THE BOCHNER INTEGRAL AND AN APPLICATION TO SINGULAR INTEGRALS

by

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Table of Contents

Abstract

In this expository thesis we describe the Bochner integral for functions taking values in a separable Banach space, and we describe how a number of standard definitions and results in real analysis can be extended for these functions, with an emphasis on Hilbert-space-valued functions. We then present a partial vector-valued version of a classical theorem on singular integrals.

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Chapter 1

Introduction

We begin our exposition by describing the Bochner integral for functions landing in a separable Banach space. This integral was first introduced by Salomon Bochner in his 1933 paper *Integration von Functionen* [2]. It is a generalization of the Lebesgue integral. After we have described the Bochner integral, we discuss how it can be used to extend a few basic results in real analysis to the vector-valued setting. We then attempt to extend a result in singular integral theory to this setting.

We begin the first chapter by discussing the space of integrable functions landing in a Banach space. We show that the simple functions are dense in this space, and use this fact to define the Bochner integral. We show how basic measure-theoretic results—such as can be found in Folland [5]—extend easily to the vector-valued setting. Having established these basic facts, we come to our first application of the Bochner integral: a vector-valued version of the Dominated Convergence Theorem. Recall that the Dominated Convergence Theorem for the Lebesgue integral states the following:

Theorem 1.1. Let $\{f_n\}$ be a sequence of integrable functions converging to f almost everywhere. Suppose further that there is a nonnegative integrable function g such that all of the f_n's are bounded almost everywhere by g. Then f is integrable, and $\int f = \lim \int f_n$.

The version that we shall prove for the Bochner integral uses this scalar-valued version, together with the important property of the Bochner integral that $\| \int f \| \leq \int \|f\|$. We conclude the first chapter by showing that linear operators can be pulled through the Bochner integral.

In the second chapter we focus primarily on functions that land in a separable Hilbert space. We discuss the notion of weak measurability for these functions, and how this relates to the usual notion of measurability (which is that preimages of measurable sets are measurable). We also discuss weak measurability of operator-valued functions.

We then turn to the L^p spaces for vector-valued functions, and describe the triangle (Minkowski) and Hölder inequalities in this context. Recall the scalar-valued versions:

Theorem 1.2 (Minkowski's inequality)**.** If $f, g \in L^p(X, \mathbb{C})$, $1 \leq p \leq \infty$, then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Theorem 1.3 (Hölder's inequality)**.** Suppose $1 \leq p \leq \infty$, where p and q are conjugate exponents (that is, $1/p + 1/q = 1$ when $1 < p < \infty$, and $p = 1$ when $q = \infty$). If f and g are measurable functions from X to $\mathbb{C}$, then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

The vector-valued versions are proven using the scalar-valued results. Hölder's inequality will be useful in showing that convolution is well-defined.

Continuing our discussion of vector-valued functions, we prove Fubini's theorem in the context of the Bochner integral and Hilbert-space-valued functions. For scalar-valued functions, Fubini's theorem is as follows:

Theorem 1.4 (The Fubini Theorem)**.** Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, and let $f : X \times Y \to \mathbb{C}$ be in $L^1(X \times Y)$. Then

$$g(x) = \int_Y f(x, y)\, \mathrm{d}y \in L^1(X),$$

$$h(y) = \int_X f(x, y)\, \mathrm{d}x \in L^1(Y),$$

and

$$\int_{X \times Y} f \, \mathrm{d}\mu \times \nu = \int_X g(x)\, \mathrm{d}\mu = \int_Y h(y)\, \mathrm{d}\nu.$$

The proof of the vector-valued version makes use of the notion of weak measurability.

We conclude Chapter 2 by discussing convolution of a Hilbert-space-valued function with an operator-valued kernel, and showing that the resulting function is continuous. We lead up to this with a discussion of the translation- and reflection-invariance of the Bochner integral. We mention in passing that with the Bochner integral, one may define the Fourier transform of a function landing in a separable Hilbert space.

In the latter half of the thesis we move toward trying to prove a vector-valued version of a classical theorem on singular integrals. There are a few important results that are needed for this major theorem; we devote the third chapter of the thesis to these results.

The first of these results is the Calderón-Zygmund lemma. It says that for a nonnegative function f, we can partition $\mathbb{R}^n$ by dyadic cubes into a closed set F on which f is essentially bounded by α, and its complement $F^c = \bigcup Q_k$, and the average value of f on each cube $Q_k \subset F^c$ is bounded below by α and above by $2^n\alpha$. This was first proven in 1952 in Calderón and Zygmund's paper *On the existence of certain singular integrals* [3]. For its proof we follow the one presented in I.3 of Stein [9], filling in certain details.

The second result included for our discussion of the singular integral theorem is a special case of the Marcinkiewicz interpolation theorem. This theorem was discovered by Józef Marcinkiewicz in 1939. The simplified special case we discuss is the one given in I.4 of Stein [9]. We repeat what is presented there in just slightly greater detail, including this primarily for convenient reference. We precede the theorem with some useful terminology.

The third and last of these results is a duality theorem for the L^p space of vector-valued functions. This theorem we take from Grafakos [6]. The analogous scalar-valued result is as follows:

Proposition 1.5. Suppose that p and q are conjugate exponents, and $1 \leq p \leq \infty$. For $f \in L^p(\mathbb{R}^n, \mathbb{C})$ we have

$$\|f\|_p = \sup\left\{ \left| \int fg \right| : g \in L^q(\mathbb{R}^n, \mathbb{C}) \ \text{ with } \ \|g\|_q = 1 \right\}.$$

This is a standard result in real analysis, and a proof can be found in Folland [5], p. 188, for example. In the case of scalar-valued functions, a stronger statement can be made:

Proposition 1.6. Let p and q be conjugate exponents. Suppose that $f : X \to \mathbb{C}$ is a measurable function on a σ-finite measure space, such that fg is integrable for every simple function $g : X \to \mathbb{C}$ supported on a set of finite measure. Suppose further that

$$M_p(f) = \sup \left\{ \left| \int fg \right| : g \text{ is simple with finite support, and } \|g\|_q = 1 \right\}$$

is finite. Then $f \in L^p(X)$ with $\|f\|_p = M_p(f)$.

A proof of this result can also be found in Folland (p. 189). A vector-valued version of Proposition 1.6 would be useful for our purposes, but all I have been able to find is the vector-valued version of Proposition 1.5 presented in Grafakos, and it may well be that the stronger result is unique to the scalar-valued setting.

In the final chapter of this thesis we attempt to give an application of the Bochner integral to the theory of singular integrals. Specifically, we attempt to give a vector-valued version of the following theorem from Stein:

Theorem 1.7 (Theorem from *Singular Integrals* [9], p. 29 and pp. 34–35)**.** Let $K \in L^2(\mathbb{R}^n, \mathbb{C})$. Suppose that

(i) The Fourier transform of K is essentially bounded, by B say.

(ii)

$$\int_{|x| \geq 2|y|} |K(x-y) - K(x)| \, \mathrm{d}x \leq B, \quad |y| > 0.$$

For $f \in L^1(\mathbb{R}^n, \mathbb{C}) \cap L^p(\mathbb{R}^n, \mathbb{C})$, $1 < p < \infty$, set

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y) f(y) \, \mathrm{d}y. \tag{1.1}$$

Then there exists a constant A_p, depending only on B, p, and n, such that

$$\|Tf\|_p \leq A_p \|f\|_p. \tag{1.2}$$

One can thus extend T to all of L^p by continuity.

Equation (1.1) defines T as a convolution operator on a dense subspace of L^p, namely $L^1 \cap L^p$. In our version we assume that T is already known to be some bounded linear operator from $L^2(\mathbb{R}^n, \mathscr{H}_1)$ to $L^2(\mathbb{R}^n, \mathscr{H}_2)$, where $\mathscr{H}_1$ and $\mathscr{H}_2$ are separable Hilbert spaces, and that T is given by convolution on a different dense subset of L^p, namely the bounded, compactly supported functions.

Assumption (i) is used to show that (1.2) holds when $p = 2$. In the vector-valued version that we present, we shall simply assume (1.2) for $p = 2$ from the outset. Assumption (ii) expresses the singularity of K at the origin.

The techniques employed in *Singular Integrals* carry over to the vector-valued setting for the $1 < p < 2$ case. The Calderón-Zygmund lemma and the interpolation theorem given in the preceding chapter are both employed. But the techniques for proving the $2 < p < \infty$ case do not carry over so nicely. For this we introduce some fairly strong hypotheses.

The reader is expected to have some familiarity with real analysis—a reasonable familiarity with the material contained in the first five chapters of Folland [5], for example. A knowledge of the elementary properties of Hilbert and Banach spaces, and of linear operators on these spaces, is also assumed; the contents of the first three chapters of Conway [4] would more than suffice. Throughout this thesis we have attempted to minimize terseness, filling in minor details in arguments taken from Stein [9] and Grafakos [6], with the hope that the reader will find our presentation perspicuous yet engaging.

Chapter 2

The Bochner Integral

2.1 Preliminaries

We begin by introducing the Bochner integral—we follow exercise 16 on p. 156 of Folland [5] for this. In order to define this integral we need a few definitions. In what follows, for any topological space X, $\mathcal{B}_X$ denotes the Borel σ-algebra on X. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\mathscr{Y}$ be a separable Banach space. For convenience, let

$$L_{\mathscr{Y}} = \{f : X \to \mathscr{Y} : f \text{ is } (\mathcal{M}, \mathcal{B}_{\mathscr{Y}})\text{-measurable}\}.$$

Since $y \mapsto \|y\|$ is continuous, it is $(\mathcal{B}_{\mathscr{Y}}, \mathcal{B}_{\mathbb{R}})$-measurable, and hence for any $f \in L_{\mathscr{Y}}$, the composition $x \mapsto \|f(x)\|$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable. We define

$$\|f\|_1 = \int \|f(x)\| \,\mathrm{d}\mu(x) \tag{2.1}$$

and we say that $f \in L_{\mathscr{Y}}$ is *integrable* if the right-hand-side of (2.1) is finite. We also set

$$L^1(X, \mathscr{Y}) = \{f : X \to \mathscr{Y} : f \text{ is integrable}\}.$$

Proposition 2.1. $L_{\mathscr{Y}}$ is a vector space which contains $L^1(X, \mathscr{Y})$ as a subspace. Moreover, $\|\cdot\|_1$ is a seminorm on $L^1(X, \mathscr{Y})$ that becomes a norm if we identify functions that are equal almost everywhere.

Proof. Since $L_{\mathscr{Y}}$ is a subset of the vector space $\mathscr{Y}^X$, to show that it is a vector space it suffices to show that it is closed under addition and scalar multiplication. Let $f, g \in L_{\mathscr{Y}}$ and define $F_1 : X \to \mathscr{Y} \times \mathscr{Y}$ by

$$F_1(x) = (f(x), g(x)),$$

and $\phi : \mathscr{Y} \times \mathscr{Y} \to \mathscr{Y}$ by

$$\phi(y_1, y_2) = y_1 + y_2.$$

So $f+g=\phi\circ F_1$. If π_1 and π_2 are the coordinate maps from $\mathscr{Y}\times\mathscr{Y}$ to $\mathscr{Y}$, then

$$\pi_1\circ F_1=f \quad \text{and} \quad \pi_2\circ F_1=g,$$

so $\pi_1\circ F_1$ and $\pi_2\circ F_1$ are $(\mathcal{M},\mathcal{B}_{\mathscr{Y}})$-measurable. It follows from a result in measure theory (see Folland [5], p. 44) that F_1 is $(\mathcal{M},\mathcal{B}_{\mathscr{Y}}\otimes\mathcal{B}_{\mathscr{Y}})$-measurable, where $\mathcal{B}_{\mathscr{Y}}\otimes\mathcal{B}_{\mathscr{Y}}$ denotes the product σ-algebra on $\mathscr{Y}\times\mathscr{Y}$. Since $\mathscr{Y}$ is separable, we have—by another standard result ([5], p. 23)—that

$$\mathcal{B}_{\mathscr{Y}}\otimes\mathcal{B}_{\mathscr{Y}}=\mathcal{B}_{\mathscr{Y}\times\mathscr{Y}}.$$

Hence, F_1 is $(\mathcal{M},\mathcal{B}_{\mathscr{Y}\times\mathscr{Y}})$-measurable. Since ϕ is continuous, it is $(\mathcal{B}_{\mathscr{Y}\times\mathscr{Y}},\mathcal{B}_{\mathscr{Y}})$-measurable, so we conclude that $f+g=\phi\circ F_1$ is $(\mathcal{M},\mathcal{B}_{\mathscr{Y}})$-measurable. This proves that $L_{\mathscr{Y}}$ is closed under addition.

Now let $f\in L_{\mathscr{Y}}$ and $\alpha\in\mathbb{F}$, where $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. Define $F_2:X\to\mathbb{F}\times\mathscr{Y}$ by

$$F_2(x)=(\alpha,f(x)),$$

and $\psi:\mathbb{F}\times\mathscr{Y}\to\mathscr{Y}$ by

$$\psi(\lambda,y)=\lambda y.$$

Let π_1,π_2 denote the coordinate maps on $\mathbb{F}\times\mathscr{Y}$. Since $\pi_1(x)=\alpha$ for all $x\in X$, we have for any $E\in\mathcal{B}_{\mathbb{F}}$ that

$$(\pi_1\circ F_2)^{-1}(E)=\begin{cases}\emptyset & \text{if } \alpha\notin E\\ X & \text{if } \alpha\in E,\end{cases}$$

and therefore $\pi_1\circ F_2$ is $(\mathcal{M},\mathcal{B}_{\mathbb{F}})$-measurable. Moreover, $\pi_2\circ F_2=f$ is $(\mathcal{M},\mathcal{B}_{\mathscr{Y}})$-measurable. Thus $\pi_1\circ F_2$ and $\pi_2\circ F_2$ are both measurable, and therefore F_2 is $(\mathcal{M},\mathcal{B}_{\mathbb{F}}\otimes\mathcal{B}_{\mathscr{Y}})$-measurable. Since $\mathscr{Y}$ and $\mathbb{F}$ are separable,

$$\mathcal{B}_{\mathbb{F}}\otimes\mathcal{B}_{\mathscr{Y}}=\mathcal{B}_{\mathbb{F}\times\mathscr{Y}},$$

and hence F_2 is $(\mathcal{M},\mathcal{B}_{\mathbb{F}\times\mathscr{Y}})$-measurable. Since ψ is $(\mathcal{B}_{\mathbb{F}\times\mathscr{Y}},\mathcal{B}_{\mathscr{Y}})$-measurable (by virtue of its being continuous), we conclude that $\alpha f=\psi\circ F_2$ is $(\mathcal{M},\mathcal{B}_{\mathscr{Y}})$-measurable. Thus we have shown that $L_{\mathscr{Y}}$ is closed under scalar multiplication.

We now show that $L^1(X,\mathscr{Y})$ is a subspace of $L_{\mathscr{Y}}$, and that $\|\cdot\|_1$ is a seminorm on $L^1(X,\mathscr{Y})$ that becomes a norm if we identify functions in $L_{\mathscr{Y}}$ that are equal almost everywhere. Let $f, g \in L^1(X, \mathscr{Y})$ and $\alpha \in \mathbb{F}$. Then since $\|f(x) + g(x)\| \leq \|f(x)\| + \|g(x)\|$ for each x, we have

$$\|f + g\|_1 = \int_X \|f(x) + g(x)\| \,\mathrm{d}x \leq \int_X \|f(x)\| \,\mathrm{d}x + \int_X \|g(x)\| \,\mathrm{d}x = \|f\|_1 + \|g\|_1.$$

Also, $\|(\alpha f)(x)\| = \|\alpha f(x)\| = |\alpha|\|f(x)\|$, so

$$\|\alpha f\|_1 = \int_X \|(\alpha f)(x)\| \,\mathrm{d}x = |\alpha| \int_X \|f(x)\| \,\mathrm{d}x = |\alpha|\|f\|_1.$$

This shows that $\|\cdot\|$ is a seminorm on $L^1(X, \mathscr{Y})$ and that $L^1(X, \mathscr{Y})$ is a subspace of $L_{\mathscr{Y}}$. If $\|f\|_1 = \int_X \|f\| \,\mathrm{d}x = 0$, then since $\|f\|$ is nonnegative and measurable, $\|f\| = 0$ a.e., and hence $f = 0$ a.e. Thus, if we identify functions on $L_{\mathscr{Y}}$ that are equal a.e., $\|\cdot\|_1$ becomes a norm on $L^1(X, \mathscr{Y})$. □

In analogy with the usual notion of scalar-valued simple functions, we now define simple functions more generally to be maps $\phi : X \to \mathscr{Y}$ of the form

$$\phi(x) = \sum_{j=1}^{m} \chi_{E_j}(x) y_j,$$

where $m \in \mathbb{N}, y_j \in \mathscr{Y}, E_j \in \mathcal{M}$, and $\mu(E_j) < \infty$. For convenience we let $F_{\mathscr{Y}}$ denote the set of simple functions. Any simple function ϕ can be written in the form $\sum_{i=1}^{m} y_i \chi_{\phi^{-1}(y_i)}$, where $\{y_1, \ldots, y_m\}$ are all the nonzero elements in the range of ϕ. We call this the *standard representation* of ϕ. The standard representation gives a unique way of writing ϕ as a finite linear combination of characteristic functions of disjoint sets, with one characteristic function for each nonzero element in the range of ϕ.

Proposition 2.2. $F_{\mathscr{Y}}$ is a subspace of $L^1(X, \mathscr{Y})$.

Proof. We start by showing that $F_{\mathscr{Y}} \subset L_{\mathscr{Y}}$. Let $\phi = \chi_E y$, where $y \in \mathscr{Y}$ and $E \in \mathcal{M}$. Elements of $F_{\mathscr{Y}}$ are finite sums of functions of this form, and we have already shown that the set of $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable functions is closed under addition (of a finite number of summands), so to show that $F_{\mathscr{Y}} \subset L_{\mathscr{Y}}$ it suffices to show that ϕ is $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable. For any $F \in \mathcal{B}_{\mathscr{Y}}$ we have

$$\phi^{-1}(F) = \begin{cases} X & \text{if } y \in F \text{ and } 0 \in F \\ E & \text{if } y \in F \text{ and } 0 \notin F \\ E^c & \text{if } y \notin F \text{ and } 0 \in F \\ \emptyset & \text{if } y \notin F \text{ and } 0 \notin F. \end{cases}$$

Therefore, $\phi^{-1}(F) \in \mathcal{M}$ for each $F \in \mathcal{B}_{\mathscr{Y}}$, so ϕ is $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable, and hence $F_{\mathscr{Y}} \subset L_{\mathscr{Y}}$. Moreover, $F_{\mathscr{Y}}$ is clearly closed under addition and scalar multiplication, so $F_{\mathscr{Y}}$ is a subspace of $L_{\mathscr{Y}}$.

