HIGHER-DIMENSIONAL VACUUM COSMOLOGIES

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To my parents,
sisters and Bahar

Contents

Abstract

Multi-dimensional spacetimes have recently been the source of many approaches towards the construction of a unified field theory. In this thesis, higher-dimensional cosmological models are discussed. In particular, 5-dimensional noncompactified vacuum solutions of Einstein's field equations are investigated. The possibility that the 4-dimensional properties of matter may be geometric in origin is discussed by studying whether the 5-dimensional vacuum fields equations reduce to Einstein's 4-dimensional theory with nonzero energy-momentum tensor constituting the material source. It is known that the 5-dimensional vacuum Einstein field equations (in which the metric is independent of the fifth dimension) give rise to the familiar radiation FRW cosmological model. However, it is of interest to study models with more general forms of matter. A variety of different higher-dimensional vacuum solutions have been found and these are discussed.

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Higher-Dimensional Vacuum Cosmologies

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Chapter 1

Introduction

1.1 General relativity and Cosmology

The aim of cosmology is to determine the large-scale structure of the physical universe. The General Theory of Relativity opened new ways of approaching the solution to problems related to the properties of the universe on a cosmic scale. In General Relativity (GR), the force of gravity is represented by the curvature of a four-dimensional Lorentzian manifold $\mathcal{M}$ endowed with a metric g_{ab}. The Einstein field equations, which relate curvature and the matter content of the universe are,

$$G_{ab} = 8\pi T_{ab}$$

where G_{ab} is the Einstein tensor derived from the metric tensor, and T_{ab} is the energy-momentum tensor which represents the energy and matter contributions and which has dependence on the metric coefficients as well. The first cosmological solution of Einstein's equations which was in good agreement with experimental observation was found by Friedmann, Robertson and Walker. This solution represents an expanding isotropic and homogeneous universe filled with a uniform distribution of matter. There have been many alternative attempts to explain the structure of the universe. Among these are Bianchi models within GR [1], models within Brans-Dicke scalar-tensor theory (variable-G) [2] and inflationary models [3]. None of these alternatives, however, has been able to replace the phenomenal success of relativity theory in satisfying most of the available observational tests. Less radical, but perhaps more enlightening, have been the extensions of General Relativity to higher dimensions [4] (eg. Kaluza-Klein models, supersymmetry theories and superstring theories). All of these theories exist in an attempt to unify the fundamental forces of nature. In addition, all of these higher-dimensional cosmologies rely on the compactification of the additional dimensions [4], which we will describe in the next section. We shall also provide some background on the original Kaluza-Klein theory and higher-dimensional vacuum cosmologies in general, paying particular attention to the theory of Wesson.

1.2 Kaluza-Klein Theory

1.2.1 Historical Overview

In 1919, Theodor Kaluza [5] proposed a generalization of general relativity from $4D$ to $5D$ in an attempt to unify the interactions of relativistic gravitation and electrodynamics. This was achieved via a weak-field approximation of an extended $5D$ metric tensor. However, the idea of higher-dimensional unification was not new. In 1912, the Finnish physicist Nordström [6] had developed a relativistic theory of gravity based on scalar fields. In 1914, before the final form of GR emerged, he utilized an extra spatial dimension to form a flat $5D$ spacetime with a 5-vector electromagnetic potential to extend maxwell's electrodynamics and found that the fifth component was equivalent to his scalar gravitational field. His scalar gravity theory couldn't explain the bending of light near the sun and was soon overtaken by the new GR. An early rival of Kaluza's theory was that of Weyl [7] who relaxed the parallel transport property of general relativity and allowed a "gauge" scaling of space and time in an attempt to have both gravity and electromagnetism arising from the geometry of $4D$ spacetime. However, electric charge is not conserved in Weyl's theory and thus it was eliminated as a candidate for a viable unified theory. In 1920, Oskar Klein [8, 9] showed that Kaluza's theory reduced rigorously to $4D$ Einstein-Maxwell theory in a full relativistic analysis[8]. He also supposed that the fifth dimension must be compact, to be curled up unobservably small, and found that this led naturally to the quantization of electric charge.

1.2.2 Kaluza-Klein Ansatz and the Transformation Law of the Fifth Dimension

We consider a generalization of $4D$ gravitation to a $5D$ spacetime, whose coordinates denoted by $\{x^A\} = \{x^\mu, x^5\}$ where $A,\ B,\ ...= \{0,1,2,3,5\}$, $\mu,\ \nu\ ...= \{0,1,2,3\}$, and the $5D$ line element is denoted as

$$d\hat{s}^2 = \hat{g}_{AB}dx^A dx^B \quad .$$

The extra dimension must be space-like to avoid causality violation due to the existence of closed time-like curves [10].

We partition the $5D$ general coordinate transformation $x^A \to x^{\bar{A}}(x^A)$ by regarding it as two-index array that we display in matrix form and a similar partition of the symmetric metric tensor $\hat{g}_{AB}$ to obtain

$$x^{\bar{A}}(x^A) = \begin{pmatrix} x^{\bar{\mu}}(x^\mu, x^5) \\ x^{\bar{5}}(x^\mu, x^5) \end{pmatrix} \quad \hat{g}_{AB} = \begin{pmatrix} \hat{g}_{\mu\nu} & \hat{g}_{\mu 5} \\ \hat{g}_{5\nu} & \hat{g}_{55} \end{pmatrix} \quad .$$

To account for the observed $4D$ character of spacetime, Kaluza introduced the cylinder condition:

"All components of the $5D$ metric and the first four coordinates x^μ must be independent of the fifth coordinate x^5, so that $\partial_5\hat{g}_{AB}$ and $\partial_5 x^\mu$ vanish identically. Straightforward analysis using this condition shows that the most general transformation of the fifth coordinate is thus of the form $x^5 \to x^{\bar{5}} = x^5 + f(x^\mu)$, where f is an arbitrary function."

1.2.3 Five-dimensional metric

In stationary $4D$ spacetime (where $\partial_0 g_{\mu\nu} = 0$), the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ may be split into

$$ds^2 = g_{00}d\lambda^2 + dl^2$$

where

$$d\lambda = dx^0 + \frac{g_{k0}}{g_{00}} dx^k$$

and

$$dl^2 = (g_{kl} - \frac{g_{k0} g_{l0}}{g_{00}}) dx^k dx^l$$

Klein [8] noted that the line element between two events in $5D$ spacetime, $d\hat{s}^2$, may be similarly split as a result of the cylinder condition into the invariant quantities

$$d\hat{\lambda} = dx^5 + \frac{\hat{g}_{\mu 5}}{\hat{g}_{55}} dx^\mu$$

and

$$d\hat{l}^2 = (\hat{g}_{\mu\nu} - \frac{\hat{g}_{\mu 5} g_{5\nu}}{g_{55}}) dx^\mu dx^\nu \quad .$$

Since the cylinder condition guarantees that $\hat{g}_{55}$ is a scalar quantity, we may define $\hat{g}_{55} \equiv \phi$ (a dimensionless scalar field) and $\hat{g}_{\mu 5} = k\phi A_\mu$, with k a constant. Taking this into account the above splittings become

$$d\hat{\lambda} = dx^5 + kA_\mu dx^\mu$$

$$d\hat{l}^2 = \hat{g}_{\mu\nu} - k^2 \phi A_\mu A_\nu dx^\mu dx^\nu \quad .$$

Now $d\hat{l}^2$ is independent of x^5 and is the $4D$ line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ if we make the identification

$$g_{\mu\nu} = \hat{g}_{\mu\nu} - k^2 \phi A_\mu A_\nu \quad .$$

Thus the $5D$ metric $d\hat{s}^2$ can be written as

$$d\hat{s}^2 = g_{\mu\nu} dx^\mu dx^\nu + \phi(kA_\mu dx^\mu + dx^5)^2 \quad ,$$

which yields the partitioned $5D$ metric

$$\hat{g}_{AB} = \begin{pmatrix} g_{\mu\nu} + \kappa^2 \phi A_\mu A_\nu & \kappa\phi A_\mu \\ \kappa\phi A_\nu & \phi \end{pmatrix}$$

which is now expressed totally in terms of the $4D$ metric, $g_{\mu\nu}$ and the fields A_μ and ϕ.

Consequently, after the identification $\hat{g}_{\mu 5} = kA_\mu \phi$ and definition $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, let us consider the $5D$ identity arising from the symmetries of the Riemann tensor, namely,:

$$\partial_D(\hat{\Gamma}_{ABC} + \hat{\Gamma}_{BCA} + \hat{\Gamma}_{CAB}) = \partial_A \hat{\Gamma}_{CDB} + \partial_B \hat{\Gamma}_{ADC} + \partial_C \hat{\Gamma}_{BDA} \quad .$$

Taking $(ABCD) = (\mu\nu\lambda 5)$ and making the substitution

$$2\hat{\Gamma}_{\mu 5\nu} = k(\phi F_{\mu\nu} + A_\mu \partial_\nu \phi - A_\nu \partial_\mu \phi) \quad ,$$

we find that

$$\partial_\mu F_{\lambda\nu} + \partial_\nu F_{\mu\lambda} + \partial_\lambda F_{\nu\mu} = 0 \quad ,$$

which is the electromagnetic Bianchi identity. Under the coordinate transformation $x^{\bar{5}} = x^5 + f(x^\mu)$, $\hat{g}_{\mu 5} = k\phi A_\mu$ transforms by

$$\hat{g}_{\bar{\mu}\bar{5}} = \hat{g}_{\mu 5} + \hat{g}_{55} \partial_{\bar{\mu}} g \quad ,$$

or

$$A_{\bar{\mu}} = A_\mu - k^{-1} \partial_\mu f \quad ,$$

which may be recognized as a gauge transformation of the vector field A_μ. This demonstrates one of the most powerful results of Kaluza's Ansatz, namely, that the gauge freedom of A_μ, previously considered "internal" in some sense, arises naturally here as a *geometric* freedom in the extra dimension.

To simplify subsequent calculations we set $\phi = 1$ as it plays no important part in this introductory description of the Kaluza-Klein theory. By analogy with the $4D$ variational principle, we consider the generalization of the Einstein-Hilbert action

$$\hat{S} = \frac{1}{16\pi\hat{G}} \int d^5x \sqrt{-\hat{g}} \hat{R}$$

where

$$\hat{R} = R - \frac{1}{4} k^2 F^{\mu\nu} F_{\mu\nu}$$

is calculated from $\hat{g}_{AB}$ and $\hat{g} = |\hat{g}_{AB}| = |g_{\mu\nu}| = g$, which is independent of x^5. The dimension of $\hat{G}$ must be of ($G\times$ length). By separating the integral into a $4D$ part and an integration over x^5 (one should assume that the fifth dimension is compact to avoid meaningless infinities), the $5D$ action becomes

$$\hat{S} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}) \ .$$

Thus our $5D$ action, $\hat{S}$, is indistinguishable from a $S_{gravity+em}$ action. Standard variational procedures yield Einstein's equations with an electromagnetic source and the vacuum Maxwell's equations $\partial_\mu F^{\mu\nu} = 0$. *This is the essence of Kaluza-Klein ansatz.* This extension to $5D$ in order to unify electromagnetism and gravity closely parallels the Minkowski extension of $3D$ space to $4D$ spacetime in special relativity to unify $\vec{E}= \{cF^{k0}\}$ and $\vec{B}= \{\frac{1}{2}\epsilon^{klm}F_{lm}\}$ in the Faraday tensor $F_{\mu\nu}$.

1.3 Higher-dimensional vacuum cosmology and induced matter theory

Multidimensional spacetime has recently been the subject of many approaches towards the construction of a unified field theory. Indeed, it is generally believed that higher dimensions must play a significant role in the early universe. There are several mechanisms known which incorporate a natural splitting of the physical and internal (higher) dimensions, including the Freund-Rubin mechanism [11], the Casimir effect associated with matter fields [12], and the effect of higher derivative terms in the gravitational action [13, 14]. Theories of this type are motivated by the original Kaluza-Klein theory [5, 9], described in previous subsections, in which the extra degrees of freedom in a five-dimensional theory were associated with an electromagnetic potential and the resulting Einstein equations mimicked the Einstein-Maxwell equations in four dimensions. Modern theories of this type include supergravity theory [15], and superstrings [16, 17].

In this thesis, we shall study the structure of vacuum Kaluza-Klein-type cosmological models. For the most part, we shall keep the discussion general so that it constitutes an analysis of the general mathematical structure of vacuum Kaluza-Klein-type solutions. We shall be particularly interested in the question of whether the properties of matter are contained in a purely geometric Kaluza-Klein-type extension of general relativity and whether matter can be completely geometric in nature.

We shall also be specifically interested in the five-dimensional Kaluza-Klein-type theory of Wesson and its physical interpretation. In Wesson's space-time-matter theory of gravity [18, 19, 20] the fifth coordinate (usually treated as spacelike to avoid the existence of closed timelike curves) is associated with a mass, m. The main idea is that, since Gmc^{-2} has the dimensions of length (where G and c are regarded as constants), it could be taken as an independent coordinate in the same way that ct is taken as a coordinate in spacetime. The resulting five-dimensional space-time-matter theory is thus a Kaluza-Klein-type extension of general relativity with a variable mass (or equivalently, since Gm is varying, a theory with a variable G); there is some astrophysical justification for studying such a theory [21]; see also [18, 19, 20, 22].

The physical identification of the fifth coordinate in a five-dimensional Kaluza-Klein-type theory is of paramount importance. For the most part, we shall not make any such identification (and regard it as a general Kaluza-Klein parameter); indeed, we shall not assume a priori that the fifth coordinate is is either unrestricted, or restricted to some interval $[0, L]$, or periodic. In the space-time-mass theory, which is of special interest, this fifth coordinate is identified with mass. However, the precise mass (e.g., an inertial or gravitational mass) is not necessarily specified a priori and is, to some extent, left open to interpretation [18, 19, 20].

One interpretation that may lead to important consequences (suggested independently by Coley, Ponce de Leon, and Wesson) is that the correct field equations for the theory are the vacuum field equations, ${}^5G_{ab} = 0$, [20]. The idea is that the extra terms present in the five-dimensional vacuum equations may play the role of matter terms that appear on the right-hand sides of the embedded four-dimensional Einstein field equations with matter. One of the aims of the present thesis is to study whether the four-dimensional properties of matter can be induced from the five-dimensional geometry in this way . Although this approach, in which the vacuum Kaluza-Klein-type equations actually contain the same physics as the four- dimensional Einstein equations with matter is new, the idea that the properties of matter might have a geometrical origin is not new and is in the spirit of the original Kaluza-Klein-type theory. It must be realized that the type of matter that can be described by a general theory of this type, and in particular in a purely gravitational theory such as one in which mass is geometrized as a fifth coordinate (whence only gravitational interactions are included), must be simple. Therefore, in general the induced matter will have no nuclear or electromagnetic structure (and more complex matter structure can only be contained within a more sophisticated theory).

Let us consider the $D = 4 + N$ dimensional metric in the form

$$ds^2 = g_{AB}dx^A dx^B = g_{\alpha\beta}dx^\alpha dx^\beta + g_{ab}dy^a dy^b, \tag{1.1}$$

where $ds_4^2 = -g_{\mu\nu}dx^\mu dx^\nu$ is given by the Robertson-Walker form, viz.,

$$ds_4^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right], \tag{1.2}$$

where k is the normalized (i.e., $k = 0, \pm 1$) curvature constant. For a perfect-fluid source with energy-momentum tensor,

$$T_{\alpha\beta} = (\mu + p)u_\alpha u_\beta + pg_{\alpha\beta}, \tag{1.3}$$

where μ and p are the energy density and pressure, respectively, and u^μ is the (comoving) fluid four-velocity, the four-dimensional Einstein equations (with matter) then yield

$$\mu = \frac{3}{R^2}(k + \dot{R}^2) \tag{1.4}$$

$$p = -2\frac{\ddot{R}}{R} - \frac{1}{R^2}(k + \dot{R}^2) \tag{1.5}$$

We shall be primarily concerned with cosmological models with matter of simple perfect-fluid type. The phenomenological μ and p are to be interpreted in terms of more fundamental geometrical quantities. We note that two issues that are usually of concern in the study of Kaluza-Klein cosmological models are that of compactification of the extra dimensions and that of dimensional reduction (i.e., the question of why a multidimensional space reduces to the product of a physical four-dimensional space-time and a static or nearly static compact [internal] space of additional dimensions of characteristic size comparable to the Planck length, thereby rendering the extra dimensions unobservable). We shall not be primarily interested in these issues here (and will not assume a priori that the extra dimensions are compact), but comments will be made in the text where appropriate. The alternative point of view (to considering the vacuum equations) in Kaluza-Klein-type cosmology, albeit contrary to the spirit of the original Kaluza-Klein theory, is to assume a higher dimensional energy-momentum tensor; indeed, in the majority of Kaluza-Klein-type cosmological models obtained, the source is assumed to be a higher dimensional comoving perfect fluid, although the pressure(s) in the higher dimensions is (are) assumed to have a variety of forms [23]. (This arises from the problem in this approach that there is no unique higher dimensional energy-momentum tensor that reduces to a given four-dimensional energy- momentum tensor; this ambiguity in the choice of higher dimensional energy-momentum tensor is clearly not present in the vacuum field equations formulation.)

Perhaps the fifth coordinate m should be identified with a gravitational mass; therefore, a constant m would only arise in the absence of a gravitational field, whence there would be no source for the four-dimensional Einstein tensor (and the above comment would not apply). Moreover, as noted above, the induced matter arising from a purely gravitational space-time-matter theory must be particularly structurally simple. Although Wesson [22] has shown that there are examples of five-dimensional vacuum equations which give rise to familiar four-dimensional cosmological models with very simple matter sources (see below), this simple matter is not necessarily consistent with a form of matter which is solely made up from the rest mass of its constituents. In Wesson and Ponce de Leon [24], coordinates were chosen so that the five-dimensional metric is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \phi^2 dy^2 \quad , \tag{1.6}$$

where $x_5 = y$ and $g_{55} = \phi^2$ (and differentiation with respect to y is denoted by an asterisk)and (here) the metric, $g_{\mu\nu}$, and ϕ are allowed to depend on x^μ and y. It was then shown that the five-dimensional Ricci tensor ${}^5R_{ab}$ (in terms of g_{ab}) can be written in terms of the four-dimensional Ricci tensor ${}^4R_{\alpha\beta}$ (in terms of $g_{\alpha\beta}$) in the following way:

$$ {}^5R_{\alpha\beta} = {}^4R_{\alpha\beta} - \frac{\phi_{\alpha;\beta}}{\phi} + \frac{1}{2\phi^2}\left(\frac{\overset{*}{\phi}}{\phi}\overset{*}{g}_{\alpha\beta} - \overset{**}{g}_{\alpha\beta} + g^{\mu\lambda}\overset{*}{g}_{\alpha\lambda}\overset{*}{g}_{\beta\mu} - \frac{1}{2}g^{\mu\nu}\overset{*}{g}_{\mu\nu}\overset{*}{g}_{\alpha\beta}\right) \tag{1.7} $$

$$ {}^5R_{4\alpha} = \sqrt{g_{44}}P^{\beta}_{\alpha;\beta} \tag{1.8} $$

$$ {}^5R_{44} = -\phi\Box\phi + \frac{1}{2}\left(-\overset{*}{g}^{\lambda\beta}\overset{*}{g}_{\lambda\beta} - g^{\lambda\beta}\overset{**}{g}_{\lambda\beta} \frac{\overset{*}{\phi}}{\phi}\overset{*}{g}^{\lambda\beta} g_{\lambda\beta} - \frac{1}{2}\overset{*}{g}^{\mu\beta} g^{\lambda\sigma}\overset{*}{g}_{\lambda\beta}\overset{*}{g}_{\mu\sigma}\right), \tag{1.9} $$

where $\phi_\alpha \equiv \phi_{,\alpha}$ and $\Box\phi \equiv g^{\mu\nu}\phi_{\mu;\nu}$. The tensor P^{α}_{β} is defined by

$$ P^{\beta}_{\alpha} = \frac{1}{2\sqrt{g_{44}}}\left(g^{\beta\sigma}\overset{*}{g}_{\sigma\alpha} - \delta^{\beta}_{\alpha}g^{\mu\nu}\overset{*}{g}_{\mu\nu}\right). \tag{1.10} $$

The five-dimensional vacuum Einstein field equations now decompose into $4D$ Einstein's field equations,

$$ {}^4R_{\alpha\beta} - \frac{1}{2}\,{}^4Rg_{\alpha\beta} = {}^4T_{\alpha\beta} \equiv \tag{1.11} $$

$$ \frac{\phi_{\alpha;\beta}}{\phi} + \frac{1}{2\phi^2}\left\{\left(\frac{\overset{*}{\phi}}{\phi}\overset{*}{g}_{\alpha\beta} - \overset{**}{g}_{\alpha\beta} + g^{\mu\lambda}\overset{*}{g}_{\alpha\lambda}\overset{*}{g}_{\beta\mu} - \frac{1}{2}g^{\mu\nu}\overset{*}{g}_{\mu\nu}\overset{*}{g}_{\alpha\beta} + \frac{1}{4}g_{\alpha\beta}\left[\overset{*}{g}^{\mu\nu}\overset{*}{g}_{\mu\nu} + (g^{\mu\nu}\overset{*}{g}_{\mu\nu})^2\right]\right\}, $$

where ${}^4T^{\beta}_{\alpha;\beta} = 0$ follows from the four-dimensional contracted Bianchi identities, and the constraints

$$ P^{\beta}_{\alpha;\beta} = 0, \tag{1.12} $$

which have the form of conservation laws, and a generalized wave equation

$$ -\phi\Box\phi + \frac{1}{2}\left(-\frac{1}{2}\overset{*}{g}^{\lambda\beta}\overset{*}{g}_{\lambda\beta} - g^{\lambda\beta}\overset{**}{g}_{\lambda\beta} \frac{\overset{*}{\phi}}{\phi}g^{\lambda\beta}\overset{*}{g}_{\lambda\beta}\right) = 0 \ . \tag{1.13} $$

Also, we note that

$$^4R = \frac{1}{4\phi^2}[\overset{*}{g}^{\mu\nu}\overset{*}{g}_{\mu\nu} + (g^{\mu\nu}\overset{*}{g}_{\mu\nu})^2] \ . \tag{1.14}$$

In particular, in the case that there is no explicit x_5 dependence, the above equations become

$$^4G_{\alpha\beta} = \frac{1}{\phi}\phi_{\alpha;\beta} \equiv T_{\alpha\beta} \ , \tag{1.15}$$

and

$$\Box\phi = 0 = T \ . \tag{1.16}$$

We note that for a perfect-fluid energy-momentum tensor, (1.16) implies a radiative equation of state (between μ and p) and that equations (1.15) have a formal interpretation in terms of a scalar field. In the case that the four-dimensional metric has the zero-curvature Robertson-Walker form and with $\phi = L(t)$; e.g.

$$ds^2 = ds_F^2 + L^2(t)dy^2 \tag{1.17}$$

in (1.1), where ds_F^2 is the flat $4D$ FRW line element, Wesson [22] obtained

$$\mu = 3\frac{\dot{R}^2}{R^2} = -3\frac{\dot{R}\dot{L}}{RL} \ , \tag{1.18}$$

$$p = -2\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} = \frac{\ddot{L}}{L} + 2\frac{\dot{R}\dot{L}}{RL} \ , \tag{1.19}$$

$$\mu = 3p \ , \tag{1.20}$$

which has solution $R = \sqrt{t}$ and $L = \frac{1}{\sqrt{t}}$ (so that the fifth dimension is shrinking), so that

$$\mu = \frac{3}{4}t^{-2} \ , \tag{1.21}$$

$$p = \frac{1}{4}t^{-2} \ , \tag{1.22}$$

and the familiar (zero-curvature) radiation Friedmann-Robertson-Walker (FRW) model is derived. So, at least in this case, the five-dimensional vacuum field equations yield the familiar four-dimensional cosmological model, with the added bonus that the properties of the matter are also prescribed by the five-dimensional geometry. Other physical properties of this model, and those of similar five-dimensional models, are discussed in Wesson [22]. This model has given rise to a very simple form for the matter (as expected and noted earlier). We note that the phenomenological equation of state $\mu = 3p$ has been deduced directly from the geometry of the five-dimensional universe through the exact cosmological solution. However, this simple equation of state is the appropriate one for cosmological models of the early universe. So, in this sense, the analysis has given a reasonable answer. Indeed, it turns out that for any four-geometry that is independent of the fifth dimension the state of matter must be radiation (i.e., independent of whether the four-geometry is anisotropic or inhomogeneous). This result is physically sensible, and it is of interest to note that the (radiation) FRW models play a central role in cosmology in the very early universe. Therefore, it will be of interest to investigate whether models exist with more general forms of matter. There are various approaches that will achieve this, one of which is to allow the metric functions $g_{\mu\nu}$ to depend on y. For example, Ponce de Leon [25] found a class of solutions of the $D = 5$ vacuum Einstein field equations in which the (spatially homogeneous and isotropic) metric components are separable functions of t and y and which can be interpreted as four-dimensional perfect-fluid solutions of Einstein's equations with $p = (\gamma - 1)\mu$ (γ constant), where $\mu = \mu(t, y)$. Fukui [26] has also studied vacuum solutions of the space-time-matter theory in which the metric components depend on both the time and fifth (mass) coordinates, which can be interpreted as four-dimensional solutions of the perfect-fluid Einstein equations with a more general (than radiative) equation of state (depending on both t and m) relating μ and p.

1.3.1 The induced matter theory of Wesson

One of the motivations for research on induced matter theory concerns the fact that the geometrical aspects of the theory (as embodied in the Einstein tensor G_{ij}) remain distinct from the physical aspects of the theory (as embodied in the energy-momentum tensor T_{ij}). Einstein is reputed to have spent considerable effort in trying to transmute the "base wood" of the T_{ij} on the right-hand side of his field equations into the "marble" of the G_{ij} on the left-hand side. This old conceptual problem is one we have become used to in $4D$ general relativity. But it is still a matter of concern because it is the source of ambiguities in the definition of the energy-momentum tensor in five- and higher-dimensional Kaluza-Klein theory. Wesson and others suggested that this ambiguity might be resolved if it were the case that the $5D$ empty field equations ${}^5G_{ij} = 0$ actually contain the same physics as the Einstein equations $G_{ij} = T_{ij}$ $(4D)$, provided appropriate definitions are made for important physical quantities. Wesson and his coworkers show that we can always go from $5D$ without matter to $4D$ with matter, provided we use a $4D$ energy-momentum tensor defined by (1.11). Therefore, the Kaluza-Klein theory is used not to unify the interactions of physics but to unify geometry and matter.

Wesson has shown that this new approach works surprisingly well for some classes of problems in gravitational physics. In recent work Wesson and his coworkers have examined a class of time-dependent soliton solutions and suggest some physical applications in particle physics and cosmology (see references under Wesson).

In particular, these soliton solutions represent an astrophysical application of induced matter theory. Because of the violation of Birkhoff's theorem in higher-dimensions, solutions of the $5D$ vacuum Einstein field equations that are spherically symmetric depend in general on a number of parameters (such as electric and scalar charge) besides mass, and in some cases these solutions are time-dependent and also non-singular. Such localized solutions of finite energy are called "solitons". Kaluza-Klein solitons were noted as early as 1951 (for a complete review of previous work see [4]). Later work showed that solitons are generic to the Kaluza-Klein theory in the same way that black holes are to ordinary general relativity. Further study of these objects reveals that Kaluza-Klein solitons must be classified as naked singularities. Wesson and his coworkers have generalized the soliton metrics to include time-dependence. In this thesis we shall present a further time-dependent generalization of these solutions (see chapter 3). These solitons were then suggested as possible dark matter candidates.

Further recent work on astrophysical applications and cosmological solutions by Wesson and coworkers have been summarized in Overduin and Wesson [4]. The observational predictions of higher-dimensional models have been studied (see [4]). The relationship between higher-dimensional gravity and scalar-tensor theory has been discussed by many authors (cf. Billyard and Coley [27]). More fundamental issues and questions of interpretation have been discussed in [4] and in Billyard and

Coley [27].

1.3.2 Outline

This thesis investigates more general cases than those studied by Wesson and others. As mentioned above, assuming metric dependence on the extra coordinate in general may produce a more general equation of state than radiation. Also, as pointed out earlier, we will make no assumptions on the compactness of the extra dimension. We will follow the main idea of taking the vacuum field equations ${}^5R_{ab} = 0$ as the correct field equations. The content of the second chapter in this thesis is to analyze possible solutions of ${}^5R_{abcd} = 0$ for a $5D$ spherically symmetric metric. One reason for doing so is that the Riemann-flat solutions can be used as an aid to find the Ricci-flat solutions. Although one could argue that these are not as interesting as Ricci-flat solutions, one has no problem in interpreting the source field. The field equations ${}^5R_{ab} = 0$ is $5D$-curved with an unknown source if one follows the spirit of GR. Indeed several well-known $5D$ solutions are in fact Riemann-flat [28]. Chapter two is divided into four possible cases, and each case is studied in its entirety.

Chapter three investigates the Ricci-flat solutions by extending the Riemann-flat solutions found in chapter one. This chapter also studies the relevant field equations under some simplifying but reasonable ansatzes. Some known solutions are recovered and qualitative analysis is applied for determining the properties of other new solutions. Chapter four uses an ansatz broad enough to encompass both the static spherically symmetric solutions, originally discussed by Gross and Perry [29] and by Davidson and Owen [30], and the cosmological solutions considered by Wesson and his co-workers. And in chapter five we seek some new power-law solutions which are not Riemann-flat. Finally, chapter six extends the idea of induced matter to Einstein-Yang-Mills theory, either Abelian or non-Abelian gauge theory, as well as to supergravity. Throughout the whole thesis, equations of state are derived wherever possible.

Chapter 2

Five Dimensional Spherically Symmetric Riemann-Flat Space-times

2.1 Preliminaries

Based on the idea presented in the introduction we wish to find solutions to ${}^5R_{ij} = 0$ for the general $5D$ spherically symmetric metric, which is given by

$$ds^2 = -e^{2f(t,r,y)}dt^2 + e^{2g(t,r,y)}(dr^2 + r^2 d\Omega^2) + e^{2k(t,r,y)}dy^2 \quad , \tag{2.1}$$

where y is the fifth coordinate.

Equations ${}^5R_{ij} = 0$ involve 7 coupled partial differential equations for f, g and h and the solutions in general are hard to find due to non-linearity. One possible strategy is to first find the solution for $5D$ Riemann flat case and and try to generalize the obtained solutions to satisfy the Ricci-flat equations. This seems to be more promising since ${}^5R_{ijkl} = 0$ involve many more equations than $R_{ij} = 0$. It is worth noting that the general solutions of the Riemann-flat equations are the same as $5D$ Minkowski space up to a diffeomorphism. That is to say, for any given solutions for f, g and h, there is always a diffeomorphism $t' = t'(t, r, y)$, $r' = r'(t, r, y)$ and

$y' = y'(t, r, y)$.

One reason for studying the Ricci-flat solutions of metric (2.1) is of mathematical interest; i.e., it is natural to ask what are the implications of generalizing the solution given by [29] or [30] to a metric depending not only on the r coordinate, but also on time and the extra coordinate. Apart from the mathematical interest of solutions of this kind, there is also a physical motivation. In the context of induced matter theory, it can be shown that metrics independent of the extra coordinate always imply a radiation-like equation of state. In order to include more general types of equations of state, like dust, vacuum and stiff matter, it is necessary to study the most general spherically symmetric metric with metric coefficients depending on radius, time and the extra coordinate. Therefore, we wish to determine all metrics of the form (2.1) that are Riemann-flat. In principle, the problem is trivial since locally all solutions are Minkowski space-time; however, one has to implement various nontrivial diffeomorphisms of the form $x^\alpha \to \bar{x}^\alpha(x^\beta)$ and $y \to \bar{y}(y)$. Thus, by restricting the permissible diffeomorphisms we ensure that the $4D$ intrinsic metric

$$ds^2 = g_{\alpha\beta}(x^\gamma, y)|_{y=const}\, dx^\alpha dx^\beta$$

are not necessarily Riemann-flat, even though the $5D$ metrics are Riemann-flat. Wesson [31] has investigated the above problem but has found the solutions only in the special case where f, g and h are separable functions of t, r and y.

As previously mentioned, there are 12 independent non-zero components of ${}^5R_{ijkl} = 0$ for the above metric (See Appendix A). Among those, the equations $R_{1323} = 0$ and $R_{2335} = 0$ are the most promising ones since they break the problem into a finite number of cases. More explicitly,

$$R_{1323} = (-g_{rt} + g_t f_r) r^2 e^{2g} = 0 \tag{2.2}$$

$$R_{2335} = (-g_{ry} + g_y k_r) r^2 e^{2g} = 0 \tag{2.3}$$

where subscripts for the functions f, g and k denote partial derivatives with respect to the specified variables. If $g_t \neq 0$ then (2.2) results $e^f = e^{h(t,y)} g_t$ where h is an arbitrary function of t and y. If $g_y \neq 0$ then (2.3) can be integrated resulting in $e^k = e^{l(t,y)} g_y$

where $l(t, y)$ is an arbitrary function to be found. Therefore, the problem of finding the general solutions $R_{ijkl} = 0$ can be broken up to the following four natural cases:

- case(I): $g_t = 0$ and $g_y = 0$;
- case(II): $g_t = 0$ and $g_y \neq 0$;
- case(III): $g_t \neq 0$ and $g_y = 0$;
- case(IV): $g_t \neq 0$ and $g_y \neq 0$.

