Universal correction to scaling amplitude ratios for inhomogeneous ferromagnets with continuous symmetry

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Critical exponents and amplitude ratios for corrections to the scaling limit are calculated for a randomly diluted, weakly inhomogeneous $O(m)$ Heisenberg model in an expansion in $\epsilon=4-d$. Calculations for the exponents and amplitude ratios are given correct to $O(\epsilon^2)$ and $O(\epsilon)$, respectively, for the heat capacity (both above and below T_C) and for the susceptibility (above T_C). The new amplitude ratios associated with dilution effects are calculated for both the first-order (associated with critical exponents $\Delta_1=-\alpha,\Delta_2=\omega\nu$) and the second-order (associated with critical exponents $\Delta_3=-2\alpha,\Delta_4=-\alpha+\omega\nu,\Delta_5=2\omega\nu$) corrections to scaling. The diluted $O(m)$ model has the feature that the correction to scaling associated with $|t|^{\Delta_2}$, which is formally of first order with respect to the perturbative coupling constants, becomes negligible for sufficiently small $|t|$ relative to $|t|^{\Delta_3}$, which is formally of higher order in the perturbative coupling constant expansion. The implications of these results for the analysis of experimental data are discussed.

I. INTRODUCTION

The critical properties of inhomogeneous magnetic systems have received a great deal of attention over the past several years. Extensive theoretical studies have been carried out both analytically and numerically on a variety of systems. Such investigations are important because diluted systems probably resemble real physical magnets more accurately than do models of pure systems.

Theoretical work based on renormalization-group[1,2] analysis often begins by considering an effective Ginzburg-Landau-Wilson Hamiltonian where only terms up to quartic coupling in the m-component order parameter $\vec{\phi}$ are explicitly included. In the much studied pure and isotropic $O(m)$ ferromagnet, for example, the starting point would be the effective Hamiltonian

$$\mathscr{H}=\int d^dx\left\{\frac{1}{2}[(\nabla\vec{\phi}(x))^2+\mu^2\vec{\phi}^2(x)]+\frac{\lambda}{4!}(\vec{\phi}^2(x))^2\right\}, \tag{1.1}$$

where $(\nabla\vec{\phi}(x))^2\equiv\Sigma_i(\nabla\vec{\phi}_i(x))^2$. This model has been studied for quite some time and its properties are well understood. For example, the leading critical exponents,[2] universal amplitude ratios,[3] as well as the correction to scaling exponents[4] and several of the universal correction to scaling amplitude ratios[5–7] are all known to at least second order in $\epsilon=4-d$.

Real physical systems are usually described by somewhat more complicated Hamiltonians than that in Eq. (1.1). Hence, terms of a lower symmetry group are often added to describe spin or spatial anisotropy present in crystalline samples, and dilution by nonmagnetic impurities is often taken into account by putting $\mu^2\rightarrow\mu^2(x)$. This essentially exploits the fact, pointed out by Harris,[8] that the crucial effect of introducing weak disorder is to cause variations in the local critical temperature.

The dilute Ising model ($m=1$) has received particular attention since it is known that dilution changes the universality class of this model.[9–11] References 12 and 13 contain field-theoretic renormalization-group calculations on the leading universal amplitude ratios, and also survey and provide references to earlier work on this model.

There are, however, many magnets with $m>1$ which are also of interest. For example, the three-dimensional (3D) diluted Heisenberg model has $m=3$. For present purposes $m=3$ disordered ferromagnets can be loosely classified into three groups. In the case of very weakly diluted systems, only a small number of the magnetic lattice ions are randomly replaced by nonmagnetic impurities. More strongly disordered ferromagnetically ordered binary alloys, especially near their percolation threshold, may be considered to be an extreme case of dilution. In the case of amorphous ferromagnets, a regular lattice does not exist. In some respects, this actually simplifies matters insofar as a number of otherwise strong perturbations on the $O(m)$ symmetry are supressed by the lack of long-range structural order and crystalline anisotropy. Thus, certain amorphous ferromagnets may well be in the $O(m)$ universality class, provided their coarse-grained average inhomogeneity is describable in terms of a local critical temperature which varies slowly in space and has suitably short-ranged autocorrelations.

The present work concerns randomly and weakly diluted, or related types of weakly inhomogeneous, ferromagnets. Our particular objective has been to carry out a theoretical study of the weakly diluted Heisenberg model using renormalization-group methods to determine critical exponents and amplitude ratios for the corrections to scaling for the specific heat (above and below T_C), and for the zero-field susceptibility (above T_C). It is hoped that this additional theoretical information may assist in the determination of reliable critical exponents and amplitude ratios from the analysis of high-quality experimental data.

To be specific, consider the zero-field susceptibility $\chi(t)$, for $t>0$

$$\chi(t)=\Gamma t^{-\gamma}(1+a_{\chi1}t^{-\alpha}+a_{\chi2}t^{\omega\nu}), \tag{1.2}$$

where t is the reduced temperature $t=(T-T_C)/T_C$. The correction to scaling term involving $t^{\omega\nu}$ is the usual thermal

correction due to the fact that $t \neq 0$.[4] This thermal correction term is independent of inhomogeneity effects, and has precisely the same value regardless of whether dilution is present or not. The other correction to scaling term involving $t^{-\alpha}$, where α is the specific-heat exponent of the pure Heisenberg model, arises from the inhomogeneity and is not present in discussions of pure $O(m)$ ferromagnets.[14,15] However, Eq. (1.2) contains only the corrections to scaling due to the first-order terms in the power-series expansions of the coupling constants about the fixed point. We must expect, in general, an infinite series of such correction terms. The second-order correction to scaling exponents, for example, are expected to be $-2\alpha, -\alpha+\omega\nu$, and $2\omega\nu$. This is verified below. We also calculate the associated amplitudes and several amplitude ratios associated with these new corrections. In particular, we find the amplitudes associated with $|t|^{-2\alpha}$ to be nonzero. Since $-2\alpha \approx 0.24$ and $\omega\nu \approx 0.55$ for the 3D Heisenberg system,[16–18] this means that part of the *second-order* $|t|^{-2\alpha}$ corrections to scaling will be dominant over part of the *first-order* $|t|^{\omega\nu}$ contribution to the corrections to scaling, for $|t|$ sufficiently small. This effect clearly generalizes to higher-order terms in the expansion of the coupling constants.

