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# Analysis of nonmetric theories of gravity. III. Summary of the analysis and its application to theories in the literature

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In this paper we summarize the analysis of nonmetric theories of gravity described in detail in some previous papers. One of the main results of this analysis is that in a spherically symmetric, static gravitational field, the class of metric-affine theories of gravity (MATG's) must take on their metric form. Some MATG's in the literature are then classified and analyzed, with the result that none of the theories investigated are viable.

## I. INTRODUCTION

This is the last in a series of papers involved in a systematic analysis of nonmetric theories of gravity. The class of metric theories of gravity (MTG's) is well defined in the literature.[1] A nonmetric theory of gravity is a theory not belonging to the class of MTG's. Although the techniques and ideas used in this analysis are quite general, they are primarily applied to a subclass of the class of all nonmetric theories of gravity, called metric-affine theories of gravity[2] (MATG's). An MATG is essentially characterized by the following:

(a) It is a geometric theory of gravity; that is, spacetime is characterized by a four-dimensional, Hausdorff, differentiable manifold of signature $-2$.

(b) The spacetime manifold is endowed with a connection $\Gamma$ and a $\binom{0}{2}$ tensor field $g$. The gravitational field is represented (completely) by $\Gamma$ and $g$. (Note: $\Gamma$ is not, in general, assumed to be the Christoffel symbol constructed from the metric.)

(c) The unique curves of freely falling test particles are associated with the natural geometric curves in the spacetime manifold, called paths. That is, the motion of freely falling test particles is governed by the (path) equation

$$\frac{d^2x^a}{d\lambda^2}+\Gamma^a{}_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda}=0 \ . \tag{1.1}$$

A theory must also specify how other fields should act in a gravitational field. In particular, the laws of electromagnetism in a gravitational field (which we shall call the gravitationally generalized laws of electromagnetism or laws of GGEM) must be given in an MATG. We find, however, that no useful results are obtained from an analysis of the laws of GGEM in their most general form.[3] Therefore, we shall restrict attention to gravitational fields that are spherically symmetric and static (SSS). In this case we find that $g$ and $\Gamma$ take on the simplified forms[3]

$$g_{00}=f \ , \qquad g_{\mu\nu}=-g\delta_{\mu\nu} \ , \tag{1.2}$$

and

$$\Gamma^\sigma{}_{\mu\nu}=(\alpha)_{,\nu}\delta^\sigma_\mu+(\bar\alpha)_{,\mu}\delta^\sigma_\nu+(\beta)_{,\sigma}\delta_{\mu\nu} \ ,$$
$$\Gamma^\sigma{}_{00}=(\gamma)_{,\sigma} \ , \tag{1.3}$$
$$\Gamma^0{}_{0\nu}=(\delta)_{,\nu} \ , \quad \Gamma^0{}_{\mu 0}=(\bar\delta)_{,\mu} \ ,$$

and the laws of GGEM can be written in terms of four electromagnetic-gravitational coupling functions $\mathscr{A}$, $\mathscr{B}$, $\mathscr{P}$, and $\mathscr{Q}$ (in addition to $g$ and $\Gamma$ in their simplified forms). In the above, $f$, $g$, $\alpha$, $\bar\alpha$, $\beta$, $\gamma$, $\delta$, $\bar\delta$, $\mathscr{A}$ $\mathscr{B}$, $\mathscr{P}$, and $\mathscr{Q}$ are arbitrary functions of the Newtonian gravitational potential $U$.

Indeed, in an SSS gravitational field the gravitationally generalized Maxwell (GGM) equations are given by[3]

$$\nabla^2\phi=\frac{g}{f}\frac{\partial^2\phi}{\partial t^2}+\mathscr{A}\vec{g}\cdot\left[\frac{\partial\vec{A}}{\partial t}+\vec\nabla\phi\right]-4\pi gJ_0 \tag{1.4a}$$

and

$$\nabla^2\vec{A}=\frac{g}{f}\frac{\partial^2\vec{A}}{\partial t^2}+\frac{f}{g}(\vec\nabla\cdot\vec{A})\vec\nabla\left[\frac{g}{f}\right]+\mathscr{B}(\vec\nabla\times\vec{A})\times\vec{g}+4\pi g\vec{J} \ , \tag{1.4b}$$

 

where $t=x^0$ is the time coordinate associated with the static nature of the gravitational field, $\vec{\nabla}$ is the usual gradient operator, $\vec{g}$ is the vector defined by $g^\mu=\partial U/\partial x^\mu$, and $A_a=(-\phi,\vec{A})$ and $J_a=(J_0,\vec{J})$ are the components of the electromagnetic four-vector potential and four-current, respectively.

The gravitationally generalized Lorentz (GGL) equations are given by[3]

$$\frac{d^2\vec{x}}{dt^2}+\vec{\nabla}(\gamma)+[\vec{\nabla}(\alpha+\bar{\alpha}-\delta-\bar{\delta})\cdot\vec{v}]\vec{v}+\vec{\nabla}(\beta)\vec{v}^2 =\frac{e}{m}L\left\{-\frac{1}{g}\left[\frac{\partial\vec{A}}{\partial t}+\vec{\nabla}\phi-\vec{\nabla}(\vec{A}\cdot\vec{v})+(\vec{v}\cdot\vec{\nabla})\vec{A}\right]+\frac{1}{f}\left[\left[\frac{\partial\vec{A}}{\partial t}+\vec{\nabla}\phi\right]\cdot\vec{v}\right]\vec{v}\right\} , \quad (1.5)$$

where $m$ is the rest mass of the test praticle, $e$ is its electromagnetic charge, $\vec{x}$ is its three-position, and $\vec{v}=d\vec{x}/dt$ is the coordinate three-velocity. $L$ is an arbitrary function of the gravitational field and $\vec{v}$ [i.e., $L=L(U,\vec{\nabla}U,\vec{v})$], and can be written in terms of the two arbitrary multiplying factors $\mathscr{P}$ and $\mathscr{Q}$ [see Eq. (2.4) in Ref. 4].

