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THE ULTIMATE CATEGORICAL INDEPENDENCE RATIO OF A GRAPH*

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Abstract. Let $\beta(G)$ denote the independence number of a graph G. We introduce $A(G) = \lim_{k\to\infty} \beta(G^k)/|V(G)|^k$, where the categorical graph product is used. This limit, surprisingly, lies in the range $(0, 1/2] \cup \{1\}$. We can show that this limit can take any such rational number, but is there any G for which $A(G)$ is irrational? A useful technique for bounding $A(G)$ is to consider special spanning subgraphs. These bounds allow us to efficiently compute $A(G)$ for many G. We give a condition which if true for G shows that $A(G) > \beta(G)/|V(G)|$. This brings up the question: for which G does $A(G) = \beta(G)/|V(G)|$? This happens if G is a Cayley graph of an Abelian group or if G is a connected graph that has an automorphism which has a single orbit.

Key words. independence number, ultimate categorical independence ratio, decomposition, self-universal

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1. Introduction.

Graph products have been used to find the *essential* value of a graph parameter (such as independence number or chromatic number) of a graph G by "multiplying" G by itself k times and examining the growth of the parameter on G^k. For example, the *Shannon capacity* [12, 15, 16] of a graph G is defined by

$$\Theta(G) = \lim_{k\to\infty} \beta(\otimes_{i=1}^{k} G)^{1/k},$$

where $\otimes$ is the strong product of graphs and $\beta(H)$ denotes the independence number of graph H (i.e., the maximum cardinality of an independent set of vertices of H). The Shannon capacity of a graph arose from a problem of transmission of words over a noisy line but has a number of other applications (see [12]).

Another such concept is the *ultimate chromatic number* of a graph G; that is,

$$\chi_u(G) = \lim_{k\to\infty} (\chi(\bullet_{i=1}^{k} G))^{1/k}.$$

(Here $\bullet$ denotes the lexicographic product and $\chi(H)$ the chromatic number of H.) This was introduced by Hilton, Rado, and Scott [11] (see also [5]) and is related to the problem of assigning radio frequencies to vehicles operating in zones (see Gilbert [6] and Roberts [13, 14]). The determination of both the Shannon capacity for some graphs and the ultimate chromatic number for all graphs can be solved using linear programming techniques. (See [15] for the former and [9] for the latter.)

In contrast to the Shannon capacity, one can investigate the parameter of the independence number by looking at the ratio of this parameter to the total number of vertices in the graph; if $|V(G)| = n$, the *ultimate independence ratio* of G is

$$I(G) = \lim_{k\to\infty} \beta(\square_{i=1}^{k} G)/n^k,$$

where $\square$ is the Cartesian product. This was introduced in [10]. We consider an analogous (but significantly different) concept.

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Let $G = (V(G), E(G))$ be a graph. We will assume that graphs are finite and simple. We write $a \sim b$ if a is adjacent to but not equal to b, and $a \perp b$ if a is neither adjacent nor equal to b. The *categorical* product $G \times H$ of G and H has the vertex set $V(G) \times V(H)$ and $(a, x) \sim (b, y)$ if both $a \sim b$ and $x \sim y$. As the categorical product is the only product we shall consider henceforth, for the rest of the paper we use $\times_{i=1}^{k} G$ and G^k interchangeably. The parameter we consider is the *ultimate categorical independence ratio* of a graph G which is defined as

$$A(G) = \lim_{k\to\infty} \beta(\times_{i=1}^{k} G)/n^k,$$

where $|V(G)| = n$. We show that this limit exists in the next section.

The next two sections deal with upper and lower bounds, respectively. The fourth section investigates disjoint unions of graphs, and we find classes of graphs for which the sequence $\langle \beta(G^k)/|V(G^k)| \rangle$ is not constant; i.e., $A(G) > \beta(G)/|V(G)|$. In the fifth section we look at graphs for which $A(G) = \beta(G)/|V(G)|$ and in the last section we pose several problems.

We follow standard graph theoretic terminology (cf. [1, 2, 8]), but we make explicit note of a few definitions. We abbreviate the disjoint union of m copies of graph G by mG. Let $S \subseteq V(G)$ and $v \in V(G)$. Then $\langle S \rangle$ denotes the induced subgraph on the vertices of S. The *neighborhood* of $v \in V(G)$ is $N(v) = \{y \mid y \sim v\}$ and $N(S) = \cup_{v\in S} N(v)$; $N[v] = N(v) \cup \{v\}$ is the *closed* neighborhood of v and $N[S] = \cup_{v\in S} N[v]$. The set S is called *independent* if $\langle S \rangle$ contains no edges and, as mentioned above, $\beta(G)$ denotes the maximum cardinality of an independent set of G. The *chromatic* number of G is denoted by $\chi(G)$. We remark that $G \times H$ is connected if and only if both G and H are connected and at least one is 3-chromatic [17]. It is easy to see as well that $G \times H \cong H \times G$, $G \times (H \cup K) = (G \times H) \cup (G \times K)$, and $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$. Whether $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$ is Hedetniemi's conjecture, an open problem that has attracted (and frustrated!) a number of mathematicians (see, for example, [4, 3]).