The conclusion can thus be drawn that the analysis of data on the diluted Heisenberg model (and certain other models as well) is somewhat delicate in that special procedures are needed to account for the $|t|^{-p\alpha}$ and $|t|^{-q\alpha+\omega\nu}$ terms (for p, q positive integers). Of course, these procedures are required in the analysis of data on amorphous ferromagnets in order to establish their universality class. The possible presence of other relevant perturbations (e.g., dipole-dipole interactions, magnetoelastic couplings) may place further restrictions on the interpretation of measurements on the correction to scaling amplitudes and exponents. It is hoped that the results derived below help in such future analyses and will also motivate further theoretical work aimed at extending the present low-order ϵ-expansion results.

The rest of the paper is organized as follows. Section II outlines the renormalization procedure used and the renormalization-group equations for the particular vertex functions of interest are derived and solved. Section III contains calculations on the correction to scaling amplitude ratios and exponents. For the most part, details of the calculations are avoided. Outlines and a few intermediate results are given, along with some general references to background material. Finally, Sec. IV concludes with a summary and some pertinent discussion. A limited comparison of theory and experiment is given in this final section.

II. RENORMALIZED PERTURBATION THEORY

The starting point of the investigation is the effective Hamiltonian

$$\mathscr{H} = \int d^d x \left\{ \frac{1}{2}[(\nabla \vec{\phi}(x))^2 + \tilde{\mu}^2(x)\vec{\phi}^2(x)] + \frac{\lambda}{4!}(\vec{\phi}^2(x))^2 \right\}, \tag{2.1}$$

where $\tilde{\mu}^2(x)$ is a random function characterizing the disorder. At each point x, the probability distribution of $\tilde{\mu}^2(x)$ is assumed to be a Gaussian centered about a mean value μ^2. For quenched randomness, the logarithm of the partition function should be averaged over the impurities.[19,20] These two facts enable the use of the replica procedure[21] to generate a new translationally invariant Hamiltonian:

$$\mathscr{H}_{\rm mn} = \int d^d x \left\{ \frac{1}{2}\sum_{i=1}^{n} [(\nabla \vec{\phi}_i(x))^2 + \mu^2 \vec{\phi}_i^2(x)] + \frac{\lambda_1}{4!}\sum_{i,j=1}^{n} \vec{\phi}_i^2(x)\vec{\phi}_j^2(x) + \frac{\lambda_2}{4!}\sum_{i=1}^{n} (\vec{\phi}_i^2(x))^2 \right\}. \tag{2.2}$$

Studying this Hamiltonian is equivalent to studying Eq. (2.1) provided the limit $n \to 0$ is taken. $\mathscr{H}_{\rm mn}$ contains two quartic couplings, each with different symmetry with respect to the replica indices [we still have $O(m)$ symmetry with respect to the Cartesian components of *each* replica vector]. The coupling λ_2 is the original from Eq. (2.1), and the new coupling λ_1 characterizes the strength of the disorder.[10,15] In this notation, $\lambda_1 < 0$ and $\lambda_2 > 0$. From Eq. (2.2), the partition function is formed in the usual way. After including the effect of an external magnetic field,

$$Z_{\rm mn}[B] = \int \prod_{i=1}^{n} \mathscr{D}\vec{\phi}_i(x) \exp\left[-\mathscr{H}_{\rm mn} + \int d^d x \vec{B}_i \cdot \vec{\phi}_i \right], \tag{2.3}$$

and the solution proceeds by way of the loop expansion.[22]

The Legendre transform of the free energy, which is also the generating functional of the one-particle irreducible Green's functions[23] (also known as the vertex functions), can be found via the definition

$$\Gamma[\overline{\phi}] \equiv \int d^d x B_i^{\alpha} \overline{\phi_i^{\alpha}} - \ln Z_{\rm mn}[B], \tag{2.4}$$

where repeated indices imply summation. Here, $\overline{\phi}$ is the field conjugate to the external field B and is thus proportional to the magnetization M. In the above, and in all that follows, Latin indices will always denote replica fields and Greek indices will always denote Cartesian components. Performing the loop expansion to first order, the free-energy density is expressed as

$$\frac{1}{nV}\Gamma(\overline{\phi}, \mu^2, \lambda_1, \lambda_2) = \frac{1}{2}\mu^2\overline{\phi^2} + \frac{1}{4!}(n\lambda_1 + \lambda_2)(\overline{\phi^2})^2 + \frac{m-1}{2}\int \frac{d^d\vec{k}}{(2\pi)^d} \ln\left(k^2 + \mu^2 + \frac{n\lambda_1\overline{\phi^2}}{6} + \frac{\lambda_2\overline{\phi^2}}{6} \right) + \frac{n-1}{2n}\int \frac{d^d\vec{k}}{(2\pi)^d} \ln\left(k^2 + \mu^2 + \frac{n\lambda_1\overline{\phi^2}}{6} + \frac{\lambda_2\overline{\phi^2}}{2} \right) + \frac{1}{2n}\int \frac{d^d\vec{k}}{(2\pi)^d} \ln\left(k^2 + \mu^2 + \frac{n\lambda_1\overline{\phi^2}}{2} + \frac{\lambda_2\overline{\phi^2}}{2} \right). \tag{2.5}$$

It is important to note that $\overline{\phi^2}$ is the magnitude squared of each bare magnetization replica vector. Replica symmetry is *not broken*. We have taken all thermally averaged replica vectors $\overline{\phi_i}$ to have equal magnitude and to also point in the same direction. From Eq. (2.5), previous calculations done on the pure isotropic $O(m)$ model[5] ($n=1$, $\lambda_1=0$), and on the dilute Ising model[24] ($m=1$) can be reproduced.

After renormalizing the free energy Γ, expressions for the specific heat and susceptibility in the perturbative regime—in contrast to the critical regime—can be found by simple differentiation.

The theory was renormalized by minimal subtraction with all integrals being dimensionally regularized.[25,26,23] Explicit ϵ expansions for most diagrams needed can be found in Ref. 27.