To see that $F_{\mathscr{Y}} \subset L^1(X, \mathscr{Y})$, let $\phi = \sum_{j=1}^m \chi_{E_j} y_j \in F_{\mathscr{Y}}$. Then $\|\phi(x)\| = \sum_{j=1}^m \|y_j\| \chi_{E_j}(x)$, and hence

$$\|\phi\|_1 = \int_X \sum_{j=1}^m \|y_j\| \chi_{E_j}(x)\, \mathrm{d}x = \sum_{j=1}^m \|y_j\| \mu(E_j).$$

Thus $\phi \in L^1(X, \mathscr{Y})$, and since $\phi \in F_{\mathscr{Y}}$ was arbitrary, this shows that $F_{\mathscr{Y}} \subset L^1(X, \mathscr{Y})$.

□

Lemma 2.3. For any normed space X, and $j \in \mathbb{N}$ with $j > 1$, if $\|x - y\| \leq \frac{1}{j}\|y\|$, then $\|y\| \leq \frac{j}{j-1}\|x\|$ (and hence $\|x - y\| \leq \frac{1}{j-1}\|x\|$).

Proof. Since $\|x - y\| \leq \frac{1}{j}\|y\|$, we have $\|y\| - \|x\| \leq \frac{1}{j}\|y\|$, which implies $(1 - \frac{1}{j})\|y\| \leq \|x\|$. This can be expressed as $\|y\| \leq \frac{j}{j-1}\|x\|$. □

Lemma 2.4. Let $\{y_n\}_1^\infty$ be a countable dense set in $\mathscr{Y}$. For each $j \in \mathbb{N}$, let $B_{n,j} = \{y \in \mathscr{Y} : \|y - y_n\| < \frac{1}{j}\|y_n\|\}$. Then for each j, $\bigcup_{n=1}^\infty B_{n,j} \supset \mathscr{Y} \setminus \{0\}$.

Proof. Let $y \neq 0$, $j \in \mathbb{N}$. Since $\{y_n\}_1^\infty$ is dense in $\mathscr{Y}$, we have

$$\|y - y_n\| \leq \frac{1}{j+1}\|y\|$$

for some n. By Lemma 2.3, this implies

$$\|y - y_n\| \leq \frac{1}{j}\|y_n\|,$$

which means that $y \in B_{n,j}$. □

Theorem 2.5. If $f \in L^1(X, \mathscr{Y})$, there is a sequence $\{\phi_n\} \subset F_{\mathscr{Y}}$ which converges to f in $L^1(X, \mathscr{Y})$ and a.e..

Proof. With the notation in Lemma 2.4, let $A_{n,j} = B_{n,j} \setminus \bigcup_{m=1}^{n-1} B_{m,j}$ and $E_{n,j} = f^{-1}(A_{n,j})$, and let $g_j = \sum_{n=1}^\infty y_n \chi_{E_{n,j}}$. Note that the $E_{n,j}$'s are disjoint, since the $A_{n,j}$'s are. Moreover, it follows from Lemma 2.4 that

$$X = \bigcup_{n=1}^\infty E_{n,j} \cup f^{-1}\{0\}, \tag{2.2}$$

for if $f(x) \neq 0$ then $f(x) \in \bigcup_{n=1}^{\infty} B_{n,j} = \bigcup_{n=1}^{\infty} A_{n,j}$, whence $x \in E_{n,j}$ for some n. The union in (2.2) is disjoint, because $0 \notin \bigcup_{n=1}^{\infty} B_{n,j} = \bigcup_{n=1}^{\infty} A_{n,j}$, and therefore if $f(x) = 0$ then $x \notin E_{n,j}$ for any n. Lastly, we note that since the $B_{n,j}$ sets are open balls in $\mathscr{Y}$, they are in $\mathcal{B}_{\mathscr{Y}}$, and hence the $A_{n,j}$ sets are in $\mathcal{B}_{\mathscr{Y}}$ as well, and since f is $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable this implies that the $E_{n,j}$'s are in $\mathcal{M}$.

Now, if $x \in E_{n,j}$ and $j > 1$, we have

$$\|f(x) - y_n\| < \frac{1}{j}\|y_n\|, \tag{2.3}$$

and hence by Lemma 2.3,

$$\|y_n\| < \frac{j}{j-1}\|f(x)\|. \tag{2.4}$$

Since

$$g_j(x) = \begin{cases} y_n & \text{if } x \in E_{n,j} \\ 0 & \text{if } x \in f^{-1}\{0\}, \end{cases}$$

we see that $\|g_j(x)\| \leq \frac{j}{j-1}\|f(x)\|$ for all $x \in X$, and hence $\|g_j\|_1 \leq \frac{j}{j-1}\|f\|_1$. Also, (2.3) and (2.4) give us $\|f(x) - y_n\| < \frac{1}{j-1}\|f(x)\|$ for $x \in E_{n,j}$, which implies that

$$\|f(x) - g_j(x)\| \leq \frac{1}{j-1}\|f(x)\|$$

for all $x \in X$. This in turn gives us

$$\|f - g_j\|_1 \leq \frac{1}{j-1}\|f\|_1. \tag{2.5}$$

We now observe that $\|g_j(x)\| = \sum_{n=1}^{\infty} \|y_n\|\chi_{E_{n,j}}(x)$, and hence

$$\|g_j\|_1 = \int_X \|g_j(x)\| \, \mathrm{d}x = \int_X \sum_{n=1}^{\infty} \|y_n\|\chi_{E_{n,j}}(x) \, \mathrm{d}x = \sum_{n=1}^{\infty} \int_X \|y_n\|\chi_{E_{n,j}}(x) \, \mathrm{d}x$$

by the Monotone Convergence Theorem. Therefore,

$$\|g_j\|_1 = \sum_{n=1}^{\infty} \|y_n\|\mu(E_{n,j}), \tag{2.6}$$

which in particular shows that $\mu(E_{n,j}) < \infty$ for each n, j. Now since $g_j \in L^1(X, \mathscr{Y})$ for $j > 1$, we can truncate the sum in (2.6) to make its tail as small as we like. In other words, for $j > 1$ there exists N_j such that

$$\sum_{n=N_j+1}^{\infty} \|y_n\|\mu(E_{n,j}) < \frac{1}{j}.$$

Now let $N_1 = 1$ and define $\phi_j = \sum_{n=1}^{N_j} y_n \chi_{E_{n,j}}$. Notice that

$$g_j - \phi_j = \sum_{n=N_j+1}^{\infty} y_n \chi_{E_{n,j}},$$

and hence

$$\|g_j - \phi_j\|_1 = \sum_{n=N_j+1}^{\infty} \|y_n\| \mu(E_{n,j}) < \frac{1}{j} \tag{2.7}$$

when $j > 1$. Now since

$$\|f - \phi_j\|_1 \leq \|f - g_j\|_1 + \|g_j - \phi_j\|_1 < \frac{1}{j-1}\|f\|_1 + \frac{1}{j}$$

when $j > 1$, we see that $\|f - \phi_j\|_1 \to 0$ as $j \to \infty$. Seen another way, this says that the nonnegative functions $\|f(x) - \phi_j(x)\|$ converge to 0 in $L^1(X, \mathbb{R})$. This implies, by a fact from real analysis (see Bartle [1], pp. 69–70, for example), that there is a subsequence $\|f(x) - \psi_j(x)\|$ which converges to 0 in $L^1(X, \mathbb{R})$ and almost everywhere. In other words, $\psi_j \to f$ in $L^1(X, \mathscr{Y})$ and a.e.. $\square$

2.2 The Integral

Theorem 2.6. There is a unique linear map $\int : L^1(X, \mathscr{Y}) \to \mathscr{Y}$ such that:

(i) $\int y\chi_E = \mu(E)y$ for $y \in \mathscr{Y}$ and $E \in \mathcal{M}$ (with $\mu(E) < \infty$), and

(ii) $\|\int f\| \leq \|f\|_1$.

Proof. We first define $\int$ on $F_{\mathscr{Y}}$ by $\int \sum_{i=1}^{n} y_i \chi_{E_i} = \sum_{i=1}^{n} y_i \mu(E_i)$. This is clearly linear on $F_{\mathscr{Y}}$. It is also well-defined, i.e., independant of representation. The proof of this fact is identical to the proof for scalar-valued simple functions, so we shall omit it, referring the reader to pp. 51–52 of Stein & Shakarchi [10], for example. Therefore, in what follows we will work exclusively with standard representations of simple functions.

Given $\phi = \sum_{j=1}^{n} y_j \chi_{E_j} \in F_{\mathscr{Y}}$, we have

$$\|\phi(x)\| = \left\| \sum_{j=1}^{n} y_j \chi_{E_j}(x) \right\| = \sum_{j=1}^{n} \|y_j\| \chi_{E_j}(x)$$

for each $x \in X$, and hence

$$\|\phi\|_1 = \int_X \|\phi\| \, \mathrm{d}x = \sum_{j=1}^{n} \|y_j\|\mu(E_j).$$

Thus,

$$\Big\| \int_X \phi \, \mathrm{d}x \Big\| \leq \sum_{j=1}^{n} \|y_j\|\mu(E_j) = \|\phi\|_1. \tag{2.8}$$

Now suppose we are given $f \in L^1(X, \mathscr{Y})$. By Theorem 2.5, there is a sequence $\{\phi_n\} \subset F_{\mathscr{Y}}$ such that $\|\phi_n - f\|_1 \to 0$. Thus $\{\phi_n\}$ is Cauchy in $L^1(X, \mathscr{Y})$, and hence

$$\Big\| \int \phi_n - \int \phi_m \Big\| = \Big\| \int (\phi_n - \phi_m) \Big\| \leq \|\phi_n - \phi_m\|_1 \to 0$$

as $m, n \to \infty$. Thus $\{\int \phi_n\}$ is Cauchy in $\mathscr{Y}$, and since $\mathscr{Y}$ is complete this means that $\{\int \phi_n\}$ converges. Define $\int f$ to be its limit. Then $\int$ is well-defined on $L^1(X, \mathscr{Y})$, for suppose that $\{\psi_n\}$ is another sequence in $F_{\mathscr{Y}}$ such that $\psi_n \to f$, and let $\varepsilon > 0$. There exists an N such that $n \geq N$ implies

(i) $\| \int \phi_n - \int f\| < \frac{\varepsilon}{2}$,

(ii) $\|\phi_n - f\|_1 < \frac{\varepsilon}{4}$,

(iii) $\|\psi_n - f\|_1 < \frac{\varepsilon}{4}$.

Thus for $n \geq N$,

$$\begin{aligned} \Big\| \int \psi_n - \int f \Big\| &\leq \Big\| \int \psi_n - \int \phi_n \Big\| + \Big\| \int \phi_n - \int f \Big\| \\ &\leq \|\psi_n - \phi_n\|_1 + \Big\| \int \phi_n - \int f \Big\| \end{aligned}$$

by (2.8). Using the triangle inequality again,

$$\begin{aligned} \Big\| \int \psi_n - \int f \Big\| &\leq \|\psi_n - f\|_1 + \|f - \phi_n\|_1 + \Big\| \int \phi_n - \int f \Big\| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2}. \end{aligned}$$

Thus $\int \psi_n \to \int f$ in $\mathscr{Y}$, so the function $\int$ is well-defined on $L^1(X, \mathscr{Y})$. It is easy to see that $\int$ is linear on $L^1(X, \mathscr{Y})$, so it remains only to show that $\| \int f\| \leq \|f\|_1$, and to verify the uniqueness statement. Given $f \in L^1(X, \mathscr{Y})$, $f = \lim \phi_n$ for a sequence $\{\phi_n\} \subset F_{\mathscr{Y}}$, so

$$\left\| \int f \right\| = \left\| \lim \int \phi_n \right\| = \lim \left\| \int \phi_n \right\| \leq \lim \|\phi_n\|_1 = \|f\|_1.$$

Thus $\| \int f\| \leq \|f\|_1$ for all $f \in L^1(X, \mathscr{Y})$.

Now, to verify the uniqueness statement, suppose that $\Psi : L^1(X, \mathscr{Y}) \to \mathscr{Y}$ is another linear map such that

(i) $\Psi(y\chi_E) = y\mu(E)$ for $y \in \mathscr{Y}, E \in \mathcal{M}$.

(ii) $\|\Psi f\| \leq \|f\|_1$ for all $f \in L^1(X, \mathscr{Y})$.

Let $f \in L^1(X, \mathscr{Y})$; then $f = \lim \phi_n$ for a sequence $\{\phi_n\} \subset F_{\mathscr{Y}}$, and since Ψ is continuous by (ii),

$$\Psi f = \Psi(\lim \phi_n) = \lim \Psi \phi_n.$$

Now for any $\phi = \sum_{i=1}^n y_i \chi_{E_i} \in F_{\mathscr{Y}}$ we have

$$\Psi\phi = \Psi\Big(\sum_{i=1}^n y_i \chi_{E_i}\Big) = \sum_{i=1}^n \Psi(y_i\chi_{E_i}) = \sum_{i=1}^n y_i \mu(E_i),$$

by (i). Then by the definition of $\int$, this is equal to

$$\int \sum_{i=1}^n y_i \chi_{E_i} = \int \phi,$$

which shows that $\int$ and Ψ agree on simple functions. Therefore, $\Psi f = \lim \Psi\phi_n = \lim \int \phi_n = \int f$, and we conclude that $\Psi = \int$ on $L^1(X, \mathscr{Y})$, proving the uniqueness of the function $\int$. □

The function $\int : L^1(X, \mathscr{Y}) \to \mathscr{Y}$ defined in Theorem 2.6 is called the *Bochner integral*. We will prove a version of the Dominated Convergence Theorem for $L^1(X, \mathscr{Y})$ functions, but we first verify that certain standard measure-theoretic results still hold for these functions.

Theorem 2.7. If f_n converges pointwise to f, and each $f_n \in L_{\mathscr{Y}}$, then $f \in L_{\mathscr{Y}}$.

Proof. Since $\mathscr{Y}$ is a separable metric space, a countable collection of open balls is a base for its topology, and it follows that $\mathcal{B}_{\mathscr{Y}}$ is generated by open balls. Thus, it suffices to show that the preimage of each open ball in $\mathscr{Y}$ under f is in $\mathcal{M}$. To do this, we first note that constant functions are in $L_{\mathscr{Y}}$, for if $F(x) = y$ for all $x \in X$, then for each $E \in \mathcal{B}_{\mathscr{Y}}$,

$$F^{-1}(E) = \begin{cases} X & \text{if } y \in E \\ \emptyset & \text{if } y \notin E, \end{cases}$$

and thus F is $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable. We now recall from Proposition 2.1 that $L_{\mathscr{Y}}$ is a vector space, and hence $f_n(x) - y \in L_{\mathscr{Y}}$ for each n. Since the norm $\|\cdot\|$ is continuous, $\|f_n(x) - y\|$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable for each n, and $\|f_n(x) - y\|$ converges to $\|f(x) - y\|$ for each x. By a standard result (see Bartle [1], p. 12), we have that $\|f(x) - y\|$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable. Thus, if we define g_y by $g_y(x) = \|f(x) - y\|$, then $g_y^{-1}\big(B(0,\delta)\big) \in \mathcal{M}$ for each $y \in \mathscr{Y}$ and $\delta > 0$. But

$$g_y^{-1}\big(B(0,\delta)\big) = \{x : \|f(x) - y\| < \delta\} = f^{-1}\big(B(y,\delta)\big).$$

We have thus shown that the preimage of any open ball in $\mathscr{Y}$ is in $\mathcal{M}$, as desired. □

Corollary 2.8. Suppose that μ is a complete measure on $\mathcal{M}$. We have the following:

(i) if f is $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable and $f = g$ μ-a.e., then g is $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable.

(ii) if f_n is $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable for each n and $f_n \to f$ μ-a.e., then f is $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable.

Proof. Suppose that $f = g$ on N^c for some null set N. Then for each $E \in \mathcal{B}_{\mathscr{Y}}$,

$$g^{-1}(E) \cap N^c = f^{-1}(E) \cap N^c \in \mathcal{M}.$$

Hence,

$$\begin{aligned} g^{-1}(E) &= g^{-1}(E) \cap (N \cup N^c) \\ &= \underbrace{\left(g^{-1}(E) \cap N\right)}_{\text{in } \mathcal{M} \text{ since } \mu \text{ is complete}} \cup \left(g^{-1}(E) \cap N^c\right). \end{aligned}$$

Thus g is $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable. This proves (i).

Suppose now that $\{f_n\}$ is a sequence of measurable functions which converges to f on N^c, where N is a null set. We define functions F_n and F by

$$F_n(x) = \chi_{X\setminus N}(x) f_n(x) \quad \text{and} \quad F(x) = \chi_{X\setminus N}(x) f(x).$$

Now F_n is measurable for each n, because for any $E \in \mathcal{B}_{\mathscr{Y}}$,

$$F_n^{-1}(E) = \begin{cases} f_n^{-1}(E) \cup N & \text{if } 0 \in E \\ f_n^{-1}(E) & \text{if } 0 \notin E. \end{cases}$$

Moreover, it is clear that $F_n(x) \to F(x)$ for each $x \in X$, so by Theorem 2.7, F is measurable. Since $f = F$ a.e., (i) gives us that f is measurable. □

Proposition 2.9. If μ is a measure on a measurable space $(X, \mathcal{M})$, and $\overline{\mu}$ is its completion, then a statement P about functions on X is true μ-a.e. if and only if it is true $\overline{\mu}$-a.e..