2. Upper bounds for $A(G)$. The first fact needed is that the ultimate categorical independence ratio really exists for any graph. We will make use of the following elementary but important fact.

LEMMA 2.1. *Let G and H be graphs. Then $\beta(G \times H) \geq \max\{\beta(G)|V(H)|, |V(G)| \beta(H)\}$.*

Proof. If I is an independent set of H then $\cup_{a\in I} G \times \{a\}$ is an independent set of $G \times H$ and thus $\beta(G \times H) \geq \beta(H)|V(G)|$. The second part follows similarly since the product is commutative. □

From this lemma it follows that $\beta(G^k)/n^k \geq n\beta(G^{k-1})/n^k = \beta(G^{k-1})/n^{k-1}$. Therefore the sequence $\beta(G^k)/n^k$ is nondecreasing and is bounded above by 1 and so the ultimate categorical independence ratio exists. We contrast this with the ultimate independence ratio of Hell, Yu, and Zhou [10], where $I(G)$ is the limit of the *nonincreasing* sequence $\beta(\square_{i=1}^{k} G)/n^k$).

The following observation is extremely useful and forms the basic idea for this and the next section.

LEMMA 2.2. *If G is a spanning subgraph of H then $A(G) \geq A(H)$.*

Proof. For all k, G^k is a spanning subgraph of H^k. Thus $\beta(G^k) \geq \beta(H^k)$ and therefore $\beta(G^k)/n^k \geq \beta(H^k)/n^k$. □

The previous result, along with the next, is quite helpful in bounding the ultimate categorical independence ratio of a graph.

THEOREM 2.3. *If G is a regular graph of degree $r > 0$ then $A(G) \leq 1/2$.*

Proof. Let $I \subseteq V(G)$ be an independent set with $|I| = \beta(G)$ and let $C = V(G) - I$. Each vertex in G has degree r so summing the degrees of the vertices in C we have $(n-\beta(G))r$. Some edges have been counted twice, but the edges between I and C have only been counted once. Therefore, since I is an independent set, $r\beta(G) \leq (n-\beta(G))r$; i.e., $\beta(G)/n \leq 1/2$. Now G^k is a regular graph of degree r^k so $\beta(G^k)/n^k \leq 1/2$ and consequently $A(G) \leq 1/2$. □

Theorem 2.3 and Lemma 2.2 give the following corollary.

COROLLARY 2.4. *If H has a regular spanning subgraph of degree at least one then*

$$\beta(H)/n \leq A(H) \leq \frac{1}{2}.$$

This result shows that $A(C_{2n}) = A(P_{2n}) = 1/2$, the latter because P_{2n} has a perfect matching as a spanning subgraph. Moreover, if G has a Hamiltonian cycle then $A(G) \leq 1/2$. However, if G has a Hamiltonian path then it is possible for $A(G)$ to be greater than $1/2$. For example, $A(P_{2n+1}) \geq (n+1)/(2n+1) > 1/2$. Indeed, this raises the question: *What is $A(P_{2n+1})$?* We answer this question later.

General upper bounds for the ultimate categorical independence ratio are few and far between. One technique that appears fruitful involves partitioning the vertex set into subgraphs. In the following, the phrase K *decomposes into* $L_1, \ldots, L_l$ means that $L_1 \cup \cdots \cup L_l$ is a spanning subgraph of K. (We denote this as well by $K \Rightarrow L_1 \cup \cdots \cup L_l$.)

LEMMA 2.5. *If G is a graph of order n and $G \times H \Rightarrow nH$ then $\beta(G \times H) = \beta(H)n$. Moreover, if also $H \cong G$ then $A(G) = \beta(G)/n$.*

Proof. From Lemma 2.1 we have that $\beta(G \times H) \geq \beta(H)|V(G)| \geq \beta(H)n$ for any graphs G and H.

Suppose that $G \times H$ can be partitioned so that the subgraphs in each partition are all isomorphic to H. Thus any independent set of $G \times H$ intersects each part in no more than $\beta(H)$ many vertices. Therefore $\beta(G \times H) \leq \beta(H)n$. Consequently, $\beta(G \times H) = \beta(H)n$.

If a graph G is of order n and G^2 decomposes into nG, then inductively $G^k \Rightarrow n^{k-1}G$ and $\beta(G^k)/n^k \leq (n^{k-1}\beta(G))/n^k = \beta(G)/n$. Thus $A(G) = \beta(G)/n$. □

This result shows that $A(K_n) = \frac{1}{n}$. This follows since K_n^2 can be decomposed into nK_n: if the vertices of K_n are $\mathbb{Z}_n$, and X_i is the subgraph of K_n^2 induced by $\{(j, j+i) : j \in \mathbb{Z}_n\}$, then K_n^2 decomposes into $X_0, \ldots, X_{n-1}$; i.e., $K_n^2 \Rightarrow nK_n$.

Decompositions can be utilized to prove the following general upper bound.

