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The Detection of Coastal-Trapped Waves

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We outline a simple method for estimating the cross-spectral matrix of coastal-trapped wave amplitudes, **A**, from a set of oceanographic observations. Specifically, we propose that **A** may be estimated by $(\mathbf{M}'\mathbf{M})^{-1}\mathbf{M}'\hat{\mathbf{U}}\mathbf{M}(\mathbf{M}'\mathbf{M})^{-1}$ where a prime denotes conjugate transpose, $\hat{\mathbf{U}}$ is the sample cross-spectral matrix of observations and **M** is a matrix which has the spatial form of the waves for columns. In general, **M** will be complex and frequency-dependent. We discuss the bias of this estimator and show how to estimate the variance of the power and cross spectra of wave amplitudes. We also outline an ad hoc scheme for assessing the predictive skill of the coastal trapped wave representation and finally give some advice on how to interpret $\hat{\mathbf{A}}$. Although the method is presented in the context of shelf circulation and coastal trapped waves, it may be applied to any linear system where the spatial forms of the waves are known and the cross-spectral matrix of their amplitudes is required.

1. INTRODUCTION

Theoreticians interested in ocean circulation often express the flow field as a linear combination of prescribed spatial modes and then determine the time-varying coefficients from the forcing and the initial conditions. The modes usually correspond to the eigenfunctions of an idealized, linear model of the real ocean. This technique has proved useful in the study of deep ocean, shelf and nearshore circulation where the eigenfunctions include vertical normal modes, coastal-trapped waves and edge waves [e.g., *LeBlond and Mysak*, 1978].

Observationalists also use linear combinations of modes to reduce large multivariate data sets down to more physically meaningful indices such as coastal-trapped wave amplitudes [e.g., *Freeland et al.*, 1986]. If the modes are real-valued and independent of frequency, their amplitudes are usually estimated by least squares fitting the modes to observations in the time domain. If the modes are complex and frequency-dependent, several techniques are available to the observationalist, and, in general, they will attribute different energies to each mode and different coherences between pairs of modal amplitudes.

In this paper we present a method for estimating the cross-spectral matrix of modal amplitudes, henceforth referred to as **A**. The expression for **A** is simple and depends only on the cross-spectral matrix of observations (**U**) and the modes (**M**), both of which may be complex and a function of frequency. The diagonal elements of **A** are the spectral densities of the modal amplitudes; they give the energy associated with each mode. The normalized off-diagonal elements of **A** give the phase lag and coherence, and hence the strength of the coupling, between the modal amplitudes. To keep the discussion focused, we present the method in a shelf circulation context where the modes correspond to coastal-trapped waves. However, the method may be applied to any linear system where the modes are known and the cross-spectral matrix of modal amplitudes is required.

There are similarities between our method and that developed by *Freeland et al.* [1986] in their search for coastal-trapped waves off the southeast coast of Australia. In essence, Freeland et al. complex demodulate the observations at a given frequency and then calculate time series of modal amplitudes. These complex series are then combined and time-averaged to give a cross-spectral matrix of modal amplitudes. In our method the time averaging is "built in" and this simplifies the estimation of **A**.

Given the simple form of **A**, it is straightforward to test its sensitivity to changes in array configuration (by modifying **M**) or instrument performance (by postulating different forms for **U**). Thus we anticipate that our method may be useful in the design of observing arrays. It is also easy to test the performance of different models (i.e., combinations of modes) on the observations, after they have been collected.

In addition to **A**, we also define a cross-spectral matrix of residuals, **R**. It too can be easily calculated from **M** and **U**. This matrix is of special interest to the oceanographer because it describes motions which cannot be represented by coastal-trapped waves.

In the following sections we outline our method for estimating **A**. We stress that the method is simple to understand, and use, as illustrated by the recent paper of *Middleton and Wright* [this issue] on the generation and propagation of coastal-trapped waves on the Labrador Shelf.

