M-CONVEXITY, EXTENSION AND EQUILIBRIUM EXISTENCE THEOREMS IN G-CONVEX SPACES

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To My Parents and My Husband

Abstract

This thesis is devoted to the study of G-convex spaces.

In Chapter 1, we introduce the new concepts of M-convex spaces and M-convexity. We present a KKM-type theorem, and two fixed point theorems which illustrate the significance of these concepts. We also define a G-convex structure on the product of a family of G-convex spaces.

In Chapter 2, we prove that any complete metric space with a continuous midpoint function is a G-convex space.

In Chapter 3, we prove several Dugundji-type extension theorems in G-convex spaces. Both cases of single and set-valued maps are considered. Important applications to the theory of games are obtained from these extension theorems.

In Chapter 4, we define M-convexity and M-concavity for real functions on an M-convex space. A continuous dual is also defined and we give solutions for some variational inequalities.

In Chapter 5, we define classes of GL_S and GL_S-majorized correspondences. We obtain some maximal element theorems for these correspondences and apply them to generalized games and minimax inequalities.

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CONTENTS

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INTRODUCTION

In 1993, Sehie Park introduced the concept of a generalized convex space or a G-convex space. Although this new concept generalizes the notion of a topological vector space, it was mainly developed in connection with fixed point theory and KKM theory. This is why it should come as no surprise that many known theorems in that field remain true in G-convex spaces, after the necessary modifications.

In 1994, Park also defined admissible multifunctions on G-convex spaces and in [PK2], Park and Kim proved a coincidence theorem and applied it to obtain an abstract variational inequality, a KKM-type theorem and a fixed point theorem. Their results included a large number of known theorems as particular cases.

As a more abstract setting for the concept of a topological vector space, this new concept comes at the top of a chain of several generalizations that can now be seen as particular forms of G-convex spaces. Michael's convex structure for a metric space [Mi] introduced in 1959 is one example. The S-contractible space [Pa] of 1980, Komiya's convex space [K] of 1981, Lassond's convex space [L] as well as the pseudo convex space [Ho1] in 1983, Bielawskie's simplicial convexity [Bi] as well as Horvath's H-space [Ho2] in 1987, and Joo's convex space [J] of 1989 are all particular forms of G-convex spaces. For the references and a detailed description of the relations between these, see [PK1].

We believe that this new concept gives rise to many questions and provides a rich area for research and study. This thesis is entirely devoted to this subject.

In Chapter One, the new concept of an $\mathcal{M}$-convex space is introduced. An $\mathcal{M}$-convex space is a G-convex space with the additional structure of a G-map system. Homogeneous G-map systems are also defined as a special case whose presence in an $\mathcal{M}$-convex space enables us to prove a selection theorem in this chapter and an extension theorem later on in Chapter Four. Other new concepts are introduced in this chapter, like $\mathcal{M}$-convexity of subsets. We believe this to be a more adequate concept of convexity in $\mathcal{M}$-convex spaces than that of G-convexity, the common notion for convexity in G-convex spaces introduced in [PK2]. The definition of $\mathcal{M}$-KKM mappings is also given. Several results are presented in this chapter which illustrate the significance of these new concepts. Among these, perhaps the definition of a product G-convex space and the related theorem presented in section three best illustrate the significance of these concepts.

In Chapter Two we introduce a new example of a G-convex space whose convexity comes from a metric. Our main motivation is Takahashi's convex metric spaces [Tak] introduced in 1970. We begin Chapter Two by defining these spaces and refer to them as G-metrically convex spaces. Then we prove that they are G-convex spaces. This involves constructing a continuous function from the standard n-dimensional simplex in $\mathbb{R}^n$ to $\Gamma(A)$, a certain subset of the metric space assigned for each finite subset A. This continuous function must also satisfy a certain condition required for the G-convex structure. Our goal is finally reached by the use of six lemmas. Although these may look tedious at first glance, only some simple analysis is needed for the proofs. Despite the fact that getting a new example of a G-convex space is well worth the task, we also find that our constructive proof of G-convexity provides us with two kinds of G-map systems which we present in the second section of Chapter Two. Thus G-metrically convex spaces turn out to be $\mathcal{M}$-convex spaces after all.

Chapter Three contains new extension theorems in G-convex spaces and we are not aware of any other work in this field so far. If X is an arbitrary metric space and A a closed subset of X, Tietze's extension theorem states that any continuous $f : A \to \mathbb{R}$ has a continuous extension $\widehat{f} : X \to \mathbb{R}$. In 1951, J.Dugundji [DJ, Theorem 4.1] proved a generalization of Tietze's theorem where $\mathbb{R}$ is replaced by any locally convex toplogical vector space.

In 1972, Tsoy-Wo Ma [Ma, Theorem 2.1] proved a generalization of Dugundji's theorem to upper semicontinuous mappings with compact convex values. Ma applied his theorem to a construction of the topological degree theory for compact convex valued vector fields in locally convex spaces.

In 1996, Tadeusz Pruszko [Psz, Theorem 1.1] proved an extension theorem for an upper semicontinuous mapping with compact convex values that is dominated by a completely continuous mapping. In Pruszko's theorem the extension obtained is completely continuous and the range is assumed to be a normed space.

In 1997, Zhou Wu [TW, Theorem 2.4] proved a version of Ma's theorem where the mapping is assumed to have star-shaped values instead of convex values. Wu applied his theorem to the study of duals of the theorems of Gale-Mas-Colell and of Shafer-Sonnenschein.

In Chapter Three we give generalizations of Dugundji's Theorem, Ma's Theorem and Pruszko's Theorem to G-convex spaces. In Section 1, we give several extension theorems for a single-valued continuous map into a G-convex space. In Section 2, we present a generalizaton of Ma's Theorem and in Section 3 we give a generalization for Pruszko's theorem. In Section 4, we adopt the method used by Wu [TW, Theorem 3.3] to obtain applications of our extension theorems to equilibrium existence theorems for qualitative games. We admit though that throughout all the new extension theorems, Dugundji's magic touch prevails, and his lemma [DJ, Lemma 2.1] is used in all the proofs except for that of the last theorem. But we must also say in fairness that Theorem 3.2.1 is a work of its own, especially where nonlinear aspects are concerned.

Chapter Four deals with the subject of variational inequalities. As variational inequality theory has many important applications in partial differential equations, operations research, mathematical programming and optimization theory, we have tried to generalize some of the known results to G-convex spaces.

The content of this chapter concerns generalizations of some variational inequalities given by K.K. Tan, E. Tarafdar, and X.Z. Yuan in [TTY] to $\mathcal{M}$-convex spaces.

We introduce $\mathcal{M}$-convexity, and $\mathcal{M}$-concavity for real functions (both set-valued and single-valued) on an $\mathcal{M}$-convex space. We also introduce the concept of an $\mathcal{M}$-affine real function. Then we construct a so-called dual space $X^*_{\mathcal{M}}$ which consists of all $\mathcal{M}$-affine continuous real-valued functions.

Our main tools for obtaining the solutions of variational and quasi-variational inequalities are a KKM-type theorem (Theorem 1.2.2) and a Fan- Glicksberg-type theorem (Theorem 1.2.3).

In Chapter Five, we define the concept of GL_S-majorized correspondences in G-convex spaces. By imposing one condition on a G-convex space, a so-called compact G-polytope property, we obtain several maximal element theorems for both compact and noncompact domains. Applying these, we obtain equilibrium existence theorems for generalized games. Although generalized games were dealt with in Chapter Three, the correspondences considered there were upper semicontinuous.

Finally the concept of GL_S-majorized families of real functions on G-convex spaces is also introduced. Minimax inequalities are also given for these in proving which we apply the maximal element theorems obtained earlier.

CHAPTER ONE

$\mathcal{M}$-CONVEX SPACES

In this chapter we define an $\mathcal{M}$-convex space, which is a G-convex space together with an additional structure, i.e. that of a G-map system. But we would like to point out here that although an $\mathcal{M}$-convex space is essentially a special case of a G-convex space, it is also possible to view the latter as a special case of the former. This is due to the fact that every G-convex space has an obvious or a trivial G-map system as we show in Proposition 1.1.1.

In section 1, we give the definition of a G-map system and also define a special kind of homogeneous G-map system. We also give a selection theorem for G-convex spaces that have such G-map systems.

In section 2, we define $\mathcal{M}$-convexity, $\mathcal{M}$-KKM mappings and present a KKM-type theorem together with some fixed point theorems.

In section 3, we define a G-convex structure on $X = \Pi_{i \in I} X_i$ where each (X_i, Γ_i) is a G-convex or an $\mathcal{M}$-convex space.

Preliminaries.

Throughout this thesis, $\langle X \rangle$ denotes the collection of all nonempty finite subsets of a given set X, Δ_n denotes the standard n-dimensional simplex in $\mathbb{R}^n$, and if A is any set, $|A|$ denotes the cardinality of A.

The following is the classical Knaster-Kuratowski-Mazurkiewicz (KKM) Theorem in [KKM].

Knaster-Kuratowski-Mazurkiewicz Theorem. *Let $F_0, F_1, \ldots, F_n$ be closed subsets of Δ_n. Assume that $co(\{e_{i_0}, \ldots, e_{i_k}\}) \subset \bigcup_{j=0}^{k} F_{i_j}$ for any $\{i_0, \ldots, i_k\} \subset \{0, 1, \ldots, n\}$. Then $\bigcap_{i=0}^{n} F_i \neq \emptyset$.*

Definition. Let X, Y be two topological spaces and $T : X \to 2^Y$ be a set-valued mapping. Then

(a) T is upper semicontinuous (USC) at $x_0 \in X$ iff whenever O is an open subset of Y containing $T(x_0)$ then there exists a neighbourhood (abbreviated as nhood) N of x_0 in X such that $T(x) \subset O$, for all $x \in N$. T is upper semicontinuous (USC) on X if it is upper semicontinuous at every $x \in X$.

(b) T is lower semicontinuous (LSC) at $x_0 \in X$ iff whenever O is an open subset of Y such that $T(x_0) \cap O \neq \emptyset$, there exists a neighbourhood N of x_0 in X such that $T(x) \cap O \neq \emptyset$, for all $x \in N$. T is lower semicontinuous (LSC) on X if it is lower semicontinuous at every $x \in X$.

1. G-Map systems.

We begin by giving the definition of a G-convex space, a new concept introduced by Sehie Park in 1993 [PK2].

Definition 1.1.1. (a) A generalized convex space or a G-convex space $(X, D; \Gamma)$ consists of a topological space X, a nonempty subset D of X, and a map $\Gamma : \langle D \rangle \to 2^X$ with nonempty values such that

(1) for each $A, B \in \langle D \rangle, A \subset B$ implies $\Gamma(A) \subset \Gamma(B)$; and

(2) for each $n \in BbbN$ and for each $A \in \langle D \rangle$ with $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that

$$J \in \langle A \rangle \Rightarrow \phi_A(\Delta_J) \subset \Gamma(J), \qquad (*)$$

where Δ_J denotes that face of Δ_n corresponding to $J \in \langle A \rangle$; i.e.
if $J = \{a_{i_0}, \ldots, a_{i_k}\} \subset A$, then $\phi_A(co\{e_{i_0}, \ldots, e_{i_k}\}) \subset \Gamma(J)$.

If $D = X$, then $(X, D; \Gamma)$ will be denoted by $(X; \Gamma)$.

(b) For a G-convex space $(X; D; \Gamma)$, a subset C of X is said to be G-convex if for each $A \in \langle C \rangle, \Gamma(A) \subset C$. The G-convex hull of a subset C, denoted by G-co(C), is the smallest G-convex subset of X that contains C.

The structure of a G-convex space $(X; \Gamma)$ gives a continuous map $\phi_A : \Delta_{|A|-1} \to \Gamma(A)$ for every finite subset A of X which satisfies the condition (*).

A G-map system is a collection of such maps assigned for each finite subset A in such a way that the maps assigned for A and those assigned for A_1 are related in a certain way whenever $A_1 \subset A$.

Only one example of a G-map system is given in this section. But in Chapter 2 we shall present two interesting examples in Theorems 2.2.2 and 2.2.3 in Section 2; where the second one of these is homogeneous.

Definition 1.1.2. Let (X, Γ) be a G-convex space. A G-map system $\mathcal{M}$ on (X, Γ) is defined as a collection $\{\mathcal{M}(A) : A \in \langle X \rangle\}$ such that

(a) $\mathcal{M}(A)$ is a nonempty collection of maps from Δ_n to $\Gamma(A)$ for each $A = \{a_0, a_1, \ldots, a_n\} \in \langle X \rangle$. Moreover each $\phi \in \mathcal{M}(A)$ is continuous and satisfies the following condition:

$$\phi(co(\{e_j : j \in J\})) \subset \Gamma(\{a_j : j \in J\}), \text{ for each } J \subset \{0, 1, \ldots, n\}.$$

(b) For each $A = \{a_0, a_1, \ldots, a_n\} \subset X$ and each $\phi \in \mathcal{M}(A)$, if $A_1 = \{a_{l_0}, a_{l_1}, \ldots, a_{l_m}\} \subset A$ then there exists $\phi^* \in \mathcal{M}(A_1)$ such that

$$\phi(\Sigma_{r=0}^{m}\lambda_r e_{l_r}) = \phi^*(\Sigma_{r=0}^{m}\lambda_r e_r) \text{ for all } \lambda_0, \ldots, \lambda_n \geq 0 \text{ with } \Sigma_{r=0}^{m}\lambda_r = 1.$$

Definition 1.1.3. An $\mathcal{M}$-convex space $(X, \Gamma, \mathcal{M})$ is defined to be a G-convex space (X, Γ) together with a G-map system $\mathcal{M}$.

Proposition 1.1.1. *Let (X, Γ) be a G-convex space. Then there exists a G-map system $\tilde{\mathcal{M}}$ on (X, Γ) such that if $\mathcal{M}$ is any other G-map system on (X, Γ) then:*

$\mathcal{M}(A) \subset \tilde{\mathcal{M}}(A)$ for any finite subset $A \subset X$.

Proof. For each finite subset $A = \{a_0, a_1, \ldots, a_k\}$ of X, let $\tilde{\mathcal{M}}(A)$ be the collection of all continuous maps ϕ from Δ_k to $\Gamma(A)$ that satisfy the following condition:

$$\phi(co(\{e_j : j \in J\})) \subset \Gamma(\{a_j : j \in J\}) \text{ for any subset } J \subset \{0, 1, \ldots, k\}. \qquad (*)$$

Then $\tilde{\mathcal{M}}(A)$ is nonempty by the definition of a G-convex space and it is obvious that $\tilde{\mathcal{M}}$ satisfies (a) in Definition 1.1.2 above. Next we show that $\tilde{\mathcal{M}}$ also satisfies (b) in Definition 1.1.2.

Let $A_1 = \{a_{i_0}, a_{i_1}, \ldots, a_{i_m}\} \subset A$ and $\phi \in \tilde{\mathcal{M}}(A)$. Let $\phi^* : \Delta_m \to \Gamma(A)$ be defined by:

$$\phi^*(\Sigma_{j=0}^{m}\lambda_j e_j) = \phi(\Sigma_{j=0}^{m}\lambda_j e_{i_j}). \qquad (**)$$

We shall show that $\phi^* \in \tilde{\mathcal{M}}(A_1)$.

First, by (*), we have:

$\phi^*(\Delta_m) = \phi(co(\{e_{i_0}, e_{i_1}, \ldots, e_{i_m}\}) \subset \Gamma(\{a_{i_0}, a_{i_1}, \ldots, a_{i_m}\}) = \Gamma(A_1)$; hence ϕ^* is indeed defined from Δ_m to $\Gamma(A_1)$. Moreover ϕ^* is clearly continuous.

Next we show that ϕ^* satisfies (*) above. So let $J \subset \{0, 1, \ldots, m\}$. Then

$$\phi^*(co(\{e_j : j \in J\})) = \phi(co(\{e_{i_j} : j \in J\})). \quad (1)$$

By (*), the R.H.S. of (1) is contained in $\Gamma(\{a_{i_j} : j \in J\}$. Therefore $\phi^* \in \tilde{\mathcal{M}}(A_1)$.

Thus we have shown that $\tilde{\mathcal{M}}$ is a G-map system on (X, Γ) and it obviously satisfies the assertion in the Proposition. □

Definition 1.1.4. Let (X, Γ) be a G-convex space. Let $\mathcal{M}$ be a G-map system on (X, Γ). Then $\mathcal{M}$ is said to be a homogeneous G-map system iff $\mathcal{M}(A)$ is a singleton for each finite subset A of X.

Remark. *Let (X, Γ) be a G-convex space with a homogeneous G-map system $\mathcal{M}$. Let $\mathcal{M}(A) = \{\phi_A\}$ for each finite subset A of X. Let $A = \{a_0, a_1, \ldots, a_n\}$ and $A_1 = \{a_{l_0}, a_{l_1}, \ldots, a_{l_m}\} \subset A$. Then*

$$\phi_A(\Sigma_{j=0}^m \lambda_j e_{l_j}) = \phi_{A_1}(\Sigma_{j=0}^m \lambda_j e_j) \text{ whenever } \lambda_0, \ldots, \lambda_m \geq 0 \text{ with } \Sigma_{j=0}^m \lambda_j = 1.$$

Example 1.1.1. *Every nonempty convex subset of a topological vector space $(V, +, \cdot, \tau)$ is an $\mathcal{M}$-convex space with a homogeneous G-map system.*

Let $\Gamma(A) = co(A)$ for every finite subset A of X. If $A = \{a_0, a_1, \ldots, a_k\} \subset X$, let $\phi_A : \Delta_{|A|-1} \to \Gamma(A)$ be defined by:

$$\phi_A(\Sigma_{j=0}^k \lambda_j e_j) = \Sigma_{j=0}^k \lambda_j a_j.$$

Then it is easy to see that $\mathcal{M} = \{\{\phi_A\} : A \in \langle X \rangle\}$ is a homogeneous G-map system on (X, Γ).

In the following we give a selection theorem that holds for a G-convex space with a homogeneous G-map system. We present it here to illustrate the importance of this new concept. One other illustration of the use of this concept will be seen in Chapter Three where an extension theorem (Theorem 1.4) is presented.

Before we present our theorem, we state a selection theorem of Tan and Zhang (Theorem 2.3 in [TZ]).

Theorem 1.1.1. *Let X be a compact topological space and $(Y;\Gamma)$ be a G-convex space . Suppose $S, T : X \to 2^Y \setminus \{\emptyset\}$ are such that*

(1) for each $x \in X$, $S(x) \subset T(x)$;

(2) for each $x \in X$, $T(x)$ is G-convex;

(3) for each $y \in Y$, $S^{-1}(y)$ is open in X.

Then there exist $A \in \langle Y\rangle$, continuous functions $g : \Delta_n \to \Gamma(A)$ and $\phi : X \to \Delta_n$ where $|A| = n+1$ such that $f = g \circ \phi$ is a continuous selection of T; i.e. $f(x) \in T(x)$ for all $x \in X$.

The following is a noncompact version of the above theorem. It is also a generalization to G-convex spaces of a selection theorem of Yennelis and Prabhakar [YP].

Theorem 1.1.2. *Let X be a paracompact topological space and $(Y;\Gamma)$ be a G-convex space with a homogeneous G-map system. Suppose $S, T : X \to 2^Y \setminus \{\emptyset\}$ are such that*

(1) for each $x \in X$, $S(x) \subset T(x)$;

(2) for each $x \in X$, $T(x)$ is G-convex;

(3) for each $y \in Y$, $S^{-1}(y)$ is open in X.

Then there exists a continuous selection of T; i.e. a continuous map $f : X \to Y$ such that $f(x) \in T(x)$ for each $x \in X$.

Proof. Using the assumption that (Y, Γ) has a homogeneous G-map system, we assign for each finite subset $A = \{a_0, \dots, a_n\}$ of Y a continuous map $\phi_A : \Delta_n \to \Gamma(A)$ which satisfies (*) and (**) in the following.

For any $0 \le i_0 < i_1 < \cdots < i_m \le n$,

$$\phi_A(co(\{e_{i_j} : 0 \le j \le k\}) \subset \Gamma(\{a_{i_j} : 0 \le j \le k\}). \tag{*}$$

For any subset $A_1 = \{a_{i_0}, a_{i_1}, \dots, a_{i_m}\}$ of A, we have

$$\phi_A(\Sigma_{j=0}^m \lambda_j e_{i_j}) = \phi_{A_1}(\Sigma_{j=0}^m \lambda_j e_j). \tag{**}$$

By (3), $\{S^{-1}(y) : y \in Y\}$ is an open cover for X. Let $\mathcal{U}$ be an open locally finite refinement for this cover. For each $U \in \mathcal{U}$, choose $y_U \in Y$ such that

$$U \subset S^{-1}(y_U). \tag{1}$$

Let $\{\beta_U\}_{U \in \mathcal{U}}$ be a continuous partition of unity on X subordinate to the covering $\mathcal{U}$. For each $x \in X$, let $A_x = \{y_U \in Y : x \in U\} = \{y_0, y_1, \dots, y_m\}$ and define $f : X \to Y$ by

$$f(x) = \phi_{A_x}(\Sigma_{i=0}^m (\Sigma_{y_U = y_i} \beta_U(x)) e_i).$$

We shall show that f is a continuous selection for T.

Indeed, let $x \in X$ be given. Then the range of $\phi_{A_x} \subset \Gamma(A_x) = \Gamma(\{y_0, \dots, y_m\})$. Moreover, for each $y_j \in A_x$, we have $x \in U \subset S^{-1}(y_j)$ for some $U \in \mathcal{U}$. i.e. $y_j \in S(x) \subset T(x)$. Therefore it follows by the G-convexity of $T(x)$ that range$(\phi_{A_x}) \subset T(x)$. Hence $f(x) \in T(x)$. Thus f is a selection of T.

To complete the proof, it remains to show that f is continuous at every $x_0 \in X$.

Let W be an open nhood of $x_0 \in X$ that intersects with finitely many elements of $\mathcal{U}$, say $C_W = \{U \in \mathcal{U} : U \cap W \neq \emptyset\}$. Let $A_W = \{y_U : U \in C_W\} = \{\grave{y}_0, \grave{y}_1, \ldots, \grave{y}_k\}$.

Define $\mu : W \to \Delta_k$ by

$$\mu(x) = \Sigma_{j=0}^{k}(\Sigma_{y_U=\grave{y}_j}\beta_U(x))e_j.$$

Then we will prove the continuity of f at x_0 through the following two steps.

Step 1. μ is well defined and continuous.

Indeed first let $x \in W$ be given. Note that

$$\beta_U(x) \neq 0 \Rightarrow U \in C_W \Rightarrow y_U \in A_W \Rightarrow y_U = \grave{y}_j \text{ for some } 0 \leq j \leq k.$$

Therefore $\Sigma_{j=0}^{k}\Sigma_{y_U=\grave{y}_j}\beta_U(x) = 1$ and μ is thus well defined.

Next we show that μ is continuous. For each $0 \leq l \leq k$, let $\{U \in C_W : y_U = \grave{y}_l\} = \{U_{j_1}, \ldots, U_{j_{n_l}}\}$.

Let $r_l : W \to [0, 1]$ be defined by :

$$r_l(x) = \Sigma_{i=1}^{n_l}\beta_{U_{j_i}}(x),$$

then r_l is obviously continuous for each $0 \leq l \leq k$. Also $\mu(x) = \Sigma_{l=0}^{k}r_l(x)e_l$. Thus it follows that μ is continuous.

Step 2. $(\phi_{A_W} \circ \mu)(x) = f(x)$, for each $x \in W$, where $\phi_{A_W} : \Delta_k \to \Gamma(A_W) = \Gamma(\{\grave{y}_0, \ldots, \grave{y}_k\})$ is the continuous map provided by the assumption that $(Y; \Gamma)$ has a homogeneous G-map system and satisfying (*) and (**).

Let $x \in W$ and let $A_x = \{y_0, y_1, \ldots, y_m\}$. Then $A_x \subset A_W$ and therefore $\{y_0, y_1, \ldots, y_m\} = \{\grave{y}_{i_0}, \grave{y}_{i_1}, \ldots, \grave{y}_{i_m}\}$ where $\grave{y}_{i_j} = y_j$ for each $0 \leq j \leq m$.

Now $f(x) = \phi_{A_x}(\Sigma_{j=0}^{m}\lambda_j e_j)$ where $\lambda_j = \Sigma_{y_U=y_j}\beta_U(x)$.

Since $A_x \subset A_W$, applying (**) to the above we have:

$$\begin{aligned} f(x) &= \phi_{A_W}(\Sigma_{j=0}^{m}\lambda_j e_{i_j}) = \phi_{A_W}(\Sigma_{j=0}^{m}(\Sigma_{y_U=y_j}\beta_U(x))e_{i_j}) \\ &= \phi_{A_W}(\Sigma_{j=0}^{m}(\Sigma_{y_U=\dot{y}_{i_j}}\beta_U(x))e_{i_j}) = \phi_{A_W}(\Sigma_{l\in\{i_0,\ldots,i_m\}}(\Sigma_{y_U=\dot{y}_l}\beta_U(x))e_l). \end{aligned} \tag{2}$$

But we notice that

$$l \in \{0,1,\ldots,k\} \setminus \{i_0, i_1, \ldots, i_m\} \Rightarrow \Sigma_{y_U=\dot{y}_l}\beta_U(x) = 0. \tag{3}$$

Let us first prove (3). Indeed, $l \in \{0,1,\ldots,k\} \setminus \{i_0,\ldots,i_m\} \Rightarrow \dot{y}_l \notin \{y_U : x \in U\} \Rightarrow x \notin U$ whenever $y_U = \dot{y}_l \Rightarrow \beta_U(x) = 0$ whenever $y_U = \dot{y}_l \Rightarrow \Sigma_{y_U=\dot{y}_l}\beta_U(x) = 0$.

Now, applying (3), the R.H.S. of (2) is equal to

$$\phi_{A_W}(\Sigma_{l=0}^{k}(\Sigma_{y_U=\dot{y}_l}\beta_U(x))e_l) = (\phi_{A_W} \circ \mu)(x).$$

Therefore it follows by (2) that $f(x) = (\phi_{A_W} \circ \mu)(x)$. □

2. $\mathcal{M}$-Convexity And A Related KKM-type Theorem.

In this section we give a definition of generalized $\mathcal{M}$-KKM mappings, which, as we shall show, is a generalization of the definition of generalized G-KKM mappings given by Tan in [T] (Definition 1.4). In [T], a G-KKM theorem is given which generalizes the celebrated Ky Fan KKM Principle whose numerous applications to minimax inequalities and variational inequalities cannot be over emphasized.

By modifying the proof of the G-KKM theorem in [T], a new generalization of the theorem is obtained. Applications to the solutions of variational inequalities will be seen in Chapter 4.

We also define a new kind of convexity in $\mathcal{M}$-convex spaces, which we believe will be very useful and may replace the concept of G-convexity in many cases. An application of this concept will be presented in Chapter 4 when we study variational inequalities for set valued mappings.

We present generalizations of two fixed point theorems due to Park [P]. We also show how a G-convex subspace is induced on any $\mathcal{M}$-convex subset. Although an obvious fact in the case of G-convex subsets, this statement needs proof in the more general case of $\mathcal{M}$-convex subsets.

Definition 1.2.1. Let X be a topological space, $(Y;\Gamma)$ a G-convex space, $\mathcal{M}$ a G-map system associated with Γ and $T : X \to 2^Y$. Then T is said to be generalized $\mathcal{M}$-KKM if for any $x_0, x_1, \dots, x_n \in X$ there exist $y_0, y_1, \dots, y_n \in Y$ such that for any subset $J \subset \{0, 1, \dots, n\}$ and for any $\phi \in \mathcal{M}(\{y_0, y_1, \dots, y_n\})$ we have $\phi(\Delta_J) \subset \bigcup_{j \in J} T(x_j)$, where Δ_J is that face of Δ_n corresponding to J.

We shall show in the following that the concept of a generalized $\mathcal{M}$-KKM mapping generalizes that of a generalized G-KKM mapping introduced in Definition 1.4 in [T], which we quote below.

Definition 1.2.2. Let X be a topological space, $(Y;\Gamma)$ be a G-convex space and $T : X \to 2^Y$. Then T is a generalized G-KKM map if for each finite subset $\{x_1, \dots, x_n\}$ of X, there exists a finite subset $\{y_1, \dots, y_n\}$ of Y such that for any subset $\{y_{i_1}, \dots, y_{i_k}\}$ of $\{y_1, \dots, y_n\}$, G-co$(\{y_{i_1}, \dots, y_{i_k}\}) \subset \cup_{j=1}^{k} T(x_{i_j})$.

Proposition 1.2.1. *Let X be a topological space, (Y,Γ) a G-convex space, $\mathcal{M}$ a G-map system associated with Γ and $T : X \to 2^Y$.*
If T is a generalized G-KKM map then T is generalized $\mathcal{M}$-KKM.

Proof. Let $x_0, \dots, x_n \in X$, then there exists $y_0, \dots, y_n \in Y$ such that for any subset $J \subset \{0, 1, \dots n\}$ we have G-co$(\{y_j : j \in J\}) \subset \bigcup_{j \in J} T(x_{i_j})$.

So if $\phi \in \mathcal{M}(\{y_0, \dots, y_n\})$ and if Δ_J is that face of Δ_n corresponding to J then:

$$\phi(\Delta_J) \subset \Gamma(\{y_j : j \in J\}) \subset \text{G-co}(\{y_j : j \in J\}) \subset \bigcup_{j \in J} T(x_{i_j}). \quad \square$$

Before we give our KKM-type theorem we quote the following from [T] (Definition 1.5, Theorem 2.5 and Theorem 2.6, respectively).

Definition 1.2.3. Let $(X;\Gamma)$ be a G-convex space and $A \subset X$. Then A is said to be finitely G-closed if for each $B \in \langle X \rangle$, $A \cap G\text{-co}\,(B)$ is closed in $G\text{-co}\,(B)$.

Theorem 1.2.1. *Let X be any non-empty set and $(Y;\Gamma)$ be a G-convex space. Suppose $T : X \to 2^Y \setminus \{\emptyset\}$ is such that each $T(x)$ is finitely G-closed.*

(1) If T is a generalized G-KKM map, then the family $\{T(x) : x \in X\}$ of subsets of Y has the finite intersection property.

(2) If the family $\{T(x) : x \in X\}$ has the finite intersection property and $\{y\}$ is G-convex for each $y \in Y$, then T is a generalized G-KKM map.

Theorem 1.2.2. *Let X be any non-empty set, $(Y;\Gamma)$ be a G-convex space. Suppose $T : X \to 2^Y \setminus \{\emptyset\}$ is such that (a) each $T(x)$ is compactly closed in Y and (b) $\cap_{j=1}^m T(x_j)$ is compact for some $x_1, \ldots, x_m \in X$.*

(1) If T is a generalized G-KKM map, then $\cap_{x \in X} T(x) \neq \emptyset$.

(2) If $\cap_{x \in X} T(x) \neq \emptyset$ and $\{y\}$ is G-convex for each $y \in Y$, then T is a generalized G-KKM map.

The following lemma generalizes (1) in Theorem 1.2.1 above.

Lemma 1.2.1. *Let X be a topological space, (Y,Γ) be a G-convex space, $\mathcal{M}$ be a G-map system associated with Γ and $T : X \to 2^Y$. Assume*
(i) $T(x)$ is finitely G-closed for each $x \in X$;
(ii) T is generalized $\mathcal{M}$-KKM.
Then the family $\{T(x) : x \in X\}$ of subsets of X has the finite intersection property.

Proof. Let $x_0, x_1, \ldots, x_n \in X$. Then there exist $y_0, y_1, \ldots, y_n \in Y$ such that whenever $0 \le i_0 < i_1 < \cdots < i_k \le n$ and $\phi \in \mathcal{M}(\{y_0, y_1, \ldots, y_n\})$,

$$\phi(\Delta_J) \subset \bigcup_{j=0}^{k} T(x_{i_j}) \text{ where } \Delta_J \text{ is that face of } \Delta_n \text{ corresponding to } J. \qquad (*)$$

Now let $\phi \in \mathcal{M}(\{y_0, y_1, \ldots, y_n\}$, let $S = \text{G-co}(\{y_0, \ldots, y_n\})$ and $G_i = \phi^{-1}(T(x_i) \cap S)$, for each $0 \le i \le n$.

Then the G_i's are closed and it is also easy to verify that $G_0, G_1, \ldots, G_n$ satisfy all the conditions of the classical KKM theorem.

For let $J = \{i_0, i_1, \ldots, i_k\} \subset \{0, 1, 2, \ldots, n\}$. Then by our assumption that T is generalized $\mathcal{M}$-KKM, we have $\phi(\Delta_J) \subset \bigcup_{j=0}^{k} T(x_{i_j})$.

So it follows that $\Delta_J \subset \bigcup_{j=0}^{k} \phi^{-1}(T(x_{i_j}) = \bigcup_{j=0}^{k} G_{i_j}$. Hence by the KKM theorem, we have $\bigcap_{i=0}^{n} G_i \neq \emptyset$ and consequently $\bigcap_{i=0}^{n} T(x_i) \neq \emptyset$. □

Applying Lemma 1.2, we get the following modification of Theorem 1.2.2 above.

Theorem 1.2.3. *Let X be a topological space, (Y, Γ) a G-convex space, $\mathcal{M}$ a G-map system associated with Γ, and $T : X \to 2^Y$. Assume*
(i) $T(x)$ is compactly closed and finitely G-closed, for each $x \in X$;
(ii) T is generalized $\mathcal{M}$-KKM;
and (iii) there exist $x_1, x_2, \ldots, x_m \in X$ such that $S = \bigcap_{i=1}^{m} T(x_i)$ is compact.
Then $\bigcap_{x \in X} T(x) \neq \emptyset$.

Proof. By Lemma 1.2.1, the family $\{T(x) : x \in X\}$ has the finite intersection property and hence $\{T(x) \cap S : x \in X\}$ has the finite intersection property also. Moreover, $T(x) \cap S$ is closed in S for each x because S is compact. So it follows that $\bigcap_{x \in X} T(x) \cap S \neq \emptyset$. □

In the following we define a new kind of convexity in $\mathcal{M}$-convex spaces. An application of this concept will be presented in Chapter 4 when we study variational inequalities for set valued mappings. We also show how a G-convex subspace is induced on any $\mathcal{M}$-convex subset. This fact will be used in proving Theorem 1.2.7, a generalization of a fixed point theorems of Park.

Definition 1.2.4. Let (X, Γ) be a G-convex space and $\mathcal{M}$ be a G-map system associated with Γ. Then a subset C of X is said to be $\mathcal{M}$-convex if
$\phi(\Delta_n) \subset C$, whenever $A = \{a_0, a_1, \ldots, a_n\} \subset C$ and $\phi \in \mathcal{M}(\{a_0, \ldots, a_n\})$.

The obvious proof of the following proposition is omitted.

Proposition 1.2.2. *The intersection of $\mathcal{M}$-convex sets is $\mathcal{M}$-convex.*

Definition 1.2.5. Let (X, Γ) be a G-convex space and $\mathcal{M}$ a G-map system on (X, Γ). Let $A \subset X$. Then $\mathcal{M}$-co(A) is defined to be the smallest $\mathcal{M}$-convex set containing A i.e. the intersection of all $\mathcal{M}$-convex sets containing A.

Definition 1.2.6. Let (X, Γ) be a G-convex space and $\mathcal{M}$ a G-map system associated with Γ. Then (X, Γ) is said to be locally $\mathcal{M}$-convex iff for each $x \in X$ and each open nhood U of x in X, there exists an open nhood V of x in X such that $\mathcal{M}$-co(V) $\subset U$.

Proposition 1.2.3. *Let (X, Γ) be a G-convex space and $\mathcal{M}$ a G-map system associated with Γ. Let C be an $\mathcal{M}$-convex subset of X and equip C with the relative topology. Define $\Gamma_C : \langle C \rangle \to 2^C$ by $\Gamma_C(A) = \bigcup\{$ image $(\phi) : \phi \in \mathcal{M}(A^*), A^* \subset A\}$. Then*

(i) (C, Γ_C) is a G-convex space.

(ii) $\Gamma_C(A) \subset \Gamma(A)$ for any finite subset A of C.

(iii) $\mathcal{M}_C = \bigcup_{A \in \langle C \rangle} \mathcal{M}(A)$ is a G-map system on (C, Γ_C).

(iv) If B is any subset of C then B is $\mathcal{M}_C$-convex iff B is $\mathcal{M}$-convex.

(v) If (X, Γ) is locally $\mathcal{M}$-convex, then (C, Γ_C) is locally $\mathcal{M}_C$-convex.

Proof. (i) First we observe that for any finite subset A of C, $\Gamma_C(A)$ is indeed a subset of C. This is so because the $\mathcal{M}$-convexity of C means that image $\phi \subset C$ whenever A is a finite subset of C and $\phi \in \mathcal{M}(A)$.
Let $A_1 \subset A_2 \subset C$. We shall prove $\Gamma_C(A_1) \subset \Gamma_C(A_2)$.
Let $y \in \Gamma_C(A_1)$, then by the definition of Γ_C it follows that $y \in$ image ϕ and $\phi \in \mathcal{M}(A^*)$ for some $A^* \subset A_1$. But then it follows that $A^* \subset A_2$ and therefore by the definition of $\Gamma_C(A_2)$ we have image $(\phi) \subset \Gamma_C(A_2)$ which implies $y \in \Gamma_C(A_2)$.

Now it only remains to show that for any finite subset $A = \{a_0, \ldots, a_n\}$ of C, there exists a continuous map $\phi : \Delta_n \to \Gamma_C(A)$ such that for any $J = \{i_0, i_1, \ldots, i_k\}$ $\subset \{0, 1, \ldots, n\}$ we have:

$$\phi(co(\{e_{i_0}, \ldots, e_{i_k}\}) \subset \Gamma_C(\{a_{i_0}, \ldots, a_{i_k}\}). \qquad (*)$$

Let ϕ be any map such that $\phi \in \mathcal{M}(A)$. Then ϕ is continuous. Moreover image$(\phi) \subset$ $\Gamma_C(A)$ by the definition of $\Gamma_C(A)$.

Also if $J = \{i_0, i_1, \ldots, i_k\} \subset \{0, 1, \ldots, n\}$ then by the definition of a G-map system, there exists $\phi^* \in \mathcal{M}(\{a_{i_0}, \ldots, a_{i_k}\})$ such that:
$\phi(\Sigma_{j=0}^k \lambda_j e_{i_j}) = \phi^*(\Sigma_{j=0}^k \lambda_j e_j)$ whenever $\lambda_0, \ldots, \lambda_k \geq 0$ with $\Sigma_{j=0}^k \lambda_j = 1$.

So the above implies that

$$\phi(co(\{e_{i_0}, \ldots, e_{i_k}\})) = \phi^*(\Delta_k). \qquad (1)$$

But $\phi^*(\Delta_k) \subset \Gamma_C(\{a_{i_0}, \ldots, a_{i_k}\}$ by the definition of Γ_C, so it follows that ϕ satisfies $(*)$, i.e. $\phi(co(\{e_{i_0}, \ldots, e_{i_k}\})) \subset \Gamma_C(\{a_{i_0}, \ldots, a_{i_k}\}$.

(ii) Let A be a finite subset of C. Let $A^* \subset A$ and $\phi \in \mathcal{M}(A^*)$. Then $\phi :$ $\Delta_{|A^*|-1} \to \Gamma(A^*)$. But $\Gamma(A^*) \subset \Gamma(A)$. So image $\phi \subset \Gamma(A)$. Thus it follows that $\Gamma_C(A) \subset \Gamma(A)$.

(iii) We notice that for any finite subset A of C, if $\phi \in \mathcal{M}(A)$ then image $\phi \subset \Gamma_C(A)$ by the definition of Γ_C. Also any $\phi \in \mathcal{M}(A)$ is continuous by definition. Now all this, together with (*) in the proof of (i) above, implies that $\mathcal{M}_C(A) = \mathcal{M}(A)$ satisfies condition (a) in the definition of a G-map system i.e. Definition 1.1 in section 1.

Condition (b) in Definition 1.1 holds immediately for $\mathcal{M}_C$ because (b) holds for $\mathcal{M}$ and $\mathcal{M}(A) = \mathcal{M}_C(A)$, for each $A \in \langle C \rangle$.

Thus $\mathcal{M}_C$ is a G-map system on (C, Γ_C).

(iv) Let B be a finite subset of C. B is $\mathcal{M}$-convex iff image $\phi \subset B$ whenever $\phi \in \mathcal{M}(A)$ and $A \subset B$ iff image $\phi \subset B$ whenever $\phi \in \mathcal{M}_C(A)$ and $A \subset B$ iff B is $\mathcal{M}_C$-convex.

(v) Let $x_0 \in C$ and $\grave{U}$ be an open nhood of x_0 in C. Then $\grave{U} = U \cap C$ where U is an open nhood of x_0 in X.

By local $\mathcal{M}$-convexity of (X, Γ), there exists an open nhood V of x_0 in X such that

$$\mathcal{M}\text{-co}(V) \subset U. \tag{1}$$

Let $\grave{V} = V \cap C$ then $\grave{V}$ is an open nhood of x_0 in C. We shall show that

$$\mathcal{M}_C\text{-co}(\grave{V}) \subset \grave{U}. \tag{**}$$

First we have

$$\grave{V} \subset \mathcal{M}\text{-co}(V) \cap C = S. \tag{2}$$

Since S is the intersection of two $\mathcal{M}$-convex sets, it is $\mathcal{M}$ -convex. Applying (iv) above, it follows that S is $\mathcal{M}_C$-convex. Then it follows from (2) that $\mathcal{M}_C$-co$(\grave{V}) \subset S$.