The laws of GGEM in MTG's are a special case of the above with

$$\alpha'=\bar{\alpha}'=-\beta'=\frac{g'}{2g} ,\qquad \delta'=\bar{\delta}'=\frac{g\gamma'}{f}=\frac{f'}{2f} , \quad (1.6)$$

and

$$\mathscr{A}=\frac{f'}{2f}-\frac{g'}{2g}=-\mathscr{B} ,\qquad \mathscr{P}=\frac{f'}{2f},\quad \mathscr{Q}=0 , \quad (1.7)$$

where the prime denotes differentiation with respect to $U$.

It is always possible to decompose $\Gamma$ according to

$$\Gamma^a{}_{bc}=\{{}_b{}^a{}_c\}+A^a{}_{bc} , \quad (1.8)$$

where $\{{}_b{}^a{}_c\}$ denotes the Christoffel symbol and $A^a{}_{bc}$ is a tensor (sometimes called the difference tensor). In an SSS gravitational field (1.8) can be realized by writing $\{\}$ according to (1.6), and denoting the part of $\Gamma$ associated with $A$ by a caret (i.e., $\alpha'=g'/2g+\hat{\alpha}'$).

We can decompose $A$ further,[5] viz.,

$$A^a{}_{bc}=(S^a{}_{bc}-S_c{}^a{}_b+S_{bc}{}^a)+\tfrac{1}{2}(Q^a{}_{bc}-Q_c{}^a{}_b+Q_{bc}{}^a) , \quad (1.9)$$

where $S$ is the torsion tensor defined by $S^a{}_{bc}=\Gamma^a{}_{[bc]}$, and $Q$ is the nonmetricity tensor defined by $Q_{abc}=-g_{bc\,|a}$. In this paper a vertical bar is used to denote covariant differentiation with respect to the affine connection $\Gamma$, and a semicolon denotes covariant differentiation with respect to the metric connection. One final comment on notation; Latin indices $(a,b,c)$ range from 0 to 3, and Greek indices $(\mu,\nu,\sigma)$ run from 1 to 3 (alternatively, three-vector notation will be used).

In Sec. II we shall summarize the analysis of MATG's outlined in the previous papers. We shall then argue that the analysis demands that $\alpha$, $\bar{\alpha}$, $\beta$, $\gamma$, $\delta$, $\bar{\delta}$, $\mathscr{A}$, $\mathscr{B}$, $\mathscr{P}$, and $\mathscr{Q}$ must take on their metric forms (so that in an SSS gravitational field MATG's must reduce to MTG's). This conclusion is reached from both a theoretical and experimental investigation. From the theoretical standpoint we find that if the theory is not an MTG, widely held principles of physics (which will be clearly stated) are violated.

However, only experimental contradiction can prove unquestionably that a theory is unviable. Unfortunately, since experiments are only valid to some predetermined accuracy we can never show categorically that a whole class of theories is not viable; effects may always be present that are undetectable to a particular observational accuracy. In the next section we will discuss solar system experiments in the framework of the class of theories under investigation. In order to do this it is appropriate to expand all the functions of the gravitational field in powers of $U$, viz.,

$$\begin{aligned} f&=1+f_1U+f_2U^2+O(U^3) ,\\ g&=1+g_1U+g_2U^2+O(U^3) ,\\ \hat{\alpha}&=\alpha_1U+\alpha_2U^2+O(U^3) ,\\ \hat{\bar{\alpha}}&=\bar{\alpha}_1U+\bar{\alpha}_2U^2+O(U^3) ,\\ \hat{\beta}&=\beta_1U+\beta_2U^2+O(U^3) ,\\ \hat{\gamma}&=\gamma_1U+\gamma_2U^2+O(U^3) ,\\ \hat{\delta}&=\delta_1U+\delta_2U^2+O(U^3) ,\\ \hat{\bar{\delta}}&=\bar{\delta}_1U+\bar{\delta}_2U^2+O(U^3) . \end{aligned} \quad (1.10)$$

In addition, $\mathscr{A}$, $\mathscr{B}$, $\mathscr{P}$, and $\mathscr{Q}$ can be expanded in powers of $U$. We shall find that the experimental evidence supports the above theoretical conclusions.

However, in this analysis we use the assumption that the gravitational field is SSS, which is appropriate for the experiments performed in the solar system, but can only be valid to some order of approximation [possibly $O(U^2)$]. Perhaps a more rigorous statement of the above results, and one that is indeed verifiable, is that in the solar system any viable theory of gravity must take on a metric form.

In Sec. III we shall outline and classify some MATG's in the literature. We shall then investigate these theories using the analysis set out in Sec. II. We shall find that none of the theories to be discussed are viable.

## II. SUMMARY OF THE ANALYSIS

### A. General structure

In this part of the analysis we ask very general questions of the theories under investigation.

First we ask whether the theory is complete. For a theory to be complete, all the fields representing gravity must be specified in the presence of matter (through the field equations of the theory). In addition, the theory must give a set of laws of GGEM.

For theories that are complete, we can calculate the magnitude of the fields in the theory representing gravity. We can then determine whether any of the fields (other than $g$) will produce effects in the solar system that can be detected by solar system experiments.

We must also ask whether the theory is internally consistent. For example, if a theory is complete, we can construct conservation laws by differentiating the field equations. We must then ask whether these conservation laws are consistent with the equations of motion in the theory.[6]

### B. The laws of GGEM

Here we shall summarize the analysis of the laws of GGEM in an SSS gravitational field. However, many theories in the literature are incomplete in the sense that they do not specify the laws of GGEM. Nevertheless, we can still obtain constraints on these theories by such an analysis. The theories can be completed by postulating their laws of GGEM; we assume that the laws of GGEM for each individual theory under investigation are a special case of the "most general" form of the laws of GGEM, with $\mathscr{A}$, $\mathscr{B}$, $\mathscr{P}$, and $\mathscr{Q}$ taking on specific forms. The analysis then yields information on the structure of each theory; the information essentially amounts to constraints on the theory in order that electromagnetism can be incorporated into the theory in a consistent and meaningful way.