2. PHYSICAL BACKGROUND TO THE PROBLEM

In a seminal paper, *Gill and Schumann* [1974] used shelf waves to explain how a narrow continental shelf sea responds to time-varying wind stress. Their model has been extended over the years to include stratification and friction [e.g., *Clarke and Van Gorder*, 1986]. The modern theory is now a useful tool for understanding and modeling large-scale, wind-driven circulation on the continental shelf.

As part of a study of nearshore and shelf circulation, we have developed a new statistical method for detecting trapped waves. In order to motivate the discussion of this method we will briefly review the Gill and Schumann theory.

Consider a narrow shelf with the x axis pointing seaward and the y axis aligned with the coast. Assume the bathymetry and wind do not change in the alongshore direction and there is no friction. If the long-wave and rigid lid approxi-

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mations are made the onshore component of the flow, u, satisfies the following vorticity equation in standard notation:

$$\frac{\partial^2}{\partial x\,\partial t}\left(\frac{1}{h}\frac{\partial(uh)}{\partial x}\right)+\frac{f}{h}\frac{dh}{dx}\frac{\partial u}{\partial y}=0 \qquad (1)$$

Taking Gill and Schumann's approach, expand u as a sum of shelf waves

$$u(x, y, t)=\sum_n a_n(y, t)\phi_n(x) \qquad (2)$$

where $\phi_n(x)$ is the nth eigenfunction derived from (1) and a_n is its amplitude. In general, the a_n are determined from the local wind forcing and the amplitudes of the shelf waves moving into the model domain across the upstream boundary [*Gill and Schumann*, 1974].

The real world is obviously more complicated than this model. For example, friction and alongshore variations in topography can cause "scattering" of energy between coastal-trapped waves. Stratification and a free surface complicate the picture by making the modes and phase speeds frequency-dependent. However, a legacy of the Gill and Schumann theory is the belief that shelf circulation can be usefully interpreted in terms of coastal-trapped waves, and, over the years, there have been attempts to detect such waves in large multivariate data sets [e.g., *Hsieh*, 1982; *Yao et al.*, 1984; *Freeland et al.*, 1986].

In the next section we outline a method for estimating the spectral density of the modal amplitudes, a_n, and the coherence and phase between pairs of them.

3. Fitting the Model to the Observations

In this section we will assume the observations are onshore currents. This will clarify the connection between the dynamical model, described in the last section, and the statistical method. Note, however, that the statistical method is quite general and would work with any other variable, or a combination of variables. We start by introducing a model for the observations in the time-domain. An estimator is then obtained for the covariance matrix of modal amplitudes. The method is then extended to cover frequency-dependent modes and cross-spectral matrices of modal amplitudes.

Fitting in the Time Domain

Suppose that p observations of onshore current are measured by a current meter array which is aligned normal to shore. Denote the onshore current measured at x_i by the ith element of $\mathbf{u}$. Suppose these observations are to be interpreted in terms of $m < p$ coastal-trapped waves passing by the array. Let the kth column of $\mathbf{M}$ denote the kth eigenfunction ϕ_k as "seen" by the array

$$\mathbf{M}=\begin{bmatrix}\phi_1(x_1) & \cdots & \phi_m(x_1)\\ \phi_1(x_2) & & \vdots\\ \vdots & & \vdots\\ \phi_1(x_p) & & \vdots\end{bmatrix} \qquad (3)$$

where $\mathbf{M}$ is a $p \times m$ matrix with real elements. Express the observation vector $\mathbf{u}$ as a sum of coastal-trapped wave contributions and a residual term:

$$\mathbf{u}=\mathbf{Ma}+\mathbf{r} \qquad (4)$$

where $\mathbf{a}$ is the $m \times 1$ vector of modal amplitudes and $\mathbf{r}$ is a $p \times 1$ vector of "residuals" corresponding to the missing physics and instrumental noise. There are many ways to estimate the covariance matrix of modal amplitudes, $\mathbf{A}$. The three obvious ones described below all lead to the same estimator for $\mathbf{A}$.