Moreover by (1), $S = \mathcal{M}\text{-co}(V) \cap C \subset U \cap C = \grave{U}$. So it follows that $\mathcal{M}_C$-co$(\grave{V}) \subset \grave{U}$. Thus (**) is proved and the proof of (v) is completed. □

Proposition 1.2.4. *(a) Let (X, Γ) be a G-convex space and let $\mathcal{M}$ be a G-map system on (X, Γ). Then any G-convex subset C of X is $\mathcal{M}$-convex.*

(b) There exists an $\mathcal{M}$-convex set that is also $\grave{\mathcal{M}}$-convex for any G-map system $\grave{\mathcal{M}}$ but is yet not G-convex.

Proof. (a) Let $A = \{a_0, a_1, \ldots, a_n\} \subset C$ and $\phi \in \mathcal{M}(A)$. Then $\phi(\Delta_n) \subset \Gamma(A) \subset C$. Hence C is $\mathcal{M}$-convex.

(b) Let $X = \mathbb{R}$. Define $\Gamma :< \mathbb{R} > \rightarrow 2^{\mathbb{R}}$ by:

$$\Gamma(A) = \begin{cases} co(A), \text{ if } \{5, 6\} \text{ is not contained in } A \\ co(A) \cup \{0\}, \text{ if } \{5, 6\} \subset A \end{cases}$$

Let $C = [5, 6]$. Then C is obviously not G-convex. But we will show that C is $\mathcal{M}$-convex for any G-map system $\mathcal{M}$ on $(\mathbb{R}, \Gamma)$.

Indeed let $\mathcal{M}$ be any G-map system on $(\mathbb{R}, \Gamma)$, $A = \{a_0, a_1, \ldots, a_k\} \subset [5, 6]$ and $\phi \in \mathcal{M}(A)$. It suffices to show that $\phi(\Delta_k) \subset [5, 6]$.

We consider two cases.

Case 1. $\{5, 6\}$ is not contained in A. In this case $\Gamma(A) = co(A) \subset [5, 6]$. And so the fact that $\phi(\Delta_k) \subset \Gamma(A)$ implies that $\phi(\Delta_k) \subset [5, 6]$.

Case 2. $\{5, 6\} \subset A$. In this case $\Gamma(A) = co(A) \cup \{0\}$ and hence $\phi(\Delta_k) \subset co(A) \cup \{0\}$. Moreover there exists j with $0 \leq j \leq k$ such that $a_j = 6$ so that $\phi(e_j) = 6$. We shall show that $\phi^{-1}(\{0\}) = \emptyset$. Indeed $\{0\}$ is both relatively open and relatively closed in $\Gamma(A)$. So by the continuity of ϕ, it follows that $\phi^{-1}(\{0\})$ is both open and closed in Δ_k. Hence $\phi^{-1}(\{0\})$ is either $\emptyset$ or Δ_k. But $\phi^{-1}(\{0\})$ cannot be Δ_k because $\phi(e_j) = 6$, so we have $\phi^{-1}(\{0\}) = \emptyset$.

Hence it follows that $\phi(\Delta_k) \subset co(A) \subset [5,6] = C$. Therefore C is $\mathcal{M}$-convex. □

In the following we give two theorems; the first is a modification of a Schauder-Tychonoff-type fixed point theorem and the second is a version of a Kakutani-type fixed point theorem due to Park[P]. The modifications are that assumptions of G-convexity are replaced by assumptions of $\mathcal{M}$-convexity.

We begin by quoting the following theorems of Park [P]:

Theorem 1.2.4. *Let (X, Γ) be a Hausdorff G-convex space such that for every $x \in X$ and every open neighbourhood U of x in X, there exists an open neighbourhood V of x in X such that G-$co(V) \subset U$. Let $g : X \to K$ be continuous, where K is a compact G-convex subset of X.*
Then g has a fixed point.

Theorem 1.2.5. *Let (X, Γ) be a Hausdorff G-convex space and assume that $\{x\}$ is G-convex for each $x \in X$. Assume also that for every compact G-convex subset A of X and every open neighbourhood V of A, there exists an open neighbourhood U of A such that G-$coU \subset V$. Let $T : X \to 2^X$ be such that:*

(i) T has nonempty compact G-convex values;

(ii) T is USC;

(iii) $T(X) \subset K$, where K is a nonempty compact G-convex subset of X.

Then T has a fixed point .

Definition 1.2.7. Let X be a topological space, $\mathcal{R}$ a cover for X, and $\text{St}(B,\mathcal{R}) = \bigcup\{V \in \mathcal{R} : B \cap V \neq \emptyset\}$ for each $B \subset X$. A cover $\mathcal{R}$ is called a star refinement (resp. barycentric refinement) of a cover $\mathcal{U}$ whenever the cover $\{\text{St}(V,\mathcal{R}) : V \in \mathcal{R}\}$ (resp. $\{\text{St}(x,\mathcal{R}) : x \in X\}$) refines $\mathcal{U}$.

The following theorem generalizes Theorem 1.2.4 above.

Theorem 1.2.6. *Let (X,Γ) be a Hausdorff locally $\mathcal{M}$-convex space and $\mathcal{M}$ be a G-map system associated with Γ. Let $g : X \to K$ be continuous, where K is a compact subset of X.*
Then g has a fixed point.

Proof. Assume g has no fixed point. Then for any $x \in X$, there exist open sets V_1 and V_2 such that $x \in V_1$ and $g(x) \in V_2$ and $V_1 \cap V_2 = \emptyset$. Since g is continuous, we may assume $V_1 \subset g^{-1}(V_2)$.

By the assumption of the theorem, there is an open neighbourhood W_x of x in X such that $\mathcal{M}\text{-co}(W_x) \subset V_1$. So it follows that:

$$\mathcal{M}\text{-co}(W_x) \cap g^{-1}(W_x) \subset V_1 \cap g^{-1}(V_1) \subset g^{-1}(V_2) \cap g^{-1}(V_1) = \emptyset \text{ for all } x \in X. \quad (1)$$

Now $\mathcal{W} = \{W_x : x \in K\}$ is an open cover for K. Hence, by the compactness of K, there exists an open star refinement $\mathcal{U}$ for $\mathcal{W}$ and an open finite subcover $\mathcal{R} = \{U_0, U_1, \ldots, U_n\}$ of $\mathcal{U}$.

For each $i = 0, 1, \ldots, n$, choose any $x_i \in U_i$ and let $X_i = X \backslash g^{-1}(U_i)$. Let $A = \{x_0, x_1, \ldots, x_n\}$ and $\phi \in \mathcal{M}(A)$.

Now we shall show that $\phi^{-1}(X_0), \ldots, \phi^{-1}(X_n)$ satisfy all the conditions of the KKM-theorem. First $\phi^{-1}(X_i)$ is obviously closed for each i.

Next let $J = \{i_0, i_1, \ldots, i_k\} \subset \{0, 1, \ldots, n\}$. And let Δ_J be that face of Δ_n corresponding to J. We shall show $\Delta_J \subset \bigcup_{j=0}^{k} \phi^{-1}(X_{i_j})$.

Assume not. Then there exists $t = \Sigma_{j=0}^{k} \lambda_j e_{i_j} \in \Delta_J$ such that $z = \phi(t) \in \bigcap_{j=0}^{k} g^{-1}(U_{i_j})$ and therefore

$$g(z) \in \bigcap_{j=0}^{k} U_{i_j}. \tag{2}$$

In view of (2), $\bigcup_{j=0}^{k} U_{i_j} \subset St(U_{i_0}, \mathcal{U}) \subset W_{\hat{x}}$ for some $\hat{x} \in X$. Therefore we have on one hand that

$$g(z) \in W_{\hat{x}}, \tag{3}$$

On the other hand,

$$\{x_{i_0}, x_{i_1}, \ldots, x_{i_k}\} \subset W_{\hat{x}}. \tag{4}$$

By the definition of a G-map system, there exists $\phi^* \in \mathcal{M}(\{x_{i_0}, \ldots, x_{i_k}\})$ such that $z = \phi^*(\Sigma_{j=0}^{k} \lambda_j e_j)$. But by (4) $\mathcal{M}$-co$(W_{\hat{x}})$ is an $\mathcal{M}$-convex set containing $x_{i_0}, \ldots, x_{i_k}$, so it follows that $\phi^*(\Delta_k) \subset \mathcal{M}$-co$(W_{\hat{x}})$ and hence that

$$z \in \mathcal{M}\text{-co}(W_{\hat{x}}). \tag{5}$$

From (5) and (3), it follows that $z \in \mathcal{M}$-co$(W_{\hat{x}}) \cap g^{-1}(W_{\hat{x}})$, which is a contradiction to (1).

Thus the classical KKM-theorem can be applied and $\bigcap_{i=0}^{n} \phi^{-1}(X_i) \neq \emptyset$, which implies that $\bigcap_{i=0}^{n} X_i \neq \emptyset$ which is a contradiction because if $w \in X_i$ for all $i = 0, 1, \ldots, n$ then $g(w) \notin U_i$ forall $i = 0, 1, \ldots, n$ which contradicts the assumption that $g(X) \subset K \subset \bigcup_{i=o}^{n} U_i$.

Thus g must have a fixed point. □

The following theorem is a modification of Theorem 1.2.5, replacing G-convexity by $\mathcal{M}$-convexity. We point out that neither of these implies the other.

Theorem 1.2.7. *Let (X, Γ) be a Hausdorff G-convex space and $\mathcal{M}$ a G-map system associated with Γ. Assume that $\{x\}$ is $\mathcal{M}$-convex for each $x \in X$. Assume also that for every compact $\mathcal{M}$-convex subset A of X and every open neighbourhood V of A, there exists an open neighbourhood U of A such that $\mathcal{M}\text{-co}U \subset V$. Let $T : X \to 2^X$ be such that:*

(i) T has nonempty compact $\mathcal{M}$-convex values;

(ii) T is USC;

(iii) $T(X) \subset K$, where K is a nonempty compact $\mathcal{M}$-convex subset of X.

Then T has a fixed point .

Proof. Assume not, i.e. for each $x \in K$, $x \notin T(x)$. Then there exist open sets U and V containing x and $T(x)$ respectively such that $U \cap V = \emptyset$.

By assumption there exists an open set V_1 containing $T(x)$ such that $\mathcal{M}\text{-co}V_1 \subset V$. By USC of T, there exists an open set U_1 containing x such that $T(U_1) = \cup_{a \in U_1} T(a) \subset V_1$.

Let $W_x = U \cap U_1$. Then

$$W_x \cap \mathcal{M}\text{-co}(T(W_x)) = \emptyset, \text{ for each } x \in X \tag{1}$$

Indeed $W_x \cap \mathcal{M}\text{-co}(T(W_x)) \subset U \cap \mathcal{M}\text{-co}T(U_1) \subset U \cap \mathcal{M}\text{-co}V_1 \subset U \cap V = \emptyset$.

Since K is normal, it follows by [DJ2, Theorem 3.2 pg.167] that the cover $\mathcal{W} = \{W_x : x \in K\}$ has a barycentric refinement $\mathcal{U}$. Let $U_0, U_1, \ldots, U_n \in \mathcal{U}$ be such that $K \subset \bigcup_{i=0}^{n} U_i$. And let $\beta_0, \beta_1, \ldots, \beta_n$ be a partition of unity on K subordinated to $\{U_0, \ldots, U_n\}$. Also pick $x_i \in T(U_i)$ for each $i \in \{0, 1, \ldots, n\}$.

Let $\phi \in \mathcal{M}(\{x_0, x_1, \ldots, x_n\})$. Note that $\phi : \Delta_n \to \Gamma(\{x_0, x_1, \ldots, x_n\})$ is continuous such that

(a) $\phi(\Delta_n) \subset \mathcal{M}\text{-co}(\{x_0, \ldots, x_n\}) \subset K$; and (b) for any subset $J = \{i_0, i_1, \ldots, i_k\}$ of $\{0, 1, \ldots, n\}$, $\phi(\Delta_J) \subset \mathcal{M}\text{-co}(\{x_{i_0}, x_{i_1}, \ldots, x_{i_k}\}$, where Δ_J is that face of Δ_n corresponding to J.

Now define $h : K \to K$ by $h(x) = \phi(\Sigma_{i=0}^{n} \beta_i(x) e_i)$. Then h is continuous.

We consider the G-convex space (K, Γ_K) together with the G-map system $\mathcal{M}_K$, where Γ_K, $\mathcal{M}_K$ are as in Proposition 1.4. By (v) in Proposition 1.4, (K, Γ_K) is locally $\mathcal{M}_K$-convex. Since K is compact and h is continuous, all the conditions of Theorem 1.2.2 are satisfied and therefore h has a fixed point $\widehat{x} \in K$. We shall show that leads to a contradiction. Let $J = \{j \in \{0, \ldots, n\} : \widehat{x} \in U_j\} = \{i_0, i_1, \ldots, i_k\} \subset \{0, 1, \ldots, n\}$. Thus

$$\widehat{x} \in U_j \text{ iff } j \in J. \tag{2}$$

(2) implies that if $\beta_j(\widehat{x}) \neq 0$, then $j \in J$. Hence it follows that $h(\widehat{x}) \in \phi(\Delta_J)$. (3)

Now by (2),

$$\bigcup_{j=0}^{k} U_{i_j} \subset St(\widehat{x}, \mathcal{U}) \subset W_{x_0}, \text{ for some } x_0 \in K. \tag{4}$$

It follows that $\{x_{i_0}, x_{i_1}, \ldots, x_{i_k}\} \subset T(W_{x_0})$.

$$\text{Now, by (ii), } h(\widehat{x}) \in \phi(\Delta_J) \subset \mathcal{M}\text{-co}(\{x_{i_0}, \ldots, x_{i_k}\}) \subset \mathcal{M}\text{-co}T(W_{x_0}). \tag{5}$$

Using (2) and (4), we have $\widehat{x} \in W_{x_0}$; so (5) would imply that $\widehat{x} = h(\widehat{x}) \in W_{x_0} \cap \mathcal{M}\text{-co}T(W_{x_0})$ which is a contradiction to (1).

Therefore T has a fixed point. □

3. Product G-Convex Spaces.

As we know every topological vector space is a G-convex space. (Define $\Gamma : \langle X \rangle \to 2^X$ by $\Gamma(\{a_0, \ldots, a_n\}) = \{\Sigma_{i=0}^n \lambda_i a_i : \lambda_0, \ldots, \lambda_n \geq 0 \text{ with } \Sigma_{i=0}^n \lambda_i = 1\}$).

Since the product of a family of topological vector spaces is a topological vector space, an obvious question is whether the product of G-convex spaces is a G-convex space.

We shall answer this question in the affirmative. Thus given a family of G-convex spaces $\{(X_i, \Gamma_i)\}_{i \in I}$, if $X = \Pi_{i \in I} X_i$ is equipped with the product topology, we shall define $\Gamma : \langle X \rangle \to 2^X$ in such a way that the resulting G-convex space coincides with the one provided by the linear space in case of topological vector spaces.

Our definition of a product G-convex space makes it possible to study generalized games and abstract economies for G-convex spaces as shall be seen in chapters Three and Five.

In the following lemma we give a construction for a certain map ϕ_A from Δ_n to $X = \Pi_{i \in I} X_i$ where A is any finite subset of X containing $n+1$ elements. The proof of the claim that this map is continuous is easy and it is therefore omitted.

Lemma 1.3.1. *Let $\{(X_i, D_i; \Gamma_i)\}_{i \in I}$ be any family of G-convex spaces. Let $X = \Pi_{i \in I} X_i$ be equipped with the product topology and $D = \Pi_{i \in I} D_i$. For each $i \in I$, let $\pi_i : X \to X_i$ be the i'th projection. Let $A = \{a_0, a_1, \ldots, a_n\}$ be a finite subset of D. For each $i \in I$ let*

$$\begin{aligned} A_i = \pi_i(A) &= \{\pi_i(a_0), \pi_i(a_1), \ldots, \pi_i(a_n)\} \\ &= \{\pi_i(a_{q_0}), \pi_i(a_{q_1}), \ldots, \pi_i(a_{q_{n_i}})\}, \end{aligned}$$

where $\pi_i(a_{q_0}), \pi_i(a_{q_1}), \ldots, \pi_i(a_{q_{n_i}})$ are all distinct and $0 \leq q_0 < q_1 < \cdots < q_{n_i} \leq n$.

Also for each $i \in I$, let $\phi_i = \phi_{A_i} : \Delta_{n_i} \to \Gamma_i(A_i)$ be continuous.

Define $\grave{\phi}_i : \Delta_n \to \Delta_{n_i}$ *by*

$$\grave{\phi}_i(\Sigma_{j=0}^{n}\lambda_j e_j) = \Sigma_{t=0}^{n_i}(\Sigma_{\pi_i(a_j)=\pi_i(a_{q_t})}\lambda_j)e_t.$$

Let $\phi_A = \tilde{\Pi}_{i\in I}\phi_i : \Delta_n \to X$ *be defined by*

$$\phi_A(\alpha) = (\phi_i \circ \phi_i'(\alpha))_{i\in I}.$$

Then $\phi_A = \tilde{\Pi}_{i\in I}\phi_i$ *is continuous.*

The following theorem provides us with a G-convex structure on the product space. We note that the proof becomes quite elaborate at the point where we show that a certain map ϕ that we construct satisfies condition (2) in Definition 1.1.1 required for a G-convex stucture. It took long hours to work through those laborious details, but we believe the outcome is worth it.

Theorem 1.3.1. *Let* $(X_i, \Gamma_i, \mathcal{M}_i)_{i\in I}$ *be any family of* $\mathcal{M}$*-convex spaces. Let* $X = \Pi_{i\in I}X_i$ *be equipped with the product topology. For each* i*, let* $\pi_i : X \to X_i$ *be the i'th projection. Given a finite subset* $A = \{a_0, a_1, \ldots, a_n\} \subset X$*, let*

$$\mathcal{M}(A) = \{\tilde{\Pi}_{i\in I}\phi_i : \{\phi_i\}_{i\in I} \in \Pi_{i\in I}\mathcal{M}_i(\pi_i(A))\},$$

(where $\tilde{\Pi}_{i\in I}\phi_i : \Delta_n \to X$ *is the continuous map constructed in Lemma 1.3.1.)*

Let $\mathcal{M} = \{\mathcal{M}(A) : A \in \langle X\rangle\}$*. Define* $\Gamma_{\mathcal{M}} : \langle X\rangle \to 2^X$ *by*

$$\Gamma_{\mathcal{M}}(A) = \cup\{\ image(\phi) : \phi \in \mathcal{M}(A^*), A^* \subset A\}.$$

Then (I) $(X, \Gamma_{\mathcal{M}})$ *is a G-convex space;*

(II) $\mathcal{M}$ *is a G-map system on* $(X, \Gamma_{\mathcal{M}})$*.*

Proof.

(I) We first notice that $\mathcal{M}(A)$ is nonempty for each $A \in \langle X \rangle$ and therefore $\Gamma_{\mathcal{M}}(A)$ is nonempty.

Suppose $A, B \in \langle X \rangle$ are such that $A \subset B$. Then $\Gamma_{\mathcal{M}}(A) = \cup\{\text{image}(\phi) : \phi \in \mathcal{M}(A^*), A^* \subset A\} \subset \cup\{\text{image}(\phi) : \phi \in \mathcal{M}(A^*), A^* \subset B\} = \Gamma_{\mathcal{M}}(B)$.

Now suppose $A = \{a_0, a_1, \ldots, a_n\} \subset X$. Let $\phi = \tilde{\Pi}_{i \in I}\phi_i \in \mathcal{M}(A)$, where $\phi_i \in \mathcal{M}_i(\pi_i(A))$ for each $i \in I$. Then $\phi : \Delta_n \to X$ is continuous by the previous lemma. Moreover image$(\phi) \subset \Gamma_{\mathcal{M}}(A)$.

To complete the proof of (I), it only remains to show that if $0 \leq l_0 < l_1 < \cdots < l_m \leq n$, then

$$\phi(co(\{e_{l_0}, e_{l_1}, \ldots, e_{l_m}\}) \subset \Gamma_{\mathcal{M}}(\{a_{l_0}, a_{l_1}, \ldots, a_{l_m}\}). \qquad (*)$$

Let $A_1 = \{a_{l_0}, a_{l_1}, \ldots, a_{l_m}\} \subset A$. Fix an arbitrary $i \in I$. Let $A_i = \pi_i(A)$ and $A_{1i} = \pi_i(A_1)$; Clearly $A_{1i} \subset A_i$. Since $\phi_i \in \mathcal{M}_i(A_i)$, it follows from (b) in the definition of a G-map system (Definition 1.1.2) that there exists $\phi_i^* \in \mathcal{M}_i(A_{1i})$ such that

$$\phi_i(\Sigma_{t=0}^{\dot{m}}\lambda_t e_{q_t}) = \phi_i^*(\Sigma_{t=0}^{\dot{m}}\lambda_t e_t), \qquad (1)$$

where $co(\{e_{q_0}, e_{q_1}, \ldots, e_{q_{\dot{m}}}\})$ is that face of $\Delta_{|A_i|-1}$ determined by A_{1i} as a subset of A_i. Let $\phi^* = \tilde{\Pi}_{i \in I}\phi_i^*$, then $\phi^* \in \mathcal{M}(A_1)$ and $\phi^* : \Delta_m \to X$ by Lemma 1.3.1. Let $\Delta_J = co(\{e_{l_0}, e_{l_1}, \ldots, e_{l_m}\})$. For each $\alpha = \Sigma_{r=0}^m \lambda_r e_{l_r} \in \Delta_J$, let $\alpha^* = \Sigma_{r=0}^m \lambda_r e_r \in \Delta_m$. We shall show that $\phi(\alpha) = \phi^*(\alpha^*)$.

Since $\phi(\alpha) = ((\phi_i \circ \grave{\phi}_i(\alpha))_{i \in I}$ and $\phi^*(\alpha^*) = ((\phi_i^* \circ \grave{\phi}^*{}_i)(\alpha^*))_{i \in I}$; it suffices to show that

$$(\phi_i \circ \grave{\phi}_i)(\alpha) = (\phi_i^* \circ \grave{\phi}_i^*)(\alpha^*), \text{ for each } i \in I. \qquad (**)$$

So let $i \in I$. Let

$$\begin{aligned} A_i = \pi_i(A) &= \{\pi_i(a_0), \pi_i(a_1), \ldots, \pi_i(a_n)\} \\ &= \{x_{0,i}, x_{1,i}, \ldots, x_{n,i}\} \\ &= \{z_0, z_1, \ldots, z_{\dot{n}}\}, \end{aligned}$$

for some $0 \leq q_0 < q_1 < \cdots < q_{\dot{m}} \leq \dot{n}$ where we also have $\dot{n} \leq n$.

Similarly let

$$A_{1i} = \pi_i(A_1) = \{\pi_i(a_{l_0}), \ldots, \pi_i(a_{l_m})\} = \{x_{l_0,i}, \ldots, x_{l_m,i}\}$$
$$= \{z_{l_0}, z_{l_1}, \ldots, z_{l_{\dot{m}}}\},$$

for some $\dot{m} \leq m$. Obviously we have $\{l_0, l_1, \ldots, l_{\dot{m}}\} \subset \{0, 1, \ldots, \dot{n}\}$.

By Lemma 1.3.1, $\dot{\phi}_i : \Delta_n \to \Delta_{\dot{n}}$ and $\phi_i : \Delta_{\dot{n}} \to \Gamma_i(A_i)$, where

$$\dot{\phi}_i(\Sigma_{j=0}^{n} \lambda_j e_j) = \Sigma_{s=0}^{\dot{n}} (\Sigma_{x_{j,i}=z_s} \lambda_j) e_s = \Sigma_{s=0}^{\dot{n}} \dot{\lambda}_s e_s.$$

Also $\dot{\phi}_i^* : \Delta_m \to \Delta_{\dot{m}}$ and $\phi_i^* : \Delta_{\dot{m}} \to \Gamma_i(A_{1i})$, where

$$\dot{\phi}^*{}_i(\Sigma_{r=0}^{m} \lambda_r e_r) = \Sigma_{t=0}^{\dot{m}} (\Sigma_{x_{l_r,i}=z_{q_t}} \lambda_r) e_t.$$

Let $\alpha = \Sigma_{r=0}^{m} \lambda_r e_{l_r} \in co(\{e_{l_0}, \ldots, e_{l_m}\}) = \Delta_J$. Then $\dot{\phi}_i(\alpha) = \Sigma_{s=0}^{\dot{n}} (\Sigma_{x_{l_r,i}=z_s} \lambda_r) e_s$.

Now since $\{x_{l_0,i}, x_{l_1,i}, \ldots, x_{l_m,i}\} = \{z_{q_0}, \ldots, z_{q_{\dot{m}}}\}$; it follows that for any $s \notin \{q_0, q_1, \ldots, q_{\dot{m}}\}$ we have $x_{l_r,i} \neq z_s$, for all $0 \leq r \leq m$.

Thus $\dot{\phi}_i(\alpha) = \Sigma_{t=0}^{\dot{m}} (\Sigma_{x_{l_r,i}=z_{q_t}} \lambda_r) e_{q_t} = \Sigma_{t=0}^{\dot{m}} \dot{\lambda}_t e_{q_t}$. Hence $(\phi_i \circ \dot{\phi}_i)(\alpha) = \phi_i(\Sigma_{t=0}^{\dot{m}} \dot{\lambda}_t e_{q_t})$, which by (1) implies

$$(\phi_i \circ \dot{\phi}_i)(\alpha) = \phi_i^*(\Sigma_{t=0}^{m} \dot{\lambda}_t e_t). \tag{2}$$

On the other hand,

$$\begin{aligned} (\phi_i^* \circ \dot{\phi}_i^*)(\alpha^*) &= \phi_i^*(\dot{\phi}_i^*(\Sigma_{r=0}^{m} \lambda_r e_r) \\ &= \phi_i^*(\Sigma_{t=0}^{\dot{m}} (\Sigma_{x_{l_r,i}=z_{q_t}} \lambda_r) e_t) = \phi_i^*(\Sigma_{t=0}^{m} \dot{\lambda}_t e_t). \end{aligned} \tag{3}$$

Therefore by (2) and (3) we have $(\phi_i \circ \dot{\phi}_i)(\alpha) = (\phi_i^* \circ \dot{\phi}_i^*)(\alpha^*)$. Thus (**) follows.

By (**), it follows that $\phi(\Delta_J) \subset$ image(ϕ^*). But $\phi^* \in \mathcal{M}(A_1)$, and hence by the definition of $\Gamma_{\mathcal{M}}$, we have image$(\phi^*) \subset \Gamma_{\mathcal{M}}(A_1)$ so that $\phi(\Delta_J) \subset \Gamma_{\mathcal{M}}(A_1)$; i.e.

$$\phi(co(\{e_{l_0}, \ldots, e_{l_m}\}) \subset \Gamma_{\mathcal{M}}(\{a_{l_0}, \ldots, a_{l_m}\}).$$

Thus (*) is proved and the proof of (I) is completed.

(II) Since in the proof of (I) above ϕ was an arbitrary map from $\mathcal{M}(A)$, where A is also an arbitrary finite subset of X, we have actually proved in (I) that

for each $A = \{a_0, a_1, \ldots, a_n\} \subset X$, for each $\phi \in \mathcal{M}(A)$ and any subset $\{l_0, l_1, \ldots, l_m\}$ of $\{0, 1, \ldots, n\}$, there exists $\phi^* \in \mathcal{M}(\{a_{l_0}, a_{l_1}, \ldots, a_{l_m}\}$ such that $\phi(\Sigma_{r=0}^m \lambda_r e_{l_r}) = \phi^*(\Sigma_{r=0}^m \lambda_r e_r)$, where $\lambda_0, \ldots, \lambda_m \geq 0$ with $\Sigma_{r=0}^m \lambda_r = 1$.

Therefore, by Definition 1.1.2, $\mathcal{M}$ is a G-map system on $(X, \Gamma_{\mathcal{M}})$ and the proof of (II) is completed. □

Definition 1.3.1. (a) Let $\{(X_i, \Gamma_i, \mathcal{M}_i)\}_{i\in I}$ be any family of $\mathcal{M}$-convex spaces. Let $X = \Pi_{i\in I} X_i$ be equipped with the product topology. Let $\mathcal{M}, \Gamma_{\mathcal{M}}$ be as in Theorem 1 above. Then $(X, \Gamma_{\mathcal{M}}, \mathcal{M})$ is said to be the product $\mathcal{M}$-convex space of the family $\{(X_i, \Gamma_i, \mathcal{M}_i)\}_{i\in I}$.

(b) Let $\{(X_i, \Gamma_i)\}_{i\in I}$ be any family of G-convex spaces. Let $X = \Pi_{i\in I} X_i$ be equipped with the product topology.

Then the product G-convex space of the family $(X_i, \Gamma_i)_{i\in I}$ is defined to be the G-convex space (X, Γ), where $\Gamma : \langle X \rangle \to 2^X$ is defined by

$$\Gamma(A) = \Gamma_{\tilde{\mathcal{M}}}(A),$$

where $(X, \Gamma_{\tilde{\mathcal{M}}}, \tilde{\mathcal{M}})$ is the product $\mathcal{M}$-convex space of the family $\{(X_i, \Gamma_i, \tilde{\mathcal{M}}_i)\}_{i\in I}$ and $\tilde{\mathcal{M}}_i$ is the natural G-map system on (X_i, Γ_i) as in Proposition 1.1.1.

Proposition 1.3.1. *Let $\{(X_i, \Gamma_i, \mathcal{M}_i)\}_{i\in I}$ be a family of $\mathcal{M}$-convex spaces. Assume further that $\mathcal{M}_i$ is a homogeneous G-map system for each $i \in I$. Let $(X, \Gamma_{\mathcal{M}}, \mathcal{M})$ be the product $\mathcal{M}$-convex space.*

Then $\mathcal{M}$ is a homogeneous G-map system on $(X, \Gamma_{\mathcal{M}})$.

Proof. Let A be a finite subset of X.
Then $\mathcal{M}_i(\pi_i(A))$ is a singleton, for each $i \in I$. Hence it follows from the construction of $\mathcal{M}(A)$ in the statement of Theorem 1.3.1 that $\mathcal{M}(A)$ is a singleton. Since $\mathcal{M}$ is a G-map system by (II) in Theorem 1.3.1, the conclusion follows. □

In the following Corollary, we see that the concept of a product $\mathcal{M}$-convex space generalizes that of a product topological vector space. The proof is obvious and is hence omitted.

Corollary 1.3.1. *Let $\{(X_i, +, \cdot, \tau_i)\}_{i \in I}$ be a family of topological vector spaces. Let $\Gamma_i : \langle X \rangle \to 2^{X_i}$ be defined by:*

$$\Gamma_i(\{a_{i0}, a_{i1}, \ldots, a_{in}\}) = co(\{a_{i0}, a_{i1}, \ldots, a_{in}\})$$

Let $\mathcal{M}_i(A_i) = \{\phi_{A_i}\}$ where $\phi_{A_i} : \Delta_n \to co(A_i)$ is defined by

$$\phi_{A_i}(\Sigma_{j=0}^n \lambda_j e_j) = \Sigma_{j=0}^n \lambda_j a_{ij}.$$

Let $(X, \Gamma_{\mathcal{M}}, \mathcal{M})$ be the product $\mathcal{M}$-convex space and let $A = \{a_0, a_1, \ldots, a_n\} \subset X$. Then

(i) $\Gamma_{\mathcal{M}}(A) = co(A)$,

(ii) If $\phi_A \in \mathcal{M}(A)$ then $\phi_A(\Sigma_{j=0}^n \lambda_j e_j) = \Sigma_{j=0}^n \lambda_j a_j$ whenever $\lambda_0, \ldots, \lambda_n \geq 0$ with $\Sigma_{j=0}^n \lambda_j = 1$.

Proposition 1.3.2. *Let $\{(X_i, \Gamma_i, \mathcal{M}_i)\}_{i \in I}$ be any family of $\mathcal{M}$-convex spaces. Let $(X, \Gamma_{\mathcal{M}}, \mathcal{M})$ be their product $\mathcal{M}$-convex space.*
Then the product of $\mathcal{M}_i$-convex sets is $\mathcal{M}$-convex.

Proof. Let $C = \Pi_{i \in I} C_i$ where $C_i \subset X_i$ is $\mathcal{M}_i$-convex for each $i \in I$.
To show that C is $\mathcal{M}$-convex, it suffices to show that:

$$\text{image}(\phi) \subset C, \text{ for any finite subset } A \text{ of } C, \text{ and any } \phi \in \mathcal{M}(A). \qquad (1)$$

Indeed let $\phi \in \mathcal{M}(A)$ for $A = \{a_0, a_1, \ldots, a_n\} \subset C$. Then $\phi = \tilde{\Pi}_{i\in I}\phi_i$ where each $\phi_i \in \mathcal{M}_i(\pi_i(A))$, and where $\phi_i : \Delta_{|\pi_i(A)|-1} \to \Gamma_i(\pi_i(A))$.

Since $\pi_i(A) \subset C_i$ and C_i is $\mathcal{M}$-convex; it follows that image$(\phi_i) \subset \Gamma_i(\pi_i(A)) \subset C_i$.

Next by the definition of ϕ, we have $\phi(\alpha) = ((\phi_i \circ \dot{\phi}_i))(\alpha))_{i\in I}$. This implies that image$(\phi) \subset \Pi_{i\in I}$image$(\phi_i) \subset \Pi_{i\in I}C_i \subset C$.

Thus (*) is proved and the conclusion follows. □

Corollary 1.3.2. *Let $\{(X_i, \Gamma_i)\}_{i\in I}$ be any family of G-convex spaces. Let (X, Γ) be their product G-convex space. Then the product of G-convex sets is G-convex.*

Proposition 1.3.3. *Let $\{(X_i, \Gamma_i, \mathcal{M}_i)\}_{i\in I}$ be any family of $\mathcal{M}$-convex spaces. Let $(X, \Gamma_{\mathcal{M}}, \mathcal{M})$ be their product $\mathcal{M}$-convex space. Assume that for each $i \in I$, singleton sets in X_i are $\mathcal{M}_i$-convex. Then singleton sets in the product space are $\mathcal{M}$-convex.*

Proof. Let $x_0 = (x_{0i})_{i\in I} \in X$.
For each $i \in I$, $\{x_{0i}\}$ is $\mathcal{M}_i$-convex, so it follows that image $(\phi_i) \subset \{x_{0i}\}$, for each $\phi_i \in \mathcal{M}_i(\{x_{0i}\})$. This implies that $\mathcal{M}_i(\{x_{0i}\})$ contains one map only, namely $\phi_{0i} : \Delta_0 \to \{x_{0i}\}$ defined by $\phi_{0i}(e_0) = x_{0i}$. Thus $\mathcal{M}(\{x_0\})$ contains one map only, namely $\phi_0 = \tilde{\Pi}_{i\in I}\phi_{0i}$ defined by $\phi_0(e_0) = (\phi_{0i}(e_0))_{i\in I} = (x_{0i})_{i\in I} = x_0$. Thus image $\phi \subset \{x_0\}$ whenever $\phi \in \mathcal{M}(\{x_0\})$.

Hence $\{x_0\}$ is $\mathcal{M}$-convex. □

Corollary 1.3.3. *Let $(X_i, \Gamma_i)_{i\in I}$ be any family of G-convex spaces. Let (X, Γ) be their product G-convex space. Assume that for each $i \in I$, singleton sets in X_i are G-convex. Then singleton sets in the product space are also G-convex.*

The following is Theorem 4.1 in [TZ]. It gives a different definition for a product

G-convex space.

Theorem 1.3.2. *Let $\{(X_i, D_i; \Gamma_i)\}_{i\in I}$ be any family of G-convex spaces. Let $X = \Pi_{i\in I} X_i$ be equipped with the product topology and $D = \Pi_{i\in I} D_i$. For each $i \in I$, let $\pi_i : X \to X_i$ be the projection. Define $\Gamma : \langle D\rangle \to 2^X \setminus \{\emptyset\}$ by*

$$\Gamma(A) = \Pi_{i\in I}\, \Gamma_i(\pi_i(A)) \text{ for each } A \in \langle D\rangle.$$

Then $(X, D; \Gamma)$ is a G-convex space.

The following proposition characterizes closed G-convex subsets when adopting the product G-convex space of [TZ].

Proposition 1.3.4. *Let I be an index set. For each $i \in I$, let (X_i, Γ_i) be a G-convex space satisfying the property that singleton sets are G-convex. Let $X = \Pi_{i\in I} X_i$ be the product G-convex space as defined in Theorem 1.3.2 above. Let A be a closed G-convex subset of X. Then $A = \Pi_{i\in I}\pi_i(A)$.*

Proof. Let $p = (p_i)_{i\in I} \in \Pi_{i\in I}\pi_i(A)$. We shall show $p \in A$.

Let $\mathcal{N}$ be a nhood base for p, and for each $N, \grave{N} \in \mathcal{N}$, let $N \le \grave{N}$ iff $\grave{N} \subset N$. Without loss of generality, we may assume that for each $N \in \mathcal{N}$, there exists a unique subset J_N of I such that
$N = \Pi_{i\in I}\grave{X}_i$, where

$$\grave{X}_i = \begin{cases} X_i, & \text{if } i \notin J_N, \\ U_i, & \text{if } i \in J_N, \\ \text{where } U_i \text{ is an open nhood of } p_j \text{ properly contained in } X_j. \end{cases} \tag{1}$$

For each $i \in I$ let $a^{(i)} \in A$ be such that $\pi_i(a^{(i)}) = p_i$, and for each finite subset J of I, let $A_J = \{a^{(j)} : j \in J\}$.
By the G-convexity of A, we have

$$\Gamma(A_J) = \Pi_{i\in I}\Gamma_i(\pi_i(A_J)) \subset A. \tag{2}$$

Since $p_j = \pi_j(a^{(j)}) \in \pi_j(A_J)$ for each $j \in J$, and since we are also assuming that singleton sets are G-convex; it follows that

$$p_j \in \Gamma_j(\pi_j(A_J)), \text{ for all } j \in J. \tag{3}$$

(2) and (3) imply that

$$\text{there exists } \tilde{a}^{(J)} \in A \text{ such that } \pi_j(\tilde{a}^{(J)}) = p_j, \text{ for all } j \in J. \tag{4}$$

It is obvious from (4) and (1) that $\tilde{a}^{(J_N)} \in N$, for all $N \in \mathcal{N}$.

For each $N \in \mathcal{N}$, we let $a_N = \tilde{a}^{(J_N)}$. Then $(a_N)_{n \in \mathcal{N}}$ is a net in A. Since $\mathcal{N}$ is a nhood basis for p, it follows immediately that this net converges to p. Hence the closedness of A implies that $p \in A$.

Since p is an arbitrary element in $\Pi_{i \in I}\pi_i(A)$, the conclusion follows. □

CHAPTER TWO

G-METRICALLY CONVEX SPACES

Inducing a convex structure on a metric space is an old idea that can be traced back as early as 1935, see [Me] and [Bu] for the related papers of K. Menger and H. Buseman.

In 1970, Takahashi also gave a concept of convexity in metric spaces (See [Tak]); he called these convex metric spaces.

Takahashi's convex metric space is a metric space X with a mapping W from $X \times X \times [0,1]$ to X such that:

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y), \text{ for all } x, y, u \in X \text{ and } 0 \leq \lambda \leq 1.$$

A Banach space is an obvious example of a convex metric space. For other examples see [Tak].

Although motivated by Takahashi's convex metric spaces, our definition of a G-metrically convex space is different from the above. We feel that the relation between these two concepts may need further study.

In presenting G-metrically convex spaces, our main interest is to offer an example of an $\mathcal{M}$-convex space that is rich with G-map systems, a new concept just introduced in Chapter One.

1. G-Convexity In Certain Metric Spaces.

In this section we present a certain type of metric spaces with a convex structure (Definition 2.1.1). In Theorem 2.1.1, (which is the main theorem of this section), we prove that these spaces are G-convex spaces. For that purpose, several lemmas (Lemma 2.1.1 to Lemma 2.1.6) are needed.

Definition 2.1.1. A G-metrically convex space (X, d, F) is a complete metric space (X, d) together with a function $F : X \times X \to X$ such that

(i) F is continuous;

(ii) For each $x, y \in X$, $d(x, F(x, y)) = d(y, F(x, y)) = (1/2)d(x, y)$.

When there is no ambiguity we refer to $F(x, y)$ as the midpoint between x and y and simply denote (X, d, F) by (X, d).

Example. *(a) Let X be a Banach space and define $F : X \times X \to X$ by*

$$F(x, y) = (x + y)/2.$$

Then (X, d, F) is obviously a G-metrically convex space, where d is the metric induced by the norm on X.

(b) Let I be the unit interval $[0, 1]$ and X be the family of closed intervals $[a_i, b_i]$ such that $0 \leq a_i \leq b_i \leq 1$. Let D be the Hausdorff distance in X.

Define $F : X \times X \to X$ by:

$$F([a_i, b_i], [a_j, b_j]) = [(a_i + a_j)/2, (b_i + b_j)/2].$$

Then (X, D, F) is a G-metrically convex space.

Proof. (b) For $[a, b] \in X$, let $O_r([a, b]) = \{y \in [0, 1] : d(x, y) < r \text{ for some } x \in [a, b]\}$. Then $D([a_i, b_i], [a_j, b_j]) = \inf_{r>0}\{r : [a_i, b_i] \in O_r([a_j, b_j])$ and $[a_j, b_j] \in$

$O_r([a_i, b_i]\}$. It is easy to verify that

$$D([a_i, b_i], [a_j, b_j]) = \max\{|a_i - a_j|, |b_i - b_j|\}.$$

We begin by showing that (X, D) is complete.

