ON THE STRUCTURE OF GAMES AND THEIR POSETS

by

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To my wonderful family that continuously supports and loves me in all that I do

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Abstract

This thesis explores the structure of games, including both the internal structure of various games and also the structure of classes of games as partially ordered sets. Internal structure is explored through consideration of juxtapositions of game positions and how the underlying games interact. We look at ordinal sums and introduce side-sums as a means of understanding this interaction, giving a full solution to a TOPPLING DOMINOES variant through its application. Loopy games in which only one player is allowed a pass move, referred to as Oslo games, are introduced and their game structure explored. The poset of Oslo games is shown to form a distributive lattice. The Oslo forms of WYTHOFF'S GAME, GRUNDY'S GAME and OCTAL .007 are introduced and full solutions given. Finally, the poset of option-closed games is given up to day 3 and all are shown to form a planar lattice. The option-closed game of CRICKET PITCH is also fully analyzed.

List of Abbreviations and Symbols Used

0	$\{\cdot\,\vert\,\cdot\}$ “zero”
$*$	$\{0\,\vert\,0\}$ “star”
$*n$	$\{0, *, \ldots, *(n-1)\,\vert\,0, *, \ldots, *(n-1)\}$ “star-n”
$\bigstar$	‘far star”
1	$\{0\,\vert\,\cdot\}$
n	$\{n-1\,\vert\,\cdot\}$
$\uparrow$	$\{0\,\vert\,*\}$ “up”
$\Uparrow$	$\{0\,\vert\uparrow *\}$ “double-up”
$\downarrow$	$\{*\,\vert\,0\}$ “down”
$\Downarrow$	$\{\downarrow *\,\vert\,0\}$ “double-down”
n	$\{0\,\vert\{0\,\vert -n\}\}$ “tiny-n”
$\mathbf{L}(G)$	the set of left options of G
G^L	a specific first left option of G
$\mathbf{R}(G)$	the set of right options of G
G^R	a specific first right option of G
$G+H$	the disjunctive sum of G and H
$G\mathbf{:}H$	the ordinal sum of G and H
$G \diamond H$	the side-sum of G and H
$G \odot H$	G side-out H
$\mathcal{N}$	the set of all first (next) player win games
$\mathcal{L}$	the set of all Left win games
$\mathcal{P}$	the set of all second (previous) player win games
$\mathcal{R}$	the set of all Right player win games

Acknowledgements

First, and foremost, I would like to thank Dr. Richard Nowakowski. My experience at Dalhousie, or as a graduate student in general, would not have been as enjoyable were it not for him. I feel lucky to have landed[1] in Halifax as his student. It seems that there is always a fun new topic to explore whenever we meet.

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[1] literally!

[2] In addition, a Ph.D. would be nice!

Chapter 1

Introduction

1.1 Overview Of Thesis

The focus of this thesis is on the structure of games. It explores the internal structure within a game, as well as the structure of classes of games as partially ordered sets.

In Chapter 1, we first introduce combinatorial games. Fundamentals are covered that will be utilized later in attaining new results. Dos Santos and Silva explored the habitat of nimbers in various games and classes of games [25]. Akin to this idea, the concept of Hackenbush-dimension, a concept that quantifies which types of numbers can arise in a game or class of games, is introduced in section 1.2.8.2. Not only does this give a means of describing the types of positions that can arise in a game, but it also becomes useful later in Chapter 2 when we take a look at ordinal sums.

Games have a rich structure that includes a partial order and equivalence classes. The latter half of Chapter 1 covers the fundamentals of partial orders and lattices. The lattice structure of games born by day n is introduced and common methodologies for exploring the partial order structure of a set of games is introduced.

In Chapter 2, we take a look at the internal structure of certain game positions. Juxtapositions, the act of two board positions being place adjacent to one another, are introduced. We look at positions in which a single position is formed from two games sitting next to each other. We explore these interactions through the use of ordinal sums and a similar function, that will be introduced, which we will call side-sums. For each, we will explore the games that can be formed by taking juxtapositions of games from day n. We will also look at applications for each as solutions to existing games. The games of LENRES, SHOVE and RESTRICTED TOPPLING DOMINOES will be explored with the help of the ordinal sum and side-sum functions.

In Chapter 3, we will venture into the realm of loopy games. We will consider the class of games, which we will call Oslo (**O**ne-**s**ided **lo**opy) games, in which only one player is allowed a pass. For these, we will explore both the structure and the form of Oslo games that can arise. We will also show that the set of Oslo games by day n form a distributive lattice. Finally, we will introduce a new concept which we will call uponic weight. We will show that this is a meaningful way to describe loopy games. Based on atomic weight, which gives an approximation of a game based on the game $\uparrow$, uponic weight gives a means of describing games with respect to the game UPON$*$.

Chapter 4 explores option-closed games. Option-closed games are those of a form such that any position that could be reached through consecutive moves could have been reached in one single move. We introduce a function that takes a game and returns it in option-closed form. We then introduce the concepts of left- and right-threats in a game. For games to have the same value under option-closure, we will require consideration of these threats. We will show that two threatbare games that have the same value will be infinitesimally close to each other when option-closed. We look at the lattice structure of option-closed games by day and show that these games form a planar lattice. We will explore three option-closed games: MAZE, ROLL THE LAWN and CRICKET PITCH. For the last, we will give a complete analysis.

Finally, in Chapter 5, we will summarize all that we have explored and conclude with the laundry list of future work that remains to be done in these areas.

Unless otherwise specified, all theorems and proofs are those of the author. A large number of results that we will make use of are taken from the book *Lessons in Play* [2]. Unless they serve a greater purpose, most proofs of theorems taken from outside sources will be omitted.

1.2 Combinatorial Games

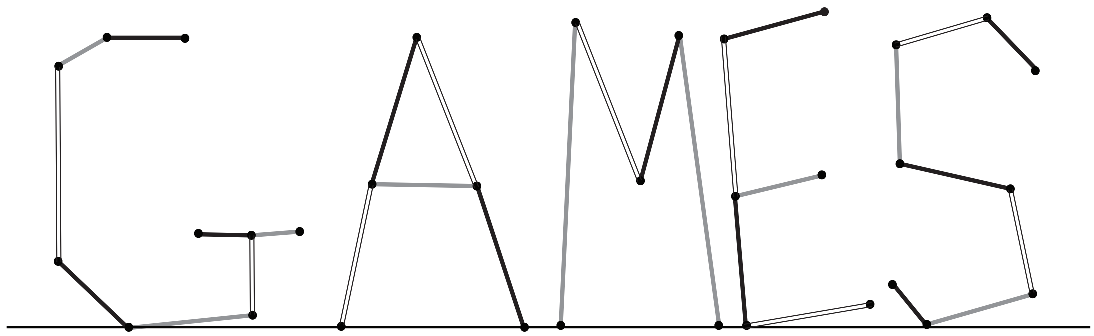


Figure 1.1: The HACKENBUSH "games" game position.

Combinatorial game theory is a mathematical theory that describes the detail and structure of games of no chance. Particularly, it studies two-player games in which players alternately take turns making defined moves in an attempt to obtain a specified ending condition.

"These young guys are playing checkers. I'm out there playing chess."
- Kobe Bryant

Not to be confused with classical game theory, used in cooperation games and the theory of economics, this theory will not help us at the casino. However, its principles can be applied to common games such as CHESS, CHECKERS and GO. In all future instances, the term game theory, used for brevity, will imply combinatorial game theory and the term game will always imply a combinatorial game.

As suggested, the games that we will look at will involve two players, who will alternate play. We shall call the two players **Left** and **Right**, as per convention.[1] At all times, in combinatorial games, both players have perfect information. That is, each player has full knowledge of the present and future moves available to both themselves and their opponent.

[1] Traditional convention also refers to Left as female and Right as male in honour of Louise and Richard Guy, the latter being one of the founding fathers of game theory and authors of *Winning Ways* [3].

All games will end with one winner. That is, there will be a defined **ending condition** such that ties or draws are not possible. We will restrict our consideration of games to those played under the **normal play convention**, whereby the first person unable to make a move loses.

We will originally consider **short games**, games having only a finite number of total positions in which repetition of positions is not allowed. In these games, due to the placed restriction, the play will always come to an end.

We will later consider games falling outside of that realm, called **loopy games**. Loopy games are interesting in that it is possible to return to the same position over and over again. In these cases, we drop the normal play ending condition and must consider the act of "winning" in a different light as, in loopy games, it is possible for a game to go on indefinitely. One such familiar example of a loopy game is the game of CHECKERS, in which certain positions can lead to an infinitely long sequence of moves.

1.2.1 Fundamentals

Based on the given ruleset for a game, each player will have a set of positions that he/she may move to. The positions that can be obtained by Left are referred to as **left options**, and those that can be obtained by Right are referred to as **right options**.

Definition 1.2.1. *[2, p. 4] For a game G, the* ***left options*** *of G are the positions that can be reached if Left moves first in G. We will denote the set of all left options of G as $\mathbf{L}(G)$ and use G^L to denote an element of $\mathbf{L}(G)$. Likewise, the* ***right options*** *of G are the positions that can be reached if Right moves first in G. Similarly, $\mathbf{R}(G)$ will denote the set of all right options and G^R will refer to an element of $\mathbf{R}(G)$.*

The **options** of a game are simply the elements in the union of these two sets. Thus, options from G are all positions in $\mathbf{L}(G) \cup \mathbf{R}(G)$. Games are defined recursively in terms of the left and right options available from a given position.

Definition 1.2.2. *[2, p. 66][9, p. 4] Let $\mathbf{L}(G)$ and $\mathbf{R}(G)$ be two arbitrary sets of games, each possibly empty or infinite. Then the pair $(\mathbf{L}(G), \mathbf{R}(G))$, which will be* *denoted as*

$$G = \{\mathbf{L}(G) \,|\mathbf{R}(G)\},$$

*is a **game** where* $\mathbf{L}(G)$ *is the set of left options and* $\mathbf{R}(G)$ *the set of right options of* G*.*

For those outside of combinatorial game theory, this may first appear to be an abuse of notation to denote the game $G = (\mathbf{L}(G), \mathbf{R}(G))$ as $G = \{\mathbf{L}(G) \,|\mathbf{R}(G)\}$. However, it is standard convention within the field.

Definition 1.2.3. *[26, 3] For a game* G*, the **positions** of* G *are* G *and all positions of any option of* G*, i.e.* G *and all positions of any* $H \in \mathbf{L}(G) \cup \mathbf{R}(G)$*.*

*The **followers** of* G *are all positions of any option of* G*, i.e. all positions of any* $H \in \mathbf{L}(G) \cup \mathbf{R}(G)$*.*

Definition 1.2.4. *[2, p. 3] A **short game** is a game in which a position may never be repeated and there are only a finite number of other positions that can be reached.*

Thus, a short game must eventually come to an end. When we refer to games, we shall assume they are short games unless specified otherwise.

In order to depict the options available to each player, we will sometimes draw a **game tree** of the position. The game tree is a diagram listing each set of options, with left options appearing below and to the left of the game, and right options appearing below and to the right, as in Figure 1.2.

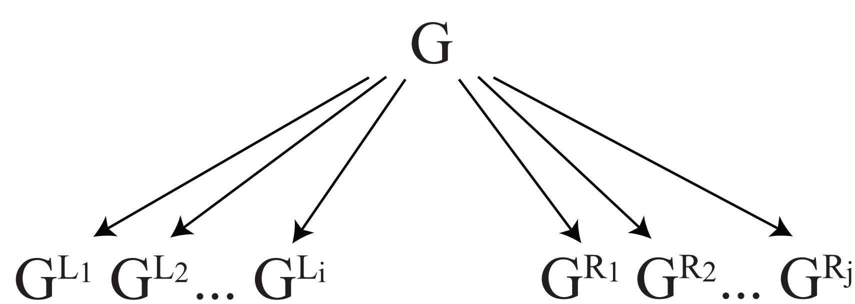


Figure 1.2: Example of a game tree.

While this figure only demonstrates one layer of depth in the game tree, the full game tree for a given position would include both left and right options from each follower.

It might first seem pointless to include right options from right options, and left from left, as we've previously specified alternating play, precluding consecutive moves by either player. However, while alternate play holds within a game, the bulk of the theory of combinatorial games comes from the analysis of the decomposition of games. That is, in play, there are situations in which games decompose into subgames. This happens when a position breaks up so that moves made in one part of the position will not affect the other parts. In such a decomposition, alternating play in the overarching game may well involve consecutive moves for a given player in one of the subgames. Thus, in that subgame (and so in any game) we must also understand the result when players do not alternate moves.

1.2.1.1 The Endgame

Prior to any game being created, the only set of games available to us is the empty set. This leads us to the definition of the important game zero, sometimes referred to as the **endgame**, which is the game in which both $\mathbf{L}(G)$ and $\mathbf{R}(G)$ are empty (i.e. $\mathbf{L}(G) = \mathbf{R}(G) = \emptyset$):

$$0 = \{\emptyset \,|\, \emptyset\}\,.$$

For brevity, we will sometimes use "$\cdot$" as a placeholder for $\emptyset$, such as $0 = \{\cdot\,|\,\cdot\}$. Note that in the game zero, neither player has a legal move.

1.2.1.2 Induction Basis

As the definition of a game G is recursive, the game zero becomes the base case for the recursion. Many of the following definitions are also recursive. It may not at first glance be obvious, but most require no basis, as ultimately they are reduced to using only members of the empty set.

Conway has given us a simple induction principle that we will utilize in almost all game theoretic proofs [9].

Theorem 1.2.5 (Conway Induction)**.** *[3, 9] Let P be a property which games might have, such that any game G has property P whenever all left and right options of G have this property. Then every game has property P.*

Proof. Suppose that there exists a game G that does not satisfy property P. If all options of G satisfy P, then G must also by hypothesis. Thus, there must exist a left or right option H of G that does not satisfy P. Likewise, since the game H does not satisfy property P, there must exist an option H' of H that does not satisfy property P. Similarly then, there must be an option H'' of H' not satisfying property P. We can inductively continue this argument and obtain an infinite sequence $H, H', H'', \ldots$ of games which are each an option of their predecessor. However, this contradicts G being a short game. Thus, the original game G must satisfy the property P. □

The fact that the endgame zero, $\{\cdot\,|\cdot\}$, satisfies property P, no matter what it might be, is vacuously true since there is no option of 0 which might fail property P. For this reason, Conway Induction does not require a specific induction base as is required with ordinary induction.

1.2.1.3 Disjunctive Sum

It was previously mentioned that decomposition of games plays a large role in their analysis. As such, we need to have an understanding of the addition of games, so that once a game has decomposed into sums of smaller subgames, we can deal with them in a meaningful way. We do this through disjunctive sums.

Definition 1.2.6. *[9, p. 5][2, p. 68] For games G and H, we define their* ***disjunctive sum****, $G + H$, as*

$$G + H = \{\mathbf{L}(G) + H, G + \mathbf{L}(H) \,|\mathbf{R}(G) + H, G + \mathbf{R}(H)\},$$

where, for a game K and a set of games $\mathbf{S}$, we define

$$K + \mathbf{S} = \{K + s\}_{s \in \mathbf{S}}.$$

Thus, if $\mathbf{S} = \emptyset$, then $K + \mathbf{S}$ is also empty. It is conventional notation to use "$\mathbf{A}, \mathbf{B}$" within a set of left (right) options to imply the set union $\mathbf{A} \cup \mathbf{B}$. This notation has been adopted by game theorists for its less cumbersome nature.

Proposition 1.2.7. *[2, Thm. 4.4, p. 69]*

$$G + 0 = G.$$

Proof. Since $\mathbf{L}(0) = \mathbf{R}(0) = \emptyset$,

$$G + 0 = \{\mathbf{L}(G) \,|\mathbf{R}(G)\} = G.$$

□

The addition of games is both commutative and associative with $0 \cong \{\cdot\,|\cdot\}$ as the identity element.

Proposition 1.2.8. *[2, Thm. 4.5][9, p.17] For games G, H and K,*

(i) [Commutativity] $G + H = H + G$, *and*

(ii) [Associativity] $(G + H) + K = G + (H + K)$.

As we have considered $G+H$, it is natural to wonder about $G-H$. To understand this, we require understanding of the negative of a game. The negative of a game G, denoted $-G$, corresponds to reversing the roles of Left and Right. That is, the sets of left and right options are recursively swapped in all of the options.

Definition 1.2.9. *[2, p. 69] The **negative** of a game G is defined to be*

$$-G = \{-\mathbf{R}(G) \,|-\mathbf{L}(G)\},$$

where

$$-\mathbf{L}(G) = \{-G^L\}_{G^L \in \mathbf{L}(G)}, \text{ and } -\mathbf{R}(G) = \{-G^R\}_{G^R \in \mathbf{R}(G)}.$$

Definition 1.2.10. *[2, p. 69]*

$$G - H = G + (-H).$$

Hence, we have the following two results.

Proposition 1.2.11. *[2, p. 69]*

$$-(-G) = G.$$

This is easy to see, as

$$-(-G) = -(\{-\mathbf{R}(G) \,|-\mathbf{L}(G)\}) = \{\mathbf{L}(G) \,|\mathbf{R}(G)\} = G.$$

Proposition 1.2.12. *[2, p. 69]*

$$-(G + H) = (-G) + (-H).$$

1.2.2 HACKENBUSH

One classic example of a combinatorial game is the game of HACKENBUSH, introduced in Winning Ways [3]. The game of HACKENBUSH is played with a picture such as that in Figure 1.1. In the game of Hackenbush, Left is allowed to move by deleting any black edge together with any edges no longer connected to the ground. Similarly, Right is allowed to remove any white edges along with any edges disconnected from the ground. Edges that are gray can be removed by either player. After a period of play[2], one player will find him or herself in a position where he or she has no available move. At that point, under the normal play convention, that player is the loser.

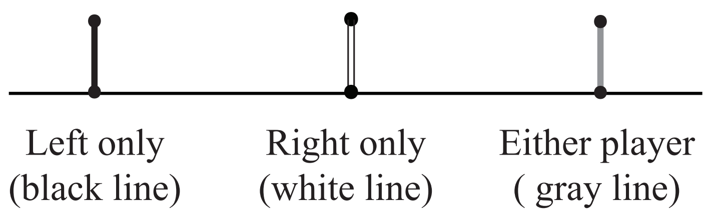


Figure 1.3: Example of the edges used in HACKENBUSH.

We introduce this game now, so as to make use of it in demonstrating further properties of games. However, we will return to HACKENBUSH on many an occasion as we look for a map taking other more complicated games to the simple and understood positions within HACKENBUSH.

1.2.3 Outcome Classes

"Winning is a habit. Unfortunately, so is losing."
- Vince Lombardi

As combinatorial games involve no chance and the players each have perfect information, it is possible to determine who will win the game prior to any move being made. It should be noted that while combinatorial game theory concerns itself with determining who will win a game, one major assumption is in play[3]; it is assumed that both players play sensibly. That is, we assume optimal play by both Left and

[2]Gratuitous hockey reference.
[3]Pun intended.

Right. If a strategy exists for a player to guarantee a win no matter how his or her opponent chooses to play, we refer to this as a **winning strategy**.

In discussing who has the possibility of winning a game, we refer to the **outcome class** of a game. In these outcome classes are determined by which player has a winning strategy based on which player moves first. The four outcome-classes are described in Table 1.2.3.

Class	**Name**	**Definition**
$\mathcal{P}$	zero	The $\mathcal{P}$revious player (or 2^{nd} to play) can force a win.
$\mathcal{N}$	fuzzy	The $\mathcal{N}$ext player to play (i.e. 1^{st} to play) can force a win.
$\mathcal{L}$	positive	$\mathcal{L}$eft can force a win, regardless of who moves first.
$\mathcal{R}$	negative	$\mathcal{R}$ight can force a win, regardless of who moves first.

Table 1.1: Outcome classes of short games.

For short games, these outcome classes are driven by the Fundamental Theorem of Combinatorial Games.

Theorem 1.2.13 (Fundamental Theorem of Combinatorial Games)**.** *[2, Thm. 2.1, p. 35] In a game G with Left moving first, either Left can force a win moving first, or Right can force a win moving second, but not both.*

In reference to the Fundamental Theorem of Combinatorial Games (Theorem 1.2.13), Table 1.2.3 offers another way of looking at outcome class.

If regardless who starts, the first (or next) player wins, the game is called a **first player win** or an $\mathcal{N}$-position. Conversely, if the first player loses and the second (or previous) player has a winning strategy, it is called a **second player win** or a

<table>
<tr><td colspan="2" rowspan="2">OUTCOME
CLASSES</td><td colspan="2">Right starts</td></tr>
<tr><td>L wins</td><td>R wins</td></tr>
<tr><td rowspan="2">Left starts</td><td>L wins</td><td>Left win
$\mathcal{L}$-position</td><td>1st player win
$\mathcal{N}$-position</td></tr>
<tr><td>R wins</td><td>2nd player win
$\mathcal{P}$-position</td><td>Right win
$\mathcal{R}$-position</td></tr>
</table>

Table 1.2: An alternate view of outcome classes.

$\mathcal{P}$-position. In these two outcome classes, the winner may be either Left or Right and depends only on their order of play.

If Left has a winning strategy no matter who starts, we call the game a **left win** or $\mathcal{L}$-position. Likewise, if Right can win no matter who starts, we call the game a **right win** or $\mathcal{R}$-position.

It is important to reiterate that if we say a game is left win (or first player win, etc.), we mean that he can force a win through optimal play. However, he will not necessarily win if he plays poorly. More importantly, it simply means that a winning strategy exists and not that it is actually known.

We will use $\mathcal{P}$, $\mathcal{N}$, L and $\mathcal{R}$ to denote the set of all $\mathcal{P}$-, $\mathcal{N}$-, $\mathcal{L}$- and $\mathcal{R}$-positions, respectively.

Notation 1.2.14. *For a game G, we represent the* ***outcome*** *of G as $o\,(G)$ with $o\,(G) \in \{\mathcal{L}, \mathcal{N}, \mathcal{P}, \mathcal{R}\}$.*

Each of the four outcome classes can be depicted by simple HACKENBUSH positions. This fact is witnessed in Figure 1.4, which depicts the smallest such HACKENBUSH positions embodying each of the four outcome classes. The positions represented in this Figure are the games $0 = \{\cdot\,|\cdot\}$, $* = \{0\,|0\}$, $1 = \{0\,|\cdot\}$ and $-1 = \{\cdot\,|0\}$, respectively, with $o(0) \in \mathcal{P}$, $o(*) \in \mathcal{N}$, $o(1) \in \mathcal{L}$ and $o(-1) \in \mathcal{R}$.

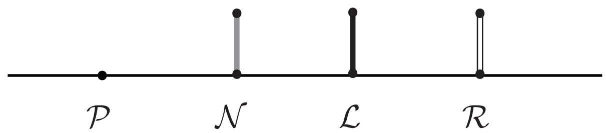


Figure 1.4: Examples of each outcome class as HACKENBUSH positions.

1.2.3.1 Equivalence

Now that we have a means of understanding outcome, we can discuss equivalence classes of games. We next introduce the concept of game equality, an equivalence relation. If two games G and H are equal, then in any sum of games, G and H act alike. As a side effect, in most contexts, you may substitute G for H when desired. However, while the games are equal in value, their game trees may differ. When we later discuss operations requiring the literal form of a game, this substitution will no longer give the desired results.

Definition 1.2.15. *[2, p.70] For games G, H and K, games G and H are said to be equal, denoted $G = H$ if*

$$\forall K,\ o(G+K) = o(H+K).$$

As noted, game equality is, in fact, an equivalence relation. It is reflexive ($G = G$), symmetric ($G = H \implies H = G$) and transitive ($G = H$ and $H = K \implies G = K$).

In the next section, we will introduce the canonical form of a game. We will see that equality between games is the same as those games having the same canonical form. We will explore games such as $\{-1\,|1\}$ and $\{*\,|*\}$ which both behave as the game 0.

If we consider play in a sum of games, $G + K$, where K is a second-player win, the outcome of the sum is the same as that when playing in G alone. Essentially, a player can ensure the same outcome of G by always playing in G unless his opponent moves in K. In that case, he simply replies by playing the second player winning strategy in K.

Proposition 1.2.16. *[2, Prop. 3.1, p. 54] For games G and K with $o(K) = \mathcal{P}$,*

$$o(G) = o(G + K).$$

Theorem 1.2.17. *[2, Thm. 4.11, p. 70] For a game G,*

$$G = 0 \iff o(G) = \mathcal{P}.$$

Thus, if a game is a win for the second player, it has value 0.

Corollary 1.2.18. *[2, Cor. 4.14] $G - G = 0$.*

Proposition 1.2.19. *[2, Thm. 4.15, p.71] For games G, H and K,*

$$G = H \iff G + K = H + K.$$

The following Corollary provides one of the most useful tools in the combinatorial game theorist's toolbox. It gives us a simple means of determining equality between two games. To check whether or not $G = H$, we consider the game $G - H$. If the second player can always win in $G - H$, then we know that the two games are equal.

Corollary 1.2.20. *[2, Cor. 4.18, p. 72] For games G and H,*

$$G = H \iff G - H = 0.$$

That is,

$$G = H \iff o(G - H) = \mathcal{P}.$$

1.2.3.2 Comparison Of Games

Once we have examined equality, it makes sense to consider whether or not one game is better for a player than another. The following definition of $G \geq H$ gives us that replacing a game H with G can never hurt Left, assuming optimal play.

Definition 1.2.21. *[2, p. 73] For game G and H,*
$G \geq H$ if $\forall K, o(H + K) = \mathcal{L} \implies o(G + K) = \mathcal{L}$, and
$G \leq H$ if $\forall K, o(H + K) = \mathcal{R} \implies o(G + K) = \mathcal{R}$.

Theorem 1.2.22. *[2, Thm. 4.23, p. 74] For games G and H,*

$$G \geq H \iff o(G-H) \in \mathcal{L} \cup \mathcal{P}.$$

That is, $G \geq H$ if and only if Left can win moving second in $G-H$. Symmetrically,

$$G \leq H \iff o(G-H) \in \mathcal{R} \cup \mathcal{P}.$$

Notation 1.2.23. *1. $G > H \implies G \geq H$ and $G \neq H$, and similarly,*
$G < H \implies G \leq H$ and $G \neq H$,

2. $G \parallel H \implies$ neither $G \leq H$ nor $G \geq H$,

3. $G \lhd H \implies G < H$ or $G \parallel H$, and similarly,
$G \rhd H \implies G > H$ or $G \parallel H$,

The combinations of Corollary 1.2.20 and Theorem 1.2.22 give the blueprint for how we should compare two games. When comparing games G and H, we consider the game $G-H$:

(i) If $o(G-H) = \mathcal{P}$, then $G = H$;

(ii) If $o(G-H) = \mathcal{L}$, then $G > H$;

(iii) If $o(G-H) = \mathcal{R}$, then $G < H$;

(iv) If $G \ngeq H$ and $G \nleq H$, then $o(G-H) = \mathcal{N}$ and $G \parallel H$.

The summary of relationships between games G and H can be found in Table 1.2.3.2, all of which we determine based on the outcome of play in $G-H$.

Play in $G-H$		*Right moves first:*	
		Right wins	Left wins
Left moves first:	Left wins	$G \parallel H$	$G > H$
	Right wins	$G < H$	$G = H$

Table 1.3: Relationship between G and H given outcome in $G-H$.

In the event that $G \parallel H$, we say that G is **incomparable** to or **confused** with H. Otherwise, we say that the two games are **comparable**. For example, consider play in the HACKENBUSH position pictured in Figure 1.5.

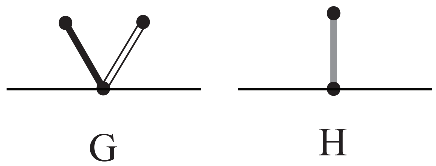


Figure 1.5: The HACKENBUSH positions 0 and $*$.

In this game, both players can move in H to the position $G - 0 = G$. From G, their respective opponents have only one move, which happens to be bad. Thus, the first player can always win.

As per convention, we sometimes draw a game tree to depict the moves available from position, with left options to the left and right-options to the right. To indicate that a move is bad or good for a player, we will adopt a notation from CHESS and use "!" to denote a good move and "?" to denote a bad move.

The good moves, of our example game $G - H$, are pictured in Figure 1.6.

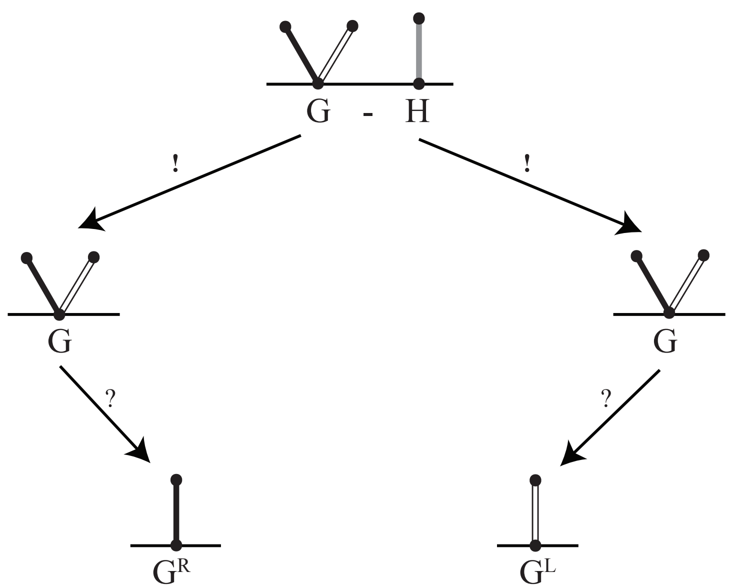


Figure 1.6: The game tree of the HACKENBUSH position $0 - *$.

Since the first player can always win in $G - H$, G and H are incomparable and $G \parallel H$. In this case, the games G and H under consideration were in fact the games 0 and $*$, respectively. We can see that $G = 0$; from G, neither player has a good move. In H, we see that each player has a move to 0 and so $H = \{0\,|0\} = *$. Hence this example demonstrates that $0 \parallel *$.

1.2.4 Canonical Form

"Everything should be as simple as it is, but not simpler."
- Albert Einstein

Consider the HACKENBUSH position of Figure 1.7.

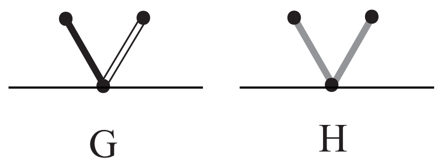


Figure 1.7: The HACKENBUSH positions $\{-1\,|1\}$ and $\{*\,|*\}$.

We can check that these two games, G and H, have the same value despite having different game trees. In this example, $G = \{-1\,|1\}$ and $H = \{*\,|*\}$. In both games, the first player (no matter who) does not have a good move. Hence, despite all of their differences, both of these games have value 0.

In this case, it is useful to be able to refer to the **canonical form** of the respective games. Every game has a unique smallest game that is equal to it, which is what we refer to as the canonical form of the game.

Definition 1.2.24. *[2] In a game G, if $A \geq B \in \mathbf{L}(G)$, then option A is said to* ***dominate*** *option B. Likewise, if $C \leq D \in \mathbf{R}(G)$, then option C is said to dominate option D.*

Theorem 1.2.25 (Removal of dominated options)**.** *[2, Thm 4.30, p. 79] If*

$$G = \left\{G^{L_1}, G^{L_2}, G^{L_3}, \ldots \middle| G^{R_1}, G^{R_2}, G^{R_3}, \ldots\right\},$$

and $G^{L_2} \geq G^{L_1}$, then $G = G'$ where

$$G' = \left\{G^{L_2}, G^{L_3}, \ldots \middle| G^{R_1}, G^{R_2}, G^{R_3}, \ldots\right\}.$$

Likewise, if $G^{R_2} \leq G^{R_1}$, then $G = G''$ where

$$G'' = \left\{G^{L_1}, G^{L_2}, G^{L_3}, \ldots \middle| G^{R_2}, G^{R_3}, \ldots\right\}.$$

Proof. We need to show that $G - G' = 0$. That is, we must show that the second player can always win in

$$\left\{G^{L_1}, G^{L_2}, G^{L_3}, \ldots \middle| G^{R_1}, G^{R_2}, G^{R_3}, \ldots\right\} - \left\{G^{L_2}, G^{L_3}, \ldots \middle| G^{R_1}, G^{R_2}, G^{R_3}, \ldots\right\}.$$

If either player makes a move to anything other than G^{L_1}, the corresponding move is available in the other component, with the response resulting in a move to either $G^{L_k} - G^{L_k}$ or $G^{R_k} - G^{R_k}$, both of which are equal to zero.

The only remaining move is for Left to play to $G^{L_1} - G'$. However, from here, Right can respond to $G^{L_1} - G^{L_2} \leq 0$ (by assumption), which Right wins. $\square$

Hence, by applying Theorem 1.2.25, we are **removing a dominated option**. Removal of dominated options is a simplification principle that will lead us toward canonical form. However, we require one move simplification which is referred to as **bypassing a reversible option**.

A **reversible option** is one that the opponent can respond to immediately so that his outcome after the exchange is at least as good as that before. By removing this intermediate step, we are able to simplify the game by simply bypassing that interchange.

Theorem 1.2.26 (Bypassing a reversible option)**.** *[2, Thm. 4.31, p. 80] For a game $G = \{\mathbf{L}(G) | \mathbf{R}(G)\}$, with $G^L \in \mathbf{L}(G)$ and $G^{LR} \in \mathbf{R}\left(G^L\right)$, if*

$$G^{LR} \leq G,$$

then $G = G'$ where

$$G' = \left\{\mathbf{L}\left(G^{LR}\right) \cup \mathbf{L}(G) \setminus G^L \,|\mathbf{R}(G)\right\}.$$

Likewise, if $G^R \in \mathbf{R}(G)$ and $G^{RL} \in \mathbf{L}\left(G^R\right)$, if

$$G^{RL} \geq G,$$

then $G = G''$ where

$$G'' = \left\{\mathbf{L}(G) \middle| \mathbf{R}\left(G^{RL}\right) \cup \mathbf{R}(G) \setminus G^R\right\}.$$

As a sketch of the proof, we note that play by either player in $\mathbf{L}(G) \setminus G^L$ or $\mathbf{R}(G)$ has a corresponding move in the other component. The rest of the proof is summarized in Figure 1.8.

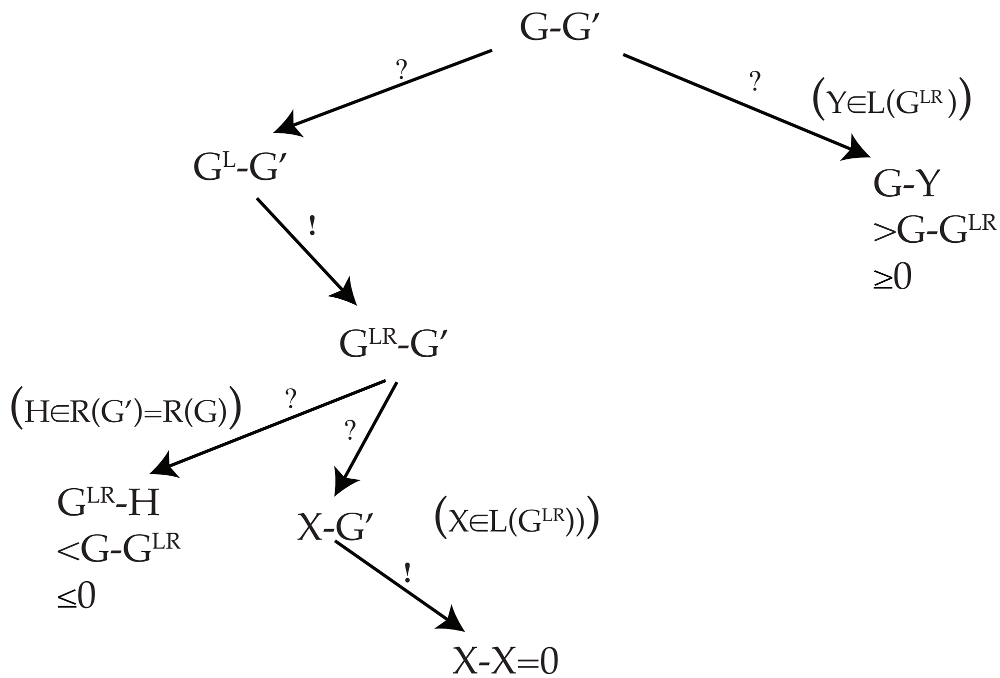


Figure 1.8: Responses to play in $G - G'$ where G^L is a reversible option and $G' = \left\{\mathbf{L}\left(G^{LR}\right) \cup \mathbf{L}\left(G\right) \setminus G^L \,|\mathbf{R}\left(G\right)\right\}$.

Definition 1.2.27. *[2] In a game G, if $G^{LR} \leq G$, then left option G^L is called a* ***reversible option*** *and the move to G^L* ***reverses through*** *G^{LR}. Likewise, if $G^{RL} \geq G$, then right option G^R is a reversible option and the move to G^R* ***reverses through*** *G^{RL}.*

Definition 1.2.28. *[2] A game G is said to be in* ***canonical form*** *if G and all of its positions have no dominated or reversible options. The canonical form of G will be denoted* $\operatorname{can}(G)$ *when it is necessary to specify.*

Both the removal of dominated options and reversal of reversible options results in a game tree that is smaller than the original, i.e. there are fewer total positions in the game. Therefore, the process of removing and reversing these options must terminate at some point. What might be surprising is the fact that the order that these processes are carried out does not matter; They will all eventually lead to the same canonical form. This is stated in Theorem 1.2.30.

1.2.4.1 Literal Form

In most cases, it is easiest to refer to the canonical form of a game. When it is not specified, we will assume that canonical form can be interchanged with the original game.

However, when we later discuss option-closed games, it becomes important to retain more information about the game. In that case, we will need to retain both dominated and reversible options in all positions of the game, a form referred to as **literal form**.

Definition 1.2.29. *The **literal form** of a game G, denoted $\mathrm{lit}\,(G)$, is the game G with all options, along with those that are dominated or reversible, included for all of its positions.*

If two games G and H have the exact same game trees, i.e. $\mathrm{lit}\,(G) = \mathrm{lit}\,(H)$, then we write $G \cong H$ and say that G and H are **isomorphic** [2, p. 66] or **identical** [9, p. 15]. If games are equal in value but have different options, they are not identical.

Theorem 1.2.30. *[2, Thm. 4.33, p. 81] If G and H are in canonical form and $G = H$, then $G \cong H$.*

In terms of literal form, this states that if $\mathrm{lit}\,(G) = \mathrm{lit}\,(\mathrm{can}\,(G))$ and $\mathrm{lit}\,(H) = \mathrm{lit}\,(\mathrm{can}\,(H))$, then $G = H$ implies that $\mathrm{lit}\,(G) = \mathrm{lit}\,(H)$.

The fact that the canonical form of a game has the same value as its literal form tells us that the dominated options do not affect the game value. Furthermore, if we add in options for either player that they would never want or need to play, it does not affect the value of the game either. We can use this principle, called the **Gift Horse Principle**, in order to guess at a value of a position and easily check our guess.

Definition 1.2.31. *[2, p. 72] For a game G, a new left option H is called a **left gift horse** if $H \lhd G$. Likewise, a new right option H' is called a **right gift horse** if $H' \rhd G$. Both H and H' are referred to as **gift horses**.*

The addition of these gift horses to a game do not affect the value.

Lemma 1.2.32 (Gift Horse Principle)**.** *[2, p. 72] If H is a left gift horse for the game G, then*

$$\{\mathbf{L}(G), H \,|\mathbf{R}(G)\} = G.$$

Likewise, if H' is a right gift horse, then $\{\mathbf{L}(G)\,|\mathbf{R}(G), H'\} = G$.

Proof. Consider the game $\{\mathbf{L}(G), H\,|\mathbf{R}(G)\} - G$. Any move by Left or Right in $\mathbf{L}(G)$ or $\mathbf{R}(G)$ has a mirror move in the other component. If Left moves to $H - G$, then since H is a left gift horse, $H \lhd G$, and so $H - G \lhd 0$. □

Definition 1.2.33. *[2, Def. 4.36, p. 83] For all games G, the* ***left incentives*** *and* ***right incentives*** *are the sets $\mathbf{L}(G) - G$ and $G - \mathbf{R}(G)$, respectively. For a particular move, the left incentive of G^L is $G^L - G$ (or, respectively, the right incentive of G^R is $G - G^R$). The* ***incentives*** *of a game are the union of the left and right incentives.*

Incentives are useful in that they can tell us where a player should like to play. The incentives tell us what is gained or lost in the making of a specific move. Since the incentives are games themselves, they are partially-ordered. If G^{L_1} and G^{L_2} are left options of a game G, then their respectively left incentives, $G^{L_1} - G$ and $G^{L_2} - G$, have an ordering. If $G^{L_1} - G > G^{L_2} - G$, then $G^{L_1} > G^{L_2}$ and so G^{L_2} is dominated by G^{L_1}.

Lemma 1.2.34. *[2] If H is an incentive of a game G, then $H \lhd 0$.*

Corollary 1.2.35. *[9, p. 16] For all games G, all left options are less than or incomparable to G, and all right options are greater than or incomparable to G. That is,*

$$\forall G^L \in \mathbf{L}(G),\ G^L \lhd G,$$

and

$$\forall G^R \in \mathbf{R}(G),\ G \lhd G^R.$$

1.2.5 Birthdays

Definition 1.2.36. *[2, Def. 4.1, p. 66] The* ***birthday*** *of a game $G = \{\mathbf{L}(G)\,|\mathbf{R}(G)\}$, denoted $b(G)$, is defined recursively as*

$$b(G) = 1 + \max\{b(H) \ \mid H \in \mathbf{L}(G) \cup \mathbf{R}(G)\},$$

with $b(G) = 0$ if $\mathbf{L}(G) = \mathbf{R}(G) = \emptyset$.

In other words, a game is **born on day** n, i.e. has its birthday on day n, if all of its options were born on day $n-1$ or earlier and if there is at least one option that was born on day $n-1$ [26, p. 21]. Thus, the birthday of a short game is equal to the height of its game tree. Hence, the only day born on day 0 is the game 0. Commonly, we will discuss games that are **born by day** n.

Definition 1.2.37. *[2, p. 66] A game G is said to be* ***born by day*** *n if*

$$b(G) \leq n.$$

From this definition, we have a means of recursively generating games using only those games created on previous days. Each game is assigned, as its birthday, an ordinal number representing the number of steps needed to create the game starting from only the empty set.

On day zero, no prior games exist and so we have the endgame $0 = \{\cdot \,|\, \cdot\}$ as the only game born on day zero. We next introduce the games born on day one. As they can have as options only those games born on a previous day, in this case only day 0, games born by day 1 can have as options only 0 or the empty set. Thus, there are $2^2 - 1 = 3$ new games born on day one. Table 1.2.5 lists the 3 new games born on day one.

1	$=$	$\{0 \,|\, \cdot\}$	
-1	$=$	$\{\cdot \,|\, 0\}$	
$*$	$=$	$\{0 \,|\, 0\}$	(pronounced “star”)

Table 1.4: Games born on day 1.

Continuing in this way, we define the 18 new games on day two, provided in Table 1.2.5, using combinations of $-1, 0, 1, *$ and $\emptyset$ as the positions that form the new game. The games that are formed by each combination of the day 1 options are listed in Table 1.5. For games $G + *$, it is standard convention to write $G*$. Games of the

G		$\mathbf{R}(G)$					
		1	$0,*$	0	$*$	-1	$\emptyset$
$\mathbf{L}(G)$	1	$1*$	$\{1\,\vert 0,*\}$	$\{1\,\vert 0\}$	$\{1\,\vert *\}$	± 1	2
	$0,*$	$\frac{1}{2}$	$*2$	$\uparrow *$	$\uparrow$	$\{0,*\,\vert -1\}$	1
	0	$\frac{1}{2}$	$\downarrow *$	$*$	$\uparrow$	$\{0\,\vert -1\}$	1
	$*$	0	$\downarrow$	$\downarrow$	0	$\{*\,\vert -1\}$	0
	-1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-1*$	0
	$\emptyset$	0	-1	-1	0	-2	0

Table 1.5: Day 2 games.

form $\{a\,|b\}$ where a and b are numbers with $a > b$ are referred to as switches. If a switch is of the form $\{a\,|-a\}$, the it is standard notation to write this as $\pm a$.

2	-2
$1*$	$-1*$
$\frac{1}{2}$	$-\frac{1}{2}$
$\{1\|0\}$	$\{0\|-1\}$
$\{1\|*\}$	$\{*\|-1\}$
$\uparrow*$	$\downarrow*$
$\uparrow$	$\downarrow$
$\{1\|0,*\}$	$\{0,*\|-1\}$
$*2$	
$\{1\|-1\}$	

Table 1.6: Games born on day 2.

All of this, of course, assumes that we are considering only short games, in which each has only a finite number of options. If we allow a game G to contain itself as either a left or right option, then the game becomes loopy. We leave consideration of loopy games to a later chapter. If we assume that each game has a finite birthday, then we reach the following.

Proposition 1.2.38. *[2, Lem. 4.2, p. 67] The number of games born by day n is finite.*

1.2.6 Impartial Games

Combinatorial game theory first considered the realm of impartial games, or games in which the available moves for both players are the same.[4] Allowable moves from a given position depend only on the position, not on the player that is moving, i.e. for any impartial game G, $\mathbf{L}(G) = \mathbf{R}(G)$ when G is in canonical form. In other words, the gains are symmetric between players and the only difference between the two is who goes first.

Given the fact that both players are allowed the same moves, there is no position in an impartial game in which there is an advantage to either Left or Right. Thus, in impartial games, there exist only $\mathcal{N}-$ and $\mathcal{P}-$positions.

Theorem 1.2.39. *[2, Thm. 2.11, p. 41] If G is an impartial game, then $G \in \mathcal{N}$ or $G \in \mathcal{P}$.*

The classical example of an impartial combinatorial game is the game of NIM, first introduced by Charles Bouton [6]. In the game of NIM, players take turns removing stones from distinct piles, called **nim-heaps**. On his turn, a player may remove any number of stones, provided he removes at least one, from any one pile. NIM is the quintessential example of an impartial game.

NIM is completely solved when played under the normal play convention, in which the winner is defined to be the person who takes the last stone. The only values that occur in NIM positions are called nimbers. A nimber is a special game denoted $*n$ for some ordinal n. The canonical form of the game $*n$ is given as

$$*n = \{0, *, \ldots, *(n-1) \,|\, 0, *, \ldots, *(n-1)\}.$$

Thus, $*0 = 0 = \{\emptyset \,|\, \emptyset\}$, $*1 = \{0 \,|\, 0\}$, $*2 = \{0, * \,|\, 0, *\}$, and so on. The name nimber comes from the fact that $*n$ corresponds to a nim-heap of n stones.

One lovely characteristic of impartial games is that all impartial games have only nimbers as values. As such, NIM has been the go-to game when considering impartial games. Independently, Sprague [31] and Grundy [15] discovered that under the normal play convention, every impartial game is equivalent to a nimber. This theorem,

[4] Although the original definition has both players having the same moves, this really should be applied to the canonical forms, since one player may have dominated moves that will never get played.

developed in the 1930's, is now refered to as the Sprague-Grundy theorem in their honor. The nim-value (or Grundy value) of an impartial game is the unique nimber that the game is equivalent to.

Theorem 1.2.40 (Sprague-Grundy theorem)**.** *[15, 31] Every impartial game is equivalent to a nim-heap. That is, for all games G,*

$$\exists n \in \mathbb{Z}^{\geq 0} \textit{ s.t. } G = *n.$$

Moreover, since NIM is a completely-solved game, this tells us that if we can find the mapping of an impartial game to its equivalent nim-heaps, then we have solved the game.

An example of an impartial game is the game of Hackenbush played with only green edges. Figure 1.9 gives the nimbers $0, *, *2$, through $*5$.

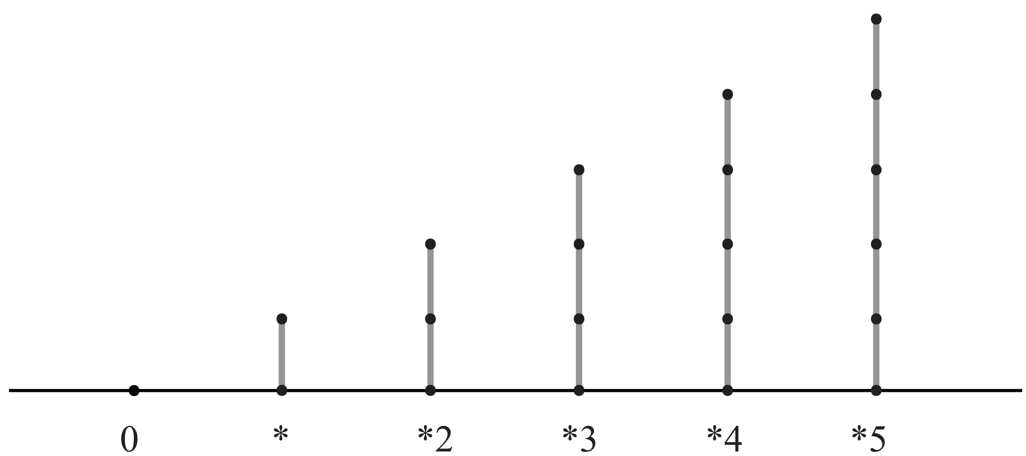


Figure 1.9: An example of the nimbers as (green only) HACKENBUSH positions.

It is clear that any nimber, $*n$, can be created as a green-only HACKENBUSH string, consisting of n green edges.

As mentioned by Theorem 1.2.39, impartial games are either first or second player wins (i.e. in $\mathcal{N}$ or $\mathcal{P}$). Since each $*n$ for $n > 0$ has a move to zero, these are all $\mathcal{N}$-positions.

Theorem 1.2.41. *[2, Thm. 2.13, p. 41]*
An impartial game is

1. *an* $\mathcal{N}$*-position if at least one of its options is a* $\mathcal{P}$*-position;*

2. *a* $\mathcal{P}$*-position if all of its options are* $\mathcal{N}$*-positions.*

We can rephrase this to say: *(i)* If you can win the game, you must have a move to a position your opponent can't win; *(ii)* you cannot win the game if all moves available to you allow your opponent to win.

1.2.7 Partizan Games

"Every truth has two sides;
it is as well to look at both, before we commit ourselves to either."
- Aesop

In many games, moves that are available to one player may not be available to the other. Most games that are commonly played fall into this category. For example, in the game of CHESS, one player is allowed to move only the white pieces and the other only the black. Unlike impartial games, the Sprague-Grundy theorem does not apply to partizan games. As such, these games remain more difficult to analyze.

In order to meaningfully discuss partizan games, we need understanding of the value of games. For instance, if a game has no moves for Right and one move for Left, it seems natural that it would have value 1 to denote that it has 1 "free" move for Left. Likewise, we should (and do) give value -1 to the game in which Right has one free move. Of course, it need not always work out as cleanly.

1.2.7.1 Numbers

In short games, the only numbers that we will encounter are the **dyadic rationals**, i.e. rationals numbers of the form $\frac{p}{2^q}$.

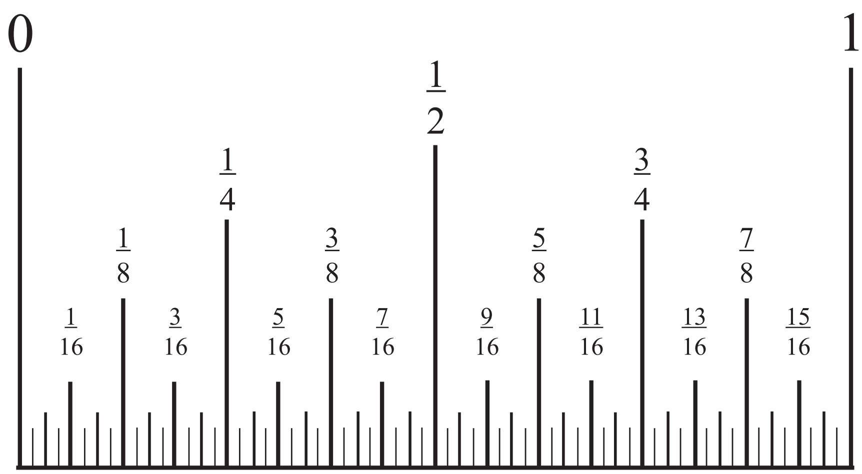


Figure 1.10: A sampling of the dyadic rationals in $[0, 1]$.

We first motivate the integers. The integers are defined such that a game of value n has n free moves for Left, and a game of value of $-n$ has n free moves for Right. As introduced earlier, the game 0 has no moves available to either player.

Definition 1.2.42. *[2, p. 88][3, p. 19]*

$$\begin{aligned} 0 &= \{\cdot \,|\, \cdot\} \\ 1 &= \{0 \,|\, \cdot\} \\ 2 &= \{1 \,|\, \cdot\} \\ &\vdots \\ n+1 &= \{n \,|\, \cdot\} \end{aligned}$$

Likewise, $-(n+1) = \{\cdot \,|\, -n\}$ *for* $n \in \mathbb{Z}^{\geq 0}$.

Figure 1.11 gives a sampling of HACKENBUSH positions that have integer values.

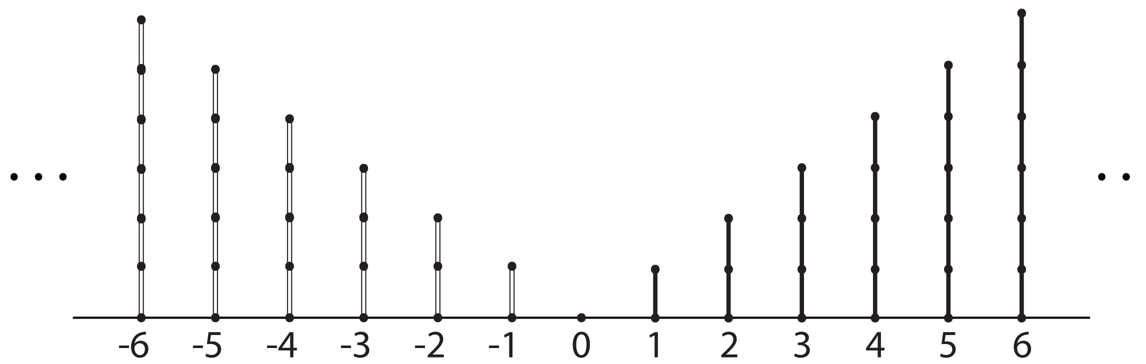


Figure 1.11: Example of HACKENBUSH positions having integer values.

