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Anisotropic fluid spherically symmetric space–times admitting a kinematic self-similarity

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Anisotropic fluid spherically symmetric space–times admitting a kinematic self-similar vector are investigated. The geodesic case is considered, and some special subcases in which the anisotropic fluid satisfies additional physical conditions are investigated in detail. A number of other special cases are studied. Particular attention is focused on the possible asymptotic behavior of the models, and it is shown that the models considered always asymptote towards an exact homothetic solution, which is in general either a perfect fluid model or a static solution. © *1999 American Institute of Physics.* [S0022-2488(99)01602-3]

I. INTRODUCTION

In a recent paper spherically symmetric space–times which admit a kinematic self-similarity of the second (or zeroth) kind were studied when the source of the gravitational field was assumed to be a perfect fluid.[1] In that paper several particular subclasses of models were studied in depth, including the subcases ''$M_1=0$'' and ''$M_2=0$'' (which includes the static models as a further subcase). Note that these particular subcases refer to specific forms for the first integral *m(r,t)* of the EFEs. The precise definitions of these subcases in terms of *m(r,t)* are not necessary here; see Benoit and Coley[1] for more details. These subclasses of models, in which exact solutions were obtained, were found to be of particular interest since their qualitative properties were representative of the asymptotic behavior of more general models.

The metric, in comoving coordinates, is given by

$$ds^2=-e^{2\phi}dt^2+e^{2\psi}dr^2+r^2S^2d\Omega^2, \tag{1.1}$$

where the functions ϕ, ψ and S and depend only on the self-similarity coordinate

$$\xi=r(\alpha t)^{-1/\alpha}, \tag{1.2}$$

where α is the self-similar index (and we shall assume henceforward that $\alpha\neq 0$). The kinematic self-similar generator is given by[2]

$$\boldsymbol{\xi}=\xi^a\frac{\partial}{\partial x^a}=\alpha t\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}. \tag{1.3}$$

It follows from (1.1) and (1.3) that

$$\mathcal{L}_{\boldsymbol{\xi}}h_{ab}=2h_{ab}, \tag{1.4}$$

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and

$$\mathcal{L}_{\xi} u_a = \alpha u_a , \tag{1.5}$$

where $h_{ab} = g_{ab} + u_a u_b$ is the projection tensor. Hence $\boldsymbol{\xi}$, as given by (1.3) with $\alpha \neq 1$, is a *kinematic self-similar* vector and corresponds to the natural relativistic counterpart of self-similarity of the more general second kind.[3] We note that in the case $\alpha = 1$, $\boldsymbol{\xi}$ is a homothetic vector, corresponding to self-similarity of the first kind. We shall adopt the notation and terminology of Benoit and Coley[1] and for brevity we shall not repeat here the motivation or the discussion of self-similarity given in that paper.

In this paper we shall generalize the perfect fluid solutions in Benoit and Coley[4] to the case of an anisotropic fluid, in which stress-energy tensor is given by

$$T_{ab} = \mu u_a u_b + p_{\parallel} n_a n_b + p_{\perp} (g_{ab} + u_a u_b - n_a n_b), \tag{1.6}$$

where u^a is the comoving fluid velocity vector and n^a is a unit spacelike vector orthogonal to u^a (i.e., $u_a n^a = 0$). The stress-energy tensor (1.6) possesses and eigenvalue degeneracy (and hence is not the most general anisotropic fluid stress-energy tensor) consistent with the assumption of spherical symmetry (see metric (1.1)). For the metric (1.1), $\mathbf{n}$ is given by

$$\mathbf{n} = n^a \frac{\partial}{\partial x^a} = e^{-\psi} \frac{\partial}{\partial r}. \tag{1.7}$$

Using Eqs. (1.1)–(1.3), it therefore follows immediately that

$$\mathcal{L}_{\xi} n_a = n_a \tag{1.8}$$

is satisfied identically, so that the form for $\mathbf{n}$ is consistent with the similarity assumption. The scalars $p_{\parallel}$ and $p_{\perp}$ are the pressures parallel to and perpendicular to n^a, respectively, and μ is the energy-density. The perfect fluid case corresponds to the case in which $p_{\parallel} = p_{\perp}$.

