Qualitative viscous cosmology

A. A. Coley and R. J. van den Hoogen*
Department of Mathematics, Statistics, and Computing Science, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5

R. Maartens†
School of Mathematical Studies, University of Portsmouth, Portsmouth, England PO1 2EG
(Received 25 October 1995)

The full (nontruncated) Israel-Stewart theory of bulk viscosity is applied to dissipative FRW spacetimes. Dimensionless variables and dimensionless equations of state are used to write the Einstein-thermodynamic equations as a plane autonomous system and the qualitative behavior of this system is determined. Entropy production in these models is also discussed. [S0556-2821(96)03314-0]

PACS number(s): 04.20.Jb, 98.80.Hw

I. INTRODUCTION

In a recent paper [1], isotropic and spatially homogeneous viscous fluid cosmological models were investigated using the truncated Israel-Stewart [2–4] theory of irreversible thermodynamics to model the bulk viscous pressure. Although it provides a causal and stable second order relativistic theory of thermodynamics, the truncated version of the theory can give rise to very different behavior than the full Israel-Stewart theory [5–8]. It can be argued that the truncated theory agrees with the full theory if one uses, instead of the local equilibrium variables, a generalized temperature and thermodynamic pressure [8,9]. However, there are difficulties in modeling these generalized variables in cosmology. Therefore the analysis of [1] can only be regarded as a first step in the study of dissipative processes in the universe utilizing the full (nontruncated) theory.

For a Friedmann-Robertson-Walker (FRW) cosmology the metric is given by

$$ds^2=-dt^2+R(t)^2\left[\frac{dr^2}{1-kr^2}+r^2(d\theta^2+\sin^2\theta d\phi^2)\right],$$
$$k=0,\pm 1,$$

and the Einstein field equations and the energy conservation equation are given by

$$\dot{H}=-H^2-\frac{1}{6}(3\gamma-2)\rho-\frac{1}{2}\Pi, \quad (1)$$

$$\dot{\rho}=-3H(\gamma\rho+\Pi), \quad (2)$$

$$H^2=\frac{1}{3}\rho-\frac{k}{R^2}, \quad (3)$$

where $H=\dot{R}/R$ is the Hubble expansion rate (we restrict ourselves to the expanding case only, i.e., $H>0$), ρ is the energy density, and the local equilibrium pressure is assumed to obey

$$P=(\gamma-1)\rho, \quad 1\leqslant\gamma\leqslant 2,$$

with γ constant.

The bulk viscous pressure Π obeys the evolution equation [5,8]

$$\Pi=-3\zeta H-\tau\dot{\Pi}-\frac{\epsilon}{2}\tau\Pi\left[3H+\frac{\dot{\tau}}{\tau}-\frac{\dot{\zeta}}{\zeta}-\frac{\dot{T}}{T}\right], \quad (4)$$

where $\zeta\geqslant 0$ is the bulk viscosity coefficient, $0\leqslant\tau$ ($\equiv\zeta\beta_0$ in [1]) is a relaxation coefficient for transient bulk viscous effects, and $T\geqslant 0$ is the temperature. Equation (4) with $\epsilon=1$ arises as the simplest way (linear in Π) to satisfy the H theorem (i.e., for entropy production to be non-negative [8]). The truncated theory effectively arises by setting $\epsilon=0$, i.e., it corresponds to the case where the term in square brackets in Eq. (4) is negligible in comparison with the other terms (see [10] for the appropriate conditions).

The Israel-Stewart theory is derived under the assumption that the thermodynamical state of the fluid is close to equilibrium, which means that the nonequilibrium bulk viscous pressure should be small when compared to the local equilibrium pressure: viz.,

$$|\Pi|<P=(\gamma-1)\rho. \quad (5)$$

If this condition is violated, then one is effectively assuming that the linear theory holds also in the nonlinear regime far from equilibrium. Such an assumption is unavoidable for viscous inflationary cosmology [8]. For a fluid description of the matter, Eq. (5) ought to be satisfied. However, note that nonlinear viscous effects may arise in a phenomenological description of particle creation in the early universe [11].