Proof. Suppose first that P is true $\overline{\mu}$-a.e.. This means that P is true on N^c for a null set $N \in \overline{\mathcal{M}}$. By definition, $N = E \cup F$, where $E \in \mathcal{M}$ and $F \subset N_0$ for some $N_0 \in \mathcal{M}$ such that $\mu(N_0) = 0$. Moreover, $\mu(E) \equiv \overline{\mu}(N) = 0$. Now since $F \subset N_0$, we have $N_0^c \subset F^c$, which implies $E^c \cap N_0^c \subset E^c \cap F^c$; in other words, $(E \cup N_0)^c \subset N^c$. Thus P is true on $(E \cup N_0)^c$, where $E \cup N_0 \subset \mathcal{M}$, and $\mu(E \cup N_0) \leq \mu(E) + \mu(N_0) = 0$; i.e., P is true μ-a.e.. Conversely, if P is true μ-a.e., then P is true on N^c, where $N \in \mathcal{M} \subset \overline{\mathcal{M}}$, which means that P is true $\overline{\mu}$-a.e.. □

In light of this proposition, we may, in speaking about a non-complete measure μ, say that something holds a.e. without specifying μ-a.e. or $\overline{\mu}$-a.e..

Proposition 2.10. Given a measure space $(X, \mathcal{M}, \mu)$ with completion $(X, \overline{\mathcal{M}}, \overline{\mu})$, and an $(\overline{\mathcal{M}}, \mathcal{B}_{\mathscr{Y}})$-measurable function f, there is an $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable function g which is equal to f a.e..

Proof. Clearly, χ_E is $(\overline{\mathcal{M}}, \mathcal{B}_{\mathscr{Y}})$-measurable for each $E \in \overline{\mathcal{M}}$. By definition, any such E can be written as $F \cup G$, where $F \in \mathcal{M}$ and $\overline{\mu}(G) = 0$. Thus $\chi_E = \chi_F$ a.e., and χ_F is $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable. It follows that the result holds for $(\overline{\mathcal{M}}, \mathcal{B}_{\mathscr{Y}})$-measurable simple functions. By Theorem 2.5, there is a sequence $\{\phi_n\}$ of $(\overline{\mathcal{M}}, \mathcal{B}_{\mathscr{Y}})$-measurable simple functions that converges a.e. to f; in other words, $\phi_n(x) \to f(x)$ for each $x \in N^c$, where $N \in \mathcal{M}$. For each ϕ_n, let ψ_n be a $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable simple function which is equal to ϕ_n on E_n^c, where $E_n \in \mathcal{M}$ and $\mu(E_n) = 0$. Letting

$$N' = N \cup \bigcup_{n=1}^{\infty} E_n \quad \text{and} \quad g(x) = \lim_{n\to\infty} \chi_{X\setminus N'}(x)\psi_n(x),$$

we have

$$N' \in \mathcal{M} \quad \text{with} \quad \mu(N') = 0,$$

and

$$g = f\chi_{X\setminus N'}.$$

The functions $\chi_{X\setminus N'}(x)\psi_n(x)$ are $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable, and therefore by Theorem 2.7, g is $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable. Since $g = f$ a.e., we are done. □

Theorem 2.11 (The Dominated Convergence Theorem)**.** If $\{f_n\}$ is a sequence in $L^1(X, \mathscr{Y})$ such that $f_n \to f$ a.e., and there exists $g \in L_1(X, \mathbb{R})$ such that $\|f_n(x)\| \leq g(x)$ for all n and almost every x, then there exists a function f^* in $L^1(X, \mathscr{Y})$ such that $\int f_n \to \int f^*$ and $f = f^*$ a.e..

Proof. We treat the general case where μ may not be complete. The f_n's are $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable, and $f_n \to f$ a.e.. Since $\mathcal{M} \subset \overline{\mathcal{M}}$, the f_n's are $(\overline{\mathcal{M}}, \mathcal{B}_{\mathscr{Y}})$-measurable. Corollary 2.8 yields that f is $(\overline{\mathcal{M}}, \mathcal{B}_{\mathscr{Y}})$-measurable. By Proposition 2.10 there exists an $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable function f^* which is equal to f a.e.. If μ were complete to begin with then f would itself be $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable, and this step would be unnecessary.

Now for each n, there exists a null set $E_n \in \mathcal{M}$ such that $\|f_n\| \leq g$ on E_n^c. Let $N_0 = \bigcup_{n=1}^{\infty} E_n$. Since $f_n \to f$ a.e. and $f = f^*$ a.e., it follows that $f_n \to f^*$ on N_1^c for some null set $N_1 \in \mathcal{M}$. Let $N = N_0 \cup N_1$; then

$$\mu(N) = 0, \qquad f_n \to f^* \text{ on } N^c, \qquad \text{and} \qquad \|f_n\| \leq g \text{ on } N^c \text{ for all } n.$$

It follows that $\|f^*\| \leq g$ a.e., and therefore f^* is in $L^1(X, \mathscr{Y})$. Now since

$$\Big\| \int f_n - \int f^* \Big\| = \Big\| \int (f_n - f^*) \Big\| \leq \|f_n - f^*\|_1,$$

it suffices to show that $\|f_n - f^*\|_1 \to 0$. We know that

$$\|f_n(x) - f^*(x)\| \to 0 \text{ for a.e. } x,$$

and

$$\|f_n(x) - f^*(x)\| \le \|f_n(x)\| + \|f^*(x)\| \le 2g(x) \quad \text{for a.e. } x \text{ and for all } n.$$

Therefore, by the scalar-valued Dominated Convergence Theorem (see Folland [5], p. 54, for example),

$$\int \|f_n - f^*\| \, \mathrm{d}x \to \int 0 \, \mathrm{d}x = 0.$$

In other words, $\|f_n - f^*\|_1 \to 0$. □

For any normed vector spaces X and Y, we let $\mathscr{B}(X, Y)$ denote the space of bounded linear transformations from X to Y.

Theorem 2.12. If $\mathscr{Z}$ is a separable Banach space, $T \in \mathscr{B}(\mathscr{Y}, \mathscr{Z})$, and $f \in L^1(X, \mathscr{Y})$, then $T \circ f \in L^1(X, \mathscr{Z})$ and $\int T \circ f = T(\int f)$.

Proof. T is continuous, so it is $(\mathcal{B}_{\mathscr{Y}}, \mathcal{B}_{\mathscr{Z}})$-measurable, and since f is $(\mathcal{M}, \mathcal{B}_{\mathscr{Y}})$-measurable we have that $T \circ f$ is $(\mathcal{M}, \mathcal{B}_{\mathscr{Z}})$-measurable. Moreover, since T is bounded, we have $\|(T \circ f)(x)\| = \|T(f(x))\| \le \|T\| \|f(x)\|$ for each $x \in X$. Therefore,

$$\|T \circ f\|_1 = \int \|(T \circ f)(x)\| \, \mathrm{d}x \le \|T\| \int \|f(x)\| \, \mathrm{d}x = \|T\| \|f\|_1, \tag{2.9}$$

so that $T \circ f \in L^1(X, \mathscr{Z})$.

Now observe that for any $\phi = \sum_{i=1}^n y_i \chi_{E_i}$, we have $T \circ \phi = \sum_{i=1}^n (Ty_i) \chi_{E_i} \in F_{\mathscr{Z}}$, and hence

$$\int T \circ \phi = \sum_{i=1}^n \mu(E_i) T y_i = T\Big(\sum_{i=1}^n \mu(E_i) y_i\Big) = T\Big(\int \phi\Big).$$

If $\{\phi_n\}$ is a sequence in $F_{\mathscr{Y}}$ such that $\|f - \phi_n\|_1 \to 0$, we see—using (2.9)—that

$$\|T \circ f - T \circ \phi_n\|_1 = \|T(f - \phi_n)\|_1 \le \|T\| \|f - \phi_n\|_1 \to 0.$$

Thus, $\{T \circ \phi_n\}$ is a sequence in $F_{\mathscr{Z}}$ such that $\|T \circ f - T \circ \phi_n\|_1 \to 0$, and therefore

$$\int T \circ f = \lim \int T \circ \phi_n = \lim T\Big(\int \phi_n\Big) = T\Big(\lim \int \phi_n\Big) = T\Big(\int f\Big).$$

□

A special variant of this theorem occurs when $T = \xi \in \mathscr{Y}^*$. Then we have

$$\xi\Big(\int_X f(x)\,\mathrm{d}x\Big) = \int_X \xi(f(x))\,\mathrm{d}x,$$

where the integral on the left is the Bochner integral, but the integral on the right is a Lebesgue integral. The proof is identical, except that after we've established $\|\xi(f) - \xi(\phi_n)\|_1 \to 0$, we note that in particular this means that $\int \xi(\phi_n) \to \int \xi(f)$, where these are Lebesgue integrals.

We will make use of this variant in Section 3.3, and refer to it simply as Theorem 2.12.

Chapter 3

The Vector-Valued Aspect

3.1 Weak measurability

Now that we have introduced the Bochner integral, we show how it can be used to extend some of the standard theory to the context of vector-valued functions. Let $\mathscr{H}$ be a separable Hilbert space. Given a measure space $(X, \mathcal{M}, \mu)$, a function $f : X \to \mathscr{H}$ is called *weakly measurable* if for each $\phi \in \mathscr{H}$, the map $x \mapsto \langle f(x), \phi \rangle$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$-measurable. Note that this is different from the definition of measurability we saw before in the more general Banach space setting, where a function was measurable if it was $(\mathcal{M}, \mathcal{B}_{\mathscr{H}})$-measurable. With the codomain of f being a separable Hilbert space, the two definitions are equivalent, as we will now show.

Lemma 3.1. If $f : X \to \mathscr{H}$ is weakly measurable then $x \mapsto \|f(x)\|$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable.

Proof. Since $\mathscr{H}$ is separable, $\dim \mathscr{H} = \aleph_0$. Let $\{e_i\}_{i=1}^{\infty}$ be a basis for $\mathscr{H}$; by Parseval's identity,

$$\|f(x)\|^2 = \sum_{i=1}^{\infty} |\langle f(x), e_i \rangle|^2.$$

Since weak measurability means that $x \mapsto \langle f(x), e_i \rangle$ is measurable for each i, it follows that $x \mapsto \sum_{i=1}^{n} |\langle f(x), e_i \rangle|^2$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable for each n. Since the pointwise limit of a sequence of measurable functions is measurable, we have that $x \mapsto \|f(x)\|^2$ is measurable, and therefore that $x \mapsto \|f(x)\|$ is measurable. $\square$

Lemma 3.2. Constant functions are weakly measurable. Moreover, if $f : X \to \mathscr{H}$ and $g : X \to \mathscr{H}$ are weakly measurable, and $c \in \mathbb{C}$, then cf and $f + g$ are weakly measurable.

Proof. Given $\psi \in \mathscr{H}$, $x \mapsto \langle \psi, \phi \rangle$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$-measurable for each $\phi \in \mathscr{H}$, since constant scalar-valued functions are measurable. Thus constant functions are weakly measurable. If f is weakly measurable and $c \in \mathbb{C}$, then the weak measurability of cf follows from the analogous property for scalar-valued measurable functions, since $\langle cf(x), \phi \rangle = c\langle f(x), \phi \rangle$. If f and g are weakly measurable, then the weak measurability of $f + g$ again follows from the analogous property for scalar-valued functions, since $\langle f(x) + g(x), \phi \rangle = \langle f(x), \phi \rangle + \langle g(x), \phi \rangle$. □

Proposition 3.3. The map $f : X \to \mathscr{H}$ is $(\mathcal{M}, \mathcal{B}_{\mathscr{H}})$-measurable if and only if it is weakly measurable.

Proof. Suppose that f is $(\mathcal{M}, \mathcal{B}_{\mathscr{H}})$-measurable. Since the inner-product $\langle \cdot, \cdot \rangle : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$ is continuous, its sections $\langle \cdot, \phi \rangle$ are continuous, and are therefore $(\mathcal{B}_{\mathscr{H}}, \mathcal{B}_{\mathbb{C}})$-measurable. It follows that the composition $x \mapsto \langle f(x), \phi \rangle$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$-measurable for each $\phi \in \mathscr{H}$.

For the converse we apply the same technique used in Lemma 2.7. Since $\mathscr{H}$ is separable it follows that $\mathcal{B}_{\mathscr{H}}$ is generated by open balls, and therefore it suffices to show that the preimage of every open ball in $\mathscr{H}$ is in $\mathcal{M}$. If f is weakly measurable, it follows from the lemmas that $g_y(x) \equiv \|f(x) - y\|$ is measurable for each $y \in \mathscr{H}$. Since

$$f^{-1}\big(B(y, \delta)\big) = \{x : \|f(x) - y\| < \delta\} = g_y^{-1}\big(B(0, \delta)\big),$$

we are done. □

Now consider $\mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$, with the norm topology, where $\mathscr{H}_1$ and $\mathscr{H}_2$ are separable Hilbert spaces. A function $K : X \to \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ is said to be weakly measurable if $K(x)\phi$ is weakly measurable for each $\phi \in \mathscr{H}_1$. To be pedantic, this amounts to $x \mapsto \langle K(x)\phi, \psi \rangle$ being $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$-measurable for each $\phi \in \mathscr{H}_1$ and each $\psi \in \mathscr{H}_2$. By Proposition 3.3 this is equivalent to requiring that $x \mapsto K(x)\phi$ be $(\mathcal{M}, \mathcal{B}_{\mathscr{H}_2})$-measurable for each $\phi \in \mathscr{H}_1$, and by the continuity of the norm it implies that $x \mapsto \|K(x)\phi\|$ is measurable for each $\phi \in \mathscr{H}_1$.

Proposition 3.4. If $K : X \to \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ is weakly measurable, then $x \mapsto \|K(x)\|$ is measurable.

Proof. Since $\mathscr{H}_1$ is separable, so is the closed unit ball $\overline{B(0,1)} \subset \mathscr{H}_1$, with countable dense subset $\{\psi_i\}_1^\infty$. Now for any x, $\|K(x)\| = \sup_{\|\phi\| \le 1} \|K(x)\phi\|$, and since $K(x)$ is continuous, this is the equal to $\sup_i \|K(x)\psi_i\|$. Since $x \mapsto \|K(x)\psi_i\|$ is measurable for each i, $x \mapsto \|K(x)\|$ is measurable. □

We note that $K : X \to \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ being weakly measurable is not the same as K being $(\mathcal{M}, \mathcal{B}_{\mathscr{B}(\mathscr{H}_1,\mathscr{H}_2)})$-measurable. However, the latter sense of measurability implies the former. To see this, observe that for each $\phi \in \mathscr{H}_1$, the map $T \mapsto T\phi$ is $(\mathcal{B}_{\mathscr{B}(\mathscr{H}_1,\mathscr{H}_2)}, \mathcal{B}_{\mathscr{H}_2})$-measurable, because it is continuous, being linear and bounded. Then $x \mapsto K(x)\phi$ is just the composition

$$x \mapsto K(x) \mapsto K(x)\phi,$$

so it is $(\mathcal{M}, \mathcal{B}_{\mathscr{H}_2})$-measurable. Thus if K is $(\mathcal{M}, \mathcal{B}_{\mathscr{B}(\mathscr{H}_1,\mathscr{H}_2)})$-measurable, it is weakly measurable. I suspect the converse does not hold. At any rate, we cannot imitate the technique used in Proposition 3.3, for $\mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ may not be separable, even when $\mathscr{H}_1$ and $\mathscr{H}_2$ are. For consider $\mathscr{B}(\mathscr{H}, \mathscr{H})$, where $\mathscr{H} = L^2([0, 1])$, the space of equivalence classes of complex-valued square-integrable functions on $[0, 1]$; this space is separable (see Folland [5], p. 178). For $t \in (0, 1]$ and $f \in \mathscr{H}$, consider the multiplication operator

$$m_t(f) = f\chi_{[0,t]}.$$

For $s, t \in (0, 1]$ with $s < t$, we have

$$\begin{aligned}
\|m_t - m_s\| &= \sup_{\|f\|=1} \|(m_t - m_s)f\| \\
&= \sup_{\|f\|=1} \|f\chi_{[s,t]}\| \\
&= \sup_{\|f\|=1} \left(\int_s^t |f(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \\
&= 1,
\end{aligned}$$

since we can always find a function supported on $[s, t]$ such that $\|f\|_{L^2} = 1$. But $\{m_t : t \in (0, 1]\}$ is an uncountable subset of $\mathscr{B}(\mathscr{H}, \mathscr{H})$. This shows that any dense subset of $\mathscr{B}(\mathscr{H}, \mathscr{H})$ must be uncountable, and therefore $\mathscr{B}(\mathscr{H}, \mathscr{H})$ is not separable.

The following proposition will be useful later.

Proposition 3.5. If $K : X \to \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ and $f : X \to \mathscr{H}_1$ are weakly measurable, then $K(\cdot)f(\cdot) : X \to \mathscr{H}_2$ is weakly measurable.

Proof. We first observe that if $f : X \to \mathscr{H}_1$ and $g : X \to \mathscr{H}_1$ are weakly measurable, then $x \mapsto \langle f(x), g(x)\rangle$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$-measurable. To see this, observe that by the polarization identity we have

$$\langle f(x), g(x)\rangle = \tfrac{1}{4}\big(\|f(x)+g(x)\|^2 - \|f(x)-g(x)\|^2 \\ + i\|f(x)+ig(x)\|^2 - i\|f(x)-ig(x)\|^2\big),$$

and each of the four summands on the right-hand-side is measurable.

We now note that if $K : X \to \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ is weakly measurable, then so is $K^* : X \to \mathscr{B}(\mathscr{H}_2, \mathscr{H}_1)$, where $K^*(x)$ is the adjoint of $K(x)$. This is because for any $\psi \in \mathscr{H}_1$, $\phi \in \mathscr{H}_2$, and $x \in X$, we have

$$\langle K^*(x)\phi, \psi\rangle = \overline{\langle \psi, K^*(x)\phi\rangle} = \overline{\langle K(x)\psi, \phi\rangle},$$

which is measurable because K is weakly measurable, and complex conjugation is measurable (being continuous).

