ONE-PARAMETER OPERATOR SEMIGROUPS AND AN APPLICATION OF DYNAMICAL SYSTEMS

by

Bassemah Alhulaimi

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Co-Supervisors: ______________________________

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Reader: ______________________________

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Signature of Author

Table of Contents

List of Figures

Abstract

This thesis consists of two parts. In the first part, which is expository, abstract theory of one-parameter operator is studied semi-groups. We develop in detail the necessary Banach space and Banach algebra theories of integration, differentiation, and series, and then give a careful rigorous proof of the exponential function characterization of continuous one-parameter operator semigroups. In the second part, which is applied and has new result, we discuss some related topics in dynamical systems. In general the linearizations give a reliable description of the non-linear orbits near the equilibrium points (the Hartman-Grobman theorem), thus illustrating the importance of linear semigroups. The aim of qualitative analysis of differential equations (DE) is to understand the qualitative behaviour (such as, for example, the long-term behaviour as $t \to \infty$) of typical solutions of the DE. The flow in the direction of increasing time defines a semigroup. As an application we study Einstein-Aether Cosmological models using dynamical systems theory.

List of Abbreviations and Symbols Used

DE Dynamical Equation

FWM Friemann Wave Model

CMB Cosmological Microwave Model

GR General relativity

$\mathcal{X}$ Banach Space

$\mathcal{X}^*$ Dual Space on $\mathcal{X}$

$\mathcal{L}(\mathcal{X})$ All Bounded Linear Operators on $\mathcal{X}$

P Partitions on the Interval $[a, b]$

$|P|$ Partitions on the Interval $[a, b]$

π Tagged Partition of The Interval $[a, b]$

$S_f(\pi)$ The Riemann Sum of f on $[a, b]$

L_z Left Multiplication by z

R_z Right Multiplication by z

$\mathcal{A}$ Banach Algebra

$\Re(\lambda)$ Real Part of λ

u_a The Aether Vector Field

g_{ab} The Metric tensor

V Self Interaction Expansion

p Pressure

ρ Energy Density

$\gamma(a)$ The Orbits Through a

γ Matter

q Deceleration Parameter

K Curvature of The Model

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Chapter 1

Introduction

A dynamical system is a collection of mappings $T_t : \mathcal{X} \to \mathcal{X}$ on a space $\mathcal{X}$ indexed by the time parameter $t \geq 0$ and satisfying

$$\begin{cases} T_{t_1} \circ T_{t_2} = T_{t_1+t_2} \\ T_0 = \text{the identity map} \end{cases}$$

Because $[0, \infty)$ is a semigroup under addition (i.e., addition is associative), and these conditions define a homomorphism of $[0, \infty)$ with mappings on $\mathcal{X}$, then mathematically $\{T_t\}$ is called a *one-parameter semigroup* on $\mathcal{X}$. The simplest example is the exponential function e^{ta} for $a \in \mathbb{C}$; defining $T_t : \mathbb{C} \to \mathbb{C}$ by

$$T_t(z) = e^{ta}\, z \quad \text{for} \;\; z \in \mathbb{C},$$

then $\{T_t\}$ is a one-parameter operator semigroup on $\mathbb{C}$. It is a familiar fact that $f(t) = e^{ta}$ is the unique solution to the initial value problem

$$\begin{cases} f'(t) = a.f(t) \\ f(0) = 1. \end{cases}$$

Furthermore, any continuous function f satisfying

$$\begin{cases} f(t_1)f(t_2) = f(t_1 + t_2) \\ f(0) = 1 \end{cases}$$

must be differentiable and satisfy the above initial value problem for some $a \in \mathbb{C}$, and therefore must be of the form e^{ta}. This exponential function characterization of the one-parameter operator semigroup on $\mathbb{C}$ generalizes perfectly to hold in the abstract setting where $\mathcal{X}$ is a Banach space, the mappings T_t are bounded linear operators on $\mathcal{X}$, and $t \to T_t$ is continuous with respect to the operator norm. In the first part of this thesis we develop in detail the necessary Banach space and Banach algebra theories of integration, differentiation, and series, and then give a careful rigorous proof of this characterization. Selection for three books [1–3] are used for developing this theory.

In the second part we discuss some related topics in dynamical systems, and present an application. The aim of qualitative analysis of differential equations (DE) is to understand the qualitative behaviour (such as, for example, the long-term behaviour as $t \to \infty$) of typical solutions of the DE. As a physical system evolves in time, the state vector can be thought of as a moving point in state space, its motion being determined by a flow. The flow of a DE partitions the state space $\mathbb{R}^n$ into subsets called orbits. The flow in the direction of increasing time defines a semigroup.

The study of exceptional solutions, such as equilibrium solutions, and their stability, is of importance. In general the linearizations give a reliable description of the non-linear orbits near the equilibrium points (the Hartman-Grobman theorem), thus illustrating the importance of linear semigroups.

In cosmology the Universe on the largest scales is studied dynamically. Therefore, the qualitative features of such models can be studied using dynamical systems theory. As an application we study Einstein-Aether Cosmological models. We establish the governing evolution equations (of a class of spatially anisotropic cosmological models in Einstein-Aether theory that include curvature and shear) as an autonomous system of first order differential equations in terms of expansion normalized variables. We study the stability of equilibrium points and we conclude that there always exists a range of values of the parameters of the model for which there exists an inflationary future attractor.

Chapter 2

Preliminaries

2.1 Banach Space

We assume familiarity with the basic theory of Banach spaces outlined in this section.

2.1.1 Definition

Let $\mathcal{X}$ be a (real or complex) vector space. A norm on $\mathcal{X}$ is a map $||.|| : \mathcal{X} \to [0, \infty)$ satisfying the following properties for all $x, y \in \mathcal{X}$, and $t \in \mathbb{C}$

(i) Non-degeneracy: $||x|| = 0 \Leftrightarrow x = 0$

(ii) Scaling: $||tx|| = |t| \ ||x||$

(iii) Triangle inequality: $||x + y|| \leq ||x|| + ||y||$

A vector space $(\mathcal{X}, ||.||)$ endowed with a norm is called a normed vector space. A normed vector space $(\mathcal{X}, ||.||)$ is always a metric space, with the norm metric d given by

$$d(x, y) = ||x - y||$$

We therefore have all metric space notions, such as Cauchy and convergent sequences, continuous functions, e.t.c. A Banach space is a complete normed vector space; that is, a normed vector space $(\mathcal{X}, ||.||)$ in which every Cauchy sequence converges.

2.1.2 Bounded Linear Maps

A linear map $T : \mathcal{X} \to \mathcal{Y}$ between normed vector spaces $\mathcal{X}$ and $\mathcal{Y}$ is bounded if there exists $C > 0$ such that

$$||Tx||_{\mathcal{Y}} \leq C||x||_{\mathcal{X}}, \quad \forall x \in \mathcal{X}.$$

The bounded linear maps form a vector space $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ under point-wise addition and scalar multiplication. The following assertions are equivalent:

(i) T is bounded.

(ii) T is continuous.

(iii) T is continuous at one point.

Given a bounded linear map $T : \mathcal{X} \to \mathcal{Y}$, we set

$$||T|| \;=\; \inf\{C : ||Tx|| \leq C||x|| \text{ for all } x\} \; .$$

Then

$$||T|| \;=\; \sup\{||Tx|| : ||x|| = 1\} \;=\; \sup\left\{\frac{||Tx||}{||x||} : x \neq 0\right\} \; .$$

This defines a norm on the space $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ of all bounded linear maps from $\mathcal{X}$ to $\mathcal{Y}$. Note that, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a Banach space if $\mathcal{Y}$ is a Banach space. We write $\mathcal{L}(\mathcal{X})$ for $\mathcal{L}(\mathcal{X}, \mathcal{X})$ and call elements $T \in \mathcal{L}(\mathcal{X})$ bounded operators on $\mathcal{X}$.

2.1.3 Invertible Operators

Let $\mathcal{X}$ be a Banach space. If $T \in \mathcal{L}(\mathcal{X})$ is one-to-one and onto, then the inverse T^{-1} is also bounded, and T is called invertible. The invertible operators form an open subset of $\mathcal{L}(\mathcal{X})$.

2.1.4 Dual Space

Moreover, if $\mathcal{X}$ is a normed vector space, the space $\mathcal{L}(\mathcal{X}, \mathbb{C})$ of bounded linear functional on $\mathcal{X}$ is called the dual space of $\mathcal{X}$ and denoted by $\mathcal{X}^*$.

The Complex Hahn-Banach Theorem: Let $\mathcal{X}$ be a complex vector space, Z a subspace of $\mathcal{X}$, and k a semi norm on $\mathcal{X}$ (that is, a norm which does not necessarily satisfy non-degeneracy). Also, let f be a complex linear functional on Z such that $|f(x)| \leq k(x)$ for all $x \in Z$. Then there exists a complex linear functional F on $\mathcal{X}$ such that $|F(x)| \leq k(x)$ for all $x \in \mathcal{X}$ and $F|Z = f$.

A consequence of this theorem which we will use later is as follows.

Corollary: Let $\mathcal{X}$ be a normed vector space. Then, the bounded linear functional on $\mathcal{X}$ separates points.

Proof. Let $x, y \in \mathcal{X}$ be distinct points and $z = x - y$. We have, $z \neq 0$ and hence $a = ||z|| \neq 0$. Then by applying the Hahn Banach theorem with $k = ||.||$, $Z = \mathbb{C}z = \{\lambda z : \lambda \in \mathbb{C}\}$ and define the map as follows:

$$f : Z \to \mathbb{C}$$
$$f(\lambda z) = \lambda a.$$

For $\lambda_1 z, \lambda_2 z \in Z$, and $t \in \mathbb{C}$ we have

$$f(\lambda_1 z + t\lambda_2 z) = f((\lambda_1 + t\lambda_2)z) = (\lambda_1 + t\lambda_2)a = \lambda_1 a + t\lambda_2 a = f(\lambda_1 z) + tf(\lambda_2 z),$$

so f is a complex linear functional on Z. f satisfies

$$|f(\lambda z)| = |\lambda a| = |\lambda|a = |\lambda|||z|| = ||\lambda z|| = k(\lambda z),$$

and hence

$$|f(\lambda z)| \leq k(\lambda z).$$

By the Hahn Banach therorem, there exists a complex linear functional F on $\mathcal{X}$ such that $|F(z)| \leq ||z|| \; \forall z \in \mathcal{X}$ and $F|Z = f$. Thus $F \in \mathcal{X}^*$. Since $f(z) = ||z|| \neq 0$, then

$$f(x) - f(y) = f(x - y) \neq 0,$$

which implies $f(x) \neq f(y)$, as required. □

2.1.5 Series

Given a sequence $\{x_n\}$ in a Banach space, we can construct a new sequence of partial sums

$$\left\{ \sum_{j=0}^{n} x_j \right\} : \; x_0 \; , \; x_0 + x_1 \; , \; x_0 + x_1 + x_2 \; , \; \ldots$$

If this sequence converges, we say the series $\sum_{n=0}^{\infty} x_n$ converges, and set

$$\sum_{n=0}^{\infty} x_n \; = \; \lim_{n \to \infty} \sum_{j=0}^{n} x_j.$$

The index can always be shifted: if a series $\sum_{n=0}^{\infty} x_n$ converges, then for all k

$$\sum_{n=0}^{\infty} x_n \;=\; \sum_{n=k}^{\infty} x_{n-k}\ .$$

Note that if $A : \mathcal{X} \to \mathcal{X}$ is a bounded linear map and $\sum_{n=0}^{\infty} x_n$ converges, then by linearity

$$A\left(\sum_{j=0}^{n} x_j\right) \;=\; \sum_{j=0}^{n} Ax_j,$$

and by continuity

$$\lim_{n\to\infty} A\left(\sum_{j=0}^{n} x_j\right) \;=\; A\left(\lim_{n\to\infty}\sum_{j=0}^{n} x_j\right)\ .$$

Therefore, $\sum_{n=0}^{\infty} Ax_n$ converges, and

$$\sum_{n=0}^{\infty} Ax_n \;=\; A\left(\sum_{n=0}^{\infty} x_n\right). \tag{2.1}$$

A series $\sum_{n=0}^{\infty} x_n$ is said to converge absolutely if the series $\sum_{n=0}^{\infty} ||x_n||$ converges. The concept of absolute convergence gives a criterion for completeness: a normed vector space is a Banach space if and only if every absolutely convergent series converges. In a Banach space, if a series $\sum_{n=0}^{\infty} x_n$ converges absolutely, then

$$\left|\left|\sum_{n=0}^{\infty} x_n\right|\right| \leq \sum_{n=0}^{\infty} ||x_n||\ .$$

2.2 Continuity and Derivatives in Banach Space

We will consider a function $f : \mathbb{R} \to \mathcal{X}$ defined on an interval (a,b).

Definitions

(i) f is weakly differentiable at $t = t_0$ with weak derivative $f'(t_0) \in \mathcal{X}$ if

$$\lim_{t\to t_0} \frac{f(t) - f(t_0)}{t - t_0} \;=\; f'(t_0)$$

with respect to the weak topology on $\mathcal{X}$. That is,

$$\lim_{t\to t_0} h\left(\frac{f(t) - f(t_0))}{t - t_0}\right) \;=\; h(f'(t_0)) \quad \forall h \in \mathcal{X}^*.$$

For each $h \in \mathcal{X}^*$, by linearity we have

$$h\left(\frac{f(t)-f(t_0)}{t-t_0}\right) \;=\; \frac{h(f(t))-h(f(t_0)))}{t-t_0},$$

so weak differentiability means that each $h \circ f : \mathbb{R} \to \mathbb{C}$ is differentiable at t_0 with derivative $h'(f(t_0))$.

(ii) f is strongly differentiable at $t = t_0$ with strong derivative $f'(t_0) \in \mathcal{X}$ if

$$\lim_{t\to t_0} \frac{f(t)-f(t_0)}{t-t_0} \;=\; f'(t_0)$$

with respect to the norm topology on $\mathcal{X}$. That is,

$$\lim_{t\to t_0} \left|\left|\frac{f(t)-f(t_0)}{t-t_0} - f'(t_0)\right|\right| = 0\ .$$

Note that strong differentiability at t_0 implies weak differentiability, with the same derivative. To see this, suppose f is strongly differentiable at t_0, and let $h \in \mathcal{X}^*$. Then

$$\lim_{t\to t_0} \frac{f(t)-f(t_0)}{t-t_0} \;=\; f'(t_0)$$

and by continuity we have that

$$\lim_{t\to t_0} h\left(\frac{f(t)-f(t_0))}{t-t_0}\right) \;=\; h(f'(t_0))\ .$$

Continuity and Differentiability

Proposition 1. *If f is (strongly) differentiable at t_0, then f is continuous at t_0.*

Proof. Since f is differentiable at t_0, there exists $r > 0$ such that

$$0 < |t-t_0| < r \Longrightarrow \left|\left|\frac{f(t)-f(t_0)}{t-t_0} - f'(t_0)\right|\right| < 1.$$

Hence

$$\left|\left|\frac{f(t)-f(t_0)}{t-t_0}\right|\right| - ||f'(t_0)|| \leq \left|\left|\frac{f(t)-f(t_0)}{t-t_0} - f'(t_0)\right|\right| < 1.$$

Therefore

$$\left|\left|\frac{f(t)-f(t_0)}{t-t_0}\right|\right| < ||f'(t_0)|| + 1,$$

and so

$$||f(t) - f(t_0)|| < (||f'(t_0)|| + 1)|t - t_0|.$$

Let $\epsilon > 0$ and take $\delta < \min(r, \frac{\epsilon}{||f'(t_0)||+1})$. Then for $0 < |t - t_0| < \delta$,

$$||f(t) - f(t_0)|| < (||f'(t_0)|| + 1)|t - t_0| < (||f'(t_0)|| + 1)\frac{\epsilon}{||f'(t_0)|| + 1} = \epsilon.$$

□

Zero Derivative

Theorem 2. *If the weak derivative of f is equal to zero everywhere in (a, b), then f is constant on (a, b).*

Proof. We fix any $t_0 \in (a, b)$ and let $t \in (a, b)$. For each $h \in \mathcal{X}^*$, the function $h \circ f : \mathbb{R} \to \mathbb{C}$ has derivative $h(0) = 0$ on (a, b). Hence, each $h \circ f$ is a constant function on (a, b). Thus

$$h(f(t_0)) = h(f(t)).$$

Since this holds for all $h \in \mathcal{X}^*$, Corollary (2.1.4) implies

$$f(t_0) = f(t),$$

which finishes the proof. □

2.3 Riemann Integral in Banach Space

2.3.1 Tagged Partitions

Definitions

(1) A partition P of an interval [a,b] is a finite ordered set of numbers $\{t_0, t_1, \cdots, t_n\}$ such that $a = t_0 < \cdots < t_n = b$. The mesh of the interval, denoted by $|P|$, is the length of its longest subinterval

$$|P| = \max_{i=1}^{n}(t_i - t_{i-1}).$$

(2) A tagged partition π of an interval [a,b] is a partition $(P : a = t_0 < \cdots < t_n = b)$ together with a set of points $\tau = \{\tau_1, \cdots, \tau_n\}$ such that

$$\tau_i \in [t_{i-1}, t_i] \quad \text{for each} \quad i = 1, ..., n.$$

Note that we define the mesh of π simply to be the mesh of P: $|\pi| = |P|$.

(3) If $\mathcal{P}$ denotes the collection of all tagged partitions of $[a, b]$, and $\mathcal{X}$ is a Banach space and $\ell \in \mathcal{X}$, then for any function $g : \mathcal{P} \longrightarrow \mathcal{X}$ we define the limit

$$\lim_{|\pi|\to 0} g(\pi) = \ell$$

to mean that given $\epsilon > 0$, there exists $\delta > 0$ such that for all $\pi \in \mathcal{P}$ we have

$$|\pi| < \delta \Longrightarrow ||g(\pi) - \ell|| < \epsilon.$$

Note that if the limit ℓ exists, then it is unique.

(4) With notation as in (3) above, the completeness of $\mathcal{X}$ gives us the following Cauchy criterion: $\lim_{|\pi|\to 0} g(\pi)$ exists if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $\pi_1, \pi_2 \in \mathcal{P}$

$$|\pi_1|, |\pi_2| < \delta \Longrightarrow ||g(\pi_1) - g(\pi_2)|| < \epsilon \ .$$

2.3.2 Riemann Sum and Riemann Integral

Definitions Let $\mathcal{X}$ be a Banach space, $f : [a, b] \longrightarrow \mathcal{X}$.
Consider a tagged partition $\pi = (P, \tau)$ of the interval [a,b] with

$$P : a = t_0 < t_1 < ..... < t_n = b \quad \text{and} \quad \tau = \{\tau_1, ...\tau_n\}.$$

The Riemann sum of f on $[a, b]$ for π is given by

$$S_{f,[a,b]}(\pi) = S_f(\pi) = \sum_{i=1}^{n} f(\tau_i)(t_i - t_{i-1}), \tag{2.2}$$

where we note $f(\tau_i) \in \mathcal{X}$ and $(t_i - t_{i-1})$ is a real scalar. Observe that for each tagged partition π of [a,b], $S_f(\pi) \in \mathcal{X}$, and so $S_f : \pi \mapsto S_f(\pi)$ defines a map $\mathcal{P} \to \mathcal{X}$. We say f is Riemann integrable if

$$\lim_{|\pi|\to 0} S_f(\pi) = S$$

in the sense defined in Section 2.3.1 (3) above. If f is Riemann integrable, the limit S is called the Riemann integral of f over $[a,b]$, and denoted $\int_a^b f(s)ds$. In fact, $\int_b^a f(s)ds = -\int_a^b f(s)ds$ and $\int_a^a f(s)ds = 0$.

Basic Propositions on Banach Space:

Proposition 3. *If $f : [a,b] \longrightarrow \mathcal{X}$ is Riemann integrable on $[a,b]$, then it is Riemann integrable on $[a,c]$ and $[c,b]$, for all $c \in (a,b)$.*

Proof. Let $\epsilon > 0$. Then by the Cauchy criterion, there exists $\delta > 0$ such that for all tagged partitions π_1, π_2 of $[a,b]$,

$$|\pi_1|, |\pi_2| < \delta \Longrightarrow ||S_f(\pi_1) - S_f(\pi_2)|| < \epsilon \ . \tag{2.3}$$

Consider any tagged partitions σ_1, σ_2 of $[a,c]$ and ρ of $[c,b]$ with $|\sigma_1|, |\sigma_2|, |\rho| < \delta$. Then $\pi_1 = \sigma_1 \cup \rho$ and $\pi_2 = \sigma_2 \cup \rho$ are tagged partitions of $[a,b]$ with mesh less than δ, and so (2.3) holds. But

$$S_f(\pi_1) = S_{f,[a,c]}(\sigma_1) + S_{f,[c,b]}(\rho).$$

and

$$S_f(\pi_2) = S_{f,[a,c]}(\sigma_2) + S_{f,[c,b]}(\rho).$$

Therefore

$$||S_{f,[a,c]}(\pi_1) - S_{f,[a,c]}(\pi_2)|| = ||S_f(\pi_1) - S_f(\pi_2)|| < \epsilon$$

So by the Cauchy criterion, f is Riemann integrable on $[a,c]$. The proof on $[c,b]$ is similar. $\square$

Proposition 4. *If $f : [a,b] \to \mathcal{X}$ is Riemann integrable on $[a,b]$, then*

$$\int_a^b f(s)ds = \int_a^c f(s)ds + \int_c^b f(s)ds \quad \forall c \in (a,b).$$

Proof. Let

$$\int_a^b f(s)ds = A \ \ , \ \ \int_a^c f(s)ds = B \ \ , \ \ \int_c^b f(s)ds = C.$$

We want to show that $A = B + C$. Let $\epsilon > 0$ be given. It suffices to show $||A - (B+C)|| < \epsilon$. Since we proved that f is Riemann integrable on $[a,b]$, $[a,c]$, and $[c,b]$, there exists $\delta > 0$ such that for all tagged partitions π, π_1, π_2 of $[a, b], [a, c]$ and $[c, b]$ respectively, with $|\pi|, |\pi_1|$, and $|\pi_2| < \delta$ we have

$$N_1 = \left\|S_{f,[a,c]}(\pi_1) - B\right\| < \frac{\epsilon}{3} \ , \quad N_2 = \left\|S_{f,[c,b]}(\pi_2) - C\right\| < \frac{\epsilon}{3} \ ,$$

and

$$N = \|A - S_f(\pi)\| < \frac{\epsilon}{3} \ .$$

Take π_1, π_2 partitions of $[a, c]$ and $[c, b]$ respectively as above. Then, $\pi = \pi_1 \cup \pi_2$ is a partition of [a,b] with $|\pi| < \delta$, and

$$S_f(\pi) = S_{f,[a,c]}(\pi_1) + S_{f,[c,b]}(\pi_2) \ .$$

Then

$$\begin{aligned} \|A - (B + C)\| &= ||A - S_f(\pi) + S_f(\pi_1) + S_f(\pi_2) - (B - C)|| \\ &\leq N + N_1 + N_2 = \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Therefore,

$$\left\| \int_a^b f(s)ds - \left(\int_a^c f(s)ds - \int_c^b f(s)ds \right) \right\| < \epsilon$$

□

Proposition 5. *If $f : [a, b] \longrightarrow \mathcal{X}$ is continuous, then f is Riemann integrable on $[a, b]$.*

Proof. If f is continuous on [a,b], then f is uniformly continuous, since $[a, b]$ is compact. Let $\epsilon > 0$ be given, then there exists $\delta > 0$ such that $||f(t) - f(s)|| < \frac{\epsilon}{2(b-a)}$ whenever $t, s \in [a, b]$ and $|t - s| < \delta$. We will use the Cauchy criterion from Section 2.3.1 (4) to prove integrability. Consider tagged partitions $\pi_1 = (P_1, \tau)$, $\pi_2 = (P_2, \sigma)$ on [a,b] with $|\pi_1|, |\pi_2| < \delta$. Suppose

$$P_1 : a = x_0 < x_1 < ..... < x_n = b \quad \text{and} \quad \tau = \{t_1, ..., t_n\},$$

$$P_2 : a = y_0 < y_1 < ..... < y_m = b \quad \text{and} \quad \sigma = \{s_1, ..., s_m\}.$$

Then the Riemann sum is

$$S_f(\pi_1) = \sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i) \quad , \quad S_f(\pi_2) = \sum_{k=0}^{m-1} f(s_k)(y_{k+1} - y_k).$$

Let $P = P_1 \cup P_2$. P is a refinement of both partitions P_1, P_2 and it has the form

$$a = z_{0,0} = x_0 < .... < z_{0,n_0} = z_{1,0} = x_1 < ... < z_{n-1,n_{n-1}} = z_{n,0} = x_n = b.$$

Let π be a tagged partition partition tagged with its left end points as in figure 2.1

