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Special conformal Killing vector space-times and symmetry inheritance

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Viscous heat-conducting fluid and anisotropic fluid space-times admitting a special conformal Killing vector (SCKV) are studied and some general theorems concerning the inheritance of the symmetry associated with the SCKV are proved. In particular, for viscous fluid space-times it is shown that (i) if the SCKV maps fluid flow lines into fluid flow lines, then all physical components of the energy-momentum tensor inherit the SCKV symmetry; or (ii) if the Lie derivative along a SCKV of the shear viscosity term $\eta\sigma_{ab}$ is zero then, again, we have symmetry inheritance. All space-times admitting a SCKV and satisfying the dominant energy condition are found. Apart from the vacuum *pp*-wave solutions, which are the only vacuum solutions that can admit a SCKV, the energy-momentum tensor associated with these space-times is shown to admit at least one null eigenvector and can represent either a viscous fluid with heat conduction or an anisotropic fluid. No perfect fluid space-times can admit a SCKV. These SCKV space-times and, also, space-times admitting a homothetic vector are used to illustrate the symmetry inheritance theorems.

I. INTRODUCTION

Homothetic vectors (HV's) and conformal Killing vectors (CKV's) have been studied at length by various authors. Cahill and Taub[1] and Taub[2] have discussed perfect fluid solutions which are self-similar, i.e., admit a HV. Wainwright and Yaremovich[3] have studied charged perfect fluids and McIntosh[4] has made a general study of HV's in general relativity, with an emphasis on vacuum and perfect fluid space-times. Herrera and co-workers[5] have studied CKV's, with particular reference to perfect fluids and anisotropic fluids; Mason and Tsamparlis[6] have investigated spacelike CKV's; and Maartens *et al.*[7] have made a study of CKV's in anisotropic fluids, in which they are particularly concerned with special conformal Killing vectors (SCKV's).

In this article we are principally interested in imperfect fluids (i.e., viscous, heat-conducting fluids) and, to a lesser extent, anisotropic fluids. The energy-momentum tensor for an imperfect fluid is

$$T_{ab} = \mu u_a u_b + p h_{ab} - 2\eta\sigma_{ab} + q_a u_b + q_b u_a\,, \qquad (1.1)$$

where μ is the energy density, p is the isotropic pressure, q^a is the heat flux vector relative to the four-velocity u^a, $\eta\,(\geqslant 0)$ is the shear viscosity coefficient, $h_{ab} = g_{ab} + u_a u_b$ is the projection tensor, and σ_{ab} is the shear tensor. The energy-momentum tensor for an anisotropic fluid is

$$T_{ab} = \mu u_a u_b + p_{\|} n_a n_b + p_{\perp} p_{ab}\,, \qquad (1.2)$$

where n_a is a unit spacelike vector orthogonal to u_a; p_{ab} is the projection tensor onto the two-plane orthogonal to u^a and n^a; and $p_{\|}, p_{\perp}$ denote the pressures parallel to and perpendicular to n^a, respectively.

The effect of a HV on space-times corresponding to the energy-momentum tensor (1.1) has been discussed by Hall and Negm,[8] but only in the case when one of η and q_a is zero. In fact, there has been no systematic study of CKV's, HV's, or even Killing vectors (KV's) in fluids of type (1.1), although fluids of type (1.2) were discussed in Refs. 5 and 7; in the latter reference the SCKV discussion was confined to the case $\mu + p_{\|} \neq 0$.

We shall consider space-times that admit a CKV ξ^a, i.e.,

$$\mathscr{L}_{\xi} g_{ab} = 2\psi g_{ab}\,, \qquad (1.3)$$

where $\mathscr{L}_{\xi}$ signifies the Lie derivative along ξ^a and $\psi(x^a)$ is the conformal factor. If $\psi_{;ab} = 0$, but $\psi_{,a} \neq 0$, then ξ^a is a SCKV; when ψ is a constant, ξ^a is a HV and $\psi = 0$ corresponds to a KV. Although our ultimate aim is to study the properties of proper CKV's (i.e., CKV's that do not degenerate into SCKV's or HV's in imperfect fluids, in this article we shall confine our attention to the simpler SCKV's and HV's.

In Sec. II, the Lie derivatives of the various kinematical quantities are calculated and the results applied to the energy-momentum tensor (1.1) and, also, to (1.2) in the special case $\mu + p_{\|} = 0$. In Sec. III, we define what we mean by *symmetry inheritance* for a SCKV and prove a number of theorems on the inheritance of the symmetry of a SCKV by the physical components of the energy-momentum tensor (1.1). In particular, we prove the results that if either $\mathscr{L}_{\xi} u_a = \psi u_a$ or $\mathscr{L}_{\xi}(\eta\sigma_{ab}) = 0$, the symmetry of ξ^a is inherited by all physical quantities. We conclude Sec. III by discussing symmetry inheritance by an anisotropic fluid in the particular case $\mu + p_{\|} = 0$, which was omitted from a similar discussion in Ref. 7.

In Sec. IV we find all space-times (irrespective of the field equations that they satisfy) which admit a proper SCKV, i.e., a SCKV for which ψ is not constant, and which satisfy the dominant energy condition. We find that (i) *there*

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are no perfect fluid space-times admitting a proper SCKV; (ii) *anisotropic fluid space-times admitting a proper SCKV must satisfy* $\mu + p_{\parallel} = 0, p_{\perp} = 0$; and (iii) *there do exist imperfect fluid space-times admitting a proper SCKV.*

Result (i) invalidates that part of Ref. 5 in which it was assumed that perfect fluid space-times admitting SCKV's do exist and result (ii) invalidates a result in Ref. 7 since it shows that SCKV anisotropic fluids are not compatible with the assumption $\mu + p_{\parallel} \neq 0$ made in Ref. 7. The space-times admitting a proper SCKV form a very restricted class in that they must admit either two null eigenvectors, or a repeated null vector, of the energy-momentum tensor. Because of the limited number of proper SCKV solutions, the work described here on symmetry inheritance has, perhaps, its greatest relevance in the study of HV's, but is couched in the language of CKV's because of our intention to extend the work to proper CKV's. In Sec. V, we illustrate the theorems of Sec. III with examples of both the SCKV and HV and in Sec. VI we make some concluding remarks.

II. KINEMATICAL AND DYNAMICAL RESULTS

In order to discuss the effect on the various kinematical quantities of the Lie derivative along a CKV, we first note the following result proved by Maartens *et al.*,[7] namely, if X^a is any unit vector (timelike or spacelike) and ξ^a is a CKV satisfying (1.3), then

$$\mathscr{L}_\xi X^a = -\psi X^a + Y^a, \tag{2.1}$$

$$\mathscr{L}_\xi X_a = \psi X_a + Y_a, \tag{2.2}$$

where Y^a is some vector orthogonal to X^a, i.e., $X^a Y_a = 0$. Applying this result to the fluid velocity vector u^a, we have

$$\mathscr{L}_\xi u^a = -\psi u^a + v^a, \tag{2.3}$$

$$\mathscr{L}_\xi u_a = \psi u_a + v_a, \tag{2.4}$$

where v^a is a spacelike vector with $u_a v^a = 0$. Note that $u_a \mathscr{L}_\xi u^a = -u^a \mathscr{L}_\xi u_a = \psi$. If $v^a = 0$, i.e., $\mathscr{L}_\xi u^a = -\psi u^a$, then fluid flow lines are mapped into fluid flow lines by the action of ξ^a.

We first consider imperfect fluids with T_{ab} of the form (1.1). The heat flux vector q^a is not a unit vector and if we define Q to be the magnitude of q^a, i.e., $q^a q_a = Q^2$, then by an argument similar to that used in establishing (2.1) it can be shown that

$$\mathscr{L}_\xi q^a = (Q^{-1} \mathscr{L}_\xi Q - \psi) q^a + w^a, \tag{2.5}$$

where $w_a q^a = 0$. Note that if ξ^a is a HV, i.e., ψ is a constant and if we require a self-similar solution (which does not automatically follow for an imperfect fluid), the dimensional requirements[9] imply that $\mathscr{L}_\xi Q = -2\psi Q$ and $w^a = 0$.