Now K^* being weakly measurable means, by definition, that $K^*(x)\phi$ is weakly measurable for each $\phi \in \mathscr{H}_2$, whence, by what we first showed,

$$x \mapsto \langle f(x), K^*(x)\phi\rangle$$

is measurable for each $\phi \in \mathscr{H}_2$. But $\langle f(x), K^*(x)\phi\rangle = \langle K(x)f(x), \phi\rangle$, so we have proven that $K(x)f(x)$ is weakly measurable. □

3.2 Vector-valued L^p-space

Let $\mathscr{B}$ be a Banach space. We define $L^p(X, \mathscr{B})$ in analogy with the usual definition, that is, with

$$\|f\|_p = \Big(\int_X \|f(x)\|_{\mathscr{B}}^p \,\mathrm{d}x\Big)^{1/p}$$

when $0 < p < \infty$, and

$$\|f\|_\infty = \operatorname{ess\,sup}_{x\in X}\|f(x)\|_{\mathscr{B}}.$$

We use $L^p(X, \mathscr{B})$ and $L^\infty(X, \mathscr{B})$ to denote the space of equivalence classes of $(\mathcal{M}, \mathcal{B}_{\mathscr{B}})$-measurable functions such that $\|f\|_p < \infty$ and $\|f\|_\infty < \infty$, respectively. When $1 \le p \le \infty$, $\|\cdot\|_p$ and $\|\cdot\|_\infty$ are norms on $L^p(X, \mathscr{B})$ and $L^\infty(X, \mathscr{B})$ respectively—that scaling and nondegeneracy hold is obvious, and the triangle inequality, or Minkowski's inequality, follows from the scalar-valued version:

Theorem 3.6 (Minkowski's inequality)**.** If $f, g \in L^p(X, \mathscr{B})$, $1 \leq p \leq \infty$, then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. Since $\|f + g\|_{\mathscr{B}} \leq \|f\|_{\mathscr{B}} + \|g\|_{\mathscr{B}}$, we simply apply the scalar-valued version to $\|f\|_{\mathscr{B}} + \|g\|_{\mathscr{B}}$. Thus,

$$\big\|\|f + g\|_{\mathscr{B}}\big\|_p \leq \big\|\|f\|_{\mathscr{B}} + \|g\|_{\mathscr{B}}\big\|_p \leq \big\|\|f\|_{\mathscr{B}}\big\|_p + \big\|\|g\|_{\mathscr{B}}\big\|_p,$$

that is,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

□

It is not hard to verify that the spaces $L^p(X, \mathscr{B})$ and $L^\infty(X, \mathscr{B})$ are Banach spaces for $1 \leq p \leq \infty$; the arguments are the same as for the scalar-valued case, but with absolute value replaced by norm. This construction works for any Banach space $\mathscr{B}$, and in the remainder of this chapter we will apply it to Hilbert space, as well as to the Banach space of linear operators between two Hilbert spaces. We note in passing that the norm on $L^2(X, \mathscr{H})$ is determined by the inner product

$$\langle f, g \rangle = \int_X \langle f(x), g(x) \rangle \,\mathrm{d}x,$$

so that $L^2(X, \mathscr{H})$ is a Hilbert space.

A vector-valued analogue of Hölder's inequality follows directly from the scalar-valued version:

Theorem 3.7 (Hölder's inequality)**.** Suppose that $1 \leq p \leq q \leq \infty$ and that q is conjugate to p (that is, $1/p + 1/q = 1$ when $1 < p < \infty$, and $p = 1$ when $q = \infty$). If $K : X \to \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ and $f : X \to \mathscr{H}_1$ are measurable, then

$$\|Kf\|_1 \leq \|K\|_p \|f\|_q.$$

Proof. Proposition 3.5 implies that $x \mapsto \|K(x)f(x)\|$ is measurable. Applying the scalar-valued Hölder's inequality to $\|K(x)\|_{op}$ and $\|f(x)\|_{\mathscr{H}_1}$ yields

$$\|Kf\|_1 \equiv \int \|K(x)f(x)\|_{\mathscr{H}_2} \,\mathrm{d}x \leq \int \|K(x)\|_{op} \|f(x)\|_{\mathscr{H}_1} \,\mathrm{d}x \leq \|K\|_p \|f\|_q,$$

as desired. □

3.3 The Fubini theorem

Fubini's theorem holds in this setting as well; we shall find it convenient to first review some properties of sections and bounded linear functionals. Recall that when we are given two sets X and Y, a subset $E \subset X \times Y$, and an element $x \in X$, we define the *x-section of E*—denoted E_x—by

$$E_x = \{y \in Y : (x, y) \in E\}.$$

We define the y-section E^y for $y \in Y$ analogously. Moreover, for a function f on $X \times Y$, we define the *x-section of f*—denoted f_x—by

$$f_x(y) = f(x, y).$$

For $y \in Y$ we define the y-section f^y analogously. Given two measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$, we have the following:

Proposition 3.8. Given $E \subset \mathcal{M} \otimes \mathcal{N}$, all of its x-sections E_x are in $\mathcal{N}$, and all of its y-sections E^y are in $\mathcal{M}$. Moreover, if $f : X \times Y \to \mathscr{H}$ is $(\mathcal{M} \otimes \mathcal{N}, \mathcal{B}_{\mathscr{H}})$-measurable, then each of its x-sections f_x is $(\mathcal{N}, \mathcal{B}_{\mathscr{H}})$-measurable, and each of its y-sections f^y is $(\mathcal{M}, \mathcal{B}_{\mathscr{H}})$-measurable.

The proof is easy and identical to that of the scalar-valued result; see Folland [5], p. 65. Now let $\mathscr{X}$ be a Banach space, and let $\mathscr{X}^*$ be its dual.

Proposition 3.9. For each $x \neq 0$ in $\mathscr{X}$, there exists $\xi \in \mathscr{X}^*$ such that $\|\xi\| = 1$ and $\xi(x) = \|x\|$.

Proof. Take $x \neq 0$. It is a useful corollary of the Hahn-Banach theorem that

$$\|x\| = \sup\{|u(x)| : u \in \mathscr{X}^* \text{ and } \|u\| \leq 1\}.$$

Moreover, this supremum is attained, at v, say. Let $\xi = \frac{\overline{v(x)}}{|v(x)|} v$; then $\|\xi\| = \|v\| = 1$, and

$$\xi(x) = \frac{\overline{v(x)}}{|v(x)|} v(x) = |v(x)| = \|x\|.$$

□

Corollary 3.10. If $\xi(x) = \xi(y)$ for all $\xi \in \mathscr{X}^*$, then $x = y$.

Proof. If $x \neq y$, select ξ so that

$$\xi(x-y) = \|x-y\|.$$

Then $\xi(x) \neq \xi(y)$. $\square$

Theorem 3.11 (The Fubini Theorem)**.** Let $\mathscr{H}$ be a separable Hilbert space, and suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces, and that $f : X \times Y \to \mathscr{H}$ is in $L^1(X \times Y, \mathscr{H})$. Then

$$g(x) = \int_Y f_x(y)\,\mathrm{d}\nu \in L^1(X, \mathscr{H}),$$
$$h(y) = \int_X f^y(x)\,\mathrm{d}\mu \in L^1(Y, \mathscr{H}),$$

and

$$\int_{X\times Y} f\,\mathrm{d}\mu \times \nu = \int_X g(x)\,\mathrm{d}\mu = \int_Y h(y)\,\mathrm{d}\nu.$$

That is,

$$\int_{X\times Y} f(x,y)\,\mathrm{d}\mu(x) \times \nu(y)$$
$$= \int_X \int_Y f(x,y)\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x) = \int_Y \int_X f(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y).$$

Proof. Since $f \in L^1(X \times Y, \mathscr{H})$, it is $(\mathcal{M} \otimes \mathcal{N}, \mathcal{B}_{\mathscr{H}})$-measurable, which implies that each of its sections is measurable. Moreover, $\|f\|$ and each of its sections is measurable. The Tonelli theorem applied to $\|f\|$ shows that

$$\int_Y \|f_x(y)\|\,\mathrm{d}\nu \in L^1(X, \mathbb{C}) \quad\text{and}\quad \int_X \|f^y(x)\|\,\mathrm{d}\mu \in L^1(Y, \mathbb{C}). \tag{3.1}$$

Then

$$\|g(x)\| = \left\| \int_Y f_x(y)\,\mathrm{d}\nu \right\| \leq \int_Y \|f_x(y)\|\,\mathrm{d}\nu, \tag{3.2}$$

so the integral defining $g(x)$ converges for a.e. x. Likewise,

$$\|h(x)\| = \left\| \int_X f^y(x)\,\mathrm{d}\mu \right\| \leq \int_X \|f^y(x)\|\,\mathrm{d}\mu, \tag{3.3}$$

so that the integral defining $h(y)$ converges for a.e. y. In fact, if we knew that g and h were measurable, then (3.1) together with (3.2) and (3.3) would yield that $g \in L^1(X, \mathscr{H})$ and $h \in L^1(Y, \mathscr{H})$.

Let $\xi \in \mathscr{H}^*$. By Theorem 2.12, $\xi(f) \in L^1(X \times Y, \mathbb{C})$. Hence we may apply the scalar-valued Fubini theorem to $\xi(f)$ to obtain that $[\xi(f)]_x \in L^1(Y, \mathbb{C})$ for a.e. $x \in X$, and $[\xi(f)]_y \in L^1(X, \mathbb{C})$ for a.e. $y \in Y$. Moreover, the a.e.-defined functions $\gamma(x) = \int_Y [\xi(f)]_x(y)\,\mathrm{d}\nu$ and $\eta(y) = \int_X [\xi(f)]_y(x)\,\mathrm{d}\mu$ are in $L^1(X, \mathbb{C})$ and $L^1(Y, \mathbb{C})$ respectively, with

$$\int_{X\times Y} \xi(f)(x,y)\,\mathrm{d}\mu \times \nu(x,y) = \int_X \gamma(x)\,\mathrm{d}\mu(x) = \int_Y \eta(y)\,\mathrm{d}\nu(y). \tag{3.4}$$

Note that by Theorem 2.12,

$$\gamma(x) = \int_Y [\xi(f)]_x(y)\,\mathrm{d}\nu = \int_Y \xi(f)(x,y)\,\mathrm{d}\nu = \xi\Big(\int_Y f(x,y)\,\mathrm{d}\nu\Big) = \xi(g(x)),$$

and similarly, $\eta(y) = \xi(\int_X f(x,y)\,\mathrm{d}\mu) = \xi(h(y))$. Since γ and η are in L^1, they are, in particular, measurable. Since $\xi \in \mathscr{H}^*$ was arbitrary, this proves that g and h are weakly measurable.

Now since $\gamma = \xi(g)$ and $\eta = \xi(h)$, and we've established that g and h are in L^1, a final application of Theorem 2.12 to (3.4) yields

$$\xi\Big(\int_{X\times Y} f\,\mathrm{d}\mu \times \nu\Big) = \xi\Big(\int_X g(x)\,\mathrm{d}\mu\Big) = \xi\Big(\int_Y h(y)\,\mathrm{d}\nu\Big).$$

Since ξ can be any element of $\mathscr{H}^*$, Corollary 3.10 implies the desired equality

$$\int_{X\times Y} f\,\mathrm{d}\mu \times \nu = \int_X g(x)\,\mathrm{d}\mu = \int_Y h(y)\,\mathrm{d}\nu.$$

□

3.4 Further properties of the Bochner integral; the Fourier transform

In what follows we specialize to $(X, \mathcal{M}, \mu) = (\mathbb{R}^n, \mathcal{L}, m)$, that is, Euclidean n-space with Lebesgue measure. For a fixed element $a \in \mathbb{R}^n$ we define translation by a on subsets and functions as follows: for a subset $E \subset \mathbb{R}^n$ the translation of E by a is

$$\tau_a(E) = \{x - a : x \in E\} = E - a,$$

and for a function f whose domain is $\mathbb{R}^n$, the translation of f by a is

$$\tau_a f(x) = f(x - a).$$

We also define the reflection of f by

$$\tilde{f}(x) = f(-x).$$

The Lebesgue measure m is translation-invariant, meaning that $E \in \mathcal{L}$ implies $\tau_a E \in \mathcal{L}$ with $m(\tau_a E) = m(E)$. It is also reflection-invariant, i.e., $E \in \mathcal{L}$ implies $-E \in \mathcal{L}$ with $m(-E) = m(E)$. It follows that the Bochner integral $\int : L^1(\mathbb{R}^n, \mathscr{H}) \to \mathscr{H}$ is also translation- and reflection-invariant. We first prove that the Lebesgue integral has these properties:

Lemma 3.12. If $f : \mathbb{R}^n \to [0, \infty)$ is measurable, or if $f \in L^1(\mathbb{R}^n, \mathbb{C})$, then $\int \tau_a f = \int f$ and $\int \tilde{f} = \int f$.

Proof. We prove the result for translation-invariance; the proof of reflection-invariance is practically identical. For any $E \in \mathcal{L}$ we have $\tau_a \chi_E = \chi_{a+E}$, whence $\int \tau_a \chi_E = m(a+E) = m(E) = \int \chi_E$. It follows by the linearity of the integral that $\int \tau_a \phi = \int \phi$ for any simple function ϕ. Since f is measurable and nonnegative, it is the limit of an increasing sequence $\{\phi_n\}$ of simple functions. Then $\tau_a \phi_n$ increases to $\tau_a f$ pointwise, and by the Monotone Convergence Theorem $\int \tau_a \phi_n \, \mathrm{d}m \to \int \tau_a f \, \mathrm{d}m$. Therefore,

$$\int \tau_a f \, \mathrm{d}m = \lim_{n\to\infty} \int \tau_a \phi_n \, \mathrm{d}m = \lim_{n\to\infty} \int \phi_n \, \mathrm{d}m = \int f \, \mathrm{d}m.$$

The result for $f \in L^1(\mathbb{R}^n, \mathbb{C})$ follows from the definition of $\int f$:

$$\int f = \int (\mathrm{Re} f)^+ - \int (\mathrm{Re} f)^- + i \int (\mathrm{Im} f)^+ - i \int (\mathrm{Im} f)^-,$$

and each of the integrands on the right-hand-side is nonnegative. □

Remark 3.13. If $f \in L^p(\mathbb{R}^n, \mathscr{H})$, $1 \leq p \leq \infty$, then $\|\tilde{f}\|_p = \|\tau_a f\|_p = \|f\|_p$. For $p = \infty$ this is obvious. For $p < \infty$ this follows directly from the definition of $\|f\|_p$ and the fact that $x \mapsto \|f(x)\|^p$ is measurable.

Proposition 3.14. The Bochner integral $\int : L^1(\mathbb{R}^n, \mathscr{H}) \to \mathscr{H}$ is translation- and reflection-invariant.

Proof. We prove translation-invariance. Let $f \in L^1(\mathbb{R}^n, \mathscr{H})$. Since $\int \chi_E = m(E)$ for $E \in \mathcal{L}$, the result holds for simple $\mathscr{H}$-valued functions. By Theorems 2.5 and 2.6, there is a sequence $\{\psi_n\}$ of simple $\mathscr{H}$-valued functions such that $\psi_n \to f$ in

$L^1(\mathbb{R}^n, \mathscr{H})$, and $\int f$ is defined to be $\lim_{n\to\infty} \int \psi_n$. Now $\tau_a\psi_n \to \tau_a f$ a.e., and by Remark 3.13,

$$\|\tau_a\psi_n - \tau_a f\|_1 = \|\tau_a(\psi_n - f)\|_1 = \|\psi_n - f\|_1 \to 0.$$

Therefore,

$$\int \tau_a f = \lim_{n\to\infty} \int \tau_a\psi_n = \lim_{n\to\infty} \int \psi_n = \int f.$$

□

It is also true that $m(\delta E) = \delta^n m(E)$ for each $\delta > 0$ and $E \in \mathcal{L}$, that is, m is homogeneous of degree n. It follows that $\int f(\delta x)\,\mathrm{d}x = \delta^{-n} \int f(x)\,\mathrm{d}x$, and this holds for the Bochner integral as well, though we shall omit the proof.

Now that we have covered the invariance properties of the Bochner integral, we are ready to discuss convolution of vector-valued functions. For conjugate exponents p and q, if

(i) $K \in L^q(\mathbb{R}^n, \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2))$

(ii) $f \in L^p(\mathbb{R}^n, \mathscr{H}_1)$

(iii) $\mathscr{H}_2$ is separable

we define the convolution of K with f—denoted $K * f$—by

$$K * f(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \tau_x\tilde{K}(y)f(y)\,\mathrm{d}y,$$

where the vector-valued integral is the Bochner integral. Since K and f are measurable and $\mathscr{H}_2$ is separable, it follows from Proposition 3.5 that $K(\cdot)f(\cdot)$ is measurable. By Hölder's Inequality we have

$$\int \|K(x-y)f(y)\|_{\mathscr{H}_2}\,\mathrm{d}y = \|\tau_x\tilde{K}f\|_1 \le \|\tau_x\tilde{K}\|_q\|f\|_p = \|K\|_q\|f\|_p < \infty. \tag{3.5}$$

Hence, $\tau_x\tilde{K}f \in L^1(\mathbb{R}^n, \mathscr{H}_2)$, so the Bochner integral $K * f(x)$ is indeed well-defined for each x. For the following proposition we follow Stein and Shakarchi [10].

Proposition 3.15. Suppose that $f \in L^1(\mathbb{R}^n, \mathbb{C})$. Then

$$\|\tau_h f - f\|_1 \to 0 \quad \text{as} \quad |h| \to 0.$$

Proof. We shall see later in Theorem 4.11 that integrable functions can be approximated in L^1 by continuous, compactly supported functions. Let g be such a function, with $\|f-g\|_1 < \frac{\varepsilon}{3}$. It follows easily from g being continuous and compactly supported that $\|\tau_h g - g\|_1 \to 0$, so choose $|h| < \delta$ small enough that $\|\tau_h g - g\|_1 < \frac{\varepsilon}{3}$. Then writing

$$\tau_h f - f = (\tau_h g - g) + (\tau_h f - \tau_h g) + (g - f)$$

and applying the triangle inequality yields the result. □

Corollary 3.16. *$K * f$ is a continuous function from $\mathbb{R}^n$ to $\mathscr{H}_2$.*

Proof. We show that for each $x \in \mathbb{R}^n$,

$$K * f(x+h) \to K * f(x) \;\text{ in }\; \mathscr{H}_2 \;\text{ as }\; |h| \to 0.$$

We have

$$\begin{aligned}
&\|\tau_{-h} K * f(x) - K * f(x)\|_{\mathscr{H}_2} \\
&\quad = \left\| \int K(x+h-y) f(y)\,\mathrm{d}y - \int K(x-y) f(y)\,\mathrm{d}y \right\|_{\mathscr{H}_2} \\
&\quad \le \int \|K(x+h-y) f(y) - K(x-y) f(y)\|_{\mathscr{H}_2}\,\mathrm{d}y \\
&\quad = \|\tau_h \tau_x \tilde{K} f - \tau_x \tilde{K} f\|_1.
\end{aligned}$$

By Proposition 3.15, this last quantity tends to 0 with h. □

Note that

$$\left\| K * f(x) \right\|_{\mathscr{H}_2} \le \int \|K(x-y) f(y)\|_{\mathscr{H}_2}\,\mathrm{d}y. \tag{3.6}$$

Together (3.5) and (3.6) show that $\|K * f\|_{unif} \le \|K\|_q \|f\|_p$, which shows that convolution with K can be regarded as a bounded linear operator from $L^p(\mathbb{R}^n, \mathscr{H}_1)$ to the space $\mathscr{C}(\mathbb{R}^n, \mathscr{H}_2)$ of continuous functions from $\mathbb{R}^n$ to $\mathscr{H}_2$. The function K is called the *kernel* of this integral operator.

