Asymptotic behavior of cosmological models in scalar-tensor theories of gravity

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We study the qualitative properties of cosmological models in scalar-tensor theories of gravity by exploiting the formal equivalence of these theories with general relativity minimally coupled to a scalar field under a conformal transformation and field redefinition. In particular, we investigate the asymptotic behavior of spatially homogeneous cosmological models in a class of scalar-tensor theories which are conformally equivalent to general relativistic Bianchi cosmologies with a scalar field and an exponential potential whose qualitative features have been studied previously. Particular attention is focused on those scalar-tensor theory cosmological models, which are shown to be self-similar, that correspond to general relativistic models that play an important role in describing the asymptotic behavior of more general models (e.g., those cosmological models that act as early-time and late-time attractors). [S0556-2821(98)03222-6]

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I. INTRODUCTION

Scalar-tensor theories of gravity are currently of great interest, partially due to the fact that such theories occur as the low-energy limit in superstring theory (see [1] and references therein). The first scalar-tensor theories to appear (with $\omega=\omega_0$) were due to Jordan [2,3], Fierz [4] and Brans and Dicke [5] and the most general scalar-tensor theories were formulated by Bergmann [6], Nordtvedt [7] and Wagoner [8]. The observational limits on scalar-tensor theories include solar system tests [9–12] and cosmological tests such as big bang nucleosynthesis constraints [13,14].

The possible isotropization of spatially homogeneous cosmological models in scalar-tensor theories has been studied previously. For example, Chauvet and Cervantes-Cota [15] have studied the possible isotropization of Bianchi models of types I, V and IX within the context of Brans-Dicke theory without a scalar potential, but with baryotropic matter, $p=(\gamma-1)\mu$, by studying exact solutions at late times. Mimoso and Wands [16] have studied Brans-Dicke theory with a variable $\omega=\omega(\phi)$ in the presence of baryotropic matter (but with no scalar field potential) and, in particular, gave forms for ω under which Bianchi type I models isotropize. We note that there is a formal equivalence between such a theory (with $\gamma\neq 2$) and a scalar-tensor theory with a potential but without matter, via the field redefinitions $V\equiv(2-\gamma)\mu$ and $\omega\nabla_a\phi\nabla_b\phi\rightarrow\omega\nabla_a\phi\nabla_b\phi-\gamma\mu\phi\delta_a^0\delta_b^0$.

In a recent paper [17] (see also [18] and [19]), cosmological models containing a scalar field with an exponential potential were studied. In particular, the asymptotic properties of the spatially homogeneous Bianchi models, and especially their possible isotropization and inflation, were investigated. Part of the motivation for studying such models is that they can arise naturally in alternative theories of gravity [20]; for example, Halliwell [21] has shown that the dimensional reduction of higher-dimensional cosmologies leads to an effective four-dimensional theory coupled to a scalar field with an exponential self-interacting potential.

The action for the general class of scalar-tensor theories (in the so-called Jordan frame) is given by [6,8]

$$S=\int\sqrt{-g}\left[\phi R-\frac{\omega(\phi)}{\phi}g^{ab}\phi_{,a}\phi_{,b}-2V(\phi)+2\mathcal{L}_m\right]d^4x. \tag{1}$$

However, under the conformal transformation and field redefinition [22,23,16]

$$g^*_{ab}=\phi g_{ab} \tag{2a}$$

$$\frac{d\varphi}{d\phi}=\frac{\pm\sqrt{\omega(\phi)+3/2}}{\phi}, \tag{2b}$$

the action becomes (in the so-called Einstein frame)

$$S^*=\int\sqrt{-g^*}\left[R^*-g^{*ab}\varphi_{,a}\varphi_{,b}-2\frac{V(\phi)}{\phi^2}+2\frac{\mathcal{L}_m}{\phi^2}\right]d^4x, \tag{3}$$

which is the action for general relativity (GR) containing a scalar field φ with the potential

$$V^*(\varphi)=\frac{V(\phi(\varphi))}{\phi^2(\varphi)}. \tag{4}$$

Our aim here is to exploit the results in previous work [17] to study the asymptotic properties of scalar-tensor theories of gravity with action (1) which under the transformations (2) transform to general relativity with a scalar field with the exponential potential given by

$$V^*=V_0e^{k\varphi}, \tag{5}$$

where V_0 and k are positive constants. That is, since we know the asymptotic behavior of spatially homogeneous Bianchi models with action (3) with the exponential potential (5), we can deduce the asymptotic properties of the corresponding scalar-tensor theories under the transformations (2) (as long as the transformations are not singular).[1] In particular, we are concerned with the possible isotropization and inflation of such scalar-tensor theories.

The outline of the paper is as follows. In Sec. II, we review the framework within which GR and a scalar field with a potential (Einstein frame) are formally equivalent to a scalar-tensor theory with a potential (Jordan frame), concentrating on both the exact and approximate forms for the parameters V and ω in the Jordan frame. In particular, we discuss the explicit example of the Brans-Dicke theory with a power-law potential and we also discuss the conditions which lead to appropriate late-time behavior as dictated by solar system and cosmological tests. In Sec. III, we then apply the conformal transformations to Bianchi models studied in the Einstein frame to produce exact solutions which represent the asymptotic behavior of more general spatially homogeneous models in the Jordan frame (for $\omega=\omega_0$, a constant). These Brans-Dicke models are self-similar and the corresponding homothetic vectors are also exhibited. We conclude with a discussion in Sec. IV.