Fluids with an anisotropic pressure have been studied for many reasons (see the discussion in Coley and Tupper).[5] For example, in several cases in which the stress-energy tensor is more general than that of a perfect fluid (due to, e.g., a two perfect fluid source, an imperfect fluid source or in the region of interaction of two colliding plane impulsive gravitation waves), the energy-momentum tensor is formally of the form (1.6). In particular, a strong magnetic field in a plasma in which the particle collision density is low can cause the pressure along and perpendicular to the magnetic field lines to be unequal.[6] If the source of the gravitational field can be represented by the sum of a perfect fluid and a local magnetic field $H^a = H n^a$ (as measured by u^a), then the stress-energy tensor can be written in the form (1.6) with

$$\mu = \overline{\mu} + \pi, \quad p_{\parallel} = \overline{p} - \pi, \quad p_{\perp} = \overline{p} + \pi, \tag{1.9}$$

where $\pi = \frac{1}{2} \lambda H^2$ and λ is the magnetic permeability. Other possible sources of anisotropic stresses, in addition to cosmological magnetic and electric fields, include, for example, populations of collisionless particles like gravitons,[7] photons[8] or relativistic neutrinos,[9] Yang–Mills fields,[10] axion fields in low-energy string theory,[11] long wavelength gravitational waves,[12] and topological defects like global monopoles, cosmic strings, and domain walls.[13–15]

Most anisotropic models that have been studied are also spherically symmetric (see references cited in Ref. 5), and have applications especially in relativistic astrophysics (e.g., stellar models); in particular, static anisotropic spheres have received much attention.[5] In addition, such models with additional symmetries, including homothetic vectors and conformal Killing vectors, have also been studied (see Refs. 6, 5, and references within).

For the metric (1.1) the Einstein field equations (EFEs) yield the following expressions for the physical variables:

$$\mu=\frac{W^1(x)}{r^2}+\frac{W^2(x)}{t^2},$$

$$p_\parallel=\frac{P_\parallel^1(x)}{r^2}+\frac{P_\parallel^2(x)}{t^2}, \tag{1.10}$$

$$p_\perp=\frac{P_\perp^1(x)}{r^2}+\frac{P_\perp^2(x)}{t^2},$$

where

$$W^1(x)=\frac{1}{S^2}-e^{-2\psi}[(1+y)^2+2y\dot{\phi}],$$

$$W^2(x)=\frac{e^{-2\phi}}{\alpha^2}y[y+2\dot{\psi}],$$

$$P_\parallel^1(x)=-\frac{1}{S^2}+e^{-2\psi}(1+y)[1+y+2\dot{\phi}],$$

$$P_\parallel^2(x)=-\frac{e^{-2\phi}}{\alpha^2}[2\dot{y}+2\alpha y+3y^2-2y\dot{\phi}],$$

$$P_\perp^1(x)=e^{-2\psi}[2y\dot{\phi}+\dot{\phi}^2+\ddot{\phi}-\dot{\phi}\dot{\psi}],$$

$$P_\perp^2(x)=-\frac{e^{-2\phi}}{\alpha^2}[(\alpha-1)y+2y\dot{\psi}+\dot{\psi}+\alpha\dot{\psi}+\dot{\psi}^2+\ddot{\psi}-\dot{\phi}\dot{\psi}], \tag{1.11}$$

and where $y\equiv\dot{S}/S$, $x\equiv\ln\xi$ and $\dot{f}=df/dx$. The final EFE (that ensures that the Einstein tensor is diagonal) becomes

$$\dot{y}=y\dot{\phi}+(\dot{\psi}-y)(1+y). \tag{1.12}$$

Clearly there exists a variety of anisotropic fluid spherically symmetric kinematic self-similar space–times satisfying Eqs. (1.10)-(1.12).

If we assume that the physical quantities also obey similarity conditions of the form

$$\mathcal{L}_{\xi}\mu=a\mu,\quad \mathcal{L}_{\xi}p_\parallel=b_\parallel p_\parallel,\quad \mathcal{L}_{\xi}p_\perp=b_\perp p_\perp, \tag{1.13}$$

where a, $b_\parallel$ and $b_\perp$ are constants, then it can easily be shown that:

$$\text{(i)}\quad W^1=0\quad \textit{or}\quad W^2=0$$

and

$$\text{(ii)}\quad P_\parallel^1=0\quad \textit{or}\quad P_\parallel^2=0$$

and

$$\text{(iii)}\quad P_\perp^1=0\quad \textit{or}\quad P_\perp^2=0.$$

The special subcases $W^i=0$ with either $P_\parallel^i\neq 0$ or $P_\perp^i\neq 0$ $(i=1,2)$ are not of physical interest. The special subcase $W^1=P_\parallel^1=P_\perp^1=0$ corresponds to the special subcase "$M_1=0$." Finally, the special subcase $W^2=P_\parallel^2=P_\perp^2=0$ is related to the special subcase "$M_2=0$," and the static models are included within this subclass of models.