Since $u_a q^a = 0$, it follows that

$$-u_a \mathscr{L}_\xi q^a = -u^a \mathscr{L}_\xi q_a = q_a \mathscr{L}_\xi u^a = q^a \mathscr{L}_\xi u_a \equiv \Delta, \tag{2.6}$$

which serves as the definition of the scalar quantity Δ. Equation (2.6) implies that

$$v_a q^a = -u_a w^a = \Delta. \tag{2.7}$$

If $\{{}_b{}^a{}_c\}$ is the metric affinity of g_{ab}, then[10]

$$\mathscr{L}_\xi \{{}_b{}^a{}_c\} = \delta^a_b \psi_{,c} + \delta^a_c \psi_{,b} - g_{bc} \psi^a \tag{2.8}$$

and (2.4) and (2.8) give

$$\mathscr{L}_\xi u_{a;b} = \psi u_{a;b} + v_{a;b} + g_{ab} \psi_{,c} u^c - \psi_{,a} u_b, \tag{2.9}$$

$$\mathscr{L}_\xi \Theta = -\psi\Theta + v^a{}_{;a} + 3\psi_{,a} u^a, \tag{2.10}$$

$$\mathscr{L}_\xi h_{ab} = 2\psi h_{ab} + 2u_{(a} v_{b)}, \tag{2.11}$$

where $\Theta = u^a{}_{;a}$ is the expansion scalar for the fluid velocity congruence.

Recalling that the shear tensor is defined by

$$\sigma_{ab} = \tfrac{1}{2}(u_{a;c} h^c{}_b + u_{b;c} h^c{}_a) - \tfrac{1}{3}\Theta h_{ab} \tag{2.12}$$

and using (2.9)–(2.13), after a long calculation we obtain

$$\mathscr{L}_\xi \sigma_{ab} = \psi\sigma_{ab} - \tfrac{1}{3} h_{ab} v^c{}_{;c} - \tfrac{2}{3}\Theta u_{(a} v_{b)} + \dot{v}_{(a} u_{b)} + v_{(a;b)} + \dot{u}_{(a} v_{b)} + u_{(a} u_{b);c} v^c, \tag{2.13}$$

where the overdot indicates the covariant derivative in the direction of the fluid flow, i.e., $\dot{X} = X_{;a} u^a$. Note that $\mathscr{L}_\xi \sigma_{ab}$ is explicitly independent of the derivatives of ψ. Also, $\sigma_{ab} g^{ab} = 0$, $\sigma_{ab} u^b = 0$, and $g^{ab} \mathscr{L}_\xi \sigma_{ab} = 0$, but

$$u^b \mathscr{L}_\xi \sigma_{ab} = -\sigma_{ab} v^b. \tag{2.14}$$

Note, also, that if $\mathscr{L}_\xi u_a = \psi u_a$, i.e., $v_a = 0$, then $\mathscr{L}_\xi \sigma_{ab} = \psi \sigma_{ab}$.

Turning now to dynamical results we note that if ξ^a is a CKV satisfying (1.3), then[7]

$$\mathscr{L}_\xi R_{ab} = -2\psi_{;ab} - g_{ab} \Box\psi, \tag{2.15}$$

$$\mathscr{L}_\xi R = -2\psi R - 6\Box\psi, \tag{2.16}$$

$$\mathscr{L}_\xi G_{ab} = 2g_{ab} \Box\psi - 2\psi_{;ab}, \tag{2.17}$$

where $\Box\psi \equiv g^{ab} \psi_{;ab}$. We consider Einstein's field equations in the form

$$G_{ab} + \Lambda g_{ab} = T_{ab} \tag{2.18}$$

and so find

$$\mathscr{L}_\xi T_{ab} = 2(\Box\psi + \Lambda\psi) g_{ab} - 2\psi_{;ab}. \tag{2.19}$$

We take T_{ab} to be of the form (1.1) and, by taking the Lie derivative of (1.1) with respect to ξ^a, we obtain

$$\mathscr{L}_\xi \mu (u_a u_b) + \mathscr{L}_\xi p (h_{ab}) + 2\psi(\mu u_a u_b + p h_{ab}) + 2(\mu + p) v_{(a} u_{b)} - 2\sigma_{ab} \mathscr{L}_\xi \eta - 2\eta \mathscr{L}_\xi \sigma_{ab} + 2(Q^{-1} \mathscr{L}_\xi Q + 2\psi) u_{(a} q_{b)} + 2q_{(a} v_{b)} + 2u_{(a} w_{b)} = 2(\Box\psi + \Lambda\psi) g_{ab} - 2\psi_{;ab}, \tag{2.20}$$

where $\mathscr{L}_\xi \sigma_{ab}$ is given by (2.13) and we have used (2.4) and (2.5).

For the remainder of this article we shall confine our attention to HV's and SCKV's, i.e., we assume that $\psi_{;ab} = 0$. We shall also assume that $\Lambda = 0$; this is not a particularly restrictive assumption since the replacements $\mu \to \mu + \Lambda$, $p \to p - \Lambda$ will reproduce the effects of including Λ. Thus (2.19) becomes $\mathscr{L}_\xi T_{ab} = 0$ and we focus our attention on (2.20) with zero rhs.

Contracting (2.20) in turn with $u^a u^b$, h^{ab}, $u^a h^b{}_c$, $h^{ac} h^{bd} - \tfrac{1}{3} h^{ab} h^{cd}$, q^b, $q^a u^b$, and $q^a q^b$, and simplifying we obtain

$$\mathscr{L}_\xi \mu + 2\psi\mu + 2\Delta = 0\,, \tag{2.21}$$

$$\mathscr{L}_\xi p + 2\psi p + \tfrac{2}{3}\Delta = 0\,, \tag{2.22}$$

$$2\eta\sigma_{ab}v^b = w_a - \Delta u_a + (\mu + p)v_a + (Q^{-1}\mathscr{L}_\xi Q + 2\psi)q_a\,, \tag{2.23}$$

$$\mathscr{L}_\xi(\eta\sigma_{ab}) = 2\eta\sigma_{c(a}u_{b)}v^c + q_{(a}v_{b)} - \tfrac{1}{3}\Delta h_{ab}\,, \tag{2.24}$$

$$2q^b\mathscr{L}_\xi(\eta\sigma_{ab}) = \tfrac{1}{3}\Delta q_a + Q^2 v_a + [(\mu+p)\Delta + Q\mathscr{L}_\xi Q + 2\psi Q^2]u_a\,, \tag{2.25}$$

$$2\eta\sigma_{ab}v^b q^a = Q(\mathscr{L}_\xi Q + 2\psi Q) + (\mu + p)\Delta\,, \tag{2.26}$$

$$2q^a q^b \mathscr{L}_\xi(\eta\sigma_{ab}) = \tfrac{4}{3}\Delta Q^2\,. \tag{2.27}$$

The case in which T_{ab} is given by (1.2) has been discussed in Ref. 7. However, Ref. 7 assumed that $\mu + p_{\parallel} \neq 0$. As we shall see, space-times admitting a SCKV and satisfying the field equations for an anisotropic fluid must have $\mu = -p_{\parallel} = \frac{1}{2}R$ and $p_\perp = 0$, so that the energy-momentum tensor is limited to the form

$$T_{ab} = \tfrac{1}{2}R(u_a u_b - n_a n_b)\,. \tag{2.28}$$

For a SCKV we have $\mathscr{L}_\xi T_{ab} = 0$, $\mathscr{L}_\xi R = -2\psi R$ and since n_a is a unit vector,

$$\mathscr{L}_\xi n_a = \psi n_a + m_a\,, \tag{2.29}$$

where $m_a n^a = 0$. From Eqs. (2.29) and (2.4), the Lie derivative of (2.28) yields

$$0 = \tfrac{1}{2}R(v_a u_b + v_b u_a - m_a n_b - m_b n_a)\,.$$

Since R cannot be zero for a nonvacuum solution, this implies that

$$v_a = \Sigma n_a, \quad m_a = \Sigma u_a\,, \tag{2.30}$$

where $\Sigma = -u^a m_a = n^a v_a$. It follows that either $v_a = m_a = 0$ or v_a, m_a are parallel to u_a, n_a, respectively. Thus the result that $v_a = m_a = 0$, given in Ref. 7 and based on the assumption $\mu + p_{\parallel} \neq 0$, is not necessarily true.