(a) First let us consider the two factors $[\hat{\beta}'+(g/f)\hat{\gamma}']$ and $(\hat{\alpha}'+\hat{\hat{\alpha}}'-\hat{\delta}'-\hat{\hat{\delta}}')$. There are theoretical reasons for requiring both these terms to be zero. If these factors are not zero then the WEP is not satisfied,[4] and the equation governing the motion of photons deduced from the optical limit of the GGM equations would not be equivalent to the mass$\rightarrow 0$, speed$\rightarrow 1$ limit of the equation of motion (1.1) governing timelike test particles.[3]

These theoretical arguments are supported by experiment. Indeed, experiments that measure the deflection of light demand that $[\hat{\beta}'+(g/f)\hat{\gamma}']$ is zero to first order in $U$ (see next subsection). Both terms are required to be zero to both first and second orders in $U$ by modern-day Eötvös experiments[4] (to initial order these experimental constraints are very severe, while to second order they are rather weaker).

We conclude that both factors must be zero (and consequently take on their "metric" form), viz.,

$$\begin{aligned} &\hat{\beta}'+\frac{g}{f}\hat{\gamma}'=0 \ , \\ &\hat{\alpha}'+\hat{\hat{\alpha}}'-\hat{\delta}'-\hat{\hat{\delta}}'=0 \ . \end{aligned} \tag{2.1}$$

(b) Next let us consider $\mathscr{A}$ and $\mathscr{B}$. The general condition for charge to be conserved is[3]

$$\mathscr{A}-\mathscr{B}-\frac{f'}{f}+\frac{g'}{g}=0 \ . \tag{2.2}$$

If we demand charge conservation in a more specific form (such as, for example, $g^{ab}J_{a;b}=0$ or $g^{ab}J_{a\,|\,b}=0$), more precise constraints on $\mathscr{A}$ and $\mathscr{B}$ are obtained (see Ref. 3).

If we demand that the WEP is theoretically satisfied,[4] then

$$\begin{aligned} &\mathscr{A}=\frac{1}{2}\frac{f'}{f}-\frac{1}{2}\frac{g'}{g} \ , \\ &\mathscr{B}=\frac{1}{2}\frac{g'}{g}-\frac{1}{2}\frac{f'}{f} \end{aligned} \tag{2.3}$$

(that is, $\mathscr{A}$ and $\mathscr{B}$ take on their metric forms).

Eötvös experiments demand that $\mathscr{A}$ and $\mathscr{B}$ take on their metric forms.[4] The experimental constraints on $\mathscr{A}$ to the first order in $U$ are very strong and they are reasonably strong on $\mathscr{A}$ to second order and $\mathscr{B}$ to first order. Unfortunately, the experiments do not constrain $\mathscr{B}$ to second order in $U$ very severely.

(c) Let us consider $\mathscr{P}$ and $\mathscr{Q}$. Theoretically the WEP demands that[4]

$$\begin{aligned} &\mathscr{P}=\frac{1}{2}\frac{f'}{f}-\frac{g\hat{\gamma}'}{f} \ , \\ &\mathscr{Q}=0 \ . \end{aligned} \tag{2.4}$$

Eötvös experiments verify these theoretical results. $\mathscr{P}$ is measured to be equal to $\frac{1}{2}f'/f-(g/f)\hat{\gamma}'$ to both first and second orders in $U$, while $\mathscr{Q}$ is measured to be zero only to first order in $U$.

An investigation of the laws of GGEM will yield information on the six quantities $(\hat{\alpha}'+\hat{\bar{\alpha}}'-\hat{\delta}'-\hat{\bar{\delta}}')$, $[\hat{\beta}'+(g/f)\hat{\gamma}']$, $\mathscr{A}$, $\mathscr{B}$, $\mathscr{P}$, and $\mathscr{Q}$. From the above analysis, it is reasonable to conclude that these six terms must take on the values given by (2.1), (2.3), and (2.4). We note that all these terms must take on their metric form except $\mathscr{P}$, which can have a nonmetric contribution depending on a nonzero $\hat{\gamma}$. We can rewrite these results in the following form:

$$\frac{1}{2}(\alpha'+\bar{\alpha}'-\delta'-\bar{\delta}')=-\mathscr{A}=\mathscr{B}=\frac{1}{2}\left[\frac{(\mu g)'}{(\mu g)}-\frac{(\mu f)'}{(\mu f)}\right],$$
$$\beta'=-\frac{1}{2}\frac{(\mu g)'}{(\mu g)},\quad \gamma'=\frac{1}{2}\frac{(\mu f)'}{(\mu g)},\quad \mathscr{P}=\frac{1}{2}\frac{(\mu^{-1}f)'}{(\mu^{-1}f)},\quad \mathscr{Q}=0, \tag{2.5}$$

where $\mu$ is a scalar defined by $\mu'/2\mu=g\hat{\gamma}'/f$. We now wish to attempt to find constraints on $\hat{\gamma}'$ (or, alternatively, $\mu$).

