Stability of asymmetric spike solutions to the Gierer-Meinhardt system

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In this paper, we study the spectra of asymmetric spike solutions to the Gierer-Meinhardt system. It has previously been shown that the spectra of such solutions may be determined by finding the generalized eigenvalues of matrices, which are determined by the positions of the spikes and various parameters from the system. We will examine the spectra of asymmetric solutions near the point at which they bifurcate off of a symmetric branch. We will confirm that all such solutions are unstable in a neighborhood of the bifurcation point and we derive an explicit expression for the leading order terms of the critical eigenvalues. © *2007 American Institute of Physics*.
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In Ref. 1 asymmetric steady-state spike solutions to a two-component reaction diffusion system are constructed. These solutions are composed of spikes of two predetermined heights arranged arbitrarily. It has been shown that such solutions may persist for long times, but are ultimately unstable. In this paper we perform a detailed analysis of the spectra of all such patterns as they bifurcate off a stable symmetric spike solution branch.

I. INTRODUCTION

Since Turing first proposed the existence of spatial patterns in reaction-diffusion systems,[2] a wide range of behaviors has been uncovered. In particular, much work has gone into the study of pattern formation for the Gierer-Meinhardt system[3] of two equations of reaction-diffusion type. The scaled Gierer-Meinhardt equations are given in (1.1), in which a represents a slowly diffusing activator and h a quickly diffusing inhibitor. Much of the early analytical results take the limit as the inhibitor diffusivity tends to infinity. This system is called the shadow system and there are a variety of results for spike formation on the boundary[4–6] and interior spike pattern solutions.[7,8] When the diffusivity of the inhibitor is order one, the two equations are strongly coupled and results for this case have revealed many more possible behaviors. The existence and stability of symmetric spike solutions are considered in Refs. 9 and 10. The dynamics of multispike solutions in a one-dimensional (1D) domain are considered in Ref. 11. A detailed analysis of spikes in a two-dimensional (2D) domain is explored in Ref. 12.

Among the more unexpected results is the formation of steady-state solutions with asymmetric spike patterns. These patterns consist of spikes of two different heights ordered arbitrarily. The detailed construction of these solutions can be found in Refs. 13 and 1, where it is shown that asymmetric spike solutions to (1.1) exist. In Ref. 14 it is shown that periodic asymmetric spike solutions to the Gierer-Meinhardt equations are unstable when posed on $\mathbb{R}$. It is shown that a consequence of this fact is that asymmetric spike solutions are unstable when posed on finite domains as well. In this paper, we will examine the spectra of the asymmetric solutions near the point of bifurcation off a symmetric branch. We will confirm that, sufficiently close to this point, all such solutions are unstable and explicitly compute the leading order term of the critical eigenvalues. A result of this calculation shows that the operator resulting from a linearization about an asymmetric spike solution with k_1 *small* spikes will result in exactly k_1 positive eigenvalues in its spectrum.

We will consider a scaled version of the system,

$$a_t = \varepsilon^2 a_{xx} - a + \frac{a^p}{h^q}, \quad -1 < x < 1, \quad t > 0, \tag{1.1a}$$

$$0 = D h_{xx} - \mu h + \frac{1}{\varepsilon}\frac{a^m}{h^s}, \quad -1 < x < 1, \quad t > 0, \tag{1.1b}$$

$$a_x(\pm 1,t) = h_x(\pm 1,t) = 0. \tag{1.1c}$$

Here $a(x,t)$, $h(x,t)$, $\varepsilon \ll 1$, $D>0$, and $\mu>0$ represent the scaled activator concentration, inhibitor concentration, activator diffusivity, inhibitor diffusivity, and inhibitor decay rate. The exponents p, q, m, and s are assumed to satisfy

$$p>1, \quad q>0, \quad m>0, \quad s \geq 0, \quad 0 < \frac{p-1}{q} < \frac{m}{s+1}. \tag{1.2}$$

The remainder of this paper will proceed as follows: In Sec. II we will give a brief review of the construction and stability results of Ref. 1. In Sec. III we find the leading order corrections to the heights and locations of asymmetric spikes near the bifurcation off the symmetric branch [see (3.9)]. In Sec. IV we find that the stability of an asymmetric spike profile is determined by the signs of the eigenvalues of a matrix. This matrix will be dependent on the arrangement and number of spikes (see proposition 4.1). In Sec. V we will show that to determine the stability, we only need to consider the eigenvalues of a simple diagonal matrix, with entries of ±1. It then follows that all asymmetric patterns will be unstable near the bifurcation point. Furthermore, the number of unstable eigenvalues (counting multiplicity) will be equal to

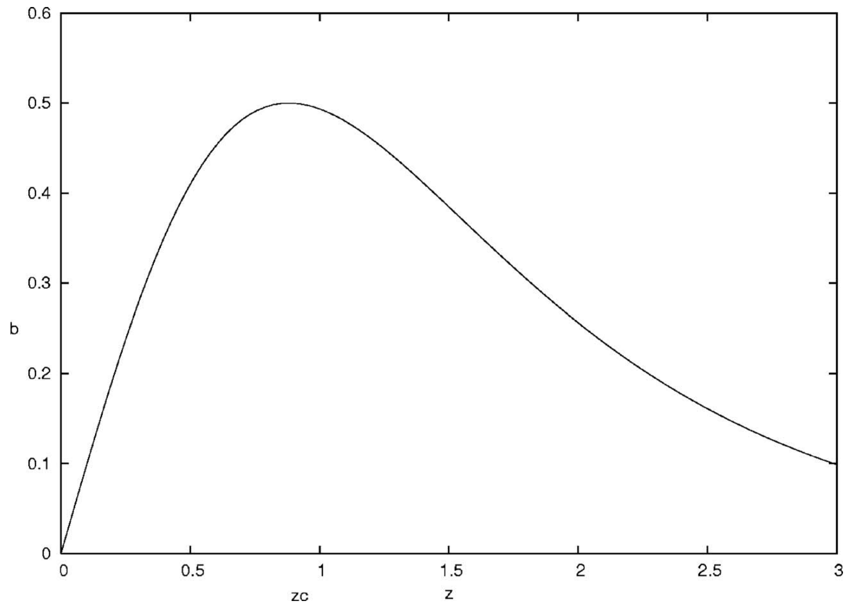

FIG. 1. Graph of $b(z)$ for the case $r=1$.

the number of small spikes in the asymmetric pattern (see proposition 5.1). Finally, in Sec. VI, we conclude.

II. REVIEW OF ASYMMETRIC SPIKE RESULTS

In this section, we will briefly review the results of Ref. 1. First, we will show that (1.1) admits solutions with spikes of at most two different heights in arbitrary arrangements. The solutions are constructed using the method of matched asymptotic expansions. We will then consider the linear stability of a given profile by examining the spectrum of the operator resulting from a linearization of (1.1) about the profile. Two types of eigenvalues, which we will refer to as the large and small eigenvalues, must be considered. The large eigenvalues determine the stability of the profile on an $O(1)$ time scale. It has been shown that if D is below a critical value, then the profile will be stable with respect to these eigenvalues. The small eigenvalues act on a much slower time scale and unstable asymmetric solutions can persist for times of duration $O(\varepsilon^{-2})$. To determine the small eigenvalues of the operator, one needs to solve for the eigenvalues of a matrix. In Ref. 1 this is done numerically on a case by case basis.

The calculations considered in this section are very involved. We include some of the main results and methods for completeness, but the subtle details are omitted. References 1 and 11 have the complete calculations with all the details.

We will begin by considering a single spike on a domain of undetermined length. Define ℓ as half the length of the "support" for a spike, and construct a one-spike equilibrium solution to

$$a_t = \varepsilon^2 a_{xx} - a + \frac{a^p}{h^q}, \quad -\ell < x < \ell, \quad t>0, \tag{2.1a}$$

$$0 = D h_{xx} - \mu h + \frac{1}{\varepsilon}\frac{a^m}{h^s}, \quad -\ell < x < \ell, \quad t>0, \tag{2.1b}$$

$$a_x(\pm\ell,t) = h_x(\pm\ell,t) = 0. \tag{2.1c}$$

We expect the solution to (2.1a) to be a single sharp spike centered at $x=0$ and exponentially small elsewhere. Thus we can approximate the last term in (2.1b) by a scaled Dirac delta function. We then find that the height $h(\ell)$ at the ends of the support interval is given by

$$h(\ell) = \left(\frac{2\sqrt{\mu D}}{\int_{-\infty}^{\infty}[u_c(y)]^m dy}\right)^r b(\ell\sqrt{\mu/D}), \tag{2.2}$$

where $u_c(y)$ is the unique positive solution to

$$\begin{gathered} u_c'' - u_c + u_c^p = 0, \quad \infty < y < \infty \quad u_c'(0)=0 \\ u_c \to 0 \ \text{ as } |y| \to \infty; \quad u_c(0) > 0, \end{gathered} \tag{2.3}$$

and u_c is given by

$$u_c(y) = \left(\frac{p+1}{2}\right)^{1/(p-1)} \left(\cosh\left[\frac{(p-1)y}{2}\right]\right)^{-2/(p-1)}.$$

We refer to u_c as the canonical spike solution.

In (2.2) the function $b(z)$, for $z>0$, and the exponent r, are defined by

$$b(z) \equiv \frac{\tanh^r z}{\cosh z}; \quad r \equiv \left(\frac{mq}{p-1} - (s+1)\right)^{-1}. \tag{2.4}$$

It can be shown that $b(z)$ has a unique maximum at $z=z_c$, where

$$z_c = \operatorname{arcsinh}\sqrt{r} = \log(\sqrt{r} + \sqrt{1+r}) \tag{2.5}$$

and thus for each $z \in (0,z_c)$ there is a unique $\tilde{z} \in (z_c,\infty)$ such that $b(z)=b(\tilde{z})$. Defining $\tilde{\ell}$ so that $b(\ell\sqrt{\mu/D})=b(\tilde{\ell}\sqrt{\mu/D})$, it follows from (2.2) that $h(\ell)=h(\tilde{\ell})$. It is thus possible to glue spike solutions of (2.1) together to form asymmetric k-spike solutions. Because each ℓ determines a unique $\tilde{\ell}$, it follows that the height of each spike in every asymmetric pattern is

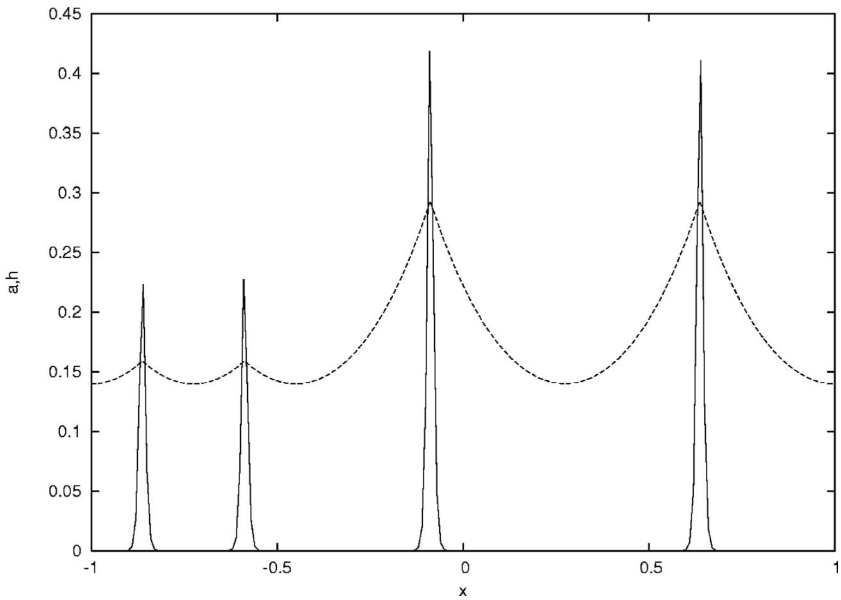

FIG. 2. Steady-state spike solution for (1.1) with parameter values $D=0.07$, $\mu=1$, $\varepsilon=0.1$. The solid curve is a_e and the dotted curve is h_e.

one of two predetermined values. For $r=1$ the plot of $b(z)$ is given in Fig. 1.

The equality $b(z)=b(\tilde{z})$ for $0<z<z_c<\tilde{z}$ establishes a function $f(z)=\tilde{z}$ between z and $\tilde{z}$. For any $r>0$, $f(z)$ is convex on $(0,z_c)$, $f'(z)<-1$ on $(0,z_c)$ and $f'(z_c)=-1$.

For any asymmetric spike pattern, there are exactly two support lengths 2ℓ and $2\tilde{\ell}$. For k_1 small spikes and $k_2=k-k_1$ large spikes, the support lengths must fit in the interval $[-1,1]$,

$$\sum_{j=1}^{k} 2\ell_j = 2\sqrt{\frac{D}{\mu}}(k_1 z + k_2\tilde{z}) = 2,$$

where

$$\ell_j = \begin{cases} \ell & \text{if } j \text{ is for a small spike} \\ \tilde{\ell} & \text{if } j \text{ is for a large spike.} \end{cases} \tag{2.6}$$

This asymmetric pattern will exist if both $k_1 z + k_2\tilde{z} = \sqrt{\mu/D}$ and $f(z)=\tilde{z}$.

In Ref. 1 it is shown that there are four combinations of k_1, k_2, D, and D_m for which asymmetric spike patterns exist:

(i) Exactly one solution: $k_1 < k_2, \quad D < D_m;$ (2.7a)

(ii) Exactly one solution: $k_1 = k_2, \quad D < D_m;$ (2.7b)

(iii) Exactly one solution: $k_1 > k_2, \quad D < D_m;$ (2.7c)

(iv) Exactly two solutions: $k_1 > k_2, \quad D_{m_1} > D > D_m;$ (2.7d)

where

$$D_m = \frac{\mu}{k^2 z_c^2} \tag{2.8}$$

and D_{m_1} is defined as the tangency solution of the system

$$-\frac{k_1}{k_2}z + \frac{1}{k_2}\sqrt{\frac{\mu}{D}} = f(z), \qquad -\frac{k_1}{k_2} = f'(z). \tag{2.9}$$

When $D=D_m$, then $z=\tilde{z}=z_c$, and the solution is k equal height and equally spaced spikes, the "symmetric" case, which is analyzed in Ref. 11.

The equilibrium result in Ref. 1 states that, for $r>0$, $D<D_m$, and $\varepsilon\to 0$, there exists an asymmetric equilibrium solution (a_e,h_e) to (1.1) of the form

$$a_e(x) \sim \sum_{j=1}^{k} [h_{\ell_j}]^{q/(p-1)} u_c\left(\frac{x-x_j}{\varepsilon}\right), \tag{2.10}$$

where x_j is the location of the center of the jth spike, the value of h at x_j satisfies

$$h_{\ell_j} = \left(\frac{2\sqrt{\mu D}\tanh(\ell_j\sqrt{\mu/D})}{\int_{-\infty}^{\infty}[u_c(y)]^m dy}\right)^r, \tag{2.11}$$

and the equilibrium h_e is

$$h_e(x) \sim \sum_{j=1}^{k} 2\sqrt{\mu D}\tanh(\ell_j\sqrt{\mu/D}) h_{\ell_j} G(x;x_j), \tag{2.12}$$

where $G(x;x_j)$ satisfies

$$DG_{xx} - \mu G = -\delta(x-x_j), \qquad -1<x<1, \tag{2.13a}$$

$$G_x(\pm 1;x_j) = 0. \tag{2.13b}$$

For $k_1=k_2=2$ and $r=1$ a plot of the activator $a_e(x)$ and inhibitor $h_e(x)$ is given in Fig. 2.