In the next sections we will study each case separately.

2.2 Case I $[g_y = 0$ and $g_t = 0]$

In this case the metric has the following form:

$$ds^2 = -e^{2f(t,r,y)}dt^2 + e^{2g(r)}(dr^2 + r^2 d\Omega^2) + e^{2k(t,r,y)}dy^2. \tag{2.4}$$

Now, $R_{1313} = 0$ has the following form (see Appendix A):

$$rf_r(rg_r + 1)e^{2g} = 0,$$

which suggests that either $f = f(t, y)$ or $rg_r + 1 = 0$. The latter can be integrated resulting in

$$g(r) = ln(\frac{d}{r}), \tag{2.5}$$

where d is a constant. Also, $R_{2323} = 0$ has the form

$$re^{2g}(rg_{rr} + g_r) = 0. \tag{2.6}$$

The general solution for $rg_{rr} + g_r = 0$ is

$$g(r) = ln(ar^c) \quad , \tag{2.7}$$

where $a \neq 0$ is a constant and can be set equal to one. Now (2.5) and (2.6) together imply that $c = -1$; i.e. $e^{2g} = a^2/r^2$ $(a \neq 0)$. Plugging this expression for g back into the equations ${}^5R_{ijkl} = 0$ reveals that

$$R_{3434} = a^2 sin\theta^2$$

which is non-zero (since $a \neq 0$) and hence this case is not possible.

To summarize, we thus obtain $g(r) = ln(ar^c)$ with $c \neq -1$ and $f = f(t, y)$. Taking this into account, R_{3434} becomes

$$R_{3434} = a^2 cr^{2c+2} sin^2\theta(-c-2) = 0 \quad ,$$

which results in either $c = 0$ or $c = -2$. In addition,

$$R_{3535} = -k_r e^{2k} r(c+1) = 0$$

implies that $k_r = 0$, or $k = k(t, y)$, since $c + 1 \neq 0$. When $c = -2$ we arrive at the metric

$$ds^2 = -e^{2f(t,y)}dt^2 + \frac{1}{r^4}(dr^2 + r^2 d\Omega^2) + e^{2k(t,y)}dy^2, \tag{2.8}$$

where $R_{1515} = 0$ and the only non-trivially vanishing component of ${}^5R_{ijkl}$ leads to the following differential relation between f and k :

$$f_{yy}e^{2f} + f_y^2 e^{2f} + f_t k_t e^{2k} - k_{tt}e^{2k} - k_t^2 e^{2k} - f_y k_y e^{2f} = 0 \quad . \tag{2.9}$$

To simplify, let $R \equiv -\frac{1}{r}$ and $e^{2f(t,y)} = F(t, y)$ and $e^{2k(t,y)} = K(t, y)$, which then renders the metric as

$$ds^2 = -F^2 dt^2 + (dR^2 + R^2 d\Omega^2) + K^2 dy^2 \quad , \tag{2.10}$$

and (2.9) becomes

$$\frac{\partial}{\partial y}(K^{-1}\frac{\partial F}{\partial y}) = \frac{\partial}{\partial t}(F^{-1}\frac{\partial K}{\partial t}). \tag{2.11}$$

This metric is clearly $4D$ flat over each slice $y = const.$

When $c = 0$ we arrive at exactly the same metric (2.10) with the equation (2.11) after an appropriate coordinate transformation.

2.3 Case II $[g_t = 0$ and $g_y \neq 0]$

As we saw earlier in this chapter, in this case $e^k = g_y e^h$, and the metric has the form

$$ds^2 = -e^{2f(t,r,y)}dt^2 + e^{2g(r,y)}(dr^2 + r^2 d\Omega^2) + g_y^2 e^{2h(t,y)}dy^2 \tag{2.12}$$

Now, the equation $R_{1225} = -g_y h_t e^{2g} = 0$ readily gives $h_t = 0$, or $h = h(y)$, since $g_y \neq 0$. Having taken this into account, we arrive at the following equations:

$$R_{2323} = 0 \Rightarrow rg_{rr} + g_r^2 + re^{2g-2h} = 0 \quad , \tag{2.13}$$

$$R_{3434} = 0 \Rightarrow rg_r^2 + 2g_r + re^{2g-2h} = 0 \quad . \tag{2.14}$$

Subtracting (2.14) from (2.13) gives

$$rg_{rr} - rg_r^2 - g_r = 0 \quad . \tag{2.15}$$

Now, $g_r \neq 0$, since $g_r = 0$ contradicts(2.14), and therefore this relation can be written as

$$\frac{g_{rr}}{g_r} - g_r - \frac{1}{r} = 0 \quad ,$$

which can be easily integrated giving

$$g_r = rF(y)e^g \quad , \tag{2.16}$$

where $F(y)$ is an arbitrary function. Substituting this back into relation (2.14) results in

$$e^{2g(r,y)} = \frac{-2F(y)}{r^2F^2(y) + e^{-2h(y)}} \quad . \tag{2.17}$$

In addition,

$$R_{1212} = 0 \quad \Rightarrow \quad f_{rr}g_y + f_r^2 g_y - f_r g_r g_y + f_y e^{2(g-h)} = 0 \quad , \tag{2.18}$$

$$R_{1313} = 0 \quad \Rightarrow \quad f_y = -e^{2(h-g)} g_y f_r (g_r + \frac{1}{r}) = 0. \tag{2.19}$$

Substituting (2.19) into (2.18) yields

$$f_{rr} = -f_r^2 + f_r(2g_r + \frac{1}{r}). \tag{2.20}$$

Now, if $f_r = 0$, then (2.19) shows that f_y must vanish as well. Hence $f = f(t)$ which can be set equal to zero by applying a time transformation resulting in $g_{11} = -1$. If $f_r \neq 0$, then (2.20) can be written as

$$\frac{f_{rr}}{f_r} = -f_r + 2g_r + \frac{1}{r},$$

which can be integrated giving

$$f_r = re^{-f}e^{2g}J(t,y) \quad , \tag{2.21}$$

where $J(t,y)$ is an arbitrary function. Equation (2.21) can again be integrated to give

$$e^f = J(t,y)\int re^{2g}dr + q(t,y), \tag{2.22}$$

where $q(t,y)$ is another arbitrary function. By using (2.17) we arrive at

$$e^f = 4J(t,y)\int \frac{rF^2}{(r^2F^2 + e^{-2h})^2}dr + q(t,y) = \frac{J(t,y)}{r^2F^2(y) + e^{-2h(y)}} + q(t,y) \quad . \tag{2.23}$$

Using the expressions found for g and f in the set of equations ${}^5R_{ijkl} = 0$, we observe that they all have a power series expansion form with respect to r. In particular, ${}^5R_{1212} = 0$ has the form:

$$r^4(JF^4F_ye^{2h} - 2F^5q_y) + r^2(2JF^2F_y - 2F^3Jh_y - 2F^3J_y - 4F^3q_ye^{-2h}) +$$

$$r^0(JF_ye^{-2h} - 2FJh_ye^{-2h} - 2FJ_ye^{-2h} - 2Fq_ye^{-4h}) = 0, \tag{2.24}$$

which, since each term in r^{2n} must be separately zero, gives rise to the following system:

$$JF_ye^{2h} - 2Fq_y = 0, \tag{2.25}$$

$$JF_y - FJh_y - FJ_y - 2Fq_ye^{-2h} = 0, \tag{2.26}$$

$$JF_y - 2FJh_y - 2FJ_y - 2Fq_ye^{-2h} = 0. \tag{2.27}$$

Using (2.25) and (2.27) we get $J_y = Jh_y$, which can be integrated, resulting in

$$J(t,y) = l(t)e^{-h(y)} \quad .$$

Substituting this back into (2.25) yields

$$l(t)F_y e^h = 2q_y F \quad ,$$

and after integrating

$$q(t,y) = \frac{1}{2}\int \frac{F_y}{F} e^h dy + p(t).$$

Hence the final form of the metric (2.12) for the solution of Riemann-flat equations is:

$$\begin{aligned} ds^2 = & - \left[\frac{e^{-h} l(t)}{r^2 F^2(y) + e^{-2h(y)}} + \frac{1}{2} l(t) \int \frac{F_y}{F} e^h dy + p(t)\right]^2 dt^2 \\ & + \frac{4F^2(y)}{r^2 F^2(y) + e^{-2h(y)}} (dr^2 + r^2 d\Omega^2) \\ & + \frac{(r^2 F F_y - F_y e^{-2h} - 2F h_y e^{-2h})^2}{(r^2 F^2 + e^{-2h})^2 F^2} dy^2. \end{aligned} \quad (2.28)$$

This solution includes the solution $g_{11} = -1$ when $l(t) \equiv 0$ and $p(t) \equiv 1$. The above metric is also non-separable. On any slice $y = const$ with the choice of $l \equiv 0$ and $h \equiv 0$ and $F \equiv 1$, this $4D$ metric is nothing but a static FRW metric with positive constant curvature($\kappa = +1$); i.e.

$$ds^2 = -dt^2 + \frac{dr^2 + r^2 d\Omega^2}{(1+r^2)^2} \quad .$$

2.4 Case III $[g_t \neq 0$ and $g_y = 0]$

As seen earlier, we have $e^f = g_t e^{h(t,y)}$ with the metric,

$$ds^2 = -e^{2f(t,r,y)} dt^2 + e^{2g(t,r)} (dr^2 + r^2 d\Omega^2) + e^{2k(t,r,y)} dy^2 \quad (2.29)$$

Having all the components of ${}^5R_{ijkl}$ calculated shows that (See Appendix A)

$$r^2 R_{1225} = R_{1335} = -g_t h_y e^{2g} r^2 = 0, \quad (2.30)$$

which implies that $h_y = 0$, or $h = h(t)$, since $g(t) \neq 0$ in this case. Having this taken into account, $R_{2323} = 0$ gives

$$r g_{rr} + g_r - r e^{2(g-h)} = 0 \quad , \quad (2.31)$$

$$rg_r^2 + 2g_r - re^{2(g-h)} = 0 \quad . \tag{2.32}$$

Subtracting (2.32) from (2.31) can be written as

$$\frac{g_{rr}}{g_r} - g_r - \frac{1}{r} = 0 \quad ,$$

since $g_r \neq 0$ ($g_r = 0$ is impossible due to (2.32)).

This relation can be easily integrated to give

$$\frac{g_r}{re^g} = F(t), \tag{2.33}$$

where $F(t) \neq 0$ is an arbitrary function. Substituting (2.33) back into (2.32) gives the following expression for g:

$$e^{g(r,t)} = \frac{-2F(t)}{r^2F^2 - e^{-2h(t)}} \quad . \tag{2.34}$$

Now, using the above expression for g results in the following forms for R_{3535} and R_{2525}:

$$R_{3535} = 0 \Rightarrow re^{2g}k_t = rk_rg_rg_te^{2h} + k_rg_te^{2h} \quad , \tag{2.35}$$

$$R_{2525} = 0 \Rightarrow e^{2h}k_{rr}g_t = -k_r^2g_te^{2h} + k_te^{2g} + g_rg_tk_re^{2h} \quad . \tag{2.36}$$

Substituting k_t into (2.36) from (2.35) yields

$$\frac{k_{rr}}{k_r} = -k_r + 2g_r + \frac{1}{r} \quad . \tag{2.37}$$

In (2.37) we assumed $k_r \neq 0$. If $k_r \neq 0$, then $k_t = 0$ due to (2.35), which implies $k = k(y)$ and hence which can be set to zero by a diffeomorphism $y = \overline{y}(y)$. This case will result in a special case of the general case. Now, (2.37) can be easily integrated giving:

$$\frac{\partial}{\partial r}(e^k) = rJ(t,y)e^{2g} \quad , \tag{2.38}$$

where J(t,y) is an arbitrary function. By substituting e^g from (2.34) into (2.38) and then integrating with respect to r, we get

$$e^{k(t,r,y)} = \frac{J(t,y)}{r^2F^2(t) - e^{-2h(t)}} + q(t,y) \quad . \tag{2.39}$$

Now, substituting (2.39) into (2.35) give rises to a power series expansion for $R_{3535} = 0$ with respect to r, and putting the corresponding coefficients identically to zero results in the following system of equations:

$$JF_te^{2h} + 2Fq_t = 0, \tag{2.40}$$

$$-2JF_t + 2FJh_t + 2FJ_t - 4Fq_te^{-2h} = 0, \tag{2.41}$$

$$JF_t - 2FJh_t - 2FJ_t + 2Fq_te^{-2h} = 0. \tag{2.42}$$

Adding (2.41) and (2.42) yields

$$JF_t = 2Fq_te^{-2h}, \tag{2.43}$$

and substituting JF_t from (2.43) in (2.40) results in $q_t(t, y) = 0$, since $F \neq 0$. Equation (2.40) now breaks into two cases, either $J(t, y) \equiv 0$ or $F_t = 0$. If $J = 0$, then the g_{44} component of the metric becomes a function of y alone, which in turn could be absorbed by a y -coordinate transformation. The $5D$ Riemann-flat metric can then be written as

$$\begin{aligned} ds^2 = & - \frac{e^{2h(t)}(r^2F^2F_t^2 + F_te^{-2h} + 2Fh_te^{-2h})^2}{F^2(e^{-2h} - r^2F^2)^2}dt^2 \\ & + \frac{4F^2}{(e^{-2h} - r^2F^2)^2}(dr^2 + r^2d\Omega^2) + dy^2 \ . \end{aligned} \tag{2.44}$$

On each hypersurface $y = const$, the above $5D$ flat metric is also $4D$ flat and thus uninteresting from an induced matter theory viewpoint. On the other hand, if $F_t = 0$, i.e., $F = F_0 = constant$, then (2.42) yields

$$Jh_t + J_t = 0 \ , \tag{2.45}$$

which can be integrated again resulting in

$$J(t, y) = p(y)e^{-h(t)} \ ,$$

where $p(y)$ is a non-zero, arbitrary function ($p(y) = 0$ results in $J = 0$ which has already been studied). With this expression for J, we get another metric for the $5D$ Riemann-flat case:

$$\begin{aligned} ds^2 = -\frac{4h_t^2e^{-2h(t)}}{(e^{-2h(t)} - r^2F_0^2)^2}dt^2 + \frac{4}{(r^2F_0^2 - e^{-2h})^2}(dr^2 + r^2d\Omega^2) \\ +(\frac{p(y)e^{-h(t)}}{r^2F_0^2 - e^{-2h}} + q(y))^2dy^2, \end{aligned} \tag{2.46}$$

where F_0 is a constant. On each hypersurface $y = constant$, the above $5D$ flat metric is also a $4D$ flat, but unlike the previous case (since $\partial_r g_{tt} \neq 0$, due to $F \neq 0$) it doesn't include FRW positive constant curvature as a special case.

2.5 Case IV $[g_t \neq 0$ and $g_y \neq 0]$

In this case the metric has the following form

$$ds^2 = -e^{h(t,y)} g_t^2 dt^2 + e^{2g(t,r,y)}(dr^2 + r^2 d\Omega^2) + e^{2l(t,y)} g_y^2 dy^2. \tag{2.47}$$

Now, $R_{3434} = 0$ and $R_{2323} = 0$ imply, respectively,

$$r g_r^2 + 2g_r + r e^{2g}(e^{-2l} - e^{-2h}) = 0 \ , \tag{2.48}$$

$$r g_{rr} + g_r + r e^{2g}(e^{-2l} - e^{-2h}) = 0 \ . \tag{2.49}$$

Subtracting (2.49) from (2.48) results in

$$r g_{rr} - r g_r^2 - g_r = 0 \ , \tag{2.50}$$

If $g_r \neq 0$ then (2.50) can be integrated giving (refer to solution in case II)

$$\frac{g_r}{r e^g} = F(t,y) \ . \tag{2.51}$$

Substituting back (2.51) into (2.48) gives the following expression for $g(t,r,y)$:

$$e^g = \frac{-2F(t,y)}{r^2 F^2 + e^{-2l} - e^{-2h}} \ . \tag{2.52}$$

Using this expression in $R_{ijkl} = 0$ will leave us with two independent non-trivial components, namely, R_{1225} and R_{1515}, with power series form with respect to r as follows:

$$\begin{aligned} R_{1225} &= \\ & -g_{ty} + g_t g_y - h_y g_t - l_t g_y = r^2[F^3(F_{yt} + F_t h_y + l_t F_y)] \\ & +r[e^{-2h}(-2FF_t h_y + 2F^2 l_t h_y - 2F^2 h_t h_y + 2F^2 h_{yt} - 2FF_y h_t - 2F_y F_t) \\ & +e^{-2l}(2F^2 l_t l_y - 2F^2 l_{ty} - 2F^2 h_y l_t + 2FF_y l_t + 2FF_t l_y + 2F_y F_t)] = 0 \ . \end{aligned} \tag{2.53}$$

Vanishing of expansion coefficients gives rise to the following partial differential equations:

$$F_{yt} + F_t h_y + F_y l_t = 0 \ , \tag{2.54}$$

$$e^{-2l}(F^2 l_t l_y - F^2 l_{yt} - F^2 h_y l_t + F F_t l_y + F F_y l_t + F_y F_t) -$$

$$e^{-2h}(F^2 h_t h_y - F^2 h_{yt} - F^2 h_y h_t + F F_t h_y + F F_y h_t + F_y F_t) = 0. \qquad (2.55)$$

The equation $R_{1515} = 0$ has a large number of terms including third order derivative terms like g_{tty}. Once again, R_{1225} has the form

$$-g_{yt} + g_t g_y - h_y g_t - l_t g_y = 0 \ . \qquad (2.56)$$

By taking derivatives with respect to t and y from the above equation, g_{tty} can be expressed in terms of second order derivatives, namely

$$\begin{aligned} g_{tty} &= g_{tt} g_y + g_t^2 g_y - h_y g_t^2 - 2 l_t g_y g_t - h_{ty} g_t \\ &\quad - h_y g_{tt} - l_{tt} g_y - l_{tt} g_y + l_t h_y g_t + l_t^2 g_y \ . \end{aligned} \qquad (2.57)$$

With substitution of (2.57) into (2.56) we obtain

$$\begin{aligned} R_{1515} &= e^{2(g+h)}[g_t g_y - l_t g_y - l_{ty} - l_t h_y + l_y l_t - g_t l_y] + \\ &\quad e^{2(g+l)}[-g_t g_y + h_y g_t + h_{yt} - l_t h_y - h_t h_y + g_y h_t] + \\ &\quad e^{2(h+l)} g_{rt} g_{ry} \ . \end{aligned} \qquad (2.58)$$

Now, by substituting (2.52) into (2.58), R_{1515} is expressed by the following power series expansion with respect to r as

$$\begin{aligned} R_{1515} &= r^4\{4F^4[e^{2h}(-l_t h_y - l_{yt} F^2 + l_y l_t + F l_y F_t + F l_t F_y + F_t F_y) + \\ &\quad e^{2l}(F^2 h_{yt} + l_t h_y - h_t h_y - F h_t F_y - F h_y F_t - F_t F_y)]\} + \\ &\quad r^2(8F^2)[e^{2(h-l)}(-F^2 l_t h_y - F^2 l_{yt} + F^2 l_t l_y + F F_t l_y + F l_t F_y + F_t F_y) + \\ &\quad e^{2(l-h)}(-F^2 h_{yt} + F^2 h_t h_y - F^2 l_t h_y + F h_y F_t + F h_t F_y + F_t F_y) + \\ &\quad (-F^2 h_t h_y + F^2 h_{yt} + F^2 l_{yt} + 2F^2 l_t h_y - l_y l_t F^2 - \\ &\quad F F_y h_t - F F_t h_y - F F_t l_y - F l_t F_y - 2F_t F_y) + r^0(T) = 0 \ , \end{aligned} \qquad (2.59)$$

where

$$r^0(T) = 4(-e^{(-2h)} + e^{(-2l)})^2 \Big(l_t F^2 h_y e^{(2l-8h)} - F F_t h_y e^{(2l-8h)} + F^2 h_{ty} e^{(-6l)}$$
$$- 4 F^2 h_{ty} e^{(-2h-4l)} - F h_t F_y e^{(2l-8h)} - 6 F_t F h_y e^{(-4h-2l)}$$

$$
\begin{aligned}
&- 6\,F\,h_t\,F_y\,e^{(-4h-2l)} + F^2\,l_t\,l_y\,e^{(2h-8l)} + F\,l_t\,F_y\,e^{(-6h)} \\
&- F_t\,F_y\,e^{(2l-8h)} - 10\,F_t\,F_y\,e^{(-4h-2l)} + 5\,F_t\,F_y\,e^{(-6h)} - l_{ty}\,F^2\,e^{(2h-8l)} \\
&+ F_t\,F_y\,e^{(2h-8l)} - 5\,F_t\,F_y\,e^{(-6l)} + 4\,l_{ty}\,F^2\,e^{(-6l)} + 6\,F^2\,l_t\,l_y\,e^{(-2h-4l)} \\
&- 6\,l_{ty}\,F^2\,e^{(-2h-4l)} - l_{ty}\,F^2\,e^{(-6h)} - 4\,l_y\,F\,F_t\,e^{(-6l)} \\
&- 10\,F^2\,l_t\,h_y\,e^{(-2h-4l)} + 4\,F_t\,F\,h_y\,e^{(-6h)} + 5\,F^2\,l_t\,h_y\,e^{(-6l)} \\
&- 5\,F^2\,l_t\,h_y\,e^{(-6h)} + 4\,F^2\,h_t\,h_y\,e^{(-2h-4l)} + 4\,F^2\,h_t\,h_y\,e^{(-6h)} \\
&+ 10\,F_t\,F_y\,e^{(-2h-4l)} + F\,l_t\,F_y\,e^{(2h-8l)} + 4\,F\,h_t\,F_y\,e^{(-6h)} \\
&- F\,h_t\,F_y\,e^{(-6l)} - 4\,F\,l_t\,F_y\,e^{(-6l)} + 4\,F\,h_t\,F_y\,e^{(-2h-4l)} \\
&+ 4\,F_t\,F\,h_y\,e^{(-2h-4l)} + l_y\,l_t\,F^2\,e^{(-6h)} - h_y\,l_t\,F^2\,e^{(2h-8l)} \\
&+ 10\,F^2\,l_t\,h_y\,e^{(-4h-2l)} - F_t\,F\,h_y\,e^{(-6l)} - 4\,l_y\,l_t\,F^2\,e^{(-4h-2l)} \\
&- F^2\,h_y\,h_t\,e^{(-6l)} + l_y\,F\,F_t\,e^{(2h-8l)} - F^2\,h_y\,h_t\,e^{(2l-8h)} \\
&+ 6\,l_y\,F\,F_t\,e^{(-2h-4l)} - 4\,l_y\,l_t\,F^2\,e^{(-6l)} + l_y\,F\,F_t\,e^{(-6h)} \\
&- 4\,F\,l_t\,F_y\,e^{(-4h-2l)} - 4\,l_y\,F\,F_t\,e^{(-4h-2l)} - 6\,F^2\,h_t\,h_y\,e^{(-4h-2l)} \\
&+ 4\,l_{ty}\,F^2\,e^{(-4h-2l)} + 6\,F^2\,h_{ty}\,e^{(-4h-2l)} + F^2\,h_{ty}\,e^{(2l-8h)} \\
&- 4\,F^2\,h_{ty}\,e^{(-6h)} + 6\,F\,l_t\,F_y\,e^{(-2h-4l)}\Big) \quad .
\end{aligned}
$$

All coefficients of this expansion vanish on account of (2.54) and (2.55). As seen, the vanishing of R_{1515} was by no means trivial.

To summarize, the Riemann-flat solution for this case reduces to a system of two partial differential equations for three unknown functions, $F(t,y)$, $h(t,y)$ and $l(t,y)$, and in general these equations do not have unique solutions.

Finally, in the particular subcase where $g_r = 0$ (i.e., $l = h$ from (2.48)) then the only non-trivial independent equation is $R_{1212} = 0$, which reduces to the following partial differential equation for the two unknowns $g(t,y)$ and $h(t,y)$:

$$
g_t g_y - g_y h_t - h_y g_t - g_{ty} = 0 \quad ,
$$

which again has no unique solution. $4D$ slices $y = constant$ in this case (IV) are not flat in general. In particular, in the subcase $g_r = 0$ the $4D$ slices are Riemann-curved.

To conclude, in this chapter we have comprehensively studied all the possible Riemann-flat solutions of the metric (2.1).

Chapter 3

Attempts towards finding the general Ricci-flat solutions of $5D$ spherically symmetric Kaluza-Klein theories

3.1 An ansatz

As discussed in the introduction, following [31] we are interested in finding solutions of ${}^5G_{ij} = 0$ and therefore, ${}^5R_{ij} = 0$. We assume the same $5D$ spherically symmetric metric as before

$$ds^2 = -e^{2f(t,r,y)}dt^2 + e^{2g(t,r,y)}(dr^2 + r^2d\Omega^2) + e^{2k(t,r,y)}dy^2. \tag{3.1}$$

Calculation of ${}^5R_{ij}$ shows that there are only seven independent non-trivial Ricci tensor components [see Appendix B]. Unlike the Riemann-flat case, we have less hope of breaking the equations into natural subcases for two reasons; first, they are highly coupled and second, the number of relevant equations are less than in the Riemann-flat case. As mentioned before, one possible approach would be to generalize the Riemann-flat solutions to obtain Ricci-flat solutions. One way to do this is by replacing any separated forms like $f_1(t)f_2(y)$ with a general function $f_3(t, y)$ or any term r to be replaced by a function of r.

One promising ansatz, based on the Riemann-flat solutions, is the metric of the form

$$ds^2 = -\frac{(C(t,y)+J(t,y)r^2)^2}{(r^2F^2(y)+e^{-2h(y)})^2}dt^2 + \frac{(2F(y)+L(y)r^2)^2}{(r^2F^2+e^{-2h(y)})^2}(dr^2+r^2d\Omega^2)$$

$$+\frac{(A(y)+B(y)r^2)^2}{(r^2F^2+e^{-2h(y)})^2}dy^2. \qquad (3.2)$$

This metric is only one possible generalization of the Riemann-flat metric in the case in which $g_t = 0$ and $g_y \neq 0$ (see equation (2.28)). Of course, assuming $F = F(t,y)$, $L = L(t,y)$, $A = A(t,y)$ and $B = B(t,y)$ would be even more general, but the corresponding ${}^5R_{ij} = 0$ have proved to be too complicated to solve.

There are seven non-trivial Ricci tensor components corresponding to metric (3.2). Each one is a power series expansion (with respect to r), and putting all the coefficients equal to zero gives rise to 23 coupled partial differential equations for the unknown functions C, J, F, L, A and B. Most of the equations have a huge number of terms (some over 60 terms). Upon examination of the equations, we find that $R_{25} = 0$ contains the smallest number of terms and these involve only first order derivatives. R_{25} is given by

$$\begin{aligned} R_{25} = \quad & +2r^2\,J\,e^{(-2h)}\,L_y\,A\,F - 24r^2\,C\,h_y\,e^{(-2h)}\,L\,B\,F \\ & -4r^2\,C_y\,e^{(-2h)}\,L\,B\,F + 4r^2\,J_y\,e^{(-2h)}\,L\,A\,F \\ & -4\,F_y\,e^{(-2h)}\,A\,J\,F - 4r^6\,F^2\,L\,F_y\,B\,J \\ & -4\,A\,L\,F_y\,e^{(-2h)}\,C + 8r^4\,A\,L_y\,F^3\,J \\ & -2r^4\,J\,e^{(-2h)}\,L_y\,B\,F + 4\,F\,A\,L_y\,e^{(-2h)}\,C \\ & -12r^2\,J\,F_y\,e^{(-2h)}\,B\,F + 2r^6\,F^2\,A\,L\,L_y\,J \\ & -4\,F^2\,C_y\,e^{(-2h)}\,B + 4\,F^2\,J_y\,e^{(-2h)}\,A \\ & -24r^4\,J\,h_y\,e^{(-2h)}\,L\,B\,F - 6\,F_y\,e^{(-2h)}\,A\,J\,L\,r^2 \\ & -24\,F^2\,C\,h_y\,e^{(-2h)}\,B - r^4\,J\,e^{(-2h)}\,L_y\,A\,L \\ & +r^6\,F^2\,C\,L_y\,B\,L - 4r^4\,F^3\,C_y\,L\,B - 4r^2\,F^4\,C_y\,B \\ & -24r^4\,J\,F_y\,L\,A\,F^2 - r^6\,F^2\,C_y\,L^2\,B + 4r^4\,F^3\,J_y\,L\,A \end{aligned}$$

$$-12F^3CF_yA+4r^2F^3CF_yB-6r^4FCF_yL^2A$$
$$-6r^4Ch_ye^{(-2h)}L^2B-r^4C_ye^{(-2h)}L^2B$$
$$+r^4J_ye^{(-2h)}L^2A-6r^6JF_yL^2AF$$
$$-6r^6Jh_ye^{(-2h)}L^2B-24r^2JF^2h_ye^{(-2h)}B$$
$$-3r^6Je^{(-2h)}L_yBL+r^6F^2J_yL^2A$$
$$-16r^2JF^3F_yA-10r^4JF_ye^{(-2h)}BL$$
$$-2r^4F^2CF_yBL-22r^2F^2CF_yLA$$
$$-2r^4Be^{(-2h)}LL_yC-8r^2Be^{(-2h)}LF_yC$$
$$-8FBe^{(-2h)}F_yC+4r^2F^4J_yA+3r^4F^2CL_yAL$$
$$+4r^6L_yBF^3J+6r^4F^3CL_yB+10r^2F^3CL_yA$$

The following method is used to reduce our problem to a finite number of possible cases. The coefficient of r in the power series expansion defines J_y in terms of C_y, F_y, L_y and h_y. The vanishing of the r^3 coefficient defines h_y in terms of L_y and F_y. Now, the vanishing of the coefficients for r^5 and r^7 (with the expressions for J_y and h_y taken into account) will give rise to the following simple equation:

$$\frac{\partial L}{\partial y}=\frac{L}{F}\frac{\partial F}{\partial y} \quad . \tag{3.3}$$

Now, we recognize 2 cases: either $L=0$, or (3.3) can be integrated resulting in

$$L(y)=cF(y) \quad ,$$

where c is a constant. The case $L=0$ can be included in the case $c=0$. If c is nonzero it can be set equal to ±1 for the following reason. If $L=cF$ then

$$\sqrt{g_{22}}dr=\frac{F(1+cr^2)dr}{r^2F^2+e^{-2h}} \quad .$$

Now if $r\to\bar{r}$ and $F\to\bar{F}$ such that $rF=\bar{r}\bar{F}$, then if $\bar{r}=\lambda r$ then $\bar{F}=\frac{1}{\lambda}F$. After this transformation

$$\sqrt{g_{22}}dr=\frac{\bar{F}(1+\frac{c}{\lambda^2}\bar{r}^2)}{\bar{r}^2\bar{F}^2+e^{-2h}}d\bar{r} \quad ,$$

and if λ is taken as $\lambda=\sqrt{c}$ ($c>0$), then

$$dr\sqrt{g_{22}}=\frac{\bar{F}(1+\bar{r}^2)d\bar{r}}{\bar{r}^2\bar{F}^2+e^{-2h}} \quad .$$

If $c<0$, we take $\lambda=\sqrt{-c}$. Therefore a transformation can be made to set $c=\pm1$.

- **Case $L = F$**

The calculations below are only highlights of the whole calculation. The second order derivatives C_{yy}, F_{yy}, L_{yy} and J_{yy} can be defined by the r^0 coefficient of R_{33} and the r^{16} coefficient of R_{11}. Now the r^{14} coefficient of R_{55} with the above expressions for the second derivatives and with $L = F$ give rise to

$$F^5 B F_y(-3JF_y + J_y F) = 0. \tag{3.4}$$

Since $F_y \neq 0$ (F must remain as a function of y as a generalization of Riemann-flat case), the solution of (3.4) is either $B = 0$ (which turns out not to be possible) or $J = F^3\omega(t)$, where ω is an arbitrary function. Since $\omega = 0$ results in $J = 0$ (which is not desired), we have that $\omega \neq 0$, and so ω can be absorbed by a time transformation.