II. DYNAMICAL SYSTEM

Equations of state for ζ and τ and a temperature law for T are needed in order for the above system of equations to be closed. Belinskii *et al.* [12] take ζ and τ to be proportional to

*Current address: Department of Mathematics and Computer Science, St. Francis Xavier University, Antigonish, Nova Scotia, B2G 2W5.

†Affiliated with Center for Nonlinear Studies, Witwatersand University, South Africa 2050.

powers of ρ, and this assumption is extended to T in [6]. We shall follow [1] and adopt ''dimensionless'' equations of state. That is, defining the dimensionless density parameter

$$x=\Omega\equiv\frac{\rho}{3H^2}, \tag{6}$$

we shall assume that ζ/H and τH are proportional to powers of x: namely,

$$\frac{\zeta}{H}=3\zeta_0x^m \quad \text{and} \quad \frac{\tau^{-1}}{H}=bx^n, \tag{7}$$

where m and n are constants which are assumed to be nonnegative and ζ_0 and b are positive parameters. Clearly the equations of state employed will determine the qualitative properties of the models [1,5–8,12,13]. Equations of state (7), which ensure that the asymptotic limit points represent self-similar models [14], are phenomological in nature and are no less appropriate than the equations of state used by Belinskii *et al.* [12]. We note that the equations of state chosen in [12] and those above coincide in the important case $m=1/2=n$ ($q=1/2$ in [8]).

From now on we shall take

$$n=0, \quad a\equiv b\zeta_0.$$

(Note that a,b are precisely the parameters used in [1].) When $n=0$, it follows that the relaxation rate is determined by the expansion rate:

$$\tau^{-1}=bH. \tag{8}$$

As argued in [8], for viscous expansion to be nonthermalizing, we should have $\tau^{-1}<H$, for otherwise the basic interaction rate for viscous effects could be sufficiently rapid to restore equilibrium as the fluid expands. Therefore we impose the constraint

$$b<1$$

on the relaxation parameter.

Defining the dimensionless viscous pressure y and the new time variable $\overline{t}$ by

$$y=\frac{\Pi}{H^2} \quad \text{and} \quad \frac{d\overline{t}}{dt}=H, \tag{9}$$

and using Eq. (1), Eqs. (2) and (4) become

$$x'=(x-1)[(3\gamma-2)x+y], \tag{10}$$

$$y'=y[2-b+y+(3\gamma-2)x]-9ax^m-\frac{\epsilon}{2}y\Psi, \tag{11}$$

where

$$\Psi\equiv3-2\frac{H'}{H}-m\frac{x'}{x}-\frac{T'}{T}, \tag{12}$$

and the prime denotes a derivative with respect to $\overline{t}$. Note that the linear condition (5) becomes

$$|y|<3(\gamma-1)x.$$

Equations (10) and (11) constitute a plane autonomous system of ordinary differential equations (ODE's) for x and y. In the truncated theory $\epsilon=0$, whence the final term in Eq. (11) is absent and there is no need to specify an equation for T. Hereafter we shall set $\epsilon=1$, and adopt the temperature power law [6–8]:

$$T=T_0\rho^r=T_03^rx^rH^{2r} \quad \text{with } r=\frac{\gamma-1}{\gamma}, \tag{13}$$

where the form of the exponent r follows from the integrability condition of the Gibbs equation when $P=(\gamma-1)\rho$ [10,15]. When the local equilibrium state of the expanding viscous fluid is thermalized radiation, then $r=1/4$, in line with the standard Stefan-Boltzmann relation. Consequently,

$$\Psi=3-2(1+r)\frac{H'}{H}-(m+r)\frac{x'}{x}=c_0+c_1y+c_2x+c_3\frac{y}{x},$$

where

$$c_0=5+2r+(3\gamma-2)(m+r), \quad c_1=1-m,$$

$$c_2=(3\gamma-2)c_1, \quad c_3=m+r.$$

A. Flat universe

All of the FRW models are governed by Eqs. (10) and (11) together with Eq. (3). We note from (10) that $x=1$ is an invariant set, where from Eq. (3) we see that this set represents the flat FRW models. Let us study this physically important zero-curvature case first. When $x=1$, the thermodynamic laws are simplified to Eqs. (8) and (13) and, by Eq. (7), to

