lemma 3.2.6, we have that either $x_2 <_1 a_1 <_1 y_2 <_1 y_1$ or $x_2 <_1 y_2 <_1 a_1 <_1 y_1$. Thus $x_2 <_1 a_1 <_1 y_1$. Again since $x_2, y_1 \in N(b_1)$, then by part (1) of Lemma 3.2.2, we have that $a_1b_1 \in \mathcal{C}_1$. If $a_1y_2 \in \mathcal{C}_2$, then the same argument with $<_1$ replaced by $<_2$ shows $a_1b_1 \in \mathcal{C}_2$. This finishes the proof. $\square$

Corollary 3.2.15. *Suppose that G is a type-1 square geometric graph as in Assumption 3.2.1. Then the structures, (1) and (2), of Figure 3.8 cannot both occur. More precisely, if there is a rigid pair $\{a_1y_1, x_2y_2\}$ with $y_1, y_2 \in N_Y(b_1)$ then there is no rigid pair $\{b_1y_1', x_2'y_2'\}$ with $y_1', y_2' \in N_Y(a_1)$. Equivalently, if there is a rigid pair $\{b_1y_1', x_2'y_2'\}$ with $y_1', y_2' \in N_Y(a_1)$ then there is no rigid pair $\{a_1y_1, x_2y_2\}$ with $y_1, y_2 \in N_Y(b_1)$.*

Proof. Suppose by contradiction that there are rigid pairs $\{a_1y_1, x_2y_2\}$ with $y_1, y_2 \in N_Y(b_1)$ and $\{b_1y_1', x_2'y_2'\}$ with $y_1', y_2' \in N_Y(a_1)$. Then $a_1y_2 \sim^* x_2y_1$ and $b_1y_2' \sim^* x_2'y_1'$, and thus a_1y_2 and b_1y_2' have neighbors in $\tilde{G}_a \cap \tilde{G}_b$. Therefore $a_1y_2 \in \mathcal{A}$ and $b_1y_2' \in \mathcal{B}$. By part (1) of Lemma 3.2.12, we know that a_1y_2 and b_1y_2' belong to different completions $\mathcal{C}_1$ and $\mathcal{C}_2$. By Lemma 3.2.14, this implies that a_1b_1 belongs to both completions $\mathcal{C}_1$ and $\mathcal{C}_2$. This contradicts $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. □

We now prove that structures (3) and (4), as shown in Figure 3.9, do not occur simultaneously.

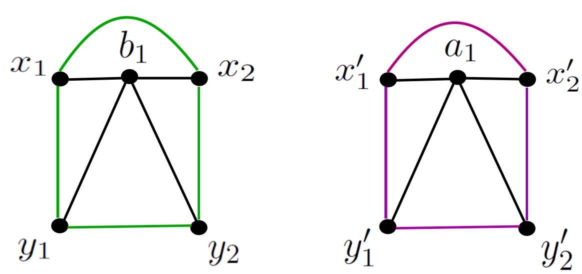


Figure 3.9: In subgraph (3), the vertex b_1 is not rigid-free with respect to $\{x_1', x_2'\}$, and in subgraph (4), the vertex a_1 is not rigid-free with respect to $\{x_1, x_2\}$.

Lemma 3.2.16. *Let $\{x_1y_1, x_2y_2\}$ be a rigid pair with $x_1, x_2 \in X_a \cap X_b$. For $v \in X_a \cup X_b$, if $y_1, y_2 \in N_Y(v)$ then, for all $i \in \{1, 2\}$, either $x_1 <_i v <_i y_2$ or $y_1 <_i v <_i x_2$. Moreover, if $x_1 <_i v <_i y_2$ then $x_1 <_i Y$ and $X_a \cap X_b <_i y_2$, and if $y_1 <_i v <_i x_2$ then $y_1 <_i X_a \cap X_b$ and $Y <_i x_2$.*

Proof. Assume without loss of generality that $i = 1$. We know that x_1y_2 and x_2y_1 belong to different completions. By symmetry of x_1y_2 and x_2y_1, we may assume that $x_2y_1 \in \mathcal{C}_1$ and $x_1y_2 \in \mathcal{C}_2$. Since $v \in X_a \cup X_b$ and $x_1 \in X_a \cap X_b$ then we have that $v \in N(x_1)$. Moreover $v \in N(y_2)$. First let $x_1 <_1 x_2$. Then by (1) of Lemma 3.2.6, we have that $x_1 <_1 v <_1 y_2$, and $x_1 <_1 Y$. Also, since $X_a \cap X_b \subseteq N(x_1)$, by Lemma 3.2.6, we have that $X_a \cap X_b <_1 y_2$.

If $x_2 <_1 x_1$ then a similar argument, and (2) of Lemma 3.2.6, proves that $y_1 <_1 v <_1 x_2$, $y_1 <_1 X_a \cap X_b$ and $Y <_1 x_2$. $\square$

Proposition 3.2.17. *Suppose G is a type-1 square geometric graph as in Assumption 3.2.1. Then the structures (3) and (4), of Figure 3.9, cannot both occur. More precisely, if there is a rigid pair $\{x_1y_1, x_2y_2\}$ with $y_1, y_2 \in N_Y(b_1)$ and $x_1, x_2 \in X_a \cap X_b$ then there is no rigid pair $\{x'_1y'_1, x'_2y'_2\}$ with $y'_1, y'_2 \in N_Y(a_1)$ and $x'_1, x'_2 \in X_a \cap X_b$.*

Proof. Suppose that there is a rigid pair $\{x_1y_1, x_2y_2\}$ with $y_1, y_2 \in N_Y(b_1)$ and $x_1, x_2 \in X_a \cap X_b$. By contradiction suppose that there is a rigid pair $\{x'_1y'_1, x'_2y'_2\}$ with $y'_1, y'_2 \in N_Y(a_1)$ and $x'_1, x'_2 \in X_a \cap X_b$. Then, by Lemma 3.2.16, either $x'_1 <_1 a_1 <_1 y'_2$ or $y'_1 <_1 a_1 <_1 x'_2$. Also, either $x_1 <_1 b_1 <_1 y_2$ or $y_1 <_1 b_1 <_1 x_2$. Thus we have the following cases.

- Let $x_1 <_1 b_1 <_1 y_2$ and $x'_1 <_1 a_1 <_1 y'_2$. Since $x_1 \in N(a_1)$ and $x'_1 \in N(b_1)$, by (4) of Lemma 3.2.2, we have that $a_1b_1 \in \mathcal{C}_1$. Similarly, if $y_1 <_1 b_1 <_1 x_2$ and $y'_1 <_1 a_1 <_1 x'_2$ then, by (4) of Lemma 3.2.2, we have that $a_1b_1 \in \mathcal{C}_1$.

- Let $x_1 <_1 b_1 <_1 y_2$ and $y'_1 <_1 a_1 <_1 x'_2$. Since $x_1 \in N(x'_2)$ and $y'_1 \in N(y_2)$, by (5) of Lemma 3.2.2, we have that $a_1b_1 \in \mathcal{C}_1$. Similarly, if $y_1 <_1 b_1 <_1 x_2$ and $x'_1 <_1 a_1 <_1 y'_2$, by (5) of Lemma 3.2.2, we have that $a_1b_1 \in \mathcal{C}_1$.

Therefore, for all cases, $a_1b_1 \in \mathcal{C}_1$. A similar argument for $i = 2$ proves that $a_1b_1 \in \mathcal{C}_2$, and thus $a_1b_1 \in \mathcal{C}_1 \cap \mathcal{C}_2$, which contradicts Proposition 3.2.4. $\square$

We prove in the next Proposition that structures (2) and (3), as shown in Figure 3.10, do not occur simultaneously.

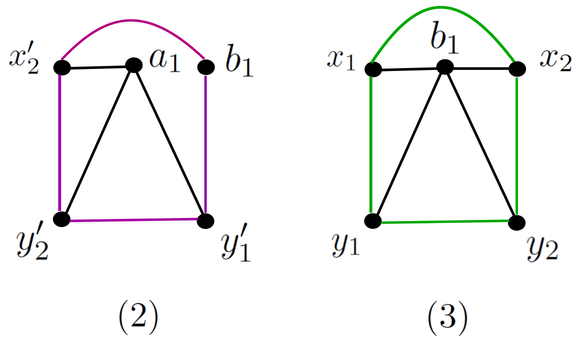


Figure 3.10: Structures (2) and (3) of Figures 3.8 and 3.9, respectively.

Proposition 3.2.18. *Suppose that G is a type-1 square geometric graph as in Assumption 3.2.1. Then the structures (2) and (3), of Figure 3.10 cannot both occur. More precisely, if there is a rigid pair $\{x_1y_1, x_2y_2\}$ with $y_1, y_2 \in N_Y(b_1)$ and $x_1, x_2 \in X_a \cap X_b$ then there is no rigid pair $\{b_1y_1', x_2'y_2'\}$ with $y_1', y_2' \in N_Y(a_1)$ and $x_2' \in X_a \cap X_b$. Similarly, if there is a rigid pair $\{x_1'y_1', x_2'y_2'\}$ with $y_1', y_2' \in N_Y(a_1)$ and $x_1', x_2' \in X_a \cap X_b$ then there is no rigid pair $\{a_1y_1, x_2y_2\}$ with $y_1, y_2 \in N_Y(b_1)$ and $x_2 \in X_a \cap X_b$.*

Proof. Suppose that the graph G contains both of the structures of Figure 3.10. Precisely, assume that there is a rigid pair $\{x_1y_1, x_2y_2\}$ with $y_1, y_2 \in N_Y(b_1)$ and $x_1, x_2 \in X_a \cap X_b$, and there is a rigid pair $\{b_1y_1', x_2'y_2'\}$ with $y_1', y_2' \in N_Y(a_1)$ and $x_1', x_2' \in X_a \cap X_b$. By Proposition 3.2.4, we know that the chords b_1y_2' and $x_2'y_1'$ belong to different completions. Without loss of generality, assume that $b_1y_2' \in \mathcal{C}_2$ and $x_2'y_1' \in \mathcal{C}_1$. Then, by Lemma 3.2.14, we have that $a_1b_1 \in \mathcal{C}_2$.

By Lemma 3.2.16, we know that either $x_1 <_1 b_1 <_1 y_2$ or $y_1 <_1 b_1 <_1 x_2$. Consider the following cases.

- Let $x_1 <_1 b_1 <_1 y_2$. Since $\{b_1y_1', x_2'y_2'\}$ is a rigid pair with $b_1y_2' \in \mathcal{C}_2$ then $x_2'y_1 \in \mathcal{C}_1$. Then, by Lemma 3.2.6, we know that either $y_2' <_1 x_2' <_1 b_1$ or $b_1 <_1 x_2' <_1 y_2'$. First let $y_2' <_1 x_2' <_1 b_1$. Moreover $b_1 <_1 y_2$, and thus $y_2' <_1 b_1 <_1 y_2$. Since $y_2' \in N(y_2)$, by definition of completion, we have that $b_1y_2' \in \mathcal{C}_1$. This contradicts our assumption, $b_1y_2' \in \mathcal{C}_2$. Now let $b_1 <_1 x_2' <_1 y_2'$. Then we have that $x_1 <_1 b_1 <_1 x_2'$. Since $x_1, x_2' \in N(a_1)$, by (1) of Lemma 3.2.2, we have that $a_1b_1 \in \mathcal{C}_1$.

- Let $y_1 <_1 b_1 <_1 x_2$. Then a similar argument to the case $x_1 <_1 b_1 <_1 y_2$ proves that $a_1b_1 \in \mathcal{C}_1$.

Therefore, for all cases we have that $a_1b_1 \in \mathcal{C}_1$. But we already know that $a_1b_1 \in \mathcal{C}_2$. This implies that $a_1b_1 \in \mathcal{C}_1 \cap \mathcal{C}_2$, which contradicts $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ (Proposition 3.2.4). □

The next proposition proves that the structures (1) and (4), as shown in Figure 3.11, cannot both occur.

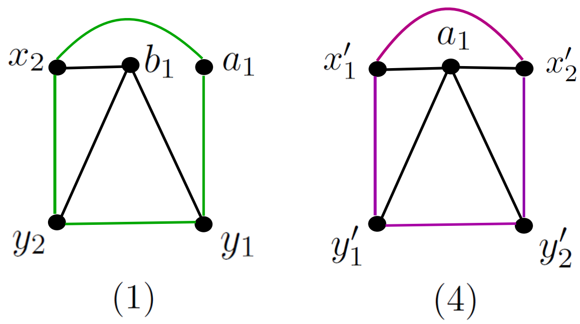


Figure 3.11: Structures (1) and (4) of Figures 3.8 and 3.9, respectively.

Proposition 3.2.19. *Suppose that G is a type-1 square geometric graph as in Assumption 3.2.1. Then the structures, (1) and (4), of Figure 3.11 cannot both occur. More precisely, if there is a rigid pair $\{x'_1y'_1, x'_2y'_2\}$ with $y'_1, y'_2 \in N_Y(a_1)$ and $x'_1, x'_2 \in X_a \cap X_b$ then there is no rigid pair $\{a_1y_1, x_2y_2\}$ with $y_1, y_2 \in N_Y(b_1)$ and $x_2 \in X_a \cap X_b$.*

Proof. By symmetry of X_a and X_b in type-1 graphs, an analogous discussion to the proof of Proposition 3.2.18 proves the proposition. □

Now we have all the required results to prove (i) and (ii) of the rigid-free conditions (Definition 3.1.4) for a square geometric $B_{a,b}$-graph of Assumption 3.2.1.

Proposition 3.2.20. *Suppose that G is a $B_{a,b}$-graph as in Assumption 3.2.1.*

(i) If G is type-1 then either a_1 is rigid-free with respect to X_b or b_1 is rigid-free with respect to X_a.

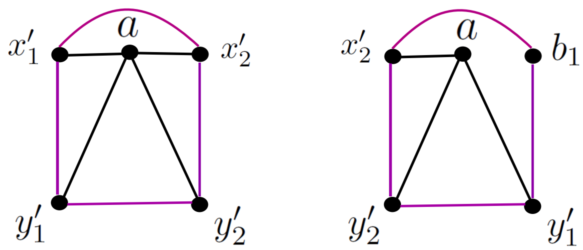
(ii) If G is type-2 and $N_Y(a_1) \subseteq N_Y(a_2)$ then either a_2 is rigid-free with respect to X_b or b_1 is rigid-free with respect to X_a.

Proof. We first prove (i). If a_1 is not rigid-free with respect to X_b then the graph G contains either structure (2) of Figure 3.8 or structure (4) of Figure 3.9. Also, if b_1 is not rigid-free then the graph G contains either structure (1) of Figure 3.8 or structure (3) of Figure 3.9. By Corollary 3.2.15, the structures (1) and (2) cannot both occur, and by Proposition 3.2.17, the structures (3) and (4) cannot occur simultaneously. Moreover, by Propositions 3.2.18 and 3.2.19, we know that the structures (2) and (3), and the structures (1) and (4) cannot occur simultaneously. This implies that either a_1 is rigid-free with respect to X_b, or b_1 is rigid-free with respect to X_a.

We now prove (ii). Suppose that G is a type-2 $\mathcal{B}_{a,b}$-graph and $N_Y(a_1) \subseteq N_Y(a_2)$. Then the graph $H = G - \{a_1\}$ is an induced type-1 $\mathcal{B}_{a,b}$-subgraph of G. Since G is square geometric then H is square geometric. By part (i), we have that either a_2 is rigid-free with respect to X_b or b_1 is rigid-free with respect to $X_a \setminus \{a_1\}$. Let b_1 be rigid-free with respect to $X_a \setminus \{a_1\}$. Since $N_Y(a_1) \subseteq N_Y(a_2)$, we have that b_1 is rigid-free with respect to X_a. This proves that either a_2 is rigid-free with respect to X_b or b_1 is rigid-free with respect to X_a. □

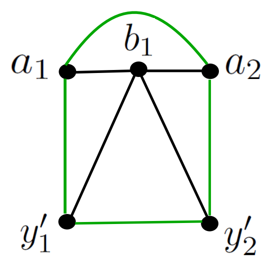
Type-2 $B_{a,b}$-graph

Assume that the graph G is a type-2 $B_{a,b}$-graph, as given in Assumption 3.2.1, and $\{a_1y_1, a_2y_2\}$ is a rigid pair (the neighborhoods of a_1 and a_2 are not nested in Y). In what follows, we prove that if G is a square geometric graph then both a_1 and a_2 are rigid-free with respect to X_b. Moreover, if $N_Y(a) \not\subseteq N_Y(b_1)$, for all $a \in \{a_1, a_2\}$, then b_1 is rigid-free with respect to $\{a_1, a_2\}$ (Condition (iii) of the rigid-free conditions). Indeed, for all $a \in \{a_1, a_2\}$, we prove that the structures of Figure 3.12 are forbidden. Moreover, we prove that if, for all $a \in \{a_1, a_2\}$, we have that $N_Y(a) \not\subseteq N_Y(b_1)$, then the structure of Figure 3.13 cannot occur.



(1) (2)

Figure 3.12: Forbidden structures of a type-2 $B_{a,b}$-graph G, as in Assumption 3.2.1, which contains a rigid pair $\{a_1y_1, a_2y_2\}$.



(3)

Figure 3.13: Forbidden structure of a type-2 $B_{a,b}$-graph G, as in Assumption 3.2.1, which contains a rigid pair $\{a_1y_1, a_2y_2\}$, and $N_Y(a) \not\subseteq N_Y(b_1)$, for all $a \in \{a_1, a_2\}$.

Throughout the rest of this section, we assume, without loss of generality, that $a_1 <_i a_2$, for all $i \in \{1, 2\}$.

First we show that, if structure (1) of Figure 3.12 occurs then, for all $x \in X_a \cup X_b$, we have that $N_Y(b_1) \subseteq N_Y(x)$ (Lemma 3.2.23). This is one of the excluded cases of Assumption 3.2.1. Note that, we prove the results for a_1. By symmetry of a_1 and a_2, all the obtained results are true if we switch a_1 and a_2 in the statements of the results. The next two lemma prepare us for the proof of Lemma 3.2.23.

Lemma 3.2.21. *Let $\{a_1y_1, a_2y_2\}$ and $\{x'_1y'_1, x'_2y'_2\}$ be rigid pairs with $y'_1, y'_2 \in N_Y(a_1)$. For all $i \in \{1, 2\}$, if $a_2y_1 \in \mathcal{C}_i$ then $x'_1 <_i a_2 <_i y_2$.*

Proof. Without loss of generality let $i = 1$. Let $a_2y_1 \in \mathcal{C}_1$. Then, by Lemma 3.2.6, either $a_1 <_1 a_2 <_1 y_2$ or $y_2 <_1 a_2 <_1 a_1$. By assumption $a_1 <_1 a_2$, so we have that $a_1 <_1 a_2 <_1 y_2$.

Moreover, $\{x_1'y_1', x_2'y_2'\}$ is a rigid pair with $y_1', y_2' \in N_Y(a_1)$, and thus, by Lemma 3.2.16, either $x_1' <_1 a_1 <_1 y_2'$ or $y_1' <_1 a_1 <_1 x_2'$. We know that $a_1 <_1 a_2 <_1 y_2$. Since $y_1' \in N(y_2)$, by (1) of Lemma 3.2.6, we know that $a_1 <_1 y_1'$, and thus $y_1' <_1 a_1 <_1 x_2'$ does not occur. Therefore, $x_1' <_1 a_1 <_1 y_2'$. This together with $a_1 <_1 a_2 <_1 y_2$, implies that $x_1' <_1 a_2 <_1 y_2$.

□

Lemma 3.2.22. *Let G be a $\mathcal{B}_{a,b}$-graph as in Assumption 3.2.1. Assume that $a_1b_1 \notin \mathcal{C}_1$.*

(1) Suppose that there are $u \in X_a \cap X_b$ and $v \in Y$ such that $u <_1 a_1 <_1 v$ and $u <_1 Y$. Then $b_1 <_1 u <_1 N_Y(b_1) <_1 a_1 <_1 v$. Moreover, for all $x \in X_a$ and for all $y_b \in N_Y(b_1) \setminus N_Y(x)$, we have that $xy_b \in \mathcal{C}_1$, and, for all $y \in Y \setminus N_Y(a_1)$, we have that $a_1y \in \mathcal{C}_1$.

(2) Suppose that there are $u \in X_a \cap X_b$ and $v \in Y$ such that $v <_1 a_1 <_1 u$ and $Y <_1 u$. Then $v <_1 a_1 <_1 N_Y(b_1) <_1 u <_1 b_1$. Moreover, for all $x \in X_a$ and for all $y_b \in N_Y(b_1) \setminus N_Y(x)$, we have that $xy_b \in \mathcal{C}_1$, and, for all $y \in Y \setminus N_Y(a_1)$, we have that $a_1y \in \mathcal{C}_1$.

If $a_1b_1 \notin \mathcal{C}_2$ then (1) and (2) hold when $<_1$ is replaced by $<_2$.

Proof. We only prove (1). The proof of (2) is analogous. Suppose that $a_1b_1 \notin \mathcal{C}_1$, and there are $u \in X_a \cap X_b$ and $v \in Y$ such that $u <_1 a_1 <_1 v$ and $u <_1 Y$. Since $u \in X_a \cap X_b$ then $b_1 \in N(u) \setminus N(a_1)$. Moreover, $u <_1 a_1$, and thus, by (2) of Lemma 3.2.2, we have that $b_1 <_1 u <_1 a_1 <_1 v$. Now, by (1) of Lemma 3.2.9, we know that either $b_1 <_1 Y$ or $Y <_1 a_1$. Since $v \in Y$ and $a_1 <_1 v$ then $Y <_1 a_1$ cannot occur, and thus $b_1 <_1 Y$. This implies that $b_1 <_1 N_Y(b_1)$. Moreover, by (1) of Lemma 3.2.8, we know that $N_Y(b_1) <_1 a_1$, and thus $b_1 <_1 N_Y(b_1) <_1 a_1$. Moreover, since $u <_1 Y$ then $u <_1 N_Y(b_1)$. This implies that $b_1 <_1 u <_1 N_Y(b_1) <_1 a_1 <_1 v$.

Since $a_1, u \in X_a$ then, for any $x \in X_a$, we have that $a_1, u \in N(x)$. Moreover, $u <_1 N_Y(b_1) <_1 a_1$, and thus, by (1) of Lemma 3.2.2, for all $x \in X_a$ and for all $y_b \in N_Y(b_1) \setminus N_Y(x)$, we have that $xy_b \in \mathcal{C}_1$. Also, since $N_Y(b_1) <_1 a_1 <_1 v$, where $v \in Y$, then by (3) of Lemma 3.2.2, for all $y \in Y \setminus N_Y(a_1)$, we have that $a_1y \in \mathcal{C}_1$. □

We now prove that the structure (1) of Figure 3.12 cannot occur.

Lemma 3.2.23. *Let G be a type-2 $\mathcal{B}_{a,b}$-graph as in Assumption 3.2.1. Suppose that there is a rigid pair $\{a_1y_1, a_2y_2\}$. Then G does not contain the structure (1) of Figure 3.12. More precisely, the graph G contains no rigid pair of the form $\{x'_1y'_1, x'_2y'_2\}$ with $x'_1, x'_2 \in X_a \cap X_b$ and $y'_1, y'_2 \in N_Y(a)$, where $a \in \{a_1, a_2\}$.*

Proof. We assume, without loss of generality, that $a = a_1$. Let $\{a_1y_1, a_2y_2\}$ be a rigid pair. Suppose, by contradiction, that G contains the structure (1) of Figure 3.12. Indeed, let $\{x'_1y'_1, x'_2y'_2\}$ be a rigid pair with $x'_1, x'_2 \in X_a \cap X_b$ and $y'_1, y'_2 \in N_Y(a_1)$.

Since a_1b_1 is a non-edge of G, by the definition of completion, we know that either $a_1b_1 \notin \mathcal{C}_1$ or $a_1b_1 \notin \mathcal{C}_2$. Without loss of generality, assume that $a_1b_1 \notin \mathcal{C}_1$.

Since $\{x'_1y'_1, x'_2y'_2\}$ is a rigid pair with $y'_1, y'_2 \in N_Y(a_1)$, by Lemma 3.2.16, we have that either $x'_1 <_1 a_1 <_1 y'_2$ or $y'_1 <_1 a_1 <_1 x'_2$. First, we assume that $x'_1 <_1 a_1 <_1 y'_2$. Then, by Lemma 3.2.16, we know that $x'_1 <_1 Y$. Therefore, by (1) of Lemma 3.2.22, we have that $b_1 <_1 x'_1 <_1 N_Y(b_1) <_1 a_1 <_1 y'_2$. Moreover, for all $x \in X_a \cup X_b$ and for all $y_b \in N_Y(b_1) \setminus N_Y(x)$, we have that $xy_b \in \mathcal{C}_1$, and for all $y \in Y \setminus N_Y(a_1)$, we have that $a_1y \in \mathcal{C}_1$. Similarly, if $y'_1 <_1 a_1 <_1 x'_2$ then, by Lemma 3.2.16, we know that $Y <_1 x'_2$. Therefore, by (2) of Lemma 3.2.22, we have that $y'_1 <_1 a_1 <_1 N_Y(b_1) <_1 x'_2 <_1 b_1$. Moreover, for all $x \in X_a \cup X_b$ and for all $y_b \in N_Y(b_1) \setminus N_Y(x)$, we have that $xy_b \in \mathcal{C}_1$, and for all $y \in Y \setminus N_Y(a_1)$, we have that $a_1y \in \mathcal{C}_1$. Therefore, for both cases $x'_1 <_1 a_1 <_1 y'_2$ and $y'_1 <_1 a_1 <_1 x'_2$, we have that $a_1y_2 \in \mathcal{C}_1$, and

$$\{xy_b |\ y_b \in N_Y(b_1) \setminus N_Y(x) \text{ and } x \in X_a \cup X_b\} \subseteq \mathcal{C}_1.$$

We now prove that $\{xy_b |\ y_b \in N_Y(b_1) \setminus N_Y(x) \text{ and } x \in X_a \cup X_b\} \subseteq \mathcal{C}_2$. To show this, we first prove that $a_2b_1 \notin \mathcal{C}_2$.

Since $a_1y_2 \in \mathcal{C}_1$, by Lemma 3.2.6, either $a_2 <_1 a_1 <_1 y_1$ or $y_1 <_1 a_1 <_1 a_2$. We assumed that $a_1 <_1 a_2$, and thus $y_1 <_1 a_1 <_1 a_2$. By (2) of Lemma 3.2.6, we have that $Y <_1 a_2$ and $y_1 <_1 X_a \cap X_b$. Since Consider the following cases:

- $b_1 <_1 x'_1 <_1 N_Y(b_1) <_1 a_1 <_1 y'_2$ and $x'_1 <_1 Y$: Since $x'_1 \in X_a \cap X_b$ and $y_1 <_1 X_a \cap X_b$, we have that $y_1 <_1 x'_1$ which contradicts $x'_1 <_1 Y$. Therefore, this case cannot happen.

- $y'_1 <_1 a_1 <_1 N_Y(b_1) <_1 x'_2 <_1 b_1$ and $Y <_1 x'_2$: Since $Y <_1 a_2$ then $N_Y(b_1) <_1 a_2$. Then either $a_1 <_1 N_Y(b_1) <_1 a_2 <_1 b_1$ or $a_1 <_1 N_Y(b_1) <_1 b_1 <_1 a_2$. Then, by the definition of completion, we have that $a_2b_1 \in \mathcal{C}_1$, and thus $a_2b_1 \notin \mathcal{C}_2$.

Since $a_1y_2 \in \mathcal{C}_1$, by Proposition 3.2.4, we have that $a_2y_1 \in \mathcal{C}_2$. Then, by Lemma 3.2.21, we know that $x_1' <_2 a_2 <_2 y_2$ and $x_1' <_1 Y$. This together with $a_2b_1 \notin \mathcal{C}_2$ and Lemma 3.2.22, proves that for all $x \in X_a \cup X_b$ and for all $y_b \in N_Y(b_1) \setminus N_Y(x)$, we have that $xy_b \in \mathcal{C}_2$. Indeed, $\{xy_b | \, y_b \in N_Y(b_1) \setminus N_Y(x) \text{ and } x \in X_a \cup X_b\} \subseteq \mathcal{C}_2$. Since $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ we have that $\{xy_b | \, y_b \in N_Y(b_1) \setminus N_Y(x) \text{ and } x \in X_a \cup X_b\} = \emptyset$. This implies that, for all $x \in X_a \cup X_b$, we have that $N_Y(x) \cap N_Y(b_1) = N_Y(b_1)$. But this structure is excluded in Assumption 3.2.1. Therefore, a type-2 $\mathcal{B}_{a,b}$-graph as in Assumption 3.2.1, which contains a rigid pair $\{a_1y_1, a_2y_2\}$, does not contain the structure (1) of Figure 3.12. $\square$

We now prove that, the structure (2) of Figure 3.12 cannot occur. We first need some auxiliary lemmas.

Lemma 3.2.24. *Let G be a type-2 $\mathcal{B}_{a,b}$-graph as in Assumption 3.2.1. Suppose that there are rigid pairs $\{a_1y_1, a_2y_2\}$ and $\{b_1'y_1', x_2'y_2'\}$ with $y_1', y_2' \in N_Y(a_1)$. Let $i \in \{1, 2\}$. If $b_1y_2' \in \mathcal{C}_i$ then either $a_2b_1 \in \mathcal{C}_i$ or $a_1y_2 \in \mathcal{C}_i$.*

Proof. Without loss of generality let $i = 1$. Let $\{b_1y_1', x_2'y_2'\}$ be a rigid pair with $y_1', y_2' \in N_Y(a_1)$ and $b_1y_2' \in \mathcal{C}_1$. Then, by Lemma 3.2.6, we have that either $x_2' <_1 b_1 <_1 y_1'$ or $y_1' <_1 b_1 <_1 x_2'$. We know that $Y \subseteq N(y_1')$. So if $x_2' <_1 b_1 <_1 y_1'$ then, by Lemma 3.2.6, we have that $x_2' <_1 Y$. Moreover, $a_1 \in N(x_2')$ and again, by Lemma 3.2.6, we have that $x_2' <_1 a_1$. Similarly, if $y_1' <_1 b_1 <_1 x_2'$ then, by Lemma 3.2.6, we have that $Y <_1 x_2'$, and since $a_1 \in N(x_2')$, by Lemma 3.2.6, we have that $a_1 <_1 x_2'$. We consider the case where $x_2' <_1 Y$ and $x_2' <_1 a_1$. The proof for the case $Y <_1 x_2'$ and $a_1 <_1 x_2'$ follows by the symmetry of the two cases. Consider the following.

(1) Let $a_2 <_1 x_2'$. Since $x_2' <_1 Y$ we have that $a_2 <_1 Y$. Therefore, $a_2 <_1 y_2$. Moreover $x_2' <_1 a_1$, and thus $a_2 <_1 a_1$. Since $a_2 <_1 y_2$ then, by Lemma 3.2.6, for rigid pair $\{a_1y_1, a_2y_2\}$ we have that either $a_2 <_1 a_1 <_1 y_2 <_1 y_1$ or $a_2 <_1 y_2 <_1 a_1 <_1 y_1$. This implies that $a_1y_2 \in \mathcal{C}_1$.

(2) Let $x_2' <_1 a_2$. Since $x_2' \in X_a \cap X_b$, we know that $x_2' \in N(b_1)$. Moreover, $x_2' <_1 b_1$, and thus, by part (2) of Lemma 3.2.2, we have that $a_2b_1 \in \mathcal{C}_1$.

This implies that either $a_2b_1 \in \mathcal{C}_1$, or $a_1y_2 \in \mathcal{C}_1$, and we are done. $\square$

Lemma 3.2.25. *Let G be a type-2 $\mathcal{B}_{a,b}$-graph as in Assumption 3.2.1. Suppose that there is a rigid pair $\{a_1y_1, a_2y_2\}$. Then G does not contain the structure (2) of Figure 3.12. More precisely, there exists no rigid pair $\{b_1y_1', x_2'y_2'\}$ with $y_1', y_2' \in N_Y(a_1)$ and $x_2' \in X_a \cap X_b$.*

Proof. Suppose that $\{b_1y_1', x_2'y_2'\}$ is a rigid pair with $y_1', y_2' \in N_Y(a_1)$. By Proposition 3.2.4, we know that b_1y_2' and $x_2'y_1'$ belong to different completions. Suppose, without loss of generality, that $b_1y_2' \in \mathcal{C}_1$, and thus $b_1y_2' \notin \mathcal{C}_2$. Then, by Lemma 3.2.6, either $b_1 <_2 x_2' <_2 y_2'$ or $y_2' <_2 x_2' <_2 b_1$. First suppose that $b_1 <_2 x_2' <_2 y_2'$. Then, by Lemma 3.2.6, we have that $b_1 <_2 Y$, and since $a_1 \in N(y_2')$ then $b_1 <_2 a_1$.