Thus, we have

$$J = F^3 \quad .$$

Then the r^{16} coefficient of R_{11} gives

$$-FF_{yy}B + BF_y^2 + FB_yF_y = 0, \tag{3.5}$$

which can be easily integrated to give

$$B(y) = k\frac{F_y}{F}, \tag{3.6}$$

where k is a constant. Given the above expression for B, the coefficient of r^3 minus the coefficient of r for $R_{25} = 0$ give the result

$$e^{2h}C(t,y)(F^3A + 2kh_yFe^{-2h} + kF_ye^{-2h}) = F(F^3A + 2kh_yFe^{-2h} + kF_ye^{-2h}).$$

If $F^3A + 2kh_yFe^{-2h} + kF_ye^{-2h} \neq 0$ (the vanishing case will actually lead to the same result), then

$$C(t,y) = F(y)e^{-2h(y)} \equiv C(y) \quad , \tag{3.7}$$

which implies C is only a function of y. Taking (3.7) taken into account, the r^0 coefficient of R_{11} yields

$$FF_yA_y - 3AF_y^2 - FAF_{yy} - 4AFF_yh_y = 0,$$

which in turn can be easily integrated to give

$$A(y) = qF^3F_ye^{4h}, \tag{3.8}$$

where q is a constant. Considering this, the r^4 coefficient of R_{11} yields

$$qF^6e^{4h}(F_y + Fh_y) = ke^{-2h}(F_y + Fh_y) \ . \tag{3.9}$$

If $F_y + Fh_y = 0$ then integration with respect to y gives

$$h(y) = \ln(\frac{p}{F(y)}) \ , \tag{3.10}$$

where p is another constant that can be set to unity from the vanishing of the r^0 coefficient of R_{22}. On the other hand, if $F_y + Fh_y \neq 0$ then (3.9) also leads to (3.10). With the expressions obtained for $J(t,y)$, $C(t,y)$, $A(y)$, $B(y)$ and $h(y)$, the Ricci-flat solution is given by

$$ds^2 = -F^2(y)dt^2 + \frac{1}{F^2(y)}(dr^2 + r^2d\Omega^2) + \frac{F_y^2}{F^6}dy^2 \ . \tag{3.11}$$

The above metric is Ricci-flat but not Riemann-flat. It is of course $4D$ flat on each slice $y = const$. The metric (3.11) can be taken to a simpler form by making the transformation $y \to \psi$ where $\psi = \dfrac{1}{F^2(y)}$, whence we obtain

$$ds^2 = -\psi^{-1}dt^2 + \psi(dr^2 + r^2d\Omega^2) + d\psi^2 \ . \tag{3.12}$$

Assuming a metric of the general form

$$ds^2 = -\psi^{2p}dt^2 + \psi^{2q}dx^2 + \psi^{2r}dy^2 + \psi^{2s}dz^2 + \psi^{2u}d\psi^2, \tag{3.13}$$

metric (3.12) has the form of (3.13) with

$$p = -\frac{1}{2}, \quad q = \frac{1}{2}, \quad r = \frac{1}{2}, \quad s = \frac{1}{2}, \quad u = 0 \ .$$

The above parameters satisfy the relations

$$p + q + r + s + u = 1 \tag{3.14}$$

$$p^2 + q^2 + r^2 + s^2 + u^2 = 1. \tag{3.15}$$

Metric (3.13) with relations (3.14) is a generalized Kasner solution [23]. Kasner solutions have been used to demonstrate why the fifth dimension ψ is so small; imagine an ever expanding universe with p, q, r, s all positive, then because of (3.14) u must be negative and hence, the fifth dimension is contracting.

- **case** $L = -F$

This case turns out to give rise the same result as $L = F$. We shall omit the details since the steps essentially follow the same path as in the previous case.

- **case** $L = 0$

Even in this case, the number of equations, as well as the number of terms in each equation, are still very large. The strategy here is as before; i.e., solving equations for first order derivatives, say J_y, from the first order PDEs. Substituting this expression back into the other first order PDEs results in

$$(\frac{J}{C} - F^2 e^{2h})(F_y F^2 A e^{2h} + 2h_y BF + F_y B) = 0 \quad . \tag{3.16}$$

Vanishing of either factors will both lead to

$$J(t, y) = C(t, y) F^2(y) e^{2h} \quad . \tag{3.17}$$

Using (3.17), the other PDEs yield

$$A(y) = -\frac{2Fh_y + F_y}{Fe^h} \tag{3.18}$$

$$B(y) = FF_y e^h \quad . \tag{3.19}$$

Substituting (3.17) - (3.19) into the remaining equations will yield the non-trivial PDE:

$$4JF_y^3 - 3FF_y^2 J_y + 2FF_y^2 h_y J + F_y J_{yy} F^2 - F^2 F_y h_y J_y - J_y F_{yy} F^2 = 0 \quad . \tag{3.20}$$

In order to solve (3.20) for $J(t, y)$ we make the simplification

$$J(t, y) = F^2(y) \phi(t, y) \quad , \tag{3.21}$$

where $\phi(t,y)$ is an unknown function ($\phi_t \neq 0$ and $\phi_y \neq 0$) to be found by (3.20). Substituting (3.21) into (3.20) yields

$$F_y^2\phi_y - FF_yh_y\phi_y - FF_{yy}\phi_y + FF_y\phi_{yy} = 0 \quad . \tag{3.22}$$

Equation (3.22) can be integrated after dividing both sides by $FF_y\phi_y$ (F_y is nonzero due to the ansatz we have chosen), resulting in

$$\phi_y = l(t)\frac{F_y}{F}e^h \quad , \tag{3.23}$$

where $l(t)$ is an arbitrary function. Another integration results

$$\phi(t,y) = l(t)\int \frac{F_y}{F}e^h + p(t) \quad , \tag{3.24}$$

where $p(t)$ is another arbitrary function. Using (3.19) - (3.24), we find that the $5D$ Riemann tensor is zero and we arrive at the same Riemann-flat metric we found in case II of chapter two (see (2.28)) .

3.2 Another Ansatz for finding Ricci-flat solutions

The ansatz here is made so that the field equations become integrable. The $5D$ spherically symmetric metric is again assumed to be

$$ds^2 = -f^2(t,r,y)dt^2 + g^2(t,r,y)(dr^2 + r^2d\Omega^2) + k^2(t,r,y)dy^2$$

As mentioned before, there are seven non-trivial components of ${}^5R_{ij} = 0$ and they cannot be integrated in general. We note that $R_{15} = 0$ has the least number of terms, and can be written as

$$3g_{yt} + 3g_tg_y - 3f_yg_t - 3k_tg_y = 0 \quad . \tag{3.25}$$

Now, assuming that $g_t = 0$ and $g_y = 0$, then this equation is trivially satisfied and makes for much simplification. In this case, the most natural generalization of the Riemann-flat solution obtained (case II, see (2.28)) with the above simplifying condition is hence

$$ds^2 = -f^2(t,r,y)dt^2 + g^2(r)(dr^2 + r^2d\Omega^2) + k^2(r,y)dy^2 \quad . \tag{3.26}$$

Now, $R_{25} = 0$ implies (see Appendix B for ${}^5R_{ij} = 0$)

$$f_{yr}k - f_y k_r = 0, \tag{3.27}$$

which can be readily integrated (assuming $f_y \neq 0$) giving

$$f_y = k\psi(t, y), \tag{3.28}$$

where ψ is an arbitrary function. Taking this into account, $R_{33} = 0$ can be integrated to give

$$rkf_r g_r + f_r gk + 3g_r fk + rfkg_{rr} + frg_r k_r + k_r fg = 0 \quad . \tag{3.29}$$

Dividing (3.29) by $rfgk$ and rearranging terms gives

$$\frac{f_r}{f} + \frac{k_r}{k}(\frac{g_r}{g} + \frac{1}{r}) = \text{ a function of } r \quad .$$

If $g_r + gr^{-1} = 0$, i.e., $g(r) = \frac{1}{r}$, then $R_{33} = -1$ which is not possible. Now, since $g_r + gr^{-1} \neq 0$, integrating (3.29) yields

$$f(t, r, y) = F(r, y)G(t, y) \quad , \tag{3.30}$$

where F is a certain functions of g and k and $G(t, y)$ is an arbitrary function. With (E.8) plugged into the remaining equations, $R_{33} = 0$ yields

$$rkF_r g_r + F_r gk + 3Fkg_r + rFkg_{rr} + rFg_r k_r + Fgk_r = 0 \quad ,$$

which can be integrated (after dividing both sides by Fk) to

$$F(r, y)k(r, y) = A(r)B(y) \quad , \tag{3.31}$$

where A is a certain function of g, g_r and g_{rr} and $B(y)$ is an arbitrary function. Now $R_{22} = 0$ and (3.31) imply that $k(r, y) = R(r)\phi(y)$, where ϕ can be set equal unity by a y-coordinate transformation, therefore

$$k(r, y) = k(r) \quad .$$

Because of (3.31), $F(r, y)$ is now separable where the y-dependence can be absorbed into $G(t, y)$ and hence

$$f(t, r, y) = F(r)G(t, y) \quad . \tag{3.32}$$

With (3.32), $R_{25} = 0$ leads to

$$-G_y(-kF_r + k_rF) = 0 \quad , \tag{3.33}$$

leaving two possibilities: either $G_y = 0$ or $F = ck$ (or after normalizing to set $c = 1$, since $F \neq 0$, $F = k$). In the case $G = G(t)$, we have $g_{00} = G(t)F(r)$ where $G(t)$ can be absorbed by a time coordinate transformation. Hence all three metric functions are only functions of the r-coordinate. Such a case has been investigated by [29] and also [30] which gives the Ricci-flat solution

$$ds^2 = -A^2(r)dt^2 + B^2(r)(dr^2 + r^2d\Omega^2) + C^2(r)dy^2 \tag{3.34}$$

where

$$A(r) = (\frac{ar-1}{ar+1})^{\epsilon k} \quad , \tag{3.35}$$

$$B(r) = \frac{(ar+1)^{\epsilon(k-1)+1}}{a^2r^2(ar-1)^{\epsilon(k-1)-1}} \quad , \tag{3.36}$$

$$C(r) = C_0(\frac{ar+1}{ar-1})^{\epsilon} \tag{3.37}$$

and ε and k are related through

$$\varepsilon^2(k^2 - k + 1) = 1 \quad .$$

To investigate the second case $F = k$, we first note that $R_{11} = 0$ has the form

$$rk(r)g(r)G(t,y)k_{rr} + rk(r)G(t,y)k_rg_r + 2k(r)g(r)G(t,y)k_r + rg(r)^3G_{yy} + rg(r)G(t,y)k_r^2 = 0 \quad , \tag{3.38}$$

which is of the following form

$$\frac{G_{yy}(t,y)}{G(t,y)} = R(r) \equiv C \quad . \tag{3.39}$$

Since the left-hand side is a function of t and y while the right-hand side is a function only of r, then both must be a constant; i.e., C is constant. The equation $G_{yy} = CG$

has the following solutions

$$G(t,y) = \begin{cases} \psi(t)\sin y + \phi(t)\cos y & \text{if } C < 0 \\ y + \psi(t) & \text{if } C = 0 \\ \psi(t)\sinh y + \phi(t)\cosh y & \text{if } C > 0 \end{cases}$$

case $C = 0$

In this case, $R_{11} = 0$ has the following form

$$rkk_{rr}g + rkg_r k_r + 2kk_r g + rgk_r^2 = 0 \quad ,$$

and it may be integrated giving

$$k_r kgr^2 = \eta = constant \quad . \tag{3.40}$$

Now $R_{33} = 0$, with (3.40), can be written as

$$\frac{\partial}{\partial r}(r^3 k^2 g_r) + 2\eta = 0 \quad ,$$

or

$$g_r = \frac{1}{k^2}\left(-\frac{2\eta}{r^2} + \frac{\mu}{r^3}\right) \quad . \tag{3.41}$$

where μ is another integration constant. Substituting (3.40) and (3.41) back into $R_{22} = 0$ now yields

$$g(r) = \frac{3\eta^2 r^2 - \beta^2}{3\mu k^2 r^2} \quad . \tag{3.42}$$

Using (3.42) in (3.40) gives expressions for $f(r)$, $g(r)$ and $k(r)$ as follows:

$$f(r) = k(r) = \left(\frac{\mu + \eta\sqrt{3}r}{\mu - \eta\sqrt{3}r}\right)^{\frac{\mp\sqrt{3}}{3}} \tag{3.43}$$

$$g(r) = \frac{3\eta^2 r^2 - \mu^2}{2r^2\mu}\left(\frac{\mu + \eta\sqrt{3}r}{-\mu + \eta\sqrt{3}r}\right)^{\pm\frac{2}{\sqrt{3}}} \quad . \tag{3.44}$$

Taking $m \equiv \frac{2\mu}{\sqrt{3}\eta}$, the Ricci-flat metric becomes

$$ds^2 = -\left(\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right)^{\pm\frac{2}{\sqrt{3}}}(y+\psi(t))^2dt^2 + (1+\frac{m}{2r})^4\left(\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right)^{\frac{2}{3}(3\mp2\sqrt{3})}(dr^2+r^2d\Omega^2) +$$
$$\left(\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right)^{\pm\frac{2}{\sqrt{3}}}dy^2 \quad . \tag{3.45}$$

The metric (3.45) is, in one sense, a subcase of the one-parameter family of solutions given by [29] (where $\alpha = \sqrt{3}$ and $\beta = 1$ in [29]) and [30] (where $k = -1$ and $\varepsilon = \frac{1}{\sqrt{3}}$), except for the fact that we have the extra term $(y + \psi(t))^2$ in g_{00} which cannot be transformed away, and so in another sense it is a more general solution (depending on the arbitrary function $\psi(t)$). This solution is also very much like that given by [32] except that ours is more general due to the term $y + \psi(t)$. The solution obtained above is a *new solution depending on an arbitrary function* $\psi(t)$. However, on each hypersurface $y = constant$ the induced matter has the same form as for the metric given by [29] without the term $(y + \psi(t))^2$ i.e., $\rho = P_{\|} + 2P_{\perp}$. An interesting astrophysical implication of the metric (3.45) is that since it is non-static it shows that Birkhoff's theorem is no longer valid in Kaluza-Klein theory. In ordinary $4D$ GR, Birkhoff's theorem states that the Schwarzschild metric, which is static, is the unique spherically symmetric asymptotically flat solution of the Einstein field equations. In addition, unlike the $4D$ Schwarzschild metric, (3.45) is singular at $r = m/2$. This can be seen by investigating the Kretschmann scalar $R_{abcd}R^{abcd}$ (which is a complicated function of r *alone*) which turns out to diverge at $r = m/2$.

case $C \neq 0$: Qualitative Analysis

In this case

$$G_{yy} = CG \ , \qquad \text{(C is a non-zero constant)} \tag{3.46}$$

and the remaining non-trivial ODEs are:

$$rgkk_{rr} + rkk_r g_r + 2kgk_r + rgk_r^2 + Crg^3 = 0 \ , \tag{3.47}$$

$$rg^2k_{rr} - rgk_r g_r - rkg_r^2 + kgg_r + rkgg_{rr} = 0 \ , \tag{3.48}$$

$$2rk_r g_r + 2gk_r + 3kg_r + rkg_{rr} = 0 \ . \tag{3.49}$$

Finding solutions for the above system of ODEs is a hard task, so instead we investigate the asymptotic behavior of above system using a qualitative analysis. Once again, the metric concerned in this case is

$$ds^2 = -(k(r)G(t,y))^2 dt^2 + g^2(r)(dr^2 + r^2 d\Omega^2) + k^2(r)dy^2, \tag{3.50}$$

where $k(r)$ and $g(r)$ and $G(t,y)$ satisfy (3.46) - (3.49). By making the transformation $\rho = \ln r$, $L = \ln g$ and $N = \ln k$, (3.47) - (3.49) take the following forms (after having g_{rr} isolated from (3.49) and substituting it back to (3.47) and (3.48)):

$$L_{\rho\rho} = -L_\rho^2 - 2N_\rho L_\rho - 2N_\rho - 2L_\rho \quad , \tag{3.51}$$

$$N_{\rho\rho} = -N_\rho^2 + 3N_\rho L_\rho + L_\rho^2 + 2L_\rho + 3N_\rho \quad , \tag{3.52}$$

$$4N_\rho L_\rho + L_\rho^2 + 2L_\rho + 4N_\rho + N_\rho^2 + Ce^{(-2N+2L+2\rho)} = 0 \quad . \tag{3.53}$$

Now, differentiating (3.53) with respect to ρ and substituting $L_{\rho\rho}$ and $N_{\rho\rho}$ into the expression obtained results in the same equation (3.53), which means that (3.53) is the first integral of equations (3.51) and (3.52). Equations (3.51) and (3.52) are an autonomous system (i.e., independent of ρ explicitly) and we shall investigate it below. Note that if we assume that

$$N = L + \rho \quad , \tag{3.54}$$

in the above, equations (3.51) and (3.52) become, respectively,

$$L_{\rho\rho} = -3L_\rho^2 - 6L_\rho - 2 \quad ,$$

$$L_{\rho\rho} = 3L_\rho^2 + 6L_\rho + 2 \quad ,$$

which results in $L_{\rho\rho} = 0$ or $L_\rho = \beta = constant$ and also $N_\rho = 1 + L_\rho = \alpha = const.$ Integrating $L_\rho = \beta$ gives $L = \rho\beta$ where the constant of integration is absorbed. With (3.54) the constraint equation (3.53) gives the following quadratic equation:

$$6\beta^2 + 12\beta + (5 + C) = 0 \quad ,$$

with solutions $\beta = -1 \pm \frac{\sqrt{1-C}}{\sqrt{6}}$ and $\alpha = 1 + \beta = \pm\frac{\sqrt{1-C}}{\sqrt{6}}$. Having obtained these we get the following power-law solutions:

$$g(r) \equiv e^L = e^{\beta\rho} = e^{\beta \ln r} = r^\beta - r^{-1\pm\frac{\sqrt{1-C}}{\sqrt{6}}}$$

and

$$k(r) = r^{\pm\frac{\sqrt{1-C}}{\sqrt{6}}}$$

Substituting the above solutions back into the field equations reveals that the constant C in the constraint equation must be -1 for the assumption we made above to be valid. These special solutions will appear below as the fixed point solutions of the above autonomous system which we are now going to study by using the geometric techniques of dynamical systems (see Appendix D).

By defining $L_\rho = l$ and $N_\rho = n$ and taking $\cdot \equiv \frac{d}{d\rho}$, equations (3.51) and (3.52) reduce to the two-dimensional dynamical system

$$\dot{l} = -2nl - l^2 - 2n - 2l \tag{3.55}$$

$$\dot{n} = 3nl + l^2 - n^2 + 2l + 3n \tag{3.56}$$

The system (3.55) and (3.56) has 4 fixed points at finite (n, l) and 6 at infinity. They are classified as follows:

- **A:** $(l = 0, n = 0)$

This solution corresponds to $g(r) = constant$ and $k(r) = constant$, and in order for this to be a solution of the system (3.55) and (3.56) C must be zero and the corresponding metric would be:

$$ds^2 = -(y + \sigma(t))^2 dt^2 + (dr^2 + r^2 d\Omega^2) + dy^2 \ . \tag{3.57}$$

This is clearly the asymptotic solution when $r \to \infty$ of the metric (3.45). This metric is $5D$ flat and on any slice $y = constant$ is $4D$ flat. Linear analysis around this point shows that it is of saddle-type with eigenvalues $\lambda = 2$ and -1. Therefore the set of solutions having this solution as an asymptote is of measure zero.

- **B:** $(l = -2, n = 0)$

This solution corresponds to $g(r) = \frac{1}{r^2}$ and $k(r) = constant$ with $C \equiv 0$, and the corresponding metric, after making the transformation $R \equiv -\frac{1}{r}$, is the same as (3.57). It is also of saddle-type with eigenvalues $\lambda = -2$ and 1.

- **C:** $(l = \frac{1}{\sqrt{3}} - 1, n = \frac{1}{\sqrt{3}})$

This solution corresponds to $g(r) = r^{\frac{1}{\sqrt{3}}-1}$ and $k(r) = r^{\frac{1}{\sqrt{3}}}$ with $C \equiv -1$. The metric obtained is

$$ds^2 = -r^{\frac{2}{\sqrt{3}}}(\sin y + \psi(t)\cos y)^2 dt^2 + r^{2(\frac{1}{\sqrt{3}}-1)}(dr^2 + r^2 d\Omega^2) + r^{\frac{2}{\sqrt{3}}} dy^2 \quad . \qquad (3.58)$$

This metric is $5D$ curved and also $4D$ curved on each slice $y = constant$. It is not static (another illustration of the violation of Birkhoff's theorem in $5D$). It does not necessarily have any time singularity and it does not have an event horizon. The Kretschmann scalar is independent of $\psi(t)$ and is given by $K = \frac{16}{3} r^{-\frac{4}{\sqrt{3}}}$ which diverges at $r = 0$. The fixed point **C** is an attracting focus with eigenvalues $\lambda_j = -\frac{\sqrt{3}}{2} \pm i\frac{\sqrt{5}}{2}$.

- **D:** $(l = -\frac{1}{\sqrt{3}} - 1, n = -\frac{1}{\sqrt{3}})$

This solution corresponds to $g(r) = r^{-\frac{1}{\sqrt{3}}-1}$ and $k(r) = r^{-\frac{1}{\sqrt{3}}}$ with $C \equiv -1$. The metric is given by

$$ds^2 = -r^{-\frac{2}{\sqrt{3}}}(\sin y + \psi(t)\cos y)^2 dt^2 + r^{2(\frac{1}{-\sqrt{3}}-1)}(dr^2 + r^2 d\Omega^2) + r^{-\frac{2}{\sqrt{3}}} dy^2 \quad . \qquad (3.59)$$

This fixed point if of type repelling focus with eigenvalues $\lambda_j = \frac{\sqrt{3}}{2} \pm i\frac{\sqrt{5}}{2}$.

The analysis of singular points at infinity of the autonomous system (3.55) and (3.56) requires the compactification of (l,n) space. Using standard polar coordinates defined by $l = rcos\theta$ and $n = rsin\theta$, one can transform the system (3.55) and (3.56) to the following system for the phase-space variables (r,θ):

$$\begin{cases} \dot{r} = -r^2cos^2\theta sin\theta + 3r^2sin^2\theta cos\theta - r^2cos^2\theta - r^2sin^3\theta - 2rcos^2\theta + 3rsin^2\theta \;, \\ \\ \dot{\theta} = 4rcos^2\theta sin\theta + rcos\theta sin^2\theta + 5sin\theta cos\theta + rcos^2\theta + 2 \;. \end{cases}$$

To compactify the phase space we make use of the transformations:

$$\overline{r} = \frac{r}{r+1}$$

$$\overline{\theta} = \theta$$

$$\frac{d\rho}{d\overline{\rho}} = (1-\overline{r})$$

where $r \to \infty$ corresponds to the circle $\overline{r} = 1$ in the compactified space. In this new compactified phase space $(\overline{r}, \overline{\theta})$ the system has the form:

$$\begin{cases} \dfrac{d\overline{r}}{d\overline{\rho}} = (1-\overline{r})[-\overline{r}^2 cos^2\overline{\theta} sin\overline{\theta} + 3\overline{r}^2 sin\overline{\theta} cos\overline{\theta} - \overline{r}^2 cos^3\overline{\theta} - \overline{r}^2 sin^3\overline{\theta} \\ \qquad -2\overline{r}(1-\overline{r})cos^2\overline{\theta} + 3\overline{r}(1-\overline{r})sin^2\overline{\theta} \ , \\ \\ \dfrac{d\overline{\theta}}{d\overline{\rho}} = 4\overline{r}cos^2\overline{\theta} sin\overline{\theta} + \overline{r} sin^2\overline{\theta} cos\overline{\theta} + 2(1-\overline{r}) + 5(1-\overline{r}) sin\overline{\theta} cos\overline{\theta} + \overline{r} cos^3\overline{\theta} \ . \end{cases}$$

We now look for the fixed points on the circle $\overline{r} = 1$ which corresponds to $r = \infty$. At $\overline{r} = 1$, $\frac{d\overline{r}}{d\overline{\rho}} = 0$, but to have $\frac{d\overline{\theta}}{d\overline{\rho}} = 0$ one must solve the trigonometric equation $cos\overline{\theta}(1 + 2sin(2\overline{\theta}) = 0$, which has six distinct solutions:

$$\overline{\theta} = \pi/2 \ , \ 3\pi/2 \ , \ \pi/2 + \pi/12 \ , \ \pi - \pi/12 \ , \ 3\pi/2 + \pi/12 \ , \ 2\pi - \pi/12$$

on the circle $\overline{r} = 1$. Each of above rays are the direction of a single trajectory approaching the fixed point on the circle.

To analyze the type of fixed points at infinity we make the transformation

$$x = \frac{1}{n} \ , \qquad y = \frac{l}{n} \ , \qquad \frac{d\rho}{d\tau} = x \ .$$

Therefore (3.55) and (3.56) take the form

$$\frac{dx}{d\tau} = x(-3y - y^2 + 1 - 2yx - 3x) \ ,$$

$$\frac{dy}{d\tau} = -y^3 - 2xy^2 - 4y^2 - 5yx - 2x - y \ .$$

At $x = 0$ (infinity of (n,l)-plane), $\frac{dx}{d\tau} = 0$ automatically, but solutions of

$$\frac{dy}{d\tau}|_{x=0} = -y^3 - 4y^2 - y = 0$$

give rise to the three fixed points $M(x=0,\ y=0)$, $N(x=0,\ y=-2+\sqrt{3})$ and $P(x=0,\ y=-2-\sqrt{3})$. Using a phase portrait plotter we find out that M is a saddle, N is a source and P is a sink. To investigate the asymptotic behavior of the solutions near infinity ($r\to\infty$) we notice that at the fixed points at infinity we have $y=\frac{l}{n}=\alpha=constant$, and substituting $l=\alpha n$ into (3.55) and (3.56), we obtain

$$\dot{n}=(-2-\alpha)n^2+(\frac{-2}{\alpha}-2)n \quad ,$$

$$\dot{n}=(\alpha^2+3\alpha-1)n^2+(2\alpha+3)n \quad .$$

At large values of n (when terms involving n are negligible comparing terms with n^2), we require that $\alpha^2+3\alpha-1=-2-\alpha$, with solutions $\alpha=-2\pm\sqrt{3}$. Now at large n,

$$\dot{n}\approx(-2-\alpha)n^2=\mp\sqrt{3}n^2 \quad ,$$

and after integration

$$n\approx\pm\frac{1}{\sqrt{3}}\frac{1}{\rho} \quad .$$

This in turn determines the asymptotic forms for $k(r)$ and $g(r)$ as follows:

$$g(r)\simeq(\ln r)^{\mp\frac{2}{\sqrt{3}}+1} \tag{3.60}$$

$$k(r)\simeq(\ln r)^{\pm\frac{1}{\sqrt{3}}} \tag{3.61}$$

In summary, the analysis of singular points at infinity shows that there are six points at infinity consisting of two sinks, two sources and two saddles. The two sources at infinity turn out to have asymptotic forms:

$$g(r)\simeq(\ln r)^{-\frac{2}{\sqrt{3}}+1}$$

$$k(r)\simeq(\ln r)^{\frac{1}{\sqrt{3}}}$$

so that $k(r)\to\infty$ and $g(r)\to 0$ (while the $5D$ volume element$\to\infty$) as $r\to\infty$. The two sinks at infinity have the asymptotic forms

$$g(r)\simeq(\ln r)^{\frac{2}{\sqrt{3}}+1}$$

$$k(r)\simeq(\ln r)^{-\frac{1}{\sqrt{3}}}$$

The two saddle points are not of primary interest because the set of solutions approaching them form a set of measure zero. Figure (3.1) gives the phase portrait of the system (3.55) and (3.56). Finally, here are a few notes on the exact solutions at finite and infinite values of the phase space.

- The solutions corresponding to the fixed points (0,0), (−2,0) are $4D$ and $5D$ flat, and therefore are asymptotically flat too.
- The solution corresponding to the point $(\frac{1}{\sqrt{3}} - 1,\ \frac{1}{\sqrt{3}})$ is asymptotically flat in the sense that $\lim_{r\to\infty} R_{ijkl} = 0$
- Asymptotic functions $g(r) \simeq (\ln r)^{\mp\frac{2}{\sqrt{3}}+1}$ and $k(r) \simeq (\ln r)^{\pm\frac{1}{\sqrt{3}}}$ corresponding to the fixed points at infinite values of N_ρ and L_ρ are not exact solutions of the corresponding Ricci-flat equations, since they are only first order approximation to the field equations.

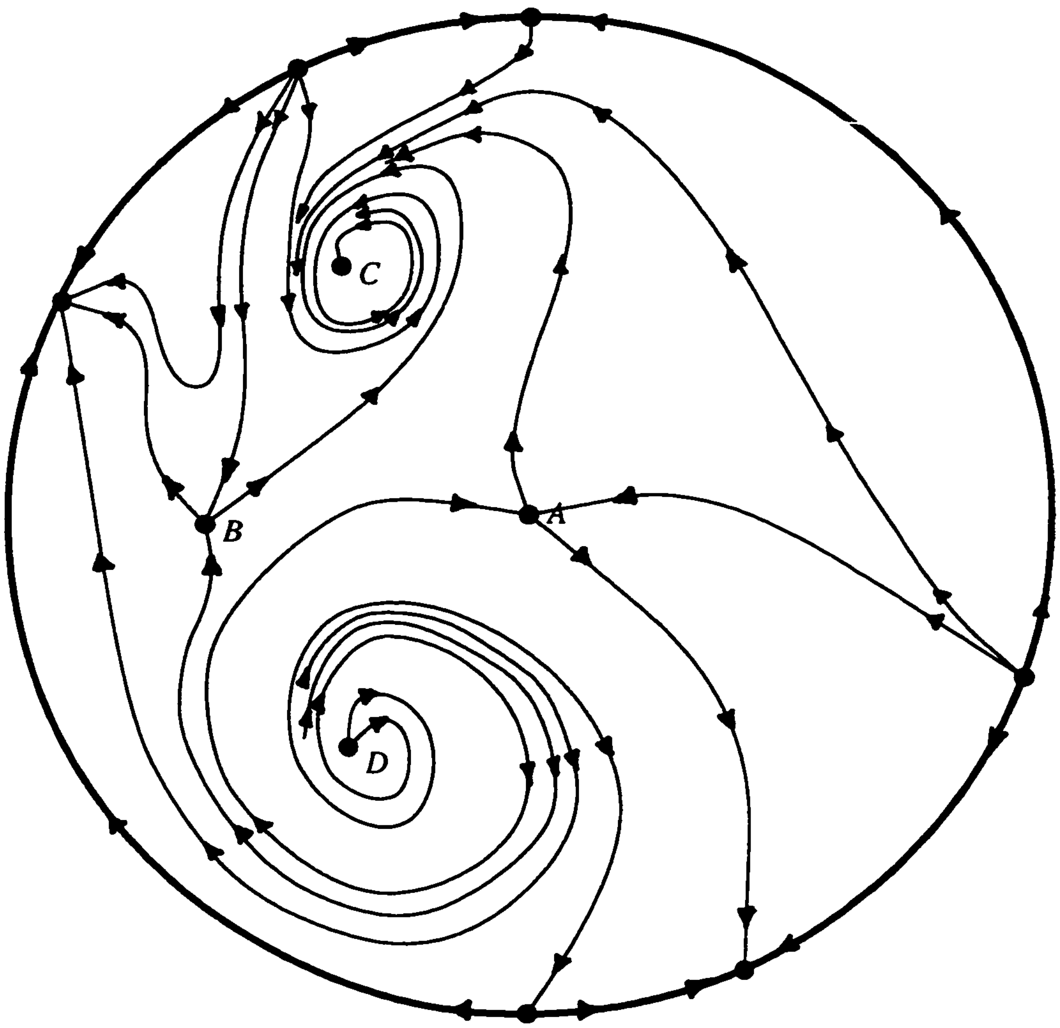

Figure 3.1: The complete phase portrait of the 2-dimensional system of ODEs (3.55) and (3.56)

3.3 The equation of state

As discussed in the introduction, the idea of induced matter theory is based on the splitting of the five-dimensional vacuum Einstein field equations ${}^5R_{ij} = 0$ into

$$ {}^{(4)}R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\ {}^{(4)}T = {}^{(4)}T_{\alpha\beta} \ , $$

where the $T_{\alpha\beta}$ terms are functions of all five coordinates and derivatives with respect to the coordinates (specifically with respect to the fifth coordinate) and these terms can be interpreted for example as density and pressure of the induced matter. The metric we are dealing with is

$$ ds^2 = -k(r)^2G(t,y)^2dt^2 + g(r)^2(dr^2 + r^2d\Omega^2) + k(r)^2dy^2 \ , $$

where k and g are governed by equations (3.47), (3.48) and (3.49). On the hypersurface $y = constant$ we calculate ${}^4G_{\alpha\beta}$ from above metric and we define

$$ -\rho \equiv G^0_0 = \frac{1}{rg(r)^4}[-rg_r^2 + 4gg_r + 2rgg_{rr}] \ , \tag{3.62} $$

$$ P_1 \equiv G^1_1 = \frac{1}{rk(r)g(r)^4}[2rgg_rk_r + rkg_r^2 + 2kgg_r + 2g^2k_r] \ , \tag{3.63} $$

$$ P_2 = P_3 \equiv G^2_2 = G^3_3 = \frac{1}{rk(r)g(r)^4}[g^2h_rkgg_r + rkgg_{rr} + rg^2k_{rr} - rkg_r^2] \ . \tag{3.64} $$

Substituting g_{rr} from (3.49) into (3.48) gives the following equation for k_{rr}

$$ rg^2k_{rr} - 3rgk_rg_r - rkg_r^2 - 2kgg_r - 2g^2k_r = 0 \ . \tag{3.65} $$

By substituting (3.65) and (3.49) in the expressions for ρ, P_1 and $P_2 \equiv P_3$ we get

$$ \rho = \frac{1}{rkg^4}(rkg_r^2 + 2gkg_r + 4rgg_rk_r + 4g^2k_r) \ , $$

$$ P_{\|} \equiv P_1 = \frac{1}{rkg^4}(2rgk_rg_r + rkg_r^2 + 2kgg_r + 2g^2k_r) \ , $$

$$ P_{\perp} \equiv P_2 = P_3 = \frac{1}{rkg^3}k_r(rg_r + g) \ . $$

This suggests an anisotropic fluid source with orthogonal pressures. By using the above equations we easily get the corresponding equation of state as

$$ \rho = P_{\|} + 2P_{\perp} \ , \tag{3.66} $$

which simplifies to the radiation equation of state in the perfect fluid case $P_{\|} = P_{\perp}$.