$$\zeta\propto H. \tag{14}$$

Thus the bulk viscosity coefficient, like the relaxation rate, is also determined by the expansion rate. Furthermore,

$$\Psi=(c_0+c_2)+(c_1+c_3)y, \tag{15}$$

whence Eq. (11) becomes

$$y'=-\frac{(r-1)}{2}y^2-by-9a. \tag{16}$$

That is, the equations governing the evolution of the flat FRW viscous fluid models reduce to a single autonomous ODE in y. Since $0\leqslant r\leqslant 1/2$, Eq. (16) is a Riccati equation with constant coefficients and its solutions can be found in implicit form.

Defining the positive parameter

$$B_1\equiv b^2+18a(1-r),$$

it follows that there are two equilibrium points, one positive and one negative:

$$y^{\pm}=\gamma(b\pm\sqrt{B_1}), \tag{17}$$

where one is a sink and the other is a source (with respect to the invariant set $x=1$, not the full set of all FRW models). The points (17) correspond to the special solutions found in [16] and rediscovered in [17].

Therefore, the behavior of the flat models using the full (nontruncated) theory is qualitatively the same as the behavior in the truncated theory [1]. Of course, this qualitative similarity only holds for the restrictive thermodynamic laws (8), (13), and (14).

B. Curved universes

Let us now return to the general curvature case $x \neq 1$ [see Eqs. (10) and (11)]. Equation (11) can be written as

$$y' = -y\left[\left(b-2+\frac{c_0}{2}\right)+x(3\gamma-2)\left(\frac{c_1}{2}-1\right)+y\left(\frac{c_1}{2}-1\right)+\frac{c_3}{2}yx^{-1}\right]-9ax^m. \tag{18}$$

There are two equilibrium points lying in the invariant set $x=1$, namely, $(1,y^{\pm})$ where $y^{\pm}$ is given by Eq. (17). Previously (in the case of the flat models) we considered the stability of the equilibrium points only with respect to the invariant set $x=1$; let us now discuss the stability of these equilibrium points with respect to the curved FRW models. The equilibrium point $(1,y^+)$ is a source with the invariant set $x=1$ as one of its primary eigendirections. If $y^-+3\gamma>2$, then the equilibrium point $(1,y^-)$ is a saddle with $x=1$ as the stable manifold. If $y^-+3\gamma<2$, then the equilibrium point $(1,y^-)$ is a sink and hence it represents a future asymptotic attractor. From Eq. (10), the equilibrium points $(\bar{x},\bar{y})$ not lying in the invariant set $x=1$ satisfy $\bar{y}=-(3\gamma-2)\bar{x}$, and hence from Eq. (18) we obtain

$$9a\bar{x}^m-\frac{1}{2}(3\gamma-2)(2b+2r+1)\bar{x}=0. \tag{19}$$

For $m>0$, there exists a singular point at the origin $(0,0)$. [Note, however, that the system of ODEs as given by Eqs. (10) and (18) is not defined at $x=0$ except when $c_3=0$ and therefore the point (0,0) may not be a well-defined equilibrium point of the system.] Changing to polar coordinates, it can be shown that this singular point is saddlelike in nature (hyperbolic sectors) if $m<1$. If $m>1$, then the point (0,0) has parabolic and hyperbolic sectors.

If $m \neq 1$, then there is a second equilibrium point at

$$(\bar{x},\bar{y})=\left(\left[\frac{(3\gamma-2)}{18a}(2b+2r+1)\right]^{1/(m-1)},-(3\gamma-2)\bar{x}\right). \tag{20}$$

If

$$B_2 \equiv (3\gamma-2)(2b+2r+1)-18a>0,$$

then $\bar{x}>1$, and when $m<1$, this point is a saddle. If $B_2<0$, then $\bar{x}<1$, and when $m>1$, this equilibrium point is again a saddle. There is a variety of other possible behaviors.