We can define the Fourier transform $\mathscr{F}$ on $L^1(\mathbb{R}^n, \mathscr{H})$ as usual: $\mathscr{F} f = \hat{f}$, where

$$\hat{f}(y) \equiv \int_{\mathbb{R}^n} e^{2\pi i x \cdot y} f(x)\,\mathrm{d}x,$$

and the integral here is again the Bochner integral. This is well-defined, because

$$\|e^{2\pi i x \cdot y} f(x)\|_{\mathscr{H}} = |e^{2\pi i x \cdot y}| \|f(x)\|_{\mathscr{H}} = \|f(x)\|_{\mathscr{H}},$$

and $f \in L^1(\mathbb{R}^n, \mathscr{H})$. Moreover, $\hat{f} \in L^\infty(\mathbb{R}^n, \mathscr{H})$, since by Theorem 2.6 we have

$$\|\hat{f}(y)\|_{\mathscr{H}} \leq \int_{x\in\mathbb{R}^n} \|e^{2\pi i x\cdot y} f(x)\|_{\mathscr{H}} \,\mathrm{d}x = \|f\|_1,$$

and hence

$$\|\hat{f}\|_\infty = \sup_{y\in\mathbb{R}^n} \|\hat{f}(y)\|_{\mathscr{H}} \leq \|f\|_1.$$

Chapter 4

Important Results for the Singular Integral Theorem

Before we can attempt to give an application of vector-valued integration to singular integral theory, we need a few important results, which we shall include in this chapter. We follow Stein closely in the first two sections, which correspond to sections I.3 and I.4 in *Singular Integrals* [9]. We follow Grafakos [6] in the third section.

4.1 The Calderón-Zygmund lemma

The following theorem is interesting in itself, but will also play a part in the proof of the theorem in the next chapter. For its proof we need the Lebesgue Differentiation Theorem, which we shall discuss presently. We say that a family $\mathscr{F}$ of measurable sets in $\mathbb{R}^n$ is *regular at* x, or that it *shrinks nicely to* x, if there is some constant $c > 0$ such that each $S \in \mathscr{F}$ is contained in an open ball B centred at x, with $m(S) > cm(B)$. For example, using the fact that Lebesgue measure on $\mathbb{R}^n$ is homogeneous of degree n (i.e., for each $\delta > 0$, $m(\delta E) = \delta^n m(E)$), one may easily verify that the family of dilations δE of some bounded set E with positive measure is regular at the origin. We will see another example in the proof of the next theorem. Lebesgue's differentiation theorem says that for almost every x (specifically, for every x in a special set called the *Lebesgue set of* f),

$$\lim_{\substack{S\in\mathscr{F} \\ m(S)\to 0}} \frac{1}{m(S)} \int_S f(y)\,\mathrm{d}y = f(x)$$

for every family $\mathscr{F}$ that is regular at x. See page 98 of Folland and §1.8 of Stein for proofs and discussion.

Theorem 4.1. Given any nonnegative integrable function f on $\mathbb{R}^n$, and any $\alpha > 0$, there exists a closed set $F \subset \mathbb{R}^n$ such that $f(x) \leq \alpha$ for almost every $x \in F$, and such that F^c is a union of closed cubes whose interiors are disjoint, and with the property that each such cube Q_k satisfies

$$\alpha < \frac{1}{m(Q_k)} \int_{Q_k} f \leq 2^n \alpha. \tag{4.1}$$

Proof. Throughout the proof we take all cubes to be closed. Since f is nonnegative and integrable, $\int_{\mathbb{R}^n} f < \infty$, and hence for any cube Q such that $m(Q) \geq \frac{1}{\alpha} \int_{\mathbb{R}^n} f$, we have

$$\frac{1}{m(Q)} \int_Q f(x)\,\mathrm{d}x \leq \frac{1}{m(Q)} \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x \leq \alpha. \tag{4.2}$$

Partition $\mathbb{R}^n$ into cubes of common diameter sufficiently large for (4.2) to hold. Let Q' be a fixed cube in this partition, and divide it into 2^n congruent cubes by bisecting each of its sides. For each of these new cubes Q'', either

$$\frac{1}{m(Q'')} \int_{Q''} f(x)\,\mathrm{d}x \leq \alpha \tag{4.3}$$

or

$$\frac{1}{m(Q'')} \int_{Q''} f(x)\,\mathrm{d}x > \alpha. \tag{4.4}$$

In the second case Q'' becomes one of the cubes in the statement of the theorem; the condition (4.1) holds for it because $m(Q'') = \frac{m(Q')}{2^n}$, and hence

$$\alpha < \frac{1}{m(Q'')} \int_{Q''} f(x)\,\mathrm{d}x \leq \frac{2^n}{m(Q')} \int_{Q'} f(x)\,\mathrm{d}x \leq 2^n \alpha.$$

In the first case we subdivide again, and continue this process *ad infinitum*, if necessary. We do this for each of the cubes in our partition of $\mathbb{R}^n$. We let $\Omega = \bigcup_k Q_k$ be the union of all the cubes for which the second case (4.4) holds. Let $F = \Omega^c$; it remains only to show that $f \leq \alpha$ a.e. in F.

Let $x \in F$, and notice that each cube in our decomposition which contains x is a cube for which (4.3) holds. Set $\mathscr{F}(x)$ to be the set of all cubes in our decomposition containing x. We claim that $\mathscr{F}(x)$ is regular at x. To see this, fix $Q \in \mathscr{F}(x)$. Let a denote its sidelength, and let $d = \sqrt{n}a$ denote its diagonal length (the maximum distance between any two points in Q). Let Q^* be the cube centred at x with sidelength equal to d (this cube need not be in the decomposition). Note that $B(x, d)$—the open ball centred at x of radius d—is contained in Q^*; for if $y \in B(x, d)$, then $|x - y| < d$, whence $y \in Q^*$. Now,

$$m(Q^*) = (2d)^n = 2^n(\sqrt{n}a)^n = 2^n n^{n/2} a^n = 2^n n^{n/2} m(Q).$$

Therefore,

$$m(B) < m(Q^*) = 2^n n^{n/2} m(Q),$$

which proves our claim that $\mathscr{F}(x)$ is regular at x. We can now apply the Lebesgue differentiation theorem: for every x in the Lebesgue set of f, we have

$$f(x) = \lim_{\substack{Q\in\mathscr{F}(x)\\ m(Q)\to 0}} \frac{1}{m(Q)} \int_Q f(y)\,\mathrm{d}y.$$

But each $Q \in \mathscr{F}(x)$ satisfies condition (4.3), and therefore $f(x) \leq \alpha$, as desired.

$\square$

Remark 4.2. The set F^c in the above theorem satisfies $m(F^c) < \frac{1}{\alpha}\|f\|_1$. To see this, we first observe that $f\chi_{\bigcup_1^n Q_k}$ increases to $f\chi_{\bigcup_1^\infty Q_k}$, and therefore by the Monotone Convergence theorem,

$$\int_{\bigcup_1^\infty Q_k} f = \lim_{n\to\infty} \int f\chi_{\bigcup_1^n Q_k}.$$

But

$$\int f\chi_{\bigcup_1^n Q_k} = \int f \sum_1^n \chi_{Q_k} = \sum_1^n \int_{Q_k} f.$$

Therefore,

$$\int_{\bigcup_1^\infty Q_k} f = \sum_1^\infty \int_{Q_k} f.$$

Using this and (4.1), we have

$$m(F^c) = \sum_k m(Q_k) < \frac{1}{\alpha}\sum_k \int_{Q_k} f = \frac{1}{\alpha}\int_{\bigcup_k Q_k} f = \frac{1}{\alpha}\int_{F^c} f \leq \frac{1}{\alpha}\|f\|_1.$$

4.2 Some terminology, and an interpolation theorem

We begin this section with some terminology:

Definition 4.3. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces, and let T be a map from $L^p(\mathbb{R}^n, \mathscr{H}_1)$ to the set of $(\mathcal{L}, \mathcal{B}_{\mathscr{H}_2})$-measurable functions on $\mathbb{R}^n$, where $1 \leq p \leq \infty$. For $1 \leq q < \infty$, we say that T is *of weak-type* (p, q) if there is a constant A such that

$$m\{x : \|Tf(x)\| > \alpha\} \leq \left(\frac{A\|f\|_p}{\alpha}\right)^q$$

for all $f \in L^p(\mathbb{R}^n, \mathscr{H}_1)$ and all $\alpha > 0$.

Definition 4.4. Suppose that T is a map from $L^p(\mathbb{R}^n, \mathscr{H}_1)$ to $L^q(\mathbb{R}^n, \mathscr{H}_2)$, where $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. We say that T is *of type* (p, q) if there is a constant A such that

$$\|Tf\|_q \leq A\|f\|_p$$

for all $f \in L^p(\mathbb{R}^n, \mathscr{H}_1)$.

We also say that T is of weak-type (p, ∞) if it is of type (p, ∞), so that the two definitions coincide when $q = \infty$. Note that if T is of type (p, q), then it is of weak-type (p, q), for

$$\alpha^q m\{x : \|Tf(x)\| > \alpha\} \leq \int_{\mathbb{R}^n} \|Tf(x)\|^q \,\mathrm{d}x = \|Tf\|_q^q \leq (A\|f\|_p)^q.$$

Lastly, we define $L^{p_1} + L^{p_2}$, for $p_1, p_2 > 0$, to be the space of all functions of the form $f = f_1 + f_2$, where $f_1 \in L^{p_1}$ and $f_2 \in L^{p_2}$.

Lemma 4.5. For $p_1 \leq p \leq p_2$, we have $L^p \subset L^{p_1} + L^{p_2}$.

Proof. Let $f \in L^p$ and fix $\gamma > 0$. Now set

$$f_1(x) = \begin{cases} f(x) & \text{if } \|f(x)\| > \gamma \\ 0 & \text{if } \|f(x)\| \leq \gamma \end{cases}$$

$$f_2(x) = \begin{cases} f(x) & \text{if } \|f(x)\| \leq \gamma \\ 0 & \text{if } \|f(x)\| > \gamma \end{cases}.$$

We have

$$\int_{\mathbb{R}^n} \|f_1(x)\|^{p_1} \,\mathrm{d}x = \int_{\|f(x)\|>\gamma} \|f(x)\|^{p_1} \,\mathrm{d}x = \int_{\|f(x)\|>\gamma} \|f(x)\|^p \|f(x)\|^{p_1 - p} \,\mathrm{d}x,$$

and since $p_1 - p \leq 0$, this implies

$$\int_{\mathbb{R}^n} \|f_1(x)\|^{p_1} \,\mathrm{d}x \leq \gamma^{p_1 - p} \int_{\|f(x)\|>\gamma} \|f(x)\|^p \,\mathrm{d}x,$$

which is finite because $f \in L^p$. Also,

$$\int_{\mathbb{R}^n} \|f_2(x)\|^{p_2} \,\mathrm{d}x = \int_{\mathbb{R}^n} \|f_2(x)\|^p \|f_2(x)\|^{p_2 - p} \,\mathrm{d}x \leq \gamma^{p_2 - p} \int_{\mathbb{R}^n} \|f(x)\|^p \,\mathrm{d}x,$$

which again is finite since $f \in L^p$. Thus, $f = f_1 + f_2$, where $f_1 \in L^{p_1}$ and $f_2 \in L^{p_2}$. $\square$

Theorem 4.6. Suppose that $1 < r \le \infty$, and suppose that T is a map from $(L^1 + L^r)(\mathbb{R}^n, \mathscr{H}_1)$ to the space of all $(\mathcal{L}, \mathcal{B}_{\mathscr{H}_2})$-measurable functions, such that

(i) T is subadditive, i.e., $\|T(f+g)(x)\| \le \|Tf(x)\| + \|Tg(x)\|$ for all $f, g \in L^1 + L^r$, $x \in \mathbb{R}^n$.

(ii) T is of weak-type $(1,1)$, i.e., $m\{x : \|Tf(x)\| > \alpha\} \le \frac{A_1\|f\|_1}{\alpha}$ for all $f \in L^1$.

(iii) T is of weak-type (r,r), i.e., $m\{x : \|Tf(x)\| > \alpha\} \le \left(\frac{A_r\|f\|_r}{\alpha}\right)^r$ for all $f \in L^r$, when $r < \infty$, and $\|Tf\|_r \le A_r\|f\|_r$ if $r = \infty$.

Then T is of type (p,p) for $1 < p < r$, i.e.,

$$\|Tf\|_p \le A_p\|f\|_p \quad \text{for all } f \in L^p,$$

where A_p depends only on A_1, A_r, p, and r.

Proof. We prove the result for $p < \infty$. Let $f \in L^p(\mathbb{R}^n, \mathscr{H}_1)$. A useful fact from real analysis is that

$$\int_{\mathbb{R}^n} \|Tf(x)\|^p \,\mathrm{d}x = p\int_0^\infty \alpha^{p-1}\lambda(\alpha)\,\mathrm{d}\alpha, \tag{4.5}$$

where $\lambda(\alpha) = m\{x : \|Tf(x)\| > \alpha\}$ (λ is called the *distribution function associated with* Tf; see Folland, pp. 197–198 for a discussion and a proof of the above equation). We will estimate $\lambda(\alpha)$ in terms of an L^1 function and an L^r function, for both of which T has weak-type inequalities by the hypotheses.

As in the lemma, we set

$$f_1(x) = \begin{cases} f(x) & \text{if } \|f(x)\| > \alpha \\ 0 & \text{if } \|f(x)\| \le \alpha \end{cases}$$

$$f_2(x) = \begin{cases} f(x) & \text{if } \|f(x)\| \le \alpha \\ 0 & \text{if } \|f(x)\| > \alpha \end{cases},$$

so that $f = f_1 + f_2$, where $f_1 \in L^1$ and $f_2 \in L^r$. Now by subadditivity, we have

$$\|Tf(x)\| \le \|Tf_1(x)\| + \|Tf_2(x)\| \quad \text{for all } x.$$

Therefore,

$$\{x : \|Tf(x)\| > \alpha\} \subset \{x : \|Tf_1(x)\| > \alpha/2\} \cup \{x : \|Tf_2(x\|) > \alpha/2\},$$

whence

$$\lambda(\alpha) \leq m\{x : \|Tf_1(x)\| > \alpha/2\} + m\{x : \|Tf_2(x)\| > \alpha/2\}.$$

Therefore by the hypotheses (ii) and (iii),

$$\begin{aligned}\lambda(\alpha) &\leq \frac{A_1}{\alpha/2}\int_{\mathbb{R}^n} \|f_1(x)\|\,\mathrm{d}x + \frac{A_r^r}{(\alpha/2)^r}\int_{\mathbb{R}^n} \|f_2(x)\|^r\,\mathrm{d}x \\ &= \frac{2A_1}{\alpha}\int_{\|f\|>\alpha} \|f(x)\|\,\mathrm{d}x + \frac{(2A_r)^r}{\alpha^r}\int_{\|f\|\leq\alpha} \|f(x)\|^r\,\mathrm{d}x.\end{aligned}$$

Hence by (4.5),

$$\begin{aligned}\int_{\mathbb{R}^n} \|Tf(x)\|^p\,\mathrm{d}x \leq 2A_1 p\int_0^\infty \alpha^{-1}\alpha^{p-1}\int_{\|f\|>\alpha} \|f(x)\|\,\mathrm{d}x\,\mathrm{d}\alpha \\ + (2A_r)^r p\int_0^\infty \alpha^{-r}\alpha^{p-1}\int_{\|f\|\leq\alpha} \|f(x)\|^r\,\mathrm{d}x\,\mathrm{d}\alpha. \quad (4.6)\end{aligned}$$

We treat each of the terms on the right-hand-side of (4.6) separately. Ignoring the constant in front for now, we see that the first term can be written as

$$\int_0^\infty \alpha^{p-2}\int_{\mathbb{R}^n} \|f(x)\|\chi_{\|f\|^{-1}(\alpha,\infty)}(x)\,\mathrm{d}x\,\mathrm{d}\alpha.$$

It is easy to verify that $\chi_{\|f\|^{-1}(\alpha,\infty)}(x) = \chi_{(0,\|f(x)\|)}(\alpha)$, and using this together with the Tonelli theorem we get

$$\int_{\mathbb{R}^n} \|f(x)\|\int_0^\infty \alpha^{p-2}\chi_{(0,\|f(x)\|)}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x,$$

which is

$$\int_{\mathbb{R}^n} \|f(x)\|\int_0^{\|f(x)\|} \alpha^{p-2}\,\mathrm{d}\alpha\,\mathrm{d}x.$$

This works out to

$$\frac{1}{p-1}\int_{\mathbb{R}^n} \|f(x)\|\|f(x)\|^{p-1}\,\mathrm{d}x,$$

or simply $\frac{1}{p-1}\|f\|_p^p$.

The second term in (4.6) is dealt with similarly: ignoring the constant in front, we write it as

$$\int_0^\infty \alpha^{-r}\alpha^{p-1}\int_{\mathbb{R}^n} \|f(x)\|^r\chi_{\|f\|^{-1}[0,\alpha]}(x)\,\mathrm{d}x\,\mathrm{d}\alpha.$$

Then noting that $\chi_{\|f\|^{-1}[0,\alpha]}(x) = \chi_{[\|f(x)\|,\infty)}(\alpha)$ and applying the Tonelli theorem, we get

$$\int_{\mathbb{R}^n} \|f(x)\|^r\int_{\|f(x)\|}^\infty \alpha^{p-1-r}\,\mathrm{d}\alpha\,\mathrm{d}x.$$

Since $p < r$, the inner improper integral works out to $\frac{1}{r-p}\|f(x)\|^{p-r}$, and hence the second term becomes $\frac{1}{r-p}\|f\|_p^p$.