III. SYMMETRY INHERITANCE

If a perfect fluid space-time is self-similar, i.e., admits a HV ξ^a, the density, pressure, and fluid velocity must satisfy

$$\mathscr{L}_\xi \mu = -2\psi\mu, \quad \mathscr{L}_\xi p = -2\psi p, \quad \mathscr{L}_\xi u_a = \psi u_a\,,$$

and we say that these quantities inherit the space-time symmetry defined by ξ^a. In contrast, if a space-time admitting a HV satisfies Einstein's field equations with T_{ab} given by (1.1), then in general, the symmetry is not inherited by the dynamical and kinematical quantities appearing in T_{ab}. However, if we impose self-similarity on the complete solution, dimensional considerations[9] will imply that the following set of equations will hold:

$$\mathscr{L}_\xi \mu = -2\psi\mu, \quad \mathscr{L}_\xi p = -2\psi p, \quad \mathscr{L}_\xi u_a = \psi u_a\,,$$
$$\mathscr{L}_\xi q_a = -\psi q_a, \quad \mathscr{L}_\xi \sigma_{ab} = \psi\sigma_{ab}, \quad \mathscr{L}_\xi \eta = -\psi\eta\,. \tag{3.1}$$

For a SCKV, there is no self-similarity unless the SCKV is in fact a HV. However, we will now make the following definition.

Definition: If the space-time solution of Einstein's field equations with T_{ab} given by (1.1) admits a SCKV ξ^a given by (1.3), then the solution will be said to inherit the symmetry corresponding to ξ^a if the set of equations (3.1) holds.

In this section we investigate the conditions under which an imperfect fluid given by (1.1) will inherit the symmetries corresponding to a SCKV ξ^a. We also comment on the symmetry inheritance for an anisotropic fluid (1.2), thus extending the work of Maartens *et al.*[7] to a crucial case which they omitted. Throughout this investigation we require that the fluid satisfies the dominant energy condition.

We consider a number of possible cases.

Case 1: In Ref. 7 it is shown that when $q^a = 0$, Eqs. (3.1) will hold provided that $\mu + p \neq 0$ and either $\mathscr{L}_\xi u_a = \psi u_a$ (i.e., $v_a = 0$) or $\mathscr{L}_\xi(\eta\sigma_{ab}) = 0$. We now complete this result by considering the exceptional case $\mu + p = 0$. Equation (2.23) becomes $2\eta\sigma_{ab}v^b = 0$, so that σ_{ab} must be of the form $\sigma_{ab} = \sigma(x_a x_b - y_a y_b)$, where x_a, y_a are orthogonal unit spacelike vectors which are also orthogonal to u_a and v_a. By applying the dominant energy condition to the resulting T_{ab}, we find that $\eta\sigma = 0$, so that the fluid degenerates into a perfect fluid with $\mu + p = 0$. Hence, Eqs. (3.1) hold for an imperfect fluid and we have the following theorem.

Theorem 1: If $q^a = 0$ and if either $\mathscr{L}_\xi u_a = \psi u_a$ or $\mathscr{L}_\xi(\eta\sigma_{ab}) = 0$, the symmetries of a SCKV ξ^a are inherited.

Case 2: Suppose that $\mathscr{L}_\xi(\eta\sigma_{ab}) = 0$. Contracting Eq. (2.25) with q^a yields $\frac{4}{3}\Delta Q^2 = 0$, so that either $\Delta = 0$ or $Q = 0$. The latter case immediately leads to inheritance from Theorem 1, so we consider $\Delta = 0$. Equation (2.25) then becomes

$$Q^2 v_a + Q(\mathscr{L}_\xi Q + 2\psi Q)u_a = 0\,,$$

which implies that $v_a = 0$ and $\mathscr{L}_\xi Q + 2\psi Q = 0$. Equation (2.23) then implies that $w_a = 0$, and so we have the following theorem.

Theorem 2: If $\mathscr{L}_\xi(\eta\sigma_{ab}) = 0$, the symmetries of a SCKV ξ^a are inherited.

Corollary 1: If $\eta\sigma_{ab} = 0$, the symmetries of a SCKV ξ^a are inherited.

Case 3: Suppose that $\mathscr{L}_\xi u_a = \psi u_a$, i.e., fluid flow lines are mapped onto fluid flow lines. In this case $v_a = 0$, $\Delta = 0$, and Eq. (2.24) becomes $\mathscr{L}_\xi(\eta\sigma_{ab}) = 0$, so that Theorem 2 leads to the following theorem.

Theorem 3: If $\mathscr{L}_\xi u_a = \psi u_a$, the symmetries of a SCKV ξ_a are inherited.

Corollary 2: If a SCKV ξ^a is parallel to u^a, then the symmetries of ξ^a are inherited.

Theorems 2 and 3 are the primary results in this paper; they are new results which generalize the work of Ref. 7 to viscous fluids with nonzero heat conduction. Theorems 2 and 3 are definitive in that they give a complete characterization of the SCKV inheritance problem for the fluid (1.1). Note that from Theorems 1 and 2, the vanishing of the shear viscosity is sufficient to ensure inheritance, whereas the vanishing of the heat conduction is not sufficient. However, as we shall see in Case 4, there are conditions on q^a which will ensure inheritance.

Case 4: Suppose that q^a is an eigenvector of the shear tensor, i.e.,

$$2\eta\sigma_{ab}q^b = \lambda q_a \,. \quad (3.2)$$

Equation (3.2) implies that, geometrically, q^a and u^a span a timelike invariant two-space of T_{ab}[8] and, physically, there exist no shear velocities between neighborhood surface elements orthogonal to the direction of the heat flux.[11] Taking the Lie derivative of (3.2) and using Eqs. (2.21)–(2.24) we obtain

$$[Q(\mathscr{L}_\xi Q + 2\psi Q) - (\mu + p)\Delta]u_a - \Delta q_a + Q^2 v_a - \lambda w_a + 2\eta\sigma_{ab}w^b = 0\,, \quad (3.3)$$

$$\mathscr{L}_\xi \lambda + 2\lambda\psi = \tfrac{4}{3}\Delta \,. \quad (3.4)$$

Relations (3.3) and (3.4) do not imply inheritance. However, if we make the additional assumption

$$\mathscr{L}_\xi q_a = -\psi q_a \,, \quad (3.5)$$

i.e., $w_a = 0$, $\Delta = 0$, and $\mathscr{L}_\xi Q = -2\psi Q$, then (3.3) shows that $v_a = 0$, so that from Theorem 3, we have complete inheritance.

Conversely, if we first assume (3.5), the Lie derivative of (2.23) contracted with $q^a v^b$ leads to

$$v_b v^b Q^2 + 2\eta\sigma_{ab}q^a \mathscr{L}_\xi v^b = (\mu + p)q^a \mathscr{L}_\xi v_a \quad (3.6)$$

and since (3.5) implies that $q^a \mathscr{L}_\xi v_a = 0$, this shows that the imposition of (3.2) leads to inheritance. On the other hand, when $v_a \neq 0$, i.e., if we have noninheritance, then (3.6) shows that q^a cannot be an eigenvector of σ_{ab}. Thus we have proved the following theorem.

Theorem 4: If q_a is an eigenvector of σ_{ab} and, also, $\mathscr{L}_\xi q_a = -\psi q_a$, then the symmetries of a SCKV ξ^a are inherited. If either of these conditions does not hold, then the symmetries are not inherited. Furthermore, if the symmetries are not inherited q^a cannot be an eigenvector of σ_{ab}.

Theorem 4 is a new result since it requires $q^a \neq 0$, a situation that has not been investigated for SCKV's.

It should be emphasized that the results given here apply not only to SCKV's, but also to HV's. Indeed, when ψ is a constant, Theorems 2 and 3 give the definitive conditions under which the physical quantities constituting the energy-momentum tensor (1.1) can inherit the self-similar symmetry associated with a HV.

These results apply also to KV's ($\psi = 0$) and are again new since no investigation has been made of KV's in a viscous, heat-conducting fluid. The major results for KV's may be summarized in the following theorem.

Theorem 5: If ξ^a is a KV of space-time satisfying Einstein's field equations for an imperfect fluid with the energy-momentum tensor (1.1), then the necessary and sufficient condition for the symmetry defined by ξ^a to be inherited by the physical quantities associated with the fluid, i.e., for the quantities

$$\mathscr{L}_\xi \mu = \mathscr{L}_\xi p = \mathscr{L}_\xi u_a = \mathscr{L}_\xi \eta = \mathscr{L}_\xi \sigma_{ab} = 0$$

to hold, is that $\mathscr{L}_\xi(\eta\sigma_{ab}) = 0$ or, equivalently, $\mathscr{L}_\xi u_a = 0$.