II. ANALYSIS

For scalar field Bianchi models the conformal factor in Eq. (2a) is a function of t only [i.e., $\phi=\phi(t)$], and hence under (non-singular) transformations (2) the Bianchi type of the underlying models does not change (i.e., the metrics g_{ab} and g^*_{ab} admit three space-like Killing vectors acting transitively with the same group structure). In general, in the class of scalar-tensor theories represented by Eq. (1) there are two arbitrary (coupling) functions $\omega(\phi)$ and $V(\phi)$. The models which transform under Eqs. (2) to an exponential potential model, in which the two arbitrary functions ω and V are constrained by Eqs. (2b) and (4), viz.,

$$\frac{\phi}{V}\frac{dV}{d\phi}=2\pm k\sqrt{\frac{3}{2}+\omega(\phi)}, \tag{6}$$

make up a special subclass with essentially one arbitrary function. Although only a subclass of models obeys this constraint, this subclass is no less general than massless scalar field models ($V=0$; see, for example, [16]) or Brans-Dicke models with a potential ($\omega=\omega_0$, const), which are often studied in the literature. Indeed, the asymptotic analysis in this paper is valid not only for ''exact'' exponential models, but also for scalar-tensor models which transform under Eqs. (2) to a model in which the effective potential is a linear combination of terms involving exponentials in which the dominant term asymptotically is a leading exponential term; hence the analysis here is rather more general (we shall return to this in the next section). For the remainder of this paper we shall not explicitly consider ordinary matter; i.e., we shall set the matter Lagrangians in Eqs. (1) and (3) to zero. Matter can be included in a straightforward way [24,16,25].

A. Exact exponential potential models

Scalar-tensor models which transform under Eqs. (2) to a model with an exact exponential potential satisfy Eqs. (2b) and (4) with Eq. (5), viz.,

$$\frac{d\varphi}{d\phi}=\pm\frac{\sqrt{\omega(\phi)+3/2}}{\phi} \tag{7}$$

$$V_0e^{k\varphi}=\frac{V(\phi)}{\phi^2}. \tag{8}$$

As long as the transformations (2) remain non-singular we can determine the asymptotic properties of the underlying scalar-tensor theories from the asymptotic properties of the exact exponential potential model. These properties were studied in [17]. We recall that the asymptotic behavior depends crucially on the parameter k [in Eq. (5)] which will be related to the various physical parameters in the scalar-tensor theory (1).

In particular, in [17] it was shown that all scalar field Bianchi models with an exponential potential (5) (except a subclass of Bianchi type IX models which recollapse) isotropize to the future if $k^2\leq 2$ and, furthermore, inflate if $k^2<2$; if $k=0$, these models inflate towards the de Sitter solution and in all other cases they experience power-law inflationary behavior. If $k^2>2$, then the models cannot inflate, and can only isotropize to the future if the Bianchi model is of type I, V, VII, or IX. Those models that do not isotropize typically asymptote towards a Feinstein-Ibáñez anisotropic model [27]. Bianchi type VII_h models with $k^2>2$ can indeed isotropize [17] but do not inflate, while generically the ever-expanding Bianchi type IX models do not isotropize [26].

Therefore, at late times and for each specific choice of $\omega(\phi)$ both the asymptotic behavior of the models and the character of the conformal transformation (2) may be determined by the behavior of the scalar field φ at the equilibrium points of the system in the Einstein frame. Recently this behavior has been thoroughly investigated [17]. We shall summarize only those aspects relevant to our study. The existence of GR as an asymptotic limit at late times is also determined by the asymptotic behavior of the scalar field; we shall return to this issue in Sec. II C.

For spatially homogeneous space-times the scalar field φ is formally equivalent to a perfect fluid, and so expansion-normalized variables can be used to study the asymptotic behavior of Bianchi models [17,28]. The scalar field variable, Ψ, is defined by

[1]The possible isotropization of spatially homogeneous scalar-tensor theories which get transformed to a model with an effective potential which passes through the origin and is concave up may be deduced from the results of Heusler [24].

$$\Psi \equiv \frac{\dot{\varphi}}{\sqrt{6}\theta^*}, \tag{9}$$

where θ^* is the expansion of the timelike congruences orthogonal to the surfaces of homogeneity.[2] At the finite equilibrium points of the reduced system of autonomous ordinary differential equations, where Ψ is a finite constant, it has been shown [28] that $\theta^* = \theta_0^*/t^*$, where t^* is the time defined in the Einstein frame:

$$dt^* = \pm\sqrt{\phi}\, dt. \tag{10}$$

From Eq. (9) it follows that $\dot{\varphi} \propto 1/t^*$, whence upon substitution into the Klein-Gordon equation

$$\ddot{\varphi} + \theta^* \dot{\varphi} + \frac{\partial V^*}{\partial \varphi} = 0, \tag{11}$$

we find that, at the finite equilibrium points,

$$\varphi(t^*) = \varphi_0 - \frac{2}{k} \ln t^*, \quad k \neq 0, \tag{12}$$

where φ_0 is a constant. Hence, from Eq. (2b) we can obtain ϕ as a function of t^*, provided a particular $\omega(\phi)$ is given. From Eq. (10) we can then find the relationship between t^* and t, consequently obtain ϕ as a function of t, and hence determine the asymptotic behavior of $\phi(t)$ for a given theory with specific $\omega(\phi)$ (in the Jordan frame). Specifically, we can determine the possible isotropization and inflation of a given scalar-tensor theory in a very straightforward way.