To study the stability of the solutions a_e and h_e, substitute

$$a(x,t) = a_e(x) + e^{\lambda t}\phi(x), \qquad h(x,t) = h_e(x) + e^{\lambda t}\eta(x) \tag{2.14}$$

into (1.1), where $\phi \ll 1$ and $\eta \ll 1$. This leads to the eigenvalue problem

$$L_\varepsilon\phi - q\frac{a_e^p}{h_e^{q+1}}\eta = \lambda\phi, \qquad -1<x<1, \tag{2.15a}$$

$$D\eta_{xx}-\mu\eta=\varepsilon^{-1}m\frac{a_e^{m-1}}{h_e^s}\phi+\varepsilon^{-1}s\frac{a_e^m}{h_e^{s+1}}\eta,\quad -1<x<1, \tag{2.15b}$$

$$\phi_x(\pm 1)=\eta_x(\pm 1)=0, \tag{2.15c}$$

where

$$L_\varepsilon\phi\equiv\varepsilon^2\phi_{xx}-\phi+p\frac{a_e^{p-1}}{h_e^q}\phi \tag{2.15d}$$

and a_e and h_e are given by (2.10) and (2.12), respectively.

The spectrum of (2.15) contains large eigenvalues that are $O(1)$ and small eigenvalues that are $O(\varepsilon^2)$. For the large eigenvalues, Ref. 1 constructs an eigenfunction of (2.15) of the form

$$\phi(x)\sim\sum_{j=1}^{k}\phi_j\left(\frac{x-x_j}{\varepsilon}\right), \tag{2.16}$$

where $\phi_j(y)\to 0$ as $|y|\to\infty$. Define the matrix of Green's functions of (2.13)

$$\mathcal{G}\equiv\begin{pmatrix} G(x_1;x_1) & \cdots & G(x_1;x_k)\\ \vdots & \ddots & \vdots\\ G(x_k;x_1) & \cdots & G(x_k;x_k)\end{pmatrix} \tag{2.17}$$

and the diagonal matrices

$$\mathcal{H}\equiv\begin{pmatrix} h_{\ell_1} & 0 & \cdots & 0\\ 0 & \ddots & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & h_{\ell_k}\end{pmatrix} \tag{2.18}$$

and

$$\mathcal{C}\equiv\begin{pmatrix} 2\tanh z_1 & 0 & \cdots & 0\\ 0 & \ddots & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 2\tanh z_k\end{pmatrix},\quad z_j\equiv\ell_j\sqrt{\mu/D}. \tag{2.19}$$

Then, the matrix $\mathcal{E}$ defined by

$$\mathcal{E}\equiv\sqrt{\mu D}\mathcal{H}^{\gamma-1}\mathcal{G}\mathcal{C}\mathcal{H}^{1-\gamma},\quad \gamma\equiv\frac{q}{p-1} \tag{2.20}$$

has real positive eigenvalues. Write $\mathcal{E}=S^{-1}\Lambda_e S$ for some invertible matrix S and diagonal matrix Λ_e, and define $\boldsymbol{\psi}=S\boldsymbol{\phi}$, where the jth component of the column vector $\boldsymbol{\phi}$ is ϕ_j. The eigenvalue problem (2.15) for the case in which $s=0$ reduces to k uncoupled problems,

$$\boldsymbol{\psi}''-\boldsymbol{\psi}+pu_c^{p-1}\boldsymbol{\psi}-mqu_c^p\left(\frac{\int_{-\infty}^{\infty}u_c^{m-1}\Lambda_e\boldsymbol{\psi}dy}{\int_{-\infty}^{\infty}u_c^m dy}\right)=\lambda\boldsymbol{\psi},\quad -\infty<y<\infty, \tag{2.21a}$$

$$\boldsymbol{\psi}\to 0\quad\text{as}\quad |y|\to\infty. \tag{2.21b}$$

The conditions for which $\mathrm{Re}(\lambda)<0$ in (2.21) can be obtained by using a key result of Wei.[15]

Theorem 2.1 (Wei[15]). *Let $\beta>0$ and consider the nonlocal eigenvalue problem for $\phi(y)$,*

$$\phi'-\phi+pu_c^{p-1}\phi-\beta(p-1)u_c^p\left(\frac{\int_{-\infty}^{\infty}u_c^{m-1}\phi dy}{\int_{-\infty}^{\infty}u_c^m dy}\right)=\lambda\phi,\quad -\infty<y<\infty, \tag{2.22a}$$

$$\phi\to 0\quad as\quad |y|\to\infty, \tag{2.22b}$$

corresponding to eigenpairs for which $\lambda\neq 0$. Here $u_c(y)$ satisfies (2.3). Let $\lambda_0\neq 0$ be the eigenvalue of (2.21) with the largest real part. Then, if $\beta<1$, we have

$$\mathrm{Re}(\lambda_0)>0. \tag{2.23}$$

Alternatively, if $\beta>1$ and either of the following two conditions hold:

$$\text{(i) } m=2,\ 1<p\le 5,\quad \textit{or}\quad \text{(ii) } m=p+1,\ p>1, \tag{2.24}$$

then

$$\mathrm{Re}(\lambda_0)<0. \tag{2.25}$$

By comparing (2.21) with (2.22) (Ref. 1), obtain the following result:

Proposition 2.2. *Let $\lambda\neq 0$ be the eigenvalue of (2.21) with the largest real part and assume that condition (2.24) holds. Let α_1 be the minimum eigenvalue of the matrix $\mathcal{E}$ defined in (2.20). Then* $\mathrm{Re}(\lambda_0)>0$ *when*

$$\alpha_1<\frac{p-1}{qm} \tag{2.26}$$

and $\mathrm{Re}(\lambda_0)<0$ *when* $\alpha_1>(p-1)/(qm)$.

Thus the eigenvalue problem (2.15) is converted to a problem of finding the eigenvalues of the matrix $\mathcal{E}$.

When $s>0$ the matrix $\mathcal{E}$ is defined by

$$\mathcal{E}=\mathcal{H}^{\gamma-1}(\mathcal{B}+s\mathcal{C})^{-1}\mathcal{C}\mathcal{H}^{1-\gamma}, \tag{2.27}$$

where $\mathcal{B}$ is the tridiagonal matrix

$$\mathcal{B}=\begin{pmatrix} c_1 & d_1 & 0 & \cdots & 0 & 0 & 0\\ d_1 & c_2 & \ddots & \ddots & \ddots & 0 & 0\\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0\\ 0 & 0 & \ddots & \ddots & \ddots & c_{k-1} & d_{k-1}\\ 0 & 0 & 0 & \cdots & 0 & d_{k-1} & c_k\end{pmatrix} \tag{2.28a}$$

with matrix entries defined by

$$c_1=\coth(z_1+z_2)+\tanh z_1;\qquad c_k=\coth(z_k+z_{k-1})+\tanh z_k, \tag{2.28b}$$

$$c_j=\coth(z_{j+1}+z_j)+\coth(z_j+z_{j-1}),\quad j=2,\ldots,k-1;\qquad d_j=-\mathrm{csch}(z_j+z_{j+1}), j=1,\ldots,k-1. \tag{2.28c}$$

The stability criterion in (2.26) still holds if α_1 in proposition 3.1 is identified as the minimum eigenvalue of the matrix

$(\mathcal{B}+s\mathcal{C})^{-1}\mathcal{C}$. Alternatively, by computing $\mathcal{E}^{-1}$, Ref. 1 expresses the following stability criterion.

Corollary 2.3. *Let $\lambda_0 \neq 0$ be the eigenvalue of (2.21) with the largest real part and assume that condition (2.24) holds. Let e_m be the maximum eigenvalue of the tridiagonal matrix $\tilde{\mathcal{E}}$ defined by $\tilde{\mathcal{E}} \equiv \mathcal{C}^{-1}\mathcal{B}$. Then,* $\text{Re}(\lambda_0)>0$ *when*

$$e_m > 1 + \frac{1}{r}. \tag{2.29}$$

Also, $\text{Re}(\lambda_0)<0$ *when $e_m < 1+r^{-1}$. Here r is defined in (2.4).*

Thus to leading order, the eigenvalues of (2.15) are negative given that corollary 2.3 is satisfied. Since the eigenvalues are $O(1)$, this is sufficient for stability with respect to the large eigenvalues. However, (2.15) also has eigenvalues of $O(\varepsilon^2)$. To determine the sign of these eigenvalues, we must consider the higher order terms.

The analysis of the small eigenvalues also involves reducing (2.15) to a matrix eigenvalue problem. If we differentiate $u_c(\varepsilon^{-1}(x-x_j))$ with respect to x, we find that $L_\varepsilon u_c'=0$. Thus to find the small eigenvalues of (2.15), we consider eigenfunctions of the form

$$\phi = \phi_0 + \varepsilon\phi_1 + \cdots, \quad \eta = \varepsilon\eta_0 + \cdots, \tag{2.30a}$$

where

$$\phi_0 \equiv \sum_{j=1}^{k} c_j u_j'\left(\frac{x-x_j}{\varepsilon}\right), \quad \phi_1 \equiv \sum_{j=1}^{k} c_j \phi_{1j}\left(\frac{x-x_j}{\varepsilon}\right) \tag{2.30b}$$

and the c_j are arbitrary coefficients. Both Refs. 1 and 11 show that η_0 satisfies

$$D\eta_{0xx} - \mu\eta_0 = -\varepsilon^{-2} m \frac{a_e^{m-1}}{h_e^s}(\phi_0 + \varepsilon\phi_1) + \varepsilon^{-1} s \frac{a_e^m}{h_e^{s+1}}\eta_0, \quad -1 < x < 1. \tag{2.31}$$

Since ϕ_0 is a linear combination of u_j', it follows that the term multiplied by ϕ_0 in (2.31) behaves like a dipole. References 1 and 11 also show that ϕ_{1j} is continuous across $x=x_j$ and has the form of a spike. This implies that the term in (2.31) proportional to ϕ_1 behaves like a linear combination of $\delta(x-x_j)$ when $\varepsilon \ll 1$, and is of the same order in ε as the dipole term proportional to ϕ_0. Thus it is necessary to approximate the eigenfunction for ϕ to both the $O(1)$ and $O(\varepsilon)$ terms in order to calculate an eigenvalue of order $O(\varepsilon^2)$.

Substitute (2.30) into (2.15a) to obtain the following result:[1]

Proposition 2.4. *The eigenvalues of $O(\varepsilon^2)$ for (2.15) satisfy*

$$\lambda c_j \int_{-\infty}^{\infty} [u_c'(y)]^2 dy \sim \frac{\varepsilon^2 q}{p+1}\int_{-\infty}^{\infty}[u_c(y)]^{p+1}dy\left(\langle\eta_{0x}\rangle_j - \frac{c_j\mu}{D}\right), \quad j=1,\ldots,k. \tag{2.32}$$

Here $\langle\eta_{0x}\rangle_j$ is to be calculated from

$$D\eta_{0xx} - \mu\eta_0 = 0; \quad -1<x<1; \quad \eta_{0x}(\pm 1) = 0, \tag{2.33a}$$

$$[D\eta_0]_j = -2\sqrt{\mu D}c_j \tanh(z_j) h_{\ell_j}^{1-\gamma}; \quad [D\eta_{0x}]_j = 2\sqrt{\mu D}\tilde{s}\tanh(z_j)\langle\eta_0\rangle_j, \tag{2.33b}$$

$$\tilde{s} \equiv s - \frac{qm}{p-1}, \tag{2.33c}$$

where, defining $\zeta(x_{j\pm})$ as the one-sided limits of $\zeta(x)$ as $x \to x_{j\pm}$, $\langle\zeta\rangle_j \equiv (\zeta(x_{j+}) + \zeta_0(x_{j-}))/2$ and $[\zeta]_j \equiv \zeta(x_{j+}) - \zeta(x_{j-})$.

To convert (2.32) and (2.33) to a matrix eigenvalue problem,[1] define the matrices $\mathcal{D}$, $\mathcal{PB}$, $\mathcal{P}_g\mathcal{B}_g$, $\mathcal{Q}$, and $\mathcal{B}_g$ as follows:

$$\mathcal{D} \equiv \tilde{s}D^2(\mathcal{K} + \tilde{s}I)^{-1}, \tag{2.34}$$

where $\mathcal{K}$ is the diagonal matrix of eigenvalues of $\mathcal{C}^{-1}\mathcal{B}$,

$$\mathcal{PB} = \frac{1}{2D}\begin{pmatrix} c_1 & d_1 & 0 & \cdots & 0 & 0 & 0 \\ -d_1 & c_2 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & c_{k-1} & d_{k-1} \\ 0 & 0 & 0 & \cdots & 0 & -d_{k-1} & c_k \end{pmatrix}, \tag{2.35}$$

where

$$c_1 = \tanh z_1 - \coth(z_1+z_2), \quad c_k = \coth(z_k + z_{k-1}) - \tanh z_k, \tag{2.36a}$$

$$c_\ell = \coth(z_\ell + z_{\ell-1}) - \coth(z_\ell + z_{\ell+1}), \quad \ell = 2,\ldots,k-1, \tag{2.36b}$$

$$d_\ell = \text{csch}(z_\ell + z_{\ell+1}), \quad \ell = 1,\ldots,k-1. \tag{2.36c}$$

The tridiagonal matrix $\mathcal{P}_g\mathcal{B}_g$ is defined to be the same as $\mathcal{PB}$ except that (2.36a) is replaced by

$$\tilde{c}_1 = \coth z_1 - \coth(z_1+z_2); \quad \tilde{c}_k = \coth(z_k + z_{k-1}) - \coth z_k. \tag{2.37}$$

The matrix $\mathcal{Q}$ is defined to be the matrix whose columns are the orthonormal eigenvectors of $\mathcal{C}^{-1}\mathcal{B}$. The tridiagonal matrix $\mathcal{B}_g$ is defined to be the same as $\mathcal{B}$ in (2.28) except that (2.28b) is replaced by

$$\tilde{c}_1 = \coth(z_1+z_2) + \coth z_1; \quad \tilde{c}_k = \coth(z_k+z_{k-1}) + \coth z_k \tag{2.38}$$

(see Ref. 1), to obtain the following result:

Proposition 2.5. *For $\varepsilon \ll 1$, the eigenvalues of (2.15) of order $\lambda = O(\varepsilon^2)$ satisfy*

$$\lambda_j \sim \frac{\varepsilon^2 q\mu}{D(p+1)}\left(\frac{\int_{-\infty}^{\infty}[u_c(y)]^{p+1}dy}{\int_{-\infty}^{\infty}[u_c'(y)]^2 dy}\right)\left(\frac{1}{\omega_j} - 1\right), \quad j=1,\ldots,k, \tag{2.39}$$

where ω_j is an eigenvalue of the generalized eigenvalue problem

$$\mathcal{C}^{-1}\mathcal{B}_g\boldsymbol{u} = \omega(I+\mathcal{R})\boldsymbol{u}. \tag{2.40a}$$

Here $\mathcal{R}$ is given by

$$\mathcal{R} \equiv -\mathcal{P}\mathcal{B}\mathcal{Q}\mathcal{D}\mathcal{Q}^{-1}\mathcal{P}_g\mathcal{B}_g. \tag{2.40b}$$

The eigenvector $\boldsymbol{\phi}$ is given by (2.30), where

$$\boldsymbol{c}_j = \mathcal{H}^{\gamma-1}\mathcal{C}^{-1}\mathcal{B}_g\boldsymbol{u}_j \tag{2.40c}$$

and $\boldsymbol{u}_j$ is an eigenvector of (2.40a).

For a symmetric k-spike pattern, ω_j can be calculated analytically from (2.40), since $\mathcal{C}$ is a constant multiple of the identity matrix, and $\mathcal{B}_g$ and $\mathcal{R}$ were found to have exactly the same eigenspace. This analysis was done in Sec. 4.2 of Ref. 11, and the following result was given in proposition 11 of Ref. 11.

Proposition 2.6 (from Ref. 11). *Consider a symmetric k-spike equilibrium solution where $z_1=z_2=\cdots=z_k=z_c$. Then for $\varepsilon \ll 1$, the eigenvalues λ of (2.15) of $O(\varepsilon^2)$ are all real, and they are negative when*

$$D < D_m = \frac{\mu}{k^2 z_c^2}, \tag{2.41}$$

where z_c is given in (2.5). When $D > D_m$, then $k-1$ small eigenvalues are positive. When $D=D_m$, $\lambda=0$ is a small eigenvalue of algebraic multiplicity $k-1$. Furthermore, $D_m < D_k$, where D_k is the largest value of D for which the symmetric branch is stable with respect to the large $O(1)$ eigenvalues.