It turns out (Benoit and Coley,[1] in particular, see the Appendix therein) that all *static* spherically symmetric kinematic self-similar solutions belong to the subclass "$M_2=0$," regardless of the form of the stress-energy tensor, and, moreover, that all such static spacetimes necessarily admit a *homothetic vector*. Consequently, no new static anisotropic solutions can be obtained that admit a proper kinematic self-similarity. Hence we shall concentrate here on the special subcase "$M_1=0$."

II. GEODESIC MODELS

The geodesic case, in which the acceleration of the comoving fluid velocity vector is zero, is characterized by $\dot{\phi}=0$ and is equivalent to the special subcase "$M_1=0$" considered in Benoit and Coley.[1] In this model, Eq. (1.12) gives (for $S+\dot{S}\neq 0$)

$$e^{2\phi}=1, \quad \dot{\psi}=\frac{\dot{S}+\ddot{S}}{S+\dot{S}}=\frac{\dot{y}+y^2+y}{1+y}, \tag{2.1}$$

whence the metric (1.1) becomes

$$ds^2=-dt^2+(S+\dot{S})^2dr^2+r^2S^2d\Omega^2. \tag{2.2}$$

Assuming the first of conditions (2.1), the second condition guarantees the resulting Einstein tensor is diagonal and hence the remaining EFEs simply yield the following expressions for μ, $p_\parallel$ and $p_\perp$:

$$\mu=W(x)t^{-2}, \quad p_\parallel=P_\parallel(x)t^{-2}, \quad p_\perp=P_\perp(x)t^{-2}, \tag{2.3}$$

(where we have now omitted the index "2" for convenience), so that Eqs. (1.13) are automatically satisfied with $a=b_\parallel=b_\perp=-2\alpha$, where

$$\begin{aligned}
W(x)&\equiv\frac{y}{\alpha^2(1+y)}(3y+3y^2+2\dot{y}),\\
P_\parallel(x)&\equiv-(3y^2+2\alpha y+2\dot{y})/\alpha^2,\\
P_\perp(x)&\equiv-\frac{(1+y)(2\dot{y}+3y^2+2\alpha y)+3y\dot{y}+\alpha\dot{y}+\ddot{y}}{\alpha^2(1+y)}.
\end{aligned} \tag{2.4}$$

Equations (2.2)–(2.4) represent a class of anisotropic fluid solutions depending upon the arbitrary function $S(x)$.

We note that the following relationships result from the definitions given in Eqs. (2.3):

$$P_\perp=P_\parallel+\frac{\dot{P}_\parallel}{2(1+y)},$$

$$W=\frac{-y[(2\alpha-3)y+\alpha^2P_\parallel]}{\alpha^2(1+y)}.$$

A. Perfect fluid models

In the perfect fluid case we have that $P_{\parallel}=P_{\perp}$, and hence from Eqs. (2.4) we obtain the following differential equation for the function $y(x)$ [and hence $S(x)$] in the metric (2.2):

$$2\dot{y}+3y^2+2\alpha y+\alpha^2 p_0=0. \tag{2.5}$$

In Eq. (2.5) p_0 is an arbitrary integration constant. In the perfect fluid case μ is obtained from Eqs. (2.3) and (2.4) and we have that

$$p=p_0 t^{-2}, \tag{2.6}$$

and hence the significance of p_0 is that it constitutes a dimensional constant (appearing in the pressure) characterizing the physical problem; this property is characteristic of self-similarity of the second kind.[3] It can be shown that these perfect fluid solutions (for $\alpha\neq 1$) cannot, in general, admit any homothetic vectors.[4]

The perfect fluid solutions were studied in detail in Benoit and Coley;[1] in fact, exact solutions were obtained and the qualitative properties of the whole class of models were studied. In particular, in the pressure-free case we obtain the exact dust solution of the Tolman family studied by Lynden-Bell and Lemos[16] and Carter and Henriksen,[2] and we found that all solutions are asymptotic to exact, power-law (flat) FRW models (which admit a homothety).