As mentioned above, the behavior determined from the key equation (12) is not necessarily valid for all Bianchi models. For Bianchi models in which the phase space is compact, the equilibrium points represent models that do have the behavior described by Eq. (12), as do the finite equilibrium points in Bianchi models with non-compact phase spaces. It is possible that the infinite equilibrium points in these non-compact phase spaces also share this behavior, although this has not been proved. Finally, from Eqs. (2) we note that since the asymptotic behavior is governed by Eq. (12), the corresponding transformations are non-singular and this technique for studying the asymptotic properties of spatially homogeneous scalar-tensor theories is valid.

B. Example

Suppose we consider a Brans-Dicke theory with a power-law potential, viz.,

$$\omega(\phi) = \omega_0 \tag{13}$$

$$V = \beta \phi^{\alpha} \tag{14}$$

[2]Note that $\theta^* > 0$ for all Bianchi models except those of type IX.

(where β and α are positive constants); then Eq. (2b) integrates to yield

$$\phi = \phi_0 \exp\left(\frac{\varphi - \varphi_0}{\bar{\omega}}\right), \tag{15}$$

where

$$\bar{\omega} \equiv \pm\sqrt{\omega_0 + 3/2}, \tag{16}$$

and hence Eq. (4) yields

$$V^* = V_0 e^{\bar{k}\varphi}, \tag{17}$$

where the critical parameter $\bar{k}$ is given by

$$\bar{k} = \frac{\alpha - 2}{\bar{\omega}}. \tag{18}$$

From [17] we can now determine the asymptotic behavior of the models in the Einstein frame, as discussed in Sec. II A, for a given model with specific values of α and $\bar{\omega}$ (and hence a particular value for $\bar{k}$).

The possible isotropization of the given scalar-tensor theory can now be obtained directly (essentially by reading off from the proceeding results—see Sec. III A). For example, the inflationary behavior of the theory can be determined from Eqs. (2a), (10) and (12). Let us further discuss the asymptotic behavior of the corresponding scalar-tensor theories (in the Jordan frame). From Eqs. (10), (12) and (15) we have that, asymptotically,

$$\phi = \tilde{\phi}_0 \left[\pm (t - t_0) \left(1 + \frac{1}{k\bar{\omega}}\right) \right]^{-2/(1 + k\bar{\omega})}, \tag{19}$$

where the $\pm$ sign is determined from Eq. (10). Both this sign and the signs of $\bar{\omega}$ and $1 + k\bar{\omega}$ are crucial in determining the relationship between t^* and t; i.e., as $t^* \to \infty$ either $t \to \pm\infty$ or $t \to t_0$ and hence either $\phi \to 0$ or $\phi \to \infty$, respectively, as $\varphi \to -\infty$.

1. Generalization

Suppose again that $\omega = \omega_0$, so that Eq. (15) also follows, but now V is a sum of power-law terms of the form

$$V = \sum_{n=0}^{m} \beta_n \phi^{\alpha_n}, \tag{20}$$

where $m > 1$ is a positive integer. Then Eq. (4) becomes

$$V^* = \sum_{n=0}^{m} \beta_n \phi^{\alpha_n - 2} = \sum_{n=0}^{m} \bar{\beta}_n \exp(\bar{k}_n \varphi), \quad \bar{k}_n = \frac{\alpha_n - 2}{\bar{\omega}}. \tag{21}$$

For example, if

$$V = V_0 + \frac{1}{2} m \phi^2 + \lambda \phi^4,$$

then

$$V^* = \bar{V}_0 e^{-2\varphi/\bar{\omega}} + \frac{1}{2}\bar{m} + \bar{\lambda} e^{2\varphi/\bar{\omega}}$$

(with obvious definitions for the new constants), which is a linear sum of exponential potentials. Asymptotically one of these potentials will dominate (e.g., as $\varphi \to +\infty$, $V^* \to \bar{\lambda} e^{2\varphi/\bar{\omega}}$) and the asymptotic properties can be deduced as in the previous section.

2. Approximate forms

In the last subsection we commented upon the asymptotic properties of a scalar-tensor theory with the forms for ω and V given by Eqs. (13) and (14). Let us now consider a scalar-tensor theory with forms for ω and V which are approximately given by Eqs. (13) and (14) (asymptotically in some well-defined sense) in order to discuss whether both theories will have the same asymptotic properties. In doing so, we hope to determine whether the techniques discussed in this paper have a broader applicability.

We assume that ω and V are analytic at the asymptotic values of the scalar field in the Jordan frame in an attempt to determine whether their values correspond to the appropriate forms for φ and V^* in the Einstein frame, namely whether $\varphi \to -\infty$ and the leading term in V^* is of the form $e^{k\varphi}$.