Altogether, we have shown that

$$\begin{aligned}\|Tf\|_p^p = \int_{\mathbb{R}^n} \|Tf(x)\|^p \,\mathrm{d}x &\leq \frac{2A_1 p}{p-1}\|f\|_p^p + \frac{(2A_r)^r p}{r-p}\|f\|_p^p \\ &= \left(\frac{2A_1}{p-1} + \frac{(2A_r)^r}{r-p}\right)p\|f\|_p^p.\end{aligned}$$

Taking the pth root of both sides, we are done. □

4.3 A duality theorem

Let $\mathscr{X}$ be a Banach space, and let $\mathscr{X}^*$ denote its dual. Let $1 \leq p \leq \infty$, and let q be the conjugate exponent of p. For $f \in L^p(\mathbb{R}^n, \mathscr{X})$ we define θ_f on $L^q(\mathbb{R}^n, \mathscr{X}^*)$ by

$$\theta_f(g^*) = \int_{\mathbb{R}^n} g^*(x)f(x)\,\mathrm{d}x.$$

Then θ_f is well-defined since

$$f \in L^p(\mathbb{R}^n, \mathscr{X}) \;\text{ implies }\; \|f(x)\|_{\mathscr{X}} \in L^p(\mathbb{R}^n, \mathbb{C}),$$

and

$$g^* \in L^q(\mathbb{R}^n, \mathscr{X}^*) \;\text{ implies }\; \|g^*(x)\|_{op} \in L^q(\mathbb{R}^n, \mathbb{C});$$

thus Hölder's inequality gives us

$$\begin{aligned}\int_{\mathbb{R}^n} |g^*(x)f(x)|\,\mathrm{d}x &\leq \int_{\mathbb{R}^n} \|g^*(x)\|_{op}\|f(x)\|_{\mathscr{X}}\,\mathrm{d}x \\ &\leq \|g^*\|_{L^q(\mathbb{R}^n,\mathscr{X}^*)}\|f\|_{L^p(\mathbb{R}^n,\mathscr{X})} \\ &< \infty. \end{aligned} \tag{4.7}$$

Therefore θ_f is well-defined on $L^q(\mathbb{R}^n, \mathscr{X}^*)$.

The main objective for this section is proving the following theorem:

Theorem 4.7. The map $f \mapsto \theta_f$ is an isometric embedding of $L^p(\mathbb{R}^n, \mathscr{X})$ into $L^q(\mathbb{R}^n, \mathscr{X}^*)^*$. In other words,

$$\|f\|_p = \sup_{\|g^*\|_{L^q(\mathbb{R}^n,\mathscr{X}^*)} \leq 1} |\theta_f(g^*)|.$$

In order to prove this theorem, we need some preliminary theory. Given a Banach space $\mathscr{X}$, recall that the set $F_{\mathscr{X}}$ of simple functions in $\mathscr{X}$ is the set of functions of the form $\sum_{j=1}^{m} \chi_{E_j} x_j$, where $m(E_j) < \infty$ and $x_j \in \mathscr{X}$ for all $j = 1, 2, \ldots, m$. In Theorem 2.5 we showed that when $\mathscr{X}$ is separable, $F_{\mathscr{X}}$ is dense in $L^1(\mathbb{R}^n, \mathscr{X})$. We now prove that the density of $F_{\mathscr{X}}$ is not specific to L^1, but is true of L^p for $p < \infty$. We follow the proof given in Grafakos ([6], pp. 323–324).

Proposition 4.8. Let $\mathscr{X}$ be a separable Banach space. The set of simple functions $F_{\mathscr{X}}$ is dense in $L^p(\mathbb{R}^n, \mathscr{X})$, $0 < p < \infty$. When $p = \infty$, the set of functions of the form $\sum_{j=1}^{\infty} \chi_{E_j} x_j$, where $\{E_j\}_{j=1}^{\infty}$ is a partition of $\mathbb{R}^n$ and $x_j \in \mathscr{X}$, is dense in $L^\infty(\mathbb{R}^n, \mathscr{X})$.

Proof. We first treat the case $p < \infty$. Let $f \in L^p$, and let $\varepsilon > 0$. Since $f \in L^p$, there exists a compact subset $K \subset \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n \setminus K} \|f(x)\|_{\mathscr{X}}^p \, \mathrm{d}x < \frac{\varepsilon^p}{3}.$$

Let $\{x_j\}_{j=1}^{\infty}$ be a countable, dense subset of $\mathscr{X}$, and let B_j denote the open ball of radius $\varepsilon\big(3m(K)\big)^{-1/p}$ centred at x_j. Now set $A_1 = B_1$, and for $j > 1$ set $A_j = B_j \setminus \bigcup_{i=1}^{j-1} B_i$. Thus the A_j are pairwise disjoint, and

$$\bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} B_j = \mathscr{X}.$$

We now set $E_j = f^{-1}(A_j) \cap K$. Then $\{E_j\}_{j=1}^{\infty}$ is pairwise disjoint and $K = \bigcup_{j=1}^{\infty} E_j$, whence

$$\sum_{j=1}^{\infty} m(E_j) = m(K) < \infty.$$

It follows that for some $m \in \mathbb{N}$,

$$\int_{\bigcup_{j=m+1}^{\infty} E_j} \|f(x)\|_{\mathscr{X}}^p \, \mathrm{d}x < \frac{\varepsilon^p}{3}.$$

Note that

$$\|f(x) - x_j\|_{\mathscr{X}} < \varepsilon\big(3m(K)\big)^{-1/p} \quad \text{whenever} \quad x \in E_j. \tag{4.8}$$

Consider the quantity

$$\int_{\bigcup_{j=1}^{m} E_j} \Big\| f(x) - \sum_{j=1}^{m} \chi_{E_j}(x) x_j \Big\|_{\mathscr{X}}^p \, \mathrm{d}x. \tag{4.9}$$

For $x \in \bigcup_{j=1}^m E_j$ we have $f(x) = f(x)\sum_{j=1}^m \chi_{E_j}(x)$, and hence

$$\Big\|f(x) - \sum_{j=1}^m \chi_{E_j}(x)x_j\Big\|_{\mathscr{X}} = \Big\|\sum_{j=1}^m \chi_{E_j}(x)[f(x) - x_j]\Big\|_{\mathscr{X}} = \sum_{j=1}^m \chi_{E_j}(x)\|f(x) - x_j\|_{\mathscr{X}}.$$

Thus we have

$$\int_{\bigcup_{j=1}^m E_j} \Big\|f(x) - \sum_{j=1}^m \chi_{E_j}(x)x_j\Big\|_{\mathscr{X}}^p \,\mathrm{d}x = \int_{\bigcup_{j=1}^m E_j} \Big(\sum_{j=1}^m \chi_{E_j}(x)\|f(x) - x_j\|_{\mathscr{X}}\Big)^p \,\mathrm{d}x,$$

and by (4.8) the right-hand-side of this is bounded by

$$\int_{\bigcup_{j=1}^m E_j} \Big(\sum_{j=1}^m \chi_{E_j}(x) \cdot \varepsilon\big(3m(K)\big)^{-1/p}\Big)^p \,\mathrm{d}x. \tag{4.10}$$

Since the E_j are disjoint, $\sum_{j=1}^m \chi_{E_j}(x) = \chi_{\bigcup_{j=1}^m E_j}(x)$, and hence (4.10) becomes

$$\frac{\varepsilon^p}{3m(K)} \int_{\bigcup_{j=1}^m E_j} \chi_{\bigcup_{j=1}^m E_j}(x)\,\mathrm{d}x = \frac{\varepsilon^p}{3m(K)} \int_{\bigcup_{j=1}^m E_j} 1 \,\mathrm{d}x = \frac{\varepsilon^p m(\bigcup_{j=1}^m E_j)}{3m(K)},$$

and thus we have shown that (4.9) is less than or equal to $\frac{\varepsilon^p}{3}$. Now since $\bigcup_{j=1}^\infty E_j \cup K^c$ is a partition of $\mathbb{R}^n$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \Big\|f(x) - \sum_{j=1}^m \chi_{E_j}(x)x_j\Big\|_{\mathscr{X}}^p \,\mathrm{d}x &= \int_{\mathbb{R}^n\setminus K} \|f(x)\|_{\mathscr{X}}^p \,\mathrm{d}x \\ &\quad + \int_{\bigcup_{j=m+1}^\infty E_j} \|f(x)\|_{\mathscr{X}}^p \,\mathrm{d}x \\ &\quad + \int_{\bigcup_{j=1}^m E_j} \Big\|f(x) - \sum_{j=1}^m \chi_{E_j}(x)x_j\Big\|_{\mathscr{X}}^p \,\mathrm{d}x \\ &< \frac{\varepsilon^p}{3} + \frac{\varepsilon^p}{3} + \frac{\varepsilon^p}{3} = \varepsilon^p.\end{aligned}$$

It remains only to verify that $\sum_{j=1}^m \chi_{E_j}(x)x_j$ is $(\mathcal{L}, \mathcal{B}_{\mathscr{X}})$-measurable. Note that each B_j is in $\mathcal{B}_{\mathscr{X}}$, and so, therefore, is each A_j. Since K is closed, it is in $\mathcal{L}$, and therefore by the measurability of f, $E_j = f^{-1}(A_j) \cap K$ is in $\mathcal{L}$. Thus $\sum_{j=1}^m \chi_{E_j}(x)x_j$ is in $F_{\mathscr{X}}$, and hence it is measurable.

Now we turn to the statement for $p = \infty$. Set $A_1 = B(x_1, \varepsilon/2)$, the open ball centred at x_1 with radius $\varepsilon/2$, and for $j > 2$ set $A_j = B(x_j, \varepsilon/2) \setminus \bigcup_{i=1}^{j-1} B(x_i, \varepsilon/2)$. Then the A_j are disjoint, and

$$\mathscr{X} = \bigcup_{j=1}^\infty A_j = \bigcup_{j=1}^\infty B(x_j, \varepsilon/2).$$

Let $E_j = f^{-1}(A_j)$; then $\{E_j\}_{j=1}^{\infty}$ is a partition of $\mathbb{R}^n$. Consider the function $\sum_{j=1}^{\infty} \chi_{E_j} x_j$. It follows from the measurability of f that $E_j \in \mathcal{L}$ for each $j \in \mathbb{N}$. This implies that $\chi_{E_j} x_j$ is $(\mathcal{L}, \mathcal{B}_{\mathscr{X}})$-measurable for each j, and since the set of $(\mathcal{L}, \mathcal{B}_{\mathscr{X}})$-measurable functions is a vector space, we have that $\sum_{j=1}^{m} \chi_{E_j} x_j$ is measurable for each m. Since $\sum_{j=1}^{\infty} \chi_{E_j} x_j$ is the pointwise limit of $\sum_{j=1}^{m} \chi_{E_j} x_j$ as m tends to infinity, we conclude, by Theorem 2.7, that it is measurable.

Finally, since $\|f(x) - x_j\|_{\mathscr{X}} < \frac{\varepsilon}{2}$ for $x \in E_j$, $j \geq 1$, we have

$$\Big\| f - \sum_{j=1}^{\infty} \chi_{E_j} x_j \Big\|_{L^{\infty}(\mathbb{R}^n, \mathscr{X})} = \Big\| \sum_{j=1}^{\infty} \chi_{E_j} [f - x_j] \Big\|_{L^{\infty}(\mathbb{R}^n, \mathscr{X})} \leq \frac{\varepsilon}{2} < \varepsilon.$$

□

For any vector space X, we let $C_c(\mathbb{R}^n, X)$ denote the space of continuous, compactly supported functions from $\mathbb{R}^n$ into X. We will show that for any Banach space $\mathscr{X}$, $C_c(\mathbb{R}^n, \mathscr{X})$ is dense in $L^p(\mathbb{R}^n, \mathscr{X})$ whenever $1 \leq p < \infty$. It follows that for $p > 1$ we can just as well take the supremum in Theorem 4.7 over all $\phi \in C_c(\mathbb{R}^n, \mathscr{X}^*)$ which satisfy $\|\phi\|_q \leq 1$. To see this, let $g^* \in L^q(\mathbb{R}^n, \mathscr{X}^*)$ with $\|g^*\|_q = 1$. We can approximate it in $L^q(\mathbb{R}^n, \mathscr{X}^*)$ by a sequence $\phi_n \in C_c(\mathbb{R}^n, \mathscr{X}^*)$. Furthermore, if we let $\psi_n = \phi_n / \|\phi_n\|_q$, then $\|\psi_n\|_q = 1$, and since $\|\phi_n\|_q \to \|g\|_q = 1$, we have

$$\|\psi_n - \phi_n\|_q = \Big| \tfrac{1}{\|\phi_n\|_q} - 1 \Big| \|\phi_n\|_q \to 0.$$

Hence,

$$\|\psi_n - g^*\|_q \leq \|\psi_n - \phi_n\|_q + \|\phi_n - g^*\|_q \to 0.$$

Thus g^* can be approximated by a sequence $\psi_n \in C_c(\mathbb{R}^n, \mathscr{X}^*)$ with $\|\psi_n\|_q = 1$. Now

$$\begin{aligned} |\theta_f(g^* - \psi_n)| &\leq \int_{\mathbb{R}^n} \|g^*(x) - \psi_n(x)\|_{op} \|f(x)\|_{\mathscr{X}} \, \mathrm{d}x \\ &\leq \|g^* - \psi_n\|_{L^q(\mathbb{R}^n, \mathscr{X}^*)} \|f\|_{L^p(\mathbb{R}^n, \mathscr{X})}, \end{aligned}$$

and this tends to 0. Thus, $\theta_f(\psi_n) \to \theta_f(g^*)$, and therefore

$$\|f\|_p = \sup_{\|g^*\|_{L^q(\mathbb{R}^n, \mathscr{X}^*)} \leq 1} |\theta_f(g^*)| = \sup_{\substack{\psi \in C_c(\mathbb{R}^n, \mathscr{X}^*) \\ \|\psi\|_q = 1}} |\theta_f(\psi)|.$$

To show that $C_c(\mathbb{R}^n, \mathscr{X})$ is dense in $L^p(\mathbb{R}^n, \mathscr{X})$ we require a preliminary proposition, and for this proposition we need a version of Urysohn's lemma:

Lemma 4.9 (Urysohn's lemma)**.** If $K \subset \mathcal{U} \subset \mathbb{R}^n$, where K is compact and $\mathcal{U}$ is open, then there exists a continuous function $f : \mathbb{R}^n \to [0, 1]$ such that $f = 1$ on K and $f = 0$ outside a compact subset of $\mathcal{U}$.

For a proof of Urysohn's lemma see Folland [5], p. 131.

Proposition 4.10. Characteristic functions of Lebesgue measurable sets can be approximated in $L^p(\mathbb{R}^n, \mathbb{C})$ by continuous, compactly supported functions.

Proof. Let E be Lebesgue measurable, and let $\varepsilon > 0$. By the outer regularity of the Lebesgue measure there is an open set $\mathcal{U}$ such that

$$m(\mathcal{U}) - m(E) < \frac{\varepsilon^p}{2}, \tag{4.11}$$

and by the inner regularity there is a compact set K such that

$$m(E) - m(K) < \frac{\varepsilon^p}{2}. \tag{4.12}$$

Adding (4.11) and (4.12) we get

$$m(\mathcal{U}) - m(K) < \varepsilon^p.$$

Now by Urysohn's lemma there exists a continuous, compactly supported function ψ such that

$$\chi_K \leq \psi \leq \chi_{\mathcal{U}}.$$

It follows that

$$\chi_K - \chi_{\mathcal{U}} \leq \psi - \chi_E \leq \chi_{\mathcal{U}} - \chi_K,$$

that is,

$$|\psi - \chi_E| \leq \chi_{\mathcal{U}} - \chi_K = \chi_{\mathcal{U} \setminus K}.$$

Hence,

$$\|\psi - \chi_E\|_p \leq \|\chi_{\mathcal{U} \setminus K}\|_p = (m(\mathcal{U}) - m(K))^{1/p} < \varepsilon.$$

□

Theorem 4.11. When $1 \leq p < \infty$, $C_c(\mathbb{R}^n, \mathscr{X})$ is dense in $L^p(\mathbb{R}^n, \mathscr{X})$.

Proof. Let $f \in L^p(\mathbb{R}^n, \mathscr{X})$. By Proposition 4.8 there exists $\phi = \sum_{j=1}^m \chi_{E_j} x_j \in F_{\mathscr{X}}$ such that

$$\|f - \phi\|_p < \frac{\varepsilon}{2}. \tag{4.13}$$

By Proposition 4.10 we may, for each $j = 1, 2, \ldots, m$, choose a function $\psi_j \in C_c(\mathbb{R}^n, \mathbb{C})$ such that

$$\|\chi_{E_j} - \psi_j\|_p < \frac{\varepsilon}{2\|x_j\|m}.$$

Let $g = \sum_{j=1}^m \psi_j x_j$. We have

$$\begin{aligned}\|\phi - g\|_p &= \Big\| \sum_{j=1}^m \chi_{E_j} x_j - \sum_{j=1}^m \psi_j x_j \Big\|_p \\ &\leq \sum_{j=1}^m \|(\chi_{E_j} - \psi_j) x_j\|_p.\end{aligned}$$

Now

$$\begin{aligned}\|(\chi_{E_j} - \psi_j) x_j\|_p &= \Big(\int \|(\chi_{E_j} - \psi_j) x_j\|_{\mathscr{X}}^p \Big)^{1/p} \\ &= \Big(\int |(\chi_{E_j} - \psi_j)|^p \|x_j\|_{\mathscr{X}}^p \Big)^{1/p} \\ &= \|x_j\|_{\mathscr{X}} \|\chi_{E_j} - \psi_j\|_p.\end{aligned}$$

Therefore,

$$\|\phi - g\|_p \leq \sum_{j=1}^m \|x_j\|_{\mathscr{X}} \|\chi_{E_j} - \psi_j\|_p < \sum_{j=1}^m \frac{\varepsilon}{2\|x_j\|m} \cdot \|x_j\| = \frac{\varepsilon}{2}. \tag{4.14}$$

Putting (4.13) and (4.14) together we get

$$\|f - g\|_q \leq \|f - \phi\|_q + \|\phi - g\|_q < \varepsilon.$$

□

The next proposition is also needed for the proof of Theorem 4.7. For a proof, see Folland [5], p. 188.