B. Solutions with $S+\dot{S}=0$

In Benoit and Coley (1998) we showed that the case $S+\dot{S}=0$, which implies that $S=s_0 e^{-x}$, could be factored out of the analysis as it could not lead to a perfect fluid solution. For that reason, we consider it as a special case here. (This case is not contained in the geodesic models studied above.)

When $S=s_0 e^{-x}$ (i.e., $y=-1$), the EFEs yield

$$\dot{\phi}=0, \tag{2.7}$$

whence we can choose coordinates so that $e^{2\phi}=1$, and

$$\mu=s_0^{-2}e^{2x}r^{-2}+(1-2\dot{\psi})\alpha^{-2}t^{-2}, \tag{2.8}$$

$$p_{\parallel}(x)=-s_0^2 e^{2x}r^{-2}+(2\alpha-3)\alpha^{-2}t^{-2}, \tag{2.9}$$

$$p_{\perp}(x)=-[(1-\alpha)(1-\dot{\psi})+\dot{\psi}^2+\ddot{\psi}]\alpha^{-2}t^{-2}. \tag{2.10}$$

The fluid described by these equations will further satisfy Eq. (1.13) in one of two cases. Either (i) $\alpha=1$, and the solution admits a homothetic vector, or (ii) $\dot{\psi}=1/2$, $\alpha=3/2$.

In the first case, i.e., $\alpha=1$, the solution is given by

$$ds^2=-dt^2+e^{2\psi}dr^2+s_0 t^{-2}d\Omega^2, \tag{2.11}$$

with

$$\mu=(s_0^{-2}+1-2\dot{\psi})t^{-2}, \tag{2.12}$$

$$p_{\parallel}=-(s_0^{-2}+1)t^{-2}, \tag{2.13}$$

$$p_{\perp}=-(\dot{\psi}^2+\ddot{\psi})t^{-2}, \tag{2.14}$$

where the function $\psi(x)$ is arbitrary.

In the second case the solution is given (after a coordinate redefinition) by

$$ds^2=-dt^2+t^{-2/3}dr^2+t^{4/3}d\Omega^2, \tag{2.15}$$

with

$$\mu=\mu_0 t^{-4/3}, \tag{2.16}$$

$$p_\parallel=-\mu, \tag{2.17}$$

$$p_\perp=0, \tag{2.18}$$

where μ_0 is a constant. It can be easily shown that the metric (2.15) does not admit a proper homothetic vector. Curiously, cosmic strings satisfy ''equations of state'' of the form $\mu+p_\parallel=0$, $p_\perp=0$.[17]

III. SPECIAL CASES

There are a variety of models which satisfy additional constraints. We consider here two such models.

A. Case A: Dimensional constants

If we assume that $P_\perp=p_0$, a constant, then Eqs. (2.4) yield

$$\dot{P}_\parallel(x)=2(1+y)(p_0-P_\parallel(x)). \tag{3.1}$$

This equation can be integrated to yield

$$P_\parallel(x)=p_0+ce^{-2x}S^{-2}, \tag{3.2}$$

where c is an arbitrary constant. Using this expression for $P_\parallel$, we obtain

$$W(x)=\frac{y}{\alpha^2(1+y)}[y(3-2\alpha)-\alpha^2 p_0-c\alpha^2 e^{-2x}S^{-2}], \tag{3.3}$$

and the differential equation

$$2\dot{y}+3y^2+2\alpha y=-\alpha^2 p_0-\alpha^2 ce^{-2x}S^{-2}. \tag{3.4}$$

Note that when $c=0$ (i.e., $P_\parallel=P_\perp=p_0$, corresponding to a perfect fluid) Eq. (3.4) is related to Eq. (2.56) in Benoit and Coley.[1]

If we had begun the analysis of this section with the assumption that $P_\parallel=p_0$, then Eqs. (2.4) automatically imply that $P_\parallel=P_\perp=p_0$, the perfect fluid case considered in Benoit and Coley.[1]

The pressures $p_\parallel$ and $p_\perp$ are positive if the constants p_0 and c are non-negative. The energy conditions will constrain these constants further (for a given value of α) through (3.3).