Define $\omega_{0,j}$ as ω_j in (2.39) when $D=D_m$. From proposition 2.6, it follows that $k-1$ of the $\omega_{0,j}$'s are equal to one, and from Ref. 11 it is possible to compute that the remaining $\omega_{0,j}$ is two. Thus we can label the $\omega_{0,j}$ so that

$$\omega_{0,j} = 1, \quad j = 1, \ldots, k-1, \tag{2.42a}$$

$$\omega_{0,k} = 2. \tag{2.42b}$$

As in Ref. 1, define ω^* by

$$\omega^* \equiv \min(\omega_j) \quad \text{such that } \omega_j > 0 \quad \text{for } j = 1, \ldots, k. \tag{2.43}$$

From proposition 2.5, it follows that an asymmetric small spike pattern will be unstable when $\omega^* < 1$, or when

$$\omega^* - \omega_{0,j} < 0, \quad j = 1, \ldots, k-1. \tag{2.44}$$

From proposition 2.6, the symmetric branch will be stable with respect to both the large and small eigenvalues when $D < D_m$. This paper will consider asymmetric patterns with D near D_m. Such solutions are near a symmetric solution, for which analytic results exist in Ref. 11, and are thus amenable to perturbation methods.

III. HEIGHTS AND POSITIONS NEAR THE BIFURCATION

We begin the analysis by approximating D near D_m,

$$D = D_m \pm \delta, \tag{3.1}$$

where the bifurcation parameter $\delta \ll 1$ and the sign in (3.1) depend on the case in (2.7).

Using the fact that $z = z_c = \tilde{z}$ at $D = D_m$ we form asymptotic expansions for z and $\tilde{z}$,

$$z = z_c + z_1\nu_1(\delta) + z_2\nu_2(\delta) + \cdots, \tag{3.2a}$$

$$\tilde{z} = z_c + \tilde{z}_1\nu_1(\delta) + \tilde{z}_2\nu_2(\delta) + \cdots, \tag{3.2b}$$

where

$$1 \gg \nu_1(\delta) \gg \nu_2(\delta) \gg \cdots. \tag{3.3}$$

Expanding $b(z)=b(\tilde{z})$ about $z=z_c=\tilde{z}$ and using $b'(z_c)=0$, gives

$$\begin{aligned}
&\tfrac{1}{2}b''(z_c)(z_1^2\nu_1^2 + 2z_1z_2\nu_1\nu_2 + z_2^2\nu_2^2 + \cdots) \\
&\quad + \tfrac{1}{6}b'''(z_c)(z_1^3\nu_1^3 + 3z_1^2z_2\nu_1^2\nu_2 + 3z_1z_2^2\nu_1\nu_2^2 + \cdots) \\
&\quad = \tfrac{1}{2}b''(z_c)(\tilde{z}_1^2\nu_1^2 + 2\tilde{z}_1\tilde{z}_2\nu_1\nu_2 + \tilde{z}_2^2\nu_2^2 + \cdots) \\
&\quad + \tfrac{1}{6}b'''(z_c)(\tilde{z}_1^3\nu_1^3 + 3\tilde{z}_1^2\tilde{z}_2\nu_1^2\nu_2 + 3\tilde{z}_1\tilde{z}_2^2\nu_1\nu_2^2 + \cdots).
\end{aligned} \tag{3.4}$$

From (3.3), the lowest order terms in (3.4) are the ν_1^2 terms. Equating these gives $z_1^2=\tilde{z}_1^2$. Because $0 < z < z_c < \tilde{z}$, it follows that $z_1 < 0 < \tilde{z}_1$. We define

$$\alpha \equiv \tilde{z}_1 = -z_1 > 0. \tag{3.5}$$

Thus (3.2) can be written as

$$z = z_c - \alpha\nu_1(\delta) + z_2\nu_2(\delta) + \cdots, \tag{3.6a}$$

$$\tilde{z} = z_c + \alpha\nu_1(\delta) + \tilde{z}_2\nu_2(\delta) + \cdots. \tag{3.6b}$$

Note that all that will be required for the main stability result is that $\alpha > 0$, and this is satisfied by (3.5). However, we continue with this calculation for completeness.

Next we expand the condition $k_1z + k_2\tilde{z} = \sqrt{\mu/D}$ about $\delta=0$,

$$k_1z + k_2\tilde{z} = \sqrt{\frac{\mu}{D_m}} \mp \frac{\sqrt{\mu}}{2}D_m^{-3/2}\delta + \cdots, \quad D = D_m \pm \delta. \tag{3.7}$$

Substituting (3.6) into (3.7) and using $(k_1+k_2)z_c = \sqrt{\mu/D_m}$ gives

$$(k_2-k_1)\alpha\nu_1 + (k_1z_2 + k_2\tilde{z}_2)\nu_2 = \mp\frac{\sqrt{\mu}}{2}D_m^{-3/2}\delta + \cdots, \quad D = D_m \pm \delta. \tag{3.8}$$

For the case (2.7a), with $D = D_m - \delta$, (3.8) gives the leading order correction as

$$(k_2-k_1)\alpha\nu_1(\delta) = \frac{\sqrt{\mu}}{2}D_m^{-3/2}\delta.$$

Thus we can set

$$\alpha = \frac{1}{2(k_2-k_1)}\sqrt{\frac{\mu}{D_m^3}}, \quad \nu_1(\delta) = \delta. \tag{3.9}$$

Each of the other three cases will involve ν_2, z_2, and $\tilde{z}_2$. To find a relation among these, we equate terms in (3.4) of equal order. *A priori*, it is not known whether $\nu_2 \gg \nu_1^2$, $\nu_2 \sim \nu_1^2$, or $\nu_2 \ll \nu_1^2$. However, an assumption other than $\nu_2 \sim \nu_1^2$ will lead to $z_i = \tilde{z}_i = 0$ for $i > 1$ and satisfying (3.8) will no longer be possible. Thus we set $\nu_2 = \nu_1^2$ and use (3.5) and (3.4) to get

$$\tilde{z}_2 = -z_2 - \alpha^2 \frac{b'''(z_c)}{3b''(z_c)} = -z_2 + \alpha^2 \frac{2r+1}{3\sqrt{r(1+r)}}. \tag{3.10}$$

For the case (2.7b) the leading order term on the left-hand side of (3.8) is $O(\nu_2)$. Thus we require $\nu_2 \sim \delta$ and $\nu_1 \sim \delta^{1/2}$. Substituting (3.10) into (3.8) with $D = D_m - \delta$ gives the leading order correction as

$$k\alpha^2 \frac{2r+1}{3\sqrt{r(1+r)}} \nu_2(\delta) = \sqrt{\mu} D_m^{-3/2} \delta.$$

Since $\alpha > 0$,

$$\alpha = \sqrt{\frac{3\sqrt{\mu r(1+r)}}{k(2r+1)D_m^{3/2}}}, \quad \nu_2(\delta) = \delta, \quad \nu_1(\delta) = \delta^{1/2}. \tag{3.11}$$

For the cases (2.7c) and (2.7d) substituting (3.6) into (3.7) can give two solutions. For (2.7c), one solution is that $z > z_c$ and so is extraneous. In both cases, for the two solutions to exist we must have, since $f''(z) > 0$, that

$$f'(z) < -\frac{k_1}{k_2} < f'(z_c). \tag{3.12}$$

Expanding about $z = z_c$ and using $f'(z_c) = -1$,

$$\alpha f''(z_c)\nu_1(\delta) + O(\nu_2(\delta)) > \frac{k_1 - k_2}{k_2} > 0. \tag{3.13}$$

Thus, given k_1 and k_2, we can define a positive constant $c < \alpha f''(z_c)$ of $O(1)$ such that

$$\frac{k_1 - k_2}{k_2} = c\nu_1(\delta) + O(\nu_2(\delta)). \tag{3.14}$$

Because k_1 and k_2 are integers, and $\alpha f''(z_c)$ is $O(1)$, Eq. (3.14) will hold only if $k_1 \sim k_2$ and $k_2 \gg 1$. It also follows that $k_1 - k_2$ is $O(1)$ and, hence, from (3.14) that

$$\frac{1}{k_2} \quad \text{is } O(\nu_1(\delta)). \tag{3.15}$$

For (2.7c), substituting (3.6) and (3.14) into (3.7) with $D = D_m - \delta$ gives, to leading order

$$\left(\alpha^2 \frac{2r+1}{3\sqrt{r(1+r)}} - \alpha c\right)\nu_2(\delta) = \frac{\sqrt{\mu}}{2} D_m^{-3/2} \frac{\delta}{k_2}, \tag{3.16}$$

or, since from (3.14), $(1/k_2) = [c/(k_1 - k_2)]\nu_1(\delta) + O(\nu_2(\delta))$,

$$(2(k_1-k_2)(2r+1)\alpha^2 - 6c(k_1-k_2)\sqrt{r(1+r)}\alpha)\nu_2(\delta)/\nu_1(\delta)$$
$$= 3c\sqrt{\mu r(1+r)} D_m^{-3/2} \delta, \tag{3.17}$$

giving

$$\alpha = \tilde{c} + \sqrt{\tilde{c}^2 + \frac{\tilde{c}}{k_1 - k_2}\sqrt{\frac{\mu}{D_m^3}}}, \quad \frac{\nu_2(\delta)}{\nu_1(\delta)} = \delta, \tag{3.18}$$

where $\tilde{c} = 3c\sqrt{r(1+r)}/(4r+2)$ and where the negative square root is rejected since it will give a negative value for α.

Similarly, for (2.7d) substituting (3.6) and (3.14) into (3.7) with $D = D_m + \delta$ gives

$$\alpha = \tilde{c} \pm \sqrt{\tilde{c}^2 - \frac{\tilde{c}}{k_1 - k_2}\sqrt{\frac{\mu}{D_m^3}}}, \quad \frac{\nu_2(\delta)}{\nu_1(\delta)} = \delta. \tag{3.19}$$

Here, both values of α will be positive. Thus for cases (3.18) and (3.19) we may choose $\nu_1(\delta) = \delta$ and then it follows that $\nu_2(\delta) = \delta^2$.

Now define

$$\mathsf{s}_\ell \equiv \begin{cases} -1 & \text{if } z_\ell = z, \text{ the scaled support for a small spike,} \\ +1 & \text{if } z_\ell = \tilde{z}, \text{ the scaled support for a large spike.} \end{cases} \tag{3.20}$$

Then we can write (3.6) as

$$z_\ell = z_c + \mathsf{s}_\ell \alpha \nu_1(\delta) + O(\nu_2(\delta)), \quad \ell = 1, \ldots, k, \tag{3.21}$$

where α and $\nu_1(\delta)$ are defined above for each of the four cases.

IV. CORRECTION TO THE SMALL EIGENVALUES

In this section, we will find the leading order correction to the small eigenvalues of (2.39). We begin by expanding (2.40a) in a δ asymptotic series. The $O(\delta)$ term will force a solvability condition on the corrections to the eigenvalues. This solvability condition will itself be in the form of an eigenvalue problem.

Let

$$A = (I + \mathcal{R})^{-1}(\mathcal{C}^{-1}\mathcal{B}_g), \tag{4.1a}$$

$$A_0 = (I + \mathcal{R}_0)^{-1}(\mathcal{C}_0^{-1}\mathcal{B}_{g0}), \tag{4.1b}$$

where the zero subscript denotes the $\delta = 0$ case. Let ω, $\boldsymbol{u}$ and ω_0, $\boldsymbol{u}_0$ be the eigenpairs of $A\boldsymbol{u} = \omega\boldsymbol{u}$ and $A_0\boldsymbol{u}_0 = \omega_0\boldsymbol{u}_0$, respectively. Define $\mathcal{U}$ as the $k \times k$ matrix with columns $\boldsymbol{u}_{0,j}$, and order the columns so that the first $k-1$ columns are the eigenvectors associated with the eigenvalue $\omega_0 = 1$. We expand ω and A as follows:

$$\omega = \omega_0 + \delta\omega_1 + O(\delta^2), \tag{4.2a}$$

$$A = A_0 + \delta A_1 + O(\delta^2). \tag{4.2b}$$

Because $\omega_0 = 1$ has multiplicity $k-1$, define $\boldsymbol{b}$ and $\boldsymbol{u}_1$ so that

$$\boldsymbol{u} = \mathcal{U}\boldsymbol{b} + \delta\boldsymbol{u}_1 + O(\delta^2). \tag{4.3}$$

For $\omega_0 = 1$, we have $\boldsymbol{b}^t = (b_1, \ldots, b_{k-1}, 0)$, and for $\omega_0 = 2$, $\boldsymbol{b}^t = (0, \ldots, 0, b_k)$.

Substituting (4.2) and (4.3) into $A\boldsymbol{u} = \omega\boldsymbol{u}$, the first-order correction terms satisfy

$$(A_0 - \omega_0 I)\boldsymbol{u}_1 = -(A_1 - \omega_1 I)\mathcal{U}\boldsymbol{b}. \tag{4.4}$$

Because, for every j,

$$\langle (A_0 - \omega_{0,j}I)\boldsymbol{u}_1, \boldsymbol{u}_{0,j}\rangle = \langle \boldsymbol{u}_1, (A_0 - \omega_{0,j}I)\boldsymbol{u}_{0,j}\rangle = 0, \tag{4.5}$$

where $\langle a, b\rangle$ is the inner product of a and b, it follows that $(A_1 - \omega_1 I)\mathcal{U}\boldsymbol{b}$ is orthogonal to each $\boldsymbol{u}_{0,j}$. For each ω_1 and $\boldsymbol{b}$, this gives k equations,

$$\boldsymbol{u}_{0,j}^t A_1 \mathcal{U}\boldsymbol{b} = \omega_1 \boldsymbol{u}_{0,j}^t \mathcal{U}\boldsymbol{b}, \quad j = 1, \ldots, k, \tag{4.6}$$

which could be written (using $\mathcal{U}^t\mathcal{U} = I$) as follows:

$$\mathcal{U}^t A_1 \mathcal{U} \boldsymbol{b} = \omega_1 \boldsymbol{b}. \tag{4.7}$$

Equation (4.7) is a standard eigenvalue problem with k eigenpairs. Write $\omega_{1,j}$ for the jth ω_1. For $\delta \to 0$, it follows from (4.2a) that the condition (2.44) will be met if

$$\omega_{1,j} < 0 \quad \text{for at least one } j \text{ such that } j = 1, \dots, k-1. \tag{4.8}$$

To expand A, we define

$$\mathcal{R} = \mathcal{R}_0 + \delta \mathcal{R}_1 + \cdots, \tag{4.9a}$$

$$\mathcal{C}^{-1}\mathcal{B}_g = \mathcal{C}_0 \mathcal{B}_{g0} + \delta(\mathcal{C}^{-1}\mathcal{B}_g)_1 + \cdots, \tag{4.9b}$$

$$(I+\mathcal{R})^{-1} = (I+\mathcal{R}_0)^{-1} - \delta(I+\mathcal{R}_0)^{-1}\mathcal{R}_1(I+\mathcal{R}_0)^{-1} + \cdots, \tag{4.9c}$$

and substitute into (4.1) to give

$$\begin{aligned} A &= A_0 + \delta[(I+\mathcal{R}_0)^{-1}(\mathcal{C}^{-1}\mathcal{B}_g)_1 \\ &\quad - (I+\mathcal{R}_0)^{-1}\mathcal{R}_1(I+\mathcal{R}_0)^{-1}\mathcal{C}_0^{-1}\mathcal{B}_{g0}] + \cdots \\ &= A_0 + \delta(I+\mathcal{R}_0)^{-1}[(\mathcal{C}^{-1}\mathcal{B}_g)_1 - \mathcal{R}_1 A_0] + \cdots. \end{aligned} \tag{4.10}$$

Hence,

$$A_1 = (I+\mathcal{R}_0)^{-1}[(\mathcal{C}^{-1}\mathcal{B}_g)_1 - \mathcal{R}_1 A_0]. \tag{4.11}$$