Proposition 4.12. Suppose that p and q are conjugate exponents, and $1 \leq p \leq \infty$. For $f \in L^p(\mathbb{R}^n, \mathbb{C})$ we have

$$\|f\|_p = \sup \Big\{ \Big| \int fg \Big| : g \in L^q(\mathbb{R}^n, \mathbb{C}) \ \text{ with } \ \|g\|_q = 1 \Big\}. \tag{4.15}$$

The denseness of $C_c(\mathbb{R}^n, \mathbb{C})$ in $L^q(\mathbb{R}^n, \mathbb{C})$ has the consequence that for $p > 1$ in Proposition 4.12, we may take the supremum instead over continuous, compactly supported functions of unit norm. The argument is simply a repetition of the one preceding Theorem 4.11: let $g \in L^q(\mathbb{R}^n, \mathbb{C})$ with $\|g\|_q = 1$. As before, there is a sequence $\{\psi_n\}$ in $C_c(\mathbb{R}^n, \mathbb{C})$ with $\|\psi_n\|_q = 1$ such that $\psi_n \to g$ in $L^q(\mathbb{R}^n, \mathbb{C})$. Then by Hölder's inequality

$$\Big|\int f(g - \psi_n)\Big| \leq \int |f||g - \psi_n| \leq \|f\|_p \|g - \psi_n\|_q \to 0.$$

Thus $\int fg = \lim_{n\to\infty} \int f\psi_n$, which implies that

$$\|f\|_p = \sup_{\|g\|_{L^q(\mathbb{R}^n,\mathbb{C})} \leq 1} \Big|\int fg\Big| = \sup_{\substack{\psi \in C_c(\mathbb{R}^n,\mathbb{C}) \\ \|\psi\|_q = 1}} \Big|\int f\psi\Big|.$$

We now come to the proof of Theorem 4.7. For this too we follow Grafakos ([6], pp. 324–325).

Proof. We wish to show that for any $f \in L^p(\mathbb{R}^n, \mathscr{X})$, $1 \leq p \leq \infty$,

$$\|f\|_p = \sup_{\|g^*\|_{L^q(\mathbb{R}^n,\mathscr{X}^*)} \leq 1} \Big|\int_{\mathbb{R}^n} g^*(x)f(x)\,\mathrm{d}x\Big|. \tag{4.16}$$

For any $g^* \in L^q(\mathbb{R}^n, \mathscr{X}^*)$ with $\|g^*\|_q \leq 1$, (4.7) shows that

$$\int_{\mathbb{R}^n} |g^*(x)f(x)|\,\mathrm{d}x \leq \|g^*\|_q \|f\|_p \leq \|f\|_p,$$

whence we see that the right-hand-side of (4.16) is controlled by the left-hand-side.

Now to establish the more difficult inequality. Let $f \in L^p(\mathbb{R}^n, \mathscr{X})$, and let $\varepsilon > 0$. We know from Proposition 4.8 that there exists a function $\phi = \sum_{j=1}^m \chi_{E_j} x_j \in L^p(\mathbb{R}^n, \mathscr{X})$—where $m = \infty$ when $p = \infty$, and the E_j are disjoint, measurable subsets of $\mathbb{R}^n$—such that $\|f - \phi\|_p < \varepsilon/3$. Since $\|\phi\|_{\mathscr{X}} \in L^p(\mathbb{R}^n, \mathbb{C})$, we may—by Proposition 4.12—choose a nonnegative function $h \in L^q(\mathbb{R}^n, \mathbb{C})$ such that $\|h\|_q = 1$ and

$$\|\phi\|_p < \int_{\mathbb{R}^n} \|\phi(x)\|_{\mathscr{X}} h(x)\,\mathrm{d}x + \frac{\varepsilon}{6}. \tag{4.17}$$

When $p < \infty$, we may—by the remarks preceding Theorem 4.11—choose h to be continuous and compactly supported, which ensures that h is integrable. The integrability of h when $p = \infty$ is given, since $h \in L^q(\mathbb{R}^n, \mathbb{C})$. It follows from Proposition 3.9

that for each x_j there exists an $x_j^* \in \mathscr{X}^*$ satisfying $\|x_j^*\|_{op} = 1$ and $x_j^* x_j = \|x_j\|$; in particular

$$\|x_j\| < x_j^* x_j + \frac{\varepsilon}{6(\|h\|_1 + 1)}. \tag{4.18}$$

Set $G(x) = \sum_{j=1}^m h(x)\chi_{E_j}(x)x_j^*$. Then $G(x) \in \mathscr{X}^*$ for each x. The measurability of E_j implies that $\chi_{E_j}x_j^*$ is $(\mathcal{L}, \mathcal{B}_{\mathscr{X}^*})$-measurable for each j, and it follows that G is $(\mathcal{L}, \mathcal{B}_{\mathscr{X}^*})$-measurable. Moreover, when $q < \infty$,

$$\begin{aligned}
\|G\|_q &= \int_{\mathbb{R}^n} \Big\| \sum_{j=1}^m h(x)\chi_{E_j}(x)x_j^* \Big\|_{op}^q \,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \sum_{j=1}^m \chi_{E_j}(x)\|h(x)x_j^*\|_{op}^q \,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \sum_{j=1}^m \chi_{E_j}(x)h(x)^q \,\mathrm{d}x \\
&\le \|h\|_q^q,
\end{aligned}$$

whence $\|G\|_q \le 1$. When $q = \infty$,

$$\|G(x)\|_{op} = \sum_{j=1}^\infty \chi_{E_j}(x)\|h(x)x_j^*\|_{op} = \sum_{j=1}^\infty \chi_{E_j}(x)h(x) \le \|h\|_\infty = 1,$$

whence $\|G\|_\infty \le 1$.

Now observe that

$$\begin{aligned}
G(x)\phi(x) &= \sum_{j=1}^m h(x)\chi_{E_j}(x)x_j^* \Big(\sum_{i=1}^m \chi_{E_i}(x)x_i \Big) \\
&= \sum_{j=1}^m \sum_{i=1}^m h(x)\chi_{E_j}(x)\chi_{E_i}(x)x_j^* x_i \\
&= \sum_{j=1}^m h(x)\chi_{E_j}(x)x_j^* x_j \\
&> \sum_{j=1}^m h(x)\chi_{E_j}(x)\left(\|x_j\| - \frac{\varepsilon}{6(\|h\|_1 + 1)} \right) \quad \text{by} \quad (4.18).
\end{aligned}$$

Then

$$\begin{aligned}
G(x)\phi(x) &> h(x)\sum_{j=1}^m \chi_{E_j}(x)\|x_j\| - \frac{h(x)\varepsilon}{6(\|h\|_1 + 1)} \sum_{j=1}^m \chi_{E_j}(x) \\
&= h(x)\|\phi(x)\|_{\mathscr{X}} - \frac{h(x)\varepsilon}{6(\|h\|_1 + 1)} \sum_{j=1}^m \chi_{E_j}(x),
\end{aligned}$$

which implies that

$$\begin{aligned}\int_{\mathbb{R}^n} G(x)\phi(x)\,\mathrm{d}x &\geq \int_{\mathbb{R}^n} h(x)\|\phi(x)\|_{\mathscr{X}}\,\mathrm{d}x - \frac{\varepsilon}{6(\|h\|_1+1)}\int_{\mathbb{R}^n} h(x)\sum_{j=1}^{m}\chi_{E_j}(x)\,\mathrm{d}x \\ &> \|\phi\|_p - \frac{\varepsilon}{6} - \frac{\varepsilon}{6(\|h\|_1+1)}\|h\|_1 \quad \text{by} \quad (4.17) \\ &> \|\phi\|_p - \frac{\varepsilon}{3}.\end{aligned}$$

Recall that ϕ was chosen such that $\|\phi - f\|_p < \frac{\varepsilon}{3}$. This implies that $\|\phi\|_p \geq \|f\|_p - \frac{\varepsilon}{3}$. Therefore, the preceding calculation shows that

$$\int_{\mathbb{R}^n} G(x)\phi(x)\,\mathrm{d}x > \|f\|_p - \frac{2\varepsilon}{3}. \tag{4.19}$$

Furthermore, since

$$\begin{aligned}\int_{\mathbb{R}^n} \left|G(x)\big(\phi(x) - f(x)\big)\right|\mathrm{d}x &\leq \int_{\mathbb{R}^n} \|G(x)\|_{op}\|\phi(x) - f(x)\|_{\mathscr{X}}\,\mathrm{d}x \\ &\leq \|G\|_q\|\phi - f\|_p \\ &\leq \|\phi - f\|_p \\ &< \frac{\varepsilon}{3},\end{aligned}$$

we have

$$\left|\int_{\mathbb{R}^n} G(x)\phi(x)\,\mathrm{d}x - \int_{\mathbb{R}^n} G(x)f(x)\,\mathrm{d}x\right| < \frac{\varepsilon}{3}.$$

Bearing in mind that $\int_{\mathbb{R}^n} G(x)\phi(x)\,\mathrm{d}x$ is nonnegative, this means that

$$\begin{aligned}\int_{\mathbb{R}^n} G(x)\phi(x)\,\mathrm{d}x &< \left|\int_{\mathbb{R}^n} G(x)f(x)\,\mathrm{d}x\right| + \frac{\varepsilon}{3} \\ &\leq \sup_{\|g^*\|_{L^q(\mathbb{R}^n,\mathscr{X}^*)}\leq 1}\left|\int_{\mathbb{R}^n} g^*(x)f(x)\,\mathrm{d}x\right| + \frac{\varepsilon}{3}.\end{aligned} \tag{4.20}$$

Putting (4.19) and (4.20) together, we get

$$\|f\|_p < \sup_{\|g^*\|_{L^q(\mathbb{R}^n,\mathscr{X}^*)}\leq 1}\left|\int_{\mathbb{R}^n} g^*(x)f(x)\,\mathrm{d}x\right| + \varepsilon.$$

Letting $\varepsilon \to 0$ yields the desired inequality. □

Now that we have proven Theorem 4.7, we discuss more specifically what it means in Hilbert space. For $f \in L^p(\mathbb{R}^n, \mathscr{H})$, we define F_f on $L^q(\mathbb{R}^n, \mathscr{H})$ by

$$F_f(g) = \int_{\mathbb{R}^n} \langle f(x), g(x) \rangle \, \mathrm{d}x.$$

Recall the Riesz Representation Theorem:

Theorem 4.13. Given $\xi \in \mathscr{H}^*$, there is a unique vector $y \in \mathscr{H}$ such that $\xi = \langle \cdot, y \rangle$, and moreover, $\|\xi\|_{op} = \|y\|_{\mathscr{H}}$.

Suppose we are given $g^* \in L^q(\mathbb{R}^n, \mathscr{H}^*)$. By the Riesz Representation Theorem there exists, for each $x \in \mathbb{R}^n$, a $g(x) \in \mathscr{H}$ such that

$$g^*(x) = \langle \cdot, g(x) \rangle \quad \text{and} \quad \|g^*(x)\|_{op} = \|g(x)\|_{\mathscr{H}}.$$

It follows that

$$F_f(g) = \int_{\mathbb{R}^n} \langle f(x), g(x) \rangle \, \mathrm{d}x = \theta_f(g^*),$$

and

$$\|g\|_{L^q(\mathbb{R}^n, \mathscr{H})} = \|g^*\|_{L^q(\mathbb{R}^n, \mathscr{H}^*)}.$$

It follows that for Hilbert spaces Theorem 4.7 may be expressed as follows:

Theorem 4.14. The map $f \mapsto \overline{F_f}$ is an isometric embedding of $L^p(\mathbb{R}^n, \mathscr{H})$ into $L^q(\mathbb{R}^n, \mathscr{H})^*$. In other words,

$$\|f\|_p = \sup_{\|g^*\|_{L^q(\mathbb{R}^n, \mathscr{H}^*)} \leq 1} |\theta_f(g^*)| = \sup_{\|g\|_{L^q(\mathbb{R}^n, \mathscr{H})} \leq 1} |F_f(g)|.$$

As usual, by the remarks preceding Theorem 4.11, we may take the supremum in the above equation over the continuous, compactly supported functions whose L^q norms are bounded by 1:

$$\|f\|_p = \sup_{\substack{\psi^* \in C_c(\mathbb{R}^n, \mathscr{H}^*) \\ \|\psi^*\|_q = 1}} |\theta_f(\psi^*)| = \sup_{\substack{\psi \in C_c(\mathbb{R}^n, \mathscr{H}) \\ \|\psi\|_q = 1}} |F_f(\psi)|.$$

Chapter 5

L^p Boundedness of a Type of Singular Integral

In this concluding chapter, we present a modified version of Theorem 1 from chapter II of Stein's book *Singular Integrals* [9]. In *Singular Integrals* there is a section devoted to showing that a type of singular integral is bounded on L^p. We attempt to imitate this result and extend it to the vector-valued setting. We follow Stein closely in the proof of Theorem 5.1.

Suppose that we have a linear transformation T of type $(2, 2)$ from $L^2(\mathbb{R}^n, \mathscr{H}_1)$ to $L^2(\mathbb{R}^n, \mathscr{H}_2)$, where $\mathscr{H}_1$ and $\mathscr{H}_2$ are separable. Suppose further that for bounded, compactly supported, measurable functions, T may be written as a convolution

$$Tf(x) = K * f(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,\mathrm{d}y \tag{5.1}$$

for $x \notin \operatorname{supp} f$. The kernel K in (5.1) takes values in $\mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ and is required to satisfy

(i) K is $(\mathcal{L}, \mathcal{B}_{\mathscr{B}(\mathscr{H}_1,\mathscr{H}_2)})$-measurable.

(ii) K is integrable on any compact set that excludes the origin.

(iii) there is a constant $B > 0$ such that

$$\int_{\|x\|\geq 2\|y\|} \|K(x-y) - K(x)\|_{op}\,\mathrm{d}x \leq B; \quad \|y\| > 0.$$

Thus T is a vector-valued function which can be expressed as a convolution operator on certain functions, with kernel having a singularity at the origin. Condition (iii) expresses this singularity. Condition (ii) ensures that (5.1) is well-defined for $x \notin \operatorname{supp} f$.

We would like to be able to assert that T is of type (p, p) for all $p \in (1, \infty)$. We were able show that T is of type (p, p) for $p \in (1, 2)$, but due to the fact that our vector-valued L^p duality theorem is not as strong as the scalar-valued result used in

Singular Integrals, we cannot prove L^p boundedness for the case $p \in (2, \infty)$ without a fairly strong hypothesis. Specifically, to prove L^p boundedness for $p > 2$ we need the hypothesis

(iv) $Tf \in L^p(\mathbb{R}^n, \mathscr{H}_2)$ when f is continuous and compactly supported.

Lastly, in order to prove L^p boundedness for $p > 2$, the differences that arise from working in the setting of vector-valued functions motivate us to require that

(v) T^*—the adjoint of T when T is regarded as a bounded linear operator from $L^2(\mathbb{R}^n, \mathscr{H}_1)$ to $L^2(\mathbb{R}^n, \mathscr{H}_2)$—is given in the same way as T but for some kernel K^* which satisfies the same requirements as K, say with B' instead of B in condition (iii).

Theorem 5.1. Suppose that T is a linear transformation with the properties described in the second paragraph above—that is, with kernel satisfying (i) to (iii). Then for each $p \in (1, 2)$ there exists a constant A_p such that

$$\|Tf\|_p \leq A_p \|f\|_p \tag{5.2}$$

for all bounded, compactly supported f. Each A_p depends only on p, B, and the dimension n. One can thus extend T to all of $L^p(\mathbb{R}^n, \mathscr{H}_2)$ by continuity, thereby making T of type (p, p) on $L^p(\mathbb{R}^n, \mathscr{H}_2)$.

Proof. **Getting oriented:** The first and largest part of the proof consists of showing that T is of weak-type $(1, 1)$ on the bounded, compactly supported functions. In other words, we wish to find a constant C, independant of f and α, such that

$$m\{x : \|Tf(x)\| > \alpha\} \leq \frac{C}{\alpha} \|f\|_1.$$

Step 1: Splitting up f into g and b. Establishing a weak-type $(1, 1)$ inequality for Tg. Let f be a bounded, compactly supported, $\mathscr{H}_1$-valued measurable function on $\mathbb{R}^n$. Then clearly $f \in L^1(\mathbb{R}^n, \mathscr{H}_1)$. We apply Theorem 4.1 to the nonnegative integrable function $\|f\|$: we have a set $F \subset \mathbb{R}^n$ such that $\|f\| \leq \alpha$ a.e. on F, $\Omega = F^c = \bigcup_{j=1}^{\infty} Q_j$, and

$$m(\Omega) < \frac{1}{\alpha}\|f\|_1, \tag{5.3}$$

$$\frac{1}{m(Q_k)}\int_{Q_k} \|f(x)\|\,\mathrm{d}x \leq 2^n\alpha. \tag{5.4}$$

We set

$$g(x) = \begin{cases} f(x) & \text{if } x \in F \\ \frac{1}{m(Q_j)}\int_{Q_j} f(x)\,\mathrm{d}x & \text{if } x \in Q_j^\circ, \end{cases}$$

and

$$b(x) = f(x) - g(x).$$

This defines $g(x)$ and $b(x)$ for a.e. x. Observe that g is bounded, since $\|f\| \leq \alpha$ on F, and on F^c we have (5.4). Moreover, we can show that g is compactly supported. Its support is contained in $K_0 = \operatorname{supp} f \cup \{Q_k : Q_k \cap \operatorname{supp} f \neq \emptyset\}$. Recall the partition of $\mathbb{R}^n$ into cubes in the proof of Theorem 4.1; consider the cubes at the top level in the partition—the largest cubes, that is. It follows easily from f being compactly supported that K_0 will be contained in a finite union of these cubes, which will be closed and bounded. Since $\operatorname{supp} g = \overline{\{x \in \mathbb{R}^n : g(x) \neq 0\}}$ is a closed subset of a compact set, it is compact.