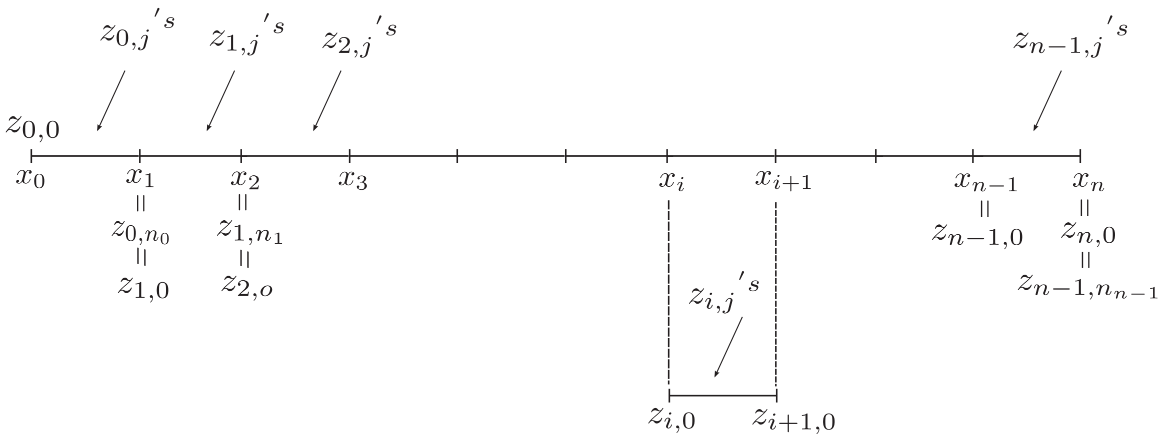


Figure 2.1: tagged partition of [a,b]

Now taking one subinterval of the partition P, we have

$$x_{i+1} - x_i = \sum_{j=0}^{n_i-1} (z_{i,j+1} - z_{i,j}).$$

Since $x_{i+1} = z_{i+1,0}$, $x_i = z_{i,0}$ we want to find the difference between $S_f(\pi_1)$ and $S_f(\pi_2)$:

$$\begin{aligned} ||S_f(\pi_1) - S_f(\pi_2)|| &= ||S_f(\pi_1) - S_f(\pi) + S_f(\pi) - S_f(\pi_2)|| \\ &\leq ||S_f(\pi_1) - S_f(\pi)|| + ||S_f(\pi) - S_f(\pi_2)||. \end{aligned}$$

We will study each term separately. For the first term, we have

$$\begin{aligned} ||S_f(\pi_1) - S_f(\pi)|| &= \left\| \sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i) - \sum_{i=0}^{n-1}\sum_{j=0}^{n_i-1} f(z_{i,j})(z_{i,j+1} - z_{i,j}) \right\| \\ &= \left\| \sum_{i=0}^{n-1} f(t_i) \sum_{j=0}^{n_i-1} (z_{i,j+1} - z_{i,j}) - \sum_{i=0}^{n-1}\sum_{j=0}^{n_i-1} f(z_{i,j})(z_{i,j+1} - z_{i,j}) \right\| \\ &= \left\| \sum_{i=0}^{n-1}\sum_{j=0}^{n_i-1} (f(t_i) - f(z_{i,j}))(z_{i,j+1} - z_{i,j}) \right\| \\ &\leq \sum_{i=0}^{n-1}\sum_{j=0}^{n_i-1} ||f(t_i) - f(z_{i,j})|| (z_{i,j+1} - z_{i,j}). \end{aligned}$$

But each $||f(t_i) - f(z_{i,j})|| < \frac{\epsilon}{2(b-a)}$, since $t_i, z_{i,j} \in [x_i, x_{i+1}]$ and $|\pi_1| < \delta$. So,

$$|t_i - z_{i,j}| \leq x_{i+1} - x_i \leq |\pi| < \delta.$$

Hence

$$||S_f(\pi_1) - S_f(\pi)|| \leq \sum_{i=0}^{n-1} \sum_{j=0}^{n_i-1} \left(\frac{\epsilon}{2(b-a)} \right) (z_{i,j+1} - z_{i,j}) = \frac{\epsilon}{2(b-a)}(b-a) = \frac{\epsilon}{2}.$$

Similarly, for the second part,

$$||S_f(\pi) - S_f(\pi_2)|| \leq \frac{\epsilon}{2}.$$

By adding the last two equations,

$$||S_f(\pi_1) - S_f(\pi)|| \leq \epsilon.$$

□

Proposition 6. *If $f : [a,b] \longrightarrow \mathcal{X}$ is Riemann integrable on $[a,b]$ and continuous at $c \in [a,b]$, then*

$$W(t) = \int_a^t f(s)ds$$

is differentiable at c with $W'(c) = f(c)$.

Proof. Suppose $h > 0$ and $c + h \in (a,b)$. By Proposition (4), we have

$$\int_a^{c+h} f(s)ds = \int_a^c f(s)ds + \int_c^{c+h} f(s)ds.$$

Therefore,

$$\begin{aligned} W(c+h) - W(c) &= \int_a^c f(s)ds + \int_c^{c+h} f(s)ds - \int_a^c f(s)ds \\ &= \int_c^{c+h} f(s)ds. \end{aligned}$$

If $h < 0$ and $c + h \in [a,b]$, then by Proposition (4),

$$\int_a^c f(s)ds = \int_a^{c+h} f(s)ds + \int_{c+h}^c f(s)ds.$$

Therefore,

$$\begin{aligned} W(c+h) - W(c) &= \int_a^{c+h} f(s)ds - (\int_a^{c+h} f(s)ds + \int_{c+h}^{c} f(s)ds) \\ &= -\int_{c+h}^{c} f(s)ds = \int_c^{c+h} f(s)ds. \end{aligned}$$

Then, we want to show that

$$\lim_{h\to 0} \frac{1}{h}\int_c^{c+h} f(s)ds = f(c).$$

Let $\epsilon > 0$. Since f is continuous at $c \in [a,b]$, there exists $\delta > 0$ such that

$$s \in (c-\delta, c+\delta) \Rightarrow ||f(s) - f(c)|| < \frac{\epsilon}{2}. \tag{2.4}$$

Let $0 < h < \delta$. Since f is Riemann integrable on $[c, c+h]$, there exists $r > 0$ such that for any tagged partition π of $[c, c+h]$ with $|\pi| < r$,

$$\left\| \int_c^{c+h} f(s)ds - S_f(\pi) \right\| < \frac{\epsilon h}{2}.$$

Let $\pi = (P, \tau)$ be such a tagged partition, with $P : c = x_0 < x_1 < ..... < x_n = c+h$ and $\tau = \{t_1, ...t_n\}$. Then, we have

$$S_f(\pi) = \sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i).$$

This implies

$$\begin{aligned} \left\| \frac{1}{h}\int_c^{c+h} f(s)ds - f(c) \right\| &= \left\| \frac{1}{h}\int_c^{c+h} f(s)ds - \frac{1}{h}S_f(\pi) + \frac{1}{h}S_f(\pi) - f(c) \right\| \\ &\le \left\| \frac{1}{h}\int_c^{c+h} f(s)ds - \frac{1}{h}S_f(\pi) \right\| + \left\| \frac{1}{h}S_f(\pi) - f(c) \right\|. \end{aligned}$$

For the first term, we have

$$\left\| \frac{1}{h}\int_c^{c+h} f(s)ds - \frac{1}{h}S_f(\pi) \right\| = \left| \frac{1}{h} \right| \left\| \int_c^{c+h} f(s)ds - S_f(\pi) \right\| \le \frac{1}{h}(\frac{\epsilon h}{2}) = \frac{\epsilon}{2}. \tag{2.5}$$

For the second term, we have

$$\left\| \frac{1}{h}S_f(\pi) - f(c) \right\| = \left| \frac{1}{h} \right| . \left|\left| S_f(\pi) - hf(c) \right|\right| . \tag{2.6}$$

Moreover, we know that

$$\begin{aligned} h &= c+h-c \\ &= x_n - x_0 \\ &= (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + ... + (x_1 - x_0) \\ &= \sum_{i=0}^{n-1} (x_{i+1} - x_i) \end{aligned}$$

By replacing h in (2.6) with $\sum_{i=0}^{n-1}(x_{i+1} - x_i)$, we obtain

$$\begin{aligned} \left\| \frac{1}{h} S_f(\pi) - f(c) \right\| &= \left| \frac{1}{h} \right| \left\| \sum_{i=0}^{n-1} f(t_i)\,(x_{i+1} - x_i) - \sum_{i=0}^{n-1} (x_{i+1} - x_i)\, f(c) \right\| \\ &= \frac{1}{h} \left\| \sum_{i=0}^{n-1} (f(t_i) - f(c))\,(x_{i+1} - x_i) \right\| \\ &\leq \frac{1}{h} \sum_{i=0}^{n-1} ||f(t_i) - f(c)||\,(x_{i+1} - x_i)\,. \end{aligned}$$

But $t_i \in (x_{i+1}, x_i) \subset [c, c+h]$ and $h < \delta$, so $t_i \in (c-\delta, c+\delta)$. Therefore by (2.3.2),

$$||f(t_i) - f(c)|| < \frac{\epsilon}{2}\ .$$

Hence,

$$\left\| \frac{1}{h} S_f(\pi) - f(c) \right\| \leq \frac{\epsilon}{2h} \sum_{i=0}^{n-1} (x_{i+1} - x_i) = \frac{\epsilon}{2}. \tag{2.7}$$

By adding (2.5) and (2.7), we obtain

$$\left\| \frac{1}{h} \int_c^{c+h} f(s)ds - f(c) \right\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

The proof is similar for $h < 0$. Hence

$$W(t) = \int_a^t f(s)ds$$

is differentiable at $c \in (a,b)$ with $W'(c) = f(c)$. $\square$

Proposition 7. *If $f : [a,b] \to \mathcal{X}$ is continuous and $f = F'$, then*

$$\int_a^b f(s)ds = F(b) - F(a).$$

Proof. Let

$$W(t) = \int_a^t f(s)ds$$

Then $(W - F)'(t) = W'(t) - F'(t) = f(t) - f(t) = 0$ for all t. Therefore, $W - F$ is constant. Hence

$$W(b) - F(b) = W(a) - F(a) = 0 - F(a) = -F(a).$$

Therefore

$$\int_a^b f(s)ds = W(b) = F(b) - F(a).$$

□

Proposition 8. *Let $f : [a, b] \longrightarrow \mathcal{X}$ be a continuous map and $A \in \mathcal{L}(\mathcal{X})$, then*

$$A \int_a^b f(s)ds = \int_a^b Af(s)ds.$$

Proof. Since $A : \mathcal{X} \longrightarrow \mathcal{X}$ and $f : [a, b] \longrightarrow \mathcal{X}$ are continuous maps, it follows that the composition $(A \circ f)$ is continuous map. Hence, by Proposition (5), both f and $A \circ f$ are integrable with

$$\int_a^b f(s)ds = \lim_{|\pi| \to 0} S_f(\pi) \qquad \text{and} \qquad \int_a^b A(f(s))ds = \lim_{|\pi| \to 0} S_{A\circ f}(\pi).$$

Let $x = \lim_{|\pi|\to 0} S_f(\pi)$ and $y = \lim_{|\pi|\to 0} S_{A\circ f}(\pi)$. We want to show $A(x) = y$. By linearity, for each tagged partition $\pi = (P, \tau)$ of $[a, b]$, we have

$$\begin{aligned} A(S_f(\pi)) &= A \sum_{i=1}^n f(\tau_i)(t_i - t_{i-1}) \\ &= \sum_{i=1}^n A(f(\tau_i))(t_i - t_{i-1}) \\ &= S_{A\circ f}(\pi). \end{aligned}$$

Thus,

$$y = \lim_{|\pi| \to 0} S_{A\circ f}(\pi) = \lim_{|\pi| \to 0} A(S_f(\pi)),$$

and so we must show $A(x) = \lim_{|\pi|\to 0} A(S_f(\pi))$. This follows from the fact that A is continuous and $x = \lim_{|\pi|\to 0} S_f(\pi)$. To see this, let $\epsilon > 0$ be given. We want to find $\delta > 0$ such that for all tagged partitions π of $[a, b]$

$$|\pi| < \delta \Longrightarrow ||A(S_f(\pi) - A(x)|| < \epsilon.$$

Since A is continuous, we have

$$\lim_{z \to x} A(z) = A(x).$$

Hence, there exists $\delta_1 > 0$ such that for all $z \in \mathcal{X}$

$$||z - x|| < \delta_1 \Longrightarrow ||A(z) - A(x)|| < \epsilon.$$

Since $x = \lim_{|\pi| \to 0} S_f(\pi)$, there exists $\delta > 0$ such that for all tagged partitions π of $[a, b]$,

$$|\pi| < \delta \Longrightarrow ||S_f(\pi) - x|| < \delta_1.$$

This implies $||A(S_f(\pi)) - A(x)|| < \epsilon$, as required. $\square$

Proposition 9. *Change of variables: If $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable on $[a, b]$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous on $g([a, b])$, then*

$$\int_{g(a)}^{g(b)} f(u)du = \int_a^b f(g(t))g'(t)dt. \tag{2.8}$$

Proof. Because f is continuous on $g([a, b])$, then by the fundamental theorem of calculus, the function

$$F(y) = \int_{g(a)}^{y} f(u)du$$

is differentiable on $g([a, b])$ and $F'(y) = f(y)$. Since f is continuous on $g([a, b])$ and g, g' are continuous on $[a, b]$, the function

$$H(s) = \int_a^s f(g(t))g'(t)dt \quad \forall s, t \in [a, b].$$

is differentalbe on $[a, b]$, and

$$\begin{aligned} H'(s) &= f(g(s))g'(s) \;\; \forall s \in [a, b] \\ H'(s) &= F'(g(s))g'(s) = (F \circ g)'(s) \;\; \text{by chain rule.} \end{aligned}$$

Hence $H'(s)-(F\circ g)'(s)=0$ which implies that $H(s)-F\circ g(s)$ is a constant function. To see this, let $a\in[a,b]$ then $H(a)-F(g(a))=0$ from the definition of the $F(y)$ and $H(s)$. Therefore we conclude that $H=F\circ g$. Also, we have

$$\int_a^b f(g(t))g'(t)dt = H(b) = F(g(b)) = \int_{g(a)}^{g(b)} f(u)du$$

$\square$

Chapter 3

Banach Algebra

3.1 Definition

A Banach algebra is a Banach space $\mathcal{A}$ together with a map

$$\begin{array}{rccc} \cdot \ : & \mathcal{A}\times\mathcal{A} & \longrightarrow & \mathcal{A} \\ & (x,y) & \longmapsto & xy \end{array}$$

called the product, which satisfies the following properties for all $x_1, x_2, x, y, z \in \mathcal{A}$ and $a \in \mathbb{C}$

(i) Associativity: $x(yz) = (xy)z$,

(ii) Bilinearity: $(ax_1 + x_2)y = ax_1y + x_2y$, $y(ax_1 + x_2) = ayx_1 + yx_2$,

(iii) Triangle inequality: $||xy|| \leq ||x||\,||y||$.

3.1.1 Examples

(a) The complex numbers $\mathbb{C}$ form a (one dimensional) complex Banach space, with the absolute value as norm: $|x + iy| = \sqrt{x^2 + y^2}$. The usual product of two complex numbers is bilinear and associative, and

$$|zw| = |z|.|w| \ \forall \ \ z, w \in \mathbb{C}.$$

Hence, the complex numbers form a Banach algebra.

(b) Given any Banach space $\mathcal{X}$, the space $\mathcal{L}(\mathcal{X})$ of all bounded linear operators on $\mathcal{X}$ is also a Banach space. For $S, T \in \mathcal{L}(\mathcal{X})$, the composition $ST = S \circ T$ is in $\mathcal{L}(\mathcal{X})$, with

$$||ST|| \leq ||S||\,||T|| \ . \tag{3.1}$$

The composition operation is bilinear and associative, $\mathcal{L}(\mathcal{X})$ is a Banach algebra with composition as the product.

3.2 Basic Properties

Continuity of the product: The product map for a Banach algebra, as we defined it above, is continuous.

Proof. We have the product map

$$\begin{array}{rcl} \cdot\ :\ \mathcal{A}\times\mathcal{A} & \longrightarrow & \mathcal{A} \\ (x,y) & \longmapsto & xy \end{array}$$

Let $(x,y)\in\mathcal{A}\times\mathcal{A}$ and consider any sequence $\{(x_n,y_n)\}\subset\mathcal{A}\times\mathcal{A}$ such that

$$(x_n,y_n)\to(x,y)\in\mathcal{A}\times\mathcal{A}.$$

This implies that: $x_n\to x$ in $\mathcal{A}$ and $y_n\to y$ in $\mathcal{A}$. Furthermore, since any convergent sequence is bounded, there exists a constant $C>0$ such that $||x_n||\leq C$. From the bilinearity of the product and the triangle inequality, we obtain

$$\begin{aligned} ||x_ny_n-xy|| &\leq ||x_ny_n-x_ny||+||x_ny-xy|| \\ &\leq ||x_n||.||y_n-y||+||x_n-x||.||y|| \\ &\leq C||y_n-y||+||x_n-x||||y||. \end{aligned}$$

But since $\lim_{n\to\infty}||x_n-x||=\lim_{n\to\infty}||y_n-y||=0$, this implies

$$\lim_{n\to\infty}||x_ny_n-xy||=0.$$

Equivalently, $x_ny_n\to xy$, as required. □

Left and right multiplication in a Banach algebra: Given any $z\in\mathcal{A}$, we define left and right multiplication by z, respectively by

$$L_z:x\longmapsto zx\in\mathcal{A}\quad\text{and}\quad R_z:x\longmapsto xz\in\mathcal{A}\ .$$

These are bounded linear maps on $\mathcal{A}$ with

$$||L_z||\leq||z||\quad\text{and}\quad||R_z||\leq||z||. \tag{3.2}$$

Proof. The linearity of L_z and R_z follows directly from the bilinearity of the product. The boundedness of these maps and (3.2) are immediate from the triangle inequality. □

Product Rule:

Proposition 10. *If $f, g : \mathbb{R} \to \mathcal{A}$ are (strongly) differentiable at t_0, then fg is differentiable at t_0 with*

$$(fg)'(t_0) = f'(t_0)g(t_0) + f(t_0)g'(t_0).$$

3.3 Series in a Banach Algebra

Left and Right Multiplication: because left and right multiplication are bounded linear maps, with operator norms satisfying (3.2), if $\sum_{n=0}^{\infty} x_n$ converges in $\mathcal{A}$ then the left and right multiplication by $a \in \mathcal{A}$ converge as well. In that case, we have the equalities

$$\begin{aligned}\sum_{n=0}^{\infty} x_n a &= \left(\sum_{n=0}^{\infty} x_n\right) a \\ \sum_{n=0}^{\infty} a.x_n &= a\left(\sum_{n=0}^{\infty} x_n\right).\end{aligned}$$

Cauchy Product:

Proposition 11. *If $\sum a_n$ and $\sum b_n$ converge and one of them is absolutely convergent, then*

$$\begin{aligned}\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} a_k b_{n-k} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k}.\end{aligned}$$

Proof. Consider the first equality. For each k, since $\sum b_n$ converges, we obtain

$$\begin{aligned}a_k \sum_{n=0}^{\infty} b_n &= a_k \sum_{n=k}^{\infty} b_{n-k} && \text{(by index shift)} \\ &= \sum_{n=k}^{\infty} a_k b_{n-k}. && \text{(by left multiplication)}\end{aligned}$$

Since $\sum a_k$ converges, the right multiplication is given by

$$\left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{k=0}^{\infty}\left(a_k \sum_{n=0}^{\infty} b_n\right).$$

For the second equality, suppose that $\sum a_n$ converges absolutely. Let $A = \sum a_n$, $B = \sum b_n$, and define

$$A_n = \sum_{j=0}^{n} a_j \quad , B_n = \textstyle\sum_{j=0}^{n} b_j \quad , C_n = \displaystyle\sum_{i=0}^{n}\sum_{k=0}^{i} a_k b_{i-k}.$$

We want to show that C_n converges to AB. Hence, let $I = \{(i,k) \in \{0,...,n\}^2$, where $k \leq i\}$. Then,

$$C_n = \sum_{(i,k)\in I} a_k b_{i-k} = \sum_{k=0}^{n}\sum_{i=k}^{n} a_k b_{i-k}.$$

Letting $j = i - k$. Then,

$$\begin{aligned}
C_n &= \sum_{k=0}^{n}\sum_{j=0}^{n-k} a_k b_j \\
&= \sum_{k=0}^{n} a_k \sum_{j=0}^{n-k} b_j \\
&= \sum_{k=0}^{n} a_k B_{n-k} \\
&= \sum_{k=0}^{n} a_k B_{n-k} - A_n B + A_n B \\
&= \left(\sum_{k=0}^{n} a_k B_{n-k}\right) - \left(\sum_{k=0}^{n} a_k B\right) + A_n B \\
&= \sum_{k=0}^{n} a_k (B_{n-k} - B) + A_n B
\end{aligned}$$

Hence

$$\begin{aligned}
||AB - C_n|| &= \left\|(A - A_n)\,B - \sum_{k=0}^{n} a_k\,(B_{n-k} - B)\right\| \\
&\leq ||A - A_n||\ ||B|| + \sum_{k=0}^{n} ||a_k||\ ||B_{n-k} - B||
\end{aligned}$$

Since $B_n \to B$ and $\sum a_n$ converges absolutely, there exists $c > 0$ such that $||B|| \leq c$, and $\forall n$, $||B_n - B|| \leq c$ and $\sum_{n=0}^{\infty} ||a_k|| \leq c$. Let $\epsilon > 0$ be given. Since $\sum a_n$ converges, there exists $N_a > 0$ such that for $n \geq N_a$, $||A_n - A|| < \frac{\epsilon}{3c}$. This implies

$$||A_n - A||||B|| < \frac{\epsilon}{3}.$$

$\sum a_n$ converges absolutely, there exists $M > 0$ such that $\sum_{k=M}^{\infty} ||a_k|| < \frac{\epsilon}{3c}$. Then, for $n \geq M$, we have $\sum_{k=M}^{n} ||a_k|| < \frac{\epsilon}{3c}$, which implies

$$\sum_{k=M}^{n} ||a_k|| \; ||B_{n-k} - B|| < c\frac{\epsilon}{3c} = \frac{\epsilon}{3}.$$

Since $\sum b_n$ converges, there exists $N_b > 0$ such that for $n \geq N_b$, $||B_n - B|| < \frac{\epsilon}{3c}$. If $n \geq N_b + M$, $n - k \geq N_b$ for all $0 < k \leq M - 1$, and hence

$$\sum_{k=0}^{M-1} ||a_k|| \quad ||B_{n-k} - B|| < c\ \tfrac{\epsilon}{3c} = \tfrac{\epsilon}{3}.$$

By taking $n \geq \max(N_a, M, N_b + M)$,then

$$\begin{aligned} ||AB - C_n|| &\leq ||A - A_n||\,.\,||B|| \\ &\quad + \sum_{k=0}^{M-1} ||a_k||\,.\,||B_{n-k} - B|| + \sum_{k=M}^{n} ||a_k||\,.\,||B_{n-k} - B|| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

□

3.4 Exponential Function in Operator Algebras

Let $\mathcal{X}$ be a Banach space, and consider the Banach algebra $\mathcal{L}(\mathcal{X})$.

Powers: Let $A \in \mathcal{L}(\mathcal{X})$. Then we define

$$A^0 = I\ ,\ A^1 = A\ ,\ A^2 = AA\ ,\ A^3 = A^2A\ ,\ \ldots\ ,\ A^n = A^{n-1}A\ ,\ \cdots \in \mathcal{L}(\mathcal{X})$$

Then, by induction, for all $m, n = 0, 1, 2, \ldots$, we have $A^mA^n = A^{m+n}$ and $||A^n|| \leq ||A||^n$

Binomial expansion: If $A, B \in \mathcal{L}(\mathcal{X})$ commute, then

$$(A + B)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k}B^k. \tag{3.3}$$

Proof. We proceed by induction onn. When $n = 0$, both sides of the equation areequal the identity, since $(A + B)^0 = I$, $\binom{0}{0} = 1$, $A^0 = I$, and $B^0 = I$. Now, assuming (3.3) holds for n, then we will prove it for $n + 1$.

$$\begin{aligned}
(A+B)^{n+1} &= (A+B)^n(A+B) \\
&= \left(\sum_{k=0}^{n}\binom{n}{k}A^{n-k}B^k\right)(A+B) \\
&= \sum_{k=0}^{n}\binom{n}{k}A^{n-k}B^kA + \sum_{k=0}^{n}\binom{n}{k}A^{n-k}B^kB.
\end{aligned}$$

But if A, B commute (i.e, $AB = BA$), then $B^kA = AB^k$. Hence

$$\begin{aligned}
(A+B)^{n+1} &= \sum_{k=0}^{n}\binom{n}{k}A^{n-k+1}B^k + \sum_{k=0}^{n}\binom{n}{k}A^{n-k}B^{k+1} \\
&= \sum_{k=0}^{n}\binom{n}{k}A^{n-k+1}B^k + \sum_{j=1}^{n+1}\binom{n}{j-1}A^{n-j+1}B^j \\
&= A^{n+1}B^0 + \sum_{k=1}^{n}\left[\binom{n}{k} + \binom{n}{k-1}\right]A^{n-k+1}B^k + A^0B^{n+1} \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}A^{n-k+1}B^k.
\end{aligned}$$

□

Exponential series: Let $A \in \mathcal{L}(\mathcal{X})$. We define

$$e^A = \sum_{n=0}^{\infty}\frac{A^n}{n!} \in \mathcal{L}(\mathcal{X}).$$

We note that this series converges absolutely, since

$$\left|\left|\frac{A^n}{n!}\right|\right| = \frac{1}{n!}\,||A^n|| \leq \frac{1}{n!}||A||^n,$$

and the real series $\sum_{n=0}^{\infty}\frac{||A||^n}{n!}$ converges to the real number $e^{||A||}$. Thus

$$\left|\left|e^A\right|\right| \leq e^{||A||}.$$

Note that $e^0 = \sum_{n=0}^{\infty}\frac{0^n}{n!} = I + 0 + 0 + \cdots = I$.