THEOREM 2.6. *Suppose $G \Rightarrow H \cup K_{m_1} \cup \cdots \cup K_{m_p}$ where $|V(H)| = n$ and $\chi(H) \leq m_1 \leq \cdots \leq m_p$. Then $A(G) \leq A(H)$.*

Proof. Suppose $H \cup K_{m_1} \cup \cdots \cup K_{m_p}$ is a spanning subgraph of G, where $\chi(H) \leq m_1 \leq \cdots \leq m_p$. Then G^k decomposes into

$$\bigcup_{i=0}^{k} \binom{k}{i} H^i \times (K_{m_1} \cup \cdots \cup K_{m_p})^{k-i}.$$

We observe that if $\chi(H) \leq r$ then $H \times K_r$ has a decomposition into r copies of H, as follows. Let $\chi(H) = j$ and the color classes of H be $C_1, C_2, \ldots, C_j$. Let $V(K_r) = \{a_1, a_2, \ldots, a_r\}$. The ith copy of H would be $(C_1 \times \{a_i\}) \cup (C_2 \times \{a_{i+1}\}) \cup \cdots \cup (C_j \times \{a_{i+j-1}\})$ for $i = 1, 2, \ldots, r$ where the subscripts are taken modulo r.

Since $\chi(H^i) \leq \chi(H)$ for all i, using the previous remark we have (for $i < k$)

$$H^i \times (K_{m_1} \cup \cdots \cup K_{m_p})^{k-i}$$

$$
\begin{aligned}
&= H^i \times (K_{m_1} \cup \cdots \cup K_{m_p}) \times (K_{m_1} \cup \cdots \cup K_{m_p})^{k-i-1} \\
&= (H^i \times K_{m_1} \cup H^i \times K_{m_2} \cup \cdots \cup H^i \times K_{m_p}) \times (K_{m_1} \cup \cdots \cup K_{m_p})^{k-i-1} \\
&\Rightarrow (m_1 + \cdots + m_p) H^i \times (K_{m_1} \cup \cdots \cup K_{m_p})^{k-i-1},
\end{aligned}
$$

and so by induction,

$$H^i \times (K_{m_1} \cup \cdots \cup K_{m_p})^{k-i} \Rightarrow (m_1 + \cdots + m_p)^{k-i} H^i$$

for any $i \in \{0, \ldots, k\}$. (We interpret the 0th power of a graph to be K_1.)

Thus G^k has a decomposition into

$$\left(\bigcup_{i>0} \binom{k}{i} (m_1 + \cdots + m_p)^{k-i} H^i \right) \quad \cup \quad (K_{m_1} \cup \cdots \cup K_{m_p})^k.$$

Now as $\beta((K_{m_1} \cup \cdots \cup K_{m_p})^k) \leq (m_1 + \cdots + m_p)^k$ we have

$$\beta(G^k) \leq \left(\sum_{i>0} \binom{k}{i} (m_1 + \cdots + m_p)^{k-i} \beta(H^i) \right) + (m_1 + \cdots + m_p)^k.$$

Since $\beta(H^i) \leq A(H) n^i$,

$$
\begin{aligned}
\beta(G^k) &\leq \left(\sum_{i>0} \binom{k}{i} (m_1 + \cdots + m_p)^{k-i} A(H) n^i \right) + (m_1 + \cdots + m_p)^k \\
&= A(H) \left((n + m_1 + \cdots + m_p)^k - (m_1 + \cdots + m_p)^k \right) \\
&\quad + (m_1 + \cdots + m_p)^k.
\end{aligned}
$$

Dividing through by $(n + m_1 + \cdots + m_p)^k$ yields

$$\frac{\beta(G^k)}{(n + m_1 + \cdots + m_p)^k} \leq A(H) \left(1 - \frac{(m_1 + \cdots + m_p)^k}{(n + m_1 + \cdots + m_p)^k} \right) + \frac{(m_1 + \cdots + m_p)^k}{(n + m_1 + \cdots + m_p)^k}.$$

Letting $k \to \infty$ we get $A(G) \leq A(H)$. □

Suppose we have a coloring π of $\overline{G}$, the complement of G, and $c_0 \leq c_1 \leq \cdots \leq c_p$ are the sizes of the color classes. Then we have a decomposition of the original graph G into cliques:

$$G \Rightarrow K_{c_0} \cup K_{c_1} \cup \cdots \cup K_{c_p}.$$

Taking $H = K_{c_0}$ and applying the previous theorem, we deduce that $A(G) \leq 1/c_0$. Of course the bound is improved when c_0 is as large as possible. We summarize as follows.

COROLLARY 2.7. *For each coloring π of $\overline{G}$, let $c(\pi)$ denote the minimum cardinality of a color class in π, and let c denote the maximum of $c(\pi)$ over all colorings. Then $A(G) \leq \frac{1}{c}$.*

3. Lower bounds for $A(G)$. If one takes an independent set I in G, then the subset $I \times G^{k-1}$ is an independent set in G^k (this is inherent in Lemma 2.1), and it follows that $A(G) \geq \beta(G)/|V(G)|$. There are graphs (such as the complete graphs) for which equality holds (we will have more to say about such graphs later). The following lemma is often useful in showing when independent sets larger than $\beta(G)|V(G)|$ exist in the product of G and H.