Moreover, since $b_1y_2' \in \mathcal{C}_1$ then by Lemma 3.2.14 we have $a_1b_1 \in \mathcal{C}_1$. Also, by Lemma 3.2.24 either $a_2b_1 \in \mathcal{C}_1$ or $a_2y_1 \in \mathcal{C}_1$. Consider the two following cases: "$a_1b_1 \in \mathcal{C}_1$ and $a_2b_1 \in \mathcal{C}_1$ ", and "$a_1b_1 \in \mathcal{C}_1$ and $a_1y_2 \in \mathcal{C}_1$ ".

First note that, since $a_1b_1 \in \mathcal{C}_1$ we have that $a_1b_1 \notin \mathcal{C}_2$. Also, we know that $b_1 <_2 a_1$. Since $a_1b_1 \notin \mathcal{C}_2$ we have that $b_1 <_2 X_a \cap X_b <_2 a_1$. Moreover, $b_1 <_2 Y$, and thus $b_1 <_2 N_Y(b_1)$. Since $a_1b_1 \notin \mathcal{C}_2$, we have that $b_1 <_2 N_Y(b_1) <_2 a_1$. We now consider the two cases we mentioned in the previous paragraph.

- Let $a_1b_1 \in \mathcal{C}_1$ and $a_2b_1 \in \mathcal{C}_1$. We know that $b_1 <_2 X_a \cap X_b <_2 a_1$ and $b_1 <_2 N_Y(b_1) <_2 a_1$. Since $b_1 <_2 a_1$ and $a_2b_1 \notin \mathcal{C}_2$, then, by part (2) of Lemma 3.2.2, we have that $b_1 <_2 a_2$. Therefore, $b_1 <_2 N_Y(b_1) <_2 a_2$ and $b_1 <_2 (X_a \cap X_b) <_2 a_2$. We know that $a_1 <_1 a_2$. Then, we have that $N_Y(b_1) <_1 a_1 <_1 a_2$. If there is $y \in Y$ such that $N_Y(b_1) <_1 a_1 <_1 a_2 <_1 y$, then, by part (3) of Lemma 3.2.2, we have that $a_1y_2 \in \mathcal{C}_2$ and $a_2y_1 \in \mathcal{C}_2$ which contradicts the fact that a_1y_2 and a_2y_1 belong to different completions. Therefore, $Y <_2 a_2$ and thus $b_1 <_1 Y <_1 a_2$. Moreover, for all $x \in X_a \cap X_b$, we know that $b_1, a_2 \in N(x)$. Then, by part (1) of Lemma 3.2.2, for all $x \in X_a \cap X_b$, we have that $xy \in \mathcal{C}_2$ for all $y \in Y \setminus N_Y(x)$. This implies that, if there exists a rigid pair $\{x_1v_1, x_2v_2\}$ with $x_1, x_2 \in X_a \cap X_b$, then both chords of the rigid pair, x_1y_2 and x_2y_1, belong to $\mathcal{C}_2$, which is a contradiction. Therefore, G cannot contain a rigid pair $\{x_1v_1, x_2v_2\}$ with $x_1, x_2 \in X_a \cap X_b$. This implies that $E(\tilde{G}_a \cap \tilde{G}_b) = \emptyset$.

- Let $a_1b_1 \in \mathcal{C}_1$ and $a_1y_2 \in \mathcal{C}_1$. Then, by Proposition 3.2.4, we have that $a_1y_2 \notin \mathcal{C}_2$ and $a_1b_1 \notin \mathcal{C}_2$. Moreover, we know that $b_1 <_2 X_a \cap X_b <_2 a_1$ and $b_1 <_2$

$N_Y(b_1) <_2 a_1$. Now suppose that there exists $y \in Y$ such that $a_1 <_2 y$. Since $N_Y(b_1) <_2 a_1$ then we have $N_Y(b_1) <_2 a_1 <_2 y$, and thus by part (3) of Lemma 3.2.2 we have $a_1y_2 \in \mathcal{C}_2$, which contradicts $a_1y_2 \notin \mathcal{C}_2$. This implies that $Y <_2 a_1$. We also have $b_1 <_2 Y$, and therefore $b_1 <_2 Y <_2 a_1$. Moreover, for all $x \in X_a \cap X_b$, we know that $b_1, a_2 \in N(x)$. Then, by part (1) of Lemma 3.2.2, for all $x \in X_a \cap X_b$, we have that $xy \in \mathcal{C}_2$, for all $y \in Y \setminus N_Y(x)$. This implies that, if there exists a rigid pair $\{x_1v_1, x_2v_2\}$ with $x_1, x_2 \in X_a \cap X_b$, then both chords of the rigid pair, x_1y_2 and x_2y_1, belong to $\mathcal{C}_2$, which is a contradiction. Therefore, G cannot contain a rigid pair $\{x_1v_1, x_2v_2\}$ with $x_1, x_2 \in X_a \cap X_b$. This implies that $E(\tilde{G}_a \cap \tilde{G}_b) = \emptyset$.

For both of the cases, we have that $E(\tilde{G}_a \cap \tilde{G}_b) = \emptyset$. But, for a graph G as in Assumption 3.2.1, $E(\tilde{G}_a \cap \tilde{G}_b) \neq \emptyset$. Therefore, the structure (2) of Figure 3.12, does not occur in G. $\square$

In the next two lemmas, we prove that the structure of Figure 3.13 cannot occur in a graph G of Assumption 3.2.1.

Lemma 3.2.26. *Let G be a type-2 $\mathcal{B}_{a,b}$-graph as in Assumption 3.2.1. Suppose that there is a rigid pair $\{a_1y_1, a_2y_2\}$ with $y_1, y_2 \in N_Y(b_1)$. For all $i \in \{1,2\}$, if $a_1b_1 \notin \mathcal{C}_i$ then $b_1y \in \mathcal{C}_i$, for all $y \in Y \setminus N_Y(b_1)$.*

Proof. First recall that we assume that $a_1 <_i a_2$, for all $i \in \{1,2\}$. Suppose, without loss of generality, that $a_1b_1 \notin \mathcal{C}_1$. Consider the following cases:

(1) $a_1y_2 \notin \mathcal{C}_1$. Then, by Lemma 3.2.6, we have that $a_1 <_1 a_2 <_1 y_2$, $a_1 <_1 Y$, and $X_a \cap X_b <_1 y_2$. Moreover, since $y_2 \in N_Y(b_1)$ and $a_1y_2 \notin \mathcal{C}_1$, then $a_1 <_1 b_1$. Therefore, by Lemma 3.2.8, we have that $a_1 <_1 X_a \cap X_b <_1 b_1$ and $a_1 <_1 N_Y(a_1) <_1 b_1$. Since $a_1 <_1 Y$, if $Y <_1 b_1$ then $a_1 <_1 Y <_1 b_1$. This, together with $a_1 <_1 X_a \cap X_b <_1 b_1$ and Lemma 3.2.10, implies that $E(\tilde{G}_a \cap \tilde{G}_b) = \emptyset$. But for a graph G as in Assumption 3.2.1, this cannot occur. Therefore, $Y <_1 b_1$ cannot happen, and thus there exists $y' \in Y$ with $b_1 <_1 y'$. Then $N_Y(a_1) <_1 b_1 <_1 y'$, and thus, by (3) of Lemma 3.2.2, we have that $b_1y \in \mathcal{C}_1$, for all $y \in Y \setminus N_Y(b_1)$.

(2) $a_2y_1 \notin \mathcal{C}_1$. The discussion in this case is slightly different from case (1). Since $a_2y_1' \notin \mathcal{C}_1$, by Lemma 3.2.6, we have that $y_1 <_1 a_1 <_1 a_2$, $Y <_1 a_2$, and $y_1 <_1 X_a \cap X_b$. Since $a_1b_1 \notin \mathcal{C}_1$ then $b_1 <_1 y_1 <_1 a_1$. Therefore, by Lemma 3.2.8, we have that $b_1 <_1 X_a \cap X_b <_1 a_1 <_1 a_2$ and $b_1 <_1 N_Y(a_1) <_1 a_1$. Since $Y <_1 a_2$, if $b_1 <_1 Y$ then $b_1 <_1 Y <_1 a_2$. This, together with $b_1 <_1 X_a \cap X_b <_1 a_2$ and Lemma 3.2.10, implies that $E(\tilde{G}_a \cap \tilde{G}_b) = \emptyset$. But for a graph G as in Assumption 3.2.1, this cannot occur. Therefore, $b_1 <_1 Y$ cannot happen, and thus there exists $y' \in Y$ with $y' <_1 b_1$. Then $y <_1 b_1 <_1 y_1$, and thus, by (3) of Lemma 3.2.2, we have that $b_1y \in \mathcal{C}_1$, for all $y \in Y \setminus N_Y(b_1)$.

Therefore, for all $y \in Y \setminus N_Y(b_1)$, we have that $b_1y \in \mathcal{C}_1$. □

Lemma 3.2.27. *Let G be a type-2 $\mathcal{B}_{a,b}$-graph as in Assumption 3.2.1. Suppose that there is a rigid pair $\{a_1y_1, a_2y_2\}$, and $N_Y(a) \not\subseteq N_Y(b_1)$, for all $a \in \{a_1, a_2\}$. Then G does not contain the structure of Figure 3.13. More precisely, there exists no rigid pair $\{a_1y_1, a_2y_2\}$ with $y_1, y_2 \in N_Y(b_1)$.*

Proof. Suppose, without loss of generality that $a_1b_1 \notin \mathcal{C}_1$. Then, by Lemma 3.2.26, we have that $b_1y \in \mathcal{C}_1$, for all $y \in Y \setminus N_Y(b_1)$. Moreover, by assumption $N_Y(a) \not\subseteq N_Y(b_1)$, for all $a \in \{a_1, a_2\}$. Then, there exist $w_1 \in N_Y(a_1) \setminus N_Y(b_1)$ and $w_2 \in N_Y(a_2) \setminus N_Y(b_1)$. We know that b_1w_1 and b_1w_2 are in $\mathcal{C}_1$. Since $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, $b_1w_1, b_1w_2 \notin \mathcal{C}_2$. Moreover, since $\{a_1y_1, a_2y_2\}$ is a rigid pair, by Lemma 3.2.6, either $a_1 <_2 a_2 <_2 y_2$ and $a_1 <_2 Y$, or $y_1 <_2 a_1 <_2 a_2$ and $Y <_2 a_2$. First let $a_1 <_2 a_2 <_2 y_2$ and $a_1 <_2 Y$. Then $a_1 <_2 w_1$. Moreover, since $b_1 \in N_Y(y_2')$, by Lemma 3.2.6, we know that $a_1 <_2 b_1$. Since $b_1w_1 \notin \mathcal{C}_2$, we have that $a_1 <_2 w_1 <_2 b_1$. Moreover, $Y <_2 b_1$ and $w_1 <_2 X_a \cap X_b <_2 b_1$. This, together with $a_1 <_2 Y$, implies that $a_1 <_2 Y <_2 b_1$ and $a_1 <_2 X_a \cap X_b <_2 b_1$. Then, by (3) of Lemma 3.2.10, we have that $E(\tilde{G}_a \cap \tilde{G}_b) = \emptyset$. But this cannot occur for a graph G as in Assumption 3.2.1. An analogous discussion, for $y_1' <_2 a_1 <_2 a_2$ and $Y <_2 a_2$, proves that $E(\tilde{G}_a \cap \tilde{G}_b) = \emptyset$, which is not possible. Therefore, the graph G does not contain the structure of Figure 3.13. □

Proposition 3.2.28. *Let G be a type-2 $\mathcal{B}_{a,b}$-graph as in Assumption 3.2.1. Suppose that there is a rigid pair $\{a_1y_1, a_2y_2\}$. Then a_1 and a_2 are rigid-free with respect to X_b. Moreover, if $N_Y(a) \not\subseteq N_Y(b_1)$, for all $a \in \{a_1, a_2\}$, then b_1 is rigid-free with respect to $\{a_1, a_2\}$.*

Proof. Suppose that G is a type-2 $\mathcal{B}_{a,b}$-graph as in Assumption 3.2.1. Let $a \in \{a_1, a_2\}$. If a is not rigid-free with respect to X_b, then one of the structures (1) and (2) of Figure 3.12 occur. But by Lemmas 3.2.23 and 3.2.25, we know that such structures cannot occur. Now assume that $N_Y(a) \not\subseteq N_Y(b_1)$, for all $a \in \{a_1, a_2\}$. If b_1 is not rigid-free with respect to $\{a_1, a_2\}$ then G contains the structure of Figure 3.13. But by Lemma 3.2.27, we know that G cannot contain the structure of Figure 3.13. This finishes the proof of the proposition. $\square$

3.3 Sufficiency of the Conditions of Theorem 3.1.9

We devote this section to the proof of sufficiency of Theorem 3.1.9. We assume that for a graph G, as given in Assumption 3.1.3, the conditions of Theorem 3.1.9 hold. We construct two linear orders $<_1$ and $<_2$ for G which satisfy Equation 2.2. This proves that G is square geometric.

First note that, for a type-1 $B_{a,b}$-graph which satisfies the rigid-free conditions (Definition 3.1.4), by symmetry of X_a and X_b, we can always assume that a_1 is rigid-free with respect to X_b. Therefore, throughout this section we state definitions and results according to the assumption that a_1 is rigid-free with respect to X_b.

Assumption 3.3.1. *Let G be as in Assumption 3.1.3. Assume that the conditions of Theorem 3.1.9 hold. Moreover, if G is a type-1 $B_{a,b}$-graph then let a_1 be rigid-free with respect to X_b. Let $f : V(\tilde{G}_a \cup \tilde{G}_b) \to \{red, blue\}$ be a 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$ whose corresponding relations $<_X$ and $<_Y$, as in Definition 3.1.6, form a proper bi-ordering, and $f(\tilde{u}) = red$ for all $\tilde{u} \in \mathcal{A}$ and $f(\tilde{u}) = blue$ for all $\tilde{u} \in \mathcal{B}$.*

Throughout this section, we assume that G is a $B_{a,b}$-graph as in Assumption 3.3.1. In what follows, we collect some immediate properties of the graph G. First we prove that the cobipartite subgraphs G_a and G_b of a graph G are square geometric.

Proposition 3.3.2. *Let G be a $B_{a,b}$-graph as in Assumption 3.3.1. Then G_a and G_b are square geometric cobipartite graphs.*

Proof. By Assumption 3.3.1, $\tilde{G}_a \cup \tilde{G}_b$ is 2-colorable, and thus the graphs $\tilde{G}_a$ and $\tilde{G}_b$ are 2-colorable. Then, by Corollary 2.2.7, we know that G_a and G_b are square geometric cobipartite graphs. $\square$

We will use the relations $<_X$ and $<_Y$ to define our desired linear orders $<_1$ and $<_2$ for the graph G, that is, orders that satisfy Equation 2.2. So, first, we investigate how vertices of the graph G relate in the relations $<_X$ and $<_Y$. Recall that, according to Definition 3.1.6, for x_1, x_2 both in X_a or both in X_b, $x_1 <_X x_2$ if and only if there is a rigid pair $\{x_1y_1, x_2y_2\}$ such that in a 2-coloring f, as in Assumption 3.3.1, x_1y_2 is colored red. Specifically, two vertices x_1, x_2 both X_a or both in X_b are related in $<_X$ if they are part of a rigid pair. Otherwise their neighborhoods are nested. Similarly, two vertices $y_1, y_2 \in Y$ are related in Y if they are part of a rigid pair. Otherwise, they have nested neighborhoods.

The next two propositions list some useful properties of the relations $<_X$ and $<_Y$ of the graph G, as given in Assumption 3.3.1. First note that a graph G as in Assumption 3.3.1 is either a type-1 or a type-2 $B_{a,b}$-graph, and thus $X_a \backslash X_b \subseteq \{a_1, a_2\}$, and $X_b \setminus X_a = \{b_1\}$.

Proposition 3.3.3. *Let G, f, and $<_X$ be as in Assumption 3.3.1. Suppose $x \in X_a \cup X_b$. Then*

(1) For any $x_1, x_2 \in X_a$ or $x_1, x_2 \in X_b$, either x_1 and x_2 are related in $<_X$, or they have nested neighborhoods.

(2) For all $a \in X_a \setminus X_b$, either $a <_X x$, or a and x have nested neighborhoods in Y. Similarly, either $x <_X b_1$, or x and b_1 have nested neighborhoods in Y.

(3) Vertices a in $X_a \setminus X_b$ and b in $X_b \setminus X_a$, are not related in $<_X$.

Proof. Part (1) follows directly from the fact that, for any two vertices x_1, x_2 in X_a or X_b, either they are part of a rigid pair or they have nested neighborhoods.

We now prove (2). Let $x \in X_a \cap X_b$. Suppose that $a \in X_a \setminus X_b$, and the neighborhoods of a and x are not nested in Y. This implies that there are $y_1, y_2 \in Y$ such that $\{ay_1, xy_2\}$ is a rigid pair of G. Therefore, $ay_2 \sim^* xy_1$, and thus $ay_2 \in \mathcal{A}$. We know that the 2-coloring f of Assumption 3.3.1 colors all vertices of $\mathcal{A}$ red. Then $f(ay_2) =$ red, and so $a <_X x$. Similarly, if the neighborhoods of b_1 and x in Y are not nested then there is a rigid pair $\{b_1y_1', xy_2'\}$. This implies that $b_1y_2' \in \mathcal{B}$, and since f colors all vertices of $\mathcal{B}$ blue then $f(b_1y_2') =$ blue. Therefore, $x <_X b_1$.

To prove (3), let $a \in X_a \setminus X_b$ and $b_1 \in X_b \setminus X_a$. since $a \nsim b$, by the definition of a rigid pair, there exist no $y_1, y_2 \in Y$ such that $\{ay_1, b_1y_2\}$ is a rigid pair. This implies that a and b_1 are not related in $<_X$. $\square$

Proposition 3.3.4. *Let G, f, and $<_Y$ be as in Assumption 3.3.1. Suppose $y_1, y_2 \in Y$. Then one of the following cases occurs.*

(1) The neighborhoods of y_1 and y_2 are nested in G.

(2) There is a rigid pair $\{x_1y_1, x_2y_2\}$ in G, and thus y_1 and y_2 are related in $<_Y$.

(3) $N_X(y_1) \setminus N_X(y_2) \subseteq X_a \setminus X_b$ and $N_X(y_2) \setminus N_X(y_1) \subseteq X_b \setminus X_a$, see Figure 3.14.

Proof. Let $y_1, y_2 \in Y$. Suppose that the neighborhoods of y_1 and y_2 are not nested in G. Then there are $x_1, x_2 \in X_a \cup X_b$ such that $x_1 \in N_X(y_1) \setminus N_X(y_2)$ and $x_2 \in N_X(y_2) \setminus N_X(y_1)$. If $x_1 \sim x_2$ then $\{x_1y_1, x_2y_2\}$ forms a rigid pair of G, and thus y_1 and y_2 are related in $<_Y$. If $x_1 \nsim x_2$, then $x_1 \in X_a \setminus X_b$ and $x_2 \in X_b \setminus X_a$. This implies that $N_X(y_1) \setminus N_X(y_2) \subseteq X_a \setminus X_b$ and $N_X(y_2) \setminus N_X(y_1) \subseteq X_b \setminus X_a$, and we are done. $\square$

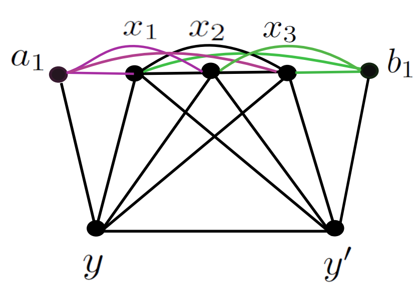


Figure 3.14: The vertices y and y' that are not related in $<_Y$ and their neighborhoods are not nested in $X_a \cup X_b$

We now define two relations $<_1$ and $<_2$ for a graph G of Assumption 3.3.1.

Definition 3.3.5. *Let G be as in Assumption 3.3.1. Let $<_X$ and $<_Y$ be as in Assumption 3.3.1. Define*
Ordering $<_1$:

1.1. $x <_1 x'$ if $x <_X x'$ or $N_Y(x) \subseteq N_Y(x')$ for all $x, x' \in X_a \cap X_b$.

1.2. $a_1 <_1 a_2 <_1 x <_1 b_1$ for all $x \in X_a \cap X_b$.

1.3. $y <_1 y'$ *if* $y <_Y y'$ *or* $N_X(y') \subseteq N_X(y)$ *for all* $y, y' \in Y$.

1.4. $y <_1 y'$ *if* $N_X(y) \setminus N_X(y') \subseteq X_a \setminus X_b$ *and* $N_X(y') \setminus N_X(y) \subseteq X_b \setminus X_a$.

1.5. $y_a <_1 b_1$ *and* $b_1 <_1 y$ *for all* $y_a \in N_Y(a)$ *and all* $y \in Y \setminus N_Y(a)$, *where* $a \in \{a_1, a_2\}$.

1.6. $x <_1 y$ *for all* $x \in X_a$ *and all* $y \in Y$.

Ordering $<_2$*:*

2.1. $x <_2 x'$ *if* $x' <_X x$ *or* $N_Y(x) \subseteq N_Y(x')$ *for all* $x, x' \in X_a \cup X_b$.

2.2. $b_1 <_2 a$ *for* $a \in \{a_1, a_2\}$ *with* $N_Y(a) \nsubseteq N_Y(b_1)$ *and* $N_Y(b_1) \nsubseteq N_Y(a)$.

2.3. $y <_2 y'$ *if* $y' <_Y y$ *or* $N_X(y') \subseteq N_X(y)$ *in* G.

2.4. $y <_2 y'$ *if* $N_X(y') \setminus N_X(y) \subseteq X_a \setminus X_b$ *and* $N_X(y) \setminus N_X(y') \subseteq X_b \setminus X_a$.

2.5. $x <_2 y$ *for all* $x \in X_a \cup X_b$ *and all* $y \in Y$.

We now briefly discuss the reasoning behind the Definition of 3.3.5. Recall that, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are completions of linear orders $<_1$ and $<_2$, then $<_1$ and $<_2$ satisfy Equation 2.2 if and only if $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ (Proposition 3.2.4). Also, recall from Section 3.1 that for a type-1 or type-2 $B_{a,b}$-graph there are three types of non-edges of G: isolated vertices of $\tilde{G}_a \cup \tilde{G}_b$, chords of rigid pairs, and a_1b_1 if G is type-1, and $\{a_1b_1, a_2b_2\}$ if G is type-2. To maintain $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ for relations $<_1$ and $<_2$ of Definition 3.3.5, we require that non-edges of these three categories belong to at most one of the completions $\mathcal{C}_1$ and $\mathcal{C}_2$.

Let us now have a look at the relations of Definition 3.3.5. We can divide the terms of definition $<_1$ into three groups: (i) terms 1.1-1.2 which determine how vertices of $X_a \cup X_b$ relate to each other in $<_1$, (ii) terms 1.3 and 1.4 which determine how vertices of Y relate to each other in $<_1$, and (iii) terms 1.5 and 1.6 which determine how vertices of $X_a \cup X_b$ and Y relate to each other in $<_1$.

Similarly, the terms of definition $<_2$ can be divided into three groups: (i) terms 2.1-2.2 which determine how vertices of $X_a \cup X_b$ relate to each other in $<_2$, (ii) terms 2.3 and 2.4 which determine how vertices of Y relate to each other in $<_2$, and (iii) term 2.4 which determines how vertices of $X_a \cup X_b$ and Y relate to each other in $<_2$.

As we can see, the definitions of $<_1$ and $<_2$ are symmetric on Y, *i.e.* group (ii). However $<_1$ and $<_2$ are not completely symmetric in groups (i) and (iii). The reason for this difference between $<_1$ and $<_2$ is that we want the non-edges of the form ab_1, for $a \in \{a_1, a_2\}$, not to belong to $\mathcal{C}_1$ (completion of $<_1$).

The way in which $<_1$ is defined in Definition 3.3.5 guarantees that $a_1b_1 \notin \mathcal{C}_1$ and $a_2b_1 \notin \mathcal{C}_1$ (for a type-2 graph). More precisely, terms 1.2 and 1.5 rule out the possibility of a_1 and b_1 (similarly a_2 and b_1) being located between two adjacent vertices in $<_1$. We will prove that other non-edges of G also belong to at most one completion.

In the rest of this section, the goal is to prove that the relations $<_1$ and $<_2$ of Definition 3.3.5 are linear orders satisfying Equation 2.2. Our main approach includes cobipartite graphs G_a and G_b, and the results of chapter 2 for cobipartite graphs (specifically Definition 2.2.12, Lemma 2.2.13 and).

Suppose that the graphs G, G_a, and G_b are as in Assumption 3.3.1. Consider a 2-coloring f as in Assumption 3.3.1 with corresponding proper bi-ordering $(<_X, <_Y)$. Then the restrictions of f to $V(\tilde{G}_a)$ and $V(\tilde{G}_b)$ are proper 2-colorings of $\tilde{G}_a$ and $\tilde{G}_b$, respectively. Let $<_X^a$ and $<_Y^a$ be the relations of Definition 2.2.10 corresponding to the restriction of f to $V(\tilde{G}_a)$. Similarly, let $<_X^b$ and $<_Y^b$ be the relations of Definition 2.2.10 corresponding to the restriction of f to $V(\tilde{G}_b)$. The next proposition states how $(<_X, <_Y)$ is related to $(<_X^a, <_Y^a)$ and $(<_X^b, <_Y^b)$.

Proposition 3.3.6. *Let the graphs G, G_a, and G_b be as in Assumption 3.3.1.*

(i) For all $x, x' \in X_a$, $x <_X^a x'$ if and only if $x <_X x'$. Similarly, for all $x, x' \in X_b$, $x <_X^b x'$ if and only if $x <_X x'$.

(ii) If $y <_Y^a y'$ then $y <_Y y'$. Similarly, If $y <_Y^b y'$ then $y <_Y y'$.

(iii) If $N_{X_a}(y') \subseteq N_{X_a}(y)$ then there are three possible cases: $N_X(y') \subseteq N_X(y)$, $y <_Y y'$, or $N_X(y) \setminus N_X(y') \subseteq \{a_1, a_2\}$ and $N_X(y') \setminus N_X(y) = \{b_1\}$. Similarly, if $N_{X_b}(y') \subseteq N_{X_b}(y)$ then there are three possible cases: $N_X(y') \subseteq N_X(y)$, $y' <_Y y$, or $N_X(y') \setminus N_X(y) \subseteq \{a_1, a_2\}$ and $N_X(y) \setminus N_X(y') = \{b_1\}$.

Proof. We only prove the proposition for G_a and $(<_X^a, <_Y^a)$. The proof for G_b and $(<_X^b, <_Y^b)$ follows by symmetry of G_a and G_b and an analogous discussion.

We first prove (i). We know that, for all $x, x' \in X_a$, a rigid pair $\{xy, x'y'\}$ in G_a is a rigid pair in G and vice versa. Therefore, $x <_X^a x'$ if and only if $x <_X x'$. To prove (ii), let $y, y' \in Y$. A rigid pair $\{xy, x'y'\}$ in G_a is a rigid pair in G. This implies that, if $y <_Y^a y'$ then $y <_Y y'$.

Now suppose that $y, y' \in Y$ and $N_{X_a}(y') \subseteq N_{X_a}(y)$. By Proposition 3.3.4, we know that there are three possible cases:

- $N_X(y') \subseteq N_X(y)$.
- There is a rigid pair $\{xy, x'y'\}$ in G. Since $N_{X_a}(y') \subseteq N_{X_a}(y)$, we know that $x' \in X_b \subseteq X_a$. Therefore, $x' = b_1$. By (2) of Proposition 3.3.3, we have that $x <_X x'$, and thus $y <_Y y'$.
- $N_X(y) \setminus N_X(y') \subseteq \{a_1, a_2\}$ and $N_X(y') \setminus N_X(y) = \{b_1\}$.

□

Corollary 3.3.7. *Suppose that we are in the settings of Proposition 3.3.6. Then* $(<_X^a, <_Y^a)$ *and* $(<_X^b, <_Y^b)$ *are proper bi-relations of cobipartite graphs* G_a *and* G_b*, respectively.*

In what follows, we will see how the relations $<_1$ and $<_2$ are related to linear orders of cobipartite graphs G_a and G_b.

Lemma 3.3.8. *Suppose that* G_a *is a cobipartite graph, as in Assumption 3.3.1, with proper bi-ordering* $(<_X^a, <_Y^a)$*. Let* $<_1$ *be the relation as in Definition 3.3.5, and* $<_1^a$ *be the relation as in Definition 2.2.12, i.e.*

(1) $x <_1^a x'$ *if* $x <_X^a x'$ *or* $N_Y(x) \subseteq N_Y(x')$ *for all* $x, x' \in X_a$*.*

(2) $y <_1^a y'$ *if* $y <_Y^a y'$ *or* $N_{X_a}(y') \subseteq N_{X_a}(y)$ *for all* $y, y' \in Y$*.*

(3) $x <_1^a y$ *for all* $x \in X_a$ *and all* $y \in Y$*.*

Then, $<_1^a$ *is a linear order on the set of vertices of the graph* G_a*. Moreover, for any pair of vertices* $v_1, v_2 \in V(G_a) \setminus \{a_1, a_2\}$*, we have that* $v_1 <_1 v_2$ *if and only if* $v_1 <_1^a v_2$*.*

Proof. First note that, G_a is a cobipartite graph with proper bi-ordering $(<_X^a, <_Y^a)$. Then, by Lemma 2.2.13, we know that $<_1^a$ is a linear order on $V(G_a)$. Now let $v_1, v_2 \in V(G_a) \setminus \{a_1, a_2\}$. First suppose that $v_1, v_2 \in X_a \cap X_b$. Then, by 1.1 of Definition 3.3.5, (1) of the definition of $<_1^H$, and (i) of Proposition 3.3.6, it immediately follows that, $v_1 <_1 v_2$ if and only if $v_1 <_1^a v_2$. Moreover, if $v_1 \in X_a \cap X_b$ and $v_2 \in Y$, then, by 1.6 of Definition 3.3.5 and (3) of the definition of $<_1^a$, we have that $v_1 <_1 v_2$ if and only if $v_1 <_1^a v_2$. So let $v_1, v_2 \in Y$. Then, either $v_1 <_Y^a v_2$ or they have nested neighborhoods in G_a. If $v_1 <_Y^a v_2$, then, by (ii) of Proposition 3.3.6, we know that $v_1 <_Y v_2$. Therefore, by (2) of the definition $<_1^a$ and 1.3 of Definition 3.3.5, we have that $v_1 <_1^a v_2$ and $v_1 <_1 v_2$. Now suppose that $N_{X_a}(v_2) \subseteq N_{X_a}(v_1)$, and thus $v_1 <_1^a v_2$. By Proposition 3.3.6, there are three possible cases:

- $N_X(v_2) \subseteq N_X(v_1)$. Then, by 1.3 of Definition 3.3.5, we have that $v_1 <_1 v_2$.
- $v_1 <_Y v_2$. Then $v_1 <_1 v_2$.
- $N_X(v_1) \setminus N_X(v_2) \subseteq \{a_1, a_2\}$ and $N_X(v_2) \setminus N_X(v_1) = \{b_1\}$. Then, by 1.4 of Definition 3.3.5, we have that $v_1 <_1 v_2$.

This finishes the proof of the lemma. □

Lemma 3.3.9. *Suppose that G_b is a cobipartite graph, as in Assumption 3.3.1, with proper bi-ordering $(<_X^b, <_Y^b)$. Let $<_2$ be the relation as in Definition 3.3.5, and $<_2^b$ be the relation as in Definition 2.2.12, i.e.*

(1) $x <_2^b x'$ if $x' <_X^b x$ or $N_Y(x) \subseteq N_Y(x')$ for all $x, x' \in X_b$.

(2) $y <_2^b y'$ if $y' <_Y^b y$ or $N_{X_b}(y') \subseteq N_{X_b}(y)$ for all $y, y' \in Y$.

(3) $x <_2^b y$ for all $x \in X_b$ and all $y \in Y$.

Then, $<_2^b$ is a linear order on the set of vertices of the graph G_b. Moreover, for any pair of vertices $v_1, v_2 \in V(G_b)$, we have that $v_1 <_2 v_2$ if and only if $v_1 <_2^b v_2$.