The basic statement is that apart from a small subset (see below), all vertex functions $\Gamma^{(N,L)}$, containing N $\overline{\phi}$ fields and L $\overline{\phi^2}$ fields, can be multiplicatively renormalized:

$$\Gamma_R^{(N,L)}(k_i,p_i;t,M,u_i,\kappa)=Z_\phi^{N/2}Z_{\phi^2}^{L}\Gamma^{(N,L)}(k_i,p_i;\mu^2,\overline{\phi},\lambda_i). \tag{2.6}$$

Renormalizing at T_C is sufficient to renormalize these functions away from T_C as well, since the vertex functions are essentially expansion coefficients[23] and can be written as

$$\Gamma^{(N,L)}(k_i,p_i;\mu^2,\overline{\phi},\lambda_i) = \sum_{I,J}\frac{(\overline{\phi})^I(\mu^2)^J}{I!J!}\Gamma^{(N+I,L+J)}(k_i,\not{\epsilon}_i=0,p_i,q_i=0;0,0,\lambda_i). \tag{2.7}$$

The connection between the bare and renormalized parameters is

$$\lambda_i=\kappa^\epsilon Z_i u_i \quad (i=1,2), \tag{2.8a}$$

$$\mu^2=Z_{\phi^2}t, \tag{2.8b}$$

$$\overline{\phi}=Z_\phi^{1/2}M. \tag{2.8c}$$

Here, $\epsilon=4-d$, t is the reduced temperature, M is the magnetization, u_i are the (dimensionless) renormalized coupling constants, and κ is an arbitrary wave vector useful for carrying the dimensions of the bare coupling constants. The renormalization constants Z_1, Z_2, Z_ϕ, and Z_{ϕ^2} are all found by minimally subtracting the poles of the primitively divergent vertex functions $\Gamma^{(4,0)}$, $\Gamma^{(2,0)}$, and $\Gamma^{(2,1)}$. The vertex functions themselves are evaluated using the standard Feynman rules and procedures.[22,23] The results are

$$Z_1=1+\frac{mn+8}{6\epsilon}u_1+\frac{m+2}{3\epsilon}u_2+\left[\frac{(mn+8)^2}{36\epsilon^2}-\frac{3mn+14}{24\epsilon}\right]u_1^2 +\left[\frac{(mn+12)(m+2)}{12\epsilon^2}-\frac{11(m+2)}{36\epsilon}\right]u_1u_2 +\left[\frac{(m+4)(m+2)}{12\epsilon^2}-\frac{5(m+2)}{72\epsilon}\right]u_2^2+O(u^3), \tag{2.9a}$$

$$Z_2=1+\frac{2}{\epsilon}u_1+\frac{m+8}{6\epsilon}u_2+\left[\frac{mn+20}{6\epsilon^2}-\frac{5mn+82}{72\epsilon}\right]u_1^2 +\left[\frac{5m+28}{6\epsilon^2}-\frac{11m+58}{36\epsilon}\right]u_1u_2 +\left[\frac{(m+8)^2}{36\epsilon^2}-\frac{3m+14}{24\epsilon}\right]u_2^2+O(u^3), \tag{2.9b}$$

$$Z_\phi=1-\frac{mn+2}{144\epsilon}u_1^2-\frac{m+2}{72\epsilon}u_1u_2-\frac{m+2}{144\epsilon}u_2^2+O(u^3), \tag{2.9c}$$

$$Z_{\phi^2}=1+\frac{mn+2}{6\epsilon}u_1+\frac{m+2}{6\epsilon}u_2 +\left[\frac{(mn+5)(mn+2)}{36\epsilon^2}-\frac{5(mn+2)}{144\epsilon}\right]u_1^2 +\left[\frac{(mn+5)(m+2)}{18\epsilon^2}-\frac{5(m+2)}{72\epsilon}\right]u_1u_2 +\left[\frac{(m+5)(m+2)}{36\epsilon^2}-\frac{5(m+2)}{144\epsilon}\right]u_2^2+O(u^3). \tag{2.9d}$$

A small subset[22,23] of the vertex functions diverge at zeroth order in the coupling constants for an infinite cutoff. These cannot be multiplicatively renormalized but require instead a further additive renormalization. The one of interest here is $\Gamma^{(0,2)}$, related to the specific heat. This vertex function is renormalized as

$$\Gamma_R^{(0,2)}(p;0,0,u_i,\kappa) = Z_{\phi^2}^2[\Gamma^{(0,2)}(p;0,0,\lambda_i)-\Gamma^{(0,2)}(p;0,0,\lambda_i)|_{p^2=\kappa^2}]. \tag{2.10}$$

The nonperturbative character of the renormalization group can be thought of as a mapping[28,29] from the critical regime—where naive perturbation theory breaks down—to the perturbative regime. Thus, it is necessary to derive some perturbative results for the noncritical regime. The specific heat and the susceptibility are of particular importance at present. These are of course obtained by differentiating the free energy, which must first be renormalized. This is accomplished by inserting the definitions of Eq. (28) into Eq. (2.5). The additive piece from $\Gamma^{(0,2)}$ must also be included. Performing these substitutions, the renormalized free energy density is given by

$$\frac{1}{nV}\Gamma_R(t,M,u_1,u_2,\kappa)=\frac{1}{2}tM^2+\frac{1}{4!}(nu_1+u_2)M^4+\frac{m-1}{2}\int\frac{d^d\vec{k}}{(2\pi)^d}\ln\left[k^2+t+\frac{1}{6}(nu_1+u_2)M^2\right]$$
$$+\frac{n-1}{2n}\int\frac{d^d\vec{k}}{(2\pi)^d}\ln\left[k^2+t+\frac{1}{6}(nu_1+3u_2)M^2\right]+\frac{1}{2n}\int\frac{d^d\vec{k}}{(2\pi)^d}\ln\left[k^2+t+\frac{1}{2}(nu_1+u_2)M^2\right]$$
$$+\frac{m}{4}t^2\kappa^{-\epsilon}\left(\frac{1}{\epsilon}+\frac{1}{2}+O(\epsilon)\right)+\frac{1}{2}\left(\frac{mn+2}{6\epsilon}u_1+\frac{m+2}{6\epsilon}u_2\right)tM^2+\frac{n}{4!}\left(\frac{mn+8}{6\epsilon}u_1+\frac{m+2}{3\epsilon}u_2\right)u_1M^4$$
$$+\frac{1}{4!}\left(\frac{2}{\epsilon}u_1+\frac{m+8}{6\epsilon}u_2\right)u_2M^4+O(u). \tag{2.11}$$

Above T_C and in zero external field (hence $M=0$), the above expression can be immediately differentiated to yield both the specific heat and the susceptibility. The specific heat is given by

$$C_B(t,0,u_1,u_2)=\lim_{n\to 0}\left(-\frac{1}{nV}\frac{\partial^2\Gamma_R}{\partial t^2}\right)\quad(t>0)$$
$$=-\frac{m}{4}(2+\ln t)+O(u). \tag{2.12}$$