LEMMA 3.1. *Let G and H be graphs and let I be an independent set of G where $|G - N[I]| = k$. Then $\beta(G \times H) \geq k\beta(H) + |I||V(H)|$.*

Proof. Let P be a maximum-sized independent set of H. Let $Q = (V(G) - N[I]) \times P$ and $R = I \times V(H)$. Then $Q \cup R$ is independent in $G \times H$ and $|Q \cup R| = k\beta(H) + |I||V(H)|$. □

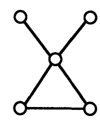

FIG. 1.

If G is the graph in Figure 1, then $\beta(G)/5 = 3/5$, but $\beta(G^2)/25 = 16/25 > 3/5$. Moreover, let $I = \{a, c\}$ and $H = G^k$. Then $|G - N[I]| = 2$, and applying the lemma to $G \times H$, we have $A(G) \geq 2/3$.

Yet for the graph G in Figure 1, what is $A(G)$? The next result provides the answer as well as giving $A(P_{2n+1})$ by proving the surprising fact that if the independence number of any graph is more than half the number of vertices, then the ultimate categorical independence ratio is 1.

THEOREM 3.2. *Let G be a graph with n vertices. If $\beta(G)/n > 1/2$ then $A(G) = 1$.*

Proof. Let I be an independent set of G with $|I| = \beta(G) > n/2$. Note that $\beta(G) > n - \beta(G)$.

In G^k denote the factors as G_i, $i = 1, 2, \ldots, k$ with I_i as the copy of I in the ith factor. Now form the set

$$J = \bigcup_P (\times_{i \in P} I_i) \times \left(\times_{j \notin P} (V(G_j) - I_j)\right)$$

where the union is taken over all $P \subseteq \{1, 2, \ldots, k\}$ with $|P| > k/2$. For any two vertices $\mathbf{x} = (x_1, \ldots, x_k), \mathbf{y} = (y_1, \ldots, y_k)$ in J, as each has more than half its coordinates in I, there is an i such that $x_i, \ y_i \in I$, and hence $\mathbf{x}$ and $\mathbf{y}$ are not adjacent. Thus J is an independent set.

Counting gives us the following:

$$|J| = \sum_{j > k/2} \binom{k}{j} \beta(G)^j (n - \beta(G))^{k-j}.$$

Completing the summation and taking it away again gives

$$|J| = \sum_{j=0}^{k} \binom{k}{j} \beta(G)^j (n - \beta(G))^{k-j} - \sum_{j \leq k/2} \binom{k}{j} \beta(G)^j (n - \beta(G))^{k-j},$$

and thus

$$|J| = n^k - \sum_{j \leq k/2} \binom{k}{j} (\beta(G))^j (n - \beta(G))^{k-j}.$$

Now since $\beta(G) > (n - \beta(G))$, $f(x) = \left(\frac{\beta(G)}{n - \beta(G)}\right)^x$ is an increasing function of x. It follows that for $j \leq k/2$,

$$\beta(G)^j (n - \beta(G))^{k-j} \leq \beta(G)^{k/2} (n - \beta(G))^{k/2}.$$

Therefore we get

$$|J| = n^k - \sum_{j \le k/2} \binom{k}{j} \beta(G)^{k/2}(n-\beta(G))^{k/2}$$
$$\ge n^k - 2^k \beta(G)^{k/2}(n-\beta(G))^{k/2}.$$

Dividing by n^k gives

$$|J|/n^k \ge 1 - \left(\frac{4\beta(G)(n-\beta(G))}{n^2}\right)^{k/2}.$$

Now as $4\beta(G)(n-\beta(G)) < n^2$, the right side tends to 1 as k goes to infinity, and since $A(G) \ge |J|/n^k$ for all k, we are done. □

We have seen that $A(C_{2n+1}) = n/(2n+1) < 1/2$. Thus the addition of a single edge to a graph (here to P_{2n+1}), while changing the ratio of independence number to order by an arbitrarily small amount, may greatly affect the ultimate categorical independence ratio.

We now consider a lower bound which is a companion result to Theorem 2.6.

THEOREM 3.3. *Let I be an independent subset of G. Then $A(G) \ge |I|/|N[I]|$*

Proof. Let $H = N[I]$ and $F = G - H$ and put $m = |V(F)|$ and $n = |V(H)|$. Let $J \subseteq G^k - F^k = (H \cup F)^k - F^k$ (that is, for $i = 1, 2, \ldots, k$ choose i of the coordinates and in these coordinates the entries will be taken from H and in the others the entries will be taken from F), with the extra condition that in the first coordinate in which the entries are taken from H they will be restricted to vertices from I. Then J is an independent set of G. This follows since if $\mathbf{x}, \mathbf{y} \in J$ then let i and j, respectively, be the least indices such that $x_i \in I$ and $y_j \in I$. If $i = j$ then $x_i = y_i$ or $x_i \perp y_i$; if $i < j$ then $y_i \in F$ thus $x_i \perp y_i$ and so $\mathbf{x} \perp \mathbf{y}$. In any event, $\mathbf{x}$ and $\mathbf{y}$ are nonadjacent. Now

$$|J| = \sum_{i>0} \binom{k}{i} |I| n^{i-1} m^{k-i} = \frac{|I|}{n} \sum_{i>0} \binom{k}{i} n^i m^{k-i} = \frac{|I|}{n}\left((n+m)^k - m^k\right).$$