We now turn to the case of an anisotropic fluid given by (1.2) and, in particular, to the special case $\mu + p_\parallel = 0$, $p_\perp = 0$, for which T_{ab} is given by (2.28). Since $\mu = -p_\parallel = \frac{1}{2}R$, it follows that $\mathscr{L}_\xi \mu = -2\psi\mu$ and $\mathscr{L}_\xi p_\parallel = -2\psi p_\parallel$, so that μ and $p_\parallel$ always inherit the symmetry of the SCKV ξ^a. However, as shown in Sec. II, v_a and m_a [as defined by (2.29)] are either zero or nonzero and parallel to n_a, u_a, respectively, so that the symmetry of the SCKV may or may not be inherited. Maartens *et al.*[7] showed that in general, v_a is given by

$$v_a = 2\omega_{ab}\xi^b + \alpha\dot{u}_a - \alpha_{,b}h^a{}_b \,, \quad (3.7)$$

where $\alpha = -\xi_a u^a$ and ω_{ab} is the vorticity tensor. Complete inheritance occurs when $v_a = 0$; two special cases of this are when ξ^a is parallel to u^a and, also, when ξ^a is orthogonal to u^a and the fluid is vorticity-free, i.e., $\xi^a u_a = 0$ and $\omega_{ab} = 0$. Hence we have the following theorem.

Theorem 6: For an anisotropic fluid of the form (1.2) with $\mu + p_\parallel = 0$, $p_\perp = 0$, the symmetries of a SCKV ξ^a are inherited if and only if expression (3.7) for v_a is zero.

IV. SPACE-TIMES ADMITTING A PROPER SCKV

We now turn to the problem of determining those space-times that admit a proper SCKV. A SCKV is defined by (1.3) with $\psi_{;ab} = 0$. This implies that $\psi_{,a}$ is a covariantly constant, hypersurface orthogonal, geodesic vector, resulting in considerable simplification of the space-time metric.[12] It also implies that $\psi_{,a}$ is globally timelike, globally spacelike, or globally null. We consider these three possibilities in turn.

A. The vector $\psi_{,a}$ timelike

We can choose coordinates in which

$$\psi_{,a} = (-1,0,0,0) \quad (4.1)$$

and the metric is of the form

$$ds^2 = -dt^2 + g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta \equiv -dt^2 + d\Omega^2 \,, \quad (4.2)$$

where $\alpha, \beta, \gamma = 1, 2, 3$.

Equation (4.1) implies that $\psi = -t$ and the metric (4.2) implies that $\{_j{}^i{}_k\} = 0$ if any of $i, j, k = 0$. Equations (1.3) take the form

$$\xi_{0,0} = t\,, \quad (4.3)$$

$$\xi_{0,\alpha} + \xi_{\alpha,0} = 0\,, \quad (4.4)$$

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = -2tg_{\alpha\beta} \,. \quad (4.5)$$

Integrating (4.3) yields

$$\xi_0 = \tfrac{1}{2}t^2 + A(x^\gamma)\,, \quad (4.6)$$

where A is a scalar function and (4.4) and (4.6) yield

$$\xi_{\alpha,0} = -A_{,\alpha}\,,$$

i.e.,

$$\xi_\alpha = -A_{,\alpha}t + B_\alpha(x^\gamma)\,, \quad (4.7)$$

where B_α is a vector function. Substituting (4.7) into (4.5) yields

$$-2A_{;\alpha\beta}t + B_{\alpha;\beta} + B_{\beta;\alpha} = -2tg_{\alpha\beta}$$

and equating coefficients of t we obtain

$$B_{\alpha;\beta} + B_{\beta;\alpha} = 0\,, \quad (4.8)$$

so that B_α is a KV of the three-dimensional space and

$$A_{;\alpha\beta} = g_{\alpha\beta} \,. \quad (4.9)$$

Petrov[13] quotes a result due to Sinyukov,[14] namely that if a V_n admits a vector field ϕ_α satisfying $\phi_{\alpha;\beta} = \rho g_{\alpha\beta}$, where ρ is a nonzero scalar function, a system of coordinates exists in which the metric takes the form

$$ds_n^2 = g_{11}(dx^1)^2 + (1/g_{11})\Gamma_{pq}(x^2,...,x^n)dx^p dx^q, \quad (4.10)$$

where $p, q \neq 1$; $g_{11} = [2\int\rho(x^1)dx^1 + C]^{-1}$; and ρ is now an arbitrary function of x^1 only. Applying the result (4.10) to Eq. (4.9), in which $\phi_\alpha = A_{,\alpha}$ and $\rho = 1$, we find that for the three-dimensional metric $d\Omega^2$, $g_{11} = (2x^1 + C)^{-1}$ and the transformation $\sqrt{2x^1 + C} \to x$ yields

$$d\Omega^2 = dx^2 + x^2\Gamma_{AB}(x^C)dx^A dx^B,$$

where A, B, C take the values 2, 3. The two-dimensional metric $\Gamma_{AB}\, dx^A dx^B$ can be transformed into $dy^2 + f^2(y,z)dz^2$, so that the space-time metric takes the final form

$$ds^2 = -dt^2 + dx^2 + x^2[dy^2 + f^2(y,z)dz^2]. \quad (4.11)$$

After excluding linear combinations with the KV admitted by this metric, we find that only one SCKV exists, namely

$$\xi^a = (-\tfrac{1}{2}t^2 - \tfrac{1}{2}x^2, -tx, 0, 0). \quad (4.12)$$

which is timelike.

B. The vector $\psi_{,a}$ spacelike

We can choose coordinates in which

$$\psi_{,a} = (0,1,0,0), \quad (4.13)$$

so that $\psi = x$ and the metric will be of the form

$$ds^2 = dx^2 + g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta,$$

where, in this case, $\alpha, \beta, \gamma = 0, 2, 3$. Following precisely the same argument as in the timelike case, we obtain two possible solutions, namely

$$ds^2 = dx^2 - dt^2 + t^2[dy^2 + g^2(y,z)dz^2] \quad (4.14)$$

and

$$ds^2 = dx^2 + dy^2 + y^2[-dt^2 + h^2(t,z)dz^2]. \quad (4.15)$$

However, the metric (4.15) does not satisfy the dominant energy condition and so will be discarded.

The metric (4.14) admits only one proper SCKV, namely

$$\xi^a = (xt, \tfrac{1}{2}x^2 + \tfrac{1}{2}t^2, 0, 0); \quad (4.16)$$

this SCKV is spacelike.

C. The vector $\psi_{,a}$ null

Since $\psi_{,a}$ is a gradient vector and a null KV, it follows that we have a generalized *pp*-wave space-time[15] with a metric of the form

$$ds^2 = P^{-2}(dx^2 + dy^2) - 2\,du(dv - m\,dx + H\,du), \quad (4.17)$$

where H, P, and m are arbitrary functions of u, x, and y only. We label the coordinates $(u,v,x,y) \equiv (x^0, x^1, x^2, x^3)$ and then the null KV $k^a = \psi^{,a}$ is given by $k^a = (0,1,0,0)$, i.e., $k_a = (-1,0,0,0)$, so that

$$\psi = -u. \quad (4.18)$$

When $R = 0$, it can be shown that the general metric admitting a covariantly constant null gradient vector is (4.17) with $P = 1$ and $m = 0$,[15] i.e.,

$$ds^2 = dx^2 + dy^2 - 2\,du\,dv - 2H\,du^2. \quad (4.19)$$

However, in general, the imperfect and anisotropic fluids considered in this article will not have zero Ricci scalar, so we will use the metric (4.17). We require those metrics of this form which admit a SCKV.