B. Case B: Equations of state

We can also consider the subclass of solutions which satisfy equations of state of the form:

$$p_\parallel=f_\parallel(\mu), \quad p_\perp=f_\perp(\mu), \tag{3.5}$$

for arbitrary functions $f_\parallel$ and $f_\perp$. From Eqs. (2.3), conditions (3.5) automatically yield

$$p_\parallel=c_\parallel\mu \quad \text{and} \quad p_\perp=c_\perp\mu, \tag{3.6}$$

where $c_\parallel$ and $c_\perp$ are constants. Substituting these conditions into the definitions (2.4) then yields

$$\mu=\mu_0 t^{-2}[Se^x]^{-2(1-c_\perp/c_\parallel)} \tag{3.7}$$

and the differential equation for y:

$$2\dot{y}+3y^2+2\alpha y=-\alpha^2 c_\parallel \mu_0 [Se^x]^{-2(1-c_\perp/c_\parallel)}. \tag{3.8}$$

Once again we note that when $c_\parallel = c_\perp$ (i.e., the perfect fluid case), we recover Eq. (2.5) as expected.

A positive value for the constant μ_0 guarantees that the energy density is positive. If $|c_\parallel| \leqslant \mu_0$ and $|c_\perp| \leqslant \mu_0$, the energy conditions are satisfied. The pressures are non-negative if $c_\parallel \geqslant 0$ and $c_\perp \geqslant 0$.

IV. ANALYSIS OF SPECIAL CASES

The behavior of each of the special cases derived in Sec. III can be studied qualitatively since each of the ordinary differential equations governing the model is autonomous.

The special cases A(dimensional constants) and B(equations of state) can be considered simultaneously using the following change of variables:

$$\nu = b[Se^x]^{-2n}, \tag{4.1}$$

where b is a non-negative constant. The resulting system is then

$$\dot{y} = -\tfrac{1}{2}(3y^2+2\alpha y+k+\nu), \tag{4.2}$$

$$\dot{\nu} = -2n\nu(1+y). \tag{4.3}$$

Using these definitions, case A is characterized by $n=1$, $k=\alpha^2 p_0$, and case B is characterized by $n=1+c_\perp/c_\parallel$ and $k=0$.

It is important to note that the invariant set $\nu=0$ of Eqs. (4.2)/(4.3) defines the perfect fluid solutions. We also note that $y=0$ represents the static solutions. Each of these cases is examined in detail in Benoit and Coley.[1]

If we consider only the case of positive pressures and positive energy density, we can impose the necessary (though not necessarily sufficient) condition that the parameters in our equations must satisfy $k \geqslant 0$, $n \geqslant 1$ and $\nu \geqslant 0$. With these restrictions, we find that there are at most three singular points at finite values. We note that $\nu=0$ is an invariant set of the system (4.2)/(4.3), as is the set $\nu>0$. As a result we need only consider the dynamics (and hence the singular points) in the half-plane $\nu \geqslant 0$.

The finite singular points (y_0, ν_0) are given by:

$$Q_1 = (\tfrac{1}{3}(-\alpha+(\alpha^2-3k)^{1/2}, 0),$$

$$Q_2 = (\tfrac{1}{3}(-\alpha-(\alpha^2-3k)^{1/2}, 0),$$

$$Q_3 = (-1, 2\alpha-3-k).$$

The nature of these singular points, which can be determined using standard techniques,[18] depends upon the relationship between the parameters α and k. The results are summarized in Table I. Note that only those singular points which are located in the physical phase space are listed in this table. It is important to note that each of the cases I–IV is possible when considering the Eqs. (4.2)/(4.3) in case A. In case B, however, we find that only the cases labeled (I) and (II) in Table I yield consistent constraints on the parameter α.

TABLE I. Summary of the nature of the finite singular points for the system (4.2)/(4.3). ''N/A'' indicates that the given point is not located in the physical region $\nu \geqslant 0$. The two cases (i) $\alpha^2=3k$, $2\alpha-3>k$ and (ii) $\alpha^2<3k$, $2\alpha-3\geqslant k$ are omitted since they do not give any real solutions for k and α.