Therefore, we turn to the purely gravitational laws of the theories under investigation. The equation of motion of test bodies in a gravitational field (i.e., the path equation) is written in terms of $\hat{\alpha}'$, $\hat{\bar{\alpha}}'$, $\hat{\beta}'$, $\hat{\gamma}'$, $\hat{\delta}'$, and $\hat{\bar{\delta}}'$. However, these six functions only occur in (1.1) in terms of the three factors $\hat{\gamma}'$, $(\hat{\alpha}'+\hat{\bar{\alpha}}'-\hat{\delta}'-\hat{\bar{\delta}}')$, and $\hat{\beta}'$, since the antisymmetric and projectively related parts of $\Gamma$ do not contribute to the path equation. We have chosen to write the path equation in terms of the six functions rather than the above three factors to highlight the fact that we are specifically concerned with theories in which an affine connection appears. Moreover, although the remaining three functions (or other three "degrees of freedom") do not occur in the path equation, they may couple to other laws of physics (e.g., they may occur in a theory of weak interactions in a gravitational field). However, only an analysis of such laws will enable us to determine whether these functions can occur in this way. Therefore, the analysis here will only yield information on $\hat{\gamma}'$, $(\hat{\alpha}'+\hat{\bar{\alpha}}'-\hat{\delta}'-\hat{\bar{\delta}}')$, and $\hat{\beta}'$. As a consequence, we note that the results of our analysis can only be applied to those laws of physics specifically under consideration.

From the above we see that we have already obtained constraints on the two factors $(\hat{\alpha}'+\hat{\bar{\alpha}}'-\hat{\delta}'-\hat{\bar{\delta}}')$ and $[\hat{\beta}'+(g/f)\hat{\gamma}']$. We shall therefore investigate the purely gravitational laws in order to obtain information on $\hat{\gamma}'$. This investigation will be based on the following: (i) The equivalence principle. (ii) Solar system tests. (iii) The principle of universality of gravitational red-shift,[1] and clock measurements in general. In the next subsection we shall discuss experiments in the solar system; we will return to this analysis in Sec. II D.

### C. Solar system experiments

We shall now consider the experiments performed in the solar system that test the nature of the gravitational field. Solar system experiments (other than Eötvös experiments, which were dealt with in Ref. 3 and the results of which were outlined above) can be essentially divided into two classes: those that measure test particle motion (such as the Newtonian limit, perihelion shifts, and light deflection), and those that measure the effects of gravity on clocks (such as the gravitational red-shift and the time-delay in radar propagation). We note that this division is not consciously made when dealing with MTG's, where all experiments are regarded as measuring the same thing (i.e., the metric). In particular, we note that in nonmetric theories of gravity (such as, for example, MATG's) the light-deflection experiment and the time-delay experiment measure two different manifestations of the gravitation field. All the observational results to be quoted here are described in Refs. 1 and 2, within which the references to the actual experiments can be found. the methods used to calculate the experimental constraints are, in the main, standard, and can be found in Ref. 2 and many standard textbooks.

#### 1. The Newtonian limit

The motion of test bodies is governed by Eq. (1.1). We can rewrite this equation in terms of the coordinate time $t$. In the solar system gravity is weak and the typical velocity $v$ of a solar body is small; consequently we can perform a weak-field—low-velocity expansion of the equation of motion, where formally $U\sim v^2\sim\epsilon^2$ ($\epsilon$ is a parameter that qualifies the expansion scheme). Assuming that the gravitational field of the Sun is SSS, we can use (1.3) and expand all functions of the gravitational field in powers of $U=M_s/r$ according to (1.10), so that to second order in $U$ (i.e., $\epsilon^4$) the equation of motion becomes

$$\begin{aligned}\vec{a}&=\frac{d^2\vec{r}}{dt^2}\\&=(\tfrac{1}{2}f_1+\gamma_1)\frac{M_S}{r^3}\vec{r}+(f_2-\tfrac{1}{2}g_1f_1+2\gamma_2)\frac{M_S{}^2}{r^4}\vec{r}\\&\quad+(g_1+\alpha_1+\bar{\alpha}_1-f_1-\delta_1-\bar{\delta}_1)\frac{M_S}{r^3}(\vec{r}\cdot\vec{v})\vec{v}\\&\quad+(\beta_1-\tfrac{1}{2}g_1)\frac{M_S}{r^3}(\vec{v}^2)\vec{r}\ ,\end{aligned}\tag{2.6}$$

where $M_S$ is the mass of the Sun, $\vec{v}=d\vec{r}/dt$, and $\vec{v}^2=(\vec{v}\cdot\vec{v})$.

The Newtonian limit demands that to first order in $\epsilon^2$ the equation of motion takes on its Newtonian form, $d^2\vec{r}/dt^2=\vec{\nabla}U$; consequently

$$\tfrac{1}{2}f_1+\gamma_1=-1\ . \tag{2.7}$$

### 2. *Perihelion shifts*

Assuming that the first-order term in (2.6) takes on its Newtonian form, the first-order approximation gives rise to the Keplerian solutions that state that test particles move on ellipses around the Sun, with the Sun as a focal point. The second-order terms in (2.6) cause a perihelion shift of these orbits. We calculate this shift by regarding the second-order terms as a perturbation on the initial-order Keplerian orbits.

Working in a spherical polar coordinate system $(t,r,\theta,\phi)$, we find that the motion is confined to a plane of constant $\theta$, which we choose as $\theta=\pi/2$. The rate of change of $\omega$ (the angle of perihelion relative to the equinox) is given by

$$\frac{d\omega}{dt}=-\frac{p}{he}a_r\cos\phi+\frac{(p+r)}{he}a_\phi\sin\phi\ , \tag{2.8}$$

where $e$, $p$, and $h$ are, respectively, the eccentricity, semilatus rectum, and angular momenta per unit mass of the (Keplerian) orbit; $a_r$ denotes the $r$th component of (2.6), and the coordinate $\phi$ is the angle of the test body measured from the perihelion.