The nonzero components of the Ricci tensor for the metric (4.17) are

$$\begin{aligned} R_{00} &= P^2(H_{xx} + H_{yy} + m_{ux} + \tfrac{1}{2}m_y^2 P^2) \\ &\quad + 2P^{-2}(PP_{uu} - 2P_u^2), \\ R_{02} &= -\tfrac{1}{2}m_{yy}P^2 - m_y PP_y + P^{-2}(PP_{ux} - P_u P_x), \\ R_{03} &= \tfrac{1}{2}m_{xy}P^2 + m_y PP_x + P^{-2}(PP_{uy} - P_u P_y), \\ R_{22} &= R_{33} = P^{-2}(PP_{xx} + PP_{yy} \\ &\quad - P_x^2 - P_y^2) = \tfrac{1}{2}P^{-2}R. \end{aligned} \quad (4.20)$$

Recalling (4.18), the SCKV equations are

$$\begin{aligned} \xi_{0,0} &= (H_u + mm_u P^2 + mH_x P^2)\xi_1 \\ &\quad + (m_u + H_x)P^2\xi_2 + H_y P^2\xi_3 + 2Hu, \\ \xi_{0,1} + \xi_{1,0} &= 2u, \\ \xi_{0,2} + \xi_{2,0} &= 2(H_x - mP^{-1}P_u)\xi_1 \\ &\quad - 2P^{-1}P_u\xi_2 - m_y P^2\xi_3 - 2mu, \\ \xi_{0,3} + \xi_{3,0} &= 2(H_y + \tfrac{1}{2}m_y mP^2)\xi_1 \\ &\quad + m_y P^2\xi_2 - 2P^{-1}P_u\xi_3, \\ \xi_{1,1} &= \xi_{1,2} + \xi_{2,1} = \xi_{1,3} + \xi_{3,1} = 0, \\ \xi_{2,2} &= -(P^{-3}P_u + mP^{-1}P_x + m_x)\xi_1 \\ &\quad - P^{-1}P_x\xi_2 + P^{-1}P_y\xi_3 - uP^{-2}, \\ \xi_{2,3} + \xi_{3,2} &= -(m_y + 2mP^{-1}P_y)\xi_1 \\ &\quad - 2P^{-1}P_y\xi_2 - 2P^{-1}P_x\xi_3, \\ \xi_{3,3} &= -(P^{-3}P_u - mP^{-1}P_y)\xi_1 \\ &\quad + P^{-1}P_x\xi_2 - P^{-1}P_y\xi_3 - uP^{-2}. \end{aligned} \quad (4.21)$$

Solving equations (4.21), we find that the most general form of the SCKV when $R \neq 0$ is

$$\begin{aligned} \xi^a = [&-(u^2 + \alpha u + \beta), \alpha v - D(u,x,y) + (2H + m^2P^2) \\ &\times (u^2 + \alpha u + \beta) + mP^2B(u,x,y), \\ &mP^2(u^2 + \alpha u + \beta) + P^2B(u,x,y), P^2C(u,x,y)], \end{aligned} \quad (4.22)$$

where α and β are arbitrary constants and B, C, and D are three functions satisfying the differential equations

$$\begin{aligned} D_u &= (H_u + mm_u P^2 + mH_x P^2)(u^2 + \alpha u + \beta) \\ &\quad + (m_u + H_x)P^2B + H_y P^2C + 2Hu, \\ B_x &= -(P^{-3}P_u + mP^{-1}P_x + m_x)(u^2 + \alpha u + \beta) \\ &\quad - P^{-1}P_x B + P^{-1}P_y C - uP^{-2}, \\ C_y &= -(P^{-3}P_u - mP^{-1}P_x)(u^2 + \alpha u + \beta) \end{aligned}$$

$$+P^{-1}P_xB - P^{-1}P_yC - uP^{-2}, \quad (4.23)$$

$$D_x + B_u = 2(H_x - mP^{-1}P_u)(u^2 + \alpha u + \beta) - 2P^{-1}P_uB - m_yP^2C - 2mu,$$

$$D_y + C_u = 2(H_y + \tfrac{1}{2}m_y mP^2)(u^2 + \alpha u + \beta) + m_yP^2B - 2P^{-1}P_uC,$$

$$B_y + C_x = -(m_y + 2mP^{-1}P_y)(u^2 + \alpha u + \beta) - 2P^{-1}P_yB - 2P^{-1}P_xC.$$

By eliminating α, β, B, C, and D from Eqs. (4.23), an expression connecting H, P, m, and their derivatives will be obtained which delineates those members of the general set of space-times with the metric (4.17) which admit a SCKV.

In the special case of the metric (4.19), i.e., when $R = 0$, the SCKV is of the form

$$\xi^a = [-(u^2 + \alpha u + \beta), \alpha v - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 + J_u x + K_u y + L(u), -ux + \gamma y + J(u), -uy - \gamma x + K(u)], \quad (4.24)$$

where α, β, and γ are arbitrary constants and J, K, and L are arbitrary functions of u only. In order to admit a SCKV, the function H in the metric must satisfy

$$H_u(u^2 + \alpha u + \beta) + H_x(ux - \gamma y - J) + H_y(uy + \gamma x - K) + 2H(u + \alpha) - J_{uu}x - K_{uu}y + L_u = 0. \quad (4.25)$$

Thus far in this section, we have found all space-times admitting a SCKV irrespective of the field equations that they satisfy. These are the space-times with metrics given by (4.11), (4.14), (4.15), and those metrics (4.17) that satisfy Eqs. (4.23). Of these, (4.15) and some members of the set (4.17) do not satisfy the dominant energy condition; we shall confine our attention only to those space-times that do satisfy this condition.

The integrability conditions for the existence of a covariantly constant vector $\psi_{,a}$ are

$$R^a{}_{bcd}\psi_{,a} = 0; \quad (4.26)$$

by contraction, this implies

$$T^a{}_b\psi_{,a} = -\tfrac{1}{2}R\psi_{,b}, \quad (4.27)$$

so that $\psi_{,a}$ is an eigenvector of the energy-momentum tensor. In the cases of the metrics (4.11) and (4.14), by calculating the Einstein tensor, each of these solutions possesses a timelike and a spacelike eigenvector in the tx plane which have the same eigenvalues, so that there are two independent null eigenvectors in the tx plane, namely

$$k_a = (1/\sqrt{2})(-1,1,0,0), \quad l_a = (1/\sqrt{2})(1,1,0,0), \quad (4.28)$$

where we have normalized the null vectors to satisfy $k_a l^a = 1$. Furthermore, there exist two spacelike eigenvectors in the yz plane, each of which has a zero eigenvalue. Hence, it follows that for the two metrics (4.11) and (4.16), T_{ab} is of Segré type $\{(1,1)(11)\}$[16] and can be written in the form

$$T_{ab} = -\tfrac{1}{2}R(k_a l_b + k_b l_a). \quad (4.29)$$

In the case of the space-times with the metric (4.17), Eq. (4.27) shows that $\psi_{,a}$ is a null eigenvector of the energy-momentum tensor. Since we also require T_{ab} to satisfy the dominant energy condition, this implies[16] that T_{ab} must be either of Segré type $\{(1,1)11\}$ or $\{2,11\}$ and so can be written, respectively, in the forms

$$T_{ab} = -A(k_a l_b + k_b l_a) + Cx_a x_b + Dy_a y_b \quad (4.30)$$

or

$$T_{ab} = -A(k_a l_b + k_b l_a) + Bk_a k_b + Cx_a x_b + Dy_a y_b, \quad (4.31)$$

where k_a, l_a are null vectors with $k_a l^a = 1$ and x_a, y_a are mutually orthogonal unit spacelike vectors which are also orthogonal to k_a and l_a. The quantities A, B, C, and D are scalar functions of the coordinates. The eigenvectors are k_a ($\equiv \psi_{,a}$), x_a, and y_a in the second case and, additionally, l_a in the first case.

From (4.20), the nonzero components of the Einstein tensor are G_{00}, G_{01} ($= \tfrac{1}{2}R$), G_{02}, and G_{03}. Equating G_{ab} with T_{ab} given by the more general expression (4.31) and using the fact that $k^a = (0,1,0,0)$, so that $l_1 = 1, x_1 = y_1 = 0$, we find that

$$A = \tfrac{1}{2}R, \quad C = D = 0,$$

so that T_{ab} is given by

$$T_{ab} = -\tfrac{1}{2}R(k_a l_b + k_b l_a) + Bk_a k_b, \quad (4.32)$$

which includes the form (4.29) when $B = 0$. Thus T_{ab} is either of Segré type $\{(1,1)(11)\}$ or $\{2(11)\}$ and we have proved the following theorems.

Theorem 7: A space-time that admits a SCKV and satisfies the dominant energy condition has an energy-momentum tensor which admits two independent spacelike eigenvectors with zero eigenvalues and, also, admits either two independent null eigenvectors with the same eigenvalue or a repeated null eigenvector, i.e., the Segré type of the energy-momentum tensor is either $\{(1,1)(11)\}$ or $\{2(11)\}$. The energy-momentum tensor has the form (4.32), where $B = 0$ or $B \neq 0$ according to whether T_{ab} is $\{(1,1)(11)\}$ or $\{2(11)\}$, respectively.

Theorem 8: There exist no perfect fluid space-times which admit a SCKV.

Having established the Segré type of the energy-momentum tensor, we shall investigate the field equations that are satisfied by the SCKV space-times.

