THE UNIFORMITY SPACE OF HYPERGRAPHS

by

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Abstract

For a hypergraph $H = (V, E)$ and a field $\mathbb{F}$, a weighting of H is a map $f : V \to \mathbb{F}$. A weighting is called *stable* if there is some $k \in \mathbb{F}$ such that the sum of the weights on each edge of H is equal to k. The set of all stable weightings of H forms a vector space over $\mathbb{F}$. This vector space is termed the *uniformity space* of H over $\mathbb{F}$, denoted $U(H, \mathbb{F})$, and its dimension is called the *uniformity dimension* of H over $\mathbb{F}$.

This thesis is concerned with several problems relating to the uniformity space of hypergraphs. For several families of hypergraphs, simple ways of computing their uniformity dimension are found. Also, the uniformity dimension of random l-uniform hypergraphs is investigated. The stable weightings of the spanning trees of a graph are determined, and lastly, a notion of *critical* uniformity dimension is introduced and explored.

List of Abbreviations and Symbols Used

C_n	the cycle of order n
$E(\mathcal{U})$	the expected value of random variable $\mathcal{U}$
$G \cdot e$	the contraction of edge e from G
$H - e$	the deletion of edge e from H
H^*	the hypergraph-complement of H
H_Δ	the facet hypergraph of complex Δ
K_n	the complete graph on n vertices
$K_{n,l}$	the complete l-uniform hypergraph on n vertices
$M(G)$	the graphic matroid of G
$M \cdot U$	the contraction of the matroid M to the set U
$M\|U$	the restriction of the matroid M to the set U
$P(\mathcal{U} = i)$	the probability that the random variable $\mathcal{U}$ is equal to i
$U \oplus V$	the direct sum of vector spaces U and V
$V^{(l)}$	the set of all l-subsets of the set V
W^t	the transpose of matrix W
W_H	the solution matrix of hypergraph H
$X \backslash Y$	the set difference of X and Y
$\Delta - X$	the deletion of the face X from the complex Δ
Δ_H	the associated complex of hypergraph H

$\mathbb{N}$	the natural numbers
$\mathbb{Z}_n$	the integers modulo n
$\mathrm{Prob}(E)$	the probability of event E
SVD	the singular value decomposition
$\mathrm{U}(H, \mathbb{F})$	the uniformity space of H over $\mathbb{F}$
$\mathrm{WC}(G, \mathbb{F})$	the space of well-covered weightings of G over $\mathbb{F}$
$\mathrm{char}(\mathbb{F})$	the characteristic of $\mathbb{F}$
$\deg(v)$	the degree of vertex v
$\gcd(l, n)$	the greatest common divisor of l and n
$\mathrm{lk}_\Delta X$	the link of the face X from the complex Δ
$\mathrm{nullity}(W)$	the nullity of matrix W
$\mathrm{rank}(W)$	the rank of matrix W
$\mathrm{udim}(H, \mathbb{F})$	the uniformity dimension of H over $\mathbb{F}$
$\mathrm{wcdim}(G, \mathbb{F})$	the well-covered dimension of G over $\mathbb{F}$
$\lvert X \rvert$	the order of set X

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Chapter 1

Introduction

Often in combinatorics, we are interested in weighting the vertices or edges of a graph or hypergraph with elements from a field $\mathbb{F}$. One well-known problem of this type is the determination of a minimum cost spanning tree for a graph with weighted edges, the history of which can be found in [10]. The problem that we wish to consider is when the weight is constant on all substructures of a particular kind.

In [7], the authors introduced and studied the *well-covered weightings* of a graph. The motivation comes from well-covered graphs – those in which every maximal independent set has the same size. For example, complete graphs are well-covered since every maximal independent set in these graphs has size 1, and the cycles C_4 and C_5 are well-covered since every maximal independent set in both C_4 and C_5 has size 2.

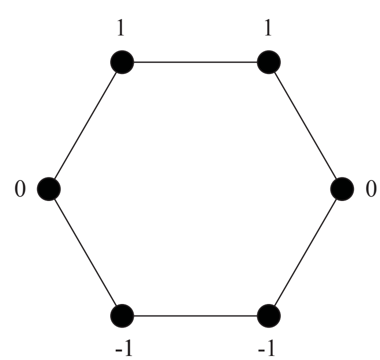

Figure 1.1: A well-covered weighting of C_6

A weighting of a graph G by elements of a field $\mathbb{F}$ is a *well-covered weighting* if the sum of the weights of the vertices of G on every maximal independent set of

G is constant. A well-covered weighting of the cycle C_6 is shown in Figure 1.1. In this example, the sum of the weights of the vertices of every maximal independent set of C_6 is 0. However, note that C_6 is not itself well-covered since it has maximal independent sets of size 2 and size 3. We observe that if a graph G is well-covered, then weighting every vertex of G with a 1 is a well-covered weighting of G. It was noted in [7] that the set of well-covered weightings of any graph G with elements of a field $\mathbb{F}$ forms a vector space over the field $\mathbb{F}$. This vector space of well-covered weightings was further studied by Caro and Yuster in [8], and by Brown et al. in [4–6]. The computational complexity of the problem has been studied more recently by Levit and Tankus in [15, 16].

We wish to study the more general problem introduced by Caro and Yuster in [8], which takes place on a hypergraph $H = (V, E)$. We consider all weightings of V by elements of a field $\mathbb{F}$ whose sums are equal on every edge e of E. We call these weightings *stable*. We see that the stable weightings of a given hypergraph form a vector space over the field $\mathbb{F}$, which we call the *uniformity space* of H over $\mathbb{F}$, denoted $U(H, \mathbb{F})$. We can then ask questions which we would ask of any vector space. We are chiefly interested in the dimension of $U(H, \mathbb{F})$, which we call the *uniformity dimension* of H over $\mathbb{F}$, denoted $\operatorname{udim}(H, \mathbb{F})$. We will also be interested in finding a basis for $U(H, \mathbb{F})$ at times.

We can see immediately that the uniformity space is a generalization of the vector space of well-covered weightings of a graph. Given a graph $G = (V, E)$, we can define the hypergraph $H = (V, E')$ on the same vertex set and with edges the maximal independent sets of G. It is clear that $U(H, \mathbb{F})$ is the same as the space of well-covered weightings of G, i.e. that the stable weightings of H correspond exactly to the well-covered weightings of G.

There are other substructures of graphs that we may want to have equal weight. For example we may take a graph $G = (V, E)$ and a subgraph H and consider weightings of the edge set E under which all isomorphic copies of H in G have the same weight (the weight of a copy being the sum of the weights on the edges of the copy). This can be viewed as a uniformity space problem by considering the hypergraph whose vertices correspond to edges of G and whose edges correspond to isomorphic copies of H in the obvious manner (they contain exactly the same edges as an isomorphic copy of H). This problem was studied in [8], primarily for the case $H = K_n$.

Weightings of the edge sets of graphs provide many more examples of uniformity spaces. For a graph $G = (V, E)$, we will study the weightings of E which are stable on the maximal acyclic subgraphs of E (i.e. spanning forests of G). In particular these are the bases of the graphic matroid, and we prove something in general about uniformity spaces of hypergraphs whose edges are the bases of a matroid.

In general, we can weight the vertices or edges of a *hypergraph* and require that all subhypergraphs of a certain type have the same weight. We can create a master hypergraph that represents the problem and try to find its uniformity space. This type of problem is studied little here, because of its increased complexity and decreased applicability.

We will be interested in finding the uniformity spaces and dimensions of several families of hypergraphs. We consider l-uniform hypergraphs for small values of l, and random l-uniform hypergraphs in general. We also consider several highly structured types of hypergraphs. These include certain types of l-uniform cycles, matroids, and block designs.

We introduce a notion of criticality for hypergraphs in terms of their uniformity dimension. A hypergraph is *critical* if removing any edge of the hypergraph increases its uniformity dimension. We characterize the *graphs* that are critical in this sense, and provide results on the criticality of some other families of hypergraphs.

Before proceeding with the formal treatment of our problem, we provide the reader with some background on hypergraphs, complexes, and block designs.

Chapter 2

Background

2.1 Graphs and Hypergraphs

The material in this section can be found in any introductory textbook on graphs and hypergraphs, such as [1]. Also see [21] for an introduction to graph theory. The definitions contained in this section vary slightly over different books, but most are fairly standard. We begin with the definition of a hypergraph.

Definition 2.1.1. A *hypergraph* is a pair $H = (V, E)$, where V is a set of objects called *vertices* and E, the set of *edges*, is a set of subsets of V. Edges containing exactly one vertex are called *loops*.

Note that our definition of hypergraph does not allow for multiple edges, because E is a set and therefore cannot contain repetition. For our purposes, any multiple edges would not make any difference, so we ignore them and do not include them in our definition. We say that a hypergraph is *finite* if it has finite vertex set (and hence finite edge set). Here we assume that all hypergraphs are finite, unless otherwise noted. We now define a special type of hypergraph.

Definition 2.1.2. A *graph* $G = (V, E)$ is a hypergraph in which every edge contains either 1 or 2 vertices. A *simple graph* is a graph $G = (V, E)$ which contains no loops. That is, every edge contains exactly 2 vertices. In general, an *l-uniform* hypergraph is a hypergraph $H = (V, E)$ whose edges all contain exactly l vertices, i.e. $E \subseteq V^{(l)}$, the set of all l-subsets of V.

Usually when we refer to a graph we will mean a simple graph. However, most results that we prove will hold for graphs with loops.

We now present some definitions for general hypergraphs which of course apply to graphs as well.

Definition 2.1.3. Let $H = (V, E)$ be a hypergraph. For an edge $e = \{v_1, v_2, ..., v_l\}$ we say each vertex in e is *incident* to e. We also say that e is *incident* to each of the vertices $v_1, v_2, ..., v_l$. We will sometimes say that e *joins* the vertices $v_1, v_2, ..., v_l$. Distinct vertices v_1 and v_2 contained in a common edge are said to be *adjacent*, and distinct edges e_1 and e_2 with nonempty intersection are said to be *adjacent* as well. For any vertex $v \in V$, we define the *degree* of v, denoted $\deg(v)$, to be the number of distinct edges of H containing v. A vertex of degree 0 is called an *isolated vertex* of H.

The reader will be familiar with drawings of graphs. For drawings of hypergraphs, we can represent an edge in two ways. We can draw a loop that encloses the vertices of that edge, or we can draw an arc that passes through the vertices of that edge. In any individual drawing, we use the same type of representation for all of the edges. We often draw each edge with a different colour to make the picture clearer.

Definition 2.1.4. A *subhypergraph* of a hypergraph $H = (V, E)$ is a hypergraph $H' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$, and for every edge $e \in E'$, $e \subseteq V'$. A subhypergraph of a graph is called a *subgraph*.

Next we define a common operation on hypergraphs which produces a subhypergraph.

Definition 2.1.5. Let $H = (V, E)$ be a hypergraph and let $e \in E$. The *deletion* of e from H, denoted $H - e$, is defined to be $H - e = (V, E \backslash \{e\})$. We say $H - e$ is obtained from H by *deleting* e. The deletion of more than one edge is defined similarly.

We now present several graph-specific definitions. Some have an analogue for hypergraphs, and we mention these analogues when they will be of use to us.

Let $G = (V, E)$ be a graph. For the standard definitions of *walk, closed walk, path, cycle*, and *length* of a walk, we refer the reader to [21]. The definitions of *connected components* of a graph and *connected* graphs can be found there as well. For the similar definitions of *connected components* of a hypergraph and *connected* hypergraphs, we refer the reader to [1]. A *forest* is a graph with no cycles. We also call this type of graph *acyclic*. The maximal acyclic subgraphs of a graph are called its *spanning forests*. A *tree* is a connected forest, thus maximal acyclic subgraphs of a connected graph are called its *spanning trees*. A *pseudoforest* is a graph in which every component has at most one cycle.

Define the relation R' on the edges of a graph by $xR'y$ if and only if $x = y$ or there is a cycle C of the graph which contains both x and y. This is an equivalence relation, whose equivalence classes are called the *blocks* of a graph. A graph with only one block is called *biconnected*.

A subset $X \subseteq V$ is called *independent* if no two vertices of X are adjacent. A graph $G = (V, E)$ is *bipartite* if there exist two disjoint sets V_1 and V_2 of V such that $V_1 \cup V_2 = V$ and every edge in G is incident with exactly one vertex from V_1 and exactly one vertex from V_2. That is, V_1 and V_2 are independent sets which partition V. The subsets V_1 and V_2 are called the *bipartition sets* of G. A well-known characterization of bipartite graphs says that they are exactly the graphs with no odd cycles. In a bipartite graph with bipartition sets V_1 and V_2, it is easy to see that any walk between vertices in V_1 and V_2 must have odd length, and any walk between two vertices of V_1 (and likewise V_2) must have even length.

We defined the operation of deletion for hypergraphs, and we now define another operation on graphs.

Definition 2.1.6. Let $G = (V, E)$ be a graph and let $e \in E$. Define the graph $G' = (V', E')$, where $v \notin V$ is a new vertex and:

$$V' = (V \backslash e) \cup \{v\}$$
$$E' = \{f \in E \mid f \cap e = \emptyset\} \cup \{\{u, v\} \mid \exists f \in E \text{ with } u \in f \backslash e, f \cap e \neq \emptyset\}$$

Then G' is called the *contraction* of e from G, denoted $G \cdot e$.

Recall that we took the edges of a graph to be a set, so that any repetitions in the edge set created by contraction are discarded.

2.2 Simplicial Complexes and Matroids

Definition 2.2.1. A *simplicial complex* $\Delta = (V, E)$ is a hypergraph whose edge set is closed under containment. That is, if $X \in E$ and $Y \subseteq X$, then $Y \in E$.

In combinatorics, we usually shorten the name and refer to these structures simply as *complexes*. The empty set is required to be in a complex by definition, but the inclusion or exclusion of the empty set has no effect on the work that we will do with complexes.

Example 2.2.1. Let $G = (V, E)$ be a graph. Define the hypergraph $H = (V, I)$, where $X \subseteq V$ is a member of I if and only if X is an independent set of G. It is easy to see that H is a complex, called the *independence complex* of the graph G.

We now give the definition of a special kind of complex called a matroid. All of the results in this section pertaining to matroids can be found in [20].

Definition 2.2.2. A complex $M = (V, E)$ is a *matroid* if the following *exchange axiom* holds:

If $X, Y \in E$ with $|X| = |Y| + 1$ then there exists an $x \in X \backslash Y$ such that $Y \cup x \in E$.

We introduce some of the terminology that has been developed for complexes and matroids. The edges of a complex are called *faces*, and the maximal faces are called *facets*. Often the faces of a matroid are called *independent sets* and the facets are called *bases*, because of the historical connection between matroids and vector spaces (the linearly independent sets of any finite-dimensional vector space form a matroid). We will often be interested in only the facets of a complex, and to study them more easily we give the following definition.

Definition 2.2.3. Let $\Delta = (V, E)$ be a complex. Let F be the set of facets of Δ. Then the hypergraph $H_\Delta = (V, F)$ is called the *facet hypergraph* of Δ. If Δ is a matroid, then H_Δ will be referred to as the *basis hypergraph* of Δ.

A *circuit* of a complex $\Delta = (V, E)$ is a minimal non-face of Δ. More precisely, $C \subseteq V$ is a circuit of Δ if C is not a face of Δ, but for any $x \in C$, $C - \{x\}$ is a face of Δ. Any non-face of a complex must contain a circuit. The *dimension* of a complex is equal to the cardinality of its largest face, and a complex is said to be *purely d-dimensional* if every one of its facets (or bases) has cardinality d. It is well-known that every (finite) matroid is purely d-dimensional for some $d \in \mathbb{N}$.

We note that a complex is determined completely by its facets, since if $\mathscr{F}$ is the set of facets (i.e. maximal edges) of a complex $\Delta = (V, E)$, then $E = \{X \mid X \subseteq F \text{ for some } F \in \mathscr{F}\}$. A complex is also determined completely by its circuits, since if $\mathscr{C}$ is the set of circuits of a complex $\Delta = (V, E)$, then $E = \{X \subseteq V \mid C \not\subseteq X \text{ for all } C \in \mathscr{C}\}$. There are alternate axiom systems for a matroid in terms of bases and circuits that we will have occasion to use.

Theorem 2.2.1 (Base axiom)**.** *A non-empty collection $\mathscr{B}$ of subsets of V is the set of bases of a matroid on V if and only if it satisfies the following condition:*

$$\text{If } B_1, B_2 \in \mathscr{B}, \text{ and } x \in B_1 \backslash B_2, \text{ then } \exists y \in B_2 \backslash B_1 \text{ such that } (B_1 \cup y) \backslash x \in \mathscr{B}.$$

Theorem 2.2.2 (Circuit axioms)**.** *A collection $\mathscr{C}$ of subsets of V is the set of circuits of a matroid on V if and only if the following conditions are satisfied.*

(i) If $C_1, C_2 \in \mathscr{C}$ with $C_1 \neq C_2$, then $C_1 \nsubseteq C_2$.

(ii) If C_1, C_2 are distinct members of $\mathscr{C}$ and $z \in C_1 \cap C_2$ then there exists $C_3 \in \mathscr{C}$ such that $C_3 \subseteq (C_1 \cup C_2)\backslash z$.

A particularly useful and illustrative example of a matroid is the so-called *graphic matroid*. Let $G = (V, E)$ be a graph, and let $\mathscr{E}$ be the set of all acyclic subsets of E. Then $(E, \mathscr{E})$ is the *graphic matroid*, denoted $M(G)$. For a proof that this is in fact a matroid, and a more detailed description of the graphic matroid, we refer the reader to [20], where it is called the *cycle matroid of a graph*.

The independent sets of the graphic matroid for a graph G are the subsets of edges that do not contain a cycle, i.e. the subgraphs of G that are forests. The bases of the graphic matroid are the spanning forests of G (or spanning trees if G is connected), and the circuits of the graphic matroid are the cycles of G.

We next define operations on matroids that correspond naturally to deletion and contraction of edges in graphs.

Definition 2.2.4. Let $M = (V, \mathscr{E})$ be a matroid, and let $U \subseteq V$.

(i) Let $\mathscr{E}|U = \{X \mid X \subseteq U, X \in \mathscr{E}\}$. Then $\mathscr{E}|U$ is the set of independent sets of a matroid on U, called the *restriction* of M to U, and denoted $M|U$.

(ii) Let $\mathscr{E}(M \cdot U)$ be the set of subsets $X \subseteq U$ such that there exists a maximal independent set Y of $V\backslash U$ in M such that $X \cup Y \in \mathscr{E}(M)$. Then $\mathscr{E}(M \cdot U)$ is the set of independent sets of a matroid on U, called the *contraction* of M to U, and denoted $M \cdot U$.

The proof that $\mathscr{E}|U$ is the set of independent sets of a matroid is fairly trivial. The proof that $\mathscr{E}(M \cdot U)$ is the set of independent sets of a matroid is more work, and can be found in [20], pg. 62. The restriction and contraction are defined in the same way for general complexes.

We can also view restriction and contraction in another way as the deletion and link. These definitions give rise to restriction and contraction complexes respectively, but are defined in terms of the elements that are being taken away as opposed to the elements that remain.

Definition 2.2.5. Let X be a face in the complex $\Delta = (V, \mathscr{E})$.

(i) The *deletion* of X, denoted $\Delta - X$, is the complex on $V \backslash X$ whose faces are the faces of Δ not containing any element of X.

(ii) The *link* of X, denoted $lk_\Delta X$, is the complex on $V \backslash X$ whose faces are the subsets $Y \subseteq V$ disjoint from X for which $Y \cup X \in \Delta$.

It is fairly easy to prove that $\Delta - X = \Delta|(V\backslash X)$ and $lk_\Delta X = \Delta \cdot (V \backslash X)$, so the deletion and link of a matroid are again matroids. When X is a singleton set, say $\{x\}$, we simply denote the deletion and link by $\Delta - x$ and $lk_\Delta x$ respectively. These notions correspond naturally to deletion and contraction of an edge of a graph. For the graphic matroid $M(G)$ of a graph G with edge e, it can be seen that $M(G) - e = M(G - e)$ and $lk_{M(G)} e = M(G \cdot e)$.

There is no notion in matroids that corresponds exactly to connection in graphs, but there is a relation that can be defined on the vertices of a matroid that corresponds to biconnection in graphs.