Proof. First note that, G_b is a cobipartite graph with proper bi-ordering $(<_X^b, <_Y^b)$. Then, by Lemma 2.2.13, we know that $<_2^b$ is a linear order on $V(G_b)$. Now let $v_1, v_2 \in V(G_b)$. First suppose that $v_1, v_2 \in X_b$. Then, by 2.1 of Definition 3.3.5, (1) of the definition of $<_2^b$, and (i) of Proposition 3.3.6, it immediately follows that, $v_1 <_2 v_2$ if and only if $v_1 <_2^b v_2$. Moreover, if $v_1 \in X_b$ and $v_2 \in Y$, then, by 2.5 of Definition 3.3.5 and (3) of the definition of $<_2^b$, we have that $v_1 <_2 v_2$ if and only if $v_1 <_2^b v_2$. So let $v_1, v_2 \in Y$. Then, either $v_1 <_Y^b v_2$ or they have nested neighborhoods in G_b. If $v_2 <_Y^b v_1$, then, by (ii) of Proposition 3.3.6, we know that $v_2 <_Y v_1$. Therefore, by (2) of the definition $<_2^b$ and 2.3 of Definition 3.3.5, we have that $v_1 <_2^b v_2$ and $v_1 <_2 v_2$. Now suppose that $N_{X_b}(v_2) \subseteq N_{X_b}(v_1)$, and thus $v_1 <_2^b v_2$. By Proposition 3.3.6, there are three possible cases:

- $N_X(v_2) \subseteq N_X(v_1)$. Then, by 2.3 of Definition 3.3.5, we have that $v_1 <_2 v_2$.
- $v_2 <_Y v_1$. Then $v_1 <_2 v_2$.
- $N_X(v_2) \setminus N_X(v_1) \subseteq \{a_1, a_2\}$ and $N_X(v_1) \setminus N_X(v_2) = \{b_1\}$. Then, by 2.4 of Definition 3.3.5, we have that $v_1 <_2 v_2$.

This finishes the proof of the lemma. □

The following proposition presents some useful properties of the relations $<_1$ and $<_2$ of Definition 3.3.5.

Proposition 3.3.10. *Let G be as in Assumption 3.3.1 and let $<_1$ and $<_2$ be relations of Definition 3.3.5. Let $a \in \{a_1, a_2\}$. Then*

(i) For all $y_a \in N_Y(a)$ and all $y \in Y \setminus N_Y(a)$ we have $y_a <_1 y$.

(ii) For all $y_b \in N_Y(b_1)$ and all $y \in Y \setminus N_Y(b_1)$ we have $y_b <_2 y$.

Proof. We first prove (i). Let $a \in \{a_1, a_2\}$. We know by Proposition 3.3.4 that for any $y_a \in N_Y(a)$ and any $y \in Y \setminus N_Y(a)$ there are three possible cases: (1) y_a and y are related in $<_Y$. Then by Proposition 3.3.4, we have $y_a <_Y y$. This implies that $y_a <_1 y$. (2) y_a and y have nested neighborhoods in G. Since $a \in N_X(y_a) \setminus N_X(y)$ then we must have $N_X(y) \subseteq N_X(y_a)$, and thus by 1.4 of Definition 3.3.5 we have $y_a <_1 y$. (3) Assume y and y_a are not related in $<_Y$ and do not have nested neighborhoods. Since $a \in N_X(y_a) \setminus N_X(y)$ then, by 3 of Proposition 3.3.4, we must have $N_X(y_a) \setminus N_X(y) \subseteq X_a \setminus X_b$ and $N_X(y) \setminus N_X(y_a) \subseteq X_b \setminus X_a$. Therefore, by 1.5 of Definition 3.3.5, we have $y_a <_1 y$.

To prove (ii), let $y_b \in N_Y(b_1)$ and $y \in Y \setminus N_Y(b_1)$. We know by Proposition 3.3.4 that there are three cases: (1) y_b and y are related in $<_Y$ then, by Proposition 3.3.4, we have that $y <_Y y_b$. Then, by 2.4 of Definition 3.3.5, we have that $y <_2 y_b$. (2) y_b and y have nested neighborhoods in G. Since $b_1 \in N_X(y_b) \setminus N_X(y)$ then we must have $N_X(y) \subseteq N_X(y_b)$, and thus by 2.4 of Definition 3.3.5 we have $y_b <_2 y$. (3) Assume y and y_b are not related in $<_Y$ and do not have nested neighborhoods. Since $b_1 \in N_X(y_b) \setminus N_X(y)$ then, by 3 of Proposition 3.3.4, we have $N_X(y_b) \setminus N_X(y) = b_1$ and $N_X(y) \setminus N_x(y_b) \setminus \{a_1, a_2\}$. Then, by 2.5 of Definition 3.3.5, we have $y_b <_2 y$. $\square$

Remark 6. *Suppose that G is as in Assumption 3.3.1. Let $<_2$ be the relation as in Definition 3.3.5. Let $a \in \{a_1, a_2\}$ and $x \in X_a \cap X_b$.*

(1) By Proposition 3.3.3, we know that, for all $x \in X_a \cap X_b$, if x and a are related in $<_X$ then $a <_X x$, and if x and b_1 are related in $<_X$ then $x <_X b_1$. Therefore, by 2.1 of Definition 3.3.5, if $a <_2 x$ then $x <_X a$, and if $x <_2 a$ then $N_Y(x) \subseteq N_Y(a)$. Similarly, if $b_1 <_2 x$ then $b_1 <_X x$, and if $x <_2 b_1$ then $N_Y(x) \subseteq N_Y(b_1)$.

(2) If there exists $x \in X_a \cap X_b$ such that $a <_X x$ and $x <_X b_1$, then $N_Y(a) \nsubseteq N_Y(b_1)$. Since $x <_X b_1$ and $a <_X x$ then there are rigid pairs $\{b_1y_1, xy_2\}$ and $\{ay_1', xy_2'\}$ in G. If $N_Y(a) \subseteq N_Y(b_1)$ then $b_1 \sim y_1'$ and $a \nsim y_2$. Therefore, $\{ay_1', xy_2\}$ and $\{b_1y_1', xy_2\}$ are rigid pairs. This implies that $ay_2 \sim^* xy_1' \sim^* b_1y_2$. Therefore, $ay_2 \in \mathcal{A}$, $b_1y_2 \in \mathcal{B}$, and in any proper 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$, both ay_2 and by_2 receive the same color. But we know, by Assumption 3.3.1, that all the vertices of $\mathcal{A}$ are red and all the vertices of $\mathcal{B}$ are blue. This implies that $N_Y(a) \nsubseteq N_Y(b_1)$.

In the next two lemmas, we prove that the relations $<_1$ and $<_2$ are linear orders on $V(G)$.

Lemma 3.3.11. *Let G be as in Assumption 3.3.1. Suppose that the relation $<_1$ is as in Definition 3.3.5. Then $<_1$ is a linear order on $V(G)$.*

Proof. First note that, by Lemma 3.3.8, the relation $<_1$ is a linear order on $V(G_a) \setminus \{a_1, a_2\}$. We need to prove that, the relation $<_1$ remains a linear order when the vertices a_1, a_2, and b_1 are considered. It directly follows, by Definition 3.1.6, that $<_1$ is reflexive on $\{a_1, a_2, b_1\}$. Now suppose that $v, v' \in V(G)$, and at least one of v and

v' is in $\{a_1, a_2, b_1\}$. Then v and v' are related by one of 1.2, 1.5, and 1.6 of Definition 3.3.5. Moreover, by the definition, $<_1$ is antisymmetric on v, v'.

We now prove that $<_1$ is transitive. Let $v_1, v_2, v_3 \in V(G)$ such that $v_1 <_1 v_2$ and $v_2 <_1 v_3$. If $v_1, v_2, v_3 \in V(G_a) \setminus \{a_1, a_2\}$, then, by Lemma 3.3.8, we know that $v_1 <_1 v_3$. If $v_1, v_2, v_3 \in X_a \cup X_b$, then, by 1.2 of Definition 3.3.5, $<_1$ is transitive on v_1, v_2, v_3, and thus $v_1 <_1 v_3$. Moreover, if $v_1, v_2, v_3 \in X_a \cup Y$, then by 1.6 of Definition 3.3.5, we know that $<_1$ is transitive on v_1, v_2, v_3, and thus $v_1 <_1 v_3$.

So assume that among v_1, v_2, v_3, one is b_1, one is in Y, and one is in $\{a_1, a_2\}$. By Definition 3.3.5, we know that vertices of $\{a_1, a_2\}$ are minimum elements of $V(G) \setminus \{a_1, a_2\}$ under $<_1$. Therefore, $v_1 \in \{a_1, a_2\}$, and $v_1 <_1 v_3$. This proves that $<_1$ is transitive, and we are done. □

We now prove that $<_2$ is a linear order on $V(G)$.

Lemma 3.3.12. *Let G be as in Assumption 3.3.1. Suppose that the relation $<_2$ is as in Definition 3.3.5. Then $<_2$ is a linear order on $V(G)$.*

Proof. We know, by Lemma 3.3.9, that $<_2$ is a linear order on $V(G_b)$. We need to prove that $<_2$ remains a linear order when the vertices a_1, a_2, are considered. It directly follows, by Definition 3.1.6, that $<_2$ is reflexive on $\{a_1, a_2\}$. Now suppose that $v, v' \in V(G)$, and at least one of v and v' is in $\{a_1, a_2\}$. Then v and v' are related by one of 2.1, 2.2, and 2.5 of Definition 3.3.5. Moreover, by the definition, $<_2$ is antisymmetric on v, v'.

We now prove that, for any triple $v_1, v_2, v_3 \in V(G)$, the relation $<_2$ is transitive on v_1, v_2, v_3. Since $<_2$ is a linear order on $V(G_b)$, we assume that at least one of v_1, v_2, v_3 is in $\{a_1, a_2\}$. If one of v_1, v_2, and v_3 is in Y, then by 2.5 of Definition 3.3.5, we know that $<_2$ is transitive on v_1, v_2, v_3. So suppose that $v_1, v_2, v_3 \in X_a \cup X_b$.

If $v_1, v_2, v_3 \in X_a$, then any pair of vertices v_1, v_2, and v_3 are either related in $<_X$ or they have nested neighborhoods in Y. Therefore, they are related in $<_2$ by 2.1 of Definition 3.3.5. By Lemma 2.2.13, for cobipartite graphs, we know that $<_2$ is transitive on v_1, v_2, and v_3. Now let $v_1, v_2 \in X_a$ and $v_3 = b_1$. If none of v_1 and v_2 is in $\{a_1, a_2\}$, then v_1, v_2, v_3 are all in $V(G_b)$. But we assume that at least one v_1, v_2, v_3 is in $\{a_1, a_2\}$. This implies that $\{v_1, v_2, v_3\} = \{a, x, b_1\}$, where $a \in \{a_1, a_2\}$ and $x \in X_a \cap X_b$. We consider the following cases:

- $a <_2 x$ and $x <_2 b_1$. Then, by (1) of Remark 6, we have that $N_Y(a) \subseteq N_Y(x)$, and $N_Y(x) \subseteq N_Y(b_1)$. This implies that $N_Y(a) \subseteq N_Y(b_1)$, and thus, by 2.1 of Definition 3.3.5, $a <_2 b_1$.

- $b_1 <_2 x$ and $x <_2 a$. Then, by (1) of Remark 6, we have that $x <_X b_1$ and $a <_X x$. Therefore, by (2) of Remark 6, $N_Y(a) \nsubseteq N_Y(b_1)$. Then either $N_Y(b_1) \subseteq N_Y(a)$ or $N_Y(b_1) \nsubseteq N_Y(a)$. By 2.1 and 2.2 of Definition, for both cases, we have that $b_1 <_2 a$.

- $x <_2 a$ and $a <_2 b_1$. Then by (1) of Remark 6, $a <_X x$, and by 2.1 of Definition 3.3.5, $N_Y(a) \subseteq N_Y(b_1)$. If x and b_1 are related in $<_X$, then $x <_X b_1$. Then, by (2) of Remark 6, $N_Y(a) \nsubseteq N_Y(b_1)$, which is a contradiction. So $N_Y(x) \subseteq N_Y(b_1)$, and thus $x <_2 b_1$. If $a <_2 b_1$ and $b_1 <_2 x$ then an analogous discussion proves that $a <_2 x$.

- $x <_2 b_1$ and $b_1 <_2 a$. Then by (1) of Remark 6, $N_Y(x) \subseteq N_Y(b_1)$. If $N_Y(a) \subseteq N_Y(x)$ then $N_Y(a) \subseteq N_Y(b_1)$, and thus $a <_2 b_1$ which is not true. Therefore, either $N_Y(x) \subseteq N_Y(a)$ or $a <_X x$. In both cases, by 2.1 of Definition 3.3.5, we have that $x <_2 a$. If $b_1 <_2 a$ and $a <_2 x$, then an analogous discussion proves that $b_1 <_2 x$.

This finishes the proof of transitivity of $<_2$. □

Now that we know that the relations $<_1$ and $<_2$ of Definition 3.3.5 are linear orders, the next step is to show that linear orders $<_1$ and $<_2$ satisfy Equation (2.2). The strategy to prove this is similar to what we did for cobipartite graphs (see Lemmas 2.2.14 and 2.2.16, and Corollaries 2.2.15 and 2.2.17). Indeed, we assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ are completions of $<_1$ and $<_2$, respectively. We first prove that chords of a rigid pair belong to different completions $\mathcal{C}_1$ and $\mathcal{C}_2$. Then we prove that isolated vertices of $\tilde{G}_a$ and $\tilde{G}_b$ belong to at most one completion $\mathcal{C}_1$ and $\mathcal{C}_2$. Note that we already defined $<_1$ and $<_2$ in a way that non-edges a_1b_1 and a_2b_2 (if G is type-2) do not belong to $\mathcal{C}_1$. Recall that definitions of linear orders $<_1$ and $<_2$ are symmetric on Y and $X_a \cap X_b$. The next lemma gives us the required results to prove that chords of a rigid pair belong to different completions $\mathcal{C}_1$ and $\mathcal{C}_2$. This is where the rigid-free conditions show up and help us with the proofs.

Lemma 3.3.13. *Let G, $<_1$ and $<_2$ be as in Assumption 3.3.1. Suppose $\{x_1y_1, x_2y_2\}$ is a rigid pair and $x_1, x_2 \in X_a \cap X_b$. If $x_1 <_1 x_2$ then*

(i) For all $y \in N_Y(x_1)$, $y <_1 y_2$.

(ii) For all $x \in N_X(y_2)$, $x_1 <_1 x$.

(iii) For any $x \in X$ and $y \in Y$ with $x <_1 x_1$ and $y_2 <_1 y$, we have $x \nsim y$.

Similarly if $x_1 <_2 x_2$ and we replace $<_1$ by $<_2$ in the statements (i)-(iii) then statements (i), (ii), and (iii) hold.

Proof. Suppose that $\{x_1y_1, x_2y_2\}$, where $x_1, x_2 \in X_a \cap X_b$, is a rigid pair with $x_1 <_1 x_2$. Then, by 1.1 of Definition 3.3.5, we know that $x_1 <_X x_2$, and thus $y_1 <_Y y_2$. This implies that $x_1 <_1 x_2 <_1 y_1 <_1 y_2$. Similarly, if $x_1 <_2 x_2$, then $x_1 <_2 x_2 <_2 y_1 <_2 y_2$.

By Lemma 3.3.8, we know that, for all $v_1, v_2 \in V(G_a) \setminus \{a_1, a_2\}$, $v_1 <_1 v_2$ if and only if $v_1 <_1^a v_2$. Then $\{x_1y_1, x_2y_2\}$ is a rigid pair of the cobipartite graph G_a with $x_1 <_1^a x_2$ and $y_1 <_1^a y_2$.

To prove (i) for $<_1$ note that, by Lemma 2.2.14 for cobipartite graph G_a, we know that, for all $y \in N_Y(x_1)$, $y <_1^a y_2$. This together with Lemma 3.3.8, implies that, for all $y \in N_Y(x_1)$, $y <_1 y_2$. This proves (i). The proof for $<_2$ follows by an analogous discussion and Lemma 3.3.9.

We now prove (ii) for $<_1$. By Lemma 2.2.14 for cobipartite graph G_a, we know that, for all $x \in N_{X_a}(y_2)$, $x_1 <_1^a x$. Then, by Lemma 3.3.8, we have that, for all $x \in X_a \setminus \{a_1, a_2\}$ such that $x \in N_X(y_2)$, we have that $x_1 <_1 x$. Now let $x \in N_X(y_2) \cap \{a_1, a_2, b_1\}$. If $x = b_1$, then by 1.5 and 1.6 of Definition 3.3.5, $x_1 <_1 x$. Now let $x \in \{a_1, a_2\}$. Since $y_2 \in N_Y(x)$ and $y_1 <_1 y_2$, by Proposition 3.3.10, we know that $y_1 \in N_Y(x)$. This implies that the neighborhood of x contains the rigid pair $\{x_1y_1, x_2y_2\}$. But, a graph G of Assumption 3.3.1 satisfies the rigid-free conditions. This means that if G is type-1, then a_1 is rigid-free, and if G is type-2, then a_1 and a_2 are rigid-free. Therefore, y_2 has no neighbor in $\{a_1, a_2\}$. Then, for all $x \in N_X(y_2)$, we have that $x_1 <_1 x$.

The proof of (ii) for $<_2$ is slightly different. Suppose that $\{x_1y_1, x_2y_2\}$ is a rigid pair with $x_1, x_2 \in X_a \cap X_b$, and $x_1 <_2 x_2$. Then $x_1 <_2 x_2 <_2 y_1 <_2 y_2$. We know, by Lemma 3.3.9 and Lemma 2.2.14, that for all $x \in N_{X_b}(y_2)$, $x_1 <_2 x$. Now let

$x \in N_X(y) \cap \{a_1, a_2\}$. Since $y_2 \notin N_Y(x_1)$, we know that $N_Y(x) \nsubseteq N_Y(x_1)$. Then, by (1) of Remark 6, we know that $x_1 <_2 x$. This proves that for all $x \in N_X(y)$, $x_1 <_2 x$.

We now prove (iii). Assume that there are $v \in X_a \cup X_b$ and $z \in Y$ such that $v <_1 x_1 <_1 x_2 <_1 y_1 <_1 y_2 <_1 z$. Suppose, by contradiction, that $v \sim z$. By Part (i), we have that $x_1 \nsim z$, and so $N_Y(v) \nsubseteq N_Y(x_1)$. Then, by Proposition 3.3.3, we know that v and x_1 are related in $<_X$. If $v \in \{a_1, a_2\}$, then, by Proposition 3.3.3, we know that $v <_X x_1$. Now let $v \in X_a \cap X_b$. Since $v <_1 x_1$, by 1.1 of Definition 3.3.5, we have that $v <_X x_1$. This implies that, there is rigid pair $\{vz, x_1w\}$ with $w \in N_Y(x_1)$. Since $v <_X x_1$ then $z <_Y w$. Then, by 1.3 of Definition 3.3.5, we have that $z <_1 w$. This is impossible since by Part (i) for all $y \in N_Y(x_1)$, we have that $y <_1 y_2$. Therefore, $v \nsim z$, and we are done. The proof for $<_2$ follows by an analogous discussion. □

Corollary 3.3.14. *Let G be as in Assumption 3.3.1, and $<_1$ and $<_2$ be as in Definition 3.3.5. Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are completions corresponding to $<_1$ and $<_2$, respectively. Then each chord of a rigid pair $\{x_1y_1, x_2y_2\}$ with $x_1, x_2 \in X_a \cap X_b$ belong to at most one of $\mathcal{C}_1$ and $\mathcal{C}_2$.*

Proof. Suppose $\{x_1y_1, x_2y_2\}$ is a rigid pair. Without loss of generality, let $x_1 <_1 x_2$. Then, by 1.1 of Definition 3.3.5, we have that $x_1 <_X x_2$, and thus $y_1 <_Y y_2$. Then, by 1.3, and 1.6 of Definition 3.3.5, we have that $x_1 <_1 x_2 <_1 y_1 <_1 y_2$. Since $x_1 \sim y_1$ then $x_2y_1 \in \mathcal{C}_1$. By Lemma 3.3.13, we know that x_1 and y_2 are not between two adjacent vertices, and thus by definition of completion $x_1y_2 \notin \mathcal{C}_1$. An analogous discussion proves that $x_2y_1 \notin \mathcal{C}_2$. □

The next lemma proves a similar result for rigid pairs $\{xy, x'y'\}$, for which, either x or x' belongs to $\{a_1, a_2, b_1\}$.

Lemma 3.3.15. *Let G be as in Assumption 3.3.1, and $<_1$ and $<_2$ be linear orders of Definition 3.3.5 with completions $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. If there exists a rigid pair of G of one of the forms $\{a_1y_1, a_2y_2\}$, or $\{ay_1, x_2y_2\}$, for $a \in \{a_1, a_2\}$, or $\{x_1y_1, b_1y_2\}$, then different chords of the rigid pair belong to different completions $\mathcal{C}_1$ and $\mathcal{C}_2$.*

Proof. First suppose $\{a_1y_1, a_2y_2\}$ is a rigid pair. By Definition 3.3.5, we have that $a_1 <_1 a_2 <_1 y_1 <_1 y_2$. Since $a_1 \in N(y_1)$, we have that $a_2y_1 \in \mathcal{C}_1$. We now prove that $a_1y_2 \notin \mathcal{C}_1$. By 1.2 of Definition 3.3.5, for all $x \in X_a \cup X_b$ we have $a_1 <_1 a_2 <_1 x$. This implies that for all $u \in X_a \cap X_b$ and all $w \in Y$, with $u <_1 a_1$ and $y_2 <_1 w$, u and w are not adjacent.

We now show that a_1 has no neighbor $w \in Y$ with $y_2 <_1 w$. Suppose, to the contrary, that, for a $w \in Y$ with $y_2 <_1 w$, we have that $a_1 \in N_X(w)$. If $w \notin N_Y(a_2)$ then $\{a_1w, a_2y_2\}$ is a rigid pair. Since $a_1 <_X y_2$ we have that $w <_Y y_2$. Then, by 1.2 of Definition 3.3.5, we have that $w <_1 y_2$, which contradicts our assumption $y_2 <_1 w$. So let $a_2 \in N_X(w)$.

Since $a_1 \in N_X(w) \setminus N_X(y_2)$ then $N_X(w) \nsubseteq N_X(y_2)$. But $y_2 <_1 w$. Then there are two possible cases:

(1) $y_2 <_Y w$. This means that there is rigid pair $\{uy_2, zw\}$. Since $a_1 \in N_X(w) \setminus N_X(y_2)$ then $\{uy_2, a_1w\}$ is also a rigid pair. This together with the fact that $\{a_1y_1, a_2y_2\}$ is a rigid pair implies that $a_2y_1 \sim^* a_1y_2 \sim^* uw$ in $\tilde{G}$.

Then uw and a_2y_1 receive the same color in f, and thus by Definition 3.1.6 we have $a_1 <_X a_2$ if and only if $a_1 <_X u$. We know $a_1 <_X a_2$. So $a_1 <_X u$, and thus $w <_Y y_2$. This together with 1.3 of Definition 3.3.5 implies that $w <_1 y_2$, which contradicts our assumption.

(2) $N_X(w) \setminus N_X(y_2) = b_1$ and $N_X(y_2) \setminus N_X(w) \subseteq \{a_1, a_2\}$. Since $a_1, a_2 \in N_X(y_2)$ we know that $N_X(y_2) \setminus N_X(w) = \emptyset$. Therefore, $N_X(y_2) \subseteq N_X(w)$ in G. This together with 1.3 of Definition 3.3.5 implies that $w <_1 y_2$ which contradicts contradicts our assumption that $y_2 <_1 w$.

This proves that all neighbors of a_1 are smaller than y_2 in $<_1$. Therefore a_1 and y_2 are not between two adjacent vertices in $<_1$, and thus $a_1y_2 \notin \mathcal{C}_1$. This finishes the proof of (i).

Now suppose that there is a rigid pair $\{a_1y_1, x_2y_2\}$. By Proposition 3.3.3, we know that $a_1 <_X x_2$, and thus $y_1 <_Y y_2$. Then by 1.3 and 1.6 of Definition 3.3.5, we have $a_1 <_1 x_2 <_1 y_1 <_1 y_2$. Since $a_1 \in N(y_2)$ then $x_2y_1 \in \mathcal{C}_1$. We prove that $a_1y_2 \notin \mathcal{C}_1$. We know by 1.2 of Definition 3.3.5 that $a_1 <_1 x$ for all vertices $x \in X_a \cap X_b$. This implies that neither y_2 nor any vertex $w \in Y$ with $y_2 <_1 w$ has a neighbor $u \in X_a \cap X_b$ with $u <_1 a_1$. Moreover by (1) of Proposition 3.3.10 we know that a_1 has no neighbor $w \in Y$ with $y_2 <_1 w$. This proves that $a_1y_2 \notin \mathcal{C}_1$. For a rigid pair $\{a_2y_2, x_1y_1\}$ an analogous discussion proves that $a_2y_1 \notin \mathcal{C}_1$.

Now suppose that there is a rigid pair $\{x_1y_1, b_1y_2\}$. Then, by Proposition **??** we have $x_1 <_X b_1$, and thus $y_1 <_Y y_2$. By 2.1, 2.3, and 2.5 of Definition 3.3.5 we have $b_1 <_2 x_1 <_2 y_1 <_2 y_2$. Since $b_1 \in N(y_1)$ we have $x_1y_1 \in \mathcal{C}_2$. We prove $b_1y_2 \notin \mathcal{C}_2$. By Proposition 3.3.10, for all $y_b \in N_Y(b_1)$, we have that $y_b <_2 y$. Moreover, if x and b_1 are related in $<_X$ then, by Proposition 3.3.3, $x <_X b_1$. Then by 2.1 of Definition 3.3.5, $b_1 <_2 x$. Therefore, if $x <_2 b_1$ then we must have $N_Y(x) \subseteq N_Y(b_1)$. This implies that, for all vertices $x <_2 b_1$, we have $N_Y(x) <_2 y_2$. Therefore b_1 and y_2 are not between two adjacent vertices in $<_2$, and thus $b_1y_2 \notin \mathcal{C}_2$. This finishes the proof of the lemma. □

As we mentioned earlier, to prove that $<_1$ and $<_2$ satisfy Equation 2.2, ,we need to show that completions $\mathcal{C}_1$ and $\mathcal{C}_2$ have empty intersection. By Corollary 3.3.14 and Lemma 3.3.15, we know that non-edges which correspond to chords of rigid pairs belong to at most one completion $\mathcal{C}_1$ or $\mathcal{C}_2$. Moreover, recall that non-edges of form a_1b_1 and a_2b_1 do not belong to $\mathcal{C}_1$. Therefore, to prove that $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ we only need to show that non-edges corresponding to isolated vertices of $\tilde{G}_a \cup \tilde{G}_b$ belong to at most one of the completion $\mathcal{C}_1$ or $\mathcal{C}_2$.

Lemma 3.3.16. *Let G be as in Assumption 2.2.11. Suppose $<_1$ and $<_2$ are linear orders as in Definition 3.3.5. Suppose uw is an isolated vertex of $\tilde{G}$, and $u \in X_a \cap X_b$.*

(i) For all $y \in N_Y(u)$ we have $y <_2 w$.

(ii) For all $x \in X_a \cup X_b$ with $x <_2 u$ we have $N_Y(x) \subseteq N_Y(u)$.

Proof. First we prove (i). Let uw be an isolated vertex of $\tilde{G}$. By Definition of $\tilde{G}$, we know that uw is not a chord of any rigid pair of G. We first prove (i). Suppose by contradiction that $y \in N_Y(u)$ and $w <_2 y$. By Proposition 3.3.4 there are three possible cases:

(1) $N_X(y) \subseteq N_X(w)$. We have $u \in N_X(y) \setminus N_X(w)$. This implies that $N_X(y) \not\subseteq N_X(w)$ in G.

(2) $N_X(w) \setminus N_X(y) = b_1$ and $N_X(y) \setminus N_X(w) \subseteq \{a_1, a_2\}$. But $u \in N_X(y) \setminus N_X(w)$ and $u \in X_a \cap X_b$. This contradicts $N_X(y) \setminus N_X(w) \subseteq \{a_1, a_2\}$.

(3) $y <_Y w$. This implies that there is a rigid pair $\{xw, x'y\}$. Since $u \in N_X(y) \setminus N_X(w)$ then $\{xw, uy\}$ is also a rigid pair with chords xy and uw. This contradicts the fact that uw is an isolated vertex of $\tilde{G}$, and thus for all $y \in N_Y(u)$ we have $y <_1 w$.

We now prove (ii). We know that uw is not chord of any rigid pair of G. Therefore, by definition of rigid pair, for any $x \in X_a \cup X_b$ either $N_Y(x) \subseteq N_Y(u)$ or $N_Y(u) \subseteq N_Y(x)$. If $N_Y(u) \subseteq N_Y(x)$ then by 2.1 of Definition 3.3.5 $u <_2 x$. But $x <_2 u$, and thus $N_Y(x) \subseteq N_Y(u)$. □

Corollary 3.3.17. *Let G and be as in Assumption 2.2.11, and $<_1$ and $<_2$ be as in Definition 3.3.5. Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are completions corresponding to $<_1$ and $<_2$, respectively. Then isolated vertices, uw, of $\tilde{G}_a \cup \tilde{G}_b$ with $u \in X_b$ do not belong to $\mathcal{C}_2$. Moreover, isolated vertices of $\tilde{G}_a \cup \tilde{G}_b$ of form aw, with $a \in \{a_1, a_2\}$ do not belong to $\mathcal{C}_1$.*

Proof. Let uw be an isolated vertex of $\tilde{G}_a \cup \tilde{G}_b$. If $u \in X_a \cap X_b$, then, by Lemma 3.3.16, we know that u and w are not in between two adjacent vertices in linear order $<_2$. This implies that $uw \notin \mathcal{C}_2$. Now let $u = b_1$. By Proposition 3.3.10, we know that, for all $y_b \in N_Y(b)$ and for all $y \in Y \setminus N_Y(b_1)$,we have that $y_b <_2 y$, and thus $y_b <_2 w$. Moreover, for any $x <_2 b_1$, we know that $N_Y(x) \subseteq N_Y(b_1)$. This implies that b_1 and w are not in between two adjacent vertices in $<_2$, and thus $b_1w \notin \mathcal{C}_2$.

Let $a \in \{a_1, a_2\}$, and suppose that $u = a$. We know, by Proposition 3.3.10, that, for all $y_a \in N_Y(a)$ and for all $y \in Y \setminus N_Y(a)$, we have that $y_a <_2 y$, and thus $y_a <_2 w$. Moreover, by 1.2 of Definition 3.3.5, we know that for all $x \in X_a \cap X_b$, we have that $a <_1 x$. This implies that a and w are not in between two adjacent vertices in $<_1$, and thus $b_1w \notin \mathcal{C}_1$. □

Theorem 3.3.18. *Let G be as in Assumption 3.3.1, and $<_1$ and $<_2$ be as in Definition 3.3.5. Then relations $<_1$ and $<_2$ are linear orders which satisfy Equation 2.2.*

Proof. Let G be as in Assumption 3.3.1, and linear orders $<_1$ and $<_2$ be as in Definition 3.3.5. By Lemmas 3.3.11 and 3.3.12, we know that $<_1$ and $<_2$ are linear orders. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be completions of $<_1$ and $<_2$, respectively. By Corollary 3.3.14 and Lemma 3.3.15, we know that non-edges that are chords of a rigid pair belong to different completions. Moreover, by Corollary 3.3.17 we know that non-isolated vertices of $\tilde{G}_a \cup \tilde{G}_b$ belong to at most one completion $\mathcal{C}_1$ or $\mathcal{C}_2$. Also, $a_1b_1, a_2b_1 \notin \mathcal{C}_1$. This implies that $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, and we are done.

$\square$

3.4 Conditions of Theorem 3.1.9 can be checked in polynomial-time

In this section, for a $B_{a,b}$-graph G, as given in Assumption 3.1.3, we present an $\mathcal{O}(n^4)$ which checks the conditions of Theorem 3.1.9, where n is the order of the graph G. The rigid-free conditions (Condition (ii) of Theorem 3.1.9) are straightforward to check. However, the coloring condition (Condition (i) of Theorem 3.1.9) is much more complicated and needs a detailed coloring algorithm. Indeed, to check the coloring condition, we present an algorithm which obtains a proper 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$ if such a coloring exists, and otherwise the algorithm fails.

