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On the construction of quasi-exactly solvable Schrödinger operators on homogeneous spaces

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The closure conditions for a quasi-exactly solvable operator is the requirement that a second-order Lie algebraic differential operator be equivalent, up to scale-change, to a Schrödinger operator on curved space. The present work begins with an invariant characterization of the closure conditions, and after a series of steps gives a reformulation of the closure conditions in terms of finite-dimensional representations of the underlying Lie algebra. These techniques are used to give a complete solution to the closure condition for two different planar realizations of $\mathfrak{sl}(2)$. Along the way two theorems about homogeneous solutions to the closure conditions are introduced and proved. Also introduced is the class of "Abelian" solutions to the closure conditions. Such solutions appear on flat spaces and give rise to a Schrödinger operator with zero potential. The concluding remarks highlight remaining questions and give references to research that address said questions. © *1995 American Institute of Physics.*

I. INTRODUCTION

A relatively recent development in the spectral theory of Schrödinger type operators is the introduction of the class of quasi-exactly solvable (QES) operators. Such operators are interesting because a finite part of their spectrum can be determined by purely algebraic means. At the root of this phenomenon is the defining characteristic of a QES operator, namely that such an operator can be generated by the actions of some finite-dimensional Lie algebra of first-order differential operators. This is the so-called "hidden symmetry algebra" associated to the operator. Thus, a QES operator has the form

$$H = C^{ab} T_a T_b + C^a T_a ,$$

where C^{ab} are constants and the operators

$$T_a = \sum_i T_a^i \partial_i + \eta_a$$

are a basis of the hidden symmetry algebra. It is clear that a finite-dimensional function module of the algebra will be stabilized by H. The payoff lies in the fact that the action of H restricted to this module is a finite-dimensional linear transformation whose spectrum can be computed algebraically. The notion of quasi-exact solvability was introduced in Refs. 1–3. Work in this area has progressed to the extent that a coherent, mathematical framework for future research is now available. A clear and rigorous description of this framework can be found in Ref. 4. An excellent survey of current results and initiatives can also be found in Ref. 5.

In a basic sense, the goal of the above program is to generate even more examples of QES operators, perhaps even to give an exhaustive list as has been done in the one-dimensional case.[6] The first step in this process is the choice of the hidden symmetry algebra and the identification of finite-dimensional modules of functions for this algebra. Work in this direction was carried out in

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Ref. 7, which classifies Lie algebras of first-order differential operators in the plane, and in Ref. 8 which addresses the issue of the nonhomogeneous terms in these differential operators. The second of these papers worked with the convenient assumption that the action of the algebra is transitive, or equivalently that the setting is some homogeneous space, G/H. This assumption will also be in force for the present effort.

The second step is the choice of the constants C^{ab} and C^a. Note that without loss of generality we can take C^{ab} to be symmetric, and we have an induced Riemannian or pseudo-Riemannian metric

$$g^{ij}=C^{ab}T_a^i\otimes T_b^j$$

on the homogeneous space. The choice of constants is constrained by the fact that the resulting H must be equivalent, up to a scale change, to a Schrödinger-type operator, $\Delta+V$, where Δ is the Laplace–Beltrami operator associated with the metric g^{ij}. This equivalence problem was settled a long time ago.[9] A modern discussion in the context of QES theory is to be found in Refs. 1 and 4. The upshot is that a certain one-form associated to the constants must be closed. The authors of Ref. 4 named this constraint the closure condition for QES operators. This constraint is also known to the authors of Ref. 1, where it is presented as the equivalent condition that a certain family of functions be "pure gauge." As stated, the closure conditions seem to be a complicated system of first-order partial differential equations. Solutions have been found in Refs. 4 and 1 for specific examples, but the solutions are not exhaustive, not even for those particular examples.

It is the aim of the present effort to give a general theoretical framework for the study of the closure conditions and to illustrate this framework with some simple examples. Section II introduces the necessary notation. Section III is a quick summary of an isomorphism theorem which allows one to classify representations of a Lie algebra by nonhomogeneous first-order differential operators. The main work begins in Sec. IV. There are three main ideas in this section: an invariant definition and characterization of the closure conditions, two general theorems about the existence of homogeneous solutions to the closure conditions, and a theorem about the invariance of solutions under G action. In Sec. V we develop some techniques for working with "mixed" tensors. These objects are contractions of left and right invariant tensors, and figure prominently in an indices-free description of the closure conditions. As an illustration of these techniques, Sec. V concludes by introducing an interesting new class of "Abelian" solutions to the closure conditions. Abelian solutions are interesting because they engender a flat metric and give rise to a Schrödinger operator with zero potential. Thus, the existence of this class lends support to a conjecture of A. Turbiner[10] to the effect that the QES potentials on a flat background metric are separable in the physical coordinates. In Sec. VI we show how the QES closure conditions can be described in terms of the finite-dimensional modules of the corresponding Lie algebra. It turns out that the closure conditions are nothing but a certain invariant set of polynomial equations, and such equations can be solved if one understands the representation theory for the underlying Lie algebra. The section concludes with a demonstration of these ideas; complete solutions are obtained to the closure conditions for two different planar realizations of $\mathfrak{sl}(2)$. In Sec. VII we discuss the limitations of the present work, and give some pointers to ongoing research which may be useful in overcoming these limitations.

II. NOTATION AND PRELIMINARIES

Let G be a Lie group, H a closed subgroup, let $\mathfrak{g}$, $\mathfrak{h}$ denote the corresponding Lie algebras, and let G/H denote the associated homogeneous space of right cosets. The right G-action on G/H induces a representation of $\mathfrak{g}$ by vector fields, making $\mathscr{C}^\infty(G/H)$ into a $\mathfrak{g}$-module. For $a\in\mathfrak{g}$ we will denote its action on $\mathscr{C}^\infty(G/H)$ by a^H. Let C: $\mathfrak{g}^*\times\mathfrak{g}^*\to\mathbb{R}$ be a symmetric bilinear form and let $C^H = C^{ij} a_i^H \otimes a_j^H$ denote the induced tensor on G/H. If C^H is nondegenerate, it will define a pseudo-Riemannian metric on G/H, which will in turn give rise to the second-order Laplace–Beltrami operator, Δ.

Let e denote the identity element of G, $\mathfrak{g}$ the tangent space at e,

$$\mathrm{Ad}: G \to \mathrm{End}(\mathfrak{g})$$

the adjoint representation of G, and

$$\mathrm{ad}: \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$

the differential of Ad at e. We make $\mathfrak{g}$ into a Lie algebra by taking ad as the multiplication rule. Right multiplication by an element of G defines a left invariant diffeomorphism of G, and thus it makes sense to think of infinitesimal right multiplication by $a \in \mathfrak{g}$ as defining a left-invariant vector field (an infinitesimal diffeomorphism) which we will denote a^L. More generally, for a $\mathfrak{g}$-valued function $f: G \to \mathfrak{g}$, we define f^L to be the vector field

$$x \mapsto (L_x)_* f_x, \quad x \in G,$$

where $L_x: G \to G$ denotes left multiplication by $x \in G$. The map $f \mapsto f^L$ defines a linear isomorphism of $\mathfrak{g}$-valued functions, $\mathscr{C}^\infty(G;\mathfrak{g})$, and vector fields, $\Gamma(TG)$. We let

$$L^{-1}: \Gamma(TG) \to \mathscr{C}^\infty(G;\mathfrak{g})$$

denote the inverse. Thus, for a left-invariant vector field, u, on G, we let $L^{-1}(u)$ denote the corresponding constant $\mathfrak{g}$-valued function. Generalizing further, we have isomorphisms

$$L^{-1}: \Gamma(V(TG)) \to \mathscr{C}^\infty(G, V(\mathfrak{g})),$$

where $V(\mathfrak{g})$ is some tensor space of $\mathfrak{g}$, $V(TG)$ is the corresponding vector bundle over G, and $\Gamma(V(TG))$ is the space of sections of that bundle. These isomorphisms map the left-invariant sections to constant functions. Example: for $\omega \in \Lambda^k \mathfrak{g}^*$, we use ω^L to denote the left-invariant k-form with value ω at e, and define $L^{-1}(\omega^L) = \omega$. Analogously, for $a \in V(\mathfrak{g})$ we let a^R denote the right-invariant tensor field with value a at e, and use

$$R^{-1}: \Gamma(V(TG)) \to \mathscr{C}^\infty(G; V(\mathfrak{g}))$$

to denote the inverse isomorphism.

Let $\pi: G \to G/H$ denote the canonical projection. If a contravariant tensor field, u, on G is $\mathfrak{h}^R$-invariant, then $\pi_* u$ is a well-defined tensor field on G/H. In particular, for $a \in \mathfrak{g}$, we have $a^H = \pi_*(a^L)$, and $C^H = \pi_*(C^L)$. As mentioned in the Introduction, we will be concerned with local phenomena and so we work with a contractible, open neighborhood, U, of $\pi(e)$, rather than with all of G/H. We let G_0 denote $\pi^{-1}(U)$

Let $\mathfrak{h}^\perp \subset \mathfrak{g}$ denote the subspace of forms that annihilate $\mathfrak{h}$. We will on occasion identify $\mathfrak{h}^\perp$ with $(\mathfrak{g}/\mathfrak{h})^*$.

We let $C^k(\mathfrak{g}; \mathscr{C}^\infty(U)) \cong \mathrm{Hom}(\Lambda^k \mathfrak{g}, \mathscr{C}^\infty(U))$ denote the space of k-cochains with $\mathscr{C}^\infty(U)$ coefficients. The coboundary operator $\delta: C^k \to C^{k+1}$ is defined by

$$(\delta\omega)(a_1,\ldots,a_k) = \sum_i (-1)^{i-1} a_i^L \omega(\ldots,\hat{a}_i,\ldots) + \sum_{i<j} (-1)^{i+j} \omega([a_i, a_j],\ldots,\hat{a}_i,\ldots,\widehat{a_j},\ldots)$$

with $H^*(\mathfrak{g}; \mathscr{C}^\infty(U))$ denoting the cohomology groups of the resulting cochain complex. For more about Lie algebra cohomology see Ref. 11. We will identify $C^k(\mathfrak{g}; \mathscr{C}^\infty(U))$ with $\mathscr{C}^\infty(U, \Lambda^k \mathfrak{g}^*)$, and then using the pullback, π^*, identify the latter with a subspace of $\mathscr{C}^\infty(G;\Lambda^k\mathfrak{g}^*)$. Thus, for $\omega\in C^k(\mathfrak{g};\mathscr{C}^\infty(U))$, we will use ω^L to denote the differential k-form on G with the property that

$$\omega^L(a_1^L,\ldots,a_k^L)=\omega(a_1,\ldots,a_k),\quad a_i\in\mathfrak{g}.$$

Indeed, the preceding correspondence defines a cochain complex map, i.e.,

$$d\omega^L=(\delta\omega)^L,$$

where the d on the left is the ordinary exterior derivative.