Product of exponentials: If $A, B \in \mathcal{L}(\mathcal{X})$ commute, then $e^A e^B = e^{A+B}$. To see this, we use the Cauchy product and the binomial theorem

$$\begin{aligned}
e^A e^B = \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{n=0}^{\infty} \frac{B^n}{n!} &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{A^k}{k!} \frac{B^{n-k}}{(n-k)!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} A^k B^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (A+B)^n = e^{A+B}.
\end{aligned}$$

Inverses of exponentials: If $A \in \mathcal{L}(\mathcal{X})$, then e^A is invertible with inverse e^{-A}. This is immediate from above, since A and $-A$ commute and $e^0 = I$.

Chapter 4

Operator Semigroups

From now on, we consider $\mathcal{X}$ to be a Banach space.

4.1 Definition

Let $T : [0, \infty) \longrightarrow \mathcal{L}(\mathcal{X})$ be a mapping. If for all $t, s \in [0, \infty)$ the identities

$$\begin{cases} T(t+s) = T(t)T(s) \\ T(0) = I \end{cases} \tag{4.1}$$

hold, then T is called an operator semigroup.

Continuous semigroups: An operator semigroup $T : [0, \infty) \longrightarrow \mathcal{L}(\mathcal{X})$ is (uniformly) continuous if it is a continuous map from $[0, \infty)$ to $\mathcal{L}(\mathcal{X})$ with the operator norm topology.

Example: For $A \in \mathcal{L}(\mathcal{X})$, consider the mapping $t \longmapsto e^{tA}$. For any $t, s \in [0, \infty)$, where tA and sA commute, and the exponential function satisfies the following properties

$$\begin{cases} e^{tA}e^{sA} = e^{(t+s)A} \\ e^{0A} = I \end{cases} , \tag{4.2}$$

e^{tA} is an operator semigroup. The following proposition shows it is a continuous operator semigroup by Proposition (13) in Section (1).

Proposition 12. *(Differentiability of the exponential map): The mapping*

$$T : t \longmapsto e^{tA}$$

is strongly differentiable and satisfies the differential equation

$$\begin{cases} \frac{d}{dt}\left(T(t)\right) = AT(t) \quad \forall t \geq 0 \\ T(0) = I. \end{cases} \tag{4.3}$$

Proof. We know $T(0) = e^0 = I$. We must find the derivative of T

$$\begin{aligned} \lim_{h\to 0} \frac{T(t+h) - T(t)}{h} &= \lim_{h\to 0} \frac{e^{(t+h)A} - e^{tA}}{h} \\ &= \lim_{h\to 0} \frac{e^{hA}e^{tA} - e^{tA}}{h} \\ &= \lim_{h\to 0} \frac{(e^{hA} - I)e^{tA}}{h} \\ &= \lim_{h\to 0} \frac{e^{hA} - I}{h} e^{tA}. \end{aligned}$$

Now, we want to show that

$$\lim_{h\to 0} \frac{e^{hA} - I}{h} = A.$$

We notice

$$e^{hA} - I = \sum_{n=0}^{\infty} \frac{(hA)^n}{n!} - I = I + \sum_{n=1}^{\infty} \frac{(hA)^n}{n!} - I = \sum_{n=1}^{\infty} \frac{(hA)^n}{n!},$$

and hence

$$\left|\left|e^{hA} - I\right|\right| \leq \sum_{n=1}^{\infty} \left|\left|\frac{(hA)^n}{n!}\right|\right| \leq \sum_{n=1}^{\infty} \frac{||hA||^n}{n!} = \sum_{n=1}^{\infty} \frac{|h|^n \, ||A||^n}{n!} = e^{|h|||A||} - 1.$$

Therefore

$$\begin{aligned} \left|\left|\frac{e^{hA} - I}{h} - A\right|\right| &\leq \left|\left|\frac{e^{hA} - I}{h}\right|\right| - ||A|| \\ &\leq \frac{e^{|h|.||A||} - 1}{|h|} - ||A|| \\ &= \frac{f(|h|) - f(0)}{|h|} - a, \end{aligned}$$

where we are letting $a = ||A||$ and $f(t) = e^{at}$ is the exponential function on $\mathbb{R}$. Then we know f is differentiable with $f'(0) = a$, so $\lim_{t\to 0} \frac{f(t)-f(0)}{t} = a$. Hence

$$\lim_{h\to 0} \frac{f(|h|) - f(0)}{|h|} = a.$$

Therefore, $\lim_{h\to 0}\frac{e^{hA}-I}{h}$ exists and is equal to A.
Because right multiplication is continuous in a Banach algebra, it follows that

$$\lim_{h\to 0}\frac{e^{hA}-I}{h}e^{tA}=Ae^{tA}.$$

This proves $\frac{d}{dt}(T(t))=AT(t)$. $\square$

Theorem 13. *If a map $T:\mathbb{R}\longmapsto(\mathcal{L}(\mathcal{X}),||.||)$ is differentiable and satisfies (4.3), then*

$$T(t)=e^{tA}\ \textit{with}\ A=T'(0)\in\mathcal{L}(\mathcal{X}).$$

Proof. Let $T:\mathbb{R}\longrightarrow(\mathcal{L}(\mathcal{X}),||.||)$ be differentiable and satisfy (4.3). Let $A=T'(0)$. Then, by the product rule for the derivative, we have

$$\begin{aligned}\frac{d}{dt}\left(e^{-tA}T(t)\right) &= \frac{d}{dt}\left(e^{-tA}\right)T(t)+e^{-tA}\frac{d}{dt}\left(T(t)\right)\\ &= -Ae^{-tA}T(t)+e^{-tA}AT(t)\ \forall t\geq 0.\end{aligned}$$

Consider

$$e^{-tA}=\sum_{n=0}^{\infty}\frac{(-t)^nA^n}{n!}=\sum_{n=0}^{\infty}B_n.$$

Then by the right multiplication of series for all $A\in\mathcal{L}(\mathcal{X})$

$$e^{-tA}A=\left(\sum_{n=0}^{\infty}B_n\right)A=\sum_{n=0}^{\infty}\left(B_nA\right).$$

But

$$B_nA=\frac{-t^nA^n}{n!}\ A=\frac{-t^nA^{n+1}}{n!}=\frac{-t^nAA^n}{n!}=A\ \frac{-t^nA^n}{n!}=AB_n.$$

Then by the left multiplication of series for all $A\in\mathcal{L}(\mathcal{X})$

$$e^{-tA}A=\sum_{n=0}^{\infty}\left(AB_n\right)=A\sum_{n=0}^{\infty}B_n=Ae^{-tA}.$$

Therefore

$$\frac{d}{dt}\left(e^{-tA}T(t)\right)=-Ae^{-tA}T(t)+Ae^{-tA}T(t)=0\ \forall t\geq 0.$$

Since the derivative is equal to zero, by Theorem (2) the function is equal to a constant $C \in \mathcal{L}(\mathcal{X})$, $e^{-tA}T(t) = C$, for all $t \geq 0$. But then for $t = 0$

$$C = e^0 T(0) = I.$$

Because e^{-tA} has inverse e^{tA} for all $t \geq 0$

$$\begin{aligned} e^{-tA}T(t) &= I \\ e^{tA}e^{-tA}T(t) &= e^{tA} \\ T(t) &= e^{tA}. \end{aligned}$$

□

4.2 Uniformly Continuous Semigroups

Theorem 14. *Every (uniformly) continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space $\mathcal{X}$ is of the form $T(t) = e^{tA}$ $\forall t \geq 0$ for some bounded operator $A \in \mathcal{L}(\mathcal{X})$.*

Proof. We will proceed by proving that the function $(T(t))_{t\geq 0}$ is differentiable and satisfies (4.3). By Theorem 13 we concluded that it must be of the form $T(t) = e^{tA}$ with $T'(0) = A$.

First, because $T(t))_{t\geq 0}$ is continuous, we can define the operator

$$W(t) = \int_0^t T(s)ds \quad \forall t \geq 0.$$

$W(t)$ is well defined because $T(s)$ is continuous and so the integral exists for all $t \geq 0$. Because T is continuous, by Proposition 6, W is differentiable with $W'(t) = T(t)$. So

$$I = T(0) = \lim_{t\to 0} \frac{W(t) - W(0)}{t} = \lim_{t\to 0} \frac{1}{t}\, W(t).$$

From Section 2.1.3, we know that the set of invertible operators is open. Since I is invertible, $\frac{1}{t}W(t)$ is invertible for small $t \in \mathbb{R}$. Therefore, $W(t)$ is invertible for small t. Then, for such small t_0, we have

$$T(t) = W^{-1}(t_0)W(t_0)T(t).$$

So

$$T(t) = W^{-1}(t_0)\left(\int_0^{t_0} T(s)ds\right)T(t).$$

Right multiplication by $T(t)$ in the Banach algebra $\mathcal{L}(\mathcal{X})$ is a bounded linear map, so by Proposition 8

$$T(t) = W^{-1}(t_0)\int_0^{t_0} T(s)T(t)ds.$$

T is a semigroup, so

$$T(t) = W^{-1}(t_0)\int_0^{t_0} T(s+t)ds.$$

By Proposition 9, the equation becomes

$$\begin{aligned}
T(t) &= W^{-1}(t_0)\int_t^{t+t_0} T(s)ds. \\
&= W^{-1}(t_0)\left(\int_0^{t+t_0} T(s)ds + \int_t^0 T(s)ds\right) \\
&= W^{-1}(t_0)\left(\int_0^{t+t_0} T(s)ds - \int_0^t T(s)ds\right) \\
&= W^{-1}(t_0)(W(t+t_0) - W(t)).
\end{aligned}$$

But W is differentiable for all $t \geq 0$, which finishes the proof that T is differentiable. To find the derivative we use the semigroup property

$$\begin{aligned}
T'(t) &= \lim_{h\to 0}\frac{T(t+h)-T(t)}{h} \\
&= \lim_{h\to 0}\frac{T(t)T(h)-T(t)}{h} \\
&= \lim_{h\to 0}\frac{T(h)-T(0)}{h}\,T(t) \;=\; T'(0)T(t).
\end{aligned}$$

Therefore, with $A = T'(0)$, we have

$$\frac{d}{dt}(T(t)) = AT(t) \quad \forall t \geq 0$$

This shows that $(T(t))_{t\geq 0}$ is differentiable and satisfies (4.3), which finishes the proof.

$\square$

Chapter 5

Theory of Dynamical Systems

We shall consider differential equations (DEs) of the form

$$x' = f(x), \tag{5.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. Here $x = (x_1(t), \cdots, x_n(t))$, a function of time, where x' denotes its derivative. The Euclidean space $\mathbb{R}^n$ is called the *state space* or *phase space*. The DE is called *linear* if f is linear, and hence given by matrix multiplication $f(x) = Ax$, for an $n \times n$ matrix of real numbers A. However, in general, we wish to consider non-linear DEs if f is non-linear. Since we identify each vector $f(x) = (f_1(x), \dots, f_n(x)) \in \mathbb{R}^n$ with a vector $x \in \mathbb{R}^n$, then the function f is defined as a vector field. The DE is called *autonomous* if f does not depend on t explicitly.

We start by giving some basic definitions and notions:

A function of $\psi : \mathbb{R} \to \mathbb{R}^n$ is called a *solution* of the DE if

$$\psi'(t) = f\big(\psi(t)\big) \tag{5.2}$$

holds for all $t \in \mathbb{R}$ in the domain of ψ. The images of solutions ψ are called *orbits* of the DE. The vector field f is tangent to the orbits because the tangent vector to the orbit of a solution ψ at $\psi(t)$ is $\psi'(t)$, which satisfies (5.2). A point $a \in \mathbb{R}^n$ is called an *equilibrium point* or *fixed point* of the DE if it is a zero of the vector field:

$$f(a) = 0. \tag{5.3}$$

A constant function $\psi(t) = a$, for all $t \in \mathbb{R}$, is a solution of the DE if and only if a is an equilibrium point, since $\psi'(t) = 0 = f(a)$. Such a solution describes an *equilibrium state* of the physical system. We examine the behaviour of the orbits of the DE near the equilibrium points to study their stability. Since we are assuming f

is continuously differentiable,

$$f(x) = f(a) + \mathrm{D}f(a)(x-a) + R_1(x,a), \tag{5.4}$$

with

$$\lim_{x \to a} \frac{||R_1(x,a)||}{||x-a||} = 0 \ .$$

Here $\mathrm{D}f(a)$ denotes the $n \times n$ derivative matrix

$$\mathrm{D}f(x) = \left(\frac{\partial f_i}{\partial x_j} \right), \quad i, j = 1, \cdots, n, \tag{5.5}$$

and $R_1(x,a)$ is called the *error term.* If $a \in \mathbb{R}^n$ is an equilibrium point of the DE, then $f(a) = 0$ and we can rewrite (5.1) as

$$x' = f(x) = \mathrm{D}f(a)(x-a) + R_1(x,a). \tag{5.6}$$

The *linearization* of the DE (5.1) at an equilibrium point a is the linear DE

$$u' = Df(a)u.$$

Since $f(x) \approx \mathrm{D}f(a)(x-a)$ for x near an equilibrium point a, in general, solutions of the linearization approximate the solutions of the original non-linear DE near the equilibrium points.

5.1 Linear Autonomous Differential Equations

Let A be an $n \times n$ real matrix. Then the linear DE

$$x' = Ax \qquad , \qquad x(0) = a \in \mathbb{R}^n, \tag{5.7}$$

has the unique solution of the form

$$x(t) = \mathrm{e}^{tA} a \qquad , \qquad \text{for all } t \in \mathbb{R}, \tag{5.8}$$

where the exponential e^{tA} maps $a \to e^{ta}a$ for all $t \in \mathbb{R}$ and $a \in \mathbb{R}^n$. The *linear flow* of the DE is a one-parameter family of linear maps, and is denoted by

$$\mathrm{g}^t = \mathrm{e}^{tA}. \tag{5.9}$$

The linear flow (and the non-linear flow) of the DE satisfies the properties

$$\mathrm{g}^0 = I \quad \text{and} \quad \mathrm{g}^{t+s} = \mathrm{g}^t \circ \mathrm{g}^s, \quad \forall\ t, s \in \mathbb{R}. \tag{5.10}$$

Therefore, the linear flow $\{e^{tA}\}_{t\in\mathbb{R}}$ constitutes a group under the composition of maps. In terms of the physical system, the flow describes the evolution of the dynamical system in terms of time. The *orbits* are subsets of $\mathbb{R}^n$ divided by the flow of the DE:

$$\gamma(a) = \left\{\mathrm{g}^t a | t \in \mathbb{R}\right\}. \tag{5.11}$$

This is called the orbit of the DE through a and is the image of the solution curve $x(t) = e^{tA}a$. Since the solution of the DE is unique, either of these properties hold:

$$\gamma(a) = \gamma(b) \ \text{ or } \gamma(a) \cap \gamma(b) = \emptyset, \forall a, b \in \mathbb{R}^n.$$

There are three types of orbits:

1. $\gamma(a)$ is a *point orbit* if $\mathrm{g}^t a = a$ for all $t \in \mathbb{R}$.

2. $\gamma(a)$ is a *periodic orbit* if there exists a $T > 0$ such that $\mathrm{g}^T a = a$.

3. $\gamma(a)$ is a *non-periodic orbit* if $g^t a \neq a$ for all $t \neq 0$.

Definitions

1. Given a linear DE $x' = Ax$ in $\mathbb{R}^n$, define $y = Px$ to be a new function, where P is a non-singular matrix, and let $\tau = kt$ be a new variable, where k is a positive constant. Then $y' = By$, where $B = \frac{1}{k} P\, A\, P^{-1}$. Furthermore, if $A = kP^{-1}\, B\, P$, then the two linear dynamical systems $x' = Ax$ and $x' = Bx$ are *linearly equivalent* (i.e., the linear map P maps each orbit of the flow e^{tA} to an orbit of the e^{tB}).

2. If there exists a non-singular matrix P and a positive constant k such that $\forall t \in \mathbb{R}, P\, \mathrm{e}^{tA} = \mathrm{e}^{ktB}\, P$, then the two linear flows e^{tA} and e^{tB} on $\mathbb{R}^n$ are linearly equivalent.

These definitions lead to the three types of Jordan Canonical forms for any 2×2 real matrix (A), which are classified as follows:

1. If A has two real independent eigenvalues, then there exists a matrix P such that $J = P\,A\,P^{-1}$, and the flow is denoted by e^{tJ}, where

$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad e^{tA} = \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & 0 \\ 0 & \mathrm{e}^{\lambda_2 t} \end{bmatrix}$$

   The eigenvectors are $e_1 = (1,0)^T$ and $e_2 = (0,1)^T$. The resulting solution is $y(t) = \mathrm{e}^{tJ}b$, $b \in \mathbb{R}^2$ (i.e., $y_1 = \mathrm{e}^{\lambda_1 t}b_1$ and $y_2 = \mathrm{e}^{\lambda_2 t}b_2$). Note that for non-zero eigenvalues, the orbits of the DE are given by

$$\left[\frac{y_1}{b_1}\right]^{\frac{1}{\lambda_1}} = \left[\frac{y_2}{b_2}\right]^{\frac{1}{\lambda_2}}.$$

2. If A has one real eigenvalue, then there exists a matrix P such that $J = P\,A\,P^{-1}$, and the flow is denoted by e^{tJ}, where

$$J = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \qquad e^{tA} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

   The eigenvector is given by $e_1 = (1,0)^T$. Note that for non-zero eigenvalues,, then the orbits of the DE are of the form

$$y_1 = y_2\left[\frac{b_1}{b_2} + \frac{1}{\lambda}\log\frac{y_2}{y_1}\right].$$

3. If A has complex eigenvalues of the form $\alpha + i\beta$, then there exists a matrix P such that $J = P\,A\,P^{-1}$, where

$$J = \begin{bmatrix} \alpha & \beta \\ -\alpha & \beta \end{bmatrix}$$

   The simplest way to calculate the orbit in this case is to introduce the polar form (r,θ), where $y_1 = r\cos\theta$ and $y_2 = r\sin\theta$. Then the DE becomes $r' = \alpha r$ and $\theta' = -\beta$, which implies that $\frac{dr}{d\theta} = -\frac{\alpha}{\beta}r$. Without loss of generality, we can assume $\beta > 0$, since the DE is invariant under the changes $(\beta, y_1) \to (-\beta, -y_1)$. Thus $\lim_{t\to\infty}\theta = -\infty$.

5.2 Topological Equivalence of Linear Flows

Since the flow of the DE is limited by the number of distinct eigenvectors, linear equivalence acts as a filter. Hence, the linear equivalence of the DE can distinguish the behaviour of the orbits near the equilibrium points. For instance, as $t \to \infty$, the orbits in the three Jordan Canonical forms approach the origin. However, we can study the long time behaviour of the DE by eliminating more features.

Definitions

1. A *homomorphism* is a non-linear map $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that: h is bijective and continuous, and h^{-1} is continuous. Using a homomorphic map on $\mathbb{R}^2$, the orbits of one of the flows can be mapped onto the orbits of the simplest flow.

2. If there exists a homomorphism h on $\mathbb{R}^n$ and a positive constant k such that $h\left(\mathrm{e}^{tA}x\right) = \mathrm{e}^{ktB}h(x)$ for all $x \in \mathbb{R}^n$ and for all $t \in \mathbb{R}$, then two linear flows e^{tA} and e^{tB} on $\mathbb{R}^n$ are *topologically equivalent*.

The flow is called *hyperbolic* if the real part of the eigenvalues are all non zero (i.e., $\Re e(\lambda_i) \neq 0, \quad i = 1, 2$). In fact, any hyperbolic linear flow in $\mathbb{R}^2$ is topologically equivalent to the linear flow e^{tA}, where A is one of the following matrices:

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} (sink), A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (source), A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{(saddle)}.$$

5.3 Linear Stability

It is significant to conclude whether a physical system that is disturbed from an equilibrium state remains close to, or approaches, the equilibrium points as $t \to \infty$.

Definitions

1. If, for all neighbourhoods U of 0, there exists a neighbourhood V of 0 such that $\mathrm{g}^t V \subseteq U$ for all $t \geq 0$, where ($\mathrm{g}^t = \mathrm{e}^{tA}$) is the flow of the DE, then the equilibrium point of the DE is called *stable*.

2. If the equilibrium point is stable, and if for all $x \in V$, $\lim_{t\to\infty} \|g^t x\| = 0$, then the equilibrium point is called *asymptotically stable*.

Note that, for $A \in M_n(\mathbb{R})$

$$\lim_{t\to\infty} e^{tA} a = 0 \qquad \text{for all } a \in \mathbb{R}^n \tag{5.12}$$

if and only if $\Re e(\lambda) < 0$ *for all eigenvalues of* A. Thus, solutions $x(t)$ of the DE approach the equilibrium points $(0,0)$ in the long term behaviour of the dynamical system. This implies that $(0,0)$ is a sink in $\mathbb{R}^n$. Conversely, if we replace A by $-A$ and t by $-t$, we obtain that $\Re e(\lambda) > 0$ for all eigenvalues. In this case, the equilibrium point $(0,0)$ is called a *source* in $\mathbb{R}^n$.

5.4 Non-Linear Differential Equations

The essential purpose of a DE is to explain the qualitative properties of a non-linear flow, without having the exact form of the flow since it is difficult to write it down explicitly. As in the linear flow, we consider the DE, $x' = f(x)$, where f is a continuously differentiable map. This DE has a unique maximal solution which satisfies $\psi_a(0) = a$. The *flow* of the DE is defined by the one-parameter family of maps $\{g^t\}_{t\in\mathbb{R}}$ such that $g^t : \mathbb{R}^n \to \mathbb{R}^n$ and $g^t a = \psi_a(t)$, for all $a \in \mathbb{R}^n$. The flow $\{g^t\}$ is defined in terms of the solution function $\psi_a(t)$ of the DE by

$$g^t a = \psi_a(t). \tag{5.13}$$

Here, the *orbit* through a is denoted $\gamma(a)$, and defined as

$$\gamma(a) = \left\{ x \in \mathbb{R}^n \middle| x = g^t a, \text{ for all } t \in \mathbb{R} \right\}. \tag{5.14}$$

Furthermore, orbits for a non-linear flow can be classified into three types, *point orbits*, *periodic orbits*, and *non-periodic orbits*, as in a linear flow. We sometimes work with the *positive orbit through* a, denoted by $\gamma^+(a)$, and defined as

$$\gamma^+(a) = \left\{ x \in \mathbb{R}^n \middle| x = g^t a, \text{ for all } t \geq 0 \right\}, \tag{5.15}$$

which defines a semigroup.

5.5 Linearization and the Hartman-Grobman Theorem

In dynamical systems theory, the Hartman-Grobman theorem is very important since it describes the behaviour of the dynamical system near the equilibrium points. Let us state the theorem in general:

Theorem 15. *Hartman-Grobman Theorem: Let $\bar{x}$ be an equilibrium point of the DE $x' = f(x)$ in $\mathbb{R}^n$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable map. If all of the eigenvalues of the matrix $\mathrm{D}f(\bar{x})$ satisfy $\Re e(\lambda) \neq 0$, then there is a homomorphism $h : U \to \bar{U}$ of a neighbourhood U of $0 \in \mathbb{R}$ onto a neighbourhood $\bar{U}$ of $\bar{x}$ which maps orbits of the linear flow $\mathrm{e}^{t\mathrm{D}f(\bar{x})}$ onto orbits of the non-linear flow g^t of the DE, preserving the parameter t .*

In short, the Hartman-Gorbman theorem states that the flow of the DE $x' = f(x)$ and the flow of its linearization $u' = \mathrm{D}f(\bar{x})u$ are *locally* topologically equivalent if $\bar{x}$ is a *hyperbolic equilibrium point*.
Furthermore, given $\bar{x}$ as an equilibrium point, if the real parts of the eigenvalues of the matrix $\mathrm{D}f(\bar{x})$ are all non-zero and not all of one sign, then the equilibrium point $\bar{x}$ of a DE (5.1) in $\mathbb{R}^n$ is called a *saddle point*. Conversely, if the real parts of the eigenvalues of the matrix $\mathrm{D}f(\bar{x})$ are all non-zero and all of one sign, then the equilibrium point is a sink if all $\Re e(\lambda) < 0$, and a source if all $\Re e(\lambda) > 0$.

Chapter 6

Application of Dynamical Systems in Cosmology

6.1 Introduction

In cosmology, the Universe on the largest scales, in which galaxies are taken to be the constituents, is studied dynamically. Since observations indicate that galaxies are distributed fairly uniformly, it is usually assumed that cosmological models are spatially homogeneous, and that the governing evolution equations (the Einstein field equations of General Relativity) are thus ordinary differential equations. Therefore, the qualitative features of such models can be studied using dynamical systems theory.