3.4.1 The Algorithms and their correctness

We first discuss the rigid-free conditions. Let G be a $\mathcal{B}_{a,b}$-graph. For a vertex $u \in X_a \cup X_b$ and a set $S \subseteq X_a \cup X_b$ define

$$V_S(u) \quad = \quad \{v \in V(\tilde{G}_a \cup \tilde{G}_b) : v = xy, x \in S \text{ and } y \in N_Y(u)\}.$$

Recall that a vertex $u \in V(G)$ is rigid-free with respect to $S \subseteq V(G)$ if there is no rigid pair $\{x_1y_1, x_2y_2\}$ such that $x_1, x_2 \in S$ and $y_1, y_2 \in N_Y(u)$.

Lemma 3.4.1. *Let H be a cobipartite graph and $S \subseteq V(H)$. Then $x \in V(H)$ is rigid-free with respect to S if and only if the induced graph $\tilde{H}[V_S(x)]$ is an empty graph.*

Proof. We prove by contraposition. Assume that $\tilde{H}[V_S(x)]$ is not empty. Then there exist x_1y_2 and x_2y_1 in the vertex set of $\tilde{H}[V_S(x)]$ such that $x_1y_2 \sim^* x_2y_1$. This implies that $\{x_1y_1, x_2y_2\}$ is a rigid pair with $x_1, x_2 \in S$ and $y_1, y_2 \in N_Y(x)$. Then x is not rigid-free with respect to S. The backward direction of the above discussion proves the other side of the lemma. $\square$

The following corollary is an immediate result of Lemma 3.4.1 and the definition of the rigid-free conditions. First recall that, Condition (ii) of Theorem 3.1.9 states that, for a graph G as in Assumption 3.1.3, the rigid-free conditions of Definition 3.1.4 hold.

Corollary 3.4.2. *Condition (ii) of Theorem 3.1.9 holds if and only if*

(1) Either $\tilde{G}_b[V_{X_b}(a_1)]$ or $\tilde{G}_a[V_{X_a}(b_1)]$ is an empty graph.

(2) If $N_Y(a_1) \subseteq N_Y(a_2)$ then either $\tilde{G}_b[V_{X_b}(a_2)]$ or $\tilde{G}_a[V_{X_a}(b_1)]$ is an empty graph.

(3) If there is a rigid pair $\{a_1y_1, a_2y_2\}$ then $\tilde{G}_b[V_{X_b}(a_1)]$ and $\tilde{G}_b[V_{X_b}(a_2)]$ are empty graphs. Moreover, if $N_Y(a) \not\subseteq N_Y(b_1)$, for all $a \in \{a_1, a_2\}$, then $\tilde{G}_a[V_{\{a_1,a_2\}}(b_1)]$ is an empty graph.

We now discuss the algorithm which checks Condition (i) of Theorem 3.1.9. First we give a description of the algorithm. The proofs and precise definitions will follow. Recall that Condition (i) requires the graph $\tilde{G}_a \cup \tilde{G}_b$ to admit a proper 2-coloring which satisfies the following conditions: (1) its associated relations $<_X$ and $<_Y$, as in Definition 3.1.6, form a proper bi-ordering, and (2) all vertices of $\mathcal{A}$ are colored red, and all the vertices of $\mathcal{B}$ are colored blue.

Since the desired 2-coloring must satisfy (2), the 2-coloring process starts with, coloring all vertices of $\mathcal{A}$ red, and all vertices of $\mathcal{B}$ blue. This will force the 2-coloring of any component of $\tilde{G}_a \cup \tilde{G}_b$ which contains a vertex of either $\mathcal{A}$ or $\mathcal{B}$. It is possible that the 2-coloring process fails at the very beginning, while extending the coloring of $\mathcal{A}$ and $\mathcal{B}$ to the components which contains a vertex of $\mathcal{A}$ or $\mathcal{B}$. Such failures occur if there is an even path between a vertex of $\mathcal{A}$ and a vertex of $\mathcal{B}$. In this case, we can conclude that G is not square geometric.

Suppose that we can successfully extend the coloring of $\mathcal{A}$ and $\mathcal{B}$ to any component that contains a vertex of $\mathcal{A}$ or $\mathcal{B}$. Moreover, suppose that $<_X$ and $<_Y$ are the relations of Definition 3.1.6 for the obtained partial 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$. We know that the obtained partial 2-coloring satisfies (2), and thus we only need to check whether it satisfies (1) *i.e.* if $<_X$ and $<_Y$ form a proper bi-ordering.

We saw in Remark 4 that if $<_X$ and $<_Y$ are partial orders, then they form a proper bi-ordering. Moreover, we have that $<_X$ is a partial order if and only if $<_Y$ is a partial order. This implies that if $<_X$ is a partial order then $(<_X, <_Y)$ is a proper bi-ordering. By Proposition 3.1.7, for any 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$, its associated relation

$<_X$ is reflexive and antisymmetric. Therefore, we only need to check the transitivity of the relation $<_X$. If there are vertices x_1, x_2, x_3 such that $x_1 <_X x_2$, $x_2 <_X x_3$, and $x_3 <_X x_1$ then from the results in the previous sections we can conclude that the graph is not square geometric. So the algorithm stops and returns "G is not square geometric". If there are vertices x_1, x_2, x_3 such that $x_1 <_X x_2$, $x_2 <_X x_3$, and x_1 and x_3 are not related in $<_X$ then to maintain transitivity we require $x_1 <_X x_3$. Therefore, for any rigid pair of the form $\{x_1y_1, x_3y_3\}$, we must color x_1y_3 red. Therefore, the transitivity of the ordering $<_X$ forces the color of the vertex x_1y_3 in $\tilde{G}_a \cup \tilde{G}_b$.

We continue this process until there are no more vertices whose color is forced by the transitivity constraint. At this point, either the whole graph $\tilde{G}_a \cup \tilde{G}_b$ is colored, or it is partially colored. If $\tilde{G}_a \cup \tilde{G}_b$ is completely colored then we have obtained a 2-coloring which satisfies Condition (i) of Theorem 3.1.9. If $\tilde{G}_a \cup \tilde{G}_b$ is partially colored, then we can extend the partial coloring to the whole graph such that the obtained 2-coloring satisfies Condition (i). Indeed, if $\tilde{G}_a \cup \tilde{G}_b$ is partially colored, then we prove that this partial 2-coloring is a closed partial 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$. Then, we use Algorithm 2.3.15 for cobipartite graphs to color the rest of the graph $\tilde{G}_a \cup \tilde{G}_b$, and obtain our desired 2-coloring which satisfies Condition (i) of Theorem 3.1.9.

We are now ready to present the algorithm. We first state the main part of the algorithm which is the forced-coloring process.

Algorithm 3.4.3. *[Forced 2-coloring process]*
Input: A partial 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$, $f_1 : V(\tilde{H}_1) \to \{red, blue\}$ with associated relations $<_X$ and $<_Y$ as in Definition 3.1.6, where $V(\tilde{H}_1)$ is a proper subset of $V(\tilde{G}_a \cup \tilde{G}_b)$, and the set Forced.Color $\subseteq V(\tilde{G}_a \cup \tilde{G}_b)$.
Output: A partial 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$, $f_2 : V(\tilde{H}_2) \to \{red, blue\}$, where $V(\tilde{H}_1)$ is a proper subset of $V(\tilde{H}_2)$,and the restriction of f_2 to $\tilde{H}_1$ is equal to f_1, and a new set Forced.Color $\subseteq V(\tilde{G}_a \cup \tilde{G}_b)$.

1 Repeat the following until Forced.Color is empty. Let u be in Forced.Color. Remove u from Forced.Color.

  1.1 If u is uncolored then color u red, and perform a BFS with root vertex u, and color the component which contains u with red and blue.

1.2 If u is colored blue, then stop the algorithm and return "Algorithm fails".

1.3 If u is colored red, then go to next step.

2 Consider the 2-coloring obtained from 1 and obtain its associated relations $<_X$ and $<_Y$, as in Definition 3.1.6.

3 Check the transitivity of the obtained $<_X$:

3.1 If there are vertices $x_1, x_2, x_3 \in V(G)$ with $x_1 <_X x_2$, $x_2 <_X x_3$, and $x_3 <_X x_1$, then stop the algorithm and return "G is not square geometric."

3.2 If there are vertices $x_1, x_2, x_3 \in V(G)$ with $x_1 <_X x_2$, $x_2 <_X x_3$, and x_1 and x_3 are not related in $<_X$, then add x_1y_3 to Forced.Color, where $x_1y_3 \sim^* x_3y_1$ is an edge of $\tilde{G}_a \cup \tilde{G}_b$.

Proposition 3.4.4. *Let G be a $B_{a,b}$-graph as in Assumption 3.1.3. If Algorithm 3.4.3 succeeds, then the output of the algorithm is a partial 2-coloring of the graph $\tilde{G}_a \cup \tilde{G}_b$, which satisfies the following conditions:*

(1) each component is either completely colored, or completely uncolored.

(2) its associated relations $<_X$ and $<_Y$, as in Definition 3.1.6, are reflexive and antisymmetric. Moreover, for $x_1, x_2, x_3 \in X$ with $x_1 <_X x_2$ and $x_2 <_X x_3$, either the transitivity holds i.e. *$x_1 <_X x_3$ or x_1 and x_3 are incomparable in $<_X$, and for rigid pair $\{x_1y_1, x_3y_3\}$, x_1y_3 is in the output set Forced.Color.*

Proof. Assume that Algorithm 3.4.3 succeeds. Let f_2 be the partial 2-coloring obtained from Algorithm 3.4.3. Consider the input set Forced.Color. In step 1, all the vertices of Forced.Color are colored red, and all the components which contain a colored vertex are colored. This implies that when Algorithm 3.4.3 stops any component which contains a vertex from Forced.Color is completely colored, and all the other components remain uncolored. Therefore, the obtained partial 2-coloring satisfies (1).

We now prove that f_2 satisfies (2). First note that by Proposition 3.1.7, we know that for any 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$, the associated relations $<_X$ and $<_Y$, as in Definition 3.1.6, are reflexive and antisymmetric. This implies that the ordering $<_X$ associated to f_2 is reflexive and antisymmetric. Now suppose that there are vertices $x_1, x_2, x_3 \in V(G)$ such that $x_1 <_X x_2$ and $x_2 <_X x_3$. There are three possible cases:

- If $x_1 <_X x_3$ then $<_X$ is transitive on x_1, x_2, and x_3.

- If $x_3 <_X x_1$ then $<_X$ is not transitive, and the algorithm fails at 3.1. But we assume that the algorithm succeeds, and thus this case does not occur.

- If x_1 and x_3 are not related in $<_X$ then for rigid pair $\{x_1y_1, x_3y_3\}$ the chord x_1y_3 is added to Forced.Color in 3.2.

This finishes the proof. $\square$

Proposition 3.4.5. *Suppose that G is a $B_{a,b}$-graph as in Assumption 3.1.3. Consider Algorithm 3.4.3 with input: a partial 2-coloring $f_1 : V(\tilde{H}_1) \to \{red, blue\}$, where $V(\tilde{H}_1)$ is a proper subset of $V(\tilde{G}_a \cup \tilde{G}_b)$, and a set Forced.Color $\subseteq V(\tilde{G}_a \cup \tilde{G}_b)$. If there is a closed partial 2-coloring $g : V(\tilde{G}_a \cup \tilde{G}_b) \to \{red, blue\}$ which agrees with f_1 on $V(\tilde{H}_1)$, and colors the vertices of the input Forced.Color red, then*

(1) Algorithm 3.4.3 succeeds, and the partial 2-coloring obtained from Algorithm 3.4.3, denoted by f_2, agrees with g on the set of vertices colored in f_2.

(2) the vertices in the output Forced.Color are all colored red in g.

Proof. Suppose that the 2-colorings f_2 and g agree on the set of vertices of $V(\tilde{H}_1)$, and input Forced.Color *i.e.* for all $u \in V(\tilde{H}_1)$, $f_2(u) = g(u)$, and both f and g color vertices of Forced.Color red. Suppose that $x_1y_2 \in V(\tilde{G}_a \cup \tilde{G}_b)$ is colored in f_2 then we have the following cases:

- $x_1y_2 \in V(\tilde{H}_1)$. Then by assumption $f_2(x_1y_2) = g(x_1y_2)$.

- $x_1y_2 \in$ Forced.Color. Then in step 1 of the algorithm we have that $f_2(x_1y_2) =$ red. By assumption, we know that g colors all vertices of Forced.Color red, and thus $f_2(x_1y_2) = g(x_1y_2)$.

- There is a path between x_1y_2 and a vertex of the Forced.Color. Then, in step 1 of the algorithm, we obtain $f_2(x_1y_2) =$ blue, if the path is of odd length, and $f(x_1y_2) =$ red, if the path is of even length. Since g is a proper 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$, and for any vertex $v \in$ Forced.Color, we have that $f_2(v) = g(v)$ then $f_2(x_1y_2) = g(x_1y_2)$.

Therefore, the coloring f_2 and g agree on the vertices colored in f.

Now consider the output Forced.Color, and let $x_1y_3 \in$ Forced.Color. Let $<_X$ be the relation associated to f_2, the 2-coloring obtained in step 1 of Algorithm 3.4.3. Then there are rigid pairs $\{x_1y_1, x_2y_2\}$, $\{x_2y_2, x_3y_3\}$, and $\{x_1y_1, x_3y_3\}$ such that $f_2(x_1y_2) = f_2(x_2y_3) =$ red, *i.e.* $x_1 <_X x_2$, $x_2 <_X x_3$, and chords of $\{x_1y_1, x_3y_3\}$ are not colored *i.e.* x_1 and x_3 are not related in $<_X$. Since x_1y_2 and x_2y_3 are colored in f_2, we know that $f_2(x_1y_2) = g(x_1y_2)$ and $f_2(x_2y_3) = g(x_2y_3)$. Now let $<_X$ be the ordering associated to the coloring g. Then $x_1 <_X x_2$ and $x_2 <_X x_3$, and since the relation $<_X$ corresponding to g is a partial order, we have that $x_1 <_X x_3$. This implies that $g(x_1y_3) =$ red. This finishes the proof. □

The following corollary is a direct consequence of Proposition 3.4.5.

Corollary 3.4.6. *If Algorithm 3.4.3 fails, then there exists no partial 2-coloring extending f_1 which colors the input set Forced.Color red.*

We now state the algorithm which checks Condition (i) of Theorem 3.1.9.

Algorithm 3.4.7. *[A proper 2-coloring of a type-1, or type-2 $\mathcal{B}_{a,b}$-graphs whose corresponding relations $<_X$ and $<_Y$ as in Definition 3.1.6 are partial orders.]*
Input: A type-1, or type-2 $\mathcal{B}_{a,b}$-graph G as in Assumption 3.1.3 with bipartition $X_a \cup X_b$ and Y.
Output: Either partial orders $<_X$ and $<_Y$ such that $(<_X, <_Y)$ is a proper bi-ordering of G or "G is not square geometric".

1. Construct the chord graphs $\tilde{G}_a, \tilde{G}_b$. Form induced graphs $G_a = G[X_a \cup Y]$ and $G_b = G[X_b \cup Y]$.

1.1 Repeat part 1 of Algorithm 2.3.20 for cobipartite graph G_a, if the output is "G_a is not square geometric" then stop the algorithm and return "G is not square geometric since G_a is not square geometric". If not, then return $\tilde{G}_a$, its components.

1.2 Repeat part 1 of Algorithm 2.3.20 for cobipartite graph G_b, if the output is "G_b is not square geometric" then stop the algorithm and return "G is not square geometric since G_b is not square geometric". If not, then return $\tilde{G}_b$, its components.

2. Find a closed partial 2-coloring for $\tilde{G}_a \cup \tilde{G}_b$ such that all vertices of $\mathcal{A}$ are colored red, and all vertices of $\mathcal{B}$ are colored blue:
Set Forced.Color equal to $= \mathcal{A} \cup N(\mathcal{B})$, where $N(\mathcal{B})$ is the set of all vertices of $\tilde{G}_b$ which have neighbors in $\mathcal{B}$. Repeat Algorithm 3.4.3, with input Forced.Color, G, $\tilde{G}_a \cup \tilde{G}_b$ and its components, until the algorithm returns a set Forced.Color that is empty.

3. Use Algorithm 2.3.15 to color any uncolored components of $\tilde{G}_a \cup \tilde{G}_b$:
Let $\tilde{H}$ be the induced subgraph of $\tilde{G}_a$, consisting of colored components of $\tilde{G}_a$. Set $f_1 : V(\tilde{H}) \to \{\text{red, blue}\}$ to be the restriction to $\tilde{G}_a$ of the closed partial 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$, obtained in step 2. Apply Algorithm 2.3.15 with input f_1. Repeat Algorithm 2.3.15 until all non-isolated vertices of $\tilde{G}_a$ are colored.

4. Consider the 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$ obtained in step 3, and find its associated relations $<_X$ and $<_Y$ as in Definition 3.1.6.

Note that step 1 of Algorithm 3.4.7 checks if $\tilde{G}_a$ and $\tilde{G}_b$ are 2-colorable (or equivalently G_a and G_b are square geometric). This step is not an essential step for the algorithm. More precisely, if one of the graphs, $\tilde{G}_a$ and $\tilde{G}_b$, is not 2-colorable, then we can catch this in step 2 of the algorithm *i.e.* step 2 fails. But step 2 could fail, even when both $\tilde{G}_a$ and $\tilde{G}_b$ are 2-colorable. So we only keep step 1 to know the reason of the failure of the 2-coloring process.

The rest of this subsection is devoted to the proof of correctness of Algorithm 3.4.7. We prove that Condition (i) of Theorem 3.1.9 holds if and only if Algorithm 3.4.7 succeeds. First we state the required results for the proof of correctness of Algorithm 3.4.7. The following proposition provides us with some information about the components of the graph $\tilde{G}_a \cup \tilde{G}_b$.

Proposition 3.4.8. *Let G be a $B_{a,b}$-graph with cobipartite subgraphs G_a and G_b. Suppose that C is a component of $\tilde{G}_a \cup \tilde{G}_b$, and C contains no a-vertices and no b-vertices i.e. $V(C) \subseteq V(\tilde{G}_a \cap \tilde{G}_b)$. Then C is a component of $\tilde{G}_a$ and $\tilde{G}_b$.*

Proof. Suppose that C is a component of $\tilde{G}_a \cup \tilde{G}_b$ and C contains no a-vertices and no$b$-vertices. Since $V(C) \subseteq V(\tilde{G}_a \cap \tilde{G}_b)$, C is a subgraph of $\tilde{G}_a$. Now suppose that there exists a vertex $u \in V(\tilde{G}_a) \setminus V(C)$ such that there exists a path P in $\tilde{G}_a$ between

u and a vertex $v \in V(C)$. Since $u \notin V(C)$ then either u is an a-vertex or the path P contains an a-vertex. This implies that C contains an a-vertex, which is not true. Therefore, C is a component of $\tilde{G}_a$. An analogous discussion proves that C is a component of $\tilde{G}_b$. $\square$

The following lemma states that Condition (i) of Theorem 3.1.9 holds if steps 1 and 2 of Algorithm 3.4.3 succeed.

Lemma 3.4.9. *If steps 1 and 2 of Algorithm 3.4.7 succeed, then $\tilde{G}_a \cup \tilde{G}_b$ has a closed partial coloring which colors the whole graph $\tilde{G}_a \cup \tilde{G}_b$, and moreover all vertices of $\mathcal{A}$ are colored red, and all vertices of $\mathcal{B}$ are colored blue.*

Proof. Suppose that steps 1 and 2 of Algorithm 3.4.7 succeed. Since step 2 does not fail then all iterartions of Algorithm 3.4.3 succeed. In the first iteration all vertices of $\mathcal{A}$ are colored red, and all vertices of $\mathcal{B}$ are colored blue. By Proposition 3.4.4, we know that the output of step 2 is a closed partial 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$, we denote it by g_1. Moreover, by Proposition 3.4.8, we know that the uncolored components of $\tilde{G}_a \cup \tilde{G}_b$ in g_1 are components of $\tilde{G}_a$.

The restriction of g_1 to $V(\tilde{G}_a)$, f_1, is a closed partial 2-coloring for the graph $\tilde{G}_a$. In step 3 of Algorithm 3.4.7, we apply Algorithm 2.3.15 with the input f_1, and we repeat Algorithm 2.3.15 until the whole graph $\tilde{G}_a$ is colored. By Lemma 2.3.17, we know that the obtained 2-coloring, g_2, is a closed partial 2-coloring of $\tilde{G}_a$.

Now suppose that $g : V(\tilde{G}_a \cup \tilde{G}_b) \to \{\text{red, blue}\}$ is a proper 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$ which agrees with g_2 on $V(\tilde{G}_a)$, and agrees with g_1 on $V(\tilde{G}_b)$. Then g is a closed partial coloring of $\tilde{G}_a \cup \tilde{G}_b$ which colors the whole graph $\tilde{G}_a \cup \tilde{G}_b$, and moreover it colors all the vertices of $\mathcal{A}$ red, and colors all the vertices of $\mathcal{B}$ blue. $\square$

The next proposition provides a sufficient condition for Algorithm 3.4.3 to succeed.

Proposition 3.4.10. *Suppose that G is a $B_{a,b}$-graph as in Assumption 3.1.3. Consider step 2 of Algorithm 3.4.7 with input: the graph G, the graph $\tilde{G}_a \cup \tilde{G}_b$, and set Forced.Color $= \mathcal{A} \cup N(\mathcal{B})$. If there is a closed partial 2-coloring $g : V(\tilde{G}_a \cup \tilde{G}_b) \to$ $\{$red, blue$\}$ which colors the vertices of the input Forced.Color red, then step 2 of Algorithm 3.4.7 succeeds, and the obtained partial 2-coloring, denoted by f, agrees with g on the set of vertices colored in f.*

Proof. Suppose that g is a closed partial 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$ which colors all the vertices of the input set Forced.Color red. Step 2 of Algorithm 3.4.7 is a repetition of Algorithm 3.4.3. Then, by induction and Proposition 3.4.5, we have that the partial 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$ obtained at each iteration of step 2 of Algorithm 3.4.7 agrees with g on the set of vertices of $\tilde{G}_a \cup \tilde{G}_b$ that are colored in that iteration. Then, each iteration of step 2 succeeds, as g is a proper 2-coloring of $\tilde{G}_a \cup \tilde{G}_b$. This finishes the proof. $\square$

We are now ready to prove the correctness of Algorithm 3.4.7. First recall that Condition (i) of Theorem 3.1.9 states that, for a graph G as in Assumption 3.1.3, there is a closed partial coloring $f : V(\tilde{G}_a \cup \tilde{G}_b) \to \{\text{red, blue}\}$ such that all vertices of $\mathcal{A}$ are colored red and all vertices of $\mathcal{B}$ are colored blue.

Lemma 3.4.11. *Suppose that Condition (ii) of Theorem 3.1.9 holds. Then Condition (i) of Theorem 3.1.9 holds if and only if Algorithm 3.4.7 succeeds.*

Proof. First we prove that if Condition (i) of Theorem 3.1.9 holds, then Algorithm 3.4.7 succeeds. So suppose that $f : V(\tilde{G}_a \cup \tilde{G}_b) \to \{\text{red, blue}\}$ is a closed partial 2-coloring that colors all vertices of $\mathcal{A}$ red and all vertices of $\mathcal{B}$ blue.

We prove that step 1 of Algorithm 3.4.7 succeeds. Since G_a and G_b are induced subgraphs of G, and G is square geometric then G_a and G_b are square geometric. Then by Corollary 2.2.7, we know that $\tilde{G}_a$ and $\tilde{G}_b$ are bipartite. Therefore, step 1 of Algorithm 3.4.7 succeeds, and returns the chord graphs $\tilde{G}_a$ and $\tilde{G}_b$, and their components.

Now we prove that step 2 does not fail. We know that f is a closed partial coloring of $\tilde{G}_a \cup \tilde{G}_b$ with all the vertices of $\mathcal{A}$ colored red, and all the vertices of $\mathcal{B}$ colored blue. Then by Proposition 3.4.10, we know that step 2 of Algorithm 3.4.7 does not fail, and moreover the partial coloring of $\tilde{G}_a \cup \tilde{G}_b$ obtained form step 2, g, agrees with f on the set of colored vertices in 2. Since steps 1 and 2 succeeds then, by Lemma 3.4.9, we know that the Algorithm 3.4.7 succeeds.

Now suppose that Algorithm 3.4.7 succeeds. Then steps 1 and 2 of Algorithm 3.4.7 succeeds. By Lemma 3.4.9, we know that the output Algorithm 3.4.7 is a closed partial coloring of $\tilde{G}_a \cup \tilde{G}_b$ which colors the whole graph $\tilde{G}_a \cup \tilde{G}_b$, and moreover all vertices of $\mathcal{A}$ are colored red, and all vertices of $\mathcal{B}$ are colored blue. Therefore, Condition (i) of Theorem 3.1.9 holds. $\square$

3.4.2 Complexity

We now prove that, for a $B_{a,b}$-graph as in Assumption 3.1.3, the conditions of Theorem 3.1.9 can be checked in $\mathcal{O}(n^4)$ steps, where n is the order of the graph G. Note that since G_a and G_b are subgraphs of G their order is at most n.

1. rigid-free conditions. Suppose that G is a type-1 or a type-2 $B_{a,b}$-graph. Then $\tilde{G}_b[V_{X_b}(a_1)]$, $\tilde{G}_b[V_{X_b}(a_2)]$, and $\tilde{G}_a[V_{X_a}(b_1)]$ are chord graphs of cobipartite graphs $G_b[X_b \cup N_Y(a_1)]$, $G_b[X_b \cup N_Y(a_2)]$, and $G_a[X_a \cup N_Y(b_1)]$, respectively. By Corollary 3.4.2, to check Condition (ii) of Theorem 3.1.9, we have to form the chord graphs $\tilde{G}_b[V_{X_b}(a_1)]$, $\tilde{G}_b[V_{X_b}(a_2)]$, and $\tilde{G}_a[V_{X_a}(b_1)]$. Moreover, we know from the cobipartite algorithm, Algorithm 2.3.20, that the chord graph of a cobipartite graph of order n can be constructed in $\mathcal{O}(n^4)$ steps. Therefore, the rigid-free conditions (Condition (ii) of Theorem 3.1.9) can be checked in $\mathcal{O}(n^4)$ steps.

2: The coloring condition (Condition (i) of Theorem 3.1.9). By Lemma 3.4.11, we know that Condition (i) of Theorem 3.1.9 holds if and only if Algorithm 3.4.7 succeeds. Therefore, we need to determine the complexity of Algorithm 3.4.7.

Step 1 of Algorithm 3.4.7 applies part 1 of Algorithm 2.3.20 to cobipartite graphs G_a and G_b. Therefore, the complexity of step 1 is $\mathcal{O}(n^4)$.

Step 2 of Algorithm 3.4.7 is a repetition of Algorithm 3.4.3. Step 1 of Algorithm 3.4.3 contains a BFS on components of the graph $\tilde{G}_a \cup \tilde{G}_b$ which contain a vertex of Forced.Color. Once a component is colored in this step, we never perform a BFS on it through any other iterations of Algorithm 3.4.3 in step 2 of Algorithm 3.4.7. Therefore, when step 2 of Algorithm 3.4.7 stops, we performed BFS on each component of $\tilde{G}_a \cup \tilde{G}_b$ at most once. This implies that the complexity of this part is $\mathcal{O}(n^4)$. In step 3 of Algorithm 3.4.3, we check the transitivity of the new relations of $<_X$ (obtained in step 2 of Algorithm 3.4.3). Once two relations $x_1 <_X x_2$ and $x'_1 <_X x'_2$ are compared for the transitivity, we never compare them again through the algorithm. Therefore, when step 2 of Algorithm 3.4.7 stops, we compared each relation $x_1 <_X x_2$ with at most all relations $x <_X x'$ of $<_X$. Since there are at most n^2 relations $x <_X x'$ then the complexity of these steps is $\mathcal{O}(n^4)$.

Step 3 of Algorithm 3.4.7 is the same as Algorithm 2.3.15 for cobipartite graphs. Since the complexity of Algorithm 2.3.15 is $\mathcal{O}(n^4)$ then the complexity of step 3 of Algorithm 3.4.7 is $\mathcal{O}(n^4)$. This implies that the complexity of Algorithm 3.4.7 is $\mathcal{O}(n^4)$.

Chapter 4

Uniform Linear Embedding of Random Graphs

A linear random graph is a graph whose vertices can be embedded in metric space $([0,1], |.|_\infty)$ such that the probability of a link between vertices decreases as the metric distance increases. Recall that, for $w \in \mathcal{W}_0$, a w-random graph with n vertices, denoted by $G(n,w)$, is constructed by the following process: a set P of n points is chosen uniformly from the metric space $[0,1]$. Any two points $x, y \in P$ are then linked with probability $w(x,y)$. As we mentioned in the introduction, for a w-random graph to correspond to the notion of a linear random graph w must be diagonally increasing *i.e.* for $x < y$ the value of $w(x,y)$ decreases whenever x decreases or y increases. In a previous study, [16], a parameter Γ has been introduced which recognizes diagonally increasing functions w. Namely, $\Gamma(w) = 0$ if and only if w is diagonally increasing. Consequently, $G(n,w)$ has a linear embedding if and only if $\Gamma(w) = 0$.

As we saw in the introduction, for a w-random graph, the link probability function w may depend on the geometric position of vertices on $[0,1]$, as well as their metric distance. In this chapter we investigate which link probability functions, w, are intrinsically uniform *i.e.* are only a function of the metric distance of vertices. We say such functions, w, admit a *uniform linear embedding.*

Throughout this chapter we assume that $w \in \mathcal{W}_0$ is a diagonally increasing function with finite range. We give necessary and sufficient conditions for the existence of uniform linear embeddings for functions in $\mathcal{W}_0$. We remark that understanding the structure and behaviour of functions in $\mathcal{W}_0$ is important, due to their deep connection to the study of graph sequences. Functions in $\mathcal{W}_0$ are referred to as *graphons*, and they play a crucial role in the emerging theory of limits of sequences of dense graphs as developed through work of several authors (see [6], [7], [8], [33], and also the book [32] and the references therein). This theory gives a framework for convergence of sequences of graphs of increasing size that exhibit similar structure. Structural similarity in this theory is seen in terms of *homomorphism densities*. For a given graph

G, the homomorphism counts are the number of homomorphisms from each finite graph F into G, and homomorphism densities are normalized homomorphism counts. It was shown in [7] that the homomorphism densities of the random graph $G(n, w)$ *asymptotically almost surely* approach those of the function w. Therefore, w encodes the structure exhibited by the random graph model $G(n, w)$.

4.1 Definitions and Main Result

In this section we present our main result which is the necessary and sufficient conditions for a w-random graph with finite range to admit a uniform linear embedding. We consider only functions of finite range. Such functions have a specific structure, as formulated in the following definition.

Definition 4.1.1. *Let $w \in \mathcal{W}_0$ be a diagonally increasing function with* $\mathrm{range}(w) = \{\alpha_1, \dots, \alpha_N\}$, *where $\alpha_1 > \alpha_2 > \dots > \alpha_N$. For $1 \leq i \leq N$, the upper boundary r_i and the lower boundary ℓ_i are functions from $[0, 1]$ to $[0, 1]$ defined as follows. Fix $x \in [0, 1]$. Then*

$$\ell_i(x) = \inf\{y \in [0, 1] : \, w(x, y) \geq \alpha_i\},$$

and

$$r_i(x) = \sup\{y \in [0, 1] : \, w(x, y) \geq \alpha_i\}.$$

Also, for $1 \leq i < N$, define $r_i^* = r_i|_{[0,\ell_i(1)]}$ and $l_i^* = \ell_i|_{[r_i(0),1]}$. Note that r_i^* has domain $[0, \ell_i(1)]$ and range $[r_i(0), 1]$, and ℓ_i^* has domain $[r_i(0), 1]$ and range $[0, \ell_i(1)]$.

We now briefly explain the reason we defined the functions r_i^* and ℓ_i^* in the above definition. First note that the domain of the function r_i^*, $[0, \ell_i(1)]$, is where the function r_i is increasing (this is proved in Proposition 4.2.2). Similarly, the domain of the function ℓ_i^*, $[r_i(0), 1]$, is where the function ℓ_i is increasing. We will see later in Remark 9 that if we assume that the boundary functions r_i and ℓ_i are continuous then they are strictly increasing on $[0, \ell_i(1)]$ and $[r_i(0), 1]$, respectively. Therefore, the restriction of r_i to $[0, \ell_i(1)]$, r_i^*, and the restriction of ℓ_i to $[r_i(0), 1]$, ℓ_i^*, are invertible functions. The invertibility of boundary functions is essential for our discussions in this thesis.

We can see from Definition 4.1.1 that the function w is determined by the boundary points where the function changes values. See Figure 4.1.