We will use the symbol $\langle;\rangle$ to denote contraction of tensors and tensor fields, although when ambiguity is not a danger we will denote the contraction by simply writing the objects next to one another. For a vector field, u, we will use $\mathscr{L}u$ to denote the Lie derivative with respect to u, and for $a\in\mathfrak{g}$, we will use $\mathscr{L}a$ to denote the adjoint action of a on tensor spaces of $\mathfrak{g}$. The two notions are related, in as much as, for $\omega\in\mathscr{C}^\infty(G,V(\mathfrak{g}))$, we have

$$(\mathscr{L}a^L)\omega^L=((\mathscr{L}a)\omega)^L.$$

As for denoting the derivative of a function, f, with respect to a vector field, u, we have several notations at our disposal:

$$uf,\quad (\mathscr{L}u)f,\quad \langle df;u\rangle.$$

III. PRELIMINARY DISCUSSION OF NONHOMOGENOUS REPRESENTATIONS

The following is a condensed version of the discussion in Refs. 7 and 8. The representation $\mathfrak{g}^H$ by vector fields on G/H can be modified to a representation by nonhomogeneous first-order operators

$$a_\eta^H=a^H+\eta(a),\quad a\in\mathfrak{g},\eta\in C^1(\mathfrak{g};\mathscr{C}^\infty(U)).$$

The condition that the Lie algebra operation is preserved is equivalent to the condition that η is a cocycle. We define a change of scale to be an endomorphism of $\mathscr{C}^\infty(U)$ given by a nonzero multiplication operator,

$$g\mapsto\mu g,\quad g\in\mathscr{C}^\infty(U),$$

where $\mu=e^f$ for some $f\in\mathscr{C}^\infty(U)$. Such a change of scale operates on differential operators by conjugation;

$$\mu^{-1}\circ(a^H+\eta(a))\circ\mu=a^H+\eta(a)+a^H(f).$$

Thus, the net result is an addition of a coboundary term, δf. We will call two nonhomogeneous representations equivalent if there they are related by a change of scale. Therefore the space of inequivalent representations is given by $H^1(\mathfrak{g},\mathscr{C}^\infty(U))$.

There is a very convenient isomorphism theorem which allows us to compute $H^1(\mathfrak{g};\mathscr{C}^\infty(U))$. A version of this isomorphism was first described in Ref. 12, and a fuller discussion with generalizations can be found in Ref. 8. Let ω be a representative cocycle of an element of $H^1(\mathfrak{g};\mathscr{C}^\infty(U))$. For all $a\in\mathfrak{g}$ and $b\in\mathfrak{h}$ we have the following, easy to verify, identity:

$$a^L(\omega^Lb^R)=b^R(\omega^La^L)=0.$$

Consequently, ω^Lb^R is a constant function, and we define $P\omega$ to be the corresponding cochain of $C^1(\mathfrak{h})$. In other words, $(P\omega)(b)$ is the constant ω^Lb^R. Note that for $a,b\in\mathfrak{h}$,

$$0 = d\omega^L(a^R, b^R) = a^R(\omega^L b^R) - b^R(\omega^L a^R) - \omega^L[a^R, b^R] = 0 + 0 - \omega^L[a^R, b^R].$$

This shows that $P\omega$ annihilates commutators of $\mathfrak{h}$ and, hence, must be a cocycle. Furthermore, for $f \in \mathscr{C}^\infty(U)$ we have $P(\delta f) = 0$. Therefore, P induces a cohomology homomorphism, which we will also call P, from $H^1(\mathfrak{g};\mathscr{C}^\infty(U))$ to $H^1(\mathfrak{h})$.

Proposition 3.1: P is an isomorphism.

Proof: Suppose $\omega^L b^R = 0$ for all $b \in \mathfrak{h}$. Since ω^L is closed we can always integrate it locally to a function. This function will be constant along the directions $\mathfrak{h}^R$. Since U is contractible we can perform an integration on all of G_0 to get an $f \in \mathscr{C}^\infty(U)$ such that $\delta f = \omega$. Therefore, P must be injective.

Now, let a $\rho \in H^1(\mathfrak{h})$ be given. We identify ρ with the corresponding right-invariant one-form on H. Since U is contractible we can choose a decomposition $G_0 = U \times H$ and pull ρ back along the second projection to get an $\omega \in \Omega^1(G_0)$. It is not hard to verify that $\rho = P(L^{-1}(\omega))$, and thus we have shown that P must be surjective as well. □

Since the infinitesimal left and right actions of an element of $\mathfrak{g}$ coincide at the identity we have the following simple characterization of the isomorphism.

Proposition 3.2: For all $a \in \mathfrak{h}$, we have $(P\omega)(a) = (\omega a)_e$.

IV. CLOSURE CONDITIONS AND THE INDUCED METRIC

The purpose of the present section is to give a naive definition of the closure conditions and then to restate them as the equation

$$\delta\psi = 0,$$

where ψ is a certain one-cochain. As mentioned earlier, C induces a pseudo-Riemannian metric on the homogeneous space, and this metric plays a vital role in the discussion of the closure conditions. We will therefore investigate some important properties of this induced metric. First, we need a criterion for the nondegeneracy of C^H.

Proposition 4.3: The symmetric tensor, C^H, is nondegenerate near $\pi(e)$ if and only if the restriction of the bilinear form C to $\mathfrak{h}^\perp$ is nondegenerate.

Proof. Nondegeneracy is an open condition and so it suffices to consider the nondegeneracy of $C^H_{\pi(e)}$. The proposition follows from the fact that the pullback of $T^*_{\pi(e)}U$ to $T^*_e G \cong \mathfrak{g}^*$ is precisely $\mathfrak{h}^\perp$. □

Definition 4.4: We will call a symmetric form, C, that satisfies the conditions of the above proposition, nondegenerate with respect to $\mathfrak{h}$.

For the rest of this section we let C be such a symmetric form. We take U, a neighborhood of $\pi(e)$, sufficiently small so that C^H is nondegenerate there. Let Δ, grad, and div denote the usual differential operators corresponding to the metric induced by C^H on U. A related second-order operator engendered by C is $\Gamma_0 = C^{ij} a_i^H a_j^H$. Indeed, Δ and Γ_0 have the same second-order part. More generally, let $\eta \in Z^1(\mathfrak{g};\mathscr{C}^\infty(U))$ be a representative cocycle for a nonhomogeneous representation of $\mathfrak{g}$. We put

$$\Gamma_\eta = C^{ij}(a_i)^H_\eta (a_j)^H_\eta.$$

The reader should verify that the definition of Γ_η is independent of the choice of $\mathfrak{g}$-basis.

Definition 4.5: Let $C \in S^2\mathfrak{g}$ be nondegenerate with respect to $\mathfrak{h}$, and $a \in \mathfrak{g}$. The pair (C,a) is called a solution to the QES closure conditions whenever the differential operator $\Gamma_\eta + a^H_\eta$ is equal, after a scale change, to an operator of the form $\Delta + V$, for some potential function $V \in \mathscr{C}^\infty(U)$. If $a = 0$, we will call C a homogeneous solution.

We now take the first in a series of steps to reformulate and simplify the QES closure conditions. Define $\phi \in C^1(\mathfrak{g};\mathscr{C}^\infty(U))$ to be the one-cochain

$$a \mapsto \mathrm{div}(a^H), \quad a \in \mathfrak{g}.$$

Proposition 4.6: The Laplacian and Γ_0 have the same second-order part. They are related by: $\Delta = \Gamma_0 + (C\phi)^H$.

Proof: Let $f \in \mathscr{C}^\infty(U)$ be given. We have

$$\Delta f = \mathrm{div}\ \mathrm{grad}\ f = \mathrm{div}(C^{ij}(a_i^H f)a_j^H) = C^{ij}a_j^H(a_i^H f) + C^{ij}(\mathrm{div}\ a_j^H)(a_i^H f) = \Gamma_0 f + (C\phi)^H f. \quad \square$$

We are now in a position to state the closure conditions without referring to the Laplacian.

Proposition 4.7: (C,a) is a solution to the closure conditions if and only if there exists an $f \in \mathscr{C}^\infty(U)$ such that

$$(C(\phi - 2\eta + \delta f))^H - a^H = 0.$$

Proof: The effect of a scale change on the Laplacian by $\mu = e^{f/2}$ is given by

$$\mu^{-1} \circ \Delta \circ \mu = \Delta + \mathrm{grad}\ f + 1/2 \Delta f + 1/4(\mathrm{grad}\ f)(f), \quad f \in \mathscr{C}^\infty(U).$$

A simple calculation shows that

$$\Gamma_\eta = \Gamma_0 + 2(C\eta)^H + V_\eta,$$

where $(C\eta)^H$ denotes the vector field $C^{ij}\eta_i a_j^H$, and where $V_\eta \in \mathscr{C}^\infty(U)$ is a certain function whose exact form we do not need to consider here. Hence, by the preceding proposition, (C,a) is a solution if and only if

$$\Gamma_0 + (C\phi)^H + (C\delta f)^H = \Gamma_0 + 2(C\eta)^H + a^H. \qquad \square$$

Corollary 4.8: Suppose that C is a homogeneous solution to the closure conditions. Then, if (C,a) is a solution so is (C,ka) for all $k \in \mathbb{R}$.

The cochain ϕ appears to play an intrinsic role in the investigation of the closure conditions, and so it should not be a surprise that it has a surprisingly uncomplicated, intrinsic characterization.

Proposition 4.9: $\delta\phi = 0$, i.e., ϕ is a actually a cocycle.

Proof: For A, B, vector fields on U, define

$$S(A,B) = A(\mathrm{div}\ B) - B(\mathrm{div}\ A) - \mathrm{div}[A, B].$$

A calculation shows that for $f \in \mathscr{C}^\infty(U)$ we have

$$S(fA,B) = S(A,fB) = fS(A,B),$$

i.e., S is a type (2,0) tensor. Now choose local coordinates about $\pi(e)$ and express C^H as $g^{ij}\partial_i \otimes \partial_j$. We have then

$$\mathrm{div}\ \partial_i = -\partial_i(\bar{g}),$$

where $\bar{g} = \log\sqrt{|\det(g^{ij})|}$. Therefore

$$S_{ij} = S(\partial_i, \partial_j) = -\partial_i\partial_j(\bar{g}) + \partial_j\partial_i(\bar{g}) = 0.$$

The desired conclusion follows by taking A and B to be $\mathfrak{g}$ actions and recalling that $S(A,B)$ is just the definition of $\delta\phi(A,B)$. $\square$

The choice of subalgebra $\mathfrak{h}\subset\mathfrak{g}$ singles out a certain element of $H^1(\mathfrak{h})$. The adjoint action naturally makes $\mathfrak{g}/\mathfrak{h}$ into an $\mathfrak{h}$ module. We let $\chi\in\mathfrak{h}^*$ denote the character of this representation. Since χ kills all commutators of $\mathfrak{h}$, we can regard χ as an element of $H^1(\mathfrak{h})$.