Several models of early universe cosmology, including the Einstein-Aether theory [4, 5] and the IR limit of Horava gravity [6, 7], violate Lorentz invariance. Lorentz invariant theories are those where the physical laws are measured to be the same for all observers that are moving uniformly with respect to each other. Einstein-Aether theory has a preferred rest frame, and when these theories are studied, it is assumed that the rest frame coincides with the Hubble expansion of the universe and the CMB which are the same, (however, see [8]).

Einstein-Aether theory combines general relativity with the aether, a dynamic unit time-like vector field. When the aether vector field is hyper surface orthogonal, the Einstein-Aether solution is a Horava solution. The reason for that is that in Horava gravity the aether vector is assumed to be hyper surface-orthogonal. In Einstein-Aether theory, the local structure consists of the aether vector field u_a and the metric tensor g_{ab}. The impact of inflation on the Lorentz violation can describe much of the physics of the early universe in conventional cosmology [9, 10]. In this chapter, we examine the late time behaviour of the dynamics of Einstein-Aether cosmological models, focussing on the influence of Lorentz violation on inflationary behaviour. In particular, we are concerned with the inflationary behaviour in the scalar vector tensor theory in which the vector field is time like and of unit length. Analysis predicts that in aether theory the Lorentz violation vector can cause inflation even without the scalar field potential, altering the dynamics of the chaotic inflationary model [11–13].

6.1.1 Einstein-Aether Cosmology

Cosmological models in aether theories of gravity are becoming increasingly popular. In [4, 14–18], an Einstein-Aether gravity theory is developed with a Lorentz-violating dynamic field that preserves locality and covariance with additional aether vector field. The aether vector field in an isotropic and homogeneous Friedmann universe with the expansion scale factor $a(t)$ and the proper time t will coincide with the cosmic frame and the expansion rate of the universe. We generalize the Einstein equations by including an additional stress tensor for the aether field. If the universe contains a self-interaction potential V, which is dependent on a self interacting scalar field ϕ, together with the expansion rate $\theta = \frac{3\dot{a}}{a} = 3H$, the modified stress tensor [4, 5] is given by

$$T_{ab} = \nabla_a\phi\nabla_b\phi - (\frac{1}{2}\nabla_c\phi\nabla^c\phi - V + \theta V_\theta)g_{ab} + \dot{V}_\theta(u_a u_b - g_{ab}). \tag{6.1}$$

This corresponds to an effective fluid, which is given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + V - \theta V_\theta \quad \text{and} \quad p = \frac{1}{2}\dot{\phi}^2 - V + \theta V_\theta + \dot{V}_\theta,$$

where ρ is pressure, p is density, V and θ are defined as before. The energy-momentum conservation law or Klein-Gordon equation is

$$\ddot{\phi} + \theta\dot{\phi} + V_\phi = 0, \tag{6.2}$$

the augmented Friedmann equation is given by

$$\frac{1}{3}\theta^2 = \rho + \frac{1}{2}\dot{\phi}^2 + V - \theta V_\theta - \frac{k}{a^2}, \tag{6.3}$$

where k is the curvature parameter, and the Friedmann matrices are defined as follows:

$$ds^2 = dt^2 - a^2(t)\{\frac{dr^2}{1-kr^2} + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2\}. \tag{6.4}$$

When we differentiate the Friedmann equation, we get the Raychaudhuri equation. Since the fluid in the cosmological frame is aligned with the aether, for a perfect matter fluid with density ρ and pressure p we assume $p = p(t)$ and $p = p(\rho) = p(t)$. Therefore, the energy momentum conservation law is

$$\dot{\rho} + \theta(\rho + p) = 0. \tag{6.5}$$

6.1.2 Exponential Potentials

Exponential potentials $V_0 e^{-\lambda\phi}$ occur in high dimensional frameworks, Kaluza-Klein theories, and super gravity [19–23]. Although the exponential potential of the scalar field in GR does not have exponential inflation [9, 10], it can have a power law inflation if the potential is not too steep. In order to have assisted inflation [24–28], we restrict the steep potentials by using multiple fields. A late time attractor is a scaling solution for exponential potentials with sufficiently flat potentials [29–33]. In the case that the parameter satisfies the ($\lambda^2 < 2$), the dynamical system with positive exponential potential is inflationary. The classical solution for the scale factor can be written as a power law, $a \; \alpha \; t^n$, with $n = \frac{2}{k^2}$. The dynamical system with negative exponential leads to rich physics, such as that which is found in Ekpyrotic behaviour [34, 35].

6.2 The Model

We look for a general scale invariant solution of (6.2)-(6.3) in which

$$V(\theta, \phi) = V_0 \exp[-\lambda\phi] + \sum_{r=0}^{n} a_r \theta^r \exp[(r-2)\lambda\phi/2], \tag{6.6}$$

where V_0, λ and $\{a_r\}$ are constants. Note that the series could be extended to negative r if required. This choice of potential subsumes the simple cases in [12, 13]. In particular, we shall study the case

$$V(\theta, \phi) = V_0 e^{-\lambda\phi} + a_1 \sqrt{V_0}\theta e^{-\frac{1}{2}\lambda\phi} + a_2\theta^2, \tag{6.7}$$

where, for convenience, we have renormalized the constant a_1 (the constant a_2 can be absorbed [4, 5] and will not play an essential role in the dynamical analysis). The constants V_0, a_1 and a_2 are expected to be positive, or at least the potential $V(\theta, \phi)$ can be assumed to be positive definite. However, as noted above, negative constants are permitted. If a_1 and a_2 are small, the potential can be thought of as a perturbation of the standard exponential potential. For very large constants a_1 and a_2, we can study non-perturbation generalizations. Note that for the positive definite $(e^{-\frac{1}{2}\lambda\phi} - \frac{1}{2}a_1\theta)^2$, a_1 is negative and $a_2 = {a_1}^2/4$.

For the potential (7.8), the augmented Friedmann equation (6.3) becomes

$$1 = \frac{3}{2(1+3a_2)}\left(\frac{\dot{\phi}}{\theta}\right)^2 + \frac{3V_0}{1+3a_2}\frac{e^{-\lambda\phi}}{\theta^2} - \frac{3k}{(1+3a_2)}\frac{1}{a^2\theta^2}, \tag{6.8}$$

where we have normalized the equation with a factor proportional to the square of the expansion. The normalized Friedmann equation suggests a suitable set of expansion normalized variables:

$$\Psi; = \sqrt{\frac{3}{2(1+3a_2)}}\frac{\dot{\phi}}{\theta}, \quad \Phi; = \sqrt{\frac{3V_0}{(1+3a_2)}}\frac{e^{-\lambda\phi/2}}{\theta}, \quad K; = \frac{3k}{(1+3a_2)}\frac{1}{a^2\theta^2},$$

assuming that V_0 is a positive constant and that $a_2 > -1/3$. In terms of these variables, the Friedmann equation assumes the simple form

$$1 = \Psi^2 + \Phi^2 - K. \tag{6.9}$$

The Raychaudhuri equation (expressed in terms of the deceleration parameter q) becomes

$$q; = -3(\frac{\dot{\theta}}{\theta^2} + \frac{1}{3}) = 2\Psi^2 - \Phi^2 - \frac{3\lambda a_1}{2\sqrt{2}}\Psi\Phi.$$

Using the Raychaudhuri equation, one can express the Klein-Gordon equation as a first order ordinary differential equation completely in terms of the expansion normalized variables, and an expansion normalized time $\frac{d\tau}{dt} = 3\theta^{-1}$:

$$\frac{d\Psi}{d\tau} = -(2 - 2\Psi^2 + \Phi^2)\Psi + \bar{\lambda}\Phi^2 + \bar{a}(1-\Psi^2)\Phi, \tag{6.10}$$

where the constants $\bar{a}$ and $\bar{\lambda}$ are defined by $\bar{a} = \frac{3\lambda a_1}{2\sqrt{2}}, \quad \bar{\lambda} = \lambda\sqrt{\frac{3(1+3a_2)}{2}}$.

The evolution equation for Φ is directly given from the definition of Φ and the Klein-Gordon and Raychaudhuri equations:

$$\frac{d\Phi}{d\tau} = (1 + 2\Psi^2 - \Phi^2 - \bar{a}\Psi\Phi - \bar{\lambda}\Psi)\Phi. \tag{6.11}$$

The equations ((6.10)) and ((6.11)) constitute an autonomous system of first order differential equations. $\Phi = 0$ is an invariant set, and the physical region corresponds to $\Phi \geq 0$. The curvature K is determined from the Friedmann equation (6.9), and the condition $K = 0$ defines an invariant set of (6.10) and (6.11) (and is preserved by the evolution equations $dK/d\tau = [4\Psi^2 - 2\Phi^2 - 2\bar{a}\Psi\Phi]K$ and a partition of the two-dimensional state space into a bounded negative curvature region ($\Psi^2 + \Phi^2 < 1$), and an unbounded positive curvature region ($\Psi^2 + \Phi^2 > 1$). Note that a_2 is absent from the equations. It corresponds to a term in the potential proportional to the square of the expansion and can be absorbed by a rescaling of the expansion scalar, but the expansion normalized system is invariant under such rescaling, which is why a_2 does not appear in such systems.

Defining $|\bar{A}| = \sqrt{(9 - \bar{\lambda}^2 + \bar{a}^2)}$, the equilibrium points of the system p_i are given by

Equilibria			$K,\ q$
$p_1:$	$\Psi = 0,$	$\Phi = 0,$	$-1,\ 0$
$p_2:$	$\Psi = \pm 1,$	$\Phi = 0,$	$0,\ 2$
$p_3:$	$\Psi = \dfrac{3\bar{\lambda} + \lvert\bar{a}\bar{A}\rvert}{9 + \bar{a}^2},$	$\Phi = \dfrac{-\bar{\lambda}\bar{a}^2 + 3\lvert\bar{a}\bar{A}\rvert}{(9 + \bar{a}^2)\bar{a}},$	$0,\ \dfrac{\bar{\lambda}\bar{a}\lvert\bar{A}\rvert - 9 + 3\bar{\lambda}^2 - \bar{a}^2}{(9 + \bar{a}^2)}$
$p_4:$	$\Psi = \dfrac{3\bar{\lambda} - \lvert\bar{a}\bar{A}\rvert}{9 + \bar{a}^2},$	$\Phi = -\dfrac{\bar{\lambda}\bar{a}^2 + 3\lvert\bar{a}\bar{A}\rvert}{(9 + \bar{a}^2)\bar{a}},$	$0,\ -\dfrac{\bar{\lambda}\bar{a}\lvert\bar{A}\rvert - 9 + 3\bar{\lambda}^2 - \bar{a}^2}{(9 + \bar{a}^2)}$
$p_5:$	$\Psi = \dfrac{1}{\bar{\lambda}},$	$\Phi = \dfrac{-\bar{a} + \sqrt{\bar{a}^2 + 8}}{2\bar{\lambda}},$	$-\dfrac{2\bar{\lambda}^2 - 6 - \bar{a}^2 + \bar{a}\sqrt{\bar{a}^2 + 8}}{2\bar{\lambda}^2},\ 0$
$p_6:$	$\Psi = \dfrac{1}{\bar{\lambda}},$	$\Phi = \dfrac{-\bar{a} - \sqrt{\bar{a}^2 + 8}}{2\bar{\lambda}},$	$\dfrac{2\bar{\lambda}^2 - 6 - \bar{a}^2 + \bar{a}\sqrt{\bar{a}^2 + 8}}{2\bar{\lambda}^2},\ 0$

The stationary solution p_6 has $\Phi < 0$, $K > 0$ for all values of $\bar{a} \in \mathbb{R}$, $\bar{\lambda} \in \mathbb{R}^+$, and corresponds to a contracting universe with positive spatial curvature. We shall in the following only consider expanding solutions with vanishing or negative curvature and will therefore ignore this point since it lies outside the region of interest $K \leq 0,\ \Phi \geq 0$. Points p_1 and p_2 always satisfy these conditions. p_1 is always a saddle, and p_2 is either a source (+) or a sink (−). We are most interested in the equilibrium points p_3, p_4, and p_5. The points p_3, p_4, and p_5 are contained in this region only for a restricted, partially overlapping, range of values in $(\bar{\lambda},\ \bar{a})$-space.

6.2.1 p_3

Range of validity:

$$\Phi_{p_3} \geq 0 \text{ when } (\bar{\lambda} \leq 3,\ \bar{a} \geq 0),\ \text{or } (\bar{\lambda} \geq 3,\ \bar{a} \leq -\sqrt{\bar{\lambda}^2 - 9}), \tag{6.12}$$

Eigenvalues:

$$\lambda_1 = \frac{-5|\bar{A}|^2 + 4\bar{\lambda}^2 + 3\bar{\lambda}|\bar{a}\bar{A}| + \sqrt{\bar{B} + \bar{C}}}{2(9 + \bar{a}^2)}$$
$$\lambda_2 = \frac{-5|\bar{A}|^2 + 4\bar{\lambda}^2 + 3\bar{\lambda}|\bar{a}\bar{A}| - \sqrt{\bar{B} + \bar{C}}}{2(9 + \bar{a}^2)}$$

where

$$\bar{B} = \bar{a}^4 + 18\bar{a}^2 + 15\bar{a}^2\bar{\lambda}^2 + 81 + 54\bar{\lambda}^2 - \bar{\lambda}^4\bar{a}^2 + \bar{\lambda}^2\bar{a}^4 + 9\bar{\lambda}^4, \quad \bar{C} = 2\bar{\lambda}(3\bar{\lambda}^2 + \bar{a}^2 + 9)|\bar{a}\bar{A}|.$$

Discussion: The point p_3 is a sink and inflationary when $0 < \bar{a}$ and $\bar{\lambda}^2 < \frac{1}{2}(\bar{a}^2 + 6 - \bar{a}\sqrt{\bar{a}^2 + 8})$. To analyse the eigenvalues of p_3, we can make the transformations

$$\bar{a} = \frac{(A+6)\Lambda}{2\sqrt{A^2 - \Lambda^2}}, \quad \tilde{\lambda} = \frac{\Lambda}{2}, \tag{6.13}$$

to define the new parameters A and Λ ($A^2 > \Lambda^2$), whence the equilibrium point p_3 is defined by

$$\tilde{\Phi} = -\frac{1}{A}\sqrt{A^2 - \Lambda^2}\ (A < 0), \Psi = -\frac{\Lambda}{A},$$

and the eigenvalues become $\left\{(1 - 3B)B^{-1}, 2(1 - B)B^{-1}\right\}$, where $B = \frac{2}{\lambda^2}\frac{(1 - \frac{1}{4}a_1\lambda^2)}{(1 + 3a_2)}$, which is a sink for $B < 1$ and corresponds to the zero-curvature inflationary solution P with $\theta = 3Bt^{-1}$, $\phi = \phi_0 + \frac{2}{\lambda}\ln t$ (cf. [18,33]). Unfortunately, since the transformations (6.13) are not one-to-one, the analysis of the other equilibria is not so simple [e.g., the ranges of applicability for the equilibrium points p_4 and p_5 (i.e., ranges of values of A and Λ for the points to be physical, such as $\Phi \geq 0$) are very complicated, and there can be 2 representations of the same equilibrium points].

6.2.2 p_4

Range of validity:

$$\Phi_{p_4} \geq 0 \text{ when } (\bar{a} \leq 0,\ \bar{a}^2 \geq \bar{\lambda}^2 - 9). \tag{6.14}$$

Eigenvalues:

$$\lambda_1 = \frac{-5|\bar{A}|^2 + 4\bar{\lambda}^2 + 3\bar{\lambda}|\bar{a}\bar{A}| + \sqrt{\bar{B} - \bar{C}}}{2(9 + \bar{a}^2)}$$
$$\lambda_2 = \frac{-5|\bar{A}|^2 + 4\bar{\lambda}^2 + 3\bar{\lambda}|\bar{a}\bar{A}| - \sqrt{\bar{B} - \bar{C}}}{2(9 + \bar{a}^2)}.$$

Discussion: The point p_4 is a sink when $\bar{a} < 0$ and $\bar{\lambda}^2 < \frac{1}{2}(\bar{a}^2 + 6 - \bar{a}\sqrt{\bar{a}^2 + 8})$. It is inflationary only for the subset of this region where $\bar{\lambda}^2 < \frac{1}{2}(\bar{a}^2 + 6 + \bar{a}\sqrt{\bar{a}^2 + 8})$.

6.2.3 p_5

Range of validity:

$$\Phi_{p_5} \geq 0 \text{ always, but } K_{p_5} \leq 0 \text{ only when } 2\bar{\lambda}^2 \geq 6 + \bar{a}^2 - \bar{a}\sqrt{\bar{a}^2 + 8}. \tag{6.15}$$

Eigenvalues:

$$\lambda_1 = -1 + \tfrac{1}{2\bar{\lambda}}\sqrt{48 + 22\bar{a}^2 - 12\bar{\lambda}^2 + 2\bar{a}^4 - 2\bar{a}^2\bar{\lambda}^2 - 2\bar{a}(7 + \bar{a}^2 - \bar{\lambda}^2)\sqrt{\bar{a}^2 + 8}},$$
$$\lambda_2 = -1 - \tfrac{1}{2\bar{\lambda}}\sqrt{48 + 22\bar{a}^2 - 12\bar{\lambda}^2 + 2\bar{a}^4 - 2\bar{a}^2\bar{\lambda}^2 - 2\bar{a}(7 + \bar{a}^2 - \bar{\lambda}^2)\sqrt{\bar{a}^2 + 8}}.$$

Discussion: The point p_5 is a sink when $\bar{\lambda}^2 > \frac{1}{2}(\bar{a}^2 + 6 - \bar{a}\sqrt{\bar{a}^2 + 8})$.

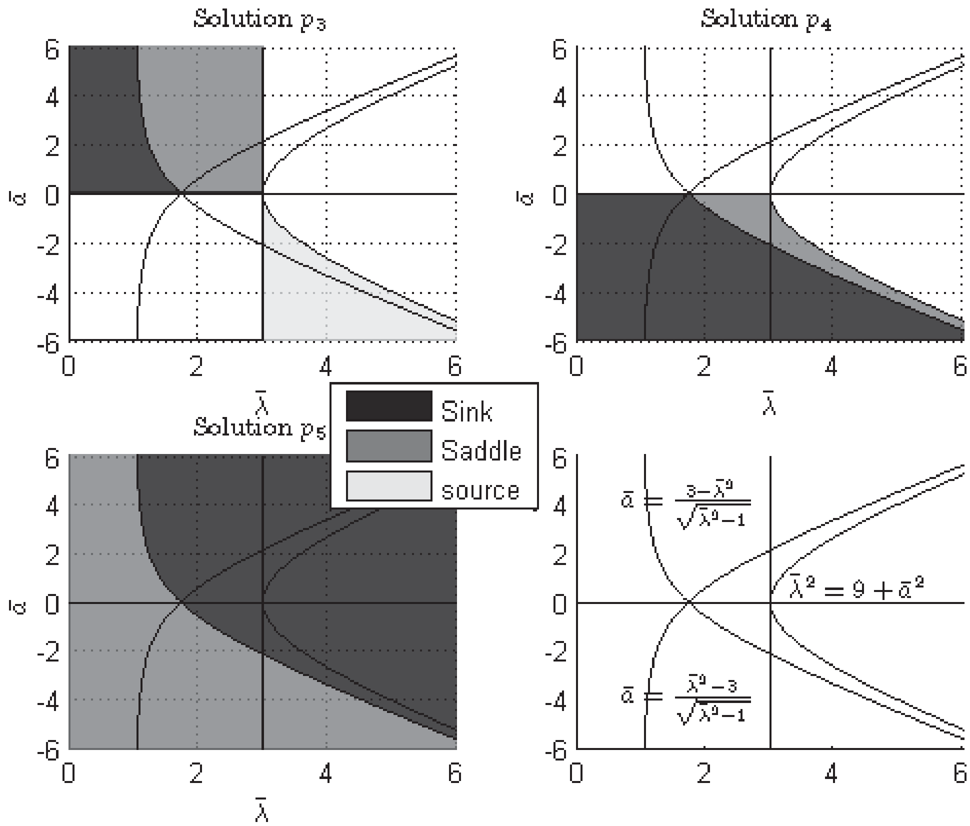


Figure 6.1: Bifurcation diagrams for the three points p_3, p_4, and p_5.

6.3 The Model with Matter:

We investigate the dynamical properties of a class of spatially homogeneous and isotropic cosmological models containing a barotropic perfect fluid and a scalar field with an exponential potential in Einstein-Aether theory. With matter, the augmented Friedmann equation becomes

$$1 = \Omega + \frac{3}{2(1+3a_2)}\left(\frac{\dot{\phi}}{\theta}\right)^2 + \frac{3V_0}{1+3a_2}\frac{e^{-\lambda\phi}}{\theta^2} - \frac{3k}{(1+3a_2)}\frac{1}{a^2\theta^2}, \tag{6.16}$$

where

$$\Omega := \frac{3\rho}{(1+3a_2)\theta^2}.$$

In terms of these variables the Friedmann equation assumes the simple form

$$1 = \Omega + \Psi^2 + \Phi^2 - K, \tag{6.17}$$

the Raychaudhuri equation becomes

$$q := -3(\frac{\dot{\theta}}{\theta^2} + \frac{1}{3}) = \frac{1}{2}(3\gamma - 2)\Omega + 2\Psi^2 - \Phi^2 - \bar{a}\Psi\Phi, \tag{6.18}$$

the evolution equations become

$$\frac{d\Psi}{d\tau} = -(2 - 2\Psi^2 + \Phi^2 - \frac{1}{2}(3\gamma - 2)\Omega)\Psi + \bar{\lambda}\Phi^2 + \bar{a}(1 - \Psi^2)\Phi \tag{6.19}$$

and

$$\frac{d\Phi}{d\tau} = (1 + 2\Psi^2 - \Phi^2 - \bar{a}\Psi\Phi - \bar{\lambda}\Psi + \frac{1}{2}(3\gamma - 2)\Omega)\Phi, \tag{6.20}$$

and the matter conservation equation becomes

$$\frac{d\Omega}{d\tau} = (-(3\gamma - 2)(1 - \Omega) + 4\Psi^2 - 2\Phi^2 - 2\bar{a}\Psi\Phi)\Omega. \tag{6.21}$$

Note that $\Omega = 0$ defines an invariant set of the three-dimensional autonomous system of first order differential equations. There are equilibrium points $\bar{p_i}$ with $\Omega = 0$. The additional eigenvalues are:

$$\begin{aligned}
&\bar{p_1}: && -(3\gamma - 2)\\
&\bar{p_2}(\pm): && -3(\gamma - 2)\\
&\bar{p_3}: && \frac{-3\gamma\bar{a}^2 - 27\gamma + 6\bar{\lambda}^2 + 2\bar{\lambda}\sqrt{\bar{a}^2(\bar{a}^2 + 9 - \bar{\lambda}^2)}}{9 + \bar{a}^2}\\
&\bar{p_4}: && -\frac{3\gamma\bar{a}^2 + 27\gamma - 6\bar{\lambda}^2 + 2\bar{\lambda}\sqrt{\bar{a}^2(\bar{a}^2 + 9 - \bar{\lambda}^2)}}{9 + \bar{a}^2}\\
&\bar{p_5}: && -(3\gamma - 2)\\
&\bar{p_6}: && -(3\gamma - 2).
\end{aligned}$$

6.3.1 $\bar{p_3}$

Range of validity:

$$\Phi_{\bar{p_3}} \geq 0 \text{ when } (\bar{\lambda} \leq 3,\ \bar{a} \geq 0),\ \text{ or } (\bar{\lambda} \geq 3,\ \bar{a} \leq -\sqrt{\bar{\lambda}^2 - 9}). \tag{6.22}$$

The point $\bar{p_3}$ is a sink and inflationary when $\bar{a} > 0$ and $\bar{\lambda}^2 < \frac{1}{2}(\bar{a}^2 + 6 - \bar{a}\sqrt{\bar{a}^2 + 8})$. In addition, this point remains a sink when

$$2\bar{\lambda}^2 < \gamma\bar{a}^2 + 9\gamma - \frac{2}{3}\bar{\lambda}|\bar{a}\bar{A}|. \tag{6.23}$$

For example, for the special case when $\gamma = 1, \bar{\lambda} = 1$, and for any value of $\bar{a}$, the equation (6.23) becomes

$$5\bar{a}^4 + 94\bar{a}^2 + 441 > 0, \tag{6.24}$$

which is always true for any value of $\bar{a}$, and thus the conditions on $\bar{a}$ in this case are the same as in the matter-free case.

6.3.2 $\bar{p_4}$

Range of validity:

$$\Phi_{\bar{p_4}} \geq 0 \text{ when } (\bar{a} \leq 0,\ \bar{a}^2 \geq \bar{\lambda}^2 - 9). \tag{6.25}$$

The point $\bar{p_4}$ is a sink and inflationary when $\bar{a} < 0$ and when

$$\bar{\lambda}^2 < \frac{1}{2}(\bar{a}^2 + 6 \pm \bar{a}\sqrt{\bar{a}^2 + 8}), \tag{6.26}$$

which implies that

$$2\bar{\lambda}^2 < \bar{a}^2 + 9, \tag{6.27}$$

since $\gamma \geq 1$ and $\frac{2}{3}\bar{\lambda}|\bar{a}\bar{A}|$ is positive. Thus, we can add them to the right side of the equation (6.27). Finally, we obtain

$$\bar{\lambda}^2 < \frac{1}{2}(\gamma\bar{a}^2 + 9\gamma + \frac{2}{3}\bar{\lambda}|\bar{a}\bar{A}|),$$

which is true for all values of λ and $\bar{a}$, and the third eigenvalue is always negative.