Games having integer values are easy to compare since they immediately reflect the number of moves that each player has to their advantage. This idea can be extended to the classes of games called **numbers**.

Definition 1.2.43. *[9, p. 4] A* ***number*** *is a game* $x = \{\mathbf{L}(x) | \mathbf{R}(x)\}$ *where*

$$\forall x^L \in \mathbf{L}(x), x^R \in \mathbf{R}(x), \quad x^L < x^R.$$

Lemma 1.2.44. *[9, p. 10] In canonical form, every number* $x = \left\{x^L \middle| x^R\right\}$ *satisfies* $x^L < x < x^R$*.*

Theorem 1.2.45. *[26, p.13] Equivalence classes of numbers form an abelian subgroup of games.*

Thus, if x and y are numbers, then $x + y$ and $-x$ are numbers.

Theorem 1.2.46. *[26, p. 13] Numbers are totally ordered.*

That is, for all numbers x and y, exactly one of the following is satisfied: (i) $x < y$, (ii) $x = y$, or (iii) $x > y$.

In order to determine whether or not a game is a number, we may look to incentives.

Theorem 1.2.47. *[2, Thm. 6.19, p. 126] If all incentives of a game* G *are negative, then* G *is a number.*

If a game is not in canonical form, yet is a number, it can still be readily identified as such. To determine which number, we require the concept of the **simplest number**.

Definition 1.2.48. *[2, Def. 5.22, p. 93] The **simplest number** between numbers $x^L < x^R$ is the unique number x having the smallest birthday that satisfies*

$$x^L < x < x^R.$$

$$i.e. \quad x \quad s.t. \quad b(x) = \min \left\{ b(y) : x^L < y < x^R \right\}.$$

Theorem 1.2.49 (Simplest Number)**.** *[2, Thm. 5.29, p. 93] If there is some number x such that $\mathbf{L}(G) \lhd x \lhd \mathbf{R}(G)$, then $G = \{\mathbf{L}(G) \,|\mathbf{R}(G)\}$ is the simplest such x.*

Definition 1.2.50. *[2, Def. 5.12, p. 91][3, p. 21] For $p \in \mathbb{Z}^{\geq 0}$ and $q \in \mathbb{Z}^{>0}$, we define*

$$\frac{2p+1}{2^q} = \left\{ \frac{2p}{2^q} \middle| \frac{2p+2}{2^q} \right\} = \left\{ \frac{p}{2^{q-1}} \middle| \frac{p+1}{2^{q-1}} \right\}.$$

If the game $G = \{\mathbf{L}(x) \,|\mathbf{R}(x)\}$ is a number x that is not an integer, then $x = \frac{2p+1}{2^q}$ for some $p \in \mathbb{Z}^{\geq 0}, q \in \mathbb{Z}^{>0}$ for which q is minimal. The definition of simplest number is well-defined. To see this, we require the fact that if we had two numbers $x < z$ of the same birthday, then there is another number y having smaller birthday with $x < y < z$. This can be visualized as ruler marks on a measuring stick, as in Figure 1.10. Between any two ruler marks of the same length, there exists a longer, and so a longest, ruler mark between them. Our simplest number is that which would be the longest ruler mark.

Figure 1.12 gives a sampling of numbers in the interval from 0 to 1 as HACKENBUSH positions.

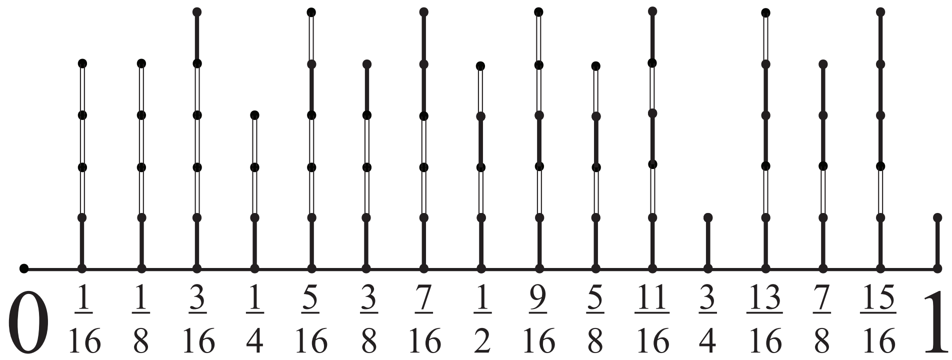


Figure 1.12: A sampling of dyadic rationals in $[0, 1]$ presented as HACKENBUSH strings.

In terms of play, the following two theorems tell us that it is best to avoid playing on numbers.

Theorem 1.2.51 (Weak Number Avoidance)**.** *[2, p. 93] Suppose G and x are games such that x is a number and G is not. If Left can win moving first on $G + x$, then he can do so by moving in G. That is,*

$$G + x \rhd 0 \quad \Longrightarrow \quad \exists G^L \in \mathbf{L}(G) \;\; s.t. \;\; G^L + x \geq 0.$$

We now state the strong version of this theorem, which we will simply refer to as the **Number Avoidance Theorem**.

Theorem 1.2.52 (Number Avoidance)**.** *[2, p. 125] Suppose G and x are games such that x is a number and G is not. Then,*

$$\forall x^L \in \mathbf{L}(x), \; \exists G^L \in \mathbf{L}(G) \qquad s.t. \qquad G^L + x > G + x^L.$$

1.2.7.2 Confusion Intervals And Stops

Much information can be revealed about a game G by comparing it with numbers. How G relates to a number can sometimes be enough to establish a winner.

We will look at some number bounds on G that will give us an interval of confusion. The bounds will consist of a left stop and a right stop, for which the game G will be confused with all numbers falling between them. The definitions that follow assume short games, as part of the argument is not applicable to infinite games in general.

Proposition 1.2.53 (Archimedian Principle)**.** *[9, Thm. 55, p. 98] For any short game G, there is some integer n with $-n < G < n$.*

The existence of this initial bound can be motivated by birthdays. To see this, consider the birthday of G, $b(G)$, and recall that the largest game born on day n is the game n. Thus, if $b(G) = n$, then $-n \leq G \leq n$. While there may be smaller n for which this holds, birthdays assure us that some integer must exist for which n and $-n$ bound G.

Definition 1.2.54. *[2, Def. 6.9, p. 123][3] Denote the* left stop *and* right stop *of a game G by $\mathrm{L}_0(G)$ and $\mathrm{R}_0(G)$, respectively. They are defined in a mutually recursive fashion:*

$$\begin{aligned}
\mathrm{L}_0(G) &= \begin{cases} G & \text{if } G \text{ is a number,} \\ \max\left\{\mathrm{R}_0\left(G^L\right) \;:\; G^L \in \mathbf{L}(G)\right\} & \text{if } G \text{ is not a number ;} \end{cases} \\
\mathrm{R}_0(G) &= \begin{cases} G & \text{if } G \text{ is a number,} \\ \min\left\{\mathrm{L}_0\left(G^R\right) \;:\; G^R \in \mathbf{R}(G)\right\} & \text{if } G \text{ is not a number.} \end{cases}
\end{aligned}$$

Intuitively, these stops are the best numbers that a player can achieve in alternating play. We will see later that in option-closed games, these stops play an important role.

Corollary 1.2.55. *[2, Thm. 6.11, p. 123] For any game G,*

$$\mathrm{R}_0(G) \leq \mathrm{L}_0(G)$$

with equality when G is a number.

Proof. If G is a number, then by definition $\mathrm{L}_0(G) = \mathrm{R}_0(G) = G$.

Suppose G is not a number and $\mathrm{R}_0(G) > \mathrm{L}_0(G)$. Then, since $\mathrm{R}_0(G)$ and $\mathrm{L}_0(G)$ are both numbers, there exists a number x such that $\mathrm{L}_0(G) \leq x \leq \mathrm{R}_0(G)$. We claim $G = x$. Consider the game $G - x$. Since x is a number, by the Weak Number Avoidance Theorem (Thm. 1.2.51), if either player has a winning move, it is in G. Hence, both players play in G until it reaches a number. If Left moved first, they eventually reach $\mathrm{L}_0(G) - x < 0$. If Right started, they reach $\mathrm{R}_0(G) - x > 0$. So $G - x = 0$, which implies $G = x$ is a number, which is a contradiction. □

Proposition 1.2.56. *[9, p. 97] A game G is confused with all numbers x falling between the left and right stop of G. That is, for all numbers x,*

$$\mathrm{R}_0(G) < x < \mathrm{L}_0(G) \implies G \parallel x.$$

Corollary 1.2.57. *If G and H are games such that $G < H$, then $\mathrm{L}_0(G) \le \mathrm{R}_0(H)$.*

1.2.8 Game Dimension

"Dimension regulated the general scale of the work,
so that the parts may all tell and be effective."
- Marcus V. Pollio

We now introduce the concept of game dimension. The concept of the Nim-dimension of a game was introduced by dos Santos and Silva [25]. The Nim-dimension of a game gives a bound on the nimbers that can occur in a particular position or set of positions. We later introduce the related concept of Hackenbush-dimension, giving a bound on the form of numbers that can occur within a game.

1.2.8.1 Nim-Dimension

In a list of unsolved problems in combinatorial games by Richard Guy [16], one of the listed open problems[5] asks the reader to determine which partizan games have the position $*2$. Dos Santos and Silva [25] generalize this problem and ask: *given a partizan game, which nimbers occur in the game?* They offer the following definition as a means of discussing this aspect of a partizan game, which they refer to as the **Nim-dimension**.

Definition 1.2.58. *[25] The **Nim-dimension** of a game G will be denoted $\mathrm{Ndim}(G)$. $\mathrm{Ndim}(G) = n$ if it contains a position $*2^{n-1}$ but does not contain a position $*2^n$.*

If the game 0 is the only nimber that occurs as a position in the game, we say that $\mathrm{Ndim}(G) = 0$. If no nimber can be constructed in the game, we say that the game has null Nim-dimension, and write $\mathrm{Ndim}(G) = \emptyset$. If all nimbers can be constructed, we say it has infinite Nim-dimension, and write $\mathrm{Ndim}(G) = \infty$.

[5] submitted by Elwyn Berlekamp (Open Problem # 45)

We deviate from dos Santos and Silva [25] in that they define the Nim-dimension of games only containing 0 and those without nimbers to have null Nim-dimension. However, this can be an important distinction. In Chapter 3, we will be introduced to the loopy game of ON which in canonical form is the game $\{\text{ON}\,|\cdot\}$ in which Left is allowed to pass and right has no move. Thus, by our definition, $\text{Ndim}\,(G) = \emptyset$.

We can employ Nim-dimension to refer to a specific game, such as the game $\uparrow = \{0\,|\,*\}$ which is

$$\text{Ndim}\,(\uparrow) = \text{Ndim}\,(\{0\,|\,*\}) = 1$$

since there is an option to $* = *2^0$ but not to $*2 = *2^1$. Similarly, we can give the Nim-dimension of the game $*5$ which is

$$\text{Ndim}\,(*5) = \text{Ndim}\,(\{0, *, *2, *3, *4\,|\,0, *, *2, *3, *4\}) = 3$$

as $*5$ contains the position $*4 = *2^2$ but does not contain the position $*8 = *2^3$.

However, as was done by dos Santos and Silva [25], we can also use Nim-dimension to discuss all possible values obtained for any starting position in a game. For instance, the game of NIM, as expected, has infinite Nim-dimension. That is, all nimbers exist as NIM positions. Hence, $\text{Ndim}\,(\text{NIM}) = \infty$. Dos Santos and Silva [25] note that $\text{Ndim}\,(\text{SHOVE}) = 0$, since all values in SHOVE are numbers[6], which was shown in [2]. Later, we will formally introduce the game of SHOVE and give motivation to its Nim-dimension.[7] They also state that $\text{Ndim}\,(\text{TOPPLING DOMINOES}) = \infty$ by noting that all nimbers can be created as TOPPLING DOMINOES positions, as in Figure 1.13, as it was shown in [2].

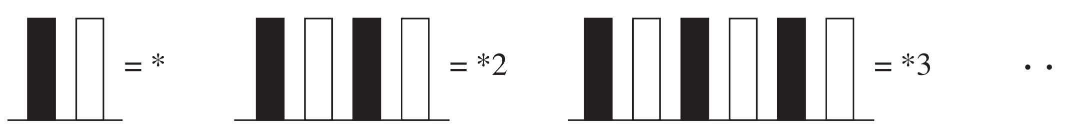


Figure 1.13: The infinite Nim-dimension of TOPPLING DOMINOES: the construction of all nimbers as TOPPLING DOMINOES positions[2].

[6]Their result actually gave $\text{Ndim}\,(\text{SHOVE}) = \emptyset$. However, by our definition change, this implies that it has Nim-dimension zero since the game of SHOVE does contain the position 0.

[7]For those too curious to wait, see Appendix A.12.

1.2.8.2 Hackenbush-Dimension

The concept of Nim-dimension can clearly be applied to partizan games, as we see with both SHOVE and TOPPLING DOMINOES. However, in order to also gather information about the types of numbers (i.e. denominator powers) that can be found in a game, we introduce a similar dimension. Thus, we define the Hackenbush-dimension of a game.

Definition 1.2.59. *For a game G, we denote the* ***Hackenbush-dimension*** *of G by* $\mathrm{Hdim}\,(G)$*. We define this as* $\mathrm{Hdim}\,(G) = q+1$ *if G contains positions of the form $\frac{p}{2^q}$ but does not contain a position of the form $\frac{p'}{2^{q+1}}$, for odd $p, p' \in \mathbb{Z}$ and $q \in \mathbb{Z}^{\geq 0}$.*

If zero is the only number that can be constructed in the game, we say that the game has Hackenbush-dimension zero, and write $\mathrm{Hdim}\,(G) = 0$*. If all numbers can be constructed, we say it has* ***infinite*** *Hackenbush-dimension, and write* $\mathrm{Hdim}\,(G) = \infty$*.*

We note that for any game G, $\mathrm{Hdim}\,(G) \geq 0$. We will use Hackenbush-dimension both to describe the dimension of a specific position and also of a general game. Hence, we could consider both $\mathrm{Hdim}\left(\frac{3}{16}\right)$, which is 5 as $\frac{3}{16} = \frac{3}{2^4}$ and all options in this game have smaller denominator, or we could ask the Hackenbush-dimension of, say, the game of GO.

From [3], we know that all numbers can be created as HACKENBUSH strings. Thus, $\mathrm{Hdim}\,(\textsc{hackenbush}) = \infty$. However, in the game of NIM, all values are nimbers and so 0 is the only number reached in the game. Thus, $\mathrm{Hdim}\,(\textsc{nim}) = 0$.

Numbers and Hackenbush-dimension

For a number, we can use Hackenbush-dimension to express the canonical options of the game, as well as the incentives.

Lemma 1.2.60. *Let x be a non-integer number. Then, in canonical form,*

$$x = \left\{ x - 2^{1-\mathrm{Hdim}(x)} \,\middle|\, x + 2^{1-\mathrm{Hdim}(x)} \right\}$$

with left and right incentives $-2^{1-\mathrm{Hdim}(x)}$.

Proof. Let $x = \frac{2p+1}{2^q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^{>0}$. Since q is minimum, $\mathrm{Hdim}\,(x) = q + 1$. Then, in canonical form,

$$\begin{aligned} x =& \left\{ \tfrac{2p}{2^q} \,\middle|\, \tfrac{2p+2}{2^q} \right\} \\ =& \left\{ x - \tfrac{1}{2^q} \,\middle|\, x + \tfrac{1}{2^q} \right\} \\ =& \left\{ x - 2^{-q} \,\middle|\, x + 2^{-q} \right\} \\ =& \left\{ x - 2^{1-(q+1)} \,\middle|\, x + 2^{1-(q+1)} \right\} \\ =& \left\{ x - 2^{1-\mathrm{Hdim}(x)} \,\middle|\, x + 2^{1-\mathrm{Hdim}(x)} \right\} \end{aligned}$$

Then, the left incentive is $\left(x - 2^{1-\mathrm{Hdim}(x)}\right) - x = -2^{1-\mathrm{Hdim}(x)}$, and the right incentive is $x - \left(x + 2^{1-\mathrm{Hdim}(x)}\right) = -2^{1-\mathrm{Hdim}(x)}$. □

1.2.9 Reduced Canonical Form

Often games, even relatively small or benign-looking positions, can have complex canonical forms. In 1996, Calistrate [7] noticed that in certain situations, we can look to a game infinitesimally close to the original game in determining how to proceed. He introduced a reduction of the original game G that he called the **reduced canonical form**. In 2009, Grossman and Siegel [14] provided a new proof correcting a flaw in Calistrate's algorithm. They show us that for a game G, the reduced canonical form of G is the simplest game infinitesimally close to G and that this concept is well-defined.

1.2.9.1 All-Small Games

A game is considered to be **all-small** if from any position in the game, either both players have a move or neither do.

Definition 1.2.61. *[2, p. 101] A game G is* ***all-small*** *if either*

(i) $G \cong \{\cdot\,|\,\cdot\}$, *or*

(ii) $\mathbf{L}\,(G)$ *and* $\mathbf{R}\,(G)$ *are both non-empty and all-small.*

For example, the games $\uparrow$ and $\downarrow$ are both all-small. All nimbers are also all-small.

1.2.9.2 Infinitesimal Games

We have looked at nimbers and have seen these values are incomparable with 0. We now consider a class of games, called **infinitesimals**, that are greater than all negative numbers but less than all positive numbers. In these games, it is much more difficult to determine whether a game is positive, negative or confused with zero.

Definition 1.2.62. *[2, p. 100] A game G is an **infinitesimal** if for every positive number x, $-x < G < x$.*

Clearly, the game 0 and all nimbers $*n$ are infinitesimal.

Theorem 1.2.63. *[2, Thm. 5.40, p. 101] If G is an all-small game, then G is infinitesimal.*

Proof. Let G be an all-small game and x a positive number. It suffices to prove that $G < x$. This is trivially true when $G \cong 0$. If $G \ncong 0$, then left and right options exist that are both all-small and therefore infinitesimal, by induction. From $G - x$, Left loses by induction when playing to any $G^L - x$. If $\exists x^R$, then it is a number $x^R > 0$, so Left also loses by induction when playing to $G - x^R$. However, Right can win playing to $G^R - x < 0$, by induction.

□

We can also determine whether or not a game is infinitesimal based on the value of its left and right stops. If these are both zero, then the game is infinitesimal.

Theorem 1.2.64. *[2, Thm. 6.12, p. 124] A game G is infinitesimal if and only if $\mathrm{L}_0(G) = 0 = \mathrm{R}_0(G)$.*

Definition 1.2.65. *We define* **Inf** *as the set of all infinitesimals.*

The all-small games are then a subset of the set **Inf** of infinitesimal games.

Definition 1.2.66. *[14] We say that games G and H are **infinitesimally close** when $G - H$ is infinitesimal. We denote this as $G \equiv_{\mathbf{Inf}} H$ or say that H is G-ish (G* i*nfinitesimally* sh*ifted).*

Grossman and Siegel then give us the following definitons to develop the reduction.

Definition 1.2.67. *[14, Defn. 3.6]*

For games G and H, we say $G \geq_{\mathbf{Inf}} H$ if $G \geq H + \epsilon$ for some infinitesimal ϵ. Likewise, $G \leq_{\mathbf{Inf}} H$ if $G \leq H + \epsilon$ for some infinitesimal ϵ.

The following are relatively trivial, but necessary, results about the transitivity of $\geq_{\mathbf{Inf}}$.

Lemma 1.2.68. *[21, Lem. 6] If $x \leq_{\mathbf{Inf}} y$ and $y \leq_{\mathbf{Inf}} z$, then $x \leq_{\mathbf{Inf}} z$.*

Lemma 1.2.69. *[21, Lem. 7] For numbers y and z, if $x \leq_{\mathbf{Inf}} y$ and $y < z$, then $x < z$.*

Definition 1.2.70. *[14] Two games G and H are* ***equalish*** *if $G \equiv_{\mathbf{Inf}} H$, if $G - H$ is an infinitesimal. A game G is* ***numberish*** *if there is a dyadic rational x such that $G \equiv_{\mathbf{Inf}} x$, i.e. $G - x$ is an infinitesimal.*

Thus, $G \equiv_{\mathbf{Inf}} H$ if and only if $G \geq_{\mathbf{Inf}} H$ and $G \leq_{\mathbf{Inf}} H$. So if $G = H$, then $G \equiv_{\mathbf{Inf}} H$.

Definition 1.2.71. *[14, Def. 4.1] For a game G,*

1. *A left option G^L is* **Inf-*dominated*** *or* ***infinitesimally-dominated*** *if $G^L \leq_{\mathbf{Inf}} G^{L'}$ for some other left option $G^{L'}$.*

2. *A left option G^L is* **Inf-*reversible*** *or* ***infinitesimally-reversible*** *if $G^{LR} \leq_{\mathbf{Inf}} G$ for some G^{LR}.*

Analogous definitions hold for right options.

For example, consider the game $G = \{3, \{3\,|2\}\,|2\}$. Since $\{3\,|2\} = 3 + \{0\,|{-1}\} < 3+\uparrow$[8] where $\uparrow \in \mathbf{Inf}$, then $\{3\,|2\} \leq_{\mathbf{Inf}} 3$. If we remove the **Inf**-dominated left option $\{3\,|2\}$ from G, we are left with the simpler game $\{3\,|2\}$. In reduced canonical form, both the infinitesimally-dominated and Infinitesimally-reversible moves are removed.

Definition 1.2.72. *[14, Defn. 4.2]*

A game G is said to be in ***reduced canonical form****, denoted* $\mathrm{rcf}\,(G)$*, if every follower of G is either*

[8] Note that $\{0\,|{-1}\} <\uparrow$. In the game $\{0\,|{-1}\} + \downarrow$, Right has a good move to $-1+\downarrow < 0$ and Left has none, either playing to $\downarrow < 0$ or $\{0\,|{-1}\} + *$, from which Right can respond to $-1* < 0$.

(i) a number in canonical form, or

(ii) not a number nor infinitesimally close to a number, and contains no **Inf***-dominated or* **Inf***-reversible options.*

The following results are included for completeness and as examples of how one proves things about reduced canonical form. The next result shows that the reduced canonical form of a game is well-defined.

Theorem 1.2.73. *[14, Thm. 4.3] For any game G, there exists a game* $\mathrm{rcf}(G)$ *in reduced canonical form where* $G \equiv_{\mathbf{Inf}} \mathrm{rcf}(G)$.

Lemma 1.2.74. *[14, Thm 4.8] If G is not numberish, then* $\mathrm{rcf}(G)$ *is obtained by*

(i) replacing options with simpler options infinitesimally close to the original option;

(ii) eliminating infinitesimally-dominated options;

(iii) bypassing infinitesimally-reversible options.

Furthermore, the reduced canonical form of a game is unique.

Theorem 1.2.75. *[14, Thm. 4.4] If games G and H are both in reduced canonical form and $G \equiv_{\mathbf{Inf}} H$, then $G = H$.*

At times when even the canonical form of a game can be complicated, we can make use of reduced canonical forms to express them in simpler terms. For example, a position with canonical form

$$G = \left\{ 2, \{2\,|0\}, \{2, \{2\,|1\}\,|0, \{1\,|0\}\} \,\middle|\, 0, \left\{\frac{1}{2}\,|0\right\}, \left\{2, \{2\,|0\} \,\middle|\, 0, \left\{\frac{1}{2}\,|0\right\}\right\}\right\}$$

can be expressed as $G = \{2\,|0\} + \epsilon$ where ϵ is an infinitesimal. That is, the difference $G - \{2\,|0\}$ is an infinitesimal. Thus, this game has reduced canonical form $\mathrm{rcf}(G) = \{2\,|0\}$.

We will see that this reduction can be enlightening in certain situations. Many fundamental qualities of a game, such as mean and temperature, are not affected by the addition or removal of an infinitesimal. While infinitesimals play an important role, there are some situations where we can primarily concern ourselves with the reduced canonical form of a game. In these situations, a player may ignore the infinitesimals as they simply determine the parity of moves remaining once the associated non-infinitesimal has played out. We will see the merit of reduced canonical form when we look option-closed games in Chapter 4.

1.2.10 Atomic Weight

The theory of atomic weights relates to the approximation of infinitesimal games by multiples of the unit $\uparrow$. One can think of the atomic weight of a game as an approximation its "uppitiness." In infinitesimal games, it is often difficult to determine the outcome of a game. We will use as example the games of $\uparrow$ and $\uparrow *$; the former is positive, while the latter is not. In order to determine outcome class of these games without having to play out the entire game, we can employ the use of atomic weights. The atomic weight of a game is defined recursively and considers the play in the game when in the presence of a NIM-heap of "infinite" size.

Recall that the game of *up* is given by $\uparrow = \{0\,|*\}$ and its negative, *down*, by $\downarrow = \text{-}\uparrow = \{*\,|0\}$. These games are positive and negative, respectively, with

$$\downarrow < 0 < \uparrow .$$

However, each is incomparable with $*$, as

$$\downarrow \;\; \| \;\; * \;\; \| \;\; \uparrow .$$

If we add $*$ to $\uparrow$, we have the game $\uparrow *$, which is confused with 0, i.e.

$$\downarrow * \;\; \| \;\; 0 \;\; \| \;\; \uparrow *.$$

1.2.10.1 Norton Products Of Up

For the games of $\uparrow$ (or $\downarrow$) and integer n, we can compute $n\,{\scriptstyle\blacksquare}\uparrow$ (and $n\,{\scriptstyle\blacksquare}\downarrow$) as follows.

Definition 1.2.76. *[2, p. 103] For a game G and integer n, we define*

$$n\,{\scriptstyle\blacksquare}G = \begin{cases} 0 & \textit{if } n = 0 \\ G + (n-1)\,{\scriptstyle\blacksquare}G & \textit{if } n > 0 \\ (-n)\,{\scriptstyle\blacksquare}(-G) & \textit{if } n < 0 \end{cases}$$

Thus, $2 \centerdot \uparrow = \uparrow + \uparrow = \Uparrow$ and $3 \centerdot \uparrow = \uparrow + \uparrow + \uparrow$, etc. The canonical forms of $n \centerdot \uparrow$ and $n \centerdot \uparrow * = n \centerdot \uparrow + *$ are given below.

Theorem 1.2.77. *[2, Thm. 5.43,p. 104] For integer $n \geq 1$, the canonical forms of $n \centerdot \uparrow$ and $n \centerdot \uparrow *$ are given by*

$$n \centerdot \uparrow \;=\; \{0 \,|(n-1) \centerdot \uparrow *\}$$

$$n \centerdot \uparrow * \;=\; \begin{cases} \{0, * \,|0\} & \textit{if } n = 1 \\ \{0 \,|(n-1) \centerdot \uparrow\} & \textit{if } n > 1 \end{cases}$$

Symmetrically, the canonical forms of $n \centerdot \downarrow$ and $n \centerdot \downarrow *$ are given by

$$n \centerdot \downarrow \;=\; \{(n-1) \centerdot \downarrow * \,|0\}$$

$$n \centerdot \downarrow * \;=\; \begin{cases} \{0 \,|0, *\} & \textit{if } n = 1 \\ \{(n-1) \centerdot \downarrow |0\} & \textit{if } n > 1 \end{cases}$$

While we can consider $n \centerdot \uparrow$ for integer values of n, in order to understand atomic weight, we require knowledge of non-integer multiples of $\uparrow$. For this, we require the following definition.

Definition 1.2.78. *[2, p. 197] For non-integer games G, the* ***Norton product*** *of G and $\uparrow$, denoted $G \centerdot \uparrow$, is given as*

$$G \centerdot \uparrow = \{\mathbf{L}(G) \centerdot \uparrow + \Uparrow * |\mathbf{R}(G) \centerdot \uparrow + \Downarrow *\}.$$

1.2.10.2 Far Star

In a game G, we can define a nimber that is large enough such that it exceeds all other nimbers in the game. The following definition introduces a symbol that will serve to act as such a nimber, behaving as $*N$ for sufficiently large N.

Definition 1.2.79. *[2, p. 195] The game* ***far star****, denoted $\bigstar$, will be the game such that from $\bigstar$, each player may move to $*n$ for all $n \geq 0$.*

Far star has the property that it is an idempotent: $\bigstar + \bigstar = \bigstar$. For every n, the move to $G + *n$ exists in the game $G + \bigstar$. We note that $\bigstar$ is not a short game. Its purpose is really to serve as a placeholder for a nim-heap $*N$ which is large enough such that any move to $*n$ in G is also available as a move in $*N$. We require $\bigstar$ for definition of the following equivalence relation.

Definition 1.2.80. *[2, Def. 9.28, p. 195] For games G and H, we say that G and H are **equivalent under** $\bigstar$, denoted $G \sim_{\bigstar} H$, if for all games X, $G + X + \bigstar$ and $H + X + \bigstar$ have the same outcome.*

Theorem 1.2.81. *[2, Thm. 9.29, p. 195] The relation $\sim_{\bigstar}$ is an equivalence relation that respects addition.*

Since actually checking every possible combination of G and H with $X + \bigstar$ would be impossible, we require the following for a more practical means of validating equivalence under $\bigstar$.

Theorem 1.2.82. *[2, Thm. 9.30, p. 195]*

$$G \sim_{\bigstar} H \quad \Longleftrightarrow \quad \downarrow \bigstar \ < \ G - H \ < \ \uparrow \bigstar.$$

In practice, we recall that $\bigstar$ is simply a placeholder for $*N$ for sufficiently large $*N$. The nimber $*N$ can serve as a far star for a game if no position of the game has value $*N$, including the game itself.

1.2.10.3 Atomic Weight Defined

At long last, we have all the definitions required to define atomic weight.

Definition 1.2.83. *[2, Def. 9.31, p. 197] For games G and g, if $G \sim_{\bigstar} g \centerdot \uparrow$, then we say that G has **atomic weight** g. We denote this by*

$$\mathrm{aw}(G) = g.$$

For example, we can check that $\downarrow + \bigstar \ < \ * \ < \uparrow + \bigstar$. Hence, $* \sim_{\bigstar} 0$ and so $\mathrm{aw}(*) = 0$. We may also note that $\mathrm{aw}(G \centerdot \uparrow) = G$ since $G \centerdot \uparrow \ \sim_{\bigstar} \ G \centerdot \uparrow$.

For all-small games, the atomic weight of a game is well-defined [2, Thm. 9.32, p. 197]. Atomic weights formalize the concept of a game being approximately some measure of $\uparrow$. Atomic weight is also additive.

Theorem 1.2.84. *[2, Thm. 9.33, p. 198] If G and H are games, then*

$$\operatorname{aw}(G+H) = \operatorname{aw}(G) + \operatorname{aw}(H).$$

For infinitesimal games, this gives us a meaningful way of determining the outcome of a game. The following theorem gives us that atomic weights are a good approximation of $\uparrow$ in a game. For example, while $\uparrow > 0$, $\uparrow * \parallel 0$; however, both $\Uparrow$ and $\Uparrow *$ are greater than 0.

Theorem 1.2.85 (Two-Ahead Rule)**.** *[2, Thm. 9.38, p. 199]*

(i) If $\operatorname{aw}(G) \geq 2$, *then* $G > 0$;

(ii) If $\operatorname{aw}(G) \geq 1$, *then* $G \triangleright 0$;

(iii) If $\operatorname{aw}(G) \leq -1$, *then* $G \triangleleft 0$;

(iv) If $\operatorname{aw}(G) \leq -2$, *then* $G < 0$.

The Two-Ahead Rule tells us that with sufficently large advantage in atomic weight, it can reveal the outcome of a game. However, it is possible to have games of equal atomic weight, but with different outcome. The simplest examples are the games of 0 and $*$, which both have atomic weight zero, but clearly have different outcomes.

So far, we only have a means of testing whether or not a game G has atomic weight g. The following gives a method for computing atomic weight for a game.

Theorem 1.2.86. *[2, Thm. 9.39, p. 199] For a game $G = \{\mathbf{L}(G) \,|\mathbf{R}(G)\}$, define*

$$\operatorname{aw}_0(G) = \left\{ \{\operatorname{aw}\left(G^L\right) - 2\}_{G^L \in \mathbf{L}(G)} \,\middle|\, \{\operatorname{aw}\left(G^R\right) + 2\}_{G^R \in \mathbf{R}(G)} \right\}.$$

The atomic weight of G, $\operatorname{aw}(G)$, *is* $\operatorname{aw}_0(G)$ *unless* $\operatorname{aw}_0(G)$ *is an integer. In that case:*

(i) If $G \parallel \bigstar$, then $\operatorname{aw}(G) = 0$;

(ii) If $G < \bigstar$, then $\operatorname{aw}(G) = \min\left\{x \in \mathbb{Z} \;:\; \forall G^L \in \mathbf{L}(G), x \triangleright \operatorname{aw}\left(G^L\right) - 2\right\}$;

(iii) If $G > \bigstar$, then $\operatorname{aw}(G) = \max\left\{y \in \mathbb{Z} \;:\; \forall G^R \in \mathbf{R}(G), y \triangleleft \operatorname{aw}\left(G^R\right) + 2\right\}$;

We will apply this to the previously considered example of $*$. In this case, $\mathrm{aw}_0(*) = \{0-2\,|0+2\} = 0$. Since $* \parallel \bigstar$, then $\mathrm{aw}(*) = 0$. If we consider $\uparrow * = \{0, *\,|0\}$, we see that

$$\mathrm{aw}_0(\uparrow *) = \{\mathrm{aw}(0)-2, \mathrm{aw}(*)-2\,|\mathrm{aw}(0)+2\} = \{0-2\,|0+2\} = 0.$$

However, $\uparrow > \bigstar$, so $\mathrm{aw}(\uparrow *)$ will be the largest integer less than or incomparable to $\mathrm{aw}(0)+2 = 2$. Thus, $\mathrm{aw}(\uparrow *) = 1$.

When we consider loopy games in chapter 3, we will revisit atomic weight and make use of these results.

1.3 Partial Orders & Lattices

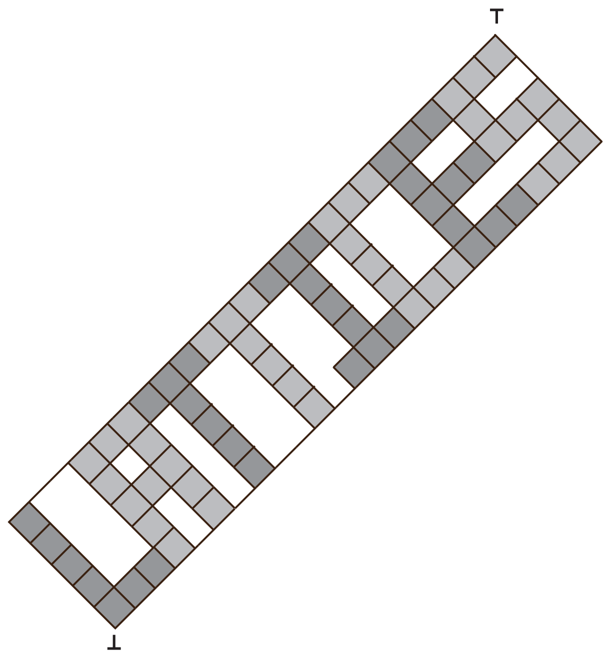


Figure 1.14: The partial order of "LATTICES".

Many sets that we consider have an intuitive ordering, such as that of the natural numbers.

$$1 < 2 < 3 < 4 < 5 < \cdots$$

The partially ordered set (or poset) formalizes this concept of ordering within the set. It consists of a set together with a partial order. The partial order is a binary relation indicating whether one element in the set precedes another. Most familiar in practice are sets, such as the set of natural numbers, that are linearly ordered, in which every pair of elements within the set are comparable. In the case of the natural numbers, if we consider the relation $<$, it is easy to see that for any pair of distinct natural numbers, one is less than the other. In other sets, such as a group of people ordered by genealogical descendancy, each pair within the set need not be related. For this reason, in a poset, the relation is referred to as a partial order to indicate that not every pair of elements within the set must be related. An example of this is given in Figure 1.15. While some pairs of family members, such as Joey and Gibson in this example, have an ancestor/descendant relationship, others, such as Eric and Angela, do not.

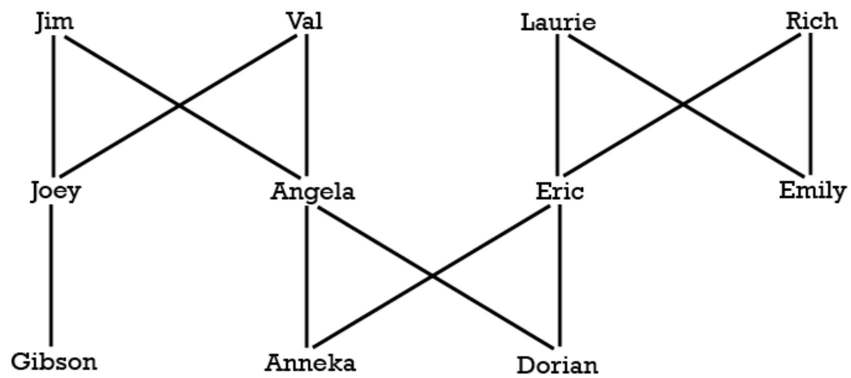


Figure 1.15: Partially ordered set: Family ordered by genealogical descendancy.

Definition 1.3.1. *[10, p. 2] For a set S, a* ***partial order*** *(or ordering) is a binary relation $\preceq \subset S \times S$ that satisfies the properties of*

(1) reflexivity: if $x \in S$, then $x \preceq x$,

(2) antisymmetry: if $x, y \in S$ such that $x \preceq y$ and $y \preceq x$, then $x = y$, and

(3) transitivity: if $x, y, z \in S$ such that $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

Notationally, we may say $x \prec y$ if $x \preceq y$ but $x \neq y$.

For the set of games, we have seen that the binary relation $\leq$ satisfies all properties of a partial order. That is, for games G, H and K, (i) $G \leq G$; (ii) if $G \leq H$ and $H \leq G$ then $G = H$; and if $G \leq H$ and $H \leq K$ then $G \leq K$. However, they do not form a linear order as we have seen several examples, such as $*$ and 0, where games are incomparable with each other.

Definition 1.3.2. *[10, p. 2] A set S and a partial order $\preceq$ on that set define a* ***partially ordered set*** *or* ***poset****, denoted $\langle S; \preceq \rangle$.*

1.3.1 Games Born By Day n

If G is a set of games, $\langle \mathsf{G}; \leq \rangle$ is a poset. Some sets of interest to us are the sets of games born by a specific day. When obvious, we may refer to a poset $\langle S; \preceq \rangle$ as simply S. For instance, when we are considering a poset on a set G of games, we will assume it is with partial order $\leq$ unless otherwise specified.

Definition 1.3.3. *[8, p. 25] We will let $\mathsf{G}[n]$ denote the set of **games born by day** n and define them recursively as*

$$\mathsf{G}[0] = \{0\},$$

and for $n > 0$,

$$\mathsf{G}[n] = \{\{S_L \,|S_R\} : S_L, S_R \subseteq \mathsf{G}[n-1]\}.$$

Thus, $\mathsf{G}[n]$ is the set of games that last at most n moves, with non-alternating play taken into account. It is the set of games that can be constructed using a subset of games born on a prior day, i.e. from $\mathsf{G}[n-1]$, as its left and right options. From this definition and our choice of name, it is clear that $\mathsf{G}[n] \subset \mathsf{G}[n+1]$.

In order to visualize the ordering relation in a poset, we make use of a Hasse diagram. A Hasse diagram depicts the partial order relationship between pairs of elements (when one exists) in a finite poset. It is the least graph whose transitive closure gives all of the comparabilities. From it, one can reconstruct the whole partial order structure.

Definition 1.3.4. *[10, p. 11] A **Hasse diagram** is a graph $G = (V, E)$, representing the partial order relation of a finite poset $\langle S; \preceq\rangle$, in which $V = S$ and there is an edge from $x \in S$ to $y \in S$ if $x \prec y$ and $\nexists z \in S$ such that $x \prec z \prec y$.*

If $x \prec y$, then y is drawn higher than x. Due to this convention, the direction of edges is not indicated within a Hasse diagram.

Definition 1.3.5. *[10, p. 11] In a poset $\langle S; \preceq\rangle$, an element $b \in S$ is said to **cover** an element $a \in S$ if $a \prec b$ and there is no element $x \in S$ such that $a \prec x \prec b$.*

*In this case, we say that b is an **upper cover** of a or, likewise, that a is a **lower cover** of b.*

A Hasse diagraph is a graph whose vertices are the elements of S and the edges correspond to the covering relation.

Definition 1.3.6. *[10, p. 15] In a poset $\langle S; \preceq\rangle$, we say that S has a **top element**, denoted $\top$, if $\top \in S$ such that for all $s \in S, s \preceq \top$. Likewise, we say that S has a **bottom element**, denoted $\bot$, if $\bot \in S$ such that for all $s \in S, \bot \preceq s$.*

Example 1.3.7. *For instance, let*

$$S = \mathscr{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\},$$

and let $\preceq$ *be the subset relation* $\subseteq$*. Then the Hasse diagram of the finite poset* $\langle S; \preceq \rangle$ *is that of Figure 1.16.*

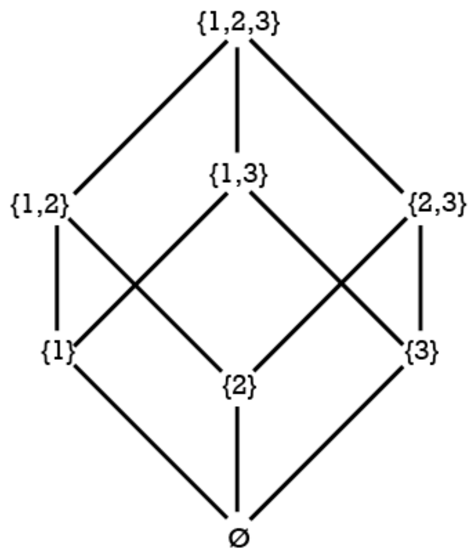


Figure 1.16: Hasse diagram of $\langle \mathscr{P}(\{1,2,3\}); \subseteq \rangle$

Note that even though $\{2\} \prec \{1,2,3\}$ (since $\{2\} \subset \{1,2,3\}$), there is no edge directly between them. This is due to the fact that there exist elements, namely $\{1,2\}$ and $\{2,3\}$, in between (i.e. $\{2\} \prec \{1,2\} \prec \{1,2,3\}$ and $\{2\} \prec \{2,3\} \prec \{1,2,3\}$). However, there is still an indirect path remaining from $\{2\}$ up to $\{1,2,3\}$.

We can readily make use of Hasse diagrams when describing the poset of games in $\mathsf{G}[n]$. We have that $\mathsf{G}[0] = \{0\}$. From Table 1.2.5, we know that

$$\mathsf{G}[1] = \mathsf{G}[0] \cup \{-1, *, 1\} = \{-1, 0, *, 1\}.$$

Table 1.2.5 lists for us all new games in $\mathsf{G}[2]$. The posets of $\mathsf{G}[1]$ and $\mathsf{G}[2]$ (with binary relation $\leq$) are given in Figures 1.17 and 1.18, respectively.

We note that the posets $\mathsf{G}[1]$ and $\mathsf{G}[2]$ represent only those games that are in canonical form. Although $\{-1, 0 \,|\, 1\} = \{0 \,|\, 1\} = \frac{1}{2}$, only their canonical form, $\frac{1}{2}$, is represented in $\mathsf{G}[2]$ (Fig. 1.18). Each element in the Hasse diagram of $\mathsf{G}[2]$ (and of

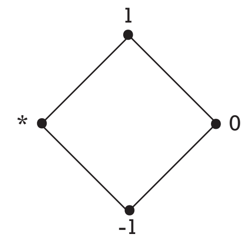


Figure 1.17: The poset of games born by day 1.

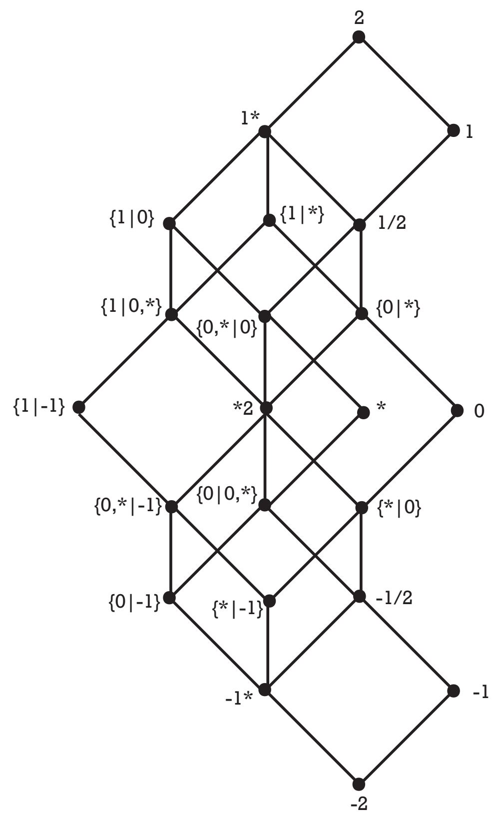


Figure 1.18: The poset of games born by day 2 [8, Fig. 1, p. 28].

similar diagrams we will present of sets of games) represents an equivalence class of games (in its canonical form), possibly depicted several different ways in literal form. Thus, in this canonical representation, if H covers G, $G < H$ (i.e. there is strict inequality).

Definition 1.3.8. *[10, p. 2] Two elements x and y in a poset $\langle S; \preceq \rangle$ are **comparable** if $x \preceq y$ or $y \preceq x$.*

*Two elements that are not comparable are said to be **incomparable**.*

Thus, two games $G, H \in \mathsf{G}[n]$ are incomparable in the poset if and only iff $G \parallel H$ in the game theoretic sense. For example, in the genealogical poset in Figure 1.15, Angela and Anneka are comparable, while Dorian and Gibson are not.

In $\mathsf{G}[2]$ (Fig. 1.18), we see that while $\frac{1}{2}$ is comparable with both 1 and $1*$, the games 1 and $1*$ are incomparable. Both 1 and $1*$ cover $\frac{1}{2}$.

1.3.2 Chains And Antichains

Definition 1.3.9. *[10, p. 3] A **chain** in a poset $\langle S; \preceq \rangle$ is a subset $C \subseteq S$ in which each pair of elements is comparable.*

Definition 1.3.10. *[10, p. 3] An **antichain** in a poset $\langle S; \preceq \rangle$ is a subset $A \subseteq S$ in which each pair of different elements is incomparable, i.e. there is no order relation between any two distinct elements in A.*

*A **maximal antichain** is an antichain that is not a proper subset of any other antichain in the poset. A **maximum antichain** is a maximal antichain that is at least as large as every other antichain.*

In the genealogical poset of Figure 1.15, both {Val, Angela, Anneka} and {Jim, Joey, Gibson}, representing grandparent, parent and child, are chains, while the set of all grandparents {Jim, Val, Laurie, Rich} or grandchildren {Gibson, Anneka, Dorian} are antichains. The set of all grandparents is a maximum antichain within this particular poset.

In $\mathsf{G}[2]$ (Fig. 1.18), we see that $\{-2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2\}$ is a chain,

$$-2 < -1 < -\frac{1}{2} < 0 < \frac{1}{2} < 1 < 2,$$

namely the chain formed by all of the numbers occuring by day 2.

1.3.3 Duality

For any relation, we can define its converse.

Definition 1.3.11. *[5, p. 3] The **converse** of a binary relation R is the relation $\overline{R}$ such that $x\overline{R}y$ if and only if yRx.*

More importantly, we know that if a relation is a partial ordering, then its converse is as well.

Theorem 1.3.12 (Duality Principle)**.** *[5, Thm. 2, p. 3] The converse of any partial ordering is itself a partial ordering.*

The converse $\overline{R}$ of a binary relation is often called the dual of R.

Definition 1.3.13. *[10, 1.19, p. 14] For a given poset $P = \langle S; \preceq \rangle$, we define the* ***dual*** *of P, denoted P^d, as*

$$P^d = \langle S; \succeq \rangle$$

where $x \succeq y$ in P^d if and only if $y \preceq x$ in P. That is, the dual of P is the poset P^d defined by the converse partial ordering on the same elements.

A poset is said to be ***self-dual*** *if the poset is isomorphic to its dual. That is, P is self-dual if $P \cong P^d$.*

This new poset is essentially formed by turning the former upside down. For example, the minimal element in a poset P is the maximal element in its dual P^d.

In general, for any statement Θ about a poset, there is a **dual statement** Θ^d made by replacing every $\preceq$ by $\succeq$ and vice versa. If a statement is true for a poset, then so is the dual statement for its dual [10, p. 15].

The concept of duality is particulary useful when it comes to posets. The usefulness of this is that with posets, many of our results will come in pairs. This fact will allow us to often give two results through proving one. We will encounter and make use of many dual concepts such as meet/join, upper/lower sets and ideals/filters, to name a few.

1.3.4 Floor And Ceiling Functions

In an approach in keeping with Calistrate, Paulhus and Wolfe [8], we define floor and ceiling functions relative to a given poset as follows:

Definition 1.3.14. *[1, p. 2][8, p. 26] For a poset $\langle S; \preceq \rangle$ and a subset $T \subseteq S$, the* ***floor*** *of T, denoted $\lfloor T \rfloor$, is defined as*

$$\lfloor T \rfloor = \bigcup_{t \in T} \{s \in S : s \not\succeq t\} = \bigcup_{t \in T} \{s \in S : s \prec t \text{ or } s \parallel t\}.$$

Dually, the ***ceiling*** *of T, denoted $\lceil T \rceil$, is defined as*

$$\lceil T \rceil = \bigcup_{t \in T} \{s \in S : s \not\preceq t\} = \bigcup_{t \in T} \{s \in S : s \succ t \text{ or } s \parallel t\}.$$

In the case where $T = \{t\}$ is a singleton, when clear, we will write $\lfloor t \rfloor$ and $\lceil t \rceil$, respectively.

Hence, when refering to a poset of games $\langle S; \leq \rangle$, for a game G,

$$\lfloor G \rfloor = \{H \in S : H \lhd G\}$$

and

$$\lceil G \rceil = \{H \in S : H \rhd G\}.$$

For example, we can see in Figure 1.18 that, in $\mathsf{G}[2]$,

$$\lfloor \{* \,|{-1}\} \rfloor = \{\{0 \,|{-1}\}, -\frac{1}{2}, -1*, -1, -2\}$$

and

$$\lceil 1 \rceil = \{\{1 \,|{-1}\}, \{1 \,|*, 0\}, \{1 \,|*\}, \{1 \,|0\}, 1*, 2\}.$$

This definition is motivated by the fact that if our poset is on the set of games with the ordering $\leq$, then for some game G, $\lfloor G \rfloor$ is the set of games smaller than or incomparable to G. We will see later that this set models the Gift Horse principle (Lemma 1.2.32) in the sense that $G = \{\lfloor G \rfloor \,|\lceil G \rceil\}$ with the set of left and right options expanded to include all games that do not affect the value of the original game G.

For example, consider the game $*$. We can see that in $\mathsf{G}[1]$ (Fig. 1.17), $\lfloor * \rfloor = \{0, -1\}$ and $\lceil * \rceil = \{0, 1\}$. If we consider the game $\{\lfloor * \rfloor \,|\lceil * \rceil\} = \{0, -1 \,|0, 1\}$, we see that the "extra" option for Left to -1 and for Right to 1 do not affect the value of the original game since they are both dominated by each of their respective moves to zero. Hence, $\{\lfloor * \rfloor \,|\lceil * \rceil\} = *$.

1.3.5 Lower And Upper Sets

Definition 1.3.15. *[10, p. 20][5, p. 25] For a poset $\langle S; \preceq \rangle$, a* ***lower set*** *(order ideal or down-set) is defined as a subset $T \subseteq S$ such that for all elements $x, y \in S$,*

$$x \preceq y \text{ and } y \in T \implies x \in T.$$

Dually, an ***upper set*** *(order filter or up-set) is a subset $U \subseteq S$ such that for all elements $x, y \in S$,*

$$x \preceq y \text{ and } x \in U \implies y \in U.$$

Thus, every poset is an upper set of itself. An intersection of upper sets is again an upper set. If we look at the complement of any upper set, it will be a lower set. Similarly, the complement of any lower set is an upper set.

For example, in $\langle \mathscr{P}(\{1,2,3\}); \subseteq \rangle$ (Fig. 1.16), the set $\{\{1\},\{1,2\},\{1,3\},\{1,2,3\}\}$ is an upper set and $\{\emptyset,\{2\},\{3\},\{2,3\}\}$ is a lower set, each being the complement of the other.

Definition 1.3.16. *[10, p. 20] For a poset $\langle S; \preceq \rangle$ and subset $T \subseteq S$, we define the* ***lower set generated by*** *T (or* ***lower set of*** *T), which we will denote as $\downarrow T$, as*

$$\downarrow T = \bigcup_{t \in T} \{s \in S : s \preceq t\}.$$

Dually, the ***upper set generated by*** *T (or* ***upper set of*** *T), which we will denote as $\uparrow T$, as*

$$\uparrow T = \bigcup_{t \in T} \{s \in S : t \preceq s\}.$$

In the case where $T = \{t\}$ is a singleton, when clear, we will write $\downarrow t$ and $\uparrow t$, respectively. In this case, the respective sets are called ***principal****.*

Thus, $x \preceq y$ implies that $\downarrow x$ is a subset of $\downarrow y$.

For example, in Figure 1.15, the lower set induced by $\{Val\}$ is

$$\downarrow Val = \{Angela, Anneka, Dorian, Gibson, Joey, Val\}.$$

Similarly, the upper set generated by $\{Eric\}$ is $\uparrow Eric = \{Eric, Laurie, Rich\}$.

When referring to a poset of games $\langle S; \leq \rangle$, for a game G,

$$\downarrow G = \{H \in S : H \leq G\}$$

and

$$\uparrow G = \{H \in S : H \geq G\}.$$

In $\mathsf{G}[2]$ (Fig. 1.18), we find that $\downarrow \{* \,|{-1}\} = \{\{* \,|{-1}\}, -1*, -2\}$ and $\uparrow 1 = \{1, 2\}$.

The following lemma gives the connection between the partial order relation and lower sets.

Lemma 1.3.17. *[10, Lemma 1.30, p. 21] Let $P = \langle S; \preceq \rangle$ be a poset and $x, y \in S$. Let $\mathcal{O}(P)$ be the set of all lower sets of P, which is itself ordered under set inclusion. Then the following are equivalent:*

(i) $x \preceq y$;

(ii) $\downarrow x \subseteq \downarrow y$;

(iii) $\forall Q \in \mathcal{O}(P),\ y \in Q \implies x \in Q$.