Let $M = (V, \mathscr{E})$ be a matroid. Define a relation R on the elements of V by xRy if and only if $x = y$ or there is a circuit C of the matroid M which contains both x and y. Then R is an equivalence relation on V, as shown in [20]. The relation R partitions V into equivalence classes, say $V_1, V_2, ..., V_k$. The matroids $M|V_1, M|V_2, ..., M|V_k$ are called the *connected components* (or more simply the *components*) of the matroid.

The components of a matroid are analogous to the blocks of a graph, and the components of the graphic matroid $M(G)$ are exactly the blocks of the graph G (since circuits of the graphic matroid are exactly the cycles of the graph). We sometimes refer to a component of a matroid by only its vertex set when no confusion results.

2.3 Block Designs

A *design* is a hypergraph that satisfies certain regularity constraints. When we are dealing with designs we call the vertices *points* and the edges *blocks*. We also often use geometric language when dealing with designs. For example, we say a block *passes through* a point if it contains that point, or that two points are *joined* by a block if they are both contained in a block. All definitions and theorems in this section can be found in any book on design theory (we refer to [2]).

Definition 2.3.1. A (finite) hypergraph $D = (V, \mathbf{B})$ is called a *block design* with parameters $v, k, \lambda \in \mathbb{N}$ if it satisfies the following conditions:

(i) $|V| = v$.

(ii) Any two distinct points are joined by exactly λ blocks.

(iii) Each block passes through exactly k points.

Projective planes are a commonly studied type of block design. We give the definition and a theorem that says that they are in fact block designs.

Definition 2.3.2. A hypergraph $D = (V, \mathbf{B})$ is called a *projective plane* if it satisfies the following axioms:

(i) Any two distinct points are joined by exactly one block.

(ii) Any two distinct blocks intersect in a unique point.

(iii) There exist four points, no three of which lie on a common block.

Proposition 2.3.1. *Let $D = (V, \mathbf{B})$ be a finite projective plane. Then there exists a natural number n, called the order of D, satisfying:*

(i) *For each point $p \in V$, p lies on exactly $n + 1$ distinct blocks of D*

(ii) *For each block $B \in \mathbf{B}$, B passes through exactly $n + 1$ distinct points.*

(iii) $|V| = |\mathbf{B}| = n^2 + n + 1$.

So we have that a finite projective plane is a block design with parameters $v = n^2 + n + 1, k = n + 1$, and $\lambda = 1$. The *Fano Plane*, a projective plane of order 2, is a commonly used example of a block design.

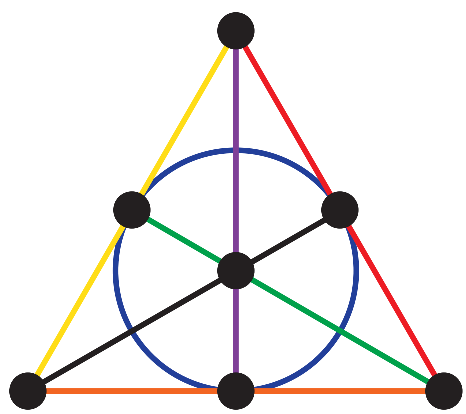

Figure 2.1: The Fano plane

We will use the following well-known results from design theory, which can be found in [2]:

Proposition 2.3.2. *In a (v, k, λ)-block design $\mathbf{D}$, we have the following:*

(i) *Each point p lies in exactly $r := \lambda(v - 1)/(k - 1)$ blocks.*

(ii) $b := |\mathbf{B}| = \lambda v(v - 1)/k(k - 1)$.

Proposition 2.3.3 (Fisher's Inequality)**.** *In a* (v, k, λ)*-block design,*

$$v \leq b.$$

Note: Fisher's Inequality holds for a more general type of design, but we only require the given result. When a block design satisfies $v = b$, it is called a *symmetric* block design.

Proposition 2.3.4. *In a symmetric* (v, k, λ)*-block design with* $v > k$*, any two distinct blocks intersect in exactly* λ *points.*

Now that we have laid the necessary framework, we can begin with the formal definition of uniformity space.

Chapter 3

The Uniformity Space of Hypergraphs

3.1 Formal Definition

We wish to consider weightings of the vertices of a hypergraph that are constant on all edges. We make this definition more precise.

Definition 3.1.1. Let $H = (V, E)$ be a hypergraph. For a field $\mathbb{F}$, a map $f : V \to \mathbb{F}$ is called a *weighting* of H. We say the *weight* of a vertex $v \in V$ is the value $f(v)$. We define the *weight* of an edge $e \in E$ to be $f(e) = \sum_{v \in e} f(v)$. More generally, for any subset $X \subseteq V$, the *weight* of X is defined to be $f(X) = \sum_{x \in X} f(x)$.

This is the natural extension of the definition of weightings for graphs used in [4–6] and [7]. In these papers, a special class of weightings was studied, called the well-covered weightings.

Definition 3.1.2 (Caro et al., [7])**.** A weighting $f : V \to \mathbb{F}$ of a graph $G = (V, E)$ is *well-covered* if there is an element $k \in \mathbb{F}$ such that the weight of every maximal independent set is equal to k.

This idea was extended by Caro and Yuster in [8], and the definition is presented here.

Definition 3.1.3. For a hypergraph $H = (V, E)$, a weighting $f : V \to \mathbb{F}$ is called *stable* if there is an element $k \in \mathbb{F}$ such that the weight of every edge of H is equal to k.

So a well-covered weighting of a graph $G = (V, E)$ is a stable weighting of the hypergraph on vertex set V whose edges are the maximal independent sets of G. The key observation in [7] is that the well-covered weightings of a graph G form a vector space over the field $\mathbb{F}$, denoted $\mathrm{WC}(G, \mathbb{F})$. The dimension of this space, denoted $\mathrm{wcdim}(G, \mathbb{F})$, is called the *well-covered dimension* of G over $\mathbb{F}$. In fact, the stable weightings of any hypergraph form a vector space, as shown below.

Observation 3.1.1 (Caro and Yuster, [8]). *The stable weightings of a hypergraph H form a vector space over the field $\mathbb{F}$, called the uniformity space of H over $\mathbb{F}$, which we denote $U(H, \mathbb{F})$. The dimension of this vector space is called the uniformity dimension of H over $\mathbb{F}$, and is denoted udim$(H, \mathbb{F})$.*

Proof. Let $H = (V, E)$ be a hypergraph. Suppose $f, g : V \to \mathbb{F}$ are both stable weightings of H, and let the weight of each edge under f be $a \in \mathbb{F}$ and the weight of each edge under g be $b \in \mathbb{F}$. So for any $e \in E$, $\sum_{v \in e} f(v) = a$ and $\sum_{v \in e} g(v) = b$. Then for the map $kf + lg : V \to \mathbb{F}$, where $k, l \in \mathbb{F}$, the weight of any edge $e \in E$ is:

$$\sum_{v \in e} (kf + lg)(v) = \sum_{v \in e} kf(v) + \sum_{v \in e} lg(v) = k \sum_{v \in e} f(v) + l \sum_{v \in e} g(v) = ka + lb$$

and so $kf + lg$ is also a stable weighting of H. $\square$

We are interested in finding the uniformity dimension of certain hypergraphs (or families of hypergraphs). We will often find a basis for the uniformity spaces of these hypergraphs in the process. Before developing any more theory, we provide several examples.

3.2 Examples

In this section we find the uniformity space of two hypergraphs derived in different ways from the graph G shown in Figure 3.1.

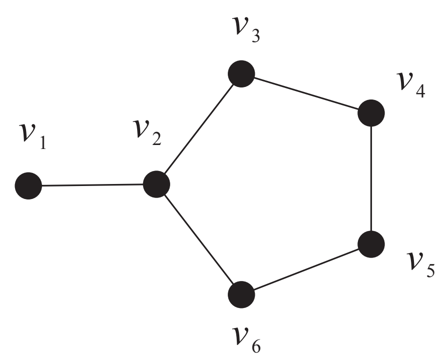

Figure 3.1: The graph G of Example 3.2.1

Example 3.2.1. Consider the graph $G = (V, E)$ pictured in Figure 3.1. In this example we will find the space of well-covered weightings of G. To do this we create the hypergraph H on vertex set V whose edges are the maximal independent sets of G. The hypergraph H is pictured in Figure 3.2. The stable weightings of H are the well-covered weightings of G, so we find $U(H, \mathbb{F})$.

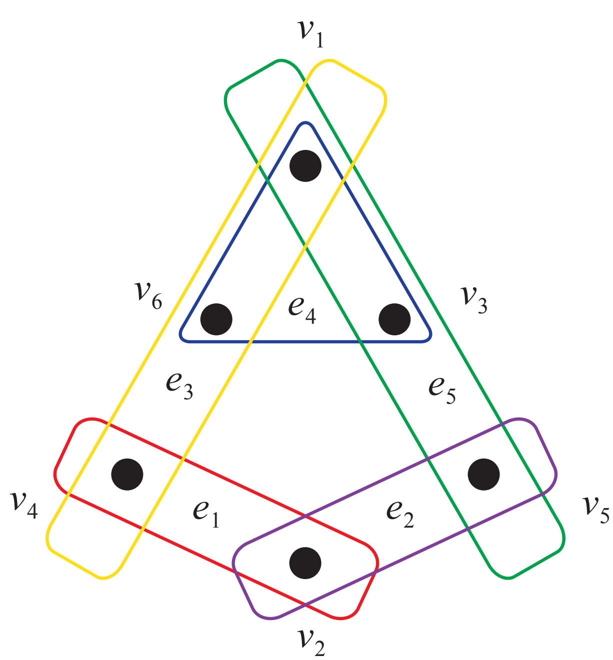

Figure 3.2: The hypergraph H whose edges are the maximal independent sets of G

We label the vertices and edges of H as in Figure 3.2, so that we have:

$$e_1 = \{v_2, v_4\}$$
$$e_2 = \{v_2, v_5\}$$
$$e_3 = \{v_1, v_4, v_6\}$$
$$e_4 = \{v_1, v_3, v_6\}$$
$$e_5 = \{v_1, v_3, v_5\}$$

which one easily verifies to be the maximal independent sets of G. We now find the stable weightings of H. By definition, $f : V \to \mathbb{F}$ is a stable weighting of H if and only if its weight on every edge is equal to some $k \in \mathbb{F}$. So we must solve the linear system:

$$v_2 + v_4 = k, \tag{3.1}$$
$$v_2 + v_5 = k, \tag{3.2}$$
$$v_1 + v_4 + v_6 = k, \tag{3.3}$$
$$v_1 + v_3 + v_6 = k, \tag{3.4}$$
$$v_1 + v_3 + v_5 = k, \tag{3.5}$$

(We are abusing notation slightly and allowing v_i to denote the weight on vertex v_i.) All we have left to do in this example is to solve this system.

From equations 3.1 and 3.2 we see that $v_4 = v_5$. Similarly from equations 3.3, 3.4, and 3.5 we get $v_3 = v_4$ and $v_5 = v_6$, so $v_3 = v_4 = v_5 = v_6$. Now letting $v_1 = a$ and $v_3 = b$, we obtain $k = a + 2b$ from equations 3.3, 3.4, and 3.5, while equations 3.1 and 3.2 both tell us that $v_2 = a + b$. Therefore, $\text{udim}(H, \mathbb{F}) \leq 2$. We now define two weightings f_1 and f_2 which we claim to be stable and linearly independent.

$$f_1(v_i) = \begin{cases} 1 & \text{if } i = 1, 2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_2(v_i) = \begin{cases} 0 & \text{if } i = 1 \\ 1 & \text{otherwise} \end{cases}$$

Under f_1, the weight on each edge of H is 1, since exactly one of the vertices v_1 and v_2 is in each edge of H. Under f_2, the weight on each edge of H is 2 by inspection. Thus f_1 and f_2 are both stable weightings of H. It is also easy to see that the set $\{f_1, f_2\}$ is linearly independent, so it must form a basis for $U(H, \mathbb{F})$.

Remark 3.2.1. We can eliminate the reference to k in the linear system of the above example by setting the left side of equations 3.1 through 3.4 equal to the left side of equation 3.5, obtaining:

$$\begin{aligned} v_2 + v_4 &= v_1 + v_3 + v_5, \\ v_2 + v_5 &= v_1 + v_3 + v_5, \\ v_1 + v_4 + v_6 &= v_1 + v_3 + v_5, \\ v_1 + v_3 + v_6 &= v_1 + v_3 + v_5, \end{aligned}$$

and then subtracting to obtain:

$$\begin{aligned} -v_1 + v_2 - v_3 + v_4 - v_5 &= 0, \\ -v_1 + v_2 - v_3 &= 0, \\ -v_3 + v_4 - v_5 + v_6 &= 0, \\ -v_5 + v_6 &= 0. \end{aligned}$$

This is a simple way to express the linear system that we wish to solve, and using Gauss-Jordan elimination we can verify that $\{f_1, f_2\}$ is a basis for the uniformity space of Example 3.2.1. This process is generalized in the next section.

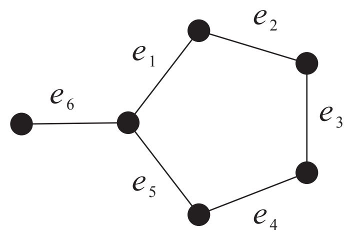

Figure 3.3: The graph G with edges labeled

Example 3.2.2. In this example we derive a different hypergraph from G. We show G in Figure 3.3 with the edges labeled, and let $M = (E, T)$ be the graph whose vertices are the edges of G, and whose edges are the spanning trees of G. M is shown in Figure 3.4.

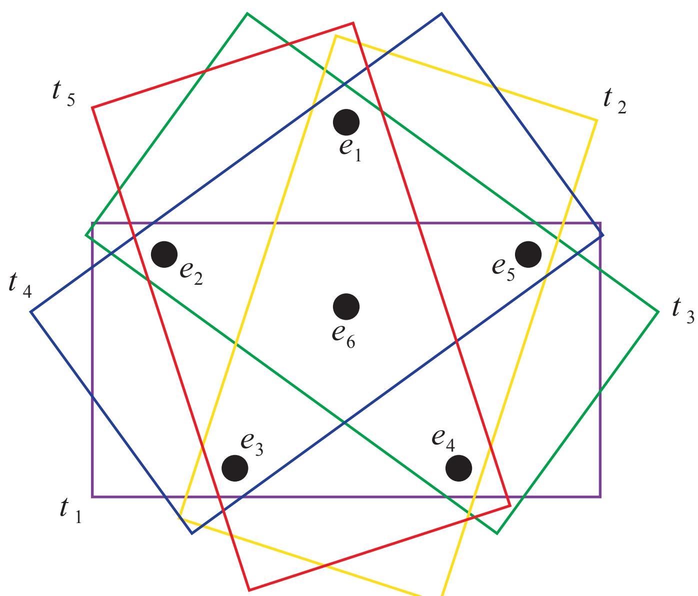

Figure 3.4: The hypergraph M whose edges are the spanning trees of G

The edges of M are exactly the bases of the graphic matroid $M(G)$ of G. We consider the uniformity space of basis hypergraphs of matroids more generally in Section 5.2. In this example, we will find the uniformity space of M. We have labeled the edges of M as t_1, t_2, t_3, t_4, and t_5 so that the index i corresponds to the only vertex e_i of M *not* contained in the edge t_i.

We first note that every edge of M (i.e. every spanning tree of G) has size 5. This means that any constant weighting of E is a stable weighting of M. Thus the weighting $f : E \to \mathbb{F}$ defined by $f(e_i) = 1$ for all $i \in \{1, 2, ..., 6\}$ is a stable weighting of M.

Next we note that e_6 is in every edge of M. Thus the weighting $f_6 : E \to \mathbb{F}$ defined by:

$$f_6(e_i) = \begin{cases} 1 & \text{if } i = 6 \\ \\ 0 & \text{otherwise} \end{cases}$$

is a stable weighting of M since the weight on every edge of M is 1.

We now claim that the weight on the remaining vertices $\{e_1, e_2, e_3, e_4, e_5\}$ must be equal under any stable weighting of M. For $i, j \in \{1, 2, 3, 4, 5\}$, the weight on edges t_i and t_j must be equal under any stable weighting of M by definition, so if $g : E \to \mathbb{F}$ is a stable weighting of M, we must have:

$$g(t_i) = g(t_j) \Leftrightarrow \left[\sum_{k=1}^{6} g(e_k)\right] - g(e_i) = \left[\sum_{k=1}^{6} g(e_k)\right] - g(e_j) \Leftrightarrow g(e_i) = g(e_j).$$

This completes the proof of our claim, and tells us that $\text{udim}(M, \mathbb{F}) \leq 2$. Since the set $\{f, f_6\}$ is clearly linearly independent, it must form a basis for $U(M, \mathbb{F})$, and $\text{udim}(M, \mathbb{F}) = 2$.

3.3 An Approach Using Matrices

We saw in Section 3.2 that given a hypergraph H, finding the uniformity space $U(H, \mathbb{F})$ involves solving a system of linear equations. We have the following approach to finding $U(H, \mathbb{F})$ using matrices.

Definition 3.3.1. Let $H = (V, E)$ be a hypergraph with $V = \{v_1, v_2, ..., v_n\}$ and $E = \{e_1, , e_2, ..., e_m\}$. Then the *incidence matrix* of H is the $n \times m$ matrix D defined as follows:

$$D_{ij} = \begin{cases} 1 & \text{if } v_i \in e_j \\ \\ 0 & \text{otherwise} \end{cases}$$

We denote the columns of D by $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_m$, so that:

$$D = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_m \end{bmatrix}$$

A weighting of H is a map $f : V \to \mathbb{F}$, so we can identify with the weighting f the vector $\mathbf{f} \in \mathbb{F}^n$ defined by $\mathbf{f}_j = f(v_j) \in \mathbb{F}$. We have:

$$\mathbf{e}_i \cdot \mathbf{f} = \sum_{j=1}^{n} \mathbf{e}_{i_j} \mathbf{f}_j = \sum_{v_j \in e_i} \mathbf{f}_j = \sum_{v_j \in e_i} f(v_j),$$

so the dot product $\mathbf{e}_i \cdot \mathbf{f}$ is the weight on edge e_i. Then the product $D^t \mathbf{f}$ gives a vector in $\mathbb{F}^m$ in which the i^{th} entry is the weight on edge e_i. Thus f is a stable weighting of H if and only if $D^t \mathbf{f} = k\mathbf{1}$, where $\mathbf{1}$ denotes the all 1's vector in $\mathbb{F}^m$, and k is any element of $\mathbb{F}$.

As mentioned in Remark 3.2.1, we can eliminate the reference to k by setting the weight on one edge equal to the weight on all other edges. Then f is a stable weighting of H if and only the following system of equations is satisfied:

$$\begin{aligned} \mathbf{e}_1 \cdot \mathbf{f} - \mathbf{e}_m \cdot \mathbf{f} &= 0 \\ \mathbf{e}_2 \cdot \mathbf{f} - \mathbf{e}_m \cdot \mathbf{f} &= 0 \\ &\vdots \\ \mathbf{e}_{m-1} \cdot \mathbf{f} - \mathbf{e}_m \cdot \mathbf{f} &= 0 \end{aligned}$$

That is, if and only if $\mathbf{f}$ is in the nullspace of the matrix W_H, defined by:

$$W_H = \begin{bmatrix} (\mathbf{e}_1 - \mathbf{e}_m)^t \\ (\mathbf{e}_2 - \mathbf{e}_m)^t \\ \vdots \\ (\mathbf{e}_{m-1} - \mathbf{e}_m)^t \end{bmatrix}$$

This means that $U(H, \mathbb{F})$ is exactly nullspace(W_H), and udim$(H, \mathbb{F}) =$ nullity(W_H). For this reason we call W_H the *solution matrix* of H. We will use both the incidence matrix and solution matrix to prove the main theorem of Section 5.3.

Given a hypergraph $H = (V, E)$, finding the uniformity space of H involves setting up the solution matrix and solving for the nullspace. The nullspace of an $n \times m$ matrix can be found in polynomial time using a number of different methods. One such method is to find the appropriate part of the singular value decomposition (SVD). The Golub-Reinsch SVD algorithm will find the nullspace of an $m \times n$ matrix using approximately $4m^2n - 8mn^2$ operations ([9], pg. 239). The problem may become more complex when we are not given a hypergraph directly, as in the examples of Section 3.2. For example, given a graph G with n vertices and m edges (we have $m \leq \binom{n}{2} < n^2/2$ necessarily), G may have up to n^{n-2} spanning trees. Thus finding all stable weightings of the spanning trees of G by the method of this section involves setting up and solving a matrix that could have up to $n^{n-2} - 1$ rows! For problems such as this we wish to develop other methods for finding the uniformity space. In the problem above we would prefer to find a characteristic of the graph itself that tells us what the stable weightings of its spanning trees look like without actually enumerating them. We do just this in Section 5.2. In the next section we present some basic results that will be important for much of our later work.