Consider an analytic expansion for ϕ about $\phi = 0$:

$$\omega = \sum_{n=0}^{\infty} \omega_n \phi^n \tag{22}$$

$$V = \sum_{n=0}^{\infty} V_n \phi^n, \tag{23}$$

where all ω_n and V_n are constants. Using Eq. (2) we find, up to leading order in ϕ, that for $\omega_0 \neq -3/2$,

$$\varphi - \varphi_0 \approx \bar{\omega} \ln \phi, \tag{24}$$

so that $\varphi \to \pm\infty$ [depending on the sign in Eq. (16)] for $\phi \to 0$. The potential in the Einstein frame is (to leading order)

$$V^* \approx \exp\left\{ -\frac{2(\varphi - \varphi_0)}{\bar{\omega}} \right\}. \tag{25}$$

Hence, the parameter k of Eq. (5) is defined here as $k \equiv -2/\bar{\omega}$. For $\omega_0 = -3/2$ we have

$$(\varphi - \varphi_0)^2 \approx 4\omega_1 \phi \tag{26}$$

$$V^* \approx \frac{16\omega_1^2}{(\varphi - \varphi_0)^4}, \tag{27}$$

so that $\varphi \not\to -\infty$ as $\phi \to 0$.

Next, let us consider an expansion in $1/\phi$, valid for $\phi \to \infty$:

$$\omega = \sum_{n=0}^{\infty} \frac{\omega_n}{\phi^n}, \tag{28}$$

$$V = \sum_{n=0}^{\infty} \frac{V_n}{\phi^n}. \tag{29}$$

For $\omega_0 \neq -3/2$, the results are similar to the $\phi = 0$ expansion:

$$\varphi - \varphi_0 \approx -\bar{\omega} \ln \phi \tag{30}$$

$$V^* \approx \exp\left\{ \frac{2(\varphi - \varphi_0)}{\bar{\omega}} \right\}, \tag{31}$$

where now $\varphi \to \mp\infty$ as $\phi \to \infty$. When $\omega_0 = -3/2$, we obtain

$$(\varphi - \varphi_0)^2 \approx \frac{4\omega_1}{\phi} \tag{32}$$

$$V^* \approx \frac{(\varphi - \varphi_0)^4}{16\omega_1^2}. \tag{33}$$

It is apparent that the sign of $\bar{\omega}$ is important in determining whether $\phi \to \infty$ or $\phi \to 0$ in order to obtain the appropriate form for φ, as was exemplified at the end of Sec. II B.

Finally, in the event that ω and V are analytic about some finite value of ϕ, namely ϕ_0, it can be shown that $\varphi \to \varphi_0$ as $\phi \to \phi_0$. Hence, if one insists that ω remain analytic as $\omega \to \omega_0$ in the limit of $\varphi \to -\infty$, then ϕ must either vanish or diverge, and the GR limit is not obtained. This would then suggest that if one imposed $\varphi \to -\infty$ for $\phi \to \phi_0$, then ω would not be analytic about $\phi = \phi_0$.

C. Constraints on possible late-time behavior

In this paper we are concerned with the possible asymptotic behavior of cosmological models in scalar-tensor theories of gravity. However, there are physical constraints on acceptable late-time behavior [as $t^* \to \infty$; see Eq. (10)]. For example, such theories ought to have GR as an asymptotic limit at late times (e.g., $\omega \to \infty$ and $\phi \to \phi_0$) in order for the theories to concur with observations such as solar system tests. In addition, cosmological models must ''isotropize'' in order to be in accordance with cosmological observations.

Nordtvedt [7] has shown that for scalar-tensor theories with no potential, $\omega(\phi) \to \infty$ and $\omega^{-3} d\omega/d\phi \to 0$ as $t \to \infty$ in order for GR to be obtained in the weak-field limit. Similar requirements for general scalar-tensor theories with a non-zero potential are not known, and as will be demonstrated from the consideration of two particular examples found in the literature, not all theories will have a GR limit.

The first example is the Brans-Dicke theory ($\omega = \omega_0 = \text{const}$) with a power-law self-interacting potential given by Eq. (14) studied earlier in Sec. II B. In this case, ϕ is given by Eqs. (15) and (16) and the potential is given by (14), viz.,

$$V(\phi)=\beta\phi^{\alpha}, \quad \alpha=2\mp k\sqrt{\omega_0+3/2}.$$

The $\alpha=1$ case for Friedmann-Robertson-Walker (FRW) metrics was studied by Kolitch [29] and the $\alpha=2$ $(k=0)$ case, corresponding to a cosmological constant in the Einstein-frame, was considered for FRW metrics by Santos and Gregory [30]. Earlier we considered whether anisotropic models in Brans-Dicke theory with a potential given by Eq. (14) will isotropize. Assuming a large value for ω_0, as suggested by solar system experiments, we conclude that for a wide range of values for α the models isotropize. However, in the low-energy limit of string theory where $\omega_0=-1$ the models are only guaranteed to isotropize for $1<\alpha<3$.

