DEVELOPMENT OF COMPLEX ENVELOPE FDTD METHODS FOR ELECTROMAGNETIC MODELING AND SIMULATION

by

Changning Ma

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This work is dedicated to my dearest wife Xuejuan Zhu and my newborn son Justin.
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List of Symbols

$e$  normalized electric field
$h$  normalized magnetic field
$\hat{e}$  envelope of $e$
$\hat{h}$  envelope of $h$
$\omega$  carrier frequency
$i, j, k$  space index
$c$  velocity of light
$\varepsilon$  permittivity
$\varepsilon_0$  permittivity of vacuum
$\varepsilon_r$  relative permittivity
$\mu$  permeability
$\Delta l$  spatial resolution of the FDTD mesh along $l$-axis, $l=x, y$ or $z$
$\Delta t$  discrete time increment in FDTD method
$J$  imaginary unit
List of Abbreviations

ABC Absorbing Boundary Condition
ADI Alternating Direction Implicit
CE Complex Envelope
CEM Computational Electromagnetics
CFL Courant-Friedrich-Levy
CFLN Courant-Friedrich-Levy Number
EM Electromagnetic
DFT Discrete Fourier Transform
FDTD Finite-Difference Time-Domain
FDM Finite-Difference Method
FEM Finite Element Method
FFT Fast Fourier Transform
FWHM Full Width Half Maximum
MoM Method of Moment
MRTD Multiresolution Time Domain
PDE Partial Differential Equation
PML Perfectly Matched Layers
TE Transverse Electric
TLM Transmission Line Method
TM Transverse Magnetic
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Abstract

The finite difference time domain (FDTD) method is a numerical method for modeling complicated electromagnetic structures. It has been used widely due to its robustness, programming simplicity and flexibility. However, this technique has the drawback of large computational expenditure, especially when dealing with narrowband signals. The main reason is that due to numerical dispersion, the marching time step has to be set small relative to the highest frequency, although useful envelope information occupies only a small bandwidth around the center or high carrier frequencies. To circumvent this problem, the concept of complex envelope (CE) was recently adapted into the FDTD method. By making use of the complex envelope representation of a narrowband signal, the high carrier frequency is absorbed into the field equations as a known quantity. Consequently, only the signal envelopes become the variants to be sampled and computed.

Investigations into the stability and numerical behaviors of the complex envelope technology are presented in this thesis. They show that the CE FDTD method offers little improvement in accuracy or efficiency over the conventional FDTD method. Therefore, in searching for a better CE-type FDTD method, attention is turned to the recently developed, unconditionally stable ADI-FDTD scheme.

By following the same procedure in formulating the CE-FDTD method, an unconditionally stable CE ADI-FDTD method is developed. The theoretical proof of the unconditional stability of CE ADI-FDTD method is given and a comprehensive analysis of the numerical dispersion is presented for a three-dimensional case. The impacts of propagation directions, ratio of carrier to envelope frequencies, temporal step size, and spatial step size on the dispersion errors are assessed. Furthermore, to validate the the exclusive advantages of the CE ADI-FDTD method in solving practical electromagnetic problems, a planar waveguide and a cavity resonator are solved. The accuracy and efficiency of the proposed CE ADI-FDTD method are then demonstrated and compared with both the conventional FDTD method and the ADI-FDTD method.

It is concluded that, in comparison with the conventional FDTD method and ADI-FDTD method, the CE ADI-FDTD method could efficiently reduce the computational burden, especially in solving bandwidth limited responses.
Chapter 1

Introduction

1.1 Computational Electromagnetics

Computational Electromagnetics (CEM) is a branch of electromagnetics that is concerned with the numerical analysis of electromagnetic phenomena, often in situations of great complexity. Several main application areas of applied electromagnetics where numerical simulation approaches are employed are listed in Figure 1.1. They include, but are not limited to, wireless communication, antenna and radar cross section (RCS) analysis, biomedical engineering, electromagnetic compatibility (EMC)/electromagnetic interference (EMI) analysis, microwave engineering, remote sensing, and integrated circuit design.

![Diagram of major complex EM problems]

Figure 1.1: Major complex EM problems

Numerical solutions of realistic EM problems started in the mid-1960s with the availability of digital computers. Since then, considerable effort has been expended in solving practical, complex EM-related problems for which closed form analytical solutions are either
intractable or do not exist.

In comparison with analytical techniques that require a high degree of ingenuity, experience and effort, numerical techniques have the advantage of allowing EM problems solving to be carried out by a person who may not have in-depth knowledge of the mathematics or physics that relate to electromagnetics. With numerical techniques, EM solutions are calculated based on numerical approximations of EM governing equations. Because of their flexibility and effectiveness to solve complex real-world EM problems, numerical techniques have evolved dramatically during the past 30 years and are still advancing at a rapid pace.

A numerical-model-based solution flow chart is given in Figure 1.2.

![Numerical-model-based solution flow chart](image)

Figure 1.2: Numerical-model-based solution flow chart

As shown, a generic numerical model is applied in the beginning and then augmented by boundary conditions and other peripheral units such as near to far-field transformations. A numerical model is deduced from the intrinsic behavior of EM fields in several steps, which include conceptualization, formulation, numerical implementation, computation and
validation.

Many numerical models have been developed. With the development of increasingly powerful computers, these models have become more sophisticated and reflect more aspects of reality. However, there is not a single model that can handle all kinds of applications accurately and efficiently. On the contrary, each model is only well-suited for the analysis of a certain class of problems. Therefore, to have an accurate, reliable, efficient and powerful model, one needs to know and compare the advantages and drawbacks of different numerical methods.

In general, these numerical methods fall into two categories based on the domains of the EM problem being solved: frequency domain, where the phase and frequency response of a system are considered; and time domain, where the time evolution of a signal or system is considered.

Frequency domain methods can be computationally more efficient when only a narrow range of frequency is of interest. However, for a large structure with complex broadband behavior, obtaining solutions at many discrete frequency points with sufficient resolution can be very time consuming. In addition, these methods cannot deal easily with nonlinear elements.

On the other hand, time domain methods solve problems in their natural dimensions, space and time. They can provide solutions of wideband frequency spectra and can handle nonlinear elements without difficulties. Moreover, transient behaviors can be visualized easily in the time domain, which yields a greater physical insight into the problem under investigation. Nevertheless, they require a large number of iterations if high frequencies are of interest or a large number of grids if a complex or large object is under study. Time domain methods are normally more time and memory consuming.

In the following sections, some well-known and widely used numerical models that are listed in Figure 1.2 will be reviewed.
1.2 Frequency-Domain Simulation Techniques

1.2.1 Finite Difference Method

The finite difference method (FDM) was first developed by A. Thom in the 1920s [1]. Since then, it has been applied to a wide range of EM problems even with irregular boundary shapes and different boundary conditions [2] [3].

The finite difference technique is simple and straightforward [4] [5]. It replaces derivatives in differential equations with corresponding finite differences. These finite difference approximations are algebraic in form; they relate the value of the dependent variable at a point in the solution region to the values at some neighboring points. The resulting algebraic equations can be solved with simple arithmetic operations.

A typical FDM involves three steps:

i) Dividing a solution region into a grid of nodes. For example, a solution region in the $x - y$ plane is divided into meshes of $\Delta x$ and $\Delta y$ as shown in Figure 1.3.

![Flow chart of FDTD simulation](image)

**Figure 1.3:** Flow chart of FDTD simulation
Thus, for a function $\Phi(x, y)$, its value at node $P$ can be expressed as

$$\Phi(x, y)_{P} = \Phi(i\Delta x, j\Delta y) = \Phi_{ij}$$  \hspace{1cm} (1.1)$$

where $i$ and $j$ are the space indices in the $x$ and $y$ directions, respectively.

ii) Approximating a given differential equation by its finite difference counterpart. The result is a system of linear equations with the function value at a node being related to the values at its adjacent nodes. For example, suppose that the governing equation for $\Phi(x, y)$ is

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = f$$  \hspace{1cm} (1.2)$$

Here $f = f(x, y)$ is the source of excitation function. Then by replacing the derivatives with their central-difference correspondences, one can obtain the approximate finite difference equation:

$$\frac{\Phi_{i+1,j} + \Phi_{i-1,j} - 2\Phi_{i,j}}{\Delta x^2} + \frac{\Phi_{i,j+1} + \Phi_{i,j-1} - 2\Phi_{i,j}}{\Delta y^2} = f_{i,j}$$ \hspace{1cm} (1.3)$$

It can be simplified when $\Delta x = \Delta y$, as

$$\Phi_{i,j} = \frac{\Phi_{i+1,j} + \Phi_{i-1,j} + \Phi_{i,j+1} + \Phi_{i,j-1} - f_{i,j}\Delta x^2}{4}$$ \hspace{1cm} (1.4)$$

As can be seen, the value of $\Phi$ at node $P$ is expressed by its values at adjacent nodes.

By varying $i$ and $j$, a system of linear equations is obtained.

iii) Solving the system of linear equations with the prescribed boundary conditions and/or initial conditions and finding the values of $\Phi(x, y)$ at all the nodes.

The major advantage of the FDM is its simplicity in conception and execution. However, if the solution region contains different materials and complex shapes, the programs become more complicated. Also, the accuracy declines if the field changes rapidly.

### 1.2.2 Method of Moments

One of the most powerful techniques, which has been in use for more than three decades in the frequency domain, is the Method of Moments (MoM). The method is documented
by Harrington in [6]. The method has been applied successfully to a wide variety of EM problems of practical interest, such as radiation due to thin-wire elements and arrays, scattering problems, microstrips and lossy structures and propagation over an inhomogeneous earth [7–11]. The MoM technique is based on solving complex integral equations (IE) by reducing them to a system of linear equations. The procedure for applying the MoM can be summarized in three steps:

The first step is to develop the appropriate integral equations (IE): the electrical field integral equation (EFIE) or the magnetic field integral equation (MFIE). Both of these equations can be derived from Maxwell’s equations by considering fields scattered by perfect conductors (or lossless dielectric). These equations are in generally in the form of

\[
\text{EFIE} : \quad \vec{E}_{\text{inc}} = fe(\vec{J})
\]

\[
\text{MFIE} : \quad \vec{H}_{\text{inc}} = fm(\vec{J})
\]

where the terms on the left-hand side are the known incident field quantities and \( \vec{J} \) is the induced current to be found.

The second step is to expand the unknown \( \vec{J} \) in terms of a set of pre-selected basis functions. Suppose that \( b_i \) is the pre-selected \( i^{th} \) basis function; then the expansion entails a sum of \( M \) terms of basis functions:

\[
\vec{J} = \sum_{i=1}^{M} \vec{J}_i b_i
\]

where \( \vec{J}_i \) are the expansion coefficients to be determined.

Substitution of the above equations into (1.5)(1.6) leads to the residual \( \hat{R} \) as

\[
\hat{R} = \vec{E}_{\text{inc}} - fe(\sum_{i=1}^{M} \vec{J}_i b_i)
\]

\[
\hat{R} = \vec{H}_{\text{inc}} - fm(\sum_{i=1}^{M} \vec{J}_i b_i)
\]

The third step is to force the residual \( \hat{R} \) to be zero in terms of the inner product with a pre-selected set of \( M \) linearly independent weighting or testing functions \( W_j \). That is

\[
< \hat{R}, W_j > = 0, \quad j = 1, 2, \ldots M
\]
In the case of the EFIE,

\[ < W_j, \hat{E}_{inc} > = \sum_{i=1}^{M} < W_j, f\epsilon(\tilde{J}_i, b_i) >, \quad j = 1, 2, ...M \]  \hspace{1cm} (1.11)

This can be written in matrix form as

\[ [\hat{E}] = [Z][\tilde{J}] \]  \hspace{1cm} (1.12)

where

\[ Z_{ij} = < W_j, f\epsilon(b_i) > \]

\[ \hat{E}_j = < W_j, \hat{E}_{inc} > \]

The column vector \( \hat{E} \) contains the known incident field quantities and the elements of \([Z]\) are functions of the geometry. By solving the matrix equation, the expansion coefficients are found and so is \( \tilde{J} \) through equation (1.7).

In general, method of moments do an excellent job of analyzing unbounded radiation problems and it excels at analyzing perfect electric conductor configurations and homogeneous dielectrics. It is not well-suited to the analysis of complex inhomogeneous geometries.

1.2.3 Finite Element Method

The finite element method (FEM) has its origin in the field of structural analysis [12], and was first applied to EM problems in 1969 [13]. Since then, the method along with its hybrid versions (such as finite element-boundary integral, finite element-absorbing boundary condition, finite element-mode matching [14–17]) has become one of the most successful frequency domain computational methods. Its applications cover semiconductor devices, antennas, microwave circuits, propagation and large scale scattering from non-metallic structures [18–20].

Most finite element methods are variational techniques. Instead of dealing with partial differential equations with certain boundary conditions, they solve for the unknown field quantities by minimizing an energy function that represents the total energy in a particular
system. For a three-dimensional, time-harmonic problem, this function may be represented as,

\[ F = \int_V \left( \frac{\mu H^2}{2} + \frac{eE^2}{2} - \frac{J \cdot E}{2j\omega} \right) \cdot dv \]  

(1.13)

In the finite element method, a solution space is first divided into a number of small homogeneous pieces or elements as Figure 1.4 shows.

![Discretization](image)

Figure 1.4: Finite element discretization

These elements are usually polygons such as triangles and rectangles for two-dimensional problems and tetrahedra for three-dimensional problems.

The next step, which makes the FEM different from others, is to derive the governing equations for a typical element. This means that analytical representation should be obtained for the elements before proceeding for the general solution. This is quite similar to deriving Green’s function in the MoM technique. For instance, in Equation (1.13), \( H \) can first be expressed in terms of \( E \). Then, by setting the derivatives of the function \( F \) with respect to \( E \) equal to zero, an equation of the following form is obtained:

\[ f(J, E) = 0 \]  

(1.14)

By expanding any unknown field quantities as a sum of basis functions, usually linear basis functions, the function \( f \) is approximated and applied at each of the \( N \) nodes, resulting in a system of equations,

\[
\begin{bmatrix}
J_1 \\
J_2 \\
\vdots \\
J_n
\end{bmatrix}
= 
\begin{bmatrix}
y_{11} & y_{12} & \cdots & \cdots \\
y_{21} & y_{22} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \cdots & y_{nn}
\end{bmatrix}
\begin{bmatrix}
E_1 \\
E_2 \\
\vdots \\
E_n
\end{bmatrix}
\]  

(1.15)
The values of $J$ on the left-hand side of this equation are referred to as the source terms. They represent the known excitations. The elements of the Y-matrix are functions of the problem geometry and boundary constraints. Since each element only interacts with elements in its own neighborhood, the Y-matrix is generally sparse. The terms of the vector on the right-hand side represent the unknown electric field at each node. These values are obtained by solving the system of equations.

The major advantage that finite element methods have over other EM modeling techniques stems from the fact that electrical and geometric properties of each element can be defined independently. This permits the problem to be set up with a large number of small elements in regions of complex geometry and fewer, larger elements in relatively open regions. In addition, tetrahedral elements are used in FEM rather than the simple cubic cells used in the finite difference method. Such tetrahedral elements can conform well with arbitrary shapes of boundaries. Therefore, FEM can solve problems that have complicated geometries and arbitrarily shaped dielectric regions in a relatively efficient manner.

Principal weaknesses of the finite element method are its difficulty in modeling open configurations, its complicated data processing and the large number of nodes it requires.

1.3 Time-Domain Simulation Techniques

1.3.1 Transmission Line Modeling

Transmission line modeling (TLM) is a numerical simulation technique that was first suggested by P. B. Johns in 1971 to solve scattering problems [21]. It is based on the analogy between the propagation of electromagnetic fields according to Huygens principle and the propagation of voltage pulses in transmission line networks [22]. Therefore by computing voltage and current in the networks, electric and magnetic fields can be solved.

The TLM method first fills a physical solution region with a network of transmission lines. The point at which the transmission lines intersect is referred to as a node, and the most commonly used node in three dimensions is the symmetrical condensed node [23] as shown
in Figure 1.5.

Figure 1.5: Symmetrical condensed node of TLM

Each node is connected to its neighboring nodes by a pair of orthogonally polarized transmission lines. Stubs maybe introduced to a node to account for dielectric and magnetic materials. Electromagnetic quantities are related to voltages at the node. For example,

$$kE_x = \frac{-V_x}{\Delta l}$$

(1.16)

and $V_x$ can be found as the average of the voltages on ports 1, 2, 9 and 12; i.e.,

$$V_x = \frac{1}{4} [(kV_1^i + kV_1^r) + (kV_2^i + kV_2^r) + (kV_9^i + kV_9^r) + (kV_{12}^i + kV_{12}^r)]$$

(1.17)

The whole TLM simulation process involves pulse scattering and transmission. At each time step, twelve pulses $V_{j,j=1..12}$ are incident onto a node from the twelve branches and then scattered along the twelve branches. The scattering follows the equation below:

$$[kV^r] = [S][kV^i]$$

(1.18)

where $[S]$ is a scattering matrix. The reflected pulses $V^r$ then travel through the transmission lines that link the neighboring nodes and become the new incident pulses at an adjacent node at next time step:

$$[k_{i+1}V^i] = [C][kV^r]$$

(1.19)
where \([C]\) is a connection matrix.

The processes of scatter and transmission are repeated for as many time steps as required. Voltage pulses are introduced into the system in a manner consistent with the desired excitation and outputs can be taken where needed.

The TLM is explicit, computationally simple and stable since it is a passive network that is solved exactly. In addition, it can be used for nonlinear or inhomogeneous problems since any variation of material properties may be updated at each time step. However, the method is limited by the amount of memory storage required, which depends on the complexity of the TLM mesh. In addition, it is subject to problems of mesh dispersion and anisotropy [24–26].

1.3.2 Finite Difference Time Domain Method

The finite-difference time domain (FDTD) method is widely regarded as one of the most popular full-wave computational electromagnetics algorithms. It was first proposed by Yee [27] in 1966 and has been advancing over past thirty years [28]. With the development of other related numerical techniques, such as perfectly matched layers absorbing boundary condition [29–32] and frequency response extraction [33], the FDTD method and its derivatives have been extended to model a wider and wider range of electromagnetics problems from Extremely Low Frequency (ELF) through optical frequency. Unlike the TLM, the FDTD method is an approximate mathematical model directly based on the discretization of Maxwell’s equations in the time and spatial domains and division of the solution domain into unit (Yee) cells. Both time and spatial derivatives are replaced with central finite difference approximations and the solution is obtained in a time-marching form. In the FDTD method, an object is characterized by permittivity, permeability and conductivity. Electric and magnetic field components are calculated at different locations of each cell. There is a half-time step difference between electric and magnetic field components, which leads to so-called leap-frog computation.

Compared with the TLM, FDTD method has unique advantages. For instance, the FDTD
model treats impulsive and nonlinear behaviors naturally with simple and systematic algorithms. Furthermore, the FDTD method needs less than one-half of the CPU time spent by the equivalent TLM under identical conditions. While the TLM scheme requires 22 real memory stores per node, the FDTD method requires only seven real memory stores in an isotropic dielectric medium [34].

For modelling a large object, however, the FDTD method still requires large amounts of computer memory and processing time in comparison to frequency domain methods. This is due to its two inherent physical constraints: one is the numerical dispersion and the other is the numerical stability. To make the numerical dispersion small, the spatial step of FDTD must be small, normally smaller than one-tenth of a wavelength. To make the time-recursion stable, the time step must also be small, smaller than what is known as the Courant-Friedich-Levy (CFL) limit.

Various schemes have been developed to relax or remove the above two constraints. Among them are the multiresolution time domain method (MRTD) [35–37] and the pseudospectral time domain method (PSTD) [38, 39]. These are higher-order numerical techniques that have significantly lower numerical dispersion. The alternating direction implicit (ADI) FDTD method [40, 41] is another technique that completely removes the CFL condition.

1.3.3 Time Domain Finite Element Method

Commonly, the time domain finite element method (TDFEM) [42–44] is based on solving the wave equations for the electric or magnetic field. For example, the general vector wave equation in a lossless bounded region \( \Omega \) is

\[
\mu^{-1} \nabla \times \nabla \times \vec{E} + \frac{\varepsilon}{\varepsilon_r} \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{\partial \vec{J}}{\partial t} \tag{1.20}
\]

Weighting the wave equation with a set of test functions \( \hat{v}_{j=1,...,N} \) and making use of Green’s first identity leads to a Galerkin form of initial value problem that can be stated as:
Find \( \tilde{E} \) such that

\[
\int_{\Omega} (\mu^{-1} \nabla \times \tilde{E} \cdot \nabla \times \tilde{w} + e \frac{\partial \tilde{E}}{\partial t} \cdot \tilde{w}) d\Omega + \oint_{\Gamma_n} \left( \frac{\mu^{-1}}{c} \left( \frac{\partial \tilde{E}}{\partial t} \times \hat{n} \right) \cdot \tilde{w} \right) d\Gamma = -\int_{\Omega} \left( \frac{\partial \tilde{J}}{\partial t} \cdot \tilde{w} \right) d\Omega
\]

(1.21)

where \( \Gamma_n \) is the truncation boundary of \( \Omega \) and \( c \) is the speed of light. Consequently, the application of the Faedo-Galerkin process [19] results in

\[
[T] \frac{\partial^2 \tilde{e}(t)}{\partial t^2} + [B] \frac{\partial \tilde{e}(t)}{\partial t} + [S] \tilde{e}(t) = \tilde{f}(t)
\]

(1.22)

where \( \tilde{e} \) is the coefficient vector of \( \tilde{E} \) expanded in terms of basis functions \( \tilde{w} \)

\[
T_{ij} = \int_{\Omega} e \tilde{w}_i \cdot \tilde{w}_j d\Omega
\]

\[
B_{ij} = \oint_{\Gamma_n} \frac{\mu^{-1}}{c} \tilde{w}_i \cdot (\tilde{w}_j \times \hat{n}) d\Gamma
\]

\[
S_{ij} = \int_{\Omega} \mu^{-1} \nabla \times \tilde{w}_i \cdot \nabla \times \tilde{w}_j d\Omega
\]

\[
f_i = -\int_{\Omega} \frac{\partial \tilde{J}}{\partial t} \cdot \tilde{w}_i d\Omega
\]

The update scheme can be easily obtained by discretizing the above equation in the time domain.