3.4 Basic Results

The first result that we present was proved before in [5] for the special case of well-covered weightings. It states a sufficient condition for two vertices of a hypergraph H to have the same weight under any stable weighting of H.

Lemma 3.4.1 (The Interchange Property)**.** *Let $H = (V, E)$ be a hypergraph, and let $x, y \in V$. If there exists a set $Z \subseteq V$ such that $x, y \notin Z$ and both $Z \cup x$ and $Z \cup y$ are edges of H, then for any stable weighting $f : V \to \mathbb{F}$ of H, $f(x) = f(y)$. In this case we say that x and y have the interchange property, with interchange set Z.*

Proof. Since $Z \cup x$ and $Z \cup y$ are both edges of H, they must have the same weight under any stable weighting by definition. So for any stable weighting $f : V \to \mathbb{F}$ we have:

$$f(x) + \sum_{t \in Z} f(t) = f(y) + \sum_{t \in Z} f(t) \quad \Longrightarrow \quad f(x) = f(y).$$

□

We will use Lemma 3.4.1 extensively, and will always refer to it as the interchange property. We see that this is the property that the vertices $e_1, e_2, ..., e_5$ of M had in Example 3.2.2. Note that we did not prove that the interchange property is necessary for two vertices to have the same weight under any stable weighting. We will see later on that it is not, in fact, necessary. We first use the interchange property to find the uniformity dimension of complete hypergraphs. We define the *complete l-uniform hypergraph $K_{n,l}$* to be the hypergraph on vertex set $V = \{v_1, v_2, ..., v_n\}$ with edge set $V^{(l)}$, i.e. with edges all l-subsets of V.

Corollary 3.4.2. *Let $K_{n,l}$ be the complete l-uniform hypergraph on n vertices, where $n \geq l > 0$, and $n, l \in \mathbb{N}$. Let $\mathbb{F}$ be any field. If $n = l$, then $udim(K_{n,l}, \mathbb{F}) = n$. Otherwise, if $n > l$, then $udim(K_{n,l}, \mathbb{F}) = 1$.*

Proof. Firstly, if $n = l$, then $K_{n,l}$ is the hypergraph on vertex set $V = \{v_1, v_2, ..., v_n\}$ with the single edge V. Any weighting is stable on a hypergraph with a single edge, so $\mathrm{udim}(K_{n,l}, \mathbb{F}) = n$ in this case.

Now suppose that $n > l$. If $n = 1$ then $l = 1$ by the assumption that $l > 0$, so we can assume that $n > 1$. Without loss of generality take two vertices v_1 and v_2 in the vertex set V of $K_{n,l}$. Now take any $(l-1)$-subset of $V \backslash \{v_1, v_2\}$ and call it Z. Such a set exists since $n > l$, so $n - 2 \geq l - 1$. Now $Z \cup v_1$ and $Z \cup v_2$ are both edges of $K_{n,l}$ since every l-subset of V is an edge of $K_{n,l}$ by definition. Therefore, v_1 and v_2 have the interchange property with interchange set Z. Since v_1 and v_2 were arbitrary, we can conclude that the weight on each vertex of $K_{n,l}$ must be the same. This tells us that $\mathrm{udim}(K_{n,l}, \mathbb{F}) \leq 1$. We now claim that the constant weighting $f : V \to \mathbb{F}$ defined by $f(v) = 1$ for all $v \in V$ is stable on $K_{n,l}$. For any edge e of $K_{n,l}$, the weight on e is $\displaystyle\sum_{v \in e} f(v) = l \cdot 1 = l$ since every edge of $K_{n,l}$ contains exactly l vertices. Thus f is stable, and since f is nonzero, this proves that $\mathrm{udim}(K_{n,l}, \mathbb{F}) = 1$. □

Lemma 3.4.3. *Let $H = (V, E)$ be a hypergraph with a set of isolated vertices $V_I = \{v_1, v_2, ..., v_s\} \subseteq V$. Define $H' = (V \backslash V_I, E)$, the same hypergraph but with the isolated vertices removed. Then $U(H, \mathbb{F}) = U(H', \mathbb{F}) \oplus \mathbb{F}^s$ for any field $\mathbb{F}$.*

Proof. Each vertex in V_I is not contained in any edge of H, so the system of linear equations whose solution is the uniformity space of H does not involve any of the vertices in V_I. Thus the system of equations for $U(H, \mathbb{F})$ is the same as the system of equations for $U(H', \mathbb{F})$. The only difference is that for $U(H, \mathbb{F})$, each of the vertices in V_I is a free variable in the solution. □

In particular, Lemma 3.4.3 tells us that $\mathrm{udim}(H, \mathbb{F}) = \mathrm{udim}(H', \mathbb{F}) + s$ in the notation of the lemma. It tells us that we can focus primarily on hypergraphs with no isolated vertices.

Definition 3.4.1. Let $H = (V, E)$ be a hypergraph. Define the set $E^* = \{V \backslash e \mid e \in E\}$. The hypergraph $H^* = (V, E^*)$ is called the *hypergraph-complement* of H.

The definition is similar to that of matroid dual. Given a matroid M, the *dual matroid* M^* is the matroid on the same underlying set that has B as a basis if and only if $V \backslash B$ is a basis of M. Note that Definition 3.4.1 does not coincide with the usual definition of the *complement* of a graph. We try to keep the two concepts separate by calling this the hypergraph-complement. It is easy to see that $(H^*)^* = H$, which makes it easy to prove that certain properties hold for a hypergraph H if and only if they hold for H^*. The next lemma tells us that the stable weightings of H are the same as the stable weightings of H^*.

Lemma 3.4.4. *Let $H = (V, E)$ be a hypergraph. A weighting $f : V \to \mathbb{F}$ is a stable weighting of H if and only if it is a stable weighting of H^*. Thus $U(H, \mathbb{F}) = U(H^*, \mathbb{F})$.*

Proof. Since $(H^*)^* = H$, it is sufficient to prove only the forward direction. So suppose $f : V \to \mathbb{F}$ is a stable weighting of H, and let the weight on each edge be $k \in \mathbb{F}$. Now the weight $\sum_{v \in V} f(v)$ on the entire set V is a constant, say $l \in \mathbb{F}$. Thus the weight on each edge of H^* must be $l - k$. Thus we conclude that f is also a stable weighting of H^*. $\square$

Using this lemma, when we were finding the uniformity space of the hypergraph M in Example 3.2.2, we could have considered the much simpler looking hypergraph-complement M^*, pictured in Figure 3.5. From now on, when we are trying to find a hypergraph's uniformity space, we can work with the hypergraph-complement if it is easier.

It is surprising how much the simple results presented in this section will help us. In the next section we study general k-uniform hypergraphs. There we describe the stable weightings of all 1-uniform and 2-uniform hypergraphs, and the stable

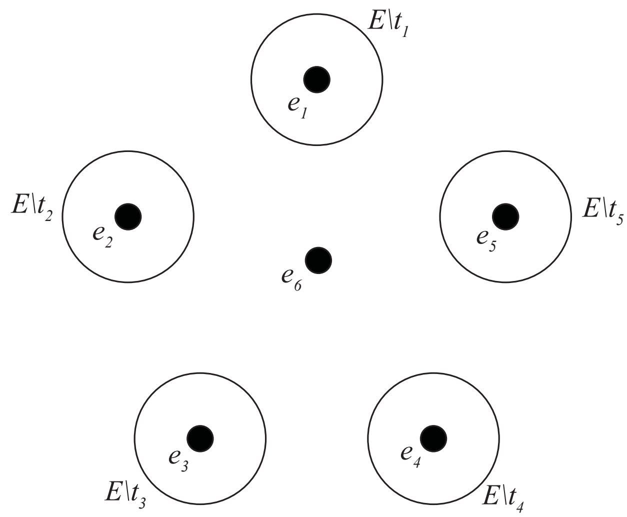

Figure 3.5: The hypergraph-complement M^* of M

weightings of all $(n-1)$-uniform and $(n-2)$-uniform hypergraphs are then described easily due to Lemma 3.4.4.

Chapter 4

l-Uniform Hypergraphs

In this chapter we consider stable weightings of l-uniform hypergraphs. We describe the uniformity space of every 1-uniform and 2-uniform hypergraph, and we present some results for 3-uniform hypergraphs with small vertex sets. We then consider the uniformity space of random l-uniform hypergraphs. First we have a result that holds for all l-uniform hypergraphs.

Proposition 4.0.1. *Let $H = (V, E)$ be an l-uniform hypergraph for some $l \in \mathbb{N}$, and let $\mathbb{F}$ be a field. Define the map $f : V \to \mathbb{F}$ by $f(v) = t \in \mathbb{F}$ for all $v \in V$. Then f is a stable weighting of H, and thus udim$(H, \mathbb{F}) \geq 1$.*

Proof. Each edge of H has l vertices, so the weight on each edge of H under f is lt. Thus f is a stable weighting of H. Taking $t \neq 0$ we conclude that $\text{udim}(H, \mathbb{F}) \geq 1$ since we have found a nonzero stable weighting of H. □

So the uniformity dimension of every l-uniform hypergraph is at least 1. We move forward by describing the uniformity dimension of 1-uniform hypergraphs in the next section.

4.1 1-Uniform Hypergraphs

For any 1-uniform hypergraph $H = (V, E)$, we wish to determine $U(H, \mathbb{F})$ over any field $\mathbb{F}$. Essentially we can weight the isolated vertices of H with anything we wish, but we must weight all vertices that are contained in some edge with the same weight.

Proposition 4.1.1. *Let $H = (V, E)$ be a 1-uniform hypergraph with n vertices and m edges. Then for any field $\mathbb{F}$, we have $udim(H, \mathbb{F}) = n - m + 1$ if $m > 0$ and $udim(H, \mathbb{F}) = n$ if $m = 0$.*

Proof. In the case $m = 0$, every weighting of H is a stable weighting, so we can conclude immediately that $\mathrm{udim}(H, \mathbb{F}) = n$. Now we assume that $m > 0$. We find a basis for the uniformity space of H over $\mathbb{F}$. The edges of H are all singleton sets, so let $\{v_1\}, \{v_2, \}, ..., \{v_m\}$ be the edges of H. If $f : V \to \mathbb{F}$ is a stable weighting of H, then $f(v_1) = f(v_2) = ... = f(v_m)$ by definition. So $\mathrm{udim}(H, \mathbb{F}) \leq n - m + 1$. We now present $n - m + 1$ linearly independent stable weightings of H, which must then form a basis for $U(H, \mathbb{F})$. Define $f_E : V \to \mathbb{F}$ by:

$$f_E(v) = \begin{cases} 1 & \text{if } \{v\} \in E \\ 0 & \text{otherwise} \end{cases}$$

and if $v_{m+1}, v_{m+2}, ..., v_n$ are the vertices not contained in any edge of H, for $i \in \{m + 1, m + 2, ..., n\}$ define $f_i : V \to \mathbb{F}$ by:

$$f_i(v) = \begin{cases} 1 & \text{if } v = v_i \\ 0 & \text{otherwise} \end{cases}$$

The set $\{f_E, f_{m+1}, f_{m+2}, ..., f_n\}$ is a set of $n - m + 1$ weightings of H, and it is clear that they are linearly independent, since each $v \in V$ has a nonzero weight in exactly one of the weightings. It is also easy to see that they are all stable weightings. The weight on each edge under f_E is 1, and the weight on each edge under all $n - m$ other weightings is 0. This means that $\{f_E, f_{m+1}, ..., f_n\}$ is a basis for $U(H, \mathbb{F})$ and $\mathrm{udim}(H, \mathbb{F}) = n - m + 1$. □

Corollary 4.1.2. *Let $H = (V, E)$ be an $(n-1)$-uniform hypergraph with n vertices and m edges. Then for any field $\mathbb{F}$, we have udim$(H, \mathbb{F}) = n - m + 1$ if $m > 0$ and udim$(H, \mathbb{F}) = n$ if $m = 0$.*

Proof. The case $m = 0$ is again trivial. Thus we assume that the hypergraph-complement H^* of H is a 1-uniform hypergraph with n vertices and $m > 0$ edges. By Lemma 3.4.4, $U(H, \mathbb{F}) = U(H^*, \mathbb{F})$. Since udim$(H^*, \mathbb{F}) = n - m + 1$ by Proposition 4.1.1, udim$(H, \mathbb{F}) = n - m + 1$ as well. A basis for $U(H, \mathbb{F})$ is also easily found by considering H^*. Each vertex contained in every edge of H is not contained in any edge of H^*, so can be weighted with any entry from $\mathbb{F}$. All vertices not contained in every edge of H must be contained in some edge of H^*, and so these vertices must all carry the same weight. $\square$

The last corollary could also have been proved without Lemma 3.4.4 by using the interchange property. Such a proof would mirror the proof of Proposition 4.1.1 closely. The proof we have presented is shorter and more practical.

Having found a simple description for the uniformity space of any 1-uniform hypergraph, we proceed to the more complicated situation of 2-uniform hypergraphs.

4.2 2-Uniform Hypergraphs

Let $G = (V, E)$ be a 2-uniform hypergraph, or in other words a simple graph. We wish to find the stable weightings of G. We begin with a lemma that will be very helpful.

Lemma 4.2.1. *Let G be as above, and suppose $f : V \to \mathbb{F}$ is a stable weighting of G. If $u, v \in V$ and there is a walk of even length (a walk with an even number of edges) between u and v in G, then $f(u) = f(v)$.*

Proof. The proof is by induction on r, where $2r$ is the length of the walk from u to v in G.

For the base case, when there is a walk of length 2 from u to v, let x be the intermediate vertex of the walk. Then $\{u, x\}$ and $\{v, x\}$ are both edges of G. So u and v have the interchange property, and thus $f(u) = f(v)$.

The inductive hypothesis is that every two vertices with a walk of length $2(r-1)$ between them have the same weight under any stable weighting of G. Now suppose there is a walk of length $2r$ between u and v. Then there is a walk of length $2(r-1)$ from u to some vertex w and a walk of length 2 from w to u. By the inductive hypothesis, $f(u) = f(w)$, and by the base case $f(w) = f(v)$, so we conclude that $f(u) = f(v)$. Therefore, the result holds by mathematical induction. $\square$

We now consider the uniformity space of connected graphs, and will extend these results to general graphs shortly afterwards.

Theorem 4.2.2. *Let $G = (V, E)$ be a connected simple graph and let $\mathbb{F}$ be a field. Then if G has order 1, udim$(G, \mathbb{F}) = 1$. If G has order at least 2, udim$(G, \mathbb{F}) = 2$ if G is bipartite, and udim$(G, \mathbb{F}) = 1$ otherwise.*

Proof. We begin by supposing that G has order at least 2. First suppose G is bipartite with bipartition sets V_1 and V_2. Since G has order at least 2 and is connected, V_1 and V_2 must be nonempty. Let $v, v' \in V_1$. Since G is connected, there is a walk from v to v'. Further, this walk must be of even length since v and v' are in the same bipartition set. So by Lemma 4.2.1, $f(v) = f(v')$ for any stable weighting $f : V \to \mathbb{F}$. Thus any two vertices of V_1 must have the same weight, and similarly for any vertices of V_2. So $\mathrm{udim}(G, \mathbb{F}) \leq 2$. For $i = 1, 2$, define $g_i : V \to \mathbb{F}$ by

$$g_i(v) = \begin{cases} 1 & \text{if } v \in V_i \\ 0 & \text{otherwise} \end{cases}$$

and g_i is a stable weighting of G for $i = 1, 2$, since the weight on each edge of G is 1. The weightings g_1 and g_2 are clearly linearly independent, so $\mathrm{udim}(G, \mathbb{F}) = 2$ and $\{g_1, g_2\}$ is a basis for the uniformity space of G.

Suppose now that G has order at least 2 but that G is not bipartite. This means that G contains an odd cycle C. There is a path of even length between any two vertices of C, so all vertices of C have the same weight under any stable weighting $f : V \to \mathbb{F}$ of G. Now if v is any vertex of G not contained in the cycle C, then take any vertex x of C. There is a walk between v and x since G is connected. If this walk has even length, then $f(v) = f(x)$ by Lemma 4.2.1. Otherwise, if the walk has odd length, then there is an even walk from v to each of the neighbours of x in C. Since all vertices in C have the same weight, $f(v) = f(x)$ again. So all vertices of G must have the same weight under any stable weighting if G is connected and not bipartite. This means that $\mathrm{udim}(G, \mathbb{F}) \leq 1$. Since the constant weightings are always stable for uniform hypergraphs, $\mathrm{udim}(G, \mathbb{F}) = 1$, with basis any single constant weighting.

The case where G has order 1 is easily handled. Since G is simple it can have no edges, so any weighting of the single vertex must be stable. Thus $\mathrm{udim}(G, \mathbb{F}) = 1$. $\square$

We now extend our results to when G is not necessarily connected. Using Lemma 3.4.3, we can ignore the isolated vertices for the moment, because we know exactly what they add to the uniformity space. So we consider components of G with nonempty edge set. That is, components of order at least 2. The weight on each edge must be constant, say $k \in \mathbb{F}$. This means that for a connected component that is not bipartite, each vertex must have weight $k/2$ (as long as $\mathrm{char}(\mathbb{F}) \neq 2$). For each connected component C that has order at least 2 and is bipartite, say with bipartition sets C_1 and C_2, without loss of generality the vertices in C_1 can be weighted with any $l_C \in \mathbb{F}$, and then the vertices of C_2 must be weighted with $k - l_C$.

So in general, the uniformity dimension of a graph G with at least one edge over any field of characteristic not equal to 2 is

$$1 + \# \text{ of bipartite components of } G \text{ of order at least } 2 + \# \text{ of isolated vertices of } G$$

$$= 1 + \# \text{ of bipartite components of } G.$$

Over a field $\mathbb{F}$ of characterstic 2, if G has only bipartite components, then the argument presented above still works. We treat the remaining case where $\text{char}(\mathbb{F}) = 2$ and G has at least one non-bipartite component separately.

Over a field $\mathbb{F}$ of characterstic 2, if G has a non-bipartite component H, the proof of Theorem 4.2.2 shows that each vertex of H must have the same weight under any stable weighting. Then since $\text{char}(\mathbb{F}) = 2$, the weight on each edge of H, and therefore each edge of G, must be 0. For a bipartite component K of G with bipartition sets K_1 and K_2, every vertex of K_1 must have the same weight, and likewise for K_2. Further, since the weight on each edge must be 0 and we are working over characteristic 2, the weight on the vertices of K_1 must be the same as the weight on the vertices of K_2. This proves that for any component of G, all of its vertices must have the same weight, so $\text{udim}(G, \mathbb{F}) \leq s$, the number of components of G. Now if $G_1, G_2, ..., G_s$ are the components of G, define the weighting $f_i : V \to \mathbb{F}$ by:

$$f_i(v) = \begin{cases} 1 & \text{if } v \in G_i \\ 0 & \text{otherwise} \end{cases}$$

Each of the f_i's is stable because the weight on every edge of G is 0, and the set $\{f_1, f_2, ..., f_s\}$ is clearly linearly independent, so it is a basis for $U(G, \mathbb{F})$.

We sum up these results in the following proposition.

Proposition 4.2.3. *Let G be a simple graph with nonempty edge set, and with s components, exactly r of which are bipartite (including isolated vertices). Let $\mathbb{F}$ be* *any field. If $char(\mathbb{F}) = 2$ and G has at least one non-bipartite component (i.e. at least one odd cycle), then $udim(G, \mathbb{F}) = s$. If $char(\mathbb{F}) = 2$ and G is bipartite, then $udim(G, \mathbb{F}) = 1 + s$. If $char(\mathbb{F}) \neq 2$ then $udim(G, \mathbb{F}) = 1 + r$.* □

Remark 4.2.1. In the notation of Proposition 4.2.3, if G has empty edge set, it has uniformity dimension n, since every weighting of G is stable in this case.

Now let $H = (V, E)$ be an $(n-2)$-uniform hypergraph on n vertices. By Lemma 3.4.4, the uniformity space of H is the same as the uniformity space of its hypergraph-complement H^*, and H^* is a simple graph. So to find the uniformity space of H, we find the hypergraph-complement H^* and can find its uniformity dimension using Proposition 4.2.3.