Let $([a_n, b_n])_{n=1}^{\infty}$ be a Cauchy sequence. Then given $\epsilon > 0$, there exists an integer K such that

$$\max\{|a_n - a_m|, |b_n - b_m|\} < \epsilon, \ \forall\ n, m \geq K.$$

It follows that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are Cauchy sequences in $[0, 1]$. Let a_0, b_0 be the respective limits. Then it is easy to show that

$$\lim_{n\to\infty} [a_n, b_n] = [a_0, b_0].$$

Next we shall show that (ii) in Definition 2.1.1 above holds.

$$D([a_i, b_i], F([a_i, b_i], [a_j, b_j])) = \max\{|(a_i + a_j)/2 - a_i)|, |(b_i + b_j)/2 - b_i|\}$$

$$= \max\{(|a_i - a_j|)/2, (|b_i - b_j|)/2\} = (1/2) \cdot (D([a_i, b_i], [a_j, b_j]))$$

$$= D([a_j, b_j], F([a_i, b_i], [a_j, b_j])).$$

Now it only remains to show that $F : X \times X \to X$ is continuous. Let $[a_1, b_1], [a_2, b_2] \in X$. It suffices to show that for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$D([a, b], [a_1, b_1]) < \delta \text{ and } D([\grave{a}, \grave{b}], [a_2, b_2]) < \delta$$

$$\Rightarrow D(F([a, b], [\grave{a}, \grave{b}]), F([a_1, b_1], [a_2, b_2]) < \epsilon.$$

Let $\delta = \epsilon/4$. Then

$$D([a, b], [a_1, b_1]) < \epsilon/4 \text{ and } D([\grave{a}, \grave{b}], [a_2, b_2]) < \epsilon/4 \text{ so that}$$

$$|a_1 - a| < \epsilon/4; \ |b_1 - b| < \epsilon/4; \ |\grave{a} - a - 2| < \epsilon/4 \text{ and } |\grave{b} - b - 2| < \epsilon/4. \quad (1)$$

Now $D(F([a, b], [\grave{a}, \grave{b}]), F([a_1, b_1], [a_2, b_2])$

$$= D([(a + \grave{a})/2, (b + \grave{b})/2], [(a_1 + a_2)/2, (b_1 + b_2)/2])$$

$$= \max\{|(a + \grave{a})/2 - (a_1 + a_2)/2|, |(b + \grave{b})/2 - (b_1 + b_2)/2|\}$$

$$= \max\{|(a - a_1)/2 + (\grave{a} - a_2)/2|, |(b - b_1)/2 + (\grave{b} - b_2)/2|\}$$

$$= \max\{|(a - a_1)/2| + |(\grave{a} - a_2)/2|, |(b - b_1)/2| + |(\grave{b} - b_2)/2|\} \leq \epsilon/2 < \epsilon.$$

Thus we have shown that $F : X \times X \to X$ is continuous. And it follows that (X, D, F) is a G-metrically convex space. □

Definition 2.1.2. Let (X, d, F) be a G-metrically convex metric space. Let $x_0, y_0 \in X$ and $S = \{m/2^n : m, n \in \mathbb{N}; 0 \leq m \leq 2^n\}$. We define $\Psi : S \to X$ as follows:

(a) $\Psi(0) = y_0$ and $\Psi(1) = x_0$.

(b) $\Psi(s) = \Psi(m/2^n)$ is defined by induction on n as follows :

(i) $\Psi(1/2)$= midpoint between x_0 and y_0.

(ii) Assume $\Psi(m/2^k)$ is defined for all $k < n$ and for all $1 \leq m \leq 2^k - 1$, then define $\Psi(m/2^n)$ as follows:

If m is even, then $\Psi(m/2^n) = \Psi((m/2)/2^{n-1})$ is already defined by the induction hypothesis.

If m is odd, then $\Psi(m - 1/2^n)$ and $\Psi(m + 1/2^n)$ are both defined, and we define $\Psi(m/2^n)$ to be the midpoint between them.

Lemma 2.1.1. *Let (X, d, F),x_0, y_0, S, and $\Psi : S \to X$ be as in Definition 2.1.2 above. Then:*

(i) $d(\Psi(m/2^n), \Psi(m + 1/2^n)) = d(x_0, y_0)/2^n$.

(ii) $d(\Psi(s), \Psi(\grave{s})) = |s - \grave{s}|d(x_0, y_0)$, for any $s, \grave{s} \in S$.

(iii) $d(\Psi(s), x_0) = |1 - s|d(x_0, y_0)$ and $d(\Psi(s), y_0) = sd(x_0, y_0)$.

Proof. (i) We shall prove by induction on n.
If $n = 1$, then $d(\Psi(0), \Psi(1/2)) = 1/2d(x_0, y_0)$ and $d(\Psi(1/2), \Psi(1)) = d(\Psi(1/2), y_0) =$

$1/2d(x_0, y_0)$.

Next assume (i) holds for $n = k$. We will prove that (i) holds for $n = k+1$ when m is odd. The case when $m+1$ is odd is essentially the same.

Now $d(\Psi(m-1/2^{k+1}), \Psi(m+1/2^{k+1})) = d(\Psi((m-1)/2/2^k), \Psi((m+1)/2/2^k)) = d(x_0, y_0)/2^k$; by the induction hypothesis.

Moreover, $\Psi(m/2^{k+1})$ is the midpoint between $\Psi((m-1)/2^{k+1})$ and $\Psi((m+1)/2^{k+1})$; so it follows that:

$$d(\Psi(m/2^{k+1}), \Psi((m+1)/2^{k+1})) = 1/2d(x_0, y_0)/2^k = d(x_0, y_0)/2^{k+1}.$$

(ii) Given $s, \grave{s} \in S$, there exists integers $m, \grave{m}$ and n such that $s = m/2^n$ and $\grave{s} = \grave{m}/2^n$.

We will use induction on n to prove (ii). If $n = 1$, then (ii) obviosly holds. So we assume (ii) is true for $n \leq k$, and let $s = m/(2^{k+1}), \grave{s} = \grave{m}/(2^{k+1})$.

If m and $\grave{m}$ are both even, then (ii) follows by the induction hypothesis.

Next we will prove that (ii) holds if one of the integers, say $\grave{m}$ is even. In this case let $s^* = m - 1/(2^{k+1})$. Then $s^* = (m-1)/2/2^k$ and $\grave{s} = \grave{m}/2/2^k$; so it follows by the induction hypothesis that:

$$d(\Psi(s^*), \Psi(\grave{s})) = |s^* - \grave{s}|d(x_0, y_0) = (\grave{m} - m + 1)/2^{k+1}d(x_0, y_0). \tag{1}$$

Also by the triangle inequality and (i); we have

$$\begin{aligned}
&d(\Psi(s), \Psi(\grave{s})) \\
&\leq d(\Psi(m/2^{k+1}), \Psi(m + 1/2^{k+1})) + d(\Psi(m + 1/(2^{k+1})), \Psi(m + 2/(2^{k+1})) \\
&+ \cdots + d(\Psi(\grave{m} - 1/(2^{k+1})), \Psi(\grave{m}/(2^{k+1}))) \qquad (2) \\
&\leq d(x_0, y_0)/(2^{k+1}) + \cdots + d(x_0, y_0)/(2^{k+1}) \\
&= (\grave{m} - m)/(2^{k+1}) \cdot d(x_0, y_0) = |\grave{s} - s|d(x_0, y_0).
\end{aligned}$$

Now, by the triangle inequality:

$$d(\Psi(s^*), \Psi(\grave{s})) \leq d(\Psi(s^*), \Psi(s)) + d(\Psi(s), \Psi(\grave{s})) \leq d(x_0, y_0)/(2^{k+1}) + d(\Psi(s), \Psi(\grave{s})).$$

The above implies that

$$d(\Psi(s), \Psi(\grave{s})) \geq d(\Psi(s^*), \Psi(\grave{s})) - d(x_0, y_0)/(2^{k+1}).$$

By (1) and the inequality above, we have

$$d(\Psi(s), \Psi(\grave{s})) \geq (\grave{m} - m)/2^{k+1} \cdot d(x_0, y_0). \tag{3}$$

Thus by (2) and (3), we have

$$d(\Psi(s), \Psi(\grave{s})) = |s - \grave{s}| \cdot d(x_0, y_0).$$

And so (ii) is proved in this case.

In case $\grave{m}$ and m are both odd, we also let $s^* = m - 1/(2^{k+1})$. Then we obtain an inequality analogous to (1) by our proof of Case 1 above (Since $m - 1$ is even).

Also since both inequalities (2) and (3) are always true (regardless of either m or $\grave{m}$ being odd or even); we conclude that (ii) holds in this case also.

(iii) $x_0 = \Psi(1)$ and $y_0 = \Psi(0)$. So applying (ii) above, the conclusion follows. □

Definition 2.1.3. Let (X, d, F) be a G-metrically convex metric space. Let $x_0, y_0 \in X$.

Define $\Psi_{(x_0,y_0)} : [0,1] \to X$ as follows:

$$\Psi_{(x_0,y_0)}(t) = \begin{cases} \Psi(t) \text{ as in Definition 2.1.2, if } t \in S; \\ \lim_{n\to\infty} \psi(s_n), \text{ if } t \notin S \text{ where } (s_n)_{n=1}^{\infty} \text{ is a sequence in S converging to t.} \end{cases}$$

Proposition 2.1.1. $\Psi_{(x_0,y_0)}$ *above is well defined.*

Proof. First we notice that if $t_0 \notin S$, then there exists a sequence in S, say $(s_n)_{n=1}^{\infty}$, such that $\lim_{n\to\infty} s_n = t_0$. By (ii) in Lemma 2.1.1, $(\Psi(s_n))_{n=1}^{\infty}$ is a Cauchy sequence in X. Moreover, X is complete; so this sequence must have a limit, say y, which we defined as $\Psi(t_0)$.

It remains to show that if $(s_n)_{n=1}^{\infty}$ and $(\dot{s}_n)_{n=1}^{\infty}$ are sequences in S which both converge to t_0 and $y, \dot{y}$ are such that $y = \lim_{n\to\infty} \Psi(s_n)$ and $\dot{y} = \lim_{n\to\infty} \Psi(\dot{s}_n)$, then $y = \dot{y}$.

Indeed, define $(s_n^*)_{n=1}^{\infty}$ by

$$s_n^* = \begin{cases} s_{n+1/2}, & \text{if } n \text{ is odd ;} \\ \dot{s}_{n/2}, & \text{if } n \text{ is even .} \end{cases}$$

Then it is easy to see that:

(i) $\lim_{n\to\infty} s_n^* = t_0$.

(ii) Both $(\Psi(s_n))_{n=1}^{\infty}$ and $(\Psi(\dot{s}_n))_{n=1}^{\infty}$ are subsequences of $(\Psi(s_n^*))_{n=1}^{\infty}$.

By (i), it follows that $(\Psi(s_n^*))$ has a limit point in X, call it y^*. And by (ii), it follows that $y = y^* = \dot{y}$. □

Lemma 2.1.2. *Let (X, d, F) be a G-metrically convex space. Let $x_0, y_0 \in X$, and $\Psi_{(x_0,y_0)} : [0,1] \to X$ be as in the Definition 2.1.2. Then:*

(i) $d(\Psi_{(x_0,y_0)}(t), \Psi_{(x_0,y_0)}(\dot{t})) = |t - \dot{t}|\ d(x_0, y_0)$, for any $t, \dot{t} \in [0,1]$;

(ii) $\Psi_{(x_0,y_0)}$ is continuous.

Proof. (i) Given $t, \grave{t} \in [0,1]$, let $(s_n)_{n=1}^{\infty}$ and $(\grave{s}_n)_{n=1}^{\infty}$ be sequences in S converging to t and $\grave{t}$, respectively. Then it follows that:

$$\Psi_{(x_0,y_0)}(t) = \lim_{n\to\infty} \Psi_{(x_0,y_0)}(s_n); \; \Psi_{(x_0,y_0)}(\grave{t}) = \lim_{n\to\infty} \Psi_{(x_0,y_0)}(\grave{s}_n) \text{ and}$$

$$d(\Psi_{(x_0,y_0)}(t), \Psi_{(x_0,y_0)}(\grave{t})) = \lim_{n\to\infty} d(\Psi_{(x_0,y_0)}(s_n), \Psi_{(x_0,y_0)}(\grave{s}_n))$$

By Lemma 2.1.1 (ii), we have

$$\lim_{n\to\infty} d(\Psi_{(x_0,y_0)}(s_n), \Psi_{(x_0,y_0)}(\grave{s}_n)) = \lim_{n\to\infty} |s_n - \grave{s}_n| d(x_0, y_0) = |t - \grave{t}| d(x_0, y_0)$$

Thus the conclusion follows.

(ii) follows immediately from (i). □

Lemma 2.1.3. *Let (X, d, F) be a G-metrically convex space, let $a \in X$, and let $t \in [0,1]$. Define $f_t : X \to X$ by $f_t(x) = \Psi_{(a,x)}(t)$. Then (i) f_s is continuous for each $s \in S = \{m/(2^n) : m, n \in \mathbb{N}$ and $0 \le m \le 2^n\}$.*

(ii) f_t is continuous for each $t \in [0,1]$.

Proof. (i) Let $s = m/2^n$. We shall use induction on n.

For $n = 0$ we have $f_0(x) = x$ and $f_1(x) = a$, both are continuous.

If $n = 2$, it is easy to see that $f_{1/2}, f_{1/4}$ and $f_{3/4}$ are all continuous.

For $f_{1/2}(x) = F(a,x)$, $f_{1/4}(x) = F(F(a,x),x)$ and $f_{3/4}(x) = F(a, F(a,x))$. Thus all three functions are continuous since they are compositions of continuous functions.

Next assume f_s is continuous for $s = m/2^k$ where $0 \le m \le 2^k$. We will show f_s is continuous for $s = m/2^{k+1}$ where $0 \le m \le 2^{k+1}$.

If m is even then the conclusion easily follows. So assume $n = 2l + 1$. Then

$$\begin{aligned} f_s(x) &= \Psi_{(a,x)}((2l+1)/2^{k+1}) \\ &= \text{ the midpoint between } \Psi_{(a,x)}((2l)/2^{k+1}) \text{ and } \Psi_{(a,x)}((2l+2)/2^{k+1} \\ &= F(\Psi_{(a,x)}(l/2^k), \Psi_{(a,x)}((l+1)/2^k)) = F(f_{l/2^k}(x), f_{(l+1)/2^k}(x)). \end{aligned}$$

Since both $f_{l/2^k}$ and $f_{(l+1)/2^k}$ are continuous by the induction hypothesis; it follows that f_s is continuous.

(ii) We will show $f_t : X \to X$ is continuous for any $t \in [0,1]$.

Given $x_0 \in X$ and $\epsilon > 0$, let $s \in S$ be such that:

$$|t - s| < \epsilon/3[d(a, x_0) + 1]. \tag{1}$$

Since f_s is continuous by (i), there exists $0 < \delta < 1$ such that

$$d(x, x_0) < \delta \text{ implies } d(f_s(x_0), f_s(x)) < \epsilon/3. \tag{2}$$

We will show that:

$$d(x, x_0) < \delta \text{ implies } d(f_t(x), f_t(x_0)) < \epsilon. \tag{*}$$

Let x be such that $d(x, x_0) < 1$. Applying Lemma 2.1.2 (i), we obtain the following two inequalities:

$$d(f_s(x), f_t(x)) \le |t - s| \; d(a, x) \le |t - s| \; (d(a, x_0) + 1),$$

$$d(f_s(x_0), f_t(x_0) \le |t - s| \; (d(a, x_0) + 1).$$

Applying (1) to the two inequalities above, we get

$$d(f_s(x), f_t(x)) \le \epsilon/3, \tag{3}$$

$$\text{and } d(f_s(x_0), f_t(x_0)) \le \epsilon/3. \tag{4}$$

From (2), (3) and (4); it follows that:

$d(x, x_0) < \delta$ implies $d(f_t(x), f_t(x_0)) < \epsilon$. □

Definition 2.1.4. Let (X, d, F) be a G-metrically convex space. Define $\Gamma_F : \langle X \rangle \to 2^X$ as follows:

(i) If $A = \{a_0\}$, then $\Gamma_F(A) = \{a_0\}$.

(ii) If $A = \{a_0, a_1\}$, then $\Gamma_F(A) = \Psi_{(a_0,a_1)}([0,1])$.

(iii) Assuming $\Gamma_F(S)$ is defined whenever $|S| = k$, define

$$\Gamma_F(A) = \Gamma_F(\{a_0, a_1, \ldots, a_k\}) = \bigcup_{i=0}^{k} \bigcup_{y \in \Gamma_F(A \setminus \{a_i\})} \Gamma_F(\{a_i, y\}).$$

Corollary 2.1.1. *Let (X, d, F) be a G-metrically convex space. Let $\Gamma_F : \langle X \rangle \to 2^X$ be as in Definition 2.1.4.*

Then

(a) $A \subset \Gamma_F(A)$ for any finite subset A of X.

(b) $B \subset A$ implies $\Gamma_F(B) \subset \Gamma_F(A)$.

Proof. (a) If $A = \{a_0\}$, then $\Gamma_F(A) = \Gamma_F(\{a_0\}) = \{a_0\} = A$.

(i) If $A = \{a_0, a_1\}$, then $a_1 = \Psi_{(a_0,a_1)}(0)$ and $a_0 = \Psi_{(a_0,a_1)}(1)$.

(ii) Assume $|A| > 2$, and let $a_i \in A$. Then by (i), we have $a_i \in \Gamma_F(\{a_i, y\})$ for any $y \in X$. Hence $a_i \in \Gamma_F(A)$ and the conclusion follows.

(b) It suffices to show that if $B \subset A$ and $|B| = |A| - 1$, then $\Gamma_F(B) \subset \Gamma_F(A)$.

So let $A = \{a_0, a_1, \ldots, a_k\}$ and $B = A \setminus \{a_j\}$ for some $0 \leq j \leq k$.

Since $\Gamma_F(A) = \bigcup_{i=o}^{k} \bigcup_{y\in\Gamma_F(A\setminus\{a_i\})} \Gamma_F(\{a_i, y\})$; it follows that

$$\bigcup_{y\in\Gamma_F(B)} \Gamma_F(\{a_j, y\}) \subset \Gamma_F(A).$$

But $y \in \Gamma_F(\{a_j, y\}$ by (a); so we have

$\Gamma_F(B) \subset \bigcup_{y\in\Gamma_F(B)} \Gamma(\{a_j, y\}) \subset \Gamma_F(A)$. □

Lemma 2.1.4. *Let (X, d, F) be a G-metrically convex space. Let $\Gamma_F : \langle X\rangle \to 2^X$ be as in Definition 2.1.4. Then $\Gamma_F(A)$ is bounded for any $A \in \langle X\rangle$.*

Proof. First $\Gamma_F(\{a_0\}) = \{a_0\}$ is obviously bounded. Also if $A = \{a_0, a_1\}$ and $y \in \Gamma_F(A)$, then $y = \Psi_{(a_0,a_1)}(t)$ for some $t \in [0, 1]$. By Lemma 2.1.2 (i),

$$d(\Psi_{(a_0,a_1)}(t), a_0)) = (1 - t)d(a_0, a_1) \leq d(a_0, a_1).$$

So $\Gamma_F(A)$ is bounded in this case also.

Next assume $\Gamma_F(\dot{A})$ is bounded whenever $|\dot{A}| = k$; and let $A = \{a_0, a_1, \ldots, a_k\}$.

Let $S_i = \bigcup_{y\in\Gamma_F(A\setminus\{a_i\})} \Gamma_F(\{a_i, y\})$.

Then $\Gamma_F(A) = \bigcup_{i=0}^{k} S_i$, and it suffices to show that each S_i is bounded; since the finite union of bounded sets is always bounded.

Now $\Gamma_F(A\setminus\{a_i\})$ is bounded by the induction hypothesis; so there exists $x_0 \in X$ and a real number $R > 0$ such that

$$\Gamma_F(A\setminus\{a_i\}) \subset B(x_0, R). \tag{1}$$

We will show that

$$S_i \subset B(a_i, R + d(x_0, a_i)). \tag{*}$$

Let $w \in S_i$; then $w = \Psi_{(a_i,y)}(t)$ for some $t \in [0, 1]$ and $y \in \Gamma_F(A\setminus\{a_i\})$.

By (i) in Lemma 2.1.2, we have

$d(w, a_i) \leq |1-t| d(a_i, y) \leq d(a_i, y) \leq d(a_i, x_0) + d(x_0, y)$.

Applying (1) to the inequality above we have

$d(w, a_i) \leq d(a_i, x_0) + R$. Hence (*) is proved. □

Definition 2.1.5. Let X be a topological space and $\Gamma : \langle X \rangle \to 2^X$. Let $A = \{a_0, a_1, \ldots, a_k\} \subset X$, $J \subset \{0, 1, \ldots, k\}$ and $\phi : co(\{e_j : j \in J\}) \to \Gamma(\{a_j : j \in J\})$. Then ϕ is said to satisfy the G-condition iff for any subset J^* of J, $\phi(co(\{e_j : j \in J^*\})) \subset \Gamma(\{a_j : j \in J^*\})$.

Definition 2.1.6. Let (X, d, F) be a G-metrically convex space and let $\Gamma_F : \langle X \rangle \to 2^X$ be as in Definition 2.1.4.

Let $A = \{a_0, a_1, \ldots, a_k\} \subset X$, and let $A_i = A \backslash \{a_i\}$ for some fixed $i \in \{0, 1, \ldots, k\}$.

Let $\phi : \Delta_{k-1} \to \Gamma_F(A_i)$ be a continuous map satisfying the G-condition i.e. for any subset J of $\{0, 1, \ldots, k-1\}$, we have:

$\phi(co(\{e_j : j \in J\}) \subset \Gamma_F(\{a_{\grave{j}} : j \in J\})$, where $\grave{j} = j$ if $j < i$ and $\grave{j} = j + 1$ if $j \geq i$. (b1)

Let $\alpha_{a_i, \phi} : \Delta_k \to \Gamma_F(A)$ be defined in three steps as follows:

Step 1. Let $J_i = \{0, 1, \ldots, k\} \backslash \{i\}$ and Δ_{J_i} be that face of Δ_k corresponding to J_i.

Define $\grave{\phi} : \Delta_{J_i} \to \Gamma_F(A_i)$ by

$\grave{\phi}(\Sigma_{j \in J_i} \lambda_j e_j) = \phi(\Sigma_{j=0}^{i-1} \lambda_j e_j \; + \; \Sigma_{j=i+1}^{k} \lambda_j e_{j-1})$.

Step 2. Define $p_i : \Delta_k \backslash \{e_i\} \to \Gamma_F(A_i)$ as follows:

$p_i(z) = p_i(\Sigma_{j=o}^{k}\lambda_j e_j) = \dot{\phi}(\Sigma_{j=0,j\neq i}^{k}(\lambda_j/\widehat{\lambda}_i)e_j)$ where $\widehat{\lambda}_i = \Sigma_{j\neq i}\lambda_j$.

Also let $\pi_i : \Delta_k \to [0,1]$ be the usual projection, i.e.

$\pi_i(z) = \pi_i(\Sigma_{j=0}^{k}\lambda_j e_j) = \lambda_i$.

Step 3. Define $\alpha_{a_i,\phi} : \Delta_k \to \Gamma_F(A)$ by:

$$\alpha_{a_i,\phi}(z) = \begin{cases} a_i, & \text{if } z = e_i; \\ \Psi_{(a_i,p_i(z))}(\pi_i(z)), & \text{if } z \neq e_i \text{ where } \Psi_{(a_i,p_i(z))} \text{ is as in Definition 2.1.3.} \end{cases}$$

Corollary 2.1.2. *Let (X,d,F) be a G-metrically convex space and let $\Gamma_F : \langle X\rangle \to 2^X$ be as in Definition 2.1.4. Let $A = \{a_0,\dots,a_k\}, A_i = A\setminus\{a_i\}$, $\phi : \Delta_{k-1} \to \Gamma_F(A_i)$, $\dot{\phi}$ and p_i be as in Definition 2.1.6.*

Then for $i = k$ we have $p_k|_{\Delta_{k-1}} = \phi$.

Proof. Let $z = \Sigma_{j=0}^{k-1}\lambda_j e_j \in \Delta_{k-1}$.

Then $\pi_k(z) = \lambda_k = 0$ and $\widehat{\lambda}_k = 1$; therefore,

$p_k(z) = \dot{\phi}(\Sigma_{j\in\{0,\dots,k\}\setminus\{k\}}(\lambda_j/\widehat{\lambda}_k)e_j) = \dot{\phi}(\Sigma_{j=0}^{k-1}\lambda_j e_j) = \phi(\Sigma_{j=0}^{k-1}\lambda_j e_j) = \phi(z)$. □

Lemma 2.1.5. *The map $\alpha_{a_i,\phi}$ constructed in Definition 2.1.6 is continuous.*

Proof. We shall show that $\alpha_{a_i,\phi}$ is continuous at every z_0 in Δ_k. First assume $z_0 \neq e_i$.

Then given $\epsilon > 0$, we will show that there exists $\delta > 0$ such that

$$||z - z_0|| < \delta \text{ implies } d(\alpha_{a_i,\phi}(z), \alpha_{a_i,\phi}(z_0)) < \epsilon. \qquad (*)$$

Let $p_i(z_0) = w_0 \in X$ and $p_i(z) = w \in X$. For simplicity, let $t_0 = \pi_i(z_0)$ and $t = \pi_i(z)$, then $\alpha_{a_i,\phi}(z_0) = \Psi_{(a_i,w_0)}(t_0)$ and $\alpha_{a_i,\phi}(z) = \Psi_{(a_i,w)}(t)$.

Next using the notation of Lemma 2.1.3, for $t \in [0,1]$ let $f_t : X \to X$ be defined by $f_t(w) = \Psi_{(a_i,w)}(t)$. By Lemma 2.1.3, f_t is continuous for any $t \in [0,1]$. Note that

$$\alpha_{a_i,\phi}(z_0) = f_{t_0}(w_0) \text{ and } \alpha_{a_i,\phi}(z) = f_t(w). \qquad (**)$$

By changing the notation according to (**), (*) is equivalent to ($\dot{*}$) below.

$$||z - z_0|| < \delta \text{ implies } d(f_{t_0}(w_0), f_t(w)) < \epsilon. \qquad (\dot{*})$$

Let $\epsilon > 0$ be given. By continuity of f_{t_0} at w_0, there exists $0 < \grave{\epsilon} < 1$ such that

$$d(w, w_0) < \grave{\epsilon} \text{ implies } d(f_{t_0}(w), f_{t_0}(w_0)) < \epsilon/2. \qquad (1)$$

We notice that $p_i : \Delta_k \backslash \{e_i\} \to \Gamma(A_i)$ is continuous at z_0; so there exists $\delta_1 > 0$ such that

$$||z - z_0|| < \delta_1 \text{ implies } d(p_i(z), p_i(z_0)) = d(w, w_0) < \grave{\epsilon}. \qquad (1^*)$$

Combining (1) and (1^*), we get

$$||z - z_0|| < \delta_1 \text{ implies } d(f_{t_0}(w), f_{t_0}(w_0)) < \epsilon/2. \qquad (1^{**})$$

Also by Lemma 2.1.2, we have

$$d(f_t(w), f_{t_0}(w)) \leq |t - t_0| d(a_i, w). \qquad (2)$$

Now $t = \pi_i(z)$, $t_0 = \pi_i(z_0)$ and $\pi_i : \Delta_k \to [0,1]$ is continuous, so there exists $0 < \delta < \delta_1$ such that

$$||z - z_0|| < \delta \text{ implies } |t - t_0| < \epsilon/(2[d(a_i, w_0) + 1]). \qquad (2^*)$$

Combining (2) and (2^*), we get

$$\begin{aligned} &||z - z_0|| < \delta \text{ implies} \\ &d(f_t(w), f_{t_0}(w)) \leq \epsilon/(2[d(a_i, w_0) + 1] \cdot d(a_i, w) \\ &\leq \epsilon[d(a_i, w_0) + d(w_0, w)]/(2[d(a_i, w_0) + 1]) \\ &\leq \epsilon[d(a_i, w_0) + 1]/(2[d(a_i, w_0) + 1]) = \epsilon/2. \end{aligned} \qquad (2^{**})$$

Now from (1^{**}) and (2^{**}), we get $||z - z_0|| < \delta$ implies $d(f_{t_0}(w_0), f_t(w)) < \epsilon$. In other words, by changing the notation according to (**), we get $||z - z_0|| < \delta$ implies $d(\alpha_{a_i,\phi}(z), \alpha_{a_i,\phi}(z_0)) < \epsilon$. Thus $\alpha_{a_i,\phi}$ is continuous at z_0.

Next we prove continuity at e_i.

Since $\alpha_{a_i,\phi}(e_i) = a_i$, it suffices to show that given $\epsilon > 0$, there exists $\delta > 0$ such that

$||z - e_i|| < \delta$ implies $d(\alpha_{a_i,\phi}(z), a_i) < \epsilon$.

Since $\alpha_{a_i,\phi}(z) = \Psi_{(a_i,p_i(z))}(\pi_i(z))$, by Lemma 2.1.2 (i), we have

$$d(\alpha_{a_i,\phi}(z), a_i) \leq |1 - \pi_i(z)| d(a_i, p_i(z)). \tag{3}$$

Moreover $p_i(z) \in \Gamma_F(A_i)$ and the latter is bounded by Lemma 2.1.4, so there exists a real number R such that

$$d(a_i, p_i(z)) < R, \ \forall \ z \in \Delta_k \setminus \{e_i\}. \tag{4}$$

Also $\pi_i : \Delta_k \to [0,1]$ is continuous and $\pi_i(e_i) = 1$; thus given $\epsilon > 0$, there exists $\delta > 0$ such that:

$$||z - e_i|| < \delta \text{ implies } |\pi_i(z) - 1| < (\epsilon/R). \tag{5}$$

Combining (3), (4), and (5), we have the following

$||z - e_i|| < \delta$ implies $d(\alpha_{a_i,\phi}(z), a_i) < \epsilon$.

Thus $\alpha_{a_i,\phi}$ is continuous at a_i. □

Lemma 2.1.6. *The map* $\alpha_{a_i,\phi} : \Delta_k \to \Gamma_F(\{a_0, a_1, \ldots, a_k\})$ *(as constructed in Definition 2.1.6) satisfies the G-condition .*

Proof. First we will show that $\grave{\phi} : \Delta_{J_i} \to \Gamma(A_i)$ (defined in Step 1 in Definition 2.1.6) satisfies the G-condition.

So let $J \subset \{0,1,\ldots,k\}\backslash\{i\}$ and let $z \in \Delta_J = co(\{e_j : j \in J\}$. We want to show that

$$\dot{\phi}(z) \in \Gamma_F(\{a_j : j \in J\}). \qquad (*)$$

Now

$$\begin{aligned}\dot{\phi}(z) &= \dot{\phi}(\Sigma_{j\in J}\lambda_j e_j) = \dot{\phi}(\Sigma_{j\in J\cap\{0,1,\ldots,i-1\}}\lambda_j e_j + \Sigma_{j\in J\cap\{i+1,\ldots,k\}}\lambda_j e_j) \\ &= \phi(\Sigma_{j\in J\cap\{0,1,\ldots,i-1\}}\lambda_j e_j + \Sigma_{j\in J\cap\{i+1,\ldots,k\}}\lambda_j e_{j-1}) \in \phi(\Delta_{J^*}),\end{aligned} \qquad (1)$$

where $J^* = \{j \in J : j < i\} \cup \{j-1 : j \in J \text{ and } j > i\}$.
By (b1) in Definition (6), we have:

$$\phi(\Delta_{J^*}) \subset \Gamma_F(\{a_j : j \in J^*\}) = \Gamma_F(\{a_j : j \in J\}). \qquad (2)$$

By (1) and (2), (*) follows.

Next we shall show that for any subset J of $\{0,1,\ldots,k\}$, we have:

$$\alpha_{a_i,\phi}(\Delta_J) \subset \Gamma(\{a_j : j \in J\}). \qquad (**)$$

As in Definition 2.1.6, let $\Delta_{J_i} = \{z \in \Delta_k : \pi_i(z) = 0\}$. Then for each $z \in \Delta_{J_i}$, we have:

$$\alpha_{a_i,\phi}(z) = \Psi_{(a_i,p_i(z))}(0) = p_i(z). \qquad (3)$$

Moreover, for each $z = \Sigma_{j=0}^k \lambda_j e_j \in \Delta_{J_i}$, we have $\widehat{\lambda}_i = \Sigma_{j=0,j\neq i}^k \lambda_j = 1$ so that

$$p_i(z) = \dot{\phi}(\Sigma_{j=0,j\neq i}^k(\lambda_j)/\widehat{\lambda}_i e_j) = \dot{\phi}(z). \qquad (4)$$

It follows from (3) and (4) that $\alpha_{a_i,\phi}|_{\Delta_{J_i}} = \dot{\phi}$.

But $\dot{\phi}$ satisfies the G-condition on Δ_{J_i}; so (**) is satisfied whenever $\Delta_J \subset \Delta_{J_i}$.

Next we consider Δ_J such that $i \in J$ i.e. Δ_J is not contained in Δ_{J_i}.

Let $J^* = J\backslash\{i\}$. We shall show that for each $z \in \Delta_J$, we have $\alpha_{a_i,\phi}(z) \in \Gamma(\{a_j : j \in J\}$.

First if $z = e_i$ then $\alpha_{a_i,\phi}(z) = a_i \in \{a_j : j \in J\} \subset \Gamma_F(\{a_j : j \in J\}$ (By Corollary 2.1.1(a)).

Next assume $z \neq e_i$. Then

$$\alpha_{a_i,\phi}(z) = \Psi_{(a_i,p_i(z))}(\pi_i(z)) \in \Gamma_F(\{a_i, p_i(z)\}), \tag{5}$$

and $p_i(z) = \dot{\phi}(\Sigma_{j=0,j\neq i}^{k}(\lambda_j)/(\widehat{\lambda}_i)e_j) \in \dot{\phi}(\Delta_{J^*}) \subset \Gamma_F(\{a_j : j \in J^*\})$ (since $\dot{\phi}$ satisfies the G-condition as we proved above).

Combining the above with (5), we have $\alpha_{a_i,\phi}(z) \in \Gamma_F(\{a_i, y\})$ where $y \in \Gamma_F(\{a_j : j \in J\}\backslash\{a_i\}$. By definition of $\Gamma_F(\{a_j : j \in J\}$, it follows that $\alpha_{a_i,\phi}(z) \in \Gamma_F(\{a_j : j \in J\})$. □

Theorem 2.1.1. *Let (X, d, F) be a G-metrically convex space (as in Definition 2.1.1). Let $\Gamma_F : \langle X\rangle \to 2^X$ be as in Definition 2.1.4. Then (X, Γ_F) is a G-convex space.*

Proof. By Corollary 2.1.1, we have $B \subset A$ implies $\Gamma_F(B) \subset \Gamma_F(A)$. So it only remains to show that for any subset $A = \{a_0, a_1, \ldots, a_k\}$, there exists a continuous map $\phi : \Delta_k \to \Gamma_F(A)$ such that ϕ satisfies the G-condition. We use induction on $|A|$. If $A = \{a_0\}$ then let $\phi : \Delta_0 \to \Gamma_F(\{a_0\})$ be defined by $\phi(e_0) = a_0$. If $A = \{a_0, a_1\}$, then let $\phi : \Delta_1 \to \Gamma_F(A)$ be defined by $\phi(\lambda_1 e_1 + \lambda_0 e_0) = \Psi_{(a_1,a_0)}(\lambda_1)$. Then ϕ is obviously continuous and satisfies the G-condition.

Next assume that for any subset B of X having k elements, there exists $\phi : \Delta_{k-1} \to \Gamma_F(B)$ such that ϕ is continuous and satisfies the G-condition. Let $A = \{a_0, a_1, \ldots, a_k\}$ be given. Then there exists $\phi : \Delta_{k-1} \to \Gamma_F(A\backslash\{a_k\}$ which is continuous and which satisfies the G-condition. Consider $\alpha_{a_k,\phi} : \Delta_k \to \Gamma_F(A)$ (as constructed in Definition 2.1.6). Indeed this is the required map since it is continuous and satisfies the G-condition by Lemma 2.1.5 and Lemma 2.1.6. □

Definition 2.1.7. Let (X, d, F) be a G-metrically convex space. Let Let $\Gamma_F : \langle X \rangle \to 2^X$ be as in Definition 2.1.4. Then the G-convex space (X, Γ_F) will be denoted by (X, d, Γ_F) or just (X, d, Γ) when there is no ambiguity. From now on by a G-metrically convex space we will mean the G-convex space (X, d, Γ) induced on X by d and F as was proved in Theorem 2.1.1.

Definition 2.1.8. Let (X, d, Γ) be a G-metrically convex space. For $x_0, y_0 \in X$ and $t \in [0, 1]$, let $tx_0 \oplus (1-t)y_0$ be defined by $\Psi_{(x_0,y_0)}(t)$; e.g.,

(i) $1x_0 \oplus 0y_0 = x_0$; (ii) $0x_0 \oplus 1y_0 = y_0$; and (iii) $(1/2)x_0 \oplus (1/2)y_0 =$ the midpoint between x_0 and y_0.

The following is a characterization of G-convex subsets of a G-metrically convex space.

Corollary 2.1.3. *Let (X, d, Γ) be a G-metrically convex space . Then a subset C of X is G-convex if and only if for any $x_0, y_0 \in C$ and any $t \in [0, 1]$ we have $tx_0 \oplus (1-t)y_0 \in C$.*

Proof. (i) Assume C is G-convex and let $x_0, y_0 \in C$. Then $tx_0 \oplus (1-t)y_0 \in \psi_{(x_0,y_0)}([0,1]) = \Gamma(\{x_0, y_0\}) \subset C$. Hence the required conclusion follows.

(ii) Assume that for any $x_0, y_0 \in C$ and any $t \in [0, 1]$ we have $tx_0 \oplus y_0 \in C$. We want to show that C is G-convex. It follows that for any $x_0, y_0 \in C$ we have $\Gamma(\{x_0, y_0\}) = \Psi_{(x_0,y_0)}([0, 1]) \subset C$; i.e.,

$$\Gamma(A) \subset C \text{ whenever } A \text{ is a subset of } C \text{ that consists of two elements.} \quad (1)$$

We will show by induction on $|A|$ that $\Gamma(A) \subset C$ for any finite subset A of C. Assume that $\Gamma(B) \subset C$ whenever $B \subset C$ and $|B| = k \geq 2$. Let $A = \{a_0, a_1, \ldots, a_k\}$. First we observe that the induction hypothesis immediately implies that:

$$\Gamma(A \backslash \{a_i\}) \subset C \text{ for any } a_i \in A.$$

Next applying (1) to the above we have:

$$\Gamma(\{y, a_i\}) \subset C \text{ whenever } y \in \Gamma(A\backslash\{a_i\}) \text{ and } a_i \in A. \tag{2}$$

From (2) it follows that:

$$\Gamma(A) = \bigcup_{i=0}^{k} \bigcup_{y \in \Gamma(A\backslash\{a_i\})} \Gamma(\{y, a_i\}) \subset C.$$

Hence C is G-convex. □

In the following proposition we prove that bounded G-convex subsets of a G-metrically convex space are contractible. Applications of this result will be seen in Chapter 3.

Proposition 2.1.2. *Let (X, Γ) be a G-metrically convex space. Let C be a nonempty bounded and G-convex subset of X. Then C is contractible.*

Proof. Let $a \in C$. Since C is G-convex; we have

$\Psi_{(a,x)}(t) \in C, \ \forall \ (x, t) \in C \times [0, 1]$ (where $\Psi_{(a,x)}(t)$ is as in Definition 2.1.2)

Let $H : C \times [0, 1] \to C$ be defined by $H(x, t) = \Psi_{(a,x)}(t)$. Then it follows from the definition of Ψ that:

(i) $H(x, 0) = x$, for each $x \in C$.

(ii) $H(x, 1) = a$, for each $x \in C$.

To complete the proof, it suffices to show that H is continuous.

Let $(x_0, t_0) \in C \times [0, 1]$. Let $\epsilon > 0$ be given and let R be a real number such that $d(x, \dot{x}) < R$, for all $x, \dot{x} \in C$.

$$\text{Let } 0 < \delta_1 < \epsilon/(2R). \tag{1}$$

Let $f_{t_0} : X \to X$ be as in Lemma 2.1.3; i.e., $f_{t_0}(x) = \Psi_{(a,x)}(t_0)$. Then $f_{t_0}(x) \in C$, for each $x \in C$, and moreover, $H(x, t_0) = f_{t_0}(x)$, for each $x \in C$. Since f_{t_0} is continuous by Lemma 2.1.3, there exists $\delta_2 > 0$ such that:

$$d(x, x_0) < \delta_2 \Rightarrow d(f_{t_0}(x), f_{t_0}(x_0)) = d(H(x, t_0), H(x_0, t_0)) < \epsilon/2. \qquad (2)$$

We shall show that:

$$d(x, x_0) < \delta_2 \text{ and } |t - t_0| < \delta_1 \Rightarrow d(H(x,t), H(x_0, t_0)) < \epsilon, \text{ for all } (x,t) \in C \times [0,1]. \qquad (*)$$

Indeed let $(x, t) \in C \times [0, 1]$. Then by (i) of Lemma 2.1.2, we have:

$$d(H(x,t), H(x, t_0)) = d(\Psi_{(a,x)}(t), \Psi_{(a,x)}(t_0)) \leq |t - t_0| d(a, x) < \delta_1 \cdot R < \epsilon/2. \quad (3)$$

Since (2) holds for x, applying the triangle inequality to (2) and (3) it follows that $d(H(x,t), H(x_0, t_0)) < \epsilon$. And (*) is hence proved.