TDFEM offers some important advantages over the standard FDTD method. The most obvious benefit has to do with its use of unstructured grids. Since the grid is unstructured and can be made conformal to arbitrary shapes of boundaries, it offers superior versatility in modeling complex geometries. Furthermore, the Faedo-Galerkin procedure used for the development of the weak statement provides for a very natural way of handling field and flux continuity conditions at material interfaces, thus further enhancing modeling accuracy. Galerkin’s formulation provides a unifying approach for the exploitation of a variety of choices of trial functions, some of them more appropriate for specific types of geometries than others.

Despite the aforementioned advantages, there are a couple of major reasons why TDFEM has not been used widely. The most important is the unstructured 3-D mesh generation. It requires a significant effort to develop proficiency in using 3-D mesh generator packages.
Also, it is not always possible to get a high-quality mesh for a given geometry. Another difficulty is the relative complexity of TDFEM as compared to the standard FDTD method. The time needed to learn TDFEM and write usable 3-D code is an order of magnitude greater than that for the standard FDTD method. Some research [45–49] has been focused on hybrid methods that take advantage of both finite difference and finite element time domain techniques.

1.4 Motivation for the Thesis

As discussed earlier, although the FDTD method shows robustness, programming simplicity and flexibility in the analysis of a wide range of structures, it has the drawback of large memory usage and computational time for an electrically large structure. One of the problems occurs when the FDTD method is applied to simulate structures with modulated RF or optical signals that carry band-limited envelopes. The reason is that the RF or optical signals are often of high frequency or short wavelengths. Therefore, to reduce numerical dispersion in the FDTD method, the time step has to be small, even though modulating envelope signals that contain useful information are of relatively low frequency and occupy only a small bandwidth around the carriers.

To address the issue, a circuit simulation technique called the Circuit Envelope [50, 51] was employed in time-domain field simulation by Pursel [52]. The envelope technique was first introduced for circuit simulations [53]. By discretizing the signal envelopes on defined carrier frequencies, it has proven to be much more efficient than transient simulators like SPICE for narrow-band cases. Based on a similar concept, Pursel introduced the concept into vector wave equations. The Crank-Nicolson scheme was used to solve the wave equation for a one-dimensional case, and the improvement on accuracy was demonstrated through numerical simulation [52].

The concept was also introduced into the finite-element method, leading to the envelope finite-element numerical method [54–56]. In spite of the above efforts, none have been seen on full formulations and assessments of the complex-envelope (CE) FDTD methods. This thesis aims to provide such evaluations as well as to extend the complex-envelope
concept to the unconditionally stable ADI-FDTD method.

1.5 Original Contributions

The work presented in this thesis is based on author's publications [57–63], which include the following contributions:

i) Derive full-wave CE FDTD formulations and study numerical stability and dispersion of this method.

ii) Extend the Complex-Envelope technique to a three-dimensional unconditionally stable ADI-FDTD algorithm.

iii) Study the effectiveness and efficiency of the CE ADI-FDTD method through rigorous analysis and numerical experiments.

iv) Draw up criteria on what problems can be solved more effectively and efficiently with the CE ADI-FDTD method than with other numerical methods.

1.6 Organization of the Thesis

The thesis is organized in the following manner:

Chapter 2 contains background knowledge of the the conventional three dimensional FDTD scheme. The techniques used to perform dispersion and stability analysis are discussed. Two structures are simulated for the purpose of validation.

Chapter 3 describes the Complex-Envelope FDTD method. Full-wave Complex-Envelope FDTD formulations are derived and a comprehensive analysis of the numerical stability and dispersion of this method is presented, from which a conclusion was drawn.

Chapter 4 reviews the theory of the unconditionally stable ADI FDTD method. Numerical dispersion and stability are presented. The same structures given in Chapter 2 are analyzed for the sake of comparison.
Chapter 5 presents the formulations of the three-dimensional Complex-Envelope ADI-FDTD method. Numerical stability and dispersion are studied comprehensively. Then the possible structure that will benefit from Complex-Envelope ADI-FDTD method is categorized through comparing the simulation results of Complex-Envelope ADI FDTD with those of the FDTD method and ADI-FDTD method.

Chapter 6 summarizes the conclusion and discusses the further research directions relating to this topic.
Chapter 2

Classic FDTD Technique

The basic concepts of the FDTD methods for the solution of electromagnetic field problems are reviewed in this chapter. The discretization of space using Yee cells and the leap-frog algorithm are explained. Next numerical dispersion and stability are discussed. Finally, two examples are presented for the sake of showing the FDTD computation procedure as well as for comparisons.

2.1 Maxwell’s Equations and the Yee Algorithm

Maxwell’s partial differential equations of electromagnetism describe the behaviours of electric and magnetic fields in time and space. In free space, the differential form of Maxwell’s equations can be expressed as

\[
\frac{\partial \hat{\mathbf{E}}}{\partial t} = c \nabla \times \hat{\mathbf{H}} \tag{2.1a}
\]

\[
\frac{\partial \hat{\mathbf{H}}}{\partial t} = -c \nabla \times \hat{\mathbf{E}} \tag{2.1b}
\]

where \(c\) is speed of light and \(\hat{\mathbf{E}}\) and \(\hat{\mathbf{H}}\) are respectively the electric and magnetic fields normalized by the free space impedance \(Z_0 = \sqrt{\mu_0/\varepsilon_0}\).

\[
\hat{\mathbf{E}} = \frac{\mathbf{E}}{Z_0} \quad \hat{\mathbf{H}} = \frac{\mathbf{H}}{Z_0}
\]

with \(\mathbf{E}\) and \(\mathbf{H}\) being the electric and magnetic fields. The vector curl equations given in Equation (2.1) can be rewritten as six scalar equations in the three-dimensional rectangular coordinate system:
\[
\begin{align*}
\frac{\partial e_x}{\partial t} &= c \left( \frac{\partial h_y}{\partial y} - \frac{\partial h_z}{\partial z} \right) \\
\frac{\partial e_y}{\partial t} &= c \left( \frac{\partial h_z}{\partial z} - \frac{\partial h_x}{\partial x} \right) \\
\frac{\partial e_z}{\partial t} &= c \left( \frac{\partial h_x}{\partial x} - \frac{\partial h_y}{\partial y} \right) \\
\frac{\partial h_x}{\partial t} &= c \left( \frac{\partial e_y}{\partial y} - \frac{\partial e_z}{\partial z} \right) \\
\frac{\partial h_y}{\partial t} &= c \left( \frac{\partial e_z}{\partial z} - \frac{\partial e_x}{\partial x} \right) \\
\frac{\partial h_z}{\partial t} &= c \left( \frac{\partial e_x}{\partial x} - \frac{\partial e_y}{\partial y} \right)
\end{align*}
\]

The solution domain is then discretized with the numerical grid cell proposed by Yee [27] as shown in Figure 2.1.

Figure 2.1: Yee’s grid
A space point is denoted as

\[(i, j, k) = (i\Delta x, j\Delta y, k\Delta z)\]  \hspace{1cm} (2.3)

and any function \(\Phi\) of space and time as

\[\Phi^n_{i,j,k} = \Phi(x = i\Delta x, y = j\Delta y, z = k\Delta z, t = n\Delta t)\]  \hspace{1cm} (2.4)

where \(\Delta l_{x,y,z}\) and \(\Delta t\) are the spatial and temporal increment steps, respectively. The second-order accurate central difference approximation is then used to replace both space and time derivatives. It gives the FDTD equations for the approximations of Maxwell’s equations:

\[\begin{align*}
e^x_{i\frac{1}{2},j,k}^{n+1} &= e^n_{i\frac{1}{2},j,k} + c \Delta t \\
\left[\frac{h_x^n_{i\frac{1}{2},j+\frac{1}{2},k} - h_x^n_{i\frac{1}{2},j-\frac{1}{2},k}}{\Delta y} - \frac{h_y^n_{i\frac{1}{2},j,k+\frac{1}{2}} - h_y^n_{i\frac{1}{2},j,k-\frac{1}{2}}}{\Delta z}\right] \\
\end{align*}\]  \hspace{1cm} (2.5a)

\[\begin{align*}
e^y_{i,j+\frac{1}{2},k}^{n+1} &= e^n_{i,j+\frac{1}{2},k} + c \Delta t \\
\left[\frac{h_y^n_{i\frac{1}{2},j+\frac{1}{2},k} - h_y^n_{i\frac{1}{2},j-\frac{1}{2},k}}{\Delta z} - \frac{h_z^n_{i,j\frac{1}{2},k+\frac{1}{2}} - h_z^n_{i,j\frac{1}{2},k-\frac{1}{2}}}{\Delta x}\right] \\
\end{align*}\]  \hspace{1cm} (2.5b)

\[\begin{align*}
e^z_{i,j,k+\frac{1}{2}}^{n+1} &= e^n_{i,j,k+\frac{1}{2}} + c \Delta t \\
\left[\frac{h_z^n_{i\frac{1}{2},j,k+\frac{1}{2}} - h_z^n_{i\frac{1}{2},j,k-\frac{1}{2}}}{\Delta x} - \frac{h_x^n_{i,j\frac{1}{2},k+\frac{1}{2}} - h_x^n_{i,j\frac{1}{2},k-\frac{1}{2}}}{\Delta y}\right] \\
\end{align*}\]  \hspace{1cm} (2.5c)

\[\begin{align*}
h_x^{n+\frac{1}{2},j,k+\frac{1}{2}} &= h_x^n_{i\frac{1}{2},j,k+\frac{1}{2}} + c \Delta t \\
\left[\frac{e_x^n_{i\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - e_x^n_{i\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}}}{\Delta y} - \frac{e_z^n_{i\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - e_z^n_{i\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}}}{\Delta z}\right] \\
\end{align*}\]  \hspace{1cm} (2.5d)

\[\begin{align*}
h_y^{n+\frac{1}{2},j,k+\frac{1}{2}} &= h_y^n_{i\frac{1}{2},j,k+\frac{1}{2}} + c \Delta t \\
\left[\frac{e_y^n_{i\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - e_y^n_{i\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}}}{\Delta z} - \frac{e_x^n_{i\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - e_x^n_{i\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}}}{\Delta x}\right] \\
\end{align*}\]  \hspace{1cm} (2.5e)

\[\begin{align*}
h_z^{n+\frac{3}{2},j,k+\frac{1}{2}} &= h_z^n_{i\frac{1}{2},j,k+\frac{1}{2}} + c \Delta t \\
\left[\frac{e_y^n_{i\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - e_y^n_{i\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}}}{\Delta y} - \frac{e_z^n_{i\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - e_z^n_{i\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}}}{\Delta x}\right] \\
\end{align*}\]  \hspace{1cm} (2.5f)
As can be seen from the above formulations, the field quantities at a grid point and at a time step can be found from the field quantities at a previous time step and at neighbouring grid points. Such a recursive algorithm scheme forms the so-called leap-frog time recursive scheme.

In summary, the FDTD computation procedure is comprised of the following main steps:

i) The structure is discretized into a computational Yee's lattice;

ii) A pulse, either a sine-wave, a Gaussian pulse or others is applied at certain grid points to act as the input stimulus while fields at all other points are set to zero;

iii) The leap-frog time-domain recursive computation is executed as represented by Equation (2.5). \( \vec{e} \) and \( \vec{h} \) at all the time steps are then obtained. Note that there is a half time-step difference between the electric and magnetic field components.

iv) After completing the simulation and recording the field quantities at different time steps, an FFT may be applied to extract frequency-domain information.

Figure 2.2 shows the flow chart of the FDTD simulation where \( T_{\text{max}} \) is the simulation time prescribed by users.

![Flow chart of FDTD simulation](image)

Figure 2.2: Flow chart of FDTD simulation
2.2 Numerical Stability

The stability of a finite-difference scheme refers to the stable decay of error difference between the theoretical solution and numerical solution of the finite-difference equation. When such a decay does not occur, it leads to instability. Numerical instability is an undesirable possibility. It can cause the computed results to increase spuriously without limit as time-marching continues.

Taflove was the first to derive the correct stability criteria for the original Yee's orthogonal-grid [64]. The criteria is:

\[ u_{\text{max}} \Delta t \leq \left[ \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2} \right]^{-1/2} \]  \hspace{1cm} (2.6)

where \( u_{\text{max}} \) is the maximum wave phase velocity in the solution domain.

In the following paragraphs, we derive the same stability criteria by using a different way called Fourier method: First, instantaneous values of electric and magnetic fields distributed in space across a grid are Fourier-transformed into waves in the spatial spectral domain to represent a spectrum of spatial sinusoidal modes. Then, by checking the eigenvalues associated with the spectral-domain waves in the system, the stability condition can be determined. If all the eigenvalues are less than or equal to unity in magnitude, the scheme is stable.

A complete basis set of complex-exponential plane-waves solutions are

\[
\begin{align*}
  e_x^{t,i,j,k} &= \tilde{e}_x^n \exp[-J(k_x i\Delta x + k_y j\Delta y + k_z k\Delta z)] \\
  e_y^{t,i,j,k} &= \tilde{e}_y^n \exp[-J(k_x i\Delta x + k_y j\Delta y + k_z k\Delta z)] \\
  e_z^{t,i,j,k} &= \tilde{e}_z^n \exp[-J(k_x i\Delta x + k_y j\Delta y + k_z k\Delta z)] \\
  h_x^{t,i,j,k} &= \tilde{h}_x^n \exp[-J(k_x i\Delta x + k_y j\Delta y + k_z k\Delta z)] \\
  h_y^{t,i,j,k} &= \tilde{h}_y^n \exp[-J(k_x i\Delta x + k_y j\Delta y + k_z k\Delta z)] \\
  h_z^{t,i,j,k} &= \tilde{h}_z^n \exp[-J(k_x i\Delta x + k_y j\Delta y + k_z k\Delta z)]
\end{align*}
\]  \hspace{1cm} (2.7a-f)

where \( k_{l=x,y,z} \) is the wavenumber in the \( l \) direction and \( J = \sqrt{-1} \).
Substituting Equations (2.7) into the finite difference equations of the system (2.5) and canceling the common terms leads to

\begin{align}
\tilde{\varepsilon}_x^{n+1} &= \tilde{\varepsilon}_x^n - 2JW_x\tilde{h}_x^{n+\frac{1}{2}} + 2JW_x\tilde{h}_x^{n-\frac{1}{2}} \\
\tilde{\varepsilon}_y^{n+1} &= \tilde{\varepsilon}_y^n + 2JW_y\tilde{h}_y^{n+\frac{1}{2}} - 2JW_y\tilde{h}_y^{n-\frac{1}{2}} \\
\tilde{\varepsilon}_z^{n+1} &= \tilde{\varepsilon}_z^n - 2JW_z\tilde{h}_z^{n+\frac{1}{2}} + 2JW_z\tilde{h}_z^{n-\frac{1}{2}} \\
\tilde{h}_x^{n+\frac{1}{2}} &= 2JW_x\tilde{\varepsilon}_y^n - 2JW_x\tilde{\varepsilon}_z^n + [1 - 4(W_x^2 + W_y^2)]\tilde{h}_x^{n+\frac{1}{2}} \\
&\quad + 4W_xW_y\tilde{h}_y^{n+\frac{1}{2}} + 4W_xW_z\tilde{h}_z^{n+\frac{1}{2}} \\
\tilde{h}_y^{n+\frac{1}{2}} &= -2JW_x\tilde{\varepsilon}_y^n + 2JW_x\tilde{\varepsilon}_z^n + 4W_xW_y\tilde{h}_x^{n+\frac{1}{2}} \\
&\quad + [1 - 4(W_x^2 + W_y^2)]\tilde{h}_y^{n+\frac{1}{2}} + 4W_xW_y\tilde{h}_y^{n+\frac{1}{2}} \\
\tilde{h}_z^{n+\frac{1}{2}} &= 2JW_x\tilde{\varepsilon}_x^n - 2JW_x\tilde{\varepsilon}_z^n + 4W_xW_z\tilde{h}_x^{n+\frac{1}{2}} \\
&\quad + 4W_yW_z\tilde{h}_y^{n+\frac{1}{2}} + [1 - 4(W_x^2 + W_y^2)]\tilde{h}_z^{n+\frac{1}{2}}
\end{align}

where

\[ W_l^2 = c^2(\Delta r^2/\Delta l^2) \sin^2(k_l\Delta l/2), \quad l = x, y, z \]

is related to the spatial frequency \( k_l \) along the \( x \), \( y \), or \( z \) direction, respectively. These equations form the FDTD formulations in the spectral domain and can be combined and rewritten in the following matrix form:

\[ \tilde{F}^{n+1} = [\Lambda] \tilde{F}^n \]

with

\[ \tilde{F}^n = [\tilde{\varepsilon}_x^n, \tilde{\varepsilon}_y^n, \tilde{\varepsilon}_z^n, \tilde{h}_x^{n+\frac{1}{2}}, \tilde{h}_y^{n+\frac{1}{2}}, \tilde{h}_z^{n+\frac{1}{2}}]^T \]

and the amplification matrix

\[
[\Lambda] =
\begin{bmatrix}
1 & 0 & 0 & 0 & -J2W_x & J2W_y \\
0 & 1 & 0 & J2W_x & 0 & -J2W_y \\
0 & 0 & 1 & -J2W_y & J2W_x & 0 \\
0 & J2W_x & -J2W_y & 1 - 4(W_x^2 + W_y^2) & 4W_xW_y & 4W_xW_z \\
-J2W_x & 0 & J2W_y & 4W_xW_y & 1 - 4(W_x^2 + W_y^2) & 4W_xW_z \\
J2W_y & -J2W_x & 0 & 4W_xW_y & 4W_xW_z & 1 - 4(W_x^2 + W_y^2)
\end{bmatrix}
\]
The corresponding eigenvalues of the matrix $[\Lambda]$ are

$$\lambda_{1,2} = 1$$

$$\lambda_{3,4} = 1 - 2W^2 - 2\sqrt{W^4 - W^2}$$

$$\lambda_{5,6} = 1 - 2W^2 + 2\sqrt{W^4 - W^2}$$

with

$$W^2 = W_x^2 + W_y^2 + W_z^2.$$

To ensure that the FDTD solution is stable, the eigenvalues need to have their magnitude equal to or smaller than unity. If $W > 1$. Then one of the eigenvalues becomes greater than one in magnitude:

$$|\lambda_3| = |2W^2 + 2\sqrt{W^4 - W^2} - 1| > 1$$

Thus the scheme is unstable with $W > 1$. If $0 < W \leq 1$, then

$$|\lambda_{1,2}| = 1$$

$$|\lambda_{3,6}| = \sqrt{(1 - 2W^2)^2 + 4W^2(1 - W^2)} = 1$$

So the stability condition is $0 < W \leq 1$, which is easy to be simplified as Equation (2.6).

2.3 Numerical Dispersion

Numerical modeling techniques have limits in respect to accuracy. One of the main factors that affects the accuracy of the finite-difference time-domain (FDTD) method is numerical dispersion. The numerical dispersion occurs in a FDTD grid because the space is discretized into a computational lattice, which makes the phase velocity of numerical wave traveling in a discrete medium differ from its theoretical speed of light traveling in the same medium [65]. The numerical dispersion relation for a plane wave can be found by assuming the electromagnetic fields to be monochromatic traveling waves; that is

$$\tilde{\mathbb{F}}^{n+1} = \exp[i\omega\Delta t] \tilde{\mathbb{F}}^n$$

(2.12)

By substituting this into (2.9), one obtains:

$$\det(\exp[i\omega\Delta t][I] - [\Lambda]) = 0$$

(2.13)
After simplification, it becomes

\[(\exp[J \omega \Delta t] - \lambda_1)^2(\exp[2J \omega \Delta t] - (\lambda_1 + \lambda_2) + 1)^2 = 0\]

This leads to:

\[\omega = 0\]

and

\[\exp[J \omega \Delta t] = (1 - 2W^2) \pm 2iW \sqrt{1 - W^2}\]

The first solution is a stationary solution that corresponds to the electrostatic and magnetostatic solutions. The second solution corresponds to the propagating modes and can be rewritten as

\[
\sin^2\left(\frac{\omega \Delta t}{2}\right) = W^2
\]

\[
= c^2 \left(\frac{\Delta t^2}{\Delta x^2} \sin^2\left(\frac{k_x \Delta x}{2}\right) + \frac{\Delta t^2}{\Delta y^2} \sin^2\left(\frac{k_y \Delta y}{2}\right) + \frac{\Delta t^2}{\Delta z^2} \sin^2\left(\frac{k_z \Delta z}{2}\right)\right)
\]

(2.14)

The above equation can be compared with the analytical dispersion relation Equation shown below for the assessment of the dispersion errors.

\[\omega^2 = c^2(k_x^2 + k_y^2 + k_z^2)
\]

(2.15)

It can be seen that the two dispersions are close only if the spatial and time steps are small enough. Further studies show that in order to make the numerical dispersion error negligible (< 1%), the spatial increment steps \(\Delta x, \Delta y\) and \(\Delta z\) should be less than \(\lambda_{\text{min}}/10\).

In general, the time step needs to satisfy the Nyquist criterion. As a time domain method, the simulation results of the FDTD method usually need to be transformed to the frequency domain to obtain frequency characteristics of a given EM structure. The maximum frequency of interest determines the minimum time step, i.e., \(\Delta t_{\text{FFT}} = 1/(2 \times f_{\text{max}})\). This is the hard limit for frequency analysis. However, when the CFL stability condition and requirements for the minimization of numerical dispersion effects are satisfied, the time step is very small and satisfies the Nyquist Criterion automatically. This can be seen clearly from the following examples.
2.4 Numerical Results

The FDTD method is applied to the two structures: a parallel-plate waveguide and a cavity. The results will then serve as the benchmarks for the new complex-envelope FDTD methods to be described in the next two chapters. For both structures, analytical solutions are available for easy assessment.