We are now able to find the uniformity dimension of any 1, 2, $(n-1)$, or $(n-2)$-uniform hypergraph without explicitly solving the linear system. We do not achieve this type of general result for l-uniform hypergraphs with $l \geq 3$, but we do present some computational results for 3-uniform hypergraphs with at most 6 vertices in the next section.

4.3 3-Uniform Hypergraphs

We wish to learn something about the uniformity dimension of 3-uniform hypergraphs, even if we cannot achieve a general result concerning them. In this section we work over a field of characteristic 0. We first wrote code in Maple to enumerate the edge sets of all non-isomorphic 3-uniform hypergraphs on n vertices for $n = 4, 5, 6$. The program takes as input the set of edge sets of all non-isomorphic 3-uniform hypergraphs on n vertices with m edges and returns the set of edge sets of all non-isomorphic 3-uniform hypergraphs on n vertices with $m + 1$ edges, for appropriate m. This program runs very slowly, and this is why the size of n is restricted. Then we wrote a program which takes as input the set of edge sets of all non-isomorphic

3-uniform hypergraphs on n vertices with m edges, and returns a list of numbers where the k^{th} entry is the number of non-isomorphic 3-uniform hypergraphs on n vertices with m edges and uniformity dimension k. All relevant Maple code can be found in Appendix A. The results are presented in Tables 4.1, 4.2, and 4.3. In these tables, zero entries are omitted for ease of reading.

m \ k	1	2	3	4
0				1
1				1
2			1	
3		1		
4	1			

Table 4.1: The number of non-isomorphic 3-uniform hypergraphs on 4 vertices with m edges and uniformity dimension k

m \ k	1	2	3	4	5
0					1
1					1
2				2	
3			4		
4		5	1		
5	4	2			
6	4	2			
7	4				
8	2				
9	1				
10	1				

Table 4.2: The number of non-isomorphic 3-uniform hypergraphs on 5 vertices with m edges and uniformity dimension k

We started at $n = 4$ because the 3-uniform hypergraphs on 3 or less vertices all have at most 1 edge. Therefore the uniformity dimension of any 3-uniform hypergraph on $n \leq 3$ vertices has uniformity dimension equal to n over any field $\mathbb{F}$.

Table 4.1 seems rather trivial, and could easily be computed by hand. Keeping in mind Lemma 3.4.4, we could even consider 1-uniform hypergraphs on 4 vertices, which makes the table very easy to compute indeed. For $n = 5$, Lemma 3.4.4 could again be used. The hypergraph-complement of any 3-uniform hypergraph on 5 vertices is a simple graph. It is easy to find a list of non-isomorphic graphs on 5 vertices and find their uniformity dimension using the characterization given in the previous section. So Table 4.2 could also be verified by hand.

However, Table 4.3 is not so easy to verify by hand. Lemma 3.4.4 is no longer of any use, since the hypergraph-complement of a 3-uniform hypergraph on 6 vertices is still a 3-uniform hypergraph. Given the large number of non-isomorphic 3-uniform hypergraphs on 6 vertices (there are 2136 of them!), the computation of Table 4.3 is much better left to a computer.

m \ k	1	2	3	4	5	6
0						1
1						1
2					3	
3				7		
4			19	2		
5		34	9			
6	45	42	7			
7	122	38	1			
8	223	25	1			
9	300	12				
10	345	7				
11	311	1				
12	248	1				
13	161					
14	94					
15	43					
16	21					
17	7					
18	3					
19	1					
20	1					

Table 4.3: The number of non-isomorphic 3-uniform hypergraphs on 6 vertices with m edges and uniformity dimension k

Now we consider the information contained in Tables 4.1, 4.2, and 4.3. We first notice that every 3-uniform hypergraph on n vertices ($n = 4, 5, 6$) with uniformity dimension $k < n$ contains at least $n - k + 1$ edges. This is a bound that holds for hypergraphs in general, as we show later on in Lemma 6.1.3. The 3-uniform hypergraphs of dimension $k < n$ that contain *exactly* $n - k + 1$ edges for $n = 4, 5$, and 6 lie on what we call the *critical diagonal* of Tables 4.1, 4.2, and 4.3 respectively. Any hypergraph lying on the critical diagonal satisfies a property which we study in Chapter 6, called *criticality*. That is, if we remove any edge from a hypergraph on the critical diagonal, its uniformity dimension must increase. In Chapter 6 we find out exactly when a hypergraph has critical uniformity dimension.

We can also study the rows and columns of Tables 4.1, 4.2, and 4.3 as integer sequences. The nonzero entries of the rows are all decreasing from left to right, and the nonzero entries of the columns are all unimodal sequences. (Unimodal sequences are those that are increasing and then decreasing. Formally, a sequence $(s_1, s_2, ..., s_n)$ is unimodal if there exists a $t \in \{1, 2, ..., n\}$ such that $s_1 \leq s_2 \leq ... \leq s_t$ and $s_t \geq s_{t+1} \geq ... \geq s_n$.) Are these statements true for $n > 6$? Are they true in general for l-uniform hypergraphs for any $l \in \mathbb{N}$?

For a fixed l, n, and m we define the sequence $(r_{l,n,m}(k))_{k=1}^{k=n}$ where the k^{th} term is the number of non-isomorphic l-uniform hypergraphs on n vertices with m edges and uniformity dimension k. Similarly, for a fixed l, n, and k, we define the sequence $(c_{l,n,k}(m))_{m=0}^{m=\binom{n}{l}}$ where the m^{th} term is the number of non-isomorphic l-uniform hypergraphs on n vertices with uniformity dimension k and m edges. So for example, $(r_{3,6,5}(k))_{k=1}^{k=6} = (0, 34, 9, 0, 0, 0)$, the 5^{th} row of Table 4.3, while $(c_{3,5,2}(m))_{m=0}^{m=10} = (0, 0, 0, 0, 5, 2, 2, 0, 0, 0, 0)$, the 2^{nd} column of Table 4.2. The questions that we asked above can now be rephrased as follows:

Are the nonzero entries of $(r_{l,n,m}(k))_{k=1}^{k=n}$ decreasing for all values of $l, n, m \in \mathbb{N}$ with $l \leq n$ and $m \leq \binom{n}{l}$? What if $l = 3$? Is $(c_{l,n,k}(m))_{m=0}^{m=\binom{n}{l}}$ unimodal for all values of

$l, n, k \in \mathbb{N}$ with $l, k \leq n$? What if $l = 3$? We show that $(c_{l,n,k}(m))_{m=0}^{m=\binom{n}{l}}$ has no internal zeros in Chapter 6.

Lastly, in Table 4.3 especially, we notice that the number of non-isomorphic 3-uniform hypergraphs on n vertices and m edges with *highest* uniformity dimension (the rightmost nonzero entry in each row) is relatively small. The number and structure of these hypergraphs may be of interest. We present pictures of some of these hypergraphs in Appendix B.

In the next section we explore the uniformity dimension of random l-uniform hypergraphs, and learn something about the uniformity space of almost all l-uniform hypergraphs for fixed $l \geq 1$.

4.4 Random l-Uniform Hypergraphs

We create a random l-uniform hypergraph $H = (V, E)$ of order n by taking a set V of size n, then including each subset of V of size l (each l-subset of V) in E with probability $p \in (0, 1)$. Here we deal with a fixed l and a fixed p. Recall that we let $V^{(l)}$ denote the set of all l-subsets of V. When we say that *almost all* l-uniform hypergraphs satisfy a certain property, we mean that as $n \to \infty$, a random l-uniform hypergraph on n vertices has the property with probability approaching 1.

We wish to explore the uniformity dimension of random l-uniform hypergraphs as the size of the underlying set V of vertices, n, approaches infinity. Before doing so, we show that as n approaches infinity, the edges of a random l-uniform hypergraph on n vertices are unlikely to be the facets of the independence complex of a graph on n vertices. We do this because most of the work on the uniformity space problem to date has been on well-covered weightings, the stable weightings of the maximal independent sets of a graph. We are essentially showing that the uniformity dimension of most l-uniform hypergraphs has not been determined through previous research on well-covered weightings.

First we need to derive a complex from any (simple) l-uniform hypergraph $H = (V, E)$. Since H is l-uniform, no edge of H is contained in any other edge of H. Thus the set E is the set of facets of a complex, which we call the *associated* complex of H, and denote Δ_H. In this new terminology, we wish to show that the associated complex of an l-uniform hypergraph on n vertices is the independence complex of a graph on n vertices with probability 0 as n approaches infinity. We begin with a theorem concerning the circuits of an l-uniform hypergraph's associated complex, from which the desired result follows as a corollary.

Theorem 4.4.1. *Fix an $l \geq 1$ and let $H = (V, E)$ be a random l-uniform hypergraph, with $|V| = n$. Then every circuit of the associated complex Δ_H contains at least l vertices with probability 1 as n approaches infinity.*

Proof. We want to show that as n approaches infinity, every circuit of Δ_H has size strictly greater than $l-1$ with probability 1. To do this we show that every set $X \in V^{(l-1)}$ is contained in some facet $F \in \Delta_H$ (i.e. an edge $F \in E$) with probability 1 as n approaches infinity. For $X \in V^{(l-1)}$ let E_X be the event that $X \not\subset F$ for any $F \in E$. Let P be the probability that Δ_H contains no circuit of size strictly less than l. We have:

$$\begin{aligned}
P &= \mathrm{Prob}\left(\, \forall X \in V^{(l-1)}, \exists F \in E \text{ such that } X \subset F\right) \\
&= 1 - \mathrm{Prob}\left(\, \exists X \in V^{(l-1)} \text{ such that } \forall F \in E, X \not\subset F\right) \\
&= 1 - \mathrm{Prob}\left(\bigcup_{X \in V^{(l-1)}} E_X \right)
\end{aligned}$$

Now for the event E_X to occur, none of the l-subsets chosen when H was constructed can contain X. There are $n-(l-1) = n-l+1$ such l-subsets for any fixed X. Thus for a fixed X,

$$\text{Prob}(E_X) = (1-p)^{n-l+1}.$$

Using the fact that $|V^{(l-1)}| = \binom{n}{l-1}$ and that the probability of a union is less than or equal to the sum of the probabilities, we obtain:

$$\text{Prob}\left(\bigcup_{X \in V^{(l-1)}} E_X\right) \leq \binom{n}{l-1}(1-p)^{n-l+1} \leq n^{l-1}(1-p)^{n-l+1}.$$

The natural logarithm of the right side of the previous inequality is

$$(l-1)\ln n + (n-l+1)\ln(1-p).$$

We know that $\ln(1-p)$ is a negative constant, so $(n-l+1)\ln(1-p) \to -\infty$ as $n \to \infty$. The first term is clearly dominated by the second term since l is fixed, so the sum goes to $-\infty$. Therefore, $n^{l-1}(1-p)^{n-l+1} \to 0$ as $n \to \infty$. Thus

$$\text{Prob}\left(\bigcup_{X \in V^{(l-1)}} E_X\right) \to 0$$

as well, and we conclude that $P \to 1$ as $n \to \infty$. □

Corollary 4.4.2. *Let H be a random l-uniform hypergraph. As n approaches ∞, the circuits of the associated complex Δ_H are all of size l or $l+1$ with probability* 1*.*

Proof. Theorem 4.4.1 tells us that every circuit of Δ_H has size at least l with probability 1. It is easy to see that no circuit of Δ_H can have size strictly greater than $l+1$ because removing any single vertex of a circuit must create a face of Δ_H, and no face in Δ_H has size strictly greater than l. □

Corollary 4.4.3. *The edge set of a random l-uniform hypergraph $H = (V, E)$ on n vertices with $l \geq 3$ is the set of facets of the independence complex of a graph $G = (V, E')$ with probability* 0 *as $n \to \infty$.*

Proof. By Corollary 4.4.2, every circuit of the associated complex Δ_H of H has size l or $l+1$ with probability 1 as n approaches infinity. However, any independence complex of a graph with nonempty edge set must have circuits of size 2 (the two ends of any edge). Since $l \geq 3$, H is the set of facets of an independence complex with probability 0 as $n \to \infty$. □

Corollary 4.4.3 tells us that for $l \geq 3$, the uniformity dimension of most l-uniform hypergraphs has not been studied previously as part of research on well-covered weightings. We will shortly pursue this problem. We first show that the uniformity dimension of a random 1-uniform hypergraph on n vertices lies in a small interval around $n - pn + 1$ with probability approaching 1 as n goes to infinity. We then deal with the case $l \geq 2$, and show that as n grows large, the uniformity dimension of a random l-uniform hypergraph H on n vertices is 1 with probability 1. That is, the constant weightings of V are the only stable weightings of H.

Theorem 4.4.4. *Let H be a random* 1*-uniform hypergraph on n vertices, and let $\varepsilon > 0$. As n approaches infinity, the probability that $n - (p+\varepsilon)n + 1 \leq udim(H, \mathbb{F}) \leq n - (p - \varepsilon)n + 1$ approaches* 1.

Proof. Let the random variable $\mathcal{U}_n$ be the uniformity dimension of H and let the random variable $\mathcal{M}_n$ be the number of edges of H. By Proposition 4.1.1, we have

$$n - (p+\varepsilon)n + 1 \leq \mathcal{U}_n \leq n - (p-\varepsilon)n + 1 \quad \Longleftrightarrow \quad (p-\varepsilon)n \leq \mathcal{M}_n \leq (p+\varepsilon)n$$

Thus we are now interested in the probability that $\mathcal{M}_n$, a binomial random variable, is close to its mean. By an application of Hoeffding's Inequality (see [11]), we get:

$$\text{Prob}\left[(p-\varepsilon)n \leq \mathcal{M}_n \leq (p+\varepsilon)n\right] \geq 1 - 2e^{-2\varepsilon^2 n},$$

which clearly approaches 1 as n approaches infinity. $\square$

Next we deal with random l-uniform hypergraphs for a fixed $l \geq 2$. We can prove that almost all 2-uniform hypergraphs (i.e. simple graphs) have uniformity dimension 1 as follows: Almost all simple graphs have the property that for any pair of distinct vertices $\{x, y\}$, there is a third vertex $z \notin \{x, y\}$ which is adjacent to both x and y. (This is an easy exercise, and a similar but much more general property is shown to hold for almost all graphs in [3].) Any graph on at least 3 vertices with this property is connected and contains a 3-cycle, which is easy to show directly. Thus by Proposition 4.2.3, almost all graphs have uniformity dimension 1 over any field $\mathbb{F}$. Note that the characteristic makes no difference in this case. The next theorem generalizes this result, proving that for any fixed $l \geq 2$, almost all l-uniform hypergraphs have uniformity dimension 1.

Theorem 4.4.5. *For any fixed $l \geq 2$, an l-uniform hypergraph $H = (V, E)$ on n vertices has uniformity dimension 1 with probability 1 as n approaches infinity over any field $\mathbb{F}$. That is, almost all l-uniform hypergraphs have uniformity dimension 1 for any fixed $l \geq 2$.*

Proof. We will show that every pair of vertices of H has the interchange property with probability 1 as n approaches infinity. This means that any stable weighting f of H must be constant, i.e. $f(v) = t \in \mathbb{F}$ for all $v \in V$. Conversely, we know that these constant weightings are stable by Proposition 4.0.1. So this will prove that $\text{udim}(H, \mathbb{F}) = 1$ with probability 1 as n approaches infinity.

Let P be the probability that two vertices of V do not have the interchange property. For a subset $X \subset V$, and vertices y and z not in X, let $E_{X_{y,z}}$ be the event that y and z have the interchange property with X as the interchange set. That is, $y, z \notin X$ and $X \cup y, X \cup z$ are both edges of H. So we have:

$$\begin{aligned}
P &= \text{Prob}(\ \exists\ y, z \in V \text{ that do not have the interchange property}) \\
&= \text{Prob}\left(\bigcup_{\{y,z\} \subset V} \bigcap_{\substack{X \in V^{(l-1)} \\ y,z \notin X}} (X \cup y \text{ and } X \cup z \text{ are not both edges of } H) \right) \\
&= \text{Prob}\left(\bigcup_{\{y,z\} \subset V} \bigcap_{\substack{X \in V^{(l-1)} \\ y,z \notin X}} \neg E_{X_{y,z}} \right)
\end{aligned}$$

For a fixed subset $X \in V^{(l-1)}$ not containing vertices y and z, $\text{Prob}(\neg E_{Xy,z}) = 1 - p^2$, since $X \cup y$ and $X \cup z$ were chosen as edges of H independently with probability p. Further, given a fixed pair of vertices $y, z \in V$, for each $X \in V^{(l-1)}$ with $y, z \notin X$, the event $E_{X_{y,z}}$ occurs independently from all other such events, (e.g. $E_{X'_{y,z}}$ for $X' \in V^{(l-1)}, X' \neq X$) since each event involves the inclusion of different subsets of V as edges. By an easy counting argument there are $\binom{n-2}{l-1}$ subsets in $V^{(l-1)}$ not containing y or z. We also use the fact that a union of probabilities is less than the sum of those probabilities to get:

$$P \leq \binom{n}{2} (1 - p^2)^{\binom{n-2}{l-1}} \leq n^2 (1 - p^2)^{\binom{n-2}{l-1}}.$$

We now show that as $n \to \infty$, $P \to 0$. The natural logarithm of $n^2 (1 - p^2)^{\binom{n-2}{l-1}}$ is

$$2 \ln n + \binom{n-2}{l-1} \ln(1 - p^2).$$

Now since $p > 0$, we have $\ln(1 - p^2) < 0$. Thus the second term has negative sign and clearly dominates the first term as $n \to \infty$, so this expression approaches $-\infty$ as $n \to \infty$, and we see that $\lim_{n\to\infty} n^2(1-p^2)^{\binom{n-2}{l-1}} = 0$.

We conclude that $P \to 0$ as $n \to \infty$, so that every two vertices of V have the interchange property in the limit. □

Having discovered something about the uniformity dimension of almost all l-uniform hypergraphs for any fixed $l \geq 1$, we now turn to the expected value of the uniformity dimension of a random l-uniform hypergraph. We derive a formula for the expected value of the uniformity dimension of a random 1-uniform hypergraph on n vertices, and for $l \geq 2$ we show that the expected value of the uniformity dimension of a random l-uniform hypergraph on n vertices approaches 1 as n approaches infinity.

Theorem 4.4.6. *Let $H = (V, E)$ be a random 1-uniform hypergraph on n vertices. The expected value of udim$(H, \mathbb{F})$ is $n - pn + 1 - (1-p)^n$, where each singleton subset of V was included in H with probability p.*

Proof. Let the random variable $\mathcal{M}_n$ be the number of edges of H, and let the random variable $\mathcal{U}_n$ be the uniformity dimension of H (the dimension is the same over every field). By proposition 4.1.1, $\mathcal{U}_n = n - \mathcal{M}_n + 1$ if $\mathcal{M}_n > 0$. On the other hand, if $\mathcal{M}_n = 0$, then $\mathcal{U}_n = n$. By the definition of expected value, we have:

$$\begin{aligned} E(\mathcal{U}_n) &= \sum_{i=1}^{n} i \cdot P(\mathcal{U}_n = i) \\ &= n \cdot P(\mathcal{U}_n = n) + \sum_{i=1}^{n-1} i \cdot P(\mathcal{U}_n = i) \end{aligned}$$

At this point, we see that $\mathcal{U}_n = n$ if and only if $\mathcal{M}_n = 0$ or 1. Also, in the sum on the right we are now left with $\mathcal{U}_n < n$. When this is the case, $\mathcal{M}_n > 0$, so

$\mathcal{U}_n = n - \mathcal{M}_n + 1$. Thus we have:

$$E(\mathcal{U}_n) = n \cdot [P(\mathcal{M}_n = 0) + P(\mathcal{M}_n = 1)] + \sum_{i=1}^{n-1} i \cdot P(\mathcal{M}_n = n - i + 1)$$

Now we make the substitution $j = n - i + 1$ in the index of our sum, then move the term $n \cdot P(\mathcal{M}_n = 1)$ into the sum, and finally break up the sum:

$$\begin{aligned} E(\mathcal{U}_n) &= n \cdot P(\mathcal{M}_n = 0) + n \cdot P(\mathcal{M}_n = 1) + \sum_{j=2}^{n} (n - j + 1) \cdot P(\mathcal{M}_n = j) \\ &= n \cdot P(\mathcal{M}_n = 0) + \sum_{j=1}^{n} (n - j + 1) \cdot P(\mathcal{M}_n = j) \\ &= n \cdot P(\mathcal{M}_n = 0) + (n+1) \sum_{j=1}^{n} P(\mathcal{M}_n = j) - \sum_{j=1}^{n} j \cdot P(\mathcal{M}_n = j) \end{aligned}$$

We know that $\mathcal{M}_n$ is a binomial distribution in which n objects are included with probability p, and we use the well-known formulas (see [14], for example):

$$\begin{aligned} &P(\mathcal{M}_n = 0) = (1-p)^n \\ &\sum_{j=0}^{n} P(\mathcal{M}_n = j) = 1 \\ &\sum_{j=0}^{n} j \cdot P(\mathcal{M}_n = j) = pn \end{aligned}$$

to get:

$$\begin{aligned} E(\mathcal{U}_n) &= n(1-p)^n + (n+1)\left[1 - (1-p)^n\right] - [pn - 0] \\ &= n + 1 - (1-p)^n - pn \end{aligned}$$

Rearranging slightly we obtain the desired result. $\square$

Finding an explicit formula for the expected uniformity dimension of a random 1-uniform hypergraph on n vertices proved to be fairly difficult. We do not determine such a formula for l-uniform hypergraphs with $l \geq 2$, but we do find a bound on the expected uniformity dimension from which we are able to show that the expected uniformity dimension of a random l-uniform hypergraph with $l \geq 2$ on n vertices approaches 1 as n approaches infinity.