Therefore,

$$\beta(G^k) \ge \frac{|I|}{n}\left((n+m)^k - m^k\right),$$

and dividing through by $(n+m)^k$ we get

$$\frac{\beta(G^k)}{(n+m)^k} \ge \frac{|I|}{n}\left(1 - \left(\frac{m}{n+m}\right)^k\right).$$

Thus as $k \to \infty$ we finally obtain $A(G) \ge |I|/n$. □

This theorem shows in particular that if G has an isolated vertex x, then $A(G) = 1$, since if $I = \{x\}$, then $A(G) \ge 1/1$.

The previous two theorems can be combined to yield the following corollary.

COROLLARY 3.4. *If G has an independent set I such that $|I| > |N(I)|$ then $A(G) = 1$.*

Proof. By the previous theorem $A(G) \ge |I|/|N[I]|$, but this latter term is greater than $1/2$ and thus by Theorem 3.2, $A(G) = 1$. □

We know from Theorem 3.2 that if $\beta(G)/|V(G)| > 1/2$ then $A(G) = 1$. What can we say if $\beta(G)/|V(G)| = 1/2$?

COROLLARY 3.5. *Suppose that G is a graph of order n with $\beta(G) = n/2$. Then $A(G) = 1/2$ if G has a perfect matching, and $A(G) = 1$ otherwise.*

Proof. If G has a perfect matching then G has a 1-regular spanning subgraph and so $A(G) \leq 1/2$. In fact, $A(G) = 1/2$ in this case as $A(G) \geq \beta(G)/n = 1/2$.

Now assume that G has no perfect matching. Let J be an independent set of size $n/2$ in G. Then there is no matching of J into $G - J$, and hence by Hall's theorem, there is a subset $I \subseteq J$ such that $|N(I)| < |I|$. Set $H = N[I]$. Then applying Theorem 3.3,

$$A(G) \geq \frac{|I|}{|I| + |N(I)|} > \frac{1}{2},$$

and hence by Theorem 3.2, $A(G) = 1$. □

We remark that there are graphs G for which $A(G) = 1/2$ and yet G has no perfect matching. For example, we see in the next section that for any $n \geq 1$, $A(K_2 \cup K_{2n+1}) = 1/2$ while clearly neither this graph nor any power has a perfect matching.

We can now determine quickly what the ultimate categorical independence ratio is for any bipartite graph G. Let G have order n. We can find a 2-coloring of G in polynomial time. Let the color classes be C_1 and C_2. Clearly

$$A(G) = \frac{\beta(G)}{n} \geq \frac{1}{2}.$$

If $|C_1| \neq |C_2|$, then $\beta(G) > n/2$, and hence $A(G) = 1$. Assume now that $|C_1| = |C_2|$. If there is a perfect matching in G (and this can be determined in polynomial time), then $A(G) = 1/2$. Otherwise, by Corollary 3.5, $A(G) = 1$. Thus we have the following corollary.

COROLLARY 3.6. *If G is bipartite, then $A(G)$ can be determined in polynomial time.*

Again, we contrast this result with that for the ultimate independence ratio, where it is known [7] that $I(G) = 1/2$ for *any* bipartite graph G.

4. The ultimate categorical independence ratio for disjoint unions. It is of interest to see how the ultimate categorical independence ratio can change under graph operations. The independence number of the union of graphs changes in an obvious way, namely the sum of the independence numbers of its constituent parts. It is not so clear as to how the ultimate categorical independence ratio changes under disjoint union. For a graph G, it is clear that $(G \cup G)^k = 2^k G^k$. It follows that $A(G \cup G) = A(2G) = A(G)$. The next result shows that $A(G \cup H)$ is at least the maximum of $A(G)$ and $A(H)$.

THEOREM 4.1. *If G and H are any graphs, then $A(G \cup H) \geq \max\{A(G), A(H)\}$.*

Proof. Let $n_G = |V(G)|$ and $n_H = |V(H)|$. We show first that $A(G \cup H) \geq A(G)$. The other inequality follows similarly.