For the metric (4.11), the only nonzero components of the Einstein tensor are

$$G^0_0 = G^1_1 = x^{-2}(1 + f^{-1}f_{yy}) \quad (4.33)$$

and, assuming a comoving velocity $u^a = (1, 0, 0, 0)$, the metric satisfies the field equations for an anisotropic fluid, with T_{ab} given by (1.2). We find that

$$\mu = -p_{\parallel} = -x^{-2}(1 + f^{-1}f_{yy}), \quad p_{\perp} = 0, \quad (4.34)$$

and we must have $1 + f^{-1}f_{yy} < 0$ for the dominant energy condition to be satisfied. Alternatively, if we assume a non-comoving velocity of the form

$$u^a = (\cosh\phi, \sinh\phi, 0, 0) , \qquad (4.35)$$

where $\phi = \phi(t,x)$, then the metric satisfies the viscous fluid field equations, with T_{ab} given by (1.1), with

$$\mu = -3p = 2\eta X = -x^{-2}(1 + f^{-1}f_{yy}), \quad Q = 0, \qquad (4.36)$$

where

$$X = (\phi_t - x^{-1})\sinh\phi + \phi_x \cosh\phi . \qquad (4.37)$$

Note that for $\mu > 0$, $\eta \geqslant 0$, we must have $1 + f^{-1}f_{yy} < 0$ and $X \geqslant 0$.

The space-time with the metric (4.14) has similar properties; the nonzero components of the Einstein tensor are

$$G^0_0 = G^1_1 = -t^{-2}(1 - f^{-1}f_{yy}) \qquad (4.38)$$

and the field equations for a comoving anisotropic fluid are satisfied, with

$$\mu = -p_\parallel = t^{-2}(1 - f^{-1}f_{yy}), \quad p_\perp = 0, \qquad (4.39)$$

so that $1 - f^{-1}f_{yy} > 0$ for $\mu > 0$. The space-time (4.14) also satisfies the viscous fluid field equations with u^a of the form (4.35) and

$$\mu = -3p = 2\eta X = t^{-2}(1 - f^{-1}f_{yy}), \quad Q = 0, \qquad (4.40)$$

where

$$X = \phi_t \sinh\phi + (\phi_x - t^{-1})\cosh\phi \qquad (4.41)$$

and we must have $1 - f^{-1}f_{yy} < 0$ and $X \geqslant 0$.

Thus each of the space-times (4.11) and (4.14) may represent an infinite set of viscous fluid solutions depending on the choice of the "tilt function" $\phi(t,x)$ and, in the particular case when $\phi = 0$, i.e., when u^a is comoving, the viscous fluid solution degenerates into the anisotropic solution given by (4.34) or (4.39).

We now investigate those members of the set of space-times with the metric (4.17), if any, which satisfy the field equations for an anisotropic fluid. Since the rhs of (1.2) is obviously diagonalizable, those solutions for which T_{ab} is of the type $\{2(11)\}$ cannot represent an anisotropic fluid since a T_{ab} of this Segré type is not diagonalizable. Hence, the only possibility for a SCKV space-time to satisfy the field equations with T_{ab} given by (1.2) is for the energy-momentum tensor to be of the form (4.29), i.e., we must have

$$\mu u_a u_b + p_\parallel n_a n_b + p_\perp (x_a x_b y_a y_b) = -\tfrac{1}{2}R(k_a l_b + k_b l_a) , \qquad (4.42)$$

where x_a, y_a are two mutually orthogonal spacelike unit vectors in the two-space orthogonal to that of u_a and n_a.

Contracting (4.42) with $l^b u^a$ we obtain

$$-\mu u_b l^b = -\tfrac{1}{2}R u^a l_a$$

and since u_a is timelike and l_a is null, $u^a l_a \neq 0$, so that

$$\mu = \tfrac{1}{2}R . \qquad (4.43)$$

Contracting in turn with $l^b n^a$ and $k^b n^a$ we obtain

$$p_\parallel l^b n_b = -\tfrac{1}{2}R l_a n^a, \quad p_\parallel k^b n_b = -\tfrac{1}{2}R k_a n^a ,$$

so that either $p_\parallel = -\tfrac{1}{2}R$ or $l^b n_b = k^b n_b = 0$. However, if we contract (4.42) with $n^a n^b$ we obtain

$$p_\parallel = -R k_a n^a l_b n^b ,$$

so that $l^b n_b = k^b n_b = 0$ implies that $p_\parallel = 0$. Hence, we have two possibilities, namely

(i) $p_\parallel = -\tfrac{1}{2}R, \quad l^b n_b \neq 0, \quad k^b n_b \neq 0$

or

(ii) $p_\parallel = 0, \quad l^b n_b = k^b n_b = 0 .$

These are two distinct possibilities since we discard the case when $R = 0$, which implies a vacuum solution.

Noting that the contraction of (4.42) with g^{ab} yields

$$\mu - p_\parallel - 2p_\perp = R \qquad (4.44)$$

and taking into account (4.43), possibility (i) leads to $p_\perp = 0$, while (ii) implies that $p_\perp = -\tfrac{1}{4}R$. Contracting (4.42) with $k^b x^a$ we obtain $p_\perp k^b x_b = -\tfrac{1}{2}R k^a x_a$ and since $p_\perp = -\tfrac{1}{2}R$ is not a possibility, it follows that $k^a x_a = 0$. Similarly, we can show that $k^a y_a = l^a x_a = l^a y_a = 0$, so that k^a, l^a lie entirely in the two-plane of u^a and n^a. Since k^a and l^a are null vectors this implies that $k^a n_a$ and $l^a n_a$ cannot be zero, so that (i) is the only possibility. Hence, we have proved the following theorem.

Theorem 9: If a space-time satisfies the field equations for an anisotropic fluid, with T_{ab} given by (1.2), and also admits a SCKV, then, necessarily,

$$\mu = -p_\parallel = \tfrac{1}{2}R, \quad p_\perp = 0, \qquad (4.45)$$

i.e., the energy-momentum tensor is of the form

$$T_{ab} = \tfrac{1}{2}R(u_a u_b - n_a n_b) . \qquad (4.46)$$

As we have seen, the assumption $\mu + p_\parallel \neq 0$ cannot hold for an anisotropic fluid admitting a SCKV and thus Theorem 9 invalidates some of the results given in Ref. 7, in which this specific assumption was made.

We now show that the general class of metrics (4.17), satisfying conditions (4.23) for the existence of a SCKV, does indeed contain space-times with energy-momentum tensors of the forms (1.1) and (1.2). In the case of the viscous fluid, it is known that the conformally flat null electrovac space-time, which is a special case of the simpler metric (4.19) with $H = f(u)(x^2 + y^2)$, where f is an arbitrary function of u, can be interpreted as representing a viscous fluid.[17] Here we note that the space-time (4.17), with

$$P = u^{-1}e^{x^2 + y^2}, \quad m = 0, \quad H = P^2 + 1, \qquad (4.47)$$

admits the timelike SCKV $\xi^a = (-u^2, u^2, 0, 0)$ and satisfies the viscous fluid field equations, with T_{ab} given by (1.1), with

$$\begin{aligned} \mu &= 4P^2(P^2+1)^{-1}[1 + 2P^2(1 + x^2 + y^2)] , \\ P &= \tfrac{4}{3}P^2(P^2+1)^{-1}[2P^2(x^2+y^2) - 1] , \\ \eta &= 3\times 2^{1/2}(P^2+1)^{1/2}[2P^2(x^2+y^2) - 1]u , \qquad (4.48) \\ Q &= 4P^4(P^2+1)^{-1}(1 + 2x^2 + 2y^2) , \\ u^a &= [2^{-1/2}(P^2+1)^{-1/2}, 0, 0, 0] , \\ q_a &= Q\,[0, 2^{-1/2}(P^2+1)^{-1/2}, 0, 0] , \end{aligned}$$

where, in order that $\eta \geqslant 0$, the solution is confined to that region of space-time for which $2P^2(x^2+y^2) \geqslant 1$, i.e., $2(x^2+y^2)e^{2(x^2+y^2)} \geqslant u^2$. Note that this restriction also ensures $p \geqslant 0$.