The eigenvalues of A_0 are given by (2.42). Define the $k\times k$ matrix,

$$\Omega_0 \equiv \operatorname*{diag}_{j=1,\dots,k} \omega_{0,j}. \tag{4.12}$$

Then the set of k equations $A_0 \boldsymbol{u}_{0,j} = \omega_{0,j}\boldsymbol{u}_{0,j}$ for $j=1,\dots,k$ can be written $A_0\mathcal{U} = \mathcal{U}\Omega_0$, and (4.11) can be written

$$\mathcal{U}^t A_1 \mathcal{U} = \mathcal{U}^t (I+\mathcal{R}_0)^{-1}[(\mathcal{C}^{-1}\mathcal{B}_g)_1\mathcal{U} - \mathcal{R}_1\mathcal{U}\Omega_0]. \tag{4.13}$$

Substitute this into (4.7), multiply on the left by $\mathcal{U}^t(I+\mathcal{R}_0)\mathcal{U}$, to get

$$\begin{aligned} \mathcal{U}^t[(\mathcal{C}^{-1}\mathcal{B}_g)_1\mathcal{U} - \mathcal{R}_1\mathcal{U}\Omega_0]\boldsymbol{b} &= \omega_1 \mathcal{U}^t(I+\mathcal{R}_0)\mathcal{U}\boldsymbol{b} \\ &= \omega_1(I+\mathcal{U}^t\mathcal{R}_0\mathcal{U})\boldsymbol{b}. \end{aligned} \tag{4.14}$$

As derived in Ref. 11 the eigenvectors of $\mathcal{R}_0$ are $\boldsymbol{u}_0$, so that $\mathcal{U}^t\mathcal{R}_0\mathcal{U} = \Sigma$, where, as shown in Ref. 11,

$$\Sigma_{jj} = \begin{cases} \dfrac{1-\cos(\pi j/k)}{2r} & \text{for } j = 1,\dots,k-1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.15}$$

Thus (4.14) becomes

$$[\mathcal{U}^t(\mathcal{C}^{-1}\mathcal{B}_g)_1\mathcal{U} - \mathcal{U}^t\mathcal{R}_1\mathcal{U}\Omega_0]\boldsymbol{b} = \omega_1(I+\Sigma)\boldsymbol{b}. \tag{4.16}$$

We can readily divide the eigenspace of (4.16) into two subspaces. For the first subspace $\mathcal{V}_1$, $\boldsymbol{b}$ is of the form $(b_1,\dots,b_{k-1},0)^t$. For the second subspace $\mathcal{V}_2$, $\boldsymbol{b}$ is of the form $(0,\dots,0,b_k)^t$.

From (4.8) and (4.16) we obtain:

Proposition 4.1. *For $(D_m - D) \to 0$, the asymmetric solution to (1.1) given by (2.10)* and *(2.12) will be unstable with respect to the small eigenvalues if at least one of the eigenvalues from $\mathcal{V}_1$ of*

$$(I+\Sigma)^{-1}[\mathcal{U}^t(\mathcal{C}^{-1}\mathcal{B}_g)_1\mathcal{U} - \mathcal{U}^t\mathcal{R}_1\mathcal{U}\Omega_0] \tag{4.17}$$

is negative.

The eigenvalues associated with $\mathcal{V}_2$ will have no effect on the stability of the profile. This is due to the fact that $\omega_{0,k}$ is 2 and thus λ_k in (2.39) is negative for $\delta \ll 1$. Thus for the remainder of this paper, we will only consider $\mathcal{V}_1$.

V. PROFILE INSTABILITY

In this section we prove the main result of the paper. We will explicitly evaluate the entries of the matrix given in (4.17). We do this for the case (2.7a). However, the conclusions about stability from this case will also hold for the other cases in (2.7), since it is only the definitions α and $\nu_1(\delta)$ in (3.21) which will change, and α is always positive. The details of the calculation will proceed as follows: In Sec. V A we expand the matrices in (4.17) and simplify using properties of the symmetric case. In Sec. V B we expand the entries of the matrices for the asymmetric case about $z=z_c$ in a Taylor series using (3.21) in (2.28b), (2.28c), and (2.36)–(2.38), with α and $\nu_1(\delta)$ given by (3.9). In Sec. V C we will show that the signs of the eigenvalues of (4.17) may be determined by the number of small spikes. The presence of k_1 small spikes leads to k_1 eigenvectors in $\mathcal{V}_1$ with negative eigenvalues. Thus we confirm that an asymmetric spike solution must always be unstable relative to the small eigenvalues near the bifurcation point.

A. Expansion of $\mathcal{U}^t\mathcal{R}_1\mathcal{U}$

Here we will find a key simplification, which will allow us to complete the characterization of the spectrum of (4.17). The chief difficulty is due to the presence of the matrix Q in (2.40b). The matrix Q will be expanded as follows:

$$Q = Q_0 + \delta Q_1 + O(\delta^2). \tag{5.1.1}$$

The columns $\boldsymbol{v}_1$ of Q_1 are the corrections to the eigenvectors of $\mathcal{C}^{-1}\mathcal{B}$. If we expand the eigenvalue problem $\mathcal{C}^{-1}\mathcal{B}\boldsymbol{v} = \lambda\boldsymbol{v}$, the leading order corrections to λ and $\boldsymbol{v}$, λ_1 and $\boldsymbol{v}_1$, must satisfy $(\mathcal{C}_0^{-1}\mathcal{B}_0 - \lambda_0 I)\boldsymbol{v}_1 = -[(\mathcal{C}^{-1}\mathcal{B})_1 - \lambda_1 I]\boldsymbol{v}_0$. Because $\mathcal{C}_0^{-1}\mathcal{B}_0 - \lambda_0 I$ is singular, it is difficult to solve explicitly for the $\boldsymbol{v}_1$'s. However, we show it is possible to replace expressions containing Q_1 with expressions that we may evaluate in general.

Substituting expansions of $\mathcal{PB} = \mathcal{PB}_0 + \delta\mathcal{PB}_1$, $\mathcal{D} = \mathcal{D}_0 + \delta\mathcal{D}_1$, $\mathcal{P}_g\mathcal{B}_g = (\mathcal{P}_g\mathcal{B}_g)_0 + \delta(\mathcal{P}_g\mathcal{B}_g)_1$, and (5.1.1) into (2.40b), we find

$$\begin{aligned} \mathcal{U}^t\mathcal{R}_1\mathcal{U} = &- \mathcal{U}^t(\mathcal{PB})_1 Q_0\mathcal{D}_0 Q_0^{-1}(\mathcal{P}_g\mathcal{B}_g)_0\mathcal{U} \\ &- \mathcal{U}^t(\mathcal{PB})_0 Q_1\mathcal{D}_0 Q_0^{-1}(\mathcal{P}_g\mathcal{B}_g)_0\mathcal{U} \\ &- \mathcal{U}^t(\mathcal{PB})_0 Q_0\mathcal{D}_1 Q_0^{-1}(\mathcal{P}_g\mathcal{B}_g)_0\mathcal{U} \\ &+ \mathcal{U}^t(\mathcal{PB})_0 Q_0\mathcal{D}_0 Q_0^{-1}Q_1Q_0^{-1}(\mathcal{P}_g\mathcal{B}_g)_0\mathcal{U} \\ &- \mathcal{U}^t(\mathcal{PB})_0 Q_0\mathcal{D}_0 Q_0^{-1}(\mathcal{P}_g\mathcal{B}_g)_1\mathcal{U}. \end{aligned} \tag{5.1.2}$$

In Ref. 11 the superdiagonal matrix $\mathcal{M}$ is defined as

$$\mathcal{M}_{j,j+1} = 2\sin\left(\frac{\pi j}{k}\right), \quad j = 1,\dots,k-1 \tag{5.1.3}$$

and is shown to relate to $\mathcal{P}_g\mathcal{B}_g$ and $\mathcal{PB}$ as follows:

$$(\mathcal{P}_g\mathcal{B}_g)_0\mathcal{U}=\frac{\operatorname{csch}(2z_c)}{2D}Q_0\mathcal{M}^t$$

and

$$\mathcal{U}^t(\mathcal{P}\mathcal{B})_0=-\frac{\operatorname{csch}(2z_c)}{2D}\mathcal{M}Q_0^t. \tag{5.1.4}$$

Substituting (5.1.4) into (5.1.2) (and using $Q_0^t=Q_0^{-1}$),

$$\begin{aligned}\mathcal{U}^t\mathcal{R}_1\mathcal{U}=&-\frac{\operatorname{csch}(2z_c)}{2D}[\mathcal{U}^t(\mathcal{P}\mathcal{B})_1Q_0\mathcal{D}_0\mathcal{M}^t\\&-\mathcal{M}\mathcal{D}_0Q_0^{-1}(\mathcal{P}_g\mathcal{B}_g)_1\mathcal{U}]+\frac{\operatorname{csch}^2(2z_c)}{4D^2}[\mathcal{M}(Q_0^tQ_1\mathcal{D}_0\\&-\mathcal{D}_0Q_0^tQ_1)\mathcal{M}^t+\mathcal{M}\mathcal{D}_1\mathcal{M}^t].\end{aligned} \tag{5.1.5}$$

Consider first the $\mathcal{M}\mathcal{D}_1\mathcal{M}^t$ term in (5.1.5). To determine $\mathcal{D}_1$, we expand the diagonal matrix $\mathcal{D}$ defined in (2.34). Using (2.4) and (2.33c) we can write $\tilde{s}$ in terms of r,

$$\tilde{s}=-\frac{1+r}{r}. \tag{5.1.6}$$

Substitute (5.1.6) into (2.34) to give

$$\mathcal{D}=-\left(\frac{1+r}{r}\right)D^2\left(\mathcal{K}-\left(\frac{1+r}{r}\right)I\right)^{-1}. \tag{5.1.7}$$

Since we are able to ignore the common D^2 factor when computing $\mathcal{R}$ [the matrix $\mathcal{R}$ is defined in (2.40b), and $\mathcal{D}$, defined in (2.34), is of the form D^2M_1, where M_1 is a $k\times k$ matrix, which does not depend on D. Also, (2.35)–(2.37) give $\mathcal{P}\mathcal{B}$ and $\mathcal{P}_g\mathcal{B}_g$ as $(1/D)M_2$ and $(1/D)M_3$ for $k\times k$ matrices M_2 and M_3, whose entries are hyperbolic trigonometric functions of the z_i's. Thus $\mathcal{R}=-M_2QM_1Q^{-1}M_3$, which does not depend on D] we expand $\mathcal{K}=\mathcal{K}_0+\delta\mathcal{K}_1+O(\delta^2)$ to yield

$$\begin{aligned}\mathcal{D}=&-\left(\frac{1+r}{r}\right)D^2\left(\mathcal{K}_0-\left(\frac{1+r}{r}\right)I\right)^{-1}+\delta\left(\frac{1+r}{r}\right)\\&\times D^2\left(\mathcal{K}_0-\left(\frac{1+r}{r}\right)I\right)^{-1}\mathcal{K}_1\left(\mathcal{K}_0-\left(\frac{1+r}{r}\right)I\right)^{-1}+O(\delta^2).\end{aligned} \tag{5.1.8}$$

The diagonal matrix $\mathcal{K}_0$ is the matrix of eigenvalues of $(\mathcal{C}^{-1}\mathcal{B})_0=\mathcal{C}_0^{-1}\mathcal{B}_0$, where $\mathcal{C}_0^{-1}=\frac{1}{2}\coth z_cI$, and $\mathcal{K}_1$ is the diagonal matrix of the lowest order corrections to the eigenvalues of $\mathcal{C}^{-1}\mathcal{B}$.

Proposition 2 in Ref. 11 gives the jth eigenvalue of $\mathcal{B}_0$,

$$\begin{aligned}\kappa_j&=2\coth 2z_c-2\operatorname{csch}2z_c\cos(\pi(j-1)/k)\\&=\frac{1+2r-\cos(\pi(j-1)/k)}{\sqrt{r(1+r)}},\quad j=1,\dots,k,\end{aligned} \tag{5.1.9}$$

where the second equality uses (2.5). Thus

$$\begin{aligned}\mathcal{D}_0&=-\left(\frac{1+r}{r}\right)D^2\left(\mathcal{K}_0-\left(\frac{1+r}{r}\right)I\right)^{-1}\\&=\operatorname*{diag}_{j=1,\dots,k}\left(\frac{2D^2(1+r)}{1+\cos[\pi(j-1)/k]}\right).\end{aligned} \tag{5.1.10}$$

Since $\mathcal{K}_1$ is also diagonal,

$$\begin{aligned}&\left[\left(\mathcal{K}_0-\left(\frac{1+r}{r}\right)I\right)^{-1}\mathcal{K}_1\left(\mathcal{K}_0-\left(\frac{1+r}{r}\right)I\right)^{-1}\right]_{jj}\\&=\frac{4r^2}{(1+\cos[\pi(j-1)/k])^2}(\mathcal{K}_1)_{jj},\end{aligned} \tag{5.1.11}$$

and the leading order correction to $\mathcal{D}$ is

$$\mathcal{D}_1=\operatorname*{diag}_{j=1,\dots,k}\left(\frac{4r(1+r)D^2}{(1+\cos[\pi(j-1)/k])^2}(\mathcal{K}_1)_{jj}\right). \tag{5.1.12}$$

In the $\mathcal{M}\mathcal{D}_1\mathcal{M}^t$ term, $\mathcal{D}_1$ is diagonal and $\mathcal{M}_{nm}=0$ unless $m=n+1$, so that

$$\begin{aligned}(\mathcal{M}\mathcal{D}_1\mathcal{M}^t)_{ij}&=\sum_{p,q}\mathcal{M}_{ip}(\mathcal{D}_1)_{pq}\mathcal{M}^t_{qj}\\&=\begin{cases}\mathcal{M}_{i,i+1}(\mathcal{D}_1)_{i+1,i+1}\mathcal{M}^t_{i+1,i} & \text{for } i=j,\\ \quad i,j=1,\dots,k-1,\\ 0\quad\text{otherwise}.\end{cases}\end{aligned} \tag{5.1.13}$$

Thus $\mathcal{M}\mathcal{D}_1\mathcal{M}^t$ is diagonal and

$$\begin{aligned}(\mathcal{M}\mathcal{D}_1\mathcal{M}^t)_{jj}&=(\mathcal{M}_{j,j+1})^2(\mathcal{D}_1)_{j+1,j+1}\\&=4\sin^2\left(\frac{\pi j}{k}\right)(\mathcal{D}_1)_{j+1,j+1},\\&\qquad j=1,\dots,k-1.\end{aligned} \tag{5.1.14}$$

Substituting (5.1.12) into (5.1.14) gives

$$(\mathcal{M}\mathcal{D}_1\mathcal{M}^t)_{ij}=\begin{cases}\dfrac{16r(1+r)D^2\sin^2(\pi j/k)}{(1+\cos(\pi j/k))^2}(\mathcal{K}_1)_{j+1,j+1}\\ \quad\text{for } i=j,\quad i,j=1,\dots,k-1,\\ 0\quad\text{otherwise}.\end{cases} \tag{5.1.15}$$

Now we consider the $\mathcal{M}(Q_0^tQ_1\mathcal{D}_0-\mathcal{D}_0Q_0^tQ_1)\mathcal{M}^t$ term in (5.1.5). Substituting $\mathcal{K}=\mathcal{K}_0+\delta\mathcal{K}_1+\cdots$ and $Q=Q_0+\delta Q_1+\cdots$ into $\mathcal{C}^{-1}\mathcal{B}Q=Q\mathcal{K}$ gives

$$\mathcal{C}_0^{-1}\mathcal{B}_0Q_1-Q_1\mathcal{K}_0=Q_0\mathcal{K}_1-(\mathcal{C}^{-1}\mathcal{B})_1Q_0, \tag{5.1.16}$$

so that

$$Q_0^t\mathcal{C}_0^{-1}\mathcal{B}_0Q_1-Q_0^tQ_1\mathcal{K}_0=\mathcal{K}_1-Q_0^t(\mathcal{C}^{-1}\mathcal{B})_1Q_0. \tag{5.1.17}$$