Thus H is continuous and the conclusion follows. □

2. G-Map Systems.

In this section we prove that G-metrically convex spaces are $\mathcal{M}$-convex spaces (see Definition 1.1.2). The following definition and theorem provide an interesting example of a G-map system $\mathcal{M}$ where $\mathcal{M}(A)$ is finite for each finite subset $A \subset X$.

Definition 2.2.1. Let (X, d, Γ) be a G-metrically convex space. We shall define $\mathcal{M}(A)$ for each finite subset A of X by using induction on $|A|$.

(a) If $A = \{a_0\}$, define $\mathcal{M}(A) = \{\phi_0\}$ where $\phi_0 : \Delta_0 \to \Gamma(\{a_0\})$ and $\phi_0(e_0) = a_0$.

(b) If $A = \{a_0, a_1\}$ define $\mathcal{M}(A) = \{\phi_1\}$ where $\phi_1 : \Delta_1 \to \Gamma(\{a_0, a_1\})$ is defined by $\phi_1(\lambda_0 e_0 + \lambda_1 e_1) = \Psi(a_1, a_0)(\lambda_1)$.

(c) If $A = \{a_0,\ a_1,\ a_2\}$ let $\mathcal{M}(A) = \{\phi_{a_0},\ \phi_{a_1},\ \phi_{a_2}\}$ where each $\phi_{a_i} = \alpha_{a_i,\phi_i}$ and $\phi_i \in \mathcal{M}(A \backslash \{a_i\})$.

(d) Let $A = \{a_0, a_1, \ldots, a_k\} \in \langle X \rangle$. Assume $\mathcal{M}(B)$ is defined whenever $|B| \leq k$.

Then let $\mathcal{M}(A) = \{\alpha_{a_i,\phi} : a_i \in A$ and $\phi \in \mathcal{M}(\{a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k\})\}$, where $\alpha_{a_i,\phi}$ is as in Definition 2.1.6.

Theorem 2.2.1. *Let (X, d, Γ) be a G-metrically convex space. For each finite subset A of X, let $\mathcal{M}(A)$ be as in Definition 2.2.1 above. Then $\mathcal{M} = \bigcup_{A \in \langle X \rangle} \mathcal{M}(A)$ is a G-map system on (X, d, Γ).*

Proof. Let $A = \{a_0, a_1, \ldots, a_k\}$ and $A_1 = \{a_{i_0}, a_{i_1}, \ldots, a_{i_m}\}$ be a subset of A.

Then for each $\phi \in \mathcal{M}(A)$, we want to show there exists $\phi^* \in \mathcal{M}(A_1)$ such that

$$\phi(\Sigma_{j=0}^m \lambda_j e_{i_j}) = \phi^*(\Sigma_{j=0}^m \lambda_j e_j) \text{ for any } \lambda_0, \lambda_1, \ldots, \lambda_m \geq 0 \text{ with } \Sigma_{j=0}^m \lambda_j = 1. \quad (eq_1)$$

We proceed by induction on $|A|$.

If $A = \{a_0, a_1\}, A_1 = \{a_1\}$ and $\phi \in \mathcal{M}(A)$ then let $\phi^* = \phi_0 \in \mathcal{M}(\{a_1\})$.

The L.H.S. of $(eq_1) = \phi(e_1) = a_1$, whereas the R.H.S.of $(eq_1) = \phi_0(e_0) = a_1$. Therefore (eq_1) holds for ϕ and ϕ^*. And the case when $A_1 = \{a_0\}$ is similar.

Next assume the proposition holds for any finite set B such that $|B| \leq k$. Let $A = \{a_0, a_1, \ldots, a_k\}$ and $A_1 = \{a_{i_0}, a_{i_1}, \ldots, a_{i_m}\} \subset A$ with $m < k$. Let $\phi \in \mathcal{M}(A)$. Then $\phi = \alpha_{a_l,\phi_l}$ where $\phi_l \in \mathcal{M}(A \backslash \{a_l\})$ for some $0 \leq l \leq k$. Let $B = A \backslash \{a_l\} = \{b_0, b_1, \ldots, b_{k-1}\}$ where

$$b_i = \begin{cases} a_i, & \text{if } i < l, \\ a_{i-1}, & \text{if } i \geq l. \end{cases} \qquad (1)$$

We consider two separate cases:

Case 1. $a_l \notin A_1$.

In this case we have $A_1 = \{a_{i_0}, \ldots, a_{i_m}\} \subset B = \{b_0, \ldots, b_{k-1}\}$. So we may write $A_1 = \{b_{\grave{i}_0}, b_{\grave{i}_1}, \ldots, b_{\grave{i}_m}\}$, where

$$\grave{i}_j = \begin{cases} i_j, & \text{if } i_j < l, \\ i_j - 1, & \text{if } i_j > l. \end{cases} \tag{1*}$$

Now since $\phi_l \in \mathcal{M}(\{b_0, \ldots, b_{k-1}\})$ and since $A_1 = \{b_{\grave{i}_0}, b_{\grave{i}_1}, \ldots, b_{\grave{i}_m}\} \subset B$, it follows by the induction hypothesis that there exists $\phi_l^* \in \mathcal{M}(A_1)$ such that ϕ_l and ϕ_l^* satisfy (eq_1); i.e.,

$$\phi_l(\Sigma_{j=0}^m \lambda_j e_{\grave{i}_j}) = \phi_l^*(\Sigma_{j=0}^m \lambda_j e_j). \tag{2*}$$

Let $\phi^* = \phi_l^*$. Then in the following we will show that ϕ and ϕ^* satisfy (eq_1).

So let $\lambda_0, \lambda_1, \ldots, \lambda_m \in [0,1]$ be such that $\Sigma_{j=0}^m \lambda_j = 1$, let $z = \Sigma_{j=0}^m \lambda_j e_{i_j} \in \Delta_k$, and let $z^* = \Sigma_{j=0}^m \lambda_j e_j \in \Delta_m$. We shall prove

$$\phi(z) = \phi^*(z^*). \tag{*}$$

Indeed

$$\begin{aligned}\phi(z) &= \alpha_{a_l,\phi_l}(z) = \Psi_{(a_l,p_l(z))}(\pi_l(z)) = \Psi_{(a_l,p_l(z))}(0) = p_l(z) \\ &= \grave{\phi}_l(\Sigma_{j=0,i_j\neq l}^m(\lambda_j)/(\widehat{\lambda}_l)e_{i_j}) \\ &= \grave{\phi}_l(\Sigma_{j=0}^m \lambda_j e_{i_j}) \\ &= \phi_l(\Sigma_{j=0,i_j<l}^m \lambda_j e_{i_j} + \Sigma_{j=0,i_j>l}^m \lambda_j e_{i_j-1}).\end{aligned}$$

By (1*), the R.H.S. of the equation above is equal to $\phi_l(\Sigma_{j=0}^m \lambda_j e_{\grave{i}_j})$.

Applying (2*), the expression above is equal to $\phi_l^*(\Sigma_{j=0}^m \lambda_j e_j) = \phi^*(\Sigma_{j=0}^m \lambda_j e_j) = \phi^*(z^*)$.

Thus (*) is proved; i.e., ϕ and ϕ^* satisfy (eq_1).

Case 2. $a_l \in A_1$; i.e., $l \in \{i_0, i_1, \ldots, i_m\}$.

In this case we have $l = i_s$ for some $0 \leq s \leq m$.

let $B_1 = A_1 \backslash \{a_l\}$. Then $B_1 \subset B$. So we may write $B_1 = \{b_{\grave{i}_0}, b_{\grave{i}_1}, \ldots, b_{\grave{i}_{m-1}}\}$ where $\grave{i}_j$ is defined for $0 \leq j \leq m-1$.

Then it can be shown that

$$\grave{i}_j = \begin{cases} i_j, & \text{if } j < s, \\ i_{j+1} - 1, & \text{if } j \geq s. \end{cases} \tag{2}$$

Indeed

$$\begin{aligned} B_1 = A_1 \backslash \{a_{i_s}\} &= \{a_{i_0}, a_{i_1}, \ldots, a_{i_{s-1}}, a_{i_{s+1}}, \ldots, a_{i_m}\} \\ &= \{b_{\grave{i}_0}, b_{\grave{i}_1}, \ldots, b_{\grave{i}_{s-1}}, b_{\grave{i}_s}, \ldots, b_{\grave{i}_{m-1}}\}. \end{aligned}$$

Then

$$b_{\grave{i}_j} = \begin{cases} a_{i_j}, & \text{if } j < s, \\ a_{i_{j+1}}, & \text{if } j \geq s. \end{cases}$$

But $j < s$ iff $i_j < l$ and $j > s$ iff $i_j > l$, so applying (1) we have:

$$b_{\grave{i}_j} = \begin{cases} b_{i_j}, & \text{if } j < s, \\ b_{i_{j+1}-1}, & \text{if } j \geq s. \end{cases}$$

Hence (2) follows.

Now $\phi_l \in \mathcal{M}(\{b_0, b_1, \ldots, b_{k-1}\}) = \mathcal{M}(B)$ and $B_1 = \{b_{\grave{i}_0}, b_{\grave{i}_1}, \ldots, b_{\grave{i}_{m-1}}\} \subset B$. So by the induction hypothesis, there exists $\phi_l^* \in \mathcal{M}(B_1)$ sastisfying (eq_1); i.e.,

$$\phi_l(\Sigma_{j=0}^{m-1} \lambda_j e_{\grave{i}_j}) = \phi_l^*(\Sigma_{j=0}^{m-1} \lambda_j e_j). \tag{3}$$

Now since $\phi_l^* \in \mathcal{M}(B_1) = \mathcal{M}(A_1 \backslash \{a_l\}$, it follows that $\alpha_{a_l, \phi_l^*} \in \mathcal{M}(A_1)$. Let $\phi^* = \alpha_{a_l, \phi_l^*}$.

We shall show in the following that ϕ and ϕ^* satisfy (eq_1). Let $\lambda_0, \lambda_1, \ldots, \lambda_m \geq 0$ be such that $\Sigma_{j=0}^m \lambda_j = 1$. Let $z = \Sigma_{j=0}^m \lambda_j e_{i_j} \in \Delta_k$ and $z^* = \Sigma_{j=0}^m \lambda_j e_j \in \Delta_m$.

$$\text{We shall show that } \phi(z) = \phi^*(z^*). \tag{*}$$

First, let us consider the case $\lambda_s = 1$. In this case $\phi(z) = \phi(e_{i_s}) = a_{i_s}$ (since ϕ satisfies the G-condition).

Moreover, $\phi^*(z^*) = \phi^*(e_s) = a_{i_s}$ (because $\phi^* : \Delta_m \to \Gamma(\{a_{i_0}, a_{i_1}, \ldots, a_{i_s}, \ldots, a_{i_m}\}$ satisfies the G-condition). Therefore (*) holds in this case.

Next assume $\lambda_s \neq 1$ and let $\widehat{\lambda}_s = \Sigma^m_{j=0, j\neq s}\lambda_j$.

Indeed $\phi(z) = \alpha_{a_l, \phi_l}(z) = \Psi_{(a_l, y_1)}(\pi_l(z)) = \Psi_{(a_l, y_1)}(\pi_{i_s}(z)) = \Psi_{(a_l, y_1)}(\lambda_s)$, (4)

where

$$
\begin{aligned}
y_1 &= \dot{\phi}_l(\Sigma^m_{j=0, i_j \neq l}(\lambda_j)/(\widehat{\lambda}_s)e_{i_j})\\
&= \phi_l(\Sigma^m_{j=0, i_j < l}(\lambda_j)/(\widehat{\lambda}_s)e_{i_j} + \Sigma^m_{j=0, i_j > l}(\lambda_j)/(\widehat{\lambda}_s)e_{i_j - 1})\\
&= \phi_l(\Sigma^m_{j=0, j<s}(\lambda_j)/(\widehat{\lambda}_s)e_{i_j} + \Sigma^m_{j=0, j>s}(\lambda_j)/(\widehat{\lambda}_s)e_{i_j - 1}), \text{ which by (2) is:}\\
&= \phi_l(\Sigma^m_{j=0, j<s}(\lambda_j)/(\widehat{\lambda}_s)e_{\dot{i}_j} + \Sigma^m_{j=0, j>s}(\lambda_j)/(\widehat{\lambda}_s)e_{\dot{i}_{j-1}})\\
&= \phi_l((\lambda_0)/(\widehat{\lambda}_s)e_{\dot{i}_0} + (\lambda_1)/(\widehat{\lambda}_s)e_{\dot{i}_1} + \cdots + (\lambda_{s-1})/(\widehat{\lambda}_s)e_{\dot{i}_{s-1}}\\
&\quad + (\lambda_{s+1})/(\widehat{\lambda}_s)e_{\dot{i}_s} + \cdots + (\lambda_m)/(\widehat{\lambda}_s)e_{\dot{i}_{m-1}}).
\end{aligned}
$$

Applying (3), the R.H.S. of the equation above is equal to:

$$
\begin{aligned}
&\phi^*_l((\lambda_0)/(\widehat{\lambda}_s)e_0 + (\lambda_1)/(\widehat{\lambda}_s)e_1 + \cdots + (\lambda_{s-1})/(\widehat{\lambda}_s)e_{s-1}\\
&+ (\lambda_{s+1})/(\widehat{\lambda}_s)e_s + \cdots + (\lambda_m)/(\widehat{\lambda}_s)e_{m-1})\\
&= \phi^*_l(\Sigma^m_{j=0, j<s}(\lambda_j)/(\widehat{\lambda}_s)e_j + \Sigma^m_{j=0, j>s}(\lambda_j)/(\widehat{\lambda}_s)e_{j-1}).
\end{aligned}
$$

It follows that

$$y_1 = \phi^*_l(\Sigma^m_{j=0, j<s}(\lambda_j)/(\widehat{\lambda}_s)e_j + \Sigma^m_{j=0, J>s}(\lambda_j)/(\widehat{\lambda}_s)e_{j-1}). \tag{5}$$

By equations (4) and (5), the left-hand side of (*) is equal to:

$\phi(z) = \Psi_{(a_l, y_1)}(\lambda_s)$ where $y_1 = \phi^*_l(\Sigma^m_{j=0, j<s}(\lambda_j)/(\widehat{\lambda}_s)e_j + \Sigma^m_{j=0, j>s}(\lambda_j)/(\widehat{\lambda}_s)e_{j-1})$.

On the other hand, the right-hand side of (*)=$\phi^*(z^*) = \phi^*(\Sigma\lambda_j e_j)$.

Now $\phi^* : \Delta_m \to \Gamma(\{a_{i_0}, \ldots, a_{i_s}, \ldots, a_{i_m}\})$.

Moreover, $\phi^* = \alpha_{a_{i_s}, \phi^*_l}$. So it follows from Definition 2.1.6 that

$$\phi^*(z^*) = \phi^*(\Sigma_{j=0}^m \lambda_j e_j) = \Psi_{a_{i_s}, y_2}(\pi_s(z^*)) = \Psi_{a_{i_s}, y_2}(\lambda_s), \tag{6}$$

$$y_2 = \dot{\phi}_l^*(\Sigma_{j=0, j\neq s}^m (\lambda_j)/(\widehat{\lambda}_s) e_j) = \phi_l^*(\Sigma_{j=0,j<s}^m (\lambda_j)/(\widehat{\lambda}_s) e_j + \Sigma_{j=0,j>s}^m (\lambda_j)/(\widehat{\lambda}_s) e_{j-1}). \tag{7}$$

From (5) and (7), it follows that $y_1 = y_2$. And so by (4) and (6), we have $\phi(z) = \Psi_{(a_l, y_1)}(\lambda_s) = \Psi_{(a_{i_s}, y_2)}(\lambda_s) = \phi^*(z^*)$.

Thus we have proved (*) and hence that ϕ and ϕ^* satisfy (eq_1). □

The following definition and theorem present yet another interesting and very useful G-map system in G-metrically convex spaces. As we have seen in Chapter One, for G-convex spaces with a homogeneous G-map system, certain selection theorems are true. We will also prove extension theorems for these spaces in Chapter 3.

Definition 2.2.2. Let (X, d, Γ) be a G-metrically convex space. Given a finite subset A of X, we shall define a map $\phi_A : \Delta_{|A|-1} \to \Gamma(A)$ by induction on $|A|$.

First let $\leq^*$ be a total order on X.

If $a = \{a_0\}$ then $\phi_A : \Delta_0 \to \Gamma(A) = \{a_0\}$ is defined by $\phi_A(e_0) = a_0$.

If $|A| = 2$, let $A = \{a_0, a_1\}$ where $a_0 \leq^* a_1$ and define

$\phi_A : \Delta_1 \to \Gamma(A)$ by

$\phi_A(\lambda_0 e_0 + \lambda_1 e_1) = \Psi_{(a_1, a_0)}(\lambda_1)$.

Next assume $\phi_B : \Delta_{|B|-1} \to \Gamma(B)$ is defined whenever $|B| = k$.

Let $A = \{a_0, a_1, \ldots, a_k\}$ where $a_0 \leq^* a_1 \leq^* \cdots \leq^* a_k$.

Then define $\phi_A = \alpha_{a_k,\phi_{A_k}}$ (where $A_k = A\backslash\{a_k\}$ and $\alpha_{a_k,\phi_{A_k}}$ is as in Definition 2.1.6).

Theorem 2.2.2. *Let (X, d, Γ) be a G-metrically convex space. Let $\mathcal{M} = \bigcup_{A\in\langle X\rangle}\{\phi_A\}$, where ϕ_A is as in Definition 2.2.2 above. Then $\mathcal{M}$ is a homogeneous G-map system.*

Proof. It suffices to show that whenever $A = \{a_0, a_1, \ldots, a_n\} \subset X$ is such that $a_0 \leq^* a_1 \leq^* \cdots \leq^* a_k$, $A_1 = \{a_{i_0}, a_{i_1}, \ldots, a_{i_m}\} \subset A$, and $\phi_A : \Delta_n \to \Gamma(A)$ is as in Definition 2.2.2 above, then for any $\lambda_0, \lambda_1, \ldots \lambda_m \in [0,1]$ satisfying $\Sigma_{j=0}^m \lambda_j = 1$, we have

$$\phi_A(\Sigma_{j=0}^m \lambda_j e_{i_j}) = \phi_{A_1}(\Sigma_{j=0}^m \lambda_j e_j). \qquad (eq_2)$$

We use induction on $|A|$. If $|A| = 1$ or 2, then (eq_2) obviously holds.

Next assume (eq_2) holds for any finite subset of X with k elements, let $A = \{a_0, a_1, \ldots, a_k\}$ and let $A_1 = \{a_{i_0}, a_{i_1}, \ldots, a_{i_m}\} \subset A$. We shall show that (eq_2) holds for A and A_1.

We consder two cases.

Case 1. $a_{i_m} \neq a_k$(*i.e.* $i_m \neq k$).

Let $A_k = A\backslash\{a_k\}$ and let $z = \Sigma_{j=0}^m \lambda_j e_{i_j}$. Then

$$\phi_A(z) = \alpha_{a_k,\phi_{A_k}}(z) = \Psi_{a_k,p_k(z))}(\pi_k(z)) = \Psi_{(a_k,p_k(z))}(0) = p_k(z). \qquad (1)$$

Since $z \in \Delta_{k-1}$, it follows by Corollary 2.1.2 that $p_k(z) = \phi_{A_k}(z)$. So (1) implies that

$$\phi_A(z) = \phi_{A_k}(z). \qquad (2)$$

But $A_1 = \{a_{i_0}, a_{i_1}, \ldots, a_{i_m}\} \subset \{a_0, a_1, \ldots, a_{k-1}\} = A_k$, so applying the induction hypothesis we have $\phi_A(\Sigma_{j=0}^m \lambda_j e_{i_j}) = \phi_A(z) = \phi_{A_k}(z) = \phi_{A_1}(\Sigma_{j=0}^m \lambda_j e_j)$. Thus (eq_2) holds in this case.

Case 2. $a_{i_m} = a_k$.

Let $z = \Sigma_{j=0}^{m}\lambda_j e_{i_j}$ and $z^* = \Sigma_{j=0}^{m}\lambda_j e_j$. If $\lambda_m = 1$ then $z = e_k$ and $z^* = e_m$. On the one hand we have $\phi_A(z) = \phi_A(e_k) \in \Gamma(\{a_k\}) = \{a_k\}$.

Also $\phi_{A_1}(z^*) = \phi_{A_1}(e_m) \in \Gamma(\{a_{i_m}\}) = \{a_{i_m}\} = \{a_k\}$. Thus (eq_2) holds in case $\lambda_m = 1$.

Next assume $\lambda_m \neq 1$.

$$\text{First } \phi_A(\Sigma_{j=0}^{m}\lambda_j e_{i_j}) = \Psi_{a_k,p_k(z)}(\pi_k(z)) = \Psi_{a_k,p_k(z)}(\lambda_m). \tag{1}$$

Let

$$\begin{aligned} y_1 &= p_k(z) = p_k(\Sigma_{j=0}^{m}\lambda_j e_{i_j}) = \dot{\phi}_{A_k}(\Sigma_{j\in\{0,1,\ldots,m\},i_j\neq k}(\lambda_j)/(\widehat{\lambda}_m)e_{i_j}) \\ &= \dot{\phi}_{A_k}(\Sigma_{j=0}^{m-1}\lambda_j/(\widehat{\lambda}_m)e_{i_j}) = \phi_{A_k}(\Sigma_{j=0}^{m-1}\lambda_j/(\widehat{\lambda}_m)e_{i_j}). \end{aligned} \tag{2}$$

Now consider $\phi_{A_1}(\Sigma_{j=0}^{m}\lambda_j e_j)$. First let $A_1^m = A_1\backslash\{a_{i_m}\} = A_1\backslash\{a_k\}$. Then

$$\phi_{A_1}(\Sigma_{j=0}^{m}\lambda_j e_j) = \Psi_{a_{i_m},p_m(z^*)}(\pi_m(z^*)) = \Psi_{a_{i_m},p_m(z^*)}(\lambda_m). \tag{1*}$$

Let

$$y_2 = p_m(z^*) = \dot{\phi}_{A_1^m}(\Sigma_{j=0}^{m-1}\lambda_j/(\widehat{\lambda}_m)e_j) = \phi_{A_1^m}(\Sigma_{j=0}^{m-1}\lambda_j/(\widehat{\lambda}_m)e_j). \tag{2*}$$

Now since $A_1^m = \{a_{i_0}, a_{i_1}, \ldots, a_{i_m}\}\backslash\{a_{i_m}\} \subset A_1 = A\backslash\{a_k\}$, by applying the induction hypothesis, (2) and (2*), we have $y_1 = \phi_{A_k}(\Sigma_{j=0}^{m-1}\lambda_j/(\widehat{\lambda}_m)e_{i_j}) = \phi_{A_1 m}(\Sigma_{j=0}^{m-1}\lambda_j/(\widehat{\lambda}_m)e_j) = y_2$. So substituting the above in (1) and (1*), we have $\phi_A(\Sigma_{j=0}^{m}\lambda_j e_{i_j}) = \psi_{(a_k,y_1)}(\lambda_m) = \psi_{(a_{i_m},y_2)}(\lambda_m) = \phi_{A_1}(\Sigma_{j=0}^{m}\lambda_j e_j)$. Hence (eq_2) is proved in this case also and the conclusion follows. □

CHAPTER THREE

EXTENSION THEOREMS AND APPLICATIONS

This chapter contains several extension theorems for continuous maps from a closed subset of an arbitrary metric space into a G-convex space. These are generalizations of the related theorems of Dugundji, Ma and Pruszko (See [DJ1], [Ma] and [Psz]).

In Section 1, we present four different extensions for the case of a single-valued map $f : A \to Y$ where A is a closed subset of a metric space X and Y is a G-convex space. These many versions reflect our several attempts to generalize Dugundji's theorem without imposing too many conditions on the range space, i.e., the G-convex space. It turned out however that only after assuming that the range space has a homogeneous G-map system can we obtain such a generalization. This indicates that G-map systems prove to be a useful tool.

The first theorem, Theorem 3.1.1, gives an extension $\widehat{f}$ which is not necessarily continuous on the whole space, but rather on a subset $A \cup B$, where B is an open dense subset of $X \setminus A$. Theorem 3.1.2 gives a set-valued extension F continuous on A and LSC on X, whereas Theorem 3.1.3 gives a set-valued extension G which is continuous on A and USC on X. We note that these three theorems contain no conditions on either the domain or the range other than those assumed in the original theorem of Dugundji. We also point out that, as far as we know, the condition of local convexity of the range cannot be relaxed even in the linear case i.e. the case of a topological vector space. Also note that whereas Theorems 3.1.1, 3.1.2 and 3.1.3 are only partial generalizations of Dugundji's theorem, Theorem 3.1.4 is a true generalization of the theorem.

In Section 2, we present a generalizaton of Ma's Theorem and in Section 3 we give a generalization for Pruszko's theorem.

In Section 4, we adopt the method used by Wu (Theorem 3.3 in [TW]) to obtain applications of our extension theorems to equilibrium existence theorems for qualitative games.

Finally, we point out that for all the extensions given in this chapter, as is also the case in the original theorems, we have the image of the new extension contained in the G-convex hull of the image of the original map.

1. Extension Of Single-Valued Maps.

As the results of this section are all directed towards generalizing Dugundji's extension theorem, we begin by quoting the theorem (Theorem 4.1 in [DJ]).

Theorem. *let X be an arbitrary metric space, A a closed subset of X, Y a locally convex linear space, and $f : A \to Y$ a continuous map. Then there exists a continuous extension $\tilde{f} : X \to Y$ of f; furthermore, $\tilde{f}(X) \subset co(f(A))$.*

Next we state Lemma 2.1 in [DJ], which is needed for all the extension theorems given in this Chapter. Although its simple proof was omitted from the original paper, we give it here for the sake of more clarity and precision.

Lemma 3.1.1. *Let (X, d) be a metric space and A be a closed subset of X. Then there exists an open covering $\mathcal{U}$ of $(X \backslash A)$ such that:*

(1.1) $\mathcal{U}$ is locally finite.

(1.2) If $a \in A$ and W is a nhood of a in X, then there exists a nhood $\grave{W}$ of a in X such that $\grave{W} \subset W$ and for every $U \in \mathcal{U}$, $U \cap \grave{W} \neq \emptyset$ implies $U \subset W$.

Proof. For each $x \in (X \backslash A)$, let $\epsilon_x = d(x, A)$. Let $U_x = B(x, \epsilon_x/2)$ (the open ball centered at x with radius $\epsilon_x/2$). Then $\{U_x : x \in (X \backslash A)\}$ is an open cover for $(X \backslash A)$. Since X is a metric, the cover above has an open locally finite refinement,

$\mathcal{U}$.

Next let $a_0 \in A$ and W be an open nhood containing a_0 in X. Then there exists $r > 0$ such that $B(a_0, r) \subset W$. Let $\grave{W} = B(a_0, r/3)$. We shall show that $\grave{W}$ satisfies (1.2). Let $U \in \mathcal{U}$ be such that $z \in \grave{W} \cap U$. Then by the construction of $\mathcal{U}$, there exists $x \in (X \backslash A)$ such that $U \subset B(x, \epsilon_x/2)$ where $\epsilon_x = d(x, A)$. First we shall show that

$$\epsilon_x < 2r/3. \qquad (*)$$

Since $z \in U \subset B(x, \epsilon_x/2)$, it follows that:

$$d(x, z) < \epsilon_x/2. \qquad (1)$$

$$z \in \grave{W} \text{ implies } d(z, a_0) < r/3. \qquad (2)$$

So by (1) and (2), we have $d(x, a_0) < r/3 + \epsilon_x/2$. But $\epsilon_x = d(x, A)$ and $a_0 \in A$, so the inequality above implies that $\epsilon_x \leq d(x, a_0) < r/3 + \epsilon_x/2$. Hence (*) is proved.

Now let $y \in U$, then:

$$d(y, a_0) \leq d(y, z) + d(z, a_0) < d(y, z) + r/3 \leq d(y, x) + d(x, z) + r/3. \qquad (3)$$

But both y and z belong to $U \subset B(x, \epsilon_x/2)$, so (3) implies $d(y, a_0) < \epsilon_x/2 + \epsilon_x/2 + r/3 < 2r/3 + r/3 = r$.

So it follows that $y \in W$. Therefore $U \subset W$. □

Next we present Lemma 3.1.2, which is true for any topological space in general. We need it here for the proof of Theorem 3.1.1.

Lemma 3.1.2. *Let X be a topological space and assume $\mathcal{U}$ is an open locally finite cover for X. Let $\mathcal{F}(\mathcal{U})$ denote the collection of all finite subfamilies of $\mathcal{U}$. Let $r_1 : X \to \mathcal{F}(\mathcal{U})$ be defined by: $r_1(x) = \{U \in \mathcal{U} : x \in U\}$. Then there exists an open* *dense subset B of X such that for any $b_0 \in B$, there exists an open nhood W of b_0 in X such that: $r_1(x) = r_1(b_0)$, for all $x \in W$.*

Proof. Let $r_2 : X \to \mathcal{F}(\mathcal{U})$ be defined by: $r_2(x) = \{U \in \mathcal{U} : x \in cl(U)\}$. Let $B = \{x \in X : r_1(x) = r_2(x)\}$. We shall first show that B is dense and open. Let O be a nonempty open subset of X and let $x \in O$. Then there exists an open nhood W of x in X such that $W \subset O$ and W intersects with only finitely many elements of $\mathcal{U}$, say, $U_1, U_2, \ldots, U_m$. It follows that $r_1(W) = \bigcup\{r_1(y) : y \in W\} = \{U_1, \ldots, U_m\}$.

Choose an element $b \in W$ such that $|r_1(b)| \geq |r_1(x)|$, for each $x \in W$. We shall now show that $b \in B$. Since $r_1(b) \subset r_2(b)$, it suffices to show that $r_2(b) \subset r_1(b)$. Let $U \in \mathcal{U}$, and assume $U \notin r_1(b) = \{U_{i_1}, U_{i_2}, \ldots, U_{i_k}\}$. We shall show $U \notin r_2(b)$. Let $W^* = (\cap_{j=1}^{k} U_{i_j}) \cap W$. Then W^* is an open nhood of b in X. We claim that $W^* \cap U = \emptyset$. This will imply that $b \notin cl(U)$ and hence that $U \notin r_2(b)$.

Indeed if $c \in W^* \cap U$, then $c \in \cap_{j=1}^{k} U_{i_j} \cap U \cap W$, i.e. $\{U, U_{i_1}, \ldots, U_{i_j}\} \subset r_1(c)$ and hence $|r_1(c)| > |r_1(b)|$, which contradicts the choice of b (since $c \in W$). This shows that B is dense.

To complete the proof, it remains to show that for any $b_0 \in B$, there exists an open nhood W of b_0 in X such that $W \subset B$ and $r_1(x) = r_1(b_0)$, for each $x \in W$. Indeed let $r_1(b_0) = r_2(b_0) = \{U_1, U_2, \ldots, U_m\}$. Then there exists a nhood W_1 of b_0 such that W_1 intersects with $U_1, \ldots, U_m$ only; i.e. for any $U \in \mathcal{U}$, we have

$$W_1 \cap U = \emptyset \text{ iff } U \notin \{U_1, \ldots, U_m\}. \qquad (*)$$

Let $W = W_1 \cap (\cap_{i=1}^{m} U_i)$, then W is open. We shall show $W \subset B$. If $x \in W$ then obviously,

$$\{U_1, \ldots, U_m\} \subset r_1(x). \qquad (1)$$

If $U \in \mathcal{U}$ and $x \in cl(U)$ then $W \cap U \neq \emptyset$ which implies $W_1 \cap U \neq \emptyset$ and hence by (*), $U \in \{U_1, \ldots, U_m\}$.Thus

$$r_2(x) \subset \{U_1, \ldots, U_m\}. \qquad (2)$$

So, by (1) and (2), for any $x \in W$ we have:

$$\{U_1, \ldots, U_m\} \subset r_1(x) \subset r_2(x) \subset \{U_1, \ldots, U_m\} \text{ i.e. } r_1(x) = r_2(x) = \{U_1, \ldots, U_m\}.$$

Therefore $W \subset B$ and $r_1(x) = r_1(b_0)$ for each $x \in W$. □

Definition 3.1.1. Let (Y, Γ) be a G-convex space. Then (Y, Γ) is said to be locally G-convex iff it satisfies the property that for every $y \in Y$ and every open nhood W of y in Y, there exists an open nhood V of y in Y such that G-co$(V) \subset W$.

Definition 3.1.2. A G-convex space (Y, Γ) is said to be strongly locally G-convex iff given any compact G-convex subset A of Y and any open subset U of Y containing A, there exists an open subset V of Y containing A such that G-co$(V) \subset U$.

Remark. *It is obvious that the local G-convexity defined above generalizes the usual local convexity in topological vector spaces. But it is a well-known fact that any locally convex topological vector space has the property mentioned in Definition 3.1.2, i.e. any locally convex topological vector space is strongly locally convex, if such an expression may be allowed. But it is not at all clear whether such an implication is true in the case of G-convex space. We suggest this as a problem worth further investigation.*

Theorem 3.1.1. *Let (X, d) be a metric space and A a closed subset of X. Let (Y, Γ) be a locally G-convex space. Let $f : A \to Y$ be a continuous function. Then there exists a function $\widehat{f} : X \to Y$ such that*

(i) $\widehat{f}|_A = f$.

(ii) $\widehat{f}(X) \subset$ G-co$(f(A))$.

(iii) $\widehat{f}$ is continuous at a for each $a \in A \cup B$, where B is an open dense subset of $(X \backslash A)$.

Proof. Let $\mathcal{U}$ be the locally finite open covering for $(X \backslash A)$ provided by Lemma 3.1.1. Let $(\beta_U)_{U \in \mathcal{U}}$ be a partition of unity on $X \setminus A$ subordinated to $\mathcal{U}$. And let

$\mathcal{F}(\mathcal{U})$ be the collection of all finite subfamilies of $\mathcal{U}$. Also let $r_1 : X \setminus A \to \mathcal{F}(\mathcal{U})$ be defined as in Lemma 3.1.2. Let $r_1(x)$ be denoted by $C_x \subset \mathcal{U}$.

For each $U \in \mathcal{U}$, choose an element $x_U \in U$ and $a_U \in A$ be such that:

$$d(x_U, a_U) \leq 2d(x_U, A). \tag{1}$$

For each $C = \{U_0, U_1, \ldots, U_n\} \subset \mathcal{U}$ satisfying the condition $\cap_{i=0}^{n} U_i \neq \emptyset$, define ϕ_C as follows:

$\phi_C : \Delta_n \to \Gamma(\{f(a_{U_0})), f(a_{U_1}), \ldots, f(a_{U_n})\})$ is a continuous map as provided by the Definition of a G-convex space.

(a1)

Now define $\widehat{f} : X \to Y$ as follows:

$$\widehat{f}(x) = \begin{cases} f(a), & \text{if } x \in A; \\ \phi_{C_x}(\Sigma_{U_i \in C_x} \beta_{U_i}(x) e_i), & \text{if } x \in X \backslash A. \end{cases}$$

Clearly, $\widehat{f}$ is well defined. Also $\widehat{f}$ clearly satisfies (i). Moreover if $x \in (X \backslash A)$ and $r_1(x) = C_x = \{U_0, U_1, \ldots, U_m\}$, then

$$\widehat{f}(x) \in \Gamma(\{f(a_{U_0}), \ldots, f(a_{U_m})\}). \tag{a_2}$$

By (a_2), (ii) is satisfied.

Next we shall show that $\widehat{f}$ is continuous on $A \cup B$ where B is the open dense subset of $(X \backslash A)$ provided by Lemma 3.1.2.

First let $b_0 \in B$, and let $r_1(b_0) = C_{b_0} = \{U_0, U_1, \ldots, U_m\}$. Then by Lemma 3.1.2, there exists an open nhood W of b_0 in $X \setminus A$ such that $C_x = r_1(x) = r_1(b_0) = C_{b_0} = \{U_0, \ldots, U_m\}$ for each $x \in W$. Let $\mu : W \to \Delta_m$ be defined by $\mu(x) = \Sigma_{i=0}^{m} \beta_{U_i}(x)$. Then μ is continuous.

Also for each $x \in W$ we have $\phi_{C_x} = \phi_{C_{b_0}} : \Delta_m \to \Gamma(\{f(a_{U_0}), \ldots, f(a_{U_m})\})$. So we indeed have $\widehat{f}(x) = \phi_{C_{b_0}}(\mu(x)\ \forall\ x \in W$. Since W is open in X and both $\phi_{C_{b_0}}$ and μ are continuous, it follows that $\widehat{f}$ is continuous at b_0.

Now it only remains to show that $\widehat{f}$ is continuous at any point $a_0 \in A$. Let $\dot{V}$ be an open nhood of $\widehat{f}(a_0) = f(a_0)$ in Y. Then there exists a nhood V of $f(a_0)$ such that G-co$V \subset \dot{V}$. By continuity of f at a_0, there exists $\delta > 0$ such that for each $a \in A$

$$d(a, a_0) < \delta \text{ implies } f(a) \in V. \tag{2}$$

Let $W = B(a_0, \delta/3)$ (the open ball centered at a_0 with radius $(\delta/3)$). We shall show that

$$\text{if } U \in \mathcal{U} \text{ and } U \subset W \text{ then } f(a_U) \in V. \tag{*}$$

Indeed, if $U \subset W$ then $x_U \in W$ and hence

$$d(x_U, a_0) < \delta/3. \tag{3}$$

Thus $d(a_U, a_0) \leq d(a_U, x_U) + d(x_U, a_0) < 2d(A, x_U) + \delta/3 \leq 2d(a_0, x_U) + \delta/3 < \delta$. Applying (2), we have $f(a_U) \in V$ so that (*) is proved.

Next, by Lemma 3.1.1, we choose an open nhood $\dot{W}$ of a_0 in X contained in W such that for any $U \in \mathcal{U}$,$U \cap \dot{W} \neq \emptyset$ implies $U \subset W$. We now show that

$$\widehat{f}(\dot{W}) \subset \dot{V}. \tag{**}$$

Let $x \in \dot{W}$. If $x \in A$, then $x = a \in W$ and hence, by (2), $\widehat{f}(x) = f(a) \in V \subset \dot{V}$. Next assume $x \notin A$. Let $C_x = r_1(x) = \{U_0, U_1, \ldots, U_m\}$. Then $x \in U_i \cap \dot{W}$ for all $0 \leq i \leq m$ and hence it follows by the choice of $\dot{W}$ that $U_i \subset W$, $\forall\, 0 \leq i \leq m$. Using (*), it follows that:

$$f(a_{U_i}) \in V \text{ for all } 0 \leq i \leq m. \tag{4}$$

Combining (a_2) and (4), we have $\widehat{f}(x) \in \Gamma(\{f(a_{U_0}),\ f(a_{U_1}), \ldots,\ f(a_{U_m})\}) \subset$ G-co$V \subset \dot{V}$.

Thus (**) is proved and hence $\widehat{f}$ is continuous on A. $\square$

Theorem 3.1.2. *Let (X,d) be a metric space and A a closed subset of X. Let (Y,Γ) be a locally G-convex space. Let $f : A \to Y$ be a continuous function. Then there exists $F : X \to 2^Y$ such that:*

(i) $F|_A = f$.

(ii) $F(X) \subset G\text{-}co(f(A))$.

(iii) F is continuous at a for each $a \in A$.

(iv) F is LSC on X.

Proof. Let $\mathcal{U}$ be the locally finite open covering for $(X \backslash A)$ provided by Lemma 3.1.1. Let $(\beta_U)_{U\in\mathcal{U}}$ be a partition of unity on $X \setminus A$ subordinated to $\mathcal{U}$. Let $\mathcal{F}(\mathcal{U})$ be the collection of all finite subfamilies of $\mathcal{U}$. Also let $r_1 : X \to \mathcal{F}(\mathcal{U})$ be as in Lemma 3.1.2. Let $r_1(x)$ be denoted by $C_x \subset \mathcal{U}$. For each $U \in \mathcal{U}$, choose an element $x_U \in U$ and let $a_U \in A$ be such that

$$d(x_U, a_U) \leq 2d(x_U, A). \tag{1}$$

For each $x \in X \backslash A$, let N_x be an open nhood of x in $X \setminus A$ that intersects with only finitely many elements of $\mathcal{U}$, say, $U_0, U_1, \dots, U_n$. Since Y is a G-convex space, we can find a map ϕ_{N_x} satisfying (a_1) and (a_2) below.

$$\phi_{N_x} : \Delta_n \to \Gamma(\{f(a_{U_0})), f(a_{U_1}), \dots, f(a_{U_n})\}) \text{ is continuous.} \tag{a_1}$$

For any subset J of $\{0, 1, \dots, n\}$, if Δ_J is that face of Δ_n corresponding to J then:

$$\phi_{N_x}(\Delta_J) \subset \Gamma(\{f(a_{U_j}) : j \in J\}. \tag{a_2}$$

Let $\mathcal{W} = \{N_x : x \in X \backslash A\}$. Then $\mathcal{W}$ is an open covering for $X \backslash A$.