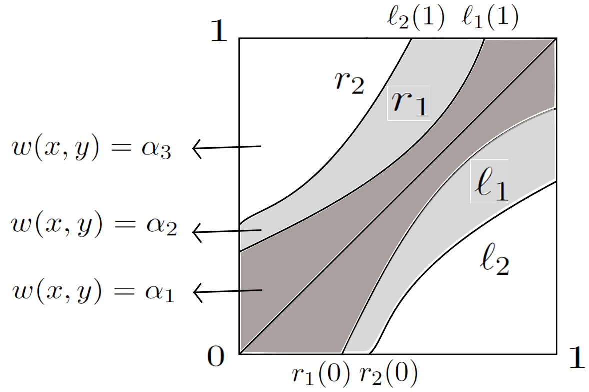


Figure 4.1: An example of a three-valued, diagonally increasing function $w \in \mathcal{W}_0$. The dark grey area is where w equals α_2, the light grey area is where w equals α_2, elsewhere w equals $\alpha_3 = 0$. The functions ℓ_1, r_1, and ℓ_2, r_2 form the boundaries of the dark grey area, and the light grey area, respectively.

Remark 7. *Let w be a finite-valued diagonally increasing function with boundaries r_i and ℓ_i, as in Definition 4.1.1. Then we have $w(x,y) \geq \alpha_i$ if $y \in (\ell_i(x), r_i(x))$. Moreover, $w(x,y) < \alpha_i$ whenever $y \in [0, \ell_i(x)) \cup (r_i(x), 1]$. Therefore w is defined by its boundaries, except the case where $y = r_i(x)$ or $y = \ell_i(x)$. In this case, both $w(x,y) \geq \alpha_i$ and $w(x,y) < \alpha_i$ are possible. The set $\{(x,y) \in [0,1]^2 | y \in \{\ell_i(x), r_i(x)\}\}$ is the set of points on the graph of r_i and ℓ_i. Therefore, the set $\{(x,y) \in [0,1]^2 | y \in \{\ell_i(x), r_i(x)\}\}$ is a curve in $[0,1]^2$, and thus is a set of measure zero in $[0,1]^2$. We modify the definition of w on this set of measure zero, and assume $w(x, r_i(x)) = w(x, \ell_i(x)) = \alpha_i$. Therefore from now on, without loss of generality, we assume the boundary functions ℓ_i, r_i, $1 \leq i \leq N$ satisfy the following property:*

$$w(x,y) \geq \alpha_i \text{ if and only if } \ell_i(x) \leq y \leq r_i(x). \tag{4.1}$$

Note also that $r_N(x) = 1$ for all x. Therefore, we usually only consider the boundary functions r_i, ℓ_i for $1 \leq i < N$.

We now adjust the definition of uniform linear embedding, Definition 1.2.6, for finite-valued functions $w \in \mathcal{W}_0$.

Definition 4.1.2. *Let $w \in \mathcal{W}_0$ be a diagonally increasing function of finite range, defined as in Equation (4.1). Then w has a uniform linear embedding if there exists a measurable injection $\pi : [0,1] \to \mathbb{R}$ and real numbers $0 < d_1 < d_2 < \cdots < d_{N-1}$ so* *that, for all* $(x, y) \in [0,1]^2$,

$$w(x,y) = \begin{cases} \alpha_1 & \text{if } |\pi(x) - \pi(y)| \leq d_1, \\ \alpha_i & \text{if } d_{i-1} < |\pi(x) - \pi(y)| \leq d_i \text{ and } 1 < i < N, \\ \alpha_N & \text{if } |\pi(x) - \pi(y)| > d_{N-1}. \end{cases} \tag{4.2}$$

We call $d_1, d_2, \ldots, d_{N-1}$ *the parameters of the uniform linear embedding* π.

As we saw earlier in Definition 4.1.1, the finite-valued diagonally increasing functions are completely determined by their boundary functions. Thus, to study functions which admit a uniform linear embedding, we need to study the boundary functions and their properties. In our study, we only consider functions with *well-separated* boundaries.

Definition 4.1.3. *Let* w *be as in Definition 4.1.2. Then boundaries* r_i, ℓ_i, $1 \leq i \leq N-1$ *are well-separated if* r_i^* *and* ℓ_i^* *are continuous, and have positive distances from the diagonal, and from each other. Precisely, there exists* $\epsilon > 0$ *so that, for all* i, j, *and for all* $x \in [0, \ell_i(1)]$, $r_i(x) - x \geq \epsilon$, *and* $|r_j(x) - r_i(x)| \geq \epsilon$.

If a function w is well-separated then $r_1(0) > 0$ and $\ell_1(1) < 1$. Throughout this chapter, we will assume that w is well-separated. We will see in Remark 8 how removing the well-separated condition will affect the existence of a uniform linear embedding. Let us now have a look at an example of a function w which has no uniform linear embedding. This gives us insight into the most important components of the necessary and sufficient conditions for a function w to admit a uniform linear embedding.

Example 9. *Let* w *be a diagonally increasing* $\{\alpha_1, \alpha_2, \alpha_3\}$*-valued function where* $\alpha_1 > \alpha_2 > \alpha_3 = 0$ *and* w *has upper boundaries* r_1 *and* r_2 *as in Figure 4.2.*

$$r_1(x) = \begin{cases} x + \frac{1}{10} & x \in [0, \frac{9}{10}] \\ 1 & x \in [\frac{9}{10}, 1] \end{cases}$$

$$r_2(x) = \begin{cases} 2x + \frac{1}{8} & x \in [0, \frac{7}{16}] \\ 1 & x \in [\frac{7}{16}, 1] \end{cases}$$

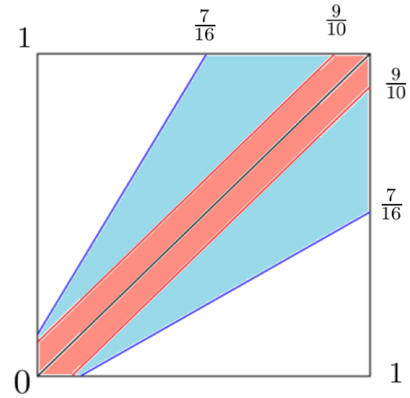


Figure 4.2: A three-valued, diagonally increasing function $w \in \mathcal{W}_0$. The red area is where w equals α_1, the blue area is where w equals α_2, elsewhere w equals α_3. The functions r_1, and r_2 form the boundaries of the red area, and the blue area, respectively.

Define sequences $\{x_i\}_{i\geq 0}$ and $\{y_j\}_{j\geq 0}$ as follows. Let $x_0 = 0$ and $x_i = r_1^i(0) = r_1(x_{i-1}) = \frac{i}{10}$ for $1 \leq i \leq 9$. Also let $y_0 = 0$ and $y_j = r_2^j(0) = r_2(x_{j-1}) = \frac{2^j-1}{8}$ for $1 \leq j \leq 3$. See Figure 4.3.

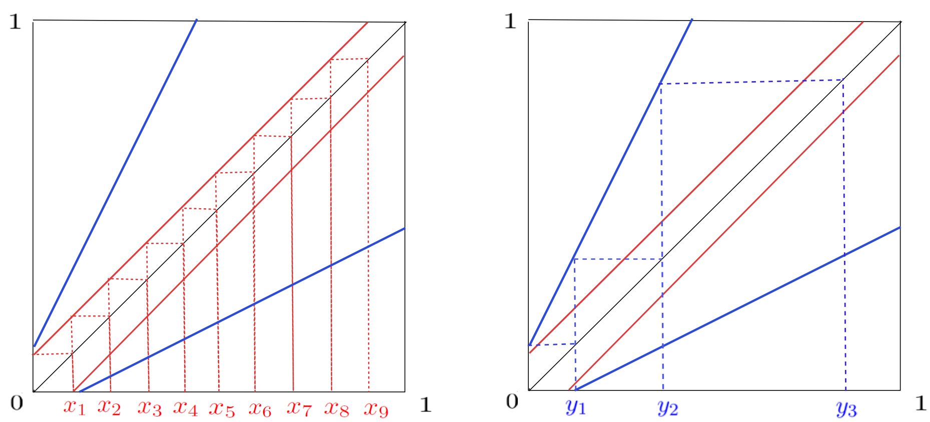


Figure 4.3: The points obtained by repeated applications of r_1 and r_2 to 0.

Suppose that w admits a uniform linear embedding π with parameters $d_2 > d_1 > 0$. We will see later (e.g. Lemma 4.3.4) that π can be assumed to be strictly increasing and $\pi(0) = 0$. The image under π of all points x_i and y_i are almost completely determined. For example, we know that $w(0, x_1) = \alpha_1$, $w(x_1, x_2) = \alpha_1$, and $w(0, x_2) = 0$. Then, by Definition 4.1.2, we have that $|\pi(0) - \pi(x_1)| \leq d_1$, $|\pi(x_1) - \pi(x_2)| \leq d_1$, and $|\pi(0) - \pi(x_2)| > d_1$. This, together with $\pi(0) = 0$, implies that $0 < \pi(x_1) \leq d_1$

and $d_1 < \pi(x_2) \leq 2d_1$. *Precisely, by Definition 4.1.2, we must have that*

$$(i-1)d_1 < \pi(x_i) \leq id_1 \quad \text{and} \quad (j-1)d_2 < \pi(y_j) \leq jd_2 \tag{4.3}$$

Equation 4.3, together with the fact that $x_8 < y_3$, *implies that* $\pi(x_2) \leq 2d_1$ *and* $7d_1 < \pi(x_8) < \pi(y_3) \leq 3d_2$. *Also, since* $x_2 > y_1 = r_2(0)$, *we have that* $\pi(x_2) > d_2$. *This gives contradicting restrictions on* d_1 *and* d_2, *and thus the function* w *has no uniform linear embedding.*

The reason for non-existence of a uniform linear embedding here, in contrast to the fact that the boundary functions behave nicely, is the uneven distribution of points obtained from repeated application of r_1 *and* r_2 *to* 0. *More precisely in the interval* $[y_1, y_2]$ *there are two points of the sequence* $\{x_i\}_{i\geq 0}$ *while in the interval* $[y_2, y_3]$ *there are five of them. See Figure 4.4. This does not allow the function* w *to have a uniform linear embedding.*

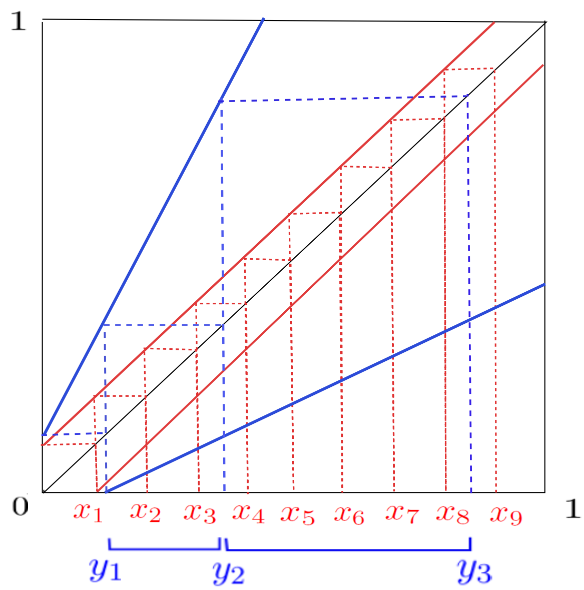


Figure 4.4: The points obtained by repeated applications of r_1 and r_2 to 0.

As we saw in Example 9, the points $\{x_i\}_{i=1}^{9}$ and $\{y_i\}_{i=1}^{3}$ played an important role in the non-existence of a uniform linear embedding for w. Indeed, these points are members of an important set of points called *constrained points* (Definition 4.1.5). The set of constrained points obtain from repeated applications of the boundary functions. This suggests that the set of constrained points and the boundary functions are essential to determine whether a uniform linear embedding exists.

We now proceed with some definitions, which formulate the properties of boundary functions. First note that the domain and range of $\ell_i^* : [r_i(0), 1] \to [0, \ell_i(1)]$ and $r_i^* : [0, \ell_i(1)] \to [r_i(0), 1]$ are closed intervals. Therefore, the domain and range of any composition of these functions are either closed intervals or an empty set.

Definition 4.1.4. *Let w be a finite-valued diagonally increasing function with boundaries r_i, ℓ_i as in Definition 4.1.1. Then $f_1 \circ \ldots \circ f_k$ is a* legal composition*, if each f_i belongs to $\{r_j^*, \ell_j^* : \ j = 1, \ldots, N-1\}$ and $\mathrm{dom}(f_1 \circ \ldots \circ f_k) \neq \emptyset$. The* signature *of the legal composition $f_1 \circ \ldots \circ f_k$ is the $(N-1)$-tuple $(m_1, \ldots, m_{N-1})$, where m_i is the number of occurrences of r_i^* minus the number of occurrences of ℓ_i^* in the composition. We use Greek letters such as $\phi, \psi, \ldots$ to denote legal compositions.*

Note that a legal composition is a function, which we denote by legal function. The reason we present legal functions as in Definition 4.1.4 is that in our approach we use the properties of legal function as a composition of boundary functions. The point is that two legal compositions may be identical as functions, but have different signatures. This can happen since legal compositions with different presentations may have different signatures, but still represent the same function. Below is the formal definition of constrained points.

Definition 4.1.5. *Let $w \in \mathcal{W}_0$ be a diagonally increasing function with finite range. Keep notations as in Definition 4.1.1, and define*

$$\begin{aligned} \mathcal{P} &= \{\phi(0) : \ \phi \textit{ is a legal composition with } 0 \in \mathrm{dom}(\phi)\}, \\ \mathcal{Q} &= \{\psi(1) : \ \psi \textit{ is a legal composition with } 1 \in \mathrm{dom}(\psi)\}. \end{aligned}$$

We refer to $\mathcal{P} \cup \mathcal{Q}$ as the set of constrained points *of w.*

The following proposition shows that a two-valued diagonally increasing function with well-separated boundaries always admits a uniform linear embedding. Once again, throughout the proof of the proposition, we will see the essential role of constrained points in the existence of uniform linear embeddings.

Proposition 4.1.6. *Let w be a well-separated two-valued diagonally increasing function with upper and lower boundaries r, ℓ respectively. Then there exists a linear uniform embedding of w.*

Proof. Note that $w(x,y)=\alpha_1$ if $\ell(x)\le y\le r(x)$, and α_2 otherwise. Let $x_0=0$ and $x_i=r^i(0)$ for $i\ge 1$. Since w is well-separated, r has positive distance from the diagonal. Therefore, there exists $\epsilon>0$ so that, for all $x\in[0,\ell(1)]$, $r(x)-x\ge\epsilon$. Now consider the sequence $x_i=r^i(0)$ for $i\in\mathbb{N}$. Since $0\in[0,\ell(1)]$ We have $r(0)-0\ge\epsilon$, and thus $r(0)\ge\epsilon$. For all $i\in\mathbb{N}$ such that $r^i(0)\in[0,\ell(1)]$, we have $r^i(0)-r^{i-1}(0)\ge\epsilon$. By induction we have $r^i(0)\ge i\epsilon$ for all $i\in\mathbb{N}$ with $r^i(0)\in[0,\ell(1)]$. Therefore, $i\epsilon\le r^i(0)\le 1$ for all $i\in\mathbb{N}$ with $r^i(0)\in[0,\ell(1)]$. Since $\epsilon>0$, there exists $k\in\mathbb{N}$ such that $r^k(0)\le\ell(1)$ and $r^i(0)>\ell(1)$ for all $i>k$. This implies that $r^k(0)<1$ and $r^{k+1}(0)=1$. Note that $\{x_i\}_{i=0}^k$ is a strictly increasing sequence, as w is well-separated. Define the function $\pi:[0,1]\to\mathbb{R}^{\ge 0}$ as follows.

$$\pi(x)=\begin{cases}\frac{x}{x_1} & \text{if } x\in[x_0,x_1],\\ \pi(\ell^i(x))+i & \text{if } x\in(x_i,x_{i+1}] \text{ for } 1\le i\le k-1\\ \pi(\ell^k(x))+k & \text{if } x\in(x_k,1].\end{cases}$$

First we prove that π is well-defined and strictly increasing. Since the function $\frac{x}{x_1}$ is well-defined and strictly increasing on $[x_0,x_1]$, the function π is also well-defined and strictly increasing on $[x_0,x_1]$. Now let $y_1,y_2\in(x_i,x_{i+1}]$ and $y_1<y_2$. Then $\ell^i(y_1),\ell^i(y_2)\in[x_0,x_1]$. Since ℓ^i is strictly increasing we have $\ell^i(y_1)<\ell^i(y_2)$. We know that π is strictly increasing on $[x_0,x_1]$, and thus $\pi(\ell^i(y_1))<\pi(\ell^i(y_2))$. This implies that $\pi(\ell^i(y_1))+1<\pi(\ell^i(y_2))+1$, and thus π is strictly increasing on $(x_i,x_{i+1}]$. A similar discussion proves that π is strictly increasing on $(x_k,1]$. We now prove that for each $y\in(x_i,x_{i+1}]$, $1\le i\le k-1$, $\pi(x_i)<\pi(y)$. Since $x_i\in(x_{i-1},x_i]$ and $\ell^{i-1}(x_i)=x_1$, $\pi(x_i)=i$. Moreover, $\pi(y)=\pi(\ell^i(y))+i$. Since ℓ^i is strictly increasing and $y>x_i$, we have $\ell^i(y)>\ell^i(x_i)=0$. Therefore, $\pi(\ell^i(y))>\pi(0)=0$. This implies that $\pi(y)>i=\pi(x_i)$. Similarly, we can prove that for $y\in(x_k,1]$, $\pi(y)>\pi(x_k)$. This proves that π is strictly increasing and well-defined and on $[0,1]$.

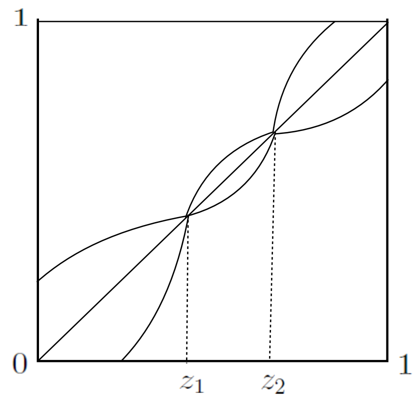
We now prove that π is a uniform linear embedding of w, namely

$$w(x,y)=\begin{cases}\alpha_1 & |\pi(x)-\pi(y)|\le 1\\ \alpha_2 & |\pi(x)-\pi(y)|>1\end{cases}. \tag{4.4}$$

To do so, partition $[0,1]$ using the intervals $J_0=[x_0,x_1], J_1=(x_1,x_2],\ldots,J_k=(x_k,1]$. Let $x,y\in[0,1]$, and $x<y$. The inequality $|\pi(x)-\pi(y)|\le 1$ holds precisely when either (i) x,y belong to the same interval, say J_i, or (ii) they belong to consecutive

intervals, say J_i and J_{i+1}, and $x \geq \ell(y)$. If case (ii) happens, clearly $w(x,y) = \alpha_1$, as $\ell(y) \leq x < y$. In case (i), $r^i(0) \leq x < y \leq r^{i+1}(0)$. Since r is increasing, $r^{i+1}(0) \leq r(x)$, which implies that $x < y \leq r(x)$. Thus $w(x,y) = \alpha_1$ in this case as well. A similar argument proves that if $|\pi(x) - \pi(y)| > 1$ then $w(x,y) = \alpha_2$. □

Remark 8. *Theorem 4.1.6 does not hold if we remove the condition that w should be well-separated. To show this, and to justify our conditions for w in Theorem 4.1.9, let us consider the following simple example. Suppose there exist $z_1 < z_2$ in $(0,1)$ such that $r(z_1) = \ell(z_1) = z_1$, $r(z_2) = \ell(z_2) = z_2$ and for every $x \in (0,1)$, $\ell(x) < r(x)$. Fix $x_0 \in (z_1, z_2)$ and $x'_0 \in (z_2, 1)$, and note that*



$$z_1 = \ell(z_1) < \ell(x_0), \quad r(x_0) < r(z_2) = z_2 \quad \text{and} \quad z_2 = \ell(z_2) < \ell(x'_0).$$

Therefore the sequences $\{x_i\}_{i\in\mathbb{N}}$, $\{y_i\}_{i\in\mathbb{N}}$ and $\{x'_i\}_{i\in\mathbb{N}}$ are all infinite sequences, where $x_i = r^i(x_0)$, $y_i = \ell^i(x_0)$ and $x'_i = \ell_i(x'_0)$ for every $i \in \mathbb{N}$. Moreover, $x_i, y_i \in (z_1, z_2)$ and $x'_i \in (z_2, 1)$. If a uniform linear embedding π exists, then for every $i \in \mathbb{N}$, we have

$$|\pi(x_i) - \pi(x_{i+1})| \leq 1, \quad |\pi(x_i) - \pi(x_{i+2})| > 1, \quad |\pi(x_i) - \pi(z_2)| > 1.$$

Therefore, all of the points of the sequence $\{\pi(x_i)\}_{i\in\mathbb{N}}$ lie on one side of $\pi(z_2)$, say inside the interval $(\pi(z_2)+1, \infty)$, in such a way that consecutive points have distance less than one. Moreover, $\pi(x_i) \to \infty$. Similar arguments, together with inequality $|\pi(x_i) - \pi(y_j)| > 1$, imply that the sequence $\{\pi(y_i)\}_{i\in\mathbb{N}}$ is distributed in a similar fashion in $(-\infty, \pi(z_1) - 1)$ and $\pi(y_i) \to -\infty$. Clearly, this process cannot be repeated for the sequence $\{\pi(x'_i)\}_{i\in\mathbb{N}}$. Hence a uniform linear embedding does not exist.

Let us now take the first step towards studying the relation between the boundary functions and the existence of a uniform linear embedding.

Proposition 4.1.7. *Let $w \in \mathcal{W}_0$ be a diagonally increasing function with finite range. Suppose $\pi : [0,1] \to \mathbb{R}$ is a continuous uniform linear embedding with parameters $0 < d_1 < d_2 < \ldots < d_{N-1}$ as in Definition 4.1.2. Then, for every $1 \leq i \leq N-1$, we have*

$$|\pi(r_i^*(x)) - \pi(x)| = d_i \quad \text{and} \quad |\pi(\ell_i^*(x)) - \pi(x)| = d_i,$$

whenever x is in the appropriate domain.

Proof. Let $1 \leq i \leq N-1$, and assume that $x \in \mathrm{dom}(r_i^*)$. Clearly $w(x, r_i^*(x)) = \alpha_i$, and $w(x,z) < \alpha_i$ whenever $z > r_i^*(x)$. Since π is a uniform linear embedding, we have $|\pi(x) - \pi(r_i^*(x))| \leq d_i$ and $|\pi(x) - \pi(z)| > d_i$ for every $z > r_i^*(x)$. These inequalities, together with the continuity of π, imply that $|\pi(x) - \pi(r_i^*(x))| = d_i$. A similar argument proves the second statement. $\square$

We consider uniform embedding π to be continuous in Proposition 4.1.7. However continuity of π is not a requirement for our results. The discontinuity of $\pi : [0,1] \to \mathbb{R}$ corresponds to empty subsets of $\mathbb{R}$. More precisely, let x and $r_i^*(x)$ are as in the proof of Proposition 4.1.7. We have that $|\pi(x) - \pi(r_i^*(x))| \leq d_i$, and for any points $z > r_i^*(x)$, we have that $|\pi(x) - \pi(z)| > d_i$. In the case where π is not continuous, it is not necessarily true that $|\pi(x) - \pi(r_i^*(x))| = d_i$. This means that, we could have $|\pi(x) - \pi(r_i^*(x))| = s < d_i$. Then $(s, d_i]$ could form an interval which contains no points of the range of π. In Section 4.3, we will state an analogous results to Proposition 4.1.7 for uniform embeddings π in general *i.e.* when π is not necessarily continuous (Proposition 4.3.9).

By repeated application of Proposition 4.1.7, we can see that the image $\pi(p)$ of a point p in $\mathcal{P}$, where $p = \phi(0)$ for a legal function ϕ, is determined by the signature of ϕ. This motivates the following definition.

Definition 4.1.8. *Assume a positive integer N and real numbers $d_{N-1} > \ldots > d_1 > 0$ are given. The* displacement *of a legal composition ϕ, denoted by $\delta(\phi)$ is defined as*

$$\delta(\phi) = d_1 m_1 + \ldots + d_{N-1} m_{N-1},$$

where $(m_1, m_2, \ldots, m_{N-1})$ is the signature of ϕ.

Note that the arrangement of the boundary functions r_i^* and ℓ_i^* in the sequence of a legal composition directly affects its displacement. More precisely, we may have two different legal compositions ϕ_1 and ϕ_2 that are identical as a function but $\delta(\phi_1)$ and $\delta(\phi_2)$ are not equal.

We now have all the required definitions to state our main result. We will see in Section 4.2 that either $\mathcal{P}$ and $\mathcal{Q}$ are disjoint, or $\mathcal{P} = \mathcal{Q}$. In the case $\mathcal{P} = \mathcal{Q}$, the conditions for the existence of a uniform linear embedding are identical to the case where $\mathcal{P}$ and $\mathcal{Q}$ are disjoint. However, in one special case, technical complications arise in the establishment of necessary conditions. This is the unlikely case where the boundary functions are too far apart from the diagonal, precisely when $r_{N-1}(0) > \ell_1(1)$. For simplicity, we will exclude this case from our main theorem, and from our discussions.

Theorem 4.1.9 (Necessary and sufficient conditions)**.** *Let w be a well-separated, finite-valued diagonally increasing function assuming values $\alpha_1 > \ldots > \alpha_N$. Let $\mathcal{P}$ and $\mathcal{Q}$ be as in Definition 4.1.5, and exclude the case "$\mathcal{P} = \mathcal{Q}$ and $r_{N-1}(0) > \ell_1(1)$". The function w has a uniform linear embedding if and only if the following conditions hold:*

1. *If ϕ is a legal function with $\phi(x) = x$ for some $x \in \mathrm{dom}(\phi)$, then ϕ is the identity function on its domain.*

2. *There exist real numbers $0 < d_1 < \ldots < d_{N-1}$ such that*

   (2a) The displacement δ as defined in Definition 4.1.8 is increasing on $\mathcal{P}$, in the sense that, for all $x, y \in \mathcal{P}$, and legal compositions ϕ, ψ so that $x = \phi(0)$ and $y = \psi(0)$, we have that, if $x < y$ then $\delta(\phi) < \delta(\psi)$.

   (2b) If $\mathcal{P} \cap \mathcal{Q} = \emptyset$ then there exists $a \in \mathbb{R}^{\geq 0}$ which satisfies the following condition: If ϕ and ψ are legal compositions with $1 \in \mathrm{dom}(\phi)$ and $0 \in \mathrm{dom}(\psi)$, and if $\phi(1) < r_i^*(0)$ for some i, $1 \leq i < N$, then $\delta(\psi) < a < d_i - \delta(\phi)$.

4.2 Properties of legal compositions

In this section we will study properties of boundary functions and legal compositions.

Assumption 4.2.1. *Let $w \in \mathcal{W}_0$ be a diagonally increasing function assuming values $\alpha_1 > \cdots > \alpha_N$. Let the boundaries of w be as in Definition 4.1.1. Assume that $w(x, y) \geq \alpha_i$ if and only if $\ell_i(x) \leq y \leq r_i(x)$. Moreover, assume that w is well-separated*

Throughout this chapter, we assume that w is as in Assumption 4.2.1. In what follows we will take a closer look at boundaries and legal compositions, which define the points in $\mathcal{P}$ and $\mathcal{Q}$.

Proposition 4.2.2. *Let w be as in Assumption 4.2.1. Then the boundaries r_i and ℓ_i, $1 \leq i < N$ are increasing.*

Proof. Assume to the contrary that there exist x, y so that $x < y$ and $\ell_i(x) > \ell_i(y)$. Then there exists z so that $\ell_i(y) < z < \ell_i(x) \leq x < y$. So $z \in [\ell_i(y), r_i(y)]$ and $z \notin [\ell_i(x), r_i(x)]$, and thus $w(x, z) < w(y, z)$. This contradicts the fact that w is diagonally increasing. $\square$

Remark 9. *Since w is a symmetric function, the lower and upper boundaries of w are closely related. Firstly, a discontinuity in ℓ_i corresponds to an interval where r_i is constant, see Figure 4.5. Thus, if $r_i^* = r_i|_{[0,\ell_i(1)]}$ and $\ell_i^* = \ell_i|_{[r_i(0),1]}$ are both strictly increasing, they are both continuous.*

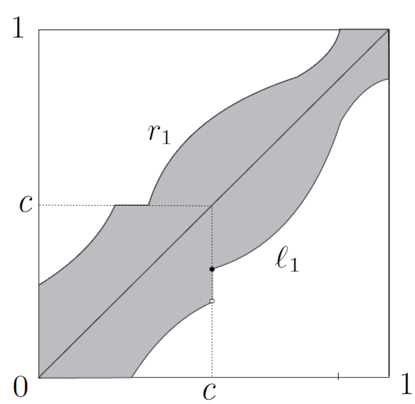


Figure 4.5: An example of a two-valued, diagonaly increasing w with discontinuous boundary.

Recall that these functions completely determine w. Next, since w is symmetric, for all $y < x$ we have that $y \geq \ell_i(x)$ if and only if $x \leq r_i(y)$. Thus, the upper boundaries are enough to completely determine w. In fact, if r_i is continuous (thus *strictly increasing) on $[0, \ell_i(1)]$, then r_i can be realized as ℓ_i^{-1} on $[0, \ell_i(1)]$, and 1 everywhere else. So if w is well-separated, then r_i^* and ℓ_i^* are bijective (hence strictly increasing and invertible) functions. Moreover, $r_i^* \circ \ell_i^*$ and $\ell_i^* \circ r_i^*$ are identity functions on their domain.*

Observation 4.2.3. *From the above remark it is clear that every legal composition $f_1 \circ \ldots \circ f_k$ is a strictly increasing function. We also remark that each term f_i of a legal composition is invertible in the sense that if $(f_1 \circ \ldots \circ f_k)(z) = x$ then $(f_2 \circ \ldots \circ f_k)(z) = f_1^{-1}(x)$. Moreover, $f_i^{-1} \in \{\ell_i^*, r_i^* : 1 \leq i < N\}$.*

Recall that a legal function is a legal composition, when considered only as a function. The following lemma shows that the domain and range of legal functions are intervals which begin at a point in $\mathcal{P}$, and end at a point in $\mathcal{Q}$.

Lemma 4.2.4. *Let w be as in Assumption 4.2.1 and, let $\phi = f_1 \circ \ldots \circ f_k$ be a legal function. Then*

(i) Suppose $z \in \mathrm{dom}(\phi)$. Then $z \in \mathcal{P}$ if and only if $\phi(z) \in \mathcal{P}$. Similarly, $z \in \mathcal{Q}$ if and only if $\phi(z) \in \mathcal{Q}$.

(ii) There is a one-to-one correspondence between $\mathrm{dom}(\phi) \cap \mathcal{P}$ and $\mathrm{range}(\phi) \cap \mathcal{P}$, and between $\mathrm{dom}(\phi) \cap \mathcal{Q}$ and $\mathrm{range}(\phi) \cap \mathcal{Q}$.

Proof. Let ϕ be a legal function, and $z \in \mathrm{dom}(\phi)$. By definition, if $z \in \mathcal{P}$ then there exists a legal function ψ such that $z = \psi(0)$. Thus, $\phi(z) = (\phi \circ \psi)(0)$ belongs to $\mathcal{P}$ as well. On the other hand, assume that $\phi(z) \in \mathcal{P}$, *i.e.* $\phi(z) = \eta(0)$ for a legal function η. By Observation 4.2.3, we have $(\phi^{-1} \circ \eta)(0) = z$, thus $z \in \mathcal{P}$. A similar argument proves the statement for $z \in \mathcal{Q}$. Part (ii) trivially follows from (i). $\square$

Note that for every legal composition $\phi = f_1 \circ \ldots \circ f_k$ applied to a point x there is a sequence of points which lead from x to $\phi(x)$, by first applying f_k, then f_{k-1} etc. This inspires the definition of "orbit" of an element in the domain of a legal composition.

Definition 4.2.5. *Under the assumption 4.2.1, let $\phi = f_1 \circ \ldots \circ f_k$ be a legal composition, and $x \in \mathrm{dom}(\phi)$. The* orbit *of x under ϕ, denoted by $\mathcal{O}_x$, is defined to be the set $\{x\} \cup \{f_t \circ \ldots \circ f_k(x) : 1 \leq t \leq k\}$. We say that $\phi(x)$* touches *0 (or 1) if the orbit of x under ϕ includes 0 (or 1).*

Proposition 4.2.6. *Under assumption 4.2.1, if $\phi = f_1 \circ \ldots \circ f_k$ is a legal composition with* $\mathrm{dom}(\phi) = [p, q]$ *then we have:*

(i) $p \in \mathcal{P}$ *and* $q \in \mathcal{Q}$.