6.3.3 $\bar{p_5}$

Range of validity:

$$\Phi_{\bar{p_5}} \geq 0 \text{ always, but } K_{\bar{p_5}} \leq 0 \text{ only when } 2\bar{\lambda}^2 \geq 6 + \bar{a}^2 - \bar{a}\sqrt{\bar{a}^2 + 8}. \tag{6.28}$$

The point $\bar{p_5}$ is a sink when $\bar{\lambda}^2 > \frac{1}{2}(\bar{a}^2 + 6 - \bar{a}\sqrt{\bar{a}^2 + 8})$. Then, since all eigenvalues are always negative for $\gamma > \frac{2}{3}$, $\bar{p_3}$ remains a sink.

6.3.4 Equilibrium Points with $\Omega \neq 0$

Moreover, there are additional equilibrium points with $\Omega \neq 0$

$$p_7 : \Psi = 0, \Phi = 0, \Omega = 1;$$

the eigenvalues for p_7 are:

$$\frac{3}{2}\gamma, 3\gamma - 2, \frac{3}{2}\gamma - 3, \tag{6.29}$$

and hence p_7 is a saddle.

There is a scaling solution, corresponding to the flat FRW solution, now represented by the equilibrium point M and

$$\Phi = \frac{1}{2\bar{\lambda}}(-\bar{a}+\sqrt{\bar{a}^2 + 9\gamma(2-\gamma)}), \Psi = \frac{3\gamma}{2\bar{\lambda}}, \Omega = -\frac{1}{2\bar{\lambda}^2}(9\gamma+\bar{a}^2-\bar{a}\sqrt{\bar{a}^2 + 9\gamma(2-\gamma)})+1. \tag{6.30}$$

In addition, M has zero curvature and

$$q_M = -\frac{1}{6}(3\gamma - 2).$$

The equilibrium point M

In order to study the stability of this equilibrium point which has zero curvature, we will first consider the sub-case when K=0, where the Friemann equation can be written as

$$\Omega = 1 - \Phi^2 - \Psi^2.$$

Therefore, we will have a two dimensional dynamical systems in terms of the two variables Φ and Ψ:

$$\frac{d\Psi}{d\tau} = -(2 - 2\Psi^2 + \Phi^2 - \frac{1}{2}(3\gamma - 2)(1 - \Phi^2 - \Psi^2))\Psi + \bar{\lambda}\Phi^2 + \bar{a}(1 - \Psi^2)\Phi \tag{6.31}$$

and

$$\frac{d\Phi}{d\tau} = (1 + 2\Psi^2 - \Phi^2 - \bar{a}\Psi\Phi - \bar{\lambda}\Psi + \frac{1}{2}(3\gamma - 2)(1 - \Phi^2 - \Psi^2))\Phi. \tag{6.32}$$

The scaling solution corresponds to the equilibrium point, M, but just given in terms of Φ and Ψ:

$$\Phi = \frac{1}{2\bar{\lambda}}(-\bar{a} + \sqrt{\bar{a}^2 + 9\gamma(2-\gamma)}), \Psi = \frac{3\gamma}{2\bar{\lambda}}. \tag{6.33}$$

The characteristic equation for point (6.33) is given by

$$-\mu^2+\frac{3}{2}(\gamma-2)\mu+\frac{1}{4\bar{\lambda}^2}(162\gamma^2+\bar{a}\sqrt{\bar{a}^2+9\gamma(2-\gamma)}(-27\gamma+2\bar{\lambda}^2-2\bar{a}^2+9\gamma^2)$$
$$+45\gamma\bar{a}^2-18\gamma^2\bar{a}^2-81\gamma^3+2\bar{a}^4-2\bar{a}^2\bar{\lambda}^2-36\gamma\bar{\lambda}^2+18\gamma^2\bar{\lambda}^2).$$

The linearization of the system (6.31)-(6.32) about the equilibrium point (6.33) thus yields the two eigenvalues are given by

$$\mu_1=\frac{3\gamma\bar{\lambda}-6\bar{\lambda}+\sqrt{\bar{D}+\bar{a}\bar{E}}}{4\bar{\lambda}}, \tag{6.34}$$

and

$$\mu_2=\frac{3\gamma\bar{\lambda}-6\bar{\lambda}-\sqrt{\bar{D}+\bar{a}\bar{E}}}{4\bar{\lambda}}, \tag{6.35}$$

where $\bar{D}=81\gamma^2\bar{\lambda}^2-108\gamma\bar{\lambda}^2+36\bar{\lambda}^2-324\gamma^3-72\gamma^2\bar{a}^2+648\gamma^2+180\gamma\bar{a}^2-8\bar{a}^2\bar{\lambda}^2+8\bar{a}^4$

$$\bar{E}=\sqrt{\bar{a}^2+18\gamma-9\gamma^2}(36\gamma^2-108\gamma+8\bar{\lambda}^2-8\bar{a}^2).$$

These eigenvalues have negative real parts for a range of the values of the parameters γ, and $\bar{\lambda}$ and for small value of $\bar{a}$ (similar to the standard model). For example, when $\gamma=1$ and $\bar{\lambda}=\sqrt{6}$, the eigenvalues are given by

$$-\frac{3}{4}\pm\sqrt{-54+\sqrt{\bar{a}^2+9}(-8\bar{a}(\bar{a}^2+3))+4\bar{a}^2(2\bar{a}^2+15)}.$$

The term in the square root is negative for $\bar{a}\gtrsim-\frac{1}{2}$ (so that the real part of the eigenvalue is negative; see Figure 6.3). In Figure 6.2, we plot the sign of the eigenvalue with the positive square root, so that there is a small range $-\frac{3}{4}\lesssim\bar{a}\lesssim-\frac{1}{2}$ such that both eigenvalues are real and negative (the graph ends when the eigenvalue becomes imaginary).

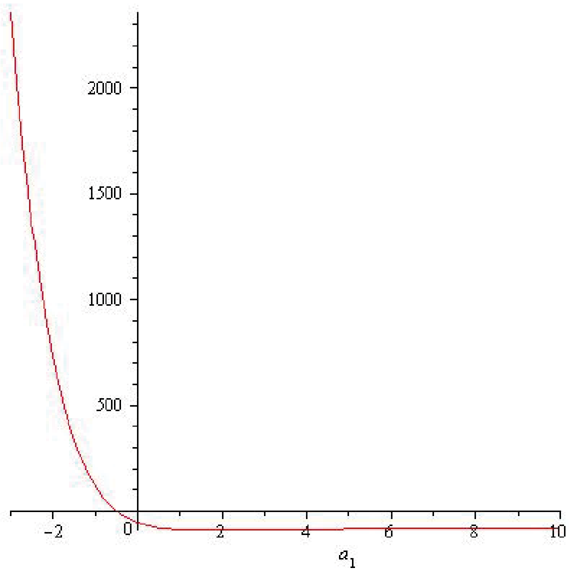


Figure 6.2: The value under the root, where $a_1 \equiv \bar{a}$.

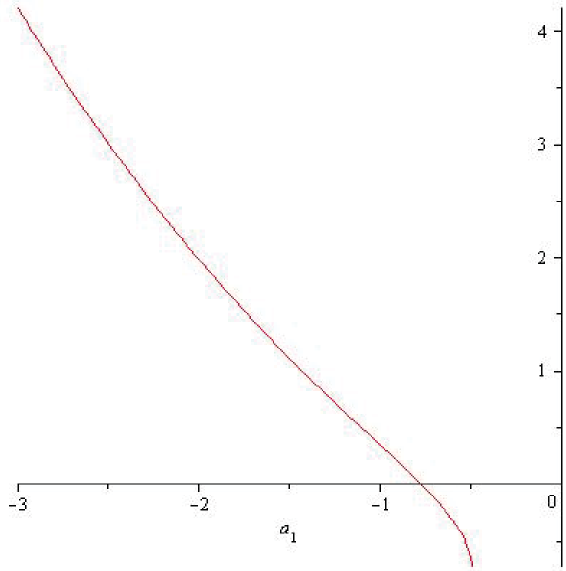


Figure 6.3: For the eigenvalues, where $a_1 \equiv \bar{a}$.

The scaling solution in 3D corresponds to the equilibrium point M. The linearization of the 3D system (6.31)-(6.21) about the equilibrium point M yields the characteristic equation

$$\mu^3 - \frac{1}{2}(10-9\gamma)\mu^2 - \frac{1}{4\bar{\lambda}^2}((27\gamma + 2\bar{a}^2 - 2\bar{\lambda}^2 - 9\gamma^2)(\bar{a}\sqrt{\bar{a}^2 + 18\gamma - 9\gamma^2}) + 2\bar{a}^2\bar{\lambda}^2 + 24\lambda^2$$

$$-45\gamma\bar{a}^2 - 12\gamma\bar{\lambda}^2 + 18\bar{a}^2\gamma^2 - 162\gamma^2 + 81\gamma^3 - 2\bar{a}^4)\mu - \frac{1}{4\bar{\lambda}^2}((\bar{a}\sqrt{\bar{a}^2 + 18\gamma - 9\gamma^2})$$

$$(-99\gamma^2 + 54\gamma - 4\bar{\lambda}^2 + 6\gamma\bar{\lambda}^2 + 4\bar{a}^2 + 27\gamma^3 - 6\gamma\bar{a}^2) - 324\gamma^2 + 72\gamma\bar{\lambda}^2 - 144\gamma^2\bar{\lambda}^2$$

$$+648\gamma^3 - 90\gamma\bar{a}^2 + 171\gamma^2\bar{a}^2 + 4\bar{a}^2\bar{\lambda}^2 - 4\bar{a}^4 + 54\gamma^3\bar{\lambda}^2 - 6\gamma\bar{a}^2\bar{\lambda}^2 - 54\gamma^3\bar{a}^2 + 6\gamma\bar{a}^4 - 243\gamma^4).$$

The linearization of system (6.31)-(6.21) about the equilibrium point (6.30) yields three eigenvalues. Two of the eigenvalues are the same as in the 2D case as given in (6.34)-(6.35) above and the third eigenvalue has the value $3\gamma - 2$. Hence, the scaling solution is only stable for $\gamma < \frac{2}{3}$. For $\gamma > \frac{2}{3}$, the equilibrium point M is a saddle with a two dimensional stable manifold and a one dimensional unstable manifold.

Chapter 7

Anisotropic Einstein-Aether Models

In this chapter we study a class of spatially anisotropic cosmological models in Einstein-Aether theory, which includes the Bianchi type I anisotropic models [11, 12]. In an anisotropic Einstein-Aether model there will be additional terms in the FE:

1: The effects on the geometry from the anisotropy (and curvature) of the actual 1-parameter subclass of Bianchi VI_h spatially homogeneous models considered.

2: The energy momentum tensor of the scalar field, due to the possible dependence of the self interaction potential V on the Lorentz violating vector field [18]; $V = V(\phi, \theta, \sigma)$, where $\theta = 3H = 3\dot{a}/a$ is the expansion rate and σ is the shear scalar, defined by $\sigma^2 \equiv \frac{1}{2}\sigma^{ab}\sigma_{ab}$. The modified stress tensor T^{ϕ}_{ab} can be written in terms of an effective fluid with density ρ_ϕ and pressure p_ϕ.

3: The Einstein FE are generalised by the contribution of an additional stress tensor, S_{ab}, for the aether field which depends on the dimensionless parameters of the aether model (e.g., "the c_i"). This has the effect of renormalizing some of the parameters in the model (e.g., the gravitational constant G, where we choose units in which $8\pi G = 1$; effectively we set $c_1 + 3c_2 + c_3 = 0$ so that the remaining parameters in the model can be characterized by the constant c^2) [12].

4: In anisotropic models, there may be a tilt between the preferred direction of the aether and that of the anisotropy (in an isotropic and spatially homogeneous Friedmann universe the aether field is aligned with the cosmic frame). This adds additional terms to the aether stress tensor S_{ab}, which can be characterized by a hyperbolic tilt angle, $\alpha(t)$, measuring the boost of the aether relative to the rest frame of the homogeneous spatial sections [11, 12]. However, it can be shown that the tilt decays ($\alpha \to 0$) to the future as is expected [12], and hence we neglect it here.

7.1 The Model

In this section we are interested in the qualitative features of cosmological models in Einstein-Aether theory in the presence of curvature and shear. Following [36], we consider a 1-parameter ($m \equiv h-1$) class of anisotropic cosmological models, which includes Bianchi types III ($m=0$), V ($m=1$), VI$_0$ ($m=-1$), and VI$_h$ (all other m). The expansion scalar and the shear scalar are given in [36]. In the case under consideration here, there is no rotation and no acceleration. In this model, $-\frac{3}{2}P = \frac{3}{a^2}N$, where $N \equiv m^2+m+1 \geq 3/4 > 0$, and $M \equiv \frac{1-m}{\sqrt{m^2+m+1}}$. The constants M and N are not independent but are related (via m); formally, we can recover the Bianchi type I case by taking $M=N=0$.

Let us now consider the forms of ρ_ϕ and p_ϕ in the specific class of geometries under consideration here. The Einstein-Aether action [4,5], which is a generalization of the Einstein-Hilbert action, also depends on the aether field (the time-like vector field u_a). In particular, the matter fields can depend on the aether field. Taking variations of the self interaction potential $V=V(\phi,\theta,\sigma)$ with respect to the metric and the aether field u_a [4,5] we obtain (for the models under consideration) the effective density ρ_ϕ and pressure p_ϕ of the form:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V - V_\theta(\theta+\sqrt{6}\sigma) - V_\sigma(\sigma - \frac{1}{\sqrt{6}}\theta), \tag{7.1}$$

$$\begin{aligned} p_\phi &= \frac{1}{2}\dot{\phi}^2 - V + V_\theta(\theta+\sqrt{6}\sigma+\sqrt{6}\frac{\dot{\sigma}}{\theta}) + V_\sigma(\sigma - \frac{1}{\sqrt{6}}\theta - \frac{1}{\sqrt{6}}\frac{\dot{\theta}}{\theta}) \\ &\quad + \dot{V}_\theta(1+\sqrt{6}\frac{\sigma}{\theta}) + \dot{V}_\sigma(\frac{\sigma}{\theta} - \frac{1}{\sqrt{6}}). \end{aligned} \tag{7.2}$$

The evolution equations follow from the field equations derived from the Einstein-Aether action [4,5]; the energy-momentum conservation law, the generalized Friedmann equation, the Raychaudhuri equation and the evolution equation for the shear. Therefore, assuming the forms of the potential discussed earlier, we obtain the following system of equations governing the Einstein-Aether cosmological models under consideration:

$$\theta^2 = 3c^2\sigma^2 + 3\rho_\phi - \frac{3}{2}P, \tag{7.3}$$

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2c^2\sigma^2 - \frac{1}{2}(\rho_\phi + 3p_\phi), \tag{7.4}$$

$$c^2\dot{\sigma} = -c^2\sigma\theta + \frac{M}{3\sqrt{3}}\,(\theta^2 - 3c^2\sigma^2 - 3\rho_\phi), \tag{7.5}$$

$$\ddot{\phi} + \theta\dot{\phi} + V_\phi = 0. \tag{7.6}$$

We now introduce new expansion-normalized variables and a new time variable:

$$\begin{aligned} \beta &= \sqrt{3}\frac{c\sigma}{\theta}, & \frac{dt}{d\tau} &= \frac{3}{\theta}, \\ \Psi &= \frac{\sqrt{3}}{\sqrt{2}}\frac{\dot{\phi}}{\theta}, & \Phi &= \sqrt{3V_0}\frac{\mathrm{e}^{-\lambda\phi/2}}{\theta}. \end{aligned} \tag{7.7}$$

For a given potential $V(\theta, \phi, \sigma)$, we can find $3\rho_\phi/\theta^2$ and $3p_\phi/\theta^2$ in terms of β, Ψ and Φ, and determine the system of differential (evolution) equations in terms of the expansion-normalized variables (see below). For convenience, we define $K \equiv \frac{3P}{2\theta^2}$, and since K is always negative the generalized Friedmann eqn. usually determines a compact region of interest (i.e., β, Ψ and Φ are bounded). Recall that the *deceleration parameter*, q, is defined by $d\theta/d\tau = -\theta(1+q)$, and inflation is defined by $q < 0$. Note that in the Bianchi I subcase we have that $K = 0$.

We are looking for a general scale invariant solution in which

$$V(\theta, \phi, \sigma) = V_0 e^{-\sqrt{6}\bar{\lambda}\phi} + \frac{a}{\sqrt{3}}\sqrt{V_0}\theta e^{-\frac{\sqrt{6}}{2}\bar{\lambda}\phi} + \sqrt{2}b\sqrt{V_0}\sigma e^{-\frac{\sqrt{6}}{2}\bar{\lambda}\phi}, \tag{7.8}$$

where we have defined $\bar{\lambda} = \frac{\lambda}{\sqrt{6}}$, and we have normalized the constants a and b appropriately.

Evolution Equations

$$\beta' = (q-2)\beta - \frac{\sqrt{2}M}{c^2}K, \tag{7.9}$$

$$\Psi' = (q-2)\Psi + \frac{3\bar{\lambda}}{2}\big(2\Phi^2 + a\Phi + b\beta\Phi\big), \tag{7.10}$$

$$\Phi' = (q+1)\Phi - 3\bar{\lambda}\Psi\Phi, \tag{7.11}$$

where

$$K = \frac{c^2}{2}\beta^2 + \Omega - 1,$$

$$\Omega = \Psi^2 + \Phi^2 - a\Phi\beta + b\Phi,$$

$$q - 2 = \frac{(a\sqrt{2}\frac{M}{c^2}\Phi - 4)K + 3(a+b)\lambda\Psi\Phi\beta + 3(a-b)\lambda\Psi\Phi + 6\Phi^2 + 3(b - a\beta)\Phi}{b\Phi - 2}.$$

Note that, $0 \leq \Omega \leq 1$ and $\frac{c^2}{2}\beta^2 \leq 1$. We also note that when $K = 0$, the equilibrium points must satisfy $\beta = 0$ or $q = -2$. In the latter case we have $\Phi = 0$ and $\Psi^2 + \frac{c^2}{2}\beta^2 = 1$, which will include the cosmological sources. When $\beta = 0$ we will find equilibrium points (with Φ and Ψ non-zero) corresponding to inflationary sinks.

7.2 2D System with Zero Curvature

By setting $K = 0$ in the equations (7.9) we obtain $K' = 0$, so that $K = 0$ is an invariant set of the system. We recall that we can obtain the Bianchi I models (with $K = 0$) when $M = N = 0$. Also, we set $c^2 = 1$. When $K = 0$, we obtain the 2D system

$$\beta' = (q-2)\beta, \tag{7.12}$$

$$\Phi' = (q+1)\Phi - 3\bar{\lambda}\Psi\Phi, \tag{7.13}$$

where

$$\Psi^2 = 1 - \frac{1}{2}\beta^2 - \Phi^2 + a\beta\Phi - b\Phi,$$
$$q - 2 = 3\frac{\Phi}{b\Phi - 2}\left[(\bar{\lambda}\Psi - 1)(a+b)\beta + (a-b)\bar{\lambda}\Psi + b(\beta+1) + 2\Phi\right].$$

7.3 Equilibrium Points

We define

$$Z = \sqrt{b^2 - 2ab\bar{\lambda}^2 + a^2\bar{\lambda}^2 + 4 - 4\bar{\lambda}^2},$$

and

$$N = N_1 + N_2,$$

$$N_1 = \sqrt{b^4\bar{\lambda}^2 + 2\bar{\lambda}^2a^2b^2 + \bar{\lambda}^2a^4 + 8\bar{\lambda}^2},$$

$$N_2 = \sqrt{-8 + 12ab\bar{\lambda}^2 - 2a^2\bar{\lambda}^2 - 8ab + 4b^2\bar{\lambda}^2 - 4b^2}.$$

The equilibrium points that are potentially of interest (i.e., possible future attractors) in the 2D system with zero curvature are denoted by l_i, given by (the associated values of Ψ are $\frac{1}{\bar\lambda}$ for $l_1, l_{4,5}$, and are given in section (7.3.1) and (7.3.2) for $l_{2,3}$):

Equilibria

$$
\begin{aligned}
l_1: &\quad \Phi = 0, \quad \beta = \pm\frac{\sqrt{2\bar\lambda^2 - 2}}{\bar\lambda} \\
l_{2,3}: &\quad \Phi = \frac{2(2\bar\lambda^2 b - b - a\bar\lambda^2 \pm Z)}{4\bar\lambda^2 b^2 - 4a\bar\lambda^2 b + a^2\bar\lambda^2 + 4}, \quad \beta = 0 \\
l_{4,5}: &\quad \Phi = -\frac{-\bar\lambda b^3 - ba^2\bar\lambda - 2a\bar\lambda \pm b\ N}{2\bar\lambda(2 + 2ab + b^2)}, \quad \beta = \frac{\bar\lambda b^2 - \bar\lambda ab - \bar\lambda a^2 \mp N}{\bar\lambda(2 + 2ab + b^2)}
\end{aligned}
$$

We cannot hope to study the stability of these equilibrium points in all generality. However, we consider some values of $\bar\lambda$ such that $\frac{1}{5} \leq \bar\lambda \leq \frac{1}{2}$. First, if $a = b = 0$ we will have the same equilibrium point p as in the single scalar model with matter. In two other cases, we will study the stability for l_i if $a = 0$ and $b = 0$, separately.

For the point l_1, which is only valid for $\bar\lambda > 1$, we have that $q = 2$, and hence l_1 is always non-inflationary. l_1 always has one zero eigenvalue. In general, this point will be a source, and we will not consider it further here.

