Motion of a point dipole in an infinite hole through a superconductor

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(Received 25 February 1993)

We consider the system of a magnetic point dipole placed in an infinite square hole through a superconductor. Using the method of images we obtain the potential and the field distribution in the hollow. Using Lagrangian mechanics, we study the motion of the point dipole in the nonrelativistic regime. Relevant applications of this problem are discussed.

The so-called Meissner effect was discovered by Meissner and Ochsenfeld[1,2] in 1933 and describes the diamagnetic behavior of a material in the superconducting state. When a superconductor is placed in a magnetic field the magnetic flux in any holes in the superconductor will be confined. If a magnetic field source (say a point dipole) is placed inside a superconducting "*void*," the confinement will cause the source (dipole) to move to an equilibrium position which corresponds to the minimum value of the free energy of the system. In this paper, we report on our investigations of a system consisting of a magnetic point dipole placed in an infinite square hole through a superconductor. Potential applications of this system will also be discussed.

Consider a magnetic point dipole with moment $\mathbf{m}$ located inside an infinite square hole through a superconductor at the coordinates $(x_0, y_0, 0)$ as illustrated in Fig. 1(a). For convenience, in the present work we consider the case of a dipole with its moment parallel to the z axis. The inclusion of the angular orientation of the dipole moment yields results which are fundamentally similar to those presented here and will be discussed in detail elsewhere.[3] In order to facilitate explicit calculations, the cross section of the hole was chosen to be a square with edge 2 (units) and we consider the ideal case with $\lambda = 0$ (λ is the penetration depth), i.e., the perfect diamagnetic model (or complete Meissner effect).

The magnetic induction, $\mathbf{B}(x, y, z)$, will be given in terms of a scalar potential by the relation $\mathbf{B}(x, y, z) = -\boldsymbol{\nabla} V(x, y, z)$. This total scalar potential, $V(x, y, z)$, inside the hole satisfies

$$\nabla^2 V(x, y, z) = \mu_0 m \delta(x - x_0)\delta(y - y_0)\delta'(z) , \tag{1}$$

where $m = |\mathbf{m}|$. The appropriate boundary conditions for the normal component of the magnetic induction on the surface of the hole, $B_n^{\text{surface}} = 0$, can be written as

$$B_x(\pm 1, y, z) = 0 \quad \text{and} \quad B_y(x, \pm 1, z) = 0 . \tag{2}$$

The scalar potential of a point dipole can be expressed as[4]

$$V_{\text{dipole}}(x, y, z) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \cdot \mathbf{R}}{R^3} , \tag{3}$$

where $R = |\mathbf{R}|$, and $\mathbf{R}$ is the spatial vector from the dipole to the point of interest. Using the image method, we obtain the induced scalar potential of the system in the hole as

$$V_{\text{ind}}(x, y, z) = \frac{\mu_0 m}{4\pi} \sum_{\substack{l=-\infty \\ n=-\infty}}^{\infty}{}' \frac{z}{\left\{[x - 2n - (-1)^n x_0]^2 + [y - 2l - (-1)^l y_0]^2 + z^2\right\}^{3/2}}, \tag{4}$$

where the distribution of image dipoles is described in Fig. 1(b). The prime on the sum indicates that the $n = l = 0$ term is excluded in the induced potential (and in the magnetic induction and interaction energy given subsequently). We have assumed that all the image dipoles are located in the same plane as the source dipole and choose $z = 0$. This is also appropriate for a dipole in motion subject to neglect of relativistic effects.

The total scalar potential is the sum of Eqs. (3) and (4). It is then easy to see that the normal components of the derivative of the total scalar potential vanish at the surface of the hole, so that the boundary conditions, Eq. (2), are satisfied.

The z component of the induced magnetic induction is given by

$$B_{\text{ind},z}(x, y, z) = -\frac{\partial V_{\text{ind}}}{\partial z} = \frac{\mu_0 m}{4\pi} \sum_{\substack{l=-\infty \\ n=-\infty}}^{\infty}{}' \frac{2z^2 - [x - 2n - (-1)^n x_0]^2 - [y - 2l - (-1)^l y_0]^2}{\left\{[x - 2n - (-1)^n x_0]^2 + [y - 2l - (-1)^l y_0]^2 + z^2\right\}^{5/2}} . \tag{5}$$

The interaction energy, U, is expressed as[5]

$$U(x_0,y_0,0) = -\frac{1}{2}\mathbf{m}\cdot\mathbf{B}_{\rm ind}(x=x_0,y=y_0,z=0)$$
$$= \frac{\mu_0 m^2}{8\pi}\sum_{\substack{l=-\infty\\ n=-\infty}}^{\infty}{}' \frac{1}{\left\{[x_0-2n-(-1)^n x_0]^2+[y_0-2l-(-1)^l y_0]^2\right\}^{3/2}} > 0\ . \tag{6}$$

The interaction energy plays the role of a potential well which confines the motions of the dipole in the x and y directions.