Using (2.6) to calculate $a_r$ and $a_\phi$ to second order and the Keplerian initial-order values of $r$, $v_r$, and $v_\phi$, we can establish (2.8) to the lowest order of approximation. We can then integrate to obtain the change $\delta\omega$ in $\omega$ in one orbit, which is given by

$$\delta\omega=\Delta\frac{\pi M_S}{p}\ . \tag{2.9}$$

Observations of the perihelion shift of Mercury yield an experimental limit on $\delta\omega$ to be $6\pi M_S/p_{\text{Mercury}}$ (or 43 sec of arc per century) to an accuracy of 1%. Consequently, we have that

$$|6-\Delta|\equiv|6+f_2-\tfrac{1}{2}g_1f_1+2\gamma_2+2f_1+2\delta_1+2\bar{\delta}_1-3g_1-2\alpha_1-2\bar{\alpha}_1+2\beta_1|\le 0.06\ . \tag{2.10}$$

### 3. *Light deflection*

We can calculate the deflection of electromagnetic waves in the presence of the gravitational field of the Sun if we assume that photons are governed by the mass→0, speed→1 limit of Eq. (1.1). We do this by considering the effect as an initial order perturbation (in $U$) on the straight-line motion defined by

$$\begin{aligned}&\frac{dx^a}{d\lambda}=(1,1,0,0)\ ,\\&\eta_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda}=0\ ,\end{aligned}\tag{2.11}$$

that is, we look for a solution of the form $(1,1+O(U),dx^y/d\lambda,0)$ to Eq. (1.1).

Using (2.11), to first order Eq. (1.1) becomes (where we assume $dx^a/d\lambda\sim 1$ for lightlike particles)

$$\begin{aligned}\frac{d^2y}{dx^2}&\simeq\frac{dx^y}{d\lambda}=-\Gamma^y{}_{00}-\Gamma^y{}_{xx}\\&=(\tfrac{1}{2}f_1+\gamma_1-\tfrac{1}{2}g_1+\beta_1)\frac{M_Sy}{r^3}\\&\simeq(\tfrac{1}{2}f_1+\gamma_1-\tfrac{1}{2}g_1+\beta_1)\\&\quad\times\frac{M_Sb}{(x^2+b^2)^{3/2}}\ ,\end{aligned}\tag{2.12}$$

where $b$ is the minimum distance between the electromagnetic wave and the Sun during its motion. Equation (2.12) integrates to

$$\begin{aligned}y&=\frac{M_S}{b}(\tfrac{1}{2}f_1+\gamma_1-\tfrac{1}{2}g_1+\beta_1)\\&\quad\times[(x^2+b^2)^{1/2}+x]+b\ ,\end{aligned}\tag{2.13}$$

where we have used the boundary conditions that $y=b$ and $dy/dx=0$ as $x \to -\infty$.

The angle of deflection $\theta$ is consequently given by

$$\theta \sim \tan\theta \sim \left.\frac{y}{x}\right|_{x\to\infty} = \frac{2M_S}{b}\left(\tfrac{1}{2}f_1+\gamma_1-\tfrac{1}{2}g_1+\beta_1\right). \tag{2.14}$$

The most recent observations yield the following experimental constraint:

$$|2+\tfrac{1}{2}f_1+\gamma_1-\tfrac{1}{2}g_1+\beta_1| \lesssim 0.02 \ . \tag{2.15}$$

#### 4. *Gravitational red-shift*

Let us suppose that the metric plays a physical role in the measuring process. Indeed, let us assume that the time measured by ideal clocks is proper time, defined by

$$d\tau^2 = g_{ij}dx^i dx^j \ . \tag{2.16}$$

If we consider an electromagnetic wave emitted in a static gravitational field which travels from an emitter to a receiver, both at rest with respect to the coordinate system, the wave undergoes a frequency shift according to

$$\frac{\lambda_{\rm rec}}{\lambda_{\rm em}} = \frac{d\tau_{\rm rec}}{d\tau_{\rm em}} = \frac{(g_{00})^{1/2}_{\rm rec}}{(g_{00})^{1/2}_{\rm em}} \ . \tag{2.17}$$

Defining the gravitational red-shift by $z=(\lambda_{\rm rec}-\lambda_{\rm em})/\lambda_{\rm em}$, and expanding $g_{00}$ in powers of $U$, we find that to initial order in $U$

$$z = \tfrac{1}{2}f_1(U_{\rm rec}-U_{\rm em}) \ . \tag{2.18}$$

Experiments that measure the gravitational red-shift effect put the following experimental constraint on $f_1$:

$$|f_1+2| \lesssim 0.02 \ . \tag{2.19}$$

#### 5. *Time delay in radar propagation*

We shall calculate the round-trip travel time, as measured by a clock on Earth, of a radio wave sent out from Earth and reflected back by a reflector elsewhere in the solar system. We choose a coordinate system so that the transmitter, reflector (both of which are assumed to be at rest), the Sun and the path of the beam are in the same plane ($z=0$), and the path of the beam is along the $x$ coordinate direction with $y=b$.

For the null ray, $g_{ij}dx^i dx^j=0$, which becomes in the SSS gravitational field of the Sun

$$[1+f_1U+O(U^2)]dt^2 + [-1-g_1U+O(U^2)]dx^2 = 0 \ . \tag{2.20}$$

Consequently, the lapse of coordinate time for the round trip from transmitter to reflector is given by (to first order in $U$)

$$t_{TR} = 2\int_{t_T}^{t_R} dt = 2\int_{-a_T}^{a_R}\left[1+\tfrac{1}{2}(g_1-f_1)\frac{M_S}{(x^2+b^2)^{1/2}}\right]dx \ , \tag{2.21}$$

where $a_R$ refers to the $x$ coordinate of the reflector in the coordinate system centered on the Sun.