7.3.1 Point l_2

The value of Ψ for l_2 is given by

$$\Psi = \frac{\bar\lambda(-ab + 2b^2 + 4 + a\ Z - b\ Z)}{4\bar\lambda^2 b^2 - 4a\bar\lambda^2 b + a^2\bar\lambda^2 + 4}.$$

We study the stability in two cases for the values of a, b between -3 and 3. The first case is when $a = 0$; the eigenvalues μ_i, where $i = 1, 2$ corresponds to $+, -$, are given by

$$\mu_i = \frac{-3(m_2\ m_1(m_3 + m_4)) \pm \sqrt{2}\ m_1\ m_5 + \sqrt{2}\bar\lambda^3 b^2\ m_6)}{m_2(m_7 \pm m_1\ m_8)},$$

(where $\sqrt{2} = +\sqrt{2}$, etc) where

$$m_1 = \sqrt{b^2 - 4\bar\lambda^2 + 4},$$

$$m_2 = \sqrt{\frac{-\bar\lambda^2(-4b^2 + 2\bar\lambda^2 b^2 + (b^3 + 2)b\ m_1 - 2 - b^4)}{(\bar\lambda^2 b^2 + 1)^2}},$$

$$m_3 = 4b\bar{\lambda}^2 + 4b^3\bar{\lambda}^4 + b^5\bar{\lambda}^4 - b,$$

$$m_4 = 4 - b^6\bar{\lambda}^4 + b^2 - 4\bar{\lambda}^2 + 4b^4\bar{\lambda}^6 2\bar{\lambda}^2 b^6 + 2b^2\bar{\lambda}^2,$$

$$m_5 = b^5\bar{\lambda}^5 + 3b^3\bar{\lambda}^5 - 3b^3\bar{\lambda}^3 - b\bar{\lambda}^3 - b^3\bar{\lambda}^5,$$

$$m_6 = 5 + b^4 - b^4\bar{\lambda}^2 + 5b^2 + 2b^2\bar{\lambda}^4 - 6\bar{\lambda}^2 - 7b^2\bar{\lambda}^2,$$

$$m_7 = 10b^2\bar{\lambda}^2 + b^6\bar{\lambda}^4 + 2b^4\bar{\lambda}^2 + 2b^6\bar{\lambda}^6 + 8b^4\bar{\lambda}^4 + 4 + b^2,$$

$$m_8 = b^5\bar{\lambda}^4 + 2b^3\bar{\lambda}^2 + b.$$

We plot the two eigenvalues μ_1, μ_2, where the horizontal axes are the values of the parameters $(b, \bar{\lambda})$ (where Λ in the figures below is equal to $\bar{\lambda}$), and the vertical axes are the values of μ_1, μ_2, and we obtain

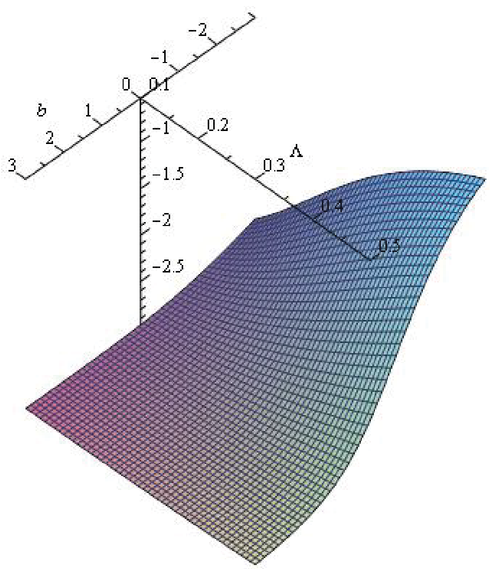


Figure 7.1: $a = 0$; μ_1-eigenvalue of l_2

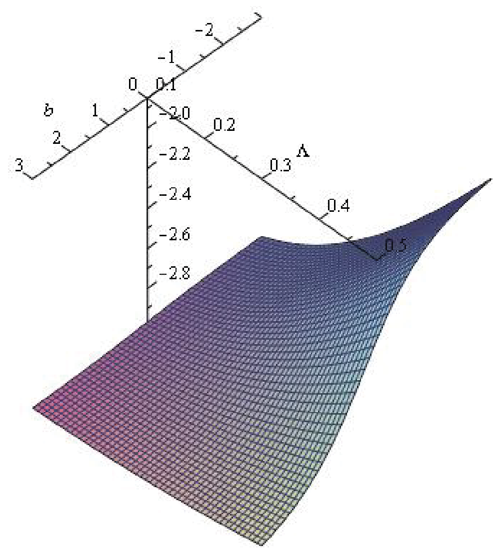


Figure 7.2: $a = 0$; μ_2-eigenvalue of l_2

For illustration, we also evaluate the two eigenvalues when $\bar{\lambda} = \frac{1}{2}$, obtaining

$$\mu_i = \frac{3(m_9(m_{10}(-16b^3 - 4b^5) - 192 + 2b^6 - 96b^2 - 4b^4) \pm m_{10}\, m_{11})}{2\, m_9(\pm m_{10}(-2b^5 - 16b^3 - 32b) + m_{12})},$$

and

$$m_9 = \sqrt{\frac{7b^2 - 2b\, m_{10}(b^2 + 2) + 4 + 2b^4}{(b^2 + 4)^2}},$$

$$m_{10} = \sqrt{b^2+3},$$

$$m_{11} = m_{10}(6b^5+18b^3+8b) - 28b^2 - 6b^6 - 27b^4,$$

$$m_{12} = 112b^2 + 3b^6 + 32b^4 + 128.$$

We plot the two eigenvalues μ_1, μ_2 when $\bar{\lambda} = \frac{1}{2}$ (where the horizontal axes are the values of $-3 \leq b \leq 3$ and the vertical axes are the value of the eigenvalues μ_1, μ_2), and we obtain

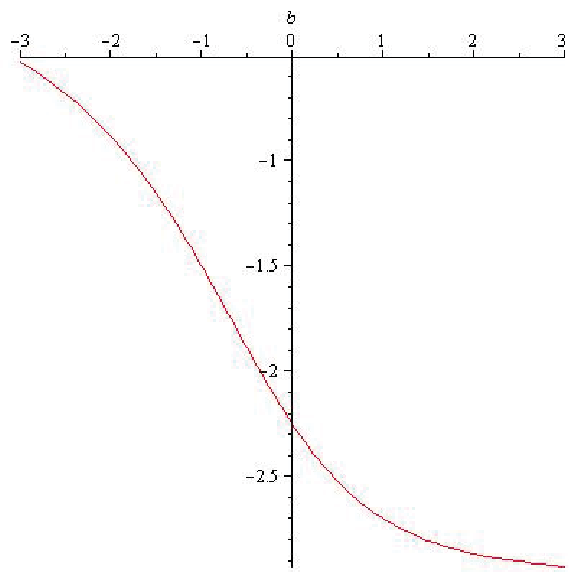


Figure 7.3: $\bar{\lambda} = \frac{1}{2}$, $a = 0$; μ_1- plot for l_2

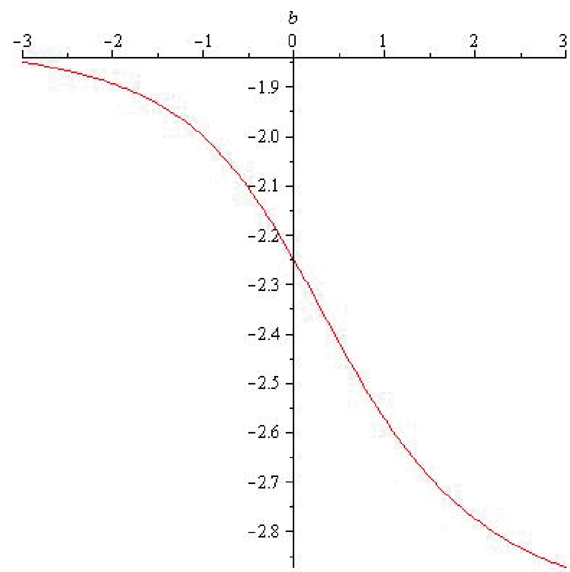


Figure 7.4: $\bar{\lambda} = \frac{1}{2}$, $a = 0$; μ_2-plot for l_2

Discussion As can be seen from the pictures above, for the parameter $-3 \leq b \leq 3$, we have two negative eigenvalues which implies that l_2 is a sink.

The second case is when $b = 0$; the eigenvalues are given by

$$\mu_i = \frac{3(m_{13}\ m_{15} \pm m_{14}\ m_{16} + m_{17})}{m_{14}(48a^2\bar{\lambda}^2 + 12a^4\bar{\lambda}^4 + a^6\bar{\lambda}^6 + 64)},$$

where

$$m_{13} = \sqrt{4 - 4\bar{\lambda}^2 + \bar{\lambda}^2 a^2},$$

$$m_{14} = \sqrt{\frac{\bar{\lambda}^2(4a^2 + a^4\bar{\lambda}^2 - 4a^2\bar{\lambda}^2 + 8a\ m_{13} + 16}{(4+a^2\bar{\lambda}^2)^2}},$$

$$m_{15} = -2a\bar{\lambda}^3\ m_{13}(16 + 4a^2 + a^2\bar{\lambda}^2 - 12\bar{\lambda}^2),$$

$$m_{16} = -64 - 32a^2\bar{\lambda}^2 - 4a^4\bar{\lambda}^4 + 64\bar{\lambda}^2 + 8a\bar{\lambda}^2\ m_{13}(a^2\bar{\lambda}^2 + 4) - 4a^4\bar{\lambda}^6,$$

$$m_{17} = -32a^2\bar{\lambda}^3 - 4a^4\bar{\lambda}^5 a^6\bar{\lambda}^7 + 48a^2\bar{\lambda}^5 - 4a^4\bar{\lambda}^7.$$

We plot the two eigenvalues μ_1, μ_2, where the horizontal axes are the values of the parameters $(a, \bar{\lambda})$ (where Λ in the figures below is equal to $\bar{\lambda}$), and the vertical axes are the values of μ_1, μ_2, and we obtain

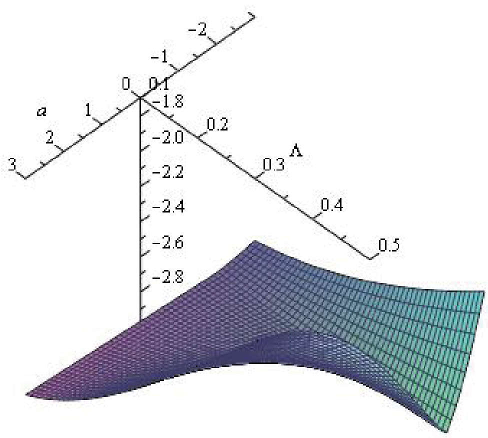


Figure 7.5: $b = 0$; μ_1-eigenvalues of l_2

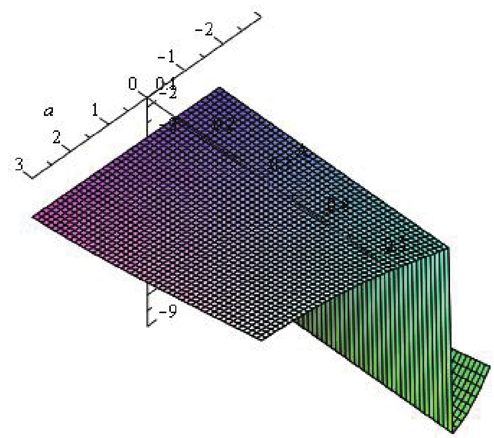


Figure 7.6: $b = 0$; μ_2-eigenvalues of l_2

For illustration, we also evaluate the two eigenvalues when $\bar{\lambda} = \frac{1}{2}$, obtaining

$$\mu_i = \frac{3(\pm m_{19}\ m_{21} + m_{20})}{2\ m_{19}(768a^2 + 48a^4 + a^6 + 4096)},$$

where

$$m_{18} = \sqrt{12 + a^2},$$

$$m_{19} = \sqrt{\frac{12a^2 + a^4 + 16a\ m_{18} + 64}{(16 + a^2)^2}},$$

$$m_{20} = -2a(64 + 4a^2 + a^4)m_{18} - 320a^2 - 20a^4 + a^6,$$

$$m_{21} = -6144 - 1024a^2 - 40a^4 + 32a^3\ m_{18} + 512a\ m_{18}.$$

We plot the two eigenvalues μ_1, μ_2 when $\bar{\lambda} = \frac{1}{2}$ (where the horizontal axes are the values of $-3 \leq a \leq 3$ and the vertical axes are the values of the eigenvalues μ_1, μ_2), and we obtain

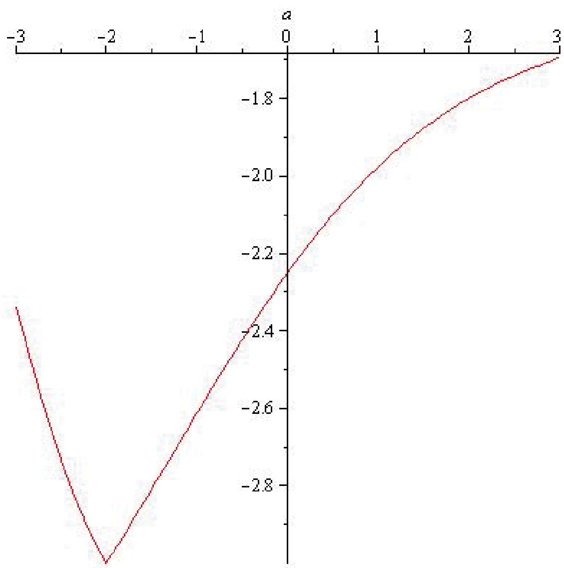


Figure 7.7: $\bar{\lambda} = \frac{1}{2}$, $b = 0$; μ_1- plot of l_2

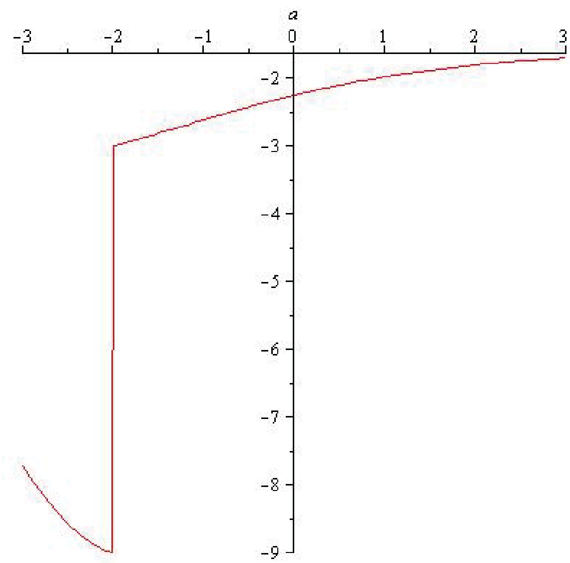


Figure 7.8: $\bar{\lambda} = \frac{1}{2}$, $b = 0$; μ_2-plot of l_2

Discussion As can be seen from the pictures above, we again have a sink.

Remark: Note that we always have a sink for the values $-3 \leq a \leq 3$ and $\frac{1}{5} \leq \bar{\lambda} \leq \frac{1}{2}$.

The deceleration parameter for l_2: In this case, the deceleration parameter can be written in the form

$$q = \frac{-4(4+b^2) + \bar{\lambda}^2(G + Z\ C) + Z\ b}{(X)(S - b\ Z)},$$

where

$$G_1 = -20ab + 12a^2\bar{\lambda}^2 + 36b^2 - a^2b^2 - 2ab^3 - 16ab^3\bar{\lambda}^2,$$

$$G_2 = 9\bar{\lambda}^2a^2b^2 + \bar{\lambda}^2a^3b - 8a^2 + 8b^4 + 48 + 4\bar{\lambda}^2b^4 - \bar{\lambda}^2a^4,$$

$$G = G_1 + G_2,$$

$$C = -8b^3 - 12b^3\bar{\lambda}^2 + 3\bar{\lambda}^2a^3 - 36b + 12a + ba62 + 2ab^2 - 15ba^2\bar{\lambda}^2 - 24\bar{\lambda}^2ab^2,$$

$$S = b^2 + 2b^2\bar{\lambda}^2 - 3ab\bar{\lambda}^2 + 4 + a^2\bar{\lambda}^2,$$

and

$$X = 4\bar{\lambda}^2 b^2 - 4a\bar{\lambda}^2 b + a^2\bar{\lambda}^2 + 4,$$

and we note that Z is defined earlier. We consider the inflationary behaviour in the same two cases as in the stability analysis.

(1) When $a = 0$ and $\bar{\lambda} = \frac{1}{2}$, then q is given by

$$q_{2a0} = \frac{-16 + 9b^4 + 20b^2 - \sqrt{b^2+3}(11b^3 + 20b)}{2(b^2+4)(3b^2 + 8 - 2b\sqrt{b^2+3})}.$$

(2) When $b = 0$ and $\bar{\lambda} = \frac{1}{2}$, then q is given by

$$q_{2b0} = \frac{-2a^2 + 3a\sqrt{12 + a^2} - 8}{2(16 + a^2)}.$$

We plot the values of q, where the horizontal axis is the value of a or b and the vertical axis is the value of q, and we obtain

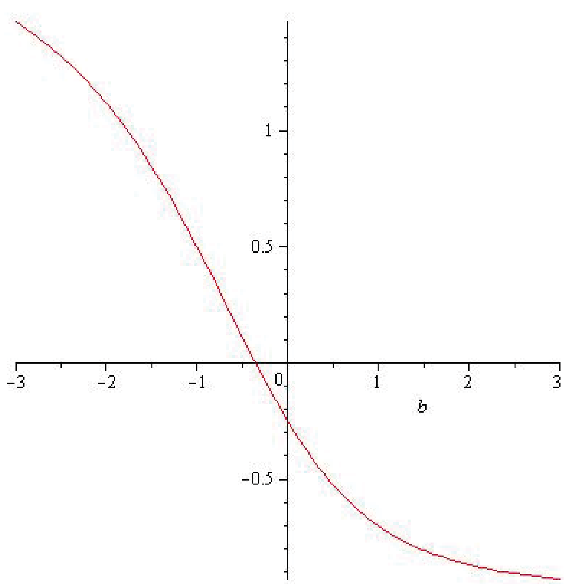


Figure 7.9: $\bar{\lambda} = \frac{1}{2}$, $a = 0$; q for l_2

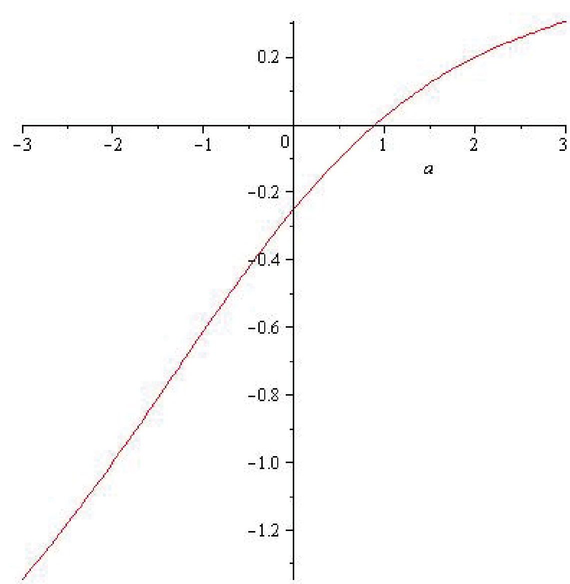


Figure 7.10: $\bar{\lambda} = \frac{1}{2}$, $b = 0$; q for l_2

Discussion For $\bar{\lambda} = \frac{1}{2}$, and the case where $a = 0$, if $b \gtrsim -\frac{1}{2}$, then q is negative; otherwise q is positive.
For the second case where $b = 0$, if $a \geq 1$, then q is positive; otherwise q is negative. That is, the model is inflationary (in the latter case) when $a < 1$.

7.3.2 Point l_3

The value of the Ψ for l_3 is given by

$$\Psi = \frac{\bar{\lambda}(-ab + 2b^2 + 4 - a\ Z + b\ Z)}{4\bar{\lambda}^2 b^2 - 4a\bar{\lambda}^2 b + a^2\bar{\lambda}^2 + 4}.$$

We study the stability in the two cases for the value of a, b between -3 and 3. The first case is when $a = 0$: the eigenvalues μ_i (where $i = 1, 2$ or -, +) are given by

$$\mu_i = \frac{-3(m_2(m_1(m_3 + m_4)) \mp \sqrt{2}m_1\ m_5 + \sqrt{2}\bar{\lambda}^3 b^2\ m_6)}{m_2(m_7 \mp m_1\ m_8)}.$$

We plot the two eigenvalues μ_1, μ_2, where the horizontal axes are the values of the parameters $(b, \bar{\lambda})$ (where Λ in the figures below is equal to $\bar{\lambda}$), and the vertical axes are the values of μ_1, μ_2, and we obtain

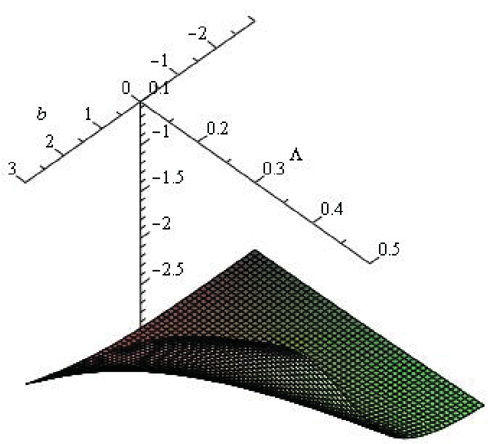


Figure 7.11: $a = 0$; μ_1-eigenvalues of l_3

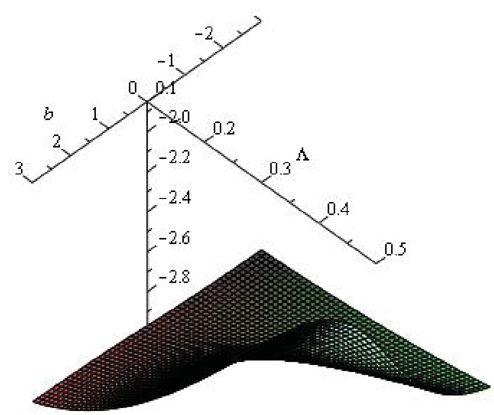


Figure 7.12: $a = 0$; μ_2-eigenvalue of l_3

For illustration, we also evaluate the two eigenvalues when $\bar{\lambda} = \frac{1}{2}$, obtaining

$$\mu_i = \frac{3(m_9(m_{10}(-16b^3 - 4b^5) - 192 + 2b^6 - 96b^2 - 4b^4) \mp m_{10}\ m_{11})}{2\ m_9(\mp m_{10}(-2b^5 - 16b^3 - 32b) + m_{12})}.$$

We plot the two eigenvalues μ_1, μ_2 $\bar{\lambda} = \frac{1}{2}$, (where the horizontal axes are the values of $-3 \leq b \leq 3$ and the vertical axes are the values of the eigenvalues μ_1, μ_2), and we obtain

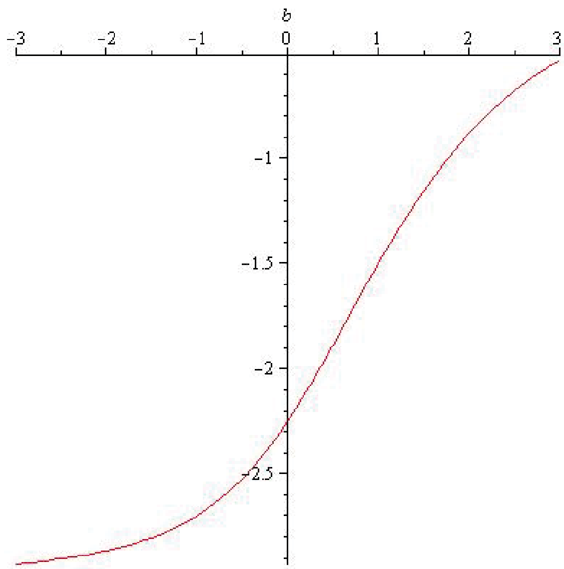


Figure 7.13: $\bar{\lambda} = \frac{1}{2}$, $a = 0$; μ_1-plot of l_3

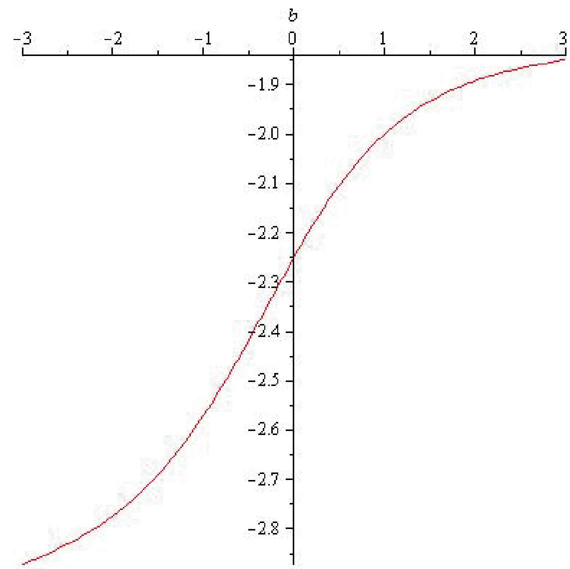


Figure 7.14: $\bar{\lambda} = \frac{1}{2}$, $a = 0$; μ_2-plot of l_3

Discussion We always have a sink.

The second case is when $b = 0$; the eigenvalues are given by

$$\mu_i = \frac{3(m_{13}\ m_{15} \mp m_{14}\ m_{16} + m_{17})}{m_{14}(48a^2\bar{\lambda}^2 + 12a^4\bar{\lambda}^4 + a^6\bar{\lambda}^6 + 64)}.$$

We plot the two eigenvalues μ_1, μ_2, where the horizontal axes are the values of the parameters $(a, \bar{\lambda})$ (where Λ in the figures below is equal to $\bar{\lambda}$), and the vertical axes are the values of μ_1, μ_2, and we obtain

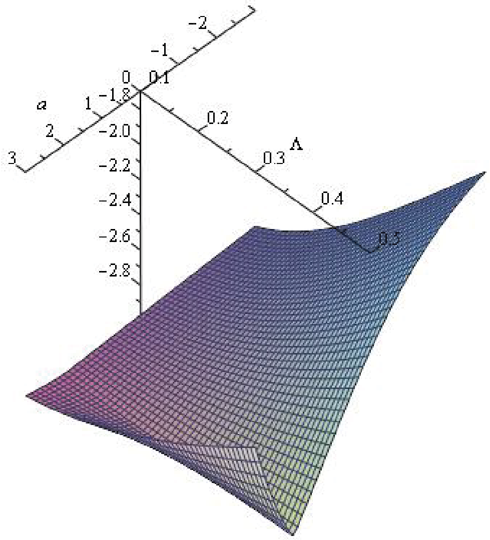


Figure 7.15: $b = 0$; μ_1-eigenvalues of l_3

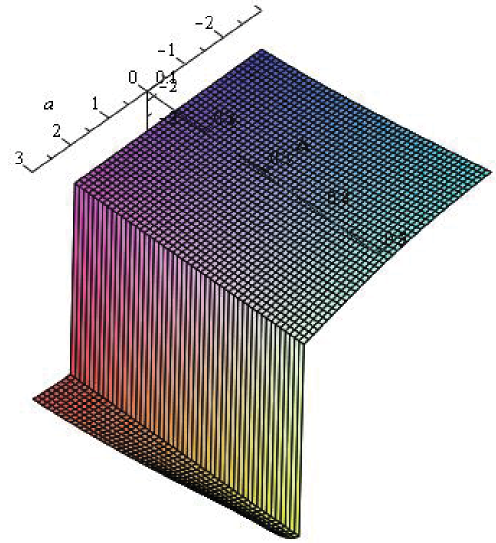


Figure 7.16: $b = 0$; μ_2-eigenvalues of l_3

For illustration, we also evaluate the two eigenvalues when $\bar{\lambda} = \frac{1}{2}$, obtaining

$$\mu_i = \frac{3(\mp m_{19}\ m_{21} + m_{20})}{2\ m_{19}(768a^2 + 48a^4 + a^6 + 4096)}.$$

We plot the two eigenvalues μ_1, μ_2 $\bar{\lambda} = \frac{1}{2}$, (where the horizontal axis are the values of $-3 \leq a \leq 3$ and the vertical axis are the values of the eigenvalues μ_1, μ_2), and we obtain

Discussion We always have a sink .

The deceleration parameter for l_3: In this case, the deceleration parameter can be written in the form

$$q = \frac{-4(4 + b^2) + \bar{\lambda}^2(G - Z\ C) - Z\ b}{(X)(S + b\ Z)}.$$

We consider the inflationary region in the same two cases as in the stability analysis.