1. The simplest approach is to require the residuals to be orthogonal to the modes:

$$\mathbf{M'r}=0 \qquad (5)$$

where a prime denotes transpose. Assuming $\mathbf{M'M}$ is nonsingular it follows from (4) and (5) that

$$\mathbf{a}=(\mathbf{M'M})^{-1}\mathbf{M'u} \qquad (6)$$

This explicit expression for the modal amplitudes has formed the basis of several earlier mode-fitting methods [e.g., *Kundu et al.*, 1975; *Freeland et al.*, 1986]. Assume that both $\mathbf{a}$ and $\mathbf{u}$ are zero mean, stationary random vectors with covariance matrices

$$\mathbf{A}=\mathscr{E}(\mathbf{aa'}) \qquad (7)$$

$$\mathbf{U}=\mathscr{E}(\mathbf{uu'}) \qquad (8)$$

where $\mathscr{E}$ denotes expectation. Combining (6), (7) and (8) we obtain the required covariance matrix of modal amplitudes:

$$\mathbf{A}=(\mathbf{M'M})^{-1}\mathbf{M'UM}(\mathbf{M'M})^{-1} \qquad (9)$$

2. A related approach is based on the assumption that the modal amplitudes can be expressed as a linear combination of the observations

$$\mathbf{a}=\mathbf{Bu} \qquad (10)$$

where $\mathbf{B}$ is an unknown, but constant, $m \times p$ matrix. From (4) we have

$$\mathbf{r}=(\mathbf{I}-\mathbf{MB})\mathbf{u} \qquad (11)$$

and hence the covariance matrix of residuals is

$$\mathbf{R}=(\mathbf{I}-\mathbf{MB})\mathbf{U}(\mathbf{I}-\mathbf{MB})' \qquad (12)$$

The diagonal elements of $\mathbf{R}$ are the residual variances, and the off-diagonal elements are the residual covariances. The trace of this covariance matrix, Tr $\mathbf{R}$, is the sum of the residual variances. If we choose $\mathbf{B}$ to minimize Tr $\mathbf{R}$ we find

$$\mathbf{B}=(\mathbf{M'M})^{-1}\mathbf{M'} \qquad (13)$$

Combining (13), (10) and (7) we obtain (9) as before. By combining (13) and (12) we obtain the covariance matrix of residuals associated with this choice of $\mathbf{A}$:

$$\mathbf{R}=(\mathbf{I}-\mathbf{M}(\mathbf{M'M})^{-1}\mathbf{M'})\mathbf{U}(\mathbf{I}-\mathbf{M}(\mathbf{M'M})^{-1}\mathbf{M'}) \qquad (14)$$

3. The contribution of the m coastal-trapped waves is $\mathbf{Ma}$ with covariance matrix

$$\tilde{\mathbf{U}}=\mathbf{MAM'} \qquad (15)$$

Our final way of defining $\mathbf{A}$ is to make $\tilde{\mathbf{U}}$ as "close" as possible to the true covariance matrix $\mathbf{U}$. More specifically we choose $\mathbf{A}$ to minimize

$$Q(\mathbf{A}) = \sum_{i=1}^{p} \sum_{j=1}^{p} (U_{ij} - \tilde{U}_{ij})^2 \quad (16)$$

This quantity may be reexpressed as

$$Q(\mathbf{A}) = \mathrm{Tr}[(\mathbf{U} - \tilde{\mathbf{U}})(\mathbf{U} - \tilde{\mathbf{U}})] \quad (17)$$

If we choose **A** to minimize $Q(\mathbf{A})$ we again recover (9).

Note that we minimized the sum of all squared differences between the elements of **U** and $\tilde{\mathbf{U}}$. If we chose instead to minimize over just one half of these symmetric matrices, arguing that the other half contained redundant information, there is no guarantee **A** would be positive semidefinite as required by any realizable covariance matrix. Simple examples can be found to prove this point.

Summarizing, (9) is a simple expression for the covariance matrix of modal amplitudes. It depends only on the modes and the covariance matrix of observations. Equation (14) is the corresponding covariance matrix of residuals.