Note that

$$(G \cup H)^k = \bigcup_{i=0}^{k} \binom{k}{i} G^i \times H^{k-i},$$

and hence

$$\beta((G \cup H)^k) = \sum_{i=0}^{k} \binom{k}{i} \beta(G^i \times H^{k-i}).$$

By dropping the $i = 0$ term, we obtain

$$\begin{aligned}\frac{\beta((G \cup H)^k)}{(n_G + n_H)^k} &\geq \sum_{i=1}^{k} \frac{\binom{k}{i}\beta(G^i \times H^{k-i})}{(n_G + n_H)^k} \\ &\geq \sum_{i=1}^{k} \frac{\binom{k}{i}\beta(G^i) n_H^{k-i}}{(n_G + n_H)^k} && \text{[Lemma 2.1]} \\ &= \sum_{i=1}^{k} \frac{\binom{k}{i} n_G^i n_H^{k-i}}{(n_G + n_H)^k} \cdot \frac{\beta(G^i)}{n_G^i}. && (*)\end{aligned}$$

There are two cases. First, suppose $A(G) = \beta(G)/n_G$. Then $A(G) = \beta(G^i)/n_G^i$ for all i. In this case

$$\begin{aligned}\frac{\beta((G \cup H)^k)}{(n_G + n_H)^k} &\geq \sum_{i=1}^{k} \frac{\binom{k}{i} n_G^i n_H^{k-i}}{(n_G + n_H)^k} \cdot \frac{\beta(G^i)}{n_G^i} \\ &= A(G) \sum_{i=1}^{k} \frac{\binom{k}{i} n_G^i n_H^{k-i}}{(n_G + n_H)^k} \\ &= A(G) \left(1 - \frac{n_H^k}{(n_G + n_H)^k}\right).\end{aligned}$$

Thus

$$A(G \cup H) = \lim_{k \to \infty} \frac{\beta((G \cup H)^k)}{(n_G + n_H)^k} \geq \lim_{k \to \infty} A(G) \left(1 - \frac{n_H^k}{(n_G + n_H)^k}\right) = A(G).$$

Now we may assume that $A(G) > \beta(G)/n_G$. We choose $\varepsilon > 0$ and we will show that $A(G \cup H) \geq A(G) - \varepsilon$. Let

$$\gamma = A(G) - \frac{\varepsilon}{2}.$$

Since $\beta(G^i)/n_G^i$ is a nondecreasing sequence we can fix $J \geq 1$ so that

$$\frac{\beta(G^j)}{n_G^j} > \gamma \text{ for all } j \geq J \qquad \text{and} \qquad \frac{\beta(G^i)}{n_G^i} \leq \gamma \text{ for all } i < J.$$

Let

$$\psi = \frac{\max\{n_G, n_H\}}{n_G + n_H}$$

and

$$\mu = \begin{cases} \gamma - \beta(G)/n_G & \text{if } J > 1, \\ 0 & \text{if } J = 1. \end{cases}$$

Note that for $J > 1$, $\mu = \max\left\{\gamma - \beta(G^i)/n_G^i \mid i < J\right\}$. Also, $\psi \in (0, 1)$ and $\mu \geq 0$ constants. Thus, taking $k > 2J$, from $(*)$ we get

$$\begin{aligned}\frac{\beta((G \cup H)^k)}{(n_G + n_H)^k} &\geq \sum_{i=1}^{k} \frac{\binom{k}{i} n_G^i n_H^{k-i}}{(n_G + n_H)^k} \cdot \gamma - \sum_{i=1}^{J-1} \frac{\binom{k}{i} n_G^i n_H^{k-i}}{(n_G + n_H)^k} \left(\gamma - \frac{\beta(G^i)}{n_G^i}\right) \\ &= \gamma \sum_{i=0}^{k} \frac{\binom{k}{i} n_G^i n_H^{k-i}}{(n_G + n_H)^k} - \gamma \frac{n_H^k}{(n_G + n_H)^k} - \sum_{i=1}^{J-1} \frac{\binom{k}{i} n_G^i n_H^{k-i}}{(n_G + n_H)^k} \left(\gamma - \frac{\beta(G^i)}{n_G^i}\right) \\ &\geq \gamma - \gamma \frac{n_H^k}{(n_G + n_H)^k} - (J - 1)\binom{k}{J-1} \psi^k \mu.\end{aligned}$$

Therefore,

$$\frac{\beta((G\cup H)^k)}{(n_G+n_H)^k} \geq \left(A(G)-\frac{\varepsilon}{2}\right)\left(1-\frac{n_H^k}{(n_G+n_H)^k}\right) - Ck^{J-1}\psi^k,$$

where $C = \frac{(J-1)\mu}{(J-1)!}$ is a nonnegative constant.

This shows that

$$\begin{aligned} A(G\cup H) &= \lim_{k\to\infty} \frac{\beta((G\cup H)^k)}{(n_G+n_H)^k} \\ &\geq \lim_{k\to\infty}\left(\left(A(G)-\frac{\varepsilon}{2}\right)\left(1-\frac{n_H^k}{(n_G+n_H)^k}\right) - Ck^{J-1}\psi^k\right) \\ &\geq A(G)-\varepsilon \end{aligned}$$

for any $\varepsilon > 0$ and hence $A(G\cup H) \geq A(G)$.

In both cases we have that $A(G\cup H) \geq A(G)$. Similarly, we also have that $A(G\cup H) \geq A(H)$, and, therefore, it follows that $A(G\cup H) \geq \max\{A(G), A(H)\}$. □

As a corollary to this theorem and Corollary 2.7, we can determine the ultimate categorical independence ratio for the disjoint union of complete graphs.