In the above, we have set $\kappa=1$. The inverse isothermal susceptibility tensor is similarly given by

$$\chi^{-1}_{\alpha\beta}=\lim_{n\to 0}\left(\frac{1}{nV}\frac{\partial^2\Gamma_R}{\partial M_\alpha\partial M_\beta}\bigg|_{M=0}\right)\quad(t>0)$$
$$=\delta_{\alpha\beta}\left\{t+\frac{1}{12}[2u_1+(m+2)u_2]t\ln t+O(u^2)\right\}. \tag{2.13}$$

Below T_C, the presence of Goldstone modes manifests itself in a divergent susceptibility. Hence, only the specific heat will be calculated below T_C. To obtain the specific heat, the (spontaneous) magnetization must now also be differentiated with respect to the reduced temperature. The calculation is simplified if the equation of state

$$B(t,M,u_1,u_2)=\frac{1}{nV}\frac{\partial\Gamma_R}{\partial M} \tag{2.14}$$

is first solved on the coexistence curve ($B=0$) in order to obtain M as a function of t. This expression can then be inserted into the renormalized free energy, Eq. (2.11), yielding an expression with no explicit M dependence. The result of these steps is

$$\frac{1}{nV}\Gamma_R(t<0,M(t),u_1,u_2)=-\frac{3}{2}(nu_1+u_2)^{-1}t^2+\frac{n-1}{2n}\int\frac{d^d\vec{k}}{(2\pi)^d}\ln[k^2-2u_2(nu_1+u_2)^{-1}t]+\frac{1}{2n}\int\frac{d^d\vec{k}}{(2\pi)^d}\ln[k^2-2t]$$
$$+\frac{m}{4}t^2\left(\frac{1}{\epsilon}+\frac{1}{2}\right)-3(nu_1+u_2)^{-1}\left(\frac{mn+2}{6\epsilon}u_1+\frac{m+2}{6\epsilon}u_2\right)t^2+\frac{3n}{2}u_1(nu_1+u_2)^{-2}\left(\frac{mn+8}{6\epsilon}u_1+\frac{m+2}{3\epsilon}u_2\right)t^2$$
$$+\frac{3}{2}u_2(nu_1+u_2)^{-2}\left(\frac{2}{\epsilon}u_1+\frac{m+8}{6\epsilon}u_2\right)t^2+O(u). \tag{2.15}$$

Using Eq. (2.15) for the free energy, the specific heat is given by

$$C_B(t,M(t),u_1,u_2)=\lim_{n\to 0}\left(-\frac{1}{nV}\frac{\partial^2\Gamma_R}{\partial t^2}\right)\quad(t<0)$$
$$=\frac{3}{u_2}-\left\{\frac{2u_1+u_2}{u_2}[1+\ln(-2t)]+\frac{u_1}{u_2}+\frac{m}{4}\right\}+O(u). \tag{2.16}$$

Equations (2.16), (2.12), and (2.13) can be expected to yield valid results only outside the critical regime. Near the critical point, reliable results are obtained from the solution to the renormalization-group equations, which can be derived by differentiating the (κ independent) bare vertex functions $\Gamma^{(N,L)}$ with respect to κ. Using the notation of Ref. 23, the renormalization-group equations can be written as

$$\left[\kappa\frac{\partial}{\partial\kappa}+\beta_1(u_1,u_2)\frac{\partial}{\partial u_1}+\beta_2(u_1,u_2)\frac{\partial}{\partial u_2}-\frac{1}{2}\gamma_\phi(u_1,u_2)\left(N+M\frac{\partial}{\partial M}\right)+\gamma_{\phi^2}(u_1,u_2)\left(L+t\frac{\partial}{\partial t}\right)\right]\Gamma_R^{(N,L)}(t,M,u_1,u_2,\kappa)$$
$$=\delta_{N0}\delta_{L2}\kappa^{-\epsilon}B(u_1,u_2), \tag{2.17}$$

where

$$\kappa^{-\epsilon}B(u_1,u_2)=-Z_{\phi^2}^2\kappa\frac{\partial}{\partial\kappa}\Gamma^{(0,2)}(\vec{p};\lambda_1,\lambda_2)|_{p^2=\kappa^2}. \tag{2.18}$$

The two β functions are found by the simultaneous solution to the two equations

$$-\epsilon=\beta_1\frac{\partial\ln(u_iZ_i)}{\partial u_1}+\beta_2\frac{\partial\ln(u_iZ_i)}{\partial u_2}\quad(i=1,2). \tag{2.19}$$

The results are

$$\beta_1(u_1,u_2)=u_1\left[-\epsilon+\frac{mn+8}{6}u_1+\frac{m+2}{3}u_2-\frac{3mn+14}{12}u_1^2-\frac{11(m+2)}{18}u_1u_2-\frac{5(m+2)}{36}u_2^2+O(u^3)\right], \tag{2.20a}$$

$$\beta_2(u_1,u_2)=u_2\left[-\epsilon+2u_1+\frac{m+8}{6}u_2-\frac{5mn+82}{36}u_1^2-\frac{11m+58}{18}u_1u_2-\frac{3m+14}{12}u_2^2+O(u^3)\right]. \tag{2.20b}$$

The Wilson functions are similarly found:

$$\gamma_\phi(u_1,u_2)=\frac{\partial\ln Z_\phi}{\partial u_1}\beta_1+\frac{\partial\ln Z_\phi}{\partial u_2}\beta_2=\frac{mn+2}{72}u_1^2+\frac{m+2}{36}u_1u_2+\frac{m+2}{72}u_2^2+O(u^3), \tag{2.21a}$$

$$\gamma_{\phi^2}(u_1,u_2)=-\left(\frac{\partial\ln Z_{\phi^2}}{\partial u_1}\beta_1+\frac{\partial\ln Z_{\phi^2}}{\partial u_2}\beta_2\right)=\frac{mn+2}{6}u_1+\frac{m+2}{6}u_2-\frac{5(mn+2)}{72}u_1^2-\frac{5(m+2)}{36}u_1u_2-\frac{5(m+2)}{72}u_2^2+O(u^3). \tag{2.21b}$$