Proposition 4.10: $P\phi=-\chi$.

Proof: Let us proceed by examining a slightly more general case. Let x be a point of a pseudo-Riemannian manifold, $X_1,\ldots,X_n$ a frame in a neighborhood of x, and $\theta^1,\ldots,\theta^n$ the dual one-form coframe. For a vector field, X, which is zero at x we have

$$(\operatorname{div} X)_x=\sum_i \langle \theta^i,[X_i, X]\rangle_x .$$

Recall that $a^H_{\pi(e)}=0$ for $a\in\mathfrak{h}$. Now, let $X=a^H$ for an $a\in\mathfrak{h}$, and take $X_i=a_i^H$ where $\{a_i\}$ is a basis of some subspace of $\mathfrak{g}$ which is complementary to $\mathfrak{h}$. The preceding formula directly implies that

$$(P\phi)(a)=(\operatorname{div} a^H)_{\pi(e)}=-\chi(a). \qquad \square$$

The space of solutions to the QES closure conditions is a subset of $S^2\mathfrak{g}\oplus\mathfrak{g}$. Since there is a natural G-action on the latter space, it is worthwhile to ask whether the subset of solutions is invariant. The answer turns out to be yes, and this fact will prove to be very significant in the search for solutions in specific instances. The correspondence

$$x\mapsto R_x^{-1}, \quad x\in G,$$

is a representation of G by diffeomorphisms of G and thus gives rise to a G-action on the various tensor fields of G. This action is closely related to the adjoint representation in as much as

$$(\mathrm{Ad}_x\, a)^L=(R_x^{-1})_* a^L, \quad a\in\mathfrak{g}, \quad x\in G.$$

The various G-actions naturally give rise to $\mathfrak{g}$-actions. For instance, the $\mathfrak{g}$-action on the space of cochains is given by the so-called homotopy formula:

$$(\mathscr{L}a)=i(a)\delta+\delta i(a), \quad a\in\mathfrak{g},$$

where $i(a)$ denotes a onefold contraction with a. The homotopy formula makes it clear that $\mathfrak{g}$-action, and hence the G-action, descends to the level of cohomology. The next proposition is also a consequence of the formula.

Proposition 4.11: The G-action on $H^*(\mathfrak{g};\mathscr{C}^\infty(U))$ is trivial.

Theorem 4.12: The solutions to the QES closure conditions form an invariant subset of $S^2\mathfrak{g}\oplus\mathfrak{g}$.

Proof: Let (C,a) be a solution to the closure conditions. By Proposition 4.7 there is an $f\in\mathscr{C}^\infty(U)$ such that for all $\alpha\in\mathfrak{h}^\perp$

$$\langle\alpha^R;(C(\phi-2\eta+df))^L-a^L\rangle=0.$$

Write C',a',f',ϕ',η' for the result of acting on C, a, f, and ϕ, η by some fixed $x\in G$. Distributing R_x^*, the action of x, on the left-hand side of the above equation, we have

$$\langle\alpha^R;(C'(\phi'-2\eta'+df'))^L-a'^L\rangle=0.$$

It's not surprising that the divergence cocycle corresponding to C' is just ϕ' and hence, (C',a') is a solution to the closure conditions if there is an f'' such that

$$\langle\alpha^R;(C'(\phi'-2\eta+df''))^L-a'^L\rangle=0.$$

However, η and η' belong to the same cohomology class and, hence, differ by a coboundary. Therefore, (C',a') is a solution to the closure conditions. □

We will now make a small diversion to consider two examples. They will serve as illustrations to the above theory and as useful references for later discussion. We will also generalize the first example into a useful theorem. A symbolic calculation package is very helpful in verifying the necessary computations.

Example 4.13: The following example is derived from the usual two-dimensional, linear representation of $\mathfrak{sl}(2,\mathrm{R})$. Note that p,q are the classical notation for the derivatives ∂_x,∂_y. We take $\mathfrak{g}=\{yp,xp-yq,-xq\}$ with basepoint $x=0$, $y=1$, and

$$C=\begin{pmatrix} 1 & 0 & -2A \\ 0 & A & 0 \\ -2A & 0 & 0 \end{pmatrix},$$

where A is an arbitrary constant. The contravariant form of the induced metric is

$$(g^{ij})=\begin{pmatrix} Ax^2+y^2 & Axy \\ Axy & Ay^2 \end{pmatrix}.$$

For the divergence cocycle we have

$$\phi_1=0, \quad \phi_2=2, \quad \phi_3=2x/y,$$

and hence,

$$(C\phi)^H=-2Axp-2yAq.$$

The reader is invited to verify that the formula $\Delta=\Gamma_0+(C\phi)^H$ is confirmed by this example. Note that $\mathfrak{h}=\{a_3\}=\{-xq\}$ is one-dimensional and hence $H^1(\mathfrak{g};\mathscr{C}^\infty(U))\cong\mathfrak{h}^*$ by Proposition 3.1. Using Proposition 3.1 we have

$$(P\phi)(a_3)=\phi(a_3)_{\substack{x=0\\y=1}}=0.$$

Therefore our theory predicts that ϕ is a coboundary and hence that $(C\phi)^H$ must be the gradient of some function. This prediction is confirmed; the function in question is

$$f=-2\log(y).$$

Hence the given C is a homogeneous solution to the closure conditions, i.e.,

$$\Gamma_0=\mu^{-1}\circ(\Delta+V)\circ\mu,$$

where $\mu=e^{f/2}$, and where the formula for the potential, V, is of no particular interest.

As an illustration of Corollary 4.8, let us take $a=ka_1$, where k is an arbitrary constant. We have

$$a^H=kyp=k\,\mathrm{grad}(x/y).$$

Thus (C,ka_1) is a solution to the closure conditions for all k. □

We have the following generalizations inspired by the above example. The second of these is a result contained in Ref. 13, although here it is proved in a much more algebraic fashion.

Theorem 4.14: Suppose that $\mathfrak{g}$ and $\mathfrak{h}$ are both reductive (in particular, $\mathfrak{h}$ could be one-dimensional, as in the preceding example), and suppose C is nondegenerate with respect to $\mathfrak{h}$. If we take $\eta=0$, then C will be a homogeneous solution of the closure conditions.

Proof: Let $a\in\mathfrak{h}$ be given. Since both Lie algebras are reductive we have $\mathrm{ad}_{\mathfrak{g}}(a)=0$ and $\mathrm{ad}_{\mathfrak{h}}(a)=0$, and hence, $\chi(a)=0$. Our conclusions follows by Propositions 4.7 and 4.10. □

Theorem 4.15: Suppose that $\mathfrak{g}$ is compact. If $\eta=0$, and if C is nondegenerate with respect to $\mathfrak{h}$, then C is a homogeneous solution to the closure conditions.

Proof: Let $\mathfrak{u}$ be the subspace of vectors that are orthogonal to $\mathfrak{h}$ with respect to the Killing form on $\mathfrak{g}$. By the invariance of the Killing form we have $[\mathfrak{h},\mathfrak{u}]\subset\mathfrak{u}$. Since we are assuming that the Killing form is negative definite we also have, $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{u}$, and $\mathrm{ad}(\mathfrak{h})|_{\mathfrak{u}}\subset\mathfrak{so}(\dim\mathfrak{u})$. Since the trace of skew-symmetric endomorphisms is zero, we conclude that $\chi=0$. □

Example 4.16: Our next example is the two-dimensional realization of $\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$ given by $\{p,xp,x^2p,q,yq,y^2q\}$. It is taken from Ref. 4 and has some interesting properties, which we will discuss later. As basepoint we take $x=0$, $y=0$, and hence $\mathfrak{h}=\{a_2,a_3,a_5,a_6\}$. We also take

$$C=\begin{pmatrix} A & 0 & 0 & 1 & 0 & 1\\ 0 & 2A & 0 & 0 & 0 & 0\\ 0 & 0 & A & 1 & 0 & 1\\ 1 & 0 & 1 & B & 0 & 0\\ 0 & 0 & 0 & 0 & 2B & 0\\ 1 & 0 & 1 & 0 & 0 & B\end{pmatrix},$$

where A and B are constants such that $AB\neq 1$. The induced metric is therefore.

$$(g^{ij})=\begin{pmatrix} A(1+x^2)^2 & (1+x^2)(1+y^2)\\ (1+x^2)(1+y^2) & B(1+y^2)^2\end{pmatrix}.$$

For the divergence cocycle we have

$$\phi_1=\frac{-2x}{1+x^2},\quad \phi_2=\frac{1-x^2}{1+x^2},\quad \phi_3=\frac{2x}{1+x^2},$$

$$\phi_4=\frac{-2y}{1+y^2},\quad \phi_5=\frac{1-y^2}{1+y^2},\quad \phi_6=\frac{2y}{1+y^2}.$$

Since $[\mathfrak{h},\mathfrak{h}]=\{a_3,a_6\}$ we must have

$$H^1(\mathfrak{h})\cong\{\alpha^2,\alpha^5\},$$

where $\alpha^1,\ldots,\alpha^6$ is the dual basis of $a_1,\ldots,a_6$. A straightforward calculation shows that $\chi=-\alpha^2-\alpha^5$. Our example behaves as it should, i.e., $P\phi=-\chi$, because

$$\phi(a_i)_{\substack{x=0\\y=0}}=\begin{cases}1 & \text{if } i=2,5,\\ 0 & \text{if } i=3,6.\end{cases}$$

Hence, ϕ is not a coboundary, although, very curiously, we have

$$(C\phi)^H=\frac{-2Ax}{1+x^2}a_1^H+\frac{2A(1-x^2)}{1+x^2}a_2^H+\frac{2Ax}{1+x^2}a_3^H+\frac{-2Ay}{1+y^2}a_4^H+\frac{2A(1-y^2)}{1+y^2}a_5^H+\frac{2Ay}{1+y^2}a_6^H=0.$$

Therefore, C is a homogeneous solution to the closure conditions. In this instance $\Gamma=\Delta$, without a change of scale. This fact is not an isolated curiosity, but is an illustration of a certain class of solutions to the closure conditions, one that we will discuss further on. □

Having completed our discussion of the cocycle ϕ, we are able to take the next step in reformulating the closure conditions. Proposition 4.7 could be restated as saying that (C,a) is a solution to the closure conditions if and only if there is a cocycle ψ, belonging to the same class of $H^1(\mathfrak{g};\mathscr{E}^\infty(U))$ as $\phi-2\eta$, and such that $(C\psi)^H=a^H$. Using the results of Sec. III and Proposition 4.10 the first condition can be restated as

$$\psi^L b^R=-(\chi+2P\eta)(b), \quad b\in\mathfrak{h}. \tag{1}$$

The second condition states that

$$\langle\psi^L;C^L\alpha^R\rangle=\langle\alpha^R;a^L\rangle \tag{2}$$

for all $\alpha\in\mathfrak{h}^\perp$.