It can be seen that Eq. (6) has four singularity conditions of order 3 which correspond to the four walls of the square hole and occur for the values $x_0 = \pm 1^{\mp}$ and $y_0 = \pm 1^{\mp}$. This means that the interaction energy goes to infinity at these surfaces. In order to visualize the behavior of the interaction energy over the entire extent of the hole, we plot Eq. (6) over the interval $x_0 \in [0,1)$ for $y_0 = 0$ in Fig. 2(a) (i.e., along the x axis). Due to the symmetry of the problem (i.e., $x_0 \to -x_0$), we have plotted only the $x_0 > 0$ portion of the curve. This curve shows a minimum at the center of the square hole and diverges at the walls of the hole. The mathematical details of this function will be described below. The general shape of the interaction energy as illustrated in Fig. 2(a) illustrates that the equilibrium position for the dipole will be at the center of the hole.

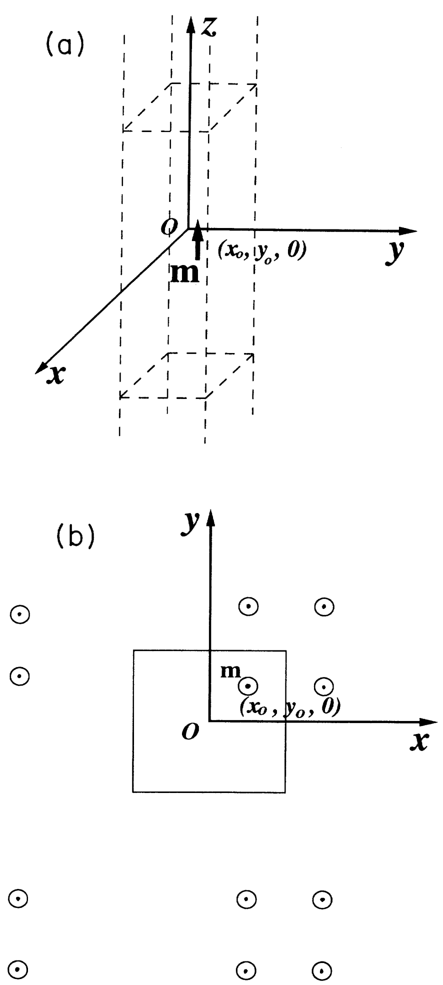

FIG. 1. (a) Diagram of a magnetic point dipole (**m**) placed in an infinite square hole through a superconductor. (b) The point dipole at $(x_0, y_0, 0)$ and some of the infinite number of image dipoles in the $z = 0$ plane.

Physically, the confined motion of the point dipole in the hole can be characterized as one of two types: (i) harmonic oscillation when the amplitude of the displacement from the equilibrium position is small and (ii) anharmonic when the amplitude of the displacement from the equilibrium position is large. These two situations will be discussed in the next two sections, respectively.

(i) *Horizontal oscillations.* Horizontal oscillations, that is, oscillations in the x-y plane, can be considered by minimizing the interaction energy as given in Eq. (6), i.e., setting

$$\frac{\partial U}{\partial x_0} = 0 \quad \text{and} \quad \frac{\partial U}{\partial y_0} = 0\ . \tag{7}$$

The equilibrium position for the dipole ($x_0 = 0$, $y_0 = 0$) follows directly from Eqs. (6) and (7). The interaction energy can be expanded in terms of x_0 and y_0 under the