To represent an anisotropic fluid, T_{ab} must be of Segré type $\{(1,1)(11)\}$; this will be the case for the metric (4.17) if the following contribution holds:

$$RR_{00} = 2P^2(R_{02}^2 + R_{03}^2), \quad R \neq 0, \tag{4.49}$$

where the Ricci tensor components are given by (4.20). As an example, condition (4.49) is satisfied, with $R_{00} = R_{02} = R_{03} = 0$, by

$$P = u^{-1}e^{x^2+y^2}, \quad m = 0, \quad H_{xx} + H_{yy} = 0; \tag{4.50}$$

and the space-time satisfies the field equations for an anisotropic fluid with $R = 8P^2$,

$$\begin{aligned} u^a &= (2^{-1/2}H^{-1/2},0,0,0), \\ n^a &= (2^{-1/2}H^{-1/2}, -2^{1/2}H^{1/2},0,0), \end{aligned} \tag{4.51}$$

and $\mu, p_\parallel$, and $p_\perp$ given by Eq. (4.45). It is interesting to note that among this class of solutions given by Eq. (4.50), there are several possible behaviors for the SCKV ξ^a. For example, the following three choices for H satisfying (4.50):

$$\text{(i)} \quad H = \ln(x^2+y^2), \tag{4.52}$$

$$\text{(ii)} \quad H = u^{-2}\ln(x^2+y^2), \tag{4.53}$$

$$\text{(iii)} \quad H = u^{-2}, \tag{4.54}$$

lead to the following SCKV:

$$\text{(i)} \quad \xi^a = [-u^2, u^2\ln(x^2+y^2),0,0], \tag{4.55}$$

$$\text{(ii)} \quad \xi^a = (-u^2,0,0,0), \tag{4.56}$$

$$\text{(iii)} \quad \xi^a = (-u^2,2,0,0), \tag{4.57}$$

respectively. In case (i), ξ^a is null; in case (ii), ξ^a is timelike and parallel to u^a; and in case (iii), ξ^a is spacelike and parallel to n^a. Note that these space-times also admit a viscous fluid interpretation with a noncomoving velocity, as in the case of solutions (4.11) and (4.14).

Finally, we note that the space-times (4.11) and (4.14) contain no nontrivial vacuum solutions since when $R_{ab} = 0$, the space-times are flat. However, the metric (4.19) does contain vacuum space-times, namely the vacuum *pp*-wave solutions which satisfy the condition

$$H_{xx} + H_{yy} = 0. \tag{4.58}$$

Thus we have the following theorem.

Theorem 10: The only vacuum space-times admitting a proper SCKV are the *pp*-wave solutions of the form (4.19) with (4.58), which also satisfy condition (4.25).

V. EXAMPLES OF INHERITANCE PROPERTIES

Having found all space-times admitting a SCKV and satisfying the dominant energy condition, we now illustrate the theorems of Sec. III by investigating the inheritance properties of these solutions.

All the SCKV space-times can be intepreted as representing either viscous or anisotropic fluids, or both of these, except for the vacuum plane-wave solutions contained in the metric (4.19). These SCKV space-times may also admit other physical interpretations, such as a null electromagnetic field [the general null electrovac conformally flat space-time is contained in (4.19)] and a perfect fluid with an electromagnetic field (it was shown in Ref. 7 that an anisotropic fluid may be so interpreted). However, none of the SCKV space-times can be interpreted as a perfect fluid solution and none can be interpreted as a non-null electrovac solution. We are concerned here only with the viscous and anisotropic fluid interpretations; possible electromagnetic interpretations and their properties will be investigated elsewhere. We note that all spacetimes admitting a SCKV admit at least one null eigenvector, so that they form a very restricted set when interpreted as fluid space-times since, in general, neither the viscous fluid energy-momentum tensor (1.1) nor the anisotropic fluid energy-momentum tensor (1.2) admit a null eigenvector. For example, the FRW models, which have been shown to be solutions of the viscous fluid field equations,[18] do not admit a null eigenvector, in general. Furthermore, while the FRW models do not admit a SCKV, the $k=0$ models with the scale factor $R(t) = t^a$ admit a HV and thus will provide us further illustrative examples of the inheritance theorems.

We first consider the viscous fluid solutions. The solution (4.11), in its viscous fluid form given by (4.35)–(4.37), has $q_a = 0$; thus from Eqs. (2.21) and (2.22) it follows that $\mathscr{L}_\xi\mu + 2\psi\mu = 0$ and $\mathscr{L}_\xi p + 2\psi p = 0$, a fact that is easily confirmed by calculating $\mathscr{L}_\xi\mu$ and $\mathscr{L}_\xi p$ with respect to the SCKV (4.12). Upon calculating $\mathscr{L}_\xi u^a$ and $\mathscr{L}_\xi(\eta\sigma_{ab})$ we find that

$$v^a = V(\sinh\phi, \cosh\phi,0,0), \tag{5.1}$$

$$\begin{aligned} \mathscr{L}_\xi(\eta\sigma_{00}) &= \tfrac{1}{3}Vx^{-2}\sinh 2\phi(1+f^{-1}f_{yy}), \\ \mathscr{L}_\xi(\eta\sigma_{01}) &= -\tfrac{1}{3}Vx^{-2}\cosh 2\phi(1+f^{-1}f_{yy}), \\ \mathscr{L}_\xi(\eta\sigma_{11}) &= \tfrac{1}{3}Vx^{-2}\sinh 2\phi(1+f^{-1}f_{yy}), \end{aligned} \tag{5.2}$$

where

$$V = \tfrac{1}{2}(t^2+x^2)\phi_t + tx\phi_x - x \tag{5.3}$$

and all other components of $\mathscr{L}_\xi(\eta\sigma_{ab}) = 0$. It follows that if $v^a = 0$, then $V = 0$ and so $\mathscr{L}_\xi(\eta\sigma_{ab}) = 0$ and vice versa. Thus the SCKV symmetry is inherited if either $\mathscr{L}_\xi u^a = -\psi u^a$ or $\mathscr{L}_\xi(\eta\sigma_{ab}) = 0$, in accordance with Theorems 1–3. Note that the condition $V=0$ for inheritance implies that only those viscous models whose tilting velocity components satisfy this condition can inherit the SCKV symmetry.

The solution given by (4.14) and (4.39) to (4.41) behaves in a similar fashion. On the other hand, the solution given by (4.17), (4.47), and (4.48) cannot inherit the symmetry of the SCKV.

In order to illustrate the inheritance theorems in the case of a viscous fluid with nonzero heat conduction we turn to the FRW models. These models have an energy-momentum tensor of the Segré type $\{1,(1\,1\,1)\}$ and are thus commonly regarded as perfect fluid solutions. However, they can satisfy Einstein's field equations with an energy-momentum tensor of the form (1.1).[18] In such viscous fluid solutions, the four-velocity is necessarily tilting. While FRW models of any curvature can satisfy the viscous fluid field equations, we will consider here only $k=0$ models which admit a HV and, in particular, the Einstein–de Sitter model.

The known viscous fluid solutions with the Einstein–de Sitter metric falls into two classes. One class, known as *radial solutions*, is obtained by writing the metric in spherical polar coordinates and taking the four-velocity to have a nonzero radial component, while the second class, known as *axial solutions*, is obtained by using cylindrical polar coordinates and taking the four-velocity to have an axial component in the z direction. For our example we shall consider only the radial case in which the metric has the form

$$ds^2 = -dt^2 + t^{4/3}(dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\Phi^2) \tag{5.4}$$

and the four-velocity has the components

$$u^a = (\cosh\phi, t^{-2/3}\sinh\phi, 0, 0)\,, \tag{5.5}$$

where $\phi = \phi(t,r)$. The field equations then give the solution in the form

$$\begin{aligned} &\mu = \tfrac{4}{3}t^{-2}\cosh^2\phi, \quad p = \tfrac{4}{3}t^{-2}\sinh^2\phi,\\ &\eta X = -\tfrac{2}{3}t^{-2}\sinh^2\phi,\\ &q_a = \tfrac{4}{3}t^{-2}\sinh\phi\cosh\phi\\ &\qquad\times(\sinh\phi, -t^{2/3}\cosh\phi, 0, 0)\,, \end{aligned} \tag{5.6}$$

where

$$X = \phi_t \sinh\phi + t^{-2/3}\phi_r\cosh\phi - r^{-1}t^{-2/3}\sinh\phi\,. \tag{5.7}$$