III. DISCUSSION

A. Exact solutions and asymptotic behaviors

The qualitative behavior of the flat FRW models has been determined completely. The unphysical flat models evolve from the equilibrium point $y=y^+$ at $t=-\infty$, where $y=y^+$ corresponds to the solution (after recoordination)

$$R(t)=R_0 t^{2/(y^++3\gamma)}, \quad H(t)=\frac{2}{y^++3\gamma}t^{-1},$$

$$\rho(t)=\frac{12}{(y^++3\gamma)^2}t^{-2}, \quad \Pi(t)=\frac{4y^+}{(y^++3\gamma)^2}t^{-2}, \tag{21}$$

towards either points at infinity or to the point $y=y^-$ (at $t=-\infty$), which, if $y^- \neq -3\gamma$, has the solution

$$R(t)=R_0(t-t_0)^{2/(y^-+3\gamma)}, \quad H(t)=\frac{2}{y^-+3\gamma}(t-t_0)^{-1},$$

$$\rho(t)=\frac{12}{(y^-+3\gamma)^2}(t-t_0)^{-2},$$

$$\Pi(t)=\frac{4y^-}{(y^-+3\gamma)^2}(t-t_0)^{-2}. \tag{22}$$

[Note that if $y^-+3\gamma>0$, then the solution (22) can be recoordinatized such that $t_0=0$.] These models and the equilibrium point $y=y^+$ are unphysical since they have positive bulk viscous pressure. Those models which evolve towards $y=y^-$ have negative bulk viscous pressure after a certain time, and may be considered as physical models after this time. The models which are physical for all times (i.e., which have $\Pi<0$ for all times) are Eqs. (22) and those which evolve from infinity at $t=-\infty$ towards $y=y^-$ at $t=\infty$.'' [Note that, by Eqs. (9), (21), and (22), $t=-\infty$ corresponds to $t=0$, while $t=\infty$ corresponds to $t=\infty$.]

If $y^-=-3\gamma$, then the solution has the form

$$R(t)=R_0 e^{H_0 t}, \quad H(t)=H_0, \quad \rho(t)=3H_0^2,$$

$$\Pi(t)=y^-H_0^2. \tag{23}$$

The exponential inflationary solution (23) clearly violates the condition (5) (cf. [8]). The solution (22) violates the condition (5) if $y^-+3\gamma<3$, when the expansion is driven by a large and effective nonlinear bulk viscous pressure. The expansion is from a big bang, and is more rapid than in the corresponding equilibrium solution $(y^-=0)$. Indeed, if

$$y^-+3\gamma<2, \tag{24}$$

then the solution represents a power-law inflationary solution. Knowing that condition (24) is also the requirement that the equilibrium point $(1,y^-)$ be stable, we can conclude that the power-law inflationary solution (23) is the future asymptotic attractor for all bulk-viscous inflationary FRW models.

We emphasize that bulk viscous inflationary solutions, such as the solution (22), violate the condition (5), so that their existence is dependent on assuming that the theory holds in the nonlinear regime. Furthermore, these inflationary solutions are limited by the simple equation of state $P=(\gamma-1)\rho$, so that, in particular, they cannot account for the processes necessary to provide an exit from inflation. The solutions are at most valid during inflation, and more realistic models would be needed to incorporate exit and reheating.

B. Entropy production

On physical grounds, one expects that $y\leqslant 0$, since the evolution of specific entropy is given by [5,8]

$$\dot{s}=-\frac{3H\Pi}{nT}, \tag{25}$$

where n is the number density. We note that solution (22) always satisfies $y\leqslant 0$. From Eq. (25), it follows that the growth of entropy in a comoving volume between times $t_0<t<t_1$ is given by

$$\Sigma(t_1)-\Sigma(t_0)=-\frac{3}{k}\int_{t_0}^{t_1}\frac{\Pi HR^3}{T}dt, \tag{26}$$

where k is the Boltzmann constant. The amount of entropy generated can be calculated for each of the solutions (21), (22), and (23). Analyzing the physically more relevant case (22), we find that (reinstating constants previously set to unity),