Substituting Eq. (12) in Eq. (15) we get

$$\phi\sim(t^*)^{\pm 2\delta}, \quad \delta=\frac{1}{k}\sqrt{\frac{2}{3+2\omega_0}}. \tag{34}$$

Now, substituting the above expression into Eq. (9), we obtain t^* as a function of t and hence we obtain

$$\phi\sim t^{\pm 2\delta/1\mp\delta}. \tag{35}$$

Depending on the sign, we deduce from this expression that for large t the scalar field tends either to zero or to infinity and so this theory, with the potential given by Eq. (14), does not have a GR limit.

In the second example we assume that

$$\omega(\phi)+\frac{3}{2}=\frac{A\phi^2}{(\phi-\phi_0)^2}, \tag{36}$$

where A is an arbitrary positive constant. This form for $\omega(\phi)$ was first considered by Mimoso and Wands [16] (in a theory without a potential). Now, we obtain

$$\phi=\phi_0+B\,e^{\mp\varphi/\sqrt{A}}, \tag{37}$$

where B is a constant, and the potential, defined by Eq. (6), is given by

$$V(\phi)=V_0\,\phi^2(\phi-\phi_0)^{\mp\sqrt{A}k}. \tag{38}$$

As before, at the equilibrium points we can express ϕ as a function of t^*, which then allows us to compute t as a function of t^*. At late times we find that

$$\phi\sim\phi_0+t^{\beta}, \tag{39}$$

where β is a constant whose sign depends on k, ω_0 and the choice of one of the signs in the theory. What is important here is that in this case, at late times, we find that the scalar field tends to a constant value for $\beta<0$, thereby yielding a GR limit. In both of the examples considered above, the conformal transformation for the equilibrium points is regular.

Of course, these are not the only possible forms for a variable $\omega(\phi)$. For example, Barrow and Mimoso [22] studied models with $2\omega(\phi)+3\propto\phi^{\alpha}$ $(\alpha>0)$ satisfying the GR limit asymptotically. (The GR limit is only obtained asymptotically as $\phi\to\infty$, although for a finite but large value of ϕ the theory can have a limit which is as close to GR as is required.) However, by studying the evolution of the gravitational ''constant'' G from the full Einstein field equations (i.e., not just the weak-field approximation), Nordtvedt [7,31] has shown that

$$\frac{\dot{G}}{G}=-\left(\frac{3+2\omega}{4+2\omega}\right)\left(1+\frac{2\omega'}{(3+2\omega)^2}\right),$$

where $\omega'=d\omega/d\phi$ (so that the correct GR limit is only obtained as $\omega\to\infty$ and $\omega'\omega^{-3}\to 0$). Torres [32] showed that when $2\omega(\phi)+3\propto\phi^{\alpha}$, $G(t)$ decreases logarithmically and hence $G\to 0$ asymptotically. In the above work, no potential was included. For a theory with $2\omega(\phi)+3\propto\phi^{\alpha}$ and with a non-zero potential satisfying Eq. (6) we have that

$$\frac{\phi}{V}\frac{dV}{d\phi}=A+B\phi^{\alpha}$$

($\alpha\neq 0$, A and B constants), so that

$$V(\phi)=V_0\phi^{A}e^{B\phi^{\alpha}/\alpha}.$$

A potential of this form was considered by Barrow [33].

Finally, Barrow and Parsons [34] have studied three parametrized classes of models for $\omega(\phi)$ which permit $\omega\to\infty$ as $\phi\to\phi_0$ (where the constant ϕ_0 can be taken as ϕ evaluated at the present time) and hence have an appropriate GR limit:

$$\text{(i)} \quad 2\omega(\phi)+3=2B_1^2|1-\phi/\phi_0|^{-\alpha} \quad \left(\alpha>\frac{1}{2}\right),$$

$$\text{(ii)} \quad 2\omega(\phi)+3=B_2^2|\ln(\phi/\phi_0)|^{-2|\delta|} \quad \left(\delta>\frac{1}{2}\right),$$

$$\text{(iii)} \quad 2\omega(\phi)+3=B_3^2|1-(\phi/\phi_0)^{|\beta|}|^{-1} \quad (\forall\ \ \beta).$$

Other possible forms for $\omega(\phi)$ were discussed in Barrow and Carr [35] and, in particular, they considered models (i) above but allowed $\alpha<0$ in order for a possible GR limit to be obtained also as $\phi\to\infty$. Schwinger [36] has suggested the form $2\omega(\phi)+3=B^2/\phi$ based on physical considerations.

III. APPLICATIONS

Let us exploit the formal equivalence of the class of scalar-tensor theories (1) with $\omega(\phi)$ and $V(\phi)$ given by

$$\omega(\phi)=\omega_0, \quad V(\phi)=\beta\phi^{\alpha}, \tag{40}$$

with that of GR containing a scalar field and an exponential potential (5). Indeed, since the conformal transformation (2a) is well-defined in all cases of interest, the Bianchi type is invariant under the transformation and we can deduce the asymptotic properties of the scalar-tensor theories from the corresponding behavior in the Einstein frame. Also, we have that

$$k \equiv \frac{\alpha-2}{\bar{\omega}}, \quad \bar{\omega}^2 \equiv \omega_0 + \frac{3}{2}. \tag{41}$$