2.4.1 Parallel-plate Waveguide

The air-filled parallel-plate waveguide considered is shown in Figure 2.3. The dispersion curves of the $TM_0$ and $TM_1$ modes are to be computed analytically and numerically.

![Parallel-plate waveguide diagram](image)

Figure 2.3: Parallel-plate waveguide

Analytical result

It is not difficult to derive the analytic expressions from Maxwell’s equations for the field components of the TM mode:

\[
\begin{align*}
H_z & = A \cos(k_y y) e^{-jk_x x} \\
E_y & = \frac{k_z}{\omega \varepsilon} A \cos(k_y y) e^{-jk_x x} \\
E_x & = -\frac{k_z}{j\omega \varepsilon} A \sin(k_y y) e^{-jk_x x}
\end{align*}
\]

where

\[
k_y = \frac{m \pi}{d}, \quad m = 0, 1, 2, ...
\]

Note that for the $TM_0$ mode, $E_x = 0$. Both electric and magnetic fields are transverse to the propagation direction. Thus, the $TM_0$ mode is also called TEM mode. It does not have a
cutoff frequency below which the mode cannot exist. For other modes, the cutoff frequency is
\[ f_{\text{cutoff}} = \frac{mc}{2d} \]
The lowest cutoff frequency is \( c/(2d) = 3.7474 \text{ GHz} \) with the \( TM_1 \) mode and the second lowest is \( c/d = 7.4948 \text{ GHz} \) with the \( TM_2 \) mode. Therefore, if the frequency of the stimulus signal is located between these two frequencies, only \( TM_1 \) will be excited. The dispersion relation of \( TM_1 \) mode can be expressed as
\[ \beta_0 = \pi \sqrt{\left( \frac{2f_c}{c} \right)^2 - \left( \frac{1}{d} \right)^2} \quad (2.16) \]

Simulation result

The FDTD method is applied to obtain the dispersion of the TM modes. In setting up the simulations, several factors need to be considered. For the FDTD source, a waveform with a finite duration in both the time and frequency domains is an ideal source. However, such a property is not physically realizable. In reality, the shorter the pulse, the higher the bandwidth, and vice versa. The best source waveform is the one with the minimum time-bandwidth product. One of the typical sources with a minimum time-bandwidth product is the Gauss function \( g(t) \) given as
\[ g(t) = A_0 \exp\left( \frac{(t - T_0)^2}{2T^2_w} \right) \]
where \( A_0, T_w \) and \( T_0 \) are the amplitude, duration parameter and time shift, respectively. However, the Gauss function has a DC component that is not desirable for a bandpass signal. In such a case, a modulated Gauss function can be used as the source:
\[ s(t) = g(t) \times \sin(2\pi f \times (t - T_0)) \quad (2.17) \]
where \( f \) is the carrier frequency.

In order to study the properties of the \( TM_1 \) mode, the main spectrum of the modulated Gaussian pulse should lie between 3.75 GHz and 7.5 GHz. Thus, the carrier frequency \( f = 4.5 \text{ GHz} \) is chosen and the \( T_{FWHM} \) (full width half maximum) of a given Gaussian
pulse is 833 ps. Consequently

\[ T_w = \frac{FWHM}{2/\sqrt{\ln(2)}} \approx 0.5\text{ns} \]

\[ T_0 = 2.5T_w \]

Figure 2.4 shows \( s(t) \) in the time domain.

![Modulated Gaussian pulse in time domain](image)

Figure 2.4: Modulated Gaussian pulse in time domain

The FWHM of the Gaussian envelope in the frequency domain is

\[ f_{FWHM} = \frac{2\ln(2)}{\pi T_{FWHM}} = 0.530\ \text{GHz} \]

which is shown in Figure 2.5. Thus the ratio of carrier to bandwidth is

\[ R = \frac{4.5}{0.530} \approx 8.5 \]

and \( f_{max} \approx 5\text{ GHz}, f_{min} \approx 4\text{ GHz}. \)

In terms of discretization, parameters such as mesh size and time step should be optimally selected in considerations of balances among simulation time, memory and accuracy requirements.
Figure 2.5: Modulated Gaussian pulse in the frequency domain

As discussed before, mesh sizes are chosen in consideration of numerical dispersion errors. The minimum wavelength $\lambda_{\text{min}}$ component has to be sampled with more than ten samples. In this case, uniform cells are used and the cell sizes are $\Delta x = \Delta y = 0.5$ mm, which equals $1/150$ of the shortest wavelength $\lambda_{\text{min}} \approx c/(4e9) = 75$ mm.

The time step is constrained by two conditions, maximum frequency and CFL condition. According to fast Fourier transform (FFT) properties, the minimum time step is

$$\Delta t_{\text{FFT}} \leq \frac{1}{2f_{\text{max}}} = 100 \text{ ps}.$$  

However the CFL condition demands

$$\Delta t_{\text{CFL}} \leq \frac{\Delta x}{\sqrt{2c}} = 1.18 \text{ ps},$$

which is much less than $\Delta t_{\text{FFT}}$. Therefore one only needs to consider $\Delta t_{\text{CFL}}$.

Simulation time is decided by two main factors: the frequency resolution and the period of the transient effect. Normally, the required frequency resolution $\Delta f$ is related to the simulation time $T_{\text{max}}$ by $\Delta f = 1/T_{\text{max}}$ and the transient time is about two or three times more
than the rising time of the source waveform. In our case, there is another consideration. Since no absorbing boundary is used to terminate the computational domains, the simulation should be ended before the wave reflected from the terminating boundaries arrives at the output points. The output points are located at 5 mm and 20 mm from the middle of the parallel-plate waveguide, respectively. Therefore, we choose the total length in the x-direction to be $2000\Delta x$ and the nearest distance from the output point to the metallic boundary is $960 \times \Delta x = 480$ mm. It takes $(500 + 480)/c = 3.3$ ns for the first wave signature of the $TM_0$ mode to travel forward and back. This is equal to approximately $2800\Delta t$. Therefore, the simulation time is chosen to be $2500\Delta t$.

Figure 2.6 shows the $E_y$ fields of the $TM_0$ mode at two different locations that are $\Delta d = 15$ mm apart, identified as $E_{y1}$ and $E_{y2}$. After Fourier transform, the propagation constant can be calculated as

$$\beta = \frac{L\hat{E}_{y2} - L\hat{E}_{y1}}{\Delta d}$$

(2.18)

where $\hat{E}_{y1}$ and $\hat{E}_{y2}$ are the Fourier transforms of $E_{y1}$ and $E_{y2}$ respectively.

![Figure 2.6: Simulated $E_y$ fields at output points of $TM_0$ mode](image)
Figure 2.7 shows the analytic and simulated dispersion curves. It can be seen they match well. Figure 2.8 and Figure 2.9 show the \( E \) field and the dispersion curves for the \( TM_1 \) mode. Again they match well.

![Figure 2.7: Numerical dispersion curve of \( TM_0 \) mode](image1)

![Figure 2.8: Simulated \( E_y \) fields at the output points of \( TM_1 \) mode](image2)
As the benchmark references, Table 2.1 gives the accuracy and the CPU time for both the $TM_0$ and $TM_1$ modes when the simulation is executed with a PC with 2.2 GHz CPU processor. Note the relative error is calculated with the following formulation:

$$Err(\%) = \max \left| \frac{\beta - \beta_o}{\beta_o} \right| \times 100$$

(2.19)

where $\beta_o$ is computed with Equation (2.16) and $\beta$ with Equation (2.18).

Table 2.1: Efficiency and accuracy of the FDTD method

<table>
<thead>
<tr>
<th>Mode</th>
<th>Iteration</th>
<th>CPU Time(ms)</th>
<th>Err(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TM_0$</td>
<td>2500</td>
<td>35310</td>
<td>0.025</td>
</tr>
<tr>
<td>$TM_1$</td>
<td>2500</td>
<td>35380</td>
<td>0.736</td>
</tr>
</tbody>
</table>

2.4.2 Metallic Cavity

The second benchmark example is a rectangular air-filled metallic cavity as shown in Figure 2.10. We will show the accuracy of FDTD method by calculating the dominant resonant frequency.
Analytical result

The dominant frequency of a given rectangular cavity can be expressed as:

\[ f_0 = \frac{c}{2} \sqrt{\frac{1}{a^2} + \frac{1}{l^2}} = 19.423 \, \text{GHz} \]  \hfill (2.20)

The second low resonant frequency is

\[ f_1 = \frac{c}{2} \sqrt{\frac{1}{a^2} + \frac{1}{(2l)^2}} = 26.016 \, \text{GHz} \]  \hfill (2.21)

Simulation result

In the FDTD simulation, the modulated Gaussian pulse is again used as the excitation, except the carrier frequency is changed to 19 GHz. The change is necessary so that the greatest energy will exist in the dominant mode. Figures 2.11 and 2.12 show the source in the time and frequency domains respectively.

Since the maximum frequency is \( f_{\text{max}} = 19.5 \, \text{GHz} \), the minimum wavelength \( \lambda_{\text{min}} = c/19.5 \, \text{GHz} = 15.4 \, \text{mm} \). \( \Delta x = \Delta y = \Delta z = 0.5 \, \text{mm} \approx 1/30 \lambda_{\text{min}} \) is chosen, and the structure then has a dimension of \( 18 \times 12 \times 30 \) cells.
Figure 2.11: Gaussian pulse in time domain

Figure 2.12: Gaussian pulse in frequency domain
The time step has to satisfy the CFL stability conditions as well as Nyquist theory. Since

\[ \Delta t_{\text{FFT}} = \frac{1}{2f_{\text{max}}} = 25.6 \text{ ps} \]
\[ \Delta t_{\text{CFL}} = \Delta x / c / \sqrt{3} = 0.96 \text{ ps} \]
\[ \Delta t_{\text{FFT}} \approx 27 \Delta t_{\text{CFL}} \]

the time step is chosen to be 0.96 ps.

As the source waveform fades away for \( 3T_w + T_{\text{off}} = 2.75 \text{ ns} = 2900\Delta t \), to be on the safe side, we chose 3000, 4000 and 5000 as the numbers of iterations, respectively.

Figure 2.13 shows the \( E_y \) fields recorded at the output points and Figure 2.14 shows the dominant frequency obtained by DFT.

![Figure 2.13: \( E_y \) fields at the reference point](image)

Table 2.2 gives errors and the computation time where

\[ \text{Err}(\%) = \left| \frac{f - f_0}{f_0} \right| \times 100 \]  
(2.22)
where $f_0$ is computed with Equation (2.20) and $f$ is detected by identifying the peak point of Figure 2.14.

Table 2.2: Errors versus stimulus time

<table>
<thead>
<tr>
<th>Iterations</th>
<th>Resonant Freq(GHz)</th>
<th>Err(%)</th>
<th>CPU time(ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3000</td>
<td>19.282</td>
<td>0.73%</td>
<td>6641</td>
</tr>
<tr>
<td>4000</td>
<td>19.343</td>
<td>0.41%</td>
<td>8890</td>
</tr>
<tr>
<td>5000</td>
<td>19.375</td>
<td>0.25%</td>
<td>10265</td>
</tr>
</tbody>
</table>

As can be seen, the longer the iterations or computation time, the more accurate the results.

2.5 Conclusions

In this chapter, the fundamentals on the FDTD method were briefly reviewed. The numerical stability and numerical dispersion were described. Two example structures were simulated to demonstrate the effectiveness of the FDTD method. These will serve as the benchmark references in the following chapters.
Chapter 3

The Complex-Envelope FDTD Method—Formulations and Properties

As pointed out in Chapter 2, both the time step and spatial step of the finite-difference time-domain (FDTD) method must be set in accordance with the maximum frequency of the signal to be simulated, regardless of the bandwidth. However, in modern wireless and optical communication systems, signals often have very narrow bandwidths relative to their carrier frequencies. For example, a typical cellular phone signal has a 30 kHz modulation bandwidth on a 1 GHz to 2 GHz carrier. The relative bandwidth is only about 0.003%. When the FDTD method is used directly in this case, it must be sampled at 1 GHz to 2 GHz instead of 30 kHz. Intuitively, much of the computation resources is wasted.

Confronted with a similar problem, HPEEsol's MDS introduced a new circuit simulation technique called the Circuit Envelope technique [51] [53]. It applies time-domain techniques on top of the frequency-domain harmonic balance solution, and has proved to be much more efficient than transient simulators like SPICE for bandpass-limited signals. A similar concept was then adopted into the FDTD method to solve wave propagation equations [52]. In it, a bandpass-limited signal is represented by a Complex-Envelope (CE) and the high frequency term (1 GHz to 2 GHz) is absorbed into the field equations as a known quantity. Consequently, only the signal envelop (30 kHz) become the variant to be sampled and computed.

In this chapter, a full-wave CE-FDTD method is derived from Maxwell's equations instead of wave propagation equations. Its stability condition and numerical dispersion are presented in an analytical form. The impacts of spatial and temporal steps on the stability and numerical dispersion are investigated. Finally, a conclusion is drawn based on the investigations.
3.1 Complex-Envelope (CE) Representation of an Analog Signal

A real band-limited signal \( v(t) \) can be expressed in the form [66]

\[
v(t) = v_p(t) \cos(\varpi t) - v_q(t) \sin(\varpi t)
\]

(3.1)

where \( v_p \) and \( v_q \) are the in-phase and quadrature portions of \( v(t) \) and \( \varpi \) is the center frequency of \( v(t) \). The bandwidth \( B \) and center frequency \( \varpi \) are related to the maximum frequency in \( v(t) \), \( \omega_{\text{max}} \) and the minimum frequency in \( v(t) \), \( \omega_{\text{min}} \) via the following relationship:

\[
B = \omega_{\text{max}} - \omega_{\text{min}}
\]

and

\[
\varpi = \frac{\omega_{\text{max}} + \omega_{\text{min}}}{2}
\]

Function \( v_p \) and \( v_q \) are real and low-pass limited with bandwidth \( B/2 \). They can be expressed in the the complex envelope notation [66]:

\[
\tilde{v}(t) = v_p(t) + Jv_q(t)
\]

(3.2)

So that

\[
v(t) = \Re\{\tilde{v}(t)e^{J\varpi t}\}
\]

(3.3)

Here \( \Re(\cdot) \) denotes the operation that takes real part of the argument inside the bracket.

The Fourier transform of \( v_p(t) \) and \( v_q(t) \) are defined as

\[
V_p(\omega) = \int_{-\infty}^{\infty} v_p(t)e^{-J\omega t} dt
\]

\[
V_q(\omega) = \int_{-\infty}^{\infty} v_q(t)e^{-J\omega t} dt
\]
The Fourier transform of the bandpass-limited signal \( v(t) \) is

\[
V(\omega) = \int_{-\infty}^{\infty} v(t)e^{-j\omega t} dt
= \int_{-\infty}^{\infty} (v_p(t) \cos(\omega t) - v_q(t) \sin(\omega t))e^{-j\omega t} dt
= 0.5(V_p(\omega + \pi) - JV_q(\omega + \pi))
+ V_p(\omega - \pi) + JV_q(\omega - \pi))
\]

(3.4)

The Fourier transform of the complex envelope \( \tilde{v}(t) \) can be obtained as

\[
\tilde{V}(\omega) = \int_{-\infty}^{\infty} \tilde{v}(t)e^{-j\omega t} dt
= \int_{-\infty}^{\infty} v_p(t)e^{-j\omega t} dt + J \int_{-\infty}^{\infty} v_q(t)e^{-j\omega t} dt
= V_p(\omega) + JV_q(\omega)
\]

(3.5)

It is not difficult to see the relationship between \( V(\omega) \) and \( \tilde{V}(\omega) \):

\[
\tilde{V}(\omega) = V(\omega + \pi) + \text{sgn}(\omega + \pi)V(\omega + \pi)
\]

(3.6)

where

\[
\text{sgn}(\omega) = \begin{cases} 
1 & \text{for } \omega > 0 \\
0 & \text{for } \omega = 0 \\
-1 & \text{for } \omega < 0 
\end{cases}
\]

The above equation indicates that \( \tilde{V}(\omega) \) can be obtained by setting the negative frequency portion of \( V(\omega) \) to zero, multiplying by two and then shifting the result to the left by \( \pi \). Because \( V(t) \) is real, the signal itself can be determined from a knowledge of the positive frequency values of \( V(\omega) \) alone. The transformation of \( V(\omega) \) to \( \tilde{V}(\omega) \) can be viewed as shown in Figure 3.1 [52]. Therefore, \( V(\omega) \) can be recovered from \( \tilde{V}(\omega) \) by reversing the steps.

3.2 The Full-Wave Complex-Envelope FDTD Formulations

In a time-invariant linear medium, if the sources are bandpass-limited, all of the electromagnetic fields resulting from the sources are also bandpass-limited. Consequently, field
Figure 3.1: The frequency-domain spectrum of (a) $V(\omega)$ and (b) $\tilde{V}(\omega)$

components can be represented in a Complex-Envelope format as:

$$(h, e) = \text{Re}((\hat{h}, \hat{e}) e^{i\omega t})$$

(3.7)

$e$ and $h$ are any of the magnetic and electric field components, while $\hat{e}$ and $\hat{h}$ represent the associated complex envelopes.

Substitution of Equation (3.7) into the scalar Maxwell’s equations (2.2) leads to the full-wave complex-envelope (CE) Maxwell’s equations. For example, for $\hat{e}_x$, the corresponding Maxwell’s equation is:

$$\partial \hat{e}_x / \partial t + J \omega \hat{e}_x = c(\partial \hat{h}_y / \partial y - \partial \hat{h}_z / \partial z)$$

(3.8)

Replacement of the derivatives with their central finite-difference counterparts in conjunction with Yee’s space lattice leads to the complex-envelope FDTD equation:

$$\hat{e}_x^{n+1}_{i+\frac{1}{2},j,k} = A \hat{e}_x^n_{i+\frac{1}{2},j,k} + B c \Delta t \left[ \frac{\hat{h}_z^n_{i+\frac{1}{2},j+\frac{1}{2},k} - \hat{h}_z^n_{i+\frac{1}{2},j-\frac{1}{2},k}}{\Delta y} - \frac{\hat{h}_y^n_{i+\frac{1}{2},j,k+\frac{1}{2}} - \hat{h}_y^n_{i+\frac{1}{2},j,k-\frac{1}{2}}}{\Delta z} \right]$$

(3.9a)

where $A = (1 - JD)/(1 + JD)$, $B = 1/(1 + JD)$, with $D = \omega \Delta t / 2$.

As can be seen, Equation (3.9a) deals only with the envelope functions $\hat{e}$, $\hat{h}$, and the carrier frequency is absorbed in the coefficients $A$ and $B$. The equations for the other components
can be obtained in a similar manner. They are as follows:

\[
\hat{e}_z^{n+1}_{i,j+\frac{1}{2},k} = A\hat{e}_z^n_{i,j+\frac{1}{2},k} + Bc\Delta t
\]
\[
\begin{bmatrix}
\hat{h}_x^{n+\frac{3}{2}}_{i+j+\frac{1}{2},k+\frac{1}{2}} - \hat{h}_x^{n+\frac{1}{2}}_{i+j+\frac{1}{2},k-\frac{1}{2}} \\
\hat{h}_y^{n+\frac{1}{2}}_{i+j+\frac{1}{2},k+\frac{1}{2}} - \hat{h}_y^{n+\frac{1}{2}}_{i+j+\frac{1}{2},k-\frac{1}{2}} \\
\hat{h}_z^{n+\frac{1}{2}}_{i+j+\frac{1}{2},k+\frac{1}{2}} - \hat{h}_z^{n+\frac{1}{2}}_{i+j+\frac{1}{2},k-\frac{1}{2}} \\
\end{bmatrix}
\Delta z
\]

(3.9b)

\[
\hat{e}_z^{n+1}_{i,j,k+\frac{1}{2}} = A\hat{e}_z^n_{i,j,k+\frac{1}{2}} + Bc\Delta t
\]
\[
\begin{bmatrix}
\hat{h}_x^{n+\frac{1}{2}}_{i+j+\frac{1}{2},k+\frac{1}{2}} - \hat{h}_x^{n+\frac{1}{2}}_{i+j-\frac{1}{2},k+\frac{1}{2}} \\
\hat{h}_y^{n+\frac{1}{2}}_{i+j+\frac{1}{2},k+\frac{1}{2}} - \hat{h}_y^{n+\frac{1}{2}}_{i+j-\frac{1}{2},k+\frac{1}{2}} \\
\hat{h}_z^{n+\frac{1}{2}}_{i+j+\frac{1}{2},k+\frac{1}{2}} - \hat{h}_z^{n+\frac{1}{2}}_{i+j-\frac{1}{2},k+\frac{1}{2}} \\
\end{bmatrix}
\Delta y
\]

(3.9c)

\[
\hat{h}_x^{n+\frac{3}{2}}_{i+j+\frac{1}{2},k+1} = A\hat{h}_x^{n+\frac{1}{2}}_{i+j+\frac{1}{2},k+\frac{1}{2}} + Bc\Delta t
\]
\[
\begin{bmatrix}
\hat{e}_y^{n+1}_{i+j+\frac{1}{2},k+\frac{1}{2}} - \hat{e}_y^{n+1}_{i+j+\frac{1}{2},k} \\
\hat{e}_z^{n+1}_{i+j+\frac{1}{2},k+\frac{1}{2}} - \hat{e}_z^{n+1}_{i+j+\frac{1}{2},k} \\
\end{bmatrix}
\Delta y
\]

(3.9d)

\[
\hat{h}_y^{n+\frac{3}{2}}_{i+j+\frac{1}{2},k+1} = A\hat{h}_y^{n+\frac{1}{2}}_{i+j+\frac{1}{2},k+\frac{1}{2}} + Bc\Delta t
\]
\[
\begin{bmatrix}
\hat{e}_x^{n+1}_{i+j+\frac{1}{2},k+\frac{1}{2}} - \hat{e}_x^{n+1}_{i+j+\frac{1}{2},k} \\
\hat{e}_z^{n+1}_{i+j+\frac{1}{2},k+\frac{1}{2}} - \hat{e}_z^{n+1}_{i+j+\frac{1}{2},k} \\
\end{bmatrix}
\Delta x
\]

(3.9e)

\[
\hat{h}_z^{n+\frac{3}{2}}_{i+j+\frac{1}{2},k+1} = A\hat{h}_z^{n+\frac{1}{2}}_{i+j+\frac{1}{2},k+\frac{1}{2}} + Bc\Delta t
\]
\[
\begin{bmatrix}
\hat{e}_x^{n+1}_{i+j+\frac{1}{2},k+\frac{1}{2}} - \hat{e}_x^{n+1}_{i+j+\frac{1}{2},k} \\
\hat{e}_y^{n+1}_{i+j+\frac{1}{2},k+\frac{1}{2}} - \hat{e}_y^{n+1}_{i+j+\frac{1}{2},k} \\
\end{bmatrix}
\Delta x
\]

(3.9f)

Note that the quantities in the above equations are complex.