$$\Sigma(t_1)-\Sigma(t_0)=\frac{\gamma 3^{(1-r)}c^2}{8\pi Gk}\left(\frac{R_0^3}{T_0}\right)\left(\frac{2}{y^-+3\gamma}\right)^{2(1-r)} \times\left(t_1^{-2y^-/[\gamma(y^-+3\gamma)]}-t_0^{-2y^-/[\gamma(y^-+3\gamma)]}\right), \tag{27}$$

where c is the speed of light and G is the gravitational constant. By Eqs. (22), we must have $y^-+3\gamma>0$ for an expanding solution. This ensures that Eq. (27) gives $\Sigma(t_1)>\Sigma(t_0)$.

The amount of entropy generated in the de Sitter model [solution (23)] is the same as that calculated by Maartens [8]: namely,

$$\Sigma(t_1)-\Sigma(t_0)=\frac{3^{(1-r)}c^2}{8\pi Gk}\left(\frac{R_0^3}{T_0}\right)H_0^{2-2r}\left(e^{3H_0t_1}-e^{3H_0t_0}\right). \tag{28}$$

It is shown that bulk viscous inflation can generate significant amounts of entropy without reheating. For the de Sitter model, using typical parameters of inflation, and assuming that almost all of the entropy is produced by inflation, one finds the following value for the amount of entropy produced during exponential inflation [8]:

$$\Sigma\approx 2.1\times 10^{87}, \tag{29}$$

which is in agreement with the expected value. The power-law inflationary solution (22) [with $y^-+3\gamma<2$, i.e., satisfying Eq. (24)] has less efficient entropy production, but nonetheless can also produce significant amounts of entropy.

In the above, we have only considered entropy production in the models corresponding to the equilibrium points of the dynamical system. By considering a simple example, we can investigate the entropy production in the more general flat FRW models. We choose parameter values $r=1/4$ (necessarily $\gamma=4/3$), $a=1/27$, and $b=1/4$. In this case the differential equation (16) reduces to

$$y'=\tfrac{3}{8}\left[\left(y-\tfrac{1}{3}\right)^2-1\right], \tag{30}$$

which has a solution of the form (neglecting the constants of integration)

$$y=\begin{cases}\tfrac{1}{3}-\tanh\left(\tfrac{3}{8}\bar{t}\right), & \left|y-\tfrac{1}{3}\right|<1,\\ \tfrac{1}{3}-\coth\left(\tfrac{3}{8}\bar{t}\right), & \left|y-\tfrac{1}{3}\right|>1.\end{cases} \tag{31}$$

The Hubble parameter is given by

$$H=\begin{cases}e^{-13\bar{t}/6}\cosh^{4/3}\left(\tfrac{3}{8}\bar{t}\right), & \left|y-\tfrac{1}{3}\right|<1,\\ e^{-13\bar{t}/6}\sinh^{4/3}\left(\tfrac{3}{8}\bar{t}\right), & \left|y-\tfrac{1}{3}\right|>1,\end{cases} \tag{32}$$

and $R=R_0e^{\bar{t}}$. The change in entropy in a comoving volume produced between times $t_0<t<t_1$ is then given by

$$\Sigma(\bar{t}_1)-\Sigma(\bar{t}_0)=\left(\frac{3^{-1/4}c^2R_0^3}{16\pi GkT_0}\right)\left[\pm 2\left(e^{-\bar{t}_1/4}-e^{-\bar{t}_0/4}\right) +\left(e^{-\bar{t}_1}-e^{-\bar{t}_0}\right)+\left(e^{\bar{t}_1/2}-e^{\bar{t}_0/2}\right)\right]. \tag{33}$$

It can be concluded in this simple model for $\bar{t}_1/\bar{t}_0>1$ that as $\bar{t}_1$ increases the entropy in a comoving volume grows exponentially with respect to t_1.