We recall that at the finite equilibrium points in the Einstein frame we have that

$$\theta^* = \theta_0^* t_*^{-1}, \tag{42}$$

$$\varphi(t^*) = \varphi_0 - \frac{2}{k}\ln(t^*), \tag{43}$$

where

$$\theta_0^* = 1 + \frac{k^2}{2} e^{k\varphi_0}. \tag{44}$$

Integrating Eq. (2b) we obtain

$$\phi(t^*) = d\,\exp[\bar{\omega}^{-1}\varphi(t^*)] = \phi_0 t_*^{-2/k\bar{\omega}}, \tag{45}$$

where the constant $\phi_0 \equiv d\,\exp(\varphi_0/\bar{\omega})$, we recall that t and t^* are related by Eq. (10), and Eq. (2a) can be written as

$$g_{ab} = \phi^{-1} g_{ab}^*. \tag{46}$$

A. Examples

(1) All initially expanding scalar field Bianchi models with an exponential potential (5) with $0<k^2<2$ within general relativity (except for a subclass of models of type IX) isotropize to the future towards the power-law inflationary flat FRW model [25], whose metric is given by

$$ds^2 = -dt_*^2 + t_*^{4/k^2}(dx^2+dy^2+dz^2). \tag{47}$$

In the scalar-tensor theory (in the Jordan frame), ϕ is given by Eq. (45) and from Eq. (46) we have that

$$ds_{ST}^2 = \phi_0^{-1} t_*^{2/k\bar{\omega}}\{ds^2\}. \tag{48}$$

Defining a new time coordinate by

$$T = c t_*^{(1+k\bar{\omega})/k\bar{\omega}}, \quad c \equiv \frac{k\bar{\omega}}{1+k\bar{\omega}} \phi_0^{-1/2} \tag{49}$$

(where $k\bar{\omega}+1 \neq 0$; i.e., $\alpha \neq 1$), we obtain, after a constant rescaling of the spatial coordinates,

$$ds_{ST}^2 = -dT^2 + T^{2K}(dX^2+dY^2+dZ^2), \tag{50}$$

where

$$K \equiv \frac{k^2+2k\bar{\omega}}{k^2(1+k\bar{\omega})}.$$

Finally, the scalar field is given by

$$\phi = \phi_0 c^{2/(1+k\bar{\omega})} T^{-2/(1+k\bar{\omega})} = \bar{\phi}_0 T^{2/(1-\alpha)}. \tag{51}$$

Therefore, all initally expanding spatially homogeneous models in scalar-tensor theories obeying Eqs. (40) with $0<(\alpha-2)^2<2\omega_0+3$ (except for a subclass of Bianchi type IX models which recollapse) will asymptote towards the exact power-law flat FRW model given by Eqs. (50) and (51), which will always be inflationary since $K=(1+\alpha+2\omega_0)/(\alpha-1)(\alpha-2)>1$ [note that whenever $2\omega_0>(\alpha-2)^2-3=\alpha^2-4\alpha+1$, we have that $1+\alpha+2\omega_0>\alpha^2-3\alpha+2=(\alpha-1)(\alpha-2)$].

When $k^2>2$, the models in the Einstein frame cannot inflate and may or may not isotropize. Let us consider two examples.

(2) Scalar field models of Bianchi type VI_h with an exponential potential (5) with $k^2>2$ asymptote to the future towards the anisotropic Feinstein-Ibáñez model [27] given by $(m\neq 1)$

$$ds^2 = -dt_*^2 + a_0^2(t_*^{2p_1}dx^2 + t_*^{2p_2}e^{2mx}dy^2 + t_*^{2p_3}e^{2x}dz^2), \tag{52}$$

where the constants obey

$$p_1 = 1,$$

$$p_2 = \frac{2}{k^2}\left(1 + \frac{(k^2-2)(m^2+m)}{2(m^2+1)}\right),$$

$$p_3 = \frac{2}{k^2}\left(1 + \frac{(k^2-2)(m+1)}{2(m^2+1)}\right). \tag{53}$$

In the scalar-tensor theory (in the Jordan frame), ϕ is given by Eq. (45) and the metric is given by Eq. (48). After defining the new time coordinate given by Eq. (49), we obtain

$$ds_{ST}^2 = -dT^2 + A_0^2(T^{2q_1}dX^2 + T^{2q_2}e^{2mX}dY^2 + T^{2q_3}e^{2X}dZ^2), \tag{54}$$

where

$$q_i \equiv \frac{1+k\bar{\omega}p_i}{1+k\bar{\omega}} \quad (i=1,2,3), \quad A_0^2 = a_0^2\phi_0^{-1}c^{-2q_1}, \tag{55}$$

and Y and Z are obtained by a simple constant rescaling (and $X=x$). Finally, the scalar field is given by Eq. (51).

The corresponding exact Bianchi type VI_h scalar-tensor theory solution is therefore given by Eqs. (51) and (54) in the coordinates (T,X,Y,Z). Consequently, all Bianchi type VI_h models in the scalar-tensor theory satisfying Eqs. (40) with $(\alpha-2)^2>2\omega_0+3$ asymptote towards the exact anisotropic solution given by Eqs. (51) and (54).