Since $\mathcal{K}_0$ is the (diagonal) matrix of eigenvalues of $\mathcal{C}_0^{-1}\mathcal{B}_0$,

$$\mathcal{C}_0^{-1}\mathcal{B}_0=Q_0\mathcal{K}_0Q_0^t\quad\text{or}\quad Q_0^t\mathcal{C}_0^{-1}\mathcal{B}_0=\mathcal{K}_0Q_0^t. \tag{5.1.18}$$

Substituting (5.1.18) into (5.1.17) gives

$$\mathcal{K}_0Q_0^tQ_1-Q_0^tQ_1\mathcal{K}_0=\mathcal{K}_1-Q_0^t(\mathcal{C}^{-1}\mathcal{B})_1Q_0. \tag{5.1.19}$$

Solve (5.1.10) for $\mathcal{K}_0$,

$$\mathcal{K}_0=\frac{1+r}{r}(I-D^2\mathcal{D}_0^{-1}), \tag{5.1.20}$$

and substitute into the left-hand side of (5.1.19),

$$\mathcal{K}_0 Q_0^t Q_1 - Q_0^t Q_1 \mathcal{K}_0 = -\left(\frac{1+r}{r}\right) D^2 (\mathcal{D}_0^{-1} Q_0^t Q_1 - Q_0^t Q_1 \mathcal{D}_0^{-1}). \tag{5.1.21}$$

Now multiply (5.1.21) on both the left and the right by $\mathcal{D}_0$,

$$\mathcal{D}_0 (\mathcal{K}_0 Q_0^t Q_1 - Q_0^t Q_1 \mathcal{K}_0) \mathcal{D}_0 = -\left(\frac{1+r}{r}\right) D^2 (Q_0^t Q_1 \mathcal{D}_0 - \mathcal{D}_0 Q_0^t Q_1). \tag{5.1.22}$$

Substitute (5.1.19) into (5.1.22) and write the result as

$$Q_0^t Q_1 \mathcal{D}_0 - \mathcal{D}_0 Q_0^t Q_1 = \left(\frac{r}{(1+r)D^2}\right) \mathcal{D}_0 (Q_0^t (\mathcal{C}^{-1}\mathcal{B})_1 Q_0 - \mathcal{K}_1) \mathcal{D}_0, \tag{5.1.23}$$

which replaces the term with Q_1 by matrices which can be evaluated. Multiplying (5.1.23) on the left by $\mathcal{M}$ and on the right by $\mathcal{M}^t$ gives

$$\begin{aligned}(\mathcal{M}(Q_0^t Q_1 \mathcal{D}_0 - \mathcal{D}_0 Q_0^t Q_1)\mathcal{M}^t)_{ij} &= \left(\frac{r}{(1+r)D^2}\right) \mathcal{M}_{i,i+1} (\mathcal{D}_0)_{i+1,i+1} (Q_0^t (\mathcal{C}^{-1}\mathcal{B})_1 Q_0 \\ &\quad - \mathcal{K}_1)_{i+1,j+1} (\mathcal{D}_0)_{j+1,j+1} \mathcal{M}_{j,j+1}.\end{aligned} \tag{5.1.24}$$

From a solvability condition, which arises from expanding $\mathcal{C}^{-1}\mathcal{B}\boldsymbol{q} = \lambda \boldsymbol{q}$,

$$(\mathcal{K}_1)_{ii} = (Q_0^t (\mathcal{C}^{-1}\mathcal{B})_1 Q_0)_{ii} \quad \text{and} \quad (\mathcal{K}_1)_{ij} = 0 \quad \text{for } i \neq j. \tag{5.1.25}$$

Substituting the expressions (5.1.3), (5.1.10), and (5.1.25) into (5.1.24) we get

$$(\mathcal{M}(Q_0^t Q_1 \mathcal{D}_0 - \mathcal{D}_0 Q_0^t Q_1)\mathcal{M}^t)_{ij} = \begin{cases} 0 \quad \text{for } i = j, \\ \dfrac{16r(1+r)D^2 \sin(\pi i/k)\sin(\pi j/k)}{[1+\cos(\pi i/k)][1+\cos(\pi j/k)]} (Q_0^t (\mathcal{C}^{-1}\mathcal{B})_1 Q_0)_{i+1,j+1} \\ \quad \text{otherwise.} \end{cases} \tag{5.1.26}$$

From (2.5) it follows that $\operatorname{csch}^2(2z_c)/(4D^2) = (16r(1+r)D^2)^{-1}$. Substitute this, Eqs. (5.1.15), (5.1.25), and (5.1.26), into the second line of (5.1.5) to get

$$\begin{aligned}&\frac{\operatorname{csch}^2(2z_c)}{4D^2} [\mathcal{M}(Q_0^t Q_1 \mathcal{D}_0 - \mathcal{D}_0 Q_0^t Q_1)\mathcal{M}^t + \mathcal{M}\mathcal{D}_1\mathcal{M}^t]_{ij} \\ &= \begin{cases} \dfrac{\sin(\pi i/k)\sin(\pi j/k)}{[1+\cos(\pi i/k)][1+\cos(\pi j/k)]} (Q_0^t (\mathcal{C}^{-1}\mathcal{B})_1 Q_0)_{i+1,j+1}, \\ \quad i,j = 1, \ldots, k-1, \\ 0, \quad i = k \text{ or } j = k. \end{cases}\end{aligned} \tag{5.1.27}$$

For the $\mathcal{U}^t(\mathcal{P}\mathcal{B})_1 Q_0 \mathcal{D}_0 \mathcal{M}^t - \mathcal{M}\mathcal{D}_0 Q_0^{-1} (\mathcal{P}_g \mathcal{B}_g)_1 \mathcal{U}$ term in (5.1.5), because $\mathcal{D}_0$ is diagonal and $\mathcal{M}$ is superdiagonal,

$$\begin{aligned}(\mathcal{U}^t(\mathcal{P}\mathcal{B})_1 Q_0 \mathcal{D}_0 \mathcal{M}^t)_{ij} &= \sum_{p,q} \mathcal{U}_{ip}^t [(\mathcal{P}\mathcal{B})_1]_{pq} (Q_0)_{q,j+1} (\mathcal{D}_0)_{j+1,j+1} \mathcal{M}_{j,j+1} \\ &= \begin{cases} (\mathcal{D}_0)_{j+1,j+1} \mathcal{M}_{j,j+1} \boldsymbol{u}_{0,i}^t (\mathcal{P}\mathcal{B})_1 \boldsymbol{v}_{0,j+1}, \\ \quad j = 1, \ldots, k-1, \; i = 1, \ldots, k, \\ 0 \quad j = k \end{cases}\end{aligned} \tag{5.1.28}$$

and

$$\begin{aligned}(\mathcal{M}\mathcal{D}_0 Q_0^{-1} (\mathcal{P}_g \mathcal{B}_g)_1 \mathcal{U})_{ij} &= \sum_{r,s} \mathcal{M}_{i,i+1} (\mathcal{D}_0)_{i+1,i+1} (Q_0^{-1})_{i+1,r} [(\mathcal{P}_g \mathcal{B}_g)_1]_{rs} \mathcal{U}_{sj} \\ &= \begin{cases} \mathcal{M}_{i,i+1} (\mathcal{D}_0)_{i+1,i+1} \boldsymbol{v}_{0,i+1}^t (\mathcal{P}_g \mathcal{B}_g)_1 \boldsymbol{u}_{0,j}, \\ \quad i = 1, \ldots, k-1, \; j = 1, \ldots, k, \\ 0, \quad i = k, \end{cases}\end{aligned} \tag{5.1.29}$$

where the ℓth column of $\mathcal{U}$ is $\boldsymbol{u}_{0,\ell}$ and the ℓth column of Q_0 is $\boldsymbol{v}_{0,\ell}$.

Substituting (5.1.3) and (5.1.10) for $\mathcal{M}$ and $\mathcal{D}_0$ gives

$$\begin{aligned}&-\frac{\operatorname{csch}(2z_c)}{2D} [\mathcal{U}^t(\mathcal{P}\mathcal{B})_1 Q_0 \mathcal{D}_0 \mathcal{M}^t - \mathcal{M}\mathcal{D}_0 Q_0^{-1} (\mathcal{P}_g \mathcal{B}_g)_1 \mathcal{U}]_{ij} \\ &= \begin{cases} D\sqrt{\dfrac{1+r}{r}} \left[\dfrac{\sin(\pi i/k)}{1+\cos(\pi i/k)} \boldsymbol{v}_{0,i+1}^t (\mathcal{P}_g \mathcal{B}_g)_1 \boldsymbol{u}_{0,j} - \dfrac{\sin(\pi j/k)}{1+\cos(\pi j/k)} \boldsymbol{u}_{0,i}^t (\mathcal{P}\mathcal{B})_1 \boldsymbol{v}_{0,j+1} \right] \quad i,j = 1, \ldots, k-1, \\ D\sqrt{\dfrac{1+r}{r}} \dfrac{\sin(\pi i/k)}{1+\cos(\pi i/k)} \boldsymbol{v}_{0,i+1}^t (\mathcal{P}_g \mathcal{B}_g)_1 \boldsymbol{u}_{0,j} \quad j = k, \; i = 1, \ldots, k-1, \\ -D\sqrt{\dfrac{1+r}{r}} \dfrac{\sin(\pi j/k)}{1+\cos(\pi j/k)} \boldsymbol{u}_{0,i}^t (\mathcal{P}\mathcal{B})_1 \boldsymbol{v}_{0,j+1} \quad i = k, \; j = 1, \ldots, k-1, \\ 0 \quad i = j = k. \end{cases}\end{aligned} \tag{5.1.30}$$

Thus, the matrix $\mathcal{U}^t\mathcal{R}_1\mathcal{U}$ may be written as the sum of the matrices computed in (5.1.30) and (5.1.27).

B. Evaluation of $Q_0^t(\mathcal{C}^{-1}\mathcal{B})_1Q_0$, $\mathcal{U}^t(\mathcal{C}^{-1}\mathcal{B}_g)_1\mathcal{U}$, $\mathcal{U}^t(\mathcal{P}\mathcal{B})_1Q_0$, and $Q_0^t(\mathcal{P}_g\mathcal{B}_g)_1\mathcal{U}$

In this section, we will find explicit expressions for the matrix $\mathcal{U}^t(\mathcal{C}^{-1}\mathcal{B}_g)_1\mathcal{U}$ and the three matrices $Q_0^t(\mathcal{C}^{-1}\mathcal{B})_1Q_0$, $\mathcal{U}^t(\mathcal{P}\mathcal{B})_1Q_0$, and $Q_0^t(\mathcal{P}_g\mathcal{B}_g)_1\mathcal{U}$, which are used to evaluate $\mathcal{U}^t\mathcal{R}_1\mathcal{U}$. These matrices will be calculated by substituting (3.21) in (2.19), (2.28b), (2.28c), and (2.36)–(2.38) and expanding about $\delta=0$. The arrangement of spikes enters the calculation by way of (3.20).

The expansions of $\mathcal{C}^{-1}\mathcal{B}$ and $\mathcal{C}^{-1}\mathcal{B}_g$ are given by

$$\mathcal{C}^{-1}\mathcal{B}=(\mathcal{C}^{-1}\mathcal{B})_0+\delta[\mathcal{C}_0^{-1}\mathcal{B}_1+\mathcal{C}_1^{-1}\mathcal{B}_0]+O(\delta^2), \tag{5.2.1}$$

$$\mathcal{C}^{-1}\mathcal{B}_g=(\mathcal{C}^{-1}\mathcal{B}_g)_0+\delta[\mathcal{C}_0^{-1}(\mathcal{B}_g)_1+\mathcal{C}_1^{-1}(\mathcal{B}_g)_0]+O(\delta^2), \tag{5.2.2}$$

where

$$(\mathcal{C}^{-1}\mathcal{B})_1=\mathcal{C}_0^{-1}\mathcal{B}_1+\mathcal{C}_1^{-1}\mathcal{B}_0, \tag{5.2.3}$$

$$(\mathcal{C}^{-1}\mathcal{B}_g)_1=\mathcal{C}_0^{-1}(\mathcal{B}_g)_1+\mathcal{C}_1^{-1}(\mathcal{B}_g)_0. \tag{5.2.4}$$

From (2.19), (3.20), and (3.21), expanding about $\delta=0$ gives

$$\mathcal{C}_{jj}^{-1}=(\mathcal{C}_0+\delta\mathcal{C}_1+\cdots)_{jj}^{-1}=\frac{1}{2}\coth z_c-\frac{1}{2}\delta\alpha \mathsf{s}_j\operatorname{csch}^2 z_c+\cdots. \tag{5.2.5}$$

Define the diagonal matrix $\mathcal{S}$ by $\mathcal{S}_{jj}\equiv\mathsf{s}_j$ and use (2.5) to get

$$\mathcal{C}_0^{-1}=\frac{1}{2}\sqrt{\frac{1+r}{r}}I,\quad \mathcal{C}_1^{-1}=-\frac{\alpha}{2r}\mathcal{S} \tag{5.2.6}$$

and

$$(\mathcal{C}^{-1}\mathcal{B})_1=\frac{1}{2}\sqrt{\frac{1+r}{r}}\mathcal{B}_1-\frac{\alpha}{2r}\mathcal{S}\mathcal{B}_0, \tag{5.2.7}$$

$$(\mathcal{C}^{-1}\mathcal{B}_g)_1=\frac{1}{2}\sqrt{\frac{1+r}{r}}(\mathcal{B}_g)_1-\frac{\alpha}{2r}\mathcal{S}(\mathcal{B}_g)_0. \tag{5.2.8}$$

Writing $\boldsymbol{v}_{0,p}$, κ_p and $\boldsymbol{u}_{0,p}$, ξ_p, respectively, for the p^{th} eigenpair of $\mathcal{B}_0$ and $(\mathcal{B}_g)_0$,

$$(Q_0^t(\mathcal{C}^{-1}\mathcal{B})_1Q_0)_{i+1,j+1}=\frac{1}{2}\sqrt{\frac{1+r}{r}}\boldsymbol{v}_{0,i+1}^t\mathcal{B}_1\boldsymbol{v}_{0,j+1}-\frac{\alpha\kappa_{j+1}}{2r}\boldsymbol{v}_{0,i+1}^t\mathcal{S}\boldsymbol{v}_{0,j+1} \tag{5.2.9}$$

and

$$(\mathcal{U}^t(\mathcal{C}^{-1}\mathcal{B}_g)_1\mathcal{U})_{ij}=\frac{1}{2}\sqrt{\frac{1+r}{r}}\boldsymbol{u}_{0,i}^t(\mathcal{B}_g)_1\boldsymbol{u}_{0,j}-\frac{\alpha\xi_j}{2r}\boldsymbol{u}_{0,i}^t\mathcal{S}\boldsymbol{u}_{0,j}. \tag{5.2.10}$$

The entries for the matrices $\mathcal{B}$ and $\mathcal{B}_g$ are given in (2.28b), (2.28c), and (2.38). The expansion of these entries about $\delta=0$ is given in the Appendix. Denoting the leading order correction by the subscript $_{,1}$, the entries for the matrices $(\mathcal{B}_g)_1$ and $\mathcal{B}_1$ are

$$c_{\ell,1}=\alpha^*\sqrt{\frac{r}{1+r}}(-\mathsf{s}_{\ell-1}-2\mathsf{s}_\ell-\mathsf{s}_{\ell+1}),\quad \ell=2,\ldots,k-1,$$

$$c_{1,1}=\alpha^*\sqrt{\frac{r}{1+r}}(-\mathsf{s}_1(1-4r)-\mathsf{s}_2),$$

$$c_{k,1}=\alpha^*\sqrt{\frac{r}{1+r}}(-\mathsf{s}_k(1-4r)-\mathsf{s}_{k-1}),$$

$$\tilde{c}_{1,1}=\alpha^*\sqrt{\frac{r}{1+r}}(-\mathsf{s}_1(5+4r)-\mathsf{s}_2),$$

$$\tilde{c}_{k,1}=\alpha^*\sqrt{\frac{r}{1+r}}(-\mathsf{s}_k(5+4r)-\mathsf{s}_{k-1}),$$

$$d_{\ell,1}=\alpha^*\sqrt{\frac{r}{1+r}}(\mathsf{s}_\ell+\mathsf{s}_{\ell+1})(1+2r),\quad \ell=1,\ldots,k-1, \tag{5.2.11}$$

where

$$\alpha^*\equiv\frac{\alpha}{4r\sqrt{r(1+r)}} \tag{5.2.12}$$

and the form of the matrices $\mathcal{B}_1$ or $(\mathcal{B}_g)_1$ is given in (2.28a).