By our nondegeneracy assumption the tangent space at a point of G_0 is the direct sum of $\mathfrak{h}^R$ and $C^L(\mathfrak{h}^\perp)^R$. Hence, Eqs. (1) and (2) completely characterize ψ. Thus, for a given $C\in S^2\mathfrak{g}$ and $a\in\mathfrak{g}$, we define $\psi\in C^1(\mathfrak{g};\mathscr{E}^\infty(U))$ to be the unique one-cochain characterized by these two equations.

Proposition 4.17: (C,a) are a solution to the closure conditions if and only if $\delta\psi=0$.

Proof: If the closure conditions are satisfied, then $\psi=\phi-2\eta+\delta f$, for some $f\in\mathscr{E}^\infty(U)$ and so the conclusion follows. If, on the other hand, $\delta\psi=0$, then for $b\in\mathfrak{g}$ and $c\in\mathfrak{h}$ we have

$$\delta\psi^L(b^L,c^R)=b^L(\psi^L c^R)-c^R(\psi^L b^L)-\psi^L[b^L,c^R]=0-c^R(\psi b)-0=0$$

because $\psi^L c^R$ is a constant. Hence,

$$c^R(\psi b)=0,$$

i.e., $\psi\in C^1(\mathfrak{g};\mathscr{E}^\infty(U))$. Furthermore, ψ is a cocycle and belongs to the same class as $\phi-2\eta$. Therefore, there exists an $f\in\mathscr{E}^\infty(U)$ such that $\psi=\phi-2\eta+\delta f$. □

Equivalent to $\delta\psi=0$ is the condition that

$$u(\psi^L v)-v(\psi^L u)-\psi^L[u,v]=0,$$

for all vector fields u,v. Because of the available decomposition of the tangent space at $x\in G_0$, mentioned above, we only need to consider three cases in order to verify the last equation:

(i) $u=b^R$, $v=c^R$ for $b,c\in\mathfrak{h}$;
(ii) $u=b^R$, $v=C^L\alpha^R$ for $b\in\mathfrak{h}$ and $\alpha\in\mathfrak{h}^\perp$;
(iii) $u=C^L\alpha^R$, $v=C^L\beta^R$ for $\alpha,\beta\in\mathfrak{h}^\perp$.

The next two Lemmas will show that we only need to consider the third case. Let ψ be the cocycle defined by Eqs. (1) and (2).

Lemma 4.18: Let $b,c\in\mathfrak{h}$ be given. Then, $d\psi^L(b^R,c^R)=0$.

Proof: This follows directly from the facts that $\psi^L(b^R)=-(\chi+2P\eta)(b)$ is a constant and that $\chi+2P\eta$ annihilates the commutators of $\mathfrak{h}$. □

Lemma 4.19: Let $b\in\mathfrak{h}$ and $\alpha\in\mathfrak{h}^\perp$ be given. Then, $d\psi^L(b^R,C^L\alpha^R)=0$.

Proof: Since $\langle\psi^L;C^L\alpha^R\rangle=\langle\alpha^R;a^L\rangle$, we have

$$b^R\langle\psi^L;C^L\alpha^R\rangle=\langle\beta^R;a^L\rangle,$$

where $\beta=-\mathscr{L}(b)\alpha$. Also recall that $\langle\psi^L;b^R\rangle$ is a constant. Hence,

$$d\psi^L(b^R,C^L\alpha^R)=b^R\langle\psi^L;C^L\alpha^R\rangle-C^L\alpha^R\langle\psi^L;b^R\rangle-\langle\psi^L;[b^R,\ C^L\alpha^R]\rangle$$
$$=\langle\beta^R;a^L\rangle-0-\langle\psi^L;C^L\beta^R\rangle=0. \qquad \square$$

We are now ready for one more restatement of the closure conditions. For $\alpha,\beta\in\mathfrak{h}$, define $v\in\mathscr{E}^\infty(G,\mathfrak{h})$ and $\xi\in\mathscr{E}^\infty(G,\mathfrak{h}^\perp)$ to be such that

$$[C^L\alpha^R,\ C^L\beta^R]=v^R+C^L\xi^R$$

is the unique vector field decomposition induced by the complementary distributions $\mathfrak{h}^R$ and $C^L(\mathfrak{h}^\perp)^R$. Based on the preceding proposition and on the last two lemmas we can state the following.

Proposition 4.20: (C,a) is a solution to the closure conditions if and only if for all $\alpha,\beta\in\mathfrak{h}^\perp$,

$$C^L\alpha^R(\beta^Ra^L)-C^L\beta^R(\alpha^Ra^L)-\xi^Ra^L+(\chi+2P\eta)v=0.$$

At first glance, this latest version of the closure conditions seems to be of theoretical interest only. However, as we will see in the subsequent sections, it can lead to a complete and concrete classification of the solutions to the QES closure conditions.

V. TENSORS OF MIXED TYPE

A central role in Proposition 4.20 was played by vector fields like $C^L\alpha^R$. Such a vector field is a contraction of a right- and a left-invariant tensor. We will call the resulting object a tensor of mixed type. In this section we will develop calculation techniques to handle these tensors and then use these techniques to describe a class of solutions to the closure conditions alluded to in Example 4.16.

First let us recall the following elementary facts.

Proposition 5.21: For $a\in\mathfrak{g}$, $\alpha\in\mathfrak{g}^*$ we have

$$a^L=(\mathrm{Ad}\ a)^R,\quad \alpha^R=(\mathrm{Ad}^*\ \alpha)^L=(\alpha\ \mathrm{Ad})^L.$$

For ease of notation we will use $\tilde{\alpha}$ to denote the $\mathfrak{g}^*$-valued function $\mathrm{Ad}^*\ \alpha$. We thus have $\alpha^R=\tilde{\alpha}^L$. The identification of tensor fields with tensor-valued functions allow us to define a modified version of the usual exterior derivative on a Lie group. Let V be a vector space and $f:G\rightarrow V$ be a smooth function. We define $D^Lf:G\rightarrow\mathrm{Hom}(\mathfrak{g},V)$ by

$$(D^Lf)(a)=a^Lf,\quad a\in\mathfrak{g}.$$

We will need a formula for the Lie bracket of vector fields in this formalism.

Proposition 5.22: Let f,g be $\mathfrak{g}$-valued functions on G. Then,

$$L^{-1}([f^L,\ g^L])=\mathrm{ad}(f,g)+(D^Lg)f-(D^Lf)g.$$

We also need a formula for the derivative of Ad.

Proposition 5.23: $(D^L\ \mathrm{Ad})_x(a)=\mathrm{Ad}_x\ \mathrm{ad}(a)$, for $a\in\mathfrak{g}$ and $x\in G$.

The preceding two propositions combine to give a formula for the Lie bracket of two vector fields of mixed type.

Proposition 5.24: Let $\alpha,\beta\in\mathfrak{g}^*$. Then,

$$L^{-1}([C^L\alpha^R,\ C^L\beta^R])=\mathrm{ad}(C\tilde{\alpha},C\tilde{\beta})+C(\tilde{\beta}\ \mathrm{ad}(C\tilde{\alpha}))-C(\tilde{\alpha}\ \mathrm{ad}(C\tilde{\beta})).$$

Let us restate the above in a more convenient notation. Since C acts as an inner product on $\mathfrak{g}^*$, it also induces and inner product, $C^{\wedge 2}$, on $\Lambda^2\mathfrak{g}^*$ which is given by

$$C^{\wedge 2}(\alpha^i\wedge\alpha^j,\alpha^k\wedge\alpha^l)=\langle\alpha^i\wedge\alpha^j;C\alpha^k\wedge C\alpha^l\rangle=\begin{vmatrix} C^{ik} & C^{jk} \\ C^{il} & C^{jl}\end{vmatrix}.$$

For convenience we will omit the C and $C^{\wedge 2}$ and use a dot to denote these inner products. Thus, for $\alpha,\beta,\gamma\in\mathfrak{g}^*$ we will write

$$\langle\gamma;\mathrm{ad}(C\alpha,C\beta)\rangle=\langle\gamma^*;C\alpha\wedge C\beta\rangle=\gamma^*\cdot\alpha\wedge\beta,$$

where we define $\gamma^*=\mathrm{ad}^*\,\gamma$, and where

$$\mathrm{ad}^*:\mathfrak{g}^*\to\Lambda^2\mathfrak{g}^*$$

denotes the transpose of the adjoint map.

Proposition 5.25: For $\alpha,\beta\in\mathfrak{g}^*$ and $\gamma\in\mathscr{C}^\infty(G,\mathfrak{g}^*)$ we have

$$\langle\gamma^L;[C^L\alpha^R,\ C^L\beta^R]\rangle=\gamma^*\cdot\tilde{\alpha}\wedge\tilde{\beta}+\tilde{\beta}^*\cdot\tilde{\alpha}\wedge\gamma-\tilde{\alpha}^*\cdot\tilde{\beta}\wedge\gamma.$$

We have now established sufficient machinery to describe a class of solutions to the closure conditions, one instance of which is given in Example 4.16.

Definition 5.26: We say that C is Abelian with respect to $\mathfrak{h}$ if for all $\alpha,\beta\in\mathfrak{h}^\perp$

$$[C^L\alpha^R,\ C^L\beta^R]=0.$$

An examination of Proposition 4.20 reveals that if C is Abelian with respect to $\mathfrak{h}$, then C is a homogeneous solution of the closure conditions for all η.