The lapse of proper time as measured by an Earth-based clock is given by $\Delta\tau = t_{TR}\{|g_{00}|^{1/2}{}_{\text{at Earth}}\}$, which becomes after integrating (2.21),

$$\Delta\tau = 2(a_T+a_R)\left[1+\tfrac{1}{2}f_1\frac{M_S}{(a_T{}^2+b^2)^{1/2}}\right] + (g_1-f_1)M_S \ln\left\{\frac{[a_R+(a_R{}^2+b^2)^{1/2}][a_T+(a_T{}^2+b^2)^{1/2}]}{b^2}\right\} \ . \tag{2.22}$$

Experiments that measure the time-delay effect therefore put an observational limit on the factor $(g_1-f_1)$, viz.,

$$|g_1-f_1-4| \lesssim 0.2 \ . \tag{2.23}$$

### D. The purely gravitational laws

In Sec. II B we found severe constraints on the form of the theories of gravity under investigation. We shall now attempt to restrict the form of these theories further by appealing to the purely gravitational laws (of the theories) as well as the laws of GGEM. In particular, we are looking to restrict the form of $\hat{\gamma}'$.

First let us ask whether the theories under investigation satisfy the equivalence principle. This is best done by quoting some results from Ref. 4. From the form of $\mathscr{P}$ and $\mathscr{Q}$ in (2.5) we find that $L$ is of the form $L=(\mu^{-1}f-\mu^{-1}g\vec{v}^{\,2})^{1/2}$. $L$ occurs in Eq. (1.5) through the terms $(1/f)L$ and $(1/g)L$, which can now be written as $[1/(\mu f)](\mu f-\mu g\vec{v}^{\,2})^{1/2}$ and

$[1/(\mu g)](\mu f - \mu g \vec{v}^{\,2})^{1/2}$. From this result, and the form of the other terms in (2.5), we find that relations (2.5) are precisely those for which the laws of GGEM take on *a* "metric" form with respect to $\mu g_{ab}$. Therefore, the analysis in Sec. II B demands that the laws of GGEM must take on a metric form with respect to a tensor conformally related to $g_{ab}$ (it does not prove however that the laws must be metric with respect to the physically important $g_{ab}$). This result is useful here, since we can now see quite clearly that unless $\mu$ is a constant (i.e., $\hat{\gamma}'=0$), the equivalence principle is indeed violated. (Moreover, a nonzero $\hat{\gamma}'$ leads to a practical breakdown in the equivalence principle in that energy is no longer conserved.[3]) Parenthetically we remark that we have therefore shown that for the theories of gravity under investigation the equivalence principle is satisfied if and only if the theories are metric.

In Sec. II B we considered the GGL equations in their general form. For any particular theory of gravity the GGL equations would be given; that is, $L$, and therefore $\mathscr{P}$ and $\mathscr{Q}$, would be specified. We could then regard (2.4) as a constraint on the form of $\Gamma$. In some cases this constraint can be shown to prove that $\hat{\gamma}'$ is zero. For example, let us consider two examples of possible GGL equations (see Ref. 4). In the first $L = d\tau/dt$, so that $\mathscr{P} = \frac{1}{2} f'/f$ and $\mathscr{Q}=0$, and (2.4) consequently yields

$$\hat{\gamma}'_{(a)}=0 \ . \tag{2.24a}$$

In the second $L = d\lambda/dt$, so that

$$\mathscr{P} = \hat{\delta}' + \hat{\hat{\delta}}' - \frac{g\hat{\gamma}'}{f} + \frac{1}{2}\frac{f'}{f}$$

and

$$\mathscr{Q} = (\hat{\delta}' + \hat{\hat{\delta}}' - \hat{\alpha}' - \hat{\hat{\alpha}}') - \left[\hat{\beta}' + \frac{g}{f}\hat{\gamma}'\right] ,$$

and (2.4) yields

$$\hat{\delta}'_{(b)} + \hat{\hat{\delta}}'_{(b)} = 0 \ . \tag{2.24b}$$

Therefore, if the GGL equations of any particular theory are of the above form we can apply the constraints represented by (2.24).

In the last subsection we described solar system tests, which give experimental values to the functions appearing in the analysis. These experimental constraints support the theoretical constraints outlined in Sec. II B. In addition, the experiments give information on the possible form of the function $\hat{\gamma}'$. This information is obtained, in part, from experiments concerned with clock measurements in the gravitational field of the Sun. Indeed, we observe from (2.7) and (2.19) that observations demand that $\gamma_1 = 0$. This result supports the assertion that $\hat{\gamma}'$ is zero. Unfortunately, we need second-order gravitational red-shift experiments in order to determine $\gamma_2$.

We conclude that the above analysis indicates that $\hat{\gamma}'$ must be zero. Consequently, we have found that MATG's must reduce to their metric form in an SSS gravitational field if the theoretical and experimental conditions outlined in this section are to be satisfied.

## III. CLASSIFICATION AND ANALYSIS OF THEORIES IN THE LITERATURE

In this section we shall consider several examples of MATG's in the literature. First we shall classify the theories, and then investigate them using the analysis described in Sec. II.

### A. Weyl-affine theories

In this class of theories we have that

$$\Gamma^a{}_{bc} = \{ {}_b{}^a{}_c \} + \psi^a{}_{bc} \ , \tag{3.1}$$

where

$$\psi^a{}_{bc} = g^{ad}(g_{bd}\phi_c + g_{cd}\phi_b - g_{bc}\phi_d) \ , \tag{3.2}$$

and $\phi_a$ is a vector field (usually constructed from a scalar field by $\phi_a = \phi_{,a}$). Using (3.1) and (3.2) we see that

$$g_{ab\,|\,c} = -2g_{ab}\phi_c \neq 0, \quad \Gamma^a{}_{[bc]} = 0 \ . \tag{3.3}$$

In an SSS gravitational field $\psi^a{}_{bc}$ takes on the following form:

$$\hat{\alpha}_{,\mu} = \hat{\hat{\alpha}}_{,\mu} = -\hat{\beta}_{,\mu} = \frac{g}{f}\hat{\gamma}_{,\mu} = \hat{\delta}_{,\mu} = \hat{\hat{\delta}}_{,\mu} = \phi_{,\mu} \ , \tag{3.4}$$

where $\phi$ is a scalar field. In a weak field we can expand $\phi$ according to

$$\phi = \phi_1 U + \phi_2 U^2 + \cdots \ . \tag{3.5}$$

In Weyl-affine theories of gravity the motion of test particles in a gravitational field is governed by the path equation (1.1), for the connection defined by (3.1). In the six theories of gravity outlined below, the above construction has been used in an attempt to produce a theory that is invariant under an arbitrary change of units.