Using the standard method of characteristics, the solution to Eq. (2.17) is given by

$$\Gamma_R^{(N,L)}(t,M,u_i,\kappa)=(\kappa\lambda)^{\Psi_{N,L}}\left(\frac{M(\lambda)}{M}\right)^N\left(\frac{t(\lambda)}{t}\right)^L\Gamma_R^{(N,L)}\times(\tilde{t}(\lambda),\tilde{M}(\lambda),u_i(\lambda),1)-\delta_{N0}\delta_{L2}\kappa^{-\epsilon}\int_1^\lambda\frac{d\lambda'}{\lambda'}\left(\frac{t(\lambda')}{t}\right)^2(\lambda')^{-\epsilon}B(u_i(\lambda')), \tag{2.22}$$

where

$$\Psi_{N,L}=d-2L-\frac{N(d-2)}{2}, \tag{2.23a}$$

$$\frac{M(\lambda)}{M}=\exp\left[-\frac{1}{2}\int_1^\lambda\frac{d\lambda'}{\lambda'}\gamma_\phi(u_1(\lambda'),u_2(\lambda'))\right], \tag{2.23b}$$

$$\frac{t(\lambda)}{t}=\exp\left[\int_1^\lambda\frac{d\lambda'}{\lambda'}\gamma_{\phi^2}(u_1(\lambda'),u_2(\lambda'))\right], \tag{2.23c}$$

$$\tilde{t}(\lambda)=\frac{t(\lambda)}{(\kappa\lambda)^2}, \tag{2.23d}$$

$$\tilde{M}(\lambda)=\frac{M(\lambda)}{(\kappa\lambda)^{(d-2)/2}}, \tag{2.23e}$$

$$\lambda\frac{\partial u_i(\lambda)}{\partial\lambda}=\beta_i(u_1(\lambda),u_2(\lambda))\quad(i=1,2)$$
$$(u_1(1),u_2(1))=(u_1,u_2). \tag{2.23f}$$

Judiciously fixing the value of the (as yet) arbitrary parameter λ allows the vertex function on the right-hand side of Eq. (2.22) to be evaluated in the perturbative regime. This was the purpose of evaluating the specific heat and susceptibility outside the critical regime. The usual choice of fixing λ is

$$\tilde{t}(\lambda)=\frac{t(\lambda)}{\kappa^2\lambda^2}=\pm1, \tag{2.24}$$

where the $+(-)$ is used when working above (below) T_C.

III. CORRECTIONS TO SCALING

This section deals with the calculation of the various quantities appearing in Eq. (2.22). The correction to scaling amplitudes will naturally appear. It can easily be shown from the results in Sec. II that the critical domain corresponds to

$\lambda \to 0$. If the initial coupling constants $(u_1(1),u_2(1))$ lie within the domain of attraction of an infrared stable fixed point (u_1^*,u_2^*), then the critical regime also corresponds to $(u_1(\lambda),u_2(\lambda)) \to (u_1^*,u_2^*)$. The fixed point is a zero of both β functions in Eqs. (2.20). It is infrared stable if the real part of the eigenvalues of the matrix

$$\left(\begin{array}{cc} \partial\beta_1/\partial u_1 & \partial\beta_1/\partial u_2 \\ \partial\beta_2/\partial u_1 & \partial\beta_2/\partial u_2 \end{array} \right) \Bigg|_{u_1^*,u_2^*} \tag{3.1}$$

are positive. To the order the β functions are currently known, there does exist such a fixed point.[30] The running coupling constants will flow to the pure (i.e., impurity free) fixed point

$$u_1^* = 0, \tag{3.2a}$$

$$u_2^* = \frac{6}{m+8}\epsilon + \frac{18(3m+14)}{(m+8)^3}\epsilon^2 + O(\epsilon^3) \tag{3.2b}$$

provided $|u_1|$ is sufficiently small relative to u_2, corresponding to a low level of inhomogeneity in the physical system. The flow diagram for the diluted Heisenberg model can be found in Ref. 15. In all that follows, we assume the fixed point in Eqs. (3.2) is the one to which the running coupling constants flow.

The explicit solution to Eq. (2.22) is mainly a matter of expanding all quantities about the fixed point (u_1^*,u_2^*). For example, $u_i(\lambda)$ is found by expanding the right-hand side of Eq. (2.23f) as

$$\lambda \frac{\partial}{\partial\lambda}[u_i(\lambda)-u_i^*] = [u_1(\lambda)-u_1^*]\partial_1\beta_i^* + [u_2(\lambda)-u_2^*]\partial_2\beta_i^* + \frac{1}{2}[u_1(\lambda)-u_1^*]^2\partial_{11}^2\beta_i^* + [u_1(\lambda)-u_1^*][u_2(\lambda)-u_2^*]\partial_{12}^2\beta_i^*$$

$$+ \frac{1}{2}[u_2(\lambda)-u_2^*]^2\partial_{22}^2\beta_i^* + O([u(\lambda)-u^*]^3) \quad (i=1,2), \tag{3.3}$$

where

$$\partial_i\beta_j^* \equiv \left(\frac{\partial\beta_j}{\partial u_i}\right)\Bigg|_{u_1^*,u_2^*}; \quad \partial_{ij}^2\beta_k^* \equiv \left(\frac{\partial\beta_k^2}{\partial u_i \partial u_j}\right)\Bigg|_{u_1^*,u_2^*}. \tag{3.4}$$

The solution to this set of coupled differential equations will be of the form

$$[u_i(\lambda)-u_i^*] = v_{i1}\lambda^{p_1} + v_{i2}\lambda^{p_2} + v_{i3}\lambda^{p_3} + v_{i4}\lambda^{p_4} + v_{i5}\lambda^{p_5} + \cdots, \tag{3.5}$$

where $p_3=2p_1$, $p_4=p_1+p_2$, and $p_5=2p_2$. The coefficients v_{ij} and the powers p_i are constants to be determined. The solution now proceeds by substituting Eq. (3.5) into Eq. (3.3) and equating coefficients of equal powers of λ on either side of the equation. In this way explicit expressions can be found for all coefficients and powers in Eq. (3.5) except for two. This is expected since, although Eq. (2.23f) has the initial conditions $u_i(1)=u_i$, they cannot be used here since the solution Eq. (3.5) is only valid near $\lambda=0$ (i.e., near the fixed point). Hence, the solutions to the differential equations must contain two constants of integration. v_{11} and v_{22} were chosen for this purpose and contain nonuniversal information on the initial coupling constants.