Following the same steps, we get the leading order corrections to $\mathcal{P}\mathcal{B}$ and $\mathcal{P}_g\mathcal{B}_g$,

$$c_{\ell,1}=\alpha^*\sqrt{\frac{r}{1+r}}(\mathsf{s}_{\ell+1}-\mathsf{s}_{\ell-1}),\quad \ell=2,\ldots,k-1, \tag{5.2.13a}$$

$$c_{1,1}=\alpha^*\sqrt{\frac{r}{1+r}}(\mathsf{s}_1(1+4r)+\mathsf{s}_2), \tag{5.2.13b}$$

$$c_{k,1}=\alpha^*\sqrt{\frac{r}{1+r}}(-\mathsf{s}_k(1+4r)-\mathsf{s}_{k-1}), \tag{5.2.13c}$$

$$\tilde{c}_{1,1}=\alpha^*\sqrt{\frac{r}{1+r}}(-\mathsf{s}_1(3+4r)+\mathsf{s}_2), \tag{5.2.13d}$$

$$\tilde{c}_{k,1}=\alpha^*\sqrt{\frac{r}{1+r}}(\mathsf{s}_k(3+4r)-\mathsf{s}_{k-1}), \tag{5.2.13e}$$

$$d_{\ell,1}=\alpha^*\sqrt{\frac{r}{1+r}}(-\mathsf{s}_\ell-\mathsf{s}_{\ell+1})(1+2r),\quad \ell=1,\ldots,k-1, \tag{5.2.13f}$$

where the form of the matrices $(\mathcal{P}\mathcal{B})_1$ or $(\mathcal{P}_g\mathcal{B}_g)_1$ is given in (2.35).

We will compute the i,j components of $Q_0^t\mathcal{B}_1Q_0$, $\mathcal{U}^t(\mathcal{B}_g)_1\mathcal{U}$, $\mathcal{U}^t(\mathcal{P}\mathcal{B})_1Q_0$, and $Q_0^t(\mathcal{P}_g\mathcal{B}_g)_1\mathcal{U}$ in terms of $\boldsymbol{v}_0$ and $\boldsymbol{u}_0$, the columns of Q_0 and $\mathcal{U}$, respectively. The calculations for these four are similar and are facilitated by defining the superdiagonal matrix,

$$\mathcal{T} \equiv \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{5.2.14}$$

From (5.2.11), (5.2.13), and (5.2.14), it is straightforward to check that

$$\sqrt{\frac{1+r}{r}}\frac{1}{\alpha^*}\mathcal{B}_1 = -2\mathcal{S} - \mathcal{T}\mathcal{S}\mathcal{T}^t - \mathcal{T}^t\mathcal{S}\mathcal{T} + (1+4r)(2\mathcal{S} - \mathcal{S}\mathcal{T}\mathcal{T}^t - \mathcal{S}\mathcal{T}^t\mathcal{T}) + (1+2r)(\mathcal{S}\mathcal{T} + \mathcal{T}\mathcal{S} + \mathcal{T}^t\mathcal{S} + \mathcal{S}\mathcal{T}^t), \tag{5.2.15a}$$

$$\sqrt{\frac{1+r}{r}}\frac{1}{\alpha^*}(\mathcal{B}_g)_1 = -2\mathcal{S} - \mathcal{T}\mathcal{S}\mathcal{T}^t - \mathcal{T}^t\mathcal{S}\mathcal{T} - (3+4r)(2\mathcal{S} - \mathcal{S}\mathcal{T}\mathcal{T}^t - \mathcal{S}\mathcal{T}^t\mathcal{T}) + (1+2r)(\mathcal{S}\mathcal{T} + \mathcal{T}\mathcal{S} + \mathcal{T}^t\mathcal{S} + \mathcal{S}\mathcal{T}^t), \tag{5.2.15b}$$

$$\sqrt{\frac{1+r}{r}}\frac{2D}{\alpha^*}(\mathcal{P}\mathcal{B})_1 = \mathcal{T}\mathcal{S}\mathcal{T}^t - \mathcal{T}^t\mathcal{S}\mathcal{T} + (1+4r)(\mathcal{S}\mathcal{T}\mathcal{T}^t - \mathcal{S}\mathcal{T}^t\mathcal{T}) + (1+2r)(-\mathcal{S}\mathcal{T} - \mathcal{T}\mathcal{S} + \mathcal{T}^t\mathcal{S} + \mathcal{S}\mathcal{T}^t), \tag{5.2.15c}$$

$$\sqrt{\frac{1+r}{r}}\frac{2D}{\alpha^*}(\mathcal{P}_g\mathcal{B}_g)_1 = \mathcal{T}\mathcal{S}\mathcal{T}^t - \mathcal{T}^t\mathcal{S}\mathcal{T} - (3+4r)(\mathcal{S}\mathcal{T}\mathcal{T}^t - \mathcal{S}\mathcal{T}^t\mathcal{T}) + (1+2r)(-\mathcal{S}\mathcal{T} - \mathcal{T}\mathcal{S} + \mathcal{T}^t\mathcal{S} + \mathcal{S}\mathcal{T}^t). \tag{5.2.15d}$$

In computing $\boldsymbol{v}_{0,i+1}^t\mathcal{B}_1\boldsymbol{v}_{0,j+1}$, $\boldsymbol{u}_{0,i}^t(\mathcal{B})_1\boldsymbol{u}_{0,j}$, $\boldsymbol{u}_{0,i}^t(\mathcal{P}\mathcal{B})_1\boldsymbol{v}_{0,j+1}$, and $\boldsymbol{u}_{0,i}^t(\mathcal{P}_g\mathcal{B}_g)_1\boldsymbol{u}_{0,j}$, it follows from (5.2.15) that we will evaluate terms such as $\mathcal{T}\boldsymbol{v}_{0,j+1}$, $\mathcal{T}^t\boldsymbol{v}_{0,j+1}$, $\mathcal{T}\boldsymbol{u}_{0,j}$, and $\mathcal{T}^t\boldsymbol{u}_{0,j}$. Propositions 2 and 9 of Ref. 11 give the components of $\boldsymbol{v}_0$ and $\boldsymbol{u}_0$,

$$u_{p,\ell} \equiv (\boldsymbol{u}_{0,p})_\ell = \sqrt{\frac{2}{k}}\sin\left(\frac{\pi p}{k}\left(\ell - \frac{1}{2}\right)\right), \quad p = 1, \ldots, k-1, \tag{5.2.16a}$$

$$u_{k,\ell} \equiv (\boldsymbol{u}_{0,k})_\ell = \frac{1}{\sqrt{k}}(-1)^{\ell+1}, \tag{5.2.16b}$$

$$v_{p+1,\ell} \equiv (\boldsymbol{v}_{0,p+1})_\ell = \sqrt{\frac{2}{k}}\cos\left(\frac{\pi p}{k}\left(\ell - \frac{1}{2}\right)\right), \quad p = 1, \ldots, k-1, \tag{5.2.16c}$$

$$v_{1,\ell} \equiv (\boldsymbol{v}_{0,1})_\ell = \frac{1}{\sqrt{k}}. \tag{5.2.16d}$$

Using standard trigonometric identities and (5.2.16) it follows that (for $j=1,\ldots,k-1$)

$$v_{j+1,\ell+1} = \cos\frac{\pi j}{k}v_{j+1,\ell} - \sin\frac{\pi j}{k}u_{j,\ell}, \quad \ell = 1, \ldots, k-1, \tag{5.2.17a}$$

$$v_{j+1,\ell-1} = \cos\frac{\pi j}{k}v_{j+1,\ell} + \sin\frac{\pi j}{k}u_{j,\ell}, \quad \ell = 2, \ldots, k, \tag{5.2.17b}$$

$$u_{j,\ell+1} = \cos\frac{\pi j}{k}u_{j,\ell} + \sin\frac{\pi j}{k}v_{j+1,\ell}, \quad \ell = 1, \ldots, k-1, \tag{5.2.17c}$$

$$u_{j,\ell-1} = \cos\frac{\pi j}{k}u_{j,\ell} - \sin\frac{\pi j}{k}v_{j+1,\ell}, \quad \ell = 2, \ldots, k. \tag{5.2.17d}$$

At $\ell = k$

$$\cos\frac{\pi j}{k}v_{j+1,\ell} - \sin\frac{\pi j}{k}u_{j,\ell} = v_{j+1,k}$$

and

$$\cos\frac{\pi j}{k}u_{j,\ell} + \sin\frac{\pi j}{k}v_{j+1,\ell} = -u_{j,k}, \tag{5.2.18a}$$

and at $\ell = 1$

$$\cos\frac{\pi j}{k}v_{j+1,\ell} + \sin\frac{\pi j}{k}u_{j,\ell} = v_{j+1,1}$$

and

$$\cos\frac{\pi j}{k}u_{j,\ell} - \sin\frac{\pi j}{k}v_{j+1,\ell} = -u_{j,1}. \tag{5.2.18b}$$

Use (5.2.17) and (5.2.18) to write

$$\mathcal{T}\boldsymbol{v}_{0,j+1} = \cos\frac{\pi j}{k}\boldsymbol{v}_{0,j+1} - \sin\frac{\pi j}{k}\boldsymbol{u}_{0,j} - v_{j+1,k}(I - \mathcal{T}\mathcal{T}^t), \tag{5.2.19a}$$

$$\mathcal{T}^t\boldsymbol{v}_{0,j+1} = \cos\frac{\pi j}{k}\boldsymbol{v}_{0,j+1} + \sin\frac{\pi j}{k}\boldsymbol{u}_{0,j} - v_{j+1,1}(I - \mathcal{T}^t\mathcal{T}), \tag{5.2.19b}$$

$$\mathcal{T}\boldsymbol{u}_{0,j} = \cos\frac{\pi j}{k}\boldsymbol{u}_{0,j} + \sin\frac{\pi j}{k}\boldsymbol{v}_{0,j+1} + u_{j,k}(I - \mathcal{T}\mathcal{T}^t), \tag{5.2.19c}$$

$$\mathcal{T}^t\boldsymbol{u}_{0,j}=\cos\frac{\pi j}{k}\boldsymbol{u}_{0,j}-\sin\frac{\pi j}{k}\boldsymbol{v}_{0,j+1}+u_{j,1}(I-\mathcal{T}^t\mathcal{T}), \tag{5.2.19d}$$

where $i,j=1,\ldots,k-1$. Then, since $(I-\mathcal{T}^t\mathcal{T})\mathcal{S}\boldsymbol{v}_{0,j+1}=(\mathsf{s}_1 v_{j+1,1},0,\ldots,0)^t$, and using (5.2.17)

$$\boldsymbol{v}_{0,i+1}^{\mathrm{t}}\mathcal{T}\mathcal{S}\mathcal{T}^t\boldsymbol{v}_{0,j+1}=\left(\cos\frac{\pi i}{k}\boldsymbol{v}_{0,i+1}+\sin\frac{\pi i}{k}\boldsymbol{u}_{0,i}\right)\times\mathcal{S}\left(\cos\frac{\pi j}{k}\boldsymbol{v}_{0,j+1}+\sin\frac{\pi j}{k}\boldsymbol{u}_{0,j}\right)-\mathsf{s}_1 v_{i+1,1}v_{j+1,1}, \tag{5.2.20a}$$

$$\boldsymbol{v}_{0,i+1}^{t}\mathcal{T}^t\mathcal{S}\mathcal{T}\boldsymbol{v}_{0,j+1}=\left(\cos\frac{\pi i}{k}\boldsymbol{v}_{0,i+1}-\sin\frac{\pi i}{k}\boldsymbol{u}_{0,i}\right)\times\mathcal{S}\left(\cos\frac{\pi j}{k}\boldsymbol{v}_{0,j+1}-\sin\frac{\pi j}{k}\boldsymbol{u}_{0,j}\right)-\mathsf{s}_k v_{i+1,k}v_{j+1,k}, \tag{5.2.20b}$$

$$\boldsymbol{u}_{0,i}^{\mathrm{t}}\mathcal{T}\mathcal{S}\mathcal{T}^t\boldsymbol{u}_{0,j}=\left(\cos\frac{\pi i}{k}\boldsymbol{u}_{0,i}-\sin\frac{\pi i}{k}\boldsymbol{v}_{0,i+1}\right)\times\mathcal{S}\left(\cos\frac{\pi j}{k}\boldsymbol{u}_{0,j}-\sin\frac{\pi j}{k}\boldsymbol{v}_{0,j+1}\right)-\mathsf{s}_1 u_{i,1}u_{j,1}, \tag{5.2.20c}$$

$$\boldsymbol{u}_{0,i}^{t}\mathcal{T}^t\mathcal{S}\mathcal{T}\boldsymbol{u}_{0,j}=\left(\cos\frac{\pi i}{k}\boldsymbol{u}_{0,i}+\sin\frac{\pi i}{k}\boldsymbol{v}_{0,i+1}\right)\times\mathcal{S}\left(\cos\frac{\pi j}{k}\boldsymbol{u}_{0,j}+\sin\frac{\pi j}{k}\boldsymbol{v}_{0,j+1}\right)-\mathsf{s}_k u_{i,k}u_{j,k}, \tag{5.2.20d}$$

$$\boldsymbol{u}_{0,i}^{t}\mathcal{T}\mathcal{S}\mathcal{T}^t\boldsymbol{v}_{0,j+1}=\left(\cos\frac{\pi i}{k}\boldsymbol{u}_{0,i}-\sin\frac{\pi i}{k}\boldsymbol{v}_{0,i+1}\right)\times\mathcal{S}\left(\cos\frac{\pi j}{k}\boldsymbol{v}_{0,j+1}+\sin\frac{\pi j}{k}\boldsymbol{u}_{0,j}\right)+\mathsf{s}_1 u_{i,1}v_{j+1,1}, \tag{5.2.20e}$$

$$\boldsymbol{u}_{0,i}^{t}\mathcal{T}^t\mathcal{S}\mathcal{T}\boldsymbol{v}_{0,j+1}=\left(\cos\frac{\pi i}{k}\boldsymbol{u}_{0,i}+\sin\frac{\pi i}{k}\boldsymbol{v}_{0,i+1}\right)\times\mathcal{S}\left(\cos\frac{\pi j}{k}\boldsymbol{v}_{0,j+1}-\sin\frac{\pi j}{k}\boldsymbol{u}_{0,j}\right)-\mathsf{s}_k u_{i,k}v_{j+1,k}. \tag{5.2.20f}$$