If we apply the analysis outlined in Sec. II to Weyl-affine theories we obtain the following results:

*(A) Theoretical:* Weyl-affine theories violate the equivalence principle. The laws of GGEM for Weyl-affine theories have been written down in a general form (i.e., $\mathscr{A}$, $\mathscr{B}$, $\mathscr{P}$, $\mathscr{Q}$, etc., have not been specified). For any particular theory the laws must be written down explicitly; we can then examine whether the given theory satisfies the WEP. In particular, if the GGL equations are of the form

represented by (2.24) the Weyl-affine theory in question would violate the WEP.

*(B) Experimental:* Solar system experiments demand that $\phi_1=0$ [see (3.4), (3.5), (2.7), and (2.19)]. If the GGL equations are of the form (2.5), Eötvös experiments demand that $\phi$ is zero to first and second orders in $U$. Consequently, the $\phi$ field can play no observable part in the solar system.

Many examples of this type of theory can be found in the literature, some of which we will briefly describe here. (i) Ross's theory[7]: $\phi_a$ is constructed from a scalar field. The field equations in Ross's theory, which are given in Ref. 7, are incomplete. The laws of GGEM are not specified. (ii) Lord's theory[8]: In this theory there is a scalar field $\sigma$ as well as the vector field $\phi_a$. The field equations are complete (but horrendously complicated). The GGM equations are given, in which

$$\mathscr{A}=-\mathscr{B}=\frac{1}{2}\frac{f'}{f}-\frac{1}{2}\frac{g'}{g}\ .$$

The GGL equations are not specified. (iii) Omote's theory[9]: This theory is a special case of Lord's theory above. (iv) Rothwell's theory[10]: $\phi_a$ is constructed from a scalar field. Neither are the field equations complete, nor the laws of GGEM specified. (v) Cohn's theory[11]: $\phi_a$ is constructed from a scalar field. The field equations are complete. The laws of GGEM, however, are not specified. (vi) Modified Brans-Dicke theory[6]: $\phi_a$ is constructed from a scalar field. The field equations are complete, but the laws of GGEM are not given. In this theory $T^{ab}{}_{;b}=0$, which leads to an internal inconsistency within the theory.[6]

Since the laws of GGEM are not given in any of the above, none of these theories are complete. In addition, in two of the theories above the field equations are not complete either. In all the cases in which the field equations are complete we find that $\phi \sim O(U)$ (i.e., $\phi_1 \neq 0$), which contradicts the experimental evidence outlined above. We conclude that none of these theories are viable.

### B. Weyl-Dirac theories[12]

These are the theories that attempt to identify the $\phi_a$ field in the Weyl-affine theories with the electromagnetic potential $A_a$, and thus unify gravity and electromagnetism. As noted by Ross,[13] these theories imply that electromagnetic field couples to all other fields, regardless of whether they are charged or not. We shall not consider these theories any further.

### C. Nonconservation theories

These are theories in which $T^{ab}{}_{;b}$ is nonzero. Smalley's theory[14] is an example of such a theory in which $T^{ab}{}_{;b}=\sigma R^{,a}$, where $R$ is the curvature scalar and $\sigma$ an arbitrary parameter. In this theory the motion of test particles is governed by the geodesic equation (therefore the theory is an MATG); this leads to an inconsistency in Smalley's theory if $\sigma$ is nonzero.[6]

### D. Torsion theories

In this class of theories we have that

$$\Gamma^a{}_{bc}=\{{}_b{}^a{}_c\}+S^a{}_{bc}-S_c{}^a{}_b+S_{bc}{}^a\ , \tag{3.6}$$

where $S^a{}_{bc}=\Gamma^a{}_{[bc]}$ is the torsion. From (3.6) we see that

$$g_{ab\,|\,c}=0\ . \tag{3.7}$$

In an SSS gravitational field $\Gamma^a{}_{bc}$ takes on the following form:

$$\hat{\delta}_{,\mu}=\frac{g}{f}\hat{\gamma}_{,\mu}=\lambda_{,\mu},\ \ \hat{\alpha}_{,\mu}=-\hat{\beta}_{,\mu}=\xi_{,\mu},\ \ \hat{\delta}_{,\mu}=\hat{\alpha}_{,\mu}=0\ , \tag{3.8}$$

where $\lambda$ and $\xi$ are scalar fields. In a weak field we expand $\lambda$ (for example) by

$$\lambda=\lambda_1 U+\lambda_2 U^2+\cdots\ . \tag{3.9}$$

In torsion theories of gravity the motion of test particles in a gravitational field is governed by the path equation for the connection given by (3.6). If we apply the analysis outlined in Sec. II to torsion theories, we obtain the following results.

*(A) Theoretical:* If $\lambda$ and $\xi$ in (3.8) are not identical, then the theory would not satisfy the WEP, nor would the equation governing the motion of photons calculated from the optical limit of the GGM equations be equivalent to the mass $\rightarrow 0$, speed $\rightarrow 1$ limit of the path equation. Let us assume that $\lambda=\xi\neq 0$, then torsion theories violate the equivalence principle, and if the GGL equations are of the form represented by (2.24), they also violate the WEP.

*(B) Experimental:* Experiments prove that $\lambda$ and $\xi$ must be equivalent. This equivalence is inferred to first order from the light-deflection experiment (i.e., $\lambda_1=\xi_1$), and to first and second orders from Eötvös experiments. Solar system experiments then demand that $\lambda_1=0$ [see (3.8), (3.9), (2.7), and (2.19)]. Depending on the precise form of the GGL equations for a particular torsion theory, Eötvös experiments may also measure $\lambda$ to first and second orders. We conclude from the observational evidence that $\lambda$ is unobservable in the solar system.