(1) When $a = 0$ and $\bar{\lambda} = \frac{1}{2}$, then q is given by

$$q_{l3a0} = \frac{-16 + 9b^4 + 20b^2 + \sqrt{b^2 + 3}(11b^3 + 20b)}{2(b^2 + 4)(3b^2 + 8 + 2b\sqrt{b^2 + 3})}.$$

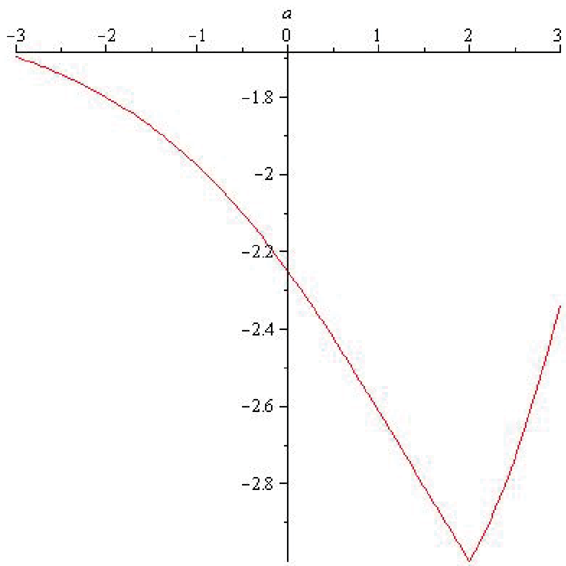


Figure 7.17: $\bar{\lambda} = \frac{1}{2}$, $b = 0$; μ_1-plot of l_3

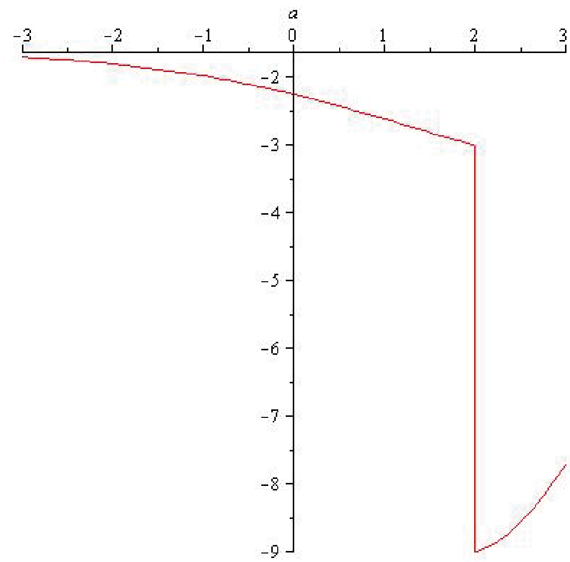


Figure 7.18: $\bar{\lambda} = \frac{1}{2}$, $b = 0$; μ_2-plot of l_3

(2) When $b = 0$ and $\bar{\lambda} = \frac{1}{2}$, then q is given by

$$q_{l3b0} = \frac{-2a^2 - 3a\sqrt{12 + a^2} - 8}{2(16 + a^2)}.$$

We plot the values of q, where the horizontal axis are the values of the either a or b and the vertical axis are the values of q, and we obtain

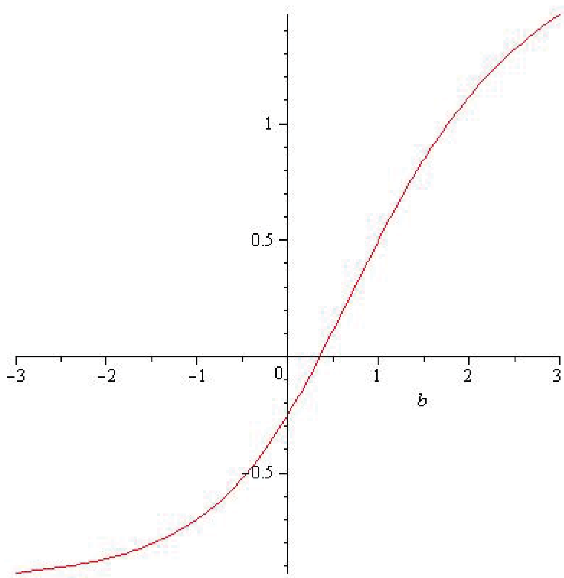


Figure 7.19: $\bar{\lambda} = \frac{1}{2}$, $a = 0$; q for l_3

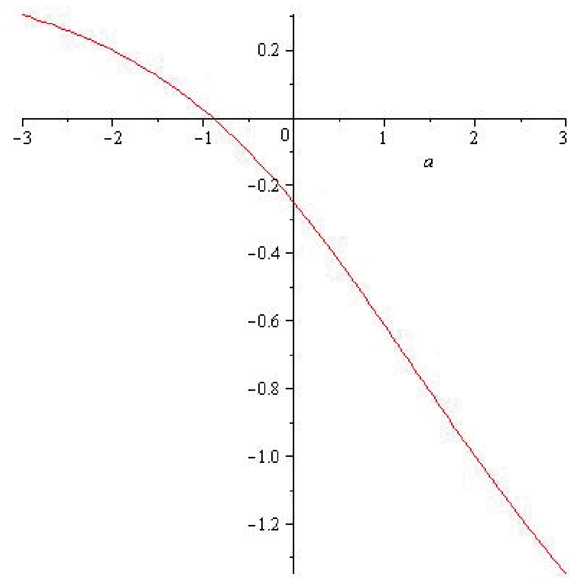


Figure 7.20: $\bar{\lambda} = \frac{1}{2}$, $b = 0$; q for l_3

Discussion For the first case when $a = 0$, for $b > \frac{1}{2}$, q is positive but if $\frac{1}{2} \gtrsim b$ it is negative. For the second case when is $b = 0$, for $a > -1$ then q is negative but if $a < -1$ it is positive. That is, the model is inflationary (in the latter case) when $a > -1$.

7.3.3 Point l_4

We define

$$J_1 = b^{11} - 7b^9 - 32b^7 - 92b^5 - 112b^3 - 32b,$$

$$J_2 = b^{13} - 13b^{11} - 20b^9 - 20b^7 - 160b^5 - 416b^3 - 384,$$

$$J = -2b^{20} - 4200b^{14} + 15b^{22} - 12576b^{12} - 34704b^{10} - b^{24} - 2b^2(1116b^{16} - 189b^{18}),$$

$$\bar{J} = +582784b^{10} + 471552b^8 + 345984b^{12} - 2b^{28} + 127584b^{14},$$

$$F_1 = \bar{J} - 894b^{22} - 124b^{24} + 42b^{26} + 28128b^{16} + 312b^{18},$$

$$J_3 = \sqrt{2b^2(-164864b^2 - 205824b^4 - 61440 - 82752b^8 - 156160b^6 + J)},$$

$$J_4 = \sqrt{-294912 - 983040b^4 - 909312b^2 - 23142b^6 + F_1},$$

$$J_5 = 18b^6 + b^{10} + 48b^4 + 48b^2 + 3b^8,$$

$$J_6 = -3b^{10} - 30b^8 + b^{12} - 80b^6 + 192b^2 + 128.$$

There are also two cases for the equilibrium point. First, when $a = 0$ and $\bar{\lambda} = \frac{1}{2}$, the eigenvalues are given by

$$\mu_i = \frac{3b(\sqrt{b^4 - 12b^2 - 24}(J_1) + J_2 \pm \sqrt{b^4 - 12b^2 - 24}\ J_3 \pm J_4)}{2(b^2+2)(\sqrt{b^4 - 12b^2 - 24}\ J_5 + J_6)}.$$

Remark From the expression above we note that

$$\mu_1(b) = +\mu_2(-b) \tag{7.14}$$

We plot the eigenvalues, where the horizontal axis is the real part of the eigenvalue and the vertical axis is the imaginary part of the eigenvalue, for different values of the parameter b (the arrow indicates the direction on the curve in which b is increasing, and the points A and B indicate where the curve crosses the vertical (imaginary) axis).

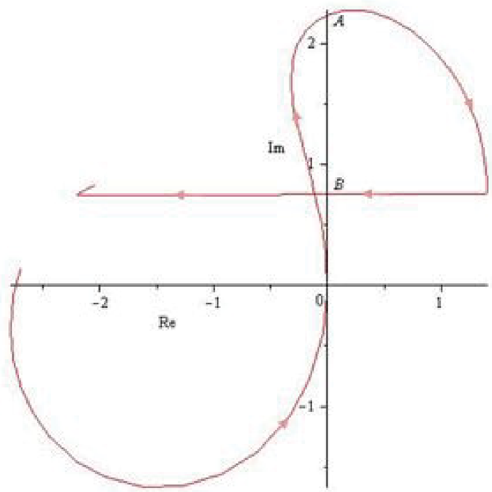


Figure 7.21: $\bar{\lambda} = \frac{1}{2}$, $a = 0$; μ_1-eigenvalue of l_4 (values of b at A and B are $b \backsimeq -2.825$,$b \backsimeq -1.25$, respectively): Plot for $-3 \leq b \leq 2$, arrows indicate the direction of increase for b

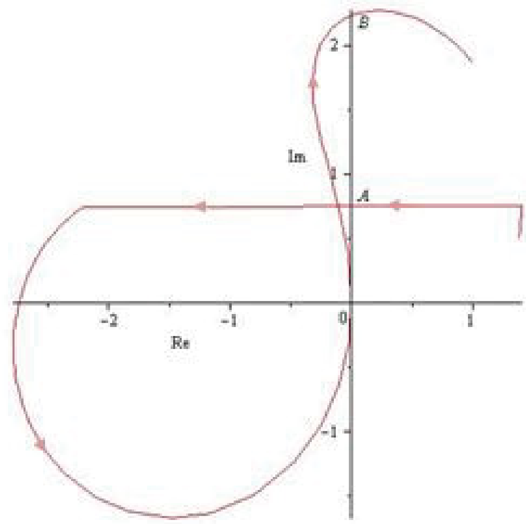


Figure 7.22: $\bar{\lambda} = \frac{1}{2}$, $a = 0$; μ_2-eigenvalue of l_4 (values of b at A and B are $b \backsimeq -2.825$, $b \backsimeq 1.25$, respectively): Plot for $-3 \leq b \leq 2$, arrows indicate the direction of increase for b

Discussion As can be seen from the figures above, the real parts of μ_1 and μ_2 are both negative when $-1.25 \lesssim b \lesssim 1.25$. Thus it is a sink in this range of values for b. But the real part of μ_1 or μ_2 changes sign at $b \simeq 1.25$ and $b \simeq -2.825$, which means l_4 is a saddle. From (7.14), there are further sign changes for μ_1 and μ_2 at $b \simeq 2.825$ and $b \simeq 1.25$, r respectively.

For comparison, let us consider $a = 0$ and $\bar{\lambda} = \frac{1}{3}$; the eigenvalues are given by

$$\mu_i = \frac{b(J_7 + \sqrt{b^4 - 32b^2 - 64}\ J_8 \pm (J_{112}) \pm \sqrt{b^4 - 32b^2 - 64}(J_{222}))}{2(b^2+2)(\sqrt{b^4 - 32b^2 - 64}\ J_{13} + J_{14})},$$

where

$$J_7 = 3b^{13} - 114b^{11} + 300b^9 + 1800b^7 + 720b^5 - 3168b^3 - 3072b,$$

$$J_8 = 3b^{11} - 66b^9 - 276b^7 - 696b^5 - 816b^3 - 96b,$$

$$J_9 = \sqrt{-8388608 + 28672000b^4 - 20512768b^2 + 103450624b^6 + 55072256b^{10}},$$

$$J_{10} = \sqrt{-8b^{28} + 1812096b^{14} + 48664b^{22} - 12216b^{24} + 578b^{26} + 664192b^{16}},$$

$$J_{111} = \sqrt{+1078087680b^{8} + 14094336b^{12} + 765088b^{18} + 413184b^{20}},$$

$$J_{112} = J_{111} + J_9 + J_{11},$$

$$J_{11} = \sqrt{-4259840b^{2} + 6047744b^{10} - 4298752b^{6} - 8927232b^{4}},$$

$$J_{12} = \sqrt{450b^{24} - 6296b^{22} + 3822208b^{12} + 118912b^{18} - 552b^{20}},$$

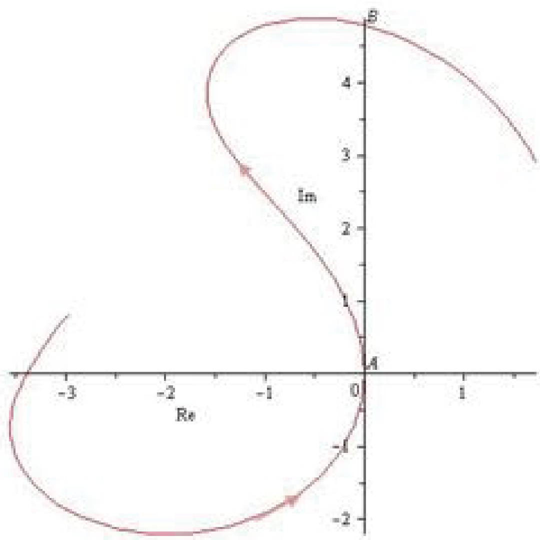
$$J_{221} = \sqrt{+3909120b^{8} - 8b^{26} + 495520b^{16} + 156782b^{14}},$$

$$J_{222} = J_{11} + J_{12} + J_{221},$$

$$J_{13} = b^{10} - 2b^{8} + 8b^{6} + 48b^{4} + 48b^{2},$$

$$J_{14} = b^{12} - 18b^{10} - 120b^{8} - 320b^{6} - 240b^{4} + 192b^{2} + 128.$$

We plot the eigenvalues above, where the horizontal axis is the real part of the eigenvalue and the vertical axis is the imaginary part of the eigenvalue, for different values of the parameter b (the arrow indicates the direction on the curve in which b is increasing, and the points A and B indicate where the curve crosses the vertical (imaginary) axis).



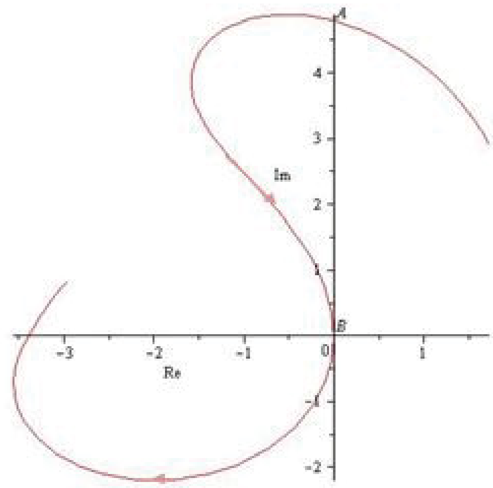


Figure 7.23: $\bar{\lambda} = \frac{1}{3}$, $a = 0$; μ_1-eigenvalue of l_4 (values of b at A and B are $b \backsimeq -1.75$ and $b \backsimeq 1.75$, respectively): Plot for $-3 \leq b \leq 3$, arrows indicate the direction of increase for b

Figure 7.24: $\bar{\lambda} = \frac{1}{3}$, $a = 0$; μ_2-eigenvalue of l_4 (values of b at A and B, are $b \backsimeq -1.75$, $b \backsimeq 1.75$, respectively): Plot for $-3 \leq b \leq 3$, arrows indicate the direction of increase for b

Discussion As can be seen from the figures above, for $-1.75 \lesssim b \lesssim 1.75$, the real parts of μ_1 and μ_2 are both negative, which implies that l_4 is a sink. But the real part of μ_1 and μ_2 change signs at $b \simeq -1.75$ and $b \simeq 1.75$, and one of the eigenvalues is positive and the other is negative, which means that l_4 is a saddle.

Lastly, when $a = 0$ and $\bar{\lambda} = \frac{1}{4}$ the eigenvalues are given by

$$\mu_i = \frac{3b(J_{15} + \sqrt{b^4 - 60b^2 - 120}\ J_{16} \pm (J_{17} + J_{18} + \sqrt{b^4 - 60b^2 - 120}(J_{212})))}{4(b^2+2)(\sqrt{b^4 - 60b^2 - 120}\ J_{21} + J_{22})},$$

where

$$J_{15} = 2b^{13} - 146b^{11} + 208b^9 + 5624b^7 + 4288b^5 - 3904b^3 - 3840b,$$

$$J_{16} = 2b^{11} - 86b^9 - 352b^7 - 856b^5 - 992b^3 - 64b,$$

$$J_{17} = J_{77} + J_{88},$$

$$J_{77} = \sqrt{1170076 16b^{10} + 424995 84b^8 - 1310390 40b^{12}},$$

$$J_{88} = \sqrt{-7372800 + 95907840b^4 - 1550720b^2 + 171548672b^6},$$

$$J_{18} = J_{99} + J_{44},$$

$$J_{99} = \sqrt{-2b^{28}70217856b^{14} + 191016b^{22} - 13288b^{24}},$$

$$J_{44} = \sqrt{+288b^{26} - 25224576b^{16} - 50976 48b^{18} + 236940b^{20}},$$

$$J_{19} = \sqrt{186634 24b^8 + 120883 20b^{10} - 3932160b^2 + 9007104b^6},$$

$$J_{20} = \sqrt{398112b^{16} + 52656b620 - 7468b^{22} + 228b^{24} + 427584b^{14}},$$

$$J_{212} = \sqrt{-5152768b^4 + 318240b^{18} - 2b^{26} + 3417984b^{12}},$$

$$J_{21} = b^{10} - 9b^8 - 6b^6 + 48b^4 + 48b^2,$$

$$J_{22} = b^{12} - 39b^{10} - 246b^8 - 656b^6 - 576b^4 + 192b^2 + 128.$$

We plot the eigenvalues above, where the horizontal axis is the real part of the eigenvalue and the vertical axis is the imaginary part of the eigenvalue, for different values of the parameter b (the arrow indicate the direction on the curve in which b is increasing, and the points A and B indicate where the curve crosses the vertical (imaginary) axis).

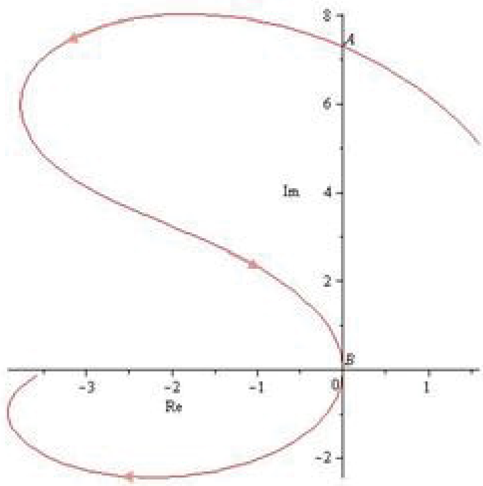


Figure 7.25: $\bar{\lambda} = \frac{1}{4}$, $a = 0$; μ_1-eigenvalues of l_4, (values of b at A and B are $b \backsimeq -2.01$, $b \backsimeq 2.01$, respectively): Plot for $-3 \leq b \leq 3$, arrows indicate the direction of increase for b

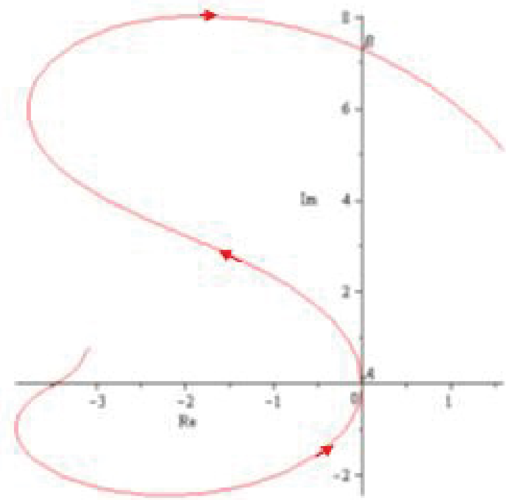


Figure 7.26: $\bar{\lambda} = \frac{1}{4}$, $a = 0$; μ_2-eigenvalues of l_4, (values of b at A and B are $b \backsimeq -2.01$,$b \backsimeq 2.01$, respectively): Plot for $-3 \leq b \leq 3$, arrows indicate the direction of increase for b

Discussion As can be seen from the figures above, for $-2.01 \lesssim b \lesssim 2.01$, the real parts of μ_1 and μ_2 are both negative, which means that l_4 is a sink in this range of the values of b. But for $b \simeq 2.01$ and $b \simeq -2,01$ the real part of μ_1 or μ_2 changes sign, so that l_4 becomes a saddle.

The second case for l_4 when $b = 0$. First, when $b = 0$ and $\bar{\lambda} = \frac{1}{2}$; the eigenvalues are given by

$$\mu_i = \frac{3}{16} a(-a^3 + a\ n_2\ \pm \bar{n_2}),$$

where

$$n_2 = \sqrt{(a^2 - 6)(a^2 + 4)},$$

$$\bar{n_2} = \sqrt{2a^6 - 2a^4\ n_2 - 20a^2 - 4a^4 + 48 + 2a^2\ n_2}.$$

We plot the eigenvalues above, where the horizontal axis is the real part of the eigenvalue and the vertical axis is the imaginary part of the eigenvalue, for different values of the parameter a (the arrow indicates the direction on the curve in which a is increasing, and the point C indicates where the curve crosses the vertical (imaginary) axis).

Remark: Again we can see that

$$\mu_2(a) = \mu_1(-a)$$

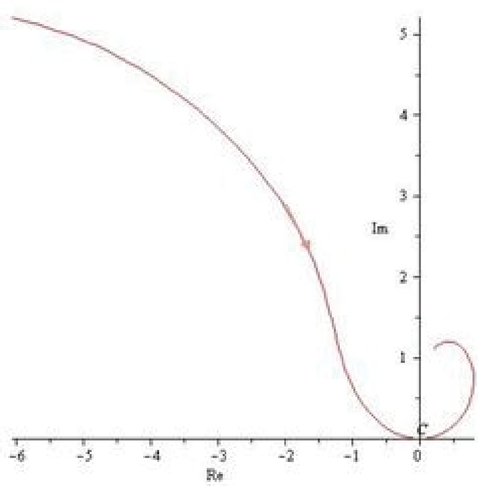


Figure 7.27: $\bar{\lambda} = \frac{1}{2}$, $b = 0$; μ_1-eigenvalue of l_4 (values of a at C is $a \backsimeq 0$): Plot for $-2 \leq b \leq 1.5$, arrows indicate the direction of increase for a

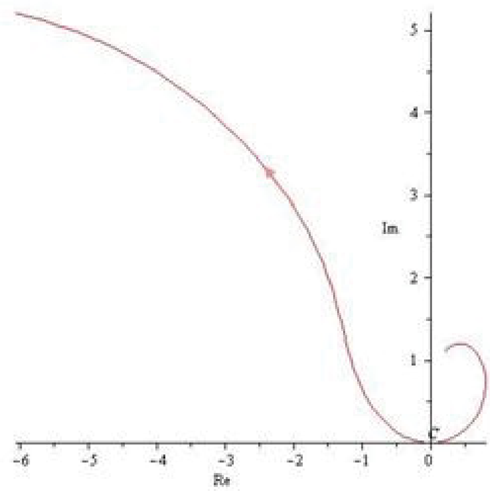


Figure 7.28: $\bar{\lambda} = \frac{1}{2}$, $b = 0$; μ_2-eigenvalue of l_4 (values of a at C is $a \backsimeq 0$): Plot for $-2 \leq b \leq 1.5$, arrows indicate the direction of increase for a

Discussion As can be seen from the figures above, the real parts of μ_1 and μ_2 changes sign at $a = 0$, which means that l_4 is a saddle.

For comparison, let us consider $b = 0$ and $\bar{\lambda} = \frac{1}{3}$; the eigenvalues are given by

$$\mu_i = \frac{1}{16} a(-3a^3 + 3\ n_3 \pm \bar{n_3}),$$

where

$$n_3 = \sqrt{a^4 - 64 - 2a^2},$$

$$\bar{n_3} = \sqrt{18a^6 - 18a^4\ n_3 - 560a^2 - 26a^4 + 512 + 8a^2\ n_3}.$$

We plot the eigenvalues above, where the horizontal axis are the real part of the eigenvalue and the vertical axis are the imaginary part of the eigenvalue, for different values of the parameter a (the arrow indicates the direction on the curve in which a is increasing , and the points C indicate where the curve crosses the vertical (imaginary) axis).