It should be noted that the equation of state for the special solutions (3.58) and (3.59) is given by

$$\rho = \frac{2}{3} r^{-\frac{2}{\sqrt{3}}}, \quad P_1 = 0, \quad P_2 = P_3 = \frac{1}{3} r^{-\frac{2}{\sqrt{3}}} \ ,$$

which of course satisfies (3.66). Note that $\rho > 0$ and $P_i \geq 0$ for all r which agrees with the energy conditions.

Chapter 4

A Unified Prescription for some Solutions in Five-dimensional Einstein Gravity

4.1 An even more general case

Currently, research in higher-dimensional cosmology has focused on examining cosmological models and spherically symmetric models. A spacetime is said to be spherically symmetric if its isometry group contains a subgroup isomorphic to the group $SO(3)$, and the orbits of this subgroup are two-dimensional spheres. In other words, a spherically symmetric spacetime is one whose metric remains invariant under rotations. The spacetime metric induces a metric on each orbit 2-sphere, which must be a multiple of metric of a unit 2-sphere. The most general spherically symmetric metric, which can be written in the form

$$ds^2 = -a^2(t,r,y) + b^2(t,r,y)(dr^2 + H^2(t,y)r^2 d\Omega^2) + c^2(t,r,y)dy^2 \quad , \tag{4.1}$$

or one of its specializations, is usually taken as the starting point. Extra assumptions need to be imposed on the form of the functions a, b and c, in order to make any further progress towards solving the $5D$ vacuum field equations.

In this chapter we consider the case where metric functions are separable in the variable r but not necessarily in the variables 't' and 'y'. Previously, investigations have been made by [28] where the case $H(t,y) = 1$ was investigated, aiming at generalizing previous work of Liu [33],Wesson [22] and Ponce de Leon [34]. The primary concern in these papers was to examine the existence of possible solutions. The field equations were taken to be the five-dimensional vacuum field equations, ${}^5R_{ij} = 0$, corresponding to the metric (4.1) with $H(t,y) = 1$. Mc Manus [28] observed that the three pivotal field equations

$$K_r \partial_t ln(C/B) = F_r \partial_t ln(CB^2)$$

$$F_r \partial_y ln(A/B) = K_r \partial_y ln(AB^2)$$

$$B_{ty} = B_t \partial_y ln(A) + B_y \partial_t ln(C)$$

divide the problem up into four distinct classes, namely (1) $F_r = K_r = 0$; (2) $F_r = 0, \ K_r \neq 0$; (3) $F_r \neq 0, \ K_r = 0$ and (4)$F_r K_r \neq 0$. The field equations were solved in all cases, either exactly or the problem was reduced to two coupled ordinary differential equations (see Appendix E). In particular, the exact solution

$$ds^2 = -A^2(t,y)(\frac{ar+1}{ar-1})^{\frac{2}{\sqrt{3}}}dt^2 + (\frac{a^2r^2-1}{a^2r^2})^2(\frac{ar-1}{ar+1})^{\frac{4}{\sqrt{3}}}[dr^2 + r^2 d\Omega^2] + C^2(t,y)(\frac{ar+1}{ar-1})^{\frac{2}{\sqrt{3}}}dy^2 \quad (4.2)$$

was obtained, where A and C satisfy

$$\partial_y(C^{-1}A_y) = \partial_t(A^{-1}C_t) \ .$$

The above metric generalizes the solution found by [35] (in which the case $C = 1$ was examined) and also a solution found by [34] (in which the case $A_t = C_t = 0$ was examined).

Our starting point in this chapter is the metric ansatz,

$$ds^2 = -e^{2F(r)}A^2(t,y)dt^2 + e^{2G(r)}B^2(t,y)(dr^2 + H^2(t,y)r^2 d\Omega^2) + e^{2K(r)}C^2(t,y)dy^2, \quad (4.3)$$

which is in an even more general form than the one studied by Mc Manus [28] in which it was assumed that $H(t, y) \equiv 1$. We wish to find all solutions of ${}^5R_{ij} = 0$ for the metric (4.3), which is the most general spherically symmetric metric with the given coordinates (t, r, θ, ϕ, y) where the dependence on r is separated. Components of the Ricci tensor for the above metric may be found in Appendix C. The results of Mc Manus [28] are briefly reviewed in Appendix E.

We observe that all components R_{11}, R_{22}, R_{33} and R_{55} have the same following forms:

$$R_{ii} = \frac{e^{2F(r)}}{e^{2G(r)}}[T_1(r)] + \frac{e^{2F(r)}}{e^{2K(r)}}[T_2(t,y)] + [T_3(t,y)] = 0. \tag{4.4}$$

We readily arrive at 2 possible cases: either $\frac{e^{2F(r)}}{e^{2K(r)}} = constant$ or $\frac{e^{2F(r)}}{e^{2K(r)}} \neq constant$. If $\frac{e^{2F(r)}}{e^{2K(r)}} = constant$, then the separability implies that

$$\text{Function of } (r) = \alpha = \text{constant} = \text{Function of } (t, y) \quad ,$$

and if $\frac{e^{2F(r)}}{e^{2K(r)}} \neq constant$ then this requires $T_2 = \alpha = constant$, $T_3 = \beta = constant$ with

$$\frac{e^{2F(r)}}{e^{2G(r)}}[T_1(r)] + \frac{e^{2F(r)}}{e^{2K(r)}}\alpha + \beta = 0 \quad .$$

Note that if $T_1 = 0$, then (4.4) implies that $e^{2(F-K)}T_2 + T_3 = 0$ which results in $\frac{e^{2F(r)}}{e^{2K(r)}} = constant$ which is not possible by assumption for this case. If

$$T_1 \frac{e^{2F(r)}}{e^{2G(r)}} \propto \frac{e^{2F(r)}}{e^{2K(r)}} \quad ,$$

then this requires

$$\frac{e^{2F(r)}}{e^{2K(r)}}(c + T_2) + T_3 = 0$$

(c is the constant of proportionality) which implies either $\frac{e^{2F(r)}}{e^{2K(r)}} = constant$ which is again impossible by assumption, or $c + T_2 = 0 = T_3$ which is contained in the case $\frac{e^{2F(r)}}{e^{2K(r)}} \neq constant$.

Due to the coordinate freedom we can rescale the t or y coordinates, and hence the possible cases are as follows:

- $F(r) = K(r)$
- $F(r) \neq K(r)$

where in the latter case we assume $F - K \neq constant$. In the following we shall look at each case in detail.

4.2 $F(r) = K(r)$

Writing $R_{11} = 0$ as

$$\text{Function of } (r) = \alpha = \text{Function of } (t, y)$$

gives the result

$$K_{rr} + 2K_r^2 + K_r G_r + 2\frac{K_r}{r} = \alpha \frac{e^{2G}}{e^{2K}} \ , \tag{4.5}$$

as well as "the rest of terms in $R_{11} = 0$ involving t and y" $= -\alpha$. Also, $R_{22} = 0$ yields

$$2K_{rr} + 2K_r^2 - 2K_r G_r + 2G_{rr} + 2\frac{G_r}{r} = \beta \frac{e^{2G}}{e^{2K}} \ . \tag{4.6}$$

Now, $R_{33} = 0$ has the following form:

$$\psi(r) + g(r)\phi(t, y) + \frac{1}{r^2}(1 - \frac{1}{H^2}) = 0 \ . \tag{4.7}$$

where $g(r) \equiv \frac{e^{2G}}{e^{2K}}$. By differentiating with respect to y, we get

$$g(r)\phi_y + \frac{1}{r^2}(\frac{2H_y}{H^3}) = 0 \ .$$

One possibility is that $H_y = 0$ and $\phi_y = 0$; i.e., $H = H(t)$ and $\phi = \phi(t)$. Since $H \neq constant$, then (4.7) implies that $r^2 e^{2(G-K)} = constant$ or

$$e^{2(G-K)} = \frac{constant}{r^2} \ . \tag{4.8}$$

Now if $H_y \neq 0$ and $\phi_y \neq 0$, the separability condition implies that

$$r^2 g(r) = \gamma = constant = \frac{-2H_y}{H^3 \phi_y} \ ,$$

which implies that $g(r) = \frac{\gamma}{r^2}$ or, in terms of the original variables,

$$e^{2(G-K)} = \frac{\gamma}{r^2} \quad .$$

That is, equation (4.8) is valid in both of these cases. We can now choose $\gamma \equiv 1$ by a rescaling in r which brings us to the following relation for the metric functions:

$$G(r) = K(r) - ln(r) \tag{4.9}$$

Now differentiating (4.9) with respect to r twice and substituting the result into (4.5) and (4.6) we get

$$K_{rr} + 3K_r^2 + \frac{K_r}{r} = \frac{\alpha}{r^2} \quad , \tag{4.10}$$

$$K_{rr} + \frac{K_r}{r} = \frac{\beta}{r^2} \quad . \tag{4.11}$$

Subtracting (4.10) from (4.11) gives the result

$$K_r = \frac{\delta}{r} \quad . \tag{4.12}$$

where δ is a constant. This case splits in two cases: either $\delta = 0$ which implies that $K(r) = constant$ and the constant could be set equal zero by a y-transformation, or $\delta \neq 0$ which yields

$$K(r) = ln(r^\sigma) \quad , \tag{4.13}$$

where σ is a constant and the constant of integration has been absorbed by a y-transformation. In the following sections we will look at the two cases $F = K = 0$ and $F = K = ln(r^\sigma)$ in more detail.

4.3 Simplifying the metric when $F(r) = K(r)$

When $F(r) = K(r)$ the original metric takes the following form:

$$ds^2 = e^{2F(r)}[-A^2(t,y)dt^2 + C^2(t,y)dy^2] + B^2(t,y)e^{2G(r)}dr^2 + B^2(t,y)H^2(t,y)r^2e^{2G(r)}d\Omega^2 \quad . \tag{4.14}$$

Taking $L \equiv BH$, one can always make a local Lorentz rotation in the (t, y) plane, setting $dtdy = 0$, and simplifying $L(t, y)$ as $L(t, y) = y$ or $L(t, y) = t$, depending upon whether $L_{,i}$ is space-like or time-like, respectively.

Let us first assume that $F = K = ln(r^{\sigma})$ $(\sigma \neq 0)$ so that

$$R_{12} = 0 \quad \Rightarrow \quad -3\sigma B_t = 0 \quad \Rightarrow \quad B_t = 0$$

$$R_{25} = 0 \quad \Rightarrow \quad -3\sigma B_y = 0 \quad \Rightarrow \quad B_y = 0,$$

which implies that

$$B(t, y) = B_0 = constant \ . \tag{4.15}$$

where B_0 can be set equal to unity by absorbing it into the function $G(r)$. Now we investigate the two cases:

Case(a): $(BH)_{,i} = space - like$, which implies that (after a coordinate transformation) one can write

$$BH \equiv y \ ,$$

or equivalently

$$H(y) = y \ .$$

Now, $R_{15} = 0$ implies that $-2C_t = 0$ and hence $C = C(y)$. The equation $R_{11} = 0$ has the form

$$-3\sigma^2 AC^3 y - 2Cb^2 A_y - yb^2 CA_{yy} + yb^2 C_y A_y = 0 \ , \tag{4.16}$$

and $R_{55} = 0$ reads

$$yA_{yy}b^2 C + 3\sigma^2 AyC^3 - yA_y C_y b^2 - 2b^2 Ac_y = 0 \ . \tag{4.17}$$

Now by adding (4.16) with (4.17) one gets

$$CA_y + AC_y = 0$$

which implies that

$$A(t, y) = \frac{\phi(t)}{C(y)} \ , \tag{4.18}$$

where $\phi(t)$ can be set equal to unity by a time transformation. Substituting these results into the $R_{11} = 0$ and $R_{55} = 0$ equations, which are in fact identical, yields

$$3\sigma^2 y C^4 - 2Cb^2 C_y + 3yb^2 (C_y)^2 - yC_{yy} Cb^2 = 0 \ , \tag{4.19}$$

and $R_{33} = 0$ yields

$$3\sigma^2 y^2 C^3 - 2yb^2 C_y - b^2 C^3 + b^2 C(y) = 0 \ . \tag{4.20}$$

By substituting C_y from (4.20) into (4.19), the latter is identically satisfied. Therefore, the only equation left is the following ODE for $C(y)$:

$$\frac{dC(y)}{dy} = (\frac{3}{2}\sigma^2) y C^3(y) - \frac{1}{2}\frac{C^3(y)}{y} + \frac{1}{2}\frac{C(y)}{y} \ . \tag{4.21}$$

It should be realized that once $\phi(t)$ is set equal to unity by a time transformation, one is not allowed to rescale $C(y)$ to unity. The differential equation (4.21) can be written as

$$\frac{d}{dy}(\frac{C(y)}{\sqrt{y}}) \ = \ (3\sigma^2 y^2 - 1)(\frac{C(y)}{\sqrt{y}})^3 \ ,$$

which can be easily integrated giving the solution for $C(y)$ as

$$C(y) = \sqrt{\frac{y}{y - \sigma^2 y^3 + \kappa}} \ ,$$

where κ is a constant of integration. With the obtained metric components, the Ricci-flat metric in this case is

$$ds^2 = -r^{2\sigma}\left(\frac{y - \sigma^2 y^3 + \kappa}{y}\right) dt^2 + r^{2(\sigma-1)}\left(dr^2 + y^2 r^2 d\Omega^2\right) + r^{2\sigma}\left(\frac{y}{y - \sigma^2 y^3 + \kappa}\right) dy^2 \ , \tag{4.22}$$

where σ is a non-zero constant. It is interesting to notice that if the integration constant κ is zero, the above metric is Riemann-flat. The Kretschmann scalar for metric (4.22) is given by $K = \frac{12\kappa^2}{r^{4\sigma} y^6}$, which diverges as $r \to 0$ (for $\sigma > 0$). Standard calculations leading to the effective density and pressures result in

$$\begin{cases} \rho = r^{-2\sigma}(\frac{1}{y^2} - \sigma^2) \\ P_1 = r^{-2\sigma}(3\sigma^2 - \frac{1}{y^2}) \\ P_2 = P_3 = r^{-2\sigma}\sigma^2 \end{cases}$$

where

$$\rho = -P_1 + P_2 + P_3 \ . \tag{4.23}$$

In the case $\sigma = 0$, after interchanging r and y and taking $\kappa = -m^2 < 0$, we obtain the Schwarzschild metric

$$ds^2 = -(1 - \frac{m^2}{r})dt^2 + (1 - \frac{m^2}{r})^{-1}dr^2 + r^2 d\Omega^2$$

on each hypersurface $y = constant$. If $\sigma \neq 0$, defining $R = \frac{r^\sigma}{\sigma}$ and interchanging R and y results in the following metric

$$ds^2 = \sigma^2 y^2 \left[-\left(\frac{R - \sigma^2 R^3 + \kappa}{R}\right) dt^2 + \left(\frac{R - \sigma^2 R^3 + \kappa}{R}\right)^{-1} dR^2 + R^2 d\Omega^2 \right] + dy^2 \ , \tag{4.24}$$

where σ^2 can be scaled to unity by a further coordinate transformation. This separable, static spherically symmetric solution was given in Mashhoon et al. [36] and also derived in [28]. The intrinsic 4- metric on the $y = constant$ hypersurfaces is the familiar Schwarzschild-de Sitter metric.

Case(b): $(BH)_{,i} = time - like$, which implies that (in the same fashion as the previous case)

$$H(t) = t \ .$$

An ODE similar to (4.21) for $A(t)$ is hence obtained:

$$\frac{dA(t)}{dt} = (-\frac{3}{2}\sigma^2)tA^3(t) + \frac{1}{2}\frac{A^3(t)}{t} + \frac{1}{2}\frac{A(t)}{t} \ , \tag{4.25}$$

with a similar solution for $A(t)$ given by

$$A(t) = \sqrt{\frac{t}{\sigma^2 t^3 - t + \kappa}} \ , \tag{4.26}$$

with the following Ricci-flat metric solution:

$$ds^2 = -r^{2\sigma}\left(\frac{t}{\sigma^2 t^3 - t + \kappa}\right) dt^2 + r^{2(\sigma-1)}\left(dr^2 + t^2 r^2 d\Omega^2\right) + r^{2\sigma}\left(\frac{\sigma^2 t^3 - t + \kappa}{t}\right) dy^2 \ , \tag{4.27}$$

with a similar equation of state to (4.23). Here again the above metric is Riemann-flat if and only if $\kappa = 0$. The Kretschmann scalar for the above metric is given by

$K = \dfrac{12\kappa^2}{r^{4\sigma}t^6}$, which has a time singularity at $t = 0$. It should be noted that the above metric doesn't compactify.

A final case (c) that we have not considered yet is that in which $(BH)_{,i} = light-like$. In this case, by definition

$$g^{ij}L_{,i}L_{,j} = 0 \quad \Rightarrow \quad -\frac{e^{-2F}}{A^2}(L_t)^2 + \frac{e^{-2F}}{C^2}(L_y)^2 = 0 \ ,$$

then

$$CL_t = AL_y$$

whence

$$C(BH)_t = A(BH)_y \ , \tag{4.28}$$

and since $B = B_0 = constant$, we get

$$CH_t = AH_y \ .$$

Although we have managed to simplify the equations somewhat, the remaining field equations have proven extremely difficult to analyse further.

4.4 $K(r) = 0 = F(r)$

In this case equation (4.9) takes the form $G(r) = -ln(r)$ (see Appendix C), in which case all r-dependent equations disappear and the metric functions A, B, C, and H satisfy five non-linear coupled partial differential equations in the variables t and y. This case is in a sense the most general case and it is difficult to make any further progress. The next step is to investigate if there are any separable solutions of ${}^5R_{ij} = 0$ for A, B, C and H.

Separability is now assumed as follows:

$$A(t,y) = a(y)$$

$$B(t,y) = b(t)\beta(y)$$

$$C(t,y) = c(t)$$

$$H(t,y) = h(t)\eta(y),$$

where the separated t-dependence in A has been absorbed by a time transformation, and the y-dependence in C has been removed by a y-transformation.

Equation $R_{33} = 0$ now takes the form

$$\frac{b^2h^2a^2}{c^2}\left[(\frac{\eta_{yy}}{\eta} + (\frac{\beta_{yy}}{\beta})^2 + 5(\frac{\beta_y}{\beta})(\frac{\eta_y}{\eta}) + 2(\frac{\beta_y}{\beta})^2 + (\frac{a_y}{a})(\frac{\eta_y}{\eta}) + (\frac{\beta_y}{\beta})(\frac{\eta_y}{\eta})\right] - b^2h^2\left[(\frac{c_t}{c})(\frac{h_t}{h}) + (\frac{c_t}{c})(\frac{b_t}{b}) + (\frac{h_t}{h})^2 + 2(\frac{b_t}{b})^2 + \frac{h_{tt}}{h} + \frac{b_{tt}}{b} + 5(\frac{b_t}{b})(\frac{h_t}{h})\right] - \frac{a^2}{\eta^2\beta^2} = 0 \quad . \tag{4.29}$$

Writing this expression in the form

$$\frac{b^2(t)h^2(t)}{c^2(t)}J(y) + T(t) + Y(y) = 0 \quad ,$$

results in two cases: either $bhc^{-1} = constant$ which, on taking derivatives with respect to t, gives rise to $\frac{dT}{dt} = 0$, or $\frac{bh}{c} \neq constant$ which, by differentiating with respect to t, results (after separation of variables) in $J(y) = constant$ and in turn yields $Y(y) = constant$, namely, $\frac{a}{\eta\beta} = constant$. To summarize, the two cases involved are

$$(i) \qquad a(y) = \eta(y)\beta(y) \quad , \tag{4.30}$$

or

$$(ii) \qquad c(t) = b(t)h(t) \quad , \tag{4.31}$$

after some rescaling. Case (ii) where $c(t) = b(t)h(t)$ turns out to be structurally quite similar to case (i).

In the following, we shall further investigate case (i) in which $a(y) = \eta(y)\beta(y)$. The equation $R_{ty} = 0$ now implies that

$$\frac{b_t}{b}\frac{\eta_y}{y} + \frac{c_t}{c}(3\frac{\beta_y}{y} + 2\frac{\eta_y}{\eta}) = 0 \quad . \tag{4.32}$$

We arrive at the following 5 separate cases to study:

- case(1) $\qquad b = const \qquad$ and $\qquad c = const$

- case(2) $b = const$ and $\beta^3\eta^2 = const$
- case(3) $c = const$ and $\eta = const$
- case(4) $\eta = const$ and $\beta = const$
- case(5) $\eta \neq const,\ c \neq const,\ b \neq const,\ \beta^3\eta^2 \neq const$.

In case (5) we immediately obtain

$$3\frac{\frac{\beta_y}{\beta}}{\frac{\eta_y}{\eta}} + 2 = m = const = \frac{\frac{-b_t}{b}}{\frac{c_t}{c}} \quad .$$

This can be easily solved to give

$$b(t) = c(t)^{-m} \tag{4.33}$$

and

$$\beta(y) = \eta(y)^{\frac{m-2}{3}} \quad , \tag{4.34}$$

where m is a constant. Then $R_{55} = 0$, after rearranging, can be written as

$$\eta^{\frac{2(m-1)}{3}}[(4m^2 - 10m + 4)(\frac{\eta_y}{\eta})^2 + 3(4m+1)\frac{\eta_{yy}}{\eta}] = p = 9\frac{c_{tt}}{c} - 27m(\frac{c_t}{c})^2 + 18\frac{c_t}{c}\frac{h_t}{h} \quad , \tag{4.35}$$

and $R_{22} = 0$ becomes

$$\eta^{\frac{2(m-1)}{3}}[(4m^2 - 10m + 4)(\frac{\eta_y}{\eta})^2 + 3(m-2)\frac{\eta_{yy}}{\eta}] = q = -9m\frac{c_{tt}}{c} + 27m^2(\frac{c_t}{c})^2 - 18m\frac{c_t}{c}\frac{h_t}{h} \quad . \tag{4.36}$$

Adding the right-hand sides of (4.35) and (4.36) gives

$$mp + q = 0 \quad ,$$

a relation between the three constants m, p and q, and subtracting the left-hand side of (4.36) from (4.35), given that $mp = -q$, gives

$$9(m+1)\frac{\eta_{yy}}{\eta} = p(m+1) \quad . \tag{4.37}$$

Now, if $m \neq -1$, we get $\eta_{yy} = (\frac{p}{9})\eta$, which in turn can be broken into the following three cases. When $p > 0$, we then have

$$\eta(y) = ae^{\frac{\sqrt{p}}{3}y} + be^{\frac{-\sqrt{p}}{3}y} \quad .$$

In the next chapter we shall find that there is no exponential solutions of this type possible. When $p < 0$, we get

$$\eta(y) = \alpha cos(\frac{\sqrt{-p}}{3}y + \beta) \quad ,$$

and no further progress can be made.

Finally, when $p = 0$, as seen easily from (4.37), we get

$$\eta(y) = c_1 y + c_2 \quad ,$$

where c_1 and c_2 are constants. Meanwhile the right-hand side of (4.36) yields

$$\frac{c_{tt}}{c_t} - 3m\frac{c_t}{c} + 2\frac{h_t}{h} = 0 \quad ,$$

which can be integrated, giving

$$c_t c^{-3m} h^2 = const \equiv 1 \quad . \tag{4.38}$$

Given this, $R_{11} = 0$ now yields

$$-6m(m-1)c^{6m} + 2(m-1)hh_t c + 2c^2h^3h_{tt} = 0 \quad , \tag{4.39}$$

and $R_{33} = 0$ implies

$$(1-3m)c^{1+m}h_t + c^{2(1-m)}hh_t^2 + c^{2(1-m)}h^2h_{tt} + c^2h = 0 \quad . \tag{4.40}$$

Cancelling h_{tt} from either of above equations and substituting the result into the other one gives rise a first order ODE which turns out not to be the first integral of the system of equations (4.39) and (4.40). This means that there are no solutions for the case $(p = 0, m \neq -1)$, since one can keep differentiating and each time get a new ODE for $h(t)$ (which has no common solutions with the rest of the equations). In other words, the system of equations is not consistent.

Now we look at the case $m = -1$. In this case, $R_{22} = R_{55} = 0$ implies that

$$2(\frac{\eta_y}{\eta})^2\eta - \frac{\eta_{yy}}{\eta} = \kappa = constant = cc_{tt} + 3c_t^2 + 2cc_t\frac{h_t}{h} \quad , \tag{4.41}$$

which gives the following non-linear ODE,

$$\eta\eta_{yy} - 2\eta_y^2 + \kappa\eta^2 = 0 \quad ,$$

which can be solved by series methods. In the special case where $p = 0$, we get

$$\eta(y) = -\frac{1}{y+\mu} \quad , \tag{4.42}$$

where μ is a constant and the other constant of integration has been transformed away. The other equation derived from (4.41) with $p = 0$ is

$$\frac{c_{tt}}{c_t} + 3\frac{c_t}{c} + 2\frac{h_t}{h} = 0 \quad , \tag{4.43}$$

which can be integrated giving

$$c_t c^3 h^2 = const \equiv 1 \quad .$$

Having taken this into account, $R_{11} = 0$ has the form

$$-2h_t c^4 h + c^8 h^3 h_{tt} - 6 = 0 \quad , \tag{4.44}$$

and $R_{33} = 0$ takes the form

$$4h_t + c^4 h_{tt} + c^2 h + c^4 h h_t^2 = 0 \quad . \tag{4.45}$$

Cancelling h_{tt} from the above equations gives the result

$$6h_t c^4 - h^2 c^6 - c^8 h^2 h_t^2 = 0 \quad .$$

If the above equation is a first integral of (4.44) and (4.45), its first derivative should result in the same equation; however, instead it gives rise to a different ODE, which means one can keep differentiating and obtain different equations. Hence, there are no solutions for the case (5) where $\eta \neq const, c \neq const, b \neq const$ and $\beta^3\eta^2 \neq constant$ when $p \geq 0$. In the case of $p < 0$ the remaining equations are intractable. We now return to the other cases:

Case(1) $b(t) = constant \equiv 1$, and $c(t) = constant \equiv 1$. $R_{22} = 0$ then takes the following form:

$$3\frac{\beta_y}{\beta} + 3\frac{\eta_y}{\eta} + \frac{\beta_{yy}}{\beta} = 0 \quad , \tag{4.46}$$

which can be integrated giving

$$\beta_y \beta^3 \eta^3 = constant \quad . \tag{4.47}$$

Given this relation, $R_{11} = 0$ can now be separated as

$$\frac{h_{tt}}{h} = const = \alpha = \text{some function of } (y) \quad ,$$

or

$$h_{tt} = \alpha h. \tag{4.48}$$

Now $R_{11} = 0$ takes the following form

$$-4\kappa\eta_y - 2\beta^4\eta_y^2 + 2\alpha\beta^2\eta^2 - \beta^4\eta^3\eta_{yy} = 0 \quad , \tag{4.49}$$

and $R_{55} = 0$ has the form

$$2\kappa\eta_y\eta^2\beta^4 - \beta^8\eta^5\eta_{yy} + 4\kappa^2 = 0 \quad . \tag{4.50}$$

These equations turn out to be incompatible since by cancelling η_{yy} it can be shown that the first order ODE obtained is not a first integral. Therefore, case (1) has no solutions.

Case(2) $b = constant$ and $\beta^3\eta^2 = constant$. Here $R_{22} = 0$ gives

$$2\frac{\eta_y}{\eta} - 3\frac{\eta_{yy}}{\eta_y} = 0 \quad ,$$

which has the general solution

$$\eta(y) = (y + \mu)^3 \quad ,$$

where the other constant of integration is absorbed. The remaining equations are

$$R_{11} = \frac{c_{tt}}{c} + 2\frac{h_{tt}}{h} = 0 \quad , \tag{4.51}$$

$$R_{33} = \frac{h_{tt}}{h} + \frac{h_t}{h}\frac{c_t}{c} + (\frac{h_t}{h})^2 + \frac{1}{h^2} = 0 \quad , \tag{4.52}$$

$$R_{55} = \frac{c_{tt}}{c} + 2\frac{h_t}{h}\frac{c_t}{c} - \frac{6}{c^2} = 0 \quad . \tag{4.53}$$

By cancelling h_{tt} and c_{tt} from two of above equations and substituting in the other equation, we get the following mixed first order ODE

$$(\frac{h_t}{h})^2 + 2\frac{h_t}{h}\frac{c_t}{c} + \frac{1}{h^2} - \frac{3}{c^2} = 0 \quad . \tag{4.54}$$

Differentiating (4.54) with respect to time and replacing h_{tt} and c_{tt} from the other equations gives rise to the same equation as (4.54), which means (4.54) is a first integral. Therefore, among equations (4.37), (4.52) and (4.53), only two of these are independent. Therefore, in this case the solutions are given by the metric

$$ds^2 \;=\; -(y+\mu)dt^2 + \frac{1}{r^2(y+\mu)^2}(dr^2 + h(t)r^2(y+\mu)^3 d\Omega^2) + c^2(t)dy^2 \quad ,$$

where $c(t)$ and $h(t)$ satisfy

$$\begin{cases} \frac{c_{tt}}{c} + 2\frac{h_{tt}}{h} = 0 \\ \frac{h_{tt}}{h} + \frac{h_t}{h}\frac{c_t}{c} + (\frac{h_t}{h})^2 + \frac{1}{h^2} = 0 \end{cases}$$

Case(3) $c \equiv constant = 1$ and $\eta \equiv constant = 1$. In this case $R_{55} = 0$ has the simple form $\beta_{yy} = 0$, which has the general solution $\beta = y + \mu$ where the other constant of integration is absorbed. The remaining equations are

$$3hb_{tt} - 3bh + 4b_t h_t + 2bh_{tt} = 0 \quad , \tag{4.55}$$

$$-bhb_{tt} + 3b^2h - 2hb_t^2 - 2bb_t h_t = 0 \quad , \tag{4.56}$$

$$5bb_t hh_t + bb_{tt}h^2 + b^2hh_{tt} - 3b^2h^2 + 2h^2b_t^2 + b^2h_t^2 + 1 = 0 \quad . \tag{4.57}$$

As before, cancelling b_{tt} and h_{tt} from two of the above equations and substituting into the third gives

$$4bhb_t h_t - 3b^2h^2 + 3b_t^2h^2 + b^2h_t^2 + 1 = 0 \quad , \tag{4.58}$$

which is the first integral of the above equations. Therefore, there are only two independent equations in this case for the functions $b(t)$ and $h(t)$, namely equations (4.55) and (4.56) or (4.55) and (4.58). Therefore, in this case the solution is given by

$$ds^2 \;=\; -(y+\mu)^2dt^2 + b^2(t)(y+\mu)^2[dr^2 + h^2(t)r^2d\Omega^2] + dy^2 \quad , \tag{4.59}$$

subject to these two ODEs.

Case(4) here $\eta = const = 1$, $\beta = const = 1$. In this case $R_{22} = 0$ implies

$$b_{tt} = -\frac{2\,b_t{}^2\,c\,h + 2\,b_t\,c\,b\,h_t + c_t\,b\,b_t\,h}{b\,h\,c} \quad , \tag{4.60}$$

and $R_{55} = 0$ yields

$$c_{tt} = \frac{-3\,c_t\,b_t\,h - 2\,c_t\,b\,h_t}{b\,h} \quad , \tag{4.61}$$

and R_{33} gives

$$h_{tt} = -\frac{3\,b\,h\,c\,b_t\,h_t + c + c\,b^2\,h_t{}^2 + c_t\,b^2\,h\,h_t}{b^2\,h\,c} \quad . \tag{4.62}$$

By substituting (4.60), (4.61) and (4.62) into $R_{11} = 0$ one obtains the following first order equation:

$$3\,b_t{}^2\,h^2\,c + 4\,b\,h\,c\,b_t\,h_t + 3\,c_t\,b\,h^2\,b_t + c + c\,b^2\,h_t{}^2 + 2\,c_t\,b^2\,h\,h_t = 0 \quad . \tag{4.63}$$

Differentiation of (4.63) results in the same equation which means that (4.63) is a first integral. Therefore, this case ends up with three second order ODEs for the unknown functions $b(t)$, $c(t)$ and $h(t)$.