3.3 Numerical Stability

The stability condition can be obtained by using the Fourier Method described in chapter 2. Considering electric and magnetic fields in the spectral domain, they can be expressed
as:

\[
\begin{align*}
\hat{e}_x |_{l,j,k}^{n} & = e_x^n e^{-i[k_i (j + \frac{1}{2}) \Delta x + k_k (k + \frac{1}{2}) \Delta z]} \\
\hat{e}_y |_{l,j,k}^{n} & = e_x^n e^{-i[k_i (j + \frac{1}{2}) \Delta x + k_k (k + \frac{1}{2}) \Delta z]} \\
\hat{e}_z |_{l,j,k}^{n} & = e_x^n e^{-i[k_i (j + \frac{1}{2}) \Delta x + k_k (k + \frac{1}{2}) \Delta z]} \\
\hat{h}_x |_{l,j,k}^{n+\frac{1}{2}} & = \hat{h}_x^{n+\frac{1}{2}} e^{-i[k_i (j + \frac{1}{2}) \Delta x + k_k (k + \frac{1}{2}) \Delta z]} \\
\hat{h}_y |_{l,j,k}^{n+\frac{1}{2}} & = \hat{h}_y^{n+\frac{1}{2}} e^{-i[k_i (j + \frac{1}{2}) \Delta x + k_k (k + \frac{1}{2}) \Delta z]} \\
\hat{h}_z |_{l,j,k}^{n+\frac{1}{2}} & = \hat{h}_z^{n+\frac{1}{2}} e^{-i[k_i (j + \frac{1}{2}) \Delta x + k_k (k + \frac{1}{2}) \Delta z]}
\end{align*}
\]

Equation (3.9a)-(3.9f) then becomes

\[
\begin{align*}
\hat{e}_x^{n+1} & = A \hat{e}_x^n - 2 J W_z B \hat{h}_y^{n+\frac{1}{2}} + 2 J W_y B \hat{h}_z^{n+\frac{1}{2}} \\
\hat{e}_y^{n+1} & = A \hat{e}_y^n + 2 J W_z B \hat{h}_y^{n+\frac{1}{2}} - 2 J W_y B \hat{h}_z^{n+\frac{1}{2}} \\
\hat{e}_z^{n+1} & = A \hat{e}_z^n - 2 J W_y B \hat{h}_z^{n+\frac{1}{2}} + 2 J W_y B \hat{h}_z^{n+\frac{1}{2}} \\
\hat{h}_x^{n+\frac{1}{2}} & = 2 J W_z A \hat{e}_z^n - 2 J W_y A \hat{e}_x^n + [A - 4(W_y^2 + W_z^2)B^2] \hat{h}_x^{n+\frac{1}{2}} \\
& + 4W_y W_z B^2 \hat{h}_x^{n+\frac{1}{2}} + 4W_y W_z B^2 \hat{h}_z^{n+\frac{1}{2}} \\
\hat{h}_y^{n+\frac{1}{2}} & = -2 J W_z A \hat{e}_z^n + 2 J W_y A \hat{e}_x^n + 4W_x W_y B^2 \hat{h}_x^{n+\frac{1}{2}} \\
& + [A - 4(W_y^2 + W_z^2)B^2] \hat{h}_y^{n+\frac{1}{2}} + 4W_x W_y B^2 \hat{h}_z^{n+\frac{1}{2}} \\
\hat{h}_z^{n+\frac{1}{2}} & = 2 J W_y A \hat{e}_x^n - 2 J W_z A \hat{e}_y^n + 4W_x W_z B^2 \hat{h}_x^{n+\frac{1}{2}} \\
& + 4W_y W_z B^2 \hat{h}_y^{n+\frac{1}{2}} + [A - 4(W_y^2 + W_z^2)B^2] \hat{h}_z^{n+\frac{1}{2}}
\end{align*}
\]

Here

\[W_y^2 = c^2(\Delta t^2/\Delta h^2) \sin^2(k_h \Delta h/2), h = x, y, z\]

and \(k_h\) is the wavenumber along the x, y, or z direction, respectively. These form the CE-FDTD formulations in the spectral domain and can be combined and rewritten in the following matrix form:

\[
\hat{F}^{n+1} = [A] \hat{F}^n
\]
with
\[ \hat{F}^n = [\hat{e}_x^n, \hat{e}_y^n, \hat{e}_z^n, \hat{h}_x^{n+\frac{1}{2}}, \hat{h}_y^{n+\frac{1}{2}}, \hat{h}_z^{n+\frac{1}{2}}]^T \]

and
\[
\begin{bmatrix}
A & 0 & 0 & 0 & -J^2 W_y B & J^2 W_y B \\
0 & A & 0 & J^2 W_y B & 0 & -J^2 W_y B \\
0 & 0 & A & -J^2 W_y B & J^2 W_y B & 0 \\
0 & J^2 W_y AB & -J^2 W_y AB & A - 4(W_x^2 + W_y^2)B^2 & 4W_x W_y B^2 & 4W_x W_y B^2 \\
-J^2 W_y AB & 0 & J^2 W_y AB & 4W_x W_y B^2 & A - 4(W_x^2 + W_y^2)B^2 & 4W_x W_y B^2 \\
\end{bmatrix}
\]

The stability condition stipulates that if the magnitudes of all the eigenvalues of \([\Lambda]\) are less than or equal to unity in magnitude, the CE-FDTD scheme represented by Equations (3.9a) to (3.9f) will be stable; otherwise it will be potentially unstable. With the help of Mathematica 4.0, the six eigenvalues of \([\Lambda]\) are found as

\[ \lambda_{1,2} = \frac{1 - JD}{1 + JD} \]  
(3.12)

\[ \lambda_{3,4} = \frac{1 + D^2 - 2W^2 - 2W\sqrt{W^2 - 1 - D^2}}{(1 + D)^2} \]  
(3.13)

\[ \lambda_{5,6} = \frac{1 + D^2 - 2W^2 + 2W\sqrt{W^2 - 1 - D^2}}{(1 + JD)^2} \]  
(3.14)

with
\[ W^2 = W_x^2 + W_y^2 + W_z^2. \]

The first and second eigenvalues \(|\lambda_{1,2}| = 1\) correspond to static solutions. They do not cause instability, as their magnitudes are unity. The other four eigenvalues correspond to propagating modes. Their magnitudes can be either equal to, or larger than unity, depending on the different scenarios.
If \((W^2 - 1 - D^2) > 0\), then

\[
|\lambda_{3,6}| = \frac{2W^2 - 1 - D^2 \mp 2W\sqrt{W^2 - 1 - D^2}}{1 + D^2} > \frac{2W^2 - 1 - D^2}{1 + D^2} > \frac{1 + D^2}{1 + D^2} = 1
\]

That means the CE-FDTD method is potentially unstable.

However, if

\[
(W^2 - 1 - D^2) \leq 0,
\]

\[
|\lambda_{3,6}| = \frac{\sqrt{(1 + D^2 - 2W^2)^2 - 4W^2(W^2 - 1 - D^2)}}{1 + D^2} = 1.
\]

Thus, the algorithm is stable. In other words, Equation (3.15) becomes the condition for the CE-FDTD method to be stable. More explicitly,

\[
W^2 - 1 - D^2 = W_x^2 + W_y^2 + W_z^2 - 1 - D^2 \leq 0
\]

Since \(W_h^2 \leq c^2(\Delta t^2/\Delta h^2)\), \(h = x, y, z\), the condition for stable CE-FDTD solutions then becomes:

\[
\Delta t^2(c^2/\Delta x^2 + c^2/\Delta y^2 + c^2/\Delta z^2 - \sigma^2/4) \leq 1
\]

The condition (3.16) can be further broken up into two cases:

\[
(A) \quad c^2/\Delta x^2 + c^2/\Delta y^2 + c^2/\Delta z^2 - \sigma^2/4 > 0 \quad (3.17)
\]

\[
(B) \quad c^2/\Delta x^2 + c^2/\Delta y^2 + c^2/\Delta z^2 - \sigma^2/4 < 0 \quad (3.18)
\]

In case (A), the carrier frequency \(\sigma\) is low or the cell size \(\Delta h(h = x, y, z)\) is small. In such a situation, to ensure the numerical stability condition given by Equation (3.16),

\[
\Delta t \leq \Delta t_{\text{max}} = \frac{1}{c\sqrt{(1/\Delta x)^2 + (1/\Delta y)^2 + (1/\Delta z)^2 - (\sigma/2c)^2}}
\]

(3.19)

In case (B), the carrier frequency \(\sigma\) is high or the spatial step size \(\Delta h(h = x, y, z)\) is large. In such an instance, condition (3.16) is automatically satisfied and the time step \(\Delta t\) is not bounded by the stability. In other words, the time step \(\Delta t\) can now be chosen arbitrarily without causing numerical instability, or simply \(\Delta t_{\text{max}} = \infty\).
For simple and easy discussions, the typical example of a uniform grid $\Delta x = \Delta y = \Delta z = \Delta l$ is used for further analysis, with the note that more general studies can always be done with condition (3.16). For a uniform grid, the number of cells per carrier wavelength can be defined as $N_{\lambda} = \frac{\lambda}{\Delta l} = \frac{2\pi c}{(\sigma \Delta l)}$. Then, it is not difficult to derive the following stability condition from Equation (3.16): when $N_{\lambda} > \frac{\pi}{\sqrt{3}}$, the time step with the CE-FDTD method is bounded by a CFL upper limit that is equal to $2/(\sigma \sqrt{3} N_{\lambda}/\pi)^2 - 1$; when $N_{\lambda} \leq \frac{\pi}{\sqrt{3}}$, however, the time step is no longer bounded by the numerical stability requirement. The condition is plotted in Figure 3.2 for better visualization. For reference purposes, the CFL number (CFLN) is used as the unit for the vertical axis. It is defined as

$$CFLN = \frac{\Delta t_{\text{max}}}{\Delta t'_{\text{max}}}$$

with $\Delta t'_{\text{max}}$ being the CFL limit of the conventional FDTD method applied with the same discretization parameters as those with the CE-FDTD method.

![Figure 3.2: CFLN of the CE-FDTD as the function of $N_{\lambda}$](image)

As can be seen from Figure 3.2, when $N_{\lambda} > \frac{\pi}{\sqrt{3}}$ (corresponds to a low carrier frequency or small spatial step sizes), the CFLN is finite but larger than one. That indicates that the
maximum time step allowed for the CE-FDTD method is larger than that for the conventional FDTD method. The smaller the $N_x$, the larger the CFLN. As $N_x$ reaches less than $\pi/\sqrt{3}$, CFLN becomes unbounded, no longer indicating any stability constraints for the time step. The time step can be taken from any value without causing stability problems. $\pi/\sqrt{3}$ is then the threshold value.

The same metallic cavity ($9 \text{ cm} \times 6 \text{ cm} \times 15 \text{ cm}$) shown in chapter 2 was simulated with the CE-FDTD method in order to numerically validate the stability condition derived. The space increments along the $x$, $y$, and $z$ directions were chosen to be $\Delta x = \Delta y = \Delta z = 5 \text{ mm}$ and a Gaussian pulse was chosen as the excitation source. Two experiments were then done, one for the bounded time step denoted as case (A) and the other for the unbounded time step denoted as case (B).

**Experiment for case (A): the bounded time step**

In this case, the carrier frequency was chosen to be $\sigma = 5.7 \times 10^{10} \text{ rad/s}$ so that $N_x = 6.6139 > \pi/\sqrt{3}$. The corresponding upper limit of the time step for stable solutions, computed from Equation (3.19), is 10 ps. Then, the time steps were taken to be 9 ps ($< 10 \text{ ps}$) and 11 ps ($> 10 \text{ ps}$) for two respective simulations. Figures 3.3(a) and 3.3(b) show the time domain response of the $e_y$ component recorded at the center of the cavity with each of the two time steps. It can be seen that the former is clearly stable while the latter explodes very quickly.

**Experiment for case (B): the unbounded time step**

In this case, the experiment was run with $\sigma = 3.14 \times 10^{11} \text{ rad/s}$ (or $N_x = 1.2006 < \pi/\sqrt{3}$). To test the time step size, a large time step $\Delta t = 1000 \text{ ps}$, which is 104 times greater than the CFL limit of the conventional FDTD, was chosen for the simulation. Figure 3.4 shows the electric field recorded at the center of the waveguide. As can be seen, the values of the CE-FDTD solutions remain between -0.4 and +0.4. This numerically illustrates the unbounded time step as proved theoretically.
Figure 3.3: Numerical results of conditional stability (a) stable case when $\Delta t = 9$ ps, and (b) unstable case when $\Delta t = 11$ ps

Figure 3.4: Numerical results of the unconditional stability $N_x = 1.200 < \pi/\sqrt{3}$

3.4 Numerical Dispersion

Assuming now that

$$\hat{F}^{n+1} = e^{j\omega\Delta t} \hat{F}^n$$  \hspace{1cm} (3.20)
in the temporal frequency domain and substituting it into Equation (3.11), one can obtain the numerical dispersion:

$$\text{det}(e^{J\omega \Delta t}[I] - [\Lambda]) = 0$$  \hspace{1cm} (3.21)

where $[I]$ is a $6 \times 6$ identity matrix and $\omega$ is the frequency of the envelope functions. Further simplification of Equation (3.21) leads to:

$$(e^{J\omega \Delta t} - \lambda_1)^2(e^{J\omega \Delta t} - \lambda_3)^2(e^{J\omega \Delta t} - \lambda_5)^2 = 0$$  \hspace{1cm} (3.22)

Here $\lambda_1, \lambda_3$, and $\lambda_5$ are the eigenvalues given by Equations (3.12)-(3.14) in the previous section. Solutions of (3.22) lead to:

$$e^{J\omega \Delta t} = (1 - J\frac{\sigma\Delta t}{2})(1 + J\frac{\sigma\Delta t}{2})$$  \hspace{1cm} (3.23)

$$e^{J\omega \Delta t} = \frac{(1 + D^2) - 2W^2 \pm 2W\sqrt{1 + D^2W^2h^2}}{(1 + JD)^2}$$  \hspace{1cm} (3.24)

These can be further reduced to

$$\tan(\frac{\omega\Delta t}{2}) = -\frac{\sigma\Delta t}{2}$$  \hspace{1cm} (3.25)

and

$$\sin(\omega\Delta t) = \frac{-2D(1 + D^2 - 2W^2) \pm 2W(1 - D^2)\sqrt{1 + D^2W^2}}{(1 + D^2)^2}$$  \hspace{1cm} (3.26)

Equation (3.26) can be rewritten as:

$$\sin^2(\omega\Delta t)(1 + D^2)^2 + 4D(1 + D^2)^3 \sin(\omega\Delta t) + 4D^2(1 + D^2)^2$$

$$= 8DW^2(1 + D^2)^2 \sin(\omega\Delta t) + 4W^2(1 + D^2)^3 - 4W^2(1 + D^2)^3$$  \hspace{1cm} (3.27)

Equation (3.25) presents a numerical spurious mode that corresponds to the static solution. It is related only to the temporal frequency and not to the spatial frequencies. Therefore, it cannot propagate. Equation (3.27) presents the numerical dispersion of the numerical waves generated with the CE-FDTD method. With it, numerical dispersion errors of the CE-FDTD method can be evaluated.

However, before the numerical dispersion of the CE-FDTD method is assessed, analytical dispersion of the CE-FDTD method needs to be derived for reference and comparison. The curl equations for the complex-envelope Maxwell’s equations can be written as:

$$\frac{\partial \hat{E}}{\partial t} + J\sigma \hat{E} = c \nabla \times \hat{H}$$  \hspace{1cm} (3.28)
\[ \frac{\partial \tilde{h}}{\partial t} + J \omega \tilde{h} = -c \nabla \times \tilde{e} \]  
(3.29)

where \( \tilde{e} = (\tilde{e}_x, \tilde{e}_y, \tilde{e}_z) \) and \( \tilde{h} = (\tilde{h}_x, \tilde{h}_y, \tilde{h}_z) \) are the envelope functions.

Assuming

\[ \tilde{e} = \tilde{e}_o e^{(\omega t - \tilde{k} \eta)} \]

and

\[ \tilde{h} = \tilde{h}_o e^{(\omega t - \tilde{k} \eta)} \]

Equations (3.28) and (3.29) become

\[ (\omega + \omega) \tilde{e}_o = -c \tilde{k} \times \tilde{h}_o \]
(3.30)

\[ (\omega + \omega) \tilde{h}_o = c \tilde{k} \times \tilde{e}_o \]
(3.31)

where \( \tilde{k} = k_x \tilde{a}_x + k_y \tilde{a}_y + k_z \tilde{a}_z \) and \( \tilde{r} = x \tilde{a}_x + y \tilde{a}_y + z \tilde{a}_z \).

The combination of (3.30) and (3.31) leads to the analytical dispersion of the CE-FDTD method:

\[ (\omega + \omega)^2 = c^2 (k_x^2 + k_y^2 + k_z^2) = (ck)^2 \]
(3.32)

The above equation can now be used to compare with the numerical dispersion of the CE-FDTD method and therefore assess the numerical dispersion errors. Several aspects of the numerical dispersion have been identified and studied, as described below. For simplicity, the typical example of the uniform grid with \( \Delta x = \Delta y = \Delta z = \Delta l \) was considered. If required, further general situations involving a non-uniform grid case can be investigated with Equation (3.27).

A. Convergence of the Numerical Dispersion

One of the key indications that the CE-FDTD method gives an approximate solution to Maxwell's equations is to examine the convergence of the numerical dispersion as the time and space steps tend to zero. If it converges to the analytical dispersion relationship as given
in Equation (3.32), the CE-FDTD method is then proved able to provide the approximate solution; otherwise, it does not.

By taking $\Delta t \to 0$, $\Delta x \to 0$, and $\Delta z \to 0$ in Equation (3.27), one obtains:

$$
(\omega \Delta t)^2 + 2\omega \sigma \Delta t^2 + (\sigma \Lambda t)^2 = 4c^2 \Delta t^2 (\frac{1}{4} k_x^2 + \frac{1}{4} k_y^2 + \frac{1}{4} k_z^2) = \Delta t^2 (ck)^2
$$

(3.33)

which is exactly the same as the analytical dispersion relationship given by Equation (3.32). That is, the numerical dispersion does converge to the analytical dispersion as the time and space steps tend to zero.

**B. Numerical Dispersion for an Unbounded Time Step**

As derived from the stability analysis, when the number of cells per carrier wavelength, $N_x$, is less than the threshold value $\pi / \sqrt{3}$, the time step is no longer a factor in the stability condition. In other words, the time step $\Delta t$ is no longer constrained by the stability condition but by modeling accuracy. Therefore, it will be very interesting and important to examine how $N_x$, spatial steps, transpire in terms of accuracy. To achieve this, the numerical dispersion relationship (3.27) was solved with the following numerical approach:

As is conventional, one can assume that a CE wave propagates at angles $\phi$ and $\theta$ in the spherical coordinates

$$
k_x/k = \sin \theta \cos \phi, \quad k_y/k = \sin \theta \sin \phi, \quad k_z/k = \cos \theta.
$$

Here $k$ is the numerical wave number of the CE-FDTD method.

By applying Newton’s iterative method to Equation (3.27), the following iterative formula is obtained:

$$
k_{i+1} - k_i = \frac{L - 8DT^2 \sin(\omega \Delta t)W^2 - 4T^3W^2 + 4T^2W^4}{-8DT^2 \sin(\sigma \Delta t)\beta - 4T^3\beta + 8T^2 W^2 \beta}
$$

(3.34)
where

\[
T = 1 + D^2
\]
\[
L = \sin^2(\omega \Delta t) T^4 + 4D^2 T^2
\]
\[
W^2 = (c\Delta t)^2 \left( \frac{\sin^2(k_x \psi_x)}{\Delta^2 x} + \frac{\sin^2(k_y \psi_y)}{\Delta^2 y} + \frac{\sin^2(k_z \psi_z)}{\Delta^2 z} \right)
\]
\[
\beta = (W^2)^2 = (c\Delta t)^2 \left( \frac{\psi_x \sin(2k_x \psi_x)}{\Delta^2 x} + \frac{\psi_y \sin(2k_y \psi_y)}{\Delta^2 y} + \frac{\psi_z \sin(2k_z \psi_z)}{\Delta^2 z} \right)
\]

and

\[
\psi_x = \frac{k_x \Delta h_x}{k}, \quad \psi_y = \frac{k_y \Delta h_y}{k}, \quad \psi_z = \frac{k_z \Delta h_z}{k}
\]

The normalized numerical phase velocity \( \frac{v_p}{c} \) can then be expressed as:

\[
\frac{v_p}{c} = \frac{k_{\text{theory}}}{k_n} = \frac{\omega + \sigma}{c k_n}
\]  \(\text{(3.35)}\)

where \( k_n \) is the approximation of numerical wavenumber \( k \) at the \( n \)th iteration of equation (3.34). In an actual computation, \( k_{\text{theory}} \), calculated by Equation (3.32), is used as the initial estimated value of (3.34). It normally leads to a fast convergence of solutions within two or three iterations. In our studies, we chose \( N_x = 1.8, \sigma = 2 \times 10^7 \text{ rad/s}, \omega = 2 \pi \times 10^4 \text{ rad/s} \) and \( \Delta t = \Delta l / c / \sqrt{3} \). Figure 3.5 shows the numerical dispersion versus different azimuthal propagation angles at \( \theta = 45^\circ \).