When considering the partial orders of games, we will often make use of the equivalence of (i) and (ii). We will look at the relationship between lower sets to show an order relation between games.

1.3.6 Join And Meet

Definition 1.3.18. *[10, p. 33-34] For a poset $\langle S; \preceq \rangle$ and $T \subseteq S$, an* ***upper bound*** *of T is an element $z \in S$ such that $x \preceq z$ for all $x \in T$. We say that T is* ***bounded from above*** *if there exists an upper bound for T. A* ***lower bound*** *and* ***bounded from below*** *are defined in a similar manner.*

The ***least upper bound*** *or* ***supremum*** *of T is an upper bound $x \in T$ such that for every upper bound $y \in T$, $x \preceq y$. If such a least upper bound of T exists, it is unique and is denoted* $\sup(T)$.

The ***greatest lower bound*** *or* ***infimum*** *of T is a lower bound $x \in T$ such that for every lower bound $y \in T$, $y \preceq x$. If such a greatest lower bound of T exists, it is unique and is denoted* $\inf(T)$.

Thus, in a poset $\langle S; \preceq \rangle$ with $x \in S$, $\sup(\downarrow x) = x$ and $\inf(\uparrow x) = x$.

Definition 1.3.19. *[10, p. 34] In a poset $P = \langle S; \preceq \rangle$, the* ***join*** $(\vee)$ *of two elements $x, y \in S$, denoted $x \vee y$, is the least upper bound of x and y if it exists, i.e.*

$$x \vee y = \sup(\{x, y\}).$$

For a set $T \subseteq S$, we will write $\bigvee T$ (the 'join of T') to denote $\sup(T)$.

If a join exists, it is unique. If j_1 and j_2 are both joins of x and y, then $j_1 \preceq j_2$ and $j_2 \preceq j_1$, so $j_1 = j_2$.

Definition 1.3.20. *[10, p. 34] In a poset $P = \langle S; \preceq \rangle$, the **meet** $(\wedge)$ of two elements $x, y \in S$, denoted $x \wedge y$, is the greatest lower bound of x and y if it exists, i.e.*

$$x \wedge y = \inf(\{x, y\}).$$

For a set $T \subseteq S$, we will write $\bigwedge T$ (the 'meet of T') to denote $\sup(T)$.

If a meet exists, it is unique. If m_1 and m_2 are both meets of x and y, then $m_1 \preceq m_2$ and $m_2 \preceq m_1$, so $m_1 = m_2$.

In a finite poset $\langle S; \preceq \rangle$,

$$x \vee y = \min\{j \in S \;:\; x \preceq j, y \preceq j\} = \min\{j \in S \;:\; x, y \in \mathord{\downarrow} j\},$$

$$x \wedge y = \max\{m \in S \;:\; m \preceq x, m \preceq y\} = \max\{m \in S \;:\; x, y \in \mathord{\uparrow} m\},$$

$$\bigvee T = \bigvee_{i \in I}\{t_i\} \quad \text{and} \quad \bigwedge T = \bigwedge_{i \in I}\{t_i\}.$$

If in that same finite poset there is a chain C containing both x and y, then $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$.

For example, in Figure 1.18, we see that the join and meet of $\frac{1}{2}$ and $\{1 \,|0, *\}$ in $\mathsf{G}[2]$ are given by $1*$ and $*2$, respectively. If we choose elements, say $\frac{1}{2}$ and 2 which are related, with $\frac{1}{2} < 2$, then $\frac{1}{2} \vee 2 = 2$ and $\frac{1}{2} \wedge 2 = \frac{1}{2}$.

1.3.7 Lattices

Definition 1.3.21. *[10, p. 34,41] A **lattice** L is a non-empty poset in which every two elements x and y have both a least upper bound, $x \vee y$, and a greatest lower bound, $x \wedge y$.*

*If $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq L$, and they do if L is a finite lattice, then L is called a **complete lattice**.*

*A **sublattice** of L is a subposet of L which is a lattice, i.e. a subposet of L which is closed under the operations of $\vee$ and $\wedge$ as defined in L.*

The join and meet operations are

(i) idempotent: $x \vee x = x \wedge x = x$,

(ii) commutative: $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$,

(iii) associative: $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, and

(iv) absorptive: $x \wedge (y \vee x) = x \vee (y \wedge x) = x$.

It should be noted that distributivity is a property that is missing from this list.

The partial order relation can be recovered from meet and join by defining $x \preceq y \iff x \wedge y = x$. Clearly, it also then holds that $x \preceq y \iff x \vee y = y$.

Lattices, like posets, can be nicely visualized by Hasse diagrams. Figure 1.19 depicts two Hasse diagrams of posets. The one on the left (a) is a lattice, while the one on the right (b) is not.

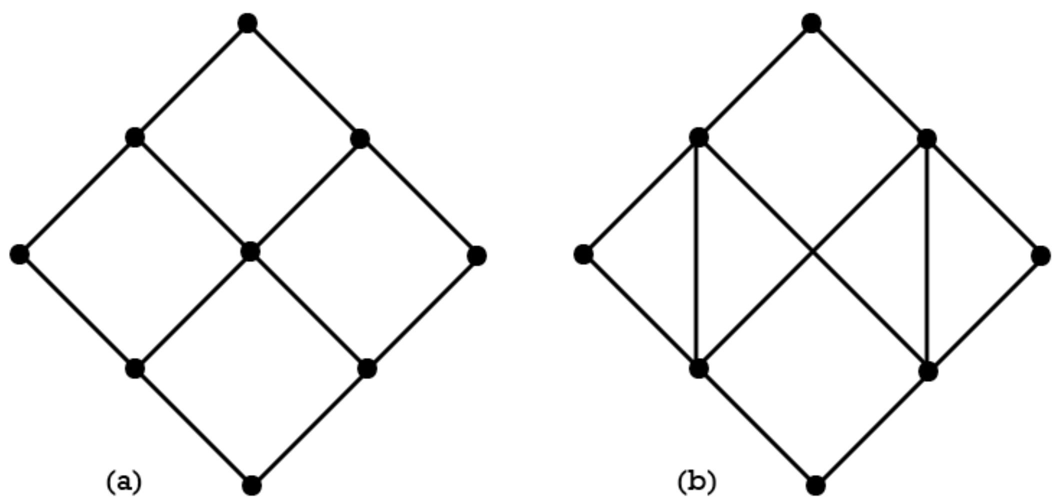


Figure 1.19: Hasse diagrams of (a) a lattice, and (b) a poset that is not a lattice.

Definition 1.3.22. *[10, p. 41] A lattice L is* ***bounded*** *if it has both a greatest (top) and least (bottom) element, $\top$ and $\bot$, respectively.*

If a top element $\top$ exists in a lattice L, then for all $s \in L$, $s = s \wedge \top$. Likewise, if a bottom element $\bot$ exists, then for all $s \in L, s = s \vee \bot$.

For example, $\langle \mathsf{G}\,[n]; \leq \rangle$ is a bounded lattice with top element n and bottom element $-n$.

Any lattice may be converted into a bounded lattice, simply by adding a greatest and least element [10, p. 41]. For example, the finite lattice $L = \langle S; \preceq \rangle$, with

$S = \{x_1, \ldots, x_n\}$, is bounded with top element $\bigvee S = x_1 \vee \cdots \vee x_n$ and bottom element $\bigwedge S = x_1 \wedge \cdots \wedge x_n$. We note that we are allowed to write iterated joins and meets unambiguously without brackets as above, thanks to the associativity of $\vee$ and $\wedge$ within a lattice.

A poset is a bounded lattice if and only if every set of elements has both a join and a meet. This is inclusive of the empty set, in which the join of an empty set of elements is defined to be the least element and the meet is defined to be the greatest element.

In the example 1.3.7, the poset considered is in fact a bounded lattice, with top element $\{1, 2, 3\}$ and bottom element $\emptyset$. The set of positive integers, in their usual order, forms a lattice under the operation *min*. The number 1 is the bottom element in this lattice, but there is no top element.

1.3.7.1 Distributivity

Since lattices include the two binary operations meet ($\wedge$) and join ($\vee$), it is natural to consider whether one of them distributes over the other.

Definition 1.3.23. *[10, p. 80] A* ***distributive lattice*** *is a lattice that satisfies either (and therefore both) of the dual distributive laws:*

(i) distributivity of $\vee$ over $\wedge$: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, *or*

(ii) distributivity of $\wedge$ over $\vee$: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

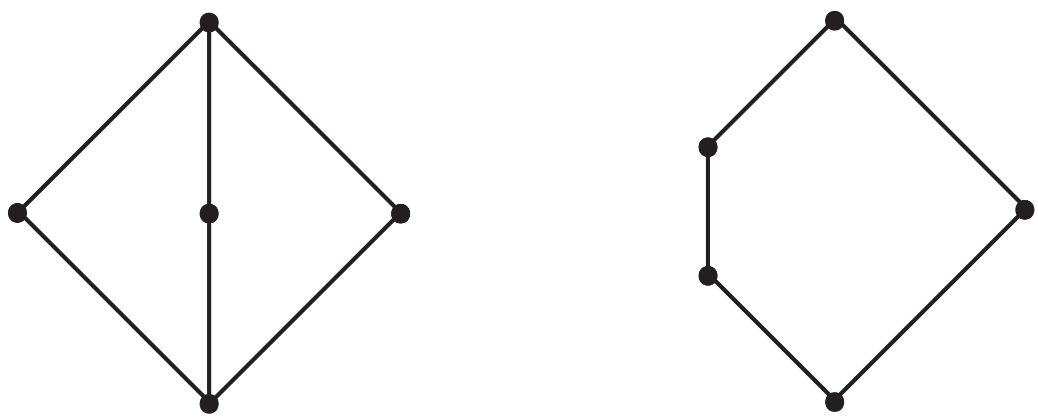
Adherence to these distributive laws ensures that the meet operation preserves nonempty finite joins. That the two laws are equivalent is a basic fact of lattice theory [10].

Among others, a totally ordered set is an example of a distributive lattice, with $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. As such, Figure 1.16 represents a distributive lattice. The natural numbers also form a distributive lattice with $x \vee y = \operatorname{lcm}(x, y)$ and $x \wedge y = \gcd(x, y)$.

Calistrate, Paulhus and Wolfe [8] showed that the poset $\langle \mathsf{G}[n]; \leq \rangle$ is a distributive lattice. While many examples of distributive lattices exist, they each have somewhat specific structures. Intuitively, a distributive lattice looks like a collection of “boxes”.

In order to show that a lattice is not distributive, we must demonstrate that there is a set of elements in the lattice for which the distributive laws fail. Thanks to Birkhoff [5], we have a means of determining whether a finite lattice is distributive from inspection of its Hasse diagram.

Theorem 1.3.24. *[5, p. 11][10, p. 89] A lattice L is non-distributive if and only if it contains either the lattice M_3 or N_5 (depicted in Figure 1.20) as a sublattice.*



(a) (b)

Figure 1.20: The nondistributive lattices (a) M_3, the "diamond", and (b) N_5, the "pentagon".

1.3.7.2 Join- And Meet-Irreducibles

For the following, let x be an element in a lattice $L = \langle S; \preceq \rangle$ and $S = \{x_i\}_{i \in I}$.

Definition 1.3.25. *[10, p. 53] An element $x \in L$ is said to be **join-irreducible** if x is not a bottom element (i.e. $x \neq \bot$) and $x = a \vee b$ implies $x = a$ or $x = b$ for all $a, b \in L$ (i.e. $a \prec x$ and $b \prec x$ imply $a \vee b \prec x$). Thus, x is join-irreducible in L if*

$$x = \bigvee_{j \in J \subseteq I} x_j \implies \exists i \in J \text{ s.t. } x = x_i.$$

The set of join-irreducible elements within the lattice L is denoted by $\mathcal{J}(L)$.

Definition 1.3.26. *[10, p. 53] An element $x \in L$ is said to be **meet-irreducible** if x is not a top element (i.e. $x \neq \top$) and $x = a \wedge b$ implies $x = a$ or $x = b$ for all $a, b \in L$ (i.e. $x \prec a$ and $x \prec b$ imply $x \prec a \wedge b$). Thus, x is meet-irreducible in L if*

$$x = \bigwedge_{j \in J \subseteq I} x_j \implies \exists i \in J \text{ s.t. } x = x_i.$$

The set of meet-irreducible elements within the lattice L is denoted by $\mathcal{M}(L)$.

Definition 1.3.27. *If x is both join- and meet-irreducible, then x is said to be **doubly-irreducible**. Thus, the set of doubly-irreducible elements of a lattice L are those elements of $\mathcal{J}(L) \cap \mathcal{M}(L)$.*

In Figure 1.21, s, t, u, v, x and y are all join-irreducible, while s, t, v, w, x and y are meet-irreducible. Since s, t, v, x and y are both meet and join-irreducible, they are doubly-irreducible. We can think of the join-irreducible elements of the lattice, with the exception of the top and bottom elements, as being those that cover only one element and, dually, the meet-irreducible elements being those covered by only one element.

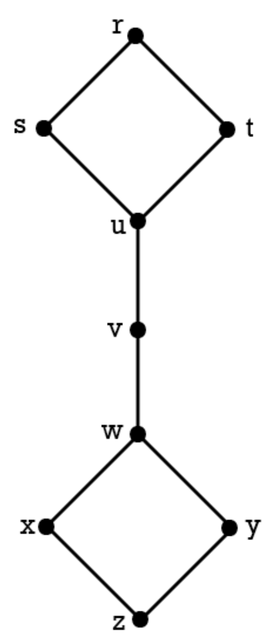


Figure 1.21: Irreducible elements.

If we refer to Figure 1.17, we see that $\mathcal{J}(\mathsf{G}[1]) = \mathcal{M}(\mathsf{G}[1]) = \{0, *\}$ and so 0 and $*$ are doubly-irreducible. If we refer to Figure 1.18, we find that the join- and meet-irreducible elements of $\mathsf{G}[2]$ are given by

$$\mathcal{J}(\mathsf{G}[2]) = \{-1, -1*, \{*\,|{-1}\}, \{0\,|{-1}\}, 0, *, \{1\,|{-1}\}, 1\}$$

and

$$\mathcal{M}(\mathsf{G}[2]) = \{-1, 0, *, \{1\,|{-1}\}, \{1\,|0\}, \{1\,|*\}, 1*, 1\},$$

respectively. Thus, the doubly-irreducible elements of $\mathsf{G}[2]$ are $-1, 0, *, \{1\,|{-1}\}$ and 1.

1.3.8 Linear Orders, Dimension & Planarity

A partial order is called a linear order if every two distinct elements are comparable.

Definition 1.3.28. *[10, p. 2] A **linear order** is a partial order on a set S such that for any $x, y \in S$, either $x \preceq y$ or $y \preceq x$.*

Definition 1.3.29. *[11] Let $P = \langle S; \preceq_P \rangle$ be a poset and $L = \langle S; \preceq_L \rangle$ be a linear order, both defined on the same set S. The linear order L is called a **linear extension** of the poset P if for every $x, y \in S$,*

$$x \preceq_P y \implies x \preceq_L y.$$

We can use a collection of linear orders to define a poset.

Definition 1.3.30. *[11] Let $\mathbf{K}$ be a collection of linear orders, $\mathbf{K} = \{L_i = \langle S; \preceq_{L_i} \rangle\}_{i \in I}$. We can define a poset $P = \langle S; \preceq_P \rangle$ as follows:*

$$\forall x, y \in S, \quad x \preceq_P y \iff \forall i \in I, \quad x \preceq_{L_i} y.$$

That is, $x \preceq y$ in P if and only if $x \preceq y$ in every linear order in the collection $\mathbf{K}$.

*A poset obtained is this way is said to be **realized** by the linear orders of $\mathbf{K}$.*

The dimension of a poset is defined to be the least number of linear orders whose intersection is the partial ordering of the poset (where the partial order is regarded as a set of ordered tuples).

Definition 1.3.31. *[11] The dimension of a poset $P = \langle S; \preceq \rangle$, denoted* $\operatorname{dimension}(P)$*, is the smallest cardinal number k such that P is realized by k linear orders on the set S.*

Thus, if we wish to show that a poset $P = \langle S; \preceq_P \rangle$ has dimension at most 2, we must show that there are two linear orders, L_1 and L_2 such that P can be mapped in an order preserving way to L_1 and L_2 by a function f_1 to L_1 and a function f_2 to L_2, such that, for all $x, y \in S$,

(i) if $x \prec_P y$, then at least one of $f_1(x) \preceq_{L_1} f_1(y)$ and $f_2(x) \preceq_{L_2} f_2(y)$ is strict; and

(ii) if $x \parallel_P y$, then either

  (a) $f_1(x) \succ_{L_1} f_1(y)$ and $f_2(x) \prec_{L_2} f_2(y)$, or

  (b) $f_1(x) \prec_{L_1} f_1(y)$ and $f_2(x) \succ_{L_2} f_2(y)$.

We will make use of poset dimension when we consider the set of option-closed games by day n in section 4.3. We will use it to show that the lattice of these games is planar.

Definition 1.3.32. *[17] A poset is called* ***planar*** *if its Hasse diagram can be drawn in such a way that none of its edges intersect.*

For example, in Figure 1.22, we see an example of a planar poset. It is presented in two forms. We see that in (a) it is drawn with intersecting edges. However, it is still a planar poset since it can be redrawn without these intersections present, as in the Hasse diagram in (b).

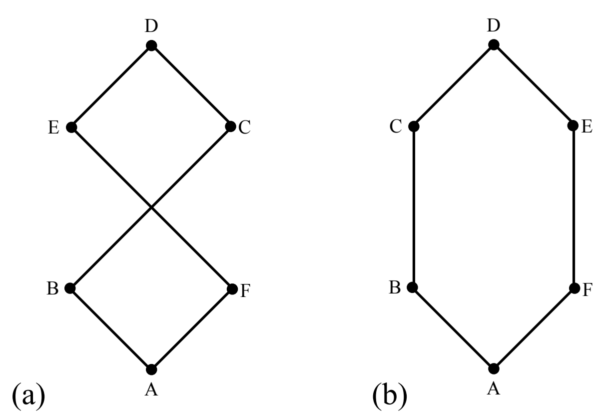


Figure 1.22: An example of a planar poset (a), drawn in its planar representation (b).

The following planarity result comes from Kelly and Rival [17]. This result, showing equivalence between planarity in lattices and dimension at most 2, was first attributed to Kirby Baker in his unpublished Honors Thesis [4], the proof only later to be published by Kelly and Rival [17]. We will make use of this result when we consider the lattices of option-closed games in Section 4.3.

Proposition 1.3.33. *[4][17] Let L be a finite lattice.*

$$L \text{ is planar if and only if } \operatorname{dimension}(L) \leq 2.$$

Chapter 2

Juxtapositions

In this chapter, we will explore the structure within a game. We will consider the positions which can be formed from two smaller positions placed adjacent to one another. We will call positions of this type juxtapositions.

jux·ta·po·si·tion [jək-stə-pə-'zi-shən]
the act or an instance of placing two or more things side by side;
also : the state of being so placed [18].

Commonly, game positions that are understood and simple were they on their own, become much more difficult to understand when placed adjacent to another simple position.

For games of this form, we will explore the ways in which the underlying games interact. We will first consider ordinal sums and later introduce a new function which we will call the side-sum. We will explore how each of the functions behave when applied.

Finally, for each function, we will look at various applications of each. For ordinal sums, we will look at the games of LENRES and SHOVE. For the game of RESTRICTED TOPPLING DOMINOES, we will require the side-sum function to gain full understanding of the game.

2.1 Juxtaposition Defined

"If I held you any closer I would be on the other side of you."
- Groucho Marx

For example, with HACKENBUSH strings, we've seen the positions $\frac{5}{8}$ and $\frac{13}{16}$. What happens if we stack one on top of the other as in figure 2.1?

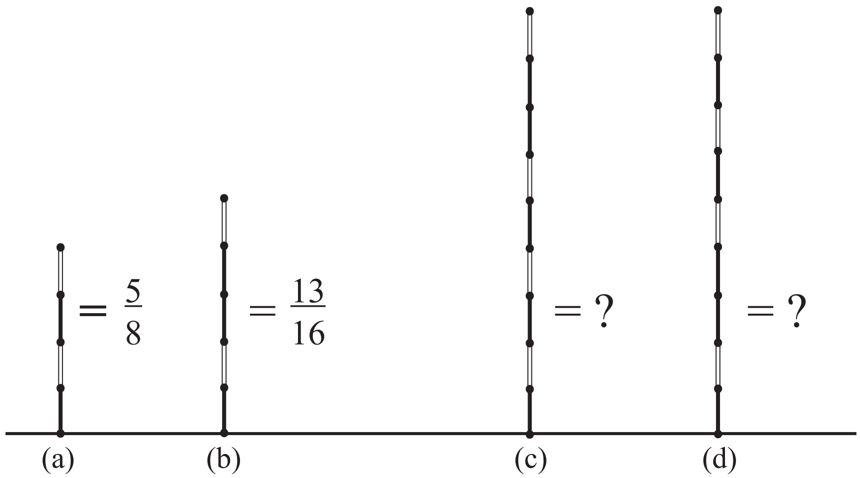


Figure 2.1: Juxtaposition of HACKENBUSH positions.

What if we do the same for two arbitrary HACKENBUSH strings? Likewise, we can ask this same question of games that are played on a board, where the adjacency is possibly to the right of (versus on top of) the other game position. The position of the adjacency must be specified with respect to the game.

Definition 2.1.1. *A* ***game adjacency*** *refers to the physical placement of two positions with their interaction defined. The reference to adjacency must be made specific with respect to the game.*

Definition 2.1.2. *A* ***juxtaposition*** *of G and H will be the game adjacency of game positions G and H and will be denoted as $G \boxplus H$ (said "G juxtaposed H").*

Consider the game of TOPPLING DOMINOES, first introduced by Albert, Nowakowski and Wolfe in *Lessons in Play* [2, p. 274]. The game of TOPPLING DOMINOES begins with a row of black and white dominoes, denoted as x and o, respectively, in text. On his turn, Left is allowed to chose any black domino and topple it either left or right. Every domino in that direction also topples and is removed from the game. On her turn, Right topples a white domino in either direction.

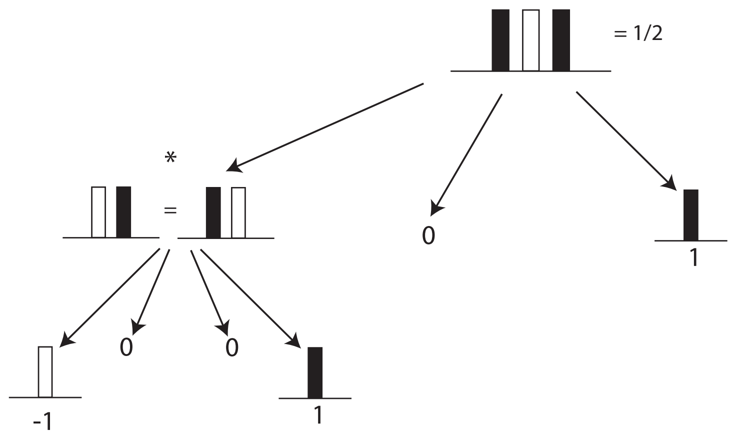


Figure 2.2: Example of a game of TOPPLING DOMINOES having value $\frac{1}{2}$.

In the game of TOPPLING DOMINOES and all variations of the game that we consider, we will use $G \boxplus H$ to mean that the row of dominoes having value G is positioned to the left of the row of dominoes having value H, as in Figure 2.3.

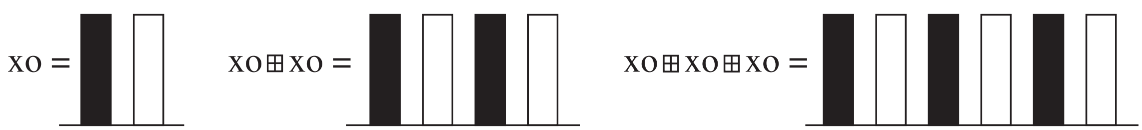


Figure 2.3: Juxtaposition of TOPPLING DOMINOES positions.

As was noted in discussion of Nim-dimension (Def. 1.2.58), it was shown in *Lessons in Play* [2] that the TOPPLING DOMINOES positions $xo \boxplus xo$ and $xo \boxplus xo \boxplus xo$ as depicted above, have value $*2$ and $*3$, respectively. In fact, they show that positions that are built as juxtapositions of n copies of the game zo have value $*n$.

In HACKENBUSH strings, we will specify that $G \boxplus H$ implies that string G sits below string H. Hence, the two positions in question in Figure 2.1, strings (c) and (d), can be represented as $\frac{5}{8} \boxplus \frac{13}{16}$ and $\frac{13}{16} \boxplus \frac{5}{8}$, respectively.

In *Winning Ways* [3, p. 190,219], consideration of similar HACKENBUSH positions introduces the concept of ordinal sum.

2.2 Ordinal Sums

In the game of HACKENBUSH, consider the position containing a HACKENBUSH string of value H atop HACKENBUSH string of value G, i.e. the position $G \boxplus H$ depicted in Figure 2.4.

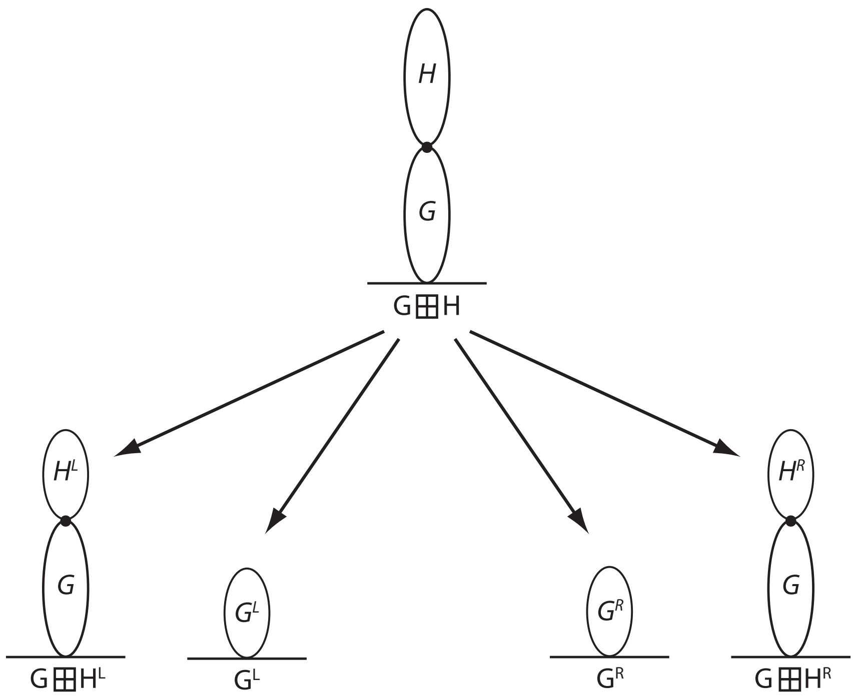


Figure 2.4: Juxtaposition, $G \boxplus H$, of HACKENBUSH strings having values G and H.

From the HACKENBUSH position $G \boxplus H$, when a player moves in H, he leaves G untouched. However, once he plays in G, everything above it, including all of H, is removed. Thus, from $G \boxplus H$, Left has moves to positions of the form G^L and $G \boxplus H^L$ and Right has moves to positions of the form G^R and $G \boxplus H^R$.

Definition 2.2.1. *[3, p. 219] For games G and H, the **ordinal sum** of G and H, denoted G:H, is defined recursively as*

$$G{:}H = \{\mathbf{L}(G), G{:}\mathbf{L}(H) \,|\, \mathbf{R}(G), G{:}\mathbf{R}(H)\}.$$

*We will refer to G as the **base** and H as the **branch** of this function.*

In positions such as those defined by the ordinal sum G:H, any move in the position G annihilates any further option to H, while moves in H do not affect G.

It is for this reason that we introduce the names "base" and "branch" for the input games for ordinal sums. The base remains intact until after the branch is chopped away.

As expected, if neither player has a move in H, then the value of $G{:}H = G$.

Lemma 2.2.2.

$$G{:}0 = \{\mathbf{L}(G) \,|\, \mathbf{R}(G)\} = G$$

Lemma 2.2.3.

$$(-G){:}(-H) = -(G{:}H)$$

Proof.

$$\begin{aligned}
(-G){:}(-H) &= \{\mathbf{L}(-G), (-G){:}\mathbf{L}(-H) \,|\, \mathbf{R}(-G), (-G){:}\mathbf{R}(-H)\} \\
&= \{-\mathbf{R}(G), (-G){:}(-\mathbf{R}(H)) \,|\, -\mathbf{L}(G), (-G){:}(-\mathbf{L}(H))\} \\
&= \{-\mathbf{R}(G), -(G{:}\mathbf{R}(H)) \,|\, -\mathbf{L}(G), -(G{:}\mathbf{L}(H))\} \qquad \text{(by induction)} \\
&= -\left(\{\mathbf{L}(G), (G{:}\mathbf{L}(H)) \,|\, \mathbf{R}(G), (G{:}\mathbf{R}(H))\}\right) \\
&= -(G{:}H)
\end{aligned}$$

□

In HACKENBUSH strings, a position $G \boxplus H$ has value $G{:}H$. Consider the HACKENBUSH position depicted in Figure 2.5.

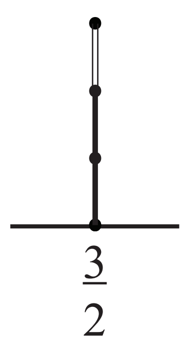


Figure 2.5: The HACKENBUSH position $\frac{3}{2}$.

We can easily check that this position has value $\frac{3}{2}$. However, we can view this as either $2 \boxplus -1$ or $1 \boxplus 1 \boxplus -1$ or $1 \boxplus \frac{1}{2}$. Hence, in HACKENBUSH strings,

$$2 \boxplus -1 \;=\; 1 \boxplus 1 \boxplus -1 \;=\; 1 \boxplus \frac{1}{2} \;=\; \frac{3}{2}$$

and, likewise,

$$2{:}-1 \;=\; 1{:}1{:}-1 \;=\; 1{:}\frac{1}{2} \;=\; \frac{3}{2}.$$

The latter of these representations is evaluated as an example of ordinal sum below.

Example 2.2.4.

$$\begin{aligned}
1 \boxplus \tfrac{1}{2} &= 1{:}\tfrac{1}{2} \\
&= \{0, 1{:}0 \,|\, 1{:}1\} \\
&= \{0, 1{:}0 \,|\, \{0, 1{:}0 \,|\, \cdot\}\} \\
&= \{0, 1 \,|\, \{0, 1 \,|\, \cdot\}\} \qquad \textit{(by Lemma 2.2.2)} \\
&= \{1 \,|\, 2\} \\
&= \tfrac{3}{2}
\end{aligned}$$

Lemma 2.2.5 (Norton's Lemma)**.** *[3, p. 219, 244][9, Thm. 93, p. 210] Let G, H and K be games such that no position of K has value G (and $K \not\cong G$). Then G and $G{:}H$ both have the same order relations with K.*

i.e.

(i) If $G < K$, then $G{:}H < K$;

(ii) If $G > K$, then $G{:}H > K$;

(iii) If $G \parallel K$, then $G{:}H \parallel K$.

Thus, when no position of value K has value G, then $o\,(G - K) = o\,(G{:}H - K)$.

For instance, consider the game $*$. Note that $-1 < * < 1$, but $* \parallel 0$. Since no positions of the games 1, -1 or 0 have value $*$, we know immediately from Norton's Lemma (2.2.5) that for all games H,

$$-1 \;<\; *{:}H \;<\; 1 \quad \text{and} \quad *{:}H \;\parallel\; 0.$$

For example, $*{:}1 = \uparrow *$, $*{:}0 = *$ and $*{:}* = *2$; all of these games fall between -1 and 1 but are confused with zero.

On the other hand, consider the game of $*2$. From $*2$, both Left and Right have options to star, so we can not apply Norton's Lemma. In fact, we can check that while $*$ is confused with $*2$, $*{:}1 = \uparrow * > *2$.

For the game $G = 0$, there is no short game for which we can apply Norton's Lemma since all must contain a 0 position. This is also clear if we consider that $0{:}H = H$, which would imply that if Norton's Lemma held in this case, $o(-K) = o(H - K)$. Clearly, we can choose a game for which this fails!

It is important to note that the value of $G{:}H$ depends on the form of G. In other words, there exists games G, G' and H such that $G = G'$ but $G{:}H \neq G'{:}H$.

Example 2.2.6.
Let $G = \{0, 1 \,|3\} = 2$, $G' = \{1 \,|\cdot\} = 2$ *and* $H = \{0 \,|\cdot\} = 1$. *Then,*

$$G{:}H = \{0, 1, G{:}0 \,|3\} = \{0, 1, 2 \,|3\} = 2\frac{1}{2}$$

and

$$G'{:}H = \{1, G'{:}0 \,|\cdot\} = \{1, 2 \,|\cdot\} = 3.$$

The value of $G{:}H$ also depends only on the value of H but not on its form. This is demonstrated by the Colon Principle.

Lemma 2.2.7. *[3, Colon Principle, p. 219]*

For games G, H and K,

$$H \geq K \implies G{:}H \geq G{:}K$$

and, in particular,

$$H = K \implies G{:}H = G{:}K.$$

Proof. Suppose that $H \geq K$. We wish to show that Left can win $G{:}H - G{:}K$ playing second. Suppose Right moves to some $G{:}H^R - G{:}K$. Since $H \lhd H^R$ and $H \geq K$, $H^R \rhd K$ and so, by induction,

$$G{:}H^R - G{:}K \rhd 0.$$

Likewise, if Right moves to some $G{:}H - G{:}K^L$ we see that $K^L \lhd K \leq H$ and so, by induction,

$$G{:}H - G{:}K^L \rhd 0.$$

Finally, if Right moves in G to either $G^R - G{:}K$ or to $G{:}H - G^L$, then Left can respond with the corresponding moves to $G^R - G^R = 0$ or $G^L - G^L = 0$, respectively. □

Corollary 2.2.8. *For games G and H,*

$$H \geq 0 \implies G{:}H \geq G.$$

Proof. By 2.2.7 and 2.2.2, we have

$$H \geq 0 \implies G{:}H \geq G{:}0 = G.$$

□

Lemma 2.2.9. *For games G and H,*

(i) $\forall G^L \in \mathbf{L}(G),\ G^L \lhd G{:}H$;

(ii) $\forall G^R \in \mathbf{R}(G),\ G{:}H \lhd G^R$;

(iii) $\forall H^L \in \mathbf{L}(H),\ G{:}H^L \lhd G{:}H$;

(iv) $\forall H^R \in \mathbf{R}(H),\ G{:}H \lhd G{:}H^R$.

Proof. This follows from Corollary 1.2.35 and the fact that by Definition 2.2.1, G^L and $G{:}H^L$ are in the set of left options while G^R and $G{:}H^R$ are in the set of right options, respectively, of $G{:}H$. □

The Colon Principle has long been established in dealing with ordinal sums [3]. We suggest that while this holds, a stronger statement can be made about the relationship between branches H and K and their ordinal sums with a base G. The following tells us that the outcome class of the branch difference $H - K$ is the same as that of $G{:}H - G{:}K$. Hence, the relationship shown in the Colon Principle becomes a much stronger if and only if statement.

Theorem 2.2.10 (Branch Outcome)**.** *For games G, H and K,*

$$H \geq K \iff G{:}H \geq G{:}K.$$

Proof. If $G = 0$, then $0{:}H - 0{:}K = H - K$.

Suppose $H \geq K$. Then by Lemma 2.2.7, $G{:}H \geq G{:}K$.

Suppose that $G{:}H \geq G{:}K$. We wish to show that Left can win playing second in the game $H - K$. Suppose that Right moves to some $H^R - K$. Since $G{:}H^R \in \mathbf{R}(G{:}H)$, $G{:}H \lhd G{:}H^R$. Hence, $G{:}K \lhd G{:}H^R$ and so by induction, $K \lhd H^R$. Thus, $H^R - K \rhd 0$. Likewise, if Right moves to some $H - K^L$, then we can show that Left can win playing first in this game since $G{:}K^L \in \mathbf{L}(G{:}K)$ and so $G{:}K^L \lhd G{:}K \leq G{:}H$. Hence, by induction, since $G{:}K^L \lhd G{:}H$, $K^L \lhd H$ and so $H - K^L \rhd 0$. □

Corollary 2.2.11. *For games G, H and K, the outcome classes of $H - K$ and $G{:}H - G{:}K$ are the same.*

Corollary 2.2.12. *(i) $H > K$ if and only if $G{:}H > G{:}K$;*

(ii) $H = K$ if and only if $G{:}H = G{:}K$;

(iii) $H \parallel K$ if and only if $G{:}H \parallel G{:}K$;

(iv) $H < K$ if and only if $G{:}H < G{:}K$;

Finally, we also note that the ordinal sum is associative.

Lemma 2.2.13.

$$(G{:}H){:}K = G{:}(H{:}K)$$

Proof.

$$\begin{aligned}
&(G{:}H){:}K \\
&= \{\mathbf{L}((G{:}H)), (G{:}H){:}\mathbf{L}(K) \,|\, \mathbf{R}((G{:}H)), (G{:}H){:}\mathbf{R}(K)\} \\
&= \{\mathbf{L}(G), G{:}\mathbf{L}(H), (G{:}H){:}\mathbf{L}(K) \,|\, \mathbf{R}(G), G{:}\mathbf{R}(H), (G{:}H){:}\mathbf{R}(K)\} \\
&= \{\mathbf{L}(G), G{:}\mathbf{L}(H), G{:}(H{:}\mathbf{L}(K)) \,|\, \mathbf{R}(G), G{:}\mathbf{R}(H), G{:}(H{:}\mathbf{R}(K))\} \quad \text{(by induction)} \\
&= \{\mathbf{L}(G), G{:}\mathbf{L}(H{:}K) \,|\, \mathbf{R}(G), G{:}\mathbf{R}(H{:}K)\} \\
&= G{:}(H{:}K)
\end{aligned}$$

□

2.2.1 Ordinal Sums Of Numbers

For positive integers k and any number x, the ordinal sum $k{:}x$ can be easily calculated. We will assume that k is in canonical form. As for x, we have already noticed that while its value can change $k{:}x$, its form does not.

For the following, we define a function $f(x) = \min\{0, \lfloor x \rfloor\}$. In this case, $\lfloor x \rfloor$ will return the largest integer that is less than or equal to x.

Lemma 2.2.14. *For $k \in \mathbb{Z}^{>0}$ and x a number,*

$$k{:}x = k - 1 + 2^{f(x)}(1 + x - f(x))$$

where $f(x) = \min\{0, \lfloor x \rfloor\}$.

Proof. Let $y = k - 1 + 2^{f(x)}(1 + x - f(x))$. We will consider play in the game $k{:}x - y$ and show that the second player can always win. We must consider Left playing to (a) $k - 1 - y$; (b) $k{:}x^L - y$; or (c) $k{:}x - y^R$; and Right playing to either (d) $k{:}x^R - y$ or (e) $k{:}x - y^L$.

We note that from Definition 1.2.59, $\operatorname{Hdim}(y) = 1 - f(x)$ and so by Lemma 1.2.60, the left and right incentives in y are $-2^{1-\operatorname{Hdim}(y)} = -2^{f(x)}$. Thus, the left and right options in y are to some

$$y^L = y - 2^{f(x)} = k - 1 + 2^{f(x)}(x - f(x))$$

and

$$y^R = y + 2^{f(x)} = k - 1 + 2^{f(x)}(2 + x - f(x)),$$

respectively.

(a) Suppose Left plays to $k - 1 - y$. Since $f(x) \geq 0$ and $x - f(x) \geq 0$, $2^{f(x)}(1 + x - f(x)) > 0$ and so $y > k - 1$. Thus $k - 1 - y < 0$.

(b) Suppose Left plays to $k{:}x^L - y$. We note that for a number x, if x^L exists, then either (i) $0 \leq x^L = x - 1$ or (ii) $\lfloor x \rfloor \leq x^L < x$. In case (i), since $x, x^L \geq 0$, $f(x) = f(x^L) = 0$. In case (ii), $f(\lfloor x \rfloor) \leq f(x^L) \leq f(x)$ and since $f(\lfloor x \rfloor) = f(x)$,

$f(x^L) = f(x)$. Thus, if this move exists, $f(x^L) = f(x)$ and so

$$\begin{aligned}
&k{:}x^L - y \\
&= k - 1 + 2^{f(x^L)}\left(1 + x^L - f(x^L)\right) - k + 1 - 2^{f(x)}\left(1 + x - f(x)\right) && \text{(by induction)} \\
&= 2^{f(x)}\left(1 + x^R - f(x)\right) - 2^{f(x)}\left(1 + x - f(x)\right) \\
&= 2^{f(x)}\left(1 + x^L - f(x) - 1 - x + f(x)\right) \\
&= 2^{f(x)}\left(x^L - x\right) \\
&< 0 && \left(\text{since } x^L < x\right)
\end{aligned}$$

which Right wins.

(c) Suppose Left plays to $k{:}x - y^R$. As noted above, $y^R = k - 1 + 2^{f(x)}\left(2 + x - f(x)\right)$ and so Right can respond to

$$\begin{aligned}
&k{:}x^R - y^R \\
&= k - 1 + 2^{f(x^R)}\left(1 + x^R - f(x^R)\right) - y^R && \text{(by induction)} \\
&= k - 1 + 2^{f(x^R)}\left(1 + x^R - f(x^R)\right) - k + 1 - 2^{f(x)}\left(2 + x - f(x)\right) \\
&= 2^{f(x^R)}\left(1 + x^R - f(x^R)\right) - 2^{f(x)}\left(2 + x - f(x)\right) \\
&\le 2^{f(x^R)}\left(1 + x^R - f(x^R) - 2 + f(x) - x\right) \\
&\le 2^{f(x^R)}\left(1 + x^R - f(x^R) - 2\right) && \text{(since } f(x) \le x\text{)} \\
&\le 2^{f(x^R)}\left(1 + f(x^R) + 1 - f(x^R) - 2\right) && \left(\text{since } x^R \le f(x^R) + 1\right) \\
&= 0
\end{aligned}$$

which Right wins.

(d) Suppose Right plays to $k{:}x^R - y$. We note that for a number x, if x^R exists, then either (i) $\lfloor x \rfloor = \lfloor x^R \rfloor$ or (ii) $\lfloor x^R \rfloor = \lfloor x \rfloor + 1 \le 0$. In case (i), $f(x^R) = f(x)$ and so

$$\begin{aligned}
&k{:}x^R - y \\
&= k - 1 + 2^{f(x^R)}\left(1 + x^R - f(x^R)\right) - k + 1 - 2^{f(x)}\left(1 + x - f(x)\right) && \text{(by induction)} \\
&= 2^{f(x)}\left(1 + x^R - f(x)\right) - 2^{f(x)}\left(1 + x - f(x)\right) \\
&= 2^{f(x)}\left(x^R - x\right) \\
&> 0 && \left(\text{since } x^R > x\right)
\end{aligned}$$

which Left can win. In case (ii), $f(x^R) = f(x) + 1$ and so

$$\begin{aligned}
&k{:}x^R - y \\
&= k - 1 + 2^{f(x^R)}\left(1 + x^R - f(x^R)\right) - k + 1 - 2^{f(x)}\left(1 + x - f(x)\right) && \text{(by induction)} \\
&= 2^{f(x)+1}\left(1 + x^R - f(x) - 1\right) - 2^{f(x)}\left(1 + x - f(x)\right) \\
&= 2^{f(x)}\left(2x^R - 2f(x) - 1 - x + f(x)\right) \\
&= 2^{f(x)}\left(x^R - x + x^R - f(x) - 1\right) \\
&= 2^{f(x)}\left(x^R - x + x^R - f(x^R)\right) \\
&\geq 2^{f(x)}\left(x^R - x\right) && \left(\text{since } x^R \geq f(x^R)\right) \\
&> 0 && \left(\text{since } x^R > x\right)
\end{aligned}$$

which Left also wins.

(e) Finally, suppose Right plays to $k{:}x - y^L$. As noted above, $y^L = k - 1 + 2^{f(x)}(x - f(x))$. If $x \in \mathbb{Z}$, then $f(x) = x$ and $y^L = k - 1$. Thus Left can play to $k - 1 - y^L = 0$. Otherwise, there exists an x^L with $f(x^L) \leq f(x)$ such that Left can respond to

$$\begin{aligned}
&k{:}x^L - y^L \\
&= k - 1 + 2^{f(x^L)}\left(1 + x^L - f(x^L)\right) - y^L && \text{(by induction)} \\
&= k - 1 + 2^{f(x^L)}\left(1 + x^L - f(x^L)\right) - k + 1 - 2^{f(x)}\left(x - f(x)\right) \\
&\geq 2^{f(x^L)}\left(1 + x^L - x + f(x) - f(x^L)\right) \\
&\geq 2^{f(x^L)}\left(1 + x^L - x\right) && \left(\text{since } f(x) \geq f(x^L)\right) \\
&\geq 2^{f(x^L)}(1 - 1) && \left(\text{since } x^L \geq x - 1\right) \\
&= 0
\end{aligned}$$

which Left wins.

□

When $x \geq 0$, $f(x)$ returns zero, while for negative x, $\lfloor x \rfloor$ is returned. The idea is that if k and x are of the same sign, then the ordinal sum is simply the sum of k and x. However, once the signs are different, this no longer holds.

Corollary 2.2.15. *For $k \in \mathbb{Z}^{>0}$, $n \in \mathbb{Z}^{\geq 0}$ and y a number with $0 \leq y < 1$,* *(i)* $k{:}(n+y) = k+n+y$*;*

(ii) $k{:}(-n+y) = k-1+\frac{1}{2^n}(1+y)$*.*

Proof. Let x be a number and $f(x) = \min\{0, \lfloor x \rfloor\}$.

(i) Consider $k{:}(n+y)$. Then $\lfloor n+y \rfloor = n \geq 0$ and so $f(n+y) = 0$. Hence, Lemma 2.2.14 gives us

$$\begin{aligned} &k{:}(n+y) \\ &\quad = k-1+2^0(1+(n+y)-0) \\ &\quad = k-1+1+n+y \\ &\quad = k+n+y. \end{aligned}$$

(ii) Consider $k{:}(-n+y)$. The case $n=0$ is covered by (i), so we can assume $-n<0$. Then $\lfloor -n+y \rfloor = -n < 0$ and so $f(n+y) = -n$. Hence, Lemma 2.2.14 gives us

$$\begin{aligned} &k{:}(-n+y) \\ &\quad = k-1+2^{-n}(1+(-n+y)-(-n)) \\ &\quad = k-1+\tfrac{1}{2^n}(1-n+y+n)) \\ &\quad = k-1+\tfrac{1}{2^n}(1+y)). \end{aligned}$$

□

2.2.2 Hackenbush-Dimension Applied

We can readily make use of ordinal sums of numbers when considering blue/red Hackenbush strings. In order to understand them, we will need to understand ordinal sums of positive numbers. We will make use of the Hackenbush-dimension of a game to approach this.

Lemma 2.2.16. *For numbers* x, y *with* $x \geq 0$ *and* $-1 \leq y \leq 1$*,*

$$x{:}y = \begin{cases} x+y & \textit{if } \operatorname{Hdim}(x) \leq 1 \textit{ and } xy \geq 0 \\ x+2^{-\operatorname{Hdim}(x)}y & \textit{otherwise} \end{cases}$$

Proof. For the base case, we note that $0{:}0 = 0$.

(i) Suppose $\operatorname{Hdim}(x) = 0$. Then $x = 0$ and so $x{:}y = 0{:}y = y$ by Lemma 2.2.2.

(ii) Second, we suppose $\operatorname{Hdim}(x) = 1$ and $xy \geq 0$. Since $\operatorname{Hdim}(x) = 1$, $\quad x \in \mathbb{Z}^{>0}$. Since $xy \geq 0$ and $x > 0$, $\quad y \geq 0$. Hence, $x{:}y = x + y$ by Corollary 2.2.15.

(iii) Third, we suppose $\operatorname{Hdim}(x) = 1$ and $xy < 0$. Again $x \in \mathbb{Z}^{>0}$, but now $-1 \leq y < 0$. Hence, when applying Lemma 2.2.14 we get

$$x{:}y = x - 1 + 2^{\lfloor y \rfloor}(1 + y - \lfloor y \rfloor).$$

Since $x \in \mathbb{Z}^{>0}$, $\quad x^L = x - 1$ and $x^R = \emptyset$. Thus,

$$x{:}y = \left\{x - 1, x{:}y^L \,\middle|\, x{:}y^R\right\}.$$

Since $-1 \leq y < 0$, $\quad -1 \leq y^L < y < y^R \leq 0$ and so by induction,

$$x{:}y^L = x + 2^{-\operatorname{Hdim}(x)} y^L = x + \frac{y^L}{2}.$$

By Lemma 2.2.2,

$$x{:}y = \left\{x - 1, x + \frac{y^L}{2} \,\middle|\, x, x{:}y^R\right\}.$$

Since $-1 \leq y^L < 0$, $\quad x - 1 < x + \frac{y^L}{2}$. By induction, $x{:}y^R = x + \frac{y^R}{2} \leq x$ since $y^R \leq 0$. Hence,

$$x{:}y = \left\{x + \frac{y^L}{2} \,\middle|\, x + \frac{y^R}{2}\right\} = x + \frac{y}{2}.$$

(iv) Finally, suppose the $\operatorname{Hdim}(x) > 1$. By definition

$$x{:}y = \left\{x^L, x{:}y^L \,\middle|\, x^R, x{:}y^R\right\}.$$

Suppose $x = \frac{p}{2^q}$. Then $\operatorname{Hdim}(x) = q+1$ and $x^L = \frac{p-1}{2^q} = x - \frac{1}{2^q}$ and $x^R = \frac{p+1}{2^q} = x + \frac{1}{2^q}$. Hence, $x^L = x - 2^{-\operatorname{Hdim}(x)+1}$ and $x^R = x + 2^{-\operatorname{Hdim}(x)+1}$. We will consider the cases when (a) $y = 1$, (b) $y = -1$ and (c) $-1 < y < 1$.

(a) If $y = 1$, then $y^L = 0$ and $y^R = \emptyset$, so that $x{:}y^L = x{:}0 = x$. Thus,

$$x{:}y = \left\{x - 2^{-\operatorname{Hdim}(x)+1}, x \,\middle|\, x + 2^{-\operatorname{Hdim}(x)+1}\right\} = \left\{x \,\middle|\, x + 2^{-\operatorname{Hdim}(x)+1}\right\} = x + 2^{-\operatorname{Hdim}(x)}.$$

(b) If $y = -1$, then $y^R = 0$ and $y^L = \emptyset$, so that $x{:}y^R = x{:}0 = x$. Thus,

$$x{:}y = \left\{x - 2^{-\operatorname{Hdim}(x)+1} \,\middle|\, x + 2^{-\operatorname{Hdim}(x)+1}, x\right\} = \left\{x - 2^{-\operatorname{Hdim}(x)+1} \,\middle|\, x\right\} = x - 2^{-\operatorname{Hdim}(x)}.$$

(c) If $-1 < y < 1$, then by induction $x{:}y^L = x + 2^{-\operatorname{Hdim}(x)} y^L$ and $x{:}y^R = x + 2^{-\operatorname{Hdim}(x)} y^R$. Thus,

$$x{:}y = \left\{x - 2^{-\operatorname{Hdim}(x)+1}, x + 2^{-\operatorname{Hdim}(x)} y^L \,\middle|\, x + 2^{-\operatorname{Hdim}(x)+1}, x + 2^{-\operatorname{Hdim}(x)} y^R\right\}.$$

But since $-1 < y < 1$, $\quad -1 \leq y^L < y < y^R \leq 1$. Hence,

$$-2^{-\mathrm{Hdim}(x)+1} < -2^{-\mathrm{Hdim}(x)} \leq 2^{-\mathrm{Hdim}(x)} y^L$$

and

$$2^{-\mathrm{Hdim}(x)+1} > 2^{-\mathrm{Hdim}(x)} \geq 2^{-\mathrm{Hdim}(x)} y^R.$$

Thus,

$$x{:}y = \left\{ x + 2^{-\mathrm{Hdim}(x)} y^L \,\middle|\, x + 2^{-\mathrm{Hdim}(x)} y^R \right\} = x + 2^{-\mathrm{Hdim}(x)} y.$$

□

The following Corollary takes a nice generalization and makes it more specific, which might first seem strange. However, this exact case will come in handy when we begin to look at both the game of LENRES later in this chapter as well as the game of HACKENBUSH.

Corollary 2.2.17. *For $p, s \in \mathbb{Z}^{>0}$ with p odd, and $t \in \mathbb{Z}^{\geq 0}$,*

$$\frac{p}{2^s}{:}\frac{1}{2^t} = \frac{p}{2^s} + \frac{1}{2^{s+t+1}}.$$

Proof.

$$\begin{aligned}
&\tfrac{p}{2^s}{:}\tfrac{1}{2^t} \\
&= \tfrac{p}{2^s} + 2^{-\mathrm{Hdim}\left(\frac{p}{2^s}\right)} \left(\tfrac{1}{2^t}\right) \quad \text{(by Lemma 2.2.16)} \\
&= \tfrac{p}{2^s} + 2^{-(s+1)} \left(\tfrac{1}{2^t}\right) \qquad \left(\text{since Hdim}\left(\tfrac{p}{2^s}\right) = s+1 \text{ by Def. 1.2.59}\right) \\
&= \tfrac{p}{2^s} + \tfrac{1}{2^{s+t+1}}.
\end{aligned}$$

□

Corollary 2.2.18. *Let $S(i) = \sum_{k=1}^{i} x_k$. Then for $x_i \in \mathbb{Z}^{\geq 0}$ with $x_1 > 0$,*

$$\frac{1}{2^{x_1}}{:}\frac{1}{2^{x_2}}{:}\cdots{:}\frac{1}{2^{x_n}} = \sum_{i=1}^{n} \left(\frac{1}{2^{S(i)+i-1}} \right).$$

Proof. We induct on n. When $n = 1$, it is clear that

$$\frac{1}{2^{x_1}} = \frac{1}{2^{S(1)}} = \frac{1}{2^{S(1)+1-1}} = \sum_{i=1}^{1} \left(\frac{1}{2^{S(i)+i-1}} \right).$$

The case where $n = 2$ is covered by Corollary 2.2.17, whereby

$$\begin{aligned}
&\tfrac{1}{2^{x_1}} : \tfrac{1}{2^{x_2}} \\
&= \tfrac{1}{2^{x_1}} + \tfrac{1}{2^{x_1+x_2+1}} && \text{(by Cor. 2.2.17)} \\
&= \tfrac{1}{2^{S(1)}} + \tfrac{1}{2^{S(2)+1}} \\
&= \tfrac{1}{2^{S(1)+1-1}} + \tfrac{1}{2^{S(2)+2-1}} \\
&= \textstyle\sum_{i=1}^{2} \left(\tfrac{1}{2^{S(i)+i-1}}\right).
\end{aligned}$$

Let $n > 2$. Then,

$$\begin{aligned}
&\tfrac{1}{2^{x_1}} : \tfrac{1}{2^{x_2}} : \cdots : \tfrac{1}{2^{x_n}} \\
&= \left(\tfrac{1}{2^{x_1}} : \cdots : \tfrac{1}{2^{x_{n-1}}}\right) : \tfrac{1}{2^{x_n}} && \text{(by Lemma 2.2.13)} \\
&= \left(\textstyle\sum_{i=1}^{n-1} \left(\tfrac{1}{2^{i-1+S(i)}}\right)\right) : \tfrac{1}{2^{x_n}} && \text{(by induction)} \\
&= y + 2^{-\mathrm{Hdim}(y)} \tfrac{1}{2^{x_n}} && \text{(by Lemma 2.2.16)}
\end{aligned}$$

where $y = \sum_{i=1}^{n-1} \left(\frac{1}{2^{S(i)+i-1}}\right) \geq 0$.