$M(\lambda)$ and $t(\lambda)$ are similarly found. Equations (2.23b) and (2.23c) can be rewritten as[5]

$$\left(\frac{M(\lambda)}{M}\right)^N = \lambda^{-N\gamma_\phi^*/2} Y^N \exp\left\{ \frac{-N}{2}\int_0^\lambda \frac{d\lambda'}{\lambda'} \times [\gamma_\phi(u_i(\lambda')) - \gamma_\phi^*] \right\}, \tag{3.6a}$$

$$\left(\frac{t(\lambda)}{t}\right)^L = \lambda^{L\gamma_{\phi^2}^*} X^L \exp\left\{ L\int_0^\lambda \frac{d\lambda'}{\lambda'}[\gamma_{\phi^2}(u_i(\lambda')) - \gamma_{\phi^2}^*] \right\}, \tag{3.6b}$$

where[31]

$$X \equiv \exp\left\{ \int_1^0 \frac{d\lambda'}{\lambda'}[\gamma_{\phi^2}(u_i(\lambda')) - \gamma_{\phi^2}^*] \right\}, \tag{3.7a}$$

$$Y \equiv \exp\left\{ \frac{-1}{2}\int_1^0 \frac{d\lambda'}{\lambda'}[\gamma_\phi(u_i(\lambda')) - \gamma_\phi^*] \right\}. \tag{3.7b}$$

The Wilson functions in the integrands of Eq. (3.6) are in a form which permits an expansion about the fixed point. The integrals can then be explicitly performed and the exponentials expanded resulting in series in powers of λ. In a similar way, $\Gamma_R^{(N,L)}$ as well as the final additive term in Eq. (2.22) can be expanded about the fixed point (u_1^*,u_2^*), which also results in series in powers of λ.[32]

Once everything is expressed in terms of λ, Eq. (2.24) can be used to convert to the reduced temperature t. This can be seen by writing (for $t>0$)

$$\frac{t(\lambda)}{t} = \frac{\kappa^2\lambda^2}{t}. \tag{3.8}$$

Assuming the left-hand side of this equation has been expressed as a power series in λ using the prescription outlined above, the series can then be inverted to obtain λ as a series in powers of t. The form of the series is

$$\lambda = \Lambda_0 t^{\Delta_0}[1 + (\Lambda_1 t^{\Delta_1} + \Lambda_2 t^{\Delta_2}) + (\Lambda_3 t^{\Delta_3} + \Lambda_4 t^{\Delta_4} + \Lambda_5 t^{\Delta_5}) + \cdots], \tag{3.9}$$

where $\Delta_0=\nu,\Delta_1=p_1\nu,\Delta_2=p_2\nu,\Delta_3=2\Delta_1,\Delta_4=\Delta_1+\Delta_2,$ $\Delta_5=2\Delta_2$. The further identification of $\Delta_1=-\alpha$ and $\Delta_2=\omega\nu$, where α is the specific-heat exponent and $\omega\nu$ the correction to scaling exponent of the pure Heisenberg model, is made on the basis of comparing the published ϵ expansions of the exponents.[22,17] In the case of $\Delta_1=-\alpha$, general arguments also apply.[14,15]

In this way, the right-hand side of Eq. (2.22) can be calculated explicitly as a series in powers of the reduced temperature t.[32] The two quantities of interest here are the specific heat and the isothermal susceptibility given by[23]

$$C_B=-\Gamma_R^{(0,2)}, \tag{3.10a}$$

$$\chi_{\alpha\beta}=(\Gamma_R^{(2,0)})^{-1}_{\alpha\beta}=\chi(t)\,\delta_{\alpha\beta}\quad (B=0,T>T_C). \tag{3.10b}$$

Performing the steps outlined above, these quantities are expressed as

$$C_B^{\pm}(t)=A_C^{\pm}t^{-\alpha}[1+(a_{c1}^{\pm}t^{-\alpha}+a_{c2}^{\pm}t^{\omega\nu}) +(a_{c3}^{\pm}t^{-2\alpha}+a_{c4}^{\pm}t^{-\alpha+\omega\nu}+a_{c5}^{\pm}t^{2\omega\nu})+\cdots], \tag{3.11a}$$

$$\chi(t)=A_\chi^{+}t^{-\gamma}[1+(a_{\chi1}^{+}t^{-\alpha}+a_{\chi2}^{+}t^{\omega\nu}) +(a_{\chi3}^{+}t^{-2\alpha}+a_{\chi4}^{+}t^{-\alpha+\omega\nu}+a_{\chi5}^{+}t^{2\omega\nu})+\cdots], \tag{3.11b}$$

where $+(-)$ denotes $t>(<)0$. The exponents

$$\alpha=\frac{4-m}{2(m+8)}\epsilon-\frac{(m+2)^2(m+28)}{4(m+8)^3}\epsilon^2+O(\epsilon^3), \tag{3.12a}$$

$$\omega\nu=\frac{1}{2}\epsilon+\frac{m^2-8m-68}{4(m+8)^2}\epsilon^2+O(\epsilon^3), \tag{3.12b}$$

have been given elsewhere.[22,17] The correction to scaling amplitudes are

$$a_{c1}^{+}=-v_{11}X^{-\alpha}\left[\frac{m(m+8)}{6(m-4)}\frac{1}{\epsilon}+\frac{m(m^3-13m^2-118m+40)}{6m^3-288m+768}+O(\epsilon)\right], \tag{3.13a}$$

$$a_{c2}^{+}=v_{22}X^{\omega\nu}\left[\frac{m-4}{6}\frac{1}{\epsilon}+\frac{m^4+26m^3+126m^2+44m-656}{6(m+2)(m+8)^2}+O(\epsilon)\right], \tag{3.13b}$$

$$a_{c3}^{+}=v_{11}^2X^{-2\alpha}\left[\frac{m(m+8)^2(m^2-22m+24)}{36(m-16)(m-4)^2}\frac{1}{\epsilon^2} +\frac{m(5m^6-202m^5+2940m^4-1248m^3-153152m^2+193536m-137216)}{72(m-16)^2(m-4)^3}\frac{1}{\epsilon}+O(\epsilon^0)\right], \tag{3.13c}$$

$$a_{c4}^{+}=-v_{11}v_{22}X^{-\alpha+\omega\nu}\left[\frac{m(m+8)}{18}\frac{1}{\epsilon^2}+\frac{m(5m^2+21m+46)}{36(m+2)}\frac{1}{\epsilon}+O(\epsilon^0)\right], \tag{3.13d}$$

$$a_{c5}^{+}=v_{22}^2X^{2\omega\nu}\left[\frac{(m-4)(m+2)}{36}\frac{1}{\epsilon^2}+\frac{5m^4+84m^3+180m^2-416m-960}{72(m+8)^2}\frac{1}{\epsilon}+O(\epsilon^0)\right], \tag{3.13e}$$

$$a_{c1}^{-}=v_{11}X^{-\alpha}\left\{-\frac{2}{3}\left(\frac{m+8}{m-4}\right)\frac{1}{\epsilon}+\frac{2}{3}\left[\frac{25m^2+118m-296}{(m-4)^2(m+8)}-\frac{1}{2}\ln 2\right]+O(\epsilon)\right\}, \tag{3.14a}$$