	$\alpha^2>3k$		$\alpha^2=3k$	$\alpha^2<3k$
	$2\alpha-3\leqslant k$ I	$2\alpha-3>k$ II	$2\alpha-3\leqslant k$ III	$2\alpha-3<k$ IV
$\mathbf{Q}_1$	sink	sink	saddle-node	N/A
$\mathbf{Q}_2$	source	saddle	($\equiv \mathbf{Q}_1$)	N/A
$\mathbf{Q}_3$	saddle	N/A	N/A	N/A

We can complete the qualitative analysis of these two cases by considering the stability of the singular points at infinity. To perform the analysis at infinity, we apply the following Poincaré transformation to our system (4.2)/(4.3) in order to compactify the phase space:

$$Y=\frac{y}{(1+y^2+\nu^2)^{1/2}}, \quad V=\frac{\nu}{(1+y^2+\nu^2)^{1/2}}. \tag{4.4}$$

In these new variables, the phase space has been compactified to the region $\Theta^2=1-(Y^2+V^2)\geqslant 0$, and all infinite points of the original system are found on the boundary $\Theta=0$. The restriction that $\nu\geqslant 0$ implies that $V\geqslant 0$, and all finite singular points remain at finite values of Y and V and are of the same sign in the new coordinate system (Y,V).

The transformed Eqs. (4.1)/(4.2) are then given by:

$$Y'=\tfrac{1}{2}(4n-3)Y^2V^2+\Theta[\tfrac{1}{2}Y+(2n-\alpha)V]-\tfrac{1}{2}\Theta^2[3Y^2+kV^2]-\Theta^3[\alpha Y+\tfrac{1}{2}V]-\tfrac{1}{2}k\Theta^4, \tag{4.5}$$

$$V'=-\tfrac{1}{2}(4n-3)Y^3V+\Theta YV[\tfrac{1}{2}V-(2n-\alpha)Y]+\tfrac{1}{2}\Theta^2YV[k-4n]-2nV\Theta^3, \tag{4.6}$$

where $f'=\Theta\dot{f}$. There are four singular points at infinity located on the boundary $Y^2+V^2=1$, which are given by

$$R_{\pm}=(0,\pm 1), \quad S_{\pm}=(\pm 1,0). \tag{4.7}$$

The points $S_{\pm}$ correspond to perfect fluid solutions, and $R_{\pm}$ correspond to static solutions. A local stability analysis shows that the points $S_{\pm}$ are both saddles. R_+ is a nonhyperbolic point containing both stable and unstable manifolds for all values of α and k. The stable manifold of R_+ lies in an elliptic sector of R_+ and corresponds to homoclinic orbits. The fixed point R_- is not in the physical phase space.

The phase portraits in the compactified phase space $(V^2+Y^2\leqslant 1, V\geqslant 0)$ are given in Fig. 1. From these portraits it is immediately evident that the only stable singular points (both to the past and the future) either lie in the $V=0$ invariant set, occur at the infinite singular point R_+, or occur at $\mathbf{Q}_3$ (when it exists in the phase space). Recall that the invariant set $V=0$ represents the perfect fluid solutions studied previously,[1] where in the equivalent ''$M_1=0$'' case the solutions were shown to asymptote towards a flat FRW model. The fixed point R_+ has $y=0$, and hence is a static solution. Finally, the fixed point $\mathbf{Q}_3$ has the property $y=-1$ (or $S+\dot{S}=0$), which was examined in Sec. II B. Since all of the solutions in the phase space, and in particular those asymptoting to the point $\mathbf{Q}_3$, have the property that $p_\parallel=P_\parallel(x)t^{-2}$, $p_\perp=P_\perp(x)t^{-2}$, and $\mu=W(x)t^{-2}$, by continuity so must the solution at $\mathbf{Q}_3$. Therefore the solution represented by the point Q_3 must be given by the metric (2.11).