Fitting in the Frequency Domain

The situation is more complicated if the array is aligned parallel to shore. To illustrate, assume a single coastal-trapped wave is propagating freely through the array with phase speed c. (The combination of free and forced waves is discussed in the final section of the paper.) If the shelf wave amplitude at the upstream boundary is denoted by $a_0(t)$, the amplitude at any downstream location ($y < 0$) is

$$a(y, t) = a_0(t + y/c) \quad (18)$$

In general, $a_0(t + y/c)$ will be a nonseparable function of y and t. Hence the time domain model (4), which separates **u** into spatial and temporal components, does not apply. We can overcome this difficulty by reexpressing (4) in the frequency domain. First note that according to the spectral representation theorem for stationary random processes [e.g., *Priestley*, 1981] we may write

$$a_0(t) = \int_{-\infty}^{\infty} e^{i\omega t} da_0(\omega) \quad (19)$$

where ω denotes frequency and da_0 is a random orthogonal increment process. In general, da_0 is complex-valued and, roughly speaking, determines the amplitudes and phases of the sinusoids which make up a_0. Combining (18) and (19),

$$a(y, t) = \int_{-\infty}^{\infty} e^{i\omega y/c} e^{i\omega t} da_0(\omega) \quad (20)$$

According to (2) the onshore current associated with the passage of this single wave is

$$u(x, y, t) = a(y, t)\phi(x) \quad (21)$$

Combining (20) and (21) we obtain the spectral representation of the onshore current associated with the passage of the free wave:

$$u(x, y, t) = \int_{-\infty}^{\infty} [\phi(x) e^{i\omega y/c}] e^{i\omega t} da_0(\omega) \quad (22)$$

Generalizing (22) to cover m coastal-trapped waves and p observations leads to

$$\mathbf{u} = \int_{-\infty}^{\infty} \mathbf{M} e^{i\omega t} d\mathbf{a}_0(\omega) \quad (23)$$

where **u** is the $p \times 1$ observation vector, $d\mathbf{a}_0(\omega)$ is the $m \times 1$ random orthogonal increment process defining the m coastal-trapped wave amplitudes at the upstream boundary and **M** is a $p \times m$ matrix which, roughly speaking, converts the sinusoidal components of $\mathbf{a}_0$ into the sinusoidal components of **u** measured by the array. In general, **M** will be complex and frequency-dependent. As in the time domain model, we add a "residual" term to our spectral representation of **u**:

$$\mathbf{u} = \int_{-\infty}^{\infty} \mathbf{M} e^{i\omega t} d\mathbf{a}_0(\omega) + \int_{-\infty}^{\infty} e^{i\omega t}\, d\mathbf{r}(\omega) \quad (24)$$

To obtain the frequency-dependent generalization of (4) note that the observation vector can also be expressed, quite generally, as

$$\mathbf{u} = \int_{-\infty}^{\infty} e^{i\omega t}\, d\mathbf{u}(\omega) \quad (25)$$

where $d\mathbf{u}$ is the random orthogonal increment processes defining the observations. Equating (24) and (25) leads to the frequency-dependent generalization of (4):

$$d\mathbf{u} = \mathbf{M}(\omega)\, d\mathbf{a}_0(\omega) + d\mathbf{r}(\omega) \quad (26)$$

Although this model was obtained through consideration of freely propagating coastal-trapped waves, it is clearly of wider applicability. Indeed it is of fundamental importance in the spectral analysis of multivariate time series [e.g., *Priestley*, 1981]. Note that if **M** is independent of frequency, (26) reduces to the time domain model (4).

It is now straightforward to obtain an explicit expression for the cross-spectral matrix of modal amplitudes, **A**, which is defined by

$$\mathbf{A}\, d\omega = \mathscr{E}(d\mathbf{a}_0\, d\mathbf{a}_0') \quad (27)$$

where a prime is interpreted henceforth as conjugate transpose. In particular, the three approaches used in the time domain can be extended to the frequency domain [*Haines*, 1987]. They all lead to the same cross-spectral matrix of modal amplitudes:

$$\mathbf{A} = (\mathbf{M}'\mathbf{M})^{-1}\mathbf{M}'\mathbf{U}\mathbf{M}(\mathbf{M}'\mathbf{M})^{-1} \quad (28)$$

where **U** is now the cross-spectral matrix of observations. Associated with **A** is a cross-spectral matrix of residuals (**R**) which, for approaches (1) and (2), is given by (14) if we interpret a prime as conjugate transpose and **U** as a cross-spectral matrix.