COROLLARY 4.2.

$$A\left(\bigcup_{i=1}^{l} K_{m_i}\right) = \frac{1}{\min\{m_i : i = 1,\ldots,l\}}.$$

This corollary yields infinitely many graphs G for which $A(G)$ is not 1, not 1/2, nor $\beta(G)/|V(G)|$; in fact, the disjoint union of complete graphs of order at least 2 where not all the cliques have the same size are such examples.

Finally, we also derive that there are graphs G with arbitrarily small $\beta(G)/|V(G)|$ for which $A(G)$ climbs up to 1. For example,

$$\frac{\beta(K_{1,2}\cup K_n)}{|V(K_{1,2}\cup K_n)|} = \frac{3}{n+3},$$

while $A(K_{1,2}\cup K_n) = 1$ as $A(K_{1,2}\cup K_n) \geq A(K_{1,2}) = 1$.

5. Universal graphs and the distribution of $A(G)$. Lemma 3.1 leads naturally to the next definition. A graph G is called *categorical-universal* if $\beta(G\times H) = \max\{\beta(G)|V(H)|, \beta(H)|V(G)|\}$ for all graphs H. A related notion of universal graphs was originally introduced in the Shannon capacity.

Of course if $G = \overline{K_n}$, then G is categorical-universal since $\overline{K_n}\times H$ contains no edges and thus $\beta(\overline{K_n}\times H) = n|V(H)| = \max\{\beta(\overline{K_n})|V(H)|, \beta(H)|V(\overline{K_n})|\}$. In fact, it can be shown that these are the *only* categorical-universal graphs.

A more restricted concept than a categorical-universal graph is the following: graph G is *self-universal* if $A(G) = \beta(G)/|V(G)|$. From Lemma 2.5 it follows that if a graph G is of order n and G^2 decomposes into nG, then G is self-universal. We have also seen that regular bipartite graphs and cliques are in the class. The next result greatly increases the known self-universal graphs by showing that it includes all Cayley graphs on an Abelian group. (A similar result holds for ultimate independence ratios [10].)

THEOREM 5.1. *If G is the Cayley graph of an Abelian group then G is self-universal.*

Proof. Let S be the generating set for G and $|V(G)| = n$. In what follows we do not distinguish between an element of the group and a vertex of the Cayley graph. The proof is by induction. Suppose that $\beta(G^i)/n^i = \beta(G)/n$ for $i = 1, 2, \ldots, k-1$.

CLAIM. *Let* $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ *be any vertex of* G^k. *Then* $\{(ga_1, ga_2, \ldots, ga_k) \mid g \in V(G)\}$ *is isomorphic to* G.

Let $T = \{(ga_1, ga_2, \ldots, ga_k) \mid g \in V(G)\}$. Define $\phi : G \to T$ by $\phi(h) = (ha_1, ha_2, \ldots, ha_k)$. Now $g \sim h$ if and only if there exists $s \in S$ such that $gs = h$. Moreover, since G is Abelian for any $a \in G$ then $gas = gsa = ha$. Thus $ga \sim ha$ if and only if $g \sim h$. Since this holds in every coordinate, we have $\phi(g) \sim \phi(h)$ if and only if $g \sim h$. Thus the claim is proved.

For each $\mathbf{x} \in V(G^k)$ set $T_{\mathbf{x}} = \{g\mathbf{x} \mid g \in V(G)\}$. If $\mathbf{c} \in T_{\mathbf{x}} \cap T_{\mathbf{z}}$ then there exist $f, g \in G$ such that $c_i = fx_i = gz_i$ for $i = 1, 2, \ldots, k$. Thus $x_i = f^{-1}gz_i$ and it follows that any vertex of $T_{\mathbf{x}}$ is in $T_{\mathbf{z}}$ and conversely. Therefore $T_{\mathbf{x}} = T_{\mathbf{z}}$ and these decompose G^k into $|V(G^{k-1})|$ many copies of G. From Lemma 2.5 the result now follows. □

As a consequence we now know that both odd and even cycles are self-universal.

We can extend the class of known self-universal graphs even further.

THEOREM 5.2. *Let G be a graph of order n with an automorphism f that has a single orbit of size n. Then G is self-universal.*

Proof. We define an equivalence relation $\equiv$ on $V(G^{k+1})$ by $\mathbf{x} \equiv \mathbf{y}$ if $y = f^l(\mathbf{x})$ for some l (where f is applied coordinatewise). Now each class is of the form $\{\mathbf{x}, f(\mathbf{x}), f^2(\mathbf{x}), \ldots, f^{n-1}(\mathbf{x})\}$. We'll show that the subgraph induced by each such class is isomorphic to G.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_{k+1})$ and $\mathbf{y} = f^l(\mathbf{x}) = (f^l(x_1), f^l(x_2), \ldots, f^l(x_{k+1}))$ (for some $1 \le l \le n-1$) be any two elements of a class. If $x_1 \perp y_1 = f^l(x_1)$ then $\mathbf{x} \perp \mathbf{y}$. If $x_1 \sim y_1 = f^l(x_1)$, then $\mathbf{x} \sim \mathbf{y}$, as if $x_i = f^j(x_1)$; then $y_i = f^l(x_i) = f^{l+j}(x_1) \sim f^j(x_1) = x_i$. It follows that the subgraph of G^{k+1} induced by the class generated by $\mathbf{x}$ is isomorphic to G, and thus G^{k+1} decomposes into copies of G, and again by Lemma 2.5 we are done. □

We now turn to the distribution of the ultimate categorical independence ratio. From Theorem 3.2, we know that there is a gap between $1/2$ and 1. Clearly 0 is not the ultimate categorical independence ratio for any graph G, so $\{A(G) : G \text{ is a graph}\} \subseteq (0, 1/2] \cup \{1\}$. While we do not know if $A(G)$ can be irrational, we can show that the closure of the set above is in fact $[0, 1/2] \cup \{1\}$.