For the remainder of this section we will assume that C is Abelian and nondegenerate with respect to $\mathfrak{h}$. The distribution $C^L(\mathfrak{h}^\perp)^R$ is thus involutive and by the nondegeneracy assumption is complementary to the distribution $\mathfrak{h}^R$. We can therefore identify U with the integral manifold of the former distribution through $e\in G$. We choose a basis $\alpha^1,\ldots,\alpha^m$ of $\mathfrak{h}^\perp$, and let $\beta^1,\ldots,\beta^m$ be the dual basis with respect to C, i.e., $\beta^i\in\mathfrak{h}^\perp$ and $C(\alpha^i,\beta^j)=\delta_{ij}$. Since the vector fields $C^L\alpha^{iL}$ commute we can choose coordinates $x^1,\ldots,x^m$ on U such that $\partial_i=C^L\alpha^{iL}|_U$.

Proposition 5.27: The pseudo-Riemannian metric $g=(C^H)^{-1}$ induced by C on U has components

$$g_{ij}=C^L(\alpha^{iL},\alpha^{jL}),\quad i,j=1,\ldots,m.$$

Proof: The linear algebra of the situation works like this. Let V be a vector space and $C\in\mathrm{Hom}(V^*,V)$ be a symmetric form. Suppose we can decompose V as $V_1\oplus V_2$ such that the decomposition respects C. In other words, V_1^* is perpendicular to V_2^* with respect to C. Furthermore, suppose that C restricted to V_2^* is nondegenerate with inverse $g\in\mathrm{Hom}(V_2,V_2^*)$. Then,

$$g(C\alpha^i,C\alpha^j)=\langle\alpha^i;C\alpha^j\rangle=C(\alpha^i,\alpha^j).$$

This is essentially what is happening with $V=T_xG$ at each $x\in U$. In this case $V_1=\mathfrak{h}_x^R$ and $V_2=C^L(\mathfrak{h}^\perp)_x^R$. Note that V_2 is spanned by the coordinate vector fields ∂_i and hence, as above,

$$g_{ij}=g(\partial_i,\partial_j)=C(\alpha^{iL},\alpha^{jL}).$$ □

Lemma 5.28: For all $\alpha,\beta,\gamma\in\mathfrak{h}^\perp$ we have

$$C^L\alpha^R(C(\tilde{\beta},\tilde{\gamma}))=0,\quad\langle\tilde{\alpha};\mathrm{ad}(C\tilde{\beta},C\tilde{\gamma})\rangle=0.$$

Proof: By Proposition 5.23 we have

$$C^L\alpha^R(C^L(\tilde{\beta},\tilde{\gamma}))=\langle\tilde{\beta};\mathrm{ad}(\tilde{\alpha}\wedge\tilde{\gamma})\rangle+\langle\tilde{\gamma};\mathrm{ad}(\tilde{\alpha}\wedge\tilde{\beta})\rangle,$$

and since we are assuming that C is Abelian with respect to $\mathfrak{h}$, Proposition 5.24 tells us that

$$C^L\alpha^R(C^L(\tilde{\beta},\tilde{\gamma}))-\langle\tilde{\alpha};\mathrm{ad}(C\tilde{\beta},C\tilde{\gamma})\rangle=0.$$

Note that the first term on the left-hand side is symmetric in β and γ while the second term is skew symmetric, and hence both terms must be zero. □

Lemma 5.29: Let $v\in\mathscr{C}^\infty(G,\mathfrak{h})$. The following identity holds everywhere on U:

$$\sum_i \langle\tilde{\beta}^i;\mathrm{ad}(C\tilde{\alpha}^i,\mathrm{Ad}^{-1}\,v)\rangle=-\chi(v).$$

Proof: Fix an $x\in U$ and let $\mathfrak{h}_x$ denote the subalgebra $\mathrm{Ad}_x^{-1}\mathfrak{h}$. Since each β^j is a constant linear combination of the α^i, Lemma 5.28 tells us that

$$\partial_k(C(\tilde{\alpha}^i,\tilde{\beta}^j))=0,\quad k=1,\ldots,m,$$

and hence,

$$C(\tilde{\alpha}^i,\tilde{\beta}^j)=\delta_{ij}$$

everywhere on U. Therefore, $\{C\tilde{\alpha}_x^i\}$ is a basis of $\mathfrak{g}/\mathfrak{h}_x$ and $\{\tilde{\beta}_x^i\}$ is the dual basis of $\mathfrak{h}_x^\perp$. Hence,

$$\sum_i \langle\tilde{\beta}^i;\mathrm{ad}(C\tilde{\alpha}^i,\mathrm{Ad}^{-1}\,v)\rangle_x=-\chi_x(\mathrm{Ad}_x^{-1}v_x),$$

where χ_x is the character of the representation of $\mathfrak{h}_x$ on $\mathfrak{g}/\mathfrak{h}_x$. Since Ad_x^{-1} is an automorphism of $\mathfrak{g}$, we must have $\chi_x(\mathrm{Ad}_x^{-1}v_x)=\chi(v_x)$. □

Theorem 5.30: The induced pseudo-Riemannian metric on U is flat and $\Delta=\Gamma_0$.

Proof: Lemma 5.28 and Proposition 5.27 tell us that $\partial_k(g_{ij})=0$ for all i,j,k. Hence, g is flat.

According to Proposition 4.6, in order to prove the second part of the theorem we must show that $(C\phi)^H=0$, or equivalently that $\langle\phi;C\tilde{\alpha}\rangle=0$ for all $\alpha\in\mathfrak{h}^\perp$. Let $a\in\mathfrak{g}$ be given. The complementary distributions $\mathfrak{h}^R$ and $C^L(\mathfrak{h}^\perp)^R$ give us the decomposition

$$a^L=v^R+C^L\xi^R,\quad v\in\mathscr{C}^\infty(G,\mathfrak{h}),\quad \xi\in\mathscr{C}^\infty(G,\mathfrak{h}^\perp).$$

Recall that we are identifying U with the integral submanifold of $C^L(\mathfrak{h}^\perp)^R$ through e and hence, $a^H=C^L\xi^H|_U$. Note that

$$C^L\xi^R=\sum_i \langle\tilde{\beta}^i;a\rangle C^L\alpha^{iR},\quad a^H=\sum_i \langle\tilde{\beta}^i;a\rangle\partial_i.$$

Since g is flat,

$$\mathrm{div}(a^H)=\sum_i \partial_i\langle\tilde{\beta}^i;a\rangle=\sum_i (C^L\alpha^{iR})\langle\tilde{\beta}^i;a\rangle=\sum_i \langle\tilde{\beta}^i;\mathrm{ad}(C\tilde{\alpha}^i,a)\rangle.\tag{3}$$

By Lemma 5.28

$$\langle\tilde{\beta}^i;\mathrm{ad}(C\tilde{\alpha}^i,C\tilde{\xi})\rangle=0.$$

Since $a=\mathrm{Ad}^{-1}\,v+C\tilde{\xi}$, we conclude from Eq. (3) that

$$\mathrm{div}(a^H)=\sum_i \langle \tilde\beta^i;\mathrm{ad}(C\tilde\alpha^i,\mathrm{Ad}^{-1}\,v)\rangle.$$

Hence, by Lemma 5.29,

$$\mathrm{div}(a^H)=-\chi(v)=\phi^L v^R.$$

However, since

$$\mathrm{div}\ a^H=\phi a=\phi^L v^R+\langle \phi;C\tilde\xi\rangle,$$

we can conclude that $\langle\phi;C\tilde\xi\rangle=0$. As we vary $a\in\mathfrak{g}$, the range of $C^L\xi^R$ spans all of $C^L(\mathfrak{h}^\perp)^R$, and hence, ϕ^L must be an annihilator of this distribution. □

Let us now use the above techniques to generalize Example 4.16. To do so we must find all C that are Abelian with respect to the given isotropy algebra. Henceforth, we take all the givens of that Example. Let us write $\mathfrak{g}\cong\mathfrak{g}_A\oplus\mathfrak{g}_B$, where both terms are equal to $\mathfrak{sl}(2,\mathbb{R})$ and agree that a_1,a_2,a_3 span $\mathfrak{g}_A$ while a_4,a_5,a_6 span $\mathfrak{g}_B$. With respect to this decomposition C breaks up into the two by two matrix

$$\begin{pmatrix} C_A & \Phi \\ \Phi^* & C_B\end{pmatrix},$$

where C_A, C_B are inner products on $\mathfrak{g}_A^*$ and $\mathfrak{g}_B^*$, while $\Phi:\mathfrak{g}_B^*\to\mathfrak{g}_A$ gives the product of heterogenous pairs. Note that $S^2(\mathfrak{sl}(2,\mathbb{R}))$ has a one-dimensional invariant subspace. We take

$$C_{\mathrm{inv}}=-a_1\otimes a_3-a_3\otimes a_1+2a_2\otimes a_2$$

as a generator.

Proposition 5.31: If C is Abelian and nondegenerate with respect to $\mathfrak{h}$, then it must have one of two forms.

(i) $\Phi=0$ and C_A, C_B are such that $C_A(\alpha^1,\alpha^1)$ and $C_B(\alpha^4,\alpha^4)$ are nonzero.

(ii) $\Phi=u_A\otimes u_B$,

$$C_A=K_A u_A\otimes u_A+L_A C_{\mathrm{inv}},\quad C_B=K_B u_B\otimes u_B+L_B C_{\mathrm{inv}},$$

where $u_A\in\mathfrak{g}_A$, $u_B\in\mathfrak{g}_B$ are such that $\alpha^1 u_A$ and $\alpha^4 u_B$ are nonzero, and K_A,K_B,L_A,L_B are constants such that $K_AK_B\neq 1$.