We notice that if $x \in N \in \mathcal{W}$ and if the set of all elements of $\mathcal{U}$ that have a nonempty intersection with N is $\{U_0, U_1, \dots, U_n\}$, then $\beta_U(x) \neq 0$ iff $U \in \{U_0, \dots, U_n\}$. Therefore we can define a continuous function $\mu_N : N \to \Delta_n$ by $\mu_N(x) = \Sigma_{i=0}^n \beta_{U_i}(x) e_i$.

Now define $F : X \to 2^Y$ as follows:

$$F(x) = \begin{cases} \{f(x)\}, & \text{if } x \in A \\ \{\phi_N(\mu_N(x)) : x \in N \in \mathcal{W}\}, & \text{if } x \in X \backslash A. \end{cases}$$

Note that (i) follows immediately.

Next we shall show that for any $x \in X \backslash A$ if $C_x = r_1(x) = \{U_0, \ldots, U_m\}$ then

$$F(x) \subset \Gamma(\{f(a_{U_0}), \ldots, f(a_{U_m}). \qquad (a_3)$$

Let $y \in F(x)$. Then $y = \phi_N(\mu_N(x))$ for some $N \in \mathcal{W}$ with $x \in N$. Note that $x \in N \cap U_i$ for each $i = 0, 1, \ldots, m$. Let $\{U \in \mathcal{U} : U \cap N \neq \emptyset\} = \{\grave{U}_0, \grave{U}_1, \ldots, \grave{U}_n\}$. Then there exists a subset J of $\{0, 1, \ldots, n\}$ such that $\{U_0, U_1, \ldots, U_m\} = \{\grave{U}_j : j \in J\}$.

It follows that $\beta_U(x) \neq 0 \Rightarrow U \in \{U_0, \ldots, U_m\} = \{\grave{U}_j : j \in J\}$. Hence we have $\mu_N(x) \in \Delta_J$, (that face of Δ_n corresponding to J).

By (a_2) it follows that $y = \phi_N(\mu_N(x)) \in \phi_N(\Delta_J) \subset \Gamma(\{f(a_{\grave{U}_j}) : j \in J\}) = \Gamma(\{f(a_{U_0}), f(a_{U_1}), \ldots, f(a_{U_m})\}$. Hence (a_3) is proved .

Now (ii) follows easily from (a_3) because the left hand side of (a_3) is always contained in G-co$(f(A))$.

(iii) F is continuous at a for each $a \in A$.

Let $a_0 \in A$, we shall first show that F is USC at a_0.

Let $\grave{V}$ be open in Y such that $F(a_0) \subset \grave{V}$. This implies that $f(a_0) \in \grave{V}$. Let V be open in Y such that $f(a_0) \in V \subset$ G-co$V \subset \grave{V}$. By continuity of f at $a_0 \in A$, there exists $\delta > 0$ such that for each $a \in A$, we have

$$d(a, a_0) < \delta \text{ implies } f(a) \in V. \qquad (2)$$

Let $W = B(a_0, \delta/3)$. Then it can be shown (See the proof of Theorem 3.1.1) that for any $U \in \mathcal{U}$ we have:

$$U \subset W \text{ implies } f(a_U) \in V. \qquad (*)$$

Now by Lemma 3.1.1, there exists an open nhood $\dot{W}$ of a_0 in X such that $\dot{W} \subset W$ and for each $U \in \mathcal{U}$ we have: $U \cap \dot{W} \neq \emptyset$ implies $U \subset W$. We shall show that

$$F(\dot{W}) \subset \dot{V}. \tag{**}$$

Indeed, let $x \in \dot{W}$. If $x = a \in A$ then $F(x) = \{f(x)\} \subset V \subset \dot{V}$. Next assume $x \notin A$. Let $C_x = r_1(x) = \{U_0, U_1, \ldots, U_m\}$, then $x \in U_i \cap \dot{W}$ for all $0 \leq i \leq m$ and therefore $U_i \subset W$ for all $0 \leq i \leq m$. It then follows by (*) that

$$\{f(a_{U_0}), f(a_{U_1}), \ldots, f(a_{U_m})\} \subset V. \tag{4}$$

Applying (a_3) to the above, we have: $F(x) \subset \Gamma(\{f(a_{U_0}), \ldots, f(a_{U_m})\} \subset$ G-co$V \subset \dot{V}$. Thus (**) is proved and F is USC at a_0.

Next we shall prove that F is LSC at a_0. So let $\dot{V}$ be open in Y such that $F(a_0) \cap \dot{V} \neq \emptyset$.

Then $F(a_0) \subset \dot{V}$ (since F is single valued at a_0), and by USC of F at a_0, there exists a nhood $\dot{W}$ of a_0 in X such that $F(\dot{W}) \subset \dot{V}$.

It follows that for each $x \in \dot{W}$, $F(x) \cap \dot{V} = F(x) \neq \emptyset$. Hence F is LSC at a_0.

(iv) F is LSC on $(X \backslash A)$.

Let $x_0 \in (X \backslash A)$, and let V be an open set in Y such that $F(x_0) \cap V \neq \emptyset$.

By definition of F, it follows that there exists an open nhood N of x_0 in $X \setminus A$ such that $N \in \mathcal{W}$ and

$$\phi_N(\mu_N(x_0)) = (\phi_N \circ \mu_N)(x_0) \in V. \tag{5}$$

Since $\phi_N \circ \mu_N : N \to Y$ is continuous, it follows from (5) that there exists a nhood N_2 of x_0 in $X \setminus A$ such that

$$N_2 \subset N \text{ and } (\phi_N \circ \mu_N)(N_2) \subset V. \tag{6}$$

We shall show that

$$F(y) \cap V \neq \emptyset \text{ for all } y \in N_2. \tag{***}$$

For let $y \in N_2$, then $y \in N$ and by definition of F at y, it follows that $(\phi_N \circ \mu_N)(y) = \phi_N(\mu_N(y)) \in F(y)$. This together with (6) implies:

$\phi_N(\mu_N(y)) \in F(y) \cap V \neq \emptyset$.

Thus we have proved (***) which implies that F is LSC at x_0. Therefore F is LSC on $(X \backslash A)$. □

Theorem 3.1.3. *Let (X, d) be a metric space and A a closed subset of X. Let (Y, Γ) be a locally G-convex space. Let $f : A \to Y$ be a continuous function.*

Then there exists $G : X \to 2^Y$ such that

(i) $G|_A = f$.

(ii) $G(X) \subset G\text{-}co(f(A))$.

(iii) G is continuous at a for each $a \in A$.

(iv) G is USC on $X \backslash A$.

Proof. Let $\mathcal{U}$ be the locally finite open covering for $(X \backslash A)$ provided by Lemma 1.

Let $(\beta_U)_{U \in \mathcal{U}}$ be a partition of unity on $X \backslash A$ subordinated to $\mathcal{U}$. Let $\mathcal{F}(\mathcal{U})$ be the collection of all finite subfamilies of $\mathcal{U}$. Also let $r_1 : X \to \mathcal{F}(\mathcal{U})$ be as in Lemma 3.1.2. Let $r_1(x)$ be denoted by $C_x \subset \mathcal{U}$. For each $U \in \mathcal{U}$ choose an element $x_U \in U$ and let $a_U \in A$ be such that

$$d(x_U, a_U) \leq 2d(x_U, A). \tag{1}$$

For each $x \in (X \backslash A)$, let N_x be an open nhood of x in $X \backslash A$ that intersects with only finitely many elements of $\mathcal{U}$, say $U_0, U_1, \ldots, U_n$.

Since Y is a G-convex space, we can find a map ϕ_{N_x} satisfying (a_1) and (a_2) below.

$$\phi_{N_x} : \Delta_n \to \Gamma(\{f(a_{U_0})), f(a_{U_1}), \ldots, f(a_{U_n})\}) \text{ is continuous.} \tag{a1}$$

For any subset J of $\{0, 1, \ldots, n\}$, if Δ_J is that face of Δ_n corresponding to J then

$$\phi_{N_x}(\Delta_J) \subset \Gamma(\{f(a_{U_j}) : j \in J\}). \tag{a_2}$$

Let $\mathcal{W} = \{N_x : x \in X\backslash A\}$. Then $\mathcal{W}$ is an open covering for $X\backslash A$.

We notice that if $x \in N \in \mathcal{W}$ and the set of all elements of $\mathcal{U}$ that have a nonempty intersection with N is $\{U_0, U_1, \ldots, U_n\}$, then $\beta_U(x) \neq \emptyset \Rightarrow U \in \{U_0, \ldots, U_n\}$.

Therefore we can define a continuous function $\mu_N : N \to \Delta_n$ by $\mu_N(x) = \Sigma_{i=0}^n \beta_{U_i}(x)e_i$. Let $f_N = \phi_N \circ \mu_N$. Then f_N is continuous from N to Y.

Next let $\mathcal{K}$ be a locally finite closed (i.e. consisting of closed sets) refinement for $\mathcal{W}$. For each $K \in \mathcal{K}$, assign $N_K \in \mathcal{W}$ such that $K \subset N_K$. Moreover for each $K \in \mathcal{K}$ let $f_K = f_{N_K}$.

Now define $G : X \to 2^Y$ as follows:

$$G(x) = \begin{cases} \{f(a)\}, & \text{if } x \in A \\ \{f_K(x)) : x \in K \in \mathcal{K}\}, & \text{if } x \in (X\backslash A) \end{cases}$$

Note that (i) follows immediately .

Next we shall show that for any $x \in X\backslash A$, if $C_x = r_1(x) = \{U_0, \ldots, U_m\}$ then

$$G(x) \subset \Gamma(\{f(a_{U_0}), \ldots, f(a_{U_m}). \qquad (a_3)$$

Let $y \in G(x)$. Then $y = f_K(x) = f_{N_K}(x) = \phi_{N_K}(\mu_{N_K}(x))$ for some $K \subset N_K \in \mathcal{W}$. Then N_K has a nonempty intersection with U_i for each $0 \leq i \leq m$.

Let the set of all elements of $\mathcal{U}$ that have a nonempty intersection with N_K be $\{\grave{U}_0, \grave{U}_1, \ldots, \grave{U}_n\}$. Then there exists a subset J of $\{0, 1, \ldots, n\}$ such that $\{U_0, U_1, \ldots, U_m\} = \{\grave{U}_j : j \in J\}$.

It follows that $\beta_U(x) \neq 0 \Rightarrow U \in \{\grave{U}_j : j \in J\}$. Hence we have $\mu_{N_K}(x) \in \Delta_J$ (that face of Δ_n corresponding to J).

By (a_2) the above implies that $y = f_K(x) = f_{N_K}(x) = \phi_{N_K}(\mu_{N_K}(x)) \in \Gamma(\{f(a_{\grave{U}_j}) : j \in J\}) = \Gamma(\{f(a_{U_0}), \ldots f(a_{U_m})\})$.

Hence (a_3) is proved.

Now (ii) follows easily from (a_3) because the left hand side of (a_3) is always contained in G-co$(f(A))$.

(iii) G is continuous on A.

Let $a_0 \in A$, we shall first show that G is USC at a_0.

Let $\dot{V}$ be open in Y such that $G(a_0) \subset \dot{V}$. This implies that $f(a_0) \in \dot{V}$. Let V be open in Y such that $f(a_0) \in V \subset$ G-co$V \subset \dot{V}$.

By continuity of f at $a_0 \in A$, there exists $\delta > 0$ such that for each $a \in A$, we have

$$d(a, a_0) < \delta \text{ implies } f(a) \in V. \tag{2}$$

Let $W = B(a_0, \delta/3)$. Then it can be shown (See the proof of Theorem 1) that for any $U \in \mathcal{U}$ we have:

$$U \subset W \text{ implies } f(a_U) \in V. \tag{*}$$

Now by Lemma 3.1.1, there exists an open nhood $\dot{W}$ of a_0 in X such that $\dot{W} \subset W$ and for each $U \in \mathcal{U}$ we have $U \cap \dot{W} \neq \emptyset$ implies $U \subset W$. We shall show that

$$G(\dot{W} \subset \dot{V}. \tag{**}$$

For let $x \in \dot{W}$. If $x = a \in A$ then $G(x) = \{f(a)\}$ and since $a \in \dot{W} \subset W$, (**) follows from (2).

Next assume $x \notin A$. Let $C_x = r_1(x) = \{U_0, U_1, \ldots, U_m\}$, then $x \in U_i \cap \dot{W}$ for all $0 \leq i \leq m$ and therefore $U_i \subset W$ for all $0 \leq i \leq m$. It then follows from (*) that

$$\{f(a_{U_0}), f(a_{U_1}), \ldots, f(a_{U_m})\} \subset V. \tag{3}$$

Applying (a_3) to the above, we have:

$G(x) \subset \Gamma(\{f(a_{U_0}), \ldots, f(a_{U_m})\} \subset$ G-co$V \subset \dot{V}$. Thus (**) is proved and G is USC at a_0.

Next we shall prove that G is LSC at a_0. So let $\dot{V}$ be open in Y such that $G(a_0) \cap \dot{V} \neq \emptyset$. Then $G(a_0) \subset \dot{V}$ (since G is single valued at a_0), and by USC of G at a_0, there exists a nhood $\dot{W}$ of a_0 in X such that $G(\dot{W}) \subset \dot{V}$. It follows that for each $x \in \dot{W}$, $G(x) \cap \dot{V} = G(x) \neq \emptyset$. Hence G is LSC at a_0.

(iv) G is USC on $X \backslash A$.

Let $x_0 \in X \backslash A$, and let V be an open subset of Y such that $G(x_0) \subset V$. Let $\{K_0, K_1, \ldots, K_m\}$ be the set of all elements of $\mathcal{K}$ that contain x_0. Then $G(x_0) = \{f_{K_0}(x_0), f_{K_1}(x_0), \ldots, f_{K_m}(x_0)\}$. And so it follows that $f_{K_j}(x_0) \in V$ for all $0 \leq j \leq m$. Now for each $0 \leq j \leq m$, $f_{K_j}(x_0) = f_{W_{K_j}}(x_0)$ where $f_{W_{K_j}} : W_{K_j} \to Y$ is continuous. So there exists an open nhood of x_0, call it O_j, such that $f_{W_{K_j}}(O_j) = f_{K_j}(O_j) \subset V$. Let $M_1 = \bigcap_{j=0}^{m} O_j$. Then M_1 is an open nhood of x_0 such that

$$f_{K_j}(M_1) \subset V \text{ for all } 0 \leq j \leq m. \tag{4}$$

Next let M_2 be a nhood of x_0 in $X \setminus A$ that intersects with only finitely many elements of $\mathcal{K}$, say $K_0, K_1, \ldots, K_m, K_{m+1}, \ldots, K_n$.

$$\text{Let } M = M_1 \cap M_2 \cap \Big(\bigcap_{j=m+1}^{n} (X \backslash K_j) \Big). \tag{5}$$

Then we shall show that

$$G(M) \subset V. \tag{***}$$

Indeed let $x \in M$. Then $x \in M_2$ and therefore

$$\{K \in \mathcal{K} : x \in K\} \subset \{K_0, K_1, \ldots, K_m, K_{m+1}, \ldots, K_n\}$$

. Moreover by (5), $x \notin K_j$ for all $m < j \leq n$. Therefore it follows that $\{K \in \mathcal{K} : x \in K\} = \{K_j : j \in J \text{ for some subset } J \text{ of } \{0, 1, \ldots, m\}\}$. This in turn implies that:

$$G(x) = \{f_{K_j}(x) : j \in J\}. \tag{6}$$

But x also belongs to M_1, so it follows from (1) that $f_{K_j}(x) \in V$for all $j \in J$, which by (3) implies that $G(x) \subset V$.

Hence (***) is proved and G is USC at x_0. □

The following theorem is a generalization of Dugundji's extension (Theorem 4.1 in [DJ]) to G-convex spaces.

Theorem 3.1.4. *Let (X, d) be a metric space and A be a closed subset of X. Let (Y, Γ) be a locally G-convex space with a homogeneous G-map system $\mathcal{M}$. Let $f : A \to Y$ be a continuous function. Then there exists a function $\widehat{f} : X \to Y$ such that*

(i) $\widehat{f}|_A = f$.

(ii) $\widehat{f}(X) \subset G\text{-}co(f(A))$.

(iii) $\widehat{f}$ *is continuous on* X.

Proof. Let $\mathcal{U}$ be the locally finite open covering for $X \backslash A$ provided by Lemma 3.1.1. Let $(\beta_U)_{U \in \mathcal{U}}$ be a partition of unity on $X \backslash A$ subordinated to $\mathcal{U}$. For each $U \in \mathcal{U}$, choose an element $x_U \in U$ and let $a_U \in A$ be such that

$$d(x_U, a_U) \leq 2d(x_U, A). \tag{1}$$

For each $x \in X \backslash A$, let $C_x = \{U \in \mathcal{U} : x \in U\}$. And let $B_x = \{f(a_U) : U \in C_x\} = \{y_0, y_1, \ldots, y_m\}$. Also let $\phi_{B_x} \in \mathcal{M}(B_x)$ i.e. $\phi_{B_x} : \Delta_{|B_x|-1} \to \Gamma(\{y_0, y_1, \ldots, y_m\}$ is the continuous map provided by the assumption of the theorem and satisfying the G-condition; i.e., such that for any subset $J \subset \{0, 1, \ldots, m\}$, we have $\phi_{B_x}(co(\{e_j : j \in J\})) \subset \Gamma(\{y_j : j \in J\})$.

Now define $\widehat{f} : X \to Y$ as follows:

$$\widehat{f}(x) = \begin{cases} f(x), & \text{if } x \in A; \\ (\phi_{B_x}(\Sigma_{y_j \in B_x}(\Sigma_{f(a_U)=y_j}\beta_U(x))e_j), & \text{if } x \in X \backslash A. \end{cases}$$

Since for any $x \in X \backslash A$ we have $\beta_U(x) \neq 0$ iff $U \in C_x$ iff $f(a_U) = y_j$ for some $y_j \in B_x$, it follows that $\Sigma_{y_j \in B_x}(\Sigma_{f(a_U)=y_j}\beta_U(x)) = \Sigma_{U \in \mathcal{U}}\beta_U(x) = 1$. Hence $\widehat{f}$ above is well defined and it clearly satisfies (i).

It is also easy to see that $\widehat{f}$ satisfies (ii), since for every $x \in X\backslash A$, we have $B_x \subset f(A)$, from which it follows that $\widehat{f}(x) \in \phi_{B_x}(\Delta_{|B_x|-1}) \subset \Gamma(B_x) \subset$ G-co$(f(A))$.

Next we will show that $\widehat{f}$ is continuous on X. Let $x_0 \in X\backslash A$. Let W be an open nhood of x_0 that intersects with finitely many elements of $\mathcal{U}$, say $C_W = \{U \in \mathcal{U} : U \cap W \neq \emptyset\}$. Let $B_W = \{f(a_U) : U \in C_W\} = \{\grave{y}_0, \grave{y}_1, \ldots, \grave{y}_k\}$.

Let $\phi_{B_W} : \Delta_{|B_W|-1} \to \Gamma(\{\grave{y}_0, \grave{y}_1, \ldots, \grave{y}_k\})$ be such that $\phi_{B_W} \in \mathcal{M}(B_W)$. Next define $\mu : W \to \Delta_{|B_W|-1}$ by

$$\mu(x) = \Sigma_{j=0}^{k}(\Sigma_{f(a_U)=\grave{y}_j}\beta_U(x))e_j.$$

To show that μ is continuous, for each $\grave{y}_j \in B_W$, let $\{U \in \mathcal{U} : f(a_U) = \grave{y}_j\} = \{U_{j_0}, U_{j_1}, \ldots, U_{j_{n_j}}\}$. Then let $r_j = \Sigma_{l=0}^{n_j}\beta_{U_{j_l}}(x)$. Obviously, r_j is continuous from W to $[0,1]$, and hence it follows that $\mu(x) = \Sigma_{j=0}^{k} r_j(x)e_j$ is continuous. It follows that $\phi_{B_W} \circ \mu : W \to \Gamma(B_W)$ is also continuous.

To prove the continuity of $\widehat{f}$ at x_0, it suffices to show that $\widehat{f}(x) = \phi_{B_W} \circ \mu(x)$ for all $x \in W$. So let $x \in W$. It follows that $B_x \subset B_W$. Let $B_x = \{\grave{y}_{i_0}, \grave{y}_{i_1}, \ldots, \grave{y}_{i_m}\} = \{y_0, y_1, \ldots, y_m\}$ where $y_j = \grave{y}_{i_j}$ for $0 \leq j \leq m$.

Now $\widehat{f}(x) = \phi_{B_x}(\Sigma_{j=0}^{m}\lambda_j e_j)$ where $\lambda_j = \Sigma_{f(a_U)=y_j}\beta_U(x)$.

But $B_x \subset B_W$ and $\{\{\phi_A\} : A \in \langle Y\rangle\}$ is a homogeneous G-map system, so it follows by the remark following Definition 1.1.3 that:

$$\begin{aligned}\widehat{f}(x) &= \phi_{B_W}(\Sigma_{j=0}^{m}\lambda_j e_{i_j}) = \phi_{B_W}(\Sigma_{j=0}^{m}(\Sigma_{f(a_U)=\grave{y}_{i_j}}\beta_U(x))e_{i_j}) \\ &= \phi_{B_W}(\Sigma_{l=i_o}^{i_m}(\Sigma_{f(a_U)=\grave{y}_l}\beta_U(x))e_l).\end{aligned} \tag{2}$$

But since whenever $l \in \{0, 1, \ldots, k\}\backslash\{i_0, \ldots, i_m\}$ and $U \in \mathcal{U}$ is such that $f(a_U) = y_l$, then we must have $\beta_U(x) = 0$, it follows that the R.H.S. of (2) is equal to:

$$\phi_{B_W}(\Sigma_{l=0}^{k}(\Sigma_{f(a_U)=\grave{y}_l}\beta_U(x))e_l) = (\phi_{B_W} \circ \mu)(x).$$

Therefore $\widehat{f}$ is continuous at x_0.

Now it only remains to show that $\widehat{f}$ is continuous at any point $a_0 \in A$.

Let $\dot{V}$ be a nhood of $\widehat{f}(a_0) = f(a_0)$ in Y. Then there exists a nhood V of $f(a_0)$ such that G-co$V \subset \dot{V}$. By continuity of f at a_0, there exists $\delta > 0$ such that for each $a \in A$:

$$d(a, a_0) < \delta \text{ implies } f(a) \in V. \tag{3}$$

Let $W = B(a_0, \delta/3)$ (the open ball centered at a_0 with radius $(\delta/3)$). We shall show that

$$\text{if } U \in \mathcal{U} \text{ and } U \subset W \text{ then } f(a_U) \in V. \tag{*}$$

For $U \subset W$ implies $x_U \in W$ and hence $d(x_U, a_0) < \delta/3$. So $d(a_U, a_0) \leq d(a_U, x_U) + d(x_U, a_0) < 2d(A, x_U) + \delta/3 \leq 2d(a_0, x_U) + \delta/3 < \delta$. Thus applying (3), we have $f(a_U) \in V$ and (*) is proved.

Next, by Lemma 3.1.1, we choose an open nhood $\dot{W}$ of a_0 in X contained in W such that $U \cap \dot{W} \neq \emptyset$ implies $U \subset W$ for each $U \in \mathcal{U}$. We shall show that

$$\widehat{f}(\dot{W}) \subset \dot{V}. \tag{**}$$

Let $x \in \dot{W}$. If $x \in A$, then $x = a \in W$ and hence, by (2), $\widehat{f}(x) = f(a) \in V \subset \dot{V}$.

Next assume $x \notin A$. Then for each $U \in C_x$ we have $U \cap \dot{W} \neq \emptyset$, which implies that $U \subset W$ and hence by (*), $f(a_U) \in V$. So it follows that $B_x \subset V$ and hence that $\Gamma(B_x) \subset$ G-co$V \subset \dot{V}$. But $\widehat{f}(x) \in \Gamma(B_x)$ by definition, so (**) follows. Therefore $\widehat{f}$ is also continuous at a_0. □

2. Extension of Set-Valued Maps.

The following extension theorem is Theorem 2.1 of Ma in [Ma].

Theorem. *Let A be a nonempty closed subset of a metrizable space X, E be a Hausdorff locally convex space, and $\mathcal{K}E$ be the family of all nonempty compact convex subsets of E. If $F : A \to \mathcal{K}E$ is an upper semicontinuous set-valued map on A, then F has an upper semicontinuous set-valued extension $G : X \to \mathcal{K}E$ such that $G(X)$ is contained in the convex hull $F(A)$.*

Theorem 3.2.1 below generalizes Ma's theorem.

Theorem 3.2.1. *Let (X,d) be a metric space and A a closed subset of X. Let (Y,Γ) be a strongly locally G-convex space such that G-co$(\{y_1, y_2, \ldots, y_n\})$ is compact for any $y_1, \ldots, y_n \in Y$. Let $F : A \to 2^Y$ be USC with nonempty compact and G-convex values. Then there exists an USC extension $\tilde{F} : X \to 2^Y$ such that:*

(i) $\tilde{F}(X) \subset$ G-co$(F(A))$.

(ii) $\tilde{F}(x)$ is nonempty, compact and G-convex, for each $x \in X$.

Proof. Let $\mathcal{U}$ be the locally finite cover for $X\backslash A$ provided by Lemma 3.1.1. Let $(\beta_U)_{U\in\mathcal{U}}$ be a partion of unity on $X \setminus A$ subordinated by $\mathcal{U}$. For each $U \in \mathcal{U}$, pick any $x_U \in U$ and let $a_U \in A$ be such that

$$d(a_U, x_U) \leq 2d(x_U, A). \tag{1}$$

Also for each $U \in \mathcal{U}$, choose any $y_U \in F(a_U)$.

Now for each $x \in X\backslash A$, let N_x be an open nhood of x in $X\backslash A$ that intersects with only finitely many elements of $\mathcal{U}$, say, $U_0, U_1, \ldots, U_n$ and such that $cl_X(N_x) \subset X\backslash A$. Let $\phi_{N_x} : \Delta_n \to \Gamma(\{y_{U_0}, \cdots y_{U_n}\})$ be a continuous map having the property that for any subset $J \subset \{0, 1, \ldots, n\}$, we have:

$$\phi_{N_x}(co(\{e_j : j \in J\}) \subset \Gamma(\{y_{U_j} : j \in J\}). \tag{a1}$$

Let $\mu_{N_x} : N_x \to \Delta_n$ be defined by $\mu_{N_x}(z) = \Sigma_{i=0}^{n} \beta_{U_i}(z) e_i$. Let $f_{N_x} : N_x \to \Gamma(\{y_{U_0}, \ldots, y_{U_n}\})$ be defined by $f_{N_x} = \phi_{N_x} \circ \mu_{N_x}$. Then f_{N_x} is obviously continuous. Now let $\mathcal{K}$ be a locally finite closed (i.e. consisting of closed sets) refinement for the open cover $\mathcal{W} = \{N_x : x \in X\backslash A\}$. For each $K \in \mathcal{K}$, assign $N_K \in \mathcal{W}$ such that $K \subset N_K$ and let $f_K = f_{N_K}$.

Define $\tilde{F} : X \to 2^Y$ by

$$\tilde{F}(x) = \begin{cases} F(x), & \text{if } x \in A; \\ \text{G-co}(\{f_K(x) : x \in K \in \mathcal{K}\}), & \text{if } x \in X\backslash A. \end{cases}$$

Obviously, $\tilde{F}|_A = F$. We shall show in the following that (I) $\tilde{F}$ is USC at every point in A; (II) $\tilde{F}$ is USC at every point in $X\backslash A$; and that $\tilde{F}$ satisfies (i) and (ii) in the statement of the theorem.

First we shall prove that for any $x \in X \backslash A$, if $\{U \in \mathcal{U} : x \in U\} = \{U_0, U_1, \dots, U_m\}$ then

$$\tilde{F}(x) \subset \text{G-co}(\{y_{U_0}, \dots, y_{U_m}\}. \tag{a2}$$

For let $K \in \mathcal{K}$ be s.t. $x \in K$ and let $\{U \in \mathcal{U} : U \cap N_K \neq \emptyset\} = \{\grave{U}_0, \grave{U}_1, \dots, \grave{U}_n\}$. Then there exists a subset J of $\{0, 1, \dots, n\}$ s.t. $\{U_0, U_1, \dots, U_m\} = \{\grave{U}_j : j \in J\}$. And, moreover, $\mu_{N_K}(x) \in co(\{e_j : j \in J\}$. This, by (a1), implies that:

$$f_K(x) = \phi_{N_K}(\mu_{N_K}(x)) \in \Gamma(\{y_{\grave{U}_j} : j \in J\}) = \Gamma(\{y_{U_0}, y_{U_1}, \dots, y_{U_m}\}. \tag{2}$$

Since K is any arbitrary element of $\mathcal{K}$ containing x, (2) implies that $\{f_K(x) : x \in K \in \mathcal{K}\} \subset \Gamma(\{y_{U_0}, \dots, y_{U_m}\}) \subset \text{G-co}(\{y_{U_0}, \dots, y_{U_m}\})$. Thus $\tilde{F}(x) = \text{G-co}(\{f_K(x) : x \in \mathcal{K}\}) \subset \text{G-co}(\{y_{U_0}, \dots, y_{U_m}\})$ and (a2) is proved.

From (a2) it follows that $\tilde{F}$ satisfies (i), since $y_U \in F(a_U) \subset F(A)$, for all $U \in \mathcal{U}$. Moreover, by the assumption that the G-convex hull of any finite subset of Y is compact, (ii) is obviously satisfied for $\tilde{F}$.

(I) $\tilde{F}$ is USC at every point in A.

Let $a_0 \in A$ and $\grave{V}$ be an open subset of Y s.t. $\tilde{F}(a_0) \subset \grave{V}$. By the local G-convexity assumption on Y, there exists an open subset V of Y such that $\tilde{F}(a_0) = F(a_0) \subset V \subset \text{G-co}(V) \subset \grave{V}$. By USC of F on A, there exists $\delta > 0$ such that

$$\text{for each } a \in A,\ d(a, a_0) < \delta \text{ implies } F(a) \subset V. \tag{3}$$

Let $W = B(a_0, \delta/3)$. And let $\grave{W}$ be a nhood of a_0 open in X as provided by Lemma 3.1.1, i.e. such that

$$\forall\, U \in \mathcal{U}, U \cap \grave{W} \neq \emptyset \text{ implies } U \subset W. \tag{4}$$

We shall show that

$$\tilde{F}(\grave{W}) \subset \grave{V}. \tag{*}$$

First we notice that W has the property that

$$\text{for each } U \in \mathcal{U}, U \subset W \text{ implies } F(a_U) \subset V. \tag{**}$$

For $U \subset W$ implies that $x_U \in W$ and hence $d(x_U, a_0) < \delta/3$. Also by (1), $d(a_U, x_U) \leq 2d(a_0, x_U) \leq 2\delta/3$. So it follows by the triangular inequality that $d(a_U, a_0) < \delta$. Applying (3), (**) follows immediately.

To show that $\tilde{F}(\dot{W}) \subset \dot{V}$, we take $x \in \dot{W}$.

Case 1. $x = a \in A$. Then $a \in W$ (since $\dot{W} \subset W$) and hence $d(a, a_0) < \delta/3$. By (3), this implies that $\tilde{F}(x) = F(a) \subset V \subset \dot{V}$.

Case 2. $x \notin A$. Let $\{U \in \mathcal{U} : x \in U\} = \{U_0, U_1, \ldots, U_m\}$. Then $U_i \cap \dot{W} \neq \emptyset$, for all $0 \leq i \leq m$. Thus by (4) we have $U_i \subset W$, for all $0 \leq i \leq m$. Applying (**), it follows that $y_{U_i} \in F(a_{U_i}) \subset V$, for all $0 \leq i \leq m$, i.e. $\{y_{U_0}, \ldots, y_{U_m}\} \subset V$. By (a2), this implies that $\tilde{F}(x) \subset$ G-co$V \subset \dot{V}$. Thus it follows that $\tilde{F}(\dot{W}) \subset \dot{V}$ and hence that $\tilde{F}$ is USC at every point in A.

(II) $\tilde{F}$ is USC at every point in $X \backslash A$.
Let $x_0 \in X\backslash A$. Let $\dot{V}$ be an open subset of Y such that $\tilde{F}(x_0) \subset \dot{V}$. By the local G-convexity assumption on Y, there exists an open subset V of Y such that $\tilde{F}(x_0) \subset V \subset$ G-co$(V) \subset \dot{V}$. Let $\{K \in \mathcal{K} : x_0 \in K\} = \{K_0, K_1, \ldots, K_m\}$. Then $\tilde{F}(x_0) =$ G-co$(\{f_{K_0}(x_0), \ldots, f_{K_m}(x_0)\}$. Let M_1 be an open nhood of x_0 contained in $X\backslash A$ that intersects with finitely many elements of $\mathcal{K}$ say $\{K \in \mathcal{K} : M_1 \cap K \neq \emptyset\} = \{K_0, \ldots, K_m, K_{m+1}, \ldots, K_n\}$. Since $\tilde{F}(x_0) \subset V$, we have $f_{K_i}(x_0) \in \tilde{F}(x_0) \subset V$, for all $0 \leq i \leq m$. And by the continuity of f_{K_i}'s, there exists an open nhood O of x_0 in $X \setminus A$ such that

$$f_{K_i}(O) \subset V, \text{ for all } 0 \leq i \leq m. \tag{5}$$

Let $M = O \cap M_1 \cap (X\backslash K_{m+1}) \cap \cdots \cap (X\backslash K_n)$. Then M is an open nhood of x_0 in $X \setminus A$. We shall show that

$$\tilde{F}(M) \subset \dot{V}. \tag{***}$$

For let $x \in M$. Then it is easy to see that

$$\{K \in \mathcal{K} : x \in K\} = \{K_j : j \in J\} \subset \{K_0, \ldots, K_m\}, \text{ where } J = \{0, 1, \ldots, m\} \tag{6}$$

It follows that $\{f_K(x) : x \in K \in \mathcal{K}\} = \{f_{K_j}(x) : j \in J\}$ where $J \subset \{0, 1, \ldots, m\}$. Since $x \in O$, it follows by (5) that $f_{K_j}(x) \in V$, for all $j \in J$, i.e., $\{f_K(x) : x \in K \in \mathcal{K}\} \subset V$. This implies that $\tilde{F}(x) = \text{G-co}(\{f_K(x) : x \in K \in \mathcal{K}\}) \subset \text{G-co}V \subset \dot{V}$.

Thus (***) is proved and therefore $\tilde{F}$ is USC at every $x \in X \setminus A$.

Therefore $\tilde{F}$ is USC on X. □

3. Completely Continuous Extensions.

Definition 3.3.1. Let (X, d) be a metric space, Y a topological space. An USC mapping $\phi : X \to 2^Y$ is said to be completely continuous if for any bounded subset N of X, the set $cl(\bigcup_{x \in N} \phi(x))$ is compact.

The following is an extension theorem of Pruszko (Theorem 1 in [Psz]).

Theorem. *Let $M \subset X$ be a nonempty closed subset of a metric space X, E be a normed space, $F : M \to cf(E)$ be an upper semicontinuous map, (where cf(E) is the family of all nonempty bounded convex and closed subsets of E), $\phi : X \to cf(E)$ be a completely continuous map such that $F(y) \subset \phi(y)$ and ϕ is continuous at y for each $y \in M$. Then there exists a completely continuous map $\tilde{F} : X \to cf(E)$ such that $\tilde{F}|M = F$ and $\tilde{F}(x) \subset \phi(x)$ for each $x \in X$.*

Theorem 3.3.1 below is a generalization of Pruszko's extension theorem to G-convex spaces.

Theorem 3.3.1. *Let (X, d) be a metric space and M be a nonempty closed subset of X. Let (Y, Γ) be a metrizable G-convex space whose topology comes from a metric ρ. Assume that Y is strongly locally G-convex and has the property that cl(G-co(A)) is compact whenever A is a compact subset of Y. Let $F : M \to 2^Y$ be USC with nonempty, closed and G-convex values. Let $\phi : X \to 2^Y$ be completely continuous such that (i) $F(y) \subset \phi(y)$, for all $y \in M$, (ii) ϕ has nonempty closed G-convex values and is continuous at every $y \in M$. Then there exists $F^* : X \to 2^Y$ such that: (i) $F^*|_M = F$, (ii) $F^*(x) \subset \phi(x)$, and (iii) F^* is completely continuous.*

Proof. The proof is divided into three steps.

Step 1.

For each $x \in X\backslash M$, we let $r(x)$ be a real number such that

$$0 < r(x) < (1/2)d(x, M). \tag{1}$$

Consider the open cover $\{B(x, r(x)) : x \in X\backslash M\}$ of $X \setminus M$. Let $\mathcal{U} = \{U_t : t \in T\}$ be a locally finite open refinement for this cover. Let $x_t \in X$ be such that $U_t \subset B(x_t, r(x_t))$. For each $U_t \in \mathcal{U}$, let $y_t \in M$ be such that

$$d(y_t, U_t) < 2d(M, U_t). \tag{2}$$

Let $\{p_t\}_{t\in T}$ be a partition of unity on $X \setminus M$ subordinated to $\mathcal{U}$. For each $x \in X\backslash M$, let $N(x) = \{y_t \in M : x \in U_t\}$. Obviously, $N(x)$ is a finite subset of M. We shall prove that for any $y \in M$, and any $x \in X\backslash M$, we have

$$N(x) \subset B(y, 5d(x, y)). \tag{3}$$

We will need to show the following:

$$d(x_t, M) \leq 2d(U_t, M), \;\; \forall x_t \in X \setminus M, \;\; \forall t \in T. \tag{a0}$$

Indeed let $\epsilon > 0$ be given. Then there exists $y \in M, z \in U_t$ such that

$$\begin{aligned} &d(y, z) < d(U_t, M) + \epsilon \\ \Rightarrow\, &d(x_t, y) \leq d(x_t, z) + d(z, y) < r(x_t) + d(U_t, M) + \epsilon \\ &\qquad < (1/2)d(x_t, M) + d(U_t, M) + \epsilon. \end{aligned}$$

But $d(x_t, M) \leq d(x_t, y)$; so it follows that

$$d(x_t, M) \leq (1/2)d(x_t, M) + d(U_t, M) + \epsilon \Rightarrow d(x_t, M) \leq 2d(U_t, M) + \epsilon.$$

Now since ϵ is arbitrary, the above implies (a0).

Next we prove (3). So let $y_t \in N(x)$. Then

$$d(y, y_t) \leq d(y_t, x) + d(y, x). \tag{a1}$$

By (2), there exists $u_t \in U_t$ such that

$$d(y_t, u_t) < 2d(M, U_t). \tag{a2}$$

Since $x \in U_t \subset B(x_t, r(x_t))$, we also have

$$d(u_t, x) \leq d(u_t, x_t) + d(x_t, x) < r(x_t) + r(x_t) < d(x_t, M). \tag{a3}$$

By $(a3)$ and (a), it follows that

$$d(u_t, x) < 2d(U_t, M). \tag{a4}$$

Now by (a1), (a2) and (a4), we have

$$\begin{aligned} d(y_t, y) &\leq d(y_t, x) + d(y, x) \leq d(y_t, u_t) + d(u_t, x) + d(y, x) \\ &< 2d(M, U_t) + 2d(U_t, M) + d(y, x) \leq 5d(y, x). \end{aligned}$$

Step 2.

For each $x \in X \backslash M$, and $t \in T$, let

$$\psi_t(x) = \begin{cases} \{w \in \phi(x) : \rho(w, F(y_t)) < 2\rho(\phi(x), F(y_t))\}, & \text{if } \phi(x) \cap F(y_t) = \emptyset; \\ \phi(x) \cap F(y_t), & \text{if } \phi(x) \cap F(y_t) \neq \emptyset. \end{cases}$$

Given $x \in X \backslash M$, let $\{U \in \mathcal{U} : x \in U\} = \{U_{t_0}, \ldots, U_{t_n}\}$. For each $A = (a_0, a_1, \ldots, a_n) \in \psi_{t_0}(x) \times \psi_{t_1}(x) \times \cdots \times \psi_{t_n}(x)$, let $f_A : \Delta_n \to \Gamma(\{a_0, a_1, \ldots, a_n\})$ be a continuous map. Note that $\{a_0, a_1, \ldots, a_n\} \subset \phi(x)$ whenever $A = (a_0, a_1, \ldots, a_n) \in \psi_{t_0}(x) \times \psi_{t_1}(x) \times \ldots \psi_{t_n}(x)$. Thus the G-convexity of $\phi(x)$ implies that image$(f_A) \subset \Gamma(\{a_0, a_1, \ldots, a_n\}) \subset \phi(x)$.

Define

$$G(x) = \{f_A(\Sigma_{i=0}^n p_{t_i}(x) e_i) : A \in \psi_{t_0}(x) \times \psi_{t_1}(x) \times \cdots \times \psi_{t_n}(x)\}.$$

We observe that $G(x) \subset \phi(x)$ for each $x \in X \backslash M$. Now define $F^* : X \to 2^Y$ as follows:

$$F^*(u) = \begin{cases} F(u), & \text{if } u \in M, \\ \bigcap_{n=1}^{\infty} \text{cl}(\text{G-co}[\bigcup_{d(x,u)<(1/n)} G(x)]), & \text{if } u \notin M. \end{cases}$$

Step 3.