Now,

$$\|g\|_2^2 = \int_{\mathbb{R}^n} \|g(x)\|^2\,\mathrm{d}x = \int_F \|g(x)\|^2\,\mathrm{d}x + \int_\Omega \|g(x)\|^2\,\mathrm{d}x. \tag{5.5}$$

For the first term on the right-hand-side,

$$\int_F \|g(x)\|^2\,\mathrm{d}x = \int_F \|f(x)\|\|f(x)\|\,\mathrm{d}x \leq \int_F \alpha\|f(x)\|\,\mathrm{d}x \leq \alpha\|f\|_1.$$

For the second term, note that if $x \in \Omega$ then $x \in Q_j$ for some j, and hence

$$\begin{aligned} \|g(x)\|^2 &= \frac{1}{m(Q_j)^2}\Big\|\int_{Q_j} f(x)\,\mathrm{d}x\Big\|^2 \\ &\leq \frac{1}{m(Q_j)^2}\Big(\int_{Q_j} \|f(x)\|\,\mathrm{d}x\Big)^2 \\ &\leq 2^{2n}\alpha^2 \quad \text{by (5.4).} \end{aligned}$$

Hence, by (5.3),

$$\int_\Omega \|g(x)\|^2\,\mathrm{d}x \leq 2^{2n}\alpha^2 m(\Omega) \leq 2^{2n}\alpha\|f\|_1.$$

So altogether the right-hand-side of (5.5) is bounded by $(2^{2n}+1)\alpha\|f\|_1$, which shows that $g \in L^2(\mathbb{R}^n, \mathscr{H}_1)$, with $\|g\|_2^2 \leq (2^{2n}+1)\alpha\|f\|_1$. Thus, since T is of type $(2,2)$ (and therefore of weak-type $(2,2)$), there exists a constant A_2 such that

$$m\{x : \|Tg(x)\| > \alpha/2\} \leq \frac{4A_2^2}{\alpha^2}\|g\|_2^2 \leq \frac{C_0}{\alpha}\|f\|_1, \tag{5.6}$$

where $C_0 = 4A_2^2(2^{2n}+1)$.

Step 2: establishing a weak-type $(1,1)$ inequality for Tb. We wish to find an analogous estimate for Tb, so that, putting the two estimates together, we will have that T is of weak-type $(1,1)$. We first make some observations about b:

$$\begin{aligned}\int_{Q_j} b(x)\,\mathrm{d}x &= \int_{Q_j} f(x)\,\mathrm{d}x - \int_{Q_j} g(x)\,\mathrm{d}x \\ &= \int_{Q_j} f(x)\,\mathrm{d}x - \int_{Q_j} \frac{1}{m(Q_j)}\int_{Q_j} f(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{Q_j} f(x)\,\mathrm{d}x - \frac{1}{m(Q_j)}\int_{Q_j} 1\,\mathrm{d}x \int_{Q_j} f(y)\,\mathrm{d}y.\end{aligned}$$

Thus,

$$b(x) = 0 \text{ for all } x \in F, \quad \text{and} \quad \int_{Q_j} b(x)\,\mathrm{d}x = 0. \tag{5.7}$$

Sub-step: A modified decomposition of the domain into cubes. Now we consider cubes Q_j^* with the same centre y^j as the cubes Q_j, but with sidelength expanded by a factor of $2\sqrt{n}$. We let $\Omega = \bigcup_j Q_j = F^c$, $\Omega^* = \bigcup_j Q_j^*$, and $F^* = (\Omega^*)^c$. Then $\Omega \subset \Omega^*$ and $F^* \subset F$. Moreover,

$$m(\Omega^*) \leq (2\sqrt{n})^n m(\Omega), \tag{5.8}$$

since

$$m(\Omega^*) \leq \sum_{j=1}^{\infty} m(Q_j^*) = \sum_{j=1}^{\infty} \big(2\sqrt{n}\,\mathrm{sidelength}(Q_j)\big)^n,$$

and

$$\sum_{j=1}^{\infty} \big(\mathrm{sidelength}(Q_j)\big)^n = \sum_{j=1}^{\infty} m(Q_j) = m(\Omega).$$

A geometric argument shows that if $x \in (Q_j^*)^c$ and $y \in Q_j$, then

$$\|x - y^j\| \geq 2\|y - y^j\|. \tag{5.9}$$

For if we let s denote the sidelength of Q_j, then $y \in Q_j$ means that

$$|y_i - y_i^j| \leq \frac{s}{2} \quad \text{for all } i = 1, 2, \cdots, n.$$

Therefore

$$\|y - y^j\| = \left(\sum_{i=1}^n |y_i - y_i^j|^2\right)^{1/2} \leq \left(\sum_{i=1}^n \left(\frac{s}{2}\right)^2\right)^{1/2} = \frac{\sqrt{n}s}{2}.$$

Moreover, if $x \notin Q_j^*$, then by definition, $|x_\ell - y_\ell^j| > \frac{2\sqrt{n}s}{2} = \sqrt{n}s$ for some $\ell = 1, \cdots, n$, and hence

$$\|x - y^j\| = \left(\sum_{i=1}^n |x_i - y_i^j|^2\right)^{1/2} \geq |x_\ell - y_\ell^j| > \sqrt{n}s.$$

Thus we have that $\|x - y^j\| \geq 2\|y - y^j\|$.

Sub-step: splitting up b. We now define b_j by

$$b_j(x) = \begin{cases} b(x) & \text{if } x \in Q_j \\ 0 & \text{if } x \notin Q_j. \end{cases}$$

Then $b(x) = \sum_{j=1}^\infty b_j(x)$ for a.e. x, and moreover,

$$Tb(x) = \sum_{j=1}^\infty Tb_j(x) \quad \text{for } x \in F^*. \tag{5.10}$$

To verify this we first observe that b is compactly supported (because g and f are), with $\operatorname{supp} b \subset \Omega \subset \Omega^*$ by (5.7). So for $x \in F^*$ we have

$$Tb(x) = \int_{\mathbb{R}^n} K(x - y) b(y)\, \mathrm{d}y.$$

Now $\sum_1^N b_j(y) \to b(y)$ for a.e. y, and since $K(x - y)$ is continuous,

$$\sum_{j=1}^N K(x - y) b_j(y) \to K(x - y) b(y)$$

for a.e. y. Now

$$\sum_{j=1}^N K(x - y) b_j(y) = \begin{cases} K(x - y) b(y) & \text{if } y \in \bigcup_1^N Q_j \\ 0 & \text{if otherwise,} \end{cases}$$

and hence $\|\sum_1^N K(x-y)b_j(y)\| \le \|K(x-y)b(y)\| \in L^1$. Thus, by the dominated convergence theorem for the Bochner integral,

$$Tb(x) = \lim_{N\to\infty} \int_{\mathbb{R}^n} \sum_{j=1}^{N} K(x-y)b_j(y)\,\mathrm{d}y = \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} K(x-y)b_j(y)\,\mathrm{d}y.$$

In other words, $Tb(x) = \sum Tb_j(x)$.

Sub-step: Estimating $\sum_{j=1}^{\infty}\int_{Q_j}\|b_j(y)\|\,\mathrm{d}y$. Observe that for $y \in Q_j$,

$$\begin{aligned}\|b_j(y)\| = \|b(y)\| &= \left\| f(y) - \frac{1}{m(Q_j)}\int_{Q_j} f(y)\,\mathrm{d}y \right\| \\ &\le \|f(y)\| + \frac{1}{m(Q_j)}\int_{Q_j}\|f(y)\|\,\mathrm{d}y \\ &\le \|f(y)\| + 2^n\alpha \quad \text{by (5.4).}\end{aligned}$$

Hence,

$$\int_{Q_j}\|b_j(y)\|\,\mathrm{d}y \le \int_{Q_j}\|f(y)\|\,\mathrm{d}y + 2^n\alpha m(Q_j),$$

and since by (5.4) again, $\int_{Q_j}\|f(y)\|\,\mathrm{d}y \le 2^n\alpha m(Q_j)$, this gives us

$$\int_{Q_j}\|b_j(y)\|\,\mathrm{d}y \le 2^{n+1}\alpha m(Q_j).$$

Therefore,

$$\sum_{j=1}^{\infty}\int_{Q_j}\|b_j(y)\|\,\mathrm{d}y \le 2^{n+1}\alpha\sum_{j=1}^{\infty} m(Q_j) = 2^{n+1}\alpha m(\Omega).$$

By (5.3), this means

$$\sum_{j=1}^{\infty}\int_{Q_j}\|b_j(y)\|\,\mathrm{d}y \le 2^{n+1}\|f\|_1.$$

Sub-step: Weak-type estimate for Tb on F^*. Now, since by (5.7) $\int_{Q_j} b_j(y)\,\mathrm{d}y = 0$, Theorem 2.12 gives us $\int_{Q_j} K(x-y^j)b_j(y)\,\mathrm{d}y = 0$, and therefore

$$Tb_j(x) = \int_{Q_j}[K(x-y) - K(x-y^j)]b_j(y)\,\mathrm{d}y, \tag{5.11}$$

for $x \notin Q_j$. Now by (5.10),

$$\int_{F^*}\|Tb(x)\|\,\mathrm{d}x = \int_{F^*}\left\|\sum_{j=1}^{\infty} Tb_j(x)\right\|\mathrm{d}x \le \int_{F^*}\sum_{j=1}^{\infty}\|Tb_j(x)\|\,\mathrm{d}x$$

where we are using the continuity of the norm. By the monotone convergence theorem,

$$\int_{F^*} \sum_{j=1}^{\infty} \|Tb_j(x)\| \,\mathrm{d}x = \sum_{j=1}^{\infty} \int_{F^*} \|Tb_j(x)\| \,\mathrm{d}x.$$

Also, since $F^* = (\Omega^*)^c = (\bigcup Q_j^*)^c = \bigcap (Q_j^*)^c \subset (Q_j^*)^c$, we have

$$\int_{F^*} \|Tb_j(x)\| \,\mathrm{d}x \leq \int_{x \notin Q_j^*} \|Tb_j(x)\| \,\mathrm{d}x,$$

and then by (5.11)

$$\int_{x \notin Q_j^*} \|Tb_j(x)\| \,\mathrm{d}x \leq \int_{x \notin Q_j^*} \int_{Q_j} \|K(x-y) - K(x-y^j)\| \|b_j(y)\| \,\mathrm{d}y \,\mathrm{d}x.$$

By Tonelli's theorem this becomes

$$\int_{Q_j} \|b_j(y)\| \int_{x \notin Q_j^*} \|K(x-y) - K(x-y^j)\| \,\mathrm{d}x \,\mathrm{d}y.$$

Recalling (5.9) we know that $\{x : x \notin Q_j^*\} \subset \{x : \|x-y^j\| \geq 2\|y-y^j\|\}$, and therefore

$$\int_{x \notin Q_j^*} \|K(x-y) - K(x-y^j)\| \,\mathrm{d}x \leq \int_{\|x-y^j\| \geq 2\|y-y^j\|} \|K(x-y) - K(x-y^j)\| \,\mathrm{d}x.$$

Make the substitutions $x' = x - y^j$ and $y' = y - y^j$. By the translation-invariance of the integral we have

$$\int_{\|x-y^j\| \geq 2\|y-y^j\|} \|K(x-y) - K(x-y^j)\| \,\mathrm{d}x = \int_{\|x'\| \geq 2\|y'\|} \|K(x'-y') - K(x')\| \,\mathrm{d}x',$$

and by our hypothesis (iii) this is bounded by B. So altogether we have

$$\int_{F^*} \|Tb(x)\| \,\mathrm{d}x \leq \sum_{j=1}^{\infty} \int_{F^*} \|Tb_j(x)\| \,\mathrm{d}x \leq B \sum_{j=1}^{\infty} \int_{Q_j} \|b_j(y)\| \,\mathrm{d}y.$$

Therefore, by the preceding sub-step,

$$\int_{F^*} \|Tb(x)\| \,\mathrm{d}x \leq B 2^{n+1} \|f\|_1. \tag{5.12}$$

It follows directly from (5.12) that

$$m\{x \in F^* : \|Tb(x)\| > \alpha/2\} \leq \frac{2}{\alpha} \cdot 2^{n+1} B \|f\|_1.$$

Sub-step: controlling the size of $(F^*)^c$. This follows easily from (5.3):

$$m((F^*)^c) = m(\Omega^*) \leq (2\sqrt{n})^n m(\Omega) \leq \frac{(2\sqrt{n})^n}{\alpha}\|f\|_1.$$

Concluding sub-step: the weak-type $(1,1)$ inequality for Tb. Clearly,

$$m\{x : \|Tb(x)\| > \alpha/2\} \leq m\{x \in F^* : \|Tb(x)\| > \alpha/2\} + m((F^*)^c).$$

We have already found estimates in terms of $\frac{1}{\alpha}\|f\|_1$ for each of the terms on the right-hand-side of this inequality, so that

$$m\{x : \|Tb(x)\| > \alpha/2\} \leq \Big(\frac{2^{n+2}B}{\alpha} + \frac{(2\sqrt{n})^n}{\alpha}\Big)\|f\|_1 = \frac{C_1}{\alpha}\|f\|_1, \tag{5.13}$$

where $C_1 = 2^{n+2}B + (2\sqrt{n})^n$.

Step 3: The weak-type $(1,1)$ inequality for Tf. Since T is linear, we have $Tf = Tg + Tb$, which implies $\|Tf\| \leq \|Tg\| + \|Tb\|$. It follows that

$$\{x : \|Tf(x)\| > \alpha\} \subset \Big\{x : \|Tg(x)\| > \frac{\alpha}{2}\Big\} \cup \Big\{x : \|Tb(x)\| > \frac{\alpha}{2}\Big\},$$

which implies

$$m\{x : \|Tf\| > \alpha\} \leq m\{x : \|Tg\| > \alpha/2\} + m\{x : \|Tb\| > \alpha/2\}. \tag{5.14}$$

This, together with steps 1 and 2, gives us

$$m\{x : \|Tf(x)\| > \alpha\} \leq \frac{C}{\alpha}\|f\|_1,$$

where $C = C_0 + C_1$. This proves that T is of weak-type $(1,1)$ on compactly supported, bounded functions.

Step 4: Interpolating to obtain the type (p,p) inequality for $1 < p < 2$. Now that we have shown that T is of weak-type $(1,1)$ and of weak-type $(2,2)$ on bounded, compactly supported functions, we may apply the interpolation theorem. First note that T is linear and hence subadditive on bounded, compactly supported functions. Given such a function f, we can write $f = f_1 + f_2$ as in the proof of Theorem 4.6, and f_1, f_2 are also bounded and compactly supported. Then T satisfies the weak-type $(1,1)$ inequality for f_1 and the weak-type $(2,2)$ inequality for f_2. Looking at the proof of Theorem 4.6, we see that this is all that is needed to conclude that $\|Tf\|_p \leq A_p\|f\|_p$ for $1 < p < 2$, where A_p depends only on B, p, and n. □

We turn to the case where $2 < p < \infty$.

Theorem 5.2. Suppose that T is as in Theorem 5.1, and additionally satisfies the conditions (iv) and (v) discussed at the beginning of this chapter. Then in addition to satisfying the conclusion of Theorem 5.1, T is of type (p, p) for $2 < p < \infty$. That is, T is of type (p, p) for all $p \in (1, \infty)$.

Proof. Let $f \in C_c(\mathbb{R}^n, \mathscr{H}_1)$. By Theorem 4.14,

$$\|Tf\|_p = \sup_{\substack{g \in C_c(\mathbb{R}^n, \mathscr{H}_2) \\ \|g\|_q = 1}} \left| \int_{\mathbb{R}^n} \langle Tf(x), g(x) \rangle \, \mathrm{d}x \right|.$$

By the Cauchy-Schwarz inequality,

$$|\langle Tf(x), g(x) \rangle| = |\langle f(x), T^* g(x) \rangle| \leq \|f(x)\| \|T^* g(x)\|.$$

Thus by Hölder's inequality,

$$\int_{\mathbb{R}^n} |\langle Tf(x), g(x) \rangle| \, \mathrm{d}x \leq \int_{\mathbb{R}^n} \|f(x)\| \|T^* g(x)\| \, \mathrm{d}x \leq \|f\|_p \|T^* g(x)\|_q.$$

Now since T satisfies (v), and $1 < q < 2$, we have

$$\|T^* g(x)\|_q \leq A_q \|g\|_q,$$

where A_q depends only on B', p, and n. Putting these facts together we get

$$\int_{\mathbb{R}^n} |\langle Tf(x), g(x) \rangle| \, \mathrm{d}x \leq A_q \|g\|_q \|f\|_p.$$

Taking the supremum yields the desired type (p, p)-inequality for continuous, compactly supported functions:

$$\|Tf\|_p \leq A_q \|f\|_p.$$

Extending by continuity to all of L^p yields the result. $\square$

Chapter 6

Conclusion

We have seen that the Bochner integral is the natural extension of the Lebesgue integral for vector-valued functions, in that it is the unique continuous map defined on simple functions in the same way as the Lebesgue integral. Important results such as Hölder's inequality, the Dominated Convergence Theorem, and Fubini's theorem were seen to carry over to the vector-valued setting. We saw that the Fourier transform can be defined in the obvious way with the Bochner integral.

The notion of weak measurability proved useful for establishing the measurability of functions of the form $K(\cdot)f(\cdot)$, where K is an operator-valued function and f is a vector-valued function. This allowed us to define convolution for vector-valued functions. Weak measurability was also useful in proving the vector-valued version of Fubini's theorem.

When we tried to prove a vector-valued version of a theorem about singular integrals with the Bochner integral, we saw that certain results do not carry over easily to the vector-valued setting. As discussed in our introduction, the L^p duality theorem invoked in the proof of the singular integral theorem may not hold for vector-valued functions; the proof presented in Chapter 4 of the weaker version of the L^p duality theorem is more involved than the proof of the analogous scalar-valued result. I do not know whether anyone has proven a stronger vector-valued L^p duality theorem, or whether a stronger version can be proved.

We could not directly imitate the proof found in Stein [9] of the singular integral theorem, and to achieve the same conclusion for vector-valued functions we needed to modify and supplement the hypotheses. I do not know whether an elegant version of the theorem exists for vector-valued functions.

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