$$a_{c2}^{-}=v_{22}X^{\omega\nu}\left\{\frac{m-4}{6}\frac{1}{\epsilon}+\left[\frac{m^4+32m^3+156m^2+344m+736}{12(m+2)(m+8)^2}+\frac{m-4}{12}\ln 2\right]+O(\epsilon)\right\}, \tag{3.14b}$$

$$a_{c3}^{-}=-v_{11}^2X^{-2\alpha}\left\{\frac{m(m+8)^2(m^2-18m+104)}{36(m-16)(m-4)^2}\frac{1}{\epsilon^2} +\left[\frac{m^7-71m^6+1098m^5-6352m^4-23520m^3+397824m^2-291840m-262144}{36(m-4)^3(m-16)^2}+\frac{m(m+8)(m^2-18m+104)}{36(m-4)(m-16)}\ln 2\right]\frac{1}{\epsilon} +O(\epsilon^0)\right\}, \tag{3.14c}$$

$$a_{c4}^{-}=-v_{11}v_{22}X^{-\alpha+\omega\nu}\left\{\frac{2(m+8)}{9}\frac{1}{\epsilon^2}+\left[\frac{m^3+16m^2-132m-416}{72(m+2)}+\frac{2(m+2)}{9}\ln 2\right]\frac{1}{\epsilon}+O(\epsilon^0)\right\}, \tag{3.14d}$$

$$a_{c5}^{-}=v_{22}X^{2\omega\nu}\left\{\frac{(m-4)(m+2)}{36}\frac{1}{\epsilon^2}+\left[\frac{3m^4+44m^3-12m^2+96m+3136}{72(m+8)^2}+\frac{(m-4)(m+2)}{36}\ln 2\right]\frac{1}{\epsilon}+O(\epsilon^0)\right\}, \tag{3.14e}$$

$$a_{\chi1}^{+}=v_{11}X^{-\alpha}\left[\frac{m(m+8)}{6(m-4)}\frac{1}{\epsilon}+\frac{2m(m^3-20m^2-164m-240)}{24m^3-1152m+3072}+O(\epsilon)\right], \tag{3.15a}$$

$$a_{\chi2}^{+}=-v_{22}X^{\omega\nu}\left[\frac{m+2}{6}\frac{1}{\epsilon}+\frac{m^3+18m^2+36m+8}{12(m+8)^2}+O(\epsilon)\right], \tag{3.15b}$$

$$a_{\chi3}^{+}=-v_{11}^2X^{-2\alpha}\left[\frac{m(m+8)^2(m^2-25m+36)}{36(m-16)(m-4)^2}\frac{1}{\epsilon^2}+\frac{m(2m^6-71m^5+1872m^4-6848m^3-68416m^2-77568m+101376)}{72(m-4)^3(m-16)^2}\frac{1}{\epsilon}+O(\epsilon^0)\right], \tag{3.15c}$$

$$a_{\chi4}^{+}=v_{11}v_{22}X^{-\alpha+\omega\nu}\left[\frac{m(m^2+7m-8)}{18(m-4)}\frac{1}{\epsilon^2}+\frac{m(m^4-3m^3-36m^2-112m+96)}{18m^3-108m^2+576}\frac{1}{\epsilon}+O(\epsilon^0)\right], \tag{3.15d}$$

$$a_{\chi5}^{+}=-v_{22}^2X^{2\omega\nu}\left[\frac{m^2+7m+10}{36}\frac{1}{\epsilon^2}+\frac{2m^4+27m^3-30m^2-848m-1392}{72(m+8)^2}\frac{1}{\epsilon}+O(\epsilon^0)\right]. \tag{3.15e}$$

Finally, the dimensionless correction to scaling amplitude ratios, which are independant of v_{11}, v_{22}, X, as well as the renormalization scheme, and are therefore universal, are given for $m=3$ by

$$\frac{a_{c1}^{+}}{a_{c1}^{-}}=\frac{3}{4}+\left(\frac{3}{4}+\frac{3\ln 2}{88}\right)\epsilon+O(\epsilon^2)\approx 0.75+0.77\epsilon+O(\epsilon^2), \tag{3.16a}$$

$$\frac{a_{c2}^{+}}{a_{c2}^{-}}=1+\left(\frac{11}{10}-\frac{\ln 2}{2}\right)\epsilon+O(\epsilon^2)\approx 1+0.75\epsilon+O(\epsilon^2), \tag{3.16b}$$

$$\frac{a_{c3}^{+}}{a_{c3}^{-}}=\frac{33}{59}+\left(\frac{87781}{76582}+\frac{3\ln 2}{59}\right)\epsilon+O(\epsilon^2)\approx 0.56+1.18\epsilon+O(\epsilon^2), \tag{3.16c}$$

$$\frac{a_{c4}^{+}}{a_{c4}^{-}}=\frac{3}{4}+\left(\frac{5619}{3520}-\frac{15\ln 2}{44}\right)\epsilon+O(\epsilon^2)\approx 0.75+1.36\epsilon+O(\epsilon^2), \tag{3.16d}$$

$$\frac{a_{c5}^{+}}{a_{c5}^{-}}=1+\left(\frac{11}{5}-\ln 2\right)\epsilon+O(\epsilon^2)\approx 1+1.51\epsilon+O(\epsilon^2), \tag{3.16e}$$

$$\frac{a_{c1}^{+}}{a_{\chi1}^{+}}=-1+\frac{7}{22}\epsilon+O(\epsilon^2)\approx -1+0.32\epsilon+O(\epsilon^2), \tag{3.17a}$$

$$\frac{a_{c2}^{+}}{a_{\chi2}^{+}}=\frac{1}{5}-\frac{281}{550}\epsilon+O(\epsilon^2)\approx 0.20-0.51\epsilon+O(\epsilon^2), \tag{3.17b}$$

$$\frac{a_{c3}^{+}}{a_{\chi3}^{+}}=-\frac{11}{10}+\frac{6977}{6600}\epsilon+O(\epsilon^2)\approx -1.1+1.06\epsilon+O(\epsilon^2), \tag{3.17c}$$

$$\frac{a_{c4}^{+}}{a_{\chi4}^{+}}=\frac{1}{2}-\frac{41}{22}\epsilon+O(\epsilon^2)\approx 0.5-1.86\epsilon+O(\epsilon^2), \tag{3.17d}$$

$$\frac{a_{c5}^{+}}{a_{\chi5}^{+}}=\frac{1}{8}-\frac{243}{1408}\epsilon+O(\epsilon^2)\approx 0.125-0.17\epsilon+O(\epsilon^2). \tag{3.17e}$$

Equations (3.16b) and (3.17b) have been previously published.[5,6] To our knowledge, the remaining eight are presented here for the first time.