We shall show first that $F^*(x) \subset \phi(x)$ for each $x \in X\backslash M$. Assume on the contrary that there exists $x_0 \in X\backslash M$ and $y \in F^*(x_0)$ such that $y \notin \phi(x_0)$. Then there exists an open nhood $\grave{V}$ of $\phi(x_0)$ in Y such that $y \notin cl(\grave{V})$. Let V be an open nhood of $\phi(x_0)$ in Y such that G-co(V) $\subset \grave{V}$. By USC of ϕ, there exists $\delta > 0$ such that $B(x_0, \delta) \subset X\backslash M$ and $\phi(B(x_0, \delta)) \subset V$. Since $G(x) \subset \phi(x)$, for all $x \in X \setminus M$, the above implies that

$$G(x) \subset V \; \forall \; x \in B(x_0, \delta) \text{ and hence } \bigcup_{d(x,x_0)<\delta} G(x) \subset V.$$

Therefore G-co($\bigcup_{d(x,x_0)<\delta} G(x)$) $\subset$ G-co(V) $\subset \grave{V}$ which in turn implies that cl(G-co($\bigcup_{d(x,x_0)<\delta} G(x)$) $\subset cl(\grave{V})$ and hence $F^*(x_0) \subset cl(\grave{V})$ which contradicts $y \in F^*(x_0)\backslash cl(\grave{V})$.

Thus we have shown that $F^*(x) \subset \phi(x)$ for each $x \in X \setminus M$. Therefore for any subset $N \subset X$, we have

$$cl(\bigcup_{x\in N} F^*(x)) \subset cl(\bigcup_{x \in N} \phi(x)).$$

Since ϕ is completely continuous,

$$cl(\bigcup_{x\in N} F^*(x)) \text{ is compact for any bounded subset } N \subset X. \tag{4}$$

Next we shall show F^* is USC at every $y \in M$. Let $y_0 \in M$ and let $\grave{V}$ be open in Y such that $F(y_0) \subset \grave{V}$. Let V be an open nhood of $F(y_0)$ such that cl(G-co(V)) $\subset \grave{V}$. By compactness of $F(y_0)$, there exists $\epsilon > 0$ such that $O_\epsilon(F(y_0)) \subset V$. Since ϕ is continuous at y_0; there exists $\delta_1 > 0$ such that

$$D(\phi(x), \phi(y_0)) < \epsilon/8 \text{ whenever } d(x, y_0) < \delta_1, \tag{b1}$$

(where $D(\phi(x), \phi(y_0)$ denotes the Hausdorff distance).
Also, by USC of F, there exists $\delta_2 > 0$ such that

$$F(y) \subset O_{\epsilon/8}(F(y_0)) \text{ whenever } y \in M \cap K(y_0, \delta_2). \tag{b2}$$

Let $\delta = \min\{\delta_1, \delta_2/5\}$. We shall show that $F^*(B(y_0), \delta/2)) \subset \dot{V}$. So let $u \in B(y_0, \delta/2)$. We consider two cases.

Case 1. $u \in M$. In this case $F^*(u) = F(u)$. Therefore by (b2) we have $F^*(u) \subset O_{\epsilon/8}(F(y_0)) \subset V \subset \dot{V}$.

Case 2. $u \in X \setminus M$.

We will first show that

$$G(x) \subset \text{G-co}(V) \text{ whenever } d(x, y_0) < \delta \text{ and } x \in X \setminus M. \tag{*}$$

Indeed let $x \in X \setminus M$ be such that $d(x, y_0) < \delta$ and let $\{U_t \in \mathcal{U} : x \in U_t\} = \{U_{t_0}, U_{t_1}, \ldots, U_{t_n}\}$. Then $N_x = \{y_{t_0}, \ldots, y_{t_n}\}$ and applying (3) and (b2) above, it follows that for every $0 \leq i \leq n$, we have

$$F(y_{t_i}) \subset O_{\epsilon/8}(F(y_0)). \tag{b3}$$

Also since $F(y_0) \subset \phi(y_0)$; (b1) implies that

$$F(y_0) \subset O_{\epsilon/8}(\phi(x)). \tag{b4}$$

From (b3) and (b4), we have $F(y_{t_i}) \subset O_{\epsilon/4}(\phi(x)) \Rightarrow \rho(F(y_{t_i}), \phi(x)) < \epsilon/4$, for all $0 \leq i \leq n$. Therefore it follows that $\psi_{t_i}(x) \subset O_{\epsilon/2}(F(y_{t_i}))$, for all $0 \leq i \leq n$, which by (b3) implies that

$$\psi_{t_i}(x) \subset O_{5\epsilon/8}(F(y_0)) \subset V, \text{ for all } 0 \leq i \leq n. \tag{b5}$$

Now let $w \in G(x)$. Then by definition of G, for each $0 \leq i \leq n$, there exists $a_i \in \psi_{t_i}(x)$ such that $w \in \Gamma(\{a_0, a_1, \ldots, a_n\})$. By (b5) we have $a_i \in V$, for all $0 \leq i \leq n$ which implies $\Gamma(\{a_0, a_1, \ldots, a_n\}) \subset$ G-co(V) so that $\in$ G-co(V). Thus (*) is proved.

Next we notice that $u \in B(y_0, \delta/2) \Rightarrow G(x) \subset$ G-co(V) whenever $x \in X \setminus M$ is such that $d(x, u) < \delta/2$. Let $0 < \delta_3 < \delta/2$ be such that $B(u, \delta_3) \subset X \setminus M$.

Then it follows that $\bigcup_{d(x,u)<\delta_3} G(x) \subset$ G-co(V). And this in turn implies that cl(G-co($\bigcup_{d(x,u)<\delta_3} G(x)$) $\subset$ cl(G-co(V)) $\subset \dot{V}$. Therefore $F^*(u) \subset \dot{V}$ whenever $d(y_0, u) < \delta/2$. Thus we have shown F^* is USC at y_0.

Next, we shall show F^* is USC at any $x_0 \in X \backslash M$.
Let $A_n =$ cl(G-co($\bigcup_{d(x,x_0)<1/n} G(x)$), $n = 1, 2, \ldots$. Then, by definition of F^*, we have

$$F^*(x_0) = \bigcap_{n=1}^{\infty} A_n. \tag{5}$$

We shall first show that A_n is compact for each n.
Indeed, $A_n \subset$ cl(G-co(cl($\bigcup_{d(x,x_0)<1/n} \phi(x)$))). But cl($\bigcup_{d(x,x_0)<1/n} \phi(x)$) is compact by our assumption that ϕ is completely continuous, and since we are also assuming that the closure of the G-convex hull of a compact set is compact, it follows that cl(G-co(cl($\bigcup_{d(x,x_0),1/n} \phi(x)$ is compact. Therefore A_n is a closed subset of a compact set and hence it is compact.

Now it follows from (5) that $F^*(x_0)$ is the Hausdorff limit of the sequence $\{A_n\}_{n=1}^{\infty}$. Let V be open in Y such that $F^*(x_0) \subset V$. Choose $\epsilon > 0$ such that $O_\epsilon(F^*(x_0) \subset V$. Then there exists an integer n_0 such that $A_m \subset O_\epsilon(F^*(x_0))$ for each $n \geq n_0$.

Choose $m > n_0$ such that $B(x_0, 1/m) \subset X \backslash M$. Let $u \in B(x_0, 1/(2m))$. Then it follows that

$$\text{cl(G-co(} \bigcup_{d(x,u)<1/(2m)} G(x))) \subset \text{cl(G-co(} \bigcup_{d(x,x_0)<1/m} G(x))) = A_m \subset O_\epsilon(F^*(x_0))$$

Thus it follows that $F^*(u) \subset O_\epsilon(F^*(x_0)) \subset V$ whenever $d(x_0, u) < 1/(2m)$. Therefore F^* is USC at x_0.

Since F^* is USC; it follows from (4) that F^* is completely continuous. □

4. Applications.

In this section we give applications for the extension theorems of the previous sections. Following the method of Tan and Wu in [TW], we obtain equilibrium existence theorems for some qualitative games and generalized games. We note here that in all these applications the G-convex structure in the product space is as in Definition 1.3.1.

Lemma 3.4.1. *(Park) Let (Y,Γ) be a G-convex space such that for any $y \in Y$, $\Gamma(y) = \{y\}$. Assume further that for any compact G-convex subset A of Y and any open neighbourhood U of A there exists an open neighbourhood V of A such that G-co$V \subset U$. Let D be a compact G-convex subset of Y, and $F : X \to 2^D$ be USC with nonempty closed G-convex values. Then there exists $y \in Y$ such that $y \in F(y)$.*

Lemma 3.4.2. *Let X and Y be topological spaces and $F : X \to 2^Y$ be USC. Then the set $\{x \in X : F(X) \neq \emptyset\}$ is a closed subset of X.*

Theorem 3.4.1. *Let $\{(X_i, \Gamma_i)\}_{i \in I}$ be any family of strongly locally G-convex compact G-convex spaces. For each $i \in I$ assume: (i) singleton subsets of X_i are G-convex, and (ii) G-co(A) is compact whenever A is a finite subset of X_i. Let (X,Γ) be their product G-convex space and assume that X is metrizable and strongly locally G-convex. For each $i \in I$ let $F_i : X \to 2^{X_i}$ be USC with closed and G-convex values. Then there exists $\widehat{x} \in X$ such that for all $i \in I$, either $F_i(\widehat{x}) = \emptyset$ or $\widehat{x}_i \in F_i(\widehat{x})$.*

Proof. For each i let $C_i = \{x \in X : F_i(x) \neq \emptyset\}$. Then C_i is closed by Lemma 3.4.2. Define $\tilde{F}_i : X \to 2^{X_i}$ as follows:

Case 1. If $C_i = \emptyset$ then let $\tilde{F}_i(x) = X_i$, for all $x \in X$.

Case 2. If $C_i = X$ then let $\tilde{F}_i(x) = F_i(x)$, for all $x \in X$.

Case 3. If C_i is a proper nonempty subset of X, let $\tilde{F}_i$ be the USC extension of F_i provided by Theorem 3.2.1.

We notice that $\tilde{F}_i(x)$ is nonempty, closed and G-convex, $\forall\, x \in X$, $\forall\, i \in I$.

Let $F : X \to 2^X$ be defined by

$$F(x) = \Pi_{i \in I} \tilde{F}_i(x).$$

Then F is USC by a lemma of Fan (Lemma 3 in [F]). Moreover, $F(x)$ is obviously nonempty and compact, for all $x \in X$. We also have $F(x)$ G-convex, for all $x \in X$ by Corollary 1.3.2. Moreover by (i) and Corollary 1.3.3, we have $\Gamma(\{x\}) = \{x\}$, for all $x \in X$.

Applying Lemma 3.4.1, it follows that there exists $\widehat{x} \in X$ such that $\widehat{x} \in F(\widehat{x})$ which implies that $\widehat{x}_i \in \tilde{F}_i(\widehat{x})$ for all $i \in I$ and hence for each $i \in I$ either $F_i(\widehat{x}) = \emptyset$ or $\widehat{x}_i \in F_i(\widehat{x})$. □

As an immediate consequence of Theorem 3.4.1, we obtain the following equilibrium existence theorem for a qualitative game.

Theorem 3.4.2. *Let $\{(X_i, \Gamma_i)\}_{i\in I}$ be any family of strongly locally G-convex compact G-convex spaces. For each $i \in I$ assume: (i) singleton subsets of X_i are G-convex, and (ii) G-co(A) is compact whenever A is a finite subset of X_i. Let (X, Γ) be their product G-convex space and assume that X is metrizable and strongly locally G-convex. For each $i \in I$ let $P_i : X \to 2^{X_i}$ be USC with closed and G-convex values such that $x_i \notin P_i(x)$, for all $i \in I$, for all $x \in X$.*

Then $(X_i, P_i)_{i\in I}$ has an equilibrium i.e. there exists $\widehat{x} \in X$ such that for all $i \in I$. $P_i(\widehat{x}) = \emptyset$.

The following is an improvement of Lemma 3.8 in [TW].

Lemma 3.4.3. *Let (Y, Γ) be a strongly locally G-convex compact G-convex space. Assume also that the closure of a G-convex set is always G-convex. Let $F : X \to 2^Y$ be USC. Then $T : X \to 2^Y$ defined by $T(x) = cl(G\text{-}co(F(x)))$ is also USC.*

Proof. Let $x_0 \in X$. Let U be open in Y such that $T(x_0) \subset U$.

By assumption, $T(x_0)$ is compact and G-convex. So $\exists$ an open nhood V_1 of $T(x_0)$ such that

$$\mathrm{cl}(\text{G-co}(V_1)) \subset U.$$

Now V_1 is an open nhood of $F(x_0)$; so by USC of $F, \exists$ an open nhood N of x_0 such that $F(x) \subset V_1, \ \forall\, x \in N \Rightarrow \mathrm{cl}(\text{G-co}(F(x))) \subset \mathrm{cl}(\text{G-co}(V_1)) \subset U, \ \forall x \in N \Rightarrow T(x) \subset U, \ \forall\, x \in N$. Therefore T is USC. □

Theorem 3.4.3. *Let $\{(X_i, \Gamma_i)\}_{i \in I}$ be any family of strongly locally G-convex compact G-convex spaces. For each $i \in I$ assume: (i) singleton subsets of X_i are G-convex, and (ii) G-co(A) is compact whenever A is a finite subset of X_i. Let (X, Γ) be their product G-convex space (As in Definition 1.3.1). Assume that X is metrizable and strongly locally G-convex. For each $i \in I$ let $P_i : X \to 2^{X_i}$ be USC such that $x_i \notin$ cl(G-co$(P_i(x))$), for all $x \in X$. Then the qualitative game $(X_i, P_i)_{i \in I}$ has an equilibrium i.e. there exists $\widehat{x} \in X$ such that $P_i(\widehat{x}) = \emptyset$, for all $i \in I$.*

Proof. For each i, let $T_i : X \to 2^{X_i}$ be defined by

$$T_i(x) = \mathrm{cl}(\text{G-co}(P_i(x))).$$

Then T_i is USC by Lemma 3.4.3. Applying Theorem 3.4.2 to $(X_i, T_i)_{i \in I}$, there exists $\widehat{x} \in X$ such that $T_i(\widehat{x}) = \emptyset$, for all $i \in I$. It follows that $P_i(\widehat{x}) = \emptyset$, for all $i \in I$. □

In the following we recall the concept of $\mathcal{U}_\theta$-majorized correspondences introduced by Tan and Yuan in 1993 [TY] and observe that these concepts can be immediately carried on to G-convex spaces which is how we present them here.

Definition 3.4.1. Let X be a topological space and Y be a nonempty subset of a G-convex space $Z, \theta : X \to Z$ be a map and $\phi : X \to 2^Y$ be a correspondence. Then

(1) ϕ is said to be of class G-$\mathcal{U}_\theta$ if (a) for each $x \in X, \theta(x) \notin \phi(x)$ and (b) ϕ is upper semicontinuous with closed and G-convex values in Y;

(2) ϕ_x is a G-$\mathcal{U}_\theta$ -majorant of ϕ at x if there is an open nhood $N(x)$ of x in X and $\phi_x : N(x) \to 2^Y$ such that (a) for each $z \in N(x), \phi(z) \subset \phi_x(z)$ and $\theta(z) \notin \phi_x(z)$ and (b) ϕ_x is upper semicontinuous with closed and G-convex values;

(3) ϕ is said to be G-$\mathcal{U}_\theta$-majorized if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists a G-$\mathcal{U}_\theta$-majorant ϕ_x of ϕ at x.

We shall deal mainly with either the case (I) $X = Y$ and X is a nonempty G-convex subset of a G-convex space Z, and $\theta = I_X$, the identity map on X, or the case (II) $X = \Pi_{i\in I} X_i$ and $\theta = \pi_j : X \to X_j$ is the projection of X onto X_j and $Y = X_j$ is a G-convex space. In both cases (I) and (II), we shall write G-$\mathcal{U}$ in place of G-$\mathcal{U}_\theta$.

The following is Lemma 2.10 of [TY]:

Lemma 3.4.4 (Tan-Yuan). *Let X and Y be two topological spaces, and let A be a subset of X. Suppose $F_1 : X \to 2^Y, F_2 : A \to 2^Y$ are LSC (respectively, USC) such that $F_2(x) \subset F_1(x)$, for all $x \in A$. Then the map $F : X \to 2^Y$ defined by*

$$F(x) = \begin{cases} F_1(x) \ , & \text{if } x \notin A; \\ F_2(x) \ , & \text{if } x \in A. \end{cases}$$

is also LSC (respectively USC).

The following Lemma 2.2 in [TY] which is an improvement of a result due to Hildenbrand (Proposition B.III.2, p.23 in [H]).

Lemma 3.4.5. *Let X be a topological space and Y be a normal space. If $F, G : X \to 2^Y$ have closed values and are USC at $x \in X$, then $F \cap G$ is also USC at x.*

Theorem 3.4.4 (Tan-Yuan). *Let X be a paracompact space and Y be a normal G-convex subset of a G-convex space Z. Let $\theta : X \to Z$ and $P : X \to 2^Y \setminus \{\emptyset\}$ be G-$\mathcal{U}_\theta$-majorized. Then there exists a correspondence $\Psi : X \to 2^Y \setminus \{\emptyset\}$ of class G-$\mathcal{U}_\theta$ such that $P(x) \subset \Psi(x)$, for all $x \in X$.*

Proof. Since P is G-$\mathcal{U}_\theta$-majorized, for each $x \in X$, let $N(x)$ be an open nhood of x in X and $\psi : N(x) \to 2^Y \setminus \{\emptyset\}$ be such that (1) for each $z \in N(x), P(z) \subset \psi_x(z)$ and $\theta(z) \notin \psi_x(z)$ and (2) ψ_x is USC with closed and G-convex values. Since X is paracompact and $X = \bigcup_{x\in X} N(x)$, by Theorem 8.1.4, p.162 in [DJ2], the open covering $\{N(x)\}$ of X has an open precise nhood-finite refinement $\{\dot{N}(x)\}$. For each $x \in X$, define $\dot{\psi}_x : X \to 2^Y \setminus \{\emptyset\}$ by

$$\dot{\psi}_x(z) = \begin{cases} \psi_x(z), & \text{if } z \in \dot{N}(x); \\ Y, & \text{if } z \notin \dot{N}(x). \end{cases}$$

Then $\dot{\psi}_x$ is also USC on X by Lemma 3.4.4 such that $P(z) \subset \dot{\psi}_x(z)$, $\forall\, z \in X$.

Now define $\Psi : X \to 2^Y \setminus \{\emptyset\}$ by $\Psi(z) = \bigcap_{x \in X} \dot{\psi}_x(z)$ for each $z \in X$. Clearly, Ψ has closed and G-convex values and $P(z) \subset \Psi(z)$, $\forall\, z \in X$.

Let $z \in X$ be given, then $z \in \dot{N}(x)$ for some $x \in X$ so that $\dot{\psi}_x(z) = \psi_x(z)$ and hence $\Psi(z) \subset \psi_x(z)$; as $\theta(z) \notin \psi_x(z)$, we must also have that $\theta(z) \notin \Psi(z)$. Thus $\theta(z) \notin \Psi(z)$, $\forall\, z \in X$.

Now we shall show that Ψ is USC. For any given $u \in X$, there exists an open nhood M_u of u in X such that the set $\{x \in X : M_u \cap \dot{N}(x) \neq \emptyset\}$ is finite, say $= \{x(u,1), \ldots, x(u,n))\}$. Thus we have

$$\Psi(w) = \bigcap_{x \in X} \dot{\psi}_x(w) = \bigcap_{i=1}^{n(u)} \dot{\psi}_{x(u,i)}(w), \ \forall\, w \in M_u.$$

For $i = 1, \ldots, n(u)$, since each $\dot{\psi}_{x(u,i)}$ is USC on X and hence on M_u with closed values and Y is normal, by Lemma 3.4.5, $\Psi : M_u \to 2^Y$ is also USC at u. Hence Ψ is of class G-$\mathcal{U}_\theta$. □

Applying Theorem 3.4.4 above and Theorem 3.2.1, we obtain the following equilibrium existence theorem for a qualitative game.

Theorem 3.4.5. *Let $(X_i, P_i)_{i \in I}$ be a qualitative game, where $\{(X_i, \Gamma_i)\}_{i \in I}$ is any family of strongly locally G-convex compact G-convex spaces. For each $i \in I$ assume: (i) singleton subsets of X_i are G-convex, and (ii) G-co(A) is compact whenever A is a finite subset of X_i. Let (X, Γ) be the product G-convex space and assume that X is metrizable and strongly locally G-convex. For each $i \in I$ let $P_i : X \to 2^{X_i}$ be G-$\mathcal{U}$-majorized such that the set $C_i = \{x \in X : P_i(x) \neq \emptyset\}$ is closed for each $i \in I$.*

Then the qualitative game $(X_i, P_i)_{i\in I}$ *has an equilibrium i.e. there exists* $\widehat{x} \in X$ *such that* $P_i(\widehat{x}) = \emptyset$, *for all* $i \in I$.

Proof. Let $i \in I$ be arbitrarily fixed. Since C_i is paracompact, X_i is normal and $P_i : C_i \to 2^{X_i} \setminus \{\emptyset\}$ is G-$\mathcal{U}$-majorized; by Theorem 3.4.4, there exists $\phi_i : X \to 2^{X_i} \setminus \{\emptyset\}$ such that ϕ_i is of class G-$\mathcal{U}$ and $P_i(x) \subset \phi_i(x)$, for all $x \in C_i$.

By Theorem 3.2.1, there exists $\Phi_i : X \to 2^{X_i}$ such that $\phi_i(x) = \Phi_i(x)$, for all $x \in C_i$, and Φ_i is USC with nonempty compact G-convex values.

Let $\Phi : X \to 2^X$ be defined by $\Phi(x) = \Pi_{i\in I}\Phi_i(x)$. Then Φ is USC by the lemma of Fan (Lemma 3 in [F]), and it obviously has nonempty compact values. It also follows that Φ has G-convex values by Corollary 1.3.2. Since X is strongly locally G-convex by assumption and since singleton subsets are G-convex by Corollary 1.3.3, it follows by Lemma 3.4.1 that Φ has a fixed point, i.e.

$$\text{there exists } \widehat{x} \in X \text{ such that } \widehat{x}_i \in \Phi_i(\widehat{x}), \text{ for all } i \in I.$$

It follows that $\widehat{x} \notin C_i$, for all $i \in I$ (For otherwise $\widehat{x}_i \in \Phi_i(\widehat{x}) = \phi_i(\widehat{x})$ which is not possible because ϕ_i is of class $\mathcal{U}$), and hence $P_i(\widehat{x}) = \emptyset$, for all $i \in I$. □

As an application of Theorem 3.3.1, we obtain the following equilibrium existence theorem of an abstract economy.

Theorem 3.4.6. *Let* $(X_i, P_i, F_i)_{i\in I}$ *be an abstract economy, where* $\{(X_i, \Gamma_i)\}_{i\in I}$ *is any family of strongly locally G-convex compact metrizable G-convex spaces. For each* $i \in I$ *assume: (i) singleton subsets of* X_i *are G-convex, (ii) cl(G-co(*A*)) is compact whenever* A *is a compact subset of* X_i, *(iii) the closure of a G-convex subset of* X_i *is always G-convex. Let* (X, Γ) *be the product G-convex space and assume that* X *is metrizable and strongly locally G-convex. For each* $i \in I$ *let* $P_i, F_i : X \to 2^{X_i}$ *satisfy the following conditions:*

(1) F_i *is continuous with nonempty closed G-convex values;*

(2) P_i *is USC such that* $x_i \notin$ *cl(G-co(*$P_i(x)$*)), for all* $x \in X$.

Then $(X_i, P_i, F_i)_{i \in I}$ *has an equilibrium i.e. there exists* $\hat{x} \in X$ *such that* $\hat{x}_i \in F_i(\hat{x})$ *and* $P_i(\hat{x}) \cap F_i(\hat{x}) = \emptyset$, *for all* $i \in I$.

Proof. Let $i \in I$ be arbitrarily fixed. Define $G_i : X \to 2^{X_i}$ by

$$G_i(x) = F_i(x) \cap \text{cl}(\text{G-co}(P_i(x))).$$

Then G_i is USC by Lemmas 3.4.3 and 3.4.5. Let $C_i = \{x \in X : G_i(x) \neq \emptyset\}$. Then C_i is closed by Lemma 3.4.2.

Notice that $G_i : C_i \to 2^{X_i} \setminus \{\emptyset\}$ is such that $G_i(x) \subset F_i(x)$, $\forall\, x \in C_i$. So all the conditions of Theorem 3.3.1 are satisfied and hence $\exists\, \tilde{G}_i : X \to 2^{X_i}$ such that $\tilde{G}_i$ is USC with nonempty compact G-convex values satisfying $\tilde{G}_i(x) \subset F_i(x)$, $\forall\, x \in X$.

Let $G : X \to 2^X$ be defined by $G(x) = \Pi \tilde{G}_i(x)$. Then G is USC by the lemma of Fan (Lemma 3 in [F]) and it obviously has nonempty compact values. It also follows that G has G-convex values by Corollary 1.3.2.

Since X is strongly locally G-convex by assumption and since singleton subsets of X are G-convex by Corollary 1.3.3, it follows by Lemma 3.4.1 that G has a fixed point $\hat{x}$ so that $\hat{x}_i \in \tilde{G}_i(\hat{x})$, for all $i \in I$ which implies that
(1) $\hat{x}_i \in F_i(\hat{x})$ and
(2) $\hat{x} \notin C_i$, for all $i \in I$, for otherwise we have $\hat{x}_i \in G_i(\hat{x}) = \text{cl}(\text{G-co}(P_i(\hat{x}))) \cap F_i(\hat{x})$ which contradicts the assumption. Thus $\hat{x}_i \in F_i(\hat{x})$, for all $i \in I$ and $P_i(\hat{x}) \cap F_i(\hat{x}) \subset G_i(\hat{x}) = \emptyset$. Thus it follows that $\hat{x}$ is an equilibrium for the generalized game $(X_i, P_i, F_i)_{i \in I}$. □

In the following we present other versions of Theorems 3.4.1, 3.4.2, 3.4.3, 3.4.5 and 3.4.6. These are true for G-metrically convex spaces and they differ from the equilibrium theorems presented so far in the fact that we do not require the product space to be strongly locally G-convex.

We shall need the following fixed point theorem due to Eilenberg-Montgomery (see [EM]).

Lemma 3.4.6. *Let X be an acyclic absolute nhood retract and $F : X \to 2^X$ be an USC correspondence such that for every $x \in X$, the set $F(x)$ is acyclic. Then F has a fixed point.*

Theorem 3.4.1. *Let $(X_i, \Gamma_i)_{i \in I}$ be a family of strongly locally G-convex compact G-metrically convex spaces where I is countable. For each i assume that (G-co(A))) is compact whenever $A \subset X_i$ is finite. Let (X, Γ) be their product G-convex space. For each $i \in I$ let $F_i : X \to 2^{X_i}$ be USC with closed and G-convex values. Then there exists $\widehat{x} \in X$ such that for all $i \in I$, either $F_i(\widehat{x}) = \emptyset$ or $\widehat{x}_i \in F_i(\widehat{x})$.*

Proof. We notice that for each $i \in I$, X_i is an absolute retract by Theorem 4, Section 1. Therefore it follows that $X = \Pi_{i \in I} X_i$ is an absolute retract and hence an absolute nhood retract (See 2.18 in [B], p. 103). For each i let $C_i = \{x \in X : F_i(x) \neq \emptyset\}$. Then C_i is closed by Lemma 3.4.2. Define $\tilde{F}_i : X \to 2^{X_i}$ as follows:

Case 1. If $C_i = \emptyset$ then let $\tilde{F}_i(x) = X_i$, for all $x \in X$.

Case 2. If $C_i = X$ then let $\tilde{F}_i(x) = F_i(x)$, for all $x \in X$.

Case 3. If C_i is a proper nonempty subset of X, let $\tilde{F}_i$ be the USC extension of F_i provided by Theorem 3.2.1. We notice that $\tilde{F}_i(x)$ is nonempty, closed and G-convex, for all $x \in X$, for all $i \in I$.

Let $F : X \to 2^X$ be defined by $F(x) = \Pi_{i \in I} \tilde{F}_i(x)$. Then F is USC by the Lemma of Fan (Lemma 3 in [F]). Moreover, $F(x)$ is obviously nonempty and compact, for all $x \in X$.

We shall also show that $F(x)$ is acyclic, for all $x \in X$. Indeed, $\tilde{F}_i(x)$ a compact and hence bounded G-convex subset of X_i. Therefore it follows by Proposition 2.1.2 that $\tilde{F}_i(x)$ is contractible. Since the product of contractible sets is contractible, it follows that $F(x)$ is contractible and hence acyclic.

Applying Lemma 3.4.6 to $F : X \to 2^X$; it follows that there exists $\widehat{x} \in X$ such that $\widehat{x} \in F(\widehat{x})$ so that $\widehat{x}_i \in \tilde{F}_i(\widehat{x})$, for all $i \in I$ which implies that for each $i \in$

I either $F_i(\widehat{x}) = \emptyset$ or $\widehat{x}_i \in F_i(\widehat{x})$. $\square$

CHAPTER FOUR

SOME VARIATIONAL INEQUALITIES IN $\mathcal{M}$-CONVEX SPACES

The content of this chapter is particular generalizations of some variational inequalities given by K.K.Tan, E.Tarafdar, and X.Z.Yuan in [TTY] to $\mathcal{M}$-convex spaces.

In chapters one and three we have seen how a homogeneous G-map system is a useful tool in proving a selection or an extension theorem. In this chapter although our study is restricted to $\mathcal{M}$-convex spaces, i.e. G-convex with a G-map system, however we do not require the G-map systems to be homogeneous.

We introduce $\mathcal{M}$-convexity, and $\mathcal{M}$-concavity for real functions (both set-valued and single-valued) on an $\mathcal{M}$-convex space. We also introduce the concept of an $\mathcal{M}$-affine real function. Then we construct a so-called dual space $X^*_{\mathcal{M}}$ which consists of all $\mathcal{M}$-affine continuous real-valued functions. As we shall see in the definition of an $\mathcal{M}$-convex, $\mathcal{M}$-concave or $\mathcal{M}$-affine map, it only seems natural that the smaller the size of the set $\mathcal{M}(A)$ the better. This indicates that it is an advantage to have a G-map system in which $\mathcal{M}(A)$ is finite for each finite subset A, as we have seen in Theorem 2.2.1. Nevertheless we feel that this subject needs further investigation and study.

Our main tools for obtaining the solutions of variational and quasi-variational inequalities are a KKM-type theorem (Theorem 1.2.3) and a Fan-Glicksberg-type theorem (Theorem 1.2.7).

As a final remark, we observe that Lemma 4.1.3 was our very first motivation for defining $\mathcal{M}$-convexity of sets (Definition 1.2.4). In this Lemma we prove that a certain set W is $\mathcal{M}$-convex, and indeed, such a set cannot be proved to be G-convex, no matter how one tries.

Now before we proceed to give our generalizations, we quote some definitions and some results from [TTY] for easier reference.

Definition 1. Let X be a nonempty convex subset of a topological vector space E and let E^* be its dual.

(1) A mapping $T : X \to 2^{E^*}$ is said to be monotone if for each $x, y \in X$, $\mathrm{Re}\langle u - v, x - y\rangle \geq 0$ for all $u \in T(x)$ and $v \in T(y)$.

(2) If $f, g : X \times X \to 2^{\mathbb{R}}$, then $\{f, g\}$ is said to be a monotone pair if for each $x, y \in X$, $u + v \geq 0$ for each $u \in f(x, y), v \in g(y, x)$. In particular when $f = g$ and is single-valued, the notion of monotone pair reduces to that of a monotone mapping defined by Mosco [M] (See Tarafdar [Ta] and also Husain and Tarafdar [HT]).

(3) $f : X \times X \to 2^{\mathbb{R}}$ is said to be hemicontinuous if for each $x, y \in X$, the mapping $k : [0, 1] \to 2^{\mathbb{R}}$ defined by: $k(t) = f((1-t)x + ty, y)$, for all $t \in [0, 1]$ satisfies the following property

For each given $s \in \mathbb{R}$ with $f(x, y) \subset (s, +\infty)$, there exists $t_0 \in (0, 1]$ such that

$$f((1-t)x + ty, y) \subset (s, +\infty), \text{ for all } t \in (0, t_0).$$

We note that if f is single-valued this definition of hemicontinuity reduces to the classical one given by Mosco [M], i.e. the function $t \to f(x + t(y - x), y)$ from $[0, 1]$ to $\mathbb{R}$ is lower semicontinuous as $t \downarrow 0$.

(4) $f : X \to 2^{\mathbb{R}}$ is said to be concave (respectively convex) if for each $n \in \mathbb{N}$, $x_1, x_2, \ldots, x_n \in X$ and each $\lambda_1, \ldots, \lambda_n \geq 0$ with $\Sigma_{i=1}^n \lambda_i = 1$ and for each $u \in f(\Sigma_{i=1}^n \lambda_i x_i)$, there exists $v_i \in f(x_i)$ for each $i = 1, \ldots, n$ such that $u \geq$ (resp. $\leq$)$\Sigma_{i=1}^n \lambda_i v_i$.

(5) $h : X \to \mathbb{R}$ is said to be lower semicontinuous (respectively upper semicontinuous) if for each $\lambda \in \mathbb{R}$, the set $\{x \in X : h(x) \leq \lambda\}$ (resp. $\{x \in X : h(x) \geq \lambda\}$)

is closed in X.

(6) $H : X \to 2^{\mathbb{R}}$ is lower (resp. upper) demicontinuous if for any $s \in \mathbb{R}$ and any $x \in X$ with $H(x) \subset (s, +\infty)$ (resp. $H(x) \subset (-\infty, s)$) there exists an open nhood N of x in X such that $H(y) \subset (s, +\infty)$ (resp $H(y) \subset (-\infty, s)$) whenever $y \in N$. We note that when H is single-valued, the notions of lower demicontinuity (resp. upper demicontinuity) and lower semicontinuity (resp. upper semicontinuity) coincide.

The following are Lemmas 2.2 and 2.4 in [TTY].

Lemma 1. *Let X be a nonempty convex subset of E and let $H : X \to 2^{\mathbb{R}}$ be lower demicontinuous. Then the mapping $h : X \to \mathbb{R} \cup \{-\infty\}$ defined by $h(x) = \inf H(x)$ for each $x \in X$ is lower semicontinuous.*

Lemma 2. *Let X be a nonempty convex subset of E. Suppose $G : X \to 2^{\mathbb{R}}$ is LSC. Then $W = \{x \in X : \inf G(x) \geq 0\}$ is closed in X.*

1. Real Set-Valued Mappings.

In this section we introduce the concepts of $\mathcal{M}$-concavity and $\mathcal{M}$-convexity for real valued mappings on an $\mathcal{M}$-convex space. We also define hemicontinuity for a mapping $f : X \times X \to 2^{\mathbb{R}}$, which generalizes the concept of hemicontinuity as defined in [TTY].

Definition 4.1.1. Let (X, Γ) be a G-convex space, and let $\mathcal{M}$ be a G-map system associated with Γ.

(i) $f : X \to \mathbb{R}$ is said to be $\mathcal{M}$-concave (respectively $\mathcal{M}$-convex) if for any $A = \{a_0, \ldots, a_n\} \subset X$ and any $\phi \in \mathcal{M}(A)$, we have

$f(\phi(\Sigma_{i=0}^{n} \lambda_i e_i)) \geq$ (respectively $\leq$) $\Sigma_{i=0}^{n} \lambda_i f(a_i)$, whenever $\lambda_0, \ldots, \lambda_n \geq 0$ with $\Sigma_{i=0}^{n} \lambda_i = 1$.

(ii) $f : X \to 2^{\mathbb{R}}$ is said to be $\mathcal{M}$-concave (respectively $\mathcal{M}$-convex) if whenever $A = \{a_0, \ldots, a_n\} \in \langle X \rangle$, $\phi \in \mathcal{M}(A)$, $\lambda_0, \ldots, \lambda_n \geq 0$ with $\Sigma_{i=0}^{n} \lambda_i = 1$ and $r \in$

$f(\phi(\Sigma_{i=0}^{n}\lambda_i.e_i))$, then there exists $r_i \in f(a_i)$ for each $i = 0, 1, \ldots, n$ such that $r \geq$ (respectively $\leq$) $\Sigma_{i=0}^{n}\lambda_i r_i$.

Remark 1. *We note that in the case where X is a nonempty convex subset of a topological vector space, if we let $\mathcal{M}$ and ϕ_A be as in Example 1.1.1, then any concave (resp. convex) set-valued function $f : X \to 2^{\mathbb{R}}$ is $\mathcal{M}$-concave (resp. $\mathcal{M}$-convex).*

Indeed for any $A = \{a_0, \ldots, a_n\} \in \langle X \rangle$, and $\phi \in \mathcal{M}(A)$, we have

$\phi(\Sigma_{i=0}^{n}\lambda_i.e_i) = \Sigma_{i=0}^{n}\lambda_i.a_i$.

Thus the two definitions i.e. (4) in Definition 1 above and Definition 4.1.1 coincide.

The following is a generalization of the definition of hemicontinuity in [TTY].

Definition 4.1.2. Let (X, Γ) be a G-convex space. Let $\mathcal{M}$ be a G-map system associated with Γ.

Let $f : X \times X \to 2^{\mathbb{R}}$. Then f is said to be $\mathcal{M}$-hemicontinuous if whenever $x_0, x_1 \in X$ are distinct, $\phi \in \mathcal{M}(\{x_0, x_1\})$ and s is a real number such that $f(x_0, x_1) \subset (s, \infty)$, there exists $t_0 \in (0, 1]$ such that $f(\phi((1-t)e_0 + te_1), x_1) \subset (s, \infty)$, for all $t \in [0, t_0)$.

Remark 2. *We note that in the case where X is a nonempty convex subset of a topological vector space, if we let $\mathcal{M}$ and ϕ_A be as in Example 1.1.1, then any hemicontinuous set-valued function $f : X \times X \to 2^{\mathbb{R}}$ is $\mathcal{M}$-hemicontinuous.*

Indeed whenever $x_0, x_1 \in X$ are distinct, and $\phi \in \mathcal{M}(\{x_0, x_1\})$ then $(\phi((1-t)e_0 + te_1), x_1) = ((1-t)x_0 + tx_1, x_1)$ for any $t \in [0, 1]$. Thus Definition 4.1.2 above and (3) in Definition 1 coincide.

The following lemma generalizes Lemma 2.1 in [TTY].

Lemma 4.1.1. *Let (X, Γ) be a G-convex space, and let $\mathcal{M}$ be a G-map system* *associated with* Γ. *Let* $f, g : X \times X \to 2^{\mathbb{R}}$.

(a) If f, g *is a monotone pair and* $\inf f(x,y) \leq 0$ *then* $\inf g(y,x) \geq 0$.

(b) Assume f *is* $\mathcal{M}$*-hemicontinuous,* $\inf f(x,x) \leq 0$ *for each* $x \in X$ *and* $y \to f(x,y)$ *is* $\mathcal{M}$*-concave for each* $x \in X$. *If there exists* $x_0 \in X$ *such that* $\inf f(y, x_0) \geq 0$ *for all* $y \in X$ *then* $\inf f(x_0, y) \leq 0$ *for all* $y \in X$.

Proof. (a) See the proof of Lemma 2.1 (1) in [TTY].

(b) Assume not, i.e. there exists $y_0 \in X$ such that $\inf f(x_0, y_0) > 0$. Then $x_0 \neq y_0$ and $f(x_0, y_0) \subset (s, \infty)$ for some $s > 0$.

Let $\phi \in \mathcal{M}(\{x_0, y_0\})$. It follows by $\mathcal{M}$-hemicontinuouity of f that there exists $t_0 \in (0,1]$ such that:

$$f(z_t, y_0) \subset (s, \infty) \text{ for all } t \in (0, t_0], \text{ where } z_t = \phi((1-t)e_0 + te_1). \tag{1}$$

Now let $r \in f(z_{t_0}, z_{t_0}) = f(z_{t_0}, \phi((1-t_0)e_0 + t_0 e_1))$. By $\mathcal{M}$-concavity of $y \to f(z_{t_0}, y)$, it follows that there exist $r_0 \in f(z_{t_0}, x_0)$ and $r_1 \in f(z_{t_0}, y_0)$ such that $r \geq (1-t_0)r_0 + t_0 r_1$. Hence $r \geq t_0 r_1$ since $\inf f(z_{t_0}, x_0) \geq 0$ by assumption.

By (1) and the inequality above, we have $r \geq t_0 s > 0$. And hence it follows that $\inf f(z_{t_0}, \phi((1-t_0)e_0 + t_0 e_1)) = \inf f(z_{t_0}, z_{t_0}) \geq t_0 s > 0$ which contradicts the assumption that $\inf f(x,x) \leq 0$ for each $x \in X$. □

Lemma 4.1.2. *Let* (X, Γ) *be a G-convex space and let* $\mathcal{M}$ *be a G-map system associated with* Γ. *Let* $f : X \times X \to 2^{\mathbb{R}}$ *be such that*

(i) $\inf f(x,x) \leq 0$ *for each* $x \in X$;

(ii) $y \to f(x,y)$ *is* $\mathcal{M}$*-concave.*

Then the mapping $T : X \to 2^X$ *defined by* $T(w) = \{x \in X : \inf f(x,w) \leq 0\}$ *is generalized* $\mathcal{M}$*-KKM.*

Proof. Let $w_0, w_1, \ldots, w_n \in X$, then we shall show that for any $0 \leq i_0 < i_1 < \cdots <$

$i_k \leq n$ and any $\phi \in \mathcal{M}(\{w_0, \ldots, w_n\})$, $\phi(\Delta_J) \subset \bigcup_{j=0}^{k} T(w_{i_j})$ where Δ_J is that face of Δ_n corresponding to $J = \{i_0, i_1, \ldots, i_k\}$.