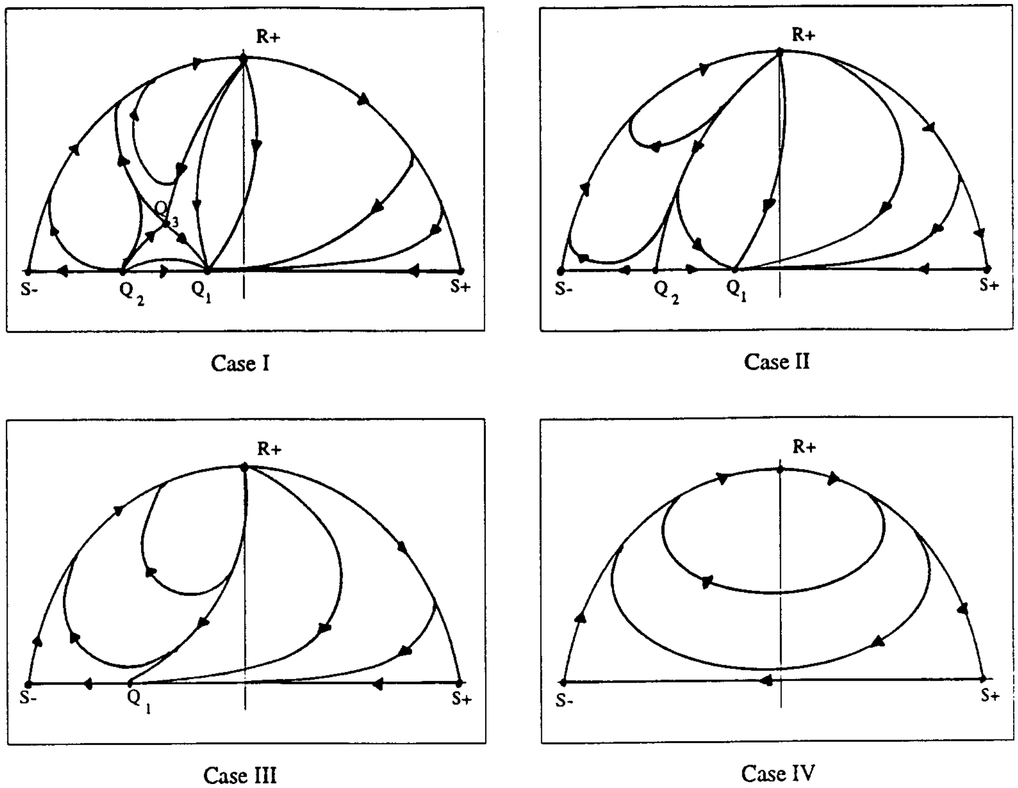

FIG. 1. Phase portraits. The phase portraits for the system (4.5)/(4.6) for various ranges of values of α and k are given, where the particular cases are as listed in Table I.

Consequently we see that in the analysis of the two cases considered in Sec. III A and III B above the asymptotic behavior is described by either a flat FRW perfect fluid model, a static model, or by that of the metric (2.11). In all cases these exact asymptotic models admit a homothetic vector.

V. DISCUSSION

We note that in the cases studied in this paper the dynamics of the models is governed by a system of the form:

$$\dot{y} = -\tfrac{1}{2}(3y^2 + 2\alpha y) + f(\nu), \tag{5.1}$$

$$\dot{\nu} = -2n\nu(1+y). \tag{5.2}$$

The variable ν is defined by Eq. (4.1) and the function $f(\nu)$ depends on the specific case being studied. In the cases considered in Sec. III we had that:

Case A: Dimensional Constants : $f(\nu) = -\tfrac{1}{2}(\nu + \alpha^2 p_0)$.

Case B: Equations of State : $f(\nu) = -\tfrac{1}{2}\nu$.

The system (5.1)/(5.2) results whenever we impose the condition

$$P_{\|}(x) = -2\alpha^{-2} f(\nu). \tag{5.3}$$

In the cases examined in Sec. III it was shown that all solutions necessarily asymptote to an exact solution admitting a homothetic vector. It is of interest to consider whether there are any possible asymptotic states for the geodesic anisotropic models which satisfy Eq. (5.3) that do *not* admit a homothetic vector.

As was the case in Sec. IV, the perfect fluid solutions are located in the invariant set $\nu=0$. The definition of ν requires that it be greater than or equal to zero. In the relevant phase space there are then (at most) three finite singular points of the system (5.1)/(5.2). These singular points, equivalent to those studied in Sec. IV, are given by:

$$Q_1=(\tfrac{1}{3}[-\alpha+(\alpha^2+6f(0))^{1/2}],0),$$

$$Q_2=(\tfrac{1}{3}[-\alpha-(\alpha^2+6f(0))^{1/2}],0),$$

$$Q_3=(-1,f^{-1}(3/2-\alpha)).$$

The singular points Q_1 and Q_2 represent perfect fluid models, and Q_3 (as in Sec. IV) is represented by the metric (2.11). In each case the model represented by the finite singular point admits a homothetic vector.

The only possibility for the asymptotic behavior not to be governed by an exact homothetic model is then (i) the model is represented asymptotically by a periodic orbit in the phase space, or (ii) the model is represented by a singular point at infinity not located on one of the coordinate axes $\nu=0$ or $y=0$.