Theorem 4.4.7. *As n approaches infinity, the expected uniformity dimension of a random l-uniform hypergraph H with $l \geq 2$ on n vertices approaches* 1*.*

Proof. Let the random variable $\mathcal{U}_n$ be the uniformity dimension of H, a random l-uniform hypergraph on n vertices. Let P be the probability that there exist two vertices of H that do not have the interchange property. In the proof of Theorem 4.4.5 we showed that $P \leq n^2(1-p^2)^{\binom{n-2}{l-1}}$. Now if every pair of vertices of H have the interchange property, then we know $\mathcal{U}_n = 1$. On the other hand, if there are two vertices of H that do not have the interchange property, we still know that $\mathcal{U}_n \leq n$. The expected uniformity dimension can thus be bounded as follows:

$$\begin{aligned} E(\mathcal{U}_n) &\leq 1 \cdot (1-P) + n \cdot P \\ &\leq 1 + n \cdot n^2(1-p^2)^{\binom{n-2}{l-1}} \\ &= 1 + n^3(1-p^2)^{\binom{n-2}{l-1}} \end{aligned}$$

Taking the limit as n approaches infinity of the natural logarithm of $n^3(1-p^2)^{\binom{n-2}{l-1}}$, we get:

$$\lim_{n\to\infty} 3\ln n + \binom{n-2}{l-1}\ln(1-p^2) = -\infty.$$

Therefore, $\lim\limits_{n\to\infty} E(\mathcal{U}_n) \leq 1$. Since $\mathcal{U}_n \geq 1$ by Proposition 4.0.1, we must have $\lim\limits_{n\to\infty} E(\mathcal{U}_n) = 1$. $\square$

Having found a variety of results on general l-uniform hypergraphs, in the next chapter we move on to hypergraphs with particular restraints on their structure, and we obtain more specific results.

Chapter 5

Highly Structured Hypergraphs

In this chapter, we consider several families of hypergraphs, all of which turn out to be l-uniform hypergraphs, but with some type of additional structure. The extra structure allows us to find some nice descriptions of their uniformity spaces. We are able to apply our work here on matroid bases to the problem of finding stable weightings of the spanning forests of a graph. The first family of hypergraphs which we study is the family of l-uniform cycles.

5.1 Cycles

We begin with the definition of l-uniform *cycles*. For graphs, cycles are easy to define, but there are several ways to extend the definition to l-uniform hypergraphs. Different definitions can be found in [1] and [13], for example. We choose to use the definition given by Kühn and Osthus in [13].

Definition 5.1.1. An l-uniform hypergraph $H = (V, E)$ is a *cycle of order* n if there exists a cyclic ordering $v_0, ..., v_{n-1}$ of V such that the pair v_i, v_{i+1} lies in an edge of H for all i modulo n, and such that every edge of H consists of l consecutive vertices (so the cyclic ordering of the vertices induces a cyclic ordering of the edges as well). Further, an l-uniform cycle is *tight* if every l consecutive vertices form an edge. An l-uniform cycle on n vertices is *loose* if it has the minimum possible number of edges among all l-uniform cycles on n vertices.

5.1.1 Loose Cycles

Here we find the uniformity dimension of all loose 3-uniform cycles. The loose 3-uniform cycle of order 3 has only one edge and therefore has uniformity dimension 3. As shown in [13], if H is a loose 3-uniform cycle of order $n \geq 4$, the number of edges of H must be $\lceil \frac{n}{2} \rceil$. Moreover, if n is even the intersection of any two consecutive edges in H must have size 1, whereas if n is odd, there exist two consecutive edges of H whose intersection is of size 2, and all other pairs of consecutive edges intersect in only one vertex.

Let $C = (V, E)$ be a loose 3-uniform cycle of order $n \geq 4$. In order to find the uniformity dimension of C, we will have to consider two cases. All subscripts in the following discussion will be modulo n.

Case i) n is odd.

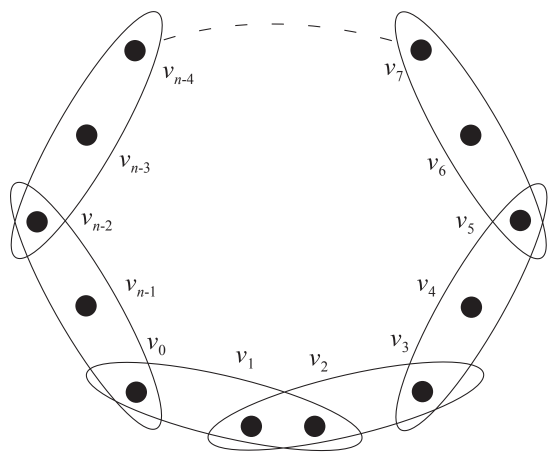

Figure 5.1: The loose cycle of odd order n

Without loss of generality let:

$$V = \{v_0, v_1, v_2, ..., v_{n-1}\},$$

$$E = \{\{v_0, v_1, v_2\}, \{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}, ..., \{v_{n-2}, v_{n-1}, v_0\}\},$$

so that C appears as in Figure 5.1. We can construct a stable weighting $f : V \to \mathbb{F}$ of C as follows: Weight the vertices v_0, v_1, v_2 arbitrarily with $a_0, a_1, a_2 \in \mathbb{F}$, and let $a_0 + a_1 + a_2 = k$. Then the weight on v_3 must be equal to a_0 since v_0 and v_3 have the interchange property with interchange set $\{v_1, v_2\}$. We next weight the vertices $v_5, v_7, v_9, ..., v_{n-2}$ arbitrarily with $a_5, a_7, a_9, ..., a_{n-2} \in \mathbb{F}$. The weight on each of the remaining vertices $v_4, v_6, ..., v_{n-1}$ is now determined by the weight of its neighbours, since the sum of the weights on each edge must be k. Thus we have defined $f : V \to \mathbb{F}$ by:

$$f(v_i) = \begin{cases} a_i & \text{if } i = 0, 1, 2, 5, 7, 9, ..., n-2, \\ a_0 & \text{if } i = 3, \\ k - a_0 - a_5 & \text{if } i = 4, \\ k - a_{i-1} - a_{i+1} & \text{if } i = 6, 8, ..., n-1. \end{cases}$$

where $a_i \in \mathbb{F}$ for all $i \in \{0, 1, 2, 5, 7, 9, ..., n-2\}$. It is easy to check that the weight on every edge of C is equal to k. Further, since any stable weighting of C is completely determined by these $\frac{n+1}{2} = \lfloor \frac{n}{2} + 1 \rfloor$ arbitrary values, we have $\text{udim}(C, \mathbb{F}) = \lfloor \frac{n}{2} + 1 \rfloor$.

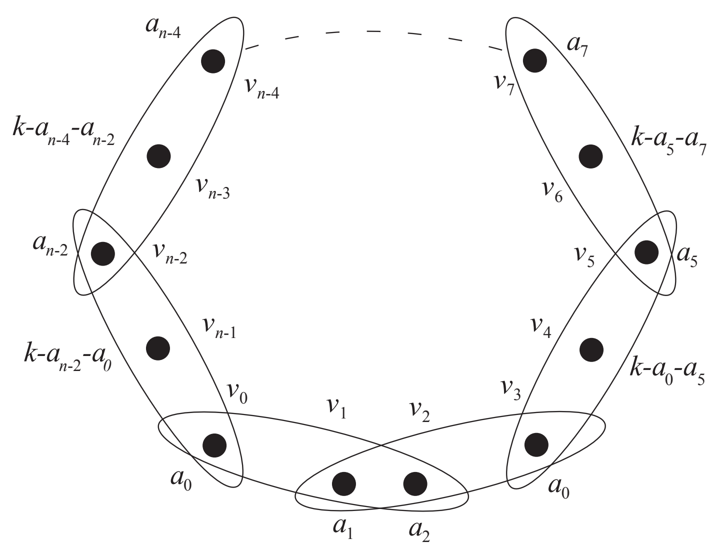

Figure 5.2: The stable weighting f of the loose cycle of odd order n

Case ii) n is even.

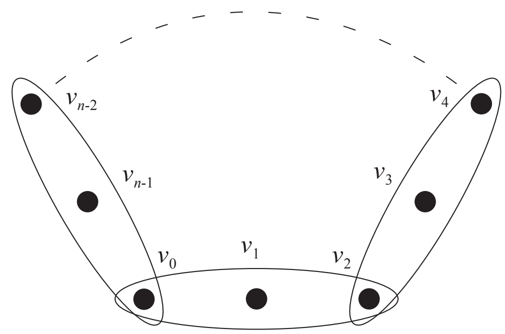

Figure 5.3: The loose cycle of even order n

Without loss of generality let:

$$V = \{v_0, v_1, v_2, ..., v_{n-1}\},$$

$$E = \{\{v_0, v_1, v_2\}, \{v_2, v_3, v_4\}, \{v_4, v_5, v_6\}, ..., \{v_{n-2}, v_{n-1}, v_0\}\},$$

so that C appears as in Figure 5.3. We can construct a stable weighting $f : V \to \mathbb{F}$ of C as follows: Assign arbitrary weights $a_0, a_1,$ and a_2 from $\mathbb{F}$ to $v_0, v_1,$ and v_2 respectively. Then let $k = a_0 + a_1 + a_2$, so that the weight on each edge of C must be equal to k. Now assign arbitrary weights $a_4, a_6, ..., a_{n-2} \in \mathbb{F}$ to the vertices $v_4, v_6, ..., v_{n-2}$ respectively. Then the weight on each of the remaining vertices $v_3, v_5, ..., v_{n-1}$ is determined by the weight of its neighbours as in the previous case. Thus we have defined $f : V \to \mathbb{F}$ by:

$$f(v_i) = \begin{cases} a_i & \text{if } i = 0, 1, 2, 4, 6, ..., n-2, \\ k - a_{i-1} - a_{i+1} & \text{if } i = 3, 5, ..., n-1, \end{cases}$$

and it is easy to check that the weight on each edge of C is equal to k under this weighting, so it is stable.

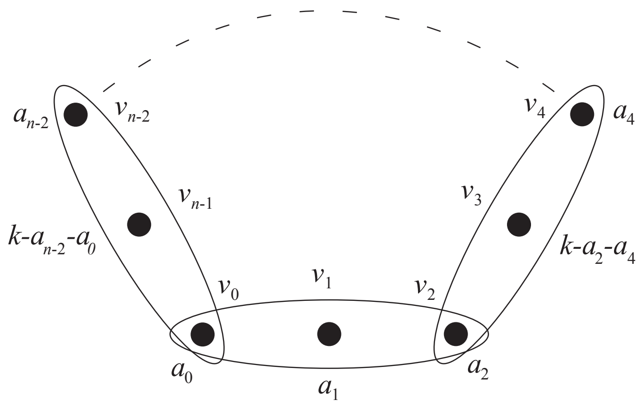

Figure 5.4: The stable weighting f of the loose cycle of even order n

The weighting f is completely determined by the placement of the weights $a_0, a_1, a_2, a_4, a_6, ..., a_{n-2}$, which can be arbitrarily chosen from $\mathbb{F}$. Thus we must have $\text{udim}(C, \mathbb{F}) = \frac{n}{2} + 1$. We summarize our results on loose 3-uniform cycles in the following proposition.

Proposition 5.1.1. *Let $C = (V, E)$ be a loose 3-uniform cycle of order $n \geq 4$. Then for any field $\mathbb{F}$, udim$(C, \mathbb{F}) = \lfloor \frac{n}{2} + 1 \rfloor$.* □

5.1.2 Tight Cycles

The uniformity dimension of loose l-uniform cycles for $l > 3$ is not so clear, as there is much more variety in forming loose l-uniform cycles of a fixed order n for any $l > 3$. However, for a given l and $n \geq l$, all *tight* l-uniform cycles of order n are isomorphic, and the uniformity dimension of a tight l-uniform cycle turns out to be very easy to determine with the next theorem.

Theorem 5.1.2. *Let $C = (V, E)$ be a tight l-uniform cycle of order $n \geq l$, and let $\mathbb{F}$ be a field. Then udim$(C, \mathbb{F}) =$ gcd(l, n).*

Proof. During this proof all subscripts are taken modulo n, the set of integers modulo n is denoted $\mathbb{Z}_n$, and we let $d = \gcd(l, n)$. In the case $n = l$, the cycle C has only one edge, so immediately we see that $\text{udim}(C, \mathbb{F}) = n = d$. Now suppose $n < l$ and let:

$$V = \{v_0, v_1, v_2, ..., v_{n-1}\},$$
$$E = \{\{v_0, v_1, ..., v_{l-1}\}, \{v_1, v_2, ..., v_l\}, \{v_2, v_3, ..., v_{l+1}\}, ..., \{v_{n-1}, v_0, ..., v_{l-2}\}\}.$$

We see immediately that v_i and v_{i+l} have the interchange property with interchange set $\{v_{i+1}, v_{i+2}, ..., v_{i+l-1}\}$ for all i modulo n. This tells us that under any stable weighting of C, each vertex in the set $V_0 = \{v_{tl} \mid t \in \mathbb{Z}_n\}$ must have the same weight. Likewise for the sets $V_1 = \{v_{1+tl} \mid t \in \mathbb{Z}_n\}, V_2 = \{v_{2+tl} \mid t \in \mathbb{Z}_n\}$ through to $V_{l-1} = \{v_{l-1+tl} \mid t \in \mathbb{Z}_n\}$. We have presented l sets whose vertices must be weighted equally, but some of them could overlap. We know that $V_0, V_1, ..., V_{d-1}$ partition V since d is the greatest common divisor of l and n. This tells us that $\text{udim}(C, \mathbb{F}) \leq d$.

For $i \in \{0, 1, 2, ..., d-1\}$, define $f_i : V \to \mathbb{F}$ by:

$$f_i(v) = \begin{cases} 1 & \text{if } v \in V_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $p \in \mathbb{N}$ be such that $l = dp$. Then we know that every edge of C contains exactly p members of each set V_i. Therefore, the weight on every edge under the weighting f_i is equal to p, and f_i is stable for all $i = 0, 1, ..., d-1$. The set $\{f_0, f_1, ..., f_{d-1}\}$ is clearly linearly independent and contains d weightings, so it forms a basis for $U(C, \mathbb{F})$. This completes the proof that $\text{udim}(C, \mathbb{F}) = \gcd(l, n)$. □

This concludes our work on l-uniform cycles. We note that more could be done with these cycles. We have found a simple description for the uniformity spaces of only two specific types of cycles. However, in the next section we move on to a different type of highly structured hypergraph.

5.2 Matroid Bases

In this section we study the uniformity spaces of hypergraphs whose edges are exactly the bases of a matroid. We can generate such a hypergraph from any matroid $M = (V, \mathscr{E})$ by defining the hypergraph $H = (V, \mathscr{B})$, where $\mathscr{B}$ is the set of bases of M. This is the *basis hypergraph* of Definition 2.2.3. Conversely, given a hypergraph $H = (V', E')$, we know that E' is the set of bases for a matroid on V' exactly when it satisfies the base axiom for a matroid (Theorem 2.2.1). The next theorem describes the uniformity space of a basis hypergraph in terms of the connected components of its associated matroid.

Theorem 5.2.1. *Let $H_M = (V, E)$ be the basis hypergraph for the matroid $M = (V, \mathscr{E})$ and let $\mathbb{F}$ be any field. Then udim$(H_M, \mathbb{F}) = k$, where k is the number of components of M. In particular, if the components of M are $M|V_1, M|V_2, ..., M|V_k$, then a basis for $U(H_M, \mathbb{F})$ is given by $\{f_1, f_2, ..., f_k\}$, where $f_i : V \to \mathbb{F}$ is defined by:*

$$f_i(v) = \begin{cases} 1 & \textit{if } v \in V_i \\ 0 & \textit{otherwise} \end{cases}$$

for all $i \in \{1, 2, ..., k\}$.

Proof. First we show that any two elements of V contained in the same component of M must have the same weight under any stable weighting of H_M. Let x and y be distinct vertices of some component of M. Then x and y are contained in some common circuit C of M. Now $C \backslash x$ must be an independent set of M, and so is contained in some base T of M (i.e. an edge of H_M), and clearly $y \in T$.

Now we claim that $(T \cup x) \backslash y$ must also be a base of M. Suppose otherwise, so that $(T \cup x) \backslash y$ contains a circuit C' of M. In particular we have $C' \subseteq T \cup x$, and we must have $x \in C'$ since $T \backslash y$ is independent in M. Now since $x \in C \cap C'$, there exists a circuit C'' of M such that $C'' \subseteq (C \cup C')\backslash x$ (by circuit axiom (ii), Theorem 2.2.2). Since $(C \cup C')\backslash x = (C\backslash x) \cup (C'\backslash x) \subseteq T$, we have $C'' \subseteq T$. This is a contradiction because T was assumed to be a base of M.

So we conclude that T and $(T \cup x)\backslash y$ are both bases of M. Since H_M is the basis hypergraph of M, T and $(T \cup x)\backslash y$ are both edges of H_M, and therefore x and y have the interchange property. Thus x and y must have the same weight under any stable weighting of H_M. Since x and y were arbitrary, any two elements contained in a common circuit of M must have the same weight under any stable weighting. This means exactly that for any component $M|V_i$ of M, all elements of V_i must have the same weight. We conclude that if k is the number of components of M, $\text{udim}(H_M, \mathbb{F}) \leq k$.

Now let $f_1, f_2, ..., f_k$ be as in the theorem statement, and claim that f_i is a stable weighting of H_M for all $i = 1, 2, ..., k$. Fix an $i \in \{1, 2, ..., k\}$. Recall that $M|V_i$ is a matroid, so the bases of $M|V_i$ are all of the same size, say n_i. We show that each base of M contains n_i elements of V_i.

Let B be a base of M. It is easy to see that B cannot contain more than n_i elements of V_i, since there are no independent sets of size greater than n_i contained in V_i. On the other hand, suppose an independent set I of M contains less than n_i elements of V_i. Then let $I_i = I \cap V_i$, and I_i is not a base of $M|V_i$, since $|I_i| < n_i$. So there exists an element $z \in V_i$ such that $z \notin I_i$ and $I_i \cup z$ is independent in $M|V_i$. Then $I \cup z$ must be independent in M because the addition of z does not create any circuits in V_i, and by the definition of V_i, no element outside of V_i can lie on a common circuit with z. This proves that I is not a base of M, and so any base of M must contain at least n_i elements of V_i.

Now since we can conclude that every base of M contains exactly n_i vertices from V_i, the weight on each base of M under f_i must be n_i. So f_i is a stable weighting of H_M. Since this holds for every $i \in \{1, 2, ..., k\}$, the weightings $f_1, f_2, ..., f_k$ are all stable. It is also easy to see that they are linearly independent. Since $\text{udim}(H_M, \mathbb{F}) \leq k$, they must form a basis for $U(H_M, \mathbb{F})$. $\square$

We have the following corollary to Theorem 5.2.1:

Corollary 5.2.2. *Let $G = (V, E)$ be a graph with graphic matroid $M(G)$. Then udim$(H_{M(G)}, \mathbb{F})$ is equal to the number of blocks of G over any field $\mathbb{F}$.*

Proof. $H_{M(G)}$ is the basis hypergraph for the graphic matroid of G, so the uniformity dimension of $H_{M(G)}$ is the number of components of $M(G)$. The number of components of $M(G)$ is equal to the number of blocks of G. Note that a basis for $U(H_{M(G)}, \mathbb{F})$ is also easily found from Theorem 5.2.1. In this basis, each vector weights the edges of a single block of G with 1's and the remaining edges with 0's. $\square$

This gives a nice solution to one problem that we mentioned in the introduction. For a graph $G = (V, E)$, we now have an efficient way to find the space of all weightings of E that are stable on the maximal acyclic subsets of E. That is, we just find the blocks of G. We compare the computational complexity of this method with the complexity of listing all spanning trees of G, then setting up the solution matrix and finding its nullspace. We restrict our attention to connected graphs because the algorithms that we reference are designed for connected graphs.