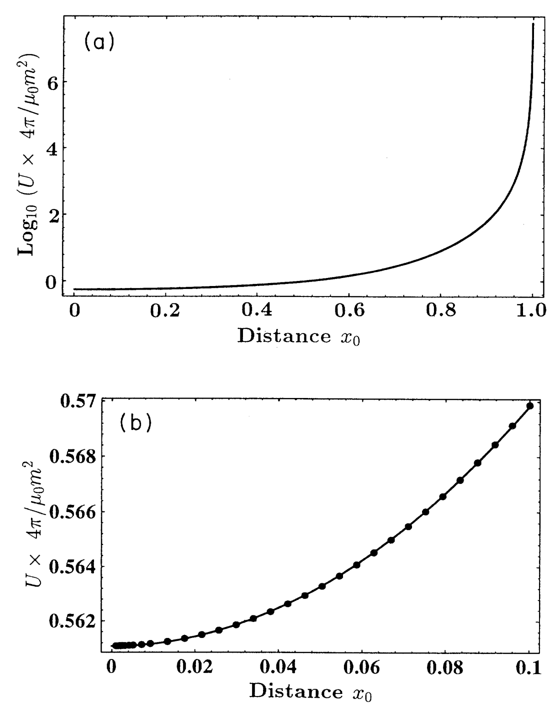

FIG. 2. The interaction energy [Eq. (6)] as a function of distance along the x axis; (a) semilog plot for $x_0 \in [0,1)$ and (b) linear plot for $x_0 \in [0, 0.1]$ (dots). The solid line is a least-squares fit with the function $u = 0.5611 + 0.8752x_0^2$.

second-order (harmonic) approximation to obtain $U = U_0 + U_{\rm eff}$, where

$$U_0 = \frac{\mu_0 m^2}{16\pi}\left[\zeta(3) + \sum_{\substack{l=1\\n=1}}^{\infty} \frac{1}{(n^2+l^2)^{3/2}}\right] \approx 2.244\frac{\mu_0 m^2}{16\pi} , \tag{8}$$

$\zeta(n)$ is the Riemann ζ function and the effective potential is given by

$$U_{\rm eff}(x_0,y_0) = \frac{3\mu_0 m^2}{512\pi}C(x_0^2+y_0^2) , \tag{9}$$

where the constant C is given by

$$C = 31\zeta(5) + 4\sum_{\substack{l=1\\n=1}}^{\infty} \frac{(4n^2-l^2)[1-(-1)^n]^2}{(n^2+l^2)^{7/2}} \approx 36.314 . \tag{10}$$

As shown in Fig. 2(b), the data obtained directly from Eq. (7) confirm the validity of the harmonic approximation for small amplitude displacements. If it assumed that the point dipole has a mass M, the effective Lagrangian in the harmonic approximation can be expressed as $L_{\rm eff} = T - U_{\rm eff}$, i.e.,

$$L_{\rm eff} = \frac{1}{2}M(\dot{x}_0^2+\dot{y}_0^2) - \frac{3\mu_0 m^2}{512\pi}C(x_0^2+y_0^2) . \tag{11}$$

The Euler-Lagrange equations can easily be derived to be

$$\ddot{x}_0 + \omega^2 x_0 = 0, \tag{12}$$

$$\ddot{y}_0 + \omega^2 y_0 = 0 , \tag{13}$$

where the degenerate oscillation frequency, ω, is

$$\omega = \sqrt{\frac{3\mu_0 m^2 C}{256\pi M}} \approx \sqrt{0.851\frac{\mu_0 m^2}{2\pi M}} . \tag{14}$$

For a hole with a rectangular cross section it can readily be shown that there are two distinct oscillation frequencies.[3]

It is straightforward to obtain solutions to Eqs. (12) and (13) of the form

$$x_0 = x_{00}\sin(\omega t + \delta_1), \tag{15}$$

$$y_0 = y_{00}\sin(\omega t + \delta_2), \tag{16}$$

where x_{00} and y_{00} are the initial displacements, and δ_1 and δ_2 are the initial phases of the oscillation.

In a real system, the motion may be dissipative for various reasons. In such cases a more detailed analysis of the physical properties of the system with the inclusion of appropriate damping terms in the equations will be necessary.

(ii) Focusing a neutron beam: a possible application. Let us now consider the focusing function of the hole. Assume a point dipole at rest and located very close to the wall at time $t = 0$, i.e., $x_{00} \approx 1^-$ and $\dot{x}_{00} = 0$. For convenience of calculation, we only need to consider the region of $x_0 \in [0, 1)$ since the potential is symmetric in x_0. For a conservative system, we can use energy conservation to obtain the equation of motion of the dipole as

$$\frac{1}{2}M\dot{x}_0^2 = U(x_{00}) - U(x_0) . \tag{17}$$

Due to the singularity at $x_0 = 1^-$, the right-hand side of Eq. (17) (i.e., the change in the potential) will be determined by the singularity term. Some straightforward algebra yields

$$\dot{x}_0 \approx -\sqrt{\frac{\mu_0 m^2}{32\pi M}}\sqrt{\frac{-\Delta x}{(1-x_{00})^3(1-x_0)}} , \tag{18}$$

where $\Delta x = x_0 - x_{00}$ the displacement of the dipole from its initial position near the wall. When the dipole is initially close to the wall, there will be an extremely strong repelling force pushing the dipole towards the equilibrium position at the center of the hole. This results in a focusing of the dipole inside the hole.