So far, our definition of the cross-spectral matrix of modal amplitudes involves the true cross-spectral matrix of observations. In practice, we do not know **U** but have an estimate, $\hat{\mathbf{U}}$. This leads us to the following estimator for **A**:

$$\hat{\mathbf{A}} = (\mathbf{M}'\mathbf{M})^{-1}\mathbf{M}'\hat{\mathbf{U}}\mathbf{M}(\mathbf{M}'\mathbf{M})^{-1} \quad (29)$$

Thus given the modal shapes, and a sample cross-spectral matrix of observations, (29) provides a straightforward way to estimate the cross-spectral matrix of modal amplitudes.

4. Assessing the Model Fit

We now examine the sampling distribution and bias of the estimator $\hat{\mathbf{A}}$. To keep the discussion as simple as possible, we will again focus on covariance matrices; the generalization to cross-spectral matrices is straightforward.

Sampling Distribution of $\hat{\mathbf{A}}$

If $\hat{\mathbf{U}}$ has a Wishart distribution based on ν degrees of freedom, then $\hat{\mathbf{A}}$ has a Wishart distribution with the same ν [e.g., *Srivastava and Khatri*, 1979]. Therefore the variances and covariances of the elements of $\hat{\mathbf{A}}$ can be calculated in the usual way. This feature of our method carries over directly to the frequency domain.

Bias

Consider now the more difficult problem of what to expect if an incomplete or inappropriate set of modes is fit to the observations. We will focus on the bias of $\hat{\mathbf{A}}$:

$$\beta = \mathscr{E}(\hat{\mathbf{A}}) - \mathbf{A} \tag{30}$$

Assuming that $\hat{\mathbf{U}}$ is an unbiased estimator of $\mathbf{U}$, the expected value of $\hat{\mathbf{A}}$ is

$$\mathscr{E}(\hat{\mathbf{A}}) = (\mathbf{M}'\mathbf{M})^{-1}\mathbf{M}'\mathbf{U}\mathbf{M}(\mathbf{M}'\mathbf{M})^{-1} \tag{31}$$

From the model for the observations

$$\mathbf{U} = \mathscr{E}(\mathbf{Ma} + \mathbf{r})(\mathbf{Ma} + \mathbf{r})' \tag{32}$$

Substituting (32) into (31), and using the above definitions of $\mathbf{A}$ and $\mathbf{R}$, gives

$$\beta = (\mathbf{M}'\mathbf{M})^{-1}\mathbf{M}'\mathbf{R}\mathbf{M}(\mathbf{M}'\mathbf{M})^{-1} + \mathbf{C} + \mathbf{C}' \tag{33}$$

where $\mathbf{C}$ depends on the covariance between the modal amplitudes and the residuals:

$$\mathbf{C} = \mathscr{E}(\mathbf{ar}')\mathbf{M}(\mathbf{M}'\mathbf{M})^{-1} \tag{34}$$

The residuals include both instrumental error and motions which cannot be described in terms of the fitted modes. In practice, our physical understanding is usually so incomplete that it will not be possible to specify a priori the covariance structure of the residuals and hence determine the bias. However, as shown below, it is possible to identify an important class of models for which the bias is zero, and also assess how the bias will be affected by instrumental noise.

If the residuals are orthogonal to the modes, as in (5), it is straightforward to show that β is zero and hence $\hat{\mathbf{A}}$ is an unbiased estimator. This would hold, for example, if the residuals were due to $(p - m)$ modes (the columns of $\mathbf{M}_1$, say) which were orthogonal to the m fitted modes (the columns of $\mathbf{M}$):

$$\mathbf{u} = \mathbf{Ma} + \mathbf{M}_1\mathbf{a}_1 \tag{35}$$

where

$$\mathbf{M}'\mathbf{M}_1 = 0 \tag{36}$$

Thus for this class of models at least, $\hat{\mathbf{A}}$ is unbiased. Let us now move on to the special case of observations which are due entirely to instrumental noise. In this case both $\mathbf{A}$ and $\mathbf{C}$ are zero and the bias is

$$\beta = (\mathbf{M}'\mathbf{M})^{-1}\mathbf{M}'\mathbf{R}\mathbf{M}(\mathbf{M}'\mathbf{M})^{-1} \tag{37}$$