To summarize case (i) where $a(y) = \eta(y)\beta(y)$, we have analyzed the field equations in detail and showed that in each tractable subcase either there are no solutions or the whole set of field equations is reduced to a system of two (three) coupled non-linear ordinary differential equations for two (three) unknowns.

In case (ii) in which $c(t) = h(t)b(t)$, the $R_{15} = 0$ equation implies that

$$\frac{\beta_y}{\beta}\frac{h_t}{h} + \frac{a_y}{a}(3\frac{b_t}{b} + 2\frac{h_t}{h}) = 0 \quad ,$$

which has the same structure as (4.32) where $a \to c$, $\eta \to h$, $\beta \to b$ and $y \to t$. The rest of the analysis follows the same as that of case (i) and results in similar conclusions. For example, the analogue of metric (4.59), given by

$$ds^2 \quad = \quad -dt^2 + \beta^2(y)(t+\mu)^2[dr^2 + \eta(y)^2 r^2 d\Omega^2] + (t+\mu)^2 dy^2$$

where $\beta(y)$ and $\eta(y)$ satisfy the two ODEs (4.55) and (4.58) (replacing b by β and h by η), is a Ricci-flat solution.

4.5 Case $F \neq K$ where $F - K \neq const.$

Here, according to the analysis done earlier, $R_{11} = 0$ takes the following form:

$$\frac{-e^{2F}}{e^{2G}}(F_{rr} + F_r^2 + F_r G_r + \frac{2F_r}{r} + F_r K_r) + \alpha\frac{e^{2F}}{e^{2K}} + \beta = 0 \ , \tag{4.64}$$

with

$$-\frac{B^2}{C^2}(\frac{A_{yy}}{A} + 3\frac{B_y}{B} - \frac{A_y}{A}\frac{C_y}{C} + 2\frac{A_y}{A}\frac{H_y}{H}) = \alpha \ , \tag{4.65}$$

and

$$\frac{B^2}{A^2}(3\frac{B_{tt}}{B} - 3\frac{A_t}{A}\frac{B_t}{B} + \frac{C_{tt}}{C} - \frac{A_t}{A}\frac{C_t}{C} + \frac{2H_{tt}}{H} + 4\frac{B_t}{B}\frac{H_t}{H} - 2\frac{A_t}{A}\frac{H_t}{H}) = \beta \ . \tag{4.66}$$

where α and β are constants. Also, $R_{22} = 0$ implies

$$F_{rr} + F_r^2 - F_r G_r + 2G_{rr} - K_r G_r + K_{rr} + K_r^2 + \frac{2}{r}G_r - \gamma\frac{e^{2G}}{e^{2F}} + \delta\frac{e^{2G}}{e^{2K}} = 0, \tag{4.67}$$

where γ and δ are constants. Now $R_{33} = 0$ has the following form:

$$\begin{aligned} & r^2\left[G_{rr} + G_r^2 + F_r G_r + K_r G_r + \frac{3}{r}G_r + \frac{F_r}{r} + \frac{K_r}{r}\right] \\ & + r^2(\frac{B}{A})^2\frac{e^{2G}}{e^{2F}}\left[-\frac{B_{tt}}{B} + \frac{A_t B_t}{AB} - 2(\frac{B_t}{B})^2 - \frac{C_t B_t}{CB} + \frac{A_t H_t}{AH} - \frac{H_{tt}}{H} - (\frac{H_t}{H})^2 - \frac{C_t H_t}{CH}\right] \\ & + r^2(\frac{B}{C})^2\frac{e^{2G}}{e^{2K}}\left[-\frac{B_{yy}}{B} + \frac{A_y B_y}{AB} + 2(\frac{B_y}{B})^2 - \frac{C_y B_y}{CB} + \frac{H_{yy}}{H} + (\frac{H_y}{H})^2 - \frac{C_y H_y}{CH}\right. \\ & \left. + 5\frac{B_y H_y}{BH} - \frac{C_y B_y}{CB}\right] + (1 - \frac{1}{H^2}) = 0 \ , \end{aligned} \tag{4.68}$$

which is of the following general form

$$f(r) + g(r)L(t,y) + h(r)M(t.y) + N(t,y) = 0, \tag{4.69}$$

where

$$g(r) \equiv r^2\frac{e^{2G}}{e^{2F}}$$

$$h(r) \equiv r^2\frac{e^{2G}}{e^{2K}},$$

$$N(t,y) \equiv 1 - \frac{1}{H^2}$$

and

$$f(r) \equiv r^2\left[G_{rr} + G_r^2 F_r G_r + K_r G_r + \frac{3}{r}G_r + \frac{F_r}{r} + \frac{K_r}{r}\right] \ .$$

A careful analysis of (4.69) consists of studying the following distinct cases:

- $L = c_1 = const$, $M = c_2 = constant$ and $N = c_3 = constant$ with $f + c_1 g + c_2 h + c_3 = 0$. Since $N = constant$ implies $H = constant$, this is the case studied by Mc Manus [28] in which $H(t, y) = 1$.

- $g = c_1 f + c_2$, $h = c_3 f + c_4$ where $f \neq constant$ (or equivalently $g = \lambda h + \mu$) with $N = c_2 L + c_4 M$ and $c_1 L + c_3 M + 1 = 0$. Here c_1 and c_3 cannot be zero simultaneously.

- $f = c_1 = constant$, $g = c_2 = constant$ and $h = c_3 = constant$ with $c_1 + c_2 L + c_3 M + N = 0$. Referring to (4.69) this implies that $F - K = const$ which is in contradiction with the case we are investigating now.

- $N = N_0 = constant$, $g(r) = c_1 h$, $M = -c_1 L + c_2$ with $f + c_2 h + N_0 = 0$. This again leads to the case studied by Mc Manus [28] in which $H(t, y) = 1$.

Therefore, assuming $H(t, y) \neq 1$, the only case which has not yet been studied is the secondcase above in which

$$g(r) = \lambda h(r) + \mu \quad ,$$

with $N = -\sigma L - \delta M$ and $\alpha L + \beta M + 1 = 0$ where $\lambda = \alpha/\beta$ and $\mu = \sigma - \alpha\delta/\beta$. This requires

$$r^2 \frac{e^{2G}}{e^{2F}} = \lambda r^2 \frac{e^{2G}}{e^{2K}} + \mu \quad ,$$

or equivalently

$$e^{2G} = \frac{\mu}{r^2} \frac{1}{e^{-2F} - \lambda e^{-2K}} \quad .$$

Since λ and μ are both non-zero (since the cases in which each of these vanishes leads to a case already studied), we can take $\mu = 1$ and $\lambda = 1$ by rescaling of r and y, respectively, to obtain

$$e^{2G(r)} = \frac{1}{r^2[e^{-2F} - e^{-2K}]}. \tag{4.70}$$

With this expression for G taken into account, $R_{12} = 0$ takes the following form

$$\begin{aligned} &F_r e^{-2F}[2CHB_t + HBC_t] + F_r e^{-2K}[-2CHB_t - 2CBH_t - HBC_t] + \\ &K_r e^{-2K}[+2CBH_t + HBC_t - HCB_t] + \\ &K_r e^{-2F}[-HBC_t + HCB_t] = 0, \end{aligned} \tag{4.71}$$

and $R_{25} = 0$ takes the form

$$F_r e^{-2F}[2HBA_y - HAB_y + 2ABH_y] + F_r e^{-2K}[-HBA_y + HAB_y] +$$
$$K_r e^{-2K}[HBA_y + 2AHB_y] +$$
$$K_r e^{-2F}[-HBA_y - 2AHB_y - 2ABH_y] = 0. \tag{4.72}$$

The functional form of (4.71) (and (4.72)) divides the problem into four distinct cases.

Case (a1) none of the terms $F_r e^{-2F}$, $F_r e^{-2K}$, $K_r e^{-2K}$, $K_r e^{-2F}$ are proportional. This case readily implies

$$2CHB_t + HBC_t = \alpha \tag{4.73}$$

$$-2CHB_t - 2CBH_t - HBC_t = \beta \tag{4.74}$$

$$2CBH_t + HBC_t - HCBt = \gamma \tag{4.75}$$

$$-HBC_t + HCB_t = \delta, \tag{4.76}$$

where α, β, γ and δ are arbitrary constants (and similar results follow from (4.72)). The case $\alpha = \beta = \gamma = \delta = 0$ must be treated as a separate case. When they are non-zero, (4.73) and (4.74) imply

$$CBH_t = constant,$$

and (4.74) and (4.75) imply

$$CHB_t = constant,$$

and (4.73) and (4.76) give

$$HBC_t = constant.$$

Dividing the above equations by each other implies

$$\frac{H_t}{H} = \sigma \frac{B_t}{B}$$

This can be integrated to give $H(t, y) = B^{\sigma} Y_1(y)$, where $Y_1(y)$ is an arbitrary function. Similarly

$$\frac{C_t}{C} = \mu \frac{B_t}{B}$$

can be integrated to yield $C(t, y) = B^{\mu} Y_2(y)$. Finally,

$$\frac{H_t}{H} = \lambda \frac{C_t}{C}$$

can be integrated to give $H(t, y) = C^{\lambda} Y_3(y)$.

Now $R_{25} = 0$, with the above conditions, yields

$$A_y H B = constant$$

$$B_y H A = constant$$

$$H_y A B = constant.$$

By similar methods we thus obtain

$$H(t, y) = B^{\nu} P(t)$$

$$H(t, y) = A^{\xi} Q(t)$$

$$A(t, y) = B^{\psi} R(t) \quad .$$

Now

$$H(t, y) = B^{\sigma} M(y)$$

$$H(t, y) = B^{\nu} P(t)$$

implies

$$B(t, y) = M(y) P(t).$$

Therefore, all of the functions H, C, A, B, are separable. Separable power-law solutions will be studied in the next chapter.

Case (a2) when $\alpha = \beta = \gamma = \delta = 0$, integration of (4.73), (4.74), (4.75) and (4.76) gives the following relations:

$$B^2 C = \alpha(y),$$

$$B C^{-1} = \beta(y),$$

$$B^2 H^2 C = \phi(y),$$

$$H^2CB^{-1} = \delta(y).$$

Combining the above relations leads to the fact that all the functions B, C and H are functions of only y. Repeating the analysis for $R_{25} = 0$ in this same case then implies that $H(t,y) = constant$, which was studied before by Mc Manus [28].

Case (b) all four terms $F_r e^{-2F}$, $F_r e^{-2K}$, $K_r e^{-2K}$, $K_r e^{-2F}$ are proportional. This implies

$$F_r e^{-2F} = \alpha F_r e^{-2K} \quad , \tag{4.77}$$

$$F_r e^{-2F} = \beta K_r e^{-2K} \quad , \tag{4.78}$$

$$F_r e^{-2F} = \gamma K_r e^{-2F} \quad . \tag{4.79}$$

Equation (4.77) now yields $e^{2(F-K)} = constant$ and hence, after a coordinate redefinition,

$$F - K = constant \quad .$$

This case has been studied before under the case $F = K$.

Case (c) three of the terms $F_r e^{-2F}$, $F_r e^{-2K}$, $K_r e^{-2K}$, $K_r e^{-2F}$ are proportional. There are four combinations of three terms and all of them lead to the previous case in which $F - K = constant$.

Case (d) two of the terms $F_r e^{-2F}$, $F_r e^{-2K}$, $K_r e^{-2K}$, $K_r e^{-2F}$ are proportional. There are six combinations of two terms which lead to either the previous case in which $F - K = constant$, or $\frac{F_r}{K_r} = constant$, or $e^{2F} = e^{2K} + constant$ or $e^{-2F} = e^{-2K} + constant$. The case where $\frac{F_r}{K_r} = constant$ can be integrated to obtain

$$F = aK + b \quad .$$

With $F = aK$ (b can be absorbed), $R_{12} = 0$ takes the following form:

$$\begin{aligned} &e^{-2aK}[-2aCHB_t - aHBC_t + BHC_t - HC_t] + \\ &e^{-2K}[-2CBH_t + \\ &2aCBH_t - HBC_t + 2aCHB_t + aHBC_t + HCB_t] = 0 \quad . \end{aligned} \tag{4.80}$$

This implies that

$$-2aCHB_t - aHBC_t + BHC_t - HCB_t = 0.$$

$$2aCHB_t + aHBC_t - BHC_t + HCB_t + H_tCB(2a-2) = 0 \quad .$$

After simplification, this reduces to

$$H_tCB(2a-2) = 0 \quad .$$

Since $a \neq 1$ [note that $a = 1$ implies $F = K$, which has been studied already] then

$$H_t = 0 \quad ; \quad H = H(y) \quad .$$

This in turn implies that,

$$[-2aCHB_t - aHBC_t + BHC_t - HCB_t] \left(e^{-2aK} - e^{-2K}\right) = 0 \quad ,$$

and hence

$$\frac{B_t}{B(1-a)} = \frac{C_t}{(1+2a)C} \quad .$$

After integration this becomes

$$B(t,y) = [C(t,y)]^{\frac{1-a}{1+2a}} \varphi(y) \quad .$$

A similar calculation for $R_{25} = 0$ (with $F = aK$) results in

$$H = H(t) \quad .$$

Therefore we have that $H = constant$, and this case was studied by McManus [28]. The two other cases $e^{2F} = e^{2K} + constant$ and $e^{-2F} = e^{-2K} + constant$ can be shown to also lead to the previously studied case $H(t,y) = constant$ in a similar way.

To summarize, in this chapter we have analyzed the most general spherically symmetric metric (4.3) in $5D$ where $H(t,y) \neq 1$. The problem broke up into two natural cases; $F = K$ and $F \neq K$. In the first case we employed methods to simplify (such as a separability assumption) the problem further. In most cases the problem was either reduced to one or a system of two nonlinear ordinary differential equations for the remaining unknown functions or shown to reduce to a case studied earlier. In the second case $F \neq K$ we showed that all solutions not studied previously are *necessarily* separable. In the next chapter we shall study the separable case further.

Chapter 5

Search for other simple solutions

In this chapter we will search for power-law and exponential solutions of the metric (4.3), as well as analysing some self-similar solutions in a $5D$ spherically symmetric spacetime.

5.1 Power-law Solutions

We assume separability and the following forms for the metric functions (some functional dependences have been absorbed by some coordinate transformations)

$$\begin{cases} A(t,y) = y^{\alpha} \\ B(t,y) = t^{\lambda} y^{\delta} \\ C(t,y) = t^{\beta} \\ H(t,y) = t^{\eta} y^{\sigma} \end{cases}$$

The Ricci-flat field equations (in the case where $F = K = 0$ and $G = -lnr$) are reduced to a set of algebraic equations for the constant exponents α, β, λ, δ, η and σ. The following are the non-trivial set of algebraic equations (corresponding to the equations $R_{00} = 0$, $R_{11} = 0$, $R_{44} = 0$, $R_{00} = 0$, $R_{22} = 0$ and $R_{04} = 0$, respectively):

$$y^2[3\lambda^2 - 3\lambda + 4\lambda\eta + 2\eta^2 - 2\eta + \beta^2 - \beta] +$$
$$y^{2\alpha}t^{2(1-\beta)}[-3\alpha\delta - 2\alpha\sigma - \alpha^2 + \alpha] = 0, \quad (5.1)$$

$$y^{2(-\alpha+\delta+1)}t^{2\lambda}[-3\lambda^2 + \lambda - 2\lambda\eta - \beta\lambda] +$$
$$y^{2\delta}t^{2(-\beta+\lambda+1)}[\alpha\delta + 3\delta^2 + 2\delta\sigma - \delta] = 0, \quad (5.2)$$

$$t^2[\alpha^2 - \alpha + 3\delta^2 - 3\delta + 4\delta\sigma + 2\sigma^2 - 2\sigma] +$$
$$t^{2\beta}y^{2(1-\alpha)}[-\beta^2 + \beta - 3\beta\lambda - 2\beta\eta] = 0 \ , \quad (5.3)$$

$$y^{2(\delta+\sigma)}t^{2(-\beta+\lambda+\eta+1)}[3\delta^2 + 5\delta\sigma + 2\sigma^2 + (\alpha-1)(\delta+\sigma)]$$
$$- \ y^{2(-\alpha+\delta+\sigma+1)}t^{2(\lambda+\eta)}[+3\lambda^2 + 5\lambda\eta + 2\eta^2 + (\beta-1)(\lambda+\eta)]$$
$$- \ y^2t^2 = 0 \ , \quad (5.4)$$

$$3\lambda\delta - 3\alpha\lambda - 3\beta\delta + 2\lambda\sigma + 2\eta\delta + 2\eta\sigma - 2\alpha\eta - 2\beta\sigma = 0 \ . \quad (5.5)$$

The above equations can be split into four cases:

- $\alpha = 1$ & $\beta = 1$

Here (5.1) takes the form

$$3\lambda^2 - 3\lambda - 3\delta + 4\lambda\eta + 2\eta^2 - 2\eta - 2\sigma \ , \quad (5.6)$$

also (5.2), (5.3), and (5.5) appear as

$$3\lambda^2 + 2\lambda\eta - 3\delta^2 - 2\delta\sigma \ , \quad (5.7)$$

$$3\delta^2 - 3\delta - 3\lambda + 4\delta\sigma + 2\sigma^2 - 2\sigma - 2\eta \ , \quad (5.8)$$

$$3\delta\lambda - 3\lambda - 3\delta + 2\lambda\sigma + 2\eta\delta + 2\eta\sigma - 2\eta - 2\sigma = 0 \ . \tag{5.9}$$

The only non-trivial solution of (5.6), (5.7), (5.8) and (5.9) is

$$\lambda = 2, \qquad \delta = 2, \qquad \eta = 0, \qquad \sigma = 0 \ . \tag{5.10}$$

This results in a $5D$ flat solution

$$ds^2 = -y^2dt^2 + y^4t^4(dr^2 + r^2d\Omega^2) + t^2dy^2 \ , \tag{5.11}$$

which is, in fact, $4D$ curved. It should be noted that the solution obtained above is special case of the solution found by Ponce de Leon [25] (with their free parameter equal to 1/2). The corresponding equation of state for this case has the form $P = -2/3\rho$.

- $\alpha = 1$ & $\beta \neq 1$

This case turns out to be impossible due to the inconsistency of the resulting system of algebraic equations (which is similar to the system in the previous case).

- $\alpha \neq 1$ & $\beta = 1$

This case also turns out to be impossible.

- $\alpha \neq 1$ & $\beta \neq 1$

In this final case the set of algebraic equations to be satisfied are :

$$3\lambda(\lambda - 1) + 2\eta(\eta - 1) + \beta(\beta - 1) + 4\lambda\eta = 0$$

$$-\alpha(3\delta + 2\sigma + \alpha - 1) = 0$$

$$\lambda(-3\lambda + 1 - 2\eta - \beta) = 0$$

$$\delta(3\delta + 2\sigma + \alpha - 1) = 0$$

$$\beta(-3\lambda + 1 - 2\eta - \beta) = 0$$

$$3\delta(\delta - 1) + 2\sigma(\sigma - 1) + \alpha(\alpha - 1) + 4\delta\alpha = 0$$

$$3\delta\lambda - 3\alpha\lambda - 3\beta\delta + 2\lambda\sigma + +2\eta\delta + 2\eta\sigma - 2\alpha\eta - 2\beta\sigma = 0$$

In addition, there are algebraic relations from the equation $R_{22} = 0$, but we shall not to consider them yet. Again, there are four cases to consider.

Case(1) $3\delta + 2\sigma + \alpha - 1 = 0$ and $3\lambda + 2\eta + \beta - 1 = 0$

Now $R_{22} = 0$ takes the form

$$\begin{aligned} & y^{2(\sigma+\delta)}t^{2(4\lambda+3\eta)}[3\delta^2 + 5\delta\sigma + 2\sigma^2 - (\delta + \sigma)(+3\delta + 2\sigma)] \\ - \; & y^{2(4\delta+3\sigma)}t^{2(\lambda+\eta)}[3\lambda^2 + 5\lambda\eta + 2\eta^2 - (\lambda + \eta)(3\lambda + 2\eta)] - y^2t^2 = 0 \quad . \end{aligned} \tag{5.12}$$

Since $\sigma + \delta = 1$ with $4\lambda + 3\eta = 0$, and $4\delta + 3\sigma = 1$ with $\lambda + \eta = 1$ implies that $R_{\theta\theta} = -1$, there is no solution in this case. Other possibilities are $\sigma + \delta = 1$ with $4\lambda + 3\eta = 0$ with either $4\delta + 3\sigma \neq 1$ or $\lambda + \eta \neq 1$. These cases turn out to be impossible as well.

Case(2) $\alpha = \lambda = \delta = \beta = 0$

In this case $R_{04} = 0 = 2\eta\sigma$ with $R_{rr} = 0 = \eta(\eta - 1)$ and $R_{yy} = 0 = \sigma(\sigma - 1)$. The solutions $\sigma = 0$ and $\eta = 0$ correspond to $H = constant$, which have been studied before, and $\sigma = 0$ and $\eta = 1$ give the result that $R_{22} = -2$ and hence there is no solution. When $\sigma = 1$ and $\eta = 0$, there is the new solution (after redefining r coordinate as $R = lnr$)

$$ds^2 = -dt^2 + dR^2 + y^2 d\Omega^2 + dy^2 \quad . \tag{5.13}$$

Calculations show that the above Ricci-flat metric is actually Riemann-flat but it is $4D$ Riemann-curved. It should be noted that the above metric doesn't compactify.

Case(3) $\alpha = 0$, $\delta = 0$ and $\beta = 1 - 3\lambda - 2\eta$.

Here $R_{yy} = 0 = \sigma(\sigma - 1)$. When $\sigma = 0$ then $R_{\theta\theta} = -1$ (i.e. there is no solution), so $\sigma = 1$ whence $R_{22} = 4\lambda + 3\eta - 1 = 0$ and $R_{00} = (2\lambda + 1)(\lambda - 1) = 0$ (i.e. $\lambda = -\frac{1}{2}$ or $\lambda = 1$). We arrive at two new solutions; the first is ($\lambda = 1$, $\eta = -1$), after redefining r coordinate as $R = lnr$,

$$ds^2 = -dt^2 + t^2 dR^2 + y^2 d\Omega^2 + dy^2, \tag{5.14}$$

which turns out to be $5D$ Riemann-flat but $4D$ curved. The other solution ($\lambda = -\frac{1}{2}$, $\eta = 1$), again after redefining r the same way,

$$ds^2 = -dt^2 + t^{-1} dR^2 + ty^2 d\Omega^2 + tdy^2, \tag{5.15}$$

which is Riemann-curved both in $5D$ and $4D$. The Kretschmann scalar for the above metric is given by $K = \frac{9}{2t^4}$, which indicates that the above metric has a big bang singularity at $t = 0$ and also, since the expansion rate along the fifth dimension is positive, it implies that the model does not compactify. On each hypersurface $y = constant$ the $4D$ metric is of type Kantowski-Sachs metric. It is also intersting to note that by defining $R = ln(r)$ and interchanging R and y in the above metric we obtain the following Kasner metric:

$$ds^2 = -dt^2 + t\left(dR^2 + R^2 d\Omega^2\right) + t^{-1} dy^2 \quad . \tag{5.16}$$

Standard analysis of induced matter theory gives the following relations for the density and pressures in this model:

$$\begin{cases} \rho = -\frac{1}{4}\frac{y^2 - 4t}{t^2 y^2} \\ p_1 = +\frac{1}{4}\frac{y^2 - 4t}{t^2 y^2} \\ p_2 = -\frac{1}{4}\frac{1}{t^2} < 0 \\ p_3 = -\frac{1}{4}\frac{1}{t^2} < 0 \quad . \end{cases}$$

The energy condition $\rho \geq 0$ implies that $p_i \leq 0$ and they satisfy the false vacuum equation of state $\rho = -p_1$ with $p_2 = p_3 < 0$.

Case(4) $\lambda = 0$, $\beta = 0$ and $\alpha = 1 - 3\delta - 2\sigma$

The equation $R_{00} = 0$ implies that $\eta(\eta - 1) = 0$; however, neither of the solutions $\eta = 0$ or $\eta = 1$ lead to a self-consistent solution.

5.2 Exponential Solutions

Exponential solutions of form

$$\begin{cases} A(t,y) = e^{\alpha y} \\ B(t,y) = e^{\lambda t} e^{\delta y} \\ C(t,y) = e^{\beta t} \\ H(t,y) = e^{\eta t} e^{\sigma y} \end{cases}$$

have also been investigated. The field equations ${}^5R_{ij} = 0$ corresponding to metric (4.3) are again a set of algebraic equations in α, β, δ, η, λ and σ. However, detailed calculations show that there are no solutions of this form.

5.3 The analysis of some self-similar solutions in five-dimensional spherically symmetric space-time

Here we are interested in finding exact self-similar solutions of the five-dimensional vacuum Einstein equations. The metric of the form

$$ds^2 = -e^{2F(r/t,y/t)}dt^2 + e^{2G(r/t,y/t)}dr^2 + e^{2L(r/t,y/t)}r^2 d\Omega^2 + e^{2H(r/t,y/t)}dy^2 \quad , \qquad (5.17)$$

admits a homothetic vector of the form

$$\xi^\alpha = (t, r, 0, 0, y) \quad . \qquad (5.18)$$

The field equations ${}^5R_{ij} = 0$ consist of a system of coupled partial differential equations in terms of the variables $u = \frac{r}{t}$ and $v = \frac{y}{t}$. However, these equations are hard to solve. Therefore, let us focus our attention on the two following special cases:

Case(1): $F = F(r/t)$, $G = G(r/t)$, $L = L(r/t)$ and $H = H(r/t)$. In this case there is always a coordinate transformation in the $(y-t)$ plane which allows the two functions L and G to be set equal. Solutions in this case correspond to self-similar solutions in the Brans-Dicke theory [27] .

Case(2): F, G, L and H are functions of y/t only. Here there is no coordinate transformation to bring L and G equal. The vector $\xi^\alpha = (t, r, 0, 0, y)$ is still a homothetic vector. The system of equations ${}^5R_{ij} = 0$ are still hard to solve.

In addition, the metric of the form

$$ds^2 = -e^{2F(r/t,y/t)}dt^2 + e^{2G(r/t,y/t)}dr^2 + e^{2L(r/t,y/t)}r^2d\Omega^2 + r^2e^{2H(r/t,y/t)}dy^2 \quad , \tag{5.19}$$

admits the homothetic vector

$$\xi^\alpha = (t, r, 0, 0, 0) \quad , \tag{5.20}$$

where the g_{44} term in metric (5.17) has been multiplied by r^2. In case (1) the field equations again become a system of coupled ordinary differential equations; it is still hard to solve this system in general even in this special case. However, there is an exact solution due Roberts [37] for this special case which is given by

$$ds^2 = (1 \pm p\frac{t}{r})^{\pm\frac{1}{\sqrt{3}}}[-dt^2 + dr^2 + r^2(1 \pm p\frac{t}{r})d\Omega^2] + (1 \pm p\frac{t}{r})^{\mp\frac{2}{\sqrt{3}}}dy^2 \tag{5.21}$$

where p is a constant. It should be mentioned that Roberts' solution is not the most general one in this class of models.

Finally, the metric

$$ds^2 = -e^{2F(r/t,y/t)}dt^2 + y^2e^{2G(r/t,y/t)}dr^2 + y^2e^{2L(r/t,y/t)}r^2d\Omega^2 + e^{2H(r/t,y/t)}dy^2 \quad , \tag{5.22}$$

admits a homothetic vector of form

$$\xi^\alpha = (t, 0, 0, 0, y) \quad . \tag{5.23}$$

Chapter 6

Einstein-Yang-Mills Extensions of Induced Matter Theory

6.1 Motivation

The aim of this chapter is to extend the idea of induced matter theory to incorporate a larger class of physical fields and hence obtain more general equation of states. In higher-dimensional theories (e.g., unifications of gravity with weak and strong interactions, as well as electromagnetism), when one chooses coordinates such that the metric's off-diagonal components are associated with gauge fields, an isometry group of internal compact manifolds generates a non-Abelian group of gauge transformations which lead to an effective four-dimensional action for Einstein gravity plus non-Abelian gauge fields. Of course, finding solutions of the resulting field equations of these theories, in particular in the spatially homogeneous and isotropic case, is of interest in its own right. Normally, the four-dimensional properties of matter are investigated by assuming that the higher-dimensional vacuum equations of general relativity reduce to Einstein's four-dimensional theory with matter [20, 22], although higher-dimensional generalized Lagrangian extensions of general relativity (with the addition of quadratic curvature invariants to the Einstein-Hilbert action) have also been studied [23]. Here we shall investigate whether Einstein's four-dimensional theory with matter can be embedded in a higher-dimensional theory of Yang-Mills-type, i.e., whether the correct field equations are the vacuum Einstein-Yang-Mills (EYM) equations. The idea is that the extra terms present in the higher-dimensional field equations may play the rôle of the matter terms that appear on the right-hand sides of the embedded four-dimensional Einstein field equations with matter. As noted earlier, the notion that the properties of matter might have a geometric origin has been developed by many authors [38, 39, 30] and is in the spirit of the original Kaluza-Klein theory [5, 8, 9].

We shall consider the $D = 4 + N$ dimensional metric in the form

$$ds^2 = g_{ab}dx^a dx^b = g_{\alpha\beta}dx^\alpha dx^\beta + g_{AB}dy^A dy^B, \tag{6.1}$$

where $ds_4^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ is given by the Friedmann-Robertson-Walker (FRW) form,

$$ds_F^2 = -dt^2 + H^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{6.2}$$

where k is the normalized (i.e. $k = 0, \pm 1$) curvature constant. In this case the matter source is a perfect fluid with energy-momentum tensor given by (1.3), where μ and p, the energy density and pressure, respectively, are given by equations (1.4) and (1.5), viz

$$\mu = \frac{3}{H^2}(k + \dot{H}^2), \qquad p = -2\frac{\ddot{H}}{H} - \frac{1}{H^2}(k + \dot{H}^2).$$

The phenomenological physical quantities μ and p are of course to be interpreted in terms of more fundamental geometric quantities.

Recall that the five-dimensional metric given by (1.17), i.e.,

$$ds^2 = ds_F^2 + L^2(t)dy^2 \quad ,$$

in the flat case ($k = 0$) gives rise to the familiar solution

$$H = t^{\frac{1}{2}}, \quad L = t^{-\frac{1}{2}} \quad , \tag{6.3}$$

$$\mu = 3p = \frac{3}{4}t^{-2} \quad , \tag{6.4}$$

which represents the familiar flat FRW radiation model (see equations (1.18) and (1.19). In higher dimensions ($N > 1$) with flat spatial curvature, the familiar generalized Kasner models are derived [23].

Here we wish to examine cosmological models in which Maxwellian and Yang-Mills terms are added to the standard Einstein Hilbert action

$$S = \int d^D V \frac{-R}{4\kappa^2}. \tag{6.5}$$

In particular, in section (6.2.1) we augment (6.5) in five dimensions with an Abelian $U(1)$ field. Next we consider an $SO(3)$ model in D-dimensions in section (6.2.2). In section (6.2.3) we derive the induced matter from a five-dimensional theory of supergravity. Section (6.3.1) gives an example of how one could generalize these examples by considering anisotropy in the three-space.

6.2 Einstein-Yang-Mills Theories

6.2.1 Abelian Gauge Fields

We begin by examining five-dimensional Einstein gravity augmented by a Maxwellian field, described by the action

$$S = \int d^5 V \left\{ -\frac{(R+2\Lambda)}{4\kappa^2} - \frac{1}{4} F_{ab} F^{ab} \right\}, \tag{6.6}$$

where $\kappa^2 = 4\pi G$ and F_{ab} is the field tensor of a $U(1)$ Abelian gauge field. This model has been extensively studied in [40] and [41], in which the Rubin-Freund ansatz

$$F = \frac{QL}{4\pi H^3} dt \wedge dy, \tag{6.7}$$

has been assumed, and where the metric is given by

$$ds^2 = -dt^2 + H^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right] + L^2(t) dy^2. \tag{6.8}$$

By varying the action (6.6) one obtains the following relevant field equations:

$$\frac{\ddot{L}}{L} + 3\frac{\ddot{H}}{H} = \frac{2\Lambda}{3} - \frac{GQ^2}{3\pi H^6} \tag{6.9}$$

$$\frac{\ddot{L}}{L} + 3\frac{\dot{H}\dot{L}}{HL} = \frac{2\Lambda}{3} - \frac{GQ^2}{3\pi H^6} \quad (6.10)$$

$$\frac{\ddot{H}}{H} + \frac{\dot{H}\dot{L}}{HL} + \frac{2}{H^2}\left(\dot{H}^2 + k\right) = \frac{2\Lambda}{3} + \frac{GQ^2}{6\pi H^6}, \quad (6.11)$$

where a dot denotes d/dt. In [40] these field equations were used to describe a N-dimensional compact internal space with two additional dimensions : one time-like and one space-like. This particular example then implies that $k > 1$ in the above field equations. However, since the internal space is space-like, one could equally view this as describing a space-time with one time-like dimension, three space-like dimensions (by setting $n = 3$) and one internal dimension, and therefore k can be 0 or ± 1.