It can be seen clearly that the numerical phase velocity error can reach 200\%, which is unacceptable. In another words, the unbounded time step case (B) has no practical use due to the occurrence of large errors.

C. Numerical Dispersion for a Bounded Time Step

Since the unbounded time step induces unacceptable errors, the bounded time step, case (A), for practical applications must then be considered. Figure 3.6 shows the effect of a spatial step on the numerical dispersion with \( N_x (> \pi / \sqrt{3}) \). The other parameters are the same as those for Figure 3.3(a) except the time step is decided by Equation (3.19). The errors shown are the dispersion errors along the coordinate axial directions, which are the
maximum errors among the different propagation directions (as will be described in the following section).

From Figure 3.6, one can see that the dispersion errors are approximately 1.7% when $N_\chi = 10$, approximately 0.4% when $N_\chi = 20$, and approximately 0.2% when $N_\chi = 30$. 

Figure 3.5: Dispersion versus propagation angle when $N_\chi = 1.8$

Figure 3.6: Dispersion versus $N_\chi$
D. Numerical Dispersion versus Propagation Direction

Like any other FDTD method, numerical dispersion of the CE-FDTD method also changes with the propagation directions. Figure 3.7 shows the numerical dispersion versus different propagation angles with $N = 15(> \pi/\sqrt{3})$. To compare the performance of the CE-FDTD method with the conventional FDTD method, the same time step is chosen for all the cases.

![Figure 3.7: Dispersion versus propagation direction](image)

It can be readily seen that the numerical phase velocity error is the largest along the axial directions and the smallest in the diagonal direction, when $\phi = 45^\circ$ and $\theta = 45^\circ$. This property is similar to the conventional FDTD method. However, the numerical dispersion of the CE-FDTD method is generally worse than that of the FDTD method, with up to 0.25% larger errors. As a result, it can be concluded that in the case of a bounded time step due to the CFL condition, the CE-FDTD method is no more advantageous than the conventional FDTD method.
3.5 Conclusions

It is found that the CE-FDTD method does have a numerical stability condition. When $N_A < \pi/\sqrt{3}$ or the spatial step is greater than $\pi/\sqrt{3}$ times the carrier wavelength, the time step can be of any large value, but the associated numerical dispersion errors are unacceptably large. In other words, the unbounded time step has no practical uses. When $N_A > \pi/\sqrt{3}$, the time step is bounded by a CFL condition. When this occurs, the numerical dispersion of the CE FDTD method does not perform better than the numerical dispersion of the conventional FDTD method. Consequently, the CE-FDTD method is not advantageous over the conventional FDTD method.

The reason for this unexpected conclusion is not hard to understand after referring to Nyquist sampling criteria in the spatial domain. The complex-envelope technique only changes the sampling target from modulated signals to envelopes in the time domain by treating the carrier frequency as a known variable. However the carrier frequency $\omega$ is not eliminated from the complex Maxwell's equations and still has its impacts which can be seen from the dispersion equation (3.32). The spatial frequency $k$ is still proportion to $\omega$ ($\omega >> \omega$), which means there is no change in spatial domain compared with the classic FDTD method and the spatial step still has to meet the Nyquist sampling criteria for the highest frequency ($\omega + \omega_{\text{max}}$). In the unbounded time step case (B), the number of cells per carrier wavelength $N_A$ is less than 2, which undoubtedly leads to large unacceptable errors. In the bounded time step case (A), the $N_A$ meets the Nyquist sampling criteria. In other words, the spatial step is bounded by the highest frequency. Because the CFL condition (3.19) has to be satisfied to obtain stable solution, the time step is also bounded by the highest frequency instead of just envelope as we expect. Therefore, to demonstrate the improvement by applying the complex-envelope technique, the CFL condition must be relaxed or even removed. For this purpose, implicit methods may be incorporated in formulating the complex-envelope FDTD method.
Chapter 4

The Unconditionally Stable ADI-FDTD Method

The alternating direction implicit (ADI) method is an implicit method that was first introduced in solving elliptic and parabolic partial differential equations [67]. It has the advantage of being unconditionally stable. An attempt was made in 1980 [68] to apply the ADI method to the FDTD scheme, but was not successful until recently [40] [69] [41]. As a result, the CFL stability restraint was removed and the choice of $\Delta t$ is now determined by the required accuracy and no longer by stability.

In this chapter, analysis and discussions on the ADI FDTD algorithm are conducted. Formulations, numerical stability and dispersion, and simulation results are presented.

4.1 The ADI-FDTD Formulations

As described in [41], ADI-FDTD formulations are obtained from scalar partial differential equations (2.2) by splitting them into two half time steps. The first step advances the fields from the $n$ time step to the $(n+1/2)$ time step and the second one advances the fields from the $(n+1/2)$ time step to the $(n+1)$ time step. In the first half time step, an implicit evaluation is applied to the first term of the right-hand side and an explicit evaluation is applied to the second. For the second half time step, the implicit and explicit evaluations are reversed in the term sequence. This yields the implicit updating equations as given below.

For the advancement from the $n$ time step to the $(n+1/2)$ time step:

\[
\begin{align*}
e_x^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k} &= e_x^n_{i+\frac{1}{2},j,k} + c\frac{\Delta t}{2}(h_z^n_{i+\frac{1}{2},j+\frac{1}{2},k} - h_z^n_{i+\frac{1}{2},j-\frac{1}{2},k})\Delta y \\
&\quad - c\frac{\Delta t}{2}(h_y^n_{i+\frac{1}{2},j,k+\frac{1}{2}} - h_y^n_{i+\frac{1}{2},j,k-\frac{1}{2}})\Delta z
\end{align*}
\]  

(4.1a)
\begin{align}
\tag{4.1b}
e_{x_{i,j,k}^{n+\frac{1}{2}}} &= e_{y_{i+\frac{1}{2},j,k}} + c\frac{\Delta t}{2}(h_{x_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}}} - h_{x_{i,j-\frac{1}{2},k}^{n+\frac{1}{2}}})\Delta z - c\frac{\Delta t}{2}(h_{x_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}}} - h_{x_{i-\frac{1}{2},j,k}^{n+\frac{1}{2}}})\Delta x
\end{align}

\begin{align}
\tag{4.1c}
e_{x_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}}} &= e_{x_{i,j,k}^{n+\frac{1}{2}}} + c\frac{\Delta t}{2}(h_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}} - h_{y_{i+\frac{1}{2},j,k-\frac{1}{2}}})\Delta x - c\frac{\Delta t}{2}(h_{y_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}}} - h_{y_{i,j-\frac{1}{2},k}^{n+\frac{1}{2}}})\Delta y
\end{align}

\begin{align}
\tag{4.1d}
h_{x_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}}} &= h_{x_{i,j,k}^{n+\frac{1}{2}}} + c\frac{\Delta t}{2}(e_{y_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}}} - e_{y_{i,j,k}^{n+\frac{1}{2}}})\Delta z - c\frac{\Delta t}{2}(e_{x_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}}} - e_{x_{i,j,k}^{n+\frac{1}{2}}})\Delta y
\end{align}

\begin{align}
\tag{4.1e}
h_{y_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}}} &= h_{y_{i,j,k}^{n+\frac{1}{2}}} + c\frac{\Delta t}{2}(e_{x_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}}} - e_{x_{i,j,k}^{n+\frac{1}{2}}})\Delta x - c\frac{\Delta t}{2}(e_{y_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}}} - e_{y_{i,j,k}^{n+\frac{1}{2}}})\Delta z
\end{align}

\begin{align}
\tag{4.1f}
h_{x_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}}} &= h_{x_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}}} + c\frac{\Delta t}{2}(e_{y_{i+\frac{1}{2},j+1,k}^{n+\frac{1}{2}}} - e_{y_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}}})\Delta y - c\frac{\Delta t}{2}(e_{x_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}}} - e_{x_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}}})\Delta x
\end{align}

For the advancement from the \((n+1)/2\) to the \((n+1)\) time step

\begin{align}
\tag{4.2a}
e_{x_{i+\frac{1}{2},j,k}^{n+1}} &= e_{x_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}}} + c\frac{\Delta t}{2}(h_{x_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}}} - h_{x_{i+\frac{1}{2},j-\frac{1}{2},k}^{n+\frac{1}{2}}})\Delta y - c\frac{\Delta t}{2}(h_{x_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}}} - h_{x_{i+\frac{1}{2},j-\frac{1}{2},k}^{n+\frac{1}{2}}})\Delta z
\end{align}

\begin{align}
\tag{4.2b}
e_{y_{i,j+\frac{1}{2},k}^{n+1}} &= e_{y_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}}} + c\frac{\Delta t}{2}(h_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}} - h_{y_{i+\frac{1}{2},j,k-\frac{1}{2}}})\Delta x - c\frac{\Delta t}{2}(h_{y_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}}} - h_{y_{i,j,k}^{n+\frac{1}{2}}})\Delta z
\end{align}

\begin{align}
\tag{4.2c}
e_{y_{i,j+\frac{1}{2},k}^{n+1}} &= e_{y_{i,j,k}^{n+\frac{1}{2}}} + c\frac{\Delta t}{2}(h_{y_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}}} - h_{y_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}}})\Delta y - c\frac{\Delta t}{2}(h_{y_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}}} - h_{y_{i,j,k}^{n+\frac{1}{2}}})\Delta x
\end{align}

\begin{align}
\tag{4.2d}
h_{x_{i,j+\frac{1}{2},k}^{n+1}} &= h_{x_{i,j,k}^{n+\frac{1}{2}}} + c\frac{\Delta t}{2}(e_{y_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}}} - e_{y_{i,j,k}^{n+\frac{1}{2}}})\Delta z - c\frac{\Delta t}{2}(e_{y_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}}} - e_{y_{i,j,k}^{n+\frac{1}{2}}})\Delta y
\end{align}

\begin{align}
\tag{4.2e}
h_{y_{i+\frac{1}{2},j,k}^{n+1}} &= h_{y_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}}} + c\frac{\Delta t}{2}(e_{x_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}}} - e_{x_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}}})\Delta x - c\frac{\Delta t}{2}(e_{x_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}}} - e_{x_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}}})\Delta y
\end{align}
\[
\frac{h_z^{n+\frac{1}{2}, j+\frac{1}{2}, k}}{h_z^{n+\frac{1}{2}, j+\frac{1}{2}, k}} = h_z^{n+\frac{1}{2}, j+\frac{1}{2}, k} + \frac{c\Delta t}{2} (e_x^{n+\frac{1}{2}, j+\frac{1}{2}, k} - e_x^{n, j+\frac{1}{2}, k})/\Delta y \\
-\frac{c\Delta t}{2} (e_y^{n+\frac{1}{2}, j+\frac{1}{2}, k} - e_y^{n, j+\frac{1}{2}, k})/\Delta x 
\] (4.2f)

All the field components are laid out on the same staggered Yee grid in space but with no staggering in time. The above equations can be further simplified by substituting the unknown variables to the right hand side of equations. For instance, by substituting Equation (4.1f) into (4.1a), the updating equation for \( e_x \) at the first sub-step is obtained:

\[
- \left( \frac{c^2\Delta t^2}{4\Delta y^2} \right) e_x^{n+\frac{1}{2}, j, k+1} + (1 + \frac{c^2\Delta t^2}{2\Delta y^2}) e_x^{n+\frac{1}{2}, j, k} - \left( \frac{c^2\Delta t^2}{2\Delta y^2} \right) e_x^{n+\frac{1}{2}, j-1, k} = \\
- \frac{c\Delta t}{\Delta y} [h_{\nu}^{n, j+\frac{1}{2}, k} - h_{\nu}^{n, j-\frac{1}{2}, k}] - \frac{c\Delta t}{\Delta z} [h_{\nu}^{n, j, k+\frac{1}{2}} - h_{\nu}^{n, j, k-\frac{1}{2}}] \\
- \frac{c^2\Delta t^2}{4\Delta x\Delta y} [e_y^{n+1, j+\frac{1}{2}, k} - e_y^{n+1, j-\frac{1}{2}, k}] - e_y^{n, j+\frac{1}{2}, k} - e_y^{n, j-\frac{1}{2}, k} + e_y^{n, j, k} 
\] (4.3)

And by substituting Equation (4.2e) into (4.2a), the updating equation for \( e_x \) at the second sub-step is:

\[
- \left( \frac{c^2\Delta t^2}{4\Delta z^2} \right) e_x^{n+\frac{1}{2}, j, k+1} + (1 + \frac{c^2\Delta t^2}{2\Delta z^2}) e_x^{n+\frac{1}{2}, j, k} - \left( \frac{c^2\Delta t^2}{2\Delta z^2} \right) e_x^{n+\frac{1}{2}, j, k-1} = \\
- \frac{c\Delta t}{\Delta y} [h_{\nu}^{n+\frac{1}{2}, j, k+\frac{1}{2}} - h_{\nu}^{n+\frac{1}{2}, j, k-\frac{1}{2}}] - \frac{c\Delta t}{\Delta z} [h_{\nu}^{n, j, k+\frac{1}{2}} - h_{\nu}^{n, j, k-\frac{1}{2}}] \\
- \frac{c^2\Delta t^2}{4\Delta x\Delta z} [e_y^{n+1, j+\frac{1}{2}, k} - e_y^{n+1, j-\frac{1}{2}, k}] - e_y^{n+\frac{1}{2}, j, k} - e_y^{n-\frac{1}{2}, j, k} + e_y^{n, j, k} 
\] (4.4)

The above equation forms a linear system of equations for \( e_x \) with its coefficient matrix being a tri-diagonal banded matrix. It can be solved quite efficiently. The same procedures can be applied to other field components, and the implementation details are shown in [70].

In general, the ADI-FDTD method requires less memory and computation time than other implicit schemes, while it still retains the simple gridding that is inherent in the traditional Yee FDTD technique.

### 4.2 Source Implementation

The standard FDTD scheme permits the use of explicit wave source conditions wherein specific electric and/or magnetic field components at source points are updated separately from the point-by-point explicit Yee updating equations. In the ADI-FDTD scheme, the
magnetic-field updating expressions remain explicit; however, the electric fields are updated using implicit updating equations along alternating directions through the grid.

Based on the work presented in [71], [72] and [73], magnetic-field sources are implemented using a standard explicit formulation for a hard source or soft source, and electric current source are embedded in the known column vector on the right-hand side of the tridiagonal matrix system for the x-, y-, z-directed lines that pass through the location of the current source. In addition, to preserve the symmetry in the field values, the ADI-FDTD method requires the same excitation value to be applied in both sub-steps. For instance, the current source $J$ is incorporated into Equation (4.3) as

$$-\left(\frac{c^2 \Delta t^2}{4 \Delta y^2}\right) e_x |_{i+\frac{1}{2}, j+\frac{1}{2}, k}^{n+1} + (1 + \frac{c^2 \Delta t^2}{4 \Delta y^2}) e_x |_{i+\frac{1}{2}, j+\frac{1}{2}, k}^{n+1} - \left(\frac{c^2 \Delta t^2}{4 \Delta y^2}\right) e_x |_{i+\frac{1}{2}, j-\frac{1}{2}, k}^{n+1} =$$

$$e_x |_{i+\frac{1}{2}, j, k}^{n+1} + \frac{c \Delta t}{2 \Delta y} [h_z |_{i+\frac{1}{2}, j+\frac{1}{2}, k} - h_z |_{i+\frac{1}{2}, j-\frac{1}{2}, k}] - \frac{c \Delta t}{2 \Delta z} [h_y |_{i+\frac{1}{2}, j, k+\frac{1}{2}} - h_y |_{i+\frac{1}{2}, j, k-\frac{1}{2}}]$$

$$- \frac{c^2 \Delta t^2}{4 \Delta x \Delta y} [e_y |_{i+1, j+\frac{1}{2}, k}^{n+1} - e_y |_{i+\frac{1}{2}, j+\frac{1}{2}, k}^{n+1} - e_y |_{i+1, j-\frac{1}{2}, k}^{n+1} + e_y |_{i+\frac{1}{2}, j-\frac{1}{2}, k}^{n+1}] - \frac{\Delta t}{2} E_p^{n+\frac{1}{2}}$$

and into Equation (4.4) as

$$-\left(\frac{c^2 \Delta t^2}{4 \Delta z^2}\right) e_y |_{i+\frac{1}{2}, j+\frac{1}{2}, k}^{n+1} + (1 + \frac{c^2 \Delta t^2}{4 \Delta z^2}) e_y |_{i+\frac{1}{2}, j+\frac{1}{2}, k}^{n+1} - \left(\frac{c^2 \Delta t^2}{4 \Delta z^2}\right) e_y |_{i+\frac{1}{2}, j-\frac{1}{2}, k}^{n+1} =$$

$$e_y |_{i+\frac{1}{2}, j, k}^{n+1} + \frac{c \Delta t}{2 \Delta y} [h_z |_{i+\frac{1}{2}, j+\frac{1}{2}, k} - h_z |_{i+\frac{1}{2}, j-\frac{1}{2}, k}] - \frac{c \Delta t}{2 \Delta z} [h_y |_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}} - h_y |_{i+\frac{1}{2}, j+\frac{1}{2}, k-\frac{1}{2}}]$$

$$- \frac{c^2 \Delta t^2}{4 \Delta x \Delta z} [e_z |_{i+1, j+\frac{1}{2}, k}^{n+1} - e_z |_{i, j+\frac{1}{2}, k+\frac{1}{2}}^{n+1} - e_z |_{i+1, j, k-\frac{1}{2}}^{n+1} + e_z |_{i, j, k-\frac{1}{2}}^{n+1}] - \frac{\Delta t}{2} E_p^{n+\frac{1}{2}}$$

(4.6)

### 4.3 Numerical Stability and Dispersion

The unconditional stability of the ADI-FDTD method has been proven with several methods [41], [74] and [75]. In [41], Fourier method was applied. The procedure is same as that described in chapter 2. By expressing the field quantities in the spectral domain, the ADI-FDTD method can be written as follows:

For advancement from the $n$ to the $(n+1/2)$ time step

$$\tilde{F}^{n+\frac{1}{2}} = [\Lambda] \tilde{F}^n$$

(4.7)
where

\[
[A_1] = \begin{bmatrix}
\frac{1}{Q_t} & \frac{w_y w_z}{Q_t} & 0 & 0 & \frac{jw_z}{Q_t} & -jw_y \\
0 & \frac{1}{Q_t} & \frac{w_y}{Q_t} & \frac{-jw_z}{Q_t} & 0 & \frac{jw_z}{Q_t} \\
w_y w_z & 0 & \frac{1}{Q_t} & \frac{jw_z}{Q_t} & -jw_y & 0 \\
-\frac{1}{Q_t} & 0 & \frac{-jw_z}{Q_t} & 0 & \frac{1}{Q_t} & \frac{-jw_z}{Q_t} \\
jw_z & 0 & \frac{-jw_z}{Q_t} & 0 & \frac{1}{Q_t} & \frac{1}{Q_t} \\
-\frac{jw_z}{Q_t} & jw_z & 0 & 0 & \frac{1}{Q_t} & \frac{1}{Q_t} \\
0 & \frac{1}{Q_t} & \frac{w_y}{Q_t} & \frac{w_z}{Q_t} & 0 & \frac{1}{Q_t}
\end{bmatrix}
\]

\[W_l = \frac{c \Delta t}{\Delta h} \cdot \sin \left( \frac{k_l \Delta h}{2} \right), l = x, y, z\]

\[Q_l = 1 + W_l, l = x, y, z\]

For advancement from the \((n+1/2)\) to the \((n+1)\) time step

\[\hat{F}^{n+1} = [A_2] \hat{F}^{n+1/2}\]  \hspace{1cm} (4.8)

where

\[
[A_2] = \begin{bmatrix}
\frac{1}{Q_t} & 0 & \frac{w_y}{Q_t} & 0 & \frac{jw_z}{Q_t} & -jw_y \\
\frac{w_y}{Q_t} & \frac{1}{Q_t} & 0 & -jw_z & 0 & \frac{jw_z}{Q_t} \\
0 & \frac{1}{Q_t} & \frac{jw_z}{Q_t} & -jw_y & 0 & \frac{jw_z}{Q_t} \\
-\frac{jw_z}{Q_t} & 0 & \frac{-jw_z}{Q_t} & 0 & \frac{1}{Q_t} & \frac{-jw_z}{Q_t} \\
jw_z & \frac{1}{Q_t} & \frac{-jw_z}{Q_t} & 0 & \frac{1}{Q_t} & \frac{1}{Q_t} \\
-\frac{jw_z}{Q_t} & \frac{jw_z}{Q_t} & 0 & \frac{1}{Q_t} & \frac{-jw_z}{Q_t} & \frac{1}{Q_t}
\end{bmatrix}
\]

Then for a complete time step, the amplification matrix is \([\Lambda] = [A_2] \ast [A_1]\). It has been proved that all the eigenvalues of \([\Lambda]\) are equal to one in magnitude, which indicates the unconditional stability of the ADI FDTD scheme [70].