If the noise has equal variance σ^2, and is uncorrelated between sensors, the bias reduces to the simple form

$$\beta = \sigma^2(\mathbf{M}'\mathbf{M})^{-1} \tag{38}$$

Note that if the modes are orthogonal (i.e., $\mathbf{M}'\mathbf{M}$ is diagonal) the cross spectra of wave amplitudes are unbiased. It follows that a well-designed array will reduce spurious coupling between wave amplitudes.

Overall Fit and Interpretation

It would be unrealistic in most oceanographic applications to assume the residuals are simply instrumental noise with a diagonal covariance matrix: we expect motions which cannot be described in terms of coastal-trapped waves to make an important contribution to $\mathbf{r}$. A good example of this is given by *Freeland et al.* [1986] who showed that a significant part of the current variability on the southeast Australian shelf was due to offshore eddies which they removed before searching for coastal-trapped waves.

Our inability to specify the covariance structure of the residuals complicates any physical interpretation of $\mathbf{A}$. This is illustrated by the following example. Suppose $\mathbf{U}$ equals $\sigma^2\mathbf{I}$. There are many physical processes which could generate such a covariance matrix. For example, the observations could be just instrumental noise. Alternatively, the observations may be noise-free, generated by p modes which are orthonormal across the array with $\mathbf{A}$ equal to $\sigma^2\mathbf{I}$. Suppose we now fit m modes to the observations. For both types of observation $\mathscr{E}(\hat{\mathbf{A}})$ is equal to $\sigma^2(\mathbf{M}'\mathbf{M})^{-1}$. For the all-noise observations $\hat{\mathbf{A}}$ is biased, and a naive interpretation of the model fit may lead to the erroneous conclusion that the modes explain some of the observed variability. Consider now the noise-free observations generated by the p orthonormal modes. If $\mathbf{M}$ is formed from a subset of these p modes then $\hat{\mathbf{A}}$ will be unbiased and we may well obtain a reasonable estimate of $\mathbf{A}$ and the true contribution of the modes. The difficulty with the physical interpretation of the fitted model stems from the generally unknown residual covariance structure; as this simple example shows, some care must be exercised in the interpretation of $\hat{\mathbf{A}}$.

So, how should we assess the overall model fit and, at the same time, guard against the misinterpretation of $\mathbf{A}$? We favor the following approach which is based on an assessment of the predictive, rather than descriptive, skill of the model. The approach is outlined in terms of covariance matrices; the extension to cross-spectral matrices is straightforward.

The first step is to select a variable, say u_i, which is to be omitted from the estimation procedure. Denote the ith row of $\mathbf{M}$ by $\mathbf{m}_i$ and delete it from $\mathbf{M}$ to leave the $(p - 1) \times m$ matrix $\mathbf{M}_i$. Let $\mathbf{u}_i$ denote the $(p - 1) \times 1$ vector of covariances between u_i and the remaining $(p - 1)$ variables. The covariance between the observed u_i and the u_i predicted by the other variables, is $\mathbf{m}_i(\mathbf{M}_i'\mathbf{M}_i)^{-1}\mathbf{M}_i'\mathbf{u}_i$. It is straightforward to test if this covariance is significantly different from zero and hence if the model has predictive skill. Repeat the above procedure for the remaining $(p - 1)$ variables and determine the overall predictive skill of the model. If $\hat{\mathbf{A}}$ and $\mathbf{M}$ cannot predict a significant proportion of the observed variance, it would be unwise to attempt a physical interpretation; this would certainly be the case when the observations are due solely to instrumental noise. On the other hand, an interpretation may be justified if a significant proportion of the variance can be predicted by modes with coupled amplitudes.