Proof: For $\alpha,\beta\in\mathfrak{g}^*$, put

$$f(\alpha,\beta)=L^{-1}[C^L\alpha^R,\ C^L\beta^R].$$

The $\mathfrak{g}$-valued function $f(\alpha^1,\alpha^4)$ must be zero in order for C to be Abelian with respect to $\mathfrak{h}$. Equivalently, we can demand that this analytic function and all of its derivatives be zero at e. This amounts to the condition that

$$a_{i_1}^L\cdots a_{i_n}^L f(\alpha^1,\alpha^4)_e=0$$

for all sequences of elements of $\mathfrak{g}$. By Proposition 5.23

$$a^L f(\alpha^1,\alpha^4)_e=f(\mathrm{ad}(a)^*\alpha^1,\alpha^4)_e+f(\alpha^1,\mathrm{ad}(a)^*\alpha^4)_e,\quad a\in\mathfrak{g}.$$

Since $\mathfrak{sl}(2,\mathbb{R})$ is simple we can conclude that C is Abelian with respect to $\mathfrak{h}$ if and only if

$$f(\mathfrak{g}_A^*,\mathfrak{g}_B^*)_e=0. \tag{4}$$

By Proposition 5.25 this means that for all $\alpha\in\mathfrak{g}_A^*$ and $\beta,\beta'\in\mathfrak{g}_B^*$

$$\beta'^*\cdot\alpha\wedge\beta+\beta^*\cdot\alpha\wedge\beta'-\alpha^*\cdot\beta\wedge\beta'=0.$$

The first two terms on the left-hand side are symmetric in β,β' and the last term is skew symmetric and hence

$$\alpha^*\cdot\beta\wedge\beta'=\langle\alpha;\mathrm{ad}(\Phi\beta,\Phi\beta')\rangle=0.$$

Hence, the image of Φ must be an Abelian subalgebra of $\mathfrak{g}_A$, and since $\mathfrak{sl}(2,\mathbb{R})$ is a rank one algebra, the image of Φ must either be zero or one-dimensional. If $\Phi=0$, then clearly Eq. (4) must hold. This gives us case one of the theorem. The restrictions on the choice of C_A and C_B assure that the resulting C is nondegenerate with respect to $\mathfrak{h}$. For the rest of the proof we assume that Φ is one-dimensional and hence, $\Phi=u_A\otimes u_B$ for some nonzero u_A and u_B. Let $\alpha\in\mathfrak{g}_A^*$, let $\beta,\beta'\in\mathfrak{g}_B^*$, and put $u=\Phi^*\alpha$. By Proposition 5.24

$$\langle\beta';f(\alpha,\beta)_e\rangle=\langle\beta';\mathrm{ad}(u,C_B\beta\rangle+\langle\beta;\mathrm{ad}(u,C_B\beta')\rangle-\langle\alpha;\mathrm{ad}(\Phi\beta,\Phi\beta')\rangle=((\mathscr{L}u)C_B)(\beta,\beta')=0.$$

Hence in order for C to be Abelian it is necessary and sufficient that $(\mathscr{L}u_A)C_A$ and $(\mathscr{L}u_B)C_B$ be zero. The reader should verify that this is possible if and only if C_A is a linear combination of $u_A\otimes u_A$ and C_{inv}, and C_B is a linear combination of $u_B\otimes u_B$ and C_{inv}. The additional constraints on u_A, u_B and the constants ensure that C is nondegenerate with respect to $\mathfrak{h}$. □

The C of Example 4.16 falls into the second category. The values of the parameters are $u_A=a_1+a_3$, $u_B=a_4+a_5$, $K_A=L_A=A$, and $K_B=L_B=B$.

VI. INVARIANT EQUATIONS

In this section we will obtain solutions to the closure conditions by translating them into $\mathfrak{g}$-invariant equations and then solving these equations. Our point of departure is Proposition 4.20 and Theorem 4.12. The first will yield the invariant equations and the second is the key to solving them. In this paper only the planar case will be analyzed. Thus, we assume that $\mathfrak{h}$ has codimension 2 and the closure conditions can be described by a single equation of the type in Proposition 4.20.

First, let us break up the equation of Proposition 4.20 into two parts:

$$C^L\alpha^R(\beta^R a^L)-C^L\beta^R(\alpha^R a^L)-\xi^R a^L=0, \tag{5}$$

$$(\chi+2P\eta)v=0. \tag{6}$$

The first equation is sufficient to describe the closure conditions whenever $\chi+2P\eta=0$. The second equation describes the homogeneous solutions to the closure conditions. The case covered by Eq. (5) was first considered in Ref. 4. That paper referred to this restriction as the simplified closure conditions, a term which we will adopt. The choice of this terminology is explained by the following.

Proposition 6.32: Every C that is nondegenerate with respect to $\mathfrak{h}$ gives a homogeneous solution to the simplified closure conditions.

Proof: An examination of Proposition 4.20 shows that, if $\chi+2P\eta=0$, then $(C,0)$ is a solution. □

Proposition 6.33: (C,a) is a solution to the simplified closure conditions if and only if

$$0=(\tilde\alpha\wedge\tilde\beta\cdot\tilde\alpha\wedge\tilde\beta)(\langle\tilde\beta^*;C\tilde\alpha\wedge a\rangle-\langle\tilde\alpha^*;C\tilde\beta\wedge a\rangle)+2(\tilde\alpha^*\cdot\tilde\alpha\wedge\tilde\beta)\langle\tilde\alpha\wedge\tilde\beta;C\tilde\beta\wedge a\rangle$$
$$-2(\tilde\beta^*\cdot\tilde\alpha\wedge\tilde\beta)\langle\tilde\alpha\wedge\tilde\beta;C\tilde\alpha\wedge a\rangle. \tag{7}$$

Proof: Let us write

$$\xi^R = AC^L\alpha^R + BC^L\beta^R, \quad A,B \in \mathscr{C}^\infty(G).$$

Since α^R and β^R annihilate v^R,

$$C^L(\alpha^R,\xi^R) = \langle \alpha^R;[C^L\alpha^R,\ C^L\beta^R]\rangle = 2\tilde{\alpha}^*\cdot\tilde{\alpha}\wedge\tilde{\beta},$$

and a similar expression holds for $C^L(\beta^R,\xi^R)$. Hence, A and B are determined by

$$\begin{pmatrix} \tilde{\alpha}\cdot\tilde{\alpha} & \tilde{\alpha}\cdot\tilde{\beta} \\ \tilde{\alpha}\cdot\tilde{\beta} & \tilde{\beta}\cdot\tilde{\beta} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 2\tilde{\alpha}^*\cdot\tilde{\alpha}\wedge\tilde{\beta} \\ 2\tilde{\beta}^*\cdot\tilde{\alpha}\wedge\tilde{\beta} \end{pmatrix}. \tag{8}$$

If C is nondegenerate with respect to $\mathfrak{h}$, then the matrix on the left is invertible. Solving for A and B we get

$$\begin{pmatrix} A \\ B \end{pmatrix} = 2/\Delta\begin{pmatrix} \tilde{\beta}\cdot\tilde{\beta} & -\tilde{\alpha}\cdot\tilde{\beta} \\ -\tilde{\alpha}\cdot\tilde{\beta} & \tilde{\alpha}\cdot\tilde{\alpha} \end{pmatrix}\begin{pmatrix} \tilde{\alpha}^*\cdot\tilde{\alpha}\wedge\tilde{\beta} \\ \tilde{\beta}^*\cdot\tilde{\alpha}\wedge\tilde{\beta} \end{pmatrix},$$

where $\Delta = \tilde{\alpha}\wedge\tilde{\beta}\cdot\tilde{\alpha}\wedge\tilde{\beta}$ is the determinant of the matrix in Eq. (8). Hence,

$$\langle \xi^R;a^L\rangle = 2/\Delta((\tilde{\beta}^*\cdot\tilde{\alpha}\wedge\tilde{\beta})\langle\tilde{\alpha}\wedge\tilde{\beta};C\tilde{\alpha}\wedge a\rangle - (\tilde{\alpha}^*\cdot\tilde{\alpha}\wedge\tilde{\beta})\langle\tilde{\alpha}\wedge\tilde{\beta};C\tilde{\beta}\wedge a\rangle).$$

To conclude, we note that by Proposition 5.23 the first two terms of Eq. (5) can be written

$$\langle\tilde{\beta};\mathrm{ad}(C\tilde{\alpha},a)\rangle - \langle\tilde{\alpha};\mathrm{ad}(C\tilde{\beta},a)\rangle = \langle\tilde{\beta}^*;C\tilde{\alpha}\wedge a\rangle - \langle\tilde{\alpha}^*;C\tilde{\beta}\wedge a\rangle. \qquad \square$$

Let f denote the right-hand side of Eq. (7). Since f is an analytic function on G it is enough to demand that f and all of its derivatives vanish at e. The value f_e is just a certain kind of contraction of $C^{\otimes 3}\otimes a$ with

$$\mu_{\mathrm{simp}} = (\alpha\wedge\beta\cdot\alpha\wedge\beta)\otimes(\beta^*\otimes\alpha - \alpha^*\otimes\beta) + 2(\alpha^*\cdot\alpha\wedge\beta)\otimes(\alpha\wedge\beta)\otimes\beta$$
$$-2(\beta^*\cdot\alpha\wedge\beta)\otimes(\alpha\wedge\beta)\otimes\alpha,$$

which belongs to the tensor space

$$S^2\Lambda^2\mathfrak{g}^*\otimes\Lambda^2\mathfrak{g}^*\otimes\mathfrak{g}^*.$$

Taking the derivative of f with respect to a^L, where $a\in\mathfrak{g}$, amounts to acting on μ_{simp} with the adjoint representation. Let $\mathfrak{g}(\mu_{\mathrm{simp}})$ denote the $\mathfrak{g}$-module generated by μ_{simp}. We will call this module the invariant equations corresponding to the simplified closure condition.

Proposition 6.34: In order for (C,a) to satisfy the simplified closure conditions it is necessary and sufficient that $C^{\otimes 3}\otimes a$ annihilate $\mathfrak{g}(\mu_{\mathrm{simp}})$.

This proposition makes clear the fact that solutions to the closure conditions are closed under G actions. After all, if a subspace of a certain tensor space is closed under $\mathfrak{g}$, then it is also closed under G actions, and hence, the annihilators of this space in the dual will also be closed under G actions.

To obtain the solutions to the simplified closure conditions it is enough to fix $C\in S^2\mathfrak{g}$ and then to ask: for which $a\in\mathfrak{g}$ is (C,a) a solution? Fixing C will turn the equations in $\mathfrak{g}(\mu_{\mathrm{simp}})$ into elements of $\mathfrak{g}^*$. It will then remain to be seen that the span of these elements possesses a nontrivial annihilator in $\mathfrak{g}$. Any such annihilator, a, will make (C,a) into a solution.

The invariance of solutions under G-actions also means that we do not have to consider all C, but merely convenient generators of the G-orbits in $S^2\mathfrak{g}$. We will illustrate all these ideas in two upcoming examples, but, first, we need to consider the homogeneous and the general solutions to the closure conditions.