(ii) $\phi(p)$ *touches 0, and* $\phi(q)$ *touches 1.*

Proof. We use induction on the number of terms of the legal composition. Clearly (i) holds for all functions r_i^* and ℓ_i^*, $1 \leq i < N$. Now let ϕ and ψ be legal functions with $\mathrm{dom}(\phi) = [p, q]$, $\mathrm{dom}(\psi) = [p', q']$, $p, p' \in \mathcal{P}$, and $q, q' \in \mathcal{Q}$. (Recall that by definition of a legal composition, $\mathrm{dom}(\phi)$ and $\mathrm{dom}(\psi)$ are nonempty.) Since ϕ and ψ are strictly increasing, we have $\mathrm{range}(\phi) = [\phi(p), \phi(q)]$ and $\mathrm{range}(\psi) = [\psi(p'), \psi(q')]$. By Lemma 4.2.4, we have $\phi(p), \psi(p') \in \mathcal{P}$, and $\phi(q), \psi(q') \in \mathcal{Q}$. We know that $\mathrm{dom}(\phi \circ \psi) = \mathrm{dom}(\psi) \cap \psi^{-1}(\mathrm{dom}(\phi) \cap \mathrm{range}(\psi))$. Since $\mathrm{range}(\psi)$, $\mathrm{dom}(\phi)$, and $\mathrm{dom}(\psi)$ are closed intervals with left bounds in $\mathcal{P}$ and right bounds in $\mathcal{Q}$, we conclude that $\mathrm{dom}(\phi \circ \psi)$ is a closed interval of the same type, *i.e.* $\mathrm{dom}(\phi \circ \psi) = [p_1, q_1]$ with $p_1 \in \mathcal{P}$ and $q_1 \in \mathcal{Q}$. This proves (i).

We use induction again to prove the statement regarding p in (ii). The proof for q is similar. First note that (ii) holds for all functions $\phi \in \{r_i^*, \ell_i^* : 1 \leq i < N\}$. Now consider a legal function $\phi = f_1 \circ \ldots \circ f_k$ with domain $[p, q]$. Assume that $0 < p$, since we are done otherwise. Let $\psi = f_2 \circ \ldots \circ f_k$, and assume that $\mathrm{dom}(\psi) = [p', q']$. First observe that $p \geq p'$ as $[p, q] \subseteq [p', q']$. By induction hypothesis, we know that $\psi(p')$ touches zero. If $p = p'$, then $\phi(p)$ touches zero as well. So suppose that $p > p'$. This means that $\psi(p') \notin \mathrm{dom}(f_1)$ but $\psi(p) \in \mathrm{dom}(f_1)$. So $f_1 = \ell_j^*$ for some j, because $\psi(p') < \psi(p)$. Moreover, we must have $p = \psi^{-1}(r_j^*(0))$. But this implies that

$$\phi(p) = \ell_j^* \circ \psi(p) = \ell_j^* \circ \psi(\psi^{-1}(r_j^*(0))) = 0,$$

which finishes the proof. □

We end this section with the following remark which states that either $\mathcal{P} = \mathcal{Q}$ or $\mathcal{P} \cap \mathcal{Q} = \emptyset$.

Remark 10. *The sets $\mathcal{P}$ and $\mathcal{Q}$ are either disjoint or identical. Namely, $\mathcal{P} \cap \mathcal{Q}$ is nonempty only when there exists a legal composition ϕ with* $\phi(0) = 1$. *Indeed, assume that there exists an element $x \in \mathcal{P} \cap \mathcal{Q}$. Let ϕ and ψ be legal compositions such that*

$$x = \phi(0) = \psi(1).$$

By Observation 4.2.3, ψ is invertible, and

$$1 = \psi^{-1} \circ \phi(0).$$

Therefore, $1 \in \mathcal{P}$, which in turn implies that $\mathcal{Q} \subseteq \mathcal{P}$. Similarly, we observe that $0 \in \mathcal{Q}$ as well, and $\mathcal{P} \subseteq \mathcal{Q}$.

Clearly if $\mathcal{P} = \mathcal{Q}$ and $\phi(0) = 1$, where ϕ is a legal function, then $\mathrm{dom}(\phi) = \{0\}$. *On the other hand, when $\mathcal{P} \cap \mathcal{Q} = \emptyset$, no legal function ψ has a singleton as its domain, because* $\mathrm{dom}(\psi) = [p, q]$ *with $p \in \mathcal{P}$ and $q \in \mathcal{Q}$.*

4.3 Necessary properties of a uniform linear embedding

This section is devoted to the study of uniform linear embeddings, and their interplay with boundary functions. Throughout this section, we assume that a uniform linear embedding π exists, and we will see how the boundaries and points in $\mathcal{P}$ severely restrict π. We show that a uniform linear embedding is necessarily strictly monotone. As we mentioned in the previous sections, a uniform linear embedding π can be discontinuous. Therefore we define the left limit π^- and the right limit π^+ at each point, and study their behaviour with respect to the boundaries. Finally, we will use these results to give the necessity proof of Theorem 4.1.9.

Assumption 4.3.1. *Let w be as in Assumption 4.2.1. Assume that w admits a uniform linear embedding $\pi : [0,1] \to \mathbb{R}$ with parameters $0 < d_1 < d_2 < \cdots < d_{N-1}$ as in Definition 4.1.2. Assume without loss of generality that $\pi(0) < \pi(r_1^*(0))$, and $\pi(0) = 0$.*

The following remark provides us with some conditions on a uniform linear embeddig π. These conditions simplify the question of existence of uniform linear embeddings.

Remark 11. *Comparing Assumption 4.2.1 and Definition 4.1.2 we see that π is a uniform linear embedding if and only if for all $x \in$* $\mathrm{dom}(r_i^*)$ *and $1 \leq i < N$,*

$$\begin{aligned} |\pi(z) - \pi(x)| &\leq d_i \quad \text{if } x \leq z \leq r_i^*(x) \text{ or } x \notin \mathrm{dom}(r_i^*) \\ |\pi(z) - \pi(x)| &> d_i \quad \text{if } z > r_i^*(x). \end{aligned} \tag{4.5}$$

Similarly, for all $x \in \mathrm{dom}(\ell_i^*)$ and $1 \leq i < N$,

$$\begin{aligned} |\pi(z) - \pi(x)| \leq d_i \quad & \text{if } \ell_i^*(x) \leq z \leq x \text{ or } x \notin \mathrm{dom}(\ell_i^*) \\ |\pi(z) - \pi(x)| > d_i \quad & \text{if } z < \ell_i^*(x). \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we obtain important properties for a uniform linear embedding π. In what follows, using (4.5) and (4.6), we will show that π is strictly monotone. To prove monotonicity of π we consider the cases $\ell_1^*(1) > r_1^*(0)$ and $\ell_1^*(1) \leq r_1^*(0)$ separately. We first consider the case where $\ell_1^*(1) > r_1^*(0)$.

Lemma 4.3.2. *Let w and π be as in Assumption 4.3.1 and $\ell_1^*(1) > r_1^*(0)$. Then we have the following. Let $x \in [0,1]$. Let $k_{\max}$ be the largest positive integer with ${r_1^*}^{k_{\max}}(x) < 1$. Then the sequence $\{\pi({r_1^*}^k(x))\}_{k=0}^{k_{\max}}$ is strictly increasing. Moreover $\pi({r_1^*}^{k_{\max}}(x)) < \pi(1)$.*

Proof. For $0 \leq k \leq k_{\max}$, let $x_k = {r_1^*}^k(x)$, and let $x_{k_{\max}+1} = 1$. First we show that the sequence $\{\pi(x_i)\}_{i=1}^{k_{\max}+1}$ is strictly increasing.

Assume that $\pi(x_1) > \pi(x_0)$. We will show by induction that for each k, $1 \leq k \leq k_{\max} + 1$, $\pi(x_k) > \pi(x_{k-1})$. The base case, $\pi(x_1) > \pi(x_0)$, is assumed. Let $k > 1$, and assume that $\pi(x_{k-2}) < \pi(x_{k-1})$. By (4.5),

$$|\pi(x_k) - \pi(x_{k-1})| \leq d_1 < |\pi(x_k) - \pi(x_{k-2})|.$$

Therefore, $\pi(x_k) > \pi(x_{k-1})$. This shows that $\{\pi(x_k)\}_{k=0}^{k_{\max}} = \{\pi({r_1^*}^k(x))\}_{k=0}^{k_{\max}}$ is strictly increasing. Moreover

$$|\pi(1) - \pi(x_k)| \leq d_1 < |\pi(1) - \pi(x_{k-1})|.$$

Therefore, $\pi(x_k) < \pi(1)$, and we conclude that the sequence $\{\pi(x_k)\}_{k=0}^{k_{\max}+1}$ is strictly increasing. If $\pi(x_1) < \pi(x_0)$, then an analogous argument shows that the sequence is strictly decreasing.

Now by Assumption 4.3.1, $\pi(r_1^*(0)) > \pi(0)$, so if we take $x = 0$, then the sequence is strictly increasing. This implies that $\pi(0) < \pi(1)$. Since for each choice of x, $x_0 = 0$ and $x_{k_{\max}+1} = 1$, we have that, for each choice of x, the sequence $\{\pi(x_k)\}_{k=0}^{k_{\max}+1}$ is increasing.

□

Lemma 4.3.3. *Let w and π be as in Assumption 4.3.1 and $\ell_1^*(1) > r_1^*(0)$. Let $x \in [0,1]$ and let $k_{\max}$ be the largest positive integer with ${r_1^*}^{k_{\max}}(x) < 1$. For every $0 \le k \le k_{\max} - 1$,*

$$\pi([{r_1^*}^k(x), {r_1^*}^{k+1}(x)]) \subseteq [\pi({r_1^*}^k(x)), \pi({r_1^*}^{k+1}(x))].$$

In addition, if $y \in [{r_1^*}^{k_{\max}}(x), 1]$ then $\pi({r_1^*}^{k_{\max}}(x)) \le \pi(y) \le \pi(1)$.

Proof. By Assumption 4.3.1 we have $r_1^*(0) < \ell_1^*(1)$. Therefore

$$\mathrm{dom}(r_1^*) \cap \mathrm{dom}(\ell_1^*) = [0, \ell_1^*(1)] \cap [r_1^*(0), 1] = [r_1^*(0), \ell_1^*(1)].$$

For $1 \le k \le k_{\max} - 2$, let $x_k = {r_1^*}^k(x)$. Suppose $y \in (x_k, x_{k+1})$. Then $y \in \mathrm{dom}(r_1^*) \cap \mathrm{dom}(\ell_1^*)$ and we have $r_1^*(y) > x_{k+1}$ and $\ell_1^*(y) < x_k$. From Lemma 4.3.2, we know that π is increasing on $\{x_k\}_{k=1}^{k_{\max}}$, in particular $\pi(x_k) < \pi(x_{k+1})$. Moreover $\pi(\ell_1^*(y)) < \pi(y) < \pi(r_1^*(y))$. To satisfy the inequalities listed in (4.5) and (4.6), we restrict the location of $\pi(y)$. Namely $x_k < x_{k+1} < r_1^*(y)$, so either $x_{k+1} \in \mathrm{dom}(r_1^*)$, and thus $x_{k+1} < r_1^*(y) < r_1^*()x_{k+1}$, or $x_{k+1} \notin \mathrm{dom}(r_1^*)$. In both cases, by inequality (4.5), we have that $|\pi(x_{k+1}) - \pi(r_1^*(y))| \le d_1$. Moreover, since $r_1^*(y) > x_{k+1} = r_1^*(x_k)$, by inequality (4.5), we have that $|\pi(x_k) - \pi(r_1^*(y))| > d_1$. Indeed,

$$|\pi(x_{k+1}) - \pi(r_1^*(y))| \le d_1 < |\pi(x_k) - \pi(r_1^*(y))|.$$

Moreover, $|\pi(y) - \pi(r_1^*(y))| \le d_1$, so $\pi(x_k) < \pi(y)$.

Similarly, $\ell_1^*(y) < x_k < x_{k+1}$, and thus

$$|\pi(x_k) - \pi(\ell_1^*(y))| \le d_1 < |\pi(x_{k+1}) - \pi(\ell_1^*(y))|.$$

Also, $|\pi(y) - \pi(\ell_1^*(y))| \le d_1$, and so $\pi(y) < \pi(x_{k+1})$. Thus $\pi(y)$ must belong to $(\pi(x_k), \pi(x_{k+1}))$.

Now let $k = k_{\max} - 1$ and $y \in (x_k, x_{k+1})$. Then either $y \le \ell_1^*(1)$ or $y > \ell_1^*(1)$. For the case $y \le \ell_1^*(1)$ by a similar discussion we obtain $\pi(y) \in (\pi(x_k), \pi(x_{k+1}))$. The case $y > \ell_1^*(1)$ can be dealt with in a similar fashion, but using 1 to take the place of $r_1^*(y)$. The remaining cases, $y \in (0, x_1)$ and $y \in (x_{k_{\max}}, 1)$, can be done similarly. □

Lemma 4.3.4. *If w has a uniform linear embedding π as in Assumption 4.3.1, and $\ell_1^*(1) > r_1^*(0)$ then π is strictly increasing. In particular, π is continuous on all except countably many points in $[0,1]$.*

Proof. Let $0 \leq x < y \leq 1$. Let $k_{\max}$ be the largest non-negative integer with ${r_1^*}^{k_{\max}}(x) < 1$. For $0 \leq k \leq k_{\max}$, let $x_k = {r_1^*}^k(x)$. By Lemma 4.3.2, we know that π is increasing on $\{x_k\}_{k=1}^{k_{\max}}$. Let l be the largest integer (possibly zero) such that $x_l \leq y$. If $l < k_{\max}$, then $y \in [x_l, x_{l+1}]$, and by Lemma 4.3.3, $\pi(y) \in [\pi(x_l), \pi(x_{l+1})]$. Therefore, $\pi(y) \geq \pi(x_l) \geq \pi(x)$, and since π is an injection, $\pi(y) > \pi(x)$. If $l = k_{\max}$, then $y \in [x_l, 1]$ and again by Lemma 4.3.3, it follows that $\pi(y) > \pi(x)$. So, π is strictly increasing. Since π is strictly increasing any discontinuity in π is of the form of a jump *i.e.* if π is discontinuous at c, then $0 < \lim_{x\to c^-} \pi(x) < \lim_{x\to c+} \pi(x) < \pi(1)$. Therefore, $[\lim_{x\to c+} \pi(x) - \lim_{x\to c^-} \pi(x)] > 0$ is a jump. So the sum of such jumps must be at most $\pi(1)$. Indeed let $D \subseteq [0,1]$ be the set of discontinuity points of π then

$$\sum_{c\in D}[\lim_{x\to c+} \pi(x) - \lim_{x\to c^-} \pi(x)] \leq \pi(1).$$

Thus, π is discontinuous in at most countably many points. $\square$

We now prove monotonicity of π when $\ell_1^*(1) < r_1^*(0)$.

Remark 12. *Let w be as in Assumption 4.2.1 and $\ell_1^*(1) < r_1^*(0)$. Then the interval $(\ell_1^*(1), r_1^*(0))$ contains no constrained points, and we have*

$$\ell_{N-1}^*(1) < \ldots < \ell_2^*(1) < \ell_1^*(1) < r_1^*(0) < r_2^*(0) < \ldots < r_{N-1}^*(0).$$

Lemma 4.3.5. *Let w and π be as in Assumption 4.3.1, and $\ell_1^*(1) < r_1^*(0)$. Then $\pi(r_{N-1}^*(0)) < \pi(1)$, and $\pi(0) < \pi(\ell_{N-1}^*(1))$. Moreover for all $1 \leq i \leq N-2$ we have*

$$\pi(\ell_{i+1}^*(1)) < \pi(\ell_i^*(1)) \quad \textit{and} \quad \pi(r_i^*(0)) < \pi(r_{i+1}^*(0)).$$

Proof. First we show that for all $1 \leq i \leq N-2$ we have $\pi(r_i^*(0)) < \pi(r_{i+1}^*(0))$.

We will show by induction that for each i, $0 \leq i \leq N-2$, $\pi(r_i^*(0)) < \pi(r_{i+1}^*(0))$. By Assumption 4.3.1 we know that $\pi(r_1^*(0)) > \pi(0)$. Let $i > 1$, and assume that $\pi(r_{i-1}^*(0)) < \pi(r_i^*(0))$. By (4.5),

$$|\pi(r_i^*(0)) - \pi(0)| \leq d_i < |\pi(r_{i+1}^*(0)) - \pi(0)|.$$

Therefore, $\pi(r_i^*(0)) < \pi(r_{i+1}^*(0))$. A similar discussion proves that $\pi(\ell_{i+1}^*(1)) < \pi(\ell_i^*(1))$. Moreover

$$|\pi(r_{N-1}^*(0)) - \pi(0)| \leq d_{N-1} < |\pi(1) - \pi(0)|,$$

and thus $\pi(r_{N-1}^*(0)) < \pi(1)$. Similarly $\pi(0) < \pi(\ell_{N-1}^*(1))$. $\square$

Lemma 4.3.6. *Let w and π be as in Assumption 4.3.1, and $\ell_1^*(1) < r_1^*(0)$. For every $1 \leq i \leq N-2$,*

$$\pi([r_i^*(0), r_{i+1}^*(0)]) \subseteq [\pi(r_i^*(0)), \pi(r_{i+1}^*(0))],$$

$$\pi([\ell_{i+1}^*(0), \ell_i^*(0)]) \subseteq [\pi(\ell_{i+1}^*(0)), \pi(\ell_i^*(0))].$$

In addition, if $y \in [0, \ell_{N-1}^*(0)]$ then $\pi(y) \in [0, \pi(\ell_{N-1}^*(1))]$, and if $y \in [r_{N-1}^*(0), 1]$ then $\pi(y) \in [\pi(r_{N-1}^*(0)), \pi(1)]$.

Proof. First we prove that if $y \in (r_i^*(0), r_{i+1}^*(0))$ then $\pi(y) > \pi(r_i^*(0))$. So let $y \in (r_i^*(0), r_{i+1}^*(0))$ then by (4.5) we have

$$|\pi(r_i^*(0)) - \pi(0)| \leq d_i < |\pi(y) - \pi(0)|$$

, and thus $\pi(y) > \pi(r_i^*(0))$. Similarly, if $y \in (\ell_{i+1}^*(1), \ell_i^*(1))$ then $\pi(y) < \pi(\ell_i^*(1))$.

Now suppose $y \in (r_i^*(0), r_{i+1}^*(0))$. From Lemma 4.3.5, we know that $\pi(r_i^*(0)) < \pi(r_{i+1}^*(0))$. Moreover $y \in \mathrm{dom}(\ell_i^*)$ and $\ell_i^*(y) < \ell_i^*(r_{i+1}^*(0))$. This, together with (4.6), implies that $d_i < |\pi(\ell_i^*(y)) - \pi(r_{i+1}^*(0))|$. Moreover, $\ell_i^*(y) \leq y$, and thus by (4.6), we have that $|\pi(\ell_i^*(y)) - \pi(y)| \leq d_i$. Therefore,

$$|\pi(\ell_i^*(y)) - \pi(y)| \leq d_i < |\pi(\ell_i^*(y)) - \pi((r_{i+1}^*(0))|.$$

This together with $\pi(\ell_i^*(y)) < \pi(y)$ implies that $\pi(y) < \pi(r_{i+1}^*(0))$. Therefore, $\pi((r_i^*(0), r_{i+1}^*(0))) \subseteq (\pi(r_i^*(0)), \pi(r_{i+1}^*(0)))$. A similar discussion proves that

$$\pi([\ell_{i+1}^*(0), \ell_i^*(0)]) \subseteq [\pi(\ell_{i+1}^*(0)), \pi(\ell_i^*(0))].$$

Now suppose that $y \in (0, \ell_{N-1}^*(0))$. Then $y < \ell_{N-1}^*(1)$, and thus by (4.6), $|\pi(y) - \pi(1)| > d_{N-1}$. Moreover, $\ell_{N-1}^*(1) \leq 1$. So, by (4.6), we have that $|\pi(1) - \pi(\ell_{N-1}(1))| \leq d_{N-1}$. We also know that $\pi(\ell_{N-1}^*(1)) < \pi(1)$. This, together with $|\pi(1) - \pi(\ell_{N-1}(1))| \leq d_{N-1} < |\pi(y) - \pi(1)|$, implies that $\pi(y) < \pi(\ell_{N-1}^*(1))$. A similar discussion proves that $\pi(0) < \pi(y)$, and thus $\pi(0, \ell_{N-1}(1)) \subset (0, \pi(\ell_{N-1}^*(1)))$. The other statement of the lemma derive with a similar discussion. $\square$

We now prove the monotonicity of π when $\ell_1^*(1) \leq r_1^*(0)$.

Lemma 4.3.7. *Let w and π be as in Assumption 4.3.1, and $\ell_1^*(1) < r_1^*(0)$. then π is strictly increasing on $[0, 1] \setminus (\ell_1^*(0), r_1^*(0))$. In particular, π is continuous on all except countably many points in $[0, 1]$.*

Proof. Let $x, y \in [0,1] \setminus (\ell_1^*(0), r_1^*(0))$, and $x < y$. If $x \leq \ell_1^*(1)$ and $y \geq r_1^*(0)$, then by Lemma 4.3.6 we know that $\pi(x) < \pi(y)$. So suppose either $x, y \geq r_1^*(0)$ or $x, y \leq \ell_1^*(0)$. First let $x, y \geq r_1^*(0)$. Let k, $1 \leq k \leq N-1$ be the largest integer with $r_k^*(0) \leq y$. If $x < r_k^*(0)$ then by Lemma 4.3.6 we have $\pi(x) < \pi(y)$. Now suppose $x, y \geq r_k^*(0)$. Since $x < y$ we have $y > r_k^*(0)$. Then $x, y \in \mathrm{dom}(\ell_k^*)$. Moreover $\ell_k^*(x) < \ell_k^*(y)$. Also by Lemma 4.3.6 we have $\pi(x) \geq \pi(r_k^*(0))$ and $\pi(y) > \pi(r_k^*(0))$. By (4.6) we have

$$|\pi(\ell_k^*(x)) - \pi(x)| \leq d_k < |\pi(\ell_k^*(x)) - \pi(y)|,$$

and thus $\pi(x) < \pi(y)$. If $x, y \leq \ell_1^*(0)$ then a similar argument proves $\pi(x) < \pi(y)$. $\square$

Definition 4.3.8. *Let π be as in Assumption 4.3.1. Let*

$$\begin{aligned} \pi^+(x) &= \inf\{\pi(z) : z > x\} \quad \text{for } x \in [0,1), \text{ and} \\ \pi^-(x) &= \sup\{\pi(z) : z < x\} \quad \text{for } x \in (0,1]. \end{aligned}$$

Note that these are limits of π at x from right and left respectively.

The following proposition is a version of Proposition 4.1.7 for discontinuous uniform linear embeddings.

Proposition 4.3.9. *If w has a uniform linear embedding π as in Assumption 4.3.1, then:*

(i) If π is continuous at $x \in (0, \ell^*(1))$ then $\pi(r_i^*(x)) = \pi(x) + d_i$. Likewise, if π is continuous at $x \in (r_i^*(0), 1)$ then $\pi(\ell_i^*(x)) = \pi(x) - d_i$.

(ii) For all x for which the limits are defined, we have that

$$\begin{aligned} \pi^+(\ell_i^*(x)) &= \pi^+(x) - d_i \text{ and } \pi^+(r_i^*(x)) &= \pi^+(x) + d_i, \text{ and} \\ \pi^-(\ell_i^*(x)) &= \pi^-(x) - d_i \text{ and } \pi^-(r_i^*(x)) &= \pi^-(x) + d_i. \end{aligned}$$

(iii) If π is continuous (from both sides) at $x \in (0, \ell^*(1))$ then π is also continuous at $r_i^*(x)$. Likewise, if π is continuous at $x \in (r_i^*(0), 1)$, then π is also continuous at $\ell_i^*(x)$.

Proof. Part (i) follows from Proposition 4.1.7 and the fact that π is increasing. We now prove $\pi^+(\ell_i^*(x)) = \pi^+(x) - d_i$. The proof of the rest is similar. Let z_n be a decreasing sequence converging to x from the right. Then $\ell_i^*(z_n)$ is a decreasing sequence converging to $\ell_i^*(x)$ from the right, which implies that $\pi(z_{n-1}) - \pi(\ell_i^*(z_n)) > d_i$. On the other hand, $\pi(z_n) - \pi(\ell_i^*(z_n)) \leq d_i$. Taking limits from both sides of these inequalities, we get $\lim_{n\to\infty} \pi(\ell_i^*(z_n)) = \lim_{n\to\infty} \pi(z_n) - d_i$. This gives us that $\pi^+(\ell_i^*(x)) = \pi^+(x) - d_i$, since π is increasing. Part (iii) follows from (ii) immediately. $\square$

Proposition 4.3.9 is a fundamental result for our approach in this chapter. When a diagonally increasing function w admits a uniform linear embedding π, this proposition illuminates how the boundary functions of w relate to a uniform linear embedding π. More precisely, Proposition 4.3.9 states that, for $f \in \{r_i^*, \ell_i^*\}$, the distance between the images of a point x and $f(x)$ under π is completely determined by d_i (displacement of f), if π is continuous at x. If π is not continuous at x, then d_i (displacement of f) determines either the distance between $\pi^-(x)$ and $\pi^-(f(x))$ or the distance between $\pi^+(x)$ and $\pi^+(f(x))$. This leads us to the fact that, the repeated application of Proposition 4.3.9 implies that the image $\pi(p)$ of a point p in $\mathcal{P}$, where $p = \phi(0)$ for a legal composition ϕ, is determined by the displacement of ϕ (see Proposition 4.3.10). This forms the foundation of our approach in this chapter.

Remark 13. *1. If π is continuous at x, neither x nor $r_i^*(x)$ (respectively $\ell_i^*(x)$) can be 0 or 1, since we need to compare continuity at these points from both sides.*

2. Let π be a uniform linear embedding for w as in Assumption 4.3.1. By Lemma 4.3.4, π is increasing. We can assume without loss of generality that π is continuous at 0. Indeed, if $0 = \pi(0) < \pi^+(0)$, then define the new function π' to be $\pi'(x) = \pi(x) - \pi^+(0)$ when $x \neq 0$ and $\pi'(0) = 0$. It is clear that π' forms a uniform linear embedding for $w|_{(0,1]}$. To show that π' is a uniform linear embedding for w, let $x > 0$, and note that there exists a decreasing sequence $\{z_n\}$ converging to 0 such that $w(x, 0) = w(x, z_n) = \alpha_i$. (This is true because w is well-separated). So for every $n \in \mathbb{N}$, we have $d_{i-1} < \pi(x) - \pi(z_n) \leq d_i$. Therefore, $d_{i-1} < \pi(x) - \pi^+(0) \leq d_i$, i.e. $d_{i-1} < \pi'(x) - \pi'(0) \leq d_i$. Therefore, *we can assume that π is continuous at 0.*

4.3.1 Properties of the displacement function

In this subsection we investigate properties of a displacement function as given in Definition 4.1.8. As we mentioned in the discussion after Proposition 4.3.9, displacement functions, δ and their relation with the uniform embedding π are crucial in our approach to study uniform linear embeddings. We explore the relation between δ and the limit behaviour of π in Proposition 4.3.10. Recall that for $x \in (0,1)$, $\pi^+(x)$ and $\pi^-(x)$ are the right and left limits of π at the point x. We start with the following remark which will be used in the proof of Proposition 4.3.10.

Remark 14. *Let ϕ be a legal function and $\mathrm{dom}(\phi) = [p,q]$, then $x \in (p,q]$ (respectively $x \in [p,q)$) precisely when $\phi(x)$ does not touch 0 (respectively 1). This implies that a point $x \in (0,1)$ belongs to the interior of the domain of a legal function ϕ if and only if $\phi(x)$ never touches either of 0 or 1.*

Proposition 4.3.10. *Let w and π satisfy assumption 4.3.1. Let $\phi = f_1 \circ \ldots \circ f_k$ be a legal function with domain $[p,q]$, where $p \in \mathcal{P}$ and $q \in \mathcal{Q}$. Then*

(i) $\pi^+(\phi(x)) - \pi^+(x) = \delta(\phi)$ *for every* $x \in [p,q)$.

(ii) $\pi^-(\phi(x)) - \pi^-(x) = \delta(\phi)$ *for every* $x \in (p,q]$.

(iii) If ψ_1 and ψ_2 are two legal functions with $\{x\} \subsetneq \mathrm{dom}(\psi_1) \cap \mathrm{dom}(\psi_2)$, and $\psi_1(x) = \psi_2(x)$ then $\delta(\psi_1) = \delta(\psi_2)$.

(iv) If $\delta(\phi) = 0$ and domain of ϕ is not a singleton, then ϕ is the identity function on its domain.

Proof. First note that ϕ is strictly increasing, therefore ϕ applied to $x \in [p,q)$ never touches 1, *i.e.* $f_l \circ \ldots \circ f_k(x) \neq 1$ for every $1 \leq l \leq k$. So, we can apply Proposition 4.3.9 (ii) to $f_l \circ \ldots \circ f_k(x) \neq 1$ for every $1 \leq l \leq k$. This means that $\pi^+(f_l \circ \ldots \circ f_k(x)(x)) = \delta(f_l \circ \ldots \circ f_k(x)) + \pi^+(x)$ for every $1 \leq l \leq k$. Therefore, $\pi^+(\phi(x)) = \delta(\phi) + \pi^+(x)$. This finishes the proof of (i). We skip the proof of (ii) as it is similar to (i).

The third statement is an easy corollary of (i), as $\psi_1^{-1} \circ \psi_2$ applied to x cannot touch both 0 and 1. Therefore, either (i) or (ii) of Proposition 4.3.10 can be applied. To see this fact, it is enough to observe that if a legal function applied to x touches both 0 and 1 then its domain is the singleton $\{x\}$. Assume, without loss of generality that $\psi_1^{-1} \circ \psi_2(x)$ does not touch 1. Therefore, $\pi^+(\psi_1^{-1} \circ \psi_2(x)) - \pi^+(x) = \delta(\psi_1^{-1} \circ \psi_2)$. Thus, $0 = \delta(\psi_1^{-1} \circ \psi_2) = -\delta(\psi_1) + \delta(\psi_2)$, and we are done.

To prove (iv), assume that $\mathrm{dom}(\phi)$ is not a singleton. Let z be a point in the interior of $\mathrm{dom}(\phi)$. By (i), we have $\pi^+(\phi(z)) - \pi^+(z) = \delta(\phi) = 0$. Also note that π^+ is strictly increasing, as π is. Therefore, we conclude that $\phi(z) = z$ on every point in the interior of $\mathrm{dom}(\phi)$. This finishes the proof of (iv), since ϕ is continuous and its domain is just a closed interval. □

The following corollary is a restatement of Proposition 4.3.10 part (iv).

Corollary 4.3.11. *Let ϕ and ψ be legal functions such that $\mathrm{dom}(\phi) \cap \mathrm{dom}(\psi)$ is nonempty and non-singleton. If $\delta(\phi) = \delta(\psi)$ then $\phi = \psi$ on the intersection of their domains.*

In what follows we extend Proposition 4.3.10 to the cases where a legal function has a singleton as its domain. Let us start with an auxiliary lemma.