These results may be applied to the study of the motion of electrically neutral particles which have magnetic moments inside a hole through a superconductor. When the size of the particle is much smaller than the spatial dimension of the hole, the chargeless particle may be modeled as a point dipole. The above results could be directly used to describe the *horizontal* motion of such particles in the hole.

The application of the above results may be extended by considering the horizontal motion of an electrically neutral particle with a magnetic dipole moment which is moving in the z direction inside the hole. Under nonrelativistic conditions, we may assume that the point dipole and all its images are in the same plane at any time. As the dipole moves along the z direction, the hole will act as a converging lens, focusing the dipole towards the symmetry axis $(0, 0, z)$.

This model has a possible practical application to the focusing of a neutron beam. To our knowledge, due to the chargeless nature of neutrons, there is no known suitable method of focusing the neutron beam from a reactor. Without such effective focusing, the usual procedures for the preparation of a neutron beam from a reactor result in the loss of a large fraction of its intensity before its utilization for experiments. Generally speaking, the thermal neutrons from a reactor have a typical speed of 10^2–10^3 m/sec. Thus, relativistic effects would not be a consideration and the calculations of the above model are applicable to the motion of neutrons. If a superconducting tube were constructed between the exit window of a neutron reactor and the experimental chamber, the intensity of the neutron beam could be largely maintained.

From Fig. 2, it is easy to see that the harmonic approximation for the interaction energy is suitable only for small displacements. In order to understand the focusing action of the hole in the superconductor on the neutron beam we have to consider the anharmonic motion which occurs for large displacements of the dipole from the equilibrium position. A detailed solution of the nonlinear system of equations necessary to describe the anharmonic behavior for large displacements is beyond the scope of the present paper. However, on the basis of the calculated interaction energy, a phenomenological description of the focusing of a neutron beam in a hole through a superconductor is possible. From Fig. 2 we see that the neutron will experience a large repulsive force as it approaches the singularity in the potential at the wall of the hole. In this respect it should be pointed out that the details of the microscale processes (i.e., quantum effects), which may be of relevance when the neutron is very close to the wall (i.e., on the length scale of the lattice parameters), are not being considered in the present work. The force which the neutron will experience will tend to confine the neutron to the interior of the hole through the superconductor and thus prevent it from penetrating the superconductor. This will allow the neutron beam to be focused in the sense that neutrons, which have a small horizontal velocity component but are generally moving in the z direction, will be guided along the interior of the hole. The general features of the present treatment may be extended to geometries which may be more suitable for guiding neutrons from a practical standpoint; i.e., a conical or tapered geometry to reduce the spatial extent of the neutron beam or a curved hole to redirect the beam. As well, the general features presented here also apply to holes with circular or elliptical cross sections. The square geometry has been presented here as it greatly simplifies the mathematical treatment.[3]

The angular orientation of the neutron magnetic moment relative to its direction of propagation defines its polarization state. In the present work we have considered only the case where the polarization axis of the neutrons is along the z axis. A more general consideration of the interactions between the neutron magnetic moment and the superconductor as a function of the polarization direction does not alter the focusing characteristics described here. This more general treatment of the problem will be reported in detail elsewhere.[3]

The present results indicate the possible potential for a superconducting tube to be used as a means of guiding a beam of neutrons on the basis of the interaction energy for a magnetic dipole moment inside a hole in a perfect diamagnet. An analogous phenomenon has been observed for a beam of charged particles (electrons) inside a superconducting tube. Matsuzawa and co-workers[6,7] have reported experimental results which have been interpreted in terms of the diamagnetic properties of a superconductor as an effective means of guiding the motion of electrons. The present work gives the theoretical basis by which this property of the interaction between moving electric charges and a superconductor may be extended to include the properties of the interaction between a magnetic dipole and a superconductor.

This research was supported in part by the Natural Science and Engineering Research Council (NSERC) of Canada and the Faculty of Graduate Studies, Dalhousie University. One of the authors (Z.J.Y.) also acknowledges support from the Killam Foundation.

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