From (5.2.16) it follows that, for $i=1,\ldots,k-1$,

$$v_{i+1,1}=-v_{i+1,k}\quad\text{and}\quad u_{i,1}=-u_{i,k}. \tag{5.2.21}$$

Use the facts that the terms on the left-hand sides in (5.2.20) are scalars and so are equal to their transposes to get

$$\boldsymbol{v}_{0,i+1}^{t}(\mathcal{T}\mathcal{S}\mathcal{T}^t+\mathcal{T}^t\mathcal{S}\mathcal{T})\boldsymbol{v}_{0,j+1}=2\cos\frac{\pi i}{k}\cos\frac{\pi j}{k}\boldsymbol{v}_{0,i+1}^{t}\mathcal{S}\boldsymbol{v}_{0,j+1}+2\sin\frac{\pi i}{k}\sin\frac{\pi j}{k}\boldsymbol{u}_{0,i}^{t}\mathcal{S}\boldsymbol{u}_{0,j}-(\mathsf{s}_1+\mathsf{s}_k)v_{i+1,1}v_{j+1,1}, \tag{5.2.22a}$$

$$\boldsymbol{u}_{0,i}^{t}(\mathcal{T}\mathcal{S}\mathcal{T}^t+\mathcal{T}^t\mathcal{S}\mathcal{T})\boldsymbol{u}_{0,j}=2\sin\frac{\pi i}{k}\sin\frac{\pi j}{k}\boldsymbol{v}_{0,i+1}^{\mathrm{t}}\mathcal{S}\boldsymbol{v}_{0,j+1}+2\cos\frac{\pi i}{k}\cos\frac{\pi j}{k}\boldsymbol{u}_{0,i}^{t}\mathcal{S}\boldsymbol{u}_{0,j}-(\mathsf{s}_1+\mathsf{s}_k)u_{i,1}u_{j,1}, \tag{5.2.22b}$$

$$\boldsymbol{u}_{0,i}^{t}(\mathcal{T}\mathcal{S}\mathcal{T}^t-\mathcal{T}^t\mathcal{S}\mathcal{T})\boldsymbol{v}_{0,j+1}=2\cos\frac{\pi i}{k}\sin\frac{\pi j}{k}\boldsymbol{u}_{0,i}^{t}\mathcal{S}\boldsymbol{u}_{0,j}-2\sin\frac{\pi i}{k}\cos\frac{\pi j}{k}\boldsymbol{v}_{0,i+1}^{t}\mathcal{S}\boldsymbol{v}_{0,j+1}+(\mathsf{s}_1-\mathsf{s}_k)u_{i,1}v_{j+1,1}, \tag{5.2.22c}$$

$$\boldsymbol{v}_{0,i+1}^{t}(\mathcal{T}\mathcal{S}\mathcal{T}^t-\mathcal{T}^t\mathcal{S}\mathcal{T})\boldsymbol{u}_{0,j}=2\sin\frac{\pi i}{k}\cos\frac{\pi j}{k}\boldsymbol{u}_{0,i}^{t}\mathcal{S}\boldsymbol{u}_{0,j}-2\cos\frac{\pi i}{k}\sin\frac{\pi j}{k}\boldsymbol{v}_{0,i+1}^{t}\mathcal{S}\boldsymbol{v}_{0,j+1}+(\mathsf{s}_1-\mathsf{s}_k)v_{i+1,1}u_{j,1}. \tag{5.2.22d}$$

For the terms multiplied by $(1+4r)$ and $-(3+4r)$,

$$\boldsymbol{v}_{0,i+1}^{t}(2\mathcal{S}-\mathcal{S}\mathcal{T}\mathcal{T}^t-\mathcal{S}\mathcal{T}^t\mathcal{T})\boldsymbol{v}_{0,j+1}=(\mathsf{s}_1+\mathsf{s}_k)v_{i+1,1}v_{j+1,1}, \tag{5.2.23a}$$

$$\boldsymbol{u}_{0,i}^{t}(2\mathcal{S}-\mathcal{S}\mathcal{T}\mathcal{T}^t-\mathcal{S}\mathcal{T}^t\mathcal{T})\boldsymbol{u}_{0,j}=(\mathsf{s}_1+\mathsf{s}_k)u_{i,1}u_{j,1}, \tag{5.2.23b}$$

$$\boldsymbol{u}_{0,i}^{t}(\mathcal{S}\mathcal{T}\mathcal{T}^t-\mathcal{S}\mathcal{T}^t\mathcal{T})\boldsymbol{v}_{0,j+1}=(\mathsf{s}_1-\mathsf{s}_k)u_{i,1}v_{j+1,1}, \tag{5.2.23c}$$

$$\boldsymbol{v}_{0,i+1}^{t}(\mathcal{S}\mathcal{T}\mathcal{T}^t-\mathcal{S}\mathcal{T}^t\mathcal{T})\boldsymbol{u}_{0,j}=(\mathsf{s}_1-\mathsf{s}_k)v_{i+1,1}u_{j,1}. \tag{5.2.23d}$$

For the terms multiplied by $(1+2r)$, we have, for example,

$$\boldsymbol{v}_{0,i+1}^{t}\mathcal{T}\mathcal{S}\boldsymbol{v}_{0,j+1}=\left(\cos\frac{\pi i}{k}\boldsymbol{v}_{0,i+1}^{t}+\sin\frac{\pi i}{k}\boldsymbol{u}_{0,i}^{t}\right)\mathcal{S}\boldsymbol{v}_{0,j+1}-\mathsf{s}_1 v_{j+i,1}v_{j+1,1}, \tag{5.2.24a}$$

$$\boldsymbol{v}_{0,i+1}^{t}\mathcal{T}^t\mathcal{S}\boldsymbol{v}_{0,j+1}=\left(\cos\frac{\pi i}{k}\boldsymbol{v}_{0,i+1}^{t}-\sin\frac{\pi i}{k}\boldsymbol{u}_{0,i}^{t}\right)\mathcal{S}\boldsymbol{v}_{0,j+1}-\mathsf{s}_k v_{i+1,k}v_{j+1,k}, \tag{5.2.24b}$$

$$\boldsymbol{v}_{0,i+1}^{t}\mathcal{S}\mathcal{T}\boldsymbol{v}_{0,j+1}=\boldsymbol{v}_{0,i+1}^{t}\mathcal{S}\left(\cos\frac{\pi j}{k}\boldsymbol{v}_{0,j+1}-\sin\frac{\pi j}{k}\boldsymbol{u}_{0,j}\right)-\mathsf{s}_1 v_{i+1,1}v_{j+1,1}, \tag{5.2.24c}$$

$$\boldsymbol{v}_{0,i+1}^t\mathcal{S}\mathcal{T}^t\boldsymbol{v}_{0,j+1} = \boldsymbol{v}_{0,i+1}^t\mathcal{S}\left(\cos\frac{\pi j}{k}\boldsymbol{v}_{0,j+1} - \sin\frac{\pi j}{k}\boldsymbol{u}_{0,j}\right) - \mathsf{s}_k v_{i+1,k} v_{j+1,k}. \tag{5.2.24d}$$

Similar expressions involving the $\boldsymbol{u}_0$ can be derived. Adding the expressions in (5.2.24), or their counterparts involving $\boldsymbol{u}_0$, and using (5.2.21),

$$\boldsymbol{v}_{0,i+1}^t(\mathcal{S}\mathcal{T}+\mathcal{T}\mathcal{S}+\mathcal{T}^t\mathcal{S}+\mathcal{S}\mathcal{T}^t)\boldsymbol{v}_{0,j+1} = 2\left(\cos\frac{\pi i}{k}+\cos\frac{\pi j}{k}\right)\boldsymbol{v}_{0,i+1}^{\mathrm{t}}\mathcal{S}\boldsymbol{v}_{0,j+1} - 2(\mathsf{s}_1+\mathsf{s}_k)v_{i+1,1}v_{j+1,1}, \tag{5.2.25a}$$

$$\boldsymbol{u}_{0,i}^t(\mathcal{S}\mathcal{T}+\mathcal{T}\mathcal{S}+\mathcal{T}^t\mathcal{S}+\mathcal{S}\mathcal{T}^t)\boldsymbol{u}_{0,j} = 2\left(\cos\frac{\pi i}{k}+\cos\frac{\pi j}{k}\right)\boldsymbol{u}_{0,i}^t\mathcal{S}\boldsymbol{u}_{0,j} + 2(\mathsf{s}_1+\mathsf{s}_k)u_{i,1}u_{j,1}, \tag{5.2.25b}$$

$$\boldsymbol{u}_{0,i}^t(-\mathcal{S}\mathcal{T}-\mathcal{T}\mathcal{S}+\mathcal{T}^t\mathcal{S}+\mathcal{S}\mathcal{T}^t)\boldsymbol{v}_{0,j+1} = 2\sin\frac{\pi i}{k}\boldsymbol{v}_{0,i+1}^t\mathcal{S}\boldsymbol{v}_{0,j+1} + 2\sin\frac{\pi j}{k}\boldsymbol{u}_{0,i}^t\mathcal{S}\boldsymbol{u}_{0,j} - 2(\mathsf{s}_1-\mathsf{s}_k)u_{i,1}v_{j+1,1}, \tag{5.2.25c}$$

$$\boldsymbol{v}_{0,i+1}^t(-\mathcal{S}\mathcal{T}-\mathcal{T}\mathcal{S}+\mathcal{T}^t\mathcal{S}+\mathcal{S}\mathcal{T}^t)\boldsymbol{u}_{0,j} = -2\sin\frac{\pi j}{k}\boldsymbol{v}_{0,i+1}^t\mathcal{S}\boldsymbol{v}_{0,j+1} - 2\sin\frac{\pi i}{k}\boldsymbol{u}_{0,i}^{\mathrm{t}}\mathcal{S}\boldsymbol{u}_{0,j} + 2(\mathsf{s}_1-\mathsf{s}_k)v_{i+1,1}u_{j,1}. \tag{5.2.25d}$$

Substitute (5.2.22), (5.2.23), and (5.2.25) into (5.2.15) to get

$$\frac{1}{2}\sqrt{\frac{1+r}{r}}\boldsymbol{v}_{0,i+1}^t\mathcal{B}_1\boldsymbol{v}_{0,j+1} = -\alpha^*\left[\left(\left(1+\cos\frac{\pi i}{k}\cos\frac{\pi j}{k} - (1+2r)\left(\cos\frac{\pi i}{k} + \cos\frac{\pi j}{k}\right)\right)\boldsymbol{v}_{0,i+1}^t\mathcal{S}\boldsymbol{v}_{0,j+1} + \sin\frac{\pi i}{k}\sin\frac{\pi j}{k}\boldsymbol{u}_{0,i}^t\mathcal{S}\boldsymbol{u}_{0,j}\right], \tag{5.2.26}$$

$$\frac{1}{2}\sqrt{\frac{1+r}{r}}\boldsymbol{u}_{0,i}^t(\mathcal{B}_g)_1\boldsymbol{u}_{0,i} = -\alpha^*\left[\left(\left(1+\cos\frac{\pi i}{k}\cos\frac{\pi j}{k} - (1+2r)\left(\cos\frac{\pi i}{k} + \cos\frac{\pi j}{k}\right)\right)\boldsymbol{u}_{0,i}^t\mathcal{S}\boldsymbol{u}_{0,j} + \sin\frac{\pi i}{k}\sin\frac{\pi j}{k}\boldsymbol{v}_{0,i+1}^t\mathcal{S}\boldsymbol{v}_{0,j+1}\right], \tag{5.2.27}$$

$$\frac{1}{2}\sqrt{\frac{1+r}{r}}\boldsymbol{u}_{0,i}^t(\mathcal{P}\mathcal{B})_1\boldsymbol{v}_{0,j+1} = -\alpha^*\left[\sin\frac{\pi j}{k}\left(1+2r+\cos\frac{\pi i/k}{k}\right)\boldsymbol{u}_{0,i}^t\mathcal{S}\boldsymbol{u}_{0,j} + \sin\frac{\pi i}{k}\left(1+2r-\cos\frac{\pi j}{k}\right)\boldsymbol{v}_{0,i+1}^t\mathcal{S}\boldsymbol{v}_{0,j+1}\right], \tag{5.2.28}$$

$$\frac{1}{2}\sqrt{\frac{1+r}{r}}\boldsymbol{u}_{0,i}^t(\mathcal{P}_g\mathcal{B}_g)_1\boldsymbol{v}_{0,j+1} = -\alpha^*\left[\sin\frac{\pi i}{k}\left(1+2r-\cos\frac{\pi i}{k}\right)\boldsymbol{u}_{0,i}^t\mathcal{S}\boldsymbol{u}_{0,j} + \sin\frac{\pi j}{k}\left(1+2r+\cos\frac{\pi i}{k}\right)\boldsymbol{v}_{0,i+1}^t\mathcal{S}\boldsymbol{v}_{0,j+1}\right]. \tag{5.2.29}$$

These are valid for $i,j=1,\ldots,k-1$.

C. Stability of asymmetric solutions

In this section, we will use the results of Sec. V B to obtain a greatly simplified expression for computing the eigenvalues from $\mathcal{V}_1$ of the matrix (4.17). With this, we show that the eigenvalues of (4.17) restricted to $\mathcal{V}_1$ will have the same signs as the eigenvalues of the matrix $\mathcal{S}$. Since the matrix $\mathcal{S}$ is diagonal with an entry of -1 for each small spike, this immediately implies the main stability result. We also use the simplified expression to compute the eigenvalues of (4.17) for some asymmetric spike patterns.

From (5.2.10) and (5.2.27),

$$(\mathcal{U}^t(\mathcal{C}^{-1}\mathcal{B}_g)_1\mathcal{U})_{ij} = -\alpha^*\left\{\left[\left(1-\cos\frac{\pi j}{k}\right)\left(3-\cos\frac{\pi i}{k}+2r\right) + 2r\left(1-\cos\frac{\pi i}{k}\right)\right]\boldsymbol{u}_{0,i}\mathcal{S}\boldsymbol{u}_{0,j} + \sin\frac{\pi i}{k}\sin\frac{\pi j}{k}\boldsymbol{v}_{0,i+1}\mathcal{S}\boldsymbol{v}_{0,j+1}\right\}, \tag{5.3.1}$$

where, from Ref. 11 it can be shown that $\xi_j=\kappa_{j+1}$, and where κ_j is given in (5.1.9).

From (5.1.5), (5.1.27), and (5.1.30),

$$(\mathcal{U}^t\mathcal{R}_1\mathcal{U})_{ij} = D\sqrt{\frac{1+r}{r}}\left[\frac{\sin(\pi i/k)}{1+\cos(\pi i/k)}\boldsymbol{v}_{0,i+1}^t(\mathcal{P}_g\mathcal{B}_g)_1\boldsymbol{u}_{0,j} - \frac{\sin(\pi j/k)}{1+\cos(\pi j/k)}\boldsymbol{u}_{0,i}^{\mathrm{t}}(\mathcal{P}\mathcal{B})_1\boldsymbol{v}_{0,j+1}\right] + \frac{\sin(\pi i/k)\sin(\pi j/k)}{\big(1+\cos(\pi i/k)\big)\big(1+\cos(\pi j/k)\big)} \times(\boldsymbol{v}_{0,i+1}^t(\mathcal{C}^{-1}\mathcal{B})_1\boldsymbol{v}_{0,j+1}). \tag{5.3.2}$$

Noting that $\boldsymbol{v}_{0,i+1}^{\mathrm{t}}(\mathcal{C}^{-1}\mathcal{B})_1\,\boldsymbol{v}_{0,j+1}=(Q_0^t(\mathcal{C}^{-1}\mathcal{B})_1Q_0)_{ij}$ we substitute (5.2.28), (5.2.29), (5.2.9), and (5.1.9) into (5.3.2) to get

$$(\mathcal{U}^t\mathcal{R}_1\mathcal{U})_{ij} = -\alpha^*\left\{\left[\left(3-\cos\frac{\pi i}{k}\right)\left(1-\cos\frac{\pi j}{k}\right) + 2r\left(2-\cos\frac{\pi i}{k}-\cos\frac{\pi j}{k}\right)\right]\boldsymbol{u}_{0,i}\mathcal{S}\boldsymbol{u}_{0,j} + \frac{\sin(\pi i/k)\sin(\pi j/k)}{(1+\cos(\pi i/k))(1+\cos(\pi j/k))} \times\left[5+\cos\frac{\pi i}{k}-3\cos\frac{\pi j}{k}+\cos\frac{\pi i}{k}\cos\frac{\pi j}{k} + 8r\right]\boldsymbol{v}_{0,i+1}\mathcal{S}\boldsymbol{v}_{0,j+1}\right\}. \quad (5.3.3)$$

Subtract (5.3.3) from (5.3.1) to give

$$(\mathcal{U}^t(\mathcal{C}^{-1}\mathcal{B}_g)_1\mathcal{U}-\mathcal{U}^t\mathcal{R}_1\mathcal{U})_{ij} = \frac{4\alpha^*\sin(\pi i/k)\sin(\pi/j)}{(1+\cos(\pi i/k))(1+\cos(\pi j/k))} \times\left(1-\cos\frac{\pi j}{k}+2r\right)v^t_{0,i+1}\mathcal{S}v_{0,j+1},\quad i,j=1,\ldots,k-1. \quad (5.3.4)$$

We are now able to determine the signs of the eigenvalues of (4.17) in proposition 4.1. We use (5.3.4), and consider only the first $k-1$ rows and columns of $\mathcal{U}^t(\mathcal{C}^{-1}\mathcal{B}_g)_1\mathcal{U}-\mathcal{U}^t\mathcal{R}_1\mathcal{U}\Omega_0$. Since $(\Omega_0)_{jj}=1$ for $j=1,\ldots,k-1$, and since, by (4.15), $\Sigma_{kk}=0$, the first $k-1$ rows and columns of (4.17) are

$$[(I+\Sigma)^{-1}[\mathcal{U}^t(\mathcal{C}^{-1}\mathcal{B}_g)_1\mathcal{U}-\mathcal{U}^t\mathcal{R}_1\mathcal{U}]]_{ij},\quad i,j=1,\ldots,k-1.$$

Thus we can use (5.3.4) and (4.15) to write

$$[(I+\Sigma)^{-1}[\mathcal{U}^t(\mathcal{C}^{-1}\mathcal{B}_g)_1\mathcal{U}-\mathcal{U}^t\mathcal{R}_1\mathcal{U}]]_{ij} = \sum_{p=1}^{k-1}(I+\Sigma)^{-1}_{ip}[\mathcal{U}^t(\mathcal{C}^{-1}\mathcal{B}_g)_1\mathcal{U}-\mathcal{U}^t\mathcal{R}_1\mathcal{U}]_{pj} = (I+\Sigma)^{-1}_{ii}[\mathcal{U}^t(\mathcal{C}^{-1}\mathcal{B}_g)_1\mathcal{U}-\mathcal{U}^t\mathcal{R}_1\mathcal{U}]_{ij} = 8\alpha^* r\frac{\sin(\pi i/k)}{(1+\cos(\pi i/k))(1-\cos(\pi i/k)+2r)} \times\frac{\sin(\pi j/k)(1-\cos(\pi j/k)+2r)}{(1+\cos(\pi j/k))}\boldsymbol{v}^t_{0,i+1}\mathcal{S}\boldsymbol{v}_{0,j+1}, \quad (5.3.5)$$

where $i,j=1,\ldots,k-1$.