The numerical dispersion relation has also been derived in terms of these eigenvalues [70]. It is

\[
\sin^2(\omega \Delta t) = \frac{4 \left( W_x^2 Q_y + W_y^2 Q_z + W_z^2 Q_x \right) (1 + W_x^2 W_y^2 W_z^2)}{Q_y^2 Q_z^2 Q_x^2} \]  \hspace{1cm} (4.9)
4.4 ADI Class Problem

Since the time step size is no longer restrained by stability, the limitation of the maximum time-step size is only dependent on accuracy. The accuracy of the ADI FDTD scheme as the time step increases beyond the CFL limit has been addressed both through numerical dispersion analysis [76] [77] [78] [79] and numerical simulations [80] [81]. It has been shown that the ADI-FDTD method has larger errors when compared to the FDTD method for the same spatial and temporal sampling. As well, errors become larger when the time step is larger. At first glance, it would appear that ADI-FDTD method is not advantageous over conventional FDTD method. For the field problems that require low spacial resolution, this is true. However, for the field problems that require high spacial resolution, the ADI-FDTD method can be much more efficient in CPU time than the conventional FDTD method because the time step can be larger without much affecting accuracy. Therefore, there exists a class of problems for which the ADI-FDTD method gives a significant advantage over the FDTD method. This class of problems is referred to as ADI-FDTD class problems.

In general, the ADI-FDTD method holds promise for analyzing problems for which the space discretization is small relative to wavelength and the time step governed by the CFL limit is unreasonably small. In such circumstances, the savings in computational time realized in reducing the time step can outweigh the additional cost of the ADI scheme, such as more memory usage (about 1.5 times or more of conventional FDTD due to auxiliary electric fields).

In a large number of electromagnetic problems, including bioelectromagnetic problems, the spatial discretization is dominated by very fine geometric details rather than the smallest wavelength of interest. The ADI method appears to be particularly attractive for these problems, in which the relatively large dispersion and phase error of the ADI method is acceptable. Examples include studies of microstrip resonators and filters with very narrow gaps and strips [82], the shielding effectiveness of thin conductive sheets [83] and power plane resonance [84].
4.5 Numerical Results

The same structures analyzed in chapter 2 are computed again using the ADI-FDTD method. For the sake of comparison, most parameters are kept the same. The results are presented in the following sections.

4.5.1 Parallel-plate Waveguide

For convenience, unchanged simulation parameters are listed again here:

i) Modulated Gaussian Pulse with \( f_0 = 4.5 \text{ GHz} \) and \( T_{FWHM} = 833 \text{ ps} \)

ii) Spatial step \( \Delta x = \Delta y = 0.5 \text{ mm} \), so \( nx = 80, ny = 2000 \)

iii) \( \Delta t_{CFL} = 1.18 \text{ ps} \)

Because the ADI-FDTD method is unconditionally stable, the time step is only constrained by modeling accuracy. With a larger time step, the total number of iterations required can be reduced. To show the overall characteristics of the ADI-FDTD method in terms of accuracy and efficiency, the CFL number (CFLN, the ratio of the time step to the CFL limit) is gradually increased with different simulations. Tables 4.1 and 4.2 list the results. For comparison, results of conventional FDTD method are also included.

Table 4.1: Efficiency and accuracy of the ADI-FDTD method for the \( TM_0 \) mode

<table>
<thead>
<tr>
<th>CFL</th>
<th>Iterations</th>
<th>CPU Time(ms)</th>
<th>Err(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FDTD</td>
<td>1</td>
<td>2500</td>
<td>36310</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2500</td>
<td>268969</td>
</tr>
<tr>
<td>ADI-FDTD</td>
<td>3</td>
<td>833</td>
<td>82719</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>500</td>
<td>48406</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>250</td>
<td>23360</td>
</tr>
</tbody>
</table>

It can be seen that with the same time step, the ADI-FDTD method performs a little worse in accuracy than the conventional FDTD method. In addition, it takes much longer computing time, almost eight times that of the conventional FDTD method. This is due to the fact that more auxiliary components are computed with the ADI-FDTD method. With a larger
Table 4.2: Efficiency and accuracy of the ADI-FDTD method for the $TM_1$ mode

<table>
<thead>
<tr>
<th></th>
<th>CFL</th>
<th>Iterations</th>
<th>CPU Time(ms)</th>
<th>Err(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FDTD</td>
<td>1</td>
<td>2500</td>
<td>35380</td>
<td>0.736</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2500</td>
<td>278344</td>
<td>0.814</td>
</tr>
<tr>
<td>ADI-FDTD</td>
<td>3</td>
<td>833</td>
<td>82719</td>
<td>1.129</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>500</td>
<td>48938</td>
<td>2.151</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>250</td>
<td>23860</td>
<td>4.041</td>
</tr>
</tbody>
</table>

time step, which is beyond the CFL constraint, accuracy deteriorates rapidly. In the case of $TM_0$ mode, accuracy falls almost 50 times when a ten times larger time step is chosen. However, the larger time step saves computing time. The actual time consumed is 1.5 times less than conventional FDTD when $CFLN = 10$.

4.5.2 Metallic Cavity

Table 4.3 shows errors of dominant frequency calculated by the ADI-FDTD method. The CPU times are shown as well.

Table 4.3: Efficiency and accuracy of the ADI-FDTD method for resonant frequency

<table>
<thead>
<tr>
<th></th>
<th>CFL</th>
<th>Iterations</th>
<th>CPU Time(ms)</th>
<th>Freq.</th>
<th>Err(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FDTD</td>
<td>1</td>
<td>5000</td>
<td>10265</td>
<td>19.375</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>5000</td>
<td>40907</td>
<td>19.35</td>
<td>0.376</td>
</tr>
<tr>
<td>ADI-FDTD</td>
<td>3</td>
<td>1666</td>
<td>13281</td>
<td>19.239</td>
<td>0.947</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>1000</td>
<td>8203</td>
<td>19.023</td>
<td>2.059</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>500</td>
<td>3828</td>
<td>18.947</td>
<td>2.451</td>
</tr>
</tbody>
</table>

As can be seen, when CFLN=10, the ADI-FDTD method consumes one third CPU time of the conventional FDTD method at the cost of approximately ten times deterioration in accuracy. The ADI-FDTD method shows advantages only if this kind of deterioration is acceptable.
4.6 Conclusions

In this chapter, the main components of the ADI-FDTD method are described. They include the formulations, unconditional stability, and numerical dispersion. In addition, by analyzing the same structures as in chapter 2 and comparing with the conventional FDTD method, it is concluded that because the error increases as the time step becomes larger, the ADI-FDTD method improves efficiency only when applied to some kinds of specific problems. How much computing time saved is dependent on the error tolerance. There is usually a trade-off between efficiency and accuracy.
Chapter 5

The Complex-Envelope ADI-FDTD Method

The analysis in chapter 3 revealed that numerical dispersion of the CE-FDTD method is not better than that of the conventional FDTD method. Therefore, in searching for a better CE-type FDTD method, attention can be turned to the recently developed, unconditionally stable ADI-FDTD scheme whose details are given in chapter 4. By following the same procedure in formulating the CE-FDTD method, the CE ADI-FDTD method is developed in this chapter. First, the full-wave CE ADI-FDTD formulations are presented. Unconditional numerical stability is proved after that. Then the numerical dispersion of the CE ADI-FDTD method is derived analytically. The impacts of propagation directions, ratio of carrier to envelope frequencies, temporal step size, and spatial step size on the dispersion errors are assessed theoretically and experimentally. Finally, discussions and conclusions are made on the implications of the results obtained.

5.1 Derivation of CE ADI-FDTD Scheme

With the same notation used in chapter 3, for a narrow bandwidth signal, the Complex-Envelope (CE) Maxwell’s equations are

\[
\frac{\partial \hat{e}_x}{\partial t} + J_0 \omega \hat{e}_x = c \left( \frac{\partial \hat{h}_x}{\partial y} - \frac{\partial \hat{h}_y}{\partial z} \right) \tag{5.1a}
\]

\[
\frac{\partial \hat{e}_y}{\partial t} + J_0 \omega \hat{e}_y = c \left( \frac{\partial \hat{h}_y}{\partial z} - \frac{\partial \hat{h}_z}{\partial x} \right) \tag{5.1b}
\]

\[
\frac{\partial \hat{e}_z}{\partial t} + J_0 \omega \hat{e}_z = c \left( \frac{\partial \hat{h}_z}{\partial x} - \frac{\partial \hat{h}_x}{\partial y} \right) \tag{5.1c}
\]

\[
\frac{\partial \hat{h}_x}{\partial t} + J_0 \omega \hat{h}_x = c \left( \frac{\partial \hat{e}_y}{\partial y} - \frac{\partial \hat{e}_z}{\partial z} \right) \tag{5.1d}
\]

\[
\frac{\partial \hat{h}_y}{\partial t} + J_0 \omega \hat{h}_y = c \left( \frac{\partial \hat{e}_z}{\partial z} - \frac{\partial \hat{e}_x}{\partial x} \right) \tag{5.1e}
\]
\[ \frac{\partial \hat{h}_z}{\partial t} + j \sigma \hat{h}_z = c \left( \frac{\partial \hat{e}_x}{\partial y} - \frac{\partial \hat{e}_y}{\partial x} \right) \]  

(5.1f)

Replacement of the derivatives with their central finite-difference counterparts in Yee's space lattice [85] and subdivision of the updating procedure into two sub-steps [69] [41] yields the following relationships.

For the advancement from the \( n \) time step to the \( (n+1/2) \) time step:

\[
(1 + D) \hat{e}_x \bigg|_{i+\frac{1}{2},j,k}^{n+\frac{1}{2}} = (1 - D) \hat{e}_x \bigg|_{i+\frac{1}{2},j,k}^{n} + c \frac{\Delta t}{2} 
\]

\[
\begin{bmatrix}
\frac{\hat{h}_x \big|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - \hat{h}_x \big|_{i+\frac{1}{2},j-\frac{1}{2},k}^{n+\frac{1}{2}}}{\Delta y} \\
\frac{\hat{h}_x \big|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - \hat{h}_x \big|_{i-\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}}}{\Delta z}
\end{bmatrix}
\]  

(5.2a)

\[
(1 + D) \hat{e}_y \bigg|_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}} = (1 - D) \hat{e}_y \bigg|_{i,j+\frac{1}{2},k}^{n} + c \frac{\Delta t}{2} 
\]

\[
\begin{bmatrix}
\frac{\hat{h}_y \big|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - \hat{h}_y \big|_{i+\frac{1}{2},j-\frac{1}{2},k}^{n+\frac{1}{2}}}{\Delta z} \\
\frac{\hat{h}_y \big|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}} - \hat{h}_y \big|_{i-\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}}}{\Delta x}
\end{bmatrix}
\]  

(5.2b)

\[
(1 + D) \hat{e}_z \bigg|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} = (1 - D) \hat{e}_z \bigg|_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n} + c \frac{\Delta t}{2} 
\]

\[
\begin{bmatrix}
\frac{\hat{h}_z \big|_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - \hat{h}_z \big|_{i-\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x} \\
\frac{\hat{h}_z \big|_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - \hat{h}_z \big|_{i,j+\frac{1}{2},k}^{n+\frac{1}{2}}}{\Delta y}
\end{bmatrix}
\]  

(5.2c)

\[
(1 + D) \hat{h}_x \bigg|_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} = (1 - D) \hat{h}_x \bigg|_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} + c \frac{\Delta t}{2} 
\]

\[
\begin{bmatrix}
\frac{\hat{e}_y \big|_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - \hat{e}_y \big|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}}}{\Delta z} \\
\frac{\hat{e}_z \big|_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - \hat{e}_z \big|_{i,\frac{1}{2},j,\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta y}
\end{bmatrix}
\]  

(5.2d)

\[
(1 + D) \hat{h}_y \bigg|_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} = (1 - D) \hat{h}_y \bigg|_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} + c \frac{\Delta t}{2} 
\]

\[
\begin{bmatrix}
\frac{\hat{e}_x \big|_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - \hat{e}_x \big|_{i+,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta z} \\
\frac{\hat{e}_z \big|_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - \hat{e}_z \big|_{i,\frac{1}{2},j,\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x}
\end{bmatrix}
\]  

(5.2e)

\[
(1 + D) \hat{h}_z \bigg|_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} = (1 - D) \hat{h}_z \bigg|_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} + c \frac{\Delta t}{2} 
\]

\[
\begin{bmatrix}
\frac{\hat{e}_x \big|_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - \hat{e}_x \big|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{2}}}{\Delta y} \\
\frac{\hat{e}_y \big|_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}} - \hat{e}_y \big|_{i+,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x}
\end{bmatrix}
\]  

(5.2f)
where $D = Jw\Delta t/4$.

For the advancement from the $(n+1/2)$ to the $(n+1)$ time step:

\[(1 + D)\hat{E}_x^{n+1/2}_{i,j,k} = (1 - D)\hat{E}_x^n_{i,j,k} + \frac{c\Delta t}{2}\]

\[
\frac{\left[\hat{h}_x^{n+1}_{i,j,\frac{k+1}{2}} - \hat{h}_x^{n+1}_{i,j,\frac{k-1}{2}} - \hat{h}_y^{n+1}_{i+\frac{1}{2},j,k+\frac{1}{2}} + \hat{h}_y^{n+1}_{i+\frac{1}{2},j,k+\frac{1}{2}}\right]}{\Delta y}
\]

\[(5.3a)\]

\[(1 + D)\hat{E}_y^{n+1/2}_{i,j,k} = (1 - D)\hat{E}_y^n_{i,j,\frac{k+1}{2}} + \frac{c\Delta t}{2}\]

\[
\frac{\left[\hat{h}_y^{n+1}_{i,j,\frac{k+1}{2}} - \hat{h}_y^{n+1}_{i,j,\frac{k-1}{2}} - \hat{h}_z^{n+1}_{i+\frac{1}{2},j,\frac{k+1}{2}} + \hat{h}_z^{n+1}_{i+\frac{1}{2},j,\frac{k-1}{2}}\right]}{\Delta z}
\]

\[(5.3b)\]

\[(1 + D)\hat{E}_z^{n+1/2}_{i,j,k} = (1 - D)\hat{E}_z^n_{i,j+\frac{1}{2},k} + \frac{c\Delta t}{2}\]

\[
\frac{\left[\hat{h}_z^{n+1}_{i,j,\frac{k+1}{2}} - \hat{h}_z^{n+1}_{i,j,\frac{k-1}{2}} - \hat{h}_x^{n+1}_{i+\frac{1}{2},j,\frac{k-1}{2}} + \hat{h}_x^{n+1}_{i+\frac{1}{2},j,\frac{k+1}{2}}\right]}{\Delta x}
\]

\[(5.3c)\]

\[(1 + D)\hat{h}_x^{n+1}_{i,j+\frac{1}{2},k+\frac{1}{2}} = (1 - D)\hat{h}_x^n_{i,j+\frac{1}{2},k+\frac{1}{2}} + \frac{c\Delta t}{2}\]

\[
\frac{\left[\hat{e}_x^{n+1}_{i+\frac{1}{2},j+1,k+\frac{1}{2}} - \hat{e}_x^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k+1} + \hat{e}_x^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k} - \hat{e}_x^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k-1}\right]}{\Delta y}
\]

\[(5.3d)\]

\[(1 + D)\hat{h}_y^{n+1}_{i+\frac{1}{2},j,k+\frac{1}{2}} = (1 - D)\hat{h}_y^n_{i+\frac{1}{2},j,k+\frac{1}{2}} + \frac{c\Delta t}{2}\]

\[
\frac{\left[\hat{e}_y^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k+1} - \hat{e}_y^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} + \hat{e}_y^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k} - \hat{e}_y^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k-1}\right]}{\Delta z}
\]

\[(5.3e)\]

\[(1 + D)\hat{h}_z^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k} = (1 - D)\hat{h}_z^n_{i+\frac{1}{2},j+\frac{1}{2},k} + \frac{c\Delta t}{2}\]

\[
\frac{\left[\hat{e}_z^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - \hat{e}_z^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k} + \hat{e}_z^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k-1} - \hat{e}_z^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}}\right]}{\Delta y}
\]

\[(5.3f)\]

By manipulating the equations obtained, one can have the final CE ADI-FDTD formulation whose coefficient matrix is tri-diagonal.
\[-(1 + D)S_{\alpha}^n \hat{\eta}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}} - \hat{\eta}_x |_{i+\frac{1}{2}, j+k-\frac{1}{2}}\]
\[-S_{\alpha, \beta}^n [\hat{\varepsilon}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}} - \hat{\varepsilon}_x |_{i+\frac{1}{2}, j-k+\frac{1}{2}} + \hat{\varepsilon}_x |_{i+\frac{1}{2}, j-k-\frac{1}{2}}] \quad (5.4a)\]

\[-S_{\gamma}^n \hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}} + [(1 + D)^2 + 2S_{\delta}^n] \hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}} - S_{\delta}^n \hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}}\]
\[= (1 - D^2) \hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}} + (1 - D)S_x [\hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}} - \hat{\xi}_x |_{i+\frac{1}{2}, j+k-\frac{1}{2}}] \]
\[-(1 + D)S_x [\hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}} - \hat{\xi}_x |_{i+\frac{1}{2}, j+k-\frac{1}{2}} + \hat{\xi}_x |_{i+\frac{1}{2}, j-k+\frac{1}{2}} - \hat{\xi}_x |_{i+\frac{1}{2}, j-k-\frac{1}{2}}] \quad (5.4b)\]

\[-S_{\gamma}^n \hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}} + [(1 + D)^2 + 2S_{\delta}^n] \hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}} - S_{\delta}^n \hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}}\]
\[= (1 - D^2) \hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}} + (1 - D)S_x [\hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}} - \hat{\xi}_x |_{i+\frac{1}{2}, j+k-\frac{1}{2}}] \]
\[-(1 + D)S_x [\hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}} - \hat{\xi}_x |_{i+\frac{1}{2}, j+k-\frac{1}{2}} + \hat{\xi}_x |_{i+\frac{1}{2}, j-k+\frac{1}{2}} - \hat{\xi}_x |_{i+\frac{1}{2}, j-k-\frac{1}{2}}] \quad (5.4c)\]

\[-S_{\gamma}^n \hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}} + [(1 + D)^2 + 2S_{\delta}^n] \hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}} - S_{\delta}^n \hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}}\]
\[= (1 - D^2) \hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}} + (1 - D)S_x [\hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}} - \hat{\xi}_x |_{i+\frac{1}{2}, j+k-\frac{1}{2}}] \]
\[-(1 + D)S_x [\hat{\xi}_x |_{i+\frac{1}{2}, j+k+\frac{1}{2}} - \hat{\xi}_x |_{i+\frac{1}{2}, j+k-\frac{1}{2}} + \hat{\xi}_x |_{i+\frac{1}{2}, j-k+\frac{1}{2}} - \hat{\xi}_x |_{i+\frac{1}{2}, j-k-\frac{1}{2}}] \quad (5.4d)\]

\[-S_{\gamma}^n \hat{\eta}_y |_{i+\frac{1}{2}, j+k+\frac{1}{2}} + [(1 + D)^2 + 2S_{\delta}^n] \hat{\eta}_y |_{i+\frac{1}{2}, j+k+\frac{1}{2}} - S_{\delta}^n \hat{\eta}_y |_{i+\frac{1}{2}, j+k+\frac{1}{2}}\]
\[= (1 - D^2) \hat{\eta}_y |_{i+\frac{1}{2}, j+k+\frac{1}{2}} + (1 - D)S_y [\hat{\eta}_y |_{i+\frac{1}{2}, j+k+\frac{1}{2}} - \hat{\eta}_y |_{i+\frac{1}{2}, j+k-\frac{1}{2}}] \]
\[-(1 + D)S_y [\hat{\eta}_y |_{i+\frac{1}{2}, j+k+\frac{1}{2}} - \hat{\eta}_y |_{i+\frac{1}{2}, j+k-\frac{1}{2}} + \hat{\eta}_y |_{i+\frac{1}{2}, j-k+\frac{1}{2}} - \hat{\eta}_y |_{i+\frac{1}{2}, j-k-\frac{1}{2}}] \quad (5.4e)\]
\[-S_x S_y \{\hat{h}_y \mid n, j+1, k+\frac{1}{2} \} - \hat{h}_y \mid n, j+1, k-\frac{1}{2} - \hat{h}_y \mid n, j+\frac{1}{2}, k+1 + \hat{h}_y \mid n, j+\frac{1}{2}, k-\frac{1}{2}\] \hspace{1cm} (5.4f)