Proposition 6.35: Choose $\rho\in\mathfrak{g}^*$ such that $\rho|_{\mathfrak{h}}=\chi+2P\eta$. Then, C is a homogeneous solution to the closure conditions if and only if

$$0=(\tilde{\alpha}\wedge\tilde{\beta}\cdot\tilde{\alpha}\wedge\tilde{\beta})(\tilde{\rho}^*\cdot\tilde{\alpha}\wedge\tilde{\beta}+\tilde{\beta}^*\cdot\tilde{\alpha}\wedge\tilde{\rho}-\tilde{\alpha}^*\cdot\tilde{\beta}\wedge\tilde{\rho})+2(\tilde{\alpha}^*\cdot\tilde{\alpha}\wedge\tilde{\beta})$$
$$\times(\tilde{\alpha}\wedge\tilde{\beta}\cdot\tilde{\beta}\wedge\tilde{\rho})-2(\tilde{\beta}^*\cdot\tilde{\alpha}\wedge\tilde{\beta})(\tilde{\alpha}\wedge\tilde{\beta}\cdot\tilde{\alpha}\wedge\tilde{\rho}). \tag{9}$$

Proof: Since $v^R=[C^L\alpha^R,\ C^L\beta^R]-C^L\xi^R$, we can write Eq. (6) as

$$\langle\rho^R;[C^L\alpha^R,\ C^L\beta^R]\rangle-C^L(\xi^R,\rho^R)=0.$$

The rest of the proof proceeds analogously to the proof for the case of simplified closure conditions. □

As before, we note that the value of the right-hand side of Eq. (9) at e is the contraction of $C^{\otimes 4}$ with

$$\mu_{\text{hom}}=(\alpha\wedge\beta\cdot\alpha\wedge\beta)\cdot(\rho^*\cdot\alpha\wedge\beta+\beta^*\cdot\alpha\wedge\rho-\alpha^*\cdot\beta\wedge\rho)$$
$$+2(\alpha^*\cdot\alpha\wedge\beta)\cdot(\alpha\wedge\beta\cdot\beta\wedge\rho)-2(\beta^*\cdot\alpha\wedge\beta)\cdot(\alpha\wedge\beta\cdot\alpha\wedge\rho),$$

which belongs to the tensor space

$$S^2S^2(\Lambda^2\mathfrak{g}^*).$$

Proposition 6.36: In order for C to be a solution to the homogeneous closure conditions it is necessary and sufficient that $C^{\otimes 4}$ annihilates $\mathfrak{g}(\mu_{\text{hom}})$.

Again, since the solutions are invariant under G actions, it suffices to check these equations for convenient generators of G orbits in $S^2\mathfrak{g}$.

Proposition 6.37: In order for (C,a) to be the solution to the general closure conditions it is necessary and sufficient that $(C^{\otimes 3}\otimes a)\oplus C^{\otimes 4}$ annihilate $\mathfrak{g}(\mu_{\text{simp}}\oplus\mu_{\text{hom}})$. Furthermore, if $\mathfrak{g}$ is semisimple, and the highest weights of $\mathfrak{g}(\mu_{\text{simp}})$ are distinct from the highest weights of $\mathfrak{g}(\mu_{\text{hom}})$, then it is necessary and sufficient that (C,a) be a solution to the simplified closure condition while simultaneously C be a homogeneous solution.

Proof: To demonstrate the first part of the proposition we need only recall that μ_{simp} and μ_{hom} derive from the two halves of the equation in Proposition 4.20.

For the second part we need to use the representation theory of semisimple Lie algebras.[11] A finite-dimensional $\mathfrak{g}$ module, M, is the direct sum of irreducible modules, each of which is generated by a certain highest weight element. Given any $u\in M$, for irreducible M, we can choose an operator X from the enveloping algebra of $\mathfrak{g}$ such that $X(u)$ gives us the highest weight generator. In particular, if M_1, and M_2 are irreducible $\mathfrak{g}$-modules with distinct highest weights we can choose an X such that Xu_1 is the highest weight generator of M_1 and such that $Xu_2=0$. Hence, for any nonzero $u_1\in M_1$ and $u_2\in M_2$, the module generated by $u_1\oplus u_2$ is all of $M_1\oplus M_2$. Applying this principle to our situation, we see that if the highest weights of $\mathfrak{g}(\mu_{\text{simp}})$ are distinct from the highest weights of $\mathfrak{g}(\mu_{\text{hom}})$, then $(C^{\otimes 3}\otimes a)\oplus C^{\otimes 4}$ must annihilate all elements of $\mathfrak{g}(\mu_{\text{simp}})\oplus\mathfrak{g}(\mu_{\text{hom}})$. Therefore, $C^{\otimes 3}\otimes a$ must annihilate $\mathfrak{g}(\mu_{\text{simp}})$ while $C^{\otimes 4}$, simultaneously, must annihilate $\mathfrak{g}(\mu_{\text{hom}})$. □

Example 6.38: Let us return to the homogeneous space presented in Example 4.13 and calculate the solutions to the closure conditions using the above method of invariant equations. We use the canonical presentation of $\mathfrak{sl}(2)$:

$$[J^+, J^-]=J^0, \quad [J^0, J^+]=2J^+, \quad [J^0, J^-]=-2J^-,$$

and take

$$J^-=a_1, \quad J^0=a_2, \quad J^+=-a_3.$$

Let us introduce the helpful notation

$$x=\alpha^3, \quad y=-2\alpha^2, \quad z=\alpha^1,$$

$$a=2\alpha^2\wedge\alpha^3, \quad b=-2\alpha^1\wedge\alpha^3, \quad c=2\alpha^1\wedge\alpha^2.$$

Recall that the canonical action of $a\in\mathfrak{g}$ on $\mathfrak{g}^*$ is given by $-\text{ad}(a)^*$. Hence, the action of J^+ is summarized by

$$x\to y\to 2z, \quad a\to b\to 2c,$$

and the action of J^- is summarized by

$$z\to y\to 2x, \quad c\to b\to 2a.$$

Also recall that in this example $\chi=0$ and hence the simplified closure conditions are those for which $\eta=0$. In terms of the above notation, y,z span $\mathfrak{h}^\perp$. Hence,

$$\mu_{\text{simp}}=(c\cdot c)(cy-bz)+2(b\cdot c)cz-2(c\cdot c)cy,$$

where the tensor product symbol, $\otimes$, has been omitted for the sake of brevity. Since $\mathfrak{h}$ is one-dimensional, $H^1(\mathfrak{h})=\mathfrak{h}^*$. Hence, by Proposition 3.1, $H^1(\mathfrak{g};\mathscr{C}^\infty(U))$ is one-dimensional. As a generator for the cocycles we take

$$\eta_1=0, \quad \eta_2=0, \quad \eta_3=1/y^2.$$

The generic nonhomogeneous representation of $\mathfrak{g}$ is therefore

$$a_1=yp, \quad a_2=xp-yq, \quad a_3=-xq+n/y^2.$$

Since $P\eta=\alpha^3$ and $\chi=0$, we take $\rho=\alpha^3$. Hence,

$$\mu_{\text{hom}}=(c\cdot c)\cdot(a\cdot c-c\cdot a+1/2b\cdot b)-(b\cdot c)\cdot(b\cdot c)+2(c\cdot c)\cdot(a\cdot c).$$

The reader should verify that $J^+\mu_{\text{simp}}=0$ and hence μ_{simp} is a highest weight generator of an irreducible $\mathfrak{g}$-module. The dimension of this module turns out to be 7. The generators, in order of descending weight, are

$$(c\cdot c)(bz+cy)-2(b\cdot c)cz,$$

$$-(c\cdot c)(cx+by+az)+(b\cdot b+2a\cdot c)cz,$$

$$3(c\cdot c)(ay+bx)+2(b\cdot c)(cx+by+az)-(b\cdot b+2a\cdot c)(bz+cy)-6(a\cdot b)cz,$$

$$-(c\cdot c)ax-(b\cdot c)(bx+ay)+(a\cdot b)(cy+bz)+(a\cdot a)cz,$$

$$6(b\cdot c)ax+(b\cdot b+2a\cdot c)(bx+ay)-2(a\cdot b)(cx+by+az)-3(a\cdot a)(cy+bz),$$

$$-(b\cdot b+2a\cdot c)ax+(a\cdot a)(cx+by+az),$$

$$2(a\cdot b)ax-(a\cdot a)(bx+ay);$$

μ_{hom} turns out to be a highest weight generator of a dimension 5 irreducible module. The generators of $\mathfrak{g}(\mu_{\text{hom}})$ are

$$(c\cdot c)\cdot(b\cdot b+4a\cdot c)-2(b\cdot c)\cdot(b\cdot c),$$

$$(b\cdot b)\cdot(b\cdot c)-4(a\cdot b)\cdot(c\cdot c),$$

$$4(a\cdot b)\cdot(b\cdot c)+8(a\cdot a)\cdot(c\cdot c)-(b\cdot b)\cdot(b\cdot b+2a\cdot c),$$

$$(a\cdot b)\cdot(b\cdot b)-4(a\cdot a)\cdot(b\cdot c),$$

$$(a\cdot a)\cdot(b\cdot b+4a\cdot c)-2(a\cdot b)\cdot(a\cdot b).$$

The reader should verify that the following four families of symmetric forms generate all of $S^2\mathfrak{g}$. The G-orbits are not entirely distinct; there are overlaps for some discrete values of the parameters. The matrix representation is taken with respect to the basis a_1,a_2,a_3:

$$C=\begin{pmatrix} J & 0 & 0\\ 0 & K & 0\\ 0 & 0 & 0\end{pmatrix}+\begin{pmatrix} 0 & 0 & -2L\\ 0 & L & 0\\ -2L & 0 & 0\end{pmatrix}, \tag{10}$$

$$C=\begin{pmatrix} J+K & 0 & J\\ 0 & 0 & 0\\ J & 0 & J\end{pmatrix}+\begin{pmatrix} 0 & 0 & -2L\\ 0 & L & 0\\ -2L & 0 & 0\end{pmatrix}, \tag{11}$$

$$C=\begin{pmatrix} K & 0 & K\\ 0 & J & 0\\ K & 0 & K\end{pmatrix}+\begin{pmatrix} 0 & 0 & -2L\\ 0 & L & 0\\ -2L & 0 & 0\end{pmatrix}, \tag{12}$$

$$C=\begin{pmatrix} J & J & -J\\ J & J+K & -J\\ -J & -J & J\end{pmatrix}+\begin{pmatrix} 0 & 0 & -2L\\ 0 & L & 0\\ -2L & 0 & 0\end{pmatrix}. \tag{13}$$

The author used the Maple V symbolic computation package to check for solutions of these four cases. The findings are given below. Let us consider the case (10) in some detail. Since solutions are closed under linear scaling it will suffice to take $J=1$. After the contraction with C, the equations in $\mathfrak{g}(\mu_{\text{simp}})$ become

$$0=(K+L)x-2K(K+L)z, \quad 0=KLy,$$

$$0=K(3K+L)x+4K^2Lz, \quad 0=0,$$

$$0=K^2Lx, \quad 0=0, \quad 0=0.$$

The above equations admit solutions when either $K=0$ or $L=0$. In the first case, the solutions are given by $x=0$. In the second case, the solution is given by $x=0$, $z=0$. Therefore, there are three types of solution generators:

$$C=\begin{pmatrix} 1 & 0 & -2L \\ 0 & L & 0 \\ -2L & 0 & 0 \end{pmatrix}, \quad a=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$C=\begin{pmatrix} 1 & 0 & -2L \\ 0 & L & 0 \\ -2L & 0 & 0 \end{pmatrix}, \quad a=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$C=\begin{pmatrix} 1 & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The generators of type (11) yield the following, additional solution:

$$C=\begin{pmatrix} 1 & 0 & 1-2L \\ 0 & L & 0 \\ 1-2L & 0 & 1 \end{pmatrix}, \quad a=\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

The solutions corresponding to the generators of type (12) are

$$C=\begin{pmatrix} -1 & 0 & -1-2L \\ 0 & 1+L & 0 \\ -1-2L & 0 & -1 \end{pmatrix}, \quad a=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

$$C=\begin{pmatrix} 0 & 0 & -2L \\ 0 & 1+L & 0 \\ -2L & 0 & \end{pmatrix}, \quad a=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

There is an additional solution that comes from a generator of type (13), but it is redundant; there exists a G-action that takes it to one of the solutions already presented.