Suppose not. Then it follows that there exist $\phi \in \mathcal{M}(\{w_0, \ldots, w_n\})$, $J = \{i_0, \ldots, i_k\} \subset \{0, 1, \ldots, n\}$ and $z = \phi(\Sigma_{j=0}^{k} \lambda_{i_j} e_{i_j})$ such that

$$z \notin T(w_{i_j}) \text{ for all } j = 0, 1, \ldots, k. \tag{*}$$

Hence there exists a real number $s > 0$ such that

$$\inf f(z, w_{i_j}) > s \text{ for all } j = 0, 1, \ldots, k. \tag{1}$$

Now let $r \in f(z, z) = f(z, \phi(\Sigma_{j=0}^{k} \lambda_{i_j} e_{i_j})) = f(z, \phi(\Sigma_{j=0}^{k} \lambda_{i_j} e_{i_j} + \Sigma_{j \notin J} 0 \cdot e_j))$. Since $\phi \in \mathcal{M}(\{w_0, \ldots, w_n\}$, it follows from (ii) that there exist $r_{i_j} \in f(z, w_{i_j})$ for each $0 \leq j \leq k$ and $r_l \in f(z, w_l)$ for each $l \notin J$ such that

$$r \geq (\Sigma_{j=0}^{k} \lambda_{i_j} r_{i_j} + \Sigma_{l \notin J} 0 r_l) = \Sigma_{j=0}^{k} \lambda_{i_j} r_{i_j}.$$

Hence by (1), we have $r \geq \Sigma_{j=0}^{k} \lambda_{i_j} s = s > 0$. And so $\inf f(z, z) \geq s > 0$, which is a contradiction to (i). □

In the following we give a sufficient condition for the existence of a solution for the variational inequality $\inf f(x, y) \leq 0$, for all $y \in X$.

Theorem 4.1.1. *Let (X, Γ) be a G-convex space and $\mathcal{M}$ be a G-map system associated with Γ. Let $f : X \times X \to 2^{\mathbb{R}}$ be such that*

(i) $\inf f(x, x) \leq 0$ for all $x \in X$;

(ii) $y \to f(x, y)$ is $\mathcal{M}$-concave for each fixed $x \in X$;

(iii) $x \to f(x, y)$ is lower demicontinuous for each fixed $y \in X$;

(iv) there exist a nonempty compact subset B of X and $w_0 \in B$ such that $\inf f(x, w_0) > 0$, for all $x \in X \backslash B$.

Then the set $S = \{x \in X : \inf f(x,w) \leq 0 \ , \ \forall\, w \in X\}$ *is a nonempty compact subset of* B.

Proof. Define $T : X \to 2^X$ by $T(w) = \{x \in X : \inf f(x,w) \leq 0\}$ for each $w \in X$. Then each $T(w)$ is a closed subset of X, by Lemma 1. Also, by Lemma 4.1.2, T is a generalized $\mathcal{M}$-KKM mapping.

Hence, by Theorem 1.2.3 , we have $S = \bigcap_{w \in X} T(w) \neq \emptyset$. Clearly S is closed. Moreover $S \subset B$, since otherwise there exists $x_0 \in X \backslash B$ such that $\inf f(x_0, w_0) \leq 0$, a contradiction to (iv).

Now that S is a nonempty closed subset of B it is compact and the proof is completed. □

Lemma 4.1.3. *Let* (X, Γ) *be a G-convex space,* $\mathcal{M}$ *be a G-map system associated with* Γ *and* $g : X \to 2^{\mathbb{R}}$ *be* $\mathcal{M}$*-concave. Then* $W = \{x \in X : \inf g(x) \geq 0\}$ *is* $\mathcal{M}$*-convex.*

Proof. Let $w_0, \ldots, w_n \in W$ and let $\phi \in \mathcal{M}(\{w_0, \ldots, w_n\}$. We shall show that $\phi(\Delta_n) \subset W$. Indeed, let $z = \phi(\Sigma_{i=0}^n \lambda_i e_i)$, where $\lambda_0, \ldots, \lambda_n \geq 0$ with $\Sigma_{i=0}^n \lambda_i = 1$. Let $r \in g(z) = g(\phi(\Sigma_{i=0}^n \lambda_i e_i))$. By $\mathcal{M}$-concavity of g, there exists $r_i \in g(w_i)$ for each i such that $r \geq \Sigma_{i=0}^n \lambda_i r_i$. Since $r_i \geq 0$ for each i, by the definition of W, it immediately follows that $r \geq 0$ and hence that $\inf g(z) \geq 0$. □

Theorem 4.1.2. *Let* (X, Γ) *be a Hausdorff G-convex space,* $\mathcal{M}$ *be a G-map system associated with* Γ *and* $f : X \times X \to 2^{\mathbb{R}}$ *be such that*

(i) f *is monotone;*

(ii) f *is* $\mathcal{M}$*-hemicontinuous;*

(iii) $\inf f(x,x) \leq 0$ *for all* $x \in X$*;*

(iv) $y \to f(x,y)$ *is* $\mathcal{M}$*-concave and LSC for each fixed* $x \in X$*;*

(v) there exist a nonempty compact subset B *of* X *and* $w_0 \in B$ *such that*

$\inf f(x, w_0) > 0$, *for all* $x \in X \backslash B$.

Then the set $S = \{x \in X : \inf f(x, w) \leq 0, \forall\ w \in X\}$ *is a nonempty compact* $\mathcal{M}$*-convex subset of* B.

Proof. Define $F, G, H : X \to 2^X$ by

$F(w) = \{x \in X : \inf f(x, w) \leq 0\}$;

$G(w) = cl_X(F(w))$;

$H(w) = \{x \in X : \inf f(w, x) \geq 0\}$.

Then by (iii) and (iv), F is $\mathcal{M}$-KKM and hence G is $\mathcal{M}$-KKM too. Moreover $F(w_0) \subset B$ which implies $G(w_0) \subset B$ since B is closed. Therefore $G(w_0)$ is compact. Thus all the conditions of Theorem 1.2.3 are satisfied for $G : X \to 2^X$ and it follows that $\bigcap_{w \in X} G(w) \neq \emptyset$.

Now $F(w) \subset H(w)$, by monotonicity of F. Moreover, $H(w)$ is closed by Lemma 2. So it follows that $F(w) \subset G(w) \subset H(w)$, $\forall w \in X$. Hence $\bigcap_{w \in X} F(w) \subset \bigcap_{w \in X} G(w) \subset \bigcap_{w \in X} H(w)$.

Next we shall show that $\bigcap_{w \in X} H(w) \subset \bigcap_{w \in X} F(w)$. So let $x_0 \in \bigcap_{w \in X} H(w)$. Then $\inf f(w, x_0) \geq 0$, for all $w \in X$, and by (ii), (iii), (iv) and lemma 1.1, it follows that $\inf f(x_0, w) \leq 0$, for all $w \in X$; i.e. $x_0 \in \bigcap_{w \in X} F(w)$. Thus it follows that $S = \bigcap_{w \in X} F(w) = \bigcap_{w \in X} G(w) = \bigcap_{w \in X} H(w) \neq \emptyset$.

The above also implies that S is closed (being an intersection of closed sets). Also $S \subset B$, since otherwise there exists $x \in S \backslash B$ such that $\inf f(w_0, x) \leq 0$ which is a contradiction to (v).

Moreover, $H(w)$ is $\mathcal{M}$-convex for each $w \in X$, by Lemma 1.3 and (iv). So S is the intersection of $\mathcal{M}$-convex sets and is hence $\mathcal{M}$-convex. □

2. Some Implicit Variational Inequalities For Monotone Mappings .

Here we define a dual space $X^*_{\mathcal{M}}$, and introduce a weak topology for it. We also define monotonicity for mappings like $T : X \to 2^{X^*_{\mathcal{M}}}$, which generalizes the usual monotonicity in the linear case.

Applying Theorem 4.1.2 and Theorem 1.2.7 , solutions of variational inequalities and quasivariational inequalities are obtained.

Definition 4.2.1. Let (X, Γ) be a G-convex space, $\mathcal{M}$ be a G-map system associated with Γ and $f : X \to \mathbb{R}$. Then f is said to be $\mathcal{M}$-affine if whenever $A = \{x_0, \ldots, x_n\}$ is a subset of X and $\phi \in \mathcal{M}(A)$ then $f(\phi(\Sigma_{i=0}^n \lambda_i e_i)) = \Sigma_{i=0}^n \lambda_i f(x_i)$ for all $\lambda_0, \ldots, \lambda_n \geq 0$ with $\Sigma_{i=0}^n \lambda_i = 1$.

Definition 4.2.2. Let (X, Γ) be a G-convex space and $\mathcal{M}$ a G-map system associated with Γ. Let $X^*_{\mathcal{M}}$ denote the collection of all continuous $\mathcal{M}$-affine real-valued functions on X. For each $x \in X$, let $h_x : X^*_{\mathcal{M}} \to \mathbb{R}$ be defined by $h_x(u) = u(x)$, for all $u \in X^*_{\mathcal{M}}$. And for each $x, y \in X$, let $h_{(x,y)} : X^*_{\mathcal{M}} \to \mathbb{R}$ be defined by $h_{(x,y)}(u) = u(x) - u(y)$, for all $u \in X^*_{\mathcal{M}}$. Let $\mathcal{H} = \{h_x : x \in X\}$. Then the topology τ_p on $X^*_{\mathcal{M}}$, is the weak topology induced on $X^*_{\mathcal{M}}$ by the elements of $\mathcal{H}$, i.e. the smallest topology on $X^*_{\mathcal{M}}$ that makes all the elements of $\mathcal{H}$ continuous.

Remark. *We note that since constant real functions on any $\mathcal{M}$-convex space are $\mathcal{M}$-affine; the space $X^*_{\mathcal{M}}$ is always nonempty.*

Definition 4.2.3. Let (X, Γ) be a G-convex space, $\mathcal{M}$ be a G-map system associated with Γ and $T : X \to 2^{X^*_{\mathcal{M}}}$. Then T is said to be monotone iff for any $x, y \in X, u \in T(x)$ and $v \in T(y)$, we have $u(x) + v(y) - u(y) - v(x) \geq 0$.

Definition 4.2.4. Let (X, Γ) be a G-convex space and $\mathcal{M}$ a G-map system on (X, Γ). Then a subset L of X is said to be an $\mathcal{M}$-line segment iff there exist $x_0, y_0 \in X$ and $\phi \in \mathcal{M}(\{x_0, y_0\})$ such that $L = \phi(\Delta_1)$.

As an application of Theorem 4.1.2, we give the following variational inequality.

Theorem 4.2.1. *Let (X, Γ) be a Hausdorff G-convex space and $\mathcal{M}$ a G-map system* *associated with Γ. Assume that $\{x\}$ is G-convex, for each $x \in X$. Let $T : X \to 2^{X^*_{\mathcal{M}}}$ be such that*

(a) T is monotone;

(b) $T(x)$ is compact w.r.t. τ_p for each $x \in X$;

(c) T is USC from $\mathcal{M}$-line segments in X to τ_p on $X^*_{\mathcal{M}}$;

(d) there exist a nonempty compact subset B of X and $w_0 \in B$ such that $u(x) - u(w_0) > 0$, for all $u \in T(x)$, for all $x \in X \backslash B$.

Then the set $S = \{x_0 \in X : \inf_{u \in T(x_0)} u(x_0) - u(x) \leq 0$, for all $x \in X\}$ is a nonempty compact $\mathcal{M}$-convex subset of B.

Proof. Let $f : X \times X \to 2^{\mathbb{R}}$ be defined by $f(x,y) = \{u(x) - u(y) : u \in T(x)\}$.

We shall prove the following:

(i) f is monotone;

(ii) $y \to f(x,y)$ is $\mathcal{M}$-concave for each fixed $x \in X$;

(iii) $y \to f(x,y)$ is LSC for each fixed $x \in X$;

(iv) f is $\mathcal{M}$-hemicontinuous.

(i) Let $r_1 \in f(x,y)$ and $r_2 \in f(y,x)$ for some $x, y \in X$. Then there exist $u \in T(x)$ and $v \in T(y)$ such that $r_1 = u(x) - u(y)$ and $r_2 = v(y) - v(x)$. Hence $r_1 + r_2 = u(x) + v(y) - u(y) - v(x) \geq 0$ by monotonicity of T. Thus f is monotone.

(ii) Let $y_0, \ldots, y_n \in X$, $\phi \in \mathcal{M}(\{y_0, \ldots, y_n\})$ and $r \in f(x, \phi(\Sigma_{i=0}^n \lambda_i e_i))$, where $\lambda_0, \ldots, \lambda_n \geq 0$ with $\Sigma_{i=0}^n \lambda_i = 1$.

We shall show that there exists $r_i \in f(x, y_i)$ for each $i \in \{0, \ldots, n\}$ such that $r \geq \Sigma_{i=0}^n \lambda_i r_i$. From the choice of r, there exists $u \in T(x)$ such that

$$r = u(x) - u(\phi(\Sigma_{i=0}^n \lambda_i e_i)).$$

Now, since u is $\mathcal{M}$-affine, we have

$$u(\phi(\Sigma_{i=0}^{n}\lambda_i e_i)) = \Sigma_{i=0}^{n}\lambda_i u(y_i).$$

So $r = \Sigma_{i=0}^{n}\lambda_i(u(x) - u(y_i)) = \Sigma_{i=0}^{n}\lambda_i r_i$, where $r_i = u(x) - u(y_i) \in f(x, y_i)$ for each i.

(iii) Fix any $x_0 \in X$. Let U be an open subset of $\mathbb{R}$ and assume $U \cap f(x_0, y) \neq \emptyset$. Then there exists $u \in T(x_0)$ such that $u(x_0) - u(y) \in U$, and hence $u(y) \in -U + u(x_0)$,(an open subset of $\mathbb{R}$). By continuity of u, there exists an open neighbourhood N of y such that $u(z) \in -U + u(x_0)$ for all $z \in N$, i.e., $u(x_0) - u(z) \in U$ for all $z \in N$. It follows that $f(x_0, z) \cap U \neq \emptyset$, $\forall\, z \in N$. Hence $y \to f(x_0, y)$ is LSC.

(iv) Let $x_0, y_0 \in X$, $\phi \in \mathcal{M}(\{x_0, y_0\})$ and s be a real number such that $f(x_0, y_0) \subset (s, \infty)$. It suffices to show that there exists $t_0 \in (0, 1]$ such that $f(\phi(z_t), y_0) \subset (s, \infty)$ for all $t \in (0, t_0)$ where $z_t = (1-t)e_0 + te_1$. First we show that $f(x_0, y_0)$ is a closed subset of $\mathbb{R}$. Indeed let $h : X^*_{\mathcal{M}} \to \mathbb{R}$ be defined by:

$$h(p) = p(x_0) - p(y_0) = h_{x_0}(p) - h_{y_0}(p).$$

Then h is continuous w.r.t. τ_p, being the difference of two continuous functions. Now $f(x_0, y_0) = \{u(x_0) - u(y_0) : u \in T(x_0)\} = h(T(x_0))$ which is compact; being the continuous image of a compact set. Therefore $f(x_0, y_0)$ is closed, and hence $\inf f(x_0, y_0) = r_0 > s$.

We shall consider two separate cases.

Case 1. $s \geq 0$. We let $r = (r_0 + s)/2$ and $t_1 = (r - s)/r$.

Case 2. $s < 0$. We choose $r \in (s, r_0)$ such that $r < 0$ and let $t_1 = 1/2$.

In each case above let $V = (r, \infty)$ and $U = (s, \infty)$. Then it follows that:

$$t_1 \in (0, 1), \quad f(x_0, y_0) \subset V \text{ and } (1-t)V \subset U \text{ for all } t \in (0, t_0). \qquad (*)$$

Indeed, the assertion that $t_0 \in (0,1)$ is obvious. Also in both cases we have $r < r_0$ which implies $f(x_0, y_0) \subset (r, \infty) = V$.

To prove the last assertion in (*), we consider each case separately.

Case 1. Let $t \in (0, t_1)$ then $(1-t) > (1-t_1)$ and hence $(1-t)r > (1-t_1)r = (1-(r-s)/r)r = s$. Therefore $((1-t)r, \infty) \subset (s, \infty)$; i.e $(1-t)V \subset U$.

Case 2. Let $t \in (0, t_1)$. Then $0 < (1-t) < 1$ and since $r < 0$ in this case, it follows that $(1-t)r > r$. But $r > s$, so we have $(1-t)r > s$ which implies that $((1-t)r, \infty) \subset (s, \infty)$; i.e. $(1-t)V \subset U$.

(*) actually means that $u(x_0) - u(y_0) \in V$ for all $u \in T(x_0)$; i.e. $h(u) \in V$, for all $u \in T(x_0)$ or, in other words, $T(x_0) \subset h^{-1}(V) = G$, where G is open by continuity of h.

Let $L_1 = \phi(\Delta_1)$. Then $x_0 \in L_1$ since $\phi(e_0) = x_0$ by the G-convexity of $\{x_0\}$. Now the upper semicontinuity of T on $\mathcal{M}$-line segments of X implies that there exists an open nhood N of x_0 in L_1 such that

$$T(x) \subset G, \text{ for all } x \in N. \tag{1}$$

But $\phi : \Delta_1 \to L_1$ is continuous, so $\phi^{-1}(N)$ is an open nhood of e_0. This implies that there exists $t_0 \in (0, t_1)$ such that

$$\phi(z_t) \in N, \text{ for all } t \in (0, t_0). \tag{2}$$

By (1) and (2), we have $T(\phi(z_t)) \subset G$, for all $t \in (0, t_0)$. In other words

$$v(x_0) - v(y_0) \in V, \text{ for all } v \in T(\phi(z_t)). \tag{3}$$

Now we shall show that $f(\phi(z_t), y_0) = \{v(\phi(z_t)) - v(y_0) : v \in T(\phi(z_t))\} \subset (s, \infty)$, whenever $t \in (0, t_0)$.

First let $v \in T(\phi(z_t))$. Since v is $\mathcal{M}$-affine, $v(\phi(z_t)) - v(y_0) = v(\phi((1-t)e_0 + te_1)) - v(y_0) = (1-t)v(x_0) + tv(y_0) - v(y_0) = (1-t)(v(x_0) - v(y_0)) \in (1-t)V$ by

(3). But $(1-t)V \subset U$ for all $t \in (0, t_0)$, so it follows that $v(\phi(z_t)) - v(y_0) \in U$ for all $v \in T(\phi(z_t))$ and for all $t \in (0, t_0)$; i.e., $f(\phi(z_t), y_0) \subset (s, \infty)$ for all $t \in (0, t_0)$. Thus f is $\mathcal{M}$-hemicontinuous.

So now by (i), (ii), (iii), (iv) and (d) in the assumption of the theorem, it is clear that f satisfies all the conditions of Theorem 4.1.2. Hence the set $S = \{x_0 \in X : \inf f(x_0, x) \leq 0 \text{ for all } x \in X\} = \{x_0 \in X : \inf_{u \in T(x_0)} u(x_0) - u(x) \text{ for all } x \in X\}$ is a nonempty compact $\mathcal{M}$-convex subset of X. □

Theorem 4.2.2. *Let (X, Γ) be a compact Hausdorff G-convex space and $\mathcal{M}$ be a G-map system associated with Γ. Assume that $\{x\}$ is $\mathcal{M}$-convex for each $x \in X$. Assume further that for any compact $\mathcal{M}$-convex subset A of X and any open subset U of X containing A, there exists an open subset V of X containing A such that $\mathcal{M}$-co$V \subset U$.*

Let $g : X \times X \times X \to 2^{\mathbb{R}}$ satisfy the following conditions:

(i) for any $u, x \in X$, $\inf g(u, x, x) \leq 0$;

(ii) for any fixed $u \in X$, $(x, y) \to g(u, x, y)$ is monotone and $\mathcal{M}$-hemicontinuous;

(iii) for any $u, x \in X$, $y \to g(u, x, y)$ is $\mathcal{M}$-concave;

(iv) for any $x \in X$, $(u, y) \to g(u, x, y)$ is LSC.

Then the set $W = \{u \in X : \inf\ g(u, u, w) \leq 0,$ for all $w \in X\}$ is a nonempty compact subset of X.

Proof. For each $u \in X$, let $f_u : X \times X \to 2^{\mathbb{R}}$ be defined by $f_u(x, y) = g(u, x, y)$. Then f_u satisfies all the conditions of Theorem 4.1.2. Thus there exists a nonempty compact $\mathcal{M}$-convex subset S_u of X such that $\inf g(u, x, w) \leq 0$, for all $w \in X$, for all $x \in S_u$. We define $S : X \to K(X)$ by $S(u) = S_u$.

Next we shall show that S is USC. It suffices to show that S has a closed graph. So let $(x_\alpha, y_\alpha)_{\alpha \in \aleph}$ be a net in $X \times X$ such that $y_\alpha \in S(x_\alpha)$ for each α. Assume $x_\alpha \to x_0$ and $y_\alpha \to y_0$. We will show $y_0 \in S(x_0)$. Let $w \in X$. For each α, since

$y_\alpha \in S(x_\alpha)$, $\inf\ g(x_\alpha, y_\alpha, w) \leq 0$. Then, applying (ii), we have:

$$\inf\ g(x_\alpha, w, y_\alpha) \geq 0. \tag{1}$$

Let $C_w = \{(x,y) \in X \times X : \inf\ g(x,w,y) \geq 0\}$. Then C_w is closed by (iv) and Lemma 2. Moreover, $(x_\alpha, y_\alpha) \in C_w$ for all $\alpha \in \aleph$. So it follows that $(x_0, y_0) \in C_w$ i.e. $\inf\ g(x_0, w, y_0) \geq 0$. And, because w is any arbitrary point in X, we actually have

$$\inf g(x_0, w, y_0) \geq 0 \text{ for all } w \in X. \tag{2}$$

Now by (i), (ii), (iii), (2) above and Lemma 4.1.1, it follows that $\inf g(x_0, y_0, w) \leq 0$, for all $w \in X$. In other words $y_0 \in S(x_0)$. Thus S has a closed graph and is hence USC.

Now by Theorem 1.2.7, it follows that S has a fixed point, and therefore, the set $W = \{u \in X : \inf g(u,u,w) \leq 0, \text{ for all } w \in X\}$ is nonempty.

It only remains to show that W is closed. Indeed let $(u_\alpha)_{\alpha\in\aleph}$ be a net in W and assume $u_\alpha \to u_0$. Then $(u_\alpha, u_\alpha) \in \text{graph}S$ for each α, and therefore $(u_0, u_0) \in \text{graph}S$, since S has a closed graph. Clearly then $u_0 \in W$ and W is closed. $\square$

The following quasi variational inequality is an application of Theorem 4.2.2 above.

Theorem 4.2.3. *Let (X, Γ) be a compact Hausdorff G-convex space and $\mathcal{M}$ be a G-map system associated with Γ. Assume $\{x\}$ is $\mathcal{M}$-convex for each $x \in X$. Assume also that for any compact $\mathcal{M}$-convex subset A of X and any open subset U of X containing A, there exists an open subset V of X containing A such that $\mathcal{M}$-co$V \subset U$. Let $S : X \to K(X)$ be USC with $\mathcal{M}$-convex values. Let $g : X \times X \times X \to 2^{\mathbb{R}}$ be such that:*

(i) for any $u, x \in X, \inf g(u,x,x) \leq 0$;

(ii) for any $u \in X$, $(x,y) \to g(u,x,y)$ is monotone and $\mathcal{M}$-hemicontinuous;

(iii) for any $u, x \in X$, both $y \to g(u,x,y)$ and $y \to g(y,x,y)$ are $\mathcal{M}$-concave;

(iv) for any $x \in X$, $(u, y) \to g(u, x, y)$ is LSC;

(v) the mapping $(u, x) \to g(u, x, u)$ is LSC.

Then (a) there exists $\widehat{y} \in X$ such that $\widehat{y} \in S(\widehat{y})$ and $\inf g(\widehat{y}, \widehat{y}, w) \leq 0$ for all $w \in S(\widehat{y})$,

and (b) the set $\{y \in X : y \in S(y)$ and $\inf g(y, y, w) \leq 0$ for all $w \in S(y)\}$ is a nonempty compact subset of X.

Proof. For any fixed $u \in X$, let $\dot{X} = S(u)$, $\dot{\Gamma} = \Gamma_{S(u)}$ and $\dot{\mathcal{M}} = \mathcal{M}_{S(u)}$ be as in Proposition 1.2.3 . Then it is easy to verify that all the conditions of Theorem 2.2 are satisfied for the mapping $g : S(u) \times S(u) \times S(u) \to 2^{\mathbb{R}}$. Therefore $T_u = \{y \in S(u) : \inf g(y, y, w) \leq 0$ for all $w \in S(u)\}$ is nonempty and compact. Define $T : X \to K(X)$ by $T(u) = T_u$. We shall now show that $T(u)$ is $\mathcal{M}$-convex, for each $u \in X$. Indeed, let $y_0, y_1, \ldots, y_n \in T(u)$, $\phi \in \mathcal{M}(\{y_0, \ldots, y_n\}$ and $z = \phi(\Sigma_{i=0}^n \lambda_i e_i) \in \phi(\Delta_n)$, where $\lambda_0, \ldots, \lambda_n \geq 0$ with $\Sigma_{i=0}^n \lambda_i = 1$.

We claim that $z \in T(u)$.

Notice that $z \in S(u)$ by the assumption of $\mathcal{M}$-convexity of $S(u)$. Let $w \in S(u)$ and let $C_w = \{y \in S(u) : \inf g(y, w, y) \geq 0\}$. Then, by Lemma 1.1(a) and (ii), we have $y_i \in C_w$ for each $i \in \{0, 1, \ldots, n\}$. Also, C_w is $\mathcal{M}$-convex by (iii) and Lemma 4.1.3; so it follows that $z \in C_w$, i.e. $\inf\ g(z, w, z) \geq 0$. And this inequality would hold for any $w \in S(u)$ so that we have:

$$\inf\ g(z, w, z) \geq 0, \text{ for all } w \in S(u). \tag{1}$$

So if we define $f : S(u) \times S(u) \to 2^{\mathbb{R}}$ by $f(x, w) = g(z, x, w)$, then by (1) above, (i), (ii), (iii), all the conditions of Lemma 4.1.1 are satisfied when taking $x_0 = z$ and we have $\inf f(z, w) = \inf g(z, z, w) \leq 0$ for all $w \in S(u)$; i.e. $z \in T(u)$. Therefore $T(u)$ is $\mathcal{M}$-convex.

Next we shall prove that T is USC by showing that it has a closed graph. Let $(u_\alpha, y_\alpha)_{\alpha \in \aleph}$ be a net in $X \times X$ such that $y_\alpha \in T(u_\alpha)$ for each α, $y_\alpha \to y_0$, and $u_\alpha \to u_0$. We shall show that $y_0 \in T(u_0)$.

Notice that $y_0 \in S(u_0)$ by USC of S. So it only remains to show that $\inf g(y_0, y_0, w) \leq 0$ for all $w \in S(u_0)$.

Let $w \in S(u_0)$. Then by LSC of S, there exists a net $(w_\alpha)_{\alpha\in\Delta}$ such that $w_\alpha \to w$ and $w_\alpha \in S(u_\alpha)$ for each α. Thus we have $\inf g(y_\alpha, y_\alpha, w_\alpha) \leq 0$ for all α. By (ii), it follows that $\inf g(y_\alpha, w_\alpha, y_\alpha) \geq 0$ for all α. Let $C = \{(y, w) \in X \times X : \inf g(y, w, y) \geq 0\}$. Then C is closed by (v) and Lemma 2. Moreover, $(y_\alpha, w_\alpha) \in C$ for all α, so it follows that:

$$(y_0, w) \in C, \text{ i.e. } \inf g(y_0, w, y_0) \geq 0. \tag{2}$$

Since (2) holds for each $w \in S(u_0)$, it follows by (i), (ii), (iii) and Lemma 4.1.1 (b) that $\inf g(y_0, y_0, w) \leq 0$ for all $w \in S(u_0)$, i.e., $y_0 \in T(u_0)$.

This shows that $T : X \to 2^X$ is USC with nonempty compact $\mathcal{M}$-convex values. Therefore all the conditions of Theorem 1.2.7 are satisfied and hence there exists $\widehat{y} \in X$ such that $\widehat{y} \in T(\widehat{y})$, i.e., $\widehat{y} \in S(\widehat{y})$ and $\inf g(\widehat{y}, \widehat{y}, w) \leq 0$ for all $w \in S(\widehat{y})$.

To complete the proof, it only remains to show that W is closed. So let $(y_\alpha)_{\alpha\in\aleph}$ be a net in W such that $y_\alpha \to y_0$. Then $(y_\alpha, y_\alpha) \in \text{graph } T$ for each $\alpha \in \aleph$, and since T has a closed graph by the argument above, $(y_0, y_0) \in \text{graph } T$. Thus $y_0 \in W$. $\square$

CHAPTER FIVE

SOME MAXIMAL ELEMENTS IN PRODUCT SPACES WITH APPLICATIONS TO GENERALIZED GAMES AND MINIMAX INEQUALITIES

In this chapter we give some maximal element theorems and fixed point theorems for GL_S -majorized correspondences in product G-convex spaces. The G-convex structure for the product, as we study it here is the one we defined in Theorem 1.3.1.

L_S-majorized correspondences were defined in [DTY], where some maximal and fixed point theorems for these correspondences were given. Here we generalize this concept to G-convex spaces and use the notation GL_S-majorized correspondences. With the help of a fixed point theorem of Tan and Zhang (Theorem 3.1 in [TZ]), in section 1 we are able to give generalizations of those maximal element and fixed point theorems of [DTY].

For the G-convex spaces studied throughout this chapter, we assume a so-called compact G-polytope property (Definition 5.1.1 (c) below). As in [DTY], maximal elements and fixed point theorems are given for both compact and noncompact domains.

In section 2, a new maximal element theorem is given, which we prove by using the G-convex generalization of Theorem 7 in [DTY] i.e., Corollary 5.1.11.

In section 3, we apply the results of the previous sections to obtain equilibrium existence theorems for generalized games.

In section 4, applications to minimax inequalities are given.

1. Maximal Elements for GL_S majorized Correspondences.

The following definition generalizes the concept of class L_S and of class M_S mappings in [DTY] to G-convex spaces.

Definition 5.1.1. Let X be a topological space and I be an index set. For each $i \in I$, let (Y_i, Γ_i) be a G-convex space. Let $Y = \Pi_{i \in I} Y_i$ be the product G-convex space and $S : Y \to X$ be a single-valued map. For each $i \in I$, let $A_i : X \to 2^{Y_i}$. Then

(a) The family $\{A_i\}_{i \in I}$ is said to be of class GL_S (or $\{A_i\}_{i \in I} \in GL_S(X, Y_i)_{i \in I}$) if for each $i \in I$,

(1) $A_i(x)$ is G-convex, for each $x \in X$;

(2) $A_i^{-1}(y_i)$ is open in X, for each $y_i \in Y_i$;

(3) $y_i \notin A_i(S(y))$, for each $y \in Y$.

(b) $\{A_i\}_{i \in I} \in GM_S(X, Y_i)_{i \in I}$ (or the family $\{A_i\}_{i \in I}$ is GL_S-majorized) if for each $i \in I$ and for each $x \in X$, there exists an open neighbourhood N_x of x in X and a mapping $B_x : X \to 2^{Y_i}$ such that

(1) B_x has G-convex values;

(2) $B_x^{-1}(y)$ is open in X for each $y_i \in Y_i$;

(3) $y_i \notin B_x(S(y))$ for each $y \in Y$;

(4) $A_i(z) \subset B_x(z)$ for each $z \in N_x$.

(c) A G-convex space (Y, Γ) is said to satisfy the compact G-polytope property iff whenever A is a compact G-convex subset of Y and $y_1, \ldots, y_n \in Y$ then G-co$(A \cup \{y_1, \ldots, y_n\})$ is contained in a compact G-convex subset of Y.

The following Lemma generalizes Theorem 3.2 in [TZ].

Lemma 5.1.2. *Let X be a compact topological space. Let (Y, Γ) be a G-convex* *space satisfying the compact G-polytope property. Let $f : Y \to X$ be continuous and $S, T : X \to 2^Y$ satisfy*

(a) $S(x) \subset T(x)$ for each $x \in X$;

(b) $S^{-1}(y)$ is open in X for each $y \in Y$;

(c) $T(x)$ is G-convex for each $x \in X$;

(d) $y \notin T(f(y))$ for each $y \in Y$.

Then there exists $x \in X$ such that $S(x) = \emptyset$.

Proof. Assume $S(x) \neq \emptyset$, for all $x \in X$. Consider the cover $\{S^{-1}(y) : y \in Y\}$. By compactness of X, there exist $y_1, \dots, y_n \in Y$ such that

$$X \subset \cup_{i=1,\dots,n} S^{-1}(y_i) \tag{*}$$

Let Y_0 be a compact G-convex subset of Y containing G-co($\{y_1, \dots, y_n\}$). Define $F, G : Y_0 \to 2^{Y_0}$ by:

$$F(y) = S(f(y)) \cap Y_0 \text{ and } G(y) = T(f(y)) \cap Y_0$$

. Obviously we have:

(1) $F(y) \subset G(y)$, for all $y \in Y_0$.

(2) $G(y)$ is G-convex, for all $y \in Y_0$.

(3) $F^{-1}(y) = f^{-1}(S^{-1}(y)) \cap Y_0$ is open in Y.

(4) $F(y) \neq \emptyset$, for all $y \in Y_0$: For let $y \in Y_0$; then $f(y) \in X$ and by (*) there exists $i \in I$ such that $f(y) \in S^{-1}(y_i)$; i.e. $y_i \in S(f(y))$. But $y_i \in Y_0$; hence $y_i \in F(y)$.

Now by Theorem 3.1 in [TZ], there exists $\widehat{y} \in Y$ such that $\widehat{y} \in G(\widehat{y})$.i.e. $\widehat{y} \in T(f(\widehat{y}))$ which is a contradiction to (d). □

Lemma 5.1.3. *Let X be a paracompact Hausdorff topological space. Let (Y, Γ) be a G-convex space. Let $S : Y \to X$ be continuous. Let $A : X \to 2^Y$ be GL_S majorized.*

For each $x \in X$, let $\phi_x : X \to 2^Y$ be a GL_S majorant of A at x i.e. there exists an open neighbourhood of x, N_x, such that:

(i) $A(z) \subset \phi_x(z)$, for all $z \in N_x$;

(ii) $\phi_x^{-1}(y)$ is open for each $y \in Y$;

(iii) $y \notin \phi_x(S(y))$ for each $y \in Y$.

(iv) $\phi_x(z)$ is G-convex, for each $z \in X$.

Then there exists $B : X \to 2^Y$ such that

(I) B is of class GL_S and (II) $A(x) \subset B(x) \subset \phi_x(x)$, for each $x \in X$.

Proof. Since X is regular, there exists an open neighbourhood V_x of x in X such that $cl(V_x) \subset N_x$. Let $\{W_x : x \in X\}$ be a locally finite refinement of $\{V_x : x \in X\}$. Then for each $x \in X$ define

$$\grave{\phi}_x(z) = \begin{cases} \phi_x(z), & \text{if } z \in cl(W_x) \\ Y, & \text{if } z \in X \backslash cl(W_x) \end{cases}$$

Then we shall show that

(a) $\grave{\phi}_x^{-1}(y)$ is open, $\forall y \in Y$.

Indeed $\grave{\phi}_x^{-1}(y) = [X \backslash cl(W_x)] \cup [\phi_x^{-1}(y) \cap cl(W_x)] = [X \backslash cl(W_x)] \cup \phi_x^{-1}(y)$ which is open.

(b) $\grave{\phi}_x(z)$ is G-convex, $\forall z \in X$.

Now define

$$B : X \to 2^Y \text{ by } : B(z) = \bigcap_{x \in X} \grave{\phi}_x(z). \qquad (*)$$

Then it can be verified that

(c_1) $y \notin B(S(y))$, for all $y \in Y$.

(c_2) $B(z)$ is G-convex, for all $z \in X$.

(c_3) $A(x) \subset B(x)$, for all $x \in X$.

(c_4) $B^{-1}(y)$ is open, for all $y \in Y$: By using the local finite property of the family $\{W_x; x \in X\}$, this can be proved in a way similar to the proof of Lemma 2 of Ding and Tan [3, p.230-232] or see the proof of Lemma 3.2.2 of Yuan [4, p.94-95].

By $(c_1), (c_2), (c_3)$ and (c_4) above, (I) is true.

Moreover, For any $x, z \in X$ we have $A(z) \subset \dot{\phi}_x(z)$, hence it follows by (*) that $A(z) \subset B(z)$, for all $z \in X$. Also by (*) $B(x) \subset \dot{\phi}_x(x)$. Hence (II) follows since $\dot{\phi}_x(x) = \phi_x(x)$. □

Theorem 5.1.4. *Let X be a compact Hausdorff topological space. Let (Y, Γ) be a Hausdorff G-convex space satisfying the compact G-polytope property. Let $S : Y \to X$ be continuous. Let $A : X \to 2^Y$ be GL_S majorized. Then there exists $\widehat{x} \in X$ such that $A(\widehat{x}) = \emptyset$.*

Proof. By Lemma 2, there exists $B : X \to 2^Y$ such that $B \in GL_S(X, Y)$ and $A(x) \subset B(x)$, for all $x \in X$. Now, applying Lemma 1 and taking $S = T = B$ and $f = S$, there exists $\widehat{x} \in X$ such that $B(\widehat{x}) = \emptyset$. Hence $A(\widehat{x}) = \emptyset$. □

Theorem 5.1.4 above generalizes Theorem 1 of [DTY] to G-convex spaces. As an immediate consequence, we obtain generalizations of Theorem 2, Theorem 3, Lemma 4, Theorem 6, Theorem 7 and Theorem 8 of [DTY] to G-convex spaces. We state these as Corollary 5.1.5, Corollary 5.1.7, Lemma 5.1.9, Corollary 5.1.10, Corollary 5.1.11 and Corollary 5.1.12, respectively. We note that the proofs of these generalizations are mere modifications of the analogous results in [DTY].

Corollary 5.1.5. *Let X be a Hausdorff topological space, (Y, Γ) be a Hausdorff G-convex space with the compact G-polytope property, $S : Y \to X$ be a continuous compact map and $A \in GM_S(X, Y)$. Then there exists $\widehat{x} \in X$ such that $A(\widehat{x}) = \emptyset$.*

Proof. Let X_0 be a compact subset of X containing $S(Y)$. Then the restriction of A to X_0 satisfies all the conditions of Theorem 5.1.4. Therefore there exists

$\widehat{x} \in X_0$ such that $A(\widehat{x}) = \emptyset$. □

Corollary 5.1.7. *Let X be a Hausdorff topological space. Let I be an index set. For each $i \in I$, let (Y_i, Γ_i) be a Hausdorff G-convex space with the compact G-polytope property. Let $Y = \Pi_{i \in I} Y_i$ be the product G-convex space and assume it satisfies the compact G-polytope property. Let $S : Y \to X$ be a continuous and compact single valued map. Let $\{A_i\}_{i \in I} \in GM_S(X, Y_i)_{i \in I}$. Assume that $\bigcup_{i \in I}\{x \in X : A_i(x) \neq \emptyset\} = \bigcup_{i \in I} int(\{x \in X : A_i(x) \neq \emptyset\}$. Then there exists $\widehat{x} \in X$ such that $A_i(\widehat{x}) = \emptyset$, for all $i \in I$.*

Proof. For each $i \in I$, define $\grave{A}_i : X \to 2^Y$ by $\grave{A}_i(x) = \pi_i^{-1}(A_i(x))$. For each $x \in X$, let $I(x) = \{i \in I : A_i(x) \neq \emptyset\}$.

If there exists $x \in X$ such that $I(x) = \emptyset$, then we have nothing to prove. So we assume that $I(x) \neq \emptyset$, for all $x \in X$. Define $A : X \to 2^Y$ by $A(x) = \bigcap_{i \in I(x)} \grave{A}_i(x)$. In the following we will show that A is GL_S-majorized.

Let $x_0 \in X$. Then by our assumption that $I(x) \neq \emptyset$, for all $x \in X$; there exists $j \in I$ such that $A_j(x) \neq \emptyset$. Since $A_j : X \to 2^{Y_j}$ is GL_S-majorized, there exists an open nhood N_{x_0} of x_0 in X and a mapping $\phi_{x_0} : X \to 2^{Y_j}$ such that

(i) ϕ_{x_0} has G-convex values;

(ii) $\phi_{x_0}^{-1}(y_j)$ is open for each $y_j \in Y_j$;

(iii) $y_j \notin \phi_{x_0}(S(y))$, for all $y \in Y$;

(iv) $A_j(z) \subset \phi_{x_0}(z)$, for all $z \in N_{x_0}$.

Without loss of generality we may assume that $N_{x_0} \subset \mathrm{int}(\{x \in X : A_j(x) \neq \emptyset\})$.

Let $\psi_{x_0} : X \to 2^Y$ be defined by $\psi_{x_0}(x) = \pi_j^{-1}(\phi_{x_0}(x))$. We shall show that ψ_{x_0} is a GL_S-majorant of A.

(I) ψ_{x_0} has G-convex values by (i) because $\pi_j^{-1}(C)$ is G- convex whenever C is G-convex.

(II) $\psi_{x_0}^{-1}(y) = \{z \in X : y \in \psi_{x_0}(z) = \pi_j^{-1}(\phi_{x_0}(z)\} = \{z \in X : y_j \in \phi_{x_0}(z)\} = \phi_{x_0}^{-1}(z)$ which is open in X by (ii).