Note that if we place y into the form $\frac{p}{2^q}$ where $q \geq 0$ is minimal, we can easily verify that $q = S(n-1) + (n-1) - 1 = S(n-1) + n - 2$. Thus, $\mathrm{Hdim}\,(y) = q + 1 = S(n-1) + n - 1$. Hence,

$$\begin{aligned}
&y + 2^{-\mathrm{Hdim}(y)} \tfrac{1}{2^{x_n}} \\
&= \textstyle\sum_{i=1}^{n-1} \left(\tfrac{1}{2^{S(i)+i-1}}\right) + 2^{-(S(n-1)+n-1)} \tfrac{1}{2^{x_n}} \\
&= \textstyle\sum_{i=1}^{n-1} \left(\tfrac{1}{2^{S(i)+i-1}}\right) + \tfrac{1}{2^{x_n+S(n-1)+n-1}} \\
&= \textstyle\sum_{i=1}^{n-1} \left(\tfrac{1}{2^{S(i)+i-1}}\right) + \tfrac{1}{2^{S(n)+n-1}} \\
&= \textstyle\sum_{i=1}^{n} \left(\tfrac{1}{2^{S(i)+i-1}}\right).
\end{aligned}$$

$\square$

Corollary 2.2.19. *For $x_i \in \mathbb{Z}^{\geq 0}$ with $x_1 > 0$,*

$$\frac{1}{2^{x_1}} : \frac{1}{2^{x_2}} : \cdots : \frac{1}{2^{x_n}} = \sum_{i=1}^{n} \left(\frac{1}{2^{i-1}} \prod_{k=1}^{i} \frac{1}{2^{x_k}} \right).$$

2.2.3 Ordinal Sums From Day n

If we consider the set of games that can be produced by ordinal sums using only those games from $\mathsf{G}\,[n]$, we can see examples of the various relationships above, including the Colon Principle and Norton's Lemma.

Definition 2.2.20. *The* ***ordinal sums from day*** n*, denoted* $\mathbf{OS}_n$*, will be the set of all games formed from ordinals sums of games born by day* n*.*

$$\mathbf{OS}_n = \{G{:}H \ \ s.t. \ \ G, H \in \mathsf{G}\,[n]\}.$$

On day 0, we can only form $0{:}0 = 0$.

Lemma 2.2.21.

$$\mathbf{OS}_0 = \{0\}.$$

However, on day 1, we can now form 13 different games. Table 2.1 lists all ordinal sums that can be formed from games by day 1. These 13 games can be represented by the poset $\langle \mathbf{OS}_1; \leq \rangle$ as pictured in Figure 2.6.

$G{:}H$		**H**			
		1	0	$*$	-1
G	1	2	1	$1*$	$\frac{1}{2}$
	0	1	0	$*$	-1
	$*$	$\uparrow*$	$*$	$*2$	$\downarrow*$
	-1	$-\frac{1}{2}$	-1	$-1*$	-2

Table 2.1: Ordinal sums that can be formed from games born by day 1.

$\mathrm{rcf}\,(G{:}H)$		**H**			
		1	0	$*$	-1
G	1	2	1	1	$\frac{1}{2}$
	0	1	0	0	-1
	$*$	0	0	0	0
	-1	$-\frac{1}{2}$	-1	-1	-2

Table 2.2: The reduced canonical forms of ordinal sums that can be formed from games born by day 1.

What is interesting to note about $\mathbf{OS}_1$ is that it appears much like $\mathsf{G}\,[1]$, but with a copy of $\mathsf{G}\,[1]$ replacing each of the elements 1, $*$ and -1, while zero remains on its own. While it might first seem strange that zero is acting differently, this is to be

	rcf $(G{:}H)$	**H**												
		2	1	$1*$	$\frac{1}{2}$	$\{1\,|\,0\}$	$\{1\,|\,*\}$	$\{1\,|\,0,*\}$	$\uparrow$	$\uparrow*$	0	$*$	$*2$	±1
G	2	4	3	3	$\frac{5}{2}$	$\{3\,|\,2\}$	$\{3\,|\,2\}$	$\{3\,|\,2\}$	2	2	2	2	2	$\{3\,|\,\frac{3}{2}\}$
	1	3	2	2	$\frac{3}{2}$	$\{2\,|\,1\}$	$\{2\,|\,1\}$	$\{2\,|\,1\}$	1	1	1	1	1	$\{2\,|\,\frac{1}{2}\}$
	$1*$	1	1	1	1	1	1	1	1	1	1	1	1	1
	$\frac{1}{2}$	$\frac{7}{8}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{5}{8}$	$\{\frac{3}{4}\,|\,\frac{1}{2}\}$	$\{\frac{3}{4}\,|\,\frac{1}{2}\}$	$\{\frac{3}{4}\,|\,\frac{1}{2}\}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\{\frac{3}{4}\,|\,\frac{1}{4}\}$
	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$
	$\{1\,|\,*\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$
	$\{1\,|\,0,*\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$
	$\uparrow$	0	0	0	0	0	0	0	0	0	0	0	0	0
	$\uparrow*$	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	2	1	1	$\frac{1}{2}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	$\{1\,|\,0\}$	0	0	0	0	0	±1
	$*$	0	0	0	0	0	0	0	0	0	0	0	0	0
	$*2$	0	0	0	0	0	0	0	0	0	0	0	0	0
	±1	±1	±1	±1	±1	±1	±1	±1	±1	±1	±1	±1	±1	±1
	$\downarrow*$	0	0	0	0	0	0	0	0	0	0	0	0	0
	$\downarrow$	0	0	0	0	0	0	0	0	0	0	0	0	0
	$\{0,*\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$
	$\{*\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$
	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$	$\{0\,|\,{-1}\}$
	$-\frac{1}{2}$	$-\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{3}{8}$	$\{-\frac{1}{4}\,|\,{-\frac{1}{2}}\}$	$\{-\frac{1}{4}\,|\,{-\frac{1}{2}}\}$	$\{-\frac{1}{4}\,|\,{-\frac{1}{2}}\}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\{-\frac{1}{4}\,|\,{-\frac{3}{4}}\}$
	$-1*$	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
	-1	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{4}$	$\{-\frac{1}{2}\,|\,{-1}\}$	$\{-\frac{1}{2}\,|\,{-1}\}$	$\{-\frac{1}{2}\,|\,{-1}\}$	-1	-1	-1	-1	-1	$\{-\,|\,{-2}\}$
	-2	$-\frac{5}{4}$	$-\frac{3}{2}$	$-\frac{3}{2}$	$-\frac{7}{4}$	$\{-\frac{3}{2}\,|\,{-2}\}$	$\{-\frac{3}{2}\,|\,{-2}\}$	$\{-\frac{3}{2}\,|\,{-2}\}$	-2	-2	-2	-2	-2	$\{-\frac{3}{2}\,|\,{-3}\}$

Table 2.3: The reduced canonical forms of ordinal sums that can be formed from games born by day 2.

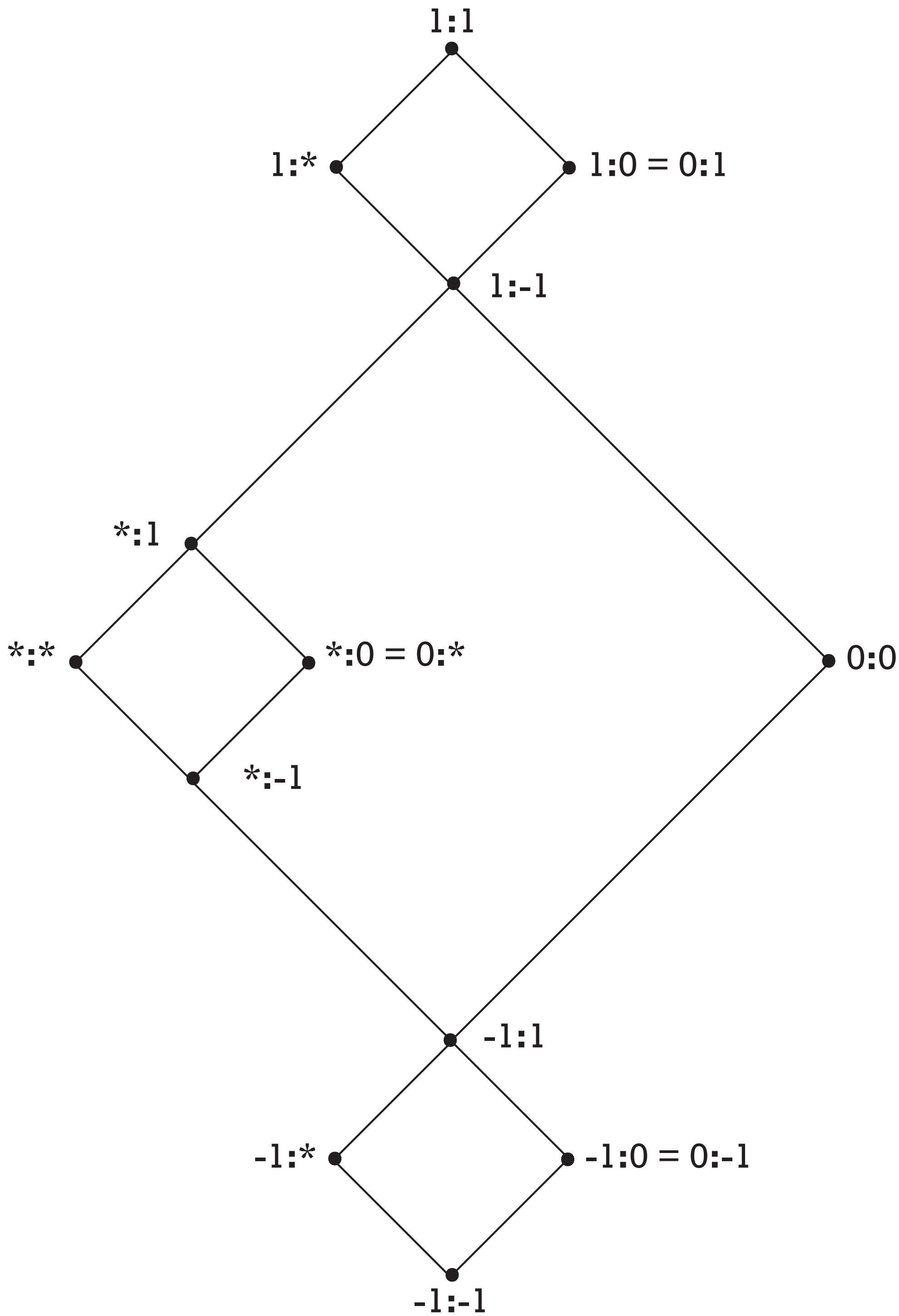


Figure 2.6: Poset of $\mathbf{OS}_1$.

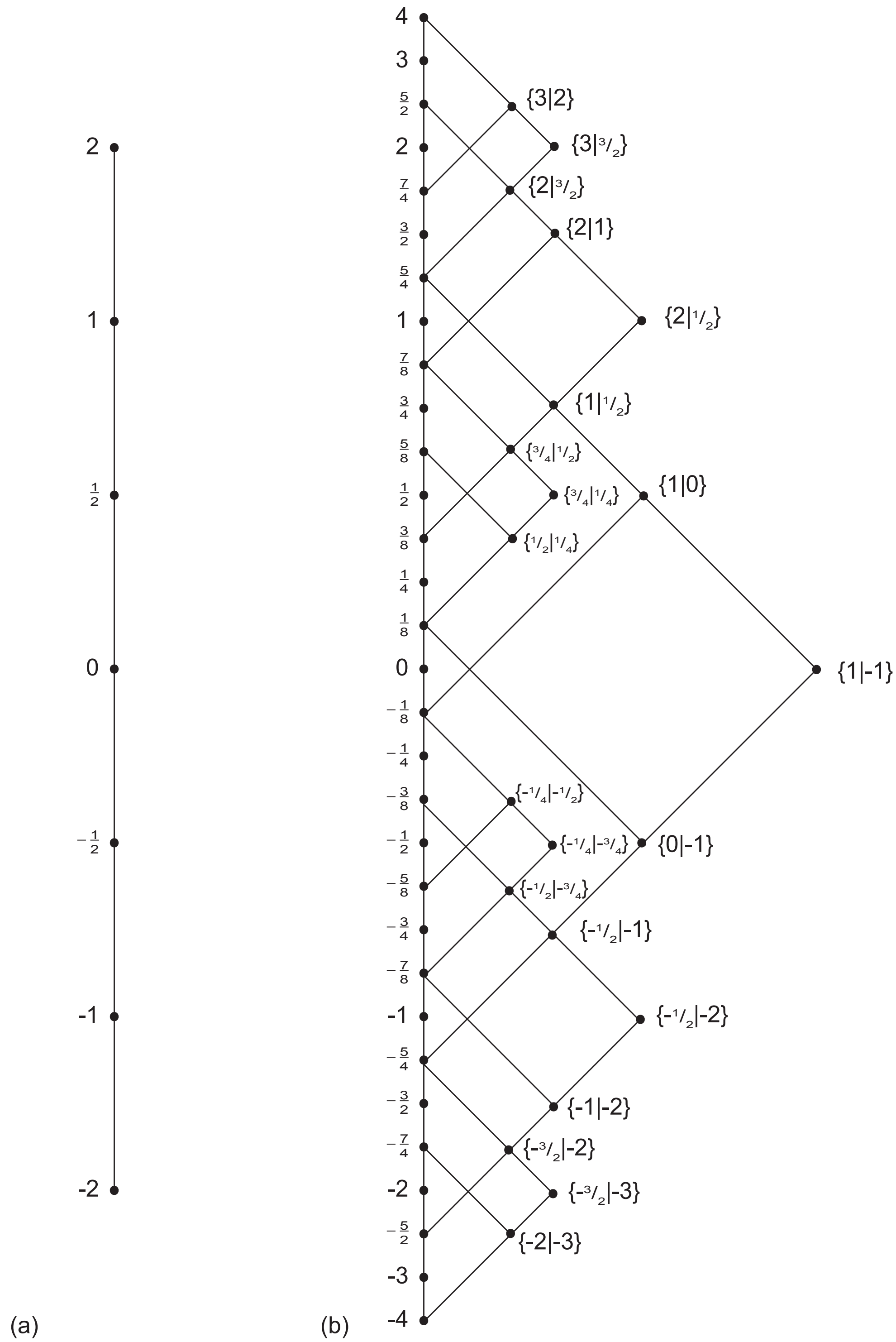


Figure 2.7: Posets of reduced canonical form of (a) $\mathbf{OS}_1$ and (b) $\mathbf{OS}_2$.

expected. Since $G{:}0 = 0{:}G = G$, all games of the form $0{:}G$, aside from $0{:}0$, appear in one of the other copies of $\mathsf{G}[1]$.

In $\mathbf{OS}_2$, there are 427 elements that are built from the 22 elements of $\mathsf{G}[2]$. If we were to look at $\mathbf{OS}_2$, we would find that it is not only zero that causes problems. Other pairs (G, H) have counterparts (G', H') that produce a similar ordinal sum. For instance,

$$*{:}{*2} = {*2}{:}{*} = {*3}$$

and

$$*{:}2 = \uparrow *{:}1.$$

Each is to be expected due to the fact that ordinal sums are associative. For the latter, we note that $2 = 1{:}1$ and $\uparrow = *{:}1$. Hence, by Lemma 2.2.13,

$$*{:}2 = *{:}(1{:}1) = (*{:}1){:}1 = \uparrow *{:}1.$$

To get a general idea for the values that crop up, we can take a look at the poset of the reduced canonical forms of the games in $\mathbf{OS}_n$. For $\mathbf{OS}_2$, this poset contains only 51 elements, of which 31 are numbers and the other 21 are switches.[1]

For any day, the largest and smallest elements of $\mathbf{OS}_n$ will be those of the form $n{:}n = 2n$ and $(-n){:}(-n) = -2n$, respectively.

Tables 2.2 and 2.3 give the reduced canonical forms of the ordinal sums produced from games with birthday 1 and 2, respectively. Their posets are depicted in Figure 2.7. While the reduced canonical forms of $\mathbf{OS}_1$ form a total order, those of $\mathbf{OS}_2$ do not even form a lattice.

[1]Recall that a switch is a game of the form $\{a\,|\,b\}$ where $a > b$ are numbers. Switches of the form $\{a\,|\,{-a}\}$ are represented as $\pm a$.

2.3 Ordinal Sum Applied

We now introduce the games of LENRES and SHOVE. In these games, we will find interesting applications of ordinal sums. Elwyn Berlekamp has often said at conferences and in private discussion that ordinal sums are used to make hard games simpler, since $G{:}H$ behaves very much like G. The challenge, of course, is finding the ordinal sum within a game. While each game that we consider is a fascinating game in and of itself, it is primarily worth considering due to the fact that its values can be expressed as ordinal sums. Most interesting is that, in each case, this is a fact which is heavily camouflaged. These example games demonstrate the fact that ordinal sums might be useful in ways that are not immediately obvious when first considering a game.

2.3.1 LENRES

The game of LENRES was created by Richard Nowakowski in a fourth year Introduction to Game Theory class. It was originally considered by Richard Nowakowski and Paul Ottaway who looked at it within a class of one dimensional games that they were interested in. They were able to analyze certain simple positions, yet no solid conjectures were formed [22]. It was utilized by Nowakowski as an in-class example as it was a game that was easy to explain, yet had interesting structure. I became interested in the game after a group of students approached me after working on it and failing to determine the value of the general game with ones and zeros.[2]

The game of LENRES is played on a sequence of integers. On his turn, Left is allowed to move any integer from its current position, to replace (cover) any other integer to the right (east) of it that is larger than, or up (north) from, it. Right is allowed to move to cover to the right (east) any integer that is less than, or down (south) from, it. Thus, the name comes from an acronym of the ruleset: Left-East-North, Right-East-South.

The general game of LENRES can be played with any string of integers. However, we will consider only the solution to games played on strings of zeros and ones. Thus,

[2] Cheers to Danielle Cox and cohorts!

the only options for Left will be to move a 0 to cover a 1 to the right of it, and the only options for Right will be to take a 1 and cover a 0 to its right.

From the position LenRes[1011], Left has a move to either LenRes[101] or LenRes[110] while Right only has one move to LenRes[111]. Figure 2.8 gives the game tree for the position LenRes[0101].

For the following lemmas, we will use Greek letters to represent arbitrary strings of zeros and ones.

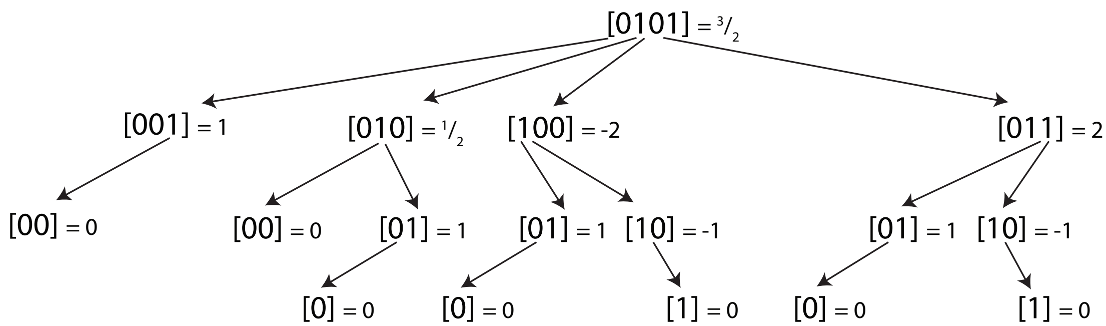


Figure 2.8: The game tree for LENRES position LenRes[0101].

We introduce LENRES in this section due to the fact that positions containing only zeros and ones can be represented as an ordinal sum. This fact is heavily masked. Through a series of reductions, from Lemma 2.3.1 through Lemma 2.3.12, we move towards a representation of any LENRES position with a leading 0 that is of the following format:

$$\text{LenRes}[010^{x_n}10^{x_{n-1}}\cdots 10^{x_2}10^{x_1}1^z].$$

As the final result in this section, Corollary 2.3.14, we then state that this can be represented as the ordinal sum

$$z + \left(\frac{1}{2^{x_1}}\colon\frac{1}{2^{x_2-1}}\colon\frac{1}{2^{x_3-1}}\colon\cdots\colon\frac{1}{2^{x_n-1}}\right).$$

We first note that since all moves are to the right within the string of integers, having a player's integer (i.e. 0 for Left, 1 for Right) positioned first (i.e. left-most) in the string is advantageous.

Lemma 2.3.1.

$$\text{LenRes}[0\alpha] \geq \text{LenRes}[\alpha] \geq \text{LenRes}[1\alpha]$$

Proof. We need only show that LenRes$[0\alpha] \geq$ LenRes$[\alpha]$. Left can win playing second in LenRes$[0\alpha] -$ LenRes$[\alpha]$ by simply ignoring the existence of the leading 0 and mirroring Right's play. Thus LenRes$[0\alpha] \geq$ LenRes$[\alpha]$.

As an immediate result of this (when restricted to strings of zeros and ones), we have by symmetry that LenRes$[1\alpha] \leq$ LenRes$[\alpha]$.

□

Next, we note that the advantage that a player obtains by having the leading integer in the string is not just due to the presence of the player's integer (i.e. zeros for Left and ones for Right), but its position. Consider the game LenRes[1] and the game LenRes[10]. The former has value 0, as neither player has a move, and the latter has value -1 as Right has the only move, to LenRes$[1] = 0$. Thus, LenRes$[1] <$ LenRes[10] despite the fact that LenRes[10] has a zero (i.e. Left's integer) in it, where the former does not. Even in positions where adding the zero to the middle of a string gives Left a move that he did not have before, such as LenRes[11] versus LenRes[101], we see that it may not help Left. In this case, LenRes$[11] = 0$ and LenRes$[101] = -\frac{1}{2}$.

Furthermore, given the presence of a player's integer, the further to the left within the string it is placed, the better. Intuitively, this is clear; since players move their integers to the right, a start position further to the left only offers more possible moves.

Lemma 2.3.2.
LenRes$[\alpha 01\beta] \geq$ LenRes$[\alpha 10\beta]$

Proof. Consider the game LenRes$[\alpha 01\beta] -$ LenRes$[\alpha 10\beta]$. No matter which component Right moves in, Left has a response that either allows him to move to the same position in the other component, giving 0, or to LenRes$[\alpha' 01\beta'] -$ LenRes$[\alpha' 10\beta']$, which is non-negative by induction. Hence, Left can always win playing second. □

Corollary 2.3.3.

(i) LenRes$[\alpha 0\beta\gamma] \geq$ LenRes$[\alpha\beta 0\gamma]$

(ii) LenRes$[\alpha\beta 1\gamma] \geq$ LenRes$[\alpha 1\beta\gamma]$

Proof. Both statements follow from repeated application of Lemma 2.3.2. □

The following two lemmas work to together to demonstrate that the left-most move is always best. Lemma 2.3.4 tells us that we should always move your left-most integer, and Lemma 2.3.6 gives us that the best move is to cover the first option available to the right. Combining these, it is always best to move your left-most integer to cover the left-most integer belonging to the opposition that is available to (i.e. to the right of) you.

Lemma 2.3.4.

$$\mathrm{LenRes}[\alpha 0\beta 0\gamma 1\delta] \geq \mathrm{LenRes}[\alpha 01\beta\gamma 0\delta]$$

Proof. By Corollary 2.3.3,

$$\mathrm{LenRes}[\alpha 0\beta 0\gamma 1\delta] \geq \mathrm{LenRes}[\alpha 01\beta 0\gamma\delta] \geq \mathrm{LenRes}[\alpha 01\beta\gamma 0\delta].$$

□

Corollary 2.3.5.

$$\mathrm{LenRes}[\alpha 1\beta 1\gamma 0\delta] \leq \mathrm{LenRes}[\alpha 10\beta\gamma 1\delta]$$

The implications of the above Lemma and Corollary are depicted in Figure 2.3.1. In the game LenRes[$\alpha 01\beta 0\gamma 1\delta$], it is always better for Left to move her left-most integer to LenRes[$\alpha 0\beta 0\gamma 1\delta$] than to move any other integer to some LenRes[$\alpha 01\beta 0\gamma 0\delta$], as shown by Lemma 2.3.4. Similarly, from LenRes[$\alpha 10\beta 1\gamma 0\delta$] it is better for Right to move his left-most integer than any later one, as shown by Corollary 2.3.5.

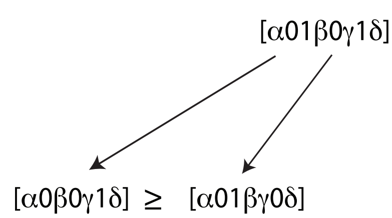


Figure 2.9: Depiction of Lemma 2.3.4, which shows that it is always best to move your left-most integer.

The following tells us that the best move is to cover the first available option. That is, from LenRes[$\alpha 01\beta 1\gamma$] it is better for Left cover the first option and move to LenRes[$\alpha 0\beta 1\gamma$] than cover a later 1 by moving to some LenRes[$\alpha 1\beta 0\gamma$], as depicted in Figure 2.3.1. Likewise, it is better for Right to cover the first available option; From LenRes[$\alpha 10\beta 0\gamma$], it is better for Right to move to LenRes[$\alpha 1\beta 0\gamma$] than to LenRes[$\alpha 0\beta 1\gamma$].

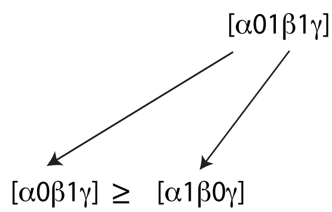


Figure 2.10: Depiction of Lemma 2.3.6, which shows that it is always best to cover the first option.

Lemma 2.3.6.

$$\text{LenRes}[\alpha 0\beta 1\gamma] \geq \text{LenRes}[\alpha 1\beta 0\gamma]$$

Proof. By Corollary 2.3.3,

$$\text{LenRes}[\alpha 0\beta 1\gamma] \geq \text{LenRes}[\alpha\beta 10\gamma] \geq \text{LenRes}[\alpha 1\beta 0\gamma].$$

□

Theorem 2.3.7. *In the game of* LENRES, *it is always best to move your left-most piece and cover the first available opponent's piece to the right of it.*

Proof. This follows directly from Lemmas 2.3.4 and 2.3.6 and Corollary 2.3.5. □

While Left is happy to play in a string with a leading 0, we note that any extra leading zeros do not improve his situation.

Lemma 2.3.8.

$$\text{LenRes}[00\alpha] = \text{LenRes}[0\alpha]$$

Proof. We must show that the second player can always win in LenRes[00α]$-$LenRes[0α]. By Theorem 2.3.7, we know that neither player will choose to move the leading zeros past any ones. Thus, they will either move a zero to cover the first one encountered (if it exists) or move within α. Thus, the second player can always respond in the other component to leave a position of the same form, which he can win by induction. □

Corollary 2.3.9.

(i) $\text{LenRes}[0^x\alpha] = \text{LenRes}[0\alpha]$

(ii) $\text{LenRes}[1^x\alpha] = \text{LenRes}[1\alpha]$

Through a series of reductions, we will show that we can always represent any LENRES position starting with a 0 in the following format:

$$\text{LenRes}[010^{x_n}10^{x_{n-1}}\cdots 10^{x_2}10^{x_1}1^z].$$

Lemma 2.3.10.

$$\text{LenRes}[01\alpha] > \text{LenRes}[\alpha]$$

Proof. Consider $\text{LenRes}[01\alpha] - \text{LenRes}[\alpha]$; We must show that Left can always win. Any move in α can be mirrored in the other component. If Right moves the one in the first component to cover some zero in α, then Left can respond by moving the leading zero to that position, thereby leaving $\text{LenRes}[\alpha] - \text{LenRes}[\alpha] = 0$. Left going first can move to $\text{LenRes}[0\alpha] - \text{LenRes}[\alpha]$ which is non-negative by Lemma 2.3.1. □

Lemma 2.3.11.

$$\text{LenRes}[010\alpha] > \text{LenRes}[0\alpha]$$

Proof. Consider the game $\text{LenRes}[010\alpha] - \text{LenRes}[0\alpha]$. In this, Left can win by playing to $\text{LenRes}[00\alpha] - \text{LenRes}[0\alpha] = 0$ by Lemma 2.3.8. If Right plays in $\text{LenRes}[0\alpha]$, Left can win by ignoring the leading 01 in the first component and mirroring Right's play, to some position $\text{LenRes}[010\alpha'] - \text{LenRes}[0\alpha']$, that she wins by induction. If Right plays in the first component to $\text{LenRes}[01\alpha] - \text{LenRes}[0\alpha]$, then Left can respond to $\text{LenRes}[00\alpha] - \text{LenRes}[0\alpha] = 0$. □

Lemma 2.3.12.

$$\text{LenRes}[0\alpha 110\beta] = \text{LenRes}[0\alpha 1\beta]$$

Proof. Consider $\text{LenRes}[0\alpha 110\beta] - \text{LenRes}[0\alpha 1\beta]$. We will consider the when (i) α is either empty (or a string of zeros), and (ii) α is non-empty. By Lemma 2.3.8, we can assume $\alpha = 1\alpha'$. By Theorem 2.3.7, we know that if a good move exists for either player, it is is by moving the left-most integer and covering the first option. Thus, these cases cover all best moves.

(i) We will first consider the case where $\alpha = \emptyset$. If either player moves in the first component, the opponent can reply in the same component to $\text{LenRes}[01\beta] -$

LenRes$[01\beta] = 0$. Suppose $\beta = 1^a0\gamma$ where $a \geq 0$, so that our game begins as LenRes$[01101^a0\gamma]$ − LenRes$[011^a0\gamma]$. A move by Left in the second component to LenRes$[01101^a0\gamma]$ − LenRes$[011^a\gamma]$ can be replied to by Right in first component to LenRes$[01101^a\gamma]$ − LenRes$[011^a\gamma] = 0$ by induction. Suppose Right moves in the second component to LenRes$[01101^a0\gamma]$−LenRes$[01^a0\gamma]$. If $a > 0$, then Left can reply in the first component to LenRes$[01101^{a-1}0\gamma]$ − LenRes$[01^a0\gamma] = 0$ by induction. If $a = 0$, then Left can play to LenRes$[0101^a0\gamma]$ − LenRes$[01^a0\gamma]$ which she wins by Lemma 2.3.11.

(ii) We next consider the case when $\alpha = 1\alpha'$. In this case, if either player moves the leading zero, mirror moves are available (moving the other leading zero in the opposite component). Thus, we only need to consider the left-most ones. Either (a) α' contains zero, or (b) $\alpha = 1^a$.

If there is a zero in α', then moving the leading one in α of either component is met with the corresponding move. Suppose $\alpha = 1^a$ with $a > 0$, then we have the case LenRes$[01^a110\beta]$ − LenRes$[01^a1\beta]$. If Right moves in the first component to LenRes$[01^a11\beta]$ − LenRes$[01^a1\beta]$, then Left can reply in the same to LenRes$[01^a1\beta]$ − LenRes$[01^a1\beta] = 0$. If $\beta = 1^b0\gamma$ with $b \geq 0$ and Left moves from LenRes$[01^a1101^b0\gamma]$− LenRes$[01^a11^b0\gamma]$ in the second component to LenRes$[01^a1101^b0\gamma]$−LenRes$[01^a11^b\gamma]$, then Right can reply in the first to LenRes$[01^a1101^b\gamma]$ − LenRes$[01^a11^b\gamma] = 0$ by induction. ∎

Theorem 2.3.13.

$$\text{LenRes}[010^{x_n}10^{x_{n-1}}\cdots 10^{x_2}10^{x_1}1^z] = z + \sum_{i=1}^{n}\left(\prod_{k=1}^{i}\frac{1}{2^{x_k}}\right).$$

Proof. Consider the game LenRes$[010^{x_n}10^{x_{n-1}}\cdots 10^{x_2}10^{x_1}1^z]$. From Theorem 2.3.7,we know that the best move for Left is to

$$\begin{aligned}
&\text{LenRes}[00^{x_n}10^{x_{n-1}}\cdots 10^{x_2}10^{x_1}1^z]\\
=\;& \text{LenRes}[010^{x_{n-1}}\cdots 10^{x_2}10^{x_1}1^z] \quad \text{(by Cor. 2.3.9)}\\
=\;& z + \textstyle\sum_{i=1}^{n-1}\left(\prod_{k=1}^{i}\frac{1}{2^{x_k}}\right) \quad \text{(by induction)}
\end{aligned}$$

and the best move for Right is to

$$\begin{aligned}
&\text{LenRes}[010^{-1+x_n}10^{x_{n-1}}\cdots 10^{x_2}10^{x_1}1^z]\\
=\;& z + \textstyle\sum_{i=1}^{n-1}\left(\prod_{k=1}^{i}\frac{1}{2^{x_k}}\right) + 2\prod_{k=1}^{n}\frac{1}{2^{x_k}} \quad \text{(by induction)}
\end{aligned}$$

Thus, our answer is the simplest value between these, which is

$$\begin{aligned} & z+\sum_{i=1}^{n-1}\left(\prod_{k=1}^{i} \tfrac{1}{2^{x_k}}\right)+2\prod_{k=1}^{n} \tfrac{1}{2^{x_k}} \\ = \quad & z+\sum_{i=1}^{n}\left(\prod_{k=1}^{i} \tfrac{1}{2^{x_k}}\right) \end{aligned}$$

□

This gives us a nice handle on the value of LENRES positions. However, we will see that it is in fact a value more easily expressed in terms of ordinal sum.

Corollary 2.3.14.

$$\mathrm{LenRes}[010^{x_n}10^{x_{n-1}}\cdots 10^{x_2}10^{x_1}1^z] = z+\left(\frac{1}{2^{x_1}}:\frac{1}{2^{x_2-1}}:\frac{1}{2^{x_3-1}}:\cdots:\frac{1}{2^{x_n-1}}\right).$$

Proof. To show that this follows from Theorem 2.3.13, we simply need to show that

$$z+\sum_{i=1}^{n}\left(\prod_{k=1}^{i}\frac{1}{2^{x_k}}\right) = z+\left(\frac{1}{2^{x_1}}:\frac{1}{2^{x_2-1}}:\frac{1}{2^{x_3-1}}:\cdots:\frac{1}{2^{x_n-1}}\right).$$

From Lemma 2.2.14, we know that

$$\frac{1}{2^q}:\frac{1}{2^t} = \frac{1}{2^q}+\frac{1}{2^{q+t+1}}.$$

□

As an example of the reductions and techniques from this section, we give the following analysis of the LENRES position LenRes[00011000111001000011].

Example 2.3.15.

$$\begin{aligned} & \mathrm{LenRes}[00011000111001000011] \\ = \quad & \mathrm{LenRes}[0001100011000011] && \textit{(by Lem. 2.3.12)} \\ = \quad & \mathrm{LenRes}[00011000100011] && \textit{(by Lem. 2.3.12)} \\ = \quad & \mathrm{LenRes}[000100100011] && \textit{(by Lem. 2.3.12)} \\ = \quad & \mathrm{LenRes}[0100100011] && \textit{(by Lem. 2.3.9)} \\ = \quad & 2+\tfrac{1}{2^3}:\tfrac{1}{2^2} && \textit{(by Cor. 2.3.14)} \\ = \quad & 2+\tfrac{1}{2^3}+\tfrac{1}{2^{3+2+1}} && \textit{(by Lem. 2.2.14)} \\ = \quad & 2+\tfrac{9}{64}. \end{aligned}$$

2.3.2 *When* PUSH *Comes To* SHOVE

The games of PUSH and SHOVE were introduced in *Lessons in Play* as tools for demonstrating various properties of games [2]. Both games are played on one or more finite strips of squares. Each square can either be empty or occupied by a black or white piece. Left moves by choosing a black piece and moving it one square to the left, Right by choosing a white piece and doing the same, with slightly different consequences based on which game one is playing.

In the game of PUSH, at most one piece can occupy a square at a time. So any pieces immediately adjacent and to the left of that which is being moved are also pushed one square to the left. Once a piece is pushed passed the left end of the strip, it is removed from play.

shove ['shəv] to push in a rough, careless or hasty manner [19].

In the game of SHOVE, the effect of the move is stronger; when a piece is moved to the left one square, all pieces to the left of it on the board are also moved over one square.

For a position of either game, in text, we will utilize an X to represent a black piece, an O for a white piece, and an underscore (_) for an empty square if not drawn on a board. As examples of the games, Figure 2.11 represents the moves from the positions $[X\_OX]$ in both the games of (a) PUSH and (b) SHOVE.

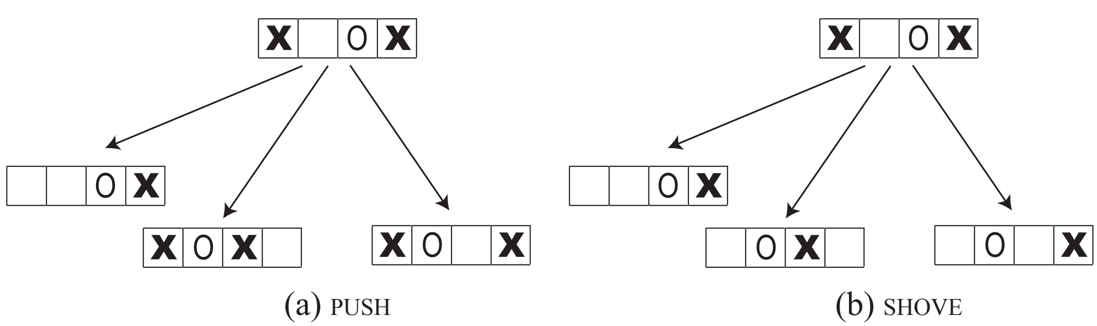


Figure 2.11: Examples of the position $[X\_OX]$ played under the rulesets for (a) PUSH and (b) SHOVE.

When these two games were introduced, no formula was known to easily solve a PUSH position. However, its variant SHOVE was solved. From the authors of *Lessons in Play* [2], we adopt the following notation.

Notation 2.3.16. *[2, p. 98] For the games of* PUSH *or* SHOVE, *we define the following for a specific position G on a single strip:*

$$
\begin{aligned}
n &= \textit{(total number of pieces) - 1} \\
i &= \textit{piece number, given by number of pieces to the right} \\
c(i) &= \begin{cases} \textit{1} & \textit{if } i^{th} \textit{ piece is black} \\ \textit{-1} & \textit{if the } i^{th} \textit{ piece is white} \end{cases} \\
k &= \max\{i \ \textit{s.t.}\ \forall j \le i, c(j) = c(0)\} \\
& \ \textit{i.e. the length of the last color block on the right} \\
p(i) &= \textit{position of the } i^{th} \textit{ piece, from the left edge of the board} \\
r(i) &= \begin{cases} \textit{0} & \textit{if } i \le k \\ \textit{i-k} & \textit{otherwise} \end{cases} \\
& \ \textit{i.e. the number of pieces to i's right not including those in} \\
& \ \textit{the last color block on the right}
\end{aligned}
$$

2.3.2.1 SHOVE

For a SHOVE position G and through application of the above notation, Albert, Nowakowski and Wolfe were able to identify that it had a value as described below.

Proposition 2.3.17. *[2, Thm. 5.32, p. 98] Adopting the notation of 2.3.16, the value of a* SHOVE *position G is*

$$G = \sum_{i=0}^{n} \left(c(i) \frac{p(i)}{2^{r(i)}} \right).$$

While this nicely solves the game, we proceed to another solution, that gives a simple reconstruction of the game as a sum of HACKENBUSH strings. As such, it also serves our purpose in embodying hidden ordinal sums. We will continue to utilize the notation of Albert, Nowakowski and Wolfe [2] and also contribute the following.

Definition 2.3.18. *For the games of* PUSH *or* SHOVE*, we define the following for a specific position G on a single strip, where i, n and the functions p, r and c are as defined in 2.3.16.*

$$\begin{aligned} b(i) &= b(i-1){:}c(i) \\ &\quad \textit{with } b(0) = 0 \\ B(i) &= B(i-1){:}c(i) = c(0){:}b(i) \\ &\quad \textit{with } B(0) = c(0) \end{aligned}$$

We claim that a SHOVE position with $n+1$ total pieces, which Albert, Nowakowski and Wolfe [2] demonstrated to have value

$$F_1(n) = \sum_{i=0}^{n} \left(c(i) \frac{p(i)}{2^{r(i)}} \right),$$

can also be expressed as a single HACKENBUSH string and that its value can be alternatively expressed as

$$F_2(n) = p(n)B(n) + \sum_{i=0}^{n-1} \left(p(i) - p(i+1) \right) B(i).$$

The latter representation comes from consideration of the former as a sum of HACKENBUSH strings and through simplification of these sums. We will refer to the former method, from Albert, Nowakowski and Wolfe [2], as the 'first method' and shall refer to the latter as the 'ordinal sum method'.

We claim that the first method inherently makes use of Hackenbush-dimension. However, it does so by making use of the Hackenbush-dimension of $B(i)$. We claim that $\mathrm{Hdim}\,(B(i)) = r(i) + 1$.

Lemma 2.3.19. *For a* SHOVE *position, for all $0 \leq k \leq n$,*

$$\mathrm{Hdim}\,(B(k)) = r(k) + 1$$

Proof. By Definition 1.2.59, $\mathrm{Hdim}\,(B(0)) = 1$ since $B(0) \in \{-1, 1\}$. Thus $\mathrm{Hdim}\,(B(0)) = r(0) + 1$ since $r(0) = 0$.

Let $p < n$ be such that $\forall k \leq p$,

$$m = c(k) \neq c(p+1) = -m$$

where $m \in \{-1, 1\}$. Then $\forall k \leq p$, $r(k) = 0$ since it and all pieces to its right are the same color. We also see that $B(0) = c(0) = m$, $B(1) = m{:}m = 2m$, and so on with $B(k) = km{:}m = (k+1)m$. Since these are all nonzero integers, then for $k \leq p$, $\operatorname{Hdim}(B(k)) = 1 = r(k) + 1$.

But $r(p+1) = 1$ and $\forall k > p$, $r(k) = k - p$. Also,

$$B(p+1) = B(p){:}c(p+1) = pm{:}{-m} = pm - \frac{m}{2} = \frac{2pm - m}{2}.$$

Since $m \in \{-1, 1\}$, $2pm - m$ is odd, so $\operatorname{Hdim}(B(p+1)) = 2$. And $r(p+1) = (p+1) - p = 1$, so that $\operatorname{Hdim}(B(p+1)) = r(p+1) + 1$.

Let $k > p + 1$ and suppose that for all $i \leq k$, $\operatorname{Hdim}(B(k)) = r(k) + 1$. We will consider the position $k + 1$ and show, by induction on k, that $\operatorname{Hdim}(B(k+1)) = r(k+1) + 1$. Here, since $r(k) = k - p > (p+1) - p = 1$ so that inductively $\operatorname{Hdim}(B(k)) > 2$, by Lemma 2.2.16,

$$B(k+1) = B(k){:}c(k+1) = B(k) + 2^{-\operatorname{Hdim}(B(k))} c(k+1).$$

By induction, we know that $\operatorname{Hdim}(B(k)) = r(k) + 1 = k + 1 - p$, so

$$B(k+1) = B(k) + \frac{c(k+1)}{2^{k+1-p}}.$$

Thus, $\operatorname{Hdim}(B(k+1)) = (k+1-p) + 1 = r(k+1) + 1$. $\square$

The following Lemma states that the first method, giving $F_1(n)$, and the ordinal sum method, giving $F_2(n)$, give values that are equivalent.

Lemma 2.3.20.
For a SHOVE *position,*

$$\sum_{i=0}^{n} \left(c(i) \frac{p(i)}{2^{r(i)}} \right) \quad = \quad p(n)B(n) + \sum_{i=0}^{n-1} \left(p(i) - p(i+1) \right) B(i).$$

Proof. Let

$$F_1(n) = \sum_{i=0}^{n} \left(c(i) \frac{p(i)}{2^{r(i)}} \right),$$

and

$$F_2(n) = p(n)B(n) + \sum_{i=0}^{n-1} \left(p(i) - p(i+1) \right) B(i).$$

We must show that for all n (i.e. for games with total number of pieces $n+1$), $F_1(n) = F_2(n)$.

If $n = 0$, then

$$F_1(0) = \frac{c(0)p(0)}{2^{r(0)}} = c(0)p(0)$$

since $r(0) = 0$. And, since $B(0) = c(0)$,

$$F_2(0) = p(0)B(0) = p(0)c(0).$$

Suppose that $n > 0$. Then

$$F_1(n) = \frac{c(n)p(n)}{2^{r(n)}} + \sum_{i=0}^{n-1} \frac{c(i)p(i)}{2^{r(i)}}.$$

By induction, this gives us that

$$F_1(n) = \frac{c(n)p(n)}{2^{r(n)}} + F_2(n-1).$$

We also have that

$$\begin{aligned} F_2(n) =& p(n)B(n) + \textstyle\sum_{i=0}^{n-1} \left(p(i) - p(i+1)\right) B(i) \\ =& p(n)B(n) + \left(p(n-1) - p(n)\right) B(n-1) + \textstyle\sum_{i=0}^{n-2} \left(p(i) - p(i+1)\right) B(i) \\ =& p(n) \left(B(n) - B(n-1)\right) + p(n-1)B(n-1) + \textstyle\sum_{i=0}^{n-2} \left(p(i) - p(i+1)\right) B(i) \\ =& p(n) \left(B(n) - B(n-1)\right) + F_2(n-1) \end{aligned}$$

Thus, we must show that

$$\frac{c(n)}{2^{r(n)}} = B(n) - B(n-1).$$

We will consider three cases: (i) where all pieces are the same color); (ii) when only the left-most piece is a different color; and (iii) that there is a color change to the right of the left-most piece.

Let p be the piece number of the first color change from the right. Then $B(0) = m$ and for all $k < p$, $B(k) = km{:}m = (k+1)m$.

(i) Suppose that all pieces are the same color, i.e. $\forall i \leq n$, $c(i) = m \in \{-1, 1\}$. As noted above, $B(n-1) = nm$ and $B(n) = (n+1)m$. Also, $c(n) = m$ and $r(n) = 0$ since it and all pieces to the right are the same color. Thus,

$$B(n) - B(n-1) = (n+1)m - nm = m = \frac{m}{2^0} = \frac{c(n)}{2^{r(n)}}.$$

(ii) Suppose that only the left-most piece is a different color, i.e. that $\forall i < n$, $c(i) = m \in \{-1, 1\}$, but $c(n) = -m$. Then $\forall i < n$, $r(i) = 0$, but $r(n) = 1$. $B(n-1) = nm$ as stated above. However, this time,

$$B(n) = B(n-1){:}{-m} = nm{:}{-m} = nm - \frac{m}{2}.$$

Thus,

$$B(n) - B(n-1) = nm - \frac{m}{2} - nm = -\frac{m}{2} = \frac{(-m)}{2^1} = \frac{c(n)}{2^{r(n)}}.$$

(iii) Suppose that there is a color change preceding $c(n)$, i.e. $\exists i, j < n$ such that $c(i) \neq c(j)$. Then $r(n) = r(n-1) + 1$ and now $\operatorname{Hdim}(B(n-1)) > 1$ since $B(n-1)$ is not an integer. By Lemma 2.2.16, we know that

$$B(n) = B(n-1){:}c(n) = B(n-1) + \frac{c(n)}{2^{\operatorname{Hdim}(B(n-1))}}$$

Then $B(n) - B(n-1) = \frac{c(n)}{2^{\operatorname{Hdim}(B(n-1))}}$ and by Lemma 2.3.19, we know that $\operatorname{Hdim}(B(n-1)) = r(n-1) + 1$. So,

$$B(n) - B(n-1) = \frac{c(n)}{2^{\operatorname{Hdim}(B(n-1))}} = \frac{c(n)}{2^{r(n-1)+1}} = \frac{c(n)}{2^{r(n)}}.$$

$\square$

In Figure 2.12, we give a representation of the SHOVE position X_O_XX. What is lovely about this configuration, is that drawing it out from a starting position is quite simple. To begin with, we will work over from the right-most piece on the board. If $c(i)$ is 1, we will be drawing blue (solid) HACKENBUSH edges and if -1 we will draw red (zig-zag) edges. We start by drawing $p(0)$ edges connected to the ground of color $c(0)$. Then, working from the left-most string across, we append one edge of color $c(1)$ to the first $p(1)$ strings. Then, again starting on the left, an edge of color $c(2)$ to the first $p(2)$ strings. We continue on in this fashion until we finally add an edge of color $c(n)$ to the top of the first $p(n)$ strings. When we are done, we will have the graphical HACKENBUSH representation of the ordinal sum method.

In our example in Figure 2.12, we first drew 6 blue lines, then added a blue line to the top of the left-most 5 strings, then red to the first 3 strings, and finally, a blue edge to the first string.

To evaluate the position, we calculate the necessary components: $n, p(i), c(i)$, and either $B(i)$ or $r(i)$. For this position, $n = 3$, we set $p(4) = 0$, and the table below gives all others:

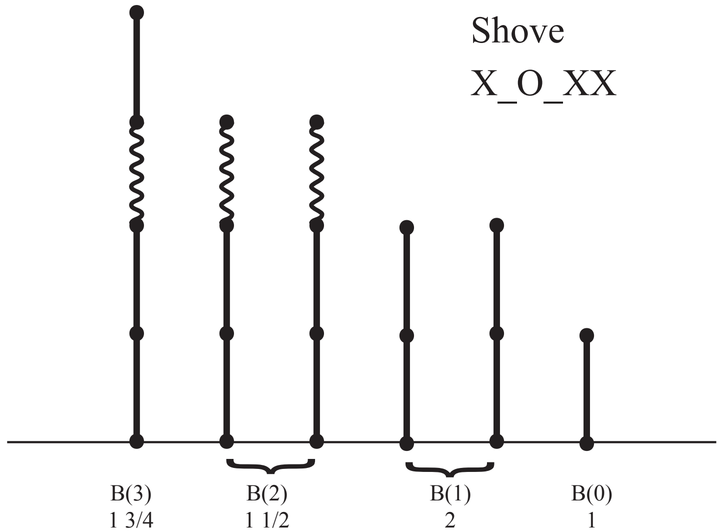


Figure 2.12: Hackenbush representation of the SHOVE position X_O_XX.

	X	_	O	_	X	X
i	3		2		1	0
$p(i)$	1		3		5	6
$r(i)$	2		1		0	0
$c(i)$	+1		-1		+1	+1
$b(i)$	$\frac{1}{2}$:$1 = \frac{3}{4}$		1:$-1 = \frac{1}{2}$		1:$0 = 1$	0
$B(i)$	$1\frac{1}{2}$:$1 = 1\frac{3}{4}$		2:$-1 = 1\frac{1}{2}$		1:$1 = 2$	1

Via the first method, we see that for $G = Shove(X\_O\_XX)$,

$$\begin{aligned} G &= \textstyle\sum_{i=0}^{3} c(i)\left(\frac{p(i)}{2^{r(i)}}\right) \\ &= 1\left(\tfrac{6}{2^0}\right) + 1\left(\tfrac{5}{2^0}\right) - 1\left(\tfrac{3}{2^1}\right) + 1\left(\tfrac{1}{2^2}\right) \\ &= 1\left(\tfrac{6}{1}\right) + 1\left(\tfrac{5}{1}\right) - 1\left(\tfrac{3}{2}\right) + 1\left(\tfrac{1}{4}\right) \\ &= 6 + 5 - 1\tfrac{1}{2} + \tfrac{1}{4} \\ &= 9\tfrac{3}{4} \end{aligned}$$

Via the ordinal sum method,

$$\begin{aligned}
G &= p(3)B(3) + \textstyle\sum_{i=0}^{2} \left(p(i) - p(i+1)\right) B(i) \\
&= 1\left(1\tfrac{3}{4}\right) + \left((6-5)(1) + (5-3)(2) + (3-1)\left(1\tfrac{1}{2}\right)\right) \\
&= 1\tfrac{3}{4} + 1(1) + 2(2) + 2\left(1\tfrac{1}{2}\right) \\
&= 1\tfrac{3}{4} + 1 + 4 + 3 \\
&= 9\tfrac{3}{4}
\end{aligned}$$

Both methods arrive at the same result. They simply provide alternate ways of considering the resulting HACKENBUSH strings. While the ordinal sum method might seem a bit cumbersome, it is the most straight-forward representation of the HACKENBUSH strings presented.

2.4 Side-Sums

Ordinal sums have served us well to describe the values of various juxtapositions. However, there are certainly games for which ordinal sums have no place. In this section, we will introduce a new function, a close relative of ordinal sums, that will handle other classes of juxtapositions in a similar way. This new function is called the side-sum. After exploring this new function, we will introduce a variant of TOPPLING DOMINOES to which this function will immediately apply.