(3) An open set of scalar field models of Bianchi type VII_h with an exponential potential with $k^2>2$ asymptote towards the isotropic (but non-inflationary) negative-curvature FRW model [17] with metric

$$ds^2 = -dt_*^2 + t_*^2 d\sigma^2, \tag{56}$$

where $d\sigma^2$ is the three-metric of a space of constant negative curvature. Again, ϕ is given by Eq. (45) and the metric is given by Eq. (48), which becomes, after the time recoordinatization (49),

$$ds^2_{ST} = -dT^2 + C^2 T^2 d\sigma^2, \tag{57}$$

where $C^2 \equiv \phi_0^{-1} c^{-2} = [(1+k\overline{\omega})/k\overline{\omega}]^2$. This negatively curved FRW metric is equivalent to that given by Eq. (56). Finally, the scalar field is given by Eq. (51).

Therefore, when $(\alpha-2)^2 > 2\omega_0 + 3$, there is an open set of (BVII_h) scalar-tensor theory solutions satisfying Eqs. (40) which asymptote towards the exact isotropic solution given by Eqs. (51) and (57).

Equations (43) and (45) and the resulting analysis are only valid for scalar-tensor theories satisfying Eqs. (40). However, the asymptotic analysis will also apply to generalized theories of the forms discussed in Secs. II B 1 and II B 2. Finally, a similar analysis can be applied in Brans-Dicke theory with $V=0$ [37].

B. Self-similarity

All three attracting scalar-tensor theory solutions in the last subsection are self-similar; metric (50) admits the homothetic vector (HV) $\boldsymbol{X} = T\,\partial/\partial T + (1-K)\{X\,\partial/\partial X + Y\,\partial/\partial Y + Z\,\partial/\partial Z\}$, metric (54) admits the HV $\boldsymbol{X} = T\,\partial/\partial T + (1-q_2) Y\,\partial/\partial Y + (1-q_3) Z\,\partial/\partial Z$, and metric (57) admits the HV $\boldsymbol{X} = T\,\partial/\partial T$. Of course, all three solutions in the corresponding general relativistic model (i.e., in the Einstein frame) are self-similar. Let us show that this is always the case; i.e., all scalar-tensor solutions obtained in this way are self-similar.

In [17] it was shown that the cosmological solutions corresponding to the finite equilibrium points of the ‘‘reduced dynamical system’’ of the spatially homogeneous scalar field models with an exponential potential are all self-similar. Let g^*_{ab} be the metric of such a solution and $\boldsymbol{X}_*$ the corresponding HV; hence we have that

$$\mathcal{L}_{\boldsymbol{X}_*} g^*_{ab} = 2 g^*_{ab}, \tag{58}$$

where $\mathcal{L}$ denotes the Lie derivative along $\boldsymbol{X}_*$. In the coordinates in which $\theta^* = \theta^*_0 t_*^{-1}$, from $\mathcal{L}_{\mathrm{X}_*}\theta^* = -\theta^*$ we find that [37]

$$\boldsymbol{X}_* = t^* \frac{\partial}{\partial t^*} + X^\mu_*(x^\gamma_*) \frac{\partial}{\partial x^\mu_*}. \tag{59}$$

Now, the metric g_{ab} in the corresponding scalar-tensor theory is given by Eq. (46), where the scalar field is given by Eq. (45), viz.,

$$\phi(t^*) = \phi_0 t_*^{-2/k\overline{\omega}} \tag{60}$$

[or by Eq. (51) in terms of the time coordinate T]. We emphasize that this *power-law* form for ϕ is only valid for scalar-tensor theories that obey conditions (40). Hence, from Eqs. (58)–(60) it follows that

$$\begin{aligned}\mathcal{L}_{\boldsymbol{X}_*} g_{ab} &= \boldsymbol{X}_*[\phi^{-1}(t^*)] g^*_{ab} + \phi^{-1} \mathcal{L}_{\boldsymbol{X}_*} g^*_{ab} \\ &= t^* \frac{\partial}{\partial t^*}(\phi_0^{-1} t_*^{2/k\overline{\omega}}) g^*_{ab} + 2\phi_0^{-1} t_*^{2/k\overline{\omega}} g^*_{ab} \\ &= \left(\frac{2}{k\overline{\omega}} + 2\right) \phi_0^{-1} t_*^{2/k\overline{\omega}} g^*_{ab} \\ &= 2 c g_{ab},\end{aligned} \tag{61}$$

where the constant c is given by $c = (1+k\overline{\omega})/(k\overline{\omega})$. That is, $\boldsymbol{X} = \boldsymbol{X}_*$ is a homothetic vector for the spacetime with metric g_{ab} and consequently the corresponding scalar-tensor theory solution is self-similar.

C. Special case $\alpha = 1$

In the analysis above we have omitted the special case $\alpha = 1$ (i.e., $k\overline{\omega} = -1$). This case is degenerate as we will now demonstrate. Let the general relativistic metric be defined by

$$ds^2 = -dt_*^2 + \gamma_{\mu\nu} dx^\mu dx^\nu. \tag{62}$$

First, suppose we take $k\overline{\omega} = -1$ in Eq. (45) and define a new time coordinate by

$$T = \phi_0^{-1/2} \ln(t_*), \tag{63}$$

then the metric (62) becomes

$$ds^2_{ST} = -dT^2 + \phi_0^{-1} \exp(-2\sqrt{\phi_0} T)\, \gamma_{\mu\nu} dx^\mu dx^\nu, \tag{64}$$

where $\phi(T) = \phi_0 \exp(2\sqrt{\phi_0} T)$. Now, from Eq. (61) we obtain

$$\mathcal{L}_{\boldsymbol{X}_*} g_{ab} = 0; \tag{65}$$

i.e., in this case $\boldsymbol{X} = \boldsymbol{X}_*$ is a Killing vector (KV) of the spacetime (64). Since the KV $\boldsymbol{X}$ is timelike, the spatially homogeneous metric (64) admits four KV's acting simply transitively and hence the resulting spacetime is (totally—i.e., four-dimensionally) homogeneous.