For the second sub-step,

\[-S_x^2 \hat{e}_x \mid i+\frac{1}{2}, j+1, k + [(1 + D)\frac{2 S_x^2}{\hat{e}_x} \mid i+\frac{1}{2}, j+\frac{1}{2}, k - S_x^2 \hat{e}_x \mid i-\frac{1}{2}, j+\frac{1}{2}, k-1]\]
\[= (1 - D^2) \hat{e}_x \mid i+\frac{1}{2}, j+\frac{1}{2}, k + (1 + D) S_x \{\hat{h}_x \mid n+\frac{1}{2}, j+\frac{1}{2}, k - \hat{h}_x \mid n+\frac{1}{2}, j-\frac{1}{2}, k\}\]
\[-(1 - D) S_x \{\hat{h}_x \mid i+\frac{1}{2}, j+\frac{1}{2}, k + \hat{h}_x \mid i+\frac{1}{2}, j-\frac{1}{2}, k\}\]
\[-S_x S_y \{\hat{e}_x \mid i, j+1, k+\frac{1}{2} - \hat{e}_x \mid i, j+\frac{1}{2}, k+1 - \hat{e}_x \mid i, j+\frac{1}{2}, k-\frac{1}{2} + \hat{e}_x \mid i, j-\frac{1}{2}, k\}\] \hspace{1cm} (5.5a)

\[-S_y^2 \hat{e}_y \mid i+\frac{1}{2}, j+\frac{1}{2}, k + [(1 + D)\frac{2 S_y^2}{\hat{e}_y} \mid i+\frac{1}{2}, j+\frac{1}{2}, k - S_y^2 \hat{e}_y \mid i-\frac{1}{2}, j+\frac{1}{2}, k-1]\]
\[= (1 - D^2) \hat{e}_y \mid i+\frac{1}{2}, j+\frac{1}{2}, k + (1 + D) S_y \{\hat{h}_y \mid n+\frac{1}{2}, j+\frac{1}{2}, k + \hat{h}_y \mid n+\frac{1}{2}, j+\frac{1}{2}, k-\frac{1}{2}\}\]
\[-(1 - D) S_y \{\hat{h}_y \mid i+\frac{1}{2}, j+\frac{1}{2}, k - \hat{h}_y \mid i+\frac{1}{2}, j+\frac{1}{2}, k\}\]
\[-S_y S_x \{\hat{e}_y \mid i, j+1, k+\frac{1}{2} - \hat{e}_y \mid i, j+\frac{1}{2}, k+1 - \hat{e}_y \mid i, j+\frac{1}{2}, k-\frac{1}{2} + \hat{e}_y \mid i, j-\frac{1}{2}, k\}\] \hspace{1cm} (5.5b)

\[-S_x^2 \hat{e}_x \mid i+\frac{1}{2}, j+\frac{1}{2}, k + [(1 + D)\frac{2 S_x^2}{\hat{e}_x} \mid i+\frac{1}{2}, j+\frac{1}{2}, k - S_x^2 \hat{e}_x \mid i-\frac{1}{2}, j+\frac{1}{2}, k-1]\]
\[= (1 - D^2) \hat{e}_x \mid i+\frac{1}{2}, j+\frac{1}{2}, k + (1 + D) S_x \{\hat{h}_x \mid n+\frac{1}{2}, j+\frac{1}{2}, k + \hat{h}_x \mid n+\frac{1}{2}, j+\frac{1}{2}, k-\frac{1}{2}\}\]
\[-(1 + D) S_x \{\hat{h}_x \mid i+\frac{1}{2}, j+\frac{1}{2}, k - \hat{h}_x \mid i-\frac{1}{2}, j+\frac{1}{2}, k\}\]
\[-S_x S_y \{\hat{e}_x \mid i, j+1, k+\frac{1}{2} - \hat{e}_x \mid i, j+\frac{1}{2}, k+1 - \hat{e}_x \mid i, j+\frac{1}{2}, k-\frac{1}{2} + \hat{e}_x \mid i, j-\frac{1}{2}, k\}\] \hspace{1cm} (5.5c)

\[-S_y^2 \hat{h}_x \mid i+\frac{1}{2}, j+\frac{1}{2}, k + [(1 + D)\frac{2 S_y^2}{\hat{h}_x} \mid i+\frac{1}{2}, j+\frac{1}{2}, k - S_y^2 \hat{h}_x \mid i-\frac{1}{2}, j+\frac{1}{2}, k-1]\]
\[= (1 - D^2) \hat{h}_x \mid i+\frac{1}{2}, j+\frac{1}{2}, k + (1 + D) S_y \{\hat{h}_x \mid n+\frac{1}{2}, j+\frac{1}{2}, k + \hat{h}_x \mid n+\frac{1}{2}, j+\frac{1}{2}, k-\frac{1}{2}\}\]
\[-(1 + D) S_y \{\hat{h}_x \mid i+\frac{1}{2}, j+\frac{1}{2}, k - \hat{h}_x \mid i-\frac{1}{2}, j+\frac{1}{2}, k\}\]
\[-S_y S_x \{\hat{h}_x \mid i, j+1, k+\frac{1}{2} - \hat{h}_x \mid i, j+\frac{1}{2}, k+1 - \hat{h}_x \mid i, j+\frac{1}{2}, k-\frac{1}{2} + \hat{h}_x \mid i, j-\frac{1}{2}, k\}\] \hspace{1cm} (5.5d)

\[-S_y^2 \hat{h}_y \mid i+\frac{1}{2}, j+\frac{1}{2}, k + [(1 + D)\frac{2 S_y^2}{\hat{h}_y} \mid i+\frac{1}{2}, j+\frac{1}{2}, k - S_y^2 \hat{h}_y \mid i-\frac{1}{2}, j+\frac{1}{2}, k-1]\]
\[= (1 - D^2) \hat{h}_y \mid i+\frac{1}{2}, j+\frac{1}{2}, k + (1 + D) S_x \{\hat{h}_y \mid n+\frac{1}{2}, j+\frac{1}{2}, k + \hat{h}_y \mid n+\frac{1}{2}, j+\frac{1}{2}, k-\frac{1}{2}\}\]
\[-(1 + D) S_x \{\hat{h}_y \mid i+\frac{1}{2}, j+\frac{1}{2}, k - \hat{h}_y \mid i-\frac{1}{2}, j+\frac{1}{2}, k\}\]
\[-S_x S_y \{\hat{h}_y \mid i, j+1, k+\frac{1}{2} - \hat{h}_y \mid i, j+\frac{1}{2}, k+1 - \hat{h}_y \mid i, j+\frac{1}{2}, k-\frac{1}{2} + \hat{h}_y \mid i, j-\frac{1}{2}, k\}\]
\[-(1 + D)S_x \hat{\varepsilon}^{n+\frac{1}{2}}_{x, i+\frac{1}{2}, j, k+1} - \hat{\varepsilon}^{n+\frac{1}{2}}_{x, i+\frac{1}{2}, j, k}\]
\[-S_x S_y [\hat{h}^{n+\frac{1}{2}}_{z, i+\frac{1}{2}, j+\frac{1}{2}, k+1} - \hat{h}^{n+\frac{1}{2}}_{z, i+\frac{1}{2}, j-\frac{1}{2}, k+1} - \hat{h}^{n+\frac{1}{2}}_{z, i+\frac{1}{2}, j+\frac{1}{2}, k} + \hat{h}^{n+\frac{1}{2}}_{z, i+\frac{1}{2}, j-\frac{1}{2}, k}] \]  
(5.5e)

\[-S_y^2 \hat{h}^{n+\frac{1}{2}}_{z, i+\frac{1}{2}, j+\frac{1}{2}, k} + [(1 + D)^2 + 2S_y^2] \hat{h}^{n+\frac{1}{2}}_{z, i+\frac{1}{2}, j+\frac{1}{2}, k} - S_y^2 \hat{h}^{n+\frac{1}{2}}_{z, i+\frac{1}{2}, j-\frac{1}{2}, k} \]
\[= (1 - D^2) \hat{h}^{n+\frac{1}{2}}_{z, i+\frac{1}{2}, j+\frac{1}{2}, k} + (1 - D)S_y [\hat{\varepsilon}^{n+\frac{1}{2}}_{x, i+\frac{1}{2}, j+\frac{1}{2}, k} - \hat{\varepsilon}^{n+\frac{1}{2}}_{x, i+\frac{1}{2}, j, k}] \]
\[-(1 + D)S_x [\hat{\varepsilon}^{n+\frac{1}{2}}_{y, i+\frac{1}{2}, j+\frac{1}{2}, k} - \hat{\varepsilon}^{n+\frac{1}{2}}_{y, i+\frac{1}{2}, j, k}] \]
\[-S_x S_y [\hat{h}^{n+\frac{1}{2}}_{z, i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}} - \hat{h}^{n+\frac{1}{2}}_{z, i+\frac{1}{2}, j-\frac{1}{2}, k+\frac{1}{2}} - \hat{h}^{n+\frac{1}{2}}_{z, i+\frac{1}{2}, j+\frac{1}{2}, k-\frac{1}{2}} + \hat{h}^{n+\frac{1}{2}}_{z, i+\frac{1}{2}, j-\frac{1}{2}, k-\frac{1}{2}}] \]  
(5.5f)

where $S_t = \frac{c \Delta t}{\Delta l} t = x, y, z$ is the Courant number.

### 5.2 Numerical Stability

Considering the field quantities in the spectral domain and applying the Fourier method to Equation (5.4), one can obtain:

\[Q_x \hat{\varepsilon}^{n+\frac{1}{2}}_{x} = (1 - D^2)\hat{\varepsilon}^{n}_{x} + W_x \hat{\varepsilon}^{n}_{y} + J(1 + D)W_y \hat{\varepsilon}^{n}_{z} - J(1 - D)W_y \hat{\varepsilon}^{n}_{z} \]  
(5.6a)

\[Q_y \hat{\varepsilon}^{n+\frac{1}{2}}_{y} = (1 - D^2)\hat{\varepsilon}^{n}_{y} + W_y \hat{\varepsilon}^{n}_{z} + J(1 + D)W_z \hat{\varepsilon}^{n}_{x} - J(1 - D)W_z \hat{\varepsilon}^{n}_{x} \]  
(5.6b)

\[Q_x \hat{\varepsilon}^{n+\frac{1}{2}}_{z} = (1 - D^2)\hat{\varepsilon}^{n}_{z} + W_z \hat{\varepsilon}^{n}_{x} + J(1 + D)W_x \hat{\varepsilon}^{n}_{y} - J(1 - D)W_x \hat{\varepsilon}^{n}_{y} \]  
(5.6c)

\[Q_y \hat{\varepsilon}^{n+\frac{1}{2}}_{z} = (1 - D^2)\hat{\varepsilon}^{n}_{y} + W_y \hat{\varepsilon}^{n}_{x} + J(1 + D)W_x \hat{\varepsilon}^{n}_{y} - J(1 - D)W_x \hat{\varepsilon}^{n}_{y} \]  
(5.6d)

\[Q_y \hat{\varepsilon}^{n+\frac{1}{2}}_{y} = (1 - D^2)\hat{\varepsilon}^{n}_{y} + W_y \hat{\varepsilon}^{n}_{z} + J(1 + D)W_z \hat{\varepsilon}^{n}_{x} - J(1 - D)W_z \hat{\varepsilon}^{n}_{x} \]  
(5.6e)

\[Q_x \hat{\varepsilon}^{n+\frac{1}{2}}_{x} = (1 - D^2)\hat{\varepsilon}^{n}_{x} + W_x \hat{\varepsilon}^{n}_{y} + J(1 + D)W_y \hat{\varepsilon}^{n}_{z} - J(1 - D)W_y \hat{\varepsilon}^{n}_{z} \]  
(5.6f)

with

\[ W_l = \frac{c \Delta t}{\Delta l} \sin \left(\frac{k \Delta l}{2} \right), l = x, y, z \]

\[Q_l = (1 + D)^2 + W_l^2, h = x, y, z \]

These can then be combined and rewritten in the following matrix form:

\[\hat{\varepsilon}^{n+\frac{1}{2}} = [\Lambda] \hat{\varepsilon}^{n} \]  
(5.7)
where

$$\begin{bmatrix} \Lambda_1 \end{bmatrix} =
\begin{bmatrix}
\frac{1-D^2}{Q_t} & \frac{W_z}{Q_t} & 0 & 0 & \frac{j(1+D)W_z}{Q_t} & \frac{-j(1-D)W_z}{Q_t} \\
0 & \frac{1-D^2}{Q_t} & \frac{W_z}{Q_t} & \frac{-j(1-D)W_z}{Q_t} & 0 & \frac{Q_t}{Q_t} \\
\frac{W_z}{Q_t} & 0 & \frac{1-D^2}{Q_t} & \frac{j(1+D)W_z}{Q_t} & \frac{-j(1-D)W_z}{Q_t} & 0 \\
0 & \frac{-j(1-D)W_z}{Q_t} & 0 & \frac{Q_t}{Q_t} & \frac{-j(1-D)W_z}{Q_t} & 0 \\
\frac{j(1+D)W_z}{Q_t} & 0 & \frac{-j(1-D)W_z}{Q_t} & 0 & \frac{Q_t}{Q_t} & \frac{-j(1-D)W_z}{Q_t} \\
\frac{-j(1-D)W_z}{Q_t} & \frac{j(1+D)W_z}{Q_t} & 0 & 0 & \frac{Q_t}{Q_t} & \frac{-j(1-D)W_z}{Q_t}
\end{bmatrix}
$$

The expression for the second half time step can also be derived in a similar way:

$$Q_z e_z^{n+\frac{1}{2}} = (1 - D^2) e_z^n + W_z W_z e_z^n + J(1 - D) W_z e_z^n - J(1 + D) W_z e_z^n$$  (5.8a)

$$Q_z e_y^{n+\frac{1}{2}} = (1 - D^2) e_y^n + W_z W_z e_y^n + J(1 - D) W_z e_y^n - J(1 + D) W_z e_y^n$$  (5.8b)

$$Q_z e_z^{n+\frac{1}{2}} = (1 - D^2) e_z^n + W_z W_z e_z^n + J(1 - D) W_z e_z^n - J(1 + D) W_z e_z^n$$  (5.8c)

$$Q_z e_x^{n+\frac{1}{2}} = (1 - D^2) e_x^n + W_z W_z e_x^n + J(1 - D) W_z e_x^n - J(1 + D) W_z e_x^n$$  (5.8d)

$$Q_z e_y^{n+\frac{1}{2}} = (1 - D^2) e_y^n + W_z W_z e_y^n + J(1 - D) W_z e_y^n - J(1 + D) W_z e_y^n$$  (5.8e)

$$Q_z e_z^{n+\frac{1}{2}} = (1 - D^2) e_z^n + W_z W_z e_z^n + J(1 - D) W_z e_z^n - J(1 + D) W_z e_z^n$$  (5.8f)

$$\tilde{\mathbf{F}}^{n+1} = [\Lambda_2][\tilde{\mathbf{F}}^{n+\frac{1}{2}}]$$  (5.9)

where

$$\begin{bmatrix} \Lambda_2 \end{bmatrix} =
\begin{bmatrix}
\frac{1-D^2}{Q_t} & 0 & \frac{W_z}{Q_t} & 0 & \frac{j(1-D)W_z}{Q_t} & \frac{-j(1+D)W_z}{Q_t} \\
0 & \frac{1-D^2}{Q_t} & 0 & \frac{-j(1+D)W_z}{Q_t} & 0 & \frac{Q_t}{Q_t} \\
\frac{W_z}{Q_t} & 0 & \frac{1-D^2}{Q_t} & \frac{j(1-D)W_z}{Q_t} & \frac{-j(1+D)W_z}{Q_t} & 0 \\
0 & \frac{-j(1+D)W_z}{Q_t} & 0 & \frac{Q_t}{Q_t} & \frac{-j(1-D)W_z}{Q_t} & 0 \\
\frac{j(1-D)W_z}{Q_t} & 0 & \frac{-j(1+D)W_z}{Q_t} & 0 & \frac{Q_t}{Q_t} & \frac{-j(1-D)W_z}{Q_t} \\
\frac{-j(1+D)W_z}{Q_t} & \frac{j(1-D)W_z}{Q_t} & 0 & 0 & \frac{Q_t}{Q_t} & \frac{-j(1-D)W_z}{Q_t}
\end{bmatrix}
$$

Subsequently, the overall amplification matrix is $[\Lambda] = [\Lambda_2] \ast [\Lambda_1]$. The CE-ADI-FDTD method will be unconditionally stable if all the eigenvalues are less than or equal unity in magnitude. Unfortunately, due to the complexity of the amplification matrix $[\Lambda]$, the eigenvalues are very difficult to find in a symbolic form. Therefore, a numerical exhausting
approach, similar to that used in [86], is employed to find the maximum eigenvalues with the aid of Mathematica 4.0.

Element values of $[\Lambda]$ are related to $D$ and $W_{l=x,y,z}$. That means, the eigenvalue $\lambda$ is a function of the following variables:

i) Carrier frequency, $\omega$.

ii) Time step, $\Delta t$.

iii) Spatial step, $\Delta l_{i=x,y,z}$.

iv) Wavenumber in $x,y,z$ direction: $k_x, k_y, k_z$.

First, a simple uniform cubic mesh $\Delta x = \Delta y = \Delta z = \Delta l$ is assumed. Instead of measuring $\Delta t$ directly, the CFLN is used to measure the size of the time step.

Since the stability condition is preferred to be independent of the angles of wave propagation, all angles must be considered. To this end, it is found through a exhaustive search that the maximum eigenvalues occur when $\sin k_i \Delta l/2 = 1, i = x, y, z$. Initially, for validation purposes, eigenvalues of $[\Lambda]$ are computed when $D = 0$, and this yields the same values as the ADI FDTD method shown in chapter 4. By using these as the initial values, the eigenvalues of $[\Lambda]$ are then computed for any CFLN and $\omega$. After examining all the results, the eigenvalues are all found to lie within or on the unit circle. More details can be found in Appendix A.

The search is extended to noncubic lattices and to nonequal choices of constitutive parameter. In all cases, it is found that $|\lambda_{i,\Lambda}| \leq 1$. Therefore, we conclude that the CE ADI-FDTD method is unconditionally stable and the time step can be chosen arbitrarily without introducing numerical instability.
5.3 Numerical Dispersion

The numerical dispersion can be obtained by assuming the field components to be a monochromatic traveling wave and by solving the following equation:

\[
\det(e^{i\omega \Delta t} I - [\Lambda]) = 0
\]  
(5.10)

here \( \omega \) is the envelope frequency (different from the frequency \( \omega_r \)).

Unfortunately, it is also difficult to get a simplified analytical expression for Equation (5.10) like those dispersion relations derived for the conventional FDTD method or ADI-FDTD method. It has to be solved numerically. In our case, Mathematica 4.0 is used for the solutions of Equation (5.10). Then the dispersion properties of the CE ADI-FDTD method can be analyzed.

Suppose that the complex envelopes propagate at angle \( \theta \) and \( \phi \) in the spherical coordinates. Then

\[
k_r/k = \sin \theta \cos \phi, \quad k_r/k = \sin \theta \sin \phi, \quad k_r/k = \cos \theta,
\]

where \( k \) is the numerical wave number. By substituting these representations into Equation (5.10), the numerical wave number \( k \) can be found and compared with the analytical result.

For convenience, we use the numerical phase velocity normalized to the speed of light (i.e. \( v_p/c \)) to measure numerical dispersion. This is related to the numerical wave number through the following equation [58]:

\[
\frac{v_p}{c} = \frac{\omega + \omega_r}{ck}
\]  
(5.11)

A. Convergence of the numerical dispersion

One of the key indications that the CE ADI-FDTD method will present an approximate solution to Maxwell's equations is to examine the convergence of its numerical dispersion when the time and space steps tend to zero. In comparisons of \([\Lambda_1]\) and \([\Lambda_2]\) with the corresponding matrices of the ADI-FDTD method [76], one can see the only difference:
the value of unity (one) in the dispersion equations of the ADI-FDTD method now becomes $1 \pm D$ or $1 - D^2$ for the CE ADI-FDTD method. Since $D = J\pi \Delta t/4$, $D$ will be zero when the temporal step $\Delta t$ goes to zero. Then, the CE ADI-FDTD method has the same matrix $[A_1]$ and $[A_2]$ as the ADI-FDTD method when $\Delta t$ goes to zero. Because the ADI-FDTD method converges to the analytical dispersion relationship, so does the CE ADI-FDTD method.

B. Numerical dispersion versus the propagation direction

Like other FDTD methods, the CE ADI-FDTD method presents variations of numerical dispersion with the propagation directions (i.e. numerical anisotropy). To make the discussion simple, the uniform cell is considered here. That is, $\Delta x = \Delta y = \Delta z$. Also, the same parameter notations as those used in [58] are employed. In particular, $N_\lambda$ is defined as number of cells per carrier wavelength and CFLN is the ratio of the time step to the CFL limit. Figure 5.1 shows the numerical dispersion versus different propagation angles with $\sigma = 2\pi \times 10^7$ rad/s for the carrier frequency, $\omega = 2\pi \times 10^4$ rad/s for the envelope frequency, $N_\lambda = 20$ for the mesh size and CFLN = 1 for the time step.

![Figure 5.1: Dispersion versus propagation angles with CFLN = 1](image)

It can be easily seen that the numerical phase velocity error is largest along the axial directions while being smallest at the diagonal direction when $\phi = 45^\circ$ and $\theta = 45^\circ$. This is the same result as in the conventional FDTD and the ADI-FDTD methods. In other words,
the CE ADI-FDTD method has numerical anisotropic behavior similar to the conventional FDTD method.

Figure 5.2 gives a comparison of the dispersion errors of the conventional FDTD, ADI-FDTD and CE ADI-FDTD methods for CFLN=1. The other parameters for Figure 5.2 are the same as those given for Figure 5.1.

![Dispersion comparison graph](image)

**Figure 5.2: Dispersions of the FDTD, ADI-FDTD and CE ADI-FDTD methods**

It can be seen that the dispersion errors of the CE ADI-FDTD method lie between conventional FDTD and ADI-FDTD methods when the time step is taken to be less than or equal to the CFL limit. The dispersion errors of the CE ADI-FDTD method are about 0.2% larger than those of the conventional FDTD method but are 0.1% less than those of the ADI-FDTD method. In the following subsection, the attention is then turned to $CFLN > 1$ (a time step beyond the CFL limit). In such a case, the conventional FDTD method becomes unstable. Therefore, the comparisons can be made only between the ADI-FDTD method and the CE ADI-FDTD method.

C. Numerical dispersion versus the time step

Since the ADI technique introduces unconditional stability, the only factor that may limit
the time step is the modeling accuracy. Therefore, it is very important to study the effect of a large time step size on the numerical dispersion. Figure 5.3 illustrates the dispersion of the CE ADI-FDTD method with CFLN = 3. The other parameters were taken the same as those used for Figure 5.2. The dispersion error of the ADI-FDTD method was also plotted in Figure 5.3.

![Dispersion versus propagation angle with CFLN = 3](image)

**Figure 5.3: Dispersion versus propagation angle with CFLN = 3**

In comparison with Figure 5.2, it can be seen that while the dispersion errors of the ADI-FDTD method deteriorate by about 2%, the dispersion of the CE ADI-FDTD method hardly changes.