5. DISCUSSION

It is often useful to model the ocean as a linear combination of prescribed spatial modes with time-varying coefficients. In this paper we have shown how to estimate the cross-spectral matrix of modal amplitudes, **A**, from observations. The estimator we propose for **A**, given by (29), has several attractive features:

1. It is easy to evaluate, requiring only the cross-spectral matrix of observations and the modes. Therefore it may be useful at the design stage of the experiment. For example, one could specify a cross-spectral matrix of observations, corresponding to a plausible mix of noise and coupled wave contributions, and then determine how accurately different array designs can recover the prescribed **A**. The robustness of an array can also be assessed by checking the sensitivity of **A** to the deletion of any instrument from the array. The performance of different combinations of modes can also be quickly tested on the observations, and subsets of them, after they have been collected.

2. $\hat{\mathbf{A}}$ is a positive semidefinite Hermitian matrix. Thus the power spectra of model amplitudes will be nonnegative, and their coherences will lie between 0 and 1. This is not guaranteed if **A** is chosen to minimize $|U_{ij} - \bar{U}_{ij}|^2$ summed over just one half of these Hermitian matrices [e.g., *Yao et al.*, 1984].

3. If $\hat{\mathbf{U}}$ has a complex Wishart distribution then $\hat{\mathbf{A}}$ has a complex Wishart distribution with the same degrees of freedom. This means that variances of the power spectra, coherences and phases of modal amplitudes can be calculated in the usual manner [e.g., *Jenkins and Watts*, 1968].

A note of caution should be sounded at this point. Our method will return an $\hat{\mathbf{A}}$ for any choice of **M**, even one based on an incorrect dynamical model. There is clearly a danger of misinterpreting $\hat{\mathbf{A}}$. This problem, which is confounded by the unknown form of the residuals, is not unique to our method: it affects methods based on fitting models to the observed cross spectra [e.g., *Yao et al.*, 1984] or fitting modes directly to the observations [e.g., *Freeland et al.*, 1986]. To help in the interpretation of $\hat{\mathbf{A}}$ we have shown how to evaluate its bias for any combination of instrumental noise and modal contributions. We have also outlined a procedure for assessing the predictive skill of the model and suggest that $\hat{\mathbf{A}}$ be taken seriously, and indeed interpreted, only if the model can predict a significant proportion of the variance.

Usually different types of data will be available for analysis, e.g., currents and sea levels. Thus how do we weight the relative importance of different variables when fitting the model? There is no easy answer to this question, although we note that weighting can be readily incorporated into the method. We could, for example, normalize the elements of $\hat{\mathbf{U}}$ by the square root of spectral density. This corresponds to replacing $\hat{\mathbf{U}}$ by the complex coherency matrix. Alternatively an estimate of the signal-to-noise ratio could be used, based on experience of instrument performance. Given that weighting will usually involve subjective judgements about data quality, it is probably best tackled on a case-by-case basis. On an encouraging note, our experience with oceanographic data shows that $\hat{\mathbf{A}}$ is relatively insensitive to the weighting scheme if the model fits well.

Coastal-trapped waves can be forced by the local wind or they can be forced remotely, by distant winds or offshore eddies for example, and then propagate into the region of interest. Our estimate of **A** does not distinguish between free and forced components. One way to isolate the free component, or at least a part of it, is to remove the forced component using a frequency domain regression model with local wind as an independent variable. We note, however, that the regression model will also remove that part of the free component which is forced by remote winds coherent with the local wind. In practice, this approach is equivalent to replacing $\hat{\mathbf{U}}$ by the partial cross-spectral matrix of observations, having allowed for local wind. This matrix is readily obtained from $\hat{\mathbf{U}}$ and the cross spectra between wind and observations [e.g., *Jenkins and Watts*, 1968]. Another approach [see *Middleton and Wright*, this issue] is to include the local wind as a "dummy mode" in the **M** matrix. An advantage of this approach is that it also gives the coherence and phase between the local wind and modal amplitudes.

Finally we reemphasize that although the method has been described in terms of free coastal-trapped waves and simple array configurations, it may be applied to any linear system where the spatial forms of the waves are known and the cross-spatial matrix of their amplitudes is required.

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