Lemma 4.3.12. *Let π and w be as in Assumption 4.3.1. Let $1 \leq i < j \leq N-1$ be fixed. Then $\pi^+(r_i^*(0)) - \pi^-(r_i^*(0)) = \pi^+(r_j^*(0)) - \pi^-(r_j^*(0))$.*

Proof. Let ϕ be a legal function such that $\phi(r_i^*(0))$ does not touch 0 or 1. Thus, by Proposition 4.3.10, we have $\pi^+(\phi(r_i^*(0))) - \pi^-(\phi(r_i^*(0))) = \pi^+(r_i^*(0)) - \pi^-(r_i^*(0))$. Thus,

$$
\begin{aligned}
\pi^+(r_j^* \circ r_i^*(0)) - \pi^-(r_j^* \circ r_i^*(0)) &= \pi^+(r_i^*(0)) - \pi^-(r_i^*(0)), \\
\pi^+(r_i^* \circ r_j^*(0)) - \pi^-(r_i^* \circ r_j^*(0)) &= \pi^+(r_j^*(0)) - \pi^-(r_j^*(0)).
\end{aligned}
$$

Moreover by Proposition 4.3.10 (i), we have

$$
\pi^+(r_i^* \circ r_j^*(x)) = \pi^+(r_j^* \circ r_i^*(x)) = d_i + d_j + \pi(x),
$$

since $r_i^* \circ r_j^*(x) < 1$ and $r_j^* \circ r_i^*(x) < 1$. This implies that $r_i^* \circ r_j^* = r_j^* \circ r_i^*$ on the intersection of their domains, as π^+ is strictly increasing and therefore injective. In particular, we have $r_i^* \circ r_j^*(0) = r_j^* \circ r_i^*(0)$, and so $\pi^+(r_i^*(0)) - \pi^-(r_i^*(0)) = \pi^+(r_j^*(0)) - \pi^-(r_j^*(0))$. $\square$

Proposition 4.3.13. *Under Assumption 4.3.1, for legal functions ϕ and ψ with* $\mathrm{dom}(\phi) = \mathrm{dom}(\psi) = \{x_0\}$ *the following holds: $\phi(x_0) = \psi(x_0)$ if and only if $\delta(\phi) = \delta(\psi)$.*

Proof. We begin the proof by considering some special cases. First, suppose ϕ and ψ are legal functions with $\phi(0) = \psi(0) = 1$, such that $\phi(0)$ and $\psi(0)$ do not touch 0 or 1 at any intermediate step. In that case, for some i, j, i', j' we have

$$\phi = r_i^* \circ \phi_1 \circ r_{i'}^* \quad \text{and} \quad \psi = r_j^* \circ \psi_1 \circ r_{j'}^*,$$

such that $\phi_1(z_1)$ and $\psi_1(z_2)$ do not touch 0 or 1, where $z_1 = r_{i'}^*(0)$ and $z_2 = r_{j'}^*(0)$. By Proposition 4.3.10, $\pi^+(\phi_1(r_{i'}^*(0))) - \pi^+(r_{i'}^*(0)) = \delta(\phi_1)$ and $\pi^-(1) - \pi^-(\phi_1(r_{i'}^*(0))) = d_i$. Recall that π is assumed to be continuous at 0, *i.e.* $\pi^+(0) = 0$. So, $\pi^+(r_{i'}^*(0)) = d_{i'}$. Thus,

$$\delta(\phi) = d_i + \delta(\phi_1) + d_{i'} = \pi^+(\phi_1 \circ r_{i'}^*(0)) - \pi^-(\phi_1 \circ r_{i'}^*(0)) + \pi^-(1).$$

Similarly, $\delta(\psi) = \pi^+(\psi_1 \circ r_{j'}^*(0)) - \pi^-(\psi_1 \circ r_{j'}^*(0)) + \pi^-(1)$. Combining (i) and (ii) of Proposition 4.3.10, we observe that $\pi^+(\psi_1 \circ r_{j'}^*(0)) - \pi^-(\psi_1 \circ r_{j'}^*(0)) = \pi^+(r_{j'}^*(0)) - \pi^-(r_{j'}^*(0))$, as ψ_1 applied to $r_{j'}^*(0)$ does not hit 0 or 1. Similarly, $\pi^+(\phi_1 \circ r_{i'}^*(0)) - \pi^-(\phi_1 \circ r_{i'}^*(0)) = \pi^+(r_{i'}^*(0)) - \pi^-(r_{i'}^*(0))$. Applying Lemma 4.3.12, we conclude that

$$\delta(\phi) = \delta(\psi) = \pi^-(1) + \pi^+(r_k^*(0)) - \pi^-(r_k^*(0)) \quad \text{for any } 1 \leq k \leq N-1. \tag{4.7}$$

Similarly, suppose ϕ and ψ are legal functions with $\phi(1) = \psi(1) = 0$, such that $\phi(1)$ and $\psi(1)$ do not touch 0 or 1 at any intermediate step. Then ϕ^{-1} and ψ^{-1} satisfy the conditions of the previous case, and we have $\delta(\phi^{-1}) = \delta(\psi^{-1})$. So,

$$\delta(\phi) = \delta(\psi) = -\pi^-(1) + \pi^-(r_k^*(0)) - \pi^+(r_k^*(0)) \quad \text{for any } 1 \leq k \leq N-1. \tag{4.8}$$

Next, consider the case where $\phi(0) = \psi(0) = 0$, such that $\phi(0)$ and $\psi(0)$ do not touch 0 or 1 at any intermediate step. Thus, by Proposition 4.3.10, we have $\delta(\phi) = \delta(\psi) = 0$. A similar argument works when $\phi(1) = \psi(1) = 1$, and $\phi(1)$ and $\psi(1)$ do not touch 0 or 1 at any intermediate step.

We can now prove the general case. Suppose $\mathrm{dom}(\phi) = \mathrm{dom}(\psi) = \{x_0\}$ and $\phi(x_0) = \psi(x_0)$. Recall that $x_0 \in \mathcal{P}$, and let η be a legal function such that $\eta(0) = x_0$. Define

$$\xi = \eta^{-1} \circ \psi^{-1} \circ \phi \circ \eta.$$

Clearly, $\xi(0) = 0$. Let $\xi = \xi_1 \circ \ldots \circ \xi_n$ be a decomposition of ξ into legal functions, where each ξ_i satisfies the conditions of one of the cases studied above. That is, the domain and the range of each ξ_i is either $\{0\}$ or $\{1\}$, and none of the ξ_i's touch 0 or 1 in any intermediate step. It is easy to observe that the number of terms ξ_i which map 0 to 1 must be the same as the number of terms which map 1 to 0. Therefore, by what we observed above, $\delta(\xi) = 0$. This proves the "only if" direction.

To prove the "if" direction, it is enough to prove that if $\delta(\phi) = 0$ then $\phi(x_0) = x_0$. We consider two possibilities: Firstly, suppose that there exists a legal function η with $\eta(0) = x_0$ such that $\eta(0)$ does not touch 1. Let $\xi = \phi \circ \eta$, and note that $\mathrm{dom}(\xi) = \{0\}$. Moreover, $\delta(\xi) = \delta(\eta)$, as $\delta(\phi) = 0$. By Proposition 4.2.6 (ii), $\xi(0)$ must touch 1. Suppose ξ decomposes into $\xi = \xi_1 \circ \xi_2 \circ \xi_3$, where $\xi_1(0) = \phi(x_0)$, $\xi_2(1) = 0$, $\xi_3(0) = 1$, and $\xi_1(0)$ does not touch 1. Applying part (i) to ξ_2 and ξ_3^{-1}, we conclude that $\delta(\xi_2) = -\delta(\xi_3)$. Therefore $\delta(\xi_1) = \delta(\eta)$. Moreover, $\{0\} \subsetneq \mathrm{dom}(\xi_1) \cap \mathrm{dom}(\eta)$. Thus, by Corollary 4.3.11 we have $\xi_1(0) = \eta(0)$, which shows that $\phi(x_0) = x_0$ in this case.

Next, suppose that ξ decomposes to $\xi = \xi_1 \circ \xi_2$ where $\xi_2(0) = 1$, $\xi_1(1) = \phi(x_0)$, and $\xi_1(1)$ does not touch 0. As argued in the proof of Lemma 4.3.12, we have $\delta(\xi_2) = \pi^-(1) + \pi^+(r_1^*(0)) - \pi^-(r_1^*(0))$. Thus,

$$\pi^+(x_0) = \delta(\eta) = \delta(\xi) = \pi^-(1) + \pi^+(r_1^*(0)) - \pi^-(r_1^*(0)) + \pi^-(\phi(x_0)) - \pi^-(1).$$

This implies that $\pi^+(x_0) \geq \pi^-(\phi(x_0))$. Observe that $x \leq y$ if and only if $\pi^-(x) \leq \pi^+(x) \leq \pi^-(y) \leq \pi^+(y)$, since π is increasing. Hence, we must have $\pi^+(x_0) \geq \pi^+(\phi(x_0))$. Since π^+ is increasing, we conclude that $x_0 \geq \phi(x_0)$. Replacing ϕ by ϕ^{-1}, we obtain $x_0 \geq \phi^{-1}(x_0)$ in a similar way. Hence, $x_0 = \phi(x_0)$ in this case as well.

Secondly, suppose that there exists a legal function η with $\eta(1) = x_0$ such that $\eta(1)$ does not touch 0. A similar argument, where the roles of 0 and 1 are switched, finishes the proof. $\square$

4.3.2 Conditions of Theorem 4.1.9 are Necessary.

We are now ready to prove the necessity of the conditions of Theorem 4.1.9.

Proof of "only if" direction of Theorem 4.1.9. Suppose w and π satisfy Assumption 4.3.1. In particular, recall that π is strictly increasing. If $\mathcal{P} \cap \mathcal{Q} = \emptyset$ then an easy application of Proposition 4.3.10 shows that Conditions (1), (2a), and (2b) must hold. Indeed, let $\phi = f_1 \circ \ldots \circ f_k$ be a legal function so that $\phi(x) = x$ for some x. Clearly, x belongs to the domain of either $\ell_1^* \circ r_1^*$ or $r_1^* \circ \ell_1^*$, which are both identity functions on their domains with zero displacement. So by Proposition 4.3.10, we have $\delta(\phi) = 0$, and therefore ϕ must be the identity function on its domain. To prove (2a), suppose that $x = \psi(0)$ and $y = \phi(0)$ where $x < y$, and ψ and ϕ are legal functions. The functions ψ and ϕ applied to 0 never touch 1, since $\mathcal{P} \cap \mathcal{Q} = \emptyset$. So $\pi^+(\phi(0)) - \pi^+(0) = \delta(\phi)$ and $\pi^+(\psi(0)) - \pi^+(0) = \delta(\psi)$. This finishes the proof of part (2a), since π^+ is a strictly increasing function, and $\pi^+(0) = 0$.

Suppose now that $y = \phi(1) < r_i^*(0)$ for some i and $x \in \mathcal{P}$ with $x = \psi(0)$. By Proposition 4.3.10 we have

$$\pi^-(y) - \pi^-(1) = \delta(\phi) \quad \text{and} \quad \pi^+(x) - \pi^+(0) = \delta(\psi).$$

On the other hand by Lemma 4.3.4, $\pi^-(y) \leq \pi(y) < \pi(r_i^*(0)) \leq d_i$. Hence $\pi^-(y) - \pi^-(1) < d_i - \pi^-(1)$, which gives $\delta(\phi) < d_i - \pi^-(1)$. Moreover by definition of π^+ and π^- we have $\pi^+(x) < \pi^-(1)$. Note that $\pi^+(0) = 0$. This implies that $\delta(\psi) < \pi^-(1)$ and therefore, $\delta(\psi) < \pi^-(1) < d_i - \delta(\phi)$. Taking $a = \pi^-(1)$ finishes the proof of (2b).

Next, consider the case where $\mathcal{P} = \mathcal{Q}$. Condition (1) holds by Proposition 4.3.10. Now suppose that $x = \psi(0)$ and $y = \phi(0)$ where $x < y$, and ψ and ϕ are legal functions. If the domains of ϕ and ψ are not singletons, then Condition (2a) follows from Proposition 4.3.10. Now assume that $\mathrm{dom}(\psi) = \{0\}$, which means that $\psi(0)$ must touch 1. Using Proposition 4.3.13 and Equations (4.7) and (4.8), note that one of the following two cases happen: Either $\psi(0)$ does not touch 0 after the last time it touches 1, in which case we have $\delta(\psi) = \pi^+(r_1^*(0)) - \pi^-(r_1^*(0)) + \pi^-(x)$. Or, $\psi(0)$ touches 0 after the last time it touches 1, in which case we have $\delta(\psi) = \pi^+(x)$. (See the proof of Proposition 4.3.13 part (ii) for a similar argument). In the latter case, ψ can be decomposed as $\psi_1 \circ \psi_2$, where $\psi_2(0) = 0$ and $\psi_1(0)$ does not touch 1. Thus, thanks to Lemma 4.3.12, we have $\pi^+(\psi_1(0)) - \pi^-(\psi_1(0)) = \pi^+(r_1^*(0)) - \pi^-(r_1^*(0))$. Clearly, $x = \psi(0) = \psi_1(0)$, and so

$$\pi^+(r_1^*(0)) - \pi^-(r_1^*(0)) + \pi^-(x) = \pi^+(x)$$

equals the value of $\delta(\psi)$ in either case. Thus $\delta(x) < \delta(y)$ as π^+ is strictly increasing. □

4.4 Sufficiency; construction of a uniform linear embedding

In this section we present a constructive proof of sufficiency of the conditions of Theorem 4.1.9. More precisely, if Conditions (1), (2a) and (2b) of Theorem 4.1.9 hold, then we give an explicit construction for the uniform linear embedding.

Assumption 4.4.1. *Let w be as in Assumption 4.2.1. Assume that the conditions of Theorem 4.1.9 hold. In particular,*

1. *If ϕ is a legal function with $\phi(x) = x$ for some $x \in \mathrm{dom}(\phi)$, then ϕ is the identity function on its domain.*

2. *There exist real numbers $0 < d_1 < \ldots < d_{N-1}$ such that*

   (2a) For all $x, y \in \mathcal{P}$, and legal compositions ϕ, ψ so that $x = \phi(0)$ and $y = \psi(0)$, we have that, if $x < y$ then $\delta(\phi) < \delta(\psi)$.

   (2b) If $\mathcal{P} \cap \mathcal{Q} = \emptyset$ then there exists $a \in \mathbb{R}^{\geq 0}$ which satisfies the following condition: If ϕ and ψ are legal compositions with $1 \in \mathrm{dom}(\phi)$ and $0 \in \mathrm{dom}(\psi)$, and if $\phi(1) < r_i^*(0)$ then $\delta(\psi) < a < d_i - \delta(\phi)$.

4.4.1 The displacement function in the case where $\mathcal{P} \cap \mathcal{Q} = \emptyset$.

In this subsection we explore properties of displacement function δ under Assumption 4.4.1. We use the results of this section to prove sufficiency of the conditions of Theorem 4.1.9 in the case where $\mathcal{P} \cap \mathcal{Q} = \emptyset$. Indeed our aim here is to show that when $\mathcal{P} \cap \mathcal{Q} = \emptyset$ then there is a nice one-to-one correspondence between the sets $\mathcal{P}$ and $\mathcal{Q}$.

Lemma 4.4.2 (δ is "increasing"). *Under Assumption 4.4.1 and $\mathcal{P} \cap \mathcal{Q} = \emptyset$, let ϕ_1 and ϕ_2 be legal functions and $x \in \operatorname{dom}(\phi_1) \cap \operatorname{dom}(\phi_2)$. Then*

(i) If $\phi_1(x) < \phi_2(x)$ then $\delta(\phi_1) < \delta(\phi_2)$.

(ii) If $\phi_1(x) = \phi_2(x)$ then $\delta(\phi_1) = \delta(\phi_2)$.

Proof. Let ϕ_1 and ϕ_2 be as above. By Lemma 4.2.4, $\operatorname{dom}(\phi_1) = [p_1, q_1]$ and $\operatorname{dom}(\phi_2) = [p_2, q_2]$ for $p_1, p_2 \in \mathcal{P}$ and $q_1, q_2 \in \mathcal{Q}$. Since $\operatorname{dom}(\phi_1) \cap \operatorname{dom}(\phi_2) \neq \emptyset$ we have $\operatorname{dom}(\phi_1) \cap \operatorname{dom}(\phi_2) = [p, q]$ with $p = \max\{p_1, p_2\}$ and $q = \min\{q_1, q_2\}$. If $x = p$ then Condition (2a) of Theorem 4.1.9 implies that $\delta(\phi_1) < \delta(\phi_2)$ and we are done. Now suppose $x \in (p, q]$. We will show that $\phi_1(p) < \phi_2(p)$. Suppose to the contrary that $\phi_1(p) \geq \phi_2(p)$. Since ϕ_1 and ϕ_2 are continuous functions on $[p, q]$, and $\phi_1(x) < \phi_2(x)$, there exists a point $y \in [p, x)$ with $\phi_1(y) = \phi_2(y)$. Hence $\operatorname{range}(\phi_1) \cap \operatorname{range}(\phi_2) \neq \emptyset$. So by Lemma 4.2.4, $\operatorname{range}(\phi_1) \cap \operatorname{range}(\phi_2) = [p', q']$ where $p' = \max\{\phi_1(p_1), \phi_2(p_2)\} \in \mathcal{P}$ and $q' = \min\{\phi_1(q_1), \phi_2(q_2)\} \in \mathcal{Q}$. In particular, since $\mathcal{P} \cap \mathcal{Q} = \emptyset$, we have

$$\max\{p_1, p_2\} < \min\{q_1, q_2\} \quad \text{and} \quad \max\{\phi_1(p_1), \phi_2(p_2)\} < \min\{\phi_1(q_1), \phi_2(q_2)\}.$$

Clearly $\phi_2^{-1} \circ \phi_1(y) = y$. Thus by Condition (1) of Theorem 4.1.9, we have that $\phi_2^{-1} \circ \phi_1$ is the identity function on its domain. Next we observe that $x \in \operatorname{dom}(\phi_2^{-1} \circ \phi_1)$. This follows from the fact that $\phi_1(x) \in \operatorname{range}(\phi_2) = \operatorname{dom}(\phi_2^{-1})$, as

$$\phi_2(p_2) \leq \phi_2(y) = \phi_1(y) < \phi_1(x) < \phi_2(x) \leq \phi_2(q_2).$$

Therefore, we must have $\phi_2^{-1} \circ \phi_1(x) = x$, *i.e.* $\phi_1(x) = \phi_2(x)$, which is a contradiction. This finishes the proof of (i).

To prove (ii), let $x \in \operatorname{dom}(\phi_1) \cap \operatorname{dom}(\phi_2)$ be such that $\phi_1(x) = \phi_2(x)$. Assume without loss of generality that $\delta(\phi_2) \geq \delta(\phi_1)$, thus $\delta(\phi_1^{-1} \circ \phi_2) \geq 0$. Clearly $\phi_1^{-1} \circ \phi_2(x) = x$, and by Condition (1) of Theorem 4.1.9, we have that the legal function $\phi_1^{-1} \circ \phi_2$ is the identity function on its domain, say $[p, q]$. In particular, $\phi_1^{-1} \circ \phi_2(p) = p$. Let η be a legal function with $\eta(0) = p$, and consider $\psi = \eta^{-1} \circ \phi_1^{-1} \circ \phi_2 \circ \eta$. Then

$$\psi^n(0) = \eta^{-1} \circ (\phi_1^{-1} \circ \phi_2)^n \circ \eta(0) = \eta^{-1} \circ (\phi_1^{-1} \circ \phi_2)^n(p) = \eta^{-1}(p) = 0 < r_1^*(0),$$

where ψ^n denotes the n-fold composition of ψ with itself. Now Condition (2a) of Theorem 4.1.9 gives that

$$0 \leq n\delta(\phi_1^{-1} \circ \phi_2) = \delta(\psi^n) < d_1,$$

for every positive integer n. So $\delta(\phi_1^{-1} \circ \phi_2)$ must be zero, and $\delta(\phi_1) = \delta(\phi_2)$. $\square$

Corollary 4.4.3. *Under Assumption 4.4.1 and* $\mathcal{P} \cap \mathcal{Q} = \emptyset$*, let* ϕ_1 *and* ϕ_2 *be legal functions which have non-disjoint domains. Then*

(i) $\phi_1(x) < \phi_2(x)$ *for some* $x \in \mathrm{dom}(\phi_1) \cap \mathrm{dom}(\phi_2)$ *if and only if* $\delta(\phi_1) < \delta(\phi_2)$ *if and only if* $\phi_1 < \phi_2$ *everywhere on* $\mathrm{dom}(\phi_1) \cap \mathrm{dom}(\phi_2)$.

(ii) $\phi_1(x) = \phi_2(x)$ *for some* $x \in \mathrm{dom}(\phi_1) \cap \mathrm{dom}(\phi_2)$ *if and only if* $\delta(\phi_1) = \delta(\phi_2)$ *if and only if* $\phi_1 = \phi_2$ *everywhere on* $\mathrm{dom}(\phi_1) \cap \mathrm{dom}(\phi_2)$.

The following lemma is a direct consequence of Lemma 4.4.2.

Lemma 4.4.4. *Under Assumption 4.4.1 and* $\mathcal{P} \cap \mathcal{Q} = \emptyset$*, let* $\phi = f_1 \circ \cdots \circ f_k$ *be a legal composition and* $x \in \mathrm{dom}(\phi)$*. For* $1 \leq t \leq k$*, let* $x_t = f_{k-t+1} \circ \cdots \circ f_k(x)$*. Then*

(1) If $x \leq \min\{x_t : 1 \leq t \leq k\}$ *then* $[0, x] \subseteq \mathrm{dom}(\phi)$.

(2) If $x \geq \max\{x_t : 1 \leq t \leq k\}$ *then* $[x, 1] \subseteq \mathrm{dom}(\phi)$.

Proof. We only prove (1). A similar argument proves part (2) of the lemma. Let ϕ and x be as stated above. To prove part (1), for $1 \leq t \leq k$, define $\phi_t = f_{k-t+1} \circ \cdots \circ f_k$, and let ϕ_0 be the identity function. Then for every i, $x_i = \phi_i(x)$, and specifically $x_0 = x$ and $x_k = \phi(x)$.

Assume $x \leq x_t$ for all $1 \leq t \leq k$. We show by induction on i that $[0, x] \subseteq \mathrm{dom}(\phi_i)$. Since the domain of a legal function is an interval, and $x \in \mathrm{dom}(\phi_i)$, it is enough to show that $0 \in \mathrm{dom}(\phi_i)$. For $i = 0$, ϕ_0 is the identity function, and so $0 \in \mathrm{dom}(\phi_i)$. For the induction step, fix t, $0 < t \leq k$, and consider $\phi_t = f_{k-t+1} \circ \phi_{t-1}$. By induction hypothesis, $0 \in \mathrm{dom}(\phi_{t-1})$. Let $y = \phi_{t-1}(0)$. We must show that $y \in \mathrm{dom}(f_{k-t+1})$. Consider two cases: First assume that $f_{k-t+1} = \ell_i^*$ for some $1 \leq i \leq N-1$. Since $x \leq x_t = \ell_i^* \circ \phi_{t-1}(x)$, we have $\phi_{t-1}(x) \geq r_i^*(x)$. So by Lemma 4.4.2 we have $\delta(\phi_{t-1}) \geq \delta(r_i^*) = d_i$. This, together with another application of Lemma 4.4.2, implies that $\phi_{t-1}(0) \geq r_i^*(0)$. Hence $0 \in \mathrm{dom}(\phi_t)$.

Next assume that $f_{k-t+1} = r_i^*$, where $1 \leq i \leq N-1$. Since $0 \leq x$, we must have that $y \leq x_{t-1}$. Observe that $\mathrm{dom}(f_{k-t+1}) = [0, \ell_i^*(1)]$ and $x_{t-1} \in \mathrm{dom}(f_{k-t+1})$. Therefore $y \in \mathrm{dom}(f_{k-t+1})$, and hence $0 \in \mathrm{dom}(\phi_t)$. This completes the proof of part (1). $\square$

Corollary 4.4.5. *Under Assumption 4.4.1 and $\mathcal{P} \cap \mathcal{Q} = \emptyset$, let $\phi_1 = f_1 \circ \cdots \circ f_k$ and $\phi_2 = g_1 \circ \cdots \circ g_l$ be legal functions with $x \in \operatorname{dom}(\phi_1) \cap \operatorname{dom}(\phi_2)$. For $1 \leq t \leq k$, let $x_t = f_{k-t+1} \circ \cdots \circ f_k(x)$. Similarly, let $y_t = g_{l-t+1} \circ \cdots \circ g_l(x)$ for $1 \leq t \leq l$.*

(i) Suppose $x \leq x_t \leq \phi_1(x)$ for every $1 \leq t \leq k$, and $x \leq y_t \leq \phi_2(x)$ for every $1 \leq t \leq l$. Then $\operatorname{dom}(\phi_i) = [0, \phi_i^{-1}(1)]$ and $\operatorname{range}(\phi_i) = [\phi_i(0), 1]$. Moreover if $\phi_2(x) < \phi_1(x)$, then $\operatorname{dom}(\phi_1) \subset \operatorname{dom}(\phi_2)$ and $\operatorname{range}(\phi_1) \subset \operatorname{range}(\phi_2)$.

(ii) Suppose $x \geq x_t \geq \phi_1(x)$ for every $1 \leq t \leq k$, and $x \geq y_t \geq \phi_2(x)$ for every $1 \leq t \leq l$. Then $\operatorname{dom}(\phi_i) = [\phi_i^{-1}(0), 1]$ and $\operatorname{range}(\phi_i) = [0, \phi_i(1)]$. Moreover if $\phi_1(x) < \phi_2(x)$, then $\operatorname{dom}(\phi_1) \subset \operatorname{dom}(\phi_2)$ and $\operatorname{range}(\phi_1) \subset \operatorname{range}(\phi_2)$.

Proof. We only prove (i). A similar argument proves part (ii). Let $\operatorname{dom}(\phi_1) = [p, q]$. By Corollary 4.4.3, for every $1 \leq t \leq k$ we have $\delta(f_{k-t+1} \circ \cdots \circ f_k) \geq 0$ and $\delta(f_{k-t+1} \circ \cdots \circ f_k) \leq \delta(\phi_1)$, since $x \leq f_{k-t+1} \circ \cdots \circ f_k(x) \leq \phi_1(x)$. Thus, $p \leq f_{k-t+1} \circ \cdots \circ f_k(p) \leq \phi_1(p)$ and $q \leq f_{k-t+1} \circ \cdots \circ f_k(q) \leq \phi_1(q)$ for every $1 \leq t \leq k$. Now Proposition 4.2.6 immediately implies that $p = 0$ and $\phi_1(q) = 1$. This proves the first part of (i). Next, observe that since $\phi_2(x) < \phi_1(x)$, by Corollary 4.4.3 we have $\delta(\phi_2) < \delta(\phi_1)$. This implies that $\delta(\phi_1^{-1}) < \delta(\phi_2^{-1})$. Then $\phi_2(0) < \phi_1(0)$ and $\phi_1^{-1}(1) < \phi_2^{-1}(1)$. This completes the proof of the second part of (i). □

The next lemma states that each element of the set $\mathcal{P}$ is paired with exactly one element of the set $\mathcal{Q}$.

Lemma 4.4.6. *(One-to-one correspondence between $\mathcal{P}$ and $\mathcal{Q}$) Let w be as in Assumption 4.4.1, where $\mathcal{P} \cap \mathcal{Q} = \emptyset$. Then for any legal function ϕ with $0 \in \operatorname{dom}(\phi)$ and signature $(m_1, \ldots, m_{N-1})$ there is a legal function ψ with $1 \in \operatorname{dom}(\psi)$ and signature $(-m_1, \ldots, -m_{N-1})$ and vice versa.*

Proof. Suppose that ϕ is a legal function with signature $(m_1, \ldots, m_{N-1})$ and $0 \in \operatorname{dom}(\phi)$. Let $\phi = \psi_k \circ \ldots \circ \psi_1$ be a decomposition of ϕ into legal functions $\psi_1, \ldots, \psi_k$ with the following property. Let $x_0 = 0, x_i = \psi_i \circ \ldots \circ \psi_1(0)$, $1 \leq i \leq k$. The legal functions $\psi_1, \ldots, \psi_k$ are chosen such that $x_i = \psi_i(x_{i-1})$ is the maximum point of orbit of x_{i-1} under $\psi_k \circ \ldots \circ \psi_i$ when i is odd, and is the minimum point of orbit of x_{i-1} under $\psi_k \circ \ldots \circ \psi_i$ when i is even.

We now prove that $\psi = \psi_k^{-1} \circ \ldots \circ \psi_1^{-1}$ is a legal function with $x_1 \in \mathrm{dom}(\psi)$. From the definition of the x_i's, it is clear that x_1 is the maximum point of the orbit of x_1 under ψ. Then, by Lemma 4.4.4, we have that $1 \in \mathrm{dom}(\psi)$. So $\psi(1) \in \mathcal{Q}$, and ψ has signature $(-m_1, \ldots, -m_{N-1})$.

To prove $x_1 \in \mathrm{dom}(\psi)$ first observe that for $1 \leq i \leq k-1$,

$$\mathrm{range}(\psi_i^{-1}) \subset \mathrm{dom}(\psi_{i+1}^{-1}).$$

Indeed, if i is odd, then the orbits of x_i under ψ_i^{-1} and ψ_{i+1} satisfy the conditions of part (ii) of Corollary 4.4.5. So $\mathrm{range}(\psi_i^{-1}) \subset \mathrm{range}(\psi_{i+1}) = \mathrm{dom}(\psi_{i+1}^{-1})$. On the other hand, if i is even, then the orbits of x_i under ψ_i^{-1} and ψ_{i+1} satisfy conditions of part (i) of Corollary 4.4.5. Therefore $\mathrm{range}(\psi_i^{-1}) \subset \mathrm{range}(\psi_{i+1}) = \mathrm{dom}(\psi_{i+1}^{-1})$, and we are done.

Finally, suppose there are legal functions ϕ and ϕ' with signature $(-m_1, \ldots, -m_{N-1})$ and $1 \in \mathrm{dom}(\phi) \cap \mathrm{dom}(\phi')$. Then $\delta(\phi) = \delta(\phi')$, and by Corollary 4.4.3, $\phi(1) = \phi'(1)$. This proves that the correspondence is one-to-one. The proof of the other side of the Lemma is analogous. □

4.4.2 Equivalence of intervals between points of $\mathcal{P}$ and $\mathcal{Q}$

As we saw earlier, in Remark 11, a uniform embedding π must satisfy certain restrictions as listed in (4.5) and (4.6) of Remark 11. For example we know that $\pi(x)$ and $\pi(r_i^*(x))$ must be within a certain distance d_i. This means that the definition of π on intervals $[x, y]$ and $[r_i^*(x), r_i^*(y)]$ are closely related. This gives rise to some complications that need to be dealt with before we can give the construction of π. Note that we still assume 4.4.1, but we cover both the case $\mathcal{P} = \mathcal{Q}$ and the case $\mathcal{P} \cup \mathcal{Q} = \emptyset$.

Lemma 4.4.7. *Let w be as in Assumption 4.4.1. Let $\overline{\mathcal{P} \cup \mathcal{Q}}$ denote the closure of $\mathcal{P} \cup \mathcal{Q}$ in the usual topology of $[0,1]$. For a countable index set I and pairwise disjoint open intervals I_i, we have $[0,1] \setminus (\overline{\mathcal{P} \cup \mathcal{Q}}) = \cup_{i \in I} I_i$. Let ϕ be a legal function. Then,*

(i) If $\phi(I_i) \cap I_i \neq \emptyset$ then ϕ is the identity function on its domain.

(ii) If $\phi(I_{i'}) \cap I_i \neq \emptyset$ then $\phi(I_{i'}) = I_i$.

Proof. First, we prove that if $x \in I_i$ and ϕ is a legal function with $\phi(x) \in I_i$, then ϕ must be the identity function on its domain. Let $I_i = (a_i, b_i)$. Then a_i and b_i belong to $\overline{\mathcal{P} \cup \mathcal{Q}}$. Fix i. Suppose that $x \in (a_i, b_i)$, and ϕ is a legal function such that $\phi(x) \in (a_i, b_i)$. Towards a contradiction, assume that $\phi(x) \neq x$. Without loss of generality, suppose $\phi(x) > x$. By Proposition 4.2.6, we have $\mathrm{dom}(\phi) = [p, q]$ with $p \in \mathcal{P}, q \in \mathcal{Q}$ and $[a_i, b_i] \subseteq [p, q]$. By Corollary 4.4.3, $\delta(\phi) > 0$ and for every $z \in \mathrm{dom}(\phi)$ we have $z < \phi(z)$. In particular, $\phi^i(p) < \phi^{i+1}(p)$, whenever $p \in \mathrm{dom}(\phi^{i+1})$.