IV. SUMMARY AND DISCUSSION

The main objective of this work has been the determination by renormalization-group methods of the *higher-order* corrections to scaling associated with inhomogeneity such as dilution by nonmagnetic impurities. We have explicitly calculated critical exponents and universal amplitude ratios for both first-order and second-order corrections to scaling. By ''first order'' and ''second order,'' we refer to the corresponding terms which occur in the formal expansion of renormalized vertex functions in powers of the deviations of the renormalized coupling constants from their values at the fixed point. The critical exponents $\Delta_1=-\alpha$ and $\Delta_2=\omega\nu$

were known previously. The relationships $\Delta_3=2\Delta_1$, $\Delta_4=\Delta_1+\Delta_2$, and $\Delta_5=2\Delta_2$ have not previously been given explicitly, but they are as expected on general grounds. Corresponding to these higher-order corrections to scaling, eight universal amplitude ratios have been calculated, six of which are absent in pure systems and arise only in the presence of inhomogeneity.

The calculations have been carried out using renormalized perturbation theory, minimal subtraction, and ϵ expansions. The critical exponents were evaluated to $O(\epsilon^2)$ while the amplitude ratios were calculated to $O(\epsilon)$. Although low-order ϵ expansions generally provide reasonable semiquantitative estimates for critical exponents, the ϵ expansions for amplitude ratios are sometimes more delicate.[7] Consequently, we regard the $O(\epsilon)$ estimates of amplitude ratios to be indicative only of the magnitudes to be expected, but not to be quantitatively accurate. Specifically, to achieve accuracy at the 10% level, it will be necessary to add at least one more term in the ϵ expansion. Alternatively, coupling constant expansions at fixed dimension would also be useful. However, irrespective of the precise numerical values of these amplitude ratios, it is important to be able to conclude that the corresponding amplitudes are indeed present (and are nonzero in general).

The major point which must be emphasized is that these new correction to scaling amplitudes associated with the dilution-induced inhomogeneity now determine not only the leading, but *also* the next-to-leading corrections to scaling. For a diluted $O(m)$ ferromagnet, the usual (first order) purely thermal correction to scaling, determined by $|t|^{\Delta_2}$, becomes negligible for sufficiently small $|t|$ relative to the formally second-order correction given by $|t|^{\Delta_3}$. This also extends to even higher order corrections to scaling. Although the *leading* correction to scaling for *pure* systems ($|t|^{\Delta_2}$) is certainly present for inhomogeneous systems, this correction is insignificant sufficiently close to T_C. It is dominated by $|t|^{p\Delta_1}$, $(p=1,2,3,4)$, since $\Delta_1=-\alpha\approx 0.12$ and $\Delta_2=\omega\nu\approx 0.55$ in the case of $m=3$.[16–18] These results clearly have significance for the analysis of experimental data. Even if the amplitudes associated with these corrections to scaling should be small in a specific range of temperature for a given system, it is necessary to consider (at the very least) the possibility of a crossover to a dilution-dominated regime as $|t|$ is reduced.

A very brief account of the experimental situation will now be given. We have previously indicated that the lack of crystalline long-range order and the possibility of a consequent suppression of strong anisotropic perturbations makes appropriate amorphous ferromagnets good candidates for the universality class of weakly inhomogeneous $O(m)$ Heisenberg systems. The same considerations apply to polycrystalline systems. Extensive studies of the influence of impurities on the critical behavior of the magnetization have been carried out by Hohenemser and co-workers using Mössbauer techniques. For example, from an analysis of their data on polycrystalline $Fe_{1-x}Al_x$ over the range $0\leqslant x\leqslant 0.04$, Collins *et al.*[33] determined $\beta=0.366(2)$ to be the mean value of five measurements. The influence of the first-order corrections to scaling was considered. Hargraves and Dunlap[34] studied the ac susceptibility of the quaternary transition-metal-based amorphous alloys $Ni_{80-x}Fe_xB_{12}Si_8$ and interpreted their results in terms of an effective exponent $\gamma_{\rm eff}$. The results of both of these experiments were consistent with the values of critical exponents predicted for three-dimensional isotropic Heisenberg systems. Critical amplitude ratios were not obtained.

Kaul and Rao[35] studied the electrical resistivity, magnetization, and ac susceptibility of a series of $Ni_{80-x}Fe_x(B,Si)_{20}$ with $(x=10,13,16,20)$ alloys and analyzed data in terms of a leading critical exponent, which was found to be independent of concentration and consistent with the $O(3)$ Heisenberg universality class, together with the first-order correction to scaling terms. In addition, estimates for several amplitude ratios for the corrections to scaling were obtained. Their results for the correction to scaling amplitude ratios for the $x=10$ case were; $a_{c1}^+/a_{c1}^-=0.08(4)$, $a_{c2}^+/a_{c2}^-=1.04(50)$, $a_{c1}^+/a_{\chi 1}^+=7.7(20)$, $a_{c2}^+/a_{\chi 2}^+=4.5(10)$. These bear little resemblance to the values obtained by simply putting $\epsilon=1$ into Eqs. (3.16a,b) and (3.17a,b), which yields; $a_{c1}^+/a_{c1}^-\approx 1.52$, $a_{c2}^+/a_{c2}^-\approx 1.75$, $a_{c1}^+/a_{\chi 1}^+\approx -0.68$, $a_{c2}^+/a_{\chi 2}^+\approx -0.31$. There does not seem to exist at present any analysis of experimental data which takes into account the strongly temperature-dependent second- and higher-order corrections to scaling. It is possible that the strong disagreement between the available theoretical and experimental estimates for correction to scaling amplitude ratios will be reduced when these additional corrections to scaling are taken into account close to the critical point. It is also possible that the effects of other perturbations will need to be taken into account.

In summary, the absence of crystalline anisotropies suggests that suitable amorphous ferromagnets may be good candidates for weakly inhomogenous, isotropic Heisenberg systems. The results available at present for the *leading* critical behavior support this suggestion. However, the corrections to scaling (particularly the amplitude ratios) are much more sensitive. Further theoretical development as well as improved procedures for the analysis of experimental data sufficiently close to the critical temperature will be useful in probing these corrections.

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