In this variant, we will play as in the game of TOPPLING DOMINOES, with the added restriction that both players be allowed to topple pieces only to one direction or the other. We will call this variation on the game X/Y-RESTRICTED TOPPLING DOMINOES, where X and Y are each either "E" (for East, i.e. to the right) or "W" (for West, i.e. to the left) and define the restriction placed on the Left and Right player, respectively (i.e. Left is only allowed to topple black dominoes in the direction X, Right only white dominoes in direction Y).

2.4.1 Side-Sum Function

Ordinal sum considers 2 possible combinations that could play out from the juxtaposition, $G \boxplus H$, of G and H. That is, those games where play in the left position G completely destroys all of H (i.e. $G\mathbf{:}H$) and where play in H destroys all of G (i.e. $H\mathbf{:}G$). However, it does not consider the case where the game rules are such that Left playing in one component removes options to the other but the reverse happens for Right. That is, rule sets that would allow Left to play in G destroying H (i.e. to $\mathbf{L}(G)$) or play in H and leave G juxtaposed $\mathbf{L}(H)$ (i.e to $G \boxplus \mathbf{L}(H)$), but would allow Right to play in G to $\mathbf{R}(G) \boxplus H$ or to play in H to $\mathbf{R}(H)$, destroying G.

In our line of TOPPLING DOMINOES variations, this possiblity is embodied in the games E/W-RESTRICTED TOPPLING DOMINOES and W/E-RESTRICTED TOPPLING DOMINOES.

In order to look at analysis of these games, we must first introduce a new definition.

Definition 2.4.1. *For games G and H, the **side-sum** of G to H, denoted $G \diamond H$ (said "G side-sum H"), will be defined recursively as*

$$G \diamond H = \{\mathbf{L}(H), \mathbf{L}(G) \diamond H \,|\mathbf{R}(G), G \diamond \mathbf{R}(H)\}.$$

The idea: Start with two positions G and H in which Left moves annihilate everything positionally to the left and Right to the right. If a position looks like G juxtaposed H ($G \boxplus H$), then Left can move in the G-component to a position leaving $\mathbf{L}(G) \boxplus H$ or in the H component leaving position $\mathbf{L}(H)$ (clear in W/E-RESTRICTED TOPPLING DOMINOES since moves in a component to the right for Left wipe out all of the left-hand component). Likewise, Right can move in the G-component (wiping out all of H) to $\mathbf{R}(G)$ or in the H-component leaving $G \boxplus \mathbf{R}(H)$.

Lemma 2.4.2.

$$G \diamond 0 = 0 \diamond G = G$$

Proof.

$$\begin{aligned} 0 \diamond 0 &= \{\mathbf{L}(0), \mathbf{L}(0) \diamond 0 \,|\mathbf{R}(0), 0 \diamond \mathbf{R}(0)\} \\ &= \{\cdot\,|\cdot\} \\ &= 0 \end{aligned}$$

$$\begin{aligned} G \diamond 0 &= \{\mathbf{L}(0), \mathbf{L}(G) \diamond 0 \,|\mathbf{R}(G), G \diamond \mathbf{R}(0)\} \\ &= \{\mathbf{L}(G) \diamond 0 \,|\mathbf{R}(G), G \diamond \mathbf{R}(0)\} \\ &= \{\mathbf{L}(G) \,|\mathbf{R}(G)\} \qquad \text{(by induction)} \\ &= G \end{aligned}$$

$$\begin{aligned} 0 \diamond G &= \{\mathbf{L}(G), \mathbf{L}(0) \diamond G \,|\mathbf{R}(0), 0 \diamond \mathbf{R}(G)\} \\ &= \{\mathbf{L}(G) \,|0 \diamond \mathbf{R}(G)\} \\ &= \{\mathbf{L}(G) \,|\mathbf{R}(G)\} \qquad \text{(by induction)} \\ &= G \end{aligned}$$

□

Lemma 2.4.3. *For all games G and H, all left options of H are less than or incomparable to $G \diamond H$, and all right options of G are greater than or incomparable to $G \diamond H$. That is,*

$$\forall H^L \in \mathbf{L}(H), \; H^L \lhd G \diamond H,$$

and

$$\forall G^R \in \mathbf{R}(G),\ G \diamond H \lhd G^R.$$

(i.e. $\mathbf{R}(G) \not\lesssim G \diamond H \not\lesssim \mathbf{L}(H)$ *.)*

Proof. This follows from Proposition 1.2.35 and the fact that, by Definition 2.4.1, $\mathbf{L}(H)$ and $\mathbf{R}(G)$ are in the set of left and right options, respectively, of $G \diamond H$. □

Lemma 2.4.4. *For* $x, y \in \mathbb{Z}^{>0}$,

(i) $x \diamond y = x + y$

(ii) $(-x) \diamond y = \{y - 1 \,|1 - x\}$

(iii) $x \diamond (-y) = x - y$

Proof.
(i) $x \diamond y = \{y - 1, x - 1 \diamond y \,|\cdot\} = \{y - 1, (x + y - 1) \,|\cdot\} = x + y$
(ii) $(-x) \diamond y = \{y - 1 \,|1 - x\}$ as $\mathbf{L}(-x) = \emptyset$ and $\mathbf{R}(y) = \emptyset$.
(iii) $x \diamond (-y) = \{x - 1 \diamond -y \,|x \diamond 1 - y\} = \{x - y - 1 \,|x - y + 1\} = x - y$

□

We note that the side-sum is associative.

Lemma 2.4.5.

$$(G \diamond H) \diamond K = G \diamond (H \diamond K)$$

Proof.

$$\begin{aligned}
&(G \diamond H) \diamond K \\
&= \{\mathbf{L}(K), \mathbf{L}(G \diamond H) \diamond K \,|\mathbf{R}(G \diamond H), (G \diamond H) \diamond \mathbf{R}(K)\} \\
&= \{\mathbf{L}(K), \mathbf{L}(H) \diamond K, (\mathbf{L}(G) \diamond H) \diamond K \,|\mathbf{R}(G), G \diamond \mathbf{R}(H), (G \diamond H) \diamond \mathbf{R}(K)\} \\
&= \{\mathbf{L}(K), \mathbf{L}(H) \diamond K, \mathbf{L}(G) \diamond (H \diamond K) \,|\mathbf{R}(G), G \diamond \mathbf{R}(H), G \diamond (H \diamond \mathbf{R}(K))\} \\
&\quad \text{(by induction)} \\
&= \{\mathbf{L}(H \diamond K), \mathbf{L}(G) \diamond (H \diamond K) \,|\mathbf{R}(G), G \diamond \mathbf{R}(H \diamond K)\} \\
&= G \diamond (H \diamond K)
\end{aligned}$$

□

However, side-sum is not commutative.

Example 2.4.6. *When* $G = 1$ *and* $H = \frac{1}{2}$ *then* $G \diamond H = 1 \neq 1* = H \diamond G$*.*

$$1 \diamond \frac{1}{2} = \left\{0, 0 \diamond \frac{1}{2} \,|\, 1 \diamond 1\right\} = \left\{0, \frac{1}{2} \,|\, \{0, 0 \diamond 1 \,|\, \cdot\}\right\} = \left\{\frac{1}{2} \,|\, \{0, 1 \,|\, \cdot\}\right\} = \left\{\frac{1}{2} \,|\, 2\right\} = 1.$$

where

$$\frac{1}{2} \diamond 1 = \{0, 0 \diamond 1 \,|\, 1\} = \{0, 1 \,|\, 1\} = 1 * .$$

2.4.2 Side-Sum From Day n

As we did with ordinal sums, let us consider the set of games that can be produced as side-sums of games from $\mathsf{G}[n]$.

Definition 2.4.7. *The* ***side-sums from day*** n*, denoted* $\mathbf{SS}_n$*, will be the set of all games formed from side-sums of games born by day* n*.*

$$\mathbf{SS}_n = \{G \diamond H \;\; s.t. \;\; G, H \in \mathsf{G}[n]\}.$$

On day 0, we can only form $0 \diamond 0 = 0$.

Lemma 2.4.8.

$$\mathbf{SS}_0 = \{0\}.$$

However, on day 1, we can now form 11 different games. Table 2.4 lists all side-sums that can be formed from games by day 1. These 11 games can be represented by the poset $\langle \mathbf{SS}_1; \leq \rangle$ as pictured in Figure 2.13.

$G \diamond H$		**H**			
		1	0	$*$	-1
G	1	2	1	$\frac{1}{2}$	0
	0	1	0	$*$	-1
	$*$	$\{1 \mid 0\}$	$*$	$*2$	$-\frac{1}{2}$
	-1	$*$	-1	$\{0 \mid -1\}$	-2

Table 2.4: Side-sums that can be formed from games born by day 1.

$\text{rcf}\,(G \diamond H)$		**H**			
		1	0	$*$	-1
G	1	2	1	$\frac{1}{2}$	0
	0	1	0	0	-1
	$*$	$\{1 \mid 0\}$	0	0	$-\frac{1}{2}$
	-1	0	-1	$\{0 \mid -1\}$	-2

Table 2.5: The reduced canonical forms of side-sums that can be formed from games born by day 1.

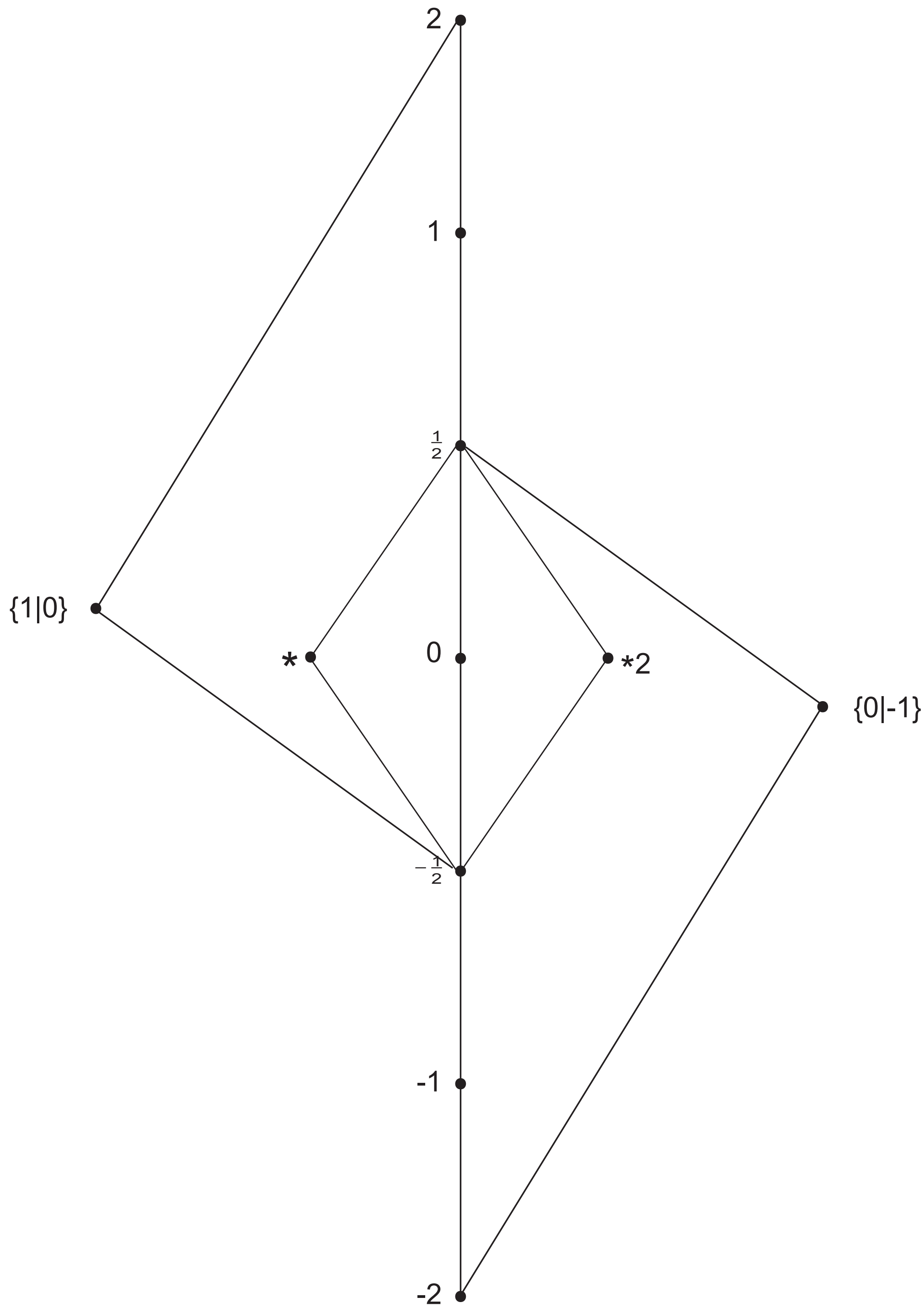


Figure 2.13: Poset of $\mathbf{SS}_1$.

2.5 Side-Sum Applied

We now revisit the game of RESTRICTED TOPPLING DOMINOES. We will give the full solution to the game through application of side sums. Unlike the examples of games that could be expressed using ordinal sums, this application is less camouflaged. However, it nicely demonstrates that the side-sum function, a cousin to ordinal sums, could prove useful in a larger class of games.

2.5.1 Restricted Toppling Dominoes

In the game of E/E-RESTRICTED TOPPLING DOMINOES, a position of value G adjacent to (to the left of) a position H, i.e. the position $G \boxplus H$, has value $G{:}H$. Should either player move in the G component of the game, all of H is annihilated as dominoes to the right (including all of H) are toppled. A move in H leaves all of G untouched and still adjacent to any remaining dominoes from H. Play in the game W/W-RESTRICTED TOPPLING DOMINOES is symmetric.

However, consider the game of W/E-RESTRICTED TOPPLING DOMINOES in which Left is only allowed to topple black dominoes to the left, and Right white dominoes to the right. In this case, ordinal sum will no longer help us. We claim that the side-sum is embodied by this game.

In considering this game, we will break the line of dominoes into strings of some number of consecutive white dominoes followed (to the right) by some number of consecutive black ones. We will refer to segments of this type as **white/black strings** and will begin with analysis of these segment types.

We will denote strings of white and strings of black dominoes in this game as $w(x)$ and $b(y)$, respectively, where x and y represent the number of dominoes in each of the strings.

2.5.1.1 White/Black Strings

In a white/black string of the form $w(x) \boxplus b(y)$ (i.e. x white dominoes to the left of y black dominoes), as in Figure 2.14, a move by Right necessarily topples all black dominoes. A move by Left topples all white dominoes. To understand the value of white/black segments, we must first know the value of single color strings of dominoes.

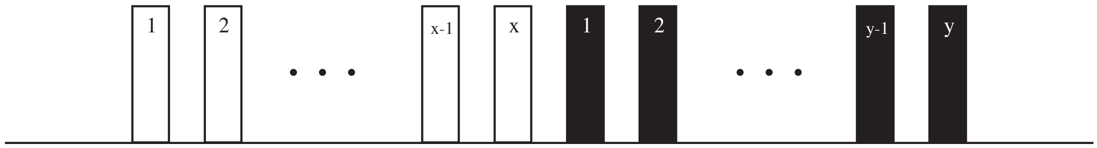


Figure 2.14: The W/E-RESTRICTED TOPPLING DOMINOES position $w(x) \boxplus b(y)$.

Lemma 2.5.1. *$b(n) = n$ and $w(n) = -n$*

Proof. Clearly, $b(0) = w(0) = \{\cdot\,|\,\cdot\} = 0$. Neither player has a move.

$b(n) = \left\{\langle\, b(i)\, \rangle_{i=0}^{n-1} \,|\, \cdot\right\} = \{0, \ldots, n-1 \,|\, \cdot\} = \{n-1 \,|\, \cdot\} = n.$

Similarly, $w(n) = -n$. □

From this, we have the value of all moves from our white/black string.

Lemma 2.5.2. *For any $x, y \in \mathbb{Z}^{>0}$, $w(x) \boxplus b(y) = \{y-1 \,|\, 1-x\}$.*

Proof.

$$w(x) \boxplus b(y) = \left\{\langle\, b(i)\, \rangle_{i=0}^{y-1} \,\middle|\, \langle\, w(j)\, \rangle_{j=0}^{x-1}\right\}.$$

From lemma 2.5.1, we know that this gives us

$$w(x) \boxplus b(y) = \{0, \ldots, (y-1) \,|\, 0, \ldots, -(x-1)\} = \{y-1 \,|\, 1-x\}.$$

□

Also, we will need the value of black/white strings. Note that in the unrestricted game of TOPPLING DOMINOES this could simply be represented as the negative of some white/black string. This would also be true if our restrictions forced both players to topple in the same direction (i.e. W/W- or E/E-restricted toppling dominoes). However, in W/E-RESTRICTED TOPPLING DOMINOES, that is not the case, as the following demonstrates.

Lemma 2.5.3. *For any $x, y \in \mathbb{Z}^{>0}$, $b(x) \boxplus w(y) = x - y$.*

Proof.

$$b(x) \boxplus w(y) = \left\{\langle\, b(i) \boxplus w(y)\, \rangle_{i=0}^{x-1} \,\middle|\, \langle\, b(x) \boxplus w(j)\, \rangle_{j=0}^{y-1}\right\}.$$

By induction on the total number of dominoes, $x + y$,

$$b(x) \boxplus w(y) = \left\{ \langle\ i - y\ \rangle_{i=1}^{x-1} \middle| \langle\ x - j\ \rangle_{j=0}^{y-1} \right\} = \{x - y - 1\,|x - y + 1\} = x - y.$$

□

2.5.1.2 Side-Sums

In order to consider lines made up of more than one of the white/black strings, we must make use of the side-sum. With that, we have all the pieces in place to look at multiple white/black strings.

Theorem 2.5.4. *Let both g and h represent series of white/black strings (in which either or both of the colors may be absent) in the game of* W/E-RESTRICTED TOPPLING DOMINOES, *where string g has value G and h has value H. Then,*

$$g \boxplus h = G \diamond H.$$

Proof.

$$\begin{aligned} g \boxplus h =& \{\mathbf{L}(g) \boxplus h, \mathbf{L}(h)\,|\mathbf{R}(g), g \boxplus \mathbf{R}(h)\} \\ =& \{\mathbf{L}(G) \diamond H, \mathbf{L}(H)\,|\mathbf{R}(G), G \diamond \mathbf{R}(H)\} \\ =& G \diamond H. \end{aligned}$$

□

Note that even if we consider the value G in canonical form, $\mathbf{L}(G) \subseteq \mathbf{L}(g)$. For any $g^L \in \mathbf{L}(g), \exists G^{L^*} \in \mathbf{L}(G)$ s.t. $g^L \leq G^{L^*}$ and $\forall G^L \in \mathbf{L}(G), \exists g^{L^*} \in \mathbf{L}(g)$ s.t. $g^{L^*} = G^L$. The same is true for h and H and symmetric results hold for Right options. Using this notation, Figure 2.15 depicts the winning second player responses.

As an example, we will look at the the following game of W/E-RESTRICTED TOPPLING DOMINOES and analyze it using side-sum.

This game can be broken down into two smaller games, game $G = 1$ shown in Figure 2.16 and game $H = *$ shown in Figure 2.17.

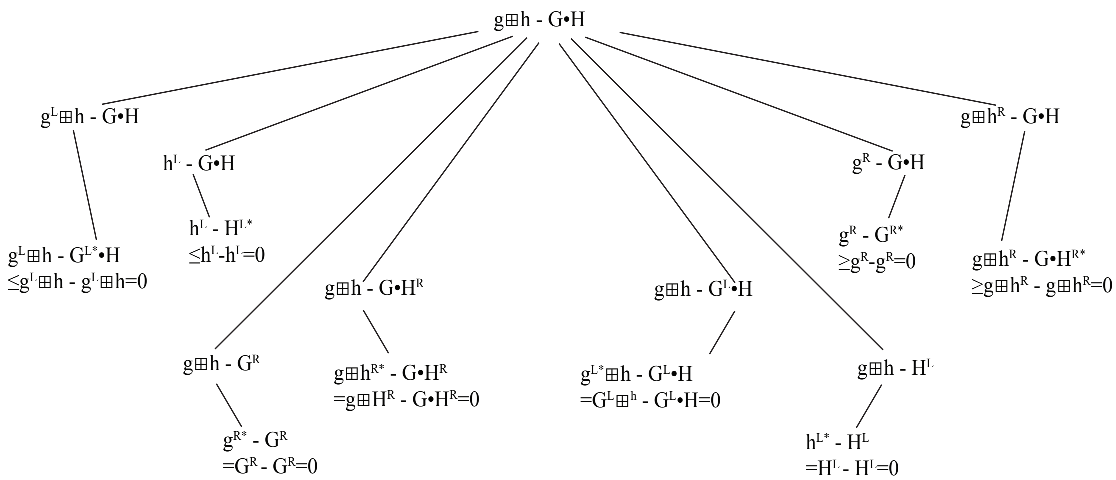


Figure 2.15: The game tree of $g \boxplus h - G \diamond H$.

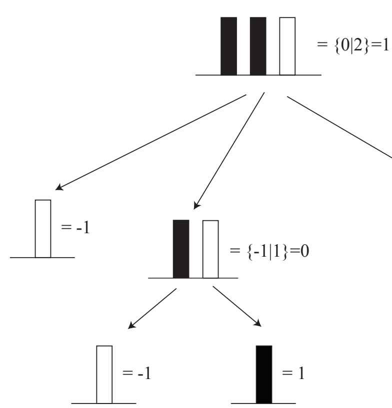


Figure 2.16: Example game $G = \{-1, 0 \,|2\} = 1$.

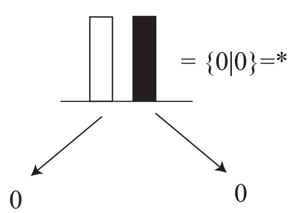


Figure 2.17: Example game $H = \{0 \,|0\} = *$.

From Theorem 2.5.4, we know that $G \boxplus H = 1 \diamond *$. We can check that

$$1 \diamond * = \{0, 0 \diamond * \,|1 \diamond 0\} = \{0, * \,|1\} = \frac{1}{2}.$$

Likewise, if we were to evaluate the entire position $G \boxplus H$, we arrive at the same result, as can be seen in its game tree in Figure 2.18.

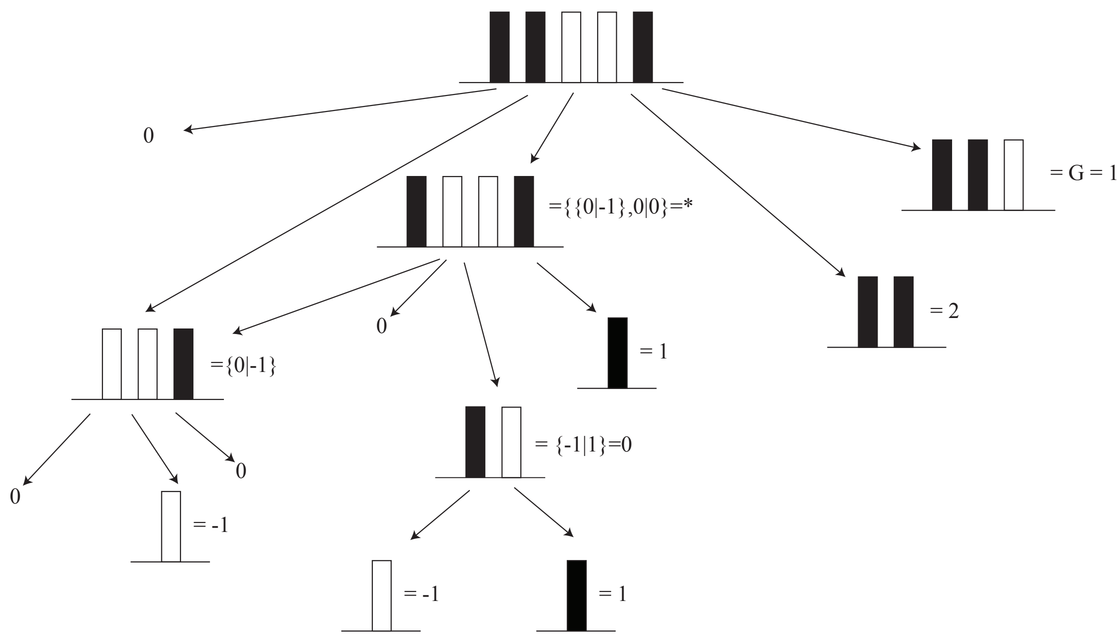


Figure 2.18: Example game $G \boxplus H = 1 \diamond *$.

Chapter 3

Loopy And Oslo Games

The traditional description of combinatorial games includes the caveat that the game will end after a finite sequence of moves and that no positions will be repeated. The problem with allowing repetition of positions is that it allows for the possibility that the game may never terminate. However, many games, such as CHECKERS, BACKSLIDING TOADS & FROGS and most-notably GO [3], exist in which certain moves can lead to an infinitely long sequence of moves.

3.1 Loopy Games

These games, in which position repetition is allowed, are referred to as **loopy games** due to the fact that their game graphs (no longer game trees) contain cycles. As the cycles complicate the structure of the game, the theory surrounding loopy games is more complicated.

Further background theory on loopy games can be found in *Winning Ways* [3], *On Numbers and Games* [9], and in the recent works of Aaron Siegel [28, 30].

Definition 3.1.1. *[3]* ***Loopy games*** *are combinatorial games in which repetition is permitted.*

When it is necessary to specify, we will refer to games that are not loopy games as **loop-free games**.

In *Winning Ways* [3] and the Ph.D. thesis of Aaron Siegel [28], a variety of idempotents were introduced that help express the detail of otherwise complicated games. We catalogue these well-behaved games in Table 3.1. The classic examples are the games of $\text{ON} = \{pass \,|\, \cdot\}$ and its negative, $\text{OFF} = \{\cdot \,|\, pass\}$, first introduced in [3]. We note that we will sometimes use "*pass*" to represent an option back to the original game. That is, if $pass \in \mathbf{L}(G)$ or $pass \in \mathbf{R}(G)$, these represent a move to G. Thus, $\text{ON} = \{\text{ON} \,|\, \cdot\}$ and $\text{OFF} = \{\cdot \,|\, \text{OFF}\}$, though we will often refer to them as we did initially, inserting *pass* moves where appropriate.

The games of ON and OFF absorb all loop-free games. That is, for any loop-free game G, $G + \text{ON} = \text{ON}$ and $G + \text{OFF} = \text{OFF}$. The game $\text{OVER} = \{0 \,|\, pass\}$ absorbs all loop-free infinitesimals. That is, if G and H are loop-free games, then $G + \text{OVER} = H + \text{OVER} \iff G \equiv_{\mathbf{Inf}} H$. The final game to note is $+_{on} = \{0 \,|\, \{0 \,|\, \text{OFF}\}\} > 0$ which does not absorb any loop-free games, and represents the smallest positive games in the universe of games.

Named loopy games	
$\text{ON}=\{pass \,|\, \cdot\}$	$\text{OFF}=\{\cdot \,|\, pass\}$
$\text{OVER}=\{0 \,|\, pass\}$	$\text{UNDER}=\{pass \,|\, 0\}$
$\text{UPON}=\{pass \,|\, *\}$	$\text{DOWNUNDER}=\{* \,|\, pass\}$
$\text{UPON}{*}=\{0, pass \,|\, 0\}$	$\text{DOWNUNDER}{*}=\{0 \,|\, 0, pass\}$
$+_{\text{OVER}}=\{0 \,|\, \{0 \,|\, \text{UNDER}\}\}$	$-_{\text{UNDER}}=\{\{\text{OVER} \,|\, 0\} \,|\, 0\}$
$+_{\text{ON}}=\{0 \,|\, \{0 \,|\, \text{OFF}\}\}$	$-_{\text{OFF}}=\{\{\text{ON} \,|\, 0\} \,|\, 0\}$
$\text{DUD} = \{\text{DUD} \,|\, \text{DUD}\}$	

Table 3.1: Named loopy games [3], [9] & [28].

Idempotent	**Loop-free games absorbed**
ON	All games
OVER	All infinitesimals
$+_{\text{OVER}}$	All tinies, but no all-smalls
$+_{\text{ON}}$	None

Table 3.2: A sampling of loopy idempotents [28].

3.2 Oslo Games

While games are clearly enjoyable, when playing against a child that never grows tired of play, there comes a time when one might opt to pass on his or her turn if given the chance[1]. Of course, at the point in which an option is made to end the game, that pass option no longer remains and the game ends. We now consider the class of games in which Left is always allowed a pass move, until that point where either player makes a move to zero. We can think of these games as those in which Left is a lazy parent, mindlessly passing and allowing their child to play on until some point where an opportunity is available to end the game.

We will call these games **O**ne-**s**ided **lo**opy games, or **Oslo** games. The following function takes a normal game and adds in a pass option for Left at all non-zero positions within the game. Effectively, it takes any game and makes it a one-sided loopy, or Oslo, game.

Definition 3.2.1.
If $\mathrm{lit}(G) = 0$*, then* $\mathrm{oslo}(G) = 0$*. Otherwise,*

$$\begin{aligned}\mathrm{oslo}(G) &= \{\mathrm{oslo}(G), \mathrm{oslo}(\mathbf{L}(G)) \,|\, \mathrm{oslo}(\mathbf{R}(G))\} \\ &= \{pass, \mathrm{oslo}(\mathbf{L}(G)) \,|\, \mathrm{oslo}(\mathbf{R}(G))\}\end{aligned}$$

For example, if we take the games $\frac{1}{2}$, $-\frac{1}{2}$ and $\{0\,|{-1}\}$ in canonical form and add in the pass moves, we have the following:

$$\begin{aligned}\mathrm{oslo}\left(\tfrac{1}{2}\right) &= \left\{\mathrm{oslo}\left(\tfrac{1}{2}\right), \mathrm{oslo}(0) \,|\, \mathrm{oslo}(1)\right\} \\ &= \left\{\mathrm{oslo}\left(\tfrac{1}{2}\right), 0 \,|\, \{\mathrm{oslo}(1), \mathrm{oslo}(0) \,|\,\}\right\} \\ &= \left\{\mathrm{oslo}\left(\tfrac{1}{2}\right), 0 \,|\, \{\mathrm{oslo}(1), 0 \,|\,\}\right\} \\ &= \{pass, 0 \,|\, \{pass, 0 \,|\,\}\} \\ &= \{pass, 0 \,|\, \textsc{on}\} \\ &= \textsc{on}\end{aligned}$$

[1] The author has *much* experience with this form of play!

$$\begin{aligned}
\text{oslo}\left(-\tfrac{1}{2}\right) &= \left\{\text{oslo}\left(-\tfrac{1}{2}\right), \text{oslo}\left(-1\right) \middle| \text{oslo}\left(0\right)\right\} \\
&= \left\{\text{oslo}\left(-\tfrac{1}{2}\right), \left\{\text{oslo}\left(-1\right) \middle| \text{oslo}\left(0\right)\right\} \middle| 0\right\} \\
&= \left\{\text{oslo}\left(-\tfrac{1}{2}\right), \left\{\mathit{pass} \,|\, 0\right\} \middle| 0\right\} \\
&= \left\{\mathit{pass}, \text{UNDER} \,|\, 0\right\} \\
&= \text{UNDER}
\end{aligned}$$

$$\begin{aligned}
\text{oslo}\left(\{0 \,|\, -1\}\right) &= \left\{\text{oslo}\left(\{0 \,|\, -1\}\right), \text{oslo}\left(0\right) \middle| \text{oslo}\left(-1\right)\right\} \\
&= \left\{\text{oslo}\left(\{0 \,|\, -1\}\right), 0 \,\middle|\, \left\{\text{oslo}\left(-1\right) \middle| \text{oslo}\left(0\right)\right\}\right\} \\
&= \left\{\mathit{pass}, 0 \,|\, \left\{\mathit{pass} \,|\, 0\right\}\right\} \\
&= \left\{\mathit{pass}, 0 \,|\, \text{UNDER}\right\} \\
&= \left\{0 \,|\, \text{UNDER}\right\}
\end{aligned}$$

These are also represented in Figure 3.1

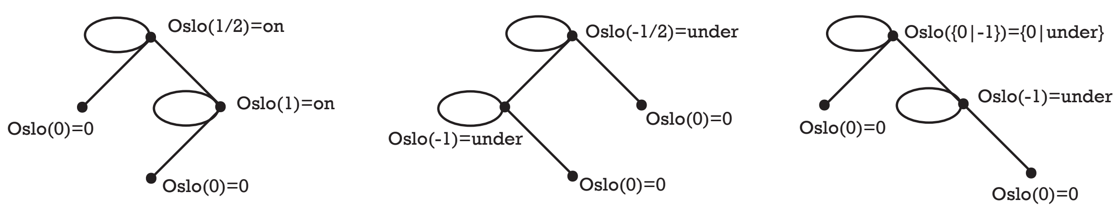


Figure 3.1: Example of oslo $\left(\frac{1}{2}\right)$, oslo $\left(-\frac{1}{2}\right)$ and oslo $\left(\{0 \,|\, -1\}\right)$.

3.2.1 Oslo Game Values

All non-zero positions in Oslo games have a pass move for Left. Hence, those positions cannot be $\mathcal{P}$-positions. Thus, the only $\mathcal{P}$-positions in Oslo games are identically zero.

Lemma 3.2.2. $\text{oslo}\,(G) = 0 \iff G \cong 0$. *Thus, zeros are the only $\mathcal{P}$-positions in Oslo games.*

Proof. If $G \cong 0$, then $\text{oslo}\,(G) = 0$ (by definition).

Suppose $G \not\cong 0$. Then

$$\text{oslo}\,(G) = \left\{\text{oslo}\,(G), \text{oslo}\,(\mathbf{L}\,(G)) \,|\, \text{oslo}\,(\mathbf{R}\,(G))\right\}.$$

If we assume that $\operatorname{oslo}(G) = 0$, then

$$\operatorname{oslo}(G) = \{0, \operatorname{oslo}(\mathbf{L}(G)) \,|\operatorname{oslo}(\mathbf{R}(G))\}\,.$$

Since $0 \in \mathbf{L}(\operatorname{oslo}(G))$, $\operatorname{oslo}(G) \neq 0$, which is a contradiction. Hence, if $G \not\cong 0$, then $\operatorname{oslo}(G) \neq 0$. □

This is essentially a strategy-stealing argument.

The game $G = \{-1\,|1\}$ is an example of a game $G \not\cong 0$ that has value zero. Under the Oslo function, Left then has a move to $\operatorname{oslo}(G)$, which is the same as a pass move for Left from $\operatorname{oslo}(G)$.

$$\begin{aligned}
\operatorname{oslo}(\{-1\,|1\}) &= \{\operatorname{oslo}(\{-1\,|1\}), \operatorname{oslo}(-1)\,|\operatorname{oslo}(1)\} \\
&= \{\mathit{pass}, \{\mathit{pass}\,|\operatorname{oslo}(0)\}\,|\{\mathit{pass}, \operatorname{oslo}(0)\,|\}\} \\
&= \{\mathit{pass}, \{\mathit{pass}\,|0\}\,|\{\mathit{pass}, 0\,|\}\} \\
&= \{\mathit{pass}, \textsc{under}\,|\textsc{on}\} \\
&= \textsc{on}
\end{aligned}$$

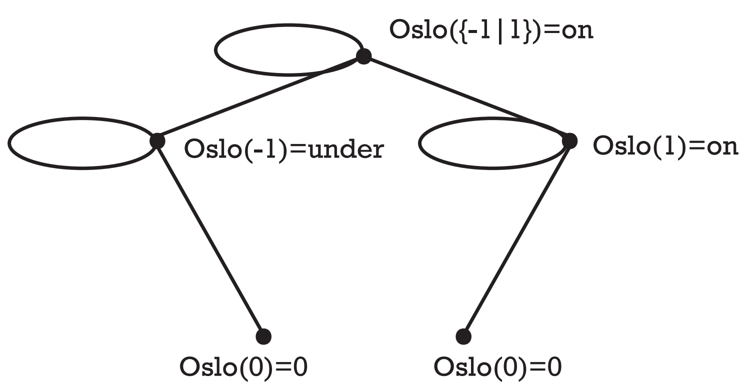


Figure 3.2: Example of $\operatorname{oslo}(\{-1\,|1\})$. This provides an example of $\operatorname{oslo}(G)$ for a game G which has value 0, but is not identically zero, i.e. $G = 0$ but $G \not\cong 0$.

It is interesting to note that outcome classes are preserved between a short game G in canonical form and $\operatorname{oslo}(G)$.

Lemma 3.2.3. *For short games G in canonical form,*

$$o(G) = o(\text{oslo}(G)).$$

Proof. Since G is in canonical form, the only $\mathcal{P}$-positions are identically 0. From this and Lemma 3.2.2, we know that $G = 0 \implies G \cong 0 \implies \text{oslo}(G) = 0$. □

For nonzero numbers, the Oslo function sends a game to either ON or UNDER, depending on its sign. Positive numbers are sent to ON and negative numbers to UNDER.

Lemma 3.2.4. *Let x be a game in canonical form, where x is a number. Then,*

$$\begin{aligned} x > 0 &\implies \text{oslo}(x) = \text{ON} \\ x = 0 &\implies \text{oslo}(x) = 0 \\ x < 0 &\implies \text{oslo}(x) = \text{UNDER} \end{aligned}$$

Proof. From Lemma 3.2.3, it is clear that when $x = 0$, $\text{oslo}(x) = 0$. Suppose that $x > 0$. Then $x = \{x^L | x^R\}$ where $x^L \geq 0$ and either $x^R = \emptyset$ (if x is an integer) or $x^R > 0$. If x^R is empty, then by induction, $\text{oslo}(x)$ is either: $\{pass, 0|\cdot\} = \text{ON}$, if $x^L = 0$; or $\{pass, \text{ON}|\cdot\} = \text{ON}$ when $x^L > 0$. If $x^R > 0$, then by induction, $\text{oslo}(x)$ is either: $\{pass, 0|\text{ON}\} = \text{ON}$ when $x^L = 0$; or $\{pass, \text{ON}|\text{ON}\} = \text{ON}$ when $x^L > 0$. Similarly, when $x < 0$, $\text{oslo}(x)$ is always of the form $\{pass|0\}$, $\{pass|\text{UNDER}\}$, $\{pass, \text{UNDER}|0\}$ or $\{pass, \text{UNDER}|\text{UNDER}\}$, which all have value UNDER. □

In the Oslo version of a game, if Left has a winning strategy in the finite version of the game, he has that same strategy available to him when repetition is allowed.

Proposition 3.2.5. $G > 0 \implies \text{oslo}(G) > 0$.

3.3 Lattice Of Oslo Games

We will now consider the structure of all Oslo games born by day n. We will use $\mathbf{Oslo}_n$ to denote the set of Oslo games $\text{oslo}\,(G)$ with birthday $b(G) \le n$.

Definition 3.3.1.

$$\mathbf{Oslo}_n = \{\text{oslo}\,(G) \;:\; b(G) \le n\} = \{\{pass, L\,|R\} \;:\; L, R \in \bigcup_{k<n} \mathbf{Oslo}_k\}.$$

Proposition 3.3.2. $\mathbf{Oslo}_0 = \{0\}$ *and* $\mathbf{Oslo}_1 = \{\text{ON}, \text{UNDER}, \text{UPON}{*}, 0\}$.

Proof. The first statement is trivially true. If $b(G) = 1$, then a pass for Left and 0 are the only options of G. Hence, the only possibilities are $\{pass, 0\,|\cdot\} = \{pass\,|\cdot\} = \text{ON}$, $\{pass\,|0\} = \text{UNDER}$ and $\{pass, 0\,|0\} = \text{UPON}{*}$, which forms a comprehensive list. $\square$

The Oslo games added on day 1 are represented in Figure 3.3. The partial order structure of $\mathbf{Oslo}_1$ is depicted in Figure 3.4.

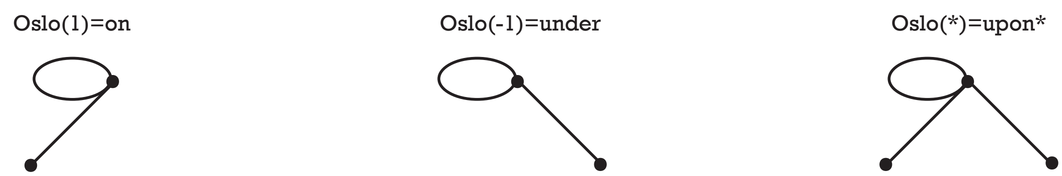


Figure 3.3: Oslo games $\text{oslo}\,(G)$ with birthday $b(G) = 1$.

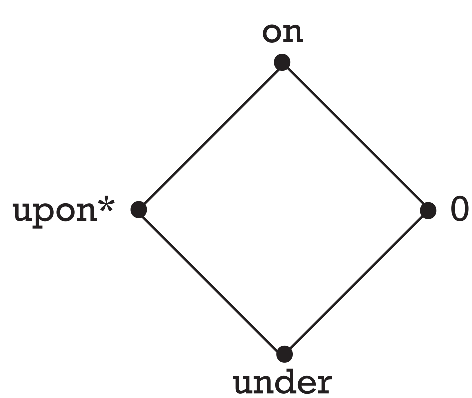


Figure 3.4: $\mathbf{Oslo}_1$: Oslo games born by day 1.

3.3.1 Closed Sets

We would like to show that $\mathbf{Oslo}_n$ is a complete distributive lattice. To do this, we will make use of the results of Albert and Nowakowski. In [1], they showed that for any set of games S, if every option of a game also lies in S, then the set of games all of whose immediate options belong to S forms a complete distributive lattice. We will make use of the set $S = \{0, \text{UPON}*, \text{ON}, \text{UNDER}\}$, satisfying this condition, to achieve the result.

For any set of games S, the children of S can be defined as follows.

Definition 3.3.3. *[1] For a set S of games, the* ***children*** *of S, which we will denote as $\mathcal{C}(S)$, are those games*

$$\mathcal{C}(S) = \{\{A \,|\, B\} \;\;:\;\; A, B \subseteq S\}.$$

Earlier, we noted that Calistrate, Paulhus and Wolfe showed in [8] that $\mathsf{G}[n]$ was a distributive lattice. They did this by beginning with a set $S = \emptyset$ and repeatedly forming $\mathcal{C}(S)$. Albert and Nowakowski furthered their investigations in [1] by considering the order structure of $\mathcal{C}(S)$ for an arbitrary set of games. Their result for the children of an arbitrary set of games follows.

Proposition 3.3.4. *[1, Prop. 2] For an arbitrary set of games S,*

$$\forall G \in \mathcal{C}(S), \quad G = \{\lfloor G \rfloor \,|\, \lceil G \rceil\}.$$

We motivate this by letting $H = \{\lfloor G \rfloor \,|\, \lceil G \rceil\}$ and considering the game $G - H$. Note that if Left moves to $G^L - H$, then $G^L \in \lfloor G \rfloor$ since $G^L \lhd G$, so Right can move to zero. Likewise, if Left plays to some $G - H'$ where $H' \in \lceil G \rceil$, then $G \lhd H'$, so $G - H' \lhd 0$. Thus, Right can win moving first in $G - H'$.

The children of an arbitrary set of games S were shown to form a complete lattice, with join and meet given below.

Theorem 3.3.5. *[1] For any set of games S, its children $\mathcal{C}(S)$ form a complete lattice with join and meet given by*

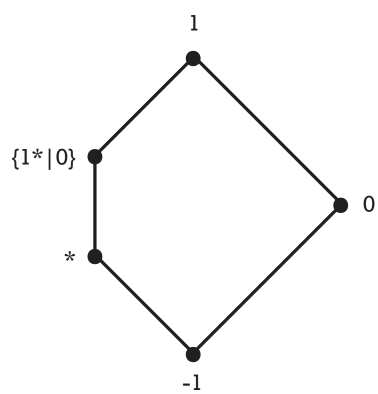
$$\bigvee_{i\in I} G_i = \{\textstyle\bigcup_{i\in I} \lfloor G_i \rfloor \,|\, \bigcap_{i\in I} \lceil G_i \rceil\}$$
$$\bigwedge_{i\in I} G_i = \{\textstyle\bigcap_{i\in I} \lfloor G_i \rfloor \,|\, \bigcup_{i\in I} \lceil G_i \rceil\}$$

For certain sets S, $\mathcal{C}(S)$ need not form a distributive lattice, as we shall see in the following example.

Example 3.3.6. *If we start with the set* $S = \{0, \{1\,|{*}\}, 1{*}\}$, *then its children are those games*

$$\mathcal{C}(S) = \{-1, *, 0, \{1{*}\,|0\}, 1\}.$$

These games form the non-distributive lattice N_5.



The result that we are most in need of from [1] regards the properties of the set S that force its children to form a distributive lattice. They note that for certain initial sets S, which they refer to as closed sets, the lattice $\mathcal{C}(S)$ is always distributive.

Definition 3.3.7. *[1] Let S be a set of games. This set is said to be a **closed set** if*

$$\forall G \in S, \quad \mathbf{L}(G), \mathbf{R}(G) \subseteq S.$$

Lemma 3.3.8. *[1, Prop. 6] If S is a closed set of games, then for $G, H \in \mathcal{C}(S)$,*

$$\lfloor G \vee H \rfloor = \lfloor G \rfloor \cup \lfloor H \rfloor$$
$$\lceil G \wedge H \rceil = \lceil G \rceil \cup \lceil H \rceil$$

Theorem 3.3.9. *[1, Thm. 7] For a closed set S of games, $\mathcal{C}(S)$ forms a complete distributive lattice.*

Proof. Let $G, H, K \in \mathcal{C}(S)$. Then,

$$\begin{aligned} G \wedge (H \vee K) &= \{ \lfloor G \rfloor \cap \lfloor H \vee K \rfloor \,|\, \lceil G \rceil \cup \lceil H \vee K \rceil \} \\ &= \{ \lfloor G \rfloor \cap (\lfloor H \rfloor \cup \lfloor K \rfloor) \,|\, \lceil G \rceil \cup (\lceil H \rceil \cap \lceil K \rceil) \} \\ &= \{ \lfloor G \wedge H \rfloor \cup \lfloor G \wedge K \rfloor \,|\, \lceil G \wedge H \rceil \cup \lceil G \wedge K \rceil \} \\ &= (G \wedge H) \vee (G \wedge K) \end{aligned}$$

□

3.3.2 Closed Set Application To Oslo Games

For the set of Oslo games, we can start with the set $S = \{\textsc{on}, 0, \textsc{upon}*, \textsc{under}\}$. It is clear that this is a closed set. For each game in our set S, each of its left and right options also belong to the set. What is interesting about this, is that the results of [1] were not initially focused on loopy games. However, under certain restrictions, their results will nicely apply.

Theorem 3.3.10. *The set of Oslo games born by day n, $\mathbf{Oslo}_n$, form a distributive lattice.*

Proof. Set $S = \mathbf{Oslo}_0 = \{\textsc{on}, 0, \textsc{upon}*, \textsc{under}\}$. Not that this set is a closed set, as all options of $\textsc{on} = \{on \,|\cdot\}$, 0, $\textsc{upon}* = \{\textsc{upon}*, 0 \,|0\}$ and $\textsc{under} = \{\textsc{under} \,|0\}$ also belong to the set S. By Theorem 3.3.9, we know that the children of these games, $\mathcal{C}(S)$, form a distributive lattice. By construction, $\mathcal{C}(S)$ is a closed set itself. If S is a closed set with $G \in S$, then $\{\mathbf{L}(G) \,|\mathbf{R}(G)\} = G \in \mathcal{C}(S)$. Hence, we note that if S is a closed set, then so too is $\mathcal{C}(S)$ since the left and right options of $\mathcal{C}(S)$ all came from the set $S \subseteq \mathcal{C}(S)$. □

Proposition 3.3.11. *There are 13 Oslo games born by day 2, so that $|\mathbf{Oslo}_2| = 13$. Figure 3.5 gives the partial-order structure of $\mathbf{Oslo}_2$.*

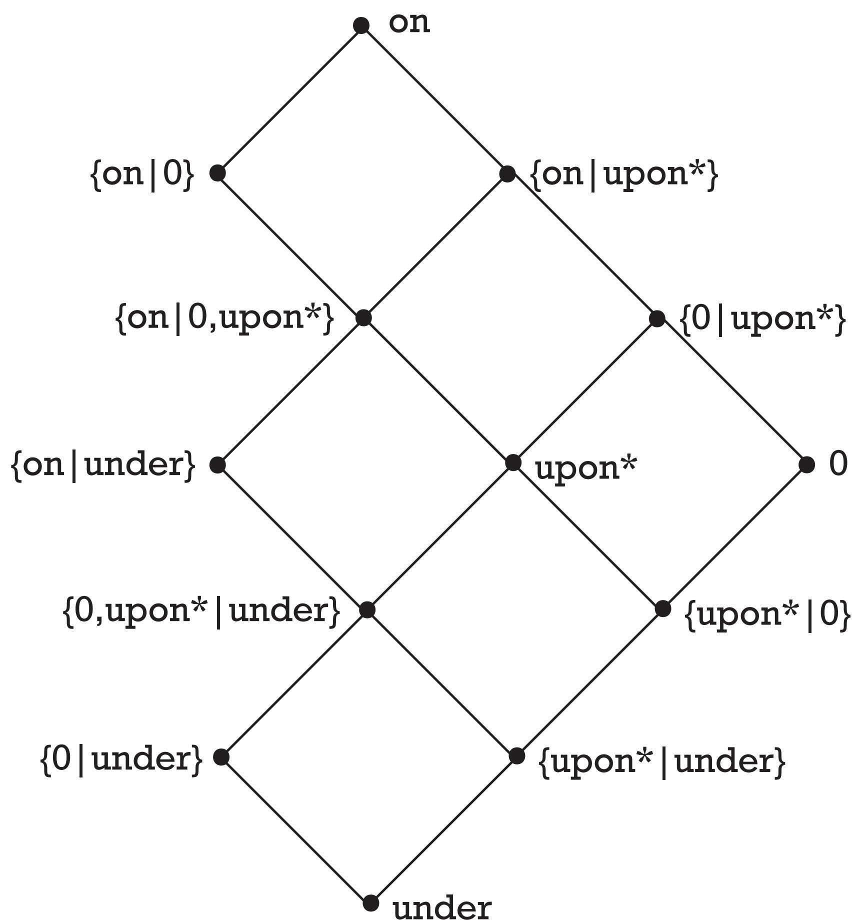


Figure 3.5: The partial-order structure of $\mathbf{Oslo}_2$.

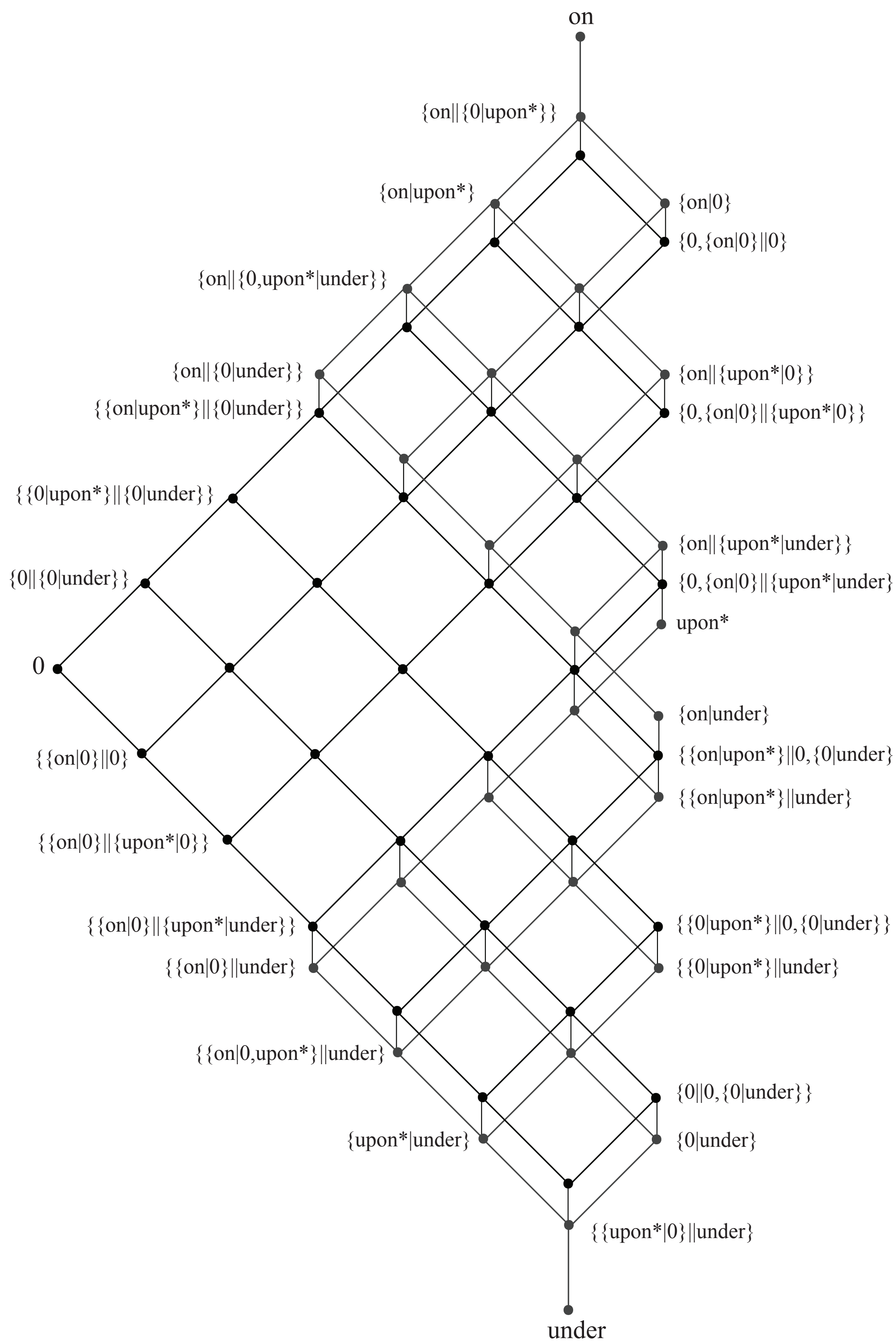


Figure 3.6: The partial-order structure of $\mathbf{Oslo}_3$.

Figure 3.6 depicts the lattice of the games composing $\mathbf{Oslo}_3$. There are 64 games that make up $\mathbf{Oslo}_3$, which form a 3-dimensional, distributive lattice. The lattice is composed of 17 levels, each containing at most 6 elements. Like the two previous days, the lattice of $\mathbf{Oslo}_3$ has a single top element, ON, and a single bottom element, UNDER.