Suppose $G = (V, E)$ is a connected graph with n vertices and m edges. In chapter 6.4 of [12], a depth-first search algorithm for finding the blocks of any connected graph is presented. The complexity of this algorithm is $O(m)$. Since the number of edges is bounded by $\binom{n}{2} = \frac{n(n-1)}{2}$, the complexity is $O(n^2)$. This is a very fast way to find the space of weightings stable on the spanning trees of G when compared with the alternate algebraic method considered next.

If we want to list all spanning trees of G, let t denote the number of spanning trees of G, which could be exponential in n. (The complete graph on n vertices has n^{n-2} spanning trees.) Any algorithm that lists the spanning trees of G will involve at least t operations. One such algorithm whose complexity is $O(n+m+t)$ is presented in [18]. So in terms of n, the complexity could be as bad as $O(n^{n-2})$. Once all spanning trees are listed, we can form the $(t-1)\times n$ solution matrix W. There are several ways to find the nullspace of this matrix, which all have approximately the same computational complexity. One such method is to find the singular value decomposition (SVD) of W. We don't actually need the entire SVD, and algorithms to find the necessary part of the SVD have complexity $O((t-1)^2 n)$ when $t \gg n$ ([9], pg. 239). So in terms of n, the complexity of this method could be as bad as $O(n^{2n-3})$. We would obviously much rather use the first method and find the blocks of the graph.

Now that we have described the stable weightings of matroid bases in terms of the blocks of the matroid, we consider the affect of some matroid operations on the uniformity dimension of its bases. In particular, we observe the affects of restriction and contraction of a matroid on its basis hypergraph's uniformity dimension. Recall that restriction corresponds to deleting certain edges of a graph, and contraction corresponds to contracting certain edges of a graph. We are interested in the number of components of the restriction and contraction of a matroid relative to the number of components of the original, because by Theorem 5.2.1, a matroid with k components has basis hypergraph with uniformity dimension k.

We deal with restriction first. Let $M=(V,\mathscr{E})$ be a matroid with k components. Let v be an element of V; we investigate the deletion of v, $M-v=M|(V\backslash v)$. We want to find a bound on the number of components of $M-v$. We have two cases:

(i) $\{v\}$ is a component of M.

In this case, $M-v$ has one less component than M, since $\{v\}$ is clearly no longer a component of $M-v$, but all other components are unchanged.

(ii) $\{v\}$ is not a component of M.

> We show that in this case, $M - v$ must have at least as many components as M. Suppose v is in the component V_1 of M, and let the other components of M be $V_2, V_3, ..., V_k$. These other components of M must still be components of $M - v$. This is because the circuits of $M - v$ are exactly the circuits of M not containing v. Since $V_1 \neq \{v\}$ by assumption, there must be some other element $u \in V'$ as well, such that $u \neq v$. Then u is in some component of $M - v$, and it cannot be in any of $V_2, V_3, ..., V_k$, so $M - v$ must have at least k components.

Let H_M be the basis hypergraph for M and H_{M-v} the basis hypergraph for $M - v$. We have shown that $\text{udim}(H_{M-v}, \mathbb{F}) \geq \text{udim}(H_M, \mathbb{F}) - 1$, with equality if and only if $\{v\}$ is a component of M. Is there an upper bound for $\text{udim}(H_{M-v}, \mathbb{F})$ in terms of $\text{udim}(H_M, \mathbb{F})$? It turns out that there is not, as the following example shows.

Example 5.2.1. Consider the graph C_n and let e be an edge of C_n. Clearly C_n has only one block, while $C_n - e$ has $n-1$ blocks. By Corollary 5.2.2, $\text{udim}(H_{M(C_n)}, \mathbb{F}) = 1$, while $\text{udim}(H_{M(C_n)-e}, \mathbb{F}) = \text{udim}(H_{M(C_n-e)}, \mathbb{F}) = n - 1$.

Next we explore contractions of a matroid $M = (V, \mathscr{E})$ with k components, by taking an element v of V and considering the link of v, $lk_M v = M \cdot (V \backslash v)$. We again want to bound the number of components of $lk_M v$. The same arguments apply to the link as applied to the deletion above. The link of a single element v changes only the component that v is part of. If $\{v\}$ is a component of M, then $lk_M v$ has $k - 1$ components. Otherwise, there is some other element $u \in V$ in the same component as v. So u is in some component of $lk_M v$ which must be distinct from the other $k - 1$ components of M. Therefore if $\{v\}$ is not a component of M, $lk_M v$ has at least k components.

Thus, as above, we have shown that $\text{udim}(H_{lk_M v}, \mathbb{F}) \geq \text{udim}(H_M, \mathbb{F}) - 1$, with equality if and only if $\{v\}$ is a component of M. We also provide an example showing that there is no upper bound for $\text{udim}(H_{M-v}, \mathbb{F})$ in terms of $\text{udim}(H_M, \mathbb{F})$.

Example 5.2.2. Consider the graph G pictured in Figure 5.5, where G is made up of n 4-cycles, all sharing exactly one edge. The graph G has only one block, so $\mathrm{udim}(H_{M(G)}, \mathbb{F}) = 1$. However, contracting edge e results in a graph that is made up of n 3-cycles all sharing exactly one vertex. Each 3-cycle is a separate block of $G \cdot e$, so $\mathrm{udim}(H_{lk_{M(G)}e}, \mathbb{F}) = \mathrm{udim}(H_{M(G\cdot e)}, \mathbb{F}) = n$.

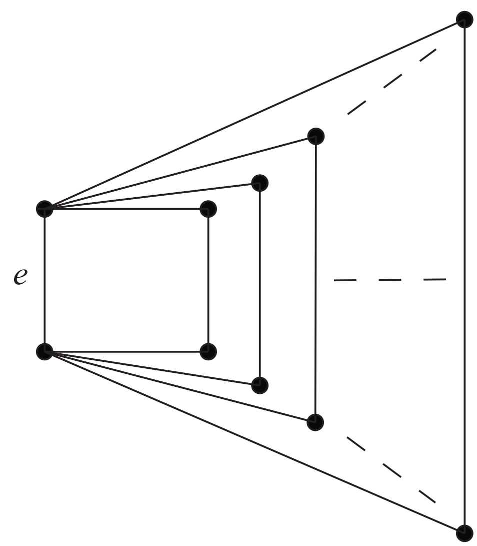

Figure 5.5: A graph G for which $G \cdot e$ has many more blocks than G

We summarize the results on the deletion and link of a single element of a matroid in the theorem below.

Theorem 5.2.3. *Let* $M = (V, \mathscr{E})$ *be a matroid with* k *blocks, so* $\mathrm{udim}(H_M, \mathbb{F}) = k$ *for any field* $\mathbb{F}$. *Then for any element* v *of* V, $\mathrm{udim}(H_{M-v}, \mathbb{F}) \geq k - 1$, *and* $\mathrm{udim}(H_{lk_M v}, \mathbb{F}) \geq k - 1$, *with equality occurring for both if and only if* $\{v\}$ *is a component of* M. □

Remark 5.2.1. We can easily extend this theorem to the deletion and link of a set $X \subseteq V$ of size r. We will have $\mathrm{udim}(H_{M-X}, \mathbb{F}) \geq k - r$, and $\mathrm{udim}(H_{lk_M X}, \mathbb{F}) \geq k - r$, with equality occurring for both if and only if every element of X is in a component of M by itself.

5.3 Symmetric Block Designs

In this section we prove that the uniformity dimension of any symmetric block design is 1 as long as the characteristic of the field we are working over does not divide certain numbers. Recall that we have defined designs as a class of hypergraphs, so in our context they have no multiple edges.

Theorem 5.3.1. *Let $D = (V, \mathbf{B})$ be a symmetric block design with parameters v, k, and λ, where $v > k > \lambda$. Then $udim(D, \mathbb{F}) = 1$ as long as $char(\mathbb{F})$ does not divide $k - \lambda$, $k + \lambda(v-1)$, or v.*

Remark 5.3.1. Requiring $v > k > \lambda$ excludes only fairly trivial block designs. Any design where $v = k$ can only contain one block which passes through every point. Further if $v > k$ and $k = \lambda$ in a symmetric design then any two blocks intersect in all of their points by Proposition 2.3.4, so again the block design can have only one block. For any design with only one block, and more generally for any hypergraph with only one block, the uniformity dimension is the number of vertices, since every weighting of the vertex set must be stable.

Proof of Theorem 5.3.1. Let M be the incidence matrix of D. We know that M is a $v \times v$ matrix since D is a symmetric block design. Now let $\mathbf{B}_i$ denote the i^{th} column of M, corresponding to the i^{th} block of D. Then consider the product $M^t M$. The ij^{th} entry of this matrix is:

$$(M^t M)_{ij} = \mathbf{B}_i \cdot \mathbf{B}_j = \begin{cases} \lambda & \text{if } i \neq j \text{ since distinct blocks intersect in exactly} \\ & \lambda \text{ points by Proposition 2.3.4.} \\ k & \text{if } i = j \text{ since each block passes through exactly} \\ & k \text{ points by Definition 2.3.1 (iii).} \end{cases}$$

Thus we have:

$$M^tM = \begin{bmatrix} k & \lambda & \cdots & \lambda \\ \lambda & k & \cdots & \lambda \\ \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \cdots & k \end{bmatrix} = (k-\lambda)I + \lambda J,$$

where I denotes the $v \times v$ identity matrix and J denotes the $v \times v$ matrix with all entries equal to 1. There are now several ways to see that M^tM has rank v. We find v linearly independent eigenvectors for J, (and hence for I and M^tM as well), none of which are in the nullspace of M^tM. This proves that M^tM has nullity 0 and rank v. We claim the eigenvectors are as follows:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \cdots \\ \cdots \\ \cdots \\ \ddots \\ \ddots \\ \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}$$

Call these vectors $\mathbf{u}_1$ and $\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, ..., \mathbf{u}_v$ respectively. It is easy to verify that they are eigenvectors for J. These vectors are linearly independent over any field of characteristic not dividing v, which we prove now. Suppose we have a dependence relation among the eigenvectors above over the field $\mathbb{F}$ as follows:

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + ... + \alpha_v\mathbf{u}_N = \mathbf{0}$$

where $\alpha_i \in \mathbb{F}$ for all $i \in \{1, 2, ..., v\}$, and at least one of the α_i is nonzero. To see that α_1 must be nonzero, suppose $\alpha_1 = 0$. Then $\alpha_i \neq 0$ for some $i \geq 2$, and then the i^{th} entry of the sum must be $-\alpha_i$, which is a contradiction. Now since $\alpha_1 \neq 0$, each entry α_i for $i = 2, 3, ..., v$ must be equal to α_1 in order for the i^{th} entry of the sum to be equal to 0. However, then the 1^{st} entry of the sum is $v\alpha_1$. As long as the characteristic of $\mathbb{F}$ does not divide v, this entry is nonzero and therefore the vectors $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_v$ must be linearly independent.

So we have a basis of eigenvectors for M^tM, and we now show that none of them have eigenvalue 0. The first vector $\mathbf{u}_1$ has eigenvalue v for J and hence eigenvalue $(k - \lambda) + v\lambda = k + \lambda(v - 1)$ for $M^tM = (k - \lambda)I + \lambda J$. The remaining eigenvectors have eigenvalue 0 for J and hence eigenvalue $k - \lambda$ for $M^tM = (k - \lambda)I + \lambda J$. We assumed $k > \lambda$ so that $k - \lambda > 0$. So the eigenvalues we have found will be nonzero over any field of characteristic not dividing $k - \lambda$ or $k + \lambda(v - 1)$.

Since we have found a basis of eigenvectors for M^tM, none of which have eigenvalue 0, nullity$(M^tM) = 0$ and so rank$(M^tM) = v$ by the rank-nullity theorem. By a well-known result in linear algebra, rank(M^tM)=rank(M). [17]

Since M is a $v \times v$ matrix, it has full column and row rank. The columns $\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_N$ of M are therefore linearly independent. Also, the solution matrix W_D for D can be written:

$$W_D = \begin{bmatrix} (\mathbf{B}_1 - \mathbf{B}_v)^t \\ (\mathbf{B}_2 - \mathbf{B}_v)^t \\ \vdots \\ (\mathbf{B}_{v-1} - \mathbf{B}_v)^t \end{bmatrix}$$

Now suppose the rows of W_D are linearly dependent. Then we would have a linear dependence relation among $\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_v$, a contradiction. Therefore, W_D has full row rank, and rank$(W_D) = v - 1$. This implies nullity$(W_D) = 1$. We conclude that if char$(\mathbb{F})$ does not divide $k - \lambda$, $k + \lambda(v - 1)$, or v then udim$(D, \mathbb{F})$ = nullity$(W_D) = 1$.

$\square$

Remark 5.3.2. It is interesting to note that Theorem 5.3.1 gives us another condition sufficient to prove that several vertices of a hypergraph must have the same weight under any stable weighting. That is, if a hypergraph H contains a subhypergraph D that satisfies the axioms for a symmetric block design with parameters $v > k > \lambda$, then every vertex of D must have the same weight under any stable weighting of H. The only other such sufficient condition we have seen up to this point is the interchange property. In particular this shows that the interchange property is not a necessary condition for two vertices of a hypergraph to have the same weight under every stable weighting. For example, in the Fano plane pictured in Figure 2.1 no two vertices have the interchange property. However, since the only stable weightings of the Fano plane are the constant weightings by Theorem 5.3.1, all vertices of the Fano plane must carry the same weight under any stable weighting.

Since projective planes are such a commonly studied type of block design, we give the following result as a corollary to Theorem 5.3.1:

Corollary 5.3.2. *Let $P = (V, \mathbf{B})$ be a projective plane of order n. Then udim$(P, \mathbb{F}) =$ 1 over any field $\mathbb{F}$ of characteristic not dividing $n, n+1,$ or n^2+n+1.*

Proof. This is a direct consequence of Theorem 5.3.1, because P is a symmetric block design with $v = n^2+n+1, k = n+1$, and $\lambda = 1$ by Proposition 2.3.1. So we have $\mathrm{udim}(P, \mathbb{F}) = 1$ as long as $\mathrm{char}(\mathbb{F})$ does not divide $k - \lambda = n$, $k + \lambda(v-1) =$ $(n+1) + (1)(n^2+n) = n^2+2n+1 = (n+1)^2$, or $v = n^2+n+1$. $\square$

Now let D be as in the statement of Theorem 5.3.1 with parameters v, k, and λ. Theorem 5.3.1 does not tell us anything about the uniformity dimension of D if the characteristic of $\mathbb{F}$ divides any of the numbers $k - \lambda$, $k + \lambda(v-1)$, or v. We now explore these situations.

First of all, suppose $\mathrm{char}(\mathbb{F})$ divides $k - \lambda$. In the proof of Theorem 5.3.1, we found $v - 1$ eigenvectors for M^tM with the eigenvalue $k - \lambda$, and they are linearly independent over a field of any characteristic, so nullity$(M^tM) = $ nullity$(M^t) \geq v-1$. In particular, anything in the nullspace of M^t is a stable weighting of D (with weight 0 on each edge), so udim$(D, \mathbb{F}) \geq v - 1$. This tells us that udim$(D, \mathbb{F}) = v - 1$ or udim$(D, \mathbb{F}) = v$. As long as the solution matrix W_D has at least one row (this means D has at least 2 blocks), rank$(W_D) \geq 1$ over a field of any characteristic, and so udim$(D, \mathbb{F}) = v - 1$. Otherwise, D has at most one block, and in this case udim$(D, \mathbb{F}) = v$.

Now suppose char$(\mathbb{F})$ does not divide $k - \lambda$. Then the $v-1$ eigenvectors for M^tM with eigenvalue $k - \lambda \neq 0$ tell us that rank$(M^tM) = $ rank$(M^t) \geq v - 1$. So for the solution matrix W_D, rank$(W_D) \geq v - 2$. This tells us that udim$(D, \mathbb{F}) \leq 2$. Since we know that the all 1's vector is a stable weighting of any symmetric block design (as it is a uniform hypergraph), we know that udim$(D, \mathbb{F}) = 1$ or udim$(D, \mathbb{F}) = 2$. We leave the exact determination of the uniformity dimension open when char$(\mathbb{F})$ divides $k + \lambda(v - 1)$ or v. A solution by cases should be achievable. We summarize our results in the following theorem.

Theorem 5.3.3. *Let $D = (V, \mathbf{B})$ be a symmetric block design with at least two blocks and parameters $v, k,$ and λ, where $v > k > \lambda$. If char$(\mathbb{F})$ divides $k - \lambda$, then udim$(D, \mathbb{F}) = v - 1$. If char$(\mathbb{F})$ does not divide $k - \lambda$, but divides v or $k + \lambda(v - 1)$, then udim$(D, \mathbb{F}) = 1$ or udim$(D, \mathbb{F}) = 2$.* □

Remark 5.3.3. In the notation of Theorem 5.3.3, we could not find an example of a block design D with udim$(D, \mathbb{F}) = 2$ over a field of characteristic not dividing $k - \lambda$, but dividing v or $k + \lambda(v - 1)$. It is possible that this never occurs.

Chapter 6

Criticality

6.1 Critical Uniformity Dimension

A notion of criticality is important in many areas in mathematics. In graph theory, the best known example relates to graph colourings. A graph G with chromatic number k is *edge k-critical* if the deletion of any edge of G decreases the chromatic number. A short section on these k-critical graphs can be found in [21]. A simple result tells us that every graph with chromatic number k has a k-critical subgraph. This means that the k-critical graphs describe the essentials of k-colourability, so these critical graphs have been studied extensively. We present a notion of when a hypergraph with uniformity dimension k is critical, and find some hypergraphs which are critical. To begin with, we show that removing an edge of a hypergraph must either leave its uniformity dimension the same or increase its uniformity dimension by 1.

Lemma 6.1.1. *Let $H = (V, E)$ be a hypergraph, and let $|V| = n$. Let e be any edge of H. Then either $udim(H - e, \mathbb{F}) = udim(H, \mathbb{F})$ or $udim(H - e, \mathbb{F}) = udim(H, \mathbb{F}) + 1$.*

Proof. Order the edges of H so that e comes first. Let $e_2, e_3, ..., e_m$ be the other edges of H, and hence the complete set of edges of $H - e$. Then the solution matrix

W_H for H is:

$$W_H = \begin{bmatrix} (\mathbf{e} - \mathbf{e}_m)^t \\ (\mathbf{e}_2 - \mathbf{e}_m)^t \\ \vdots \\ (\mathbf{e}_{m-1} - \mathbf{e}_m)^t \end{bmatrix}$$

The solution matrix W_{H-e} for $H - e$ can be obtained from W_H by removing the first row:

$$W_{H-e} = \begin{bmatrix} (\mathbf{e}_2 - \mathbf{e}_m)^t \\ (\mathbf{e}_3 - \mathbf{e}_m)^t \\ \vdots \\ (\mathbf{e}_{m-1} - \mathbf{e}_m)^t \end{bmatrix}$$

Now suppose $\text{nullity}(W_H) = \text{udim}(H, \mathbb{F}) = k$. Then $\text{rank}(W_H) = n - k$ by the rank-nullity theorem. So any basis for the row space of W_H has size $n - k$. When we remove the first row of W_H, the dimension of the row space could either stay the same or decrease by 1. So $\text{rank}(W_{H-e})$ is either $n - k$ or $n - k - 1$. This implies $\text{udim}(H - e, \mathbb{F}) = \text{nullity}(W_{H-e})$ is equal to k or $k + 1$, as desired. □

In particular, Lemma 6.1.1 tells us that removing an edge from a hypergraph cannot decrease its uniformity dimension. The following definition of criticality seems the most natural.

Definition 6.1.1. Let $H = (V, E)$ be a hypergraph. Then if $\text{udim}(H, \mathbb{F}) = k$ and $\text{udim}(H - e, \mathbb{F}) > k$ for all $e \in E$, we say that H has *k-critical* uniformity dimension.