Let M denote the positive integer such that $p \in \mathrm{dom}(\phi^M)$ and $p \notin \mathrm{dom}(\phi^{M+1})$. First, note that such an integer exists. Indeed, if $p \in \mathrm{dom}(\phi^j)$ for every positive integer j, then the increasing sequence $\{\phi^j(p)\}_{j \in \mathbb{N}}$ lies inside $[p, q]$. Therefore, $p_0 = \lim_{j \to \infty} \phi^j(p)$ lies inside $[p, q] = \mathrm{dom}(\phi)$ as well, and we have $\phi(p_0) = p_0$. But this is a contradiction with $\delta(\phi) > 0$. Clearly, for M as above, we have

$$p < \phi(p) < \phi^2(p) < \ldots < \phi^M(p),$$

with $p < a_i$ and $\phi^M(p) > q_i$. Since $\{p, \phi(p), \ldots, \phi^M(p)\} \subseteq \mathcal{P}$, none of these points lie inside (a_i, b_i). Thus, there exists $1 \leq i_0 \leq M - 1$ with $\phi^{i_0}(p) \leq a_i$ and $\phi^{i_0+1}(p) \geq b_i$. So, $\phi^{i_0}(p) < x < \phi^{i_0+1}(p)$. Since ϕ is strictly increasing, this implies that $\phi^{i_0+1}(p) < \phi(x)$, and in particular, $\phi(x) \notin (a_i, b_i)$ which is a contradiction. Thus, we must have $\delta(\phi) = 0$, and ϕ is the identity function on its domain. This proves (i).

Now suppose that ϕ is a legal function. Let $x \in I_{i'}$ and $\phi(x) \in I_i$. In particular, we have $x \in \mathrm{dom}(\phi)$. Thus, there exist $p \in \mathcal{P}$ and $q \in \mathcal{Q}$ such that $I_{i'} \subseteq [p, q] = \mathrm{dom}(\phi)$, since $I_{i'} \cap (\mathcal{P} \cup \mathcal{Q}) = \emptyset$. Since ϕ is a strictly increasing continuous function and for every j, $I_j \cap (\overline{\mathcal{P} \cup \mathcal{Q}}) = \emptyset$, we have that $\phi(a_{i'}) = a_i$ and $\phi(b_{i'}) = b_i$. Thus, $\phi(I_{i'}) = I_i$. □

We say $i \sim i'$ if there exists a legal function ϕ such that $I_i \cap \phi(I_{i'}) \neq \emptyset$ (or equivalently if $\phi(I_{i'}) = I_i$). The relation $\sim$ is an equivalence relation. Consider the equivalence classes produced by $\sim$. For each i, we denote the equivalence class of I_i by $[I_i]$.

4.4.3 Conditions of Theorem 4.1.9 are sufficient.

In this subsection we prove the sufficiency of the conditions of Theorem 4.1.9 by constructing a uniform linear embedding $\pi : [0, 1] \to \mathbb{R}$ when a function w satisfies the conditions of the theorem as given in Assumption 4.4.1.

Define π first on $\mathcal{P}$ by $\pi(x) = \delta(\phi)$, where ϕ is a legal function with $\phi(0) = x$. By Corollary 4.4.3, if there are two legal functions ϕ_1, ϕ_2 with $\phi_1(0) = \phi_2(0) = x$ then

$\delta(\phi_1) = \delta(\phi_2)$. Thus π is well-defined on $\mathcal{P}$. Moreover (2a) tells us that π is strictly increasing on $\mathcal{P}$.

Next, we will extend π to a strictly increasing function on $\mathcal{P} \cup \mathcal{Q}$. If $\mathcal{P} = \mathcal{Q}$ then there is nothing to do. So assume that $\mathcal{P} \cap \mathcal{Q} = \emptyset$. Let

$$\begin{aligned} m &= \sup\{\delta(\phi) | \phi \text{ is a legal function and } 0 \in \operatorname{dom}(\phi)\} \\ M &= \min_{1 \le i \le N-1} \inf\{d_i - \delta(\psi) | \, 1 \in \operatorname{dom}(\psi), \psi(1) < r_i^*(0)\} \end{aligned}$$

By Condition (2b) of Theorem 4.1.9, $m \le M$. Choose $\pi(1)$ as follows: If $m = M$, then let $\pi^-(1) = \pi(1) = m$. Otherwise, choose $\pi^-(1) = \pi(1) \in (m, M)$. Observe that if ϕ and ψ are legal functions with $0 \in \operatorname{dom}(\phi)$, $1 \in \operatorname{dom}(\psi)$, and $\psi(1) < r_i^*(0)$ then $\delta(\phi) < \pi(1) < d_i - \delta(\psi)$. This is clear when $m < M$. In the case where $m = M$, we have $\pi(1) = m = M = a$, and the desired inequality is given by Condition (2b).

Now define π on $\mathcal{Q}$ to be $\pi(y) = \pi(1) + \delta(\phi)$, where ϕ is a legal function with $y = \phi(1)$. Let $y \in \mathcal{Q}$, and ϕ_1, ϕ_2 be legal functions with $\phi_1(1) = \phi_2(1) = y$. Then by Corollary 4.4.3, $\delta(\phi_2) = \delta(\phi_1)$. Also, note that $\mathcal{P} \cap \mathcal{Q} = \emptyset$, so the function π as defined on $\mathcal{Q}$ is well-defined. Moreover π is strictly increasing on $\mathcal{Q}$ by Lemma 4.4.2.

Claim 4.4.8. *π is increasing on $\mathcal{P} \cup \mathcal{Q}$.*

Proof of claim. Recall that we are assuming $\mathcal{P} \cap \mathcal{Q} = \emptyset$. We consider two cases:

Case 1: Assume $x \in \mathcal{P}$, $y \in \mathcal{Q}$, $x < y$. Let ϕ and ψ be legal functions with $\phi(0) = x$ and $\psi(1) = y$. By Proposition 4.2.6, $\operatorname{dom}(\psi) = [p, 1]$ and $\operatorname{range}(\psi) = [\psi(p), \psi(1)]$ where $p \in \mathcal{P}$. If $x = \phi(0) \in \operatorname{range}(\psi)$, then $\psi^{-1} \circ \phi(0) \in \mathcal{P}$. Thus, by choice of $\pi(1)$ we have $\delta(\phi) - \delta(\psi) < \pi(1)$. This implies that $\pi(x) = \delta(\phi) < \pi(1) + \delta(\psi) = \pi(y)$ and we are done. Now assume that $x \notin \operatorname{range}(\psi)$. Then $x < \psi(p)$. By a similar argument, we have $\pi(\psi(p)) < \pi(y)$. Also, since π is strictly increasing on $\mathcal{P}$, $\pi(x) < \pi(\psi(p))$ and thus $\pi(x) < \pi(y)$. This completes the proof of the claim for this case.

Case 2: Assume $x \in \mathcal{P}$, $y \in \mathcal{Q}$, $y < x$. Let $\phi = f_1 \circ \ldots \circ f_s$ be a legal composition with $\phi(0) = x$. Since 0 belongs to the domain of ϕ, we know that f_s is an upper boundary function, say r_i^*. Let $\phi_1 := f_1 \circ \ldots \circ f_{s-1}$. If $y \in \operatorname{range}(\phi_1) = \operatorname{dom}(\phi_1^{-1})$ then $\phi_1^{-1} \circ \psi(1) < r_i^*(0)$, where ψ is a legal function with $\psi(1) = y$. By the choice of

$\pi(1)$ we have $\pi(1) < d_i - (\delta(\psi) + \delta(\phi_1^{-1}))$. Therefore, $\pi(1) + \delta(\psi) < d_i + \delta(\phi_1) = \delta(\phi)$ and thus $\pi(y) < \pi(x)$.

Let us now assume that $y \notin \mathrm{dom}(\phi_1^{-1})$, $\phi_1^{-1} = f_{s-1}^{-1} \circ \ldots \circ f_1^{-1}$. Thus there exists $0 \leq t \leq s-2$ such that $\eta_t(y) := f_t^{-1} \circ \ldots \circ f_0^{-1}(y) \notin \mathrm{dom}(f_{t+1}^{-1})$ where f_0 is the identity function. Since $y < x$ and $x \in \mathrm{dom}(\phi^{-1})$, this implies that f_{t+1}^{-1} must be a lower boundary function, say ℓ_j^*. Also $\eta_t(y) < r_j^*(0)$. Hence by the choice of $\pi(1)$ we have $\pi(1) < d_j - (\delta(\psi) + \delta(\eta_t))$. On the other hand, since $\eta_t(x) \in \mathrm{dom}(f_{t+1}^{-1})$, we have $r_j^*(0) \leq \eta_t(x)$. Therefore $d_j \leq \delta(\phi) + \delta(\eta_t)$, as π is strictly increasing on $\mathcal{P}$. Thus, $\pi(1) + \delta(\psi) < d_j \leq \delta(\phi)$, *i.e.* $\pi(y) < \pi(x)$, and we are done. □

We now extend π to $\overline{\mathcal{P} \cup \mathcal{Q}}$ in the following manner: for any $x \in \overline{\mathcal{P} \cup \mathcal{Q}} \setminus \mathcal{P} \cup \mathcal{Q}$ there exists a sequence $\{x_n\}$ in $\mathcal{P} \cup \mathcal{Q}$ converging to x. We define $\pi(x)$ to be the limit of the sequence $\{\pi(x_n)\}$. We need to prove that this is a well-defined process, *i.e.* for $x \in \overline{\mathcal{P} \cup \mathcal{Q}} \setminus \mathcal{P} \cup \mathcal{Q}$ the value of $\pi(x)$ is independent of the choice of the sequence converging to x.

Let $\{x_n\}$ be a sequence in $\mathcal{P} \cup \mathcal{Q}$ converging to a limit point x. Since π is increasing on $\mathcal{P} \cup \mathcal{Q}$ we have, $\lim_{i \in J_1} \pi(x_i) = \sup\{\pi(x_i) | i \in J_1, x_i < x\}$ and $\lim_{j \in J_2} \pi(x_j) = \inf\{\pi(x_j) | j \in J_2, x < x_j\}$, where J_1 and J_2 are the sets of positive integers i with $x_i \leq x$ and $x_i > x$, respectively. Define $\pi(x) = \sup_{i \in J_1} \pi(x_i)$, if there exists a sequence in $\mathcal{P} \cup \mathcal{Q}$ converging to x from the left. Otherwise, define $\pi(x) = \inf_{j \in J_2} \pi(x_j)$. This extension is well-defined. Indeed, let $\{x_n\}$ and $\{y_n\}$ be two sequences in $\mathcal{P} \cup \mathcal{Q}$ which converge to $x \in [0,1] \setminus (\mathcal{P} \cup \mathcal{Q})$ from left. Since π is increasing, it is easy to see that $\sup\{\pi(x_n) : n \in \mathbb{N}\} = \sup\{\pi(y_n) : n \in \mathbb{N}\}$. Using a similar argument for infimum, we conclude that the extension is well-defined. Moreover if a sequence $\{x_n\}$ in $\mathcal{P} \cup \mathcal{Q}$ converges to $x \in \overline{\mathcal{P} \cup \mathcal{Q}} \setminus \mathcal{P} \cup \mathcal{Q}$ from the left, then $\{\ell_j^*(x_n)\}$ (respectively $\{r_j^*(x_n)\}$) converges to $\ell_j^*(x)$(respectively $r_j^*(x)$) from the left as well. Therefore the function π defined on $\overline{\mathcal{P} \cup \mathcal{Q}} \setminus \mathcal{P} \cup \mathcal{Q}$ satisfies the properties of a uniform linear embedding, *i.e.* Conditions (4.5) and (4.6).

We now define π on $[0,1] \setminus \overline{\mathcal{P} \cup \mathcal{Q}}$ using the equivalence relation defined in Lemma 4.4.7. First recall that $[0,1] \setminus \overline{\mathcal{P} \cup \mathcal{Q}} = \cup_{i \in I} I_i$ for a countable index set I. As discussed in the proof of Lemma 4.4.7, $I_i = (a_i, b_i)$ where $a_i, b_i \in \overline{\mathcal{P} \cup \mathcal{Q}}$. For each equivalence class $[I_i]$, proceed as follows. First, pick a representative I_i for $[I_i]$, and define π on I_i to be the linear function with $\pi(a_i)$ and $\pi(b_i)$ as defined in the previous paragraph (definition of π on $\overline{\mathcal{P} \cup \mathcal{Q}} \setminus \mathcal{P} \cup \mathcal{Q}$). Next, for every $I_j \in [I_i]$, let ϕ be a legal function such that $\phi(I_j) = I_i$, equivalently $a_i = \phi(a_j)$ and $b_i = \phi(b_j)$. For every $x \in I_j = (a_j, b_j)$, we define $\pi(x)$ according to the definition of π on I_i, *i.e.*,

$$\pi(x) = \pi(\phi(x)) - \delta(\phi).$$

Thus π extends to a strictly increasing function on $[0, 1]$ which gives us the desired uniform linear embedding.

4.4.4 Construction of π: complexity and examples

In this subsection we discuss the algorithmic aspects of the necessary and sufficient conditions given in Theorem 4.1.9. Let w be as in Theorem 4.1.9. A natural way of producing the set $\mathcal{P}$ is to do it in stages: Let $\mathcal{P}_0 = \{0\}$, and $\mathcal{P}_i = \{r_j^*(x), \ell_j^*(x) : 1 \leq j \leq N-1, x \in \mathcal{P}_{i-1}\}$. The sets $\mathcal{P}_i$, $i \geq 0$, are generations of $\mathcal{P}$. The process of producing generations of $\mathcal{P}$ either stops after finite steps (when $\mathcal{P}$ is finite) or goes on endlessly (when $\mathcal{P}$ is infinite). Obviously when $\mathcal{P}$ is infinite, the conditions of Theorem 4.1.9 are not testable in finite time. For a finite $\mathcal{P}$, we prove in Theorem 4.4.9 that Condition (2) of Theorem 4.1.9 is testable in polynomial-time in the size of the set $\mathcal{P}$. As for Condition (1) one needs to check if a real function on $[0, 1]$ is equivalent to identity function. This is not generally decidable in finite time.

Theorem 4.4.9. *Let w be as in Assumption 4.2.1 and $\mathcal{P}$ and $\mathcal{Q}$ are as in Definition 4.1.5. Moreover suppose $\mathcal{P}$ is finite. Then Condition (2) of Theorem 4.1.9 is polynomial-time testable.*

Proof. Let $\mathcal{P}$ be a set of size $s+1$ and $\mathcal{P} = \{0 = \psi_0(0), \psi_1(0), \ldots, \psi_s(0)\}$. Condition (2a) holds if there are real numbers $d_{N-1} > \ldots > d_1 > 0$ such that

$$0 < \delta(\psi_1) < \delta(\psi_2) < \ldots < \delta(\psi_s),$$

where $\delta(\psi_j)$ for $1 \leq j \leq s$ is the displacement of ψ_j as in Definition 4.1.8. This gives us s inequalities as follow:

$$\begin{array}{rcl} 0 & < & d_1 m_{1,1} + \ldots + d_{N-1} m_{1,N-1} \\ d_1 m_{1,1} + \ldots + d_{N-1} m_{1,N-1} & < & d_1 m_{2,1} + \ldots + d_{N-1} m_{2,N-1} \\ & \vdots & \\ d_1 m_{s-1,1} + \ldots + d_{N-1} m_{s-1,N-1} & < & d_1 m_{s,1} + \ldots + d_{N-1} m_{s,N-1} \end{array}$$

Th above inequalities can be formulated as $M_1 d < 0$, where M_1 is the $s \times N - 1$ matrix of coefficients and d is the $N - 1 \times 1$ matrix of variables $d_1, \ldots, d_{N-1}$

Therefore Condition (2a) is satisfied if the system of strict linear inequalities $M_1 d < 0$ has a real solution. Note that since $r_i^*(0) \in \mathcal{P}$ and $\delta(r_i^*(0)) = d_i$ for $1 \leq i \leq N - 1$, the condition $d_{N-1} > \ldots > d_1 > 0$ has been already considered in the above system of inequalities.

Let A and b be matrices, with integer entries, of size $m \times n$ and $m \times 1$, respectively. Gács and Lovász in [23] present an algorithm to solve the system of strict linear inequalities $Ax < b$, where x is a real vector in $\mathbb{R}^n$. Their algorithm is an $\mathcal{O}(mn^2 L)$-algorithm, where L is the bit length of the input data:

$$L = \Sigma_{i,j} \log(|a_{ij}| + 1) + \Sigma_i \log(|b_i| + 1) + \log nm + 1.$$

The algorithm in [23] decides whether or not $Ax < b$ is solvable, and if it is the algorithm provides a solution. Let L_1 be the bit length of the input data for the system $M_1 d < 0$. Since $N - 1 \leq |\mathcal{P}|$ and each $m_{i,j}$ is also bounded by the size of $\mathcal{P}$ we have

$$L_1 \leq s(N-1)(2s+1) + s(N-1) + 1 \leq 2(s^3 + s^2) + 1.$$

Therefor we can check in polynomial-time in the size of $\mathcal{P}$ if the system $M_1 d < 0$ has a solution, and if so find one.

Similarly we can convert Condition (2b) to a system of linear inequalities. Indeed if ϕ is a legal composition with $1 \in \mathrm{dom}(\phi)$, and if $\phi(1) < r_i^*(0)$ then $\delta(\psi_j) < d_i - \delta(\phi)$ for any $0 \leq j \leq s$. This gives us $s+1$ inequalities. Therefore for legal composition ϕ we have at most $(N-1) \times (s+1)$ inequalities. Moreover by Lemma 4.4.6, $|\mathcal{Q}| = s + 1$. This implies that Condition (2b) gives us at most $(N-1) \times (s+1)^2$ inequalities.

We denote the system of strict linear inequalities obtained from Condition (2b) by $M_2 d < 0$, where M_2 is the $(s+1)^2 \times (N-1)$ matrix of coefficients and d represents the $N-1 \times 1$ matrix of variables, $d_1, \ldots, d_{N-1}$. Then Condition (2a) and (2b) hold if the following system of inequalities has a solution:

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} d < 0.$$

Let L_2 be the bit length of the input data for $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} d < 0$. A similar discussion shows that L_2 is also bounded with a polynomial function in terms of s. This implies that it is polynomial-time in the size of $\mathcal{P}$ to check if the system has a solution and if so, to find one.

$\square$

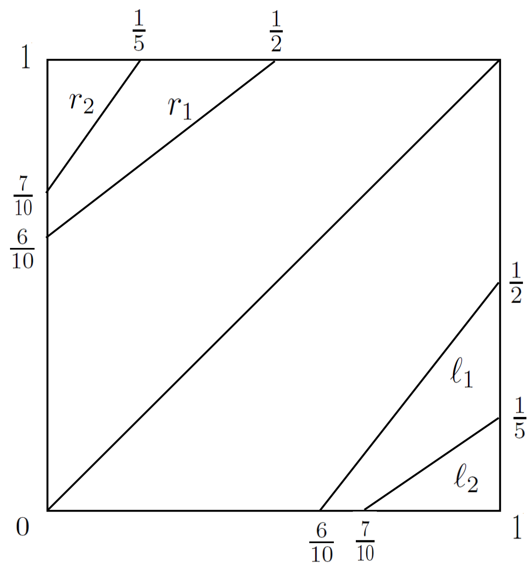
We will now present an example of a diagonally increasing function w, with nice linear boundaries, which produces infinite $\mathcal{P} \cup \mathcal{Q}$, even in the case of a three-valued function w.

Example 10. *For $i = 1, 2$ let $r_i(x) = x + b_i$, where $0 < b_1 < b_2 < \frac{1}{2}$. Moreover assume that $\frac{b_1}{b_2}$ is irrational. We produce a sequence $x_i \in \mathcal{P}$ inductively. Let $x_0 = 0$ and $x_1 = r_2(0) = b_2$. For each $i > 1$, define*

$$x_i = \begin{cases} r_2(x_{i-1}) = x_{i-1} + b_2 & \text{if } x_{i-1} < \frac{1}{2} \\ \ell_1(x_{i-1}) = x_{i-1} - b_1 & \text{if } x_{i-1} \geq \frac{1}{2} \end{cases}$$

Clearly, $x_i \in \mathcal{P}$ as it always lies in $(0, 1)$. Also each x_i is in the form of $m_i b_2 - n_i b_1$ for positive integers n_i, m_i. Moreover, at each step we increase the value of either m_i or n_i by exactly 1. Thus, $m_i + n_i = i$ for every i. The x_i's are all distinct. Namely, if $x_i = x_j$ for positive integers i, j, then $b_1(n_i - n_j) = b_2(m_i - m_j)$, which is a contradiction with the fact that $\frac{b_1}{b_2}$ is irrational. This implies that all x_i's are distinct, and thus $\mathcal{P} \cup \mathcal{Q}$ is an infinite set.

The following example consider a diagonally increasing function w with a finite set of constrained points.



Example 11. *Let w be a well-separated diagonally increasing $\{\alpha_1, \alpha_2, \alpha_3\}$-valued function with $\alpha_1 > \alpha_2 > \alpha_3$ and the following boundary functions.*

$$r_1(x) = \begin{cases} \frac{8}{10}x + \frac{6}{10} & x \in [0, \frac{1}{2}] \\ 1 & x \in [\frac{1}{2}, 1] \end{cases}, \ell_1(x) = \begin{cases} 0 & x \in [0, \frac{6}{10}] \\ \frac{10}{8}(x - \frac{6}{10}) & x \in [\frac{6}{10}, 1] \end{cases}$$

$$r_2(x) = \begin{cases} \frac{3}{2}x + \frac{7}{10} & x \in [0, \frac{1}{5}] \\ 1 & x \in [\frac{1}{5}, 1] \end{cases}, \ell_2(x) = \begin{cases} 0 & x \in [0, \frac{7}{10}] \\ \frac{2}{3}(x - \frac{7}{10}) & x \in [\frac{7}{10}, 1] \end{cases}$$

Generate $\mathcal{P}$ and $\mathcal{Q}$: *We first find the set of constrained points of w.*

$$\mathcal{P} = \{0, r_1^*(0), r_2^*(0), \ell_1^* r_2^*(0), r_2^* \ell_1^* r_2^*(0), \ell_1^* r_2^* \ell_1^* r_2^*(0)\} = \{0, \frac{6}{10}, \frac{7}{10}, \frac{1}{8}, \frac{71}{80}, \frac{23}{64}\}$$

$$\mathcal{Q} = \{1, \ell_1^*(1), \ell_2^*(1), r_1^* \ell_2^*(1), \ell_2^* r_1^* \ell_2^*(1), r_1^* \ell_2^* r_1^* \ell_2^*(1)\} = \{1, \frac{1}{2}, \frac{1}{5}, \frac{38}{50}, \frac{1}{25}, \frac{158}{250}\}$$

Check Condition (2) of Theorem 4.1.9: *We now check if Conditions (2a) and (2b) of Theorem 4.1.9 holds. The order of elements of $\mathcal{P}$ is*

$$0 < \ell_1^* r_2^*(0) < \ell_1^* r_2^* \ell_1^* r_2^*(0) < r_1^*(0) < r_2^*(0) < r_2^* \ell_1^* r_2^*(0),$$

or

$$0 < \frac{1}{8} < \frac{23}{64} < \frac{6}{10} < \frac{7}{10} < \frac{71}{80}.$$

Hence Condition (2a) holds if there are real numbers $d_2 > d_1 > 0$ such that

$$0 < d_2 - d_1 < 2d_2 - 2d_1 < d_1 < d_2 < 2d_2 - d_1$$

This gives us the following system of inequalities.

$$\begin{aligned} d_1 - d_2 &< 0 \\ 2d_2 - 3d_1 &< 0 \end{aligned}$$

A solution to the above system of inequalities is :$d_1 = 1$ *and* $d_2 = \frac{5}{4}$*. Now let us check Condition (2b). We have that*

$$\begin{aligned} \frac{1}{25}, \frac{1}{5}, \frac{1}{2} &< \frac{6}{10} = r_1^*(0) \\ \frac{1}{25}, \frac{1}{5}, \frac{1}{2}, \frac{158}{250} &< \frac{7}{10} = r_2^*(0) \end{aligned}$$

The system of inequalities obtained from the above inequalities is the following

$$\begin{aligned} d_1 - d_2 &< 0 \\ 2d_2 - 3d_1 &< 0 \end{aligned}$$

which is the same as the system of inequalities obtained from Condition (2a). Therefore $d_1 = 1$ *and* $d_2 = \frac{5}{4}$ *satisfy Conditions (2a) and (2b).*

Construct π: *We now construct the function* π *based on the proof of the Theorem 4.1.9. For* $d_1 = 1$ *and* $d_2 = \frac{5}{4}$*, we have* $m = \frac{6}{4}$ *and* $M = \frac{7}{4}$*. Pick* $\pi(1) = \frac{13}{8}$*. The equivalence classes of lemma 4.4.7 for the function* w *are:*

$$\begin{aligned} [I_1] &= \left\{(0, \frac{1}{25}), (\frac{1}{8}, \frac{1}{5}), (\frac{23}{64}, \frac{1}{2}), (\frac{6}{10}, \frac{158}{250}), (\frac{71}{80}, 1)\right\}, \\ [I_2] &= \left\{(\frac{1}{25}, \frac{1}{8}), (\frac{1}{5}, \frac{23}{64}), (\frac{158}{250}, \frac{7}{10}), (\frac{38}{50}, \frac{71}{80})\right\}, \\ [I_3] &= \left\{(\frac{1}{2}, \frac{6}{10})\right\}. \end{aligned}$$

This gives us the following embedding π.

$$\pi(x) = \begin{cases} \frac{25}{8}x & x \in [0, \frac{1}{25}] \\ \frac{25}{17}(x - \frac{1}{8}) + \frac{1}{4} & x \in [\frac{1}{25}, \frac{1}{8}] \\ \frac{5}{3}x + \frac{1}{24} & x \in [\frac{1}{8}, \frac{1}{5}] \\ \frac{5}{51}(8x - \frac{23}{8}) + \frac{1}{2} & x \in [\frac{1}{5}, \frac{23}{64}] \\ \frac{1}{9}(8x - 1) + \frac{7}{24} & x \in [\frac{23}{64}, \frac{1}{2}] \\ \frac{15}{4}(x - \frac{6}{10}) + 1 & x \in [\frac{1}{2}, \frac{6}{10}] \\ \frac{250}{64}(x - \frac{6}{10}) + 1 & x \in [\frac{6}{10}, \frac{158}{250}] \\ \frac{25}{136}(10x - 7) + \frac{5}{4} & x \in [\frac{158}{250}, \frac{7}{10}] \\ \frac{50}{24}(x - \frac{7}{10}) + \frac{5}{4} & x \in [\frac{7}{10}, \frac{38}{50}] \\ \frac{25}{51}(2x - \frac{71}{40}) + \frac{3}{2} & x \in [\frac{38}{50}, \frac{71}{80}] \\ \frac{10}{9}(x - \frac{7}{10}) + \frac{31}{24} & x \in [\frac{71}{80}, 1] \end{cases}$$

Chapter 5

Conclusion and future work

In this thesis we studied spatial embeddings of graphs and random graphs into space $\mathbb{R}^k$ equipped with $\|.\|_\infty$ metric, for $k \leq 2$. Our studies focused on two main directions, that is adressing the question "If, a given graph model G, is compatible with a notion of spatial graph? " for both graphs and random graphs. We studied the mentioned question for graphs in Chapters 2 and 3. Then in Chapter 4, we studied w-random graphs that are compatible with a notion of spatial random graph

Square geometric graphs

We know that the graphs which have a $(\mathbb{R}, \|.\|_\infty)$-geometric representation are the well-known class of unit interval graphs, for which there are several linear time recognition algorithm. However recognition of $(\mathbb{R}^2, \|.\|_\infty)$-geometric graphs is proved to be NP-hard [5, 43]. In this thesis, we addressed the problem of recognition of $(\mathbb{R}^2, \|.\|_\infty)$-geometric graphs for special classes of graphs.

In Chapter 2, we introduced the concept of proper bi-ordering $(<_X, <_Y)$ for a cobipartite graph with clique bipartition X and Y. we proved that a cobipartite graph G is square geometric if and only if G has a proper bi-ordering $(<_X, <_Y)$. Then, we presented an $\mathcal{O}(n^4)$ recognition algorithm for square geometric cobipartite graphs. In Chapter 3, we presented necessary and sufficient conditions for type-1 and type-2 $B_{a,b}$ graphs to be square geometric. We also presented an $\mathcal{O}(n^4)$ algorithm for recognition of square geometric type-1 and type-2 $B_{a,b}$-graphs.

Our results shows that, for a general $B_{a,b}$-graph G with cobipartite subgraphs G_a and G_b, a necessary condition for the graph G to be square geometric is that each cobipartite subgraph, G_a and G_b, must have a proper bi-ordering consistent with the proper bi-ordering of the other cobipartite subgraphs. Initially we believed that, for the general $B_{a,b}$-graphs, this necessary condition would be the backbone of necessary and sufficient conditions. But even for type-1 and type-2 $B_{a,b}$-graphs, as we saw in Chapter 3, this necessary condition is not sufficient. Indeed, there are further structural restrictions (such as the rigid-free conditions) that need to be characterized to obtain necessary and sufficient conditions.

One direction of future research is to study square geometric $B_{a,b}$-graphs or binate interval graphs. Our results shows that obtaining necessary and sufficient conditions for the class of square geometric binate interval graphs will be a challenging problem. However we would not comment that characterizing square geometric binate interval graphs is likely an NP-hard problem. The reason is that despite the fact that the rigid-free conditions are restrictive and not easy to obtain, they can be checked easily in polynomial-time.

We are very hopeful that the methods we developed in the study of square geometric cobipartite graphs, specifically the concept of proper bi-ordering, provide us with the required tools to study graphs that are the union of cobipartite graphs. We precisely target the union of cobipartite graphs which have a simpler structure than binate interval graphs.

One of the classes of such graphs that we are mostly interested to study is the graphs which are obtained by replacing each vertex of a path with a clique, and each edge of the path with a set of arbitrary edges between the two cliques replacing the two ends of the edge. We call them *chain cobipartite graphs*. Our intuition is that for this class of graphs the consistency of the proper bi-orderings of each of the cobipartite graphs of the chain forms the core of the necessary and sufficient conditions for the graph to be square geometric.

The other class of graphs we are interested in is the class of *ring cobipartite graphs*. Similar to the definition of chain cobipartite graphs, a ring cobipartite graph is a circle, C_n, which its vertices is replaced by cliques and each edge is replaced by an arbitrary set of edges between the two cliques replacing the two ends of the edge. We are hoping that our partial ordering approach can be adjusted to study larger family of square geometric graphs, specifically binate interval, chain cobipartite, and ring cobipartite graphs.

Another possible direction that is worthwhile to investigate, is to study the problem of characterizing $(\mathbb{R}^3, \|.\|_\infty)$-geometric cobipartite graphs.

Uniform linear embedding of random graphs

We saw in the introduction that for a random graph with a linear embedding the probability of linking to a vertex decreases as the distance increases. The rate of decrease may depend on the properties of the particular vertex v and on the direction in which the distance increases. Then we asked "how can we recognize when a random graph has a linear embedding such that the probability of a link between two vertices only depends on their distance *i.e.* a uniform linear embedding."

In Section 4.1 we gave necessary and sufficient conditions for the existence of a uniform linear embedding for spatial random graphs where the link probability can attain only a finite number of values (see Theorem 4.1.9). Then in Subsections 4.3.2 and 4.4.3 we proved the necessity and sufficiency of conditions of Theorem 4.1.9. Our findings shows that the existence of the uniform linear embedding is related to the behaviour of the boundaries of the function, w. Indeed, the conditions of Theorem 4.1.9 are mostly conditions on the set of constrained point, which are obtained from the boundary of w.

We showed in Subsection 4.4.4, when the set of constrained points is finite, and the sufficiency conditions hold, the embedding function π and probability function f_{pr} can be computed in time polynomial in the size of the set. But when the set of constrained point is not finite, the necessary and sufficient conditions of Theorem 4.1.9 are not testable in finite time. However, the proof of the sufficiency of the conditions given in Subsection 4.4.3 has a constructive structure *i.e.* assuming the conditions hold the proof suggests a step-by-step construction of a possible uniform linear embedding for w. This puts forward an approximation approach for the case when the set of constrained points is infinite using finite number of generations of constrained points. We are planning to investigate a possible approximation algorithm to find the uniform linear embedding of the functions w with an infinite set of constrained points.

Another interesting question to study is to find the uniform linear embedding of matrices. More precisely, we want to replace the function w in our discussion with a symmetric diagonally increasing $n \times n$ matrix *i.e.* a weighted graph. The concept of a diagonally increasing function w can be defined similarly for a symmetric $n \times n$ matrix. Moreover the entries of the matrix are real numbers in $[0, 1]$. Suppose that $\alpha_1 > \alpha_2 > \ldots > \alpha_{N-1} > 0$ are the entries of w. Then we say w admits a uniform linear embedding if the following conditions satisfy: There exists an embedding π of vertices of the weighted graph into the real line $\mathbb{R}$, and real numbers $0 < d_1 < d_2 < \ldots < d_{N-1}$ such that the vertices x and y are adjacent with probability α_i if and only if their metric distance is at most d_i.

We are hopeful that, in this discrete case, we can also presents necessary and sufficient condition for a matrix to admit a uniform linear embedding.

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