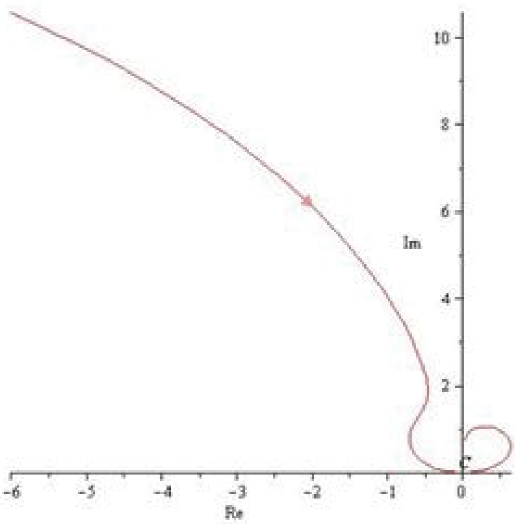


Figure 7.29: $\bar{\lambda} = \frac{1}{3}$, $b = 0$; μ_1-eigenvalue of l_4 (values of a at C is $a \backsimeq 0$): Plot for $-2 \leq b \leq 1.5$, arrows indicate the direction of increase for a

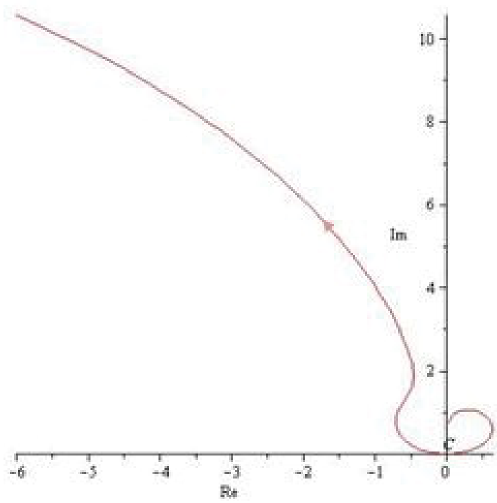


Figure 7.30: $\bar{\lambda} = \frac{1}{3}$, $b = 0$; μ_2-eigenvalue of l_4 (values of a at C is $a \backsimeq 0$): Plot for $-2 \leq b \leq 1.5$, arrows indicate the direction of increase for a

Discussion As can be seen from the figures above, the real parts of μ_1 and μ_2 changes sign at $a = 0$, which means that l_4 is a saddle.

Lastly, the eigenvalues when $\bar{\lambda} = \frac{1}{4}$ are given by

$$\mu_i = \frac{3}{16}a(-a^3 + a\ n_4 \pm \frac{1}{2}\ \bar{n_4}),$$

where

$$n_4 = \sqrt{(a^2 + 10)(a^2 - 12)},$$

$$\bar{n_4} = \sqrt{8a^6 - 8a^4\ n_4 - 476a^2 - 10a^4 + 240 + 2a^2\ n_4}.$$

We plot the eigenvalues above, where the horizontal axis is the real part of the eigenvalue and the vertical axis is the imaginary part of the eigenvalue, for different values of the parameter a (the arrow indicates the direction on the curve in which a is increasing, and the points C indicate where the curve crosses the vertical (imaginary) axis).

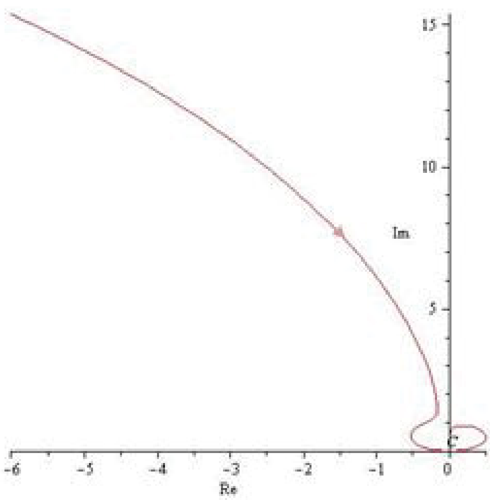


Figure 7.31: $\bar{\lambda} = \frac{1}{4}$, $b = 0$; μ_1-eigenvalue of l_4 (values of a at C is $a \backsimeq 0$): Plot for $-2 \leq b \leq 1.5$, arrows indicate the direction of increase for a

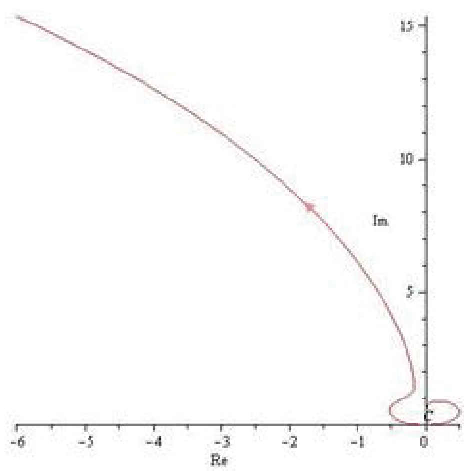


Figure 7.32: $\bar{\lambda} = \frac{1}{4}$, $b = 0$; μ_2-eigenvalue of l_4 (values of a at C is $a \backsimeq 0$): Plot for $-2 \leq b \leq 1.5$, arrows indicate the direction of increase for a

Discussion As can be seen from the figures above, the real parts of μ_1 and μ_2 changes sign at $a = 0$, which means that l_4 is a saddle.

7.3.4 Point l_5

There are also two cases for this point. Let us consider $a = 0$. When $a = 0$ and $\bar{\lambda} = \frac{1}{2}$; the eigenvalues are given by

$$\mu_i = \frac{3b(-\sqrt{b^4 - 12b^2 - 24}\ (J_1) + J_2 \mp \sqrt{b^4 - 12b^2 - 24}\ J_3 \pm J_4)}{2(b^2 + 2)(-\sqrt{b^4 - 12b^2 - 24}J_5 + J_6)}.$$

We plot the eigenvalues above, where the horizontal axis is the real part of the eigenvalue and the vertical axis is the imaginary part of the eigenvalue, for different values of the parameter b (the arrow indicates the direction on the curve in which b is increasing, and the points A and B indicate where the curve crosses the vertical (imaginary) axis).

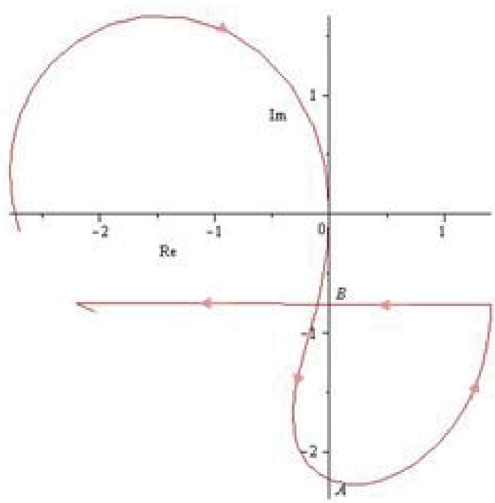


Figure 7.33: $\bar{\lambda} = \frac{1}{2}$, $a = 0$; μ_1-eigenvalue of l_5 (values of b at A and B are $b \backsimeq -2.825$, $b \backsimeq -1.25$, respectively): Plot for $-3 \leq b \leq 2$, arrows indicate the direction of increase for b

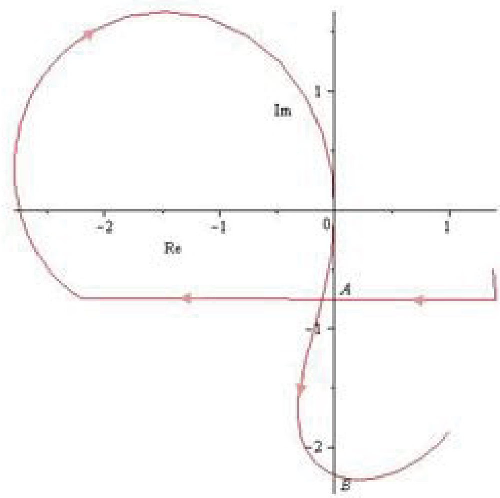


Figure 7.34: $\bar{\lambda} = \frac{1}{2}$, $a = 0$; μ_2-eigenvalue of l_5 (values of b at A and B are $b \backsimeq -2.825$, $b \backsimeq +1.25$, respectively): Plot for $-3 \leq b \leq 2$, arrows indicate the direction of increase for b

Discussion As can be seen from the figures above, the real parts of μ_1 and μ_2 are both negative when $-1.25 \lesssim b \lesssim 1.25$. Thus it is a sink in this range of values for b. But μ_1 or μ_2 change signs at $b \simeq 1.25$ and $b \simeq -2.825$, which means l_5 is a saddle. From (7.14), there are further sign changes for μ_1 and μ_2 at $b \simeq 2.825$ and $b \simeq 1.25$, respectively.

For comparison, let us consider the case when $a = 0$ and $\bar{\lambda} = \frac{1}{3}$; the eigenvalues are given by

$$\mu_i = \frac{b(J_7 - \sqrt{b^4 - 32b^2 - 64}\ J_8 \pm (J_9 + J_{10}) \mp \sqrt{b^4 - 32b^2 - 64}(J_{11} + J_{12}))}{2(b^2+2)(-\sqrt{b^4 - 32b^2 - 64}\ J_{13} + J_{14})}.$$

We plot the eigenvalues above, where the horizontal axis is the real part of the eigenvalue and the vertical axis is the imaginary part of the eigenvalue, for different values of the parameter b (the arrow indicates the direction on the curve in which b is increasing, and the points A and B indicate where the curve crosses the vertical (imaginary) axis).

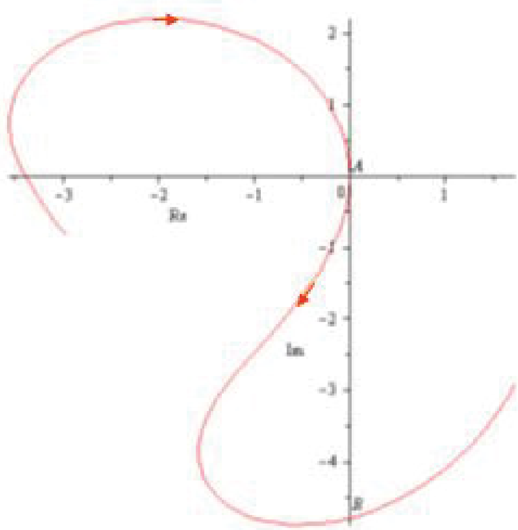


Figure 7.35: $\bar{\lambda} = \frac{1}{3}$, $a = 0$; μ_1-eigenvalue of l_5 (values of b at A and B are $b \backsimeq -1.75$ and $b \backsimeq 1.75$, respectively): Plot for $-3 \leq b \leq 3$, arrows indicate the direction of increase for b

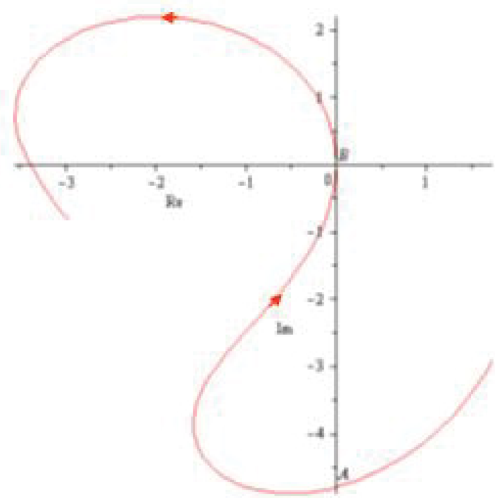


Figure 7.36: $\bar{\lambda} = \frac{1}{3}$, $a = 0$; μ_2-eigenvalue of l_5 (values of b at A and B are $b \backsimeq -1.75$ and $b \backsimeq 1.75$): Plot for $-3 \leq b \leq 3$, arrows indicate the direction of increase for b

Discussion As can be seen from the figures above, for $-1.75 \lesssim b \lesssim 1.75$, the real parts of the eigenvalues are both negative which implies a sink in this range of the values of b. But μ_1 and μ_2 change signs at $b < -1.75$ and $b > 1.75$; one of the eigenvalues is positive and the other is negative, which means that l_5 is a saddle.

Lastly, when $a = 0$ and $\bar{\lambda} = \frac{1}{4}$; the eigenvalues are given by

$$\mu_i = \frac{3b(J_{15} - \sqrt{b^4 - 60b^2 - 120}\ J_{16} \mp (J_{17} + J_{18} + \sqrt{b^4 - 60b^2 - 120}(J_{19} + J_{20})))}{4(b^2+2)(-\sqrt{b^4 - 60b^2 - 120}J_{21} + J_{22})}.$$

We plot the eigenvalues above, where the horizontal axis is the real part of the eigenvalue and the vertical axis is the imaginary part of the eigenvalue, for different values of the parameter b (the arrow indicates the direction on the curve in which b is increasing, and the points A and B indicate where the curve crosses the vertical (imaginary) axis).

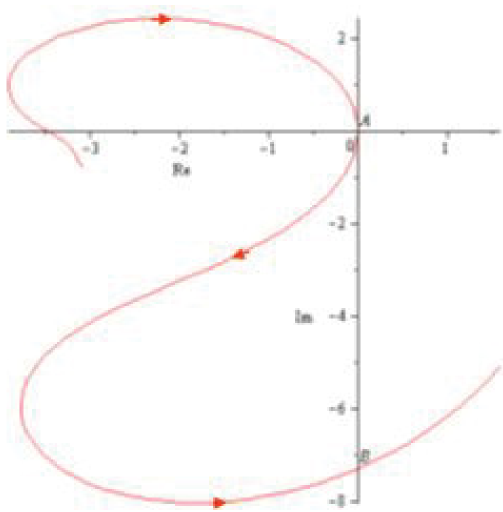


Figure 7.37: $\bar{\lambda} = \frac{1}{4}$, $a = 0$; μ_1-eigenvalue of l_5 (values of b at A and B are $b \backsimeq -2.01$, $b \backsimeq 2.01$: Plot $-3 \leq b \leq 3$, arrows indicate the direction of increase for b

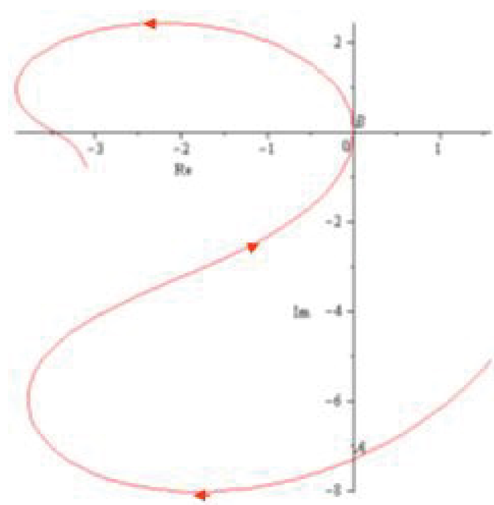


Figure 7.38: $\bar{\lambda} = \frac{1}{4}$, $a = 0$; μ_2-eigenvalue of l_5 (values of b at A and B are $b \backsimeq -2.01$, $b \backsimeq 2.01$: Plot $-3 \leq b \leq 3$, arrows indicate the direction of increase for b

Discussion As can be seen from the figures above, for $-2.01 \lesssim b \lesssim 2.01$, the real parts of μ_1 and μ_2 are both negative, which means that l_4 is a sink in this range of the values of b. But for $b > 2.01$ and $b < -2,01$ the real part of μ_1 or μ_2 changes signs, so that l_5 becomes a saddle.

The second case for l_5 when $b = 0$. First, when $b = 0$ and $\bar{\lambda} = \frac{1}{2}$; the eigenvalues are given by

$$\mu_i = \frac{3}{16}a(-a^3 - a\ n_2 \pm \bar{n_6}),$$

where

$$n_2 = \sqrt{(a^2 - 6)(a^2 + 4)}, \bar{n_6} = \sqrt{2a^6 + 2a^4\ n_2 - 20a^2 - 4a^4 + 48 - 2a^2\ n_2}.$$

We plot the eigenvalues above, where the horizontal axis are the real part of the eigenvalues and the vertical axis are the imaginary part of the eigenvalues and the curve is the values of the parameter a (the arrow indicates the direction on the curve in which a is increasing, and the points C indicate where the curve crosses the vertical (imaginary) axis).

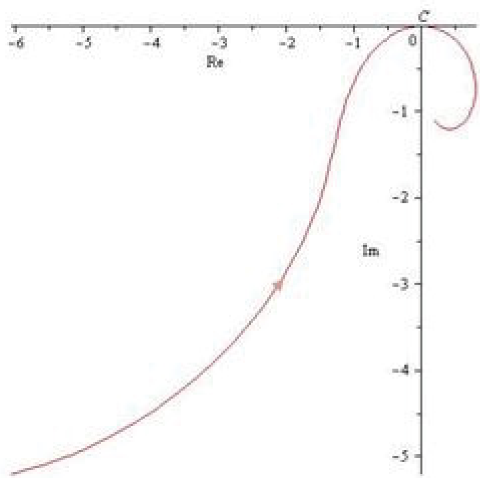


Figure 7.39: $\bar{\lambda} = \frac{1}{2}$, $b = 0$; μ_1-eigenvalue of l_5 (values of a at C is $a \backsimeq 0$): Plot for $-2 \le b \le 1.5$, arrows indicate the direction of increase for a

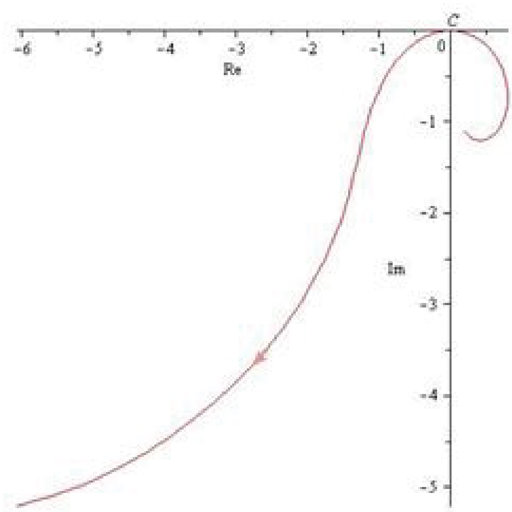


Figure 7.40: $\bar{\lambda} = \frac{1}{2}$, $b = 0$; μ_2-eigenvalue of l_5 (values of a at C is $a \backsimeq 0$): Plot for $-2 \le b \le 1.5$, arrows indicate the direction of increase for a

Discussion As can be seen from the figures above, the real parts of μ_1 and μ_2 changes signs at $a = 0$, which means that l_5 is a saddle.

For comparison, let consider when $b = 0$ and $\bar{\lambda} = \frac{1}{3}$; the eigenvalues are given by

$$\mu_i = \frac{1}{16}a(-3a^3 - 3\ n_3 \pm \bar{n_7}),$$

where

$$n_3 = \sqrt{a^4 - 64 - 2a^2}, \bar{n_7} = \sqrt{18a^6 + 18a^4\ n_3 - 560a^2 - 26a^4 + 512 - 8a^2\ n_3}.$$

We plot the eigenvalues above, where the horizontal axis are the real part of the eigenvalues and the vertical axis are the imaginary part of the eigenvalues and the curve is the values of the parameter a (the arrow indicates the direction on the curve in which a is increasing, and the points C indicate where the curve crosses the vertical (imaginary) axis).

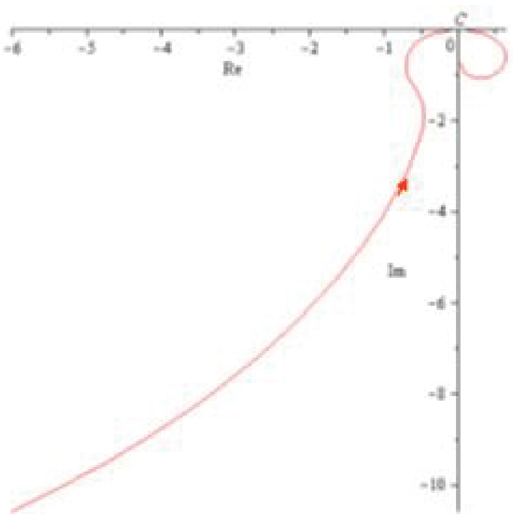


Figure 7.41: $\bar{\lambda} = \frac{1}{3}$, $b = 0$; μ_1-eigenvalue of l_5 (values of a at C is $a \backsimeq 0$): Plot for $-2 \le b \le 1.5$, arrows indicate the direction of increase for a

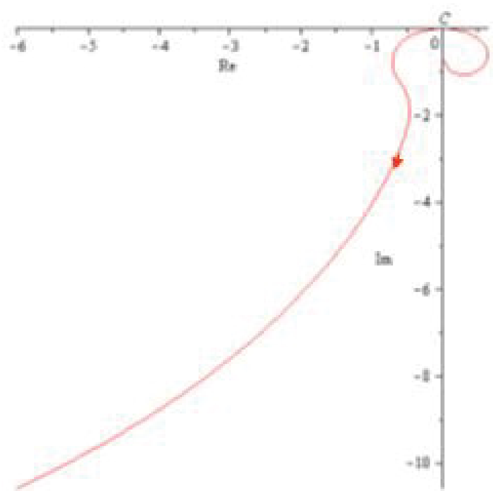


Figure 7.42: $\bar{\lambda} = \frac{1}{3}$, $b = 0$; μ_2-eigenvalue of l_5 (values of a at C is $a \backsimeq 0$): Plot for $-2 \le b \le 1.5$, arrows indicate the direction of increase for a

Discussion As can be seen from the figures above, the real parts of μ_1 and μ_2 changes signs at $a = 0$, which means that l_5 is a saddle.

Lastly, when $b = 0$ and $\bar{\lambda} = \frac{1}{4}$; the eigenvalues are given by

$$\mu_i = \frac{3}{16}a(-a^3 - a\ n_4 \pm \frac{1}{2}\ \bar{n_8}),$$

where

$$n_4 = \sqrt{(a^2+10)(a^2-12)}, \bar{n_8} = \sqrt{8a^6 + 8a^4\ n_4 - 476a^2 - 10a^4 + 240 - 2a^2\ n_4}.$$

We plot the eigenvalues above, where the horizontal axis are the real part of the eigenvalues and the vertical axis are the imaginary part of the eigenvalues and the curve is the values of the parameter a (the arrow indicates the direction on the curve in which a is increasing, and the points C indicate where the curve crosses the vertical (imaginary) axis).

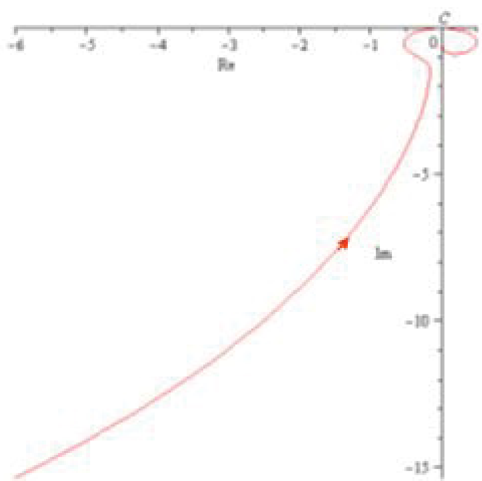


Figure 7.43: $\bar{\lambda} = \frac{1}{4}$, $b = 0$; μ_1-eigenvalue of l_5 (values of a at C is $a \backsimeq 0$): Plot for $-2 \le b \le 1.5$, arrows indicate the direction of increase for a

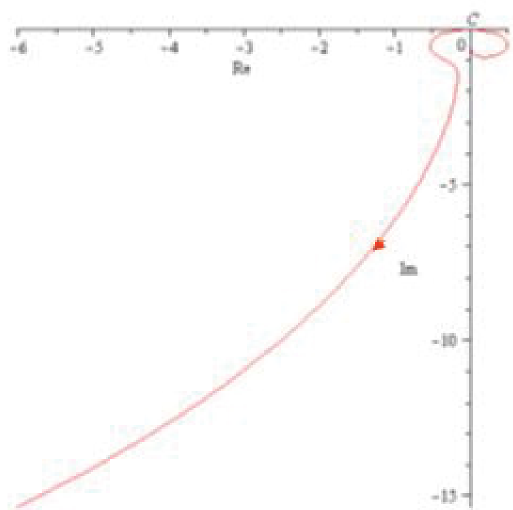


Figure 7.44: $\bar{\lambda} = \frac{1}{4}$, $b = 0$; μ_2-eigenvalue of l_5 (values of a at C is $a \backsimeq 0$): Plot for $-2 \le b \le 1.5$, arrows indicate the direction of increase for a

Discussion As can be seen from the figures above, the real parts of μ_1 and μ_2 changes signs at $a = 0$, which means that l_5 is a saddle.

Chapter 8

Conclusion

To summarize, the stability of the points l_2, l_3 is the same in both cases ($a = 0$, $b = 0$) for all relevant values; they are sinks in the range of values $-3 \le a, b \le 3$ and $\frac{1}{5} \le \bar{\lambda} \le \frac{1}{2}$. The model is inflationary for l_2 if either $-\frac{1}{2} \lesssim b$ or $a \lesssim 1$. The model is inflationary for l_3 if either $\frac{1}{2} \lesssim b$ or $a \lesssim -1$. Let us next consider the stability of $l_{4,5}$. When $b = 0$, they are both always saddles for all values of the parameters $\frac{1}{5} \le \bar{\lambda} \le \frac{1}{2}$ and $-2 \le a \le 1.5$. When $a = 0$, the eigenvalues change signs in various regions. For example, for $-1.25 \lesssim b \lesssim 1.25$, $l_{4.5}$ are sinks (but otherwise they are saddles).

We conclude that there are always ranges of values for which there exists an inflationary sink. Not all of the equilibrium points correspond to physically realistic solutions for all values of the parameters. Therefore, a complete discussion of the cosmological model would include further analysis of the viable (physical) ranges for the parameters. However, for each set of parameters in the ranges discussed, there always exists an inflationary future attractor for sufficiently small $\bar{\lambda}$.

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