The solution to (6.9) for H and L is given by

$$\dot{H}^2 = L^2 = \frac{\sqrt{2GM}}{H^2}\sqrt{H^2 - \frac{Q^2}{24\pi M}} + \frac{2\Lambda H^6}{24GM} - \frac{kH^4}{2GM}, \quad (6.12)$$

where M is an integration constant. When $\Lambda = k = 0$, and defining α and β by $\alpha = \sqrt{2GM}$, $\beta^2 = Q^2[24\pi M]^{-1}$, equation (6.12) can be further integrated to yield (after rescaling of t)

$$\alpha t = \frac{1}{2}\beta^2 \cosh^{-1}(H/\beta) + \frac{1}{2}H\sqrt{H^2 - \beta^2} \ , \quad (6.13)$$

It is straightforward to derive the equation of state in this case, although it is not necessarily needed to find H in terms of t to find the forms of μ and p. By differentiation with respect to time from (6.12) one gets

$$2\ddot{H} = -\frac{-4GM}{H^3} + \frac{GQ^2}{3\pi H^5} \quad (6.14)$$

and the following expressions for μ and p,

$$\mu = 3\frac{\dot{H}^2}{H^2} = \frac{6GM}{H^4} - \frac{GQ^2}{4\pi H^6} \quad (6.15)$$

$$p = -2\frac{\ddot{H}}{H} - (\frac{\dot{H}}{H})^2 = \frac{2GM}{H^4} - \frac{GQ^2}{4\pi H^6} \ . \quad (6.16)$$

Now by using the above expression one obtains,

$$\frac{1}{3}\mu + p = \frac{4GM}{H^4} - \frac{GQ^2}{3\pi H^6} \ ; \quad (6.17)$$

multiplying (6.15) by 4 and (6.17) by 3 and then subtracting results in

$$\frac{1}{H^2} = \sqrt{\frac{\mu - p}{4GM}} \quad . \tag{6.18}$$

Then, by inserting (6.18) in (6.17), one obtains the equation of state

$$(\frac{1}{3}\mu - p) = \omega^2(\mu - p)^3, \tag{6.19}$$

where

$$\omega^2 = \frac{Q^2}{(48\pi)^2 GM^3}.$$

The late time behavior of these solutions asymptote towards a radiation equation of state. This becomes apparent by examining (6.13) for large H. We notice that

$$\lim_{H\to\infty} \frac{\cosh^{-1}(\frac{H}{\beta})}{H\sqrt{H^2 - \beta^2}} = 0$$

Now for large values of H we get $H^2 \approx 2\alpha t$, and so $p \approx \frac{1}{3}\mu \approx \frac{1}{4}t^{-2}$ by using (6.15). The corresponding line element is

$$ds^2 \approx -dt^2 + 2\alpha t(dr^2 + r^2 d\Omega^2) + \frac{\alpha}{2t}dy^2,$$

the four-dimensional component of which is a special case of the Tolman line element.

6.2.2 Yang-Mills fields in Higher-Dimensions

We turn our attention now to *non-Abelian* fields coupled to higher-dimensional Einstein gravity. As an explicit example, we shall consider an $SO(3)$ Yang-Mills field coupled to gravity via the six-dimensional action

$$S = \int d^6V \left\{ -\frac{(R + 2\Lambda)}{4\kappa^2} - \frac{1}{4}F^{(a)}_{\alpha\beta}F^{(a)\alpha\beta} \right\} \tag{6.20}$$

(where again $\kappa^2 = 4\pi G$), with the metric described by the line interval

$$ds^2 = -dt^2 + H^2(t)\frac{dr^2 + r^2 d\Omega^2}{(1 + \frac{1}{4}kr^2)^2} + L^2(t)\left[d\xi^2 + \sin^2(\xi)d\zeta^2\right],$$

where $y^1 = \xi$ and $y^2 = \zeta$ are the two extra coordinates. We assume also that all components of the gauge field are zero except

$$A^{(a)}_\xi = (v - 1/e)\left[-\sin\zeta, \cos\zeta, 0\right], \qquad (6.21)$$
$$A^{(a)}_\zeta = (v - 1/e)\left[-\cos\zeta\cos\xi\sin\xi, \sin\zeta\cos\xi\sin\xi, 0\right], \qquad (6.22)$$

where v is an integration constant and e is the field's charge (see [42]).

The relevant field equations obtained from this action, using the above ansatz for the metric and gauge field, are

$$3\frac{\ddot{H}}{H} + 2\frac{\ddot{L}}{L} = \frac{1}{2}\Lambda - \alpha L^{-4} \qquad (6.23)$$
$$\frac{\ddot{H}}{H} + 2\frac{\dot{H}\dot{L}}{HL} + \frac{2}{H^2}\left(\dot{H}^2 + k\right) = \frac{1}{2}\Lambda - \alpha L^{-4} \qquad (6.24)$$
$$\frac{\ddot{L}}{L} + 3\frac{\dot{H}\dot{L}}{HL} + \frac{1}{L^2}\left(\dot{L}^2 + K\right) = \frac{1}{2}\Lambda + 3\alpha L^{-4} \qquad (6.25)$$
$$6\left(\frac{\dot{H}^2 + k}{H^2}\right) + 12\frac{\dot{H}\dot{L}}{HL} + 2\left(\frac{\dot{L}^2 + K}{L^2}\right) = 2\Lambda + 4\alpha L^{-4}, \qquad (6.26)$$

where $\alpha^2 = \frac{1}{4}(v^2 e - e^{-1})^2$. Cremmer and Scherk [42, 17] presented a six-dimensional Yang-Mills solution similar to the 't Hooft magnetic monopole [43, 44] where $L = L_0$ a constant and $k = 0$ (see also [45]). Their solution is a fixed point of the system (6.23) for $N = 2$, $K = 1$ and $k = 0$ (see below).

These field equations can be generalized to the (4+N)-dimensional case in a straightforward manner. In particular, we can exploit the results of Wiltshire [41] who studied an Abelian gauge field (using the Reubin-Freund ansatz) coupled to (4+N)-dimensional gravity using the line interval

$$ds^2 = -dt^2 + H^2(t)\frac{dr^2 + r^2 d\Omega^2}{(1 + \frac{1}{4}kr^2)^2} + L^2(t)\tilde{g}_{IJ}dy^I dy^J.$$

Here, the "internal" space is an N-dimensional Einstein space of constant curvature, K, described by the metric $\tilde{g}_{IJ}$; i.e., the Ricci tensor constructed from $\tilde{g}_{IJ}$ is defined by $R_{IJ} = (N-1)K\tilde{g}_{IJ}$.

The field equations in [41] are given by

$$3\frac{\ddot{H}}{H} + N\frac{\ddot{L}}{L} = \frac{2\Lambda}{N+2} - \alpha(N-1)L^{-2N} \qquad (6.27)$$

$$\frac{\ddot{H}}{H} + N\frac{\dot{H}\dot{L}}{HL} + \frac{2}{H^2}\left(\dot{H}^2 + k\right) = \frac{2\Lambda}{N+2} - \alpha(N-1)L^{-2N} \tag{6.28}$$

$$\frac{\ddot{L}}{L} + 3\frac{\dot{H}\dot{L}}{HL} + \frac{N-1}{L^2}\left(\dot{L}^2 + K\right) = \frac{2\Lambda}{N+2} + 3\alpha L^{-2N} \tag{6.29}$$

$$6\left(\frac{\dot{H}^2 + k}{H^2}\right) + 6N\frac{\dot{H}\dot{L}}{HL} + N(N-1)\left(\frac{\dot{L}^2 + K}{L^2}\right) = 2\Lambda + (N+2)\alpha L^{-2N}\,. \tag{6.30}$$

Our immediate focus here is with the case of an $SO(3)$ non-Abelian gauge field in six dimensions. However, we note that the results pertaining to this particular case follow immediately from the solutions of (6.27)-(6.30) by setting $N = 2$ (for more details about the $SO(3)$ model see appendix C). Attempts to solve (6.27)-(6.30) analytically for the most general solution may prove futile. However, the behavior of the system for all times may be obtained through qualitative analysis. This can be accomplished either by a stability analysis of the known solutions using perturbation methods (see appendix C) or by the the method used by Wiltshire. In his work, he completed a full phase-plane analysis of equations (6.27), including use of a Poincaré transformation to compactify the phase space in order to evaluated the system's fixed points at infinity (in terms of the dynamic variables used). We only highlight the solutions obtained in Wiltshire's work and refer the reader to his paper for full details of the analysis. Specifically, we will describe the non-saddle fixed points of (6.27), describe the "induced" equation of state associated with each of these fixed points and then briefly summarize the behaviour of the solutions.

The field equations (6.27) admit up to seven non-saddle fixed points, although several of these fixed points are described by the same solution. The first set of fixed points are the only fixed points at infinity and are represented by the generalized Kasner solution [23, 41, 46, 47]

$$H = H_0 t^{m_\pm}, \quad L = L_0 t^{n_\pm}, \tag{6.31}$$

where

$$m_\pm = \frac{1}{3+N}\left\{1 \pm \frac{1}{3}\sqrt{3N^2+6N}\right\} \gtrless 0 \tag{6.32}$$

$$n_\pm = \frac{1}{3+N}\left\{1 \mp \frac{1}{N}\sqrt{3N^2+6N}\right\} \lessgtr 0, \tag{6.33}$$

where two of the fixed points have (m_+, n_+) and the other two fixed points have (m_-, n_-). Although there are two solutions here, the solution with m_- and n_- are only saddle points and so solutions asymptoting towards or away from this solution are of measure zero. One of the m_+ and n_+ solutions is an attracting node whilst the other is a repelling node. Here, and throughout the rest of the paper, when the Kasner solution is mentioned it will be assumed that we are referring to the m_+ and n_+ solution unless otherwise stated.

For both solutions $(m_\pm)$, the induced matter has the equation of state (see [23])

$$p = \sigma_\pm \mu = \left\{ \frac{2N + 3 \mp \sqrt{3N^2 + 6N}}{3 \pm \sqrt{3N^2 + 6N}} \right\} \mu. \qquad (6.34)$$

For the m_+ solution, σ_+ ranges from $\frac{1}{3}$ for $N = 1$ to $\frac{2}{3}(\sqrt{3} - 1)$ for $N \to \infty$.

The next two fixed points are obtained for $k = K = \alpha = 0$ and $\Lambda > 0$, and are represented by the solutions

$$H = H_0 e^{\gamma t}, \quad L = L_0 e^{\gamma t}, \qquad (6.35)$$

where

$$\gamma = \pm \sqrt{\frac{2\Lambda}{(N + 2)(N + 3)}}. \qquad (6.36)$$

The growing mode solution is an attracting node and hence a future attractor, whilst the decaying mode solution is a repelling node (past attractor). The induced mass-energy density and pressure for this solution are respectively

$$\mu = -p = 3 \frac{2\Lambda}{(N + 2)(N + 3)} \qquad (6.37)$$

This equation of state corresponds to that of a false vacuum and so we have de Sitter-like solutions.

The next set of fixed points is another set of de Sitter-like solutions for $k = 0$, $K = 1$, $\Lambda > 0$. Although the number of solutions is either two or four depending on the value of Λ, they all have the form

$$H = H_0 e^{\pm \delta t}, \quad L = L_0. \qquad (6.38)$$

This is the form of the solution obtained by Cremmer and Scherk [42]. The integration constant L_0 is not arbitrary, but depends on the values of Λ and α. In all of these cases, the equation of state is again $p = -\mu$. If $\alpha = 0$, then there are only two solutions: $\delta^2 = \Lambda/6$ with $L_0^{-2} = \frac{1}{2}\Lambda$. When $\alpha \neq 0$, finding δ and L_0 in closed form may be quite difficult for arbitrary N. To illustrate, for these solutions the first two equations of (6.27) both yield ($N \neq 1$)

$$\frac{1}{L_0^{2N}} = \frac{2\Lambda - 4(N+2)\delta^2}{\alpha(N-1)(N+2)}$$

which isolates the value for L_0. Using this expression, one then obtains from either of the last two equations of (6.27)

$$(N+2)\alpha\left[2\Lambda - 9\delta^2\right]^N = (N-1)^{2N-1}\left[2\Lambda - 3(N+2)\delta^2\right],$$

which is the condition found in [41]. Unfortunately, one cannot analytically solve this for arbitrary N. However, to demonstrate that this does lead to either two or four solutions, we shall consider the case $N = 2$. We find that Λ is bounded by $\Lambda \leq (6\alpha)^{-1}$ for any real solution to exist, so we write $\Lambda = \Sigma/(6\alpha)$ where Σ has the range $[0, 1]$. The solution for δ and L_0 is hence

$$\delta_\pm^2 = \frac{2\Sigma - 1 \pm \sqrt{1-\Sigma}}{54\alpha} \tag{6.39}$$

$$\frac{1}{L_0^4} = \frac{1 - \frac{1}{2}\Sigma \mp \sqrt{1-\Sigma}}{18\alpha^2}. \tag{6.40}$$

It is apparent that there are no real solutions for $\Sigma > 1$ ($\Lambda > (6\alpha)^{-1}$). For $\Lambda \geq \frac{1}{8\alpha}$, we find that the δ_+ solution is an attracting node for $\delta > 0$ and a repelling node for $\delta < 0$, and the δ_- solutions are saddle points of the system. For $\Lambda < \frac{1}{8\alpha}$, the δ_-^2 solutions are not real and so there are only two fixed points (δ_+^2) which are saddles points.

The final fixed-point is given by the solution $k = K = \alpha = \Lambda = \dot{L} = \dot{H} = 0$, which is just an D-dimensional Minkowski spacetime. This fixed point is an attracting node.

From the dynamical systems analysis in [41], or from a straightforward perturbation analysis (see Appendix C), we find the following evolution of this system. With the exception of solutions of measure zero, all solutions asymptote into the past to the Kasner solution, or to the decaying de Sitter solution of (6.35), or to the decaying de Sitter solution for δ_- of (6.39) (for the correct values of Λ). The future behaviour of the solutions is that they either recollapse to the Kasner singularity (i.e. the time reverse solution of (6.31) with m_+) or they asymptote towards the growing de Sitter solution (6.35), or to the growing de Sitter solution of (6.39) for δ_-, or to a Minkowski spacetime.

Finally, Wiltshire [41] finds three exact solutions which represent separatrices in the phase portraits constructed in his analysis. The first is the Kasner solution (6.31) when $k = K = \alpha = \Lambda = 0$. The induced matter is characterized by (6.34). The next two solutions occur for $k = K = \alpha = 0$. The scale factors H and L for the first of these solution, which were obtained for $\Lambda < 0$, are

$$H = H_0 \left|\sin(\frac{1}{2}\gamma t)\right|^{m\pm} \left|\cos(\frac{1}{2}\gamma t)\right|^{\frac{2}{N+3}-m\pm} \tag{6.41}$$

$$L = L_0 \left|\sin(\frac{1}{2}\gamma t)\right|^{n\pm} \left|\cos(\frac{1}{2}\gamma t)\right|^{\frac{2}{N+3}-n\pm}, \tag{6.42}$$

where

$$\gamma^2 = 2\left[\frac{N+3}{N+2}\right]|\Lambda|.$$

The corresponding mass-energy density and pressure are, respectively,

$$\mu = \frac{|\Lambda|}{2(N+2)}\left\{(j_\mu \pm l)\cot^2(\frac{1}{2}\gamma t) + (j_\mu \mp l)\tan^2(\frac{1}{2}\gamma t) + 2(N-1)\right\} \tag{6.43}$$

$$p = \frac{|\Lambda|}{6(N+2)}\left\{(j_p \pm l)\cot^2(\frac{1}{2}\gamma t) + (j_p \mp l)\tan^2(\frac{1}{2}\gamma t) - 6(N-3)\right\}, \tag{6.44}$$

where

$$l = \frac{2\sqrt{3N^2+6N}}{N+3}$$

$$j_\mu = \frac{N^2+2N+3}{N+3}$$

$$j_p = \frac{9-3N^2}{N+3}.$$

These solutions initially expand from a Kasner singularity and recollapse back to a Kasner singularity (see [41]) and so both early time ($t \to 0^+$) and late time ($\gamma t \to \pi$) solutions will have the Kasner equation of state (6.34).

The last of the separatrix solutions has $\Lambda > 0$ and has the solution

$$H = H_0 \left|\sinh(\frac{1}{2}\gamma t)\right|^{m\pm} \left|\cosh(\frac{1}{2}\gamma t)\right|^{\frac{2}{N+3}-m\pm}, \tag{6.45}$$

$$L = L_0 \left|\sinh(\frac{1}{2}\gamma t)\right|^{n\pm} \left|\cosh(\frac{1}{2}\gamma t)\right|^{\frac{2}{N+3}-n\pm}. \tag{6.46}$$

The corresponding mass-energy density and pressure are, respectively,

$$\mu = \frac{|\Lambda|}{2(N+2)}\left\{(j_\mu \pm l)\coth^2(\frac{1}{2}\gamma t) + (j_\mu \mp l)\tanh^2(\frac{1}{2}\gamma t) - 2(N-1)\right\} \tag{6.47}$$

$$p = \frac{|\Lambda|}{6(N+2)}\left\{(j_p \pm l)\coth^2(\frac{1}{2}\gamma t) + (j_p \mp l)\tanh^2(\frac{1}{2}\gamma t) + 6(N-3)\right\}. \tag{6.48}$$

For early times, $t \to 0^+$, the equation of state approaches the Kasner equation of state, whereas at late times, $\gamma t >> 1$, the equation of state approaches the false vacuum equation (6.37).

6.2.3 Supergravity

In this section, we study an example from supergravity, in which the fermionic fields are zero, the Maxwellian potential is given by $A_\lambda = (0,0,0,0,\psi)$, and the five-dimensional line-interval is given by

$$\begin{aligned} ds^2 &= -dt^2 + \frac{H^2(t)}{1+kr^2}(dr^2 + r^2 d\Omega^2) + L^2(t)dy^2 \\ &= -H^6(\eta)d\eta^2 + \frac{H^2(\eta)}{1+kr^2}(dr^2 + r^2 d\Omega^2) + L^2(\eta)dy^2, \end{aligned} \tag{6.49}$$

where the conformal time coordinate is defined by $dt \equiv H^3 d\eta$. As given in [48, 49], all quantities considered here depend only on the four-dimensional "external" coordinates and so the five-dimensional Lagrangian can be expressed as a four-dimensional Lagrangian coupled to two scalar fields, ψ and $L^2 = g_{yy}$, namely

$$S = \int d^4V \left\{-\frac{LR}{4\kappa^2} + \frac{2}{L}D_\lambda\psi D^\lambda\psi\right\}, \tag{6.50}$$

where D_λ is the gauge covariant derivative corresponding to A_λ.

The resulting field equations

$$\frac{\ddot{H}}{H} + \frac{\dot{H}^2}{H^2} + \frac{k}{H^2} = \frac{\dot{\psi}^2}{4L^2}, \tag{6.51}$$

$$\frac{\dot{H}^2}{H^2} + \frac{k}{H^2} + \frac{\dot{H}\dot{L}}{HL} = \frac{\dot{\psi}^2}{4L^2}, \tag{6.52}$$

$$\ddot{L} + 3\frac{\dot{H}}{H}\dot{L} = -\frac{\dot{\psi}^2}{L^2}, \tag{6.53}$$

$$\ddot{\psi} + 3\frac{\dot{H}}{H}\dot{\psi} = \frac{\dot{L}}{L}\dot{\psi}, \tag{6.54}$$

were solved in [48, 49]; the solutions are given by

$$H = \frac{H_0}{\sqrt{1 - q\cos(a\eta)}} \tag{6.55}$$

$$L = -L_0 \sin(a\eta) \tag{6.56}$$

$$\psi = -L_0 \cos(a\eta), \tag{6.57}$$

where H_0, L_0 and a are integration constants and

$$q \equiv \sqrt{1 - 4kH_0^4/a^2}. \tag{6.58}$$

To ensure that t depends monotonically on η, one requires that $dt/d\eta = H^3 > 0$, which can be verified numerically for the range $0 < q < 1$. It is apparent that H oscillates for all time (and therefore μ and p will oscillate for all time). When $q \geq 1$, t and H then diverge as η approaches $1/a\cos^{-1}(1/q)$, as pointed out in [48, 49]. Except for the trivial case $H_0 = 0$, H never vanishes and so the four-dimensional space-time can be considered singular-free.

The mass-energy density and the pressure of the induced matter are given by, respectively,

$$\mu = \frac{3}{4}\frac{a^2}{H_0^6}\{1 - q\cos(a\eta)\}\left[(1-q) + q^2\sin^2(a\eta)\right] \quad , \tag{6.59}$$

$$p = \frac{a^2}{H_0^6}\{1 - q\cos(a\eta)\}\left[q\cos(a\eta)\{1 - q\cos(a\eta)\} - \frac{1}{4}q^2\sin^2(a\eta) - \frac{1}{4}(1-q)\right]. \tag{6.60}$$

We define $\bar{p} = pH_0^6a^{-2}$ and $\bar{\mu} = \mu H_0^6 a^{-2}$ and combine (6.59) and (6.60) and obtain the "equation of state"

$$\left[\frac{27(\bar{\mu} - \bar{p})^5}{(\bar{\mu} + 3\bar{p}) - 3C(\bar{\mu} - \bar{p})} + C(\bar{\mu} + 3\bar{p}) - \frac{12\bar{\mu}(\bar{\mu} - \bar{p})^2}{(\bar{\mu} + 3\bar{p}) - 3C(\bar{\mu} - \bar{p})}\right](\bar{\mu} - \bar{p}) = 0, \tag{6.61}$$

where $C = 1 - q + q^2 > 0$.

To help elucidate the nature of this equation of state, we have provided several figures of p, μ and p/μ as function of η for various values of q. In the calculations used to produce Figures 6.1 to 6.5, we defined $t = 0$ for $\eta = \pi/a$. In the plots for $q < 1$, the value of μ and p repeat themselves every $2\pi/a$ and so we only plot them from $\eta = 0$ to $\eta = 2\pi/a$. For $q \geq 1$, we only plot μ and p for the range of η which corresponds to $t \in (-\infty, \infty)$ (which are marked on the plots by the dashed lines). For these values of q, the equation of state asymptotes into the past *and* future towards the relation $p = -\frac{1}{3}\mu$.

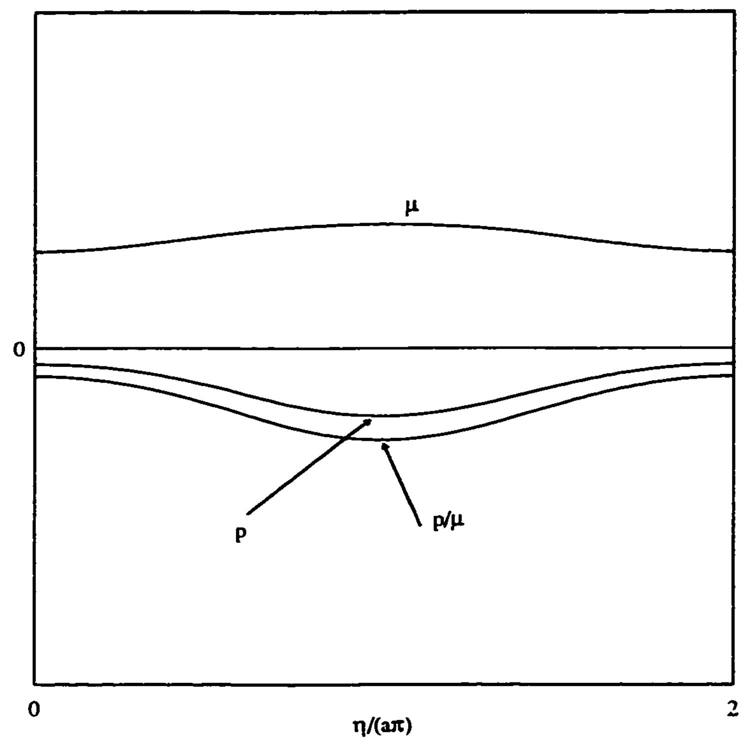

Figure 6.1: Energy density and pressure for $q < 1/4$, where q is defined by (6.58)

6.3 Generalizations

From these examples it is quite apparent that there are many different ways of obtaining equations of state different from radiation in the context of induced matter.

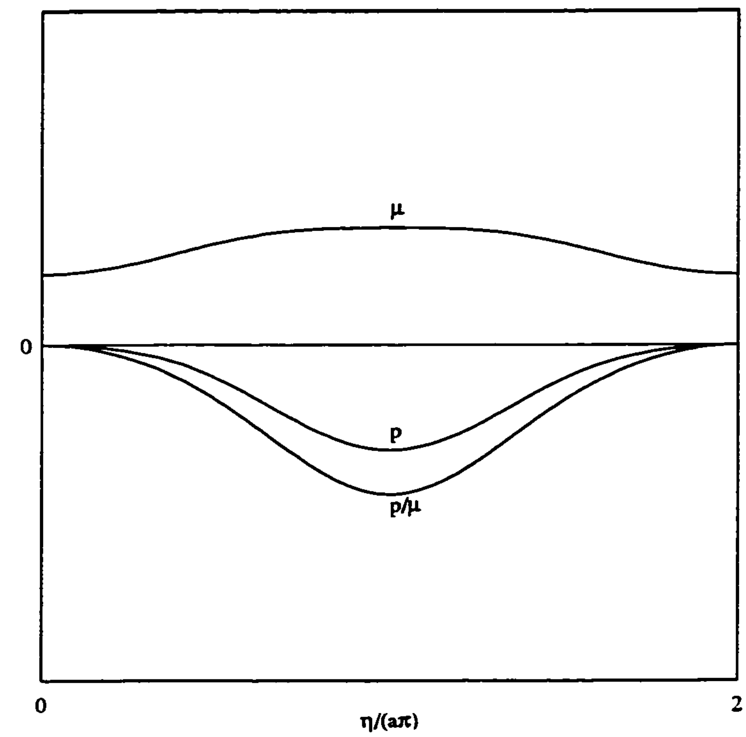

Figure 6.2: Energy density and pressure for q=1/4, where q is defined by (6.58)

Indeed, there are more examples and theories found in the literature that may be used in this manner. In the context of Einstein-Maxwell (EM) theories, Gleiser *et al.* [50] have studied ten- and eleven-dimensional spacetimes, and Freund and Rubin [11] have also found solutions in the eleven-dimensional case in which seven of the eleven dimensions compactify. Gibbons and Wiltshire [40] studied arbitrary D-dimensional spacetimes containing an EM gauge field. Similarly, Fabris [51] showed that in order to obtain a traceless electromagnetic stress-energy tensor in $D = 4 + N$ dimensions, the electromagnetic potential is required to have a $\frac{1}{2}(N-2)$-form, and hence he considered even-dimensional cosmologies. Fabris [52] also studied a $D = 6$ anisotropic model and a $D = 8$ model which contained an anti-de Sitter space-time as a solution.

In terms of Einstein-Yang-Mills (EYM) higher-dimensional theories, the literature is extensive. Kubyshin *et al.* [53] studied higher-dimensional cosmologies containing $SU(5)$ and $SU(2) \times U(1)$ gauge fields with a static compact "internal" space, as well as considering anisotropic internal spaces. Clements [54] studied a six-dimensional $SO(3)$ EYM-Higgs model, examining the stability of static solutions. Bertolami *et al.*

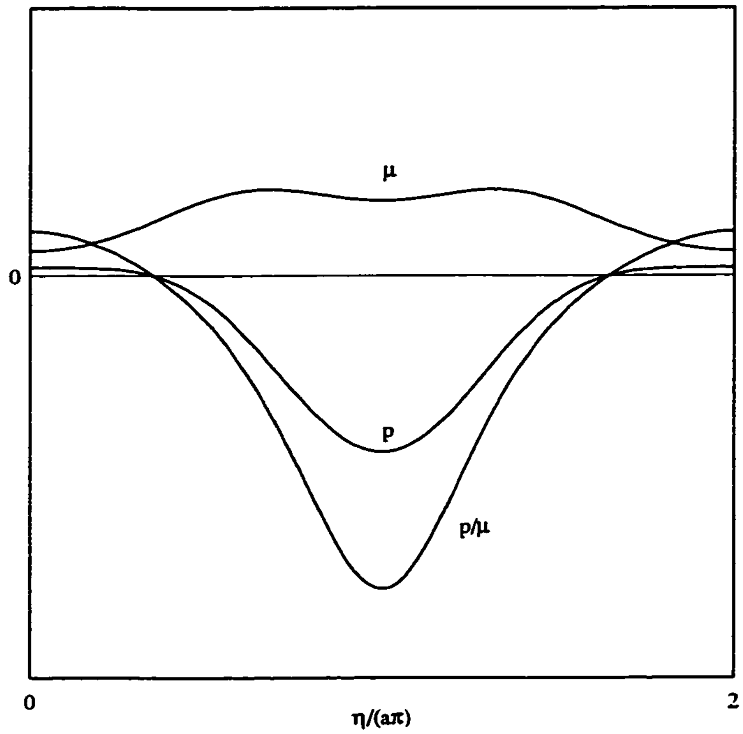

Figure 6.3: Energy density and pressure for q=1/2, where q is defined by (6.58)

[55] considered D-dimensional spacetimes in the context of compactification. Luciani [56] extended the work of Cremmer and Scherk [42, 17] by considering various symmetry groups [for example, a $(4+2PQ)$-dimensional spacetime with group $SU(P+Q)$ and subgroup $SU(P)\times SU(Q)\times U(1)$, a $(4+\frac{1}{2}(N-1)(N+2))$-dimensional spacetime with group $SU(N)$ and subgroup $SO(N)$, a $(4+N(N-1))$-dimensional spacetime with group $SO(2N)$ and subgroup $U(N)$, and a $(4+PQ)$-dimensional spacetime with group $SO(P+Q)$ and subgroup $SO(P)\times SO(Q)$].

There are several examples of supergravity theories that have been studied. The five-dimensional supergravity theory has been studied by Balbinot *et al.* [48, 49] and by Pimentel [57] (who considered a Bianchi I model for the four-dimensional part of the space-time). In addition, Duruisseau and Fabris [58] studied five-dimensional supergravity with Gauss-Bonnet terms in the action. Ten-dimensional supergravity has also been studied by Gleiser and Stein-Schabes [59], who obtained a de Sitter-type solution as a late time solution.

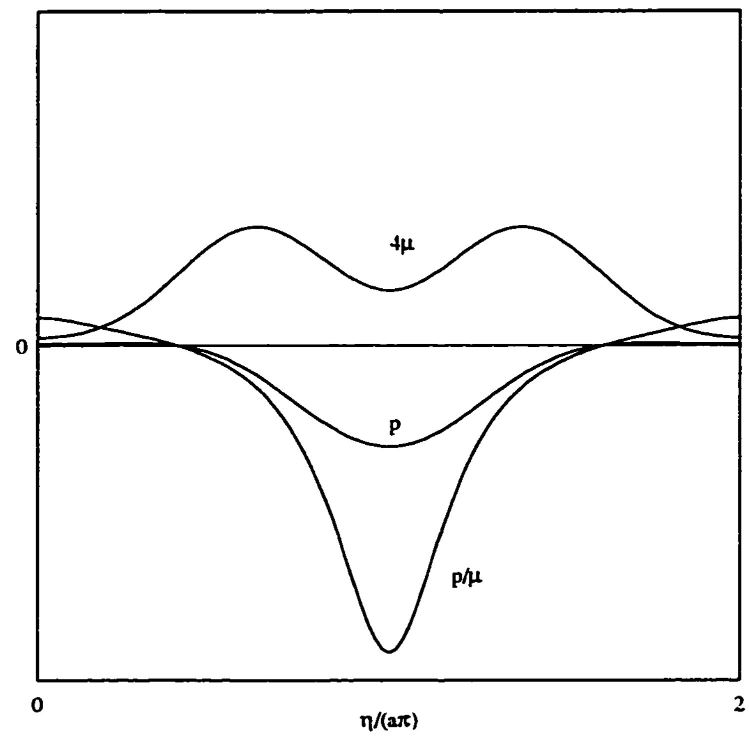

Figure 6.4: Energy density and pressure for q=3/4, where q is defined by (6.58)

Of course, there are other approaches one could consider. For instance, one could also propose a generalized Einstein theory of gravity in the context of Lovelock theory [60, 23]. Another example could be to include anisotropy into any of the aforementioned works. In all of the examples studied in section two the induced fluids were perfect; by introducing anisotropy into the three-space we would expect to induce anisotropies in the pressure and hence dissipative terms into the energy-momentum tensor. The energy-momentum tensor would be then modified from (1.3) in general to

$$T_{\alpha\beta} = (\mu + p)u_\alpha u_\beta + p g_{\alpha\beta} + \pi_{\alpha\beta} + q_\alpha u_\beta + q_\beta u_\alpha, \tag{6.62}$$

where $\pi_{\alpha\beta}$ is the anisotropic pressure tensor and q^α is the heat conduction vector, where $\pi^\alpha_\alpha = u_\alpha \pi^\alpha_\beta = q^\alpha u_\alpha = 0$ [61]. The variable p is now the pressure averaged over all three directions and the pressure in each direction is then defined as $p_i = p + \pi^i_i$ (for $i = 1, 2, 3$ with no summation implied).