(III) $y \notin \psi_{x_0}(S(y))$, for all $y \in Y$. Indeed assume the contrary. Then there exists $\bar{y} \in Y$ such that $\bar{y} \in \psi_{x_0}(S(\bar{y}) = \pi_j^{-1}(\phi_{x_0}(S(\bar{y})))$. But then $\bar{y}_j \in \phi_{x_0}(S(\bar{y}))$, which is a contradiction to (iii).

(IV) $A(z) \subset \psi_{x_0}(z)$, for all $z \in N_{x_0}$. Since $A_j(z) \neq \emptyset$, for all $z \in N_{x_0}$, it follows that

$$A(z) \subset \grave{A}_j(z) = \pi_j^{-1}(A_j(z)), \text{ for all } z \in N_{x_0}.$$

But by (iv), for each $z \in N_{x_0}$, we have $\pi_j^{-1}(A_j(z)) \subset \pi_j^{-1}(\phi_{x_0}(z)) = \psi_{x_0}(z)$. Therefore $A(z) \subset \psi_{x_0}(z)$, for all $z \in N_{x_0}$.

By (I), (II), (III), and (IV), it follows that ψ_{x_0} is a GL_S- majorant of A. Thus $A \in GM_S(X, Y)$. Thus all the conditions of Theorem 5.1.4 are satisfied for A and therefore there exists $\widehat{x} \in X$ such that $A(\widehat{x}) = \emptyset$, *i.e.* $A_i(\widehat{x}) = \emptyset$,for all $i \in I$ so that $I(\widehat{x}) = \emptyset$, which is a contradiction. □

Definition 5.1.8. Let X be a topological space. Let (Y_i, Γ_i) be a G-convex space for each i in an index set I. Let $A_i : X \to 2^{Y_i}$. Then $\{A_i\}_{i \in I}$ is in class GKF (or $\{A_i\}_{i \in I} \in GKF(X, Y_i)_{i \in I}$) iff:

(1) $A_i(x)$ is G-convex for each $x \in X$;

(2) $A_i^{-1}(y_i)$ is open in X for each $i \in I$ and each $y_i \in Y_i$;

(3) For each $x \in X$ there exists $i \in I$ such that $A_i(x) \neq \emptyset$.

Lemma 5.1.9. *Let X be a compact topological space, and I be an index set. For each $i \in I$, let (Y_i, Γ_i) be a Hausdorff G-convex space with the compact G-polytope property. Let $\{A_i\}_{i \in I} \in GKF(X, Y_i)_{i \in I}$. Then there exists a subset $C = \Pi_{i \in I} C_i$ of $Y = \Pi_{i \in I} Y_i$ such that for each $x \in X$ there exists $i \in I$ such that $A_i(x) \cap C_i \neq \emptyset$. Moreover for all but finitely many $i \in I$, C_i is a singleton and for those finitely many indices where C_i is not a singleton, it is compact and G-convex.*

Proof. Consider $\{A_i^{-1}(y_i) : y_i \in Y_i \text{ and } i \in I\}$. Since for each $x \in X$, there exists $i \in I$ such that $A_i(x) \neq \emptyset$, it follows that this collection of open subsets of X covers X. By compactness of X, there exists a finite subset J of I and for each $j \in J$ a finite subset $W_j = \{y_j^1, y_j^2, \dots, y_j^{m_j}\}$ of Y_j such that

$$X = \bigcup_{j \in J} \bigcup_{i=1}^{m_j} A_j^{-1}(y_j^i).$$

Fix $y^0 = (y_i^0)_{i \in I}$. And for each $j \in J$, let C_j be a compact G- convex subset of Y_j containing G-co(W_j). If $j \notin J$, let $C_j = \{y_j^0\}$. Then $C = \Pi_{j \in I} C_j$ is the required set. □

Corollary 5.1.10. *Let X be a Hausdorff topological space. Let I be an index set. For each i let (Y_i, Γ_i) be a Hausdorff G-convex space with the compact G-polytope property. Let $Y = \Pi_{i \in I} Y_i$ and assume it satisfies the compact G-polytope property. Let $S : X \to Y$ be continuous and compact. Assume $\{A_i\}_{i \in I} \in GKF(X, Y_i)_{i \in I}$. Then there exists $\widehat{y} \in Y$ and $i_0 \in I$ such that $\widehat{y}_{i_0} \in A_{i_0}(S(\widehat{y}))$.*

Proof. Assume the contrary. Then for each $y \in Y$ and $i \in I$ we have $y_i \notin A_i(S(y))$. Thus the family $\{A_i\}_{i \in I} \in GL_S(X, Y_i)_{i \in I}$. Moreover $\{x \in X : A_i(x) \neq \emptyset\} = \bigcup_{y_i \in Y_i} A_i^{-1}(y_i)$ is open so that the family $\{A_i\}_{i \in I}$ satisfies all the conditions of Corollary 5.1.7. Thus there exists $\widehat{x} \in X$ such that $A_i(\widehat{x}) = \emptyset$, for all $i \in I$, which contradicts the assumption that $\{A_i\}_{i \in I} \in GKF(X, Y_i)_{i \in I}$. □

Corollary 5.1.11. *Let X be a Hausdorff topological space and (Y_i, Γ_i) be a Hausdorff G-convex space with the compact G-polytope property for each i in an index set I. Let $Y = \Pi_{i \in I} Y_i$ and assume it satisfies the compact G-polytope property. Let $S : Y \to X$ be continuous. Let $\{A_i\}_{i \in I} \in GL_S(X, Y_i)_{i \in I}$. Assume there exist a nonempty compact subset K of X and a nonempty compact G-convex subset C_i of Y_i for each $i \in I$ such that for each $x \in X \backslash K$ there exists $i \in I$ such that $A_i(x) \cap C_i \neq \emptyset$. Then there exists $\widehat{x} \in K$ such that $A_i(\widehat{x}) = \emptyset$, for all $i \in I$.*

Proof. Assume the contrary. Then for each $x \in K$, there exists $\in I$ such that

$A_i(x) \neq \emptyset$. Applying Lemma 1.9 to K and $\{A_i|_K\}_{i\in I}$, it follows that there exists a subset $D = \Pi_{i\in I} D_i$ of Y having the property that for each $x \in K$, there exists $i \in I$ such that $A_i(x) \cap D_i \neq \emptyset$. Moreover the D_i's are either singletons or G-convex hulls of finite sets, also being singletons for all but finitely many indices.

Now for each $i \in I$, since Y_i satisfies the compact G-polytope property, there exists a nonempty compact G-convex subset H_i of Y_i such that $D_i \cup C_i \subset H_i$. Let $H = \Pi_{i\in I} H_i$. Then H is a compact G-convex subset of Y. Let $X_0 = S(H)$. It also follows that X_0 is compact. Let $S_1 = S|_H : H \to X_0$.

We notice that for each $x \in X$, if $x \in K$ then $\emptyset \neq A_i(x) \cap D_i \subset A_i(x) \cap H_i$. And if $x \notin K$, then $\emptyset \neq A_i(x) \cap C_i \subset A_i(x) \cap H_i$. Thus

$$A_i(x) \cap H_i \neq \emptyset, \text{ for all } x \in X.$$

Now for each $i \in I$, we define $\grave{A}_i : X_0 \to 2^{H_i}$ by

$$\grave{A}_i(x) = A_i(x) \cap H_i.$$

It is easy to see that $\{\grave{A}_i\}_{i\in I} \in GKF(X_0, H_i)_{i\in I}$. Applying Corollary 5.1.10, it follows that there exists $y \in H$ and $i_0 \in I$ such that $y_{i_0} \in \grave{A}_{i_0}(S_1(y)) = A_{i_0}(S(y)) \cap H_{i_0} \subset A_{i_0}(S(y))$, which is a contradiction to the assumption that $\{A_i\}_{i\in I} \in GL_S(X, Y_i)_{i\in I}$. This shows that there exists $\widehat{x} \in K$ such that $A_i(\widehat{x}) = \emptyset$, for all $i \in I$. □

Corollary 5.1.12. *Let X be a paracompact Hausdorff topological space and (Y_i, Γ_i) be a Hausdorff G-convex space with the compact G-polytope property for each i in an index set I. Let $Y = \Pi_{i\in I} Y_i$ and assume it satisfies the compact G-polytope property. Let $S : Y \to X$ be continuous. Let $\{A_i\}_{i\in I} \in GM_S(X, Y_i)_{i\in I}$. Assume there exist a nonempty compact subset K of X and a nonempty compact G-convex subset C_i of Y_i for each $i \in I$ such that for each $x \in X\backslash K$ there exists $i \in I$ such that $A_i(x) \cap C_i \neq \emptyset$. Then there exists $\widehat{x} \in K$ such that $A_i(\widehat{x}) = \emptyset$, for all $i \in I$.*

Proof. For each $i \in I$, (by a slight modification of Lemma 5.1.3), it is possible to construct $B_i : X \to 2^{Y_i}$ such that

(I) $A_i(x) \subset B_i(x)$, for all $x \in X$;

(II) $\{B_i\}_{i\in I} \in GL_S(X, Y_i)_{i\in I}$.

Now applying Corollary 5.1.11 to the family $\{B_i\}_{i\in I}$, we obtain $\widehat{x} \in K$ such that $B_i(\widehat{x}) = \emptyset$, for all $i \in I$, which implies that $A_i(\widehat{x}) = \emptyset$, for all $i \in I$. □

2. A Maximal Element Theorem.

Applying Corollary 5.1.11, we give the following maximal element theorem for the case when X and S are not compact. X is a G-convex space in this theorem.

Theorem 5.2.1. *Let (X, Λ) be a Hausdorff G-convex space satisfying the compact G-polytope property. Let I be an index set. Let (Y_i, Γ_i) be a Hausdorff G-convex space satisfying the compact G-polytope property for each $i \in I$. Let $Y = \Pi_{i\in I} Y_i$ and assume it satisfies the compact G-polytope property. Let $S : Y \to X$ be continuous. Let $\{A_i\}_{i\in I} \in GL_S(X, Y_i)_{i\in I}$. Assume there exist a nonempty compact subset K of X and a continuous function $h : X \times X \to X$ such that*

(i) $A_i(h(x_1, x_2)) \subset A_i(x_1)$, for each $(x_1, x_2) \in X \times X$ and each $i \in I$,

(ii) $h(X \times K) \subset K$,

(iii) K contains a nonempty compact G-convex subset $\grave{K}$.

Then there exists $\widehat{x} \in K$ such that $A_i(\widehat{x}) = \emptyset$, for each $i \in I$.

Proof. Let $J = I \cup \{I\}$. Let $Y_I = X$.

Define $A_I : X \to 2^{Y_I}$ by

$$A_I(x) = \begin{cases} \grave{K}, & \text{if } x \in X \backslash K, \\ \emptyset, & \text{if } x \in K. \end{cases}$$

Then A_I has G-convex values and

$$A_I^{-1}(x) = \begin{cases} X \backslash K, & \text{if } x \in \grave{K}, \\ \emptyset, & \text{if } x \notin \grave{K}, \end{cases}$$

so that $A_I^{-1}(x)$ is open for each $x \in X$.

Let $\grave{Y} = \Pi_{j\in J} Y_j$. Define $\grave{S} : \grave{Y} \to X$ by:

$\grave{S}((y_j)_{j\in J}) = h(S(y_j)_{j\in I}, y_I)$.Then it can be shown that $\grave{S}$ is continuous. Next we shall show that the family $\{A_j\}_{j\in J}$ satisfies all conditions of Corollary 5.1.11.

(I)$\{A_j\}_{j\in J} \in GL_{\grave{S}}(X, Y_j)_{j\in J}$. It suffices to prove that

$$y_j \notin A_j(\grave{S}(y)), \text{ for each } y \in \grave{Y}, \text{ and for each } j \in J. \qquad (*)$$

Let $y = (y_j)_{j\in J}$

Case 1. $j \in I$.

$A_j(\grave{S}(y)) = A_j(h(S((y_j)_{j\in I}, y_I))) \subset A_j(S((y_j)_{j\in I}))$ by condition (i). The latter does not contain y_j, because $\{A_j\}_{j\in I} \in L_S(X, Y_j)_{j\in I}$. Thus (*) holds.

Case 2. $j = I$.

We notice that $A_I(S(y_j)_{j\in J}) = \grave{K}$ or $\emptyset$. So if $y_I \notin K$, (*) obviously holds. Next assume that $y_I \in K$, then $S(y_j)_{j\in J} = h(S(y_j)_{j\in I}, y_I) \in K$ by (ii) and hence $A_I(S(y_j)_{j\in J}) = \emptyset$. Hence (*) holds.

(II) Let $C_j =$ G-co($\{y_j\}$) for any fixed choice $y_j \in Y_j$, if $j \in I$. And let $C_I = \grave{K}$.

Then for any $x \in X \backslash K$, we take $j = I \in J$ so that $C_I \cap A_I(x) = \grave{K} \cap \grave{K} \neq \emptyset$.

So by (I) and (II), all the conditions of Corollary 5.1.11 are satisfied so that there exists $\widehat{x} \in K$ such that $A_j(\widehat{x}) = \emptyset$, for each $j \in J$. It follows that $A_i(\widehat{x}) = \emptyset$, for each $i \in I$. □

Remark. *Condition (iii) on K in Theorem 5.2.1 above holds if X is a convex subset of a topological vector space or if X is a G-convex space with the compact G-polytope property and K contains G-co($\{x_1, \ldots, x_n\}$) for some $x_1, \ldots, x_n \in X$.*

Corollary 5.2.2. *Let (X, Λ) be a Hausdorff G-convex space. Let I be an index set. Let (Y_i, Γ_i) be a Hausdorff G-convex space satisfying the compact G-polytope* *property for each $i \in I$. Let $Y = \Pi_{i\in I} Y_i$ and assume it satisfies the compact G-polytope property. Let $S : Y \to X$ be continuous. Let $\{A_i\}_{i\in I} \in GL_S(X, Y_i)_{i\in I}$. Assume there exists $x_0 \in X$ such that $A_i(x_0) \subset A_i(x)$, for all $x \in X$ and for all $i \in I$. Then there exists $\widehat{x} \in G\text{-co}\{x_0\}$ such that $A_i(\widehat{x}) = \emptyset$, for all $i \in I$.*

Proof. Let $K = \text{G-co}\{x_0\}$. Define $h : X \times X \to X$ by $h(x_1, x_2) = x_0$ for each $x_1, x_2 \in X$. Then it is easy to see that all the conditions of Theorem 5.2.1 are satisfied. Hence there exists $\widehat{x} \in K$ such that $A_i(\widehat{x}) = \emptyset$, for all $i \in I$. □

Lemma 5.2.3. *Let X be a paracompact Hausdorff topological space and (Y, Γ) be a G-convex space with the compact G-polytope property. Let $S : Y \to X$ be continuous and $A : X \to 2^Y$ be GL_S majorized. Let $h : X \times X \to X$ be continuous such that*

For any $x_1, x_2 \in X$ and $x_0 = h(x_1, x_2)$ there exists a GL_S majorant ϕ_{x_0} of A at x_0 satisfying the condition

$$\phi_{x_0}(x_0) \subset A(x_1) \cap A(x_2). \qquad (*)$$

Then there exists $B : X \to 2^Y$ such that:

(I) B is of class GL_S;

(II) $A(x) \subset B(x)$, for all $x \in X$;

(III) $B(h(x_1, x_2)) \subset B(x_1) \cap B(x_2)$.

Proof. By Lemma 5.1.3, it follows that there exists $B : x \to 2^Y$ that satisfies (I) and (II).

Also by Lemma 1.3, $B(h(x_1, x_2) = B(x_0) \subset \phi_{x_0}(x_0)$.

So it follows from (*) that $B(h(x_1, x_2)) \subset A(x_1) \cap A(x_2)$.

Since by (II) we have $A(x_1) \cap A(x_2) \subset B(x_1) \cap B(x_2)$, (III) follows. □

Theorem 5.2.4. *Let (X, Λ) be a paracompact G-convex Space. Let (Y_i, Γ_i) be a G-convex space with the compact G-polytope property for each i in an index set*

I. Let $Y = \Pi_{i\in I}Y_i$ and assume it satisfies the compact G-polytope property. Let $S : Y \to X$ be continuous. Let $\{A_i\}_{i\in I} \in GM_S(X, Y_i)_{i\in I}$. assume there exist a continuous $h : X \times X \to X$ and a nonempty compact G-convex subset K of X such that:

(i) $h(X \times K) \subset K$;

(ii) For any $x = h(x_1, x_2)$ in X and for any $i \in I$, there exists a GL_S majorant ϕ_x of A_i at x such that $\phi_x(x) \subset A_i(x_1) \cap A_i(x_2)$.

Then there exists $\widehat{x} \in K$ such that $A_i(\widehat{x}) = \emptyset$, for all $i \in I$.

Proof. By Lemma 5.2.3, there exists $B_i : X \to 2^Y$ such that (1) $A_i(x) \subset B_i(x)$, (2) B_i is of class GL_S and (3) $B_i(h(x_1, x_2)) \subset B_i(x_1) \cap B_i(x_2)$.

It then follows that the family $\{B_i\}_{i\in I}$ satisfies all conditions of Theorem 5.2.1 and hence there exists $\widehat{x} \in X$ such that $B_i(\widehat{x}) = \emptyset$, for all $i \in I$, hence $A_i(\widehat{x}) = \emptyset$, for all $i \in I$. □

3. Generalized Games.

The following theorem is an application of Corollary 5.1.12. It gives an equilibrium point for the generalized game $(Y_i; A_i; P_i)_{i\in I}$ when $Y = \Pi_{i\in I}Y_i$ is paracompact where each (Y_i, Γ_i) is a G-convex space.

Theorem 5.3.1. *Let X be a Hausdorff topological space. Let I be an index set. For each $i \in I$, let (Y_i, Γ_i) be a G-convex space satisfying the compact G-polytope property.*

Let $Y = \Pi_{i\in I}Y_i$ be paracompact and assume it satisfies the compact G-polytope property. Assume $S : Y \to X$ is continuous. Let $A_i, P_i : X \to 2^{Y_i}$. Assume there exist a nonempty compact subset K of Y and for each $i \in I$ a nonempty compact G-convex subset C_i of Y_i such that

(1) for each $y \in Y\backslash K$, there exists $i \in I$ such that $A_i(S(y)) \cap P_i(S(y)) \cap C_i \neq \emptyset$;

(2) the family $\{(A_i \cap P_i)\}_{i\in I} \in GM_S(X, Y_i)_{i\in I}$*;*

(3) clA_i *is upper semicontinuous, for each* $i \in I$*;*

(4) $A_i^{-1}(y)$ *is open in* X*, for each* $y \in Y_i$*;*

(5) $A_i(x)$ *is nonempty and G-convex for each* $x \in X$*.*

Then there exists $\widehat{y} \in K$ *such that:*

(i) $A_i(S(\widehat{y})) \cap P_i(S(\widehat{y})) = \emptyset$*, for all* $i \in I$*, and*

(ii) $\widehat{y}_i \in clA_i(S(\widehat{y}))$*, for each* $i \in I$*.*

Proof. For each $i \in I$, let $F_i = \{y \in Y : y_i \in cl(A_i(S(y))\}$. Clearly, F_i is closed by upper semicontinuity of clA_i. Define $Q_i(y) : Y \to 2^{Y_i}$ by

$$Q_i(y) = \begin{cases} A_i(S(y)) \cap P_i(S(y)), & \text{if } y \in F_i; \\ A_i(S(y)), & \text{if } y \notin F_i. \end{cases}$$

Then we will show that the family $\{Q_i\}_{i\in I}$ satisfies all conditions of Corollary 5.1.12 with $Y = X$, and $S = I_Y$.

(I) $\{Q_i\}_{i\in I} \in M_{I_Y}(Y, Y_i)_{i\in I}$.

Let $w \in Y$.

Case 1. $w \in F_i$.

Let x=S(w). Then by (1), there exists an open neighbourhood N_x of x and an L_S majorant B_x of $A_i \cap P_i$ at x.

Let $\Psi_w : Y \to 2^{Y_i}$ be defined by

$$\Psi_w(y) = \begin{cases} B_x(S(y)) \cap A_i(S(y)), & \text{if } y \in F_i; \\ A_i(S(y)), & \text{if } y \notin F_i. \end{cases}$$

Then Ψ_w is an L_{I_Y} majorant of Q_i since

(1) Ψ_w has convex values;

(2) $y_i \notin \Psi_w(S(y))$,for each $y \in Y$: If $y \in F_i$ then $\Psi_w(y) \subset B_x(S(y))$. Moreover, $y_i \notin B_x(S(y))$, because B_x is an L_S majorant of $A_i \cap P_i$. Hence $y_i \notin \Psi_w(y)$. On the other hand, if $y \notin F_i$, then by definition of F_i, $y_i \notin A_i(S(y))$. Hence (2) obviously holds in this case also.

(3) If $y_i \in Y_i$, then $\Psi_w^{-1}(y_i)$ is open in X. Indeed,

$$\begin{aligned}\Psi_w^{-1}(y_i) &= [S^{-1}(A_i^{-1}(y_i)) \cap (Y \backslash F_i)] \cup [S^{-1}(A_i^{-1}(y_i)) \cap S^{-1}(B_x^{-1}(y_i)) \cap F_i] \\ &= [S^{-1}(A_i^{-1}(y_i)) \cap (Y \backslash F_i)] \cup [S^{-1}(A_i^{-1}(y_i)) \cap S^{-1}(B_x^{-1}(y_i))],\end{aligned}$$

which is obviously open in X.

(4) $Q_i(y) \subset \Psi_w(y)$, for each y in the open neighbourhood $S^{-1}(N_x)$ of y.

Case 2. $w \notin F_i$.

Define

$$\Psi_w(y) = \begin{cases} A_i(S(y)), & \text{if } y \notin F_i; \\ \emptyset, & \text{if } y \in F_i. \end{cases}$$

Then $Q_i(y) \subset \Psi_w(y)$, for each y in the open set $Y \backslash F_i$. Obviously, Ψ_w is an L_{I_Y} majorant of Q_i at w.

(II) For each $y \in Y \backslash K$, there exists $i \in I$ such that $Q_i(y) \cap C_i \neq \emptyset$. We notice that $Q_i(y) \supset A_i(S(y)) \cap P_i(S(y))$. The refore (II) follows from (1).

By (I) and (II), all the conditions of Corollary 5.1.12 are satisfied and hence there exists $\widehat{y} \in Y$ such that $Q_i(\widehat{y})) = \emptyset$, for all $i \in I$. It follows that $\widehat{y}_i \in cl(A_i(S(\widehat{y}))$, for all $i \in I$, and that $(A_i \cap P_i)(S(\widehat{y})) = \emptyset$, for all $i \in I$. □

Corollary 5.3.2. *Let (Y_i, Γ_i) be a compact G-convex space with the compact G-polytope property, for each $i \in I$ in an index set I. Let $Y = \Pi_{i \in I} Y_i$ be the product G-convex space and assume it satisfies the compact G-polytope property. For each $i \in I$, let $A_i, P_i : Y \to 2^{Y_i}$ be such that*

(1) the family $\{(A_i \cap P_i)\}_{i \in I} \in GM_S(X, Y_i)_{i \in I}$;

(2) clA_i is upper semicontinuous, for each $i \in I$;

(3) $A_i^{-1}(y)$ *is open in* Y, *for each* $y \in Y_i$;

(4) $A_i(x)$ *is nonempty and G-convex, for each* $x \in X$.

Then there exists $\widehat{y} \in Y$ *such that:*

(i) $A_i(\widehat{y})) \cap P_i(\widehat{y})) = \emptyset$, *for all* $i \in I$ *and*

(ii) $\widehat{y}_i \in clA_i(\widehat{y}))$, *for each* $i \in I$.

Proof. Apply Theorem 5.2.1 with $X = K = Y$, $C_i = Y_i$ and $S = I_Y$, the conclusion follows. □

Applying Theorem 5.2.4, another equilibrium existence theorem is obtained.

Theorem 5.3.3. *Let* (Y_i, Γ_i) *be a G-convex space with the compact G-polytope property for each i in an index set I. Let* $Y = \Pi_{i \in I} Y_i$ *be the product G-convex space and assume it is paracompact and satisfies the compact G-polytope property. Let* $A_i, P_i : Y \to 2^{Y_i}$. *Let K be a nonempty compact G-convex subset of Y and h be a continuous function from* $Y \times Y$ *to* Y. *Assume that*

(1) $h(Y \times K) \subset K$ *and for any* $y_1, y_2 \in Y$ *and any* $i \in I$, *we have* $A_i(h(y_1, y_2)) \subset A_i(y_1) \cap P_i(y_1) \cap A_i(y_2) \cap P_i(y_2)$;

(2) the family $\{(A_i \cap P_i)\}_{i \in I} \in GM_{I_Y}(X, Y_i)_{i \in I}$;

(3) clA_i *is upper semicontinuous, for each* $i \in I$;

(4) $A_i^{-1}(y)$ *is open in* Y, *for each* $y \in Y_i$;

(5) $A_i(x)$ *is nonempty and G-convex, for each* $x \in Y$.

Then there exists $\widehat{y} \in K$ *such that:*

(i) $A_i(\widehat{y}) \cap P_i(\widehat{y}) = \emptyset$, *for all* $i \in I$ *and*

(ii) $\widehat{y}_i \in clA_i(\widehat{y})$, *for each* $i \in I$.

Proof. For each $i \in I$, let $F_i = \{y \in Y : y_i \in cl(A_i(y)\}$. Clearly, F_i is closed by upper semicontinuity of clA_i.

Define $Q_i(y) : Y \to 2^{Y_i}$ by

$$Q_i(y) = \begin{cases} A_i(y) \cap P_i(y), & \text{if } y \in F_i; \\ A_i(y), & \text{if } y \notin F_i. \end{cases}$$

Then we shall show that the family $\{Q_i\}_{i\in I}$ satisfies all conditions of Theorem 5.2.4, with $Y = X$, and $S = I_Y$.

(I) $\{Q_i\}_{i\in I} \in GM_{I_Y}(Y, Y_i)_{i\in I}$.

Let $w \in Y$.

Case 1. $w \in F_i$.

By (2), there exist an open neighbourhood N_w of w and a GL_{I_Y} majorant B_w of $A_i \cap P_i$ at w.

Let $\Psi_w : Y \to 2^{Y_i}$ be defined by

$$\Psi_w(y) = \begin{cases} B_x(y) \cap A_i(y), & \text{if } y \in F_i; \\ A_i(y), & \text{if } y \notin F_i. \end{cases}$$

Case 2. $w \notin F_i$. Define

$$\Psi_w(y) = \begin{cases} A_i(y), & \text{if } y \notin F_i; \\ \emptyset, & \text{if } y \in F_i. \end{cases}$$

Then in each case it can be verified that ψ_w is a GL_{I_Y} majorant of Q_i at w, (see the proof of Theorem 5.3.1).

(II) For any $y_1, y_2 \in Y$, let $y_0 = h(y_1, y_2)$. Let ψ_{y_0} be a majorant of Q_i at y_0 as defined above. Then we can show that ψ_{y_0} satisfies the following condition :

$$\psi_{y_0}(y_0) \subset Q_i(y_1) \cap Q_i(y_2). \qquad (*)$$

Indeed for any $w \in Y$, it follows from the definition of ψ_w that $\psi_w(y) \subset A_i(y)$. In particular $\psi_{w_0}(w_0) \subset A_i(w_0)$.

Moreover by (1), $\psi_{w_0}(w_0) \subset A_i(w_1) \cap P_i(w_1) \cap A_i(w_2) \cap P_i(w_2)$.

So by definition of Q_i, the right hand side above is contained in $Q_i(w_1) \cap Q_i(w_2)$. Hence (*) holds.

Now by (I) and (II), the conclusion of Theorem 2.3 holds and there exists $\widehat{y} \in K$ such that $Q_i(\widehat{y}) = \emptyset$, for all $i \in I$.

It follows that $\widehat{y}_i \in cl(A_i(\widehat{y}))$, for all $i \in I$ and $A_i(\widehat{y}) \cap P_i(\widehat{y}) = \emptyset$, for all $i \in I$. □

4. Minimax Inequalities.

As an application of Theorem 5.2.1, we have the following.

Theorem 5.4.1. *Let (X, Λ) be a G-convex space. Let I be an index set. Let (Y_i, Γ_i) be a G-convex space with the compact G-polytope property for each $i \in I$. Let $Y = \Pi_{i \in I} Y_i$ be the product G-convex space and assume it has the compact G-polytope property. Let $S : Y \to X$ be continuous. For each $i \in I$, let $f_i : X \times Y_i \to \mathbb{R}$ be such that:*

(i) for each fixed $x \in X, y_i \to f(x, y_i)$ is quasi-concave;

(ii) for each fixed $y_i \in Y_i, x \to f(x, y_i)$ is lower semi-continuous;

(iii) there exist a nonempty compact G-convex subset K of X and a continuous function $h : X \times X \to X$ such that

(1) $f_i(h(x_1, x_2), y_i) \leq \min\{f_i(x_1, y_i), f_i(x_2, y_i)\}$, for each $(x_2, x_2) \in X \times X$ and each $y_i \in Y_i$; and

(2) $h(X \times K) \subset K$.

Then we have

(A) For any real number λ, one of the following is true:

(a_1) there exists $y \in Y$ and $i \in I$ such that $f_i(S(y), y_i) > \lambda$.

(a_2) there exists $\widehat{x} \in K$ such that $\sup_{i \in I} \sup_{y_i \in Y_i} f_i(\widehat{x}, y_i) \leq \lambda$.

(B) The following minimax inequality holds:

$\inf_{x\in X} \sup_{i\in I} \sup_{y_i\in Y_i} f_i(x, y_i) \leq \sup_{i\in I} \sup_{y_i\in Y_i} f_i(S(y), y)$.

Proof. (A) Assume (a_1) does not hold.

For each $i \in I$, define $A_i : X \to 2_i^Y$ by $A_i(x) = \{y \in Y_i : f(x, y_i) \geq \lambda\}$. Then the family $\{A_i\}_{i\in I}$ satisfies all conditions of Theorem 5.2.1. Hence there exists $\widehat{x} \in K$ such that $A_i(\widehat{x}) = \emptyset$, for all $i \in I$, i.e. (a_2) of (A) holds.

(B) Take $\lambda_0 = \sup_{i\in I} \sup_{y\in Y} f_i(S(y), y)$. By applying (A), obviously (a_1) does not hold. Hence (a_2) is true, i.e., there exists $x_0 \in X$ such that

$$\sup_{i\in I} \sup_{y_i\in Y_i} f_i(x_0, y_i) \leq \lambda_0.$$

Therefore $\inf_{x\in X} \sup_{i\in I} \sup_{y_i\in Y_i} f_i(x, y_i) \leq \sup_{i\in I} \sup_{y\in Y} f_i(x, S(y))$. □

For the case when I is a singleton and $X = Y$ is a convex subset of a topological vector space and $S = I_Y$, we have the following corollary.

Corollary 5.4.2. *Let X be a nonempty convex subset of a Hausdorff topological vector space. Let $f : X \times X \to \mathbb{R}$ be such that*

(i) for each fixed $x \in X, y \to f(x, y)$ is quasi-concave;

(ii) for each fixed $y \in X, x \to f(x, y)$ is lower semicontinuous;

(iii) there exists a nonempty compact convex subset K of X and a continuous function $h : X \times X \to X$ such that: (1) $f(h(x_1, x_2), y) \leq \min\{f(x_1, y_i), f(x_2, y_i)\}$, for each $(x_2, x_2) \in X \times X$ and each $y \in Y$; and (2) $h(X \times K) \subset K$.

Then we have

(A) For any real number λ, one of the following is true

(a_1) there exists $x \in X$ such that $f(x, x) > \lambda$.

(a_2) there exists $\widehat{x} \in K$ such that $\sup_{y\in Y} f(\widehat{x}, y) \leq \lambda$.

(B) The following minimax inequality holds:

$\inf_{x\in X}\sup_{y\in Y} f(x,y) \le \sup_{y\in Y} f(x,x)$.

The following is a generalization of the concept of an L_S majorized family of functions in [DTY] to G-convex spaces.

Definition 5.4.3. Let X be a Hausdorff topological space. Let (Y_i, Γ_i) be a G-convex space for each i in an index set I. For each $i \in I$, let $f_i : X \times Y_i \to \mathbb{R}$. Then the family $\{f_i\}_{i\in I}$ is said to be GL_S majorized if the following conditions are satisfied for each $i \in I$:

For each $\lambda \in \mathbb{R}$, if there exists $(x, y_i) \in X \times Y_i$ such that $f_i(x, y_i) > \lambda$, then there exists a non-empty open nhood N_x of x in X and a real-valued function $f_i^x : X \times Y_i \to \mathbb{R}$ such that

(c_1) for each fixed $z \in X$, $y_i \to f_i^x(z, y_i)$ is quasi-concave;

(c_2) for each fixed $y_i \in Y_i$, $z \to f_i^x(z, y_i)$ is lower semicontinuous;

(c_3) $f_i(z, y_i) \le f_i^x(z, y_i)$ for each $z \in X$ and each $y_i \in Y_i$;

(c_4) $f_i^x(S(y), y_i) > \lambda$ implies that $f_i(S(y), y_i) > \lambda$ for each $y \in Y$.

The following is an analytic formulation of Theorem 5.2.4.

Theorem 5.4.4. *Let (X, Λ) be a paracompact G-convex space and I be an index set. Let (Y_i, Γ_i) be a G-convex space with the compact G-polytope property for each $i \in I$. Let $Y = \Pi_{i\in I} Y_i$ be the product G-convex space and assume it satisfies the compact G-polytope property. Let $S : Y \to X$ be continuous. For each $i \in I$, let $f_i : X \times Y_i \to \mathbb{R}$. Assume the family $\{f_i\}_{i\in I}$ is GL_S majorized. Assume that*

(I) there exists a nonempty compact G-convex subset K of X and a continuous function $h : X \times X \to X$ such that $h(X \times K) \subset K$;

(II) for any $x = h(x_1, x_2) \in X$, there is a GL_S majorant $f_i^x : X \times Y_i \to \mathbb{R}$ such that $f_i^x(x, y_i) \le \min\{f_i(x_1, y_i), f_i(x_2, y_i)\}$ for each $y_i \in Y_i$.

Then we have

(A) for any real number λ, *one of the following is true*

(a_1) *there exist* $y \in Y$ *and* $i \in I$ *such that* $f_i(S(y), y_i) > \lambda$;

(a_2) *there exists* $\widehat{x} \in K$ *such that* $\sup_{i\in I} \sup_{y_i\in Y_i} f_i(\widehat{x}, y_i) \leq \lambda$.

(B) the following minimax inequality holds:

$\inf_{x\in X} \sup_{i\in I} \sup_{y_i\in Y_i} f_i(x, y_i) \leq \sup_{i\in I} \sup_{y_i\in Y_i} f_i(S(y), y)$.

Proof. (A) Assume (a_1) does not hold. For each $i \in I$, define $A_i : X \to 2_i^Y$ by: $A_i(x) = \{y \in Y_i : f(x, y_i) > \lambda\}$. Then we shall show that the family $\{A_i\}_{i\in I}$ satisfies all conditions of Theorem 5.2.4. Indeed, the family $\{A_i\}_{i\in I}$ is GL_S majorized. Since for any $x \in X$, if $A_i(x) \neq \emptyset$, then (by the assumption that the family $\{f_i\}_{i\in I}$ is GL_S majorized) there exists a real-valued function $f_i^x : X \times Y_i \to \mathbb{R}$ satisfying $(c_1), \ldots, (c_4)$. Define $\phi_x : X \to 2^{Y_i}$ by $\phi_x(z) = \{y_i \in Y_i : \phi_x(z, y_i) > \lambda\}$.

Then it is easy to show that ϕ_x is a GL_S -majorant of A_i at x.

Also if $x = h(x_1, x_2) \in X$ then (by condition (II) above), there is a GL_S majorant $f_i^x : X \times Y_i \to \mathbb{R}$ such that

$f_i^x(x, y_i) \leq \min\{f_i(x_1, y_i), f_i(x_2, y_i)\}$ for each $y_i \in Y_i$.

Again define $\phi_x : X \to 2^{Y_i}$ by $\phi_x(z) = \{y_i \in Y_i : \phi_x(z, y_i) > \lambda\}$. Then the condition (ii) of Theorem 5.2.4 is satisfied. Hence, by Theorem 5.2.4, there exists $\widehat{x} \in K$ such that $A_i(\widehat{x}) = \emptyset$, for all $i \in I$, i.e. (a_2) of (A) holds.

(B) Take $\lambda_0 = \sup_{i\in I} \sup_{y\in Y} f_i(S(y), y)$. By applying (A), obviously (a_1) does not hold. Hence (a_2) is true, i.e. there exists $x_0 \in X$ such that

$$\sup_{i\in I} \sup_{y_i\in Y_i} f_i(x_0, y_i) \leq \lambda_0.$$

Therefore $\inf_{x\in X} \sup_{i\in I} \sup_{y_i\in Y_i} f_i(x, y_i) \leq \sup_{i\in I} \sup_{y\in Y} f_i(x, S(y))$. $\square$

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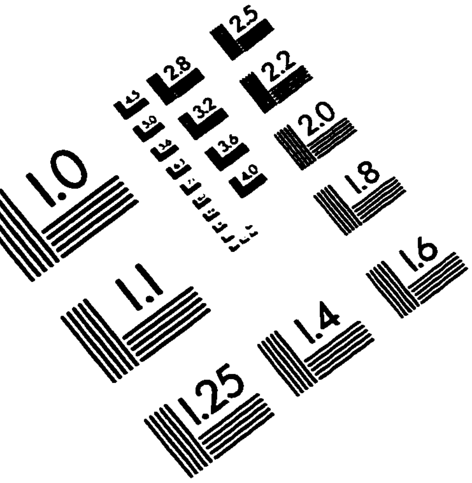
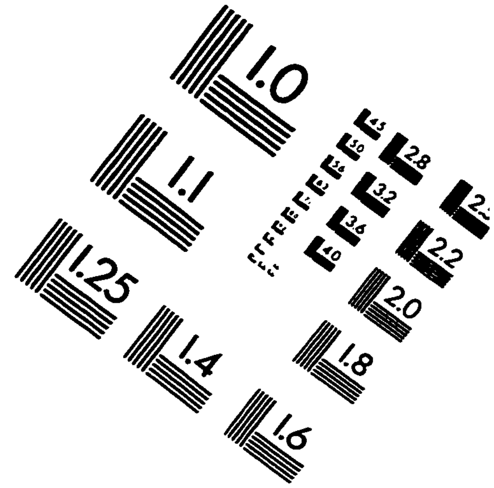
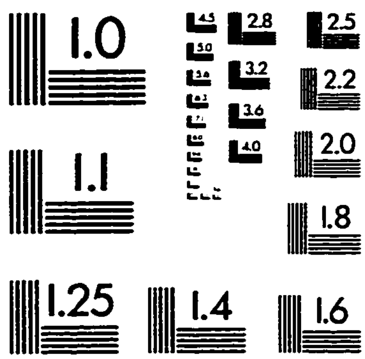
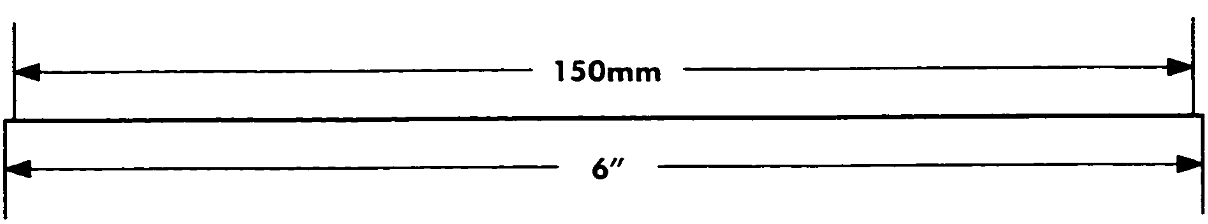
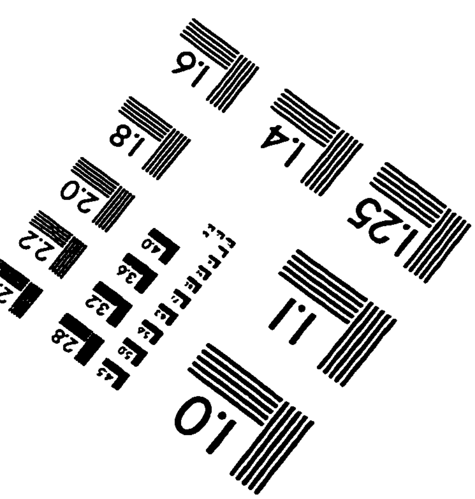
IMAGE EVALUATION TEST TARGET (QA-3)

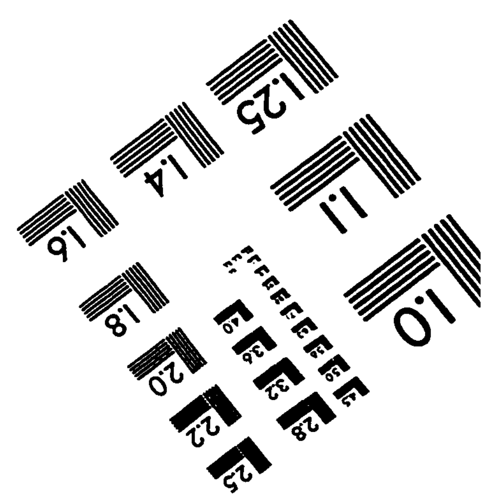












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