IV. CONCLUSIONS

The behavior of the viscous fluid FRW models where the bulk viscous pressure satisfies the full Israel-Stewart theory of irreversible thermodynamics has been analyzed. The stability of the equilibrium point (0,0) representing the Milne model depends upon the value of m which appears in the equation of state for the bulk viscosity. The equilibrium point $(\bar{x},\bar{y})$ can represent either an open, flat, or closed FRW model depending upon the value of the parameter B_2. Exact determination of the nature of this particular equilibrium point is extremely difficult. However, a partial result is possible: If $B_2(1-m)>0$, then the equilibrium point is a saddle. There exist two equilibrium points with qualitative behavior similar to that found using the truncated Israel-Stewart theory.

It can be concluded that the behavior of the FRW models in which the bulk viscous pressure satisfies the full Israel-Stewart theory can in principle be qualitatively similar to the behavior of the FRW models in the truncated theory. One cannot say, however, that the full theory has the same behavior as the truncated theory in all cases because it is not at all clear what effects the presence of anisotropies or different equations of state will have. For example, in the models studied here, it was the equations of state for the temperature and for the bulk viscosity coefficient that played major roles in determining the dynamics of the models. In the case of a relativistic Maxwell-Boltzmann gas, which has very different equations of state, the truncated and full theories can lead to very different behavior, with the truncated theory leading to pathological behavior of the temperature in many cases [5].

As stated above, the consistency condition that viscous expansion should be nonthermalising requires $b<1$. Further constraints may arise from entropy arguments. The evolution equation (4) already guarantees that entropy production is non-negative. But one may place constraints on the rate and amount of entropy production. If we impose the requirement that the specific entropy's, should increase with expansion, but at a decreasing rate, then we have $y<0$ and possibly further constraints on r (equivalently γ) and m.

A complete analysis of the asymptotic behaviors of these viscous fluid models depending on the (many) free parameters in the model (a,b,γ,m) and utilizing the energy conditions can be made. However, the next step in this research program is to attempt to use results from kinetic theory in order to motivate physically plausible equations of state or, at the very least, to limit the form of the phenomological equations of state used.

ACKNOWLEDGMENTS

This work was supported by research grants from the Natural Sciences and Engineering Research Council of Canada (A.A.C.) and Portsmouth University (R.M.) and through the Izaak Walton Killam Fund (R.V.D.H.).

[1] A. A. Coley and R. J. van den Hoogen, Class. Quantum Grav. **12**, 1977 (1995).
[2] W. Israel, Ann. Phys. (N.Y.) **100**, 310 (1976).
[3] W. Israel and J. M. Stewart, Proc. R. Soc. London **A365**, 43 (1979).
[4] W. Israel and J. M. Stewart, Ann. Phys. (N.Y.) **118**, 341 (1979).
[5] W. A. Hiscock and J. Salmonson, Phys. Rev. D **43**, 3249 (1991).
[6] M. Zakari and D. Jou, Phys. Rev. D **48**, 1597 (1993).
[7] V. Romano and D. Pavon, Phys. Rev. D **50**, 2572 (1994).
[8] R. Maartens, Class. Quantum Grav. **12**, 1455 (1995).
[9] J. Gariel and G. le Denmat, Phys. Rev. D **50**, 2560 (1994).
[10] W. Zimdahl, Phys. Rev. D **53**, 5483 (1996).
[11] G. M. Vereshkov, Yu. S. Grishkan, S. V. Ivanov, V. A. Nesterenko, and A. N. Poltavtsev, Sov. Phys. JETP **46**, 1041 (1977).
[12] V. A. Belinskii, E. S. Nikomarov, and I. M. Khalatnikov, Sov. Phys. JETP **50**, 213 (1979).
[13] R. J. van den Hoogen and A. A. Coley, Class. Quantum Grav. **12**, 2335 (1995).
[14] A. A. Coley and R. J. van den Hoogen, in *Deterministic Chaos in General Relativity*, edited by D. Hobill, A. Burd, and A. Coley (NATO Advanced Study Institute, Series B: Physics, Vol. 332B) (Plenum, New York, 1994).
[15] R. Maartens, Portsmouth University report, 1995 (unpublished).
[16] R. Maartens and A. M. Kgathi (unpublished).
[17] N. Banerjee and A. Beesham, report, 1995 (unpublished).