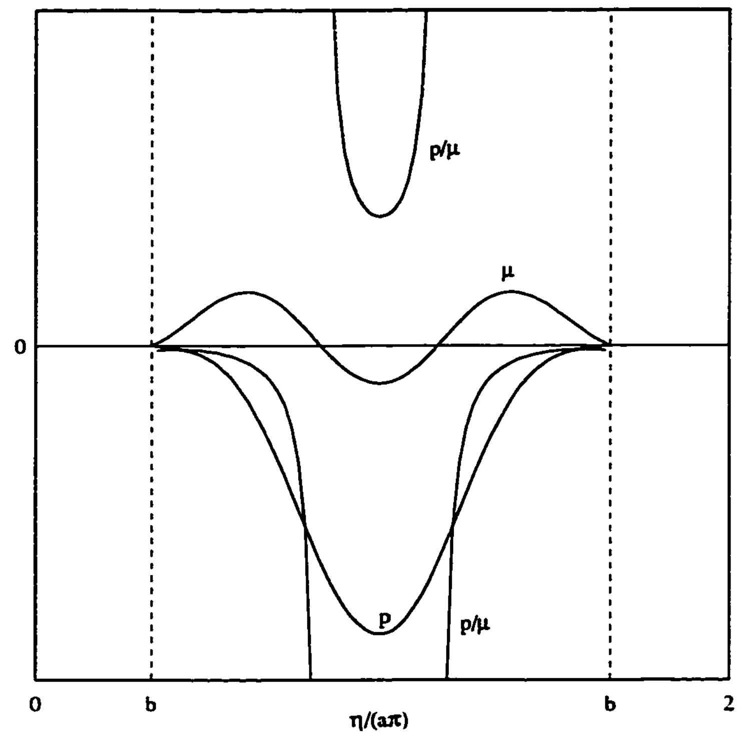

Figure 6.5: Energy density and pressure for $q > 1$, where q is defined by (6.58)

6.3.1 Anisotropic Generalizations

As an illustration, we consider anisotropy in the supergravity model of section 6.2.3, which has been previously studied in [48]. The cylindrically symmetric metric is given by

$$ds^2 = -A^2(\eta)B^4(\eta)d\eta^2 + A^2(\eta)dx^2 + B^2(\eta)(dy^2 + dz^2) + L^2(\eta)d(x^5)^2, \qquad (6.63)$$

where now conformal time, η, is defined by $dt = AB^2 d\eta$. The field equations then give rise to the following set of ordinary differential equations (see [48] for details):

$$\frac{2A'}{A}\frac{B'}{B} + \frac{(B')^2}{B^2} + \left[\frac{A'}{A} + \frac{2B'}{B}\right]\frac{L'}{L} = \frac{1}{2}\frac{\psi'}{L^2} \qquad (6.64)$$

$$\frac{2B''}{B} + \frac{L''}{L} - \frac{2B'}{B}\frac{A'}{A} - \frac{3(B')^2}{B^2} - \frac{L'}{L}\frac{A'}{A} = -\frac{1}{2}\frac{\psi'}{L^2} \qquad (6.65)$$

$$\frac{A''}{A} + \frac{B''}{B} + \frac{L''}{L} - \frac{(A')^2}{A^2} - \frac{2(B')^2}{B^2} - \frac{L'}{L}\frac{B'}{B} = -\frac{1}{2}\frac{\psi'}{L^2} \qquad (6.66)$$

$$\frac{A''}{A} + \frac{2B''}{B} - \frac{2A'}{A}\frac{B'}{B} - \frac{3(B')^2}{B^2} - \frac{(A')^2}{A^2} = \frac{1}{2}\frac{\psi'}{L^2} \qquad (6.67)$$

$$\psi'' - \frac{L'}{L}\psi' = 0 \tag{6.68}$$

where again two scalar fields ψ and $L^2 = g_{yy}$ are coupled to a $4D$ Lagrangian via (6.50). The general solution of these equations is then given by [48]:

$$\begin{aligned} A &= A_0 \left(\tan(\frac{a\eta}{2})\right)^{\frac{b}{a}} \frac{1}{\sqrt{\sin(a\eta)}} \\ B &= B_0 \left(\tan(\frac{a\eta}{2})\right)^{\frac{c}{a}} \frac{1}{\sqrt{\sin(a\eta)}} \\ L &= L_0 sin(a\eta) \\ \psi &= -L_0 cos(a\eta), \end{aligned} \tag{6.69}$$

where the integration constants a, b and c are constrained by $2bc + c^2 = 3a^2/4$. Evidently, when $b = c = \pm a/2$, the solutions found in section 6.2.3 are recovered.

To calculate the induced matter, we define the comoving fluid four-velocity to be

$$u^\alpha = \frac{\delta^\alpha_t}{A(\eta)B^2(\eta)},$$

which satisfies $u^\alpha u_\alpha = -1$. From (6.62) we then obtain

$$\mu = \frac{1}{4A_0^2B_0^4}\left[\tan\left(\frac{1}{2}a\eta\right)\right]^{-2\frac{b+2c}{a}} [2c - a\cos(a\eta)]\,[4b + 2c - 3a\cos(a\eta)]\sin(a\eta) \tag{6.70}$$

$$p = \frac{a}{12A_0^2B_0^4}\left[\tan\left(\frac{1}{2}a\eta\right)\right]^{-2\frac{b+2c}{a}} \left[9a\cos^2(a\eta) - 4(b+2c)\cos(a\eta) - 3a\right]\sin(a\eta) \tag{6.71}$$

$$\pi^x_x = \frac{-2a}{3A_0^2B_0^4}\left[\tan\left(\frac{1}{2}a\eta\right)\right]^{-2\frac{b+2c}{a}} [b - c]\sin(a\eta)\cos(a\eta) \tag{6.72}$$

$$\pi^y_y = \pi^z_z = -\frac{1}{2}\pi^x_x. \tag{6.73}$$

Notice that there are no heat conduction terms in this model. It may be verified that $\pi^\alpha_\beta = -\lambda(\eta)\sigma^\alpha_\beta$, where σ^α_β is the shear tensor defined from u^α [61], and λ is the viscosity coefficient of the fluid, given by

$$\lambda(\eta) = -\frac{a}{A_0B_0^2}\left[\tan\left(\frac{1}{2}a\eta\right)\right]^{-\frac{b+2c}{a}} \sqrt{\sin(a\eta)}\cos(a\eta) \quad , \tag{6.74}$$

where the above expression for the viscosity coefficient is only physical for a restricted range for η.

We note that the above quantities either diverge or vanish at $\eta = 0$ and $\eta = \pi/a$, but which possibility occurs depends on the values of b and c, which can be positive or negative. One finds by integration that the original time variable, defined by $t = \int AB^2 d\eta$, is monotonic in η in the interval $[0, \pi/a]$ and so we will consider these "endpoints" as early-time and late-time limits. Taking the ratio of p/μ and using the constraint $2bc + c^2 = 3a^2/4$, we obtain

$$\frac{p}{\mu} = \frac{a^2 cos(a\eta) + 2ac[1 - 3\cos^2(a\eta)] + 4c^2\cos(a\eta)}{3a[a - 2c\cos(a\eta)][a\cos(a\eta) - 2c]}.$$

For early-time ($\eta \to 0^+$) behaviour *and* for late-time behaviour ($\eta \to \pi/a$) we find that $p \to \frac{\mu}{3a}$, and hence we see that the constant a plays an important rôle in determining the equation of state.

To summarize, in this chapter our main goal has been to investigate the induced matter theory of Wesson [20] in the context of higher-dimensional Einstein-Yang-Mills cosmological models. This goal has been achieved either through exact solutions or by an examination of the asymptotic behaviours of the models by analyzing the fixed points of the underlying field equations. The appropriate equations of state were derived. In this chapter we also investigated the induced matter theory in the context of a $5D$ supergravity theory. The energy and pressure of the induced matter in this case exhibit oscillatory behaviour. Finally, by introducing anisotropy into the $4D$ part of this model we can derive anisotropic pressure and also viscosity.

Chapter 7

Conclusions

7.1 Summary

The main body of this thesis contains the study and generalization of Kaluza-Klein-type cosmological models. We have also been interested, in particular, in Wesson's five-dimensional space-time-matter type theories [18, 19, 20] and their various extensions. We have also taken as the correct interpretation of these theories that the field equations are the higher dimensional vacuum field equations. In this thesis a number of new solutions of the five and six-dimensional vacuum field equations have been found and the properties of these solutions have been studied.

The possibility that the four-dimensional properties of matter may be completely geometric in origin has been investigated by studying whether the higher dimensional vacuum field equations formally reduce to Einstein's four-dimensional theory with a non-zero energy-momentum tensor constituting the matter source. The embedding of the four-dimensional space-time in the vacuum five-dimensional space-time is interpreted as producing an effective four-dimensional stress-energy tensor. Here the four-dimensional source is taken as a cosmological fluid with energy-momentum tensor of of the form (6.15).

In chapter two we analyzed Riemann-flat solutions of a class of $5D$ spherically symmetric metrics of the form (2.1) which are natural generalizations of the Schwarzschild metric and the FRW metric on each hypersurface $y = constant$. The purpose of this chapter was twofold: not only did we find the explicit form for all the Riemann-flat solutions of the $5D$ metrics of the form (2.1), but also we employed these solutions to find new Ricci-flat solutions. In particular, we used our knowledge of the Riemann-flat solutions as an aid to construct new Ricci-flat solutions. We were able to do so by breaking down the Riemann-flat field equations into a number of cases and analyzing each case separately. In the context of induced matter theory we are interested in those $4D$ foliations of $5D$ spacetime which are not flat. Among the four cases studied, only one explicitly has a curved four-dimensional hypersurface. This metric was then generalized to a Ricci-flat metric, and we studied this metric in chapter three. Throughout this thesis we have been concerned with higher-dimensional vacuum Einstein field equations which constitute a set of coupled non-linear partial differential equations. We have assumed no boundary-initial conditions to solve the system. As expected, those solutions we have found involve various arbitrary functions, some of which may be absorbed into the metric in some cases by some coordinate transformations.

In chapter three, as mentioned above, we attempted to find Ricci-flat field equations by employing the Riemann-flat solutions found in chapter one. The Riemann-flat metrics were used as an aid to construct some Ricci-flat solutions of the metric (2.1) that are not Riemann-flat. We based our first ansatz on the form of one particular class of solutions, since the other cases were either physically uninteresting ($4D$-flat) or too complicated to analyze. Although the equations involved in this particular case were also complex to analyze, we were able to break them down into only two cases each of which then led to known solutions, namely $5D$ Minkowski space and the Riemann-flat solution we started with. In chapter three we also examined the ansatz where three-dimensional spatial metric only depends on the r coordinate while the other metric coefficients have general dependence on the coordinates. The known solution of Gross and Perry [29] was recovered as well as some new exact solutions, and some solutions were fully analysed by qualitative methods using dynamical systems theory. It should be pointed out that the dynamical system studied is not of type time-evolution but of space-evolution.

In chapter four we generalized the metric we studied in chapter two to an even more general one in which the three-dimensional spherical metric has functional dependence on the coordinates t and y which is not, in general, separable. We also suggest a more natural way of breaking the problem up into a number of special cases than the one suggested by McManus [28]. All the cases were investigated in full detail.

Chapter five was concerned with possible power-law solutions of the equations studied in chapter four. Some new power-law solutions were found as a result of this investigation, of which one class of solutions turned out to be Riemann-curved. The corresponding expression for the induced energy density turns out to be time-dependent and positive, although the induced anisotropic pressures are negative. In addition, we showed that exponential solutions are not possible, and some particular self-similar solutions were studied. The solutions found are summarized in table (7.1).

Throughout chapters two to five we have given the form of the equation of state of the induced matter obtained whenever possible. A variety of equations of state were found for perfect fluid models, and various anisotropic fluid models were also obtained.

We would like to comment upon some astrophysical and cosmological implications of the solutions obtained in this thesis. From the cosmological aspect we are basically concerned with whether or not the $5D$ solutions found exhibit an initial singularity in the finite past (big bang) or future (big crunch). Another property which is also of some concern is whether or not a model compactifies in the extra dimension. This is mainly of interest in explaining why the fifth dimension is virtually unobservable. Non-static spherically symmetric $5D$ solutions are of interest since, as noted in this thesis, they illustrate that Birkhoff's theorem is not valid in dimensions more than 4. Birkhoff's theorem in higher dimensions has been discussed in [62] and [63]. Ponce de Leon and Wesson [31] have studied static spherically symmetric solutions which are separable. A new *non-separable* static solution has been presented in this thesis (in which $l(t) = constant$ and $p(t) = constant$ in metric (2.28)). We are also interested in determining whether the new exact solutions exhibit any event horizons analogous to that of the Schwarzschild solution in $4D$. Scalar curvature invariants, such as the Kretschmann scalar, can be used to investigate the properties of such models.

Let us make some comments about the Ricci-flat solutions we have found. The metric (3.45) is a non-static version of (3.35) which does not compactify in the course of time. It also exhibits a singularity at $r = m/2$ which is a true irremovable singularity despite the fact that in the $4D$ counterpart there is a coordinate singularity (and an event horizon). The spatial metric is time independent so it does not expand; that is to say no big bang or big crunch occurs in this model. The $5D$ solution (3.58) is also non-static, non-singular at $r = 0$ and does not compactify. The metric (3.12) is a generalized Kasner solution. Kasner solutions have been used to explain the smallness of the fifth dimension; an ever expanding solution requires the fifth dimension to contract and hence to compactify. The solution (5.15) shows that no event horizon exists, although a big bang singularity is present at $t = 0$ (i.e., some of the curvature tensor components diverge); also since the expansion rate along the fifth dimension is positive, it implies that the model does not compactify. Finally, the solution (4.27) does not compactify.

In chapter six we studied Abelian and non-Abelian gauge fields coupled with gravity in $4+1$ and $4+N$ dimensions. Our main goal has been to consider induced matter theory in the context of higher-dimensional Einstein-Yang-Mills cosmological models. These gauge fields do not come from the metric as in the traditional Kaluza-Klein theory (i.e., $A_\mu \propto g_{5\mu}$). In the case of an Abelian Maxwell field the induced matter is that of a perfect fluid with the equation of state (6.19). At late times this form of matter asymptotes towards a radiative equation of state. In the case of the non-Abelian Yang-Mills model, we described the fixed point solutions of the field equations consisting of an autonomous system of ordinary differential equations, and we discussed the induced equation of state associated with these fixed point solutions. We also gave the form of the solutions explicitly for two of the fixed points whose existence was simply noted in Wiltshire's work [41]. The general behavior of the solutions is that they evolve either from an anti-de Sitter spacetime or from a Kasner singularity. The solutions asymptote either to another Kasner singularity or to a de Sitter inflationary phase or to flat D-dimensional Minkowski vacuum at late times. The induced equation of state for these fixed point solutions is linear and barotropic, but depends on the number of dimensions considered. We also investigated the induced matter theory in the context of a 5-dimensional supergravity theory. The induced matter obtained is somewhat exotic but still of a perfect fluid form. For suitable values of the parameters involved, one finds that there are no initial singularities in the four-dimensional space-time which exhibits a periodic nature. Consequently, there are no early/late time behaviors for the induced matter. Instead, the energy density and the pressure have oscillatory behavior, the latter remaining mostly negative. For other values of the model's parameters, there are indeed singularities and the matter asymptotes to $p = -\frac{1}{3}\mu$ for early and late times. In these cases there are times at which the energy density actually becomes negative. Finally, by introducing anisotropy into the four-dimensional part of the space-time, dissipation terms are added to the induced matter. Consequently, anisotropic pressures are introduced which are proportional to the fluid's shear, and the corresponding viscosity coefficient was obtained. This viscosity coefficient either diverges or vanishes at early/late times, depending on the values of the constant parameters in the model. The induced matter has the asymptotic form $p = \mu/(3a)$ for early and late times, where a is a constant.

7.2 Future Work

In this thesis we have found a number of new exact higher-dimensional solutions and we have studied other models by employing the geometric theory of differential equations. Although we could apply these methods to generate more new solutions, this is not our primary goal for future work.

Using new methods of solving the field equations we have shown that by employing more general higher-dimensional metrics, and perhaps coupling them with a Yang-Mills field or supergravity, more realistic forms for the induced matter can be derived. The work in thesis consequently supports the fact that the induced matter theory is capable of offering new fundamental physical insights.

In addition, the theorem of Campbell [64] asserts that any N-dimensional Riemannian space can be locally embedded in a Ricci-flat $N+1$-dimensional Riemannian space. This implies that all solutions to the four-dimensional Einstein field equations with arbitrary energy-momentum tensor can be embedded, at least locally, in a spacetime that is itself five-dimensional and Ricci-flat. One can always, in principle, start off from a known solution of Einstein's equations with matter in four dimensions and determine a Ricci-flat spacetime in five dimensions whose induced matter is that of the four-dimensional energy-momentum tensor. Recently some new work has been done in this direction. Rippl *et al.* [65] have generalized Wesson's procedure to arbitrary dimensions. In particular, they employ this generalization to relate the usual $(3+1)$-dimensional vacuum field equations to $(2+1)$-dimensional field equations with sources. This is of importance in establishing a relationship between lower-dimensional gravity and the usual $4D$ general relativity. An outcome of this correspondence is that the intuitions obtained in $(3+1)$ dimensions may not be automatically transportable to lower dimensions. In further work Lidsey *et al.* [66] have investigated an embedding for a class of N-dimensional Einstein spaces and the local nature of Campbell's theorem is highlighted by studying the embedding of some lower-dimensional spaces.

However, there are some fundamental questions that need to be addressed in the context of induced matter theory. Perhaps the most important one involves the field equations ${}^5R_{ab} = 0$, for example, which already represent a curved space-time (except for Riemann-flat solutions) and there is no mechanism known to explain how and why the five-dimensional spacetime is curved. This perhaps motivates the study of Riemann-flat solutions in this thesis. However, as noted above, Ricci-flat (and curved) spacetimes can always be embedded in a higher-dimensional Riemann-flat spacetime. This is very important in the interpretation of the induced matter theory. The Ricci-flat solutions always involve some arbitrary functions or constants to be interpreted in the theory. This is in analogy with the Schwarzschild solution of ${}^4R_{\mu\nu} = 0$ which involves an integrating constant C which is interpreted as the central field point mass. In conventional induced matter theory there is no such physical interpretation available. In addition, as pointed out in the Introduction, the five-dimensional analogue of Einstein's equations with source is ambiguous in the sense that there is no unique way of defining a higher-dimensional energy-momentum tensor that reduces to a given four-dimensional energy-momentum tensor. This, we believe, is further motivation for studying induced matter theory.

Another issue is that in order to have an intrinsic induced matter theory, a unique way of foliating $5D$ spacetime into $4+1$ slices (or even better a slicing-independent theory) is necessary.

It is these more fundamental issues that we should like to pursue in future work.

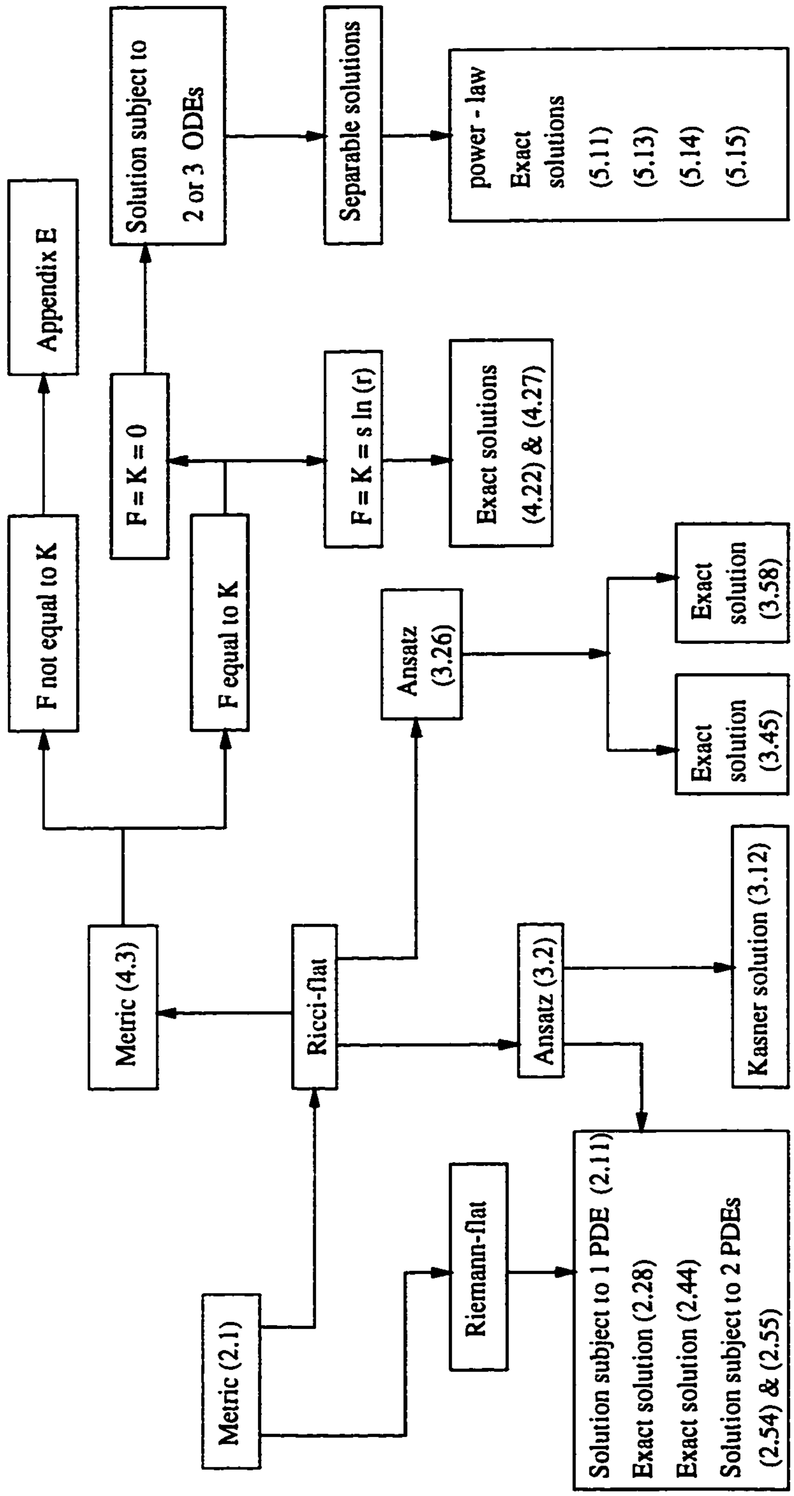

Table 7.1: Summary of solutions

Appendix A

Riemann tensor components for the general spherically symmetric 5D metric

$$ds^2 = -e^{2f(t,r,y)}dt^2 + e^{2g(t,r,y)}(dr^2 + r^2 d\Omega^2) + e^{2k(t,r,y)}dy^2$$

$$R_{1212} = \Bigg(\left(\frac{\partial^2}{\partial r^2} f\right) e^{(2f+2k)} + \left(\frac{\partial}{\partial r} f\right)^2 e^{(2f+2k)} - \left(\frac{\partial^2}{\partial t^2} g\right) e^{(2g+2k)}$$
$$- \left(\frac{\partial}{\partial t} g\right)^2 e^{(2g+2k)} + \left(\frac{\partial}{\partial t} f\right)\left(\frac{\partial}{\partial t} g\right) e^{(2g+2k)}$$
$$- \left(\frac{\partial}{\partial r} f\right)\left(\frac{\partial}{\partial r} g\right) e^{(2f+2k)} + \left(\frac{\partial}{\partial y} f\right)\left(\frac{\partial}{\partial y} g\right) e^{(2f+2g)}\Bigg) e^{(-2k)}$$

$R_{1215} =$
$$e^{(2f)} \left(\left(\frac{\partial^2}{\partial y\, \partial r} f\right) + \left(\frac{\partial}{\partial r} f\right)\left(\frac{\partial}{\partial y} f\right) - \left(\frac{\partial}{\partial r} f\right)\left(\frac{\partial}{\partial y} g\right) - \left(\frac{\partial}{\partial y} f\right)\left(\frac{\partial}{\partial r} k\right)\right)$$

$R_{1225} =$

$$e^{(2g)}\left(\left(\frac{\partial^2}{\partial y\,\partial t}g\right)+\left(\frac{\partial}{\partial t}g\right)\left(\frac{\partial}{\partial y}g\right)-\left(\frac{\partial}{\partial y}f\right)\left(\frac{\partial}{\partial t}g\right)-\left(\frac{\partial}{\partial t}k\right)\left(\frac{\partial}{\partial y}g\right)\right)$$

$$R_{1313}=r\left(-\left(\frac{\partial^2}{\partial t^2}g\right)r\,e^{(2g+2k)}-\left(\frac{\partial}{\partial t}g\right)^2 r\,e^{(2g+2k)}\right.$$
$$+\left(\frac{\partial}{\partial t}f\right)\left(\frac{\partial}{\partial t}g\right)r\,e^{(2g+2k)}+\left(\frac{\partial}{\partial r}f\right)\left(\frac{\partial}{\partial r}g\right)r\,e^{(2f+2k)}$$
$$\left.+\left(\frac{\partial}{\partial r}f\right)e^{(2f+2k)}+\left(\frac{\partial}{\partial y}f\right)\left(\frac{\partial}{\partial y}g\right)r\,e^{(2f+2g)}\right)e^{(-2k)}$$

$$R_{1323}=e^{(2g)}r^2\left(-\left(\frac{\partial^2}{\partial t\,\partial r}g\right)+\left(\frac{\partial}{\partial t}g\right)\left(\frac{\partial}{\partial r}f\right)\right)$$

$$R_{1335}=e^{(2g)}r^2$$
$$\left(\left(\frac{\partial^2}{\partial y\,\partial t}g\right)+\left(\frac{\partial}{\partial t}g\right)\left(\frac{\partial}{\partial y}g\right)-\left(\frac{\partial}{\partial y}f\right)\left(\frac{\partial}{\partial t}g\right)-\left(\frac{\partial}{\partial t}k\right)\left(\frac{\partial}{\partial y}g\right)\right)$$

$$R_{1414}=-r(\cos(\theta)-1)(\cos(\theta)+1)\left(-r\,e^{(2g)}\left(\frac{\partial}{\partial t}g\right)^2-r\,e^{(2g)}\left(\frac{\partial^2}{\partial t^2}g\right)\right.$$
$$+r\,e^{(2f)}\left(\frac{\partial}{\partial r}f\right)\left(\frac{\partial}{\partial r}g\right)+r\left(\frac{\partial}{\partial t}g\right)\left(\frac{\partial}{\partial t}f\right)e^{(2g)}$$
$$\left.+r\left(\frac{\partial}{\partial y}g\right)\left(\frac{\partial}{\partial y}f\right)e^{(-2k+2f+2g)}+\left(\frac{\partial}{\partial r}f\right)e^{(2f)}\right)$$

$$R_{1424}=r^2e^{(2g)}(\cos(\theta)-1)(\cos(\theta)+1)\left(\left(\frac{\partial^2}{\partial t\,\partial r}g\right)-\left(\frac{\partial}{\partial t}g\right)\left(\frac{\partial}{\partial r}f\right)\right)$$

$$R_{1445}=-e^{(2g)}r^2(\cos(\theta)-1)(\cos(\theta)+1)$$
$$\left(\left(\frac{\partial^2}{\partial y\,\partial t}g\right)+\left(\frac{\partial}{\partial t}g\right)\left(\frac{\partial}{\partial y}g\right)-\left(\frac{\partial}{\partial y}f\right)\left(\frac{\partial}{\partial t}g\right)-\left(\frac{\partial}{\partial t}k\right)\left(\frac{\partial}{\partial y}g\right)\right)$$

$$R_{1515}=\left(\left(\frac{\partial^2}{\partial y^2}f\right)e^{(2f+2g)}+\left(\frac{\partial}{\partial y}f\right)^2e^{(2f+2g)}-\left(\frac{\partial^2}{\partial t^2}k\right)e^{(2g+2k)}\right.$$
$$-\left(\frac{\partial}{\partial t}k\right)^2e^{(2g+2k)}+\left(\frac{\partial}{\partial t}f\right)\left(\frac{\partial}{\partial t}k\right)e^{(2g+2k)}$$

$$+\left(\frac{\partial}{\partial r} f\right)\left(\frac{\partial}{\partial r} k\right) e^{(2f+2k)} - \left(\frac{\partial}{\partial y} f\right)\left(\frac{\partial}{\partial y} k\right) e^{(2f+2g)}\Bigg) e^{(-2g)}$$

$$R_{1525} = -e^{(2k)}\left(\left(\frac{\partial^2}{\partial t\,\partial r} k\right) + \left(\frac{\partial}{\partial t} k\right)\left(\frac{\partial}{\partial r} k\right) - \left(\frac{\partial}{\partial t} k\right)\left(\frac{\partial}{\partial r} f\right) - \left(\frac{\partial}{\partial t} g\right)\left(\frac{\partial}{\partial r} k\right)\right)$$

$$R_{2323} = -r\left(\left(\frac{\partial^2}{\partial r^2} g\right) r\, e^{(2f+2k)} + \left(\frac{\partial}{\partial r} g\right) e^{(2f+2k)} - \left(\frac{\partial}{\partial t} g\right)^2 r\, e^{(2g+2k)} + \left(\frac{\partial}{\partial y} g\right)^2 r\, e^{(2f+2g)}\right) e^{(2g-2f-2k)}$$

$$R_{2335} := -e^{(2g)} r^2 \left(-\left(\frac{\partial^2}{\partial y\,\partial r} g\right) + \left(\frac{\partial}{\partial r} k\right)\left(\frac{\partial}{\partial y} g\right)\right)$$

$$R_{2424} = r\,(\cos(\theta) - 1)\,(\cos(\theta) + 1)\left(r\left(\frac{\partial^2}{\partial r^2} g\right) e^{(2g)} + \left(\frac{\partial}{\partial r} g\right) e^{(2g)} - r\, e^{(4g-2f)}\left(\frac{\partial}{\partial t} g\right)^2 + r\, e^{(4g-2k)}\left(\frac{\partial}{\partial y} g\right)^2\right)$$

$$R_{2445} = e^{(2g)} r^2 (\cos(\theta) - 1)(\cos(\theta) + 1)\left(-\left(\frac{\partial^2}{\partial y\,\partial r} g\right) + \left(\frac{\partial}{\partial r} k\right)\left(\frac{\partial}{\partial y} g\right)\right)$$

$$R_{2525} = -\Bigg(\left(\frac{\partial^2}{\partial y^2} g\right) e^{(2f+2g)} + \left(\frac{\partial}{\partial y} g\right)^2 e^{(2f+2g)} + \left(\frac{\partial^2}{\partial r^2} k\right) e^{(2f+2k)} + \left(\frac{\partial}{\partial r} k\right)^2 e^{(2f+2k)} - \left(\frac{\partial}{\partial t} g\right)\left(\frac{\partial}{\partial t} k\right) e^{(2g+2k)} - \left(\frac{\partial}{\partial r} g\right)\left(\frac{\partial}{\partial r} k\right) e^{(2f+2k)} - \left(\frac{\partial}{\partial y} g\right)\left(\frac{\partial}{\partial y} k\right) e^{(2f+2g)}\Bigg) e^{(-2f)}$$

$$R_{3434} = r^3 (\cos(\theta) - 1)(\cos(\theta) + 1)\left(-r\, e^{(4g-2f)}\left(\frac{\partial}{\partial t} g\right)^2 + r\, e^{(2g)}\left(\frac{\partial}{\partial r} g\right)^2 + 2\left(\frac{\partial}{\partial r} g\right) e^{(2g)} + r\, e^{(4g-2k)}\left(\frac{\partial}{\partial y} g\right)^2\right)$$

$$R_{3535} = r\left(-\left(\frac{\partial^2}{\partial y^2} g\right) r\, e^{(2f+2g)} - \left(\frac{\partial}{\partial y} g\right)^2 r\, e^{(2f+2g)}\right.$$
$$+ \left(\frac{\partial}{\partial t} g\right) r \left(\frac{\partial}{\partial t} k\right) e^{(2g+2k)} - \left(\frac{\partial}{\partial r} g\right) \left(\frac{\partial}{\partial r} k\right) r\, e^{(2f+2k)}$$
$$\left. - \left(\frac{\partial}{\partial r} k\right) e^{(2f+2k)} + \left(\frac{\partial}{\partial y} g\right) r \left(\frac{\partial}{\partial y} k\right) e^{(2f+2g)}\right) e^{(-2f)}$$

$$R_{4545} = -r\,(\cos(\theta) - 1)\,(\cos(\theta) + 1)\left(-r\, e^{(2g)} \left(\frac{\partial}{\partial y} g\right)^2\right.$$
$$- r\, e^{(2g)} \left(\frac{\partial^2}{\partial y^2} g\right) - r \left(\frac{\partial}{\partial r} k\right) \left(\frac{\partial}{\partial r} g\right) e^{(2k)}$$
$$+ r\, e^{(-2f+2g+2k)} \left(\frac{\partial}{\partial t} g\right) \left(\frac{\partial}{\partial t} k\right) + r \left(\frac{\partial}{\partial y} k\right) \left(\frac{\partial}{\partial y} g\right) e^{(2g)}$$
$$\left. - \left(\frac{\partial}{\partial r} k\right) e^{(2k)}\right)$$