The metric (5.4) admits the HV

$$\xi^a = (t, \tfrac{1}{3}r, 0, 0)\,, \tag{5.8}$$

corresponding to $\psi = 1$. Calculating $\mathscr{L}_\xi u^a$, $\mathscr{L}_\xi q_a$, and $\mathscr{L}_\xi(\eta\sigma_{ab})$ we find that

$$v^a = A(\sinh\phi, t^{-2/3}\cosh\phi, 0, 0)\,, \tag{5.9}$$

$$\begin{aligned} w_a = \tfrac{4}{3}At^{-2}[&(2\cosh^2\phi\\ &+\sinh^2\phi)\cosh\phi, t^{2/3}(\cosh^2\phi\\ &+2\sinh^2\phi)\sinh\phi, 0, 0]\,, \end{aligned} \tag{5.10}$$

$$\begin{aligned} &\mathscr{L}_\xi(\eta\sigma_{00}) = 4A\coth\phi\,\eta\sigma_{00}\,,\\ &\mathscr{L}_\xi(\eta\sigma_{01}) = A(3\coth\phi + \tanh\phi)\eta\sigma_{01}\,,\\ &\mathscr{L}_\xi(\eta\sigma_{11}) = 2A(\coth\phi + \tanh\phi)\eta\sigma_{11}\,,\\ &\mathscr{L}_\xi(\eta\sigma_{22}) = 2A\coth\phi\,\eta\sigma_{22}\,,\\ &\mathscr{L}_\xi(\eta\sigma_{33}) = 2A\coth\phi\,\eta\sigma_{33}\,, \end{aligned} \tag{5.11}$$

where

$$A = t\phi_t + \tfrac{1}{3}r\phi_r \tag{5.12}$$

and all other components of $\mathscr{L}_\xi(\eta\sigma_{ab}) = 0$. In addition, we find that $\mathscr{L}_\xi\mu = -2\psi\mu + 2A\mu\tanh\phi$ and $\mathscr{L}_\xi p = -2\psi p + 2A\mu\coth\phi$. It follows that if $v^a = 0$, i.e., $A = 0$, we have complete inheritance, in accordance with Theorem 3. Furthermore, in this model q_a is an eigenvector of σ_{ab}, so that if $w^a = 0$, i.e., $A = 0$, we again have complete inheritance, thus illustrating Theorem 4.

Note that the inheritance condition $t\phi_t + \tfrac{1}{3}r\phi_r = 0$ implies that $\phi = \phi(\chi, \theta, \Phi)$, where

$$\chi = t^{1/3}r^{-1} \tag{5.13}$$

is the self-similar variable associated with the space-time (5.4). In fact, any viscous fluid $k=0$ FRW model with $R(t) = t^a$ will inherit the symmetry of the HV admitted by such space-times if and only if the local Minkowskian components of the four-velocity are functions of the self-similar variable associated with the HV (as well as other coordinates not appearing in the self-similar variable), i.e., if and only if they are self-similar solutions. The perfect fluid solutions, which have comoving four-velocity, i.e., $\phi = 0$, are trivially self-similar and so inherit the symmetry.

Turning now to anisotropic fluid solutions, we first consider the solution (4.11) and its anisotropic fluid form given by (4.34). Using the SCKV ξ^a given by (4.12) we find

$$\begin{aligned} &\mathscr{L}_\xi u^a = -\psi u^a + v^a, \quad v^a = (0, x, 0, 0)\,,\\ &\mathscr{L}_\xi n^a = -\psi n^a = m^a, \quad m^a = (x, 0, 0, 0)\,. \end{aligned} \tag{5.14}$$

Note that u^a and n^a do not inherit the SCKV symmetry and that, in accordance with (2.30), v^a and m^a are indeed parallel to n^a and u^a, respectively, thus illustrating the apparent contradiction, mentioned earlier, with a result of Ref. 7.

The solution (4.14), in the form (4.39), and with the SCKV ξ^a given by (4.16), leads to expressions (5.15), but with v^a and m^a given by $v^a = (0, -t, 0, 0)$ and $m^a = (-t, 0, 0, 0)$.

The inheritance behavior of the solutions given by the metric (4.17) and Eqs. (4.50)–(4.57) is as follows: Solution (i) is noninheriting with $v^a = un^a$ and $m^a = uu^a$; solution (ii), in which ξ^a is parallel to u^a, obviously must inherit, i.e., $v^a = w^a = 0$; and solution (iii), in which ξ^a is orthogonal to u^a, is also an inheriting solution. The reason for inheritance in case (iii) is that since $\xi_a u^a = 0$, $\alpha = 0$ in Eq. (3.7) and, for this solution, the vorticity tensor $\omega_{ab} = 0$. Hence, Eq. (3.7) implies that $v_a = 0$ and, from (2.30), $m_a = 0$.

VI. CONCLUSION

This work consists essentially of two parts. In one part, namely Sec. IV, we found all space-times which admit a SCKV and satisfy the dominant energy condition. None of these space-times can represent a perfect fluid and the only vacuum solutions are given by the *pp*-wave metric. However, in general, these SCKV space-times can represent either viscous heat-conducting fluids or a special case of anisotropic fluids. In the second part, largely Sec. III, we derived theorems concerning the inheritance of the symmetries associated with a SCKV ξ^a by the physical components of a viscous imperfect fluid and also by those of the only type of anisotropic fluid that can admit a SCKV. The main results of Sec. III show that in the viscous fluid case, the SCKV symmetries are completely inherited if and only if either of the equivalent statements $\mathscr{L}_\xi(\eta\sigma_{ab}) = 0$ or $\mathscr{L}_\xi u^a = -\psi u^a$ (i.e., fluid flow lines are mapped conformally) is true. These results also apply to the symmetries associated with HV's and KV's.

Various subcases of the general imperfect fluid source are also covered by these results. Apart from an imperfect fluid ($\eta = q^a = 0$), which cannot admit a SCKV and for which the results are already known for HV's and KV's, these include a viscous fluid with no heat conduction ($q^a = 0$), the results for which are already known[7]; the models (4.11) and (4.14) are examples of such fluid space-times admitting a SCKV. Another subcase is that of a heat-conducting perfect fluid ($\eta\sigma_{ab} = 0$) which will always inherit the symmetry; this result is the generalization to nonzero heat conduction of a known result.[7]

The inheritance of symmetry results presented here can be extended to generalizations of the energy-momentum tensor (1.1). For example, the actions of KV's and HV's on an electromagnetic field and on an electromagnetic field with perfect fluid are well known[3]; an investigation of the effect of SCKV's on an electromagnetic field with imperfect fluid would be a logical extension. Such fields have been the subject of a number of cosmological investigations[18,19] and, in the same way as it has been shown that FRW models can be interpreted as electromagnetic field plus imperfect fluid models,[18] so, also, can some of the SCKV models found in Sec. IV be interpreted as such models.

Another possible extension is to multifluid models and, in particular, to two-fluid models. Models in which one fluid is a radiation perfect fluid, representing the cosmic microwave background, and the second fluid is either a perfect or an imperfect fluid, representing the galactic matter, have been studied extensively.[20] Again, as in the case of FRW models, the SCKV models of Sec. IV can also be interpreted as two-fluid models. However, whether we consider the case when one fluid is a radiation perfect fluid or the case of two general imperfect fluids, the expression for T_{ab} contains too many physical variables for the field equations to provide information on the inheritance properties of the separate physical quantities, although the inheritance theorems of Sec. III can be applied formally to suitable summed quantities. In the case when the four-velocities of the separate fluids are not parallel, the question of the symmetry inheritance is not well posed.

As stated earlier, our intended goal is the study of proper CKV's. Such a study is the natural mathematical generalization of work that has been done previously. Also, CKV's are of more physical interest than SCKV's. The results in this work will be useful in the proposed investigation. Moreover, some of the points made in this article serve to motivate the further study of CKV's and illustrate the potential problems inherent in such an investigation.

We have shown that there are very few space-times admitting SCKV's. In particular, there exist no SCKV's in FRW space-times. However, it is known that there do exist proper CKV's in FRW models[21] (nine in general) including the simple timelike CKV $\xi^a = R(\partial/\partial t)$. This indicate the greater physical significance in the study of proper CKV's.

In the case of a proper CKV with $\psi_{;ab} \neq 0$, it can be seen from Eqs. (2.19) that $\mathscr{L}_\xi T_{ab}$ is no longer zero. By studying the analog of Eq. (2.24), it can be shown that the equation for $\mathscr{L}_\xi(\eta\sigma_{ab})$ now includes the term $\psi_{;ab}$ on the rhs and cannot be shown to be zero when $\psi_{;ab} \neq 0$. Thus in the case, it is impossible for the physical quantities to satisfy (3.1); consequently, the symmetries cannot be inherited in the sense defined in Sec. III. In particular, it can be shown that even in the case of a perfect fluid source, a conformal motion will not, in general, map fluid flow conformally (i.e., $\mathscr{L}_\xi u^a \neq -\psi u^a$). Clearly, one of the starting points of future research is a notion of what is actually meant by symmetry inheritance in space-times admitting CKV's and what modifications are required to equations such as (3.1) for proper CKV's.

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