Now, let us turn to the homogeneous solutions for nonzero η. Contracting the equations of $\mathfrak{g}(\mu_{\text{hom}})$ with a generator of type (10) gives the following equations. Once again because solutions are stable under scaling we can take $J=1$:

$$0=L(L+2K)(L+K), \quad 0=0,$$

$$0=KL^3, \quad 0=0, \quad 0=0.$$

Hence, the only homogeneous solution comes about when $L=0$. A similar analysis for the other families of generators does not turn up any new solutions.

Finally, since the highest weights of $\mathfrak{g}(\mu_{\text{simp}})$ and $\mathfrak{g}(\mu_{\text{hom}})$ are distinct, Proposition 6.37 shows that for a general solution, both sets of equations must be satisfied simultaneously. From the above analysis we see, therefore, that the general solution occurs with generators of type (10) with $L=0$, and $a=Ma_2$. After acting on these solutions by an element of SL(2,R),

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad AD-BC=1,$$

we obtain the general solution

$$C^{11}=A^2(A^2J+4B^2K),$$

$$C^{12}=A(A^2CJ+2ABDK+2B^2CK),$$

$$C^{13}=AC(ACJ+4BDK),$$

$$C^{22}=A^2C^2J+(A^2D^2+2ABCD+B^2C^2)K,$$

$$C^{23}=AC^3J+(2ACD^2+2BC^2D)K,$$

$$C^{33}=C^4J+4C^2D^2K, \quad a^1=2MAB,$$

$$a^2=M(AD+BC), \quad a^3=2MCD.$$

The corresponding scale change function, Gaussian curvature, and the potential are

$$\mu=\exp\left(\frac{nC}{y(Ay+Cx)}\right)(Ay+Cx)^{(M-2K)/2K},$$

$$\kappa=-4K, \quad V=\frac{M^2-4K^2}{4K}.$$

Example 6.39: Now let us consider another representation of $\mathfrak{sl}(2,\mathbb{R})$ by planar vector fields, namely,

$$\{p-y^2q,2xp-2yq,x^2p-q\}=\{a_1,a_2,a_3\}.$$

The most convenient basepoint is $x=0$, $y=0$, and the corresponding isotropy subalgebra is spanned by a_2. Hence, α^1, α^3 span $\mathfrak{h}^\perp$, and so in terms of the notation introduced in the last example, the generating equation for the simplified closure conditions is given by

$$\mu_{\text{simp}}=(b\cdot b)(cx-az)+2(a\cdot b)bz-2(b\cdot c)bx.$$

The weight of μ_{simp} is zero and it turns out to generate a module which is a direct sum of an irreducible seven-dimensional module and an irreducible three-dimensional module. The generators for the seven-dimensional module, in order of decreasing weight, are

$$(c\cdot c)(cy+bz)-2(b\cdot c)cz,$$

$$(c\cdot c)(cx+bz+ax)-(b\cdot b+2a\cdot c)cz,$$

$$3(c\cdot c)(bx+ay)+2(b\cdot c)(cx+bz+ax)-(b\cdot b+2a\cdot c)(cy+bz)-6(a\cdot b)cz,$$

$$(c\cdot c)ax+(b\cdot c)(bx+ay)-(a\cdot b)(cy+bz)-(a\cdot a)cz,$$

$$6(b\cdot c)ax+(b\cdot b+2a\cdot c)(bx+ay)-2(a\cdot b)(cx+bz+ax)-3(a\cdot a)(cy+bz),$$

$$(b\cdot b+2a\cdot c)ax-(a\cdot a)(cx+bz+ax),$$

$$2(a\cdot b)ax-(a\cdot a)(bx+ay).$$

The generators for the three-dimensional module are

$$4(c\cdot c)(2bx-3ay)+2(b\cdot c)(-4cx+by+6az)-(b\cdot b)\times(cy+bz)+4(a\cdot c)(2cy-3bz)+4(a\cdot b)cz,$$

$$-4(c\cdot c)ax+2(b\cdot c)(3bx-2ay)+5(b\cdot b)(-cx+az)+2(a\cdot b)(2cy-3bz)+4(a\cdot a)cz,$$

$$-4(b\cdot c)ax+(b\cdot b)(bx+ay)+4(a\cdot c)(3bx-2ay)+2(a\cdot b)\times(-6cx-by+4az)+4(a\cdot a)(3cy-2bz).$$

There are more equations here than in the previous example, and so it is reasonable to expect that there will be fewer solutions to the simplified closure. Indeed, using the same method as in the preceding example, one can show that the solutions to the simplified closure conditions are generated by

$$C=\begin{pmatrix}1&0&0\\0&K&0\\0&0&0\end{pmatrix},\quad a=\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

As in the preceding example, the isotropy algebra is one-dimensional and hence so is $H^1(\mathfrak{g};\mathscr{E}^\infty(U))$. As a generator for the cocycles we take

$$\eta_1=0,\quad \eta_2=1,\quad \eta_3=x.$$

Since $P\eta=\alpha^2$, and $\chi=0$, we take $\rho=\alpha^2$. Hence, the generating equation for the homogeneous solutions is given by

$$\mu_{\text{hom}}=(b\cdot b)\cdot(2a\cdot c+1/2b\cdot b)-4(a\cdot b)\cdot(b\cdot c).$$

It generates a sum of an irreducible five-dimensional and a one-dimensional module. The six resulting equations are

$$-(c\cdot c)\cdot(b\cdot b+4a\cdot c)+2(b\cdot c)\cdot(b\cdot c),$$

$$(b\cdot c)\cdot(b\cdot b)-4(a\cdot b)\cdot(c\cdot c),$$

$$(b\cdot b)\cdot(b\cdot b+2a\cdot c)-4(a\cdot b)\cdot(b\cdot c)-8(a\cdot a)\cdot(c\cdot c),$$

$$(a\cdot b)\cdot(b\cdot b)-4(a\cdot a)\cdot(b\cdot c),$$

$$-(a\cdot a)\cdot(b\cdot b+4a\cdot c)+2(a\cdot b)\cdot(a\cdot b),$$

$$(b\cdot b)\cdot(b\cdot b+8a\cdot c)-16(a\cdot b)\cdot(b\cdot c)+16(a\cdot a)\cdot(c\cdot c).$$

The generators of the homogenous solutions are

$$C=\begin{pmatrix}1&0&0\\0&K&0\\0&0&0\end{pmatrix}.$$

The general solution of the closure conditions is therefore the same as in the preceding example. The scale change function is different, although curiously, the potential is the same constant:

$$\mu=\left[\frac{xy-1}{(A+Cx)(C+Ay)}\right]^{(M-2K)/4K}\left[\frac{A+Cx}{C+Ay}(xy-1)\right]^{n/2},$$

$$\kappa=-4K,$$

$$V=\frac{M^2-4K^2}{4K}.$$

□

VII. CONCLUDING REMARKS

There are two basic limitations to the techniques developed in this paper. First, even though the closure conditions can be restated into a more tractable form, their solution must still be done on a case by case basis. Realistically, this is feasible only when G is semi-simple, and not too complicated at that. Otherwise, the representation theory machinery is too cumbersome to be useful. The next step should be an exhaustive study of simple, low-dimensional examples (the present paper makes this possible) with the goal of spotting patterns which will generalize to more complicated algebras.

There is an aspect of the QES operator program which has not yet been discussed here; the second limitation of the present work derives from this aspect. Namely, how does one determine the finite-dimensional function modules of a given hidden symmetry algebra, and which of these functions will actually be integrable after the gauge transformation which changes H into $\Delta+V$? If these questions are not answered one will be in possession of a rather uninteresting operator; certainly it will be of Schrödinger-type, but its spectrum will, in general, remain obscured. Integrability plays a doubly important role at this point. The Schrödinger operator is self-adjoint on L^2, and hence any finite-dimensional, *integrable* function module, M, of the QES operator, H, will automatically yield dim M distinct eigenfunctions.[4] If the function module is not integrable, then the restriction of H to M may not be diagonalizable and in the worst case may yield only one eigenfunction. Example: consider the one-dimensional QES operator

$$H=\frac{\partial^2}{\partial x^2}.$$

The space of polynomials of degree n or less is a module, but there are only two eigenfunctions of H in this module.

Work on the question of existence of finite-dimensional function modules has been carried out in Refs. 14 and 10. The approach in the first of these papers is to take holomorphic line bundles with G-action, and then to consider the finite-dimensional module of holomorphic sections of these bundles. The second of these papers (in addition to other results) gives theorems that characterize those planar, linear differential operators which preserve certain modules of polynomials. There is an intriguing connection between the ideas of these two papers. There are exactly three maximal families of Lie algebras of first-order differential operators in the plane (see the concluding paragraph of Ref. 15). The finite-dimensional function modules for each of these families are polynomial and well understood. On the one hand, it seems reasonable that these polynomials will generate the algebraic portion of the spectrum of a planar QES operator (Conjecture 1 of Ref. 10). On the other hand, Ref. 14 shows that each of these three types of Lie algebras of operators corresponds to a certain global model. It is therefore to be hoped that the question of finite-dimensional function modules can be attacked by studying global realizations of Lie algebras of first-order operators.

The integrability issue was completely settled for one-dimensional operators in Ref. 6. For higher dimensions, there are some promising ideas in Section 4 of Ref. 13. This paper highlights a condition—existence of a G-invariant metric on the homogeneous space—which, in principle, makes it possible to determine the full spectrum of a homogeneous QES operator. Perhaps a better understanding of global realizations will also shed light on the question of integrability. In any case, much more work needs to be done in this area.

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