To be more explicit, Figure 5.4 presents the maximum dispersion errors (which occur in the axial direction) versus CFLN for two different ratios of carrier to envelope frequencies.
Figure 5.4: Dispersion errors versus CFLN

As can be seen, with the higher ratio of carrier to envelope frequencies, the dispersion errors of the CE ADI-FDTD method do not change very much with increasing CFLN, while the ADI-FDTD method does. For dispersion error of less than 1%, the CFLN of the ADI-FDTD can only be 1.3 or less, while the CFLN of the CE ADI-FDTD method can be 50 for ratio of carrier to envelope frequency of $10^3$, and 100 for a ratio of $10^5$. In other words, the narrower the band of the signals that are simulated, the larger the time step that can be taken, and therefore the more savings in computation time. Next, the effects of the ratio of carrier to envelope frequencies on the dispersion are discussed.

D. Numerical dispersion versus the ratio of carrier to envelope frequencies

As can be seen in Figure 5.4, the ratio of carrier to envelope frequencies also has effects on the dispersion errors. In the CE ADI-FDTD method, the envelopes of the field components, instead of the field components themselves, are sampled and computed. The carrier frequency is considered as a known variant in Maxwell’s equations. Such a known variant should have some effect on the numerical dispersion.
Figure 5.5 shows the maximum dispersion errors versus the ratio of the carrier to envelope frequencies.

![Dispersion versus ratio of carrier to envelope](image)

Figure 5.5: Dispersion versus ratio of carrier to envelope

In general, the larger the ratio, the smaller the errors. The errors level off at 0.5% as the ratio increases beyond 150. In other words, the CE ADI-FDTD is effective for narrow-band simulations.

E. Numerical dispersion versus spatial step

As it varies with the temporal step, the numerical dispersion also changes with the spatial steps. Figure 5.6 shows the maximum dispersion error versus the spatial steps represented by the number of cells per carrier wavelength (i.e. $N_{\lambda}$).

The dispersion errors become smaller as the cell size is reduced ($N_{\lambda}$ increases). For less than 1% in dispersion errors, $N_{\lambda}$ should be larger than 10. That is, the number of cells per wavelength should be at least 10. In comparisons with the conventional FDTD method, one can see that the requirement of spatial step size is almost the same as that for the conventional FDTD method.
5.4 Numerical Results

5.4.1 Parallel-plate Waveguide

The same configuration is used when verifying the ADI-FDTD method, except the following setups:

i) Source consideration

Because the carrier frequency is absorbed into the envelope Maxwell’s equations, only the envelope of the signal (expressed by equation (2.17)) is required as stimulus. That is,

\[ g(t) = A_0 \exp\left[\frac{(t - T_0)^2}{2T_w^2}\right] \]

(5.12)

where \( T_w \) and \( T_0 \) have the same values as those given in chapter 2.

• Signal recovery

The \( \hat{e}_y \) obtained is expressed by a complex number. As stated in chapter 3, the real \( e_y \) field can be recovered through the following relationship:

\[ e_y(t) = Re[\hat{e}_y(t) \ast \exp(J \omega t)] \]

(5.13)
In the frequency domain, \( e_y(f) \) can be obtained by shifting the \( \hat{e}_y(f) \) to the left and right by \( f_0 \), then dividing by two (as depicted in section 1 of chapter 3).

To verify the relationship between the envelope and the signal, we first take the same time step as in the conventional FDTD method. Figure 5.7 shows the envelopes at the two reference points for \( TM_0 \) mode. For the sake of comparison, the results of the conventional FDTD method are also plotted. It can be seen that signals match well with the envelopes.

![Graph showing envelopes and signals with CFLN=1](image)

Figure 5.7: Envelopes and signals with CFLN=1

The conventional FDTD method becomes unstable when CFLN is greater than one. In such a case, the FDTD method cannot be used for comparisons with the CE ADI-FDTD method. Therefore, we then turn to use the ADI-FDTD method as our reference. Figure 5.8 shows the envelopes solved by the CE ADI-FDTD method and the signals solved by the ADI-FDTD method when CFLN=10.
Figure 5.8: Envelopes and signals with CFLN=10

Some noticeable small differences between results of these two methods can be found in Figure 5.8. Table 5.1 and Table 5.2 present the efficiency and accuracy of the two methods for $TM_0$ and $TM_1$ modes.

Table 5.1: Efficiency and accuracy of the CE ADI-FDTD method for the $TM_0$ mode

<table>
<thead>
<tr>
<th></th>
<th>CFL</th>
<th>Iterations</th>
<th>CPU Time(ms)</th>
<th>Err(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FDTD</td>
<td>1</td>
<td>2500</td>
<td>36310</td>
<td>0.025</td>
</tr>
<tr>
<td>ADI-FDTD</td>
<td>1</td>
<td>2500</td>
<td>268969</td>
<td>0.038</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>250</td>
<td>23360</td>
<td>1.206</td>
</tr>
<tr>
<td>CE-ADIFDTD</td>
<td>1</td>
<td>2500</td>
<td>1098301</td>
<td>0.030</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>250</td>
<td>125815</td>
<td>0.073</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>125</td>
<td>60762</td>
<td>0.4553</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>63</td>
<td>24957</td>
<td>1.615</td>
</tr>
</tbody>
</table>

As can be seen from these two tables, the CE ADI-FDTD method shows better performance in accuracy than the ADI-FDTD method when the same time step is selected. Furthermore,
Table 5.2: Efficiency and accuracy of the CE-ADI-FDTD method for the $TM_1$ mode

<table>
<thead>
<tr>
<th></th>
<th>CFL</th>
<th>Iterations</th>
<th>CPU Time(ms)</th>
<th>Err(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FDTD</td>
<td>1</td>
<td>2500</td>
<td>35380</td>
<td>0.736</td>
</tr>
<tr>
<td>ADI-</td>
<td>1</td>
<td>2500</td>
<td>278344</td>
<td>0.814</td>
</tr>
<tr>
<td>FDTD</td>
<td>10</td>
<td>250</td>
<td>23860</td>
<td>4.041</td>
</tr>
<tr>
<td>CE-</td>
<td>1</td>
<td>2500</td>
<td>1521539</td>
<td>0.323</td>
</tr>
<tr>
<td>ADI-</td>
<td>10</td>
<td>250</td>
<td>150392</td>
<td>1.319</td>
</tr>
<tr>
<td>FDTD</td>
<td>20</td>
<td>125</td>
<td>65689</td>
<td>2.942</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>63</td>
<td>32350</td>
<td>4.615</td>
</tr>
</tbody>
</table>

it can have almost four times larger time step than that of the ADI-FDTD method to achieve a same level of accuracy. This merit is just what we expect from our theoretical analysis. However, this improvement in accuracy is at the cost of sacrificing efficiency. When using the same time step, the CE ADI-FDTD method takes approximately five or six times longer than the ADI-FDTD method due to operations with complex numbers. Consequently, the CE ADI-FDTD method makes trade off between accuracy and efficiency in comparison with the ADI FDTD method in this example. The reason for this can be founded from the theoretical analysis of numerical dispersion. Besides all common factors that affect the numerical dispersion of both the ADI FDTD method and CE ADI-FDTD method, the numerical dispersion of the CE ADI-FDTD method is also a function of the ratio of carrier to envelope. A larger ratio results in more accurate results and the effect is more obvious with small ratios below 150 as Figure 5.6 shows. A larger ratio also slowdowns the deterioration of accuracy caused by larger CFLNs as Figure 5.5 shows. In other words, the CE ADI-FDTD method is effective for narrowband simulation. With a larger ratio or carrier to envelope, a larger CFLN can be chosen without sacrificing much accuracy. Therefore, both efficiency and accuracy can get improvement over the ADI FDTD method. Otherwise, the accuracy will deteriorate very quickly while increasing CFLNs. Considering the ratio in this example is only 8.5, it is not surprising to get such a result. In next example, we will check the performance of the CE ADI-FDTD method in a larger ratio situation.
5.4.2 Metallic Cavity

The carrier frequency in this example is 19 GHz. Therefore the ratio of carrier to bandwidth now is about 35.9. Figure 5.9 shows the signal envelope at the reference point with CFLN=1, 10, 20, 50 and 100. The signal itself solved by the conventional FDTD method is plotted as well.

![Figure 5.9: Envelopes solved with different CFLNs](image)

It can be seen that the envelope matches well with the signal when CFLN=1, and larger CFLN results in a worse match. By recovering signals from envelopes with Equation (5.13) and applying Discrete Fourier Transform (DFT), the corresponding dominant frequency is obtained and shown in Figure 5.10.

Table 5.3 lists errors and simulation time for the conventional FDTD method, ADI-FDTD method and the CE ADI-FDTD method.

It can be seen that with as large as 100 times of CFL, the computing time decreases to 2592 milliseconds while the error is still below 2%. On the contrary, the ADI-FDTD method with CFLN=10 takes as long as 3828 milliseconds and achieves less accurate result (error of 2.45%). On both accuracy and efficiency, the CE ADI-FDTD method shows advantages
Figure 5.10: Dominant frequencies solved with different CFLNs

Table 5.3: Errors versus simulation time for the CE-ADI-FDTD method

<table>
<thead>
<tr>
<th>Method</th>
<th>CFL</th>
<th>Iterations</th>
<th>CPU Time (ms)</th>
<th>Freq.</th>
<th>Err(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FDTD</td>
<td>1</td>
<td>5000</td>
<td>10265</td>
<td>19.375</td>
<td>0.25</td>
</tr>
<tr>
<td>ADI-FDTD</td>
<td>1</td>
<td>5000</td>
<td>40907</td>
<td>19.350</td>
<td>0.376</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>500</td>
<td>3828</td>
<td>18.947</td>
<td>2.451</td>
</tr>
<tr>
<td>CE-ADI-FDTD</td>
<td>1</td>
<td>5000</td>
<td>123421</td>
<td>19.356</td>
<td>0.345</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>500</td>
<td>24581</td>
<td>19.347</td>
<td>0.391</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>250</td>
<td>12153</td>
<td>19.303</td>
<td>0.618</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>100</td>
<td>5161</td>
<td>19.162</td>
<td>1.343</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>50</td>
<td>2592</td>
<td>19.049</td>
<td>1.926</td>
</tr>
</tbody>
</table>

over the ADI-FDTD method. Combining the results obtained from the first example, the improvement on accuracy and efficiency made by the CE ADI-FDTD method is validated on narrowband simulation.

5.5 Conclusions

In this chapter, we have derived the numerical dispersion relationship for the CE ADI-FDTD method and presented a comprehensive analysis of the dispersion properties of the method. There are several conclusions that have been drawn:
i) The anisotropy of the CE ADI-FDTD dispersion is very much like that of the conventional FDTD method; the maximum dispersion errors occur in the axial directions, while the minimum occur at $\phi = 45^\circ, \theta = 45^\circ$.

ii) Within the CFL limit, the dispersion errors of the CE ADI-FDTD method are about 0.2% larger than those of the conventional FDTD method, but are about 0.1% less than those of the ADI-FDTD. In other words, the dispersion errors of the CE ADI-FDTD lie between those of the conventional FDTD and the ADI-FDTD methods.

iii) Beyond the CFL limit, the errors of the CE ADI-FDTD are much smaller than those of the ADI-FDTD for a narrowband simulation. The time step for the CE ADI-FDTD can be 50 or even 100 times larger than that for the ADI-FDTD.

iv) The CE ADI-FDTD is effective only for a narrowband simulation. For a larger bandwidth, the dispersion will be more than 1% which may lead to noticeable computation errors.

v) The requirement for the spatial resolution for the CE ADI-FDTD is almost the same as that for the conventional FDTD method; that is, the number of cells should be greater than 10.

In conclusion, based on our numerical analysis and simulations, the CE ADI-FDTD is found to be an effective method for simulating electromagnetic structures carrying narrowband signals. Also like the ADI-FDTD method, the CE ADI-FDTD is particularly efficient in analysis of electrically fine geometry, since a large time step can be taken uniformly across coarse and dense meshes [76].
Chapter 6

Conclusions and Suggestions for Future Work

6.1 Summary

With the growth of modern communication applications, time-domain (TD) computational techniques are very attractive for solving of electromagnetic (EM) problems as they are broadband techniques. The main purpose of this thesis is to study existing time-domain methods, their advantages and disadvantages and to develop new approaches that can increase computational efficiency.

Background knowledge of computational electromagnetics is reviewed in chapter 1. The primary numerical techniques, including both frequency-domain and time-domain methods, are briefly discussed. The basics of the conventional FDTD method are illustrated in chapter 2. In chapter 3, complex envelope technology is adopted into the conventional FDTD method. It is shown that the numerical dispersion of the CE-FDTD method is not better than that of the conventional FDTD method. Although it is not constrained by the well-known Courant-Friedrich-Levy (CFL) stability condition when the carrier frequency is above a threshold value, the numerical dispersion errors are not acceptable. When the carrier frequency is below the threshold value, a CFL-like condition exists and the numerical dispersion errors are not smaller than those of the conventional FDTD method. Consequently, the uncondition stable scheme, ADI-FDTD, is sought and introduced in chapter 4, and then modified with the introduction of the CE concept in chapter 5. The properties of stability and dispersion of the CE ADI-FDTD method are then analyzed. Finally, examples are given to validate the method.

In conclusion, the CE ADI-FDTD method developed in this thesis is expected to become a very efficient tool in simulating electromagnetic behavior in RF/microwave circuits and devices. However, there is still a wide range of topics yet to be researched. These are
discussed in the following sections.

6.2 Future Work

6.2.1 Absorbing Boundary Condition

As is well-known, for a type of modelling, if a problem domain is an open domain, it has to be truncated with the absorbing boundary condition (ABC). Several approximate ABC schemes has been developed for the FDTD method. Among them the Murs and PML boundary conditions are proved to be either efficient or accurate. The implementation of the ABC into the CE ADI-FDTD method is one of the tasks that needs to be completed.

6.2.2 Broadband Signal

In order to receive the benefits of the CE ADI-FDTD method, the signals used are narrowband in this thesis. Based on the numerical dispersion analyses, the higher the ratio of carrier to envelope frequency, the more accurate the results. How to remove such constraint to include broadband simulation in the CE ADI-FDTD method is another future research topic.

6.2.3 Combination with Other Numerical Methods

Since it was introduced into computational electromangetics by HPEEsol [51], the complex envelope technology until now has been adopted into the time domain finite element method and finite difference time domain method. The FEM is efficient in handling irregular shapes of structures, while the FDTD method requires less computational expenditures. Therefore, hybridizing the two methods may achieve the advantages of both methods. In addition, in the CE ADI-FDTD method, the step size is found to be still bounded by the highest frequency. One possibility of alleviating this constraint is to incorporate the higher order principle into the proposed method. Since the higher order methods allow the number of grid points to be much smaller than the conventional FDTD method, incorporation of both techniques could improve the efficiency and accuracy even further.
Bibliography


Appendix A: Proof of Numerical Stability of the Complex Envelope ADI-FDTD Method

Amplification matrix $\Lambda_1$ for a half time step from the $n$ to the $n + 1/2$:

$$\text{In[1]} := m_1 = \begin{bmatrix} \frac{1 - \alpha^2}{Q_x} W_x W_y, & 0, & 0, & \frac{\alpha (1 + \alpha)}{Q_y} W_x, & -\frac{\alpha (1 - \alpha)}{Q_y} W_y, \\ 0, & \frac{1 - \alpha^2}{Q_x} W_y W_x, & -\frac{\alpha (1 - \alpha)}{Q_x} W_x, & 0, & \frac{\alpha (1 + \alpha)}{Q_x} W_y, \\ \frac{W_x W_y}{Q_x}, & 0, & \frac{\alpha (1 + \alpha)}{Q_x} W_y, & -\frac{\alpha (1 - \alpha)}{Q_x} W_x, & 0, \\ 0, & -\frac{\alpha (1 - \alpha)}{Q_y} W_x, & \frac{\alpha (1 + \alpha)}{Q_y} W_y, & 1 - \alpha^2, & 0, \\ \frac{W_x W_y}{Q_y}, & 0, & 0, & \frac{W_x W_y}{Q_y}, & 1 - \alpha^2 \end{bmatrix}.$$ 

Amplification matrix $\Lambda_1$ for a half time step from the $n + 1/2$ to the $n + 1$:

$$\text{In[2]} := m_2 = \begin{bmatrix} \frac{1 - \alpha^2}{Q_x} W_x W_y, & 0, & 0, & \frac{\alpha (1 + \alpha)}{Q_y} W_x, & -\frac{\alpha (1 - \alpha)}{Q_y} W_y, \\ \frac{W_y W_x}{Q_x}, & 1 - \alpha^2, & 0, & \frac{\alpha (1 - \alpha)}{Q_x} W_x, & 0, \\ W_x W_y, & 0, & \frac{\alpha (1 + \alpha)}{Q_y} W_y, & -\frac{\alpha (1 - \alpha)}{Q_y} W_x, & 0, \\ 0, & -\frac{\alpha (1 + \alpha)}{Q_y} W_x, & \frac{\alpha (1 - \alpha)}{Q_y} W_y, & 1 - \alpha^2, & 0, \\ \frac{W_x W_y}{Q_y}, & 0, & 0, & \frac{W_x W_y}{Q_y}, & 1 - \alpha^2 \end{bmatrix}.$$ 

Amplification matrix $\Lambda$ for a complete time step from the $n$ to the $n + 1$:

$$\text{In[3]} := m = m_1 \cdot m_2 / \text{Together}$$

All elements of $\Lambda$ are related to $\alpha$ and $W_{l_{ax,y}}$ after $Q_{l_{ax,y}}$ are replaced.

$$\text{In[4]} := \text{m.t} = m \cdot \left( Q_x \rightarrow (1 + \alpha)^2 + W_x^2, Q_y \rightarrow (1 + \alpha)^2 + W_y^2, Q_z \rightarrow (1 + \alpha)^2 + W_z^2 \right)$$
where

\[
W = C \frac{\Delta t}{\Delta h}, W_x = W \sin \left( \frac{k_x \Delta h}{2} \right)
\]

In a special case, \( \alpha = 0 \), which happens when the carrier frequency \( \omega \) equals zero.

In[6]:= ms = mt /. (\( \alpha \to 0 \))

The eigenvalues can be solved analytically and are same as these of the ADI FDTD method.

In[7]:= \text{Eigenvalues}[\text{ms}]

Out[7]= \{1, 1, \(2 - 2 \omega_x^2 - 2 \omega_y^2 - 2 \omega_x^2 \omega_y^2 - 2 \omega_x^2 - 2 \omega_y^2\),

\[2 \omega_x^2 \omega_y^2 + 2 \omega_x^2 \omega_y^2 \omega_z^2 - 4 \sqrt{(-\omega_x^2 - \omega_y^2 - \omega_z^2 \omega_x^2 - \omega_z^2 \omega_y^2 - \omega_z^2 \omega_x^2 - \omega_z^2 \omega_y^2 - \omega_z^2 \omega_x^2 - \omega_z^2 \omega_y^2)} \]

\(2 (1 + \omega_x^2 + \omega_y^2 + \omega_z^2 + \omega_x^2 \omega_y^2 + \omega_x^2 \omega_z^2 + \omega_y^2 \omega_z^2 + \omega_x^2 \omega_y^2 \omega_z^2)\),

\(2 - 2 \omega_x^2 - 2 \omega_y^2 - 2 \omega_x^2 \omega_y^2 - 2 \omega_x^2 - 2 \omega_y^2 \omega_z^2 - 2 \omega_x^2 \omega_y^2 + 2 \omega_x^2 \omega_y^2 \omega_z^2 -

4 \sqrt{(-\omega_x^2 - \omega_y^2 - \omega_z^2 \omega_x^2 - \omega_z^2 \omega_y^2 - \omega_z^2 \omega_x^2 - \omega_z^2 \omega_y^2 - \omega_z^2 \omega_x^2 - \omega_z^2 \omega_y^2)} \]

\(2 (1 + \omega_x^2 + \omega_y^2 + \omega_z^2 + \omega_x^2 \omega_y^2 + \omega_x^2 \omega_z^2 + \omega_y^2 \omega_z^2 + \omega_x^2 \omega_y^2 \omega_z^2)\),

\(2 - 2 \omega_x^2 - 2 \omega_y^2 - 2 \omega_x^2 \omega_y^2 - 2 \omega_x^2 - 2 \omega_y^2 \omega_z^2 - 2 \omega_x^2 \omega_y^2 + 2 \omega_x^2 \omega_y^2 \omega_z^2 +

4 \sqrt{(-\omega_x^2 - \omega_y^2 - \omega_z^2 \omega_x^2 - \omega_z^2 \omega_y^2 - \omega_z^2 \omega_x^2 - \omega_z^2 \omega_y^2 - \omega_z^2 \omega_x^2 - \omega_z^2 \omega_y^2)} \]

\(2 (1 + \omega_x^2 + \omega_y^2 + \omega_z^2 + \omega_x^2 \omega_y^2 + \omega_x^2 \omega_z^2 + \omega_y^2 \omega_z^2 + \omega_x^2 \omega_y^2 \omega_z^2)\),

\(2 - 2 \omega_x^2 - 2 \omega_y^2 - 2 \omega_x^2 \omega_y^2 - 2 \omega_x^2 - 2 \omega_y^2 \omega_z^2 - 2 \omega_x^2 \omega_y^2 + 2 \omega_x^2 \omega_y^2 \omega_z^2 +

4 \sqrt{(-\omega_x^2 - \omega_y^2 - \omega_z^2 \omega_x^2 - \omega_z^2 \omega_y^2 - \omega_z^2 \omega_x^2 - \omega_z^2 \omega_y^2 - \omega_z^2 \omega_x^2 - \omega_z^2 \omega_y^2)} \]

\(2 (1 + \omega_x^2 + \omega_y^2 + \omega_z^2 + \omega_x^2 \omega_y^2 + \omega_x^2 \omega_z^2 + \omega_y^2 \omega_z^2 + \omega_x^2 \omega_y^2 \omega_z^2)\))

In other cases, assume a simple uniform cubic mesh, which leads to \( W_x = W_y = W_z = W \).

In[8]:= mp = mt /. (Wx -> W, Wy -> W, Wz -> W)

Solve eigenvalues numerically with different values of \( \alpha \) and \( W \).
All eigenvalues are equal to one. Therefore the unconditional numerical stability of complex envelope ADI-FDTD are proved.