Define the diagonal matrices A and B by

$$A_{ii}\equiv 1-\cos\frac{\pi i}{k}+2r, \quad (5.3.6)$$

$$B_{ii}\equiv\begin{cases}\dfrac{\sin(\pi i/k)}{1+\cos(\pi i/k)} & i\neq k\\ 0 & i=k,\end{cases} \quad (5.3.7)$$

so that

$$(AB)_{ii}=\frac{\sin(\pi i/k)(1-\cos(\pi i/k)+2r)}{(1+\cos(\pi i/k))},$$

$$(A^{-1}B)_{ii}=\frac{\sin(\pi i/k)}{(1+\cos(\pi i/k))(1-\cos(\pi i/k)+2r)}, \quad (5.3.8)$$

for $i\neq k$.

Then, since diagonal matrices commute, by (5.3.5) and (5.3.8),

$$(I+\Sigma)^{-1}[\mathcal{U}^t(\mathcal{C}^{-1}\mathcal{B}_g)_1\mathcal{U}-\mathcal{U}^t\mathcal{R}_1\mathcal{U}] = 8r\alpha^*A^{-1}BQ_0^t\mathcal{S}Q_0AB. \quad (5.3.9)$$

Substitute (5.3.9) into $(I+\Sigma)^{-1}[\mathcal{U}^t(\mathcal{C}^{-1}\mathcal{B}_g)_1\mathcal{U}-\mathcal{U}^t\mathcal{R}_1\mathcal{U}]\boldsymbol{b}=\omega_1\boldsymbol{b}$ to get

$$8r\alpha^*BQ_0^t\mathcal{S}Q_0B(A\boldsymbol{b})=\omega_1(A\boldsymbol{b}). \quad (5.3.10)$$

Now we will demonstrate that it always is possible to find a set of k-independent vectors $\boldsymbol{z}_i$ such that for k_1 of the vectors ω_1 in (5.3.10) is negative and for k_2-1 of the vectors ω_1 is positive. This will imply that the spectrum of (2.15) will have exactly k_1 small positive eigenvalues and exactly k_2 small negative eigenvalues. We begin by noting that for $\boldsymbol{x}\in\mathbb{R}^k$, $Q_0B\boldsymbol{x}$ will be in the space spanned by the first $k-1$ columns of Q_0. This space may also be described as the $k-1$ dimensional subspace $\mathcal{W}$ of $\mathbb{R}^k$, which is orthogonal to the vector $(1,\ldots,1)$. With this fact, we only need to find a k_1 dimensional subspace of $\mathcal{W}$ on which $\mathcal{S}$ is negative definite and a k_2-1 dimensional subspace on which it is positive definite.

We begin the construction with the two subspaces $\mathcal{W}^+$ and $\mathcal{W}^-$ for which $\boldsymbol{x}^t\mathcal{S}\boldsymbol{x}<0$ and $\boldsymbol{x}^t\mathcal{S}\boldsymbol{x}>0$, respectively. The subspace we are ultimately interested in is defined by

$$\mathcal{T}=\mathrm{proj}_{\mathcal{W}}(\mathrm{perp}(\mathrm{proj}_{\mathcal{W}}(\mathcal{W}^+))). \quad (5.3.11)$$

Here $\mathrm{proj}_{\mathcal{W}}(S)$ is the projection of the space S onto $\mathcal{W}$ and $\mathrm{perp}(S)$ is the subspace orthogonal to S. We note that $\dim(\mathrm{proj}_{\mathcal{W}}(\mathcal{W}^-))=k_1-1$ and $\dim(\mathrm{proj}_{\mathcal{W}}(\mathcal{W}^+))=k_2-1$, and thus $\dim(\mathcal{T})=k_1$. We define the index sets $I^{\pm}$ by $\mathcal{W}^+=\mathrm{span}(\{\boldsymbol{e}_i\}_{i\in I^+})$ and $\mathcal{W}^-=\mathrm{span}(\{\boldsymbol{e}_i\}_{i\in I^-})$ (note that the standard basis vectors $\boldsymbol{e}_i$ are eigenvectors of $\mathcal{S}$).

Define $i^+=\max_{i\in I^+}$. Then the k_2-1 vectors given by $\{\boldsymbol{e}_i-\boldsymbol{e}_{i^+}\}_{i\in I^+,i\neq i^+}$ form a basis of $\mathrm{proj}_{\mathcal{W}}(\mathcal{W}^+)$. For the basis of $\mathrm{perp}(\mathrm{proj}_{\mathcal{W}}(\mathcal{W}^+))$, define the vector $\boldsymbol{v}$ by

$$v_i=\begin{cases}1 & i\in I^+,\\ 0 & i\in I^-.\end{cases} \quad (5.3.12)$$

Then $\mathrm{perp}(\mathrm{proj}_{\mathcal{W}}(\mathcal{W}^+))$ is spanned by the k_1+1 vectors $\{\{\boldsymbol{e}_i\}_{i\in I^-},\boldsymbol{v}\}$. The subspace $\mathcal{T}$ is spanned by the k_1 vectors $\boldsymbol{z}_i$, with components defined by

$$(z_i)_j=\begin{cases}1 & j\in I^+,\\ -k_2 & j=i,\\ 0 & \text{otherwise},\end{cases} \quad (5.3.13)$$

where $i\in I^-$. Finally, $z_i^tSz_i=-k_2^2-k_2$, which will always be negative if $k_2>1$. For the case $k_2=1$, we can construct $\boldsymbol{z}_i$ as follows: Set $(z_i)_j=1$ for the one value of $j\in I^+$. We can then place m negative ones and $m-1$ positive ones in the remaining places provided that $m>1$. We may always form a basis in this manner and for any such basis vector $\boldsymbol{x}$, $\boldsymbol{x}^t\mathcal{S}\boldsymbol{x}=-2m+2<0$. The only case left to consider is $k_1=k_2=1$. In this case, the results of Ref. 14 imply the solution is unstable

TABLE I. Estimates of the leading order corrections, $\hat{\omega}_1$, to the eigenvalues of the generalized eigenvalue problem (2.40a) in $\mathcal{V}_1$ as a function of δ, compared to the corresponding eigenvalues ω_1 in (5.3.10).

$\hat{\omega}_1$ δ=.0001	δ=.000 01	δ=.000 001	ω_1
The "1, −1, 1, 1" pattern			
−15.52	−15.50	−15.49	−15.49
3.393	3.396	3.396	3.396
35.13	35.30	35.33	35.33
The "−1, 1, 1, −1, 1" pattern			
−61.14	−60.59	−60.52	−60.52
−9.053	−8.923	−8.914	−8.912
16.79	16.87	16.88	16.88
285.0	292.8	294.5	294.6
The "−1, −1, 1, 1, −1, 1, 1" pattern			
−400.8	−376.1	−373.7	−373.5
−171.0	−166.5	−166.1	−166.1
−3.881	−3.620	−3.594	−3.591
31.97	32.60	32.66	32.67
338.0	356.5	358.4	358.6
1005.0	1132.0	1147.0	1148.0

with respect to the small eigenvalues, and the remaining small eigenvalue, corresponding to the eigenvector in $\mathcal{V}_2$, must always be negative.

Thus there is a set of k_1 independent vectors on which the eigenvalues of (5.3.10) must be negative and a set of k_2-1 independent vectors on which they must be positive. We summarize the results in this section in the following proposition:

Proposition 5.1. *Given an asymmetric steady-state solution* (2.10) and (2.12) *with k_1 small spikes sufficiently close to the bifurcation, the eigenvalue problem* (2.15) *will possess exactly k_1 small positive eigenvalues counting multiplicity.*

Table I reports a check of these conclusions. For some specific spike patterns and for $r=1$ and $\mu=1$, we compute the eigenvalues ω_1 in (5.3.10) using Maple™ (Ref. 16). We compare them to estimates, $\hat{\omega}_1$, of corrections to ω in (2.40a). We compute these estimates as follows. From (2.40a) we use (4.2a) to approximate ω_1 as

$$\hat{\omega}_1 = \frac{\omega-\omega_0}{\delta}. \tag{5.3.14}$$

We use Maple™ to compute ω from (2.40a). The values of ω_0 are given in (2.42). Following the notation (3.20) we write spike patterns as a sequence of +1's and −1's. For example, "−1, 1, 1, −1, 1" is a pattern with five spikes, with one small spike at the left end and another small spike next to a big spike at the right end of the $[-1,1]$ interval. As $\delta\to 0$ we should get that $\hat{\omega}_1\to\omega_1$. Since we are interested only in the ω_1 for $\omega_0=1$, we report only those $k-1$ values. As expected k_1 of these $k-1$ values are negative.

VI. CONCLUSIONS

We have confirmed that all asymmetric spike solutions (1.1) are unstable as they bifurcate off of a stable symmetric branch. Moreover, the signs of the eigenvalues are determined by the eigenvalues of the diagonal matrix $\mathcal{S}$. Thus if there are k_1 small spikes, $\mathcal{S}$ will always have k_1 negative eigenvalues and there must be exactly k_1 positive eigenvalues for (2.15). Using (5.3.10) we can compute the leading order terms for the critical eigenvalues for any given pattern.

In this paper, we have restricted our analysis to the Gierer-Meinhardt system, however, similar asymmetric patterns have been observed in many other reaction-diffusion systems such as the Gray-Scott,[13] and with minor modifications, our result should carry over to all such systems. The key features of (1.1) needed to perform this analysis are the relations between the stability of a spike profile and the matrix eigenvalue problem (2.40a). This relationship is a result of the inner and outer matching of multispike solutions and similar relations should hold for reaction diffusion equations of activator and inhibitor type with a slowly diffusing inhibitor.

APPENDIX: THE CORRECTIONS $\mathcal{B}_1$ TO $\mathcal{B}$

By (2.28c) and (3.20)

$$\begin{aligned} c_\ell &= \coth(z_{\ell-1}+z_\ell)+\coth(z_\ell+z_{\ell+1}) \\ &= \coth(2z_c+\delta\alpha(\mathsf{s}_{\ell-1}+\mathsf{s}_\ell))+\coth(2z_c+\delta\alpha(\mathsf{s}_\ell+\mathsf{s}_{\ell+1})) \\ &= 2\coth(2z_c)-\delta\alpha\,\mathrm{csch}^2(2z_c)(\mathsf{s}_{\ell-1}+2\mathsf{s}_\ell+\mathsf{s}_{\ell+1})+O(\delta^2) \\ &= \frac{1+2r}{\sqrt{r(1+r)}}+\delta\left(\frac{\alpha(-\mathsf{s}_{\ell-1}-2\mathsf{s}_\ell-\mathsf{s}_{\ell+1})}{4r(1+r)}\right)+O(\delta^2), \end{aligned}$$

$$\begin{aligned} c_\ell &= \frac{1+2r}{\sqrt{r(1+r)}}+\delta\alpha^*\sqrt{\frac{r}{1+r}}(-\mathsf{s}_{\ell-1}-2\mathsf{s}_\ell-\mathsf{s}_{\ell+1}) \\ &\quad +O(\delta^2), \quad \ell=2,\ldots,k-1 \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} d_\ell &= -\mathrm{csch}(z_\ell+z_{\ell+1}) \\ &= -\mathrm{csch}(2z_c+\delta\alpha(\mathsf{s}_\ell+\mathsf{s}_{\ell+1})) \\ &= -\mathrm{csch}(2z_c)+\delta\alpha(\mathsf{s}_\ell+\mathsf{s}_{\ell+1})\mathrm{csch}(2z_c)\coth(2z_c)+O(\delta^2) \\ &= -\frac{1}{2\sqrt{r(1+r)}}+\delta\alpha(\mathsf{s}_\ell+\mathsf{s}_{\ell+1})\frac{1+2r}{4r(1+r)}+O(\delta^2) \\ &= -\frac{1}{2\sqrt{r(1+r)}}+\delta\alpha^*\sqrt{\frac{r}{1+r}}(\mathsf{s}_\ell+\mathsf{s}_{\ell+1})(1+2r)+O(\delta^2). \end{aligned} \tag{A.2}$$

The conversion from hyperbolic functions of z_c to algebraic functions of r is accomplished by using (2.5), and α^* is defined in (5.2.12).

From (2.28b),

$$\begin{aligned} c_1 &= \coth(z_1+z_2)+\tanh z_1 \\ &= \coth(2z_c+\delta\alpha(\mathsf{s}_1+\mathsf{s}_2))+\tanh(z_c+\delta\alpha\mathsf{s}_1) \\ &= \coth(2z_c)+\tanh z_c+\delta\alpha(-(\mathsf{s}_1+\mathsf{s}_2)\mathrm{csch}^2(2z_c) \\ &\quad +\mathsf{s}_1\,\mathrm{sech}^2 z_c)+O(\delta^2) \end{aligned}$$

$$= \frac{1+2r}{2\sqrt{r(1+r)}} + \sqrt{\frac{r}{1+r}} + \delta\alpha\left(-\frac{\mathsf{s}_1+\mathsf{s}_2}{4r(1+r)} + \frac{\mathsf{s}_1}{1+r}\right) + O(\delta^2)$$

$$= \frac{1+4r}{2\sqrt{r(1+r)}} + \delta\alpha\left(\frac{\mathsf{s}_1(4r-1)-\mathsf{s}_2}{4r(1+r)}\right) + O(\delta^2)$$

$$= \frac{1+4r}{2\sqrt{r(1+r)}} + \delta\alpha^* \sqrt{\frac{r}{1+r}}(\mathsf{s}_1(4r-1)-\mathsf{s}_2) + O(\delta^2) \tag{A.3}$$

and

$$c_k = \coth(z_k + z_{k-1}) + \tanh z_k = \frac{1+4r}{2\sqrt{r(1+r)}} + \delta\alpha^* \sqrt{\frac{r}{1+r}}(\mathsf{s}_k(4r-1)-\mathsf{s}_{k-1}) + O(\delta^2). \tag{A.4}$$

The corrections to $\mathcal{B}_g$, $\mathcal{PB}$, and $\mathcal{P}_g\mathcal{B}_g$ are computed in a similar fashion.

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