We shall now briefly describe three examples of torsion theories in the literature. (i) Dunn's theory[15]: In this theory the torsion is defined by $S^a{}_{bc}=\frac{1}{2}(\delta^a_c\lambda_{,b}-\delta^a_b\lambda_{,c})$ where $\lambda$ is a scalar field.

The field equations are incomplete and the laws of GGEM are not formulated. (ii) Halford's theory[16]: In this theory the torsion is defined by $S^a{}_{bc}=\frac{1}{2}(\delta^a_c\lambda_b-\delta^a_b\lambda_c)$, where $\lambda_a$ is a vector field, assumed to be cosmological in origin (and related to the cosmological constant). The field equations are incomplete and the laws of GGEM are not specified. (iii) $U_4$ theory: Here we mention the first version of the theory formulated by Hehl *et al.*,[17] in which the torsion is assumed to be related to spin angular momentum. Since the equations of motion in the theory are obtained by integrating the conservation laws, and $U_4$ is not really a "classical" theory of gravity but a theory that attempts to unify the quantum phenomenon of spin angular momentum and gravity in a consistent way, it is not clear whether the analysis can be applied to the $U_4$ in a meaningful way. (We note that in the solar system $U_4$ essentially reduces to general relativity.)

### E. Theories with both torsion and nonmetricity nonzero

A second version of $U_4$ (Ref. 18) has been developed with nonzero nonmetricity, in which $g_{ab\,|\,c}$ is assumed to be related to "hypermomentum."

A second example of this type of theory is provided by Sen and Dunn.[19,20] Their theory incorporates an arbitrary scalar field $\psi$ and is based on a generalized Riemannian geometry called a Lyra geometry. The authors attempt to produce a theory which is invariant under arbitrary units transformations.

In this theory the motion of test particles is governed by the path equation for the connection defined by

$$\Gamma^a{}_{bc}=\left\{{a \atop bc}\right\}+q\psi_{,b}\delta^a_c-\tfrac{1}{2}pg_{bc}g^{ad}\psi_{,d}\ , \qquad (3.10)$$

where the constants $p$ and $q$ may take on different values. Sen and Dunn suggest that either (i) $p=q=1$, so that test particles follow the geometric curves in the Lyra geometry called "extremals," or (ii) $p=-\frac{1}{2}, q=\frac{1}{4}$, so that test particles follow the geometric curves in the Lyra geometry called "autoparallels." Let us consider the two cases as two different theories.

In case (i) we find that $\Gamma$ takes on the following form in an SSS gravitational field:

$$(\hat{\alpha}+\hat{\bar{\alpha}})_{,\mu}=(\hat{\delta}+\hat{\bar{\delta}})_{,\mu}=-2\hat{\beta}_{,\mu}=2\frac{g}{f}\hat{\gamma}_{,\mu}=\psi_{,\mu}\ . \qquad (3.11)$$

Depending on how the antisymmetric part of $\Gamma$ is defined, this case would either be a theory of the type described in Sec. III (A) or Sec. III (D).

In case (ii) we find that $\Gamma$ takes on the following form in an SSS gravitational field:

$$(\hat{\alpha}+\hat{\bar{\alpha}})_{,\mu}=(\hat{\delta}+\hat{\bar{\delta}})_{,\mu}=\hat{\beta}_{,\mu}=-\frac{g}{f}\hat{\gamma}_{,\mu}=\frac{1}{4}\psi_{,\mu}\ . \qquad (3.12)$$

This theory is different from all the others above in that $g_{ab\,|\,c}\neq 0$ and (depending on how the antisymmetric part of the connection is defined) possibly $S^a{}_{bc}\neq 0$.

Finally, we note that although the field equations are given in the theories of Sen and Dunn, the laws of GGEM are not; the theories are therefore incomplete.

We conclude by noting that an additional consequence of the analysis outlined in this paper is that none of the theories listed here are viable.


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[1] C. M. Will, in *General Relativity: An Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel (Cambridge University Press, New York, 1979).

[2] A. A. Coley, Ph.D. Thesis, University of London, 1980 (unpublished).

[3] A. A. Coley, Phys. Rev. D 27, 728 (1983).

[4] A. A. Coley, preceding paper, Phys. Rev. D 28, 1829 (1983).

[5] J. A. Schouten, *Ricci Calculus,* 2nd ed. (Springer, Berlin, 1954).

[6] A. A. Coley, Nuovo Cimento 69B, 89 (1982).

[7] D. K. Ross, Phys. Rev. D 5, 284 (1972).

[8] E. A. Lord, Nuovo Cimento 11B, 185 (1972).

[9] E. Omote, Lett. Nuovo Cimento 2, 58 (1971).

[10] P. L. Rothwell, Nuovo Cimento 30B, 267 (1975).

[11] J. Cohn, Gen. Relativ. Gravit. 6, 143 (1974).

[12] P. A. M. Dirac, Proc. R. Soc. London A333, 403 (1973). (1973).

[13] D. K. Ross, Nuovo Cimento 33B, 449 (1976).

[14] L. L. Smalley, Phys. Rev. D 9, 1635 (1974).

[15] K. A. Dunn, J. Math. Phys. 15, 229 (1974).

[16] W. D. Halford, Austr. J. Phys. 23, 863 (1970).

[17] F. W. Hehl, P. Von der Heyde, G. Kerlick, and J. M. Nester, Rev. Mod. Phys. 48, 393 (1976).

[18] F. W. Hehl *et al.*, Phys. Lett. 63B, 446 (1976).

[19] D. K. Sen and K. A. Dunn, J. Math. Phys. 12, 578 (1971).

[20] J. S. Jeavons, C. B. G. McIntosh, and D. K. Sen, J. Math. Phys. 16, 320 (1975).