It is interesting to note that some of the smallest games appear as early as day 3: for the universe of Oslo games, $+_{\text{OVER}} = \{0 \,|\, \{0 \,|\, \text{UNDER}\}\}$; and the smallest in any universe of games, $-_{\text{OFF}} = \{\{\text{ON} \,|\, 0\} \,|\, 0\}$.

3.4 Passification And Uponic Weight

Just as a parent pacifies a child by playing with them much beyond their own patience, so too is a game passified when one person is given the non-loopy "equivalent" of a pass move (thereby pacifying a youthful opponent by allowing them to play longer).

Definition 3.4.1. *For a game G, we may **passify** G to obtain the new game* $\mathrm{p}(G)$ *defined such that* $\mathrm{p}(0) = 0$ *and for all games* $G \ncong 0$*,*

$$\mathrm{p}(G) = \{\mathrm{p}(\mathbf{L}(G)), \{\mathrm{p}(\mathbf{L}(G)) \mid \mathrm{p}(\mathbf{R}(G))\} \mid \mathrm{p}(\mathbf{R}(G))\}.$$

For example, passifying the game $*$ gives us

$$\mathrm{p}(*) = \{\mathrm{p}(0), \{\mathrm{p}(0) \mid \mathrm{p}(0)\} \mid \mathrm{p}(0)\} = \{0, * \mid 0\} = \uparrow *.$$

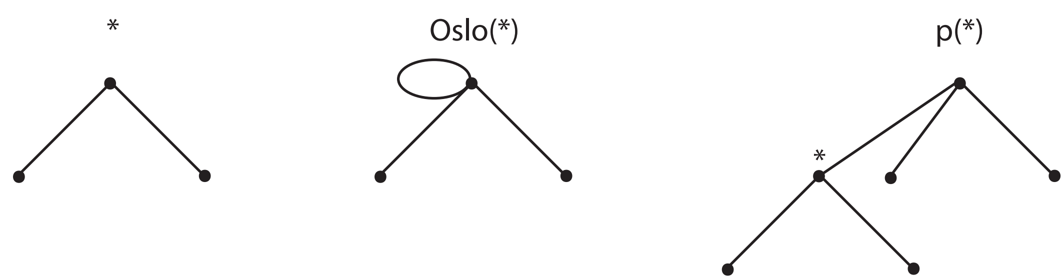


Figure 3.7: The game of $*$ along with both the Oslo and passified versions of the game, having values UPON$*$ and $\uparrow *$, respectively.

Passifying $\downarrow$, we get

$$\mathrm{p}(\downarrow) = \{\mathrm{p}(*), \{\mathrm{p}(*) \mid \mathrm{p}(0)\} \mid \mathrm{p}(0)\} = \{\uparrow *, \{\uparrow * \mid 0\} \mid 0\} = \{\uparrow *, \downarrow_2 \mid 0\}.$$

Finally, passification of $\uparrow$ gives us

$$\mathrm{p}(\uparrow) = \{\mathrm{p}(0), \{\mathrm{p}(0) \mid \mathrm{p}(*)\} \mid \mathrm{p}(*)\} = \{0, \{0 \mid \uparrow *\} \mid \uparrow *\} = \{\Uparrow \mid \uparrow *\} = \Uparrow^{[2]}.$$

The idea is that since Right cannot pass in an Oslo game, Left only has to pass once. The motivation for using this is that in finite games, atomic weight is well

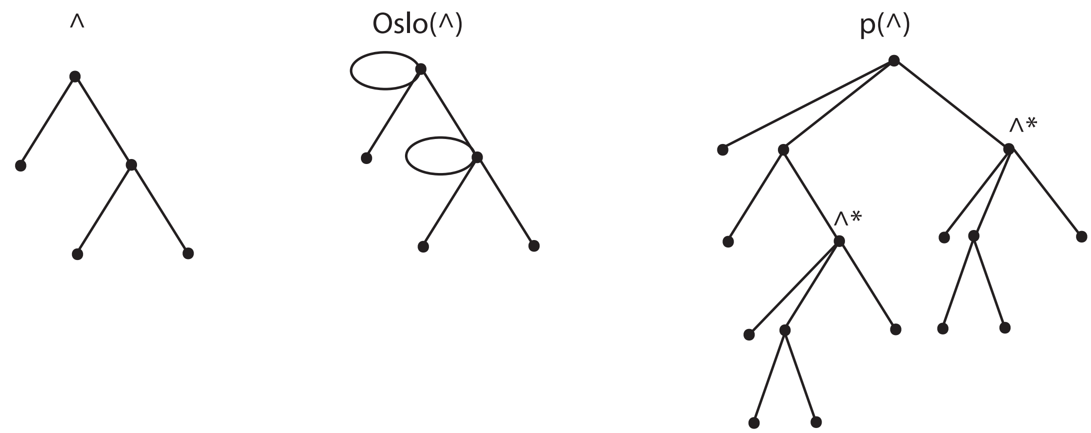


Figure 3.8: The game of $\uparrow$ along with both the Oslo and passified versions of the game, having values $2\,.\textsc{upon}*$ and $\Uparrow^{[2]}$, respectively.

understood. We will make use of the atomic weight of a passified game to define its uponic weight as an Oslo game.

Definition 3.4.2. *For an Oslo game G such that $G = \mathrm{oslo}\,(g)$, where g is in literal form, we define the* ***uponic weight*** *of G to be*

$$\mathrm{uw}\,(G) = \mathrm{aw}\,(\mathrm{p}\,(g))\,.$$

In our above examples, we see that $\mathrm{oslo}\,(*) = \textsc{upon}*$ while $\mathrm{p}\,(*) = \uparrow *$, which we know has atomic weight 1. Thus, $\mathrm{uw}\,(\mathrm{oslo}\,(*)) = \mathrm{aw}\,(\mathrm{p}\,(*)) = 1$ and so $\textsc{upon}*$ has uponic weight 1.

For the game $\downarrow$, $\mathrm{oslo}\,(\downarrow) = \{\textsc{upon}* \,|0\}$ has uponic weight 0, since $\mathrm{p}\,(\downarrow) = \{\uparrow *, \downarrow_2 |0\}$ which has atomic weight 0.

For the game $\uparrow$, $\mathrm{oslo}\,(\uparrow) = 2\,.\textsc{upon}*$ which (as we'd like to see) has uponic weight 2 since $\mathrm{p}\,(\uparrow) = \Uparrow^{[2]}$ and $\mathrm{aw}\left(\Uparrow^{[2]}\right) = 2$.

3.4.1 Loopy Subtraction Games

An example of how this might be applied would be to the class of subtraction games which we will call OSLO SUBTRACTION. Subtraction games are played on a pile of tokens, as in NIM. However, in a subtraction game, $S(A, B)$, each player is assigned a set of positive integers, set A for Left and set B for Right, and is allowed to remove a number of tokens from the pile only if that number falls in his or her assigned set. Thus, if Left has the set $\{1, 2\}$ and Right the set $\{3, 4\}$, from a pile of size 2, Left can move to a pile of size 1 or 0, while Right has no move. In OSLO SUBTRACTION, the moves are the same, except now Left is also allowed to pass.

For example, Table 3.3 gives details for the OSLO SUBTRACTION where each player is allowed to take either 1 or 2 stones on a move, and Left is always alloted a pass.

n	$g(n)$	$UW = \mathrm{aw}\,(\mathrm{p}\,(g(n)))$	$G(n)$
0	0	0	0
1	$*$	1	UPON$*$
2	$*2$	1	UPON$*$
3	$\{*, *2 \,\vert *, *2\}$	2	2.UPON$*$
4	$\{*2, \{*, *2 \,\vert *, *2\} \,\vert *2, \{*, *2 \,\vert *, *2\}\}$	2	2.UPON$*$
5	$\{g(3), g(4) \,\vert g(3), g(4)\}$	3	3.UPON$*$

Table 3.3: The Oslo subtraction game G =OSLO SUBTRACTION$(\{p, 1, 2\}, \{1, 2\})$. Rows give heap size; literal form of the underlying subtraction game $g = S\,(\{1, 2\}, \{1, 2\})$ at heap size n, $g(n)$; uponic weight (or equivalently, the atomic weight of the passified game, $\mathrm{p}\,(g(n))$; and, finally, the value of the game G at heap size n, $G(n)$.

At the end of this Chapter, we will explore open questions surrounding uponic weight and how it could be employed.

3.5 Side-Out Function

We introduce a recursive function that we will call the side-out function. It will give us another means of defining the Oslo form of a game.

Definition 3.5.1. *For all literal games G, H, we define the* ***side-out function*** *"G side-out H", denoted $G \odot H$, as follows:*

$$G \odot H = \begin{cases} 0 & \textit{if } \operatorname{lit}(H) = 0 \\ \{G \odot \mathbf{L}(H), \mathbf{L}(G) \odot H \,|\, G \odot \mathbf{R}(H), \mathbf{R}(G) \odot H\} & \textit{otherwise} \end{cases}$$

The function, at first glance, looks much like that of disjunctive sum. However, upon further inspection, we see that if $H \cong 0$, then $G \odot H = 0$, but $G + H = G$, possibly nonzero. The function takes its name from exactly this exception; once H is out of options, the game is over.

Proposition 3.5.2.
$G \odot 0 = 0.$
$0 \odot G = G.$

Proof. The first assertion follows from the definition of the side-out function, and so we know that $0 \odot 0 = 0$. For the second,

$$\begin{aligned} 0 \odot G &= \{0 \odot \mathbf{L}(G) \,|\, 0 \odot \mathbf{R}(G)\} \\ &= \{\mathbf{L}(G) \,|\, \mathbf{R}(G)\} \qquad \text{(by induction)} \\ &= G \end{aligned}$$

$\square$

In order to use the side-out function to produce an Oslo game, we must use the form $\textsc{on} \odot G$.

Recall that $\textsc{on} = \{\textsc{on} \,|\cdot\}$. In the special case of $\textsc{on} \odot G$, when $\operatorname{lit}(G) = 0$, $\textsc{on} \odot G = 0$ and otherwise,

$$\begin{aligned} \textsc{on} \odot G &= \{\textsc{on} \odot G, \textsc{on} \odot \mathbf{L}(G) \,|\, \textsc{on} \odot \mathbf{R}(G)\} \\ &= \{\textit{pass}, \textsc{on} \odot \mathbf{L}(G) \,|\, \textsc{on} \odot \mathbf{R}(G)\} \end{aligned}$$

Claim 3.5.3.

$$\operatorname{oslo}(G) = \textsc{on} \odot G$$

Proof. If $\operatorname{lit}(G) = 0$, then $\operatorname{oslo}(G) = 0 = \textsc{on} \odot 0$. If $\operatorname{lit}(G) \neq 0$, then

$$\begin{aligned}
\operatorname{oslo}(G) &= \{\operatorname{oslo}(G), \operatorname{oslo}(\mathbf{L}(G)) \,|\, \operatorname{oslo}(\mathbf{R}(G))\} \\
&= \{\operatorname{oslo}(G), \textsc{on} \odot \mathbf{L}(G) \,|\, \textsc{on} \odot \mathbf{R}(G)\} \text{ (by induction)} \\
&= \{\mathit{pass}, \textsc{on} \odot \mathbf{L}(G) \,|\, \textsc{on} \odot \mathbf{R}(G)\} \\
&= \textsc{on} \odot G
\end{aligned}$$

$\square$

3.6 Oslo Examples

We will next explore some examples of Oslo games. Initially, we will take a look at an Oslo variation on a classic game, the Oslo version of WYTHOFF'S GAME. We will then explore the Oslo versions of some common games that are considered "hard" under their non-loopy versions. While loopy games, in general, are considered to be more difficult to analyze, we will show that some games become much simpler under the Oslo condition. We will take a look at the OCTAL game .007 and GRUNDY'S GAME. Finally, we will introduce an open Oslo game, called the INDEPENDENCE GAME.

3.6.1 A Classic Oslo Variant

3.6.1.1 OSLO WYTHOFF

WYTHOFF'S GAME was introduced in 1907 by W. A. Wythoff [33]. The game is a variant of the game of NIM, played on two heaps.

In WYTHOFF'S GAME, play begins with two heaps of tokens. On a move, a player may take any number of tokens from one pile or an equal number of tokens from both piles, provided at least one token is removed on a turn. The player taking the last token wins.

For example, starting from a set of heaps of size x and $x+y$ $(x, y \geq 0)$, a position we will denote as $W(x, x+y)$, legal moves are to any (i) $W(i, x+y)$ for $0 \leq i < x$; (ii) $W(x, j)$ for $0 \leq j < x+y$; or $W(k, k+y)$ where $0 \leq k < x$.

It was shown in [33] that for this non-loopy version of the game, the first few $\mathcal{P}$-positions are given by

$$W(0,0) \qquad W(1,2) \qquad W(3,5) \qquad W(4,7) \qquad W(6,10) \qquad W(8,13) \quad \cdots$$

where the n^{th} $\mathcal{P}$-position is given by $W(a_n, b_n)$ with

$$W(a_n, b_n) = W\left(\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor\right)$$

where $\phi = \frac{1}{2}(1+\sqrt{5})$, the Golden Ratio.

We consider the Oslo variant of this game, which we will call OSLO WYTHOFF, where each player is still allowed the same moves as in the standard WYTHOFF'S GAME, but Left is also allowed to pass at any point before the game ends.

Table 3.4 gives the values of some smaller heap sizes in terms of multiples of UPON∗.

y	**5**	1	2	2	2	2	1	2
	4	1	2	2	2	1	2	2
	3	1	2	2	1	2	2	2
	2	1	2	1	2	2	2	2
	1	1	1	2	2	2	2	2
	0	0	1	1	1	1	1	1
$W(x,y)$		**0**	**1**	**2**	**3**	**4**	**5**	**6**
		x						

Table 3.4: Values of OSLO WYTHOFF for some smaller heap sizes in terms of multiples of UPON∗.

The values in the game of OSLO WYTHOFF can be summarized as follows:

Theorem 3.6.1. *In* OSLO WYTHOFF*,*

$$W(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \text{UPON}* & \text{if } x = y > 0 \\ \text{UPON}* & \text{if } xy = 0 \text{ but } x + y > 0 \\ 2.\text{UPON}* & \text{otherwise} \end{cases}$$

Proof.

$$W(0,0) = 0.$$

$$\begin{aligned} W(1,0) = W(0,1) &= \{pass, W(0,0) \,|W(0,0)\} \\ &= \{pass, 0 \,|0\} \\ &= \text{UPON}* . \end{aligned}$$

$$\begin{aligned} W(1,1) &= \{pass, W(1,0), W(0,1), W(0,0) \,|W(1,0), W(0,1), W(0,0)\} \\ &= \{pass, \text{UPON}*, 0 \,|\text{UPON}*, 0\} \\ &= \text{UPON}* . \end{aligned}$$

For $x > 1$, we consider the cases (i) Then $W(x, 0)$ and $W(0, x)$, (ii) $W(x, x)$ and (iii) $W(x, y)$ where $y > 0$.

(i) We first consider $W(x, 0)$ where $x > 1$.

$$W(x,0) = \{pass, \langle W(x-i,0)\rangle_{i=1}^{x} \,|\langle W(x-i,0)\rangle_{i=1}^{x}\}\,.$$

By induction, for $1 \leq i < x$, $W(x-i, 0) = \text{UPON}*$ and when $i = x$, $W(x-i, 0) = W(0, 0) = 0$. Thus,

$$W(x,0) = \{pass, \text{UPON}*, 0\,|\text{UPON}*, 0\} = \text{UPON}*\,.$$

Options for $W(0, x)$ are symmetric.

(ii) Consider $W(x, x)$ where $x > 1$.

$$W(x,x) = \{pass, M(x,x)\,|M(x,x)\}\,,$$

where

$$M(x,x) = \langle W(x-i,x)\rangle_{i=1}^{x} \cup \langle W(x,x-j)\rangle_{j=1}^{x} \cup \langle W(x-k,x-k)\rangle_{k=1}^{x}.$$

By induction,

$$W(x-i,x) = W(x,x-i) = \begin{cases} 2\,\text{.UPON}* & \text{when } 1 \leq i < x \\ 0 & \text{when } i = x \end{cases}$$

and

$$W(x-k,x-k) \begin{cases} \text{UPON}* & \text{when } 1 \leq k < x \\ 0 & \text{when } k = x \end{cases}$$

Hence, $M(x, x) = \{0, \text{UPON}*, 2\,\text{.UPON}*\}$ and so

$$W(x,x) = \{pass, 0, \text{UPON}*, 2\,\text{.UPON}*\,|0, \text{UPON}*, 2\,\text{.UPON}*\} = \text{UPON}*\,.$$

(iii) Finally, we consider the position $W(x, x + y)$ where $x > 1$ and $y > 0$.

$$W(x,x+y) = \{pass, M(x,x+y)\,|M(x,x+y)\}\,,$$

where

$$M(x,x+y) = \langle W(x-i,x+y)\rangle_{i=1}^{x} \cup \langle W(x,x+y-j)\rangle_{j=1}^{x+y} \cup \langle W(x-k,x+y-k)\rangle_{k=1}^{x}.$$

In this case, by induction,

$$W\left(x-i,x+y\right)=\begin{cases}2\,\textbf{.}\textsc{upon}* & \text{when } 1\le i<x\\ \textsc{upon}* & \text{when } i=x\end{cases}$$

and

$$W\left(x,x+y-j\right)=\begin{cases}2\,\textbf{.}\textsc{upon}* & \text{when } 1\le j<x+y \text{ and } j\ne y\\ \textsc{upon}* & \text{when } j=y \text{ or } j=x+y\end{cases}$$

and, finally,

$$W\left(x-k,x+y-k\right)\begin{cases}2\,\textbf{.}\textsc{upon}* & \text{when } 1\le k<x\\ \textsc{upon}* & \text{when } k=x\end{cases}$$

Thus, $M(x,x+y)=\{\textsc{upon}*, 2\,\textbf{.}\textsc{upon}*\}$, and so

$$W\left(x,x+y\right)=\left\{pass, \textsc{upon}*, 2\,\textbf{.}\textsc{upon}* \,\middle|\, \textsc{upon}*, 2\,\textbf{.}\textsc{upon}*\right\}=2\,\textbf{.}\textsc{upon}*.$$

□

3.6.2 *"Hard" Games Made Simple*

3.6.2.1 OSLO GRUNDY

We now consider the Oslo version of GRUNDY'S GAME. GRUNDY'S GAME is also played on heaps of tokens. In GRUNDY'S GAME, the only legal move is to split a single heap of tokens into two smaller heaps of different sizes. The winner is the the player who is last able to split a heap. For instance, a heap of size 3 can be split into heaps of size 1 and 2. A heap of size 4 can be split into to heaps, one of size 1 and the other of size 3. However, it could not have been split into two heaps both of size 2. Heaps of size 1 and 2 can no longer be split. We will denote a heap of size n in this game as OG$[n]$.

Definition 3.6.2. *A sequence* $\{\mathcal{G}(0),\mathcal{G}(1),\ldots,\mathcal{G}(k)\}$ *is said to be* ***arithmetic-periodic*** *if there exist* $e\ge 0$*,* s *and* $p>0$ *such that for all* $n\ge e$*,*

$$\mathcal{G}\left(n+p\right)=\mathcal{G}\left(n\right)+s.$$

*The least e and p for which this is true are called the **preperiod** length e and the **period** length p. The value s is referred to as the **saltus** and the elements of the preperiod, i.e. the sequence*

$$\mathcal{G}(0), \mathcal{G}(1), \ldots, \mathcal{G}(e),$$

*are called the **exceptional values**.*

For example, the sequence

$$5\ 4\ 3\ 2\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 2\ 2\ 2\ 3\ 3\ 3\ 4\ 4\ 4\ 5\ 5\ 5\ \cdots$$

has period 3 and saltus 1 with preperiod length 5.

In [3], the first 100 nim-values for GRUNDY'S GAME are given, with a note that there is a strong tendency to periodicity of length 3 after consideration of the first quarter-million values. Extensive calculations and the efforts of many people have taken part in tackling this game. Achim Flammenkamp has computed values of heap sizes beyond 34 billion[12]. Despite this and the fact that he has found periods for several other unsolved octal games, the periodicity of this sequence remains unknown[2, p. 144].

We will consider the Oslo version of GRUNDY'S GAME, which we will call OSLO GRUNDY, in which play is the same but with the option for Left to pass. We will denote this game with a heap of size n as $\textsc{og}[n]$.

Table 3.5 gives the first several heap sizes for OSLO GRUNDY.

We claim that for OSLO GRUNDY $n \geq 3$, $\textsc{og}[n] = \left\lfloor \frac{n}{2} \right\rfloor .\textsc{upon}*$. We will arrive at this result with the following theorem.

Theorem 3.6.3. *For* OSLO GRUNDY*,*

$$\textsc{og}[0] = \textsc{og}[1] = \textsc{og}[2] = 0$$

and $\textsc{og}[3] = \textsc{upon}*$*. For $k \geq 2$,*

$$\textsc{og}[2k] = \textsc{og}[2k+1] = k\,.\textsc{upon}*\,.$$

Proof. From heaps of size 0, 1 and 2, neither player has a move, so $\textsc{og}[0] = \textsc{og}[1] = \textsc{og}[2] = 0$. From $\textsc{og}[3]$, Left can pass and either player can move to $\textsc{og}[1]+\textsc{og}[2] = 0$. Thus, $\textsc{og}[3] = \{pass, 0\,|\,0\} = \textsc{upon}*$.

n	$\text{OG}[n]$	Left options	Right options
1	0		
2	0		
3	$1 \cdot \text{UPON}*$	$pass, 0$	0
4	$2 \cdot \text{UPON}*$	$pass, 1 \cdot \text{UPON}*$	$1 \cdot \text{UPON}*$
5	$2 \cdot \text{UPON}*$	$pass, 1 \cdot \text{UPON}*, 2 \cdot \text{UPON}*$	$1 \cdot \text{UPON}*, 2 \cdot \text{UPON}*$
6	$3 \cdot \text{UPON}*$	$pass, 2 \cdot \text{UPON}*$	$2 \cdot \text{UPON}*$
7	$3 \cdot \text{UPON}*$	$pass, 2 \cdot \text{UPON}*, 3 \cdot \text{UPON}*$	$2 \cdot \text{UPON}*, 3 \cdot \text{UPON}*$
8	$4 \cdot \text{UPON}*$	$pass, 3 \cdot \text{UPON}*$	$3 \cdot \text{UPON}*$
9	$4 \cdot \text{UPON}*$	$pass, 3 \cdot \text{UPON}*, 4 \cdot \text{UPON}*$	$3 \cdot \text{UPON}*, 4 \cdot \text{UPON}*$
10	$5 \cdot \text{UPON}*$	$pass, 4 \cdot \text{UPON}*, 5 \cdot \text{UPON}*$	$4 \cdot \text{UPON}*, 5 \cdot \text{UPON}*$
11	$5 \cdot \text{UPON}*$	$pass, 4 \cdot \text{UPON}*, 5 \cdot \text{UPON}*$	$4 \cdot \text{UPON}*, 5 \cdot \text{UPON}*$

Table 3.5: Values of OSLO GRUNDY for small heap sizes.

Let where $g_i(n) = \text{OG}[i] + \text{OG}[n-i]$. Let $k \geq 2$.

First, consider $\text{OG}[2k]$. By induction,

$$g_1(2k) = \text{OG}[1] + \text{OG}[2k-1] = 0 + \text{OG}[2(k-1)+1] = (k-1) \cdot \text{UPON}*,$$

$$g_2(2k) = \text{OG}[2] + \text{OG}[2k-2] = 0 + (k-1) \cdot \text{UPON}* = (k-1) \cdot \text{UPON}*,$$

and

$$g_3(2k) = \text{OG}[3] + \text{OG}[2k-3] = \text{UPON}* + (k-2) \cdot \text{UPON}* = (k-1) \cdot \text{UPON}*.$$

Let p be such that $1 < p < k$. Then by induction,

$$g_{2p}(2k) = \text{OG}[2p] + \text{OG}[2k-2p] = p \cdot \text{UPON}* + (k-p) \cdot \text{UPON}* = k \cdot \text{UPON}*$$

and

$$g_{2p+1}(2k) = \text{OG}[2p+1] + \text{OG}[2k-2p-1] = p \cdot \text{UPON}* + (k-p-1) \cdot \text{UPON}* = (k-1) \cdot \text{UPON}*.$$

Hence, for $i \leq 3$, $g_i(2k) = (k-1) \cdot \text{UPON}*$ and for all other $i > 3$, $g_i(2k)$ is $k \cdot \text{UPON}*$ for even i and $(k-1) \cdot \text{UPON}*$ for odd i. Thus for $k \geq 2$,

$$\begin{aligned}
\text{OG}[2k] &= \left\{ pass, \langle g_i(2k) \rangle_{i=1}^{2k-1} \,\middle|\, \langle g_i(2k) \rangle_{i=1}^{2k-1} \right\} \\
&= \{ pass, (k-1) \cdot \text{UPON}*, k \cdot \text{UPON}* \mid (k-1) \cdot \text{UPON}*, k \cdot \text{UPON}* \} \\
&= k \cdot \text{UPON}*.
\end{aligned}$$

Next, consider $\text{OG}[2k+1]$. By induction,

$$g_1(2k+1) = \text{OG}[1] + \text{OG}[2k] = 0 + k \centerdot \text{UPON}* = k \centerdot \text{UPON}*,$$

$$g_2(2k+1) = \text{OG}[2] + \text{OG}[2k-1] = 0 + (k-1) \centerdot \text{UPON}* = (k-1) \centerdot \text{UPON}*,$$

and

$$g_3(2k+1) = \text{OG}[3] + \text{OG}[2k-2] = \text{UPON}* + (k-1) \centerdot \text{UPON}* = k \centerdot \text{UPON}* .$$

Let p be such that $1 < p < k$. Then by induction,

$$g_{2p}(2k+1) = \text{OG}[2p] + \text{OG}[2k-2p+1] = p \centerdot \text{UPON}* + (k-p) \centerdot \text{UPON}* = k \centerdot \text{UPON}*$$

and

$$g_{2p+1}(2k+1) = \text{OG}[2p+1] + \text{OG}[2k-2p] = p \centerdot \text{UPON}* + (k-p) \centerdot \text{UPON}* = k \centerdot \text{UPON}* .$$

Hence, $g_2(2k+1) = (k-1) \centerdot \text{UPON}*$ and for all other i, $g_i(2k+1) = k \centerdot \text{UPON}*$. Thus for $k \geq 2$,

$$\begin{aligned} \text{OG}[2k+1] &= \left\{ pass, \langle g_i(2k+1) \rangle_{i=1}^{2k-1} \middle| \langle g_i(2k+1) \rangle_{i=1}^{2k-1} \right\} \\ &= \{ pass, (k-1) \centerdot \text{UPON}*, k \centerdot \text{UPON}* \mid (k-1) \centerdot \text{UPON}*, k \centerdot \text{UPON}* \} \\ &= k \centerdot \text{UPON}* . \end{aligned}$$

□

Thus, despite the fact that we still cannot answer the periodicity question for GRUNDY'S GAME, can say that its Oslo version is ultimately periodic, after only 2 exceptions. For $n > 2$, we see that OSLO GRUNDY is periodic with period 2 and saltus UPON$*$. It is still conjectured that the non-loopy GRUNDY'S GAME is ultimately periodic [3, p. 111], though this question has now been open for many decades.

3.6.2.2 OSLO(OCTAL .007)

The second "hard" game that we will look at is the octal game .007, again played on heaps of tokens. For a move, a player is allowed to take 3 tokens from any one pile at which point he may split that pile into 2 separate heaps, if he wishes. For the non-loopy game .007, we will denote a heap of size n as $[n]$. From heaps of size 0, 1 and 2, there are no allowed moves, and so $[0] = [1] = [2] = 0$. From $[3]$, each player has a move to 0 and so $[3] = *$. From $[4]$, one can move to $[1]$, but not split, so that $[4] = *$ as well. From $[6]$, a choice exists to split the remaining pile. Each player can move to either $[3] = *$ or $[1] + [2] = 0$. Hence, $[6] = *2$.

The nim-sequence of OCTAL .007 begins as follows, starting from a heap of size 0:

$$000111220331110433322244055222330501113335 6 \cdots [3]$$

The period of this sequence is unknown.

We will call the Oslo variant of this game OSLO(OCTAL .007), in which we will allow Left to pass, and denote heaps of size n as $O_{.007}[n]$. Once we allow Left to pass under the Oslo variant of this game, we immediately see an improvement in the periodicity of its values. Table 3.6 depicts the values of OSLO(OCTAL .007), where we can see that the sequence of game values is purely periodic with period 5 and saltus UPON$*$.

n	$O_{.007}[n]$	Heap sizes available
0	0	
1	0	
2	0	
3	$1\centerdot$UPON$*$	$O_{.007}[0]$
4	$1\centerdot$UPON$*$	$O_{.007}[1]$
5	$1\centerdot$UPON$*$	$O_{.007}[2]$, $O_{.007}[1] + O_{.007}[1]$
6	$1\centerdot$UPON$*$	$O_{.007}[3]$, $O_{.007}[1] + O_{.007}[2]$
7	$1\centerdot$UPON$*$	$O_{.007}[4]$, $O_{.007}[1] + O_{.007}[3]$, $O_{.007}[2] + O_{.007}[2]$
8	$2\centerdot$UPON$*$	$O_{.007}[5]$, $O_{.007}[1] + O_{.007}[4]$, $O_{.007}[2] + O_{.007}[3]$
9	$2\centerdot$UPON$*$	$O_{.007}[6]$, $O_{.007}[1] + O_{.007}[5]$, $O_{.007}[2] + O_{.007}[4]$, $O_{.007}[3] + O_{.007}[3]$
10	$2\centerdot$UPON$*$	$O_{.007}[7]$, $O_{.007}[1] + O_{.007}[6]$, $O_{.007}[2] + O_{.007}[5]$, $O_{.007}[3] + O_{.007}[4]$
$\vdots$	$\vdots$	
13	$3\centerdot$UPON$*$	
$\vdots$	$\vdots$	

Table 3.6: Values of OSLO(OCTAL .007) for small heap sizes.

Theorem 3.6.4. *For* OSLO .007*, where* $n = 5k - 2 + r \geq 0$ *with* $k \in \mathbb{Z}^{\geq 0}$ *and* $r \in \{0, 1, 2, 3, 4\}$,

$$O_{.007}[n] = k \,\blacksquare\, \text{UPON} * .$$

Proof. From heaps of size 0, 1 and 2, neither player has a move, so $O_{.007}[0] = O_{.007}[1] = O_{.007}[2] = 0$. We also calculate the values for $n = 3$ and $n = 4$, the last games before we are able to split the remaining heap.

$$O_{.007}[3] \quad = \{pass, O_{.007}[0] \,|\, O_{.007}[0]\} \quad = \{pass, 0 \,|\, 0\} \quad = \text{UPON}*$$

$$O_{.007}[4] \quad = \{pass, O_{.007}[1] \,|\, O_{.007}[1]\} \quad = \{pass, 0 \,|\, 0\} \quad = \text{UPON}*$$

Let $n = 5k - 2 + r \geq 5$. Then

$$O_{.007}[5k - 2 + r] = \left\{ pass, \langle s_i \rangle_{i=0}^{5(k-1)+r} \,\middle|\, \langle s_i \rangle_{i=0}^{5(k-1)+r} \right\}$$

where $s_i = O_{.007}[i] + O_{.007}[5(k-1) + r - i]$.

When $i = 0$, then by induction,

$$\begin{aligned} s_0 =& O_{.007}[0] + O_{.007}[5(k-1) + r - 0] \\ =& 0 + O_{.007}[5(k-1) + (r+2) - 2] \\ =& \begin{cases} (k-1) \,\blacksquare\, \text{UPON}* & \text{if } r < 3 \\ k \,\blacksquare\, \text{UPON}* & \text{if } r \geq 3 \end{cases} \end{aligned}$$

When $i = 1$, by induction we have

$$\begin{aligned} s_1 =& O_{.007}[1] + O_{.007}[5(k-1) + r - 1] \\ =& 0 + O_{.007}[5(k-1) + (r+1) - 2] \\ =& \begin{cases} (k-1) \,\blacksquare\, \text{UPON}* & \text{if } r < 4 \\ k \,\blacksquare\, \text{UPON}* & \text{if } r \geq 4 \end{cases} \end{aligned}$$

When $i = 2$, then by induction

$$\begin{aligned} s_2 =& O_{.007}[0] + O_{.007}[5(k-1) + r - 2] \\ =& 0 + (k-1) \,\blacksquare\, \text{UPON}* \\ =& (k-1) \,\blacksquare\, \text{UPON}* \end{aligned}$$

For games $O_{.007}[n]$ where $n \geq 5$, this move to s_2 always exists. We shall see that this move to $(k-1)\,\mathbf{.}\,\text{UPON}*$ is always the best move for right.

We now consider moves s_i when $3 \leq i \leq 5(k-1)+r-i$. Let $i = 5x+y-2$ where $x \in \mathbb{Z}^{>0}$ and $y \in \{0,1,2,3,4\}$. Note that if $r < y+1$, then $0 \leq (r-y+4) < 5$ but if $r \geq y+1$ then $5 \leq (r-y+4) \leq 8$. By induction we have

$$\begin{aligned} s_i =& O_{.007}[i] + O_{.007}[5(k-1)+r-i] \\ =& O_{.007}[5x+y-2] + O_{.007}[5(k-1)+r-5x-y+2] \\ =& x\,\mathbf{.}\,\text{UPON}* + O_{.007}[5(k-1-x)+(r-y+4)-2] \\ =& \begin{cases} (k-1)\,\mathbf{.}\,\text{UPON}* & \text{if } r < y+1 \\ k\,\mathbf{.}\,\text{UPON}* & \text{if } r \geq y+1 \end{cases} \end{aligned}$$

With this, we have everything. We know that both Left and Right have a move to $s_2 = (k-1)\,\mathbf{.}\,\text{UPON}*$. Since $(k-1)\,\mathbf{.}\,\text{UPON}* < k\,\mathbf{.}\,\text{UPON}*$ and all other potential moves are to $k\,\mathbf{.}\,\text{UPON}*$, Right will always play to $(k-1)\,\mathbf{.}\,\text{UPON}*$. While Left would prefer $k\,\mathbf{.}\,\text{UPON}*$, we need not concern ourselves with the value of r nor the relationship between r and y in order to determine if he has that option. This is simply due to the fact that for $S \subseteq \{(k-1)\,\mathbf{.}\,\text{UPON}*, k\,\mathbf{.}\,\text{UPON}*\}$, even possibly empty, then $\{pass, S \,|(k-1)\,\mathbf{.}\,\text{UPON}*\} = k\,\mathbf{.}\,\text{UPON}*$. Thus, for all $n = 5k+r-2$, $O_{.007}[n] = k\,\mathbf{.}\,\text{UPON}*$. $\square$

Thus, we find that OSLO(OCTAL .007) is purely periodic with period 5 and saltus UPON$*$. Further, we conjecture that OSLO(OCTAL $.0^n7$), the Oslo variant of the octal game $.0^n7$ in which a player is allowed to take $n+1$ tokens from any one heap and then split that heap if he wishes, is also purely periodic with period $2n+1$ and saltus UPON$*$.

3.6.3 An Open Oslo Game

While some nice results can be obtained regarding Oslo variants, we certainly don't want to lead the reader to believe that open games aren't ripe for the picking. The following is an Oslo game for which little is known.

3.6.3.1 INDEPENDENCE GAME

In order to discuss the next game, we require a few basic definitions from graph theory.

Definition 3.6.5. *[13, p. 75] For a graph G with vertices V_G, a subset $I \subseteq V_G$ is said to be an **independent set** (of vertices) if no pair of vertices in S is joined by an edge. Equivalently, I is a subset of mutually non-adjacent vertices of G.*

*An independent set I is said to be a **maximal independent set** if there is no independent set I' such that $I \subset I'$, i.e. no larger independent set contains I.*

*The **independence number**, denoted $\alpha(G)$, of the graph G is the number of vertices in a largest independent set in G.*

The INDEPENDENCE GAME is an example of a naturally occurring Oslo game. In the INDEPENDENCE GAME, we start with a graph G. Every vertex of G begins with a white token on it. On his turn, Right colors a token black, such that the vertices with black tokens form an independent set. On her turn, Left may swap the tokens of two adjacent vertices, provided that the vertices containing black tokens still form an independent set. The game ends when the vertices containing black tokens form a maximal independent set.

There are graphs such that the INDEPENDENCE GAME is confused with zero. For example, if we look at the path on 5 vertices with the first and last vertices included in the independent set I, as in Figure 3.9, we see that Left can effectively *pass* by switching any two vertices that are both not in I, and thereby "moving" to the game position she started from. She may also end the game by swapping two vertices at one end and moving to a maximal independent set. Right has a move to zero by playing the middle vertex. Thus, this position has value $\{0, \mathit{pass}|0\} = \text{UPON}* \parallel 0$.

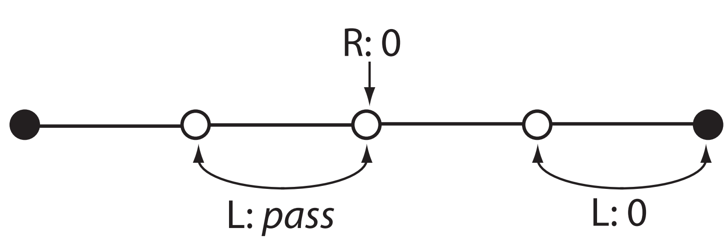


Figure 3.9: INDEPENDENCE GAME position on P_5 having value $\{0, \mathit{pass}\,|0\} = \text{UPON}*$.

In Figure 3.10, we see an example of a graph that is both positive and connected. Consider this P_9 with the first, last and middle vertices included in the independent set. From here, Left can essentially *pass* by switching any two vertices that are both not in I. Or, he may play as indicated to a position with value UPON$*$, from which Left and Right have moves to zero and Left can still choose to pass. Likewise, Right can make the indicated move to UPON$*$. Thus, the game has value $\{pass, \text{UPON}* \,|\, \text{UPON}*\} = \{0 \,|\, \text{UPON}*\} = 2\,\centerdot\,\text{UPON}*$.

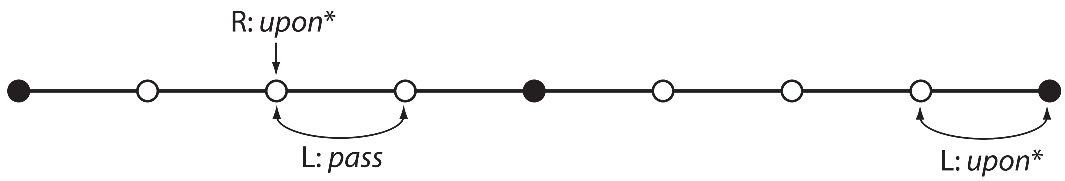


Figure 3.10: INDEPENDENCE GAME position on P_9 having value $2\,\centerdot\,\text{UPON}*$.

In Figure 3.11, we demonstrate that we can extend this notion on to a path on $4k+1$ vertices to obtain the value $k\,\centerdot\,\text{UPON}*$ for any integer $k > 0$.

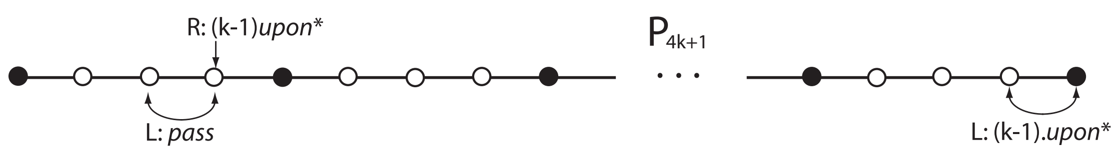


Figure 3.11: INDEPENDENCE GAME position on P_{4k+1} having value $k\,\centerdot\,\text{UPON}*$.

If the graph is well-covered (i.e. all maximal independent sets are the same size), then the INDEPENDENCE GAME is negative. It is also possible to have negative values in graphs where this is not the case, such as in the P_5 of Figure 3.12. This demonstrates a graph that is not well-covered, yet exhibits all possible outcome classes; $\mathcal{N}$-positions (UPON$*$), $\mathcal{L}$-positions ($\{0 \,|\, \text{UPON}*\}$), $\mathcal{R}$-positions (UNDER and $\{\text{UPON}* \,|\, 0\}$) and of course $\mathcal{P}$-positions (0).

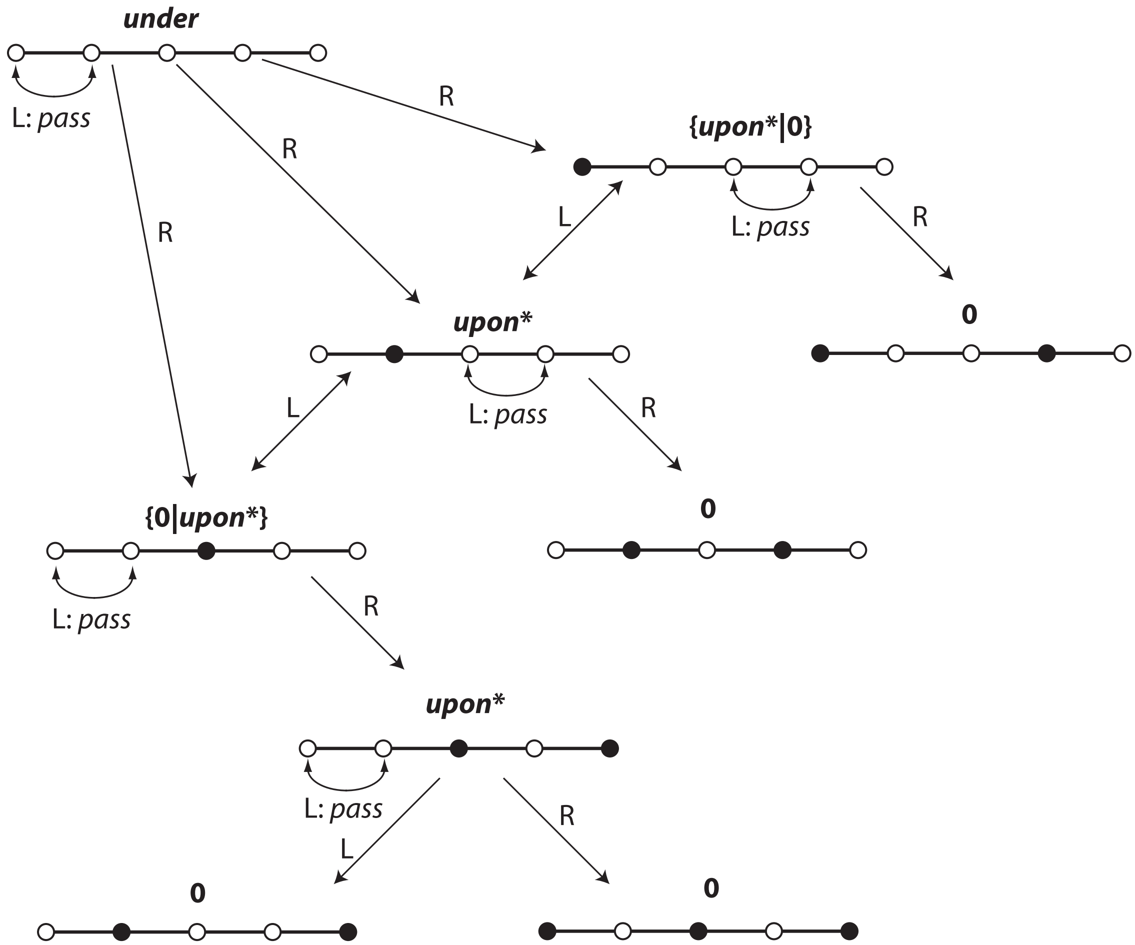


Figure 3.12: All INDEPENDENCE GAME positions on P_5 demonstrating the presence of all possible outcome classes.

3.6.4 The Next Moves

The surface of Oslo games has only barely been scratched. Two large areas that should be explored are the realms of all-small Oslo games and also those games in which both players are granted a pass.

3.6.4.1 TwoSlo Games

The obvious next game to analyze would be the one in which both players are always allowed to pass. We will call these **Two-S**ided **lo**opy games, or **TwoSlo** games. Appropriately, we say this as “too slow” games, as they may take a while!

Definition 3.6.6.
If $\operatorname{lit}(G) = 0$*, then* $\operatorname{TwoSlo}(G) = 0$*. Otherwise,*

$$\begin{aligned}\operatorname{TwoSlo}(G) &= \{\operatorname{TwoSlo}(G), \operatorname{TwoSlo}(\mathbf{L}(G)) \,|\operatorname{TwoSlo}(G), \operatorname{TwoSlo}(\mathbf{R}(G))\} \\ &= \{\mathit{pass}, \operatorname{TwoSlo}(\mathbf{L}(G)) \,|\mathit{pass}, \operatorname{TwoSlo}(\mathbf{R}(G))\}\end{aligned}$$

Similarly, we can use the side-out function to return the TwoSlo form of a game G by returning $\textsc{dud} \odot G$.

Further, recall that $\textsc{dud} = \{\textsc{dud} \,|\textsc{dud}\}$. In the special case of $\textsc{dud} \odot G$, we have the following:

$$\begin{aligned}\textsc{dud} \odot G &= \{\textsc{dud} \odot G, \textsc{dud} \odot \mathbf{L}(G) \,|\textsc{dud} \odot G, \textsc{dud} \odot \mathbf{R}(G)\} \\ &= \{\mathit{pass}, \textsc{dud} \odot \mathbf{L}(G) \,|\mathit{pass}, \textsc{dud} \odot \mathbf{R}(G)\}\end{aligned}$$

Claim 3.6.7.

$$\operatorname{TwoSlo}(G) = \textsc{dud} \odot G$$

Proof.
If $\operatorname{lit}(G) = 0$, then $\operatorname{TwoSlo}(G) = 0 = \textsc{dud} \odot 0$. If $\operatorname{lit}(G) \neq 0$, then

$$\begin{aligned}\operatorname{TwoSlo}(G) &= \{\operatorname{TwoSlo}(G), \operatorname{TwoSlo}(\mathbf{L}(G)) \,|\operatorname{TwoSlo}(G), \operatorname{TwoSlo}(\mathbf{R}(G))\} \\ &= \{\operatorname{TwoSlo}(G), \textsc{dud} \odot \mathbf{L}(G) \,|\operatorname{TwoSlo}(G), \textsc{dud} \odot \mathbf{R}(G)\} \text{ (by induction)} \\ &= \{\mathit{pass}, \textsc{dud} \odot \mathbf{L}(G) \,|\mathit{pass}, \textsc{dud} \odot \mathbf{R}(G)\} \\ &= \textsc{dud} \odot G\end{aligned}$$

□

While the side-out function might be of help in analyzing TwoSlo positions, not much is known about games of this form. After a preliminary look at the partial order of the TwoSlo form of games by day n, we can report that they do not form a lattice.

3.6.4.2 All-Small Oslo Games

Yet another field that is almost completely unexplored is that of all-small Oslo games. Motivation for this once again comes from loopy subtraction games. The example given in 3.4.1 describes the subtraction game where either player is allowed to take 1 or 2 tokens and Left is also allowed to pass. This happens to be an all-small game. What about other all-small Oslo games? Upon preliminary inspection of the poset of all-small Oslo games, the day 2, 3 and 4 posets contain 6, 17 and 72 elements, respectively. Day 4 is not distributive, as the non-distributive N_5 is present. Beyond that, the questions far outweigh the answers. Do these even always form a lattice?

Chapter 4

Option-Closed Games

We now switch gears and consider a class of combinatorial games called option-closed games. Option-closed games have the property that a player's move only eliminates options for that player and does not add any new ones. We will first explore the details of the internal structure of option-closed games, including a nice results from Nowakowski and Ottaway [23] that the reduced canonical forms of these games are either numbers or switches[1]. We will then introduce a function that takes a game and returns its option-closed form. We will introduce the concept of left- and right-threats and show that for a game to be infinitesimally close to its option-closed form, a game infinitesimally close to these threats must exist in the set of left and right options, respectively. We then turn our focus to the lattice of option-closed games by day n and show that these games form a planar lattice. As examples of option-closed games, we examine the games of MAZE, ROLL THE LAWN and CRICKET PITCH. For the last, we provide a complete analysis, neatly employing the use of ordinal sums from Section 2.2.

4.1 Structure

Option-closed games are those games in which, from any position in the game, every move that a player can reach in two moves, he could have made in one. That is, the set of options available to a player from a position in the game is a superset of the those options available to him after having made a move from that position.

[1]A **switch** is a game of the form $\{a \,|\, b\}$ for numbers $a \geq b$

Definition 4.1.1. *For a game G, we will let $\mathbf{L}^n(G)$ denote the set of all options that can be reached in n consecutive Left moves from G. Thus, $\mathbf{L}^0(G) = \{G\}$ and for $n > 0$, we define*

$$\mathbf{L}^n(G) = \mathbf{L}\left(\mathbf{L}^{n-1}(G)\right).$$

*We will call a left option G' a **first left option** of G if G' can be reached in one move by Left, but not in two, i.e. $G' \in \mathbf{L}(G) \setminus \mathbf{L}^2(G)$. A **first right option** is defined analogously.*

Most of combinatorial game theory considers the canonical forms of games, in which $\mathbf{L}(G)$ and $\mathbf{R}(G)$ have had dominated options removed and reversible options bypassed. When dealing with option-closed games, we require the literal form of the game, $\operatorname{lit}(G)$, with all the options included, even the bad ones.

Definition 4.1.2. *[23] A game G is called **option-closed** if, in literal form, $\mathbf{L}^2(G) \subset \mathbf{L}(G)$, $\mathbf{R}^2(G) \subset \mathbf{R}(G)$ and, recursively, all the followers of G are option-closed.*

It is important to note that whether or not a game is option-closed depends on its form. The canonical form of the game need not be option-closed. An equivalence class of games may contain members that are option-closed as well as some that are not. As a simple example, the game $\{0, 1\,|0\}$ is option-closed while its canonical form, $\{1\,|0\}$, is not. Likewise, $\{0\,|\{1\,|0\}, 0\}$ is option-closed but $\{0\,|\{1\,|-1\}, 0\}$ is not since Right can move to -1 in two moves but not in one. The fact that a game is option-closed is intrinsic to the game and is not necessarily identifiable from the canonical form. For example, while both $\{0\,|0, *2\}$ and $\{0\,|0\}$ have value $*$, only the latter is option-closed. Thus, when discussing an option-closed game G, we assume that we leave G in the representation that satisfies Definition 4.1.2.

We note that while the term option-closed places restrictions on the literal form of a game, we will still represent games in their canonical form when it is understood that we are dealing with option-closed games. Thus, we will represent G as $\operatorname{can}(G)$ unless necessary to specify otherwise.

Canonical forms of option-closed games can be complicated. Through the use of reduced canonical forms, we can express them much more simply when the differences between the options are infinitesimals.

For example, the following is an option-closed game in canonical form:

$$G = \{1, \{1\,|0\}\,, \{1, \{1\,|0\}\,|0, \{1\,|0\}\}\,|0, \{1\,|0\}\,, \{1, \{1\,|0\}\,|0, \{1\,|0\}\}\}\,.$$

The reduced canonical form of a game can be much less complicated than the game itself, especially in the case of option-closed games. In the example above, $\mathrm{rcf}\,(G) = \{1\,|0\}$.

The reduced canonical form of any option-closed games is simple and the difference between the game and its reduced canonical form is small. In Nowakowski and Ottaway [23], it was shown that the reduced canonical form of an option-closed game is either a number or a switch $\{a|b\}$ of numbers $a \geq b$.

Lemma 4.1.3. *[23, Lem. 10] If $a \geq b$ are both numbers, then $b \leq_{\mathbf{Inf}} \{a\,|b\} \leq_{\mathbf{Inf}} a$.*

Nowakowski and Ottaway showed this by noting that for any n,

$$\{a\,|b\} - a < n\cdot \uparrow\,.$$

For example, take numbers $a \geq b$ as above. Clearly, a and $\{a\,|b\}$ are incomparable. However, it can be shown that $a - \{a\,|b\} + 3\cdot \uparrow > 0$. Left wins by moving to $a - b + 3\cdot \uparrow \geq 3\cdot \uparrow > 0$. By the Number Avoidance Theorem (1.2.52), Right's best move is in either $\{a\,|b\}$ or in $3\cdot \uparrow$. Moving in $\{a\,|b\}$, he plays to $a - a + 3\cdot \uparrow = 3\cdot \uparrow > 0$. If he moves in $3\cdot \uparrow$, he plays to $a + \{a\,|b\} + \Uparrow *$, from which Left wins by responding to $a - b + \Uparrow * \geq \Uparrow * > 0$.

The main result of Nowakowski and Ottaway [23] is that the reduced canonical form of an option-closed game is simple.

Theorem 4.1.4. *[23, Thm. 11] If G is an option-closed game, then its reduced canonical form, $\mathrm{rcf}\,(G)$, is either a number or a switch $\{a\,|b\}$ where $a \geq b$ are both numbers.*

Thus, every option-closed game, no matter how complicated in canonical form, is either infinitesimally close to a number or a switch. While by definition the difference between a game and its reduced canonical form is an infinitesimal, Nowakowski and Ottaway were able to define even tighter bounds for the class of option-closed games.

Theorem 4.1.5. *[23, Thm. 16] If G is an option-closed game, then $\Downarrow * < G - \mathrm{rcf}\,(G) < \Uparrow *$.*

Since this result was published by Nowakowski and Ottaway [23], Neil McKay has been able to tighten these bounds [20]. He has been able to show that

$$-.11111...12* < G - \mathrm{rcf}\,(G) < .111111...12*$$

Recall definition 1.2.54 for left and right stops. As mentioned earlier, these stops represent the best numbers that a player can achieve in alternating play. Normally, this could take many plays before the game becomes a number. However, because of the structure of option-closed games, the Left and Right stops must be included in the set of first left and first right options, respectively. Hence, in option-closed games, they can be reached in only one move.

Lemma 4.1.6. *[23, Lemma 4] Let G be an option-closed game that is not a number, then $\mathrm{L}_0(G) \in \mathbf{L}(G)$ and $\mathrm{R}_0(G) \in \mathbf{R}(G)$.*

Corollary 4.1.7. *[23, Cor. 13] Suppose G is option-closed and let a, b and x be numbers with $a \geq b$.*

(i) If $\mathrm{rcf}\,(G) = x$, then $\mathrm{L}_0(G) = \mathrm{R}_0(G) = x$.