Example 6.1.1. In Section 4.3, we mentioned the *critical diagonal* of Tables 4.1, 4.2, and 4.3. The 3-uniform hypergraphs that lie on the critical diagonal are those with n vertices, $m > 1$ edges, and uniformity dimension $n - m + 1$. By considering the tables we see that these hypergraphs must all have $(n-m+1)$-critical uniformity dimension, because if we remove an edge, the uniformity dimension must increase, since there are no hypergraphs on n vertices and $m-1$ edges with uniformity dimension less than $n-m+2$. In fact, we will show in Lemma 6.1.3 that any hypergraph H with n vertices and m edges must have $\text{udim}(H,\mathbb{F}) \geq n-m+1$.

We will see later that the hypergraphs on the critical diagonal are the only hypergraphs in Tables 4.1, 4.2, and 4.3 that have critical uniformity dimension. This is not immediately obvious, and we soon begin developing the theory that will be needed to prove this result. Before doing this, we present a lemma that tells us that we are primarily interested in the critical hypergraphs with no isolated vertices.

Lemma 6.1.2. *Let $H=(V,E)$ be a hypergraph with a set of isolated vertices $V_I = \{v_1, v_2, ..., v_s\} \subseteq V$. Let $H - V_I$ denote H with the isolated vertices removed, and suppose udim$(H,\mathbb{F}) = k$ for some field $\mathbb{F}$. Then H is k-critical if and only if $H - V_I$ is $(k-s)$-critical.*

Proof. We have that $\text{udim}(H,\mathbb{F}) = k$ if and only if $\text{udim}(H - V_I,\mathbb{F}) = k-s$ by Lemma 3.4.3. Further, H is k-critical if and only if $\text{udim}(H-e,\mathbb{F}) \geq k$ for all $e \in E$, if and only if $\text{udim}(H - V_I - e,\mathbb{F}) \geq k-s$ for all $e \in E$ by Lemma 3.4.3, if and only if $H - V_I$ is $(k-s)$-critical. $\square$

We note that this lemma tells us we can find all k-critical hypergraphs on n vertices by finding all $(k-t)$-critical hypergraphs on $n-t$ vertices and adding t isolated vertices for all appropriate values of t. Thus we will be primarily interested in the critical hypergraphs which contain no isolated vertices. However, we do not include this restriction in the definition so that we get the desired result of Corollary 6.1.7 later on.

Lemma 6.1.3. *Let $H=(V,E)$ be a hypergraph with n vertices, $m \geq 1$ edges, and uniformity dimension k over some field $\mathbb{F}$. Then $m \geq n-k+1$.*

Proof. We know that any hypergraph with only one edge has uniformity dimension equal to the size of its vertex set, because every weighting of the vertex set must be stable when there is only one edge. So if we delete $m-1$ edges from H, we are left with a hypergraph H' whose uniformity dimension is n. By Lemma 6.1.1, deleting an edge from a hypergraph can increase its uniformity dimension by at most 1. This implies that the uniformity dimension of H can increase by at most $m-1$ through any $m-1$ deletions. This tells us that $\mathrm{udim}(H', \mathbb{F}) \leq \mathrm{udim}(H, \mathbb{F}) + (m-1)$, from which we obtain $n \leq \mathrm{udim}(H, \mathbb{F}) + (m-1) \Rightarrow m \geq n-k+1$. □

Remark 6.1.1. The result of Lemma 6.1.3 can also be seen from the matrix form of the uniformity space solution.

Theorem 6.1.4. *Let $H = (V, E)$ be a hypergraph with n vertices and m edges, and suppose udim$(H, \mathbb{F}) = k < n$. Then H has k-critical uniformity dimension if and only if $m = n-k+1$.*

Proof. $(\Rightarrow)$ Let H be as in the theorem statement and suppose H has k-critical uniformity dimension. We know that $m \geq n-k+1$ by Lemma 6.1.3, and we would like to show that $m = n-k+1$. Suppose otherwise that $m > n-k+1$. Rearranged slightly, this says that $m-1 > n-k$. Consider the $(m-1) \times n$ solution matrix W_H of H. We have $\mathrm{nullspace}(W_H) = k$, and therefore by the rank-nullity theorem, $\mathrm{rank}(W_H) = n-k$. Our inequality now says that the number of rows of W_H exceeds the rank of W_H. Therefore, there is some row of W_H which is a linear combination of the others. So the edge e of H that corresponds to this row can be deleted from H without changing the rank (and nullity) of W_H. This contradicts the assumption that H has k-critical uniformity dimension, and so we conclude that $m = n-k+1$.
$(\Leftarrow)$ Let H be as in the theorem statement and suppose $m = n-k+1$. Since $k < n$ this implies $m \geq 2$. For any $e \in E$, the deletion $H - e$ has $m-1 \geq 1$ edges and still lies on n vertices. By Lemma 6.1.3, we have $m-1 \geq n - \mathrm{udim}(H-e, \mathbb{F}) + 1$, so that $\mathrm{udim}(H-e, \mathbb{F}) \geq n-m+2 = k+1$. So $\mathrm{udim}(H-e, \mathbb{F}) > \mathrm{udim}(H, \mathbb{F})$. Since e was arbitrary, we conclude that H has $(n-m+1)$-critical uniformity dimension. □

As a special case, Theorem 6.1.4 tells us that the hypergraphs lying on the critical diagonal of Tables 4.1, 4.2, and 4.3 are in fact the only critical 3-uniform hypergraphs on $4, 5$, and 6 vertices respectively. It also forms the foundation of the proofs to follow.

Corollary 6.1.5. *Let $H = (V, E)$ be a hypergraph with n vertices and m edges, and let udim$(H, \mathbb{F}) = k < n$. Then there is a subhypergraph H_k of H, obtainable from H by the deletion of $m - (n - k + 1)$ edges, which has k-critical uniformity dimension.*

Proof. By Lemma 6.1.3, we know that $m \geq n - k + 1$. If $m = n - k + 1$, then H has k-critical uniformity dimension by Theorem 6.1.4, so $H_k = H$ in this case. Otherwise, $m > n - k + 1$, and by Theorem 6.1.4, H does not have k-critical uniformity dimension. So there is an edge $e_1 \in H$ which can be deleted without increasing the uniformity dimension. This process can be repeated $m - (n - k + 1)$ times, at which point we will be left with a k-critical subhypergraph H_k. □

In Section 4.3 we mentioned that we could study the sequences of numbers that make up the columns and rows of Tables 4.1, 4.2, and 4.3 more generally. We defined $(c_{l,n,k}(m))_{m=0}^{m=\binom{n}{l}}$ as the sequence whose m^{th} entry is the number of non-isomorphic l-uniform hypergraphs on n vertices with uniformity dimension k and m edges. We now show that this sequence has no internal zeros.

Corollary 6.1.6. *The sequence $(c_{l,n,k}(m))_{m=0}^{m=\binom{n}{l}}$ has no internal zeros for any fixed values of $l, n, k \in \mathbb{N}$ with $l, k \leq n$.*

Proof. Fix $l, n, k \in \mathbb{N}$ with $l, k \leq n$. Suppose $c_{l,n,k}(m^*)$ is the last nonzero term of $(c_{l,n,k}(m))_{m=0}^{m=\binom{n}{l}}$ (so $c_{l,n,k}(m) = 0$ for all $m > m^*$). Then there is an l-uniform hypergraph H on n vertices and m^* edges with uniformity dimension k. By Corollary 6.1.5, H contains a k-critical subhypergraph H_k which is obtainable from H by deleting $m^* - (n - k + 1)$ edges. We know H_k has $n - k + 1$ edges, and that no hypergraph on n vertices with fewer edges can have uniformity dimension k by Lemma 6.1.3 (so $c_{l,n,k}(m) = 0$ for all $m < n - k + 1$).

As we delete edges to go from H to H_k, the uniformity dimension must always be equal to k. Further, since H is an l-uniform hypergraph, each hypergraph along the way must also be l-uniform. Thus in the process of going from H to H_k, we obtain an l-uniform hypergraph on n vertices with uniformity dimension k and with m edges for every $m \in \{n-k+1, n-k+2, ..., m^*-1, m^*\}$. This tells us that $c_{l,n,k}(m) \neq 0$ for all $m \in \{n-k+1, n-k+2, ..., m^*-1, m^*\}$, and completes the proof that $(c_{l,n,k}(m))_{m=0}^{m=\binom{n}{l}}$ has no internal zeros. $\square$

We now strengthen Corollary 6.1.5 slightly by showing that every hypergraph with uniformity dimension k contains an i-critical subhypergraph for any $k < i < n$.

Corollary 6.1.7. *Let $H = (V, E)$ be a hypergraph with n vertices and m edges, and let udim$(H, \mathbb{F}) = k < n$. Then for any $i \in \{k, k+1, ..., n-1\}$, there is a subhypergraph H_i of H, obtainable from H by the deletion of $m-(n-i+1)$ edges, which has i-critical uniformity dimension.*

Proof. By Corollary 6.1.5, H has a subhypergraph H_k, obtainable by the deletion of $m-(n-k+1)$ edges, which has k-critical uniformity dimension. For any $i \in \{k, k+1, ..., n-1\}$, delete any $i-k$ edges from H_k, and thus $m-(n-i+1)$ edges total from H. Call the resulting hypergraph H_i. We now prove that H_i has i-critical uniformity dimension. By Lemma 6.1.1, deleting $i-k$ edges from H_k can increase the uniformity dimension by at most $i-k$. So $\text{udim}(H_i, \mathbb{F}) \leq \text{udim}(H_k, \mathbb{F}) + (i-k) = k + i - k = i$. Since H_k has $n-k+1$ edges by Theorem 6.1.4, H_i has $n-k+1-(i-k) = n-i+1$ edges. By Lemma 6.1.3 on the hypergraph H_i, we have $n-i+1 \geq n - \text{udim}(H, \mathbb{F}) + 1 \Rightarrow \text{udim}(H, \mathbb{F}) \geq i$. We conclude that $\text{udim}(H_i, \mathbb{F}) = i$, and since H_i has $n-i+1$ edges, we conclude that H_i has i-critical uniformity dimension by Theorem 6.1.4. $\square$

Observation 6.1.8. The proof of Corollary 6.1.7 essentially tells us that once we find a hypergraph H_k with k-critical uniformity dimension, when we delete any $i-k$ edges from H_k, we have reached a hypergraph H_i with i-critical uniformity dimension for any $k < i < n$. So if we extend our definition of criticality to include all hypergraphs with $m \in \{0, 1\}$, then for any hypergraph H, the edge sets of the subhypergraphs of H that have critical uniformity dimension form a complex on the underlying set of edges of H.

In the next section, we use this observation to show that any subhypergraph of a symmetric block design obtained by deleting only edges has critical uniformity dimension. We also characterize the simple graphs which have critical uniformity dimension.

6.2 Hypergraphs of Critical Uniformity Dimension

We now wish to identify specific hypergraphs with k-critical uniformity dimension. We begin by considering 2-uniform hypergraphs, or simple graphs. We found in section 4.2 that if G is a simple graph, and $\mathbb{F}$ is a field with $\text{char}(\mathbb{F}) \neq 2$, $\text{udim}(G, \mathbb{F}) = 1+r$, where r is the number of bipartite components of G (including isolated vertices).

Theorem 6.2.1. *Let $G = (V, E)$ be a graph with at least 2 edges, and let $\mathbb{F}$ be any field of characteristic not equal to 2. Then G has critical uniformity dimension over $\mathbb{F}$ if and only if G is a pseudoforest with no even cycles. In other words, G has no even cycles, and each component of G has at most one odd cycle.*

Proof. G is critical if and only if deleting any edge of G increases the uniformity dimension. That is, if and only if deleting any edge of G increases the number of bipartite components by Proposition 4.2.3. The deletion of an edge can increase the number of bipartite components in only the following three ways:

(i) By splitting a bipartite component into two.

(ii) By splitting a non-bipartite component into two, producing one bipartite and one non-bipartite component.

(iii) By changing a non-bipartite component into a bipartite component.

We first prove the backward implication of the statement.

($\Leftarrow$) Suppose that G has no even cycles, and that every component of G has at most one odd cycle. Then every bipartite component of G must be a tree. The deletion of any edge of a tree splits the tree into two smaller trees, so this splits a bipartite component of G into two. Any non-bipartite component G_k of G has exactly one odd cycle, say C, and no even cycles. The deletion of any edge not contained in C must then disconnect G_k, leaving a bipartite component and a non-bipartite component, since the unique odd cycle C remains intact. Deleting any edge contained in C leaves a tree since C was the only cycle in G_k, so this changes G_k into a bipartite component. Thus $G - e$ has more bipartite components than G for any $e \in E$, and so G must be critical.

($\Rightarrow$) For the forward implication, we first show that if G is critical then G cannot contain an even cycle. Suppose otherwise that G is critical and contains an even cycle D. We find an edge of D whose deletion does not increase the number of bipartite components of G, and thus tells us that G is not critical. Let D be contained in the component G_k of G. We consider two cases:

(i) If G_k is bipartite, let e be any edge of D, and the deletion of e leaves G_k connected, so does not increase the number of bipartite components of G.

(ii) If G_k is not bipartite, it contains an odd cycle F. Let e be an edge contained in D but not contained in F. Then deleting e leaves G_k connected since e is contained in the cycle D, but it leaves the cycle F alone. Thus $G_k - e$ remains connected and non-bipartite. So the deletion of e does not increase the number of bipartite components of G.

This completes the proof that G cannot contain an even cycle. It remains to prove that no component of G can contain more than one odd cycle. Suppose otherwise that the component G_k contains two distinct odd cycles, S and T. There must be an edge of e of S not contained in T. Deleting e does not split G_k into two components since e is part of a cycle. Nor does deleting e change G_k into a bipartite component, since the cycle T remains intact. So deleting e does not increase the number of bipartite components of G. Therefore, if G is critical, no component of G can contain more than one odd cycle. $\square$

We also characterize the graphs which have critical uniformity dimension over a field of characteristic 2.

Theorem 6.2.2. *Let $G = (V, E)$ be a graph with at least 2 edges, and let $\mathbb{F}$ be any field of characteristic equal to 2. Then G has critical uniformity dimension over $\mathbb{F}$ if and only if G has no even cycles and at most one odd cycle.*

Proof. Let G be as in the theorem statement and suppose that G has s components throughout this proof.

($\Leftarrow$) If G has no even cycles and at most one odd cycle, then either G is a forest or G has a unique cycle C, which must be odd. First suppose G is a forest. Then G has uniformity dimension $1+s$ by Proposition 4.2.3. For any edge e of G, the graph $G-e$ has at least 1 edge since G has at least 2 edges, and $G-e$ has $1+s$ components. Therefore $\text{udim}(G-e, \mathbb{F}) = 2+s$ by Proposition 4.2.3, and thus G is critical.

Now suppose that G has a unique cycle C, which must be odd. Then G is not bipartite, and by Proposition 4.2.3, we have $\text{udim}(G, \mathbb{F}) = s$. Take any edge e of G. If $e \in C$, then the graph $G - e$ is a bipartite graph with s components. Therefore $\text{udim}(G-e, \mathbb{F}) = 1+s$ by Proposition 4.2.3. On the other hand, if $e \notin C$, then $G-e$ is still not bipartite, but it has $1+s$ components, so $\text{udim}(G-e, \mathbb{F}) = 1+s$ in this case as well. Therefore, G is critical.

($\Rightarrow$) Suppose G is critical. We first prove that G contains no even cycles. Suppose that G is bipartite and contains an even cycle C. Take an edge $e \in C$. The graph

$G - e$ is bipartite and still has s components, so by Proposition 4.2.3, $\mathrm{udim}(G, \mathbb{F}) = \mathrm{udim}(G - e, \mathbb{F})$. This contradicts the criticality of G, and therefore if G is bipartite, it contains no even cycles.

Now suppose that G is not bipartite and contains an even cycle C. We know that G also contains an odd cycle C', and we take an edge $e \in C \backslash C'$. The graph $G - e$ remains non-bipartite, and still has s components. Therefore, by Proposition 4.2.3, $\mathrm{udim}(G, \mathbb{F}) = \mathrm{udim}(G - e, \mathbb{F})$. This contradicts the criticality of G, and completes the proof that G contains no even cycles.

It remains to prove that G can have only one odd cycle. Suppose G has two distinct odd cycles, C_1 and C_2. Then there is an edge $e \in C_1 \backslash C_2$. The graph $G - e$ has s components, and is not bipartite since it still contains the odd cycle C_2. Therefore, by Proposition 4.2.3, $\mathrm{udim}(G, \mathbb{F}) = \mathrm{udim}(G - e, \mathbb{F})$. This contradicts the criticality of G, and therefore G contains at most one odd cycle. $\square$

The next theorem tells us that any hypergraph obtained by deleting edges from a symmetric block design is critical as long as certain restrictions on the characteristic of the field are met.

Theorem 6.2.3. *Let $D = (V, \mathbf{B})$ be a symmetric block design with parameters v, k, and λ, where $v > k > \lambda$. Then any hypergraph obtained by deleting $0 \leq t < v$ blocks from D has critical uniformity dimension $1 + t$ over any field $\mathbb{F}$ of characteristic not dividing $v, k - \lambda$, or $k + \lambda(v - 1)$.*

Proof. First we show that D has 1-critical uniformity dimension. We know that $\mathrm{udim}(D, \mathbb{F}) = 1$ by Theorem 5.3.1. Further, D has v vertices and v blocks, so by Theorem 6.1.4, this dimension is critical.

By Observation 6.1.8, we see that deleting any t blocks from D creates a hyergraph with critical uniformity dimension $1 + t$ for any $0 < t < v - 2$. $\square$

Remark 6.2.1. In the notation of Theorem 6.2.3, if the characteristic of $\mathbb{F}$ divides $k - \lambda$, then $\mathrm{udim}(D, \mathbb{F}) = v - 1$ by Theorem 5.3.3. Thus if $v > 2$, then D is not critical by Theorem 6.1.4. If the characteristic of $\mathbb{F}$ does not divide $k - \lambda$, but divides v or $k + \lambda(v - 1)$, we do not know whether the uniformity dimension of D over $\mathbb{F}$ is 1 or 2, so we cannot make a conclusion about whether this uniformity dimension is critical.

Chapter 7

Conclusion

We conclude with a number of questions and problems suggested by our investigation of uniformity dimension. We found a way to determine the uniformity dimension of 1-uniform and 2-uniform hypergraphs in Chapter 4 without explicitly solving a linear system. Is there such a method for 3-uniform hypergraphs, or more generally for l-uniform hypergraphs? We did find the uniformity dimension of some families of highly structured hypergraphs. Are there other families of hypergraphs whose uniformity dimension can be found without explicitly solving the linear system?

In Section 4.3 we presented tables containing the number of non-isomorphic 3-uniform hypergraphs on n vertices with m edges and uniformity dimension k for $n = 4, 5, 6$ and all possible values of m and k. We defined two families of sequences, $(r_{l,n,m}(k))_{k=1}^{k=n}$ and $(c_{l,n,k}(m))_{m=0}^{m=\binom{n}{l}}$, which were generalizations of the rows and columns respectively of these tables. Corollary 6.1.6 told us that the column sequences $(c_{l,n,k}(m))_{m=0}^{m=\binom{n}{l}}$ contain no internal zeros. Are the column sequences in fact unimodal? (We know that if the generating polynomial of a sequence a_n of positive numbers has all real roots then a_n is unimodal, as shown in [19]. However, the 2$^{\text{nd}}$ column of Table 4.2, which is the sequence $(c_{3,5,2}(m))_{m=0}^{m=10}$, has generating polynomial $5x^4 + 2x^5 + 2x^6 = x^4(5 + 2x + 2x^2)$. This polynomial has non-real roots, and therefore a different approach would be necessary to prove that all column sequences are unimodal.) Do the row sequences contain internal zeros? Are the nonzero terms of each row sequence decreasing?

We found an easy way to compute the uniformity dimension of tight l-uniform cycles and loose 3-uniform cycles. Can we find an easy way to compute the uniformity dimension of loose l-uniform cycles for $l > 3$? What about other special types of l-uniform cycles? For example, those l-uniform cycles with maximum degree 2. What about l-uniform cycles in general?

We proved that a symmetric block design D with parameters v, k, and λ has uniformity dimension 1 when the characteristic of the field does not divide $k - \lambda$, $k + \lambda(v - 1)$, or v. The proof of this result was strictly algebraic, and it would be interesting to find a combinatorial proof. If the field has characteristic dividing $k-\lambda$, we found that the uniformity dimension of D must be $v - 1$ as long as D has at least 2 blocks. If the characteristic does not divide $k - \lambda$ but divides one or both of $k + \lambda(v-1)$ and v, then we proved that the uniformity dimension of D must be 1 or 2. We have left to find out exactly when the dimension is 1 and when it is 2.