All known non-flat homogeneous spacetimes are given in Table 10.1 in [38]; hence the metric (64) is given by one of those spacetimes in this table representing an orthogonal spatially homogeneous metric with a diagonal Einstein tensor (representing a perfect fluid spacetime or an Einstein spacetime with a cosmological constant); all of these metrics are indeed known [38]. In the case when metric (64) is the flat Minkowski metric, the corresponding general relativistic spacetime (62) is de Sitter spacetime. However, this corresponds to the degenerate case in which

$$\theta^* = \theta^*_0, \quad \text{a constant;}$$

this is the only possibility in which Eq. (42) is not valid and hence the resulting analysis does not follow. This degenerate case corresponds to $k=0$ in Eq. (5) (i.e., $V^* = V_0$, a constant); since $k\overline{\omega}=-1$ this corresponds to $\overline{\omega}\rightarrow\infty$ or $\omega_0\rightarrow\infty$ (in which case GR is recovered from the scalar-tensor theory under consideration).

Finally, if $\alpha=1$ in Eq. (14) (i.e., $V=\beta\phi$), then the action (1) becomes

$$S=\int\sqrt{-g}\left[\phi(R-2\beta)-\frac{\omega_0}{\phi}g^{ab}\phi_{,a}\phi_{,b}+2\mathcal{L}_m\right]d^4x,$$

which is equivalent to that for Brans-Dicke theory incorporating an additional constant β. Under the conformal transformation and field redefinition (2) the action becomes that for general relativity with a cosmological constant (and additional matter fields), and from the cosmic no-hair theorem it follows that all spatially homogeneous models (except for a subclass of Bianchi type IX) asymptote to the future towards the de Sitter model [39].

IV. CONCLUSIONS

In this paper we have studied the asymptotic behavior of a special subclass of spatially homogeneous cosmological models in scalar-tensor theories which are conformally equivalent to general relativistic Bianchi models containing a scalar field with an exponential potential by exploiting results found in previous work [17].

We illustrated the method by studying the particular example of Brans-Dicke theory with a power-law potential and various generalizations thereof, paying particular attention to the possible isotropization and inflation of such models. In addition, we discussed physical constraints on possible late-time behavior and, in particular, whether the scalar-tensor theories under consideration have a general relativistic limit at late times.

In particular, several exact scalar-tensor theory cosmological models (both inflationary and non-inflationary, isotropic and anisotropic) which act as attractors were discussed, and all such exact scalar-tensor solutions were shown to be self-similar.

This is related to the previous work of several authors. Specifically, Chauvet and Cervantes-Cota [15] studied isotropization in Brans-Dicke gravity including a perfect fluid with $p=(\gamma-1)\mu$. They examined whether the anisotropic models contain a FRW model as an asymptotic limit, which is how they defined isotropization. For Bianchi models of types I, V and IX, they found exact solutions in these cosmologies which can isotropize to the future, depending on the values of γ and ω and two other arbitrary constants. Furthermore, Mimoso and Wands [16] also studied scalar-tensor models with variable ω without a self-interacting potential V but coupled to barotropic matter. Regarding the possible isotropization of the cosmological models (meaning here that the shear of the fluid becomes negligible), they concentrated on models of Bianchi type I and first discussed constraints on a fixed $\omega=\omega_0$ model necessary for isotropization at late times. In the particular case of a false vacuum ($p=-\mu$), they showed that the de Sitter solution is the late-time attractor of the model. They then proceeded to examine arbitrary $\omega(\phi)$ Bianchi type I cosmologies and showed that if a solution is to asymptote towards a GR limit (i.e., $\omega\rightarrow\infty$), then it must also isotropize. Their paper also discussed initial singularities in models of other Bianchi types.

The work in this paper can be generalized in a number of ways. In particular, more general scalar-tensor theories can be considered and more general (than spatially homogeneous) geometries can be studied. For example, the more general class of inhomogeneous G_2 models could be considered [40,41,43] in which there exists two commuting space-like Killing vectors. The motivation for studying G_2 cosmologies is that there is some evidence that the class of *self-similar* G_2 models plays an important role in describing the asymptotic behavior of more generic general relativistic scalar field models with an exponential potential (cf. [42]; in this way, we may be able to find special scalar-tensor G_2 cosmological models that describe the asymptotic properties of more general scalar-tensor cosmologies. Some potential problems that exist in this more general context is that since ϕ, and hence the transformation (2a), depends on both time and one space variable, the transformation (2) will be singular (at least for certain values of the space variable) and the classification of G_2 models may not be preserved under such a transformation.

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