In the first case we can impose necessary conditions for the existence of a periodic orbit. Any periodic orbit in a plane must necessarily enclose a singular point. As a result we must have that the point Q_3 is in the phase space in which case we necessarily have that $f^{-1}(3/2-\alpha)$ is positive. The energy conditions requiring that the pressures and density are positive will result in the further condition that $f(\nu)\leqslant 0$, and therefore $\alpha\geqslant 3/2$ and $y\geqslant 0$. We consider the existence of a periodic orbit which encloses Q_3 by examining the horizontal and vertical isoclines of the system (5.1)/(5.2). The horizontal isoclines are located at (i) $\nu=0$, an invariant line, and (ii) $y=-1$. The second case indicates that if there exists a periodic orbit about the point Q_3 then there must be vertical isoclines on either side of the line $y=-1$. Solving Eq. (5.2), we find that the vertical isoclines are given by

$$y_{\pm}=\tfrac{1}{3}(-\alpha\pm(\alpha^2+3f(\nu))^{1/2}). \tag{5.4}$$

Imposing the energy conditions $f(\nu)\leqslant 0$ and $\alpha\geqslant 3/2$, we find that the y-values of the vertical isoclines must satisfy

$$-1\leqslant y\leqslant 0; \tag{5.5}$$

i.e., $y_{\pm}$ cannot take on values less than -1. Therefore, there can be no periodic orbits enclosing the point Q_3 if the energy conditions are to be satisfied.

If there is an asymptotic solution at infinite values of y and/or ν which is not homothetic then the corresponding singular point at infinity must be such that $y\neq 0$ or $\nu\neq 0$. This will occur when $\lim_{\nu\to\infty}f(\nu)\nu^{-2}\neq 0$. In such cases the infinite fixed point may represent a nonhomothetic asymptotic solution. Therefore, geodesic models for which Eq. (5.3) and the energy conditions are satisfied will not admit a nonhomothetic asymptotic solution whenever $\lim_{\nu\to\infty}f(\nu)\nu^{-2}$ is exactly zero.

VI. OTHER MODELS

Additional anisotropic fluid models can be investigated. For example, we can consider the case in which the source is a combination of *a perfect fluid and a magnetic field* satisfying Eqs. (1.9). Assuming $\overline{p}=(\gamma-1)\overline{\mu}$ (where γ is a constant), in the geodesic case we can immediately derive the governing system as:

$$\dot{y}=-\tfrac{1}{2}(3y^2+2\alpha y)-\tfrac{1}{2}\alpha^2\eta, \tag{6.1}$$

$$\dot{\eta}=-4(1+n)y\eta-4(n-1)(3-2\alpha)\alpha^{-2}y^2, \tag{6.2}$$

where $\eta\equiv-\alpha^{-2}(3y^2+2\alpha y+2\dot{y})=P_\parallel$ and $n\equiv1/\gamma$. The system (6.1)/(6.2) is of a similar form to Eqs. (5.1) and (5.2) and can be analyzed using similar techniques. In the special cases $\gamma=1$ ($n=1$) and $\alpha=3/2$, Eq. (6.2) can be integrated immediately and exact solutions can be obtained. We note that at the equilibrium points of the system (6.1)/(6.2), $P_\parallel=$ constant ($\dot{P}_\parallel=0$), and hence from Eqs. (1.9), (2.3) and (2.4) we have that

$$\pi=\frac{1}{2}(p_\perp-p_\parallel)=\frac{\dot{P}_\parallel}{2t^2(1+y)}=0; \tag{6.3}$$

hence these equilibrium points correspond to perfect fluid models.

However, in order to study the physics of this particular model we note that $\pi=\lambda H^2/2$ and Eqs. (6.1) and (6.2) need to be supplemented by an additional differential equation (for H, derived from Maxwell's equations) and an assumption on the form of the magnetic permeability, λ.

Finally, we note that in the case in which $\pi=\text{constant}=\pi_0$ (with an unrestricted equation of state) it can be shown that the governing equations reduce to

$$\dot{y}=-\tfrac{1}{2}(3y^2+2\alpha y)-\alpha^2\pi_0\ln(\nu), \tag{6.4}$$

$$\dot{\nu}=-\nu(1+y). \tag{6.5}$$

This system is of the same form as that of (5.1)/(5.2) with $f(\nu)=-\alpha^2\pi_0\ln(\nu)$ and with the constant $n=\frac{1}{2}$. Since (6.3)/(6.4) is of the same form we can immediately conclude that the only asymptotic states of the system necessarily admit a homothetic vector. Note that in this case $f(\nu)$ is not analytic at $\nu=0$; however the physical phase space has $\nu>0$.

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