THEOREM 5.3. *For any rational number $r \in (0, 1/2] \cup \{1\}$ there is a graph G_r with $A(G_r) = r$.*

If $r = 1$, we may take G_r to be any graph with independence number greater than half the number of vertices. Otherwise, let $r = p/l$ (p, l positive integers) and G_r be the Cayley graph $\mathbb{Z}_l$ where x is joined to y if and only if $x - y \in \{p, p+1, \ldots, l-p\}$; it can be easily seen that $\beta(G_r) = p$, and from Theorem 5.1, $A(G_r) = \beta(G_r)/|V(G_r)| = p/l = r$.

6. Open problems. There are a number of open problems concerning the ultimate categorical independence ratio. While we have shown that every rational number $r \in (0, \frac{1}{2}] \cup \{1\}$ is the ultimate categorical independence ratio of a graph, we do not know the following.

PROBLEM 6.1. *Can $A(G)$ be irrational?*

There are also families of graphs (such as complete multipartite graphs) for which the determination of $A(G)$ is unknown. (Although, for many cases, the results here will provide a solution.)

PROBLEM 6.2. *What is $A(G)$ for a random graph G? Are almost all graphs G of order n self-universal?*

In terms of algorithmic considerations, we have shown that one can determine in polynomial time the ultimate categorical independence ratio of a bipartite graph.

PROBLEM 6.3. *Is $A(G)$ computable? If so, what is its complexity?*

Finally, in many cases, equality does hold in Theorem 4.1, and we do not know of any examples where equality does not hold.

PROBLEM 6.4. *Is $A(G \cup H) = \max\{A(G), A(H)\}$?*

REFERENCES

[1] G. CHARTRAND AND L. LESNIAK, *Graphs and Digraphs*, Prindle, Weber & Schmidt International Series, London, 1986.

[2] C. BERGE, *Graphs and Hypergraphs*, North Holland, New York, 1979.

[3] D. DUFFUS, B. SANDS, AND R. E. WOODROW, *On the chromatic number of product of graphs*, J. Graph Theory, 9 (1986), pp. 487–495.

[4] M. EL–ZAHAR AND N. SAUER, *The chromatic number of the product of two* 4*-chromatic graphs is* 4, Combinatorica, 5 (1985), pp. 121–126.

[5] D. GELLER AND S. STAHL, *The chromatic number and other functions of the lexicographic product*, J. Combin. Theory Ser. B, 19 (1975), pp. 87–95.

[6] E. N. GILBERT, Technical memorandum, Bell Telephone Laboratories, Murray Hill, NJ, 1972, manuscript.

[7] G. HAHN, P. HELL, AND S. POLJAK, *On the Ultimate Independence Ratio of a Graph*, Discrete Math., 127 (1994), pp. 213–220.

[8] F. HARARY, *Graph Theory*, Addison–Wesley, London, 1972.

[9] P. HELL AND F. S. ROBERTS, *Analogues of the Shannon capacity of a graph*, in Theory and Practice of Combinatorics, North-Holland Math. Stud., Vol. 60, North–Holland, Amsterdam-New York, 1982, pp. 155–168.

[10] P. HELL, X. YU, AND H. ZHOU, *Independence ratios of graph powers*, Discrete Math.,.27 (1994), pp. 213–220.

[11] A. J. W. HILTON, R. RADO, AND S. H. SCOTT, *A ($<$ 5)-colour theorem for planar graphs*, Bull. London Math. Soc., 5 (1973), pp. 302–306.

[12] L. LOVÁSZ, *On the Shannon capacity of a graph*, IEEE Trans. Inform. Theory, 25 (1979), pp. 1–7.

[13] F. S. ROBERTS, *Graph Theory and its Applications to Problems of Society*, CBMS–NSF Monographs 29, Society for Industrial and Applied Mathematics, Philadelphia, 1978.

[14] ———, *On the mobile radio frequency assignment problem and the traffic light phasing problem*, Ann. New York Acad. Sci., 319 (1978), pp. 466–483.

[15] M. ROSENFELD, *On a Problem of C.E. Shannon in Graph Theory*, Proc. Amer. Math. Soc., 18 (1967), pp. 315–319.

[16] C. E. SHANNON, *The zero error capacity of a noisy channel*, I.R.E., Trans. on Inform. Theory, IT-2 (1956), pp. 8–19.

[17] P. M. WEICHSEL, *The Kronecker product of graphs*, Proc. Amer. Math. Soc., 13 (1962), pp. 47-52.