Finally, we characterized the space of stable weightings of the spanning forests of any graph G. This problem is similar to the problem of finding the space of well-covered weightings of a graph, because in both cases we are looking for the weightings of G for which the weight is equal on all substructures of a particular kind. We would like to consider more problems like this that involve finding the stable weightings of certain graph substructures.

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Appendix A

Maple Code

We include two worksheets created using Maple 14. Figure A.1 gives the code written to enumerate the edge sets of all non-isomorphic 3-uniform hypergraphs on 6 vertices. Figure A.2 gives the code written to determine the uniformity dimensions of any set of 3-uniform hypergraphs on 6 vertices. Together, these worksheets contain the procedures used to create Table 4.3. We also modified the procedures contained in Figures A.1 and A.2 to produce Tables 4.1 and 4.2.

The procedures can easily be adjusted to work for 3-uniform hypergraphs on 7 or more vertices. However, the algorithm for enumerating the edge sets of all non-isomorphic 3-uniform hypergraphs on n vertices is very slow even for $n = 6$.

```
> with(combinat) :
> PermAct := proc(S, T)
    # For S a permutation of n elements (for example [1, 3, 4, 6, 2, 5] represents the
    # permutation of {1,2,3,4,5,6} that sends 1 to 1, 2 to 3, 3 to 4, 4 to 6, 5 to 2, and 6 to 5)
    # and T the edge set of some hypergraph on n vertices, PermAct(S, T) returns the edge
    # set T with the vertices permuted by S.
  local N, i;
  N := NULL;
  for i from 1 to nops(T) do
    N := N, {S[T[i][1]], S[T[i][2]], S[T[i][3]]};
  od;
  {N};
  end;
PermAct := proc(S, T)                                                    (1)
  local N, i;
  N := NULL;
  for i to nops(T) do N := N, {S[T[i][1]], S[T[i][2]], S[T[i][3]]} end do;
  {N}
end proc
> IsIsoTo := proc(S, T)
    # For S and T edge sets of hypergraphs on underlying vertex set {1,2,3,4,5,6}, IsIsoTo(S,T)
    # returns 1 if S and T are isomorphic, and 0 otherwise.
  local U;
  for U in permute([1, 2, 3, 4, 5, 6]) do
    if is(PermAct(U, S) = T) then
      return 1;
    fi;
  od;
  return 0;
  end;
IsIsoTo := proc(S, T)                                                    (2)
  local U;
  for U in combinat:-permute([1, 2, 3, 4, 5, 6]) do
    if is(PermAct(U, S) = T) then return 1 end if
  end do;
  return 0
end proc
> Complement := proc(S)
    # For S the edge set of a 3-uniform hypergraph on vertex set {1,2,3,4,5,6}, Complement(S)
    # returns the edge set of the complement of S, i.e. the edge set of the 3-uniform hypergraph
    # on vertex set {1,2,3,4,5,6} with edges exactly those edges of size 3 not contained in S.
  choose({1, 2, 3, 4, 5, 6}, 3) minus S;
  end;
  Complement := proc(S) minus(combinat:-choose({1, 2, 3, 4, 5, 6}, 3), S) end proc    (3)
```

```
> NonIso := proc(Z, n)
    # For Z a set of edge sets of 3-uniform hypergraphs on vertex set {1,2,3,4,5,6} containing
    # exactly n-1 edges,the function NonIso(Z, n) returns the set of of edge sets of all
    # nonisomorphic 3-uniform hypergraphs on vertex set {1,2,3,4,5,6} which can be obtained
    # by adding a single edge to some member of Z.
    # So if Z contians the edge sets of all nonisomorphic 3-uniform hypergraphs on
    # {1,2,3,4,5,6} with n-1 edges,then NonIso(Z, n) returns the set of edge sets of all
    # nonisomorphic 3-uniform hypergraphs on {1,2,3,4,5,6} with n edges.
  local N, W, X, Y, M, V, t, U;
  M := choose(choose({1, 2, 3, 4, 5, 6}, 3), n)[1];
  N := NULL;
  for X in Z do
    for W in Complement(X) do
      N := N, X union {W};
    od;
  od;
  for V in N do
    t := 0;
    for U in M do
      if IsIsoTo(U, V) = 1 then
        t := 1;
        break
      fi;
    od;
    if t = 0 then
      M := M, V;
    fi;
  od;
  {M};
  end;
NonIso := proc(Z, n)                                                    (4)
  local N, W, X, Y, M, V, t, U;
  M := combinat:-choose(combinat:-choose({1, 2, 3, 4, 5, 6}, 3), n)[1];
  N := NULL;
  for X in Z do for W in Complement(X) do N := N, union(X, {W}) end do end do;
  for V in N do
    t := 0;
    for U in M do if IsIsoTo(U, V) = 1 then t := 1; break end if end do;
    if t = 0 then M := M, V end if
  end do;
  {M}
end proc
> # Examples

  # We know there is only one 3-uniform hypergraph on {1,2,3,4,5,6} with one edge up to
```

```
# isomorphism, so the following code returns the set of edge sets of all nonisomorphic
# 3-uniform hypergraphs on {1,2,3,4,5,6} with 2 edges.

NonIso({{{1, 2, 3}}}, 2);

            {{{1, 2, 3}, {1, 2, 4}}, {{1, 2, 3}, {1, 4, 5}}, {{1, 2, 3}, {4, 5, 6}}}      (5)
> # Now using the output of the previous line, we can find the set of edge sets of all
# nonisomorphic 3-uniform hypergraphs on {1,2,3,4,5,6} with 3 edges.

NonIso({{{1, 2, 3}, {1, 2, 4}}, {{1, 2, 3}, {1, 4, 5}}, {{1, 2, 3}, {4, 5, 6}}}, 3);
{{{1, 2, 3}, {1, 2, 4}, {1, 2, 5}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 4}}, {{1, 2, 3}, {1, 2, 4}, {1, 3,      (6)
   5}}, {{1, 2, 3}, {1, 2, 4}, {1, 5, 6}}, {{1, 2, 3}, {1, 2, 4}, {3, 4, 5}}, {{1, 2, 3}, {1, 2,
   4}, {3, 5, 6}}, {{1, 2, 3}, {1, 4, 5}, {2, 4, 6}}}
> # Similarly we find the set of edge sets of all nonisomorphic 3-uniform hypergraphs on
# {1,2,3,4,5,6} with 4 edges.

NonIso({{{1, 2, 3}, {1, 2, 4}, {1, 2, 5}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 4}}, {{1, 2, 3}, {1, 2,
      4}, {1, 3, 5}}, {{1, 2, 3}, {1, 2, 4}, {1, 5, 6}}, {{1, 2, 3}, {1, 2, 4}, {3, 4, 5}}, {{1, 2,
      3}, {1, 2, 4}, {3, 5, 6}}, {{1, 2, 3}, {1, 4, 5}, {2, 4, 6}}}, 4);
{{{1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 2, 6}}, {{1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 3, 4}}, {{1, 2,      (7)
   3}, {1, 2, 4}, {1, 2, 5}, {1, 3, 6}}, {{1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {3, 4, 5}}, {{1, 2, 3},
   {1, 2, 4}, {1, 2, 5}, {3, 4, 6}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {1, 5, 6}}, {{1, 2, 3}, {1,
   2, 4}, {1, 3, 4}, {2, 3, 4}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 5}}, {{1, 2, 3}, {1, 2,
   4}, {1, 3, 4}, {2, 5, 6}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 5}, {1, 4, 5}}, {{1, 2, 3}, {1, 2, 4},
   {1, 3, 5}, {1, 4, 6}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 5}, {2, 3, 6}}, {{1, 2, 3}, {1, 2, 4}, {1,
   3, 5}, {2, 4, 5}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 5}, {2, 4, 6}}, {{1, 2, 3}, {1, 2, 4}, {1, 3,
   5}, {2, 5, 6}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 5}, {4, 5, 6}}, {{1, 2, 3}, {1, 2, 4}, {1, 5, 6},
   {2, 5, 6}}, {{1, 2, 3}, {1, 2, 4}, {1, 5, 6}, {3, 4, 5}}, {{1, 2, 3}, {1, 2, 4}, {1, 5, 6}, {3,
   5, 6}}, {{1, 2, 3}, {1, 2, 4}, {3, 5, 6}, {4, 5, 6}}, {{1, 2, 3}, {1, 4, 5}, {2, 4, 6}, {3, 5,
   6}}}
> # We continue this process to find all nonisomorphic 3-uniform hypergraphs on
# {1,2,3,4,5,6} with k edges for k up to 10.  For k larger than 10 we can use the
# Complement function and the previous output of NonIso to find
# all nonisomorphic 3-uniform hypergraphs on {1,2,3,4,5,6} with k edges.
```

Figure A.1: Maple code used to find the edge sets of all non-isomorphic 3-uniform hypergraphs on 6 vertices

```
> with(LinearAlgebra) :
> SolutionMatrix := proc(S, n)
    # For S the edge set of a hypergraph on vertex set {1,2,3,...,n},
    # SolutionMatrix(S,n) returns the solution matrix W of S.
  local A, k, i :
  A := Matrix(nops(S) − 1 , n) :
  convert(S, list) :
  for i from 1 to nops(S) − 1 do
    for k in S[i] do
      A[i, k] := 1 :
    od:
  od:
  for i from 1 to nops(S) -1 do
    for k in S[nops(S)] do
      A[i, k] := A[i, k] − 1 :
    od:
  od:
  A;
  end;
SolutionMatrix := proc(S, n)                                                              (1)
  local A, k, i;
  A := Matrix(nops(S) − 1, n);
  convert(S, list);
  for i to nops(S) − 1 do for k in S[i] do A[i, k] := 1 end do end do;
  for i to nops(S) − 1 do for k in S[nops(S)] do A[i, k] := A[i, k] − 1 end do end do;
  A
end proc
> UniformityDimensions := proc(S)
    # For S a set of edge sets of hypergraphs on vertex set {1,2,3,4,5,6},
    # UniformityDimensions(S) returns a list [x1,x2,x3,x4,x5,x6], where xi is the
    # number of hypergraphs in S with uniformity dimension i for all i
    # in {1,2,3,4,5,6}.
  local x1, x2, x3, x4, x5, x6, i, T, A;
  x1 := 0;
  x2 := 0;
  x3 := 0;
  x4 := 0;
  x5 := 0;
  x6 := 0;
  for T in S do
    A := SolutionMatrix(T, 6);
    i := ColumnDimension(A) − Rank(A);
    if i = 1 then x1 := x1 + 1;
    elif i = 2 then x2 := x2 + 1;
    elif i = 3 then x3 := x3 + 1;
    elif i = 4 then x4 := x4 + 1;
```

```
        elif i = 5 then x5 := x5 + 1;
        elif i = 6 then x6 := x6 + 1;
        fi;
    od;
    [x1, x2, x3, x4, x5, x6];
    end;
```

UniformityDimensions := **proc**(S) **(2)**

```
    local x1, x2, x3, x4, x5, x6, i, T, A;
    x1 := 0;
    x2 := 0;
    x3 := 0;
    x4 := 0;
    x5 := 0;
    x6 := 0;
    for T in S do
        A := SolutionMatrix(T, 6);
        i := LinearAlgebra:-ColumnDimension(A) - LinearAlgebra:-Rank(A);
        if i = 1 then
            x1 := x1 + 1
        elif i = 2 then
            x2 := x2 + 1
        elif i = 3 then
            x3 := x3 + 1
        elif i = 4 then
            x4 := x4 + 1
        elif i = 5 then
            x5 := x5 + 1
        elif i = 6 then
            x6 := x6 + 1
        end if
    end do;
    [x1, x2, x3, x4, x5, x6]
end proc
```

```
> #Example

  # Taking the set of edge sets of all 3-uniform hypergraphs on {1,2,3,4,5,6}
  # with 4 edges generated in the previous example, we find the uniformity
  # dimensions of these hypergraphs.

  UniformityDimensions({{{1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 2, 6}}, {{1, 2, 3}, {1, 2, 4}, {1,
```

```
2, 5}, {1, 3, 4}}, {{1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 3, 6}}, {{1, 2, 3}, {1, 2, 4}, {1, 2,
5}, {3, 4, 5}}, {{1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {3, 4, 6}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 4},
{1, 5, 6}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2,
3, 5}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 5, 6}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 5}, {1, 4,
5}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 5}, {1, 4, 6}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 5}, {2, 3,
6}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 5}, {2, 4, 5}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 5}, {2, 4,
6}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 5}, {2, 5, 6}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 5}, {4, 5,
6}}, {{1, 2, 3}, {1, 2, 4}, {1, 5, 6}, {2, 5, 6}}, {{1, 2, 3}, {1, 2, 4}, {1, 5, 6}, {3, 4,
5}}, {{1, 2, 3}, {1, 2, 4}, {1, 5, 6}, {3, 5, 6}}, {{1, 2, 3}, {1, 2, 4}, {3, 5, 6}, {4, 5,
6}}, {{1, 2, 3}, {1, 4, 5}, {2, 4, 6}, {3, 5, 6}}});
                        [0, 0, 19, 2, 0, 0]                        (3)
> # So we see that 19 of these 21 hypergraphs have uniformity dimension 3,
  # while 2 of the 21 have uniformity dimension 4.
```

Figure A.2: Maple code used to find the uniformity dimension of all non-isomorphic 3-uniform hypergraphs on 6 vertices

Appendix B

3-Uniform Hypergraphs of High Uniformity Dimension

We include drawings of the non-isomorphic 3-uniform hypergraphs with 6 vertices and m edges that have *maximal* uniformity dimension for $m = 4, 5, 6, 7$, and 8.

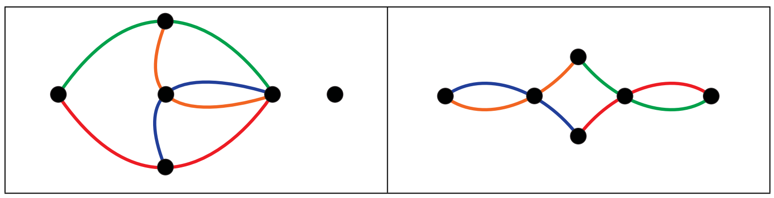

Table B.1: The 3-uniform hypergraphs on 6 vertices with 4 edges and uniformity dimension 4

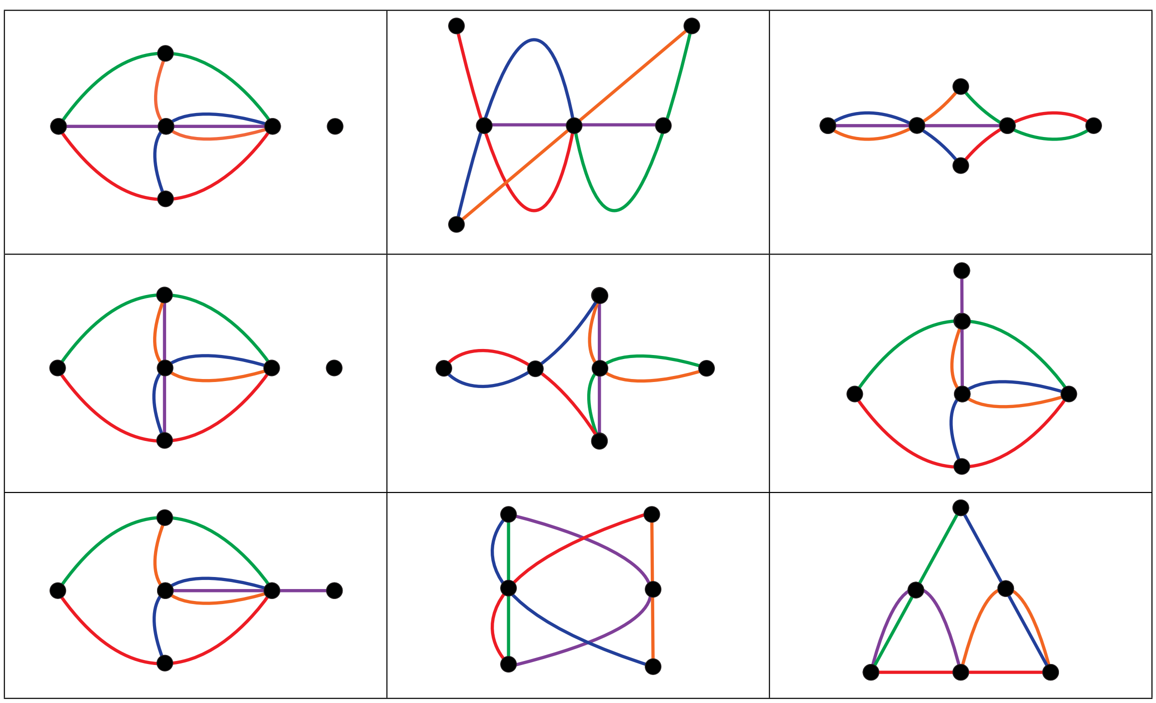

Table B.2: The 3-uniform hypergraphs on 6 vertices with 5 edges and uniformity dimension 3

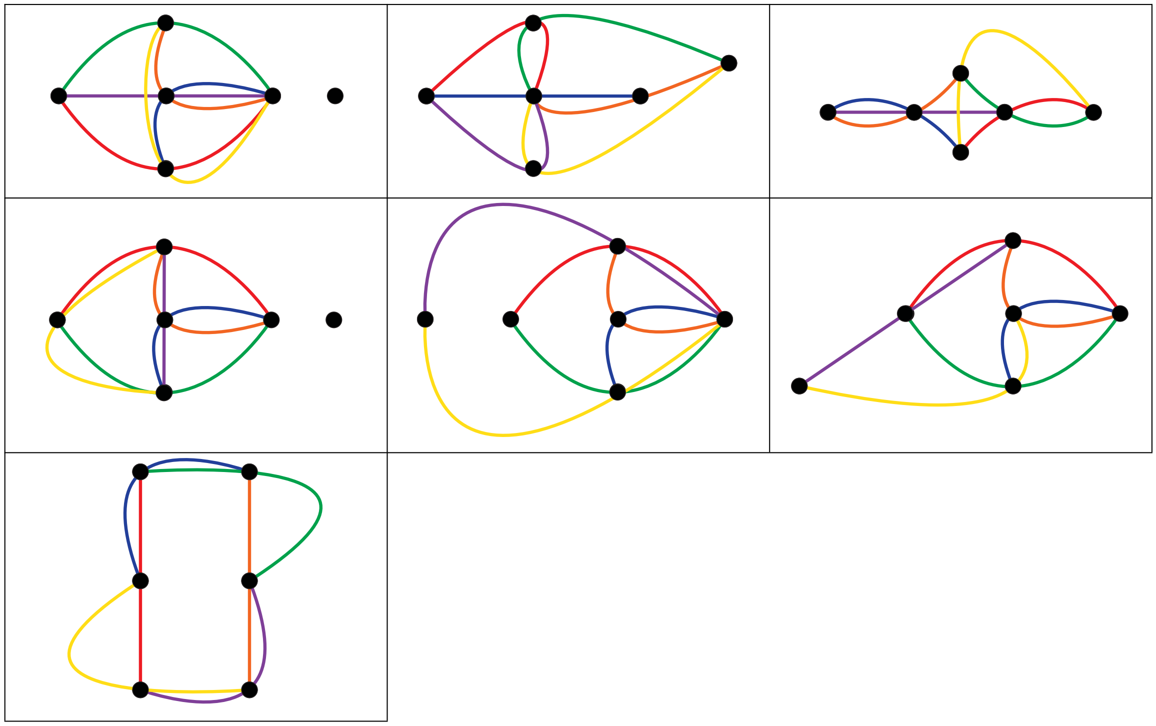

Table B.3: The 3-uniform hypergraphs on 6 vertices with 6 edges and uniformity dimension 3

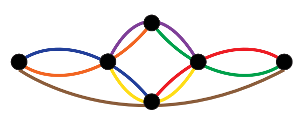

Figure B.1: The 3-uniform hypergraph on 6 vertices with 7 edges and uniformity dimension 3

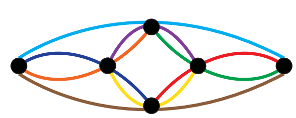

Figure B.2: The 3-uniform hypergraph on 6 vertices with 8 edges and uniformity dimension 3