(ii) If $\mathrm{rcf}\,(G) = \{a\,|b\}$, then $\mathrm{L}_0(G) = a$ and $\mathrm{R}_0(G) = b$.

In each case, there exist left and right options with $G^L = \mathrm{L}_0(G)$ and $G^R = \mathrm{R}_0(G)$, respectively.

Definition 4.1.8. *[21] For a game G, a **left-option-closed sequence** is a sequence of left options of G,*

$$\alpha = \langle x_0, x_1, x_2, \ldots, x_m \rangle$$

such that x_0 is a first left option of G; for all $0 \leq i < m$, x_{i+1} is a first left option of x_i; and x_m is a number.

*Similarly, a **right-option-closed sequence** is a sequence of right options of G,*

$$\beta = \langle y_0, y_1, y_2, \ldots, y_n \rangle$$

such that y_0 is a first right option of G; for all $0 \leq j < n$, y_{j+1} is a first right option of y_j; and y_n is a number.

Definition 4.1.9. *[21] For a game G, a* ***norm*** *of a left-option-closed sequence α, denoted $\overline{\alpha}$, is the maximum of all numbers in α. Similarly, for a right-option-closed sequence β, $\overline{\beta}$ is the minimum of all numbers in the sequence. The norm of $\langle \emptyset \rangle$ is not defined.*

Thus, the norm of an option-closed sequence represents that number in the sequence that a player would choose to play to.

Lemma 4.1.10. *[21, Lemma 9] Let G be an option-closed game and $\alpha = \langle x_i \rangle_{i=0}^{n}$ be a left-option-closed sequence. Let k be the least index such that x_k is a number. Then $\overline{\alpha} = x_k$.*

Proof. Suppose that α is a left-option-closed sequence. Let k be the least index such that x_k is a number. Suppose that there exists $j > k$ such that x_j is a number. Then x_j is a left option of x_k and so $x_j < x_k$. Similar arguments hold if α were a right-option-closed sequence. □

For example, if we had the right-option-closed sequences $\alpha_1 = \langle \{2 \,|{-1}\}, -1 \rangle$ and $\alpha_2 = \langle \{2 \,|{-1}\}, -1, \{-1 \,|0\}, 0 \rangle$, we see that the norms of these sequences are the same, that is $\overline{\alpha_1} = \overline{\alpha_2} = -1$.

In fact, we will see that in a right-option-closed sequence, the norm is actually the right stop.

Lemma 4.1.11. *Let $G = \{\mathbf{L}(G) \,|\mathbf{R}(G)\}$ be an option-closed game with $\mathrm{R}_0(G) = x$ for some number $x \in \mathbf{R}(G)$. Then $\forall G^R \in \mathbf{R}(G), x \leq_{\mathbf{Inf}} G^R$.*

Proof. If $G^R = x$, this is clearly true, so we can assume that $G^R \neq x$. Suppose $x \in G^{RR}$. Then G^R is not a number since x is a right option of G^R and if both are numbers, then $G^R < x = \mathrm{R}_0(G)$, which is impossible. If G^R is not numberish, then by Corollary 4.1.7, we have that $\mathrm{rcf}\left(G^R\right) = \left\{\mathrm{L}_0\left(G^R\right) \middle|\mathrm{R}_0\left(G^R\right) = x\right\}$. By Lemma 4.1.4, $x \leq_{\mathbf{Inf}} \left\{\mathrm{L}_0\left(G^R\right) |x\right\} \equiv_{\mathbf{Inf}} G^R$. If G^R is numberish, then $\mathrm{L}_0\left(G^R\right) = \mathrm{R}_0\left(G^R\right) = x$ by Lemma 4.1.4, and so $G^R \equiv_{\mathbf{Inf}} x$.

Suppose $x \notin G^{RR}$. If G^R is a number, then $G^R > x$ by definition of a number. If G^R is not a number, then G^R is right-option-closed with $G^R \geq_{\mathbf{Inf}} \mathrm{R}_0(G) > x$. Thus $G^R \geq_{\mathbf{Inf}} x = \mathrm{R}_0(G)$. □

The same proof, with the signs reversed, holds true for the left options.

If an option-closed game G is not a number, then $\overline{\mathbf{L}(G)}$ is the left stop of G and $\overline{\mathbf{R}(G)}$ is the right stop. We note that $\mathbf{L}(G)$ and $\mathbf{R}(G)$ may contain more than one left- and right-option-closed sequence, respectively.

Lemma 4.1.12. *Get G be an option-closed game and $\langle \mathbf{L}(G) \rangle = \langle x_i \rangle_{i=0}^{m}$ and $\langle \mathbf{R}(G) \rangle = \langle y_i \rangle_{i=0}^{n}$ be left- and right-option-closed sequences. Let $X = \overline{\langle \mathbf{L}(G) \rangle}$ and $Y = \overline{\langle \mathbf{R}(G) \rangle}$. Then*

(i) $\operatorname{rcf}(G) = \operatorname{rcf}(\{X \,|Y\})$;

(ii) If G is not a number, then $Y \leq_{\mathbf{Inf}} G \leq_{\mathbf{Inf}} X$;

(iii) If G is a number, then $G = \{X \,|Y\}$.

Proof. We induct on the birthday of the position. If the position has birthday 0 (i.e. $\mathbf{L}(G) = \mathbf{R}(G) = \emptyset$), then all claims are trivially true. Suppose that these results hold for all option-closed games with birthday at most $n-1$. Let $b(G) = n$. We may assume that $\mathbf{L}(G)$ is not empty and that $\overline{\langle \mathbf{L}(G) \rangle} = x_k$ for some $0 \leq k \leq m$.

(i) If $k = 1$, then x_1 is a number and so by Lemma 1.2.44, $\forall i$, $x_i < x_1$ since $x_i \in \mathbf{L}(x_1)$. If $k \neq 1$, then for $1 \leq i < k$, x_i is not a number and so by induction $x_i \leq_{\mathbf{Inf}} x_k$ from (ii). For $i > k$, $x_i < x_k$ by Lemma 1.2.44 since $x_i \in \mathbf{L}(x_k)$. Combining these results, we have that for all i, $x_k \geq_{\mathbf{Inf}} x_i$. Thus, $\operatorname{rcf}(G) = \operatorname{rcf}(\{x_k \,|Y\})$.

(ii) If G is not a number, then $\mathbf{L}(G) \neq \emptyset$ and so $\overline{\langle \mathbf{L}(G) \rangle} = x_j$ for some $0 \leq j \leq m$ and $\overline{\langle \mathbf{R}(G) \rangle} = y_k$ for some $0 \leq k \leq n$. Thus, $\operatorname{rcf}(G) = \operatorname{rcf}(\{x_j \,|y_k\})$. If $x_j < y_k$, then for all p and q, $x_p \leq_{\mathbf{Inf}} x_j < y_k \leq_{\mathbf{Inf}} y_q$. But then $G = \{x_j \,|y_k\}$ is a number, which is a contradiction. Hence, we can assume that $x_j \geq y_k$. Since $\operatorname{rcf}(G) = \operatorname{rcf}(\{x_j \,|y_k\})$, then $G \equiv_{\mathbf{Inf}} \{x_j \,|y_k\}$. Thus, by Lemma 4.1.3, $y_k \leq_{\mathbf{Inf}} G \leq_{\mathbf{Inf}} x_j$.

(iii) If G is a number with $\overline{\langle \mathbf{L}(G) \rangle} = x_k$, then $x_k < G$ by Lemma 1.2.44 since $x_k \in \mathbf{L}(G)$. From (i), we have that for all i, $x_i \leq_{\mathbf{Inf}} x_k < G$. Then by Lemma 1.2.69, $x_i < G$. If $\mathbf{R}(G) = \emptyset$, then $G = \{x_k \,|\cdot\}$. Otherwise, $x_k < G < \overline{\langle \mathbf{R}(G) \rangle}$, so $G = \left\{x_k \,\middle|\, \overline{\langle \mathbf{R}(G) \rangle}\right\}$. ∎

4.2 Option-Closure Function

"I always say 'don't make plans, make options.'"
- Jennifer Aniston

A natural question regarding option-closed games is whether or not we can add options to any game to make it option-closed. In fact, we can. However, this process can wreak havoc on the value of the game. We define the following function which takes a game G and returns an option-closed form of G, which we will refer to as the option-closure of G. We will later show that this function is a closure operator, which is the motivation for its name.

Definition 4.2.1. *For a game G, we define its **option-closure**, denoted $\mathrm{oc}\left(G\right)$, as follows:*

$$\mathrm{oc}\left(G\right) = \left\{\mathrm{oc}_{\mathrm{L}}\left(\mathbf{L}\left(G\right)\right)\middle|\mathrm{oc}_{\mathrm{R}}\left(\mathbf{R}\left(G\right)\right)\right\}$$

where for a set of games S,

$$\mathrm{oc}_{\mathrm{L}}\left(S\right) = \bigcup_{k\geq 0}\left\{\mathrm{oc}\left(H\right) : H \in \mathbf{L}^{k}\left(S\right)\right\},$$

and

$$\mathrm{oc}_{\mathrm{R}}\left(S\right) = \bigcup_{k\geq 0}\left\{\mathrm{oc}\left(H\right) : H \in \mathbf{R}^{k}\left(S\right)\right\}.$$

with $\mathrm{oc}_{\mathrm{L}}\left(\emptyset\right) = \mathrm{oc}_{\mathrm{R}}\left(\emptyset\right) = \emptyset$.

*We will call $\mathrm{oc}_{\mathrm{L}}\left(S\right)$ the **left-option-closure** of S and $\mathrm{oc}_{\mathrm{R}}\left(S\right)$ the **right-option-closure** of S. Similarly, we will say that a set of games S is **left-option-closed** if $S = \mathrm{oc}_{\mathrm{L}}\left(S\right)$ and, likewise, **right-option-closed** if $S = \mathrm{oc}_{\mathrm{R}}\left(S\right)$.*

For example we can consider the option-closure of the game G which has literal (and canonical) form $\{2\,|1\}$. To obtain this, we note that $\mathrm{oc}\left(0\right) = 0$, $\mathrm{oc}\left(1\right) = \{\mathrm{oc}\left(0\right)|\cdot\} = 1$ and $\mathrm{oc}\left(2\right) = \{\mathrm{oc}\left(1\right),\mathrm{oc}\left(0\right)|\cdot\} = \{1,0\,|\cdot\}$.

$$\begin{aligned}
\mathrm{oc}\left(G\right) &= \mathrm{oc}\left(\{2\,|1\}\right)\\
&= \left\{\mathrm{oc}_{\mathrm{L}}\left(2\right)|\mathrm{oc}_{\mathrm{R}}\left(1\right)\right\}\\
&= \left\{\mathrm{oc}\left(2\right),\mathrm{oc}\left(1\right),\mathrm{oc}\left(0\right)|\mathrm{oc}\left(1\right)\right\}\\
&= \left\{\{1,0\,|\cdot\},\{0\,|\cdot\},0\,|\{0\,|\cdot\}\right\}
\end{aligned}$$

We see that in canonical form $G = \mathrm{oc}\left(G\right)$, but their literal forms differ, with $\mathrm{lit}\left(\mathrm{oc}\left(G\right)\right) = \left\{\{1,0\,|\cdot\},1,0\,|1\right\}$.

4.2.1 Literal Requirements

It is important to note that two games that are canonically equivalent need not be equal under the option-closure function. In the original games, there may be reversible moves that when option-closed give Left a better move. In the following proof, we consider a game G and another equivalent game H which is, in its literal form, the canonical form of G, i.e. $\mathrm{lit}\,(H) \cong \mathrm{can}\,(G)$. We see that the left option to $\{2\,|0\}$ is a reversible move in G. However, since the option-closure of a game is taken on its literal form, that same left option is the cause of 2 being added to $\mathbf{L}\,(\mathrm{oc}\,(G))$.

Lemma 4.2.2. $G = H \not\Rightarrow \mathrm{oc}\,(G) = \mathrm{oc}\,(H)$.

Example 4.2.3. *Let* $\mathrm{lit}\,(G) = \{1, \{2\,|0\}\,|3\}$ *and* $\mathrm{lit}\,(H) = \{1\,|\cdot\}$. *Thus,* $G = H = 2$. *However,*

$$\mathrm{oc}\,(G) = \{0, 1, 2, \{0, 1, 2\,|0\}\,|\{0, 1, 2\,|\cdot\}\} = \{2, \{2\,|0\}\,|3\} = 2\frac{1}{2},$$

while

$$\mathrm{oc}\,(H) = \{0, 1\,|\cdot\} = 2.$$

Thus $G = H$ *but* $\mathrm{oc}\,(G) \neq \mathrm{oc}\,(H)$.

Likewise, two games need not be equal despite being equivalent under option-closure. The following example demonstrates this.

Lemma 4.2.4. $\mathrm{oc}\,(G) = \mathrm{oc}\,(H) \not\Rightarrow G = H$.

Example 4.2.5. *Let* $G = \{*\,|*\}$ *and* $H = \{0, *\,|0, *\}$.

Then $\mathrm{oc}\,(G) = \mathrm{oc}\,(H) = \mathrm{can}\,(H) = \{0, *\,|0, *\} = *2$, *but* $\mathrm{can}\,(G) = 0$.

Thus, $\mathrm{oc}\,(G) = \mathrm{oc}\,(H) = \mathrm{can}\,(H) \neq \mathrm{can}\,(G)$.

In Example 4.2.5, we have a game G and another game H chosen such that $\mathrm{oc}\,(G) = \mathrm{oc}\,(H)$. In G, both the left and right option to $*$ are reversible and so $\mathrm{can}\,(G) = 0$. In H, the options to $*$ are not reversible and so $H \cong \mathrm{can}\,(H) = *2$.

We need to consider the specific game with its literal options. In Example 4.2.3, we have a case where the option-closure of the game is bigger than the game itself, i.e. $G = 2 < 2\frac{1}{2} = \mathrm{oc}\,(G)$. Example 4.2.4 gives a game for which its option-closure is confused with itself, i.e. $G = 0 \parallel *2 = \mathrm{oc}\,(G)$.

At this point, it is interesting to look at what this function does to the game tree. We first consider the game $G = \{*\,|*\}$ from our example above. The game tree for G and $\mathrm{oc}\,(G)$ are given in Figure 4.1.

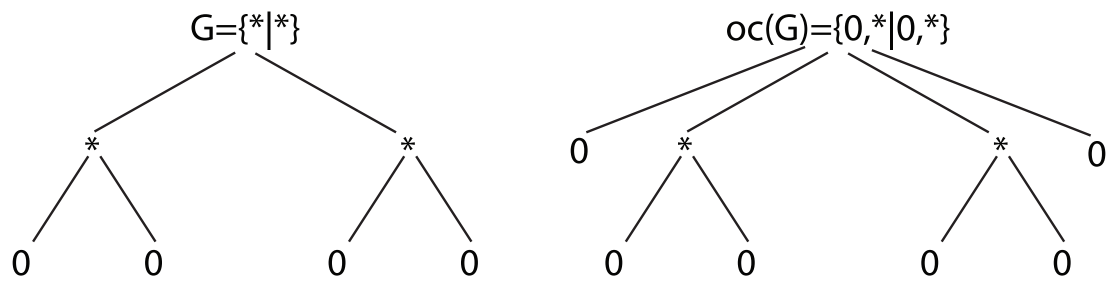


Figure 4.1: The game tree for $G = \{*\,|*\}$ (left) and $\mathrm{oc}\,(G)$ (right).

We can see that in this example, we append a new left option to zero to the root of the tree as well as a new right option to zero. In general, what we are doing is adding in new branches for each of the consecutive left and right options. In any game G, a game tree will include left branches from the root to each option in $\mathbf{L}\,(G)$. For any positions H in $\mathrm{oc}\,(G)$, the game tree will include left branches from H to $\mathbf{L}\,(H)$ as well, but will also have left branches to $\mathbf{L}^2\,(H)$, $\mathbf{L}^3\,(H)$, etc., as depicted in Figure 4.2.

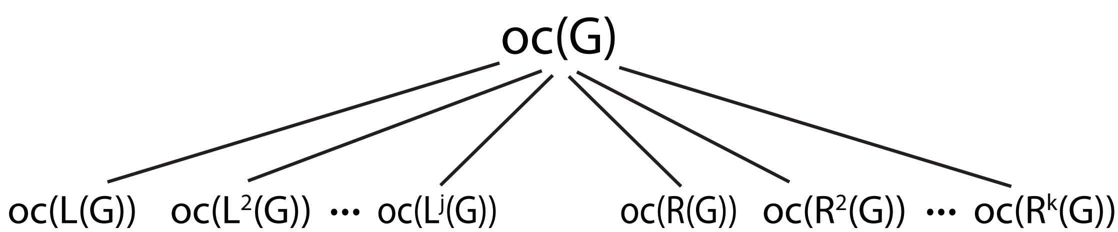


Figure 4.2: The effect of the option-closure funtion on the game tree.

While simple games produce nice results, games such as tinies, games **tiny-n** of the form $+_n = \{0\,|\{0\,|-n\}\}$, can throw a wrench in the works, as demonstrated in example below.

Example 4.2.6. *Consider* $+_n = \{0\,|\{0\,|-n\}\}$. *Then* $\text{oc}\left(+_n\right) = \{0\,|\{0\,|-n\}\,,-n\}$, *so that* $\text{rcf}\left(+_n\right) = 0$ *and* $\text{rcf}\left(\text{oc}\left(+_n\right)\right) = \{0\,|-n\}$.

4.2.2 Threatbare Games

In a game, a threat is a position that can be reached in two or more moves, but not in one. Thus, the opponent could prevent the ability to reach that threat by his or her first move choice. For instance, in the game $+_n$, Right has a threat to $-n$ in two moves; However, Left can play to zero and avoid that possibility. As seen in the example of $+_n$ above, threats in the original game can be taken immediately in the option-closure of the game.

Definition 4.2.7. *Let G be a game.*

*We define the **left-threat** of G as*

$$\text{LT}\left(G\right) = \max\left\{x \in \bigcup_{k\geq 0} \mathbf{L}^k\left(G\right) \;\; s.t. \;\; x \text{ is a number}\right\},$$

*and the **right-threat** of G as*

$$\text{RT}\left(G\right) = \min\left\{y \in \bigcup_{k\geq 0} \mathbf{R}^k\left(G\right) \;\; s.t. \;\; y \text{ is a number}\right\}.$$

We note that by definition, for any left option G^L of a game G, $\text{LT}\left(G^L\right) \leq \text{LT}\left(G\right)$, and analogously, for any right option G^R, $\text{RT}\left(G^R\right) \geq \text{RT}\left(G\right)$.

For games to have the same value under option-closure, the left threat must already be available as a first left option of the game. Likewise, the right threat must be available as a first right option of the game.

Definition 4.2.8. *For a game G, we say that G is **left-threatbare** if*

$$\exists H \in \mathbf{L}\left(G\right) \;\; s.t. \;\; H \equiv_{\mathbf{Inf}} \text{LT}\left(G\right).$$

*Likewise, we say that G is **right-threatbare** if*

$$\exists H \in \mathbf{R}\left(G\right) \;\; s.t. \;\; H \equiv_{\mathbf{Inf}} \text{RT}\left(G\right).$$

*If G is both left-threatbare and right-threatbare, then G is said to be **threatbare**.*

Intuitively, in a threatbare game, all the gains have to be made on the first move, not subsequent moves.

Lemma 4.2.9.

$$G = \mathrm{LT}(G) \iff G \text{ is a number.}$$

Proof. For $k \geq 0$, if $H \in \mathbf{L}^k(G)$, then $H^L \lhd H$ by Corollary 1.2.35. Hence, for $k > 0$, if $H \in \mathbf{L}^k(G)$, $H \lhd G$ since $\mathbf{L}^0(G) = \{G\}$. Thus, if G is a number, then by Definition 4.2.7 of left-threat, $\mathrm{LT}(G) = G$

If G is not a number, then $G \neq \mathrm{LT}(G)$ since $\mathrm{LT}(G)$ is a number by definition. □

Lemma 4.2.10. *For any game G,*

$$\mathrm{RT}(G) \trianglelefteq G \trianglelefteq \mathrm{LT}(G)$$

with equality holding only when G is a number.

Proof. If G is a number, then $G = \mathrm{LT}(G)$ by Lemma 4.2.9.

Suppose G is not a number and n is, with $G > n = \mathrm{LT}(G)$. Then $\mathbf{R}(G) > n$ since $n < G \lhd \mathbf{R}(G)$. Let $G^L \in \mathbf{L}(G)$. By induction, $G^L \trianglelefteq \mathrm{LT}\left(G^L\right)$. By Definition 4.2.7, $\mathrm{LT}\left(G^L\right) \leq \mathrm{LT}(G) = n$. Hence, $G^L \trianglelefteq n$. Then $G = \{\mathbf{L}(G) \,|\mathbf{R}(G)\}$ with $\mathbf{L}(G) \trianglelefteq n < \mathbf{R}(G)$. But then by simplicity (Thm. 1.2.49), $G = n$, contradicting the assumption it is not a number. □

Corollary 4.2.11. *Let G be a game.*

$$\forall H \in \mathbf{L}^k(G), H \trianglelefteq \mathrm{LT}(G).$$

Furthermore, if H is a number, then $H \leq \mathrm{LT}(G)$.

Proof. When $k = 0$, this follows immediately from Lemma 4.2.10 as $\mathbf{L}^0(G) = \{G\}$, so $H = G$.

Let $k > 0$. By Lemma 4.2.10, $H \trianglelefteq \mathrm{LT}(H)$. By Definition 4.2.7, $\mathrm{LT}(H) \leq \mathrm{LT}(G)$, so $H \trianglelefteq \mathrm{LT}(G)$.

If H is a number, then H and $\mathrm{LT}(G)$ are comparable since $\mathrm{LT}(G)$ is also a number by definition. Hence, $H \leq \mathrm{LT}(G)$. □

We can think of the left-threat of a game as the maximum number that Left can get to in any number of consecutive moves with Left going first. The left stop of the game is the maximum number that can be reached after alternating moves with Left going first. Thus, we should see that the left-threat of a game is at least as good for Left as the left stop.

Similarly, the right-threat of a game represents the minimum number that Right can get to in any number of consecutive moves with Right going first and the right stop of the game is the minimum number that can be reached after alternating moves when Right goes first. So, the right-threat of a game is at least as good for Right as the right stop.

Lemma 4.2.12. *For a game G,*

$$\mathrm{RT}(G) \le \mathrm{R}_0(G) \le \mathrm{L}_0(G) \le \mathrm{LT}(G).$$

Proof. If G is a number, then by Corollary 1.2.55, $\mathrm{R}_0(G) = \mathrm{L}_0(G) = G$, and $\mathrm{RT}(G) = \mathrm{LT}(G) = G$ by Lemma 4.2.9. Thus, $\mathrm{RT}(G) = \mathrm{R}_0(G) = \mathrm{L}_0(G) = \mathrm{LT}(G) = G$.

Suppose G is not a number. By Corollary 1.2.55, $\mathrm{R}_0(G) \le \mathrm{L}_0(G)$. By symmetry then, we need only show that $\mathrm{L}_0(G) \le \mathrm{LT}(G)$. Let $G^L \in \mathbf{L}(G)$. By induction, $\mathrm{R}_0\left(G^L\right) \le \mathrm{L}_0\left(G^L\right) \le \mathrm{LT}\left(G^L\right)$. By Definition 4.2.7, $\mathrm{LT}\left(G^L\right) \le \mathrm{LT}(G)$, so $\mathrm{R}_0\left(G^L\right) \le \mathrm{LT}(G)$. Since this holds for all G^L,

$$\mathrm{L}_0(G) = \max\left\{\mathrm{R}_0\left(G^L\right)\right\}_{G^L\in\mathbf{L}(G)} \le \max\left\{\mathrm{LT}(G)\right\}_{G^L\in\mathbf{L}(G)} = \mathrm{LT}(G).$$

Thus $\mathrm{L}_0(G) \le \mathrm{LT}(G)$.

Similar arguments hold for $G^R \in \mathbf{R}(G)$ giving $\mathrm{R}_0(G) \ge \mathrm{RT}(G)$. □

Lemma 4.2.13. *If G is threatbare, then*

$$H \in \mathbf{L}(G) \implies H \le_{\mathbf{Inf}} \mathrm{LT}(G).$$

Analogously,

$$H \in \mathbf{R}(G) \implies H \ge_{\mathbf{Inf}} \mathrm{RT}(G).$$

Proof. Suppose not. Say $\mathrm{LT}(G) = x$. Then $\exists H \in \mathbf{L}(G)$ such that $x < H$. Note that H is not a number, because otherwise $H = \mathrm{LT}(G)$.

By Corollary 1.2.55 and Proposition 1.2.56, $x \le \mathrm{R}_0(H) \le \mathrm{L}_0(H)$.

(i) If $x < \mathrm{R}_0(H) \le \mathrm{L}_0(H)$, then

$$x < \mathrm{R}_0(H) \le \mathrm{L}_0(H) \le \mathrm{LT}(H),$$

which is a contradiction as $\mathrm{LT}(H) \le \mathrm{LT}(G)$ since $H \in \mathbf{L}(G)$.

(ii) If $x = \mathrm{R}_0(H) \le \mathrm{L}_0(H)$, then either

$$x = \mathrm{R}_0(H) < \mathrm{L}_0(H) \le \mathrm{LT}(H),$$

which is a a contradiction, or

$$x = \mathrm{R}_0(H) = \mathrm{L}_0(H) = \mathrm{LT}(H),$$

which would imply $H \equiv_{\mathbf{Inf}} x$, again a contradiction. □

Corollary 4.2.14. *If G is threatbare, then for any k,*

$$H \in \mathbf{L}^k(G) \implies H \le_{\mathbf{Inf}} \mathrm{LT}(G).$$

Analogously,

$$H \in \mathbf{R}^k(G) \implies H \ge_{\mathbf{Inf}} \mathrm{RT}(G).$$

Corollary 4.2.15. *If G is threatbare, then*

$$\mathrm{RT}(G) \le_{\mathbf{Inf}} G \le_{\mathbf{Inf}} \mathrm{LT}(G).$$

Proof. Since $G \in \mathbf{L}^0(G)$, this follows from Corollary 4.2.14. □

Lemma 4.2.16. *If G is a threatbare game, then*

$$\mathrm{L}_0(G) = \mathrm{LT}(G) \text{ and } \mathrm{R}_0(G) = \mathrm{RT}(G).$$

Proof. Suppose G is threatbare. Then $\exists K \in \mathbf{L}(G)$ such that $K \equiv_{\mathbf{Inf}} \mathrm{LT}(G)$. We know from Lemma 4.2.13 that $\forall G^L \in \mathbf{L}(G)$,

$$G^L \le_{\mathbf{Inf}} \mathrm{LT}(G).$$

Hence, $G^L \le_{\mathbf{Inf}} K$.

Suppose $G^L \equiv_{\mathbf{Inf}} K$, then

$$G^L - K \equiv_{\mathbf{Inf}} G^L - \mathrm{LT}(G) \equiv_{\mathbf{Inf}} 0.$$

Hence, $\mathrm{L}_0\left(G^L\right) \equiv_{\mathbf{Inf}} \mathrm{R}_0\left(G^L\right) \equiv_{\mathbf{Inf}} K$. So $\mathrm{L}_0(G) \geq K \equiv_{\mathbf{Inf}} \mathrm{LT}(G)$. However, both $\mathrm{L}_0(G)$ and $\mathrm{LT}(G)$ are numbers, so $\mathrm{L}_0(G) \geq \mathrm{LT}(G)$. But by Lemma 4.2.12, $\mathrm{L}_0(G) \leq \mathrm{LT}(G)$, so $\mathrm{L}_0(G) = \mathrm{LT}(G)$.

Analogous arguments hold to show $\mathrm{R}_0(G) = \mathrm{RT}(G)$. □

Corollary 4.2.17. *If games G and H are threatbare, then*

$$G < H \implies \mathrm{LT}(G) \leq \mathrm{LT}(H).$$

Proof. By Corollary 1.2.57, since $G < H$, $\mathrm{L}_0(G) \leq \mathrm{R}_0(H)$. By Corollary 1.2.55, $\mathrm{R}_0(H) \leq \mathrm{L}_0(H)$. Thus, $\mathrm{L}_0(G) \leq \mathrm{L}_0(H)$. By Lemma 4.2.16, the left-threat and Left stop of G are equal, as are those of H. Thus,

$$\mathrm{LT}(G) = \mathrm{L}_0(G) \leq \mathrm{L}_0(H) = \mathrm{LT}(H).$$

□

Lemma 4.2.18. *For a game G, $G \equiv_{\mathbf{Inf}} \mathrm{L}_0(G)$.*

Proof. If G is a number, then $\mathrm{L}_0(G) = G$ by the definition of left stop (Def. 1.2.54).

If G is not a number, then in $G - \mathrm{L}_0(G)$, the Number Avoidance Theorem 1.2.52 says to play in G until we reach a number. If Left starts, we reach the game $\mathrm{L}_0(G) - \mathrm{L}_0(G) = 0$. If Right starts, we reach the game $\mathrm{R}_0(G) - \mathrm{L}_0(G) \leq 0$ by Corollary 1.2.55. □

Theorem 4.2.19. *If G is threatbare, then $G \equiv_{\mathbf{Inf}} \mathrm{oc}(G)$.*

Proof. Consider $G - \mathrm{oc}(G)$. By symmetry, we will consider only Right's moves. Play in G is bad because $\mathbf{R}(G) \subseteq \mathbf{R}(\mathrm{oc}(G))$. So Right must move in $\mathrm{oc}(G)$ to $G - H$ where $H \in \bigcup \mathbf{L}^k(G)$. By theorem 4.2.14, $H \leq_{\mathbf{Inf}} \mathrm{LT}(G)$ and since G is threatbare, $\exists K \in \mathbf{L}(G)$ such that $K \equiv_{\mathbf{Inf}} \mathrm{LT}(G)$. So Left answers to $K - H \equiv_{\mathbf{Inf}} \mathrm{LT}(G) - H \geq_{\mathbf{Inf}} 0$. □

Corollary 4.2.20. *If game G and H are threatbare, then*

$$G = H \implies \mathrm{oc}(G) \equiv_{\mathbf{Inf}} \mathrm{oc}(H).$$

Lemma 4.2.21. *For threatbare games G and H,* *(i)* $\mathrm{oc}(G) = \mathrm{oc}(\mathrm{oc}(G))$

(ii) $G \leq_{\mathbf{Inf}} H \implies \mathrm{oc}(G) \leq_{\mathbf{Inf}} \mathrm{oc}(H)$

Proof. (i) Since $\mathbf{L}^n(\mathrm{oc}(G)) \subseteq \mathbf{L}(\mathrm{oc}(G))$ for any n, $\mathbf{L}(\mathrm{oc}(G)) \cong \mathbf{L}(\mathrm{oc}(\mathrm{oc}(G)))$. The same is true of right options.

(ii) $\mathrm{oc}(G) \equiv_{\mathbf{Inf}} G \leq_{\mathbf{Inf}} H \equiv_{\mathbf{Inf}} \mathrm{oc}(H)$ by Theorem 4.2.19. □

Corollary 4.2.22. *The option-closure function is a closure operator.*

Definition 4.2.23. *The game G is said to be* ***closed*** *if $G = \mathrm{oc}(G)$.*

In a closed game, there is no door for either Left or Right to sneak through to better their position in the game. All threats are on the table. It should be noted that in general, if games $G < H$, this does not imply that $\mathrm{oc}(G) < \mathrm{oc}(H)$.

Claim 4.2.24. *For a game G,*

$$\{\mathrm{oc}(\mathbf{L}(G)) \,|\mathbf{R}(G)\} \geq G \geq \{\mathbf{L}(G)\,| \mathrm{oc}(\mathbf{R}(G))\}.$$

Proof. Consider the game $\{\mathrm{oc}(\mathbf{L}(G))\,|\mathbf{R}(G)\} - \{\mathbf{L}(G)\,|\mathbf{R}(G)\}$. We want to show that Left can win going second. Right has only two moves. Let $G^R \in \mathbf{R}(G)$ and $G^L \in \mathbf{L}(G)$.

If Right plays to $G^R - \{\mathbf{L}(G)\,|\mathbf{R}(G)\}$, then Left can respond to $G^R - G^R = 0$. If Right instead plays to $\{\mathrm{oc}(\mathbf{L}(G))\,|\mathbf{R}(G)\} - G^L$, then Left can respond to $G^L - G^L = 0$. □

4.3 Lattice Of Option-Closed Games

We will now consider the structure of all option-closed games born by day n. We will use $\mathbf{OC}_n$ to denote the set of option-closed games G with birthday $b(G) \leq n$ and consider the poset $\langle \mathbf{OC}_n; \leq \rangle$, which we will denote as $\mathbf{OC}_n$ as the partial order relation on games will be assumed to be $\leq$.

Definition 4.3.1. $\mathbf{OC}_n = \{G \ : \ b(G) \leq n,\ G \text{ option-closed}\}$.

Proposition 4.3.2. $\mathbf{OC}_0 = \{0\}$ *and* $\mathbf{OC}_1 = \{-1, 0, *, 1\}$.

Proof. The first statement is trivially true. If $b(G) = 1$, 0 is the only possible option and as zero has no left or right options, it can occur as either a left or right option in G by itself. Hence, the only new possibilities are $\{0\,|\cdot\} = 1$, $\{\cdot\,|0\} = -1$ and $\{0\,|0\} = *$, which forms a comprehensive list. □

The option-closed games occurring by day 1 and their partial order structure are depicted in Figure 4.3. The diagram should be familiar as it is the lattice of all games born by day 1 (Fig. 1.17), as games born by day 1 happen to all be option-closed. From day 2 forward, we will see that this is not always the case.

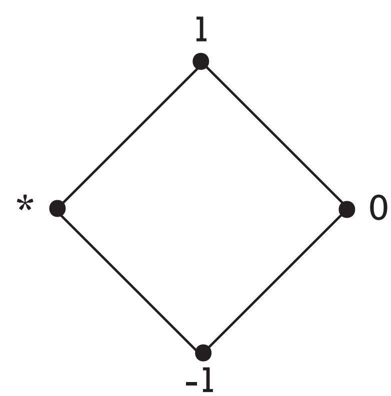


Figure 4.3: $\langle \mathbf{OC}_1; \leq \rangle$: The lattice of option-closed games born by day 1.

Definition 4.3.3. *Let* $\mathcal{A}_n = \{\mathrm{oc_L}\,(A) : A \text{ is an antichain in } \mathbf{OC}_n\}$. *That is,* $\mathcal{A}_n$ *is the set containing the left-option-closure of all antichains from the lattice of option-closed games born by day* n*, which we will call the* ***left-option-closed antichains****.*

Similarly, let $\mathcal{B}_n$ be the set of all ***right-option-closed antichains****.*

Thus, $\mathcal{B}_n = \{\mathrm{oc}_{\mathrm{R}}(B) : B \text{ is an antichain in } \mathbf{OC}_n\}$.

Observation 4.3.4. *If $A \in \mathcal{A}_n$, then $-A \in \mathcal{B}_n$ and vice-versa.*

Corollary 4.3.5. *If $x \in \mathbf{OC}_n$, then $x = \{A \,|{-B}\}$ where $A, B \in \mathcal{A}_{n-1}$.*

We will show that the set of left-option-closed antichains $\mathcal{A}_n$ forms a linear order under set containment of the lower sets that they induce. That is, we can label elements of $\mathcal{A}_n$ as $A_0, A_1, \ldots, A_k$ such that for A_i in literal form,

$$\downarrow A_0 \subset \downarrow A_1 \subset \cdots \subset \downarrow A_k.$$

We see, in Figure 4.3, that the left-option-closed antichains in $\mathbf{OC}_1$ are given in $\mathcal{A}_1 = \{\emptyset, \{-1\}, \{0\}, \{0, *\}, \{0, 1\}\}$. We note that

$$\downarrow \emptyset \subset \downarrow \{-1\}, \subset \downarrow \{0\} \subset \downarrow \{0, *\} \subset \downarrow \{0, 1\}.$$

Note that if we have $a, b \in \mathbf{OC}_n$ with $A = \mathrm{oc}_{\mathrm{L}}(a)$ and $B = \mathrm{oc}_{\mathrm{L}}(b)$, then

$$(\downarrow A \cup \downarrow B) \setminus (\downarrow A \cap \downarrow B)$$

is not empty if and only if a and b are not related, i.e. $a \parallel b$.

For example, in $\mathbf{OC}_1$ (see Fig. 4.3), $\mathrm{oc}_{\mathrm{L}}(0) = \{0\}$ and $\mathrm{oc}_{\mathrm{L}}(*) = \{0, *\}$. We can check that $\downarrow 0 = \{-1, 0\}$ and $\downarrow \{0, *\} = \{-1, 0, *\}$. Thus,

$$* \in (\downarrow 0 \cup \downarrow \{0, *\}) \setminus (\downarrow 0 \cap \downarrow \{0, *\})$$

so $0 \parallel *$, as is expected and known.

Recall from definition 1.3.15 that the lower set of an antichain A contains everything below and including the antichain A, and that $A \preceq B$ in $\mathbf{OC}_n$ implies that the lower set of A is a subset of the lower set of B. Thus, if $A \preceq B$ in $\mathbf{OC}_n$, then $\downarrow A \subseteq \downarrow B$.

Lemma 4.3.6. *If $A \in \mathcal{A}_n$ and $\overline{A} = p$, then for all $x \in A$, $x = p + \{0 \,|{-s}\} + \epsilon$ for some number $s \geq 0$ and some $\epsilon \in \mathbf{Inf}$.*

Proof. Suppose $\mathrm{L}_0(A) = p$. Then for any $x \in A$, by Lemma 4.1.4, $\mathrm{rcf}(x) = p$ or $\{p\,|q\}$ where $p \geq q$. So either $x = p + \delta$ or $x = \{p\,|q\} + \delta$ for some $\delta \in \mathbf{Inf}$. In the first case, note that $\delta = \{0\,|0\} + * + \delta$ and that $(\delta + *) \in \mathbf{Inf}$. In the second case, we see that $\{p\,|q\} = p + \{0\,|q - p\}$ and that $q - p \leq 0$. Either way, $x = p + \{0\,|-s\} + \epsilon$ for some $\epsilon \in \mathbf{Inf}$. □

Lemma 4.3.7. *Let* $\max\{\mathrm{L}_0(x)\,|x \in A_i\} = \overline{A_i} = p$ *and* $\max\{\mathrm{L}_0(y)\,|y \in A_j\} = \overline{A_j} = q$.

If $p > q$*, then* $A_i > A_j$ *and* $p \in A_i$.

Proof. Let $x \in A_i$ and $y \in A_j$. Then by Lemma 4.3.6,

$$y = q + \{0\,|-r\} + \delta$$

for some number $r \geq 0$ and $\delta \in \mathbf{Inf}$. Hence,

$$p - y = p - q + \{r\,|0\} - \delta$$

which Left wins going first or second since δ is an infinitesimal. If Left goes first, she plays to $p - q + r - \delta > r - \delta > 0$. If Right goes goes first, he can either play to $p - q + 0 - \delta > 0$ or to $p - q + \{r\,|0\} + \delta'$. Since $\mathrm{L}_0(\delta) = \mathrm{R}_0(\delta) = 0$, Left responds with her best move in δ' which is at worst an infinitesimal, i.e. $\mathrm{L}_0(\delta') \geq \mathrm{R}_0(\delta') = 0$. □

Lemma 4.3.8. *Let* $\mathcal{A}_n$ *be the set of all left-option-closed antichains in* $\mathbf{OC}_n$*. If* $A, B \in \mathcal{A}_n$*, then either* $\downarrow A \subseteq \downarrow B$ *or* $\downarrow B \subseteq \downarrow A$.

Proof. Let $\Delta(A, B) = \min\{\{\}\,|\downarrow A - \downarrow B|, |\downarrow B - \downarrow A|\}$. Suppose that there exists two left-option-closed antichains where neither is a subset of the other. Over all such pairs, choose sets A and B such that $\Delta(A, B)$ is minimum. We may also suppose that the $|\downarrow A| + |\downarrow B|$ is minimum.

Choose maximal elements $g \in \downarrow A - \downarrow B$ and $h \in \downarrow B - \downarrow A$.

Take $g' \leq g$ in $\downarrow A - \downarrow B$ such that if $k < g'$, then $k \in \downarrow B$.

Consider $B' = B \cup \{g'\}$. Since every left option of g' is in B, B' is left-option-closed. Since A and B where chosen so that one of their differences was minimum, it follows that $A \subseteq B \cup \{g'\}$, Similarly there is an $h' \leq h$ in $\downarrow B - \downarrow A$ where if $k \leq h'$, then $k \in \downarrow A$ and $B \subseteq A \cup \{h'\}$. Thus $C = \downarrow A \cup \downarrow B$ is a left-option-closed antichain whose difference with B is 1 and whose symmetric difference with B is also 1, contradicting the choice of A and B.

Note that this cannot reduce to one of A and B being empty set since the empty set is a subset of everything. □

Lemma 4.3.9. *Let $\mathcal{A}_n = \{A_1, A_2, \ldots, A_k\}$ be the set of left-option-closed antichains in $\mathbf{OC}_n$ such that $\downarrow A_i \subseteq \downarrow A_{i+1}$ in literal form. If*

$$g = \{A_i \,|B\} \quad \textit{and} \quad h = \{A_{i+1} \,|B\}\,,$$

then $g \leq h$.

Proof. Consider $h - g = \{A_{i+1} \,|B\} - \{A_i \,|B\}$. If Right plays in h to $B - \{A_i \,|B\}$, Left can respond to $B - B = 0$. If Right plays in g to $\{A_{i+1} \,|B\} - A_i$, then Left can respond to $A_i - A_i = 0$ since $\downarrow A_i \subseteq \downarrow A_{i+1}$. Thus $h - g \geq 0$. □

Corollary 4.3.10. *Let $\mathcal{A}_n = \{A_1, A_2, \ldots, A_k\}$ be the set of left-option-closed antichains in $\mathbf{OC}_n$ such that $\downarrow A_i \subseteq \downarrow A_{i+1}$ in literal form. If*

$$g = \{B\,|{-A_i}\} \textit{ and } h = \{B\,|{-A_{i+1}}\}\,,$$

then $g \geq h$.

The following lemma (Lemma 4.3.12) tells us that while there could be many antichains from day $n-1$ that give us the left and right options for a given element in day n that give the same value, there is a unique maximum and minimum antichain having that value. Thus, there exists a highest high antichain and a lowest low antichain and these determine the elements that exist in $\mathbf{OC}_n$. For each game in day n, these are defined below.

Definition 4.3.11. *For $g \in \mathbf{OC}_n$, let*

$$AC_L\,(g) = \max\{A \in \mathcal{A}_{n-1} \ : \ g = \{A\,|{-B}\} \textit{ for some } B \in \mathcal{A}_{n-1}\}\,,$$

and let

$$AC_R\,(g) = \min\{-B \in \mathcal{A}_{n-1} \ : \ g = \{A\,|{-B}\} \textit{ for some } A \in \mathcal{A}_{n-1}\}\,.$$

Lemma 4.3.12. *The left-option-closed antichains in $\mathcal{A}_n$ are linearly ordered under set containment of their lower sets.*

Proof. The left-option-closed antichains of $\mathcal{A}_0$ and $\mathcal{A}_1$ are linearly ordered. In $\mathcal{A}_0$,

$$\downarrow \emptyset \subset \downarrow \{0\}$$

and in $\mathcal{A}_1$,

$$\downarrow \emptyset \subset \downarrow \{-1\} \subset \downarrow \{0\} \subset \downarrow \{0, *\} \subset \downarrow \{0, 1\}.$$

Assume that $\mathcal{A}_{n-1}$ is linearly ordered as well. Consider $\mathcal{A}_n$.

For $g \in \mathbf{OC}_n$, $g = \{X \,|{-Y}\}$ with $X, Y \in \mathcal{A}_{n-1}$ by Corollary 4.3.5. Thus $X, Y \in \mathcal{A}_{n-1}$ and so are comparable, as $\mathcal{A}_{n-1}$ is linearly ordered by assumption. Let $AC_L\,(g)$, $AC_R\,(g)$ be as defined in Definition 4.3.11.

Let $\mathcal{A}_{n-1} = \{A_1, A_2, \ldots, A_k\}$ be the set of left-option-closed antichains from $\mathbf{OC}_{n-1}$ such that $\downarrow A_i \subset \downarrow A_{i+1}$ in literal form. Then for some $A_p, A_q \in \mathcal{A}_{n-1}$,

$$\begin{aligned} g &= \{A_p \,|AC_R\,(g)\} \qquad (p \text{ max}) \\ g &= \{AC_L\,(g)\,|{-A_q}\} \quad (q \text{ max}) \end{aligned}$$

Let $h = \{AC_L\,(g)\,|AC_R\,(g)\}$. Since $\mathcal{A}_{n-1}$ is linearly ordered, $AC_L\,(g) = A_i$ and $AC_R\,(g) = -A_j$ for some $A_i, A_j \in \mathcal{A}_{n-1}$. Thus,

$$h = \{AC_L\,(g)\,|AC_R\,(g)\} = \{A_i \,|{-A_j}\}\,.$$

We claim that $g = h$.

By Definition 4.3.11 and the fact that $\mathcal{A}_{n-1}$ is linearly ordered, $AC_L\,(g) \geq A_p$ and $AC_R\,(g) \leq -A_q$. So,

$$A_i \geq A_p \;\text{ and }\; A_j \geq A_q.$$

Thus, for some $k_1, k_2 \geq 0$, $i = p + k_1$ and $j = q + k_2$. By Lemma 4.3.9 and Corollary 4.3.10,

$$g = \{A_p \,|{-A_j}\} \leq \{A_{p+k_1} \,|{-A_j}\} = \{A_i \,|{-A_j}\} = h = \{A_i \,|{-A_{q+k_2}}\} \leq \{A_i \,|{-A_q}\} = g.$$

Thus, $g \leq h \leq g$ and so $g = h$.

Hence, in $\mathbf{OC}_n$, $g = \{AC_L\,(g)\,|AC_R\,(g)\}$.

Now suppose that $\mathcal{A}_n$ is not linearly ordered. Then there exists $g, h \in \mathbf{OC}_n$ such that $g \parallel h$. Let

$$\begin{aligned} g &= \{AC_L(g)\,|AC_R(g)\} &= \{A_p\,|-A_q\} \\ h &= \{AC_L(h)\,|AC_R(h)\} &= \{A_s\,|-A_t\} \end{aligned}$$

where $A_p, A_q, A_s, A_t \in \mathcal{A}_{n-1}$ and so are ordered.

If $p \leq s$ and $q \geq t$, then by Lemma 4.3.9 and Corollary 4.3.10,

$$g = \{A_p\,|-A_q\} \leq \{A_s\,|-A_q\} \leq \{A_s\,|-A_t\} = h.$$

Similarly, if $p \geq s$ and $q \leq t$, then again by Lemma 4.3.9 and Corollary 4.3.10,

$$g = \{A_p\,|-A_q\} \geq \{A_s\,|-A_q\} \geq \{A_s\,|-A_t\} = h.$$

So, without loss of generality, we can assume that $p \leq s$ and $q \leq t$. Then in $\mathbf{OC}_{n-1}$,

$$\downarrow A_p \subseteq \downarrow A_s \text{ and } \uparrow -A_q \subseteq \uparrow -A_t \ \ (i.e. \downarrow A_q \subseteq \downarrow A_t).$$

Consider the game $h - g$. Since both h and g are option-closed, we know that $\text{rcf}(h)$ and $\text{rcf}(g)$ are either numbers or switches. So, $\text{rcf}(h) = x$ or $\{x\,|y\}$ for numbers $x \geq y$ and $\text{rcf}(g) = u$ or $\{u\,|v\}$ for numbers $u \geq v$. If $\text{rcf}(h) > \text{rcf}(g)$, then Left can win $h - g$. Likewise, if $\text{rcf}(h) < \text{rcf}(g)$, then Right can win in $h - g$. For a number x, we can't have $\text{rcf}(h) = \text{rcf}(g) = x$, because then $h = x = g$. For numbers u and $x \geq y$, we must have: (i) $\text{rcf}(h) = \text{rcf}(g) = \{x\,|y\}$; or (ii) $\text{rcf}(h) = u$, $\text{rcf}(g) = \{x\,|y\}$ and $y \leq u \leq x$; or (iii) $\text{rcf}(h) = \{x\,|y\}$, $\text{rcf}(g) = u$ and $y \leq u \leq x$.

(i) Say $\text{rcf}(h) = \text{rcf}(g) = \{x\,|y\}$. Then $x \in \{A_p, A_s\}$ and $y \in \{-A_q, -A_t\}$. Thus $x \in \downarrow A_p \subseteq \downarrow A_s$ and $y \in \uparrow -A_q \subseteq \uparrow -A_t$. In $h - g$, Left can go to $\alpha - g$ where $\alpha \in \downarrow A_s \setminus \downarrow A_p$, which is comparable to 0.

(ii) Say $\text{rcf}(h) = u$, $\text{rcf}(g) = \{x\,|y\}$ and $y \leq u \leq x$. Then $x \in \downarrow A_p$. If $x > u$, then since $u \equiv_{\mathbf{Inf}} h$, $x > h$. So $x = u = y$. Therefore $u \in A_j$ for some $A_j \in \mathcal{A}_n$. We want to show $u \geq g$. We know that $\text{rcf}(g) = \text{rcf}(\{A_p\,|-A_q\}) = u*$. So, $u \in A_p$ and $u \in -A_q$. But $u \in A_s$ and $\downarrow A_s \supseteq \downarrow A_p$, which is a contradiction.

$\square$

Observation 4.3.13. $\mathcal{A}_n \subset \mathcal{A}_{n+1}$

Left/right options in games of $\mathbf{OC}_n$ then must come from $\mathbf{OC}_{n-1}$, or the options themselves would not have been option-closed. Thus, elements are perpetual. If an option-closed game exists in day n, then it exists in all days after day n.

Corollary 4.3.14. *If $x \in \mathbf{OC}_{n-1}$, then $x \in \mathbf{OC}_n$.*

Since the $A_i \in \mathcal{A}_n$ are linearly ordered, it is natural to wonder what the difference is between A_i and A_{i+1}, i.e. $A_{i+1} - A_i$.

To consider this, we will need the following function which takes the norm of an element within a set and returns the next highest norm that can be found within the set.

Definition 4.3.15. *For a set of S of option-closed sequences,*

$$N(x;S) = \begin{cases} \min\{\overline{y} \ : \ y \in S, \overline{y} > \overline{x}\} & \text{if } \exists y \in S \text{ s.t. } \overline{y} > \overline{x} \\ \infty & \text{otherwise} \end{cases}$$

and $N^2(x;S) = N((N(x;S));S)$.

The following gives us a way of telling what the next element is in the list of elements in $\mathbf{OC}_n$, building from the bottom up.

Definition 4.3.16. *Suppose $\mathcal{A}_{n-1} = \{A_0, A_1, \ldots, A_p\}$ with $A_0 = \emptyset$, $A_p = (n-1)$ and*

$$\downarrow A_0 \subset \ \downarrow A_1 \subset \cdots \subset \ \downarrow A_p.$$

Let $\mathcal{B}_{n-1} = \{B_0, B_1, \ldots, B_p\}$ with $B_i = -B_{p-i}$.

For $\mathbf{OC}_n$, we define the following:

Let $S_0 = \{\emptyset \,|{-(n-1)}\,\}$, which is element $\{A_0\,|B_0\,\}$ of $\mathbf{OC}_n$.

For $A_i \in \mathcal{A}_{n-1}$ and $B_j \in \mathcal{B}_{n-1}$ (so that $\{A_i\,|B_j\,\} \in \mathbf{OC}_n$),

$$F(\{A_i\,|B_j\,\}) = \begin{cases} \{A_{i+1}\,|B_0\,\} & \text{if } \overline{A_i} = \overline{A_{i+1}} = \overline{B_j} < \overline{B_{j+1}} \quad (i) \\ & \text{or } N^2(A_i;\mathcal{A}_{n-1}) = \overline{B_j} \quad (ii) \\ \{A_i\,|N(B_j;\mathcal{B}_{n-1})\,\} & \text{if } N(A_i;\mathcal{A}_{n-1}) = \overline{B_j} \\ & \text{and } N(B_j;\mathcal{B}_{n-1}) < \infty \quad (iii) \\ \{A_i\,|B_{j+1}\,\} & \text{otherwise} \quad (iv) \end{cases}$$

Define $S_k = S_{k-1} \cup F^k(S_0)$.