Appendix B

Ricci tensor components for the general spherically symmetric 5D metric

$$ds^2 = -f^2(t,r,y)dt^2 + g^2(t,r,y)(dr^2 + r^2 d\Omega^2) + k^2(t,r,y)dy^2$$

$$\begin{aligned}
R_{11} = & -k^3 r f^2 \left(\frac{\partial^2}{\partial r^2} f\right) g + 3 k^3 r g^2 \left(\frac{\partial^2}{\partial t^2} g\right) f - 3 k^3 r \left(\frac{\partial}{\partial t} f\right) g^2 \left(\frac{\partial}{\partial t} g\right) \\
& - k^3 r f^2 \left(\frac{\partial}{\partial r} f\right) \left(\frac{\partial}{\partial r} g\right) - 3 k r f^2 \left(\frac{\partial}{\partial y} f\right) g^2 \left(\frac{\partial}{\partial y} g\right) - 2 f^2 \left(\frac{\partial}{\partial r} f\right) k^3 g \\
& - g^3 r f^2 \left(\frac{\partial^2}{\partial y^2} f\right) k + g^3 r k^2 \left(\frac{\partial^2}{\partial t^2} k\right) f - g^3 r k^2 \left(\frac{\partial}{\partial t} f\right) \left(\frac{\partial}{\partial t} k\right) \\
& - g r f^2 \left(\frac{\partial}{\partial r} f\right) k^2 \left(\frac{\partial}{\partial r} k\right) + g^3 r f^2 \left(\frac{\partial}{\partial y} f\right) \left(\frac{\partial}{\partial y} k\right)
\end{aligned}$$

$$\begin{aligned}
R_{12} = & -2 k \left(\frac{\partial}{\partial r} g\right) \left(\frac{\partial}{\partial t} g\right) f + 2 k g \left(\frac{\partial^2}{\partial r\, \partial t} g\right) f - 2 k g \left(\frac{\partial}{\partial t} g\right) \left(\frac{\partial}{\partial r} f\right) \\
& + \left(\frac{\partial^2}{\partial r\, \partial t} k\right) f g^2 - \left(\frac{\partial}{\partial t} k\right) \left(\frac{\partial}{\partial r} f\right) g^2 - g \left(\frac{\partial}{\partial t} g\right) \left(\frac{\partial}{\partial r} k\right) f
\end{aligned}$$

$$R_{15} = 3\left(\frac{\partial^2}{\partial y\,\partial t} g\right) f\,k - 3\left(\frac{\partial}{\partial y} f\right)\left(\frac{\partial}{\partial t} g\right) k - 3\left(\frac{\partial}{\partial t} k\right)\left(\frac{\partial}{\partial y} g\right) f$$

$$\begin{aligned} R_{22} = {} & g^2\,k^3\,r\,f^2\left(\frac{\partial^2}{\partial r^2} f\right) - g^3\,k^3\,r\left(\frac{\partial^2}{\partial t^2} g\right) f + g^3\,k^3\,r\left(\frac{\partial}{\partial t} f\right)\left(\frac{\partial}{\partial t} g\right) \\ & - g\,k^3\,r\,f^2\left(\frac{\partial}{\partial r} f\right)\left(\frac{\partial}{\partial r} g\right) + g^3\,k\,r\,f^2\left(\frac{\partial}{\partial y} f\right)\left(\frac{\partial}{\partial y} g\right) - 2\,f^3\,k^3\left(\frac{\partial}{\partial r} g\right)^2 r \\ & + 2\,f^3\,k^3\,g\left(\frac{\partial}{\partial r} g\right) + 2\,f^3\,k^3\,g\,r\left(\frac{\partial^2}{\partial r^2} g\right) - 2\,f\,k^3\,g^2\left(\frac{\partial}{\partial t} g\right)^2 r \\ & + 2\,f^3\,k\,g^2\left(\frac{\partial}{\partial y} g\right)^2 r + f^3\,g^3\,r\left(\frac{\partial^2}{\partial y^2} g\right) k + f^3\,g^2\,r\,k^2\left(\frac{\partial^2}{\partial r^2} k\right) \\ & - f\,g^3\,r\left(\frac{\partial}{\partial t} g\right) k^2\left(\frac{\partial}{\partial t} k\right) - f^3\,g\,r\,k^2\left(\frac{\partial}{\partial r} g\right)\left(\frac{\partial}{\partial r} k\right) \\ & - f^3\,g^3\,r\left(\frac{\partial}{\partial y} g\right)\left(\frac{\partial}{\partial y} k\right) \end{aligned}$$

$$\begin{aligned} R_{25} = {} & \left(\frac{\partial^2}{\partial r\,\partial y} f\right) g^2\,k - g\left(\frac{\partial}{\partial r} f\right)\left(\frac{\partial}{\partial y} g\right) k - \left(\frac{\partial}{\partial y} f\right)\left(\frac{\partial}{\partial r} k\right) g^2 \\ & - 2\,f\left(\frac{\partial}{\partial y} g\right)\left(\frac{\partial}{\partial r} g\right) k + 2\,f\,g\left(\frac{\partial^2}{\partial r\,\partial y} g\right) k - 2\,f\left(\frac{\partial}{\partial r} k\right) g\left(\frac{\partial}{\partial y} g\right) \end{aligned}$$

$$\begin{aligned} R_{33} = {} & \Bigg(k^3\,r\,g^2\left(\frac{\partial^2}{\partial t^2} g\right) f - k^3\,r\left(\frac{\partial}{\partial t} f\right) g^2\left(\frac{\partial}{\partial t} g\right) - k^3\,r\,f^2\left(\frac{\partial}{\partial r} f\right)\left(\frac{\partial}{\partial r} g\right) \\ & - f^2\left(\frac{\partial}{\partial r} f\right) k^3\,g - k\,r\,f^2\left(\frac{\partial}{\partial y} f\right) g^2\left(\frac{\partial}{\partial y} g\right) - 3\,f^3\,k^3\left(\frac{\partial}{\partial r} g\right) \\ & - f^3\,k^3\,r\left(\frac{\partial^2}{\partial r^2} g\right) + 2\,f\,k^3\,g\left(\frac{\partial}{\partial t} g\right)^2 r - 2\,f^3\,k\,g\left(\frac{\partial}{\partial y} g\right)^2 r \\ & - f^3\,g^2\,r\left(\frac{\partial^2}{\partial y^2} g\right) k + f\,g^2\,r\left(\frac{\partial}{\partial t} g\right) k^2\left(\frac{\partial}{\partial t} k\right) - f^3\,r\,k^2\left(\frac{\partial}{\partial r} g\right)\left(\frac{\partial}{\partial r} k\right) \\ & - f^3\,g\,k^2\left(\frac{\partial}{\partial r} k\right) + f^3\,g^2\,r\left(\frac{\partial}{\partial y} g\right)\left(\frac{\partial}{\partial y} k\right) \end{aligned}$$

$$
\begin{aligned}
R_{55} = {} & g^3\, r\, f^2 \left(\frac{\partial^2}{\partial y^2} f\right) k - g^3\, r\, k^2 \left(\frac{\partial^2}{\partial t^2} k\right) f + g^3\, r\, k^2 \left(\frac{\partial}{\partial t} f\right) \left(\frac{\partial}{\partial t} k\right) \\
& + g\, r\, f^2 \left(\frac{\partial}{\partial r} f\right) k^2 \left(\frac{\partial}{\partial r} k\right) - g^3\, r\, f^2 \left(\frac{\partial}{\partial y} f\right) \left(\frac{\partial}{\partial y} k\right) + 3\, f^3\, g^2\, r \left(\frac{\partial^2}{\partial y^2} g\right) k \\
& + f^3\, r\, k^2 \left(\frac{\partial^2}{\partial r^2} k\right) g - 3\, f\, g^2\, r \left(\frac{\partial}{\partial t} g\right) k^2 \left(\frac{\partial}{\partial t} k\right) + f^3\, r\, k^2 \left(\frac{\partial}{\partial r} g\right) \left(\frac{\partial}{\partial r} k\right) \\
& - 3\, f^3\, g^2\, r \left(\frac{\partial}{\partial y} g\right) \left(\frac{\partial}{\partial y} k\right) + 2\, f^3\, g\, k^2 \left(\frac{\partial}{\partial r} k\right)
\end{aligned}
$$

Appendix C

Perturbation analysis

C.1 (a) The Kasner solution in general $SO(N+1)$ in $4+N$ dimensions for the field equations (6.27)-(6.30)

We shall investigate first order perturbations of the Kasner solution of (6.27)-(6.30) of the following form:

$$H(t) = t^{a+\frac{1}{3}b}(1+\epsilon h(t))$$

$$L(t) = t^{a-\frac{1}{N}b}(1+\epsilon l(t))$$

where $a = \frac{1}{3+N}$ and $b = \frac{\sqrt{3N^2+6N}}{3+N}$, in order for the zeroth order solution to be of Kasner-type. By substituting the above in the constraint equation (6.30) and taking only ϵ terms (the ϵ^0 terms vanish by taking above values for a and b since it corresponds to an exact solution), we get

$$\dot{h}(t) = -\frac{[aN(N+2)+b]}{[a(3N+6)-b]}\dot{l} - \frac{3kt^{1-2a-\frac{2}{3}b}}{[a(3N+6)-b]} - \frac{\frac{1}{2}N(N-1)Kt^{1-2a+\frac{2}{N}}}{[a(3N+6)-b]}$$

$$+\frac{1}{2}\frac{(N+2)t^{1-2aN+2b}}{[a(3N+6)-b]}+\frac{1}{2}\frac{(N+2)\Lambda t}{a[3(N+6)-b]} \quad \text{(C.1)}$$

and after integration one obtains

$$h(t)=h_1+\frac{1}{[a(3N+6)-b]}\{-[aN(N+2)+b]l-\frac{3kt^{2(1-a-\frac{1}{3}b)}}{2(1-a-\frac{b}{3})}$$
$$-\frac{N(N-1)\tilde{K}t^{2(1-a+\frac{b}{3})}}{4(1-a+\frac{b}{N})}+\frac{(N+2)t^{2(1-aN+b)}}{4(1-aN+b)}+\frac{(N+2)\Lambda t^2}{4}\} \ . \quad \text{(C.2)}$$

Now from (6.27) one obtains

$$\frac{3\ddot{H}}{H}+N\frac{\ddot{L}}{L}=\frac{\Lambda}{2}-(N-1)\frac{C}{L^{2N}}$$

which leads to

$$b(N+3)(t^2\ddot{l}+2t\dot{l})-\frac{1}{2}\Lambda t^2[((a+1)(3N+6)+b(5+2N)]$$
$$-\frac{3}{2}Ct^{2-2aN+2b}[N+2+2b(N+3)]+9kt^{2-2a-\frac{2}{3}b}$$
$$+\frac{1}{2}(N-1)[3N+2b(3+N)]t^{2-2a+\frac{2}{N}b}\tilde{K}=0 \ . \quad \text{(C.3)}$$

Solving the above differential equation for $l(t)$, we obtain

$$l(t)=l_1+\frac{l_2}{t}+\tilde{a}\frac{\Lambda}{2}t^2+\tilde{b}Ct^{2-2aN+2b}+\tilde{c}kt^{2-2a-\frac{2}{3}b}+\tilde{d}\tilde{K}t^{2-2a+\frac{2}{N}b}$$

where $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$ have a particular dependence on the constants a, b and N, and l_1 and l_2 are arbitrary constants.

The existence of the term $l_2\, t^{-1}$ doesn't imply that the unperturbed solution is unstable to the past. One should actually consider all perturbations to all orders in order to actually decide if the unperturbed solution is unstable. In fact, further analysis shows that the Kasner solution is stable by taking all orders of perturbation into account. This corresponds to the fact that the Kasner solution is a 1-parameter family of solutions (the parameter can be absorbed in the metric be a change of coordinates) and the term $l_2\, t^{-1}$ actually corresponds to the fact that we have perturbed the solution to another exact Kasner solution.

C.2 (b) The de Sitter solution in general $SO(N+1)$ in $4+N$ dimensions for the field equation (6.27)-(6.30)

Here we consider the first order perturbation of the form

$$H(t) = H_0 e^{pt}(1+\epsilon h(t))$$

$$L(t) = L_0 e^{pt}(1+\epsilon l(t))$$

where $p = \sqrt{\frac{\Lambda}{2(3+N)}}$. Again by substitution into the field equations, and considering the first order terms in ϵ, we get

$$\dot{h} = -\frac{N}{3}\dot{l} - \frac{ke^{-pt}}{(N+2)p} - \frac{N(N-1)}{6(N+2)p}e^{-pt} + \frac{C}{6p}e^{-ptN} \quad ,$$

and after integration, we obtain

$$h = h_1 - \frac{N}{3}l + \frac{k(N+3)}{\Lambda(N+2)}e^{-pt} + \frac{N(N+3)(N-1)K}{6\Lambda(N+2)}e^{-pt} - \frac{C(N+3)}{6\Lambda N}e^{-Npt} \quad .$$

Now from (6.27) we obtain

$$\ddot{l} + \frac{N+3}{\sqrt{2}}p\dot{l} - \frac{5}{2}Ce^{-Npt} - \frac{3k}{N+2}e^{-pt} + \frac{(N+4)(N-1)}{2(N+2)}e^{-pt} \quad .$$

Solving the above differential equation for $l(t)$ results in

$$\begin{aligned} l(t) \quad &= \quad l_1 + l_2 e^{-\frac{1}{2}\sqrt{\Lambda(N+3)t}} - \frac{3(N+3)k}{\Lambda(N+1)(N+2)}e^{-\sqrt{\frac{2\Lambda}{N+3}}t} \\ &+ \frac{(N+4)(N+3)(N-1)K}{2\Lambda(N+1)(N+2)} + \frac{5C(N+3)}{\Lambda N(N-3)}e^{-N\sqrt{\frac{2\Lambda}{N+3}}t} \quad , \end{aligned} \tag{C.4}$$

and we then obtain the following expression for $h(t)$:

$$h(t) = h_1 - \frac{N}{3}l_1 - \frac{Nl_2}{3}e^{-\sqrt{\frac{\Lambda(N+3)}{2}}t} + \frac{C(2N-1)(N+3)}{2\Lambda N(N-3)}e^{-N\sqrt{\frac{2\Lambda}{3+N}}t} +$$
$$\frac{(N+3)(2N+1)k}{\Lambda(N+1)(N+2)}e^{-\sqrt{\frac{2\Lambda}{N+3}}t} - \frac{N(N-1)(N+3)K}{2\Lambda N(N+1)(N+2)}e^{-\sqrt{\frac{2\Lambda}{N+3}}t} \quad , \qquad \text{(C.5)}$$

where l_1 and l_2 are arbitrary constants. The above analysis shows that the de Sitter solution of the field equation (6.27) is late-time stable because of the negative exponents in the terms in the perturbed functions $h(t)$ and $l(t)$.

Appendix D

Dynamical Systems Review

D.1 Preliminary Definitions

This review comes from two primary sources; the set of dynamical systems notes prepared by John Wainwright which appeared in the workshop proceedings *Deterministic Chaos in General Relativity* [67] and from the first chapter of Stephen Wiggins' book *Introduction to Applied Nonlinear Dynamical Systems and Chaos* [68].

Definition 1 *An equilibrium solution of the DE $\dot{x} = f(x)$ is a point $\bar{x} \in \mathcal{R}^n$ such that*

$$f(\bar{x}) = 0.$$

Once an equilibrium solution is found, it becomes of interest to determine the behaviour of solutions of the DE in a neighborhood of the equilibrium solution.

Definition 2 *Let $\bar{x} \in \mathcal{R}^n$ be an equilibrium point of the DE $\dot{x} = f(x)$, and let $u = x - \bar{x}$, then the nonlinear DE $\dot{x} = f(x)$ has an associated linear DE*

$$\dot{u} = \mathcal{D}f(\bar{x})u$$

which is called the linearization of the DE $\dot{x} = f(x)$ at the equilibrium point $\bar{x}$.

Definition 3 *Let $\bar{x}$ be a equilibrium point of the DE $\dot{x} = f(x)$. Then $\bar{x}$ is called a hyperbolic equilibrium point if none of eigenvalues of $\mathcal{D}f(\bar{x})$ have zero real parts.*

D.2 The Flow of a Non-Linear DE

Definition 4 *Let $x(t) = \psi_a(t)$ be a solution of the DE $\dot{x} = f(x)$ with initial condition $x(0) = a$. The flow $\{g^t\}$ is defined in terms of the solution function $\psi_a(t)$ of the DE by*

$$g^t a = \psi_a(t).$$

Definition 5 *The orbit through a, denoted by $\gamma(a)$ is defined to be*

$$\gamma(a) = \left\{ x \in \mathcal{R}^n | x = g^t a, \text{ for all } t \in \mathcal{R} \right\}$$

Orbits are classified as *point orbits*, *periodic orbits*, and *non-periodic orbits.*

Definition 6 *An ω-limit set of a point a, $\omega(a)$, is the set of points in $\mathcal{R}^n$ which are approached along the orbit through a with increasing time.*

Definition 7 *Given a DE $\dot{x} = f(x)$ in $\mathcal{R}^n$, a set $S \subseteq \mathcal{R}^n$ is called an invariant set for the DE if for any point $a \in S$, the orbit through a lies entirely in S, that is $\gamma(a) \subseteq S$.*

In order to determine an ω-limit set, it is helpful to know that an orbit enters a bounded set S and never leaves it. Such a set is called a trapping set.

Definition 8 *Given a DE $\dot{x} = f(x)$ in $\mathcal{R}^n$, with flow $\{g^t\}$, a subset $S \subset \mathcal{R}^n$ is said to be a trapping set of the DE if it satisfies*

1. *S is a closed and bounded set,*

2. *$a \in S$ implies $g^t a \in S$ for all $t \geq 0$.*

The usefulness of trapping sets lies in this result; if S is a trapping set of a DE $\dot{x} = f(x)$, then for all $a \in S$, the ω-limit set $\omega(a)$ is non-empty and is contained in S.

Definition 9

1. *The equilibrium point $\bar{x}$ of a DE $\dot{x} = f(x)$ is stable if for all neighborhoods U of $\bar{x}$, there exists a neighborhood V of $\bar{x}$ such that $g^t V \subseteq U$ for all $t \geq 0$ where g^t is the flow of the DE.*

2. *The equilibruim point $\bar{x}$ of a DE $\dot{x} = f(x)$ is asymptotically stable if it is stable and if, in addition, for all $x \in V$, $\lim_{t\to\infty} ||g^t x - \bar{x}|| = 0$.*

Theorem 1 (Lyapunov Stability) *Let $\bar{x}$ be an equilibrium point of the DE $\dot{x} = f(x)$ in $\mathcal{R}^n$. Let $V : \mathcal{R}^n \to \mathcal{R}$ be a C^1 function such that $V(\bar{x}) = 0$, $V(x) > 0$ for all $x \in U - \{\bar{x}\}$, where U is a neighborhood of $\bar{x}$.*

1. *If $\dot{V}(x) < 0$ for all $x \in U - \{\bar{x}\}$, then $\bar{x}$ is asymptotically stable.*

2. *If $\dot{V} \leq 0$ for all $x \in U - \{\bar{x}\}$, then $\bar{x}$ is stable.*

3. *If $\dot{V}(x) > 0$ for all $x \in U - \{\bar{x}\}$, then $\bar{x}$ is unstable.*

Proof. [See [68].] □

A function $V : \mathcal{R}^n \to \mathcal{R}$ which satisfies $V(\bar{x}) = 0$, $V(x) > 0$ for all $x \in U - \{\bar{x}\}$, and $\dot{V}(x) \leq 0$ (respectively, < 0) for all $x \in U - \{\bar{x}\}$, is called a Lyapunov function (respectively, a strict Lyapunov function) for the equilibrium point $\bar{x}$.

Theorem 2 (Criterion for Asymptotic Stability) *Let $\bar{x}$ be an equilibrium point of the DE $\dot{x} = f(x)$ in $\mathcal{R}^n$. If all eigenvalues of the derivative matrix $\mathcal{D}f(\bar{x})$ satisfy $\mathcal{R}e(\lambda) < 0$, then the equilibrium point $\bar{x}$ is asymptotically stable.*

Proof. [See Wiggins [68], page 13.] □

D.3 The Hartman-Grobman Theorem

Theorem 3 (Hartman-Grobman) *Let $\bar{x}$ be a hyperbolic equilibrium point of the DE $\dot{x} = f(x)$ in $\mathcal{R}^n$, where $f : \mathcal{R}^n \to \mathcal{R}^n$ is of class C^1. Then there is a homeomorphism which maps orbits of the linear flow $e^{t\mathcal{D}f(\bar{x})}$ onto orbits of the non-linear flow g^t in a neighborhood of the equilibrium point $\bar{x}$, preserving the parameter t.*

Proof. [See Hartman [69], pages 244-250.] □

A hyperbolic fixed point $\bar{x}$, is called a saddle if not all of the eigenvalues of the associated linearization are of the same sign. $\bar{x}$ is called a source if the eigenvalues are all positive, and a sink if they are all negative.

The following theorem follows from the Hartmann-Grobman theorem.

Theorem 4 (Stable Manifold Theorem) *Let $\bar{x}$ be a equilibrium point of $\dot{x} = f(x)$ in $\mathcal{R}^n$, where f is of class C^2, and let E^s be the stable subspace of the linearization at $\bar{x}$, that is the subspace spanned by the eigenvectors corresponding to the eigenvalues with $\mathcal{R}e(\lambda) < 0$. Then there exists a neighborhood U of $\bar{x}$ such that the local stable manifold $W^s(\bar{x}, U)$ is a smooth (C^1) manifold that is tangent to E^s at $\bar{x}$.*

D.4 Periodic Orbits and Limit Sets in the Plane

Theorem 5 (Dulac's Criterion) *If $D \subseteq \mathcal{R}^2$ is a simply connected open set and $\nabla(Bf) = \frac{\partial}{\partial x_1}(Bf_1) + \frac{\partial}{\partial x_2}(Bf_2) > 0$, (or < 0) for all $x \in D$ where B is a C^1 function, then the DE $\dot{x} = f(x)$ where $f \in C^1$ has no periodic orbit which is contained in D.*

Proof. Based on Green's Theorem. □

Comment: The function $B(x_1, x_2)$ is called a Dulac function for the DE in the set D.

The second criterion for excluding periodic orbits, which is valid in $\mathcal{R}^n$, $n \geq 2$, follows from the observation that if a function $V(x)$ is monotone decreasing along an orbit of a DE, then that orbit cannot be periodic.

Theorem 6 *Let $V : \mathcal{R}^n \to \mathcal{R}$ be a C^1 function. If $\dot{V}(x) = \nabla V(x) \cdot f(x) \le 0$ on a subset $D \subseteq \mathcal{R}^n$, then any periodic orbit of the DE $\dot{x} = f(x)$ which lies in D, belongs to the subset $\left\{x | \dot{V}(x) = 0\right\} \cap D$.*

Theorem 7 *Consider a DE $\dot{x} = f(x)$ in $\mathcal{R}^2$. Let $a \in \mathcal{R}^2$ be an initial point such that $\{g^t a | t \ge 0\}$ lies in a closed bounded subset $K \subset \mathcal{R}^2$. If K contains only a finite number of equilibrium points then one of the following holds:*

1. *$\omega(a)$ is an equilibrium point*
2. *$\omega(a)$ is a periodic orbit*
3. *$\omega(a)$ is a cycle graph[1].*

Proof. The proof is based on the fundamental lemma of ω-limit sets in $\mathcal{R}^2$. [See Hale [70], page 230, and Lefshetz [71], page 129]. □

Comment: This theorem does not generalize to DEs in $\mathcal{R}^n$, $n \ge 3$, or to DEs on the 2–torus. Indeed, the problem of describing all possible ω-limit sets in $\mathcal{R}^n$, $n \ge 3$, is presently unsolved.

D.5 Bifurcations of Equilibria

Consider a DE in $\mathcal{R}^n$ of the form $\dot{x} = f(x, \mu)$ where μ is a real parameter. Bifurcation theory, as applied to DEs, is the study of how the portrait of the orbits change as μ varies.

Theorem 8 (Hopf) *Consider the DE $\dot{x} = f(x, \mu)$ in $\mathcal{R}^2$, where $f \in C^3$. Suppose $f(0, \mu) = 0$ for all $\mu \in I \subset \mathcal{R}$, and that $\mathcal{D}f(0, \mu)$ has eigenvalues $\alpha(\mu) + i\beta(\mu)$. If*

H1: *there exists a $\mu_0 \in I$ such that $\alpha(\mu_0) = 0$, $\beta(\mu_0) \ne 0$, $\alpha'(\mu_0) \ne 0$*

H2: *the equilibrium point $x = 0$ is not a nonlinear center when $\mu = \mu_0$*

[1] *A cycle graph is a union of two or more whole orbits, e.g., a homoclinic orbit.*

then

C: *there exists a* $\delta > 0$ *such that for each* $\mu \in (\mu_0, \mu_0 + \delta)$ *or* $\mu \in (\mu_0 - \delta, \mu_0)$, *the DE has a unique periodic orbit (when restricted to a sufficiently small neighborhood of* $x = 0$*).*

Proof. [See Hopf [72, 73], vol. 94 , pages 1-22 and vol. 95, pages 3-22.] □

Appendix E

Previous work of McManus [28]

Mc Manus' starting point is the metric ansatz:

$$ds^2 = -e^{2F(r)} A^2(t,y)\, dt^2 + e^{2G(r)} B^2(t,y)\,(dr^2 + r^2\, d\Omega^2) + e^{2K(r)} C^2(t,y)\, dy^2 \quad . \tag{E.1}$$

First, he observed that the pivotal field equations are $R_{tr} = R_{ty} = R_{ry} = 0$, which yield

$$K_r\, \partial_t \ln(C/B) = F_r\, \partial_t \ln(CB^2) \quad , \tag{E.2}$$

$$B_{ty} = B_t\, \partial_y \ln A + B_y\, \partial_t \ln C \quad , \tag{E.3}$$

$$F_r\, \partial_y \ln(A/B) = K_r\, \partial_y \ln(AB^2) \quad . \tag{E.4}$$

Equation (E.2) immediately implies that the solutions can be classified into four cases, namely (1) $F_r = K_r = 0$; (2) $F_r = 0, K_r \neq 0$; (3) $F_r \neq 0, K_r = 0$; and (4) $F_r K_r \neq 0$. Below, we shall only list the new exact solutions found and not the known solutions nor the reduced system of ODEs obtained.

$F_r = K_r = 0$

This case has been completely studied previously by Ponce de Leon [25] and Mc Manus[28].

$F_r = 0, K_r \neq 0$

When $B_t \neq 0$ the exact solution obtained is

$$ds^2 = -dt^2 + t^2\left[(R^2 + 1 + \frac{a}{R})^{-1}dR^2 + R^2\, d\Omega^2\right] + t^2(R^2 + 1 + \frac{a}{R})\, dy^2 \quad , \quad \text{(E.5)}$$

where a is an arbtrary constant. The above metric is Riemann-flat if and only if $a = 0$. If $B_t = 0$, then the field equations reduce to a system of ODEs for two unknown functions.

$F_r \neq 0, K_r = 0$

In the case where $B_t = 0$, integration of the field equations results in the metric

$$ds^2 = -\cos^2(\ln r)\, y^2\, dt^2 + \frac{y^2}{3r^2}\left(dr^2 + r^2\, d\Omega^2\right) + dy^2 \quad , \quad \text{(E.6)}$$

when $g_s = 0$, where $g = G + lnr$ and $s = lnr$. In the case when $g_s \neq 0$ the exact solution can be written as

$$ds^2 = -\left(1 - R^2 + \frac{a}{R}\right) y^2\, dt^2 + y^2\left[\left(1 - R^2 + \frac{a}{R}\right)^{-1} dR^2 + R^2\, d\Omega^2\right] + dy^2 \quad , \quad \text{(E.7)}$$

where again a is an arbitrary constant. The above metric is Riemann-flat if and only if $a = 0$. If $B_t \neq 0$, then the field equations reduce to a system of ODEs for two unknown functions.

$F_r K_r \neq 0$

The following are two related special exact solutions found in this case after a detailed anlysis of the special case:

$$ds^2 = -(-y^2 + d)^{-1}\, r^{-\frac{2}{\sqrt{3}}}\, dt^2 + (-y^2 + d)\, r^{2(\frac{2}{\sqrt{3}}-1)}\, (dr^2 + r^2\, d\Omega^2) + r^{\frac{4}{\sqrt{3}}}\, dy^2 \quad , \quad \text{(E.8)}$$

$$ds^2 = -r^{\frac{4}{\sqrt{3}}}\, dt^2 + (t^2 + d)\, r^{2(\frac{2}{\sqrt{3}}-1)}\, (dr^2 + r^2\, d\Omega^2) + (t^2 + d)^{-1}\, r^{-\frac{2}{\sqrt{3}}}\, dy^2 \quad , \tag{E.9}$$

where a is a constant.

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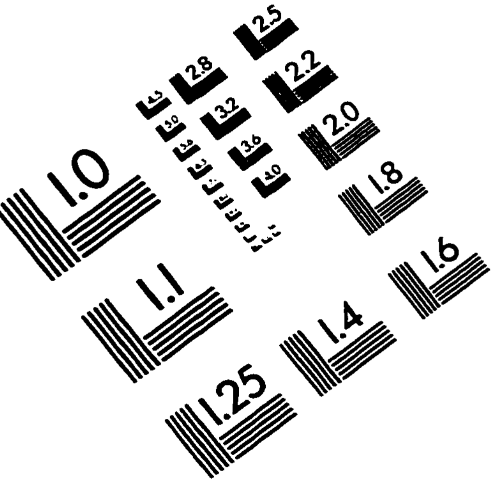
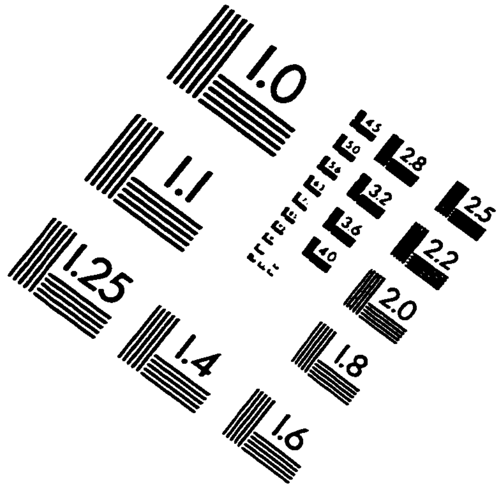
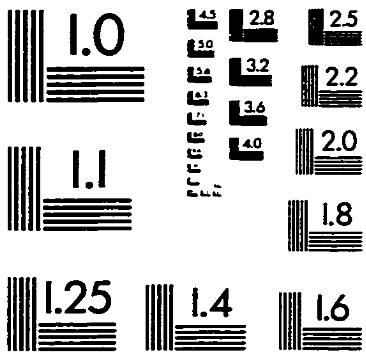
IMAGE EVALUATION TEST TARGET (QA-3)

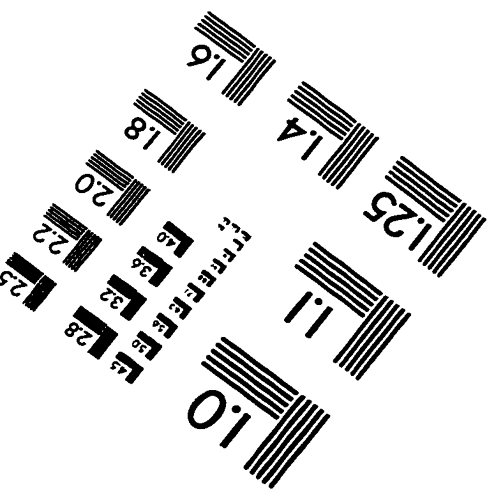






150mm

6"



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