Hence, the definition gives us that

$$\begin{aligned}
S_0 &= \{\emptyset \,|{-(n-1)}\} &&= -n \\
S_1 &= S_0 \cup F(S_0) &&= \{S_0, F(S_0)\} \\
S_2 &= S_1 \cup F^2(S_0) &&= \{S_0, F(S_0), F^2(S_0)\} \\
&\vdots \\
S_k &= S_{k-1} \cup F^k(S_0) &&= \{S_0, F(S_0), F^2(S_0), \ldots, F^k(S_0)\}
\end{aligned}$$

For example, we can consider S_k in $\mathbf{OC}_2$ for which we need $\mathcal{A}_1 = \{A_0, \ldots, A_4\}$ with $A_0 = \emptyset, A_1 = \{-1\}, A_2 = \{0\}, A_3 = \{*, 0\}, A_4 = \{1, 0\}$ and $\mathcal{B}_1 = \{B_0, \ldots, B_4\}$ with $B_i = -A_{4-i}$.

$$\begin{array}{llllllll}
S_0 & & = & \{A_0 \,|\, B_0\} & = & \{\emptyset \,|{-1}\} & = & -2 \quad \text{by definition} \\
F(S_0) & = F(\{\emptyset \,|{-1}\}) & = & \{A_0 \,|\, B_2\} & = & \{\emptyset \,|\, 0\} & = & -1 \quad \text{by case (iii)} \\
F^2(S_0) & = F(\{\emptyset \,|\, 0\}) & = & \{A_1 \,|\, B_0\} & = & \{-1 \,|{-1}\} & = & -1* \quad \text{by case (ii)} \\
F^3(S_0) & = F(\{-1 \,|{-1}\}) & = & \{A_1 \,|\, B_1\} & = & \{-1 \,|\, *, 0\} & = & -\tfrac{1}{2} \quad \text{by case (iv)} \\
F^4(S_0) & = F(\{-1 \,|\, *, 0\}) & = & \{A_1 \,|\, B_3\} & = & \{-1 \,|\, 1\} & = & 0 \quad \text{by case (iii)} \\
F^5(S_0) & = F(\{-1 \,|\, 1\}) & = & \{A_2 \,|\, B_0\} & = & \{0 \,|{-1}\} & & \quad \text{by case (ii)} \\
F^6(S_0) & = F(\{0 \,|{-1}\}) & = & \{A_2 \,|\, B_1\} & = & \{0 \,|\, 0, *\} & = & \downarrow * \quad \text{by case (iv)} \\
F^7(S_0) & = F(\{0 \,|\, 0, *\}) & = & \{A_2 \,|\, B_2\} & = & \{0 \,|\, 0\} & = & * \quad \text{by case (iv)} \\
F^8(S_0) & = F(\{0 \,|\, 0\}) & = & \{A_3 \,|\, B_0\} & = & \{0, * \,|{-1}\} & & \quad \text{by case (i)} \\
F^9(S_0) & = F(\{0, * \,|{-1}\}) & = & \{A_3 \,|\, B_1\} & = & \{0, * \,|\, 0, *\} & = & *2 \quad \text{by case (iv)} \\
F^{10}(S_0) & = F(\{0, * \,|\, 0, *\}) & = & \{A_3 \,|\, B_2\} & = & \{0, * \,|\, 0\} & = & \uparrow * \quad \text{by case (iv)} \\
F^{11}(S_0) & = F(\{0, * \,|\, 0\}) & = & \{A_3 \,|\, B_3\} & = & \{0, * \,|\, 1\} & = & \tfrac{1}{2} \quad \text{by case (iv)} \\
F^{12}(S_0) & = F(\{0, * \,|\, 1\}) & = & \{A_3 \,|\, B_4\} & = & \{0, * \,|\, \emptyset\} & = & 1 \quad \text{by case (iv)} \\
F^{13}(S_0) & = F(\{0, * \,|\, \emptyset\}) & = & \{A_4 \,|\, B_0\} & = & \{1 \,|{-1}\} & = & \pm 1 \quad \text{by case (i)} \\
F^{14}(S_0) & = F(\{1 \,|{-1}\}) & = & \{A_4 \,|\, B_1\} & = & \{1 \,|\, 0, *\} & & \quad \text{by case (iv)} \\
F^{15}(S_0) & = F(\{1 \,|\, 0, *\}) & = & \{A_4 \,|\, B_2\} & = & \{1 \,|\, 0\} & & \quad \text{by case (iv)} \\
F^{16}(S_0) & = F(\{1 \,|\, 0\}) & = & \{A_4 \,|\, B_3\} & = & \{1 \,|\, 1\} & = & 1* \quad \text{by case (iv)} \\
F^{17}(S_0) & = F(\{1 \,|\, 1\}) & = & \{A_4 \,|\, B_4\} & = & \{1 \,|\, \emptyset\} & = & 2 \quad \text{by case (iv)}
\end{array}$$

Lemma 4.3.17. *$S_i \preceq S_{i+1}$ in $\mathbf{OC}_n$.*

Proof. $S_i \subseteq S_{i+1}$. $\square$

Lemma 4.3.18. *If $A \in \mathcal{A}_n$, then $\exists i$ such that $\downarrow A = S_i$, where $\downarrow A$ is the lower set of A in $\mathbf{OC}_n$.*

Proof. Let

$$\downarrow A = \left\{ \bigcup_{i=1}^{p} \{C_i \,|D_j\} \;\; : \;\{C_1, \ldots, C_p\} \subseteq \mathcal{A}_{n-1}, \{D_1, \ldots, D_p\} \subseteq \mathcal{B}_{n-1} \right\}.$$

Let $C = \max_i \{C_i\}$ and $D = \min_j \{D_j \;:\; \{C\,|D_j\} \in \downarrow A\}$. There exists a minimum i such that $\{C\,|D\} \in S_i$.

Claim $\downarrow A = S_i$.

We know $S_i = S_{i-1} \bigcup F^i(S_0)$, so $S_{i-1} \subseteq S_i$. For all $j < i, F^j(S_0) \lhd F^i(S_0)$. Suppose $F^j(S_0) = \left\{A_{a_j} \,\middle|B_{b_j}\right\}$ and $F^i(S_0) = \{A_{a_i}\,|B_{b_i}\}$. By construction, $a_j \leq a_i$ and $b_j \leq b_i$. So $\overline{A_{a_j}} \leq \overline{A_{a_i}}$.

If $F^j(S_0)$ is a number, then

$$\overline{B_{b_j}} \geq N\big(A_{a_j}; \mathcal{A}_n\big) \leq \overline{B_{b_j}} \leq N^2\big(A_{a_j}; \mathcal{A}_n\big).$$

So $\overline{B_{b_j}} \leq N\big(A_{a_j}; \mathcal{A}_n\big)$. In reduced canonical form, we know that $F^j(S_0) = \left\{\overline{A_{a_j}}\,\middle|\overline{B_{b_j}}\right\}$ and $F^i(S_0) = \left\{\overline{A_{a_i}}\,\middle|\overline{B_{b_i}}\right\}$.

If $\overline{A_{a_i}} > \overline{A_{a_j}}$ and $\overline{A_{a_i}} - \overline{B_{b_i}} \neq \overline{A_{a_j}} - \overline{B_{b_j}}$, then Left wins by going first if he takes the largest switch. $\square$

Below we introduce two new functions on games G and H, which we will denote as $G \vee H$ and $G \wedge H$. We will show that this choice of notation is appropriate since these are the join and meet, respectively, of G and H in $\mathbf{OC}_n$.

Definition 4.3.19.
For games $G, H \in \mathbf{OC}_n$, we define

$$G \vee H = \{\max\{\mathbf{L}(G), \mathbf{L}(H)\}\,|\max\{\mathbf{R}(G), \mathbf{R}(H)\}\}$$

and

$$G \wedge H = \{\min\{\mathbf{L}(G), \mathbf{L}(H)\}\,|\min\{\mathbf{R}(G), \mathbf{R}(H)\}\}.$$

We should note that since $G, H \in \mathbf{OC}_n$, then $\mathbf{L}(G), \mathbf{L}(H) \in \mathcal{A}_{n-1}$ and by Lemma 4.3.12, $\mathcal{A}_{n-1}$ is linearly ordered.

Lemma 4.3.20.
For all $G, H \in \mathbf{OC}_n$, we have

(i) $G \vee H \geq G$

(ii) $G \vee H \geq H$

Proof. The two assertions are symmetric, so it suffices to show that Left can win in $G \vee H - G$ playing second. If Right moves to $G \vee H - \mathbf{L}(G)$, then Left can play to $\max\{\mathbf{L}(G), \mathbf{L}(H)\} - \mathbf{L}(G) \geq 0$ and win. Conversely, if Right moves to $\max\{\mathbf{R}(G), \mathbf{R}(H)\} - G$, then Left has a winning move to $\max\{\mathbf{R}(G), \mathbf{R}(H)\} - \mathbf{R}(G) \geq 0$. □

Lemma 4.3.21. *If $G, H, K \in \mathbf{OC}_n$ and $K \geq G, H$, then $K \geq G \vee H$.*

Proof. We must show that Left can win playing second in

$$K - G \vee H = K - \{\max\{\mathbf{L}(G), \mathbf{L}(H)\} | \max\{\mathbf{R}(G), \mathbf{R}(H)\}\}.$$

Suppose Right plays to some $K^R - G \vee H$. Without loss of generality, suppose that $\max\{\mathbf{R}(G), \mathbf{R}(H)\} = \mathbf{R}(G)$. Since $K \geq G$, then $\forall K^R \in \mathbf{R}(K)$, $\exists G^R \in \mathbf{R}(G)$ such that $K^R \geq G^R$. Thus, Left has a response to $K^R - G^R \geq 0$.

Right can also play in $G \vee H$. Without loss of generality, we can assume that $\max\{\mathbf{L}(G), \mathbf{L}(H)\} = \mathbf{L}(G)$. In this case, Right has a move to $K - G^L$ where $G^L \in \mathbf{L}(G)$. Since $K \geq G$, then $\forall G^L \in \mathbf{L}(G)$, $\exists K^L \in \mathbf{L}(K)$ such that $K^L \geq G^L$. Hence, Left can respond to $K^L - G^L \geq 0$. □

The above results (Lemmas 4.3.20 and 4.3.21) and their duals ($\vee$ replaced with $\wedge$) show us that $\mathbf{OC}_n$ is a lattice with join and meet given by $\vee$ and $\wedge$, respectively. Each pair of elements in the lattice of $\mathbf{OC}_n$ has a unique join. That is, there is one element, $G \vee H$, that is above both G and H and less than any other element above them.

Theorem 4.3.22. $\mathbf{OC}_n$ *is a bounded lattice with top element n and bottom element $-n$, and the join and meet of any two games in $\mathbf{OC}_n$ given by $G \vee H$ and $G \wedge H$ of Definition 4.3.19, respectively.*

The description of $\mathbf{OC}_n$ will depend on the reduced canonical form. In $\mathbf{OC}_n$, the join irreducible elements are those from day $n-1$ and all of the elements of the form $\{n-1\,|B\}$ where $B \in -\mathcal{A}_{n-1}$.

Proposition 4.3.23. $\mathcal{J}\left(\mathbf{OC}_n\right) = \mathbf{OC}_{n-1} \cup \{\{n-1\,|B\} \ : \ B \in -\mathcal{A}_{n-1}\}$.

Corollary 4.3.24. *In $\mathbf{OC}_n$, the doubly-irreducible elements are those in the set $\mathbf{OC}_{n-1} \cup \{\pm(n-1)\}$.*

Proposition 4.3.25. *There are 18 option-closed games born by day 2, so that $|\mathbf{OC}_2| =$ 18.*

Proof. There are 6 antichains of $\mathbf{OC}_1$: four one-element antichains ($\{1\}$,$\{-1\}$,$\{0\}$ and $\{*\}$); the empty antichain $\emptyset$; and finally $\{0, *\}$. By Corollary 4.3.5, if $G \in \mathbf{OC}_n$, then $B = \{A\,|{-B}\}$ where $A, B \in \mathcal{A}_{n-1}$. Thus, every element of $\mathbf{OC}_2$ is represented by a pair formed by a left- and a right-option-closed antichain from $\mathbf{OC}_1$, so $\{*\}$ cannot be utilized as it is neither left- nor right-option-closed.

So this gives us 25 possibilities to consider. To determine the exact size of $\mathbf{OC}_2$, we must determine which pairs of antichains $G = \{\mathbf{L}\left(G\right)|\mathbf{R}\left(G\right)\}$ are canonical, i.e. those that contain no reversible options.

Suppose that G^L is reversible. Then $\exists G^{LR} \in \mathbf{R}\left(G^L\right)$ such that $G^{LR} \leq G$. Since $b(G) \leq 2$, we know that $b(G^{LR}) = 0$ and as $\mathbf{OC}_0 = \{0\}$, we know that $G^{LR} = 0$. Hence $G^L = *$ or -1. So $\mathbf{L}\left(G\right) = \{-1\}$ or $\{0, *\}$. But we also have $G^{LR} = 0 \leq G$, so $0, -1 \notin \mathbf{R}\left(G\right)$. Therefore, $\mathbf{R}\left(G\right) = \emptyset$ or $\{1\}$. This gives four pairs with a reversible Left option.

Suppose that G^R is reversible. Then $\exists G^{RL} \in \mathbf{L}\left(G^R\right)$ such that $G^{RL} \geq G$. Since $b(G) \leq 2$, we know that $b(G^{RL}) = 0$, and so $G^{RL} = 0$. Hence $G^R = 1$ or $*$. So $\mathbf{R}\left(G\right) = \{1\}$ or $\{0, *\}$. But we also have $G^{RL} = 0 \geq G$, so $0, 1 \notin \mathbf{R}\left(G\right)$. Therefore, $\mathbf{L}\left(G\right) = \emptyset$ or $\{-1\}$. This gives four pairs with a reversible Left option.

The case where $\mathbf{L}\left(G\right) = \{-1\}$ and $\mathbf{R}\left(G\right) = \{1\}$ has been counted twice, so we have a total of 7 games that possess reversible options. Thus, we have $25 - 7 = 18$ games in $\mathbf{OC}_2$. □

Figure 4.4 gives the partial-order structure of $\mathbf{OC}_2$.

In all depictions of this lattice so far, we see that the Hasse diagrams have been drawn in a planar representation. An obvious question to consider is whether or not

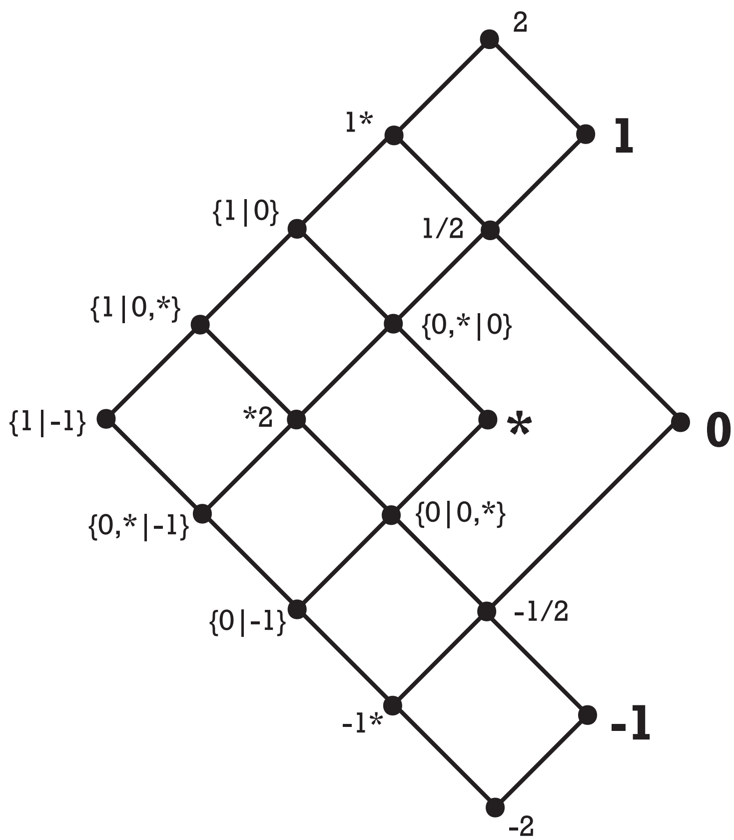


Figure 4.4: The partial-order structure of the 18 option-closed games born by day 2 that make up $\mathbf{OC}_2$.

this is the case for any day. By construction, we do have a bound on the size of $\mathbf{OC}_n$. Since the set of left and right options come from antichains of $\mathbf{OC}_n$, then we know that the following bound holds for day n.

Lemma 4.3.26. *The set of option-closed games born by day n is finite, with*

$$|\mathbf{OC}_n| \leq |\mathcal{A}_n|^2 \leq (|\mathbf{OC}_{n-1}| + 1)^2 .$$

Proof. By construction, $\forall G \in \mathbf{OC}_n$, $\mathbf{L}(G) \in \mathcal{A}_n$ and $\mathbf{R}(G) \in \mathcal{B}_n$. Also by design, $|\mathcal{A}_n| = |\mathcal{B}_n|$. Finally, $|\mathcal{A}_n| \leq |\{\emptyset\} \cup \mathbf{OC}_{n-1}|$. □

While this bound is poor, and tighter bounds remain to be found, it does give us that the lattice of $\mathbf{OC}_n$ is finite. With that, we can show that $\mathbf{OC}_n$ is a planar lattice.

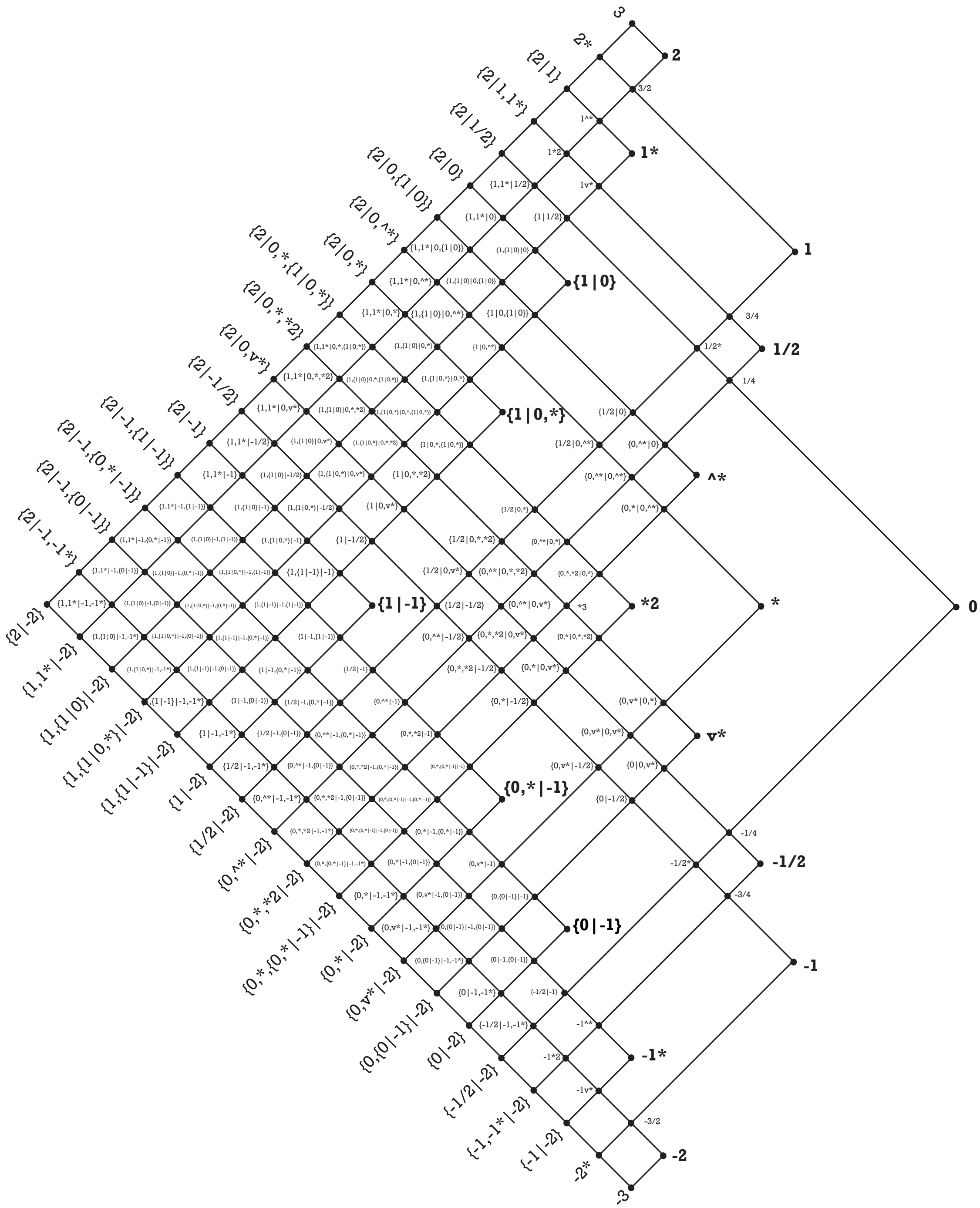


Figure 4.5: The partial-order structure of the 176 option-closed games born by day 3 that make up $\mathbf{OC}_3$.

Theorem 4.3.27. *The lattice of* $\mathbf{OC}_n$ *is planar.*

Proof. Let $G \in \mathbf{OC}_n$. By construction, $\mathbf{L}(G) \in \mathcal{A}_n$ and $\mathbf{R}(G) \in \mathcal{B}_n$. Both $\mathcal{A}_n$ and

$\mathcal{B}_n$ are linear orders and $\mathbf{OC}_n$ is realized by these 2 orders. For games $G, H \in \mathbf{OC}_n$, if $G < H$, then $\downarrow\mathbf{L}(G) \subseteq \downarrow\mathbf{L}(H)$ and $\downarrow\mathbf{R}(G) \subseteq \downarrow\mathbf{R}(H)$, and at least one of these is strict. If G and H are confused in $\mathbf{OC}_n$, then either $\downarrow\mathbf{L}(G) \subset \downarrow\mathbf{L}(H)$ and $\downarrow\mathbf{R}(G) \supset \downarrow\mathbf{R}(H)$, or vice versa.

Since dimension$(\mathbf{OC}_n) \leq 2$ and $\mathbf{OC}_n$ is a finite lattice, the result follows from Proposition 1.3.33. $\square$

4.4 An Option-Closed Compendium

We present now what is known of option-closed games. Consideration for the class of option-closed games began with the work of Nowakowski and Ottaway in 2008 [23]. As their work is quite recent, few new games have been added into this classification. However, more can now be said of the structure of those option-closed games previously introduced.

4.4.1 MAZE

The game of MAZE was introduced in 2006 [2]. At that time, it was introduced as a means of demonstrating outcome class, and little was known of the values of the game. MAZE is played on a board, with a token starting in the top-left position on the board. Solid edges on the board are *walls* that may not be crossed. On a move, Left is allowed to move the token downward any distance and Right is allowed to move the token to the right, neither moving it over or past a solid wall. An interesting feature of MAZE, is that any number of consecutive Left (Right) moves can also be accomplished in one move. As such, MAZE is an instance of an option-closed game. Nowakowski and Ottaway [23] noted that MAZE was option-closed and asked for an analysis of the game.

Since MAZE is option-closed, we know from Lemma 4.1.4 that each position has reduced canonical form equal to a number or a switch. It was conjectured that because of the 2-dimensional structure of the board, there was a bound on the denominator of the numbers that appeared as numbers or in the switches.

Through joint work with Nowakowski and McKay [21], we disprove this by constructing, for each number and each switch, a MAZE position whose reduced canonical form is that value. Surprisingly, we are able to do this with the added restriction that the positions are rectangular and such that any interior walls are vertical. We will refer to boards of this type as **vertical Maze positions**. This restriction is especially surprising as vertical interior walls, and therefore vertical Maze positions, only prevent Right moves, appearing to give an advantage to Left. This construction gives a linear time algorithm that will determine the best move up to an infinitesimal.

We will make use of the theory of reduced canonical form and the fundamentals of option-closed games, presented in sections 1.2.9 and 4.1, respectively, to prove our main result, which is a construction recipe for a MAZE board for any number or switch.

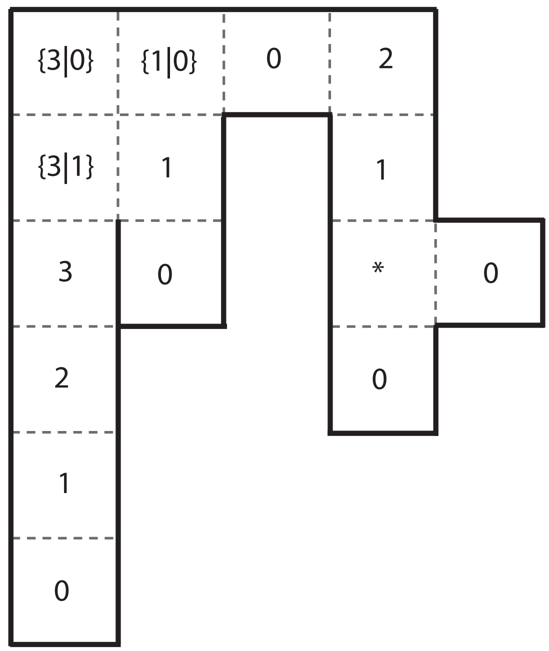


Figure 4.6: A MAZE board in which Left and Right move down and right, respectively. Reduced canonical form of values for each position is included on the board.

For a MAZE position, the left and right options form left- and, respectively, right-option-closed sequences.

Theorem 4.4.1. *[21] Let G be a MAZE position and let α, β be left-, right-option-closed sequences of G, respectively. Then* $\operatorname{rcf}(G) = \operatorname{rcf}\left(\left\{\overline{\alpha}\,\middle|\,\overline{\beta}\right\}\right)$.

4.4.1.1 The Construction

All positions will be rectangular mazes with vertical walls plus the horizontal walls on the lower edge of the rectangle. For brevity, we refer to such a position as a *vertical* position. In any MAZE layout, each square has a value corresponding to that position where the token is on that square. The value of the square at the top left of the maze is called the *value of the rectangle*. With a number a, we associate $m(a)$, any MAZE position with a reduced canonical form of value a. The following shows how to adjust the height of any MAZE position without changing its value.

Lemma 4.4.2. *[21, Lemma 11] Let M be a* MAZE *position. Let M' be the* MAZE *position obtained by: deleting the bottom and right-hand walls of the position; adding another row at the bottom and a column on the right hand side with walls on the bottom of the new row and on the right-hand side of the new column. (Each dimension has increased by 1.) The values of M and M' are equal.*

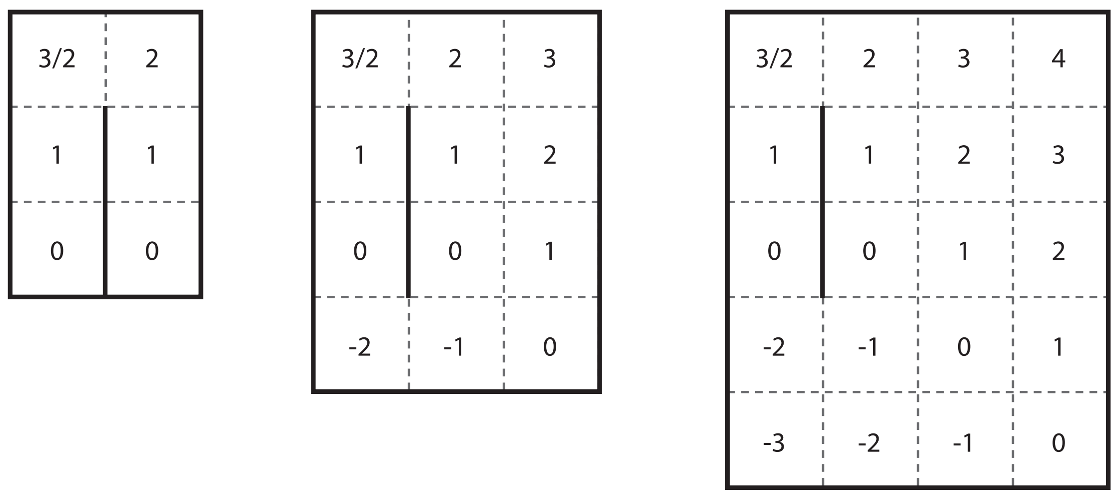


Figure 4.7: MAZE boards of equivalent value.

The idea is the following: if we wish to construct G with $\mathrm{rcf}\,(G) = \mathrm{rcf}\,(\{a\,|b\})$, a and b numbers, then take the MAZE positions $m(A)$ and $m(B)$ obtained via the Construction, adjust the heights and adjoin the two positions as in Figure 4.8.

1. Let p be an integer. If $p = 0$ then $m(0)$ is a single square; if $p > 0$ then $m(p)$ is a vertical line of $p + 1$ squares; if $p < 0$ then $m(p)$ is a horizontal line of $p + 1$ squares.

2. In order to make sure that we can accommodate values for which a is not an integer, we must construct the values of the form $p + \frac{1}{2^q}$ and $-p + \frac{1}{2^q}$. If constructing the former, set $a = p$ and if the latter, $a = -p$. In each case, use the Construction defined above. For $q = 1$, proceed to step 3, taking $b = a + \frac{1}{2}$. For $q > 1$, proceed to step 3, taking $b = a + \frac{1}{2^{q-1}}$ as defined by the Construction.

3. Let $G = \{a\,|b\}$. Take the MAZE positions obtained via the Construction for a and b, for which $m(a)$ and $m(b)$ are the smallest such Construction. Adjust

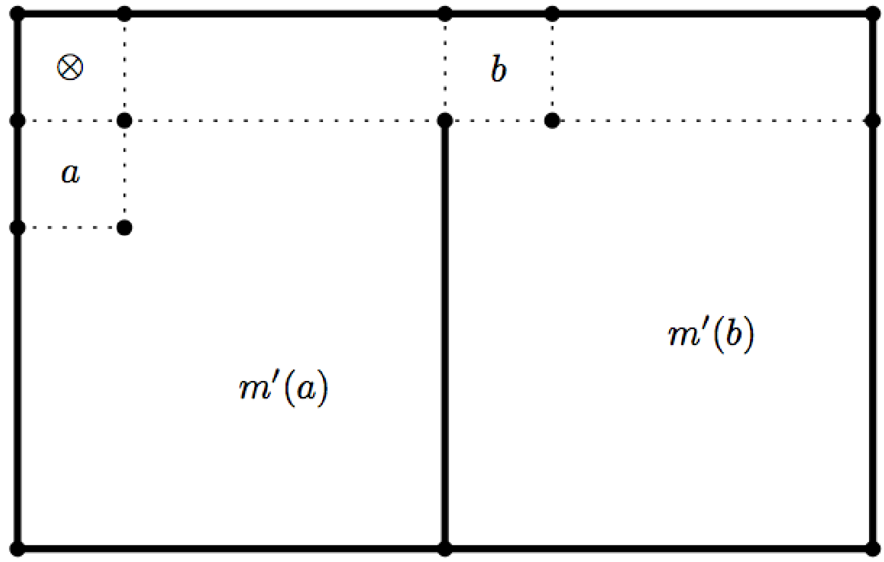


Figure 4.8: The Outline of the Construction. Black lines represent existing walls, dashed lines walls that do not exist.

the height, if necessary, so that $m(a)$ is a $p \times (q+1)$ rectangle and $m(b)$ is a $(p+1) \times (r+1)$ rectangle. Form the MAZE position M by concatenating the rows of $m(a)$ and $m(b)$ from the bottom up. In addition, the top row consist of a line of $p+r+2$ squares with no walls. (See Figure 4.8.)

Claim 4.4.3. *The value of M is $\{a \,|\, b\}$.*

4.4.1.2 Proof Of The Construction Claim

We must show that $M = \{a \,|\, b\}$. After some preliminary, general results about option-closed games, the proof falls into two parts: first show that a and b are the 'dominant' terms except for possibly the x_i; then show that each x_i is a 'switch' which are in turn dominated by b—all this modulo the reduced canonical form.

Proof. Let a and b be dyadic rationals and let M be the MAZE position whose reduced canonical form is claimed to be $\{a \,|\, b\}$. Note that both $m(a)$ and $m(b)$ were obtained by the Construction, so contain no walls in their top rows. Since there are no walls in the top row of $m(b)$, the top row of values, $\langle x_1, x_2, \ldots, x_q, b, y_1, y_2, \ldots, y_r \rangle$, is a right-option-closed sequence, and the values in the first column below a, in order, and including a itself form a left-option-closed sequence. Thus, $\langle \overline{\otimes^{\mathcal{L}}} \rangle = a$ by Lemma

4.1.10 and

$$\mathrm{rcf}\,(\otimes) = \{a|\,\overline{\langle x_1, x_2, \ldots, x_q, b, y_1, y_2, \ldots, y_r\rangle}\}$$

Since $\langle b, y_1, y_2, \ldots, y_r\rangle$ is also a right-option-closed sequence, then $y_i \geq_{\mathbf{Inf}} b$ by Lemma 4.1.12. Thus,

$$\mathrm{rcf}\,(\otimes) = \{a|\,\overline{\langle x_1, x_2, \ldots, x_q, b\rangle}\}$$

and we are left to show that

$$b = \overline{\langle x_1, x_2, \ldots, x_q, b\rangle}.$$

Now, if x_i is a not a number and there is no $j > i$ such that x_j is a number, then by Lemma 4.1.10, $x_i \geq_{\mathbf{Inf}} b$ and is infinitesimally-dominated. Therefore, let i be the greatest index such that x_i is a number. Necessarily then $x_i < b$ since b is a right option of x_i. Since there are no walls in the top row of $m(a)$, i.e. between a and any $x_i^{L_1}$, $\langle x_1^{L_1}, \ldots, x_{q-1}^{L_1}, x_q^{L_1} = p - 1\rangle$ is a right-option-closed sequence. Also, $x_i^{L_1}$ is a Right option of a and so by Lemma 1.2.44 $x_i^{L_1} > a$.

Consider the case $a > b$. Now we have the inequalities $x_i^{L_1} > a > b > x_i$ which is a contradiction since for any game G, $G^L \not> G$. Therefore, there is no i such that x_i is a number and thus $\mathrm{rcf}\,(\otimes) = \{a\,|b\}$.

Now, $\otimes = \{a\,|b, x_1, x_2, \ldots, x_q\}$. To show that $\otimes = \{a\,|b\}$, we must show that each x_i is reversible. Consider

$$x_i^{L_1} - \{a|b, x_1, x_2, \ldots, x_q\} = x_i^{L_1} + \{-b, -x_1, -x_2, \ldots, -x_q| - a\}.$$

Left wins playing first by moving to $x_i^{L_i} - b$ because $x_i^{L_1} > a > b$. Playing first, Right loses if he plays to $x_i^{L_1} - a$. If he plays to a right option of $x_i^{L_1}$ then this is equal to $x_j^{L_1}$ for some $j > i$ giving the position $x_j^{L_1} + \{-b, -x_1, -x_2, \ldots, -x_q| - a\}$ and again Left wins by playing to $x_j^{L_1} - a$, i.e. $x_i^{L_1} > \otimes$.

Since the right options from $x_i^{L_1}$ are $x_j^{L_1}$ for $j > i$ and $x_j^{L_1} > a > b$, then bypassing the reversible move to x_i which adds $x_j^{L_1}(j > i)$ to the right options, and eliminating dominated options, gives $\otimes = \{a\,|b\}$.

Suppose x_i is a number less than b. Suppose $a < b$. We know that $x_i^{L_1} > a$ by Lemma 4.1.10. Thus, $a < x_i < b$ and $x_i = \{d|b\}$ for some number d, where $d = \mathrm{L}_0\,(x_i)$ and $b > d > a$. Now d is a left option of x_i and hence is in $m(a)$.

Let $c = \{a\,|b\} = \frac{y}{2^{n+1}}$ for some integers y and n. Since c is the simplest dyadic rational between a and b then the denominator of x_i is 2^{n+j} for some $j \geq 1$. By induction, the denominator of d is less than that of a, i.e. at most 2^n but then d is simpler than c and lies between a and b which is a contradiction.

Hence no x_i is a number and thus $\mathrm{rcf}\,(\otimes) = c = \{a\,|b\}$.

Now, if $\mathrm{rcf}\,(\otimes) = c = \{a\,|b\}$ and $a < b$ then $\otimes = \{a|b, \otimes^R\}$ then only non-dominated, non-numerical options possibly remaining are those that are infinitesimally-dominated by b. Note that if n and x are numbers and δ an infinitesimal with $x = \{x^L|n, n+\delta, x^R\}$ then $x = \{x^L|n, x^R\}$. Repeated application of this fact gives $\otimes = \{a|b\}$. $\square$

4.4.1.3 Evaluating MAZE *Positions*

Grossman and Siegel [14] note that for most situations the reduced canonical form is sufficient to evaluate a position. Calculating the reduced canonical form of a MAZE position can be done in linear time with regard to the number of squares in the position [21].

Corollary 4.4.4. *[21] The reduced canonical form of an $p \times q$* MAZE *position can be calculated in $O(pq)$ time.*

4.4.1.4 Open Questions

From Nowakowski and Ottaway [23], we know that for any option closed game G, $\Downarrow +* < G - \mathrm{rcf}\,(G) < \Uparrow +*$. The construction can also build games of the form $*n$. What infinitesimals occur in MAZE? This construction is able to return MAZE positions having reduced canonical form $\{a\,|b\}$ for numbers a and b. Can we do better than coming within an infinitesimal from this value and, in fact, construct a MAZE position having value $\{a\,|b\}$?

4.4.2 Roll The Lawn

The game of ROLL THE LAWN was introduced by Nowakowski and Ottaway [23] as an example of an option-closed game. We will discuss this game under Normal play, however Ottaway looked at this game under Misère Play in his Ph.D. thesis [24].

4.4.2.1 *How To Play*

ROLL THE LAWN uses a row of nonnegative integers (or bumps) and a roller that is placed between any two bumps or at either end. Left (Right) moves the roller to the left (right), flattening each bump it passes over by 1 unless the bump has already been completely flattened to zero. At least one bump must decrease in size at each move.

The game can be represented as a path graph with weighted edges. In this visual representation, the bumps are edge-weights in the graph and the roller is located on one of the vertices. On a move, the roller is either moved to the left by Left (right by Right) to another vertex. In doing so, the edge-weight of all edges traversed having positive edge-weight is decreased by one.

For simplicity, we will represent a position by a string of nonnegative integers with a roller $\ominus$ located within the string. Since we will consider similar games, the string will be prefixed RL to denote that it is indeed a ROLL THE LAWN position in question.

Example 4.4.5.
$\mathrm{RL}[7, \ominus, 1, 4]$

$\xrightarrow{R}$ $\mathrm{RL}[7, 0, 3, \ominus]$
$\xrightarrow{L}$ $\mathrm{RL}[7, \ominus, 0, 2]$
$\xrightarrow{R}$ $\mathrm{RL}[7, 0, 1, \ominus]$
$\xrightarrow{L}$ $\mathrm{RL}[7, 0, \ominus, 0]$ *(Left has won)*

Nowakowski and Ottaway were able to determine the value of a ROLL THE LAWN position based primarily on the parity of the bumps, or edge-weights, to either side of the roller.

Theorem 4.4.6. *[23, Cor. 18] Let G be the* ROLL THE LAWN *position*

$$G = \mathrm{RL}[a_1, a_2, \ldots, a_j, \ominus, a_{j+1}, \ldots, a_k]$$

and let $b_i = a_i(\mathrm{mod} 2)$ with $b_i \in \{0, 1\}$. Then

$$G = \sum_{i \leq j} b_i - \sum_{i > j} b_i.$$

This can be seen from the fact that when the roller starts on one side of a bump, it will finish back on the same side if the bump is even and on the other side if the bump is odd. The winner will be the person who has more odd bumps on their side.

4.4.3 Cricket Pitch

4.4.3.1 How To Play

Unlike a normal lawn, certainly unlike my own, the greens on a cricket pitch are meticulously kept. The cricket pitch is the strip of the cricket field between the wickets. This tended strip of grass is kept very flat and the grass on it extremely short. Once it is made perfect, it is not to be touched.

As such, the game of CRICKET PITCH (or ROLL THE CRICKET PITCH), as introduced by Nowakowski and Ottaway [23], has the same ruleset as ROLL THE LAWN, but with the added constraint that the roller cannot go over a 0.

We will again represent a position by a string of nonnegative integers and the roller by the symbol $\ominus$. The string will be prefixed CL to denote that it is a CRICKET PITCH positions position.

Example 4.4.7.

$\mathrm{CP}[5, \ominus, 1, 2, 4]$

$\xrightarrow{R}$ $\mathrm{CP}[5, 0, 1, \ominus, 4]$

$\xrightarrow{L}$ $\mathrm{CP}[5, 0, \ominus, 0, 4]$ *(Left has won)*

For brevity, if the position includes a sequence of integers $\alpha = \langle a_i \rangle_{i=1}^{n}$, we then make use of the notation $\alpha - 1 = \langle a_i - 1 \rangle_{i=1}^{n}$. This notation is useful in that from a starting position $\mathrm{CP}[\alpha, \ominus, \beta]$, where α and β are strings of positive integers, the moves available to Left and Right are of the form $\mathrm{CP}[\alpha_1, \ominus, \alpha_2 - 1, \beta]$ and $\mathrm{CP}[\alpha, \beta_1 - 1, \ominus, \beta_2]$, respectively, where $\alpha = \alpha_1\alpha_2$ and $\beta = \beta_1\beta_2$.

4.4.3.2 Reductions

A necessary observation from Nowakowski and Ottaway [23] is that anything outside of a zero-bump from the roller is irrelevant. Since the roller can never pass over a zero, it can be removed from the position without changing the game. Thus, we may **prune** anything outside of a zero from the roller.

Lemma 4.4.8. *[23, Obs. 19] For strings α and β of nonnegative integers,*

$$\mathrm{CP}[\alpha, 0, \beta, \ominus, \gamma, 0, \delta] = \mathrm{CP}[\beta, \ominus, \gamma].$$

For example,

$$\mathrm{CP}[2,3,0,3,4,\ominus,1,2,0,98] \quad = \quad \mathrm{CP}[3,4,\ominus,1,2]$$

They also gave us an important reduction that says that we can **reduce** all bumps in play by 2 so long as we don't move any below zero. In other words, a given Cricket Pitch position is equivalent to that in which all bump sizes are increased by 2.

Lemma 4.4.9. *[23, Lem. 20] For strings α and β of nonnegative integers,*

$$\mathrm{CP}[\alpha, \ominus, \beta] = \mathrm{CP}[\alpha + 2, \ominus, \beta + 2].$$

If we repeatedly apply the act of pruning and reducing, we eventually reach a position in which there are no bumps left, or there is a bump of size 1 but none of size zero. That is, a position which can no longer be pruned or reduced further. We call a position of this type **reduced and pruned**.

For example, we can take the following game and move it to a position that has been reduced and pruned:

$$\begin{aligned}
&\mathrm{CP}[3,2,5,8,\ominus,6,2,5,7] \\
&\qquad \longrightarrow \quad \mathrm{CP}[1,0,3,6,\ominus,4,0,3,5] \quad \text{(reduce)} \\
&\qquad \longrightarrow \quad \mathrm{CP}[3,6,\ominus,4] \qquad\qquad\quad\ \ \text{(prune)} \\
&\qquad \longrightarrow \quad \mathrm{CP}[1,4,\ominus,2] \qquad\qquad\quad\ \ \text{(reduce)}
\end{aligned}$$

4.4.3.3 Low Points

Nowakowski and Ottaway were unable to determine the value of all Cricket Pitch positions, but they were able to determine the outcome class of a game based on the relative low points amongst all odd integers on either side of the roller.

Definition 4.4.10. *[23, Def. 21] In*

$$G = \mathrm{CP}[a_1, \ldots, a_m, \ominus, b_1, \ldots, b_n],$$

the ***Left odd low point****, denoted* ldip*, is*

$$\text{ldip}(G) = \min\{a_i \ : \ a_i \textit{ is odd and } \forall j > i, a_i < a_j\}$$

unless there is no such heap in which case $\text{ldip}(G) = \infty$*.*

Similarly, the ***Right odd low point****, denoted* rdip*, is*

$$\text{rdip}(G) = \min\{b_i \ : \ b_i \textit{ is odd and } \forall j < i, b_j > b_i\}$$

unless there is no such heap in which case $\text{rdip}(G) = \infty$*.*

For example, the position $\text{CP}[1, 2, 3, 4, \ominus, 1, 2, 3]$ has $ldip(G) = 1$ and $\text{rdip}(G) = 1$, while the position $\text{CP}[5, 2, 3, 4, \ominus, 2, 2, 3]$ has $\text{ldip}(G) = 3$ and $\text{rdip}(G) = \infty$.

With these tools in hand, they determine the outcome class based on the relationship between these Left and Right odd low points.

Theorem 4.4.11. *[23, Thm. 22] For* $G = \text{CP}[A, \ominus, B]$*, the outcome classes are determined by the odd low points as follows:*

(i) If $\text{ldip}(G) < \text{rdip}(G)$*, then* $G \in \mathcal{L}$

(ii) If $\text{ldip}(G) > \text{rdip}(G)$*, then* $G \in \mathcal{R}$

(iii) If $\text{ldip}(G) = \text{rdip}(G) < \infty$*, then* $G \in \mathcal{N}$

(iv) If $\text{ldip}(G) = \text{rdip}(G) = \infty$*, then* $G \in \mathcal{P}$

4.4.3.4 New Results

Switch on the Pitch

For ease, when considering strings of nonnegative integers, we will denote a number n of repeating integers by a superscript n. For example, the position $\text{CP}[1, 1, 1, \ominus, 1]$ would be denoted $\text{CP}[1^3, \ominus, 1]$.

We first note the value of games involving only strings of ones on either side of the roller. When at most one side of the roller contains any ones, the position is clearly an integer.

Lemma 4.4.12.

$$\text{CP}[1^a, \ominus] = a.$$

Proof. When $a = 0$, then $\mathrm{CP}[1^a, \ominus] = \mathrm{CP}[\ominus] = 0$.

If $a > 0$, then $\mathrm{CP}[1^a, \ominus] = \left\{ \langle \mathrm{CP}[1^{a', \ominus}] \rangle_{a'=0}^{a-1} \,|\, \cdot \right\} = \{0, \ldots, a-1 \,|\, \cdot\} = \{a-1 \,|\, \cdot\} = a$. □

Analogously, we see that $\mathrm{CP}[\ominus, 1^b] = -b$. If both sides contain at least one 1, then these values are switches.

Lemma 4.4.13.

$$\mathrm{CP}[1^a, \ominus, 1^b] = \{\ a-1 \ \mid 1-b\ \}.$$

Proof. For $a, b > 0$, then $\mathrm{CP}[1^a, \ominus, 1^b] = \left\{ \langle \mathrm{CP}[1^{a'}, \ominus, 0^{a-a'}, 1^b] \rangle_{a'=0}^{a-1} \,\middle|\, \langle \mathrm{CP}[1^a, 0^{b-b'}, \ominus, 1^{b'}] \rangle_{b'=0}^{b-1} \right\} = \left\{ \langle \mathrm{CP}[1^{a'}, \ominus] \rangle_{a'=0}^{a-1} \,\middle|\, \langle \mathrm{CP}[\ominus, 1^{b'}] \rangle_{b'=0}^{b-1} \right\} = \{0, \ldots, a-1 \,|\, 0, \ldots, 1-b\} = \{a-1 \,|\, 1-b\}$. □

For example, $\mathrm{CP}[1, 1\ominus, 1] = \{1 \,|\, 0\}$ and $\mathrm{CP}[1, 1, 1, 1, \ominus, 1, 1, 1] = \{3 \,|\, {-2}\}$.

Lemma 4.4.14. *Let α, β, γ be (possibly empty) sequences of nonnegative integers and let $n \in \mathbb{Z}^{>0}$.*

$$\mathrm{CP}[\alpha, 2n, \beta, \ominus, \gamma] \leq \mathrm{CP}[\alpha, (2n+2), \beta, \ominus, \gamma].$$

Proof. Let $G = \mathrm{CP}[\alpha, 2n, \beta, \ominus, \gamma]$ and $H = \mathrm{CP}[\alpha, (2n+2), \beta, \ominus, \gamma]$. We consider the game $G - H$ with Left moving first. We claim that Left cannot win.

Any move that Left is able to make, Right can 'mirror' in the other other component. Finally, we are reduced to the position

$$\mathrm{CP}[0, \beta', \ominus, \gamma'] - \mathrm{CP}[\alpha', 2, \beta', \ominus, \gamma']$$

with Left to move. If on his move, Left does not cross a 1, then Right can mirror that move in the other component, leaving a position of similar type. If Left crosses a 1 in γ' then Right mirrors this playing to $\mathrm{CP}[0, \ominus, \gamma''] - \mathrm{CP}[0, \ominus, \gamma'']$. If Left crosses a 1 in β', then Right mirrors again to

$$\mathrm{CP}[\beta'', \ominus, \beta'''] - \mathrm{CP}[\alpha', 2, \beta'', \ominus, \beta''']$$

which is less than or equal to zero since $\mathrm{CP}[\alpha', 2, \ominus] \geq 0$. □

An Odd Game

In order to arrive at the actual value of the game, we need two further reductions. First, we **even it out** by giving a reduction that allows us to remove all even numbered bumps leaving outside of it either a smaller odd number or a zero which could then be pruned.

Lemma 4.4.15 (Even reduction)**.** *Let α, β, γ be (possibly empty) sequences of non-negative integers and let c and n be positive integers with c odd.*

$$\textit{(i)}\ \mathrm{CP}[\alpha, c, 2n, \beta, \ominus, \gamma] = \begin{cases} \mathrm{CP}[\alpha, 2n, \beta, \ominus, \gamma] & c \geq 2n \\ \mathrm{CP}[\alpha, c, \beta, \ominus, \gamma] & c < 2n \end{cases}$$

$$\textit{(ii)}\ \mathrm{CP}[\alpha, \ominus, \beta, 2n, c, \gamma] = \begin{cases} \mathrm{CP}[\alpha, \ominus, \beta, 2n, \gamma] & c \geq 2n \\ \mathrm{CP}[\alpha, \ominus, \beta, c, \gamma] & c < 2n \end{cases}$$

Proof. We can assume that for all $b \in \beta$, $b \geq 2n$.

(i) Suppose $c \geq 2n$. Let $G = \mathrm{CP}[\alpha, c, 2n, \beta, \ominus, \gamma]$ and $H = \mathrm{CP}[\alpha, 2n, \beta, \ominus, \gamma]$. We consider the game $G - H$ and claim that it is a second player win.

If either player within β or γ, the 'mirror' response is available to the second player in the other component to a position that is zero by induction. We must consider (a) Left moving in G past $2n$, (b) Left moving in G past c, and (c) Right moving in H past $2n$.

(a) If Left moves in G just past $2n$ to

$$\mathrm{CP}[\alpha, c, \ominus, (2n-1), (\beta-1), \gamma] - H,$$

the Right can respond by moving in H to

$$\mathrm{CP}[\alpha, c, \ominus, (2n-1), (\beta-1), \gamma] - \mathrm{CP}[\alpha, 2n, \ominus, (\beta-1), \gamma].$$

From this position, Left must either move in the first component past the c or in the second component.

If Left plays in the first component to some

$$\mathrm{CP}[\alpha_1, \ominus, \alpha_2, (c-1), (2n-1), (\beta-1), \gamma] - \mathrm{CP}[\alpha, 2n, \ominus, (\beta-1), \gamma],$$

then Right can respond to

$$\mathrm{CP}[\alpha_1, \ominus, \alpha_2, (c-1), (2n-1), (\beta-1), \gamma] - \mathrm{CP}[\alpha_1, \ominus, \alpha_2, (2n-1), (\beta-1), \gamma],$$

which is zero by induction since $2n-1 \leq c-1$ with $c-1$ even.

If Right plays in the second component to a position of the form

$$\mathrm{CP}[\alpha, c, \ominus, (2n-1), (\beta-1), \gamma] - \mathrm{CP}[\alpha, 2n, (\beta-2), \gamma_1, \ominus, \gamma_2],$$

then Right can respond to

$$\mathrm{CP}[\alpha, c, (2n-2), (\beta-2), \gamma_1, \ominus, \gamma_2] - \mathrm{CP}[\alpha, 2n, (\beta-2), \gamma_1, \ominus, \gamma_2],$$

which by induction is

$$\mathrm{CP}[\alpha, (2n-2), (\beta-2), \gamma_1, \ominus, \gamma_2] - \mathrm{CP}[\alpha, 2n, (\beta-2), \gamma_1, \ominus, \gamma_2]$$

since $c \geq 2n > 2n-2$. Thus Right wins playing second since by Lemma 4.4.14, $\mathrm{CP}[\alpha, (2n-2), (\beta-2), \gamma_1, \ominus, \gamma_2] \leq \mathrm{CP}[\alpha, 2n, (\beta-2), \gamma_1, \ominus, \gamma_2]$.

(b) If Left moves in G past c to

$$\mathrm{CP}[\alpha_1, \ominus, (\alpha_2-1), (c-1), (2n-1), (\beta-1), \gamma] - H,$$

then Right responds with the 'mirror' move in H to

$$\mathrm{CP}[\alpha_1, \ominus, (\alpha_2-1), (c-1), (2n-1), (\beta-1), \gamma] - \mathrm{CP}[\alpha_1, \ominus, (\alpha_2-1), (2n-1), (\beta-1), \gamma]$$

which is zero by induction since $(2n-1) \leq (c-1)$ with $c-1$ even.

(c) If Right moves in H past $2n$, then Left can respond with the 'mirror' move in G to some

$$\mathrm{CP}[\alpha_1, \ominus, \alpha_2, (c-1), (2n-1), (\beta-1), \gamma] - \mathrm{CP}[\alpha_1, \ominus, \alpha_2, (2n-1), (\beta-1), \gamma]$$

which Left wins by induction since $2n-1 \leq c-1$ with $c-1$ even.

(ii) Suppose $c < 2n$. Let $G = \mathrm{CP}[\alpha, c, 2n, \beta, \ominus, \gamma]$ and $H = \mathrm{CP}[\alpha, c, \beta, \ominus, \gamma]$. Consider $G - H$.

Again, if either player within β or γ, the 'mirror' response is available to the second player in the other component to a position that is zero by induction. We must consider (a) Left moving in G past $2n$, (b) Left moving in G past c, and (c) Right moving in H past c.

(a) If Left moves in G just past $2n$ to

$$\mathrm{CP}[\alpha, c, \ominus, (2n-1), (\beta-1), \gamma] - H,$$

the Right can respond by moving in H to

$$\mathrm{CP}[\alpha, c, \ominus, (2n-1), (\beta-1), \gamma] - \mathrm{CP}[\alpha, c, \ominus, (\beta-1), \gamma].$$

From this position, Left must either move in the first component past the c or in the second component.

If Left plays in the first component to some

$$\mathrm{CP}[\alpha_1, \ominus, \alpha_2, (c-1), (2n-1), (\beta-1), \gamma] - \mathrm{CP}[\alpha, c, \ominus, (\beta-1), \gamma],$$

then Right can respond to

$$\mathrm{CP}[\alpha_1, \ominus, \alpha_2, (c-1), (2n-1), (\beta-1), \gamma] - \mathrm{CP}[\alpha_1, \ominus, \alpha_2, (c-1), (\beta-1), \gamma],$$

which is zero by induction since $2n-1 > c-1$ with $c-1$ even.

If Right plays in the second component to a position of the form

$$\mathrm{CP}[\alpha, c, \ominus, (2n-1), (\beta-1), \gamma] - \mathrm{CP}[\alpha, c, (\beta-2), \gamma_1, \ominus, \gamma_2],$$

then Right can respond to

$$\mathrm{CP}[\alpha, c, (2n-2), (\beta-2), \gamma_1, \ominus, \gamma_2] - \mathrm{CP}[\alpha, c, (\beta-2), \gamma_1, \ominus, \gamma_2].$$

By Lemma 4.4.14, this is at most

$$\mathrm{CP}[\alpha, c, (2n), (\beta-2), \gamma_1, \ominus, \gamma_2] - \mathrm{CP}[\alpha, c, (\beta-2), \gamma_1, \ominus, \gamma_2]$$

which is zero by induction since $c < 2n$.

(b) If Left moves in G past c to

$$\mathrm{CP}[\alpha_1, \ominus, (\alpha_2-1), (c-1), (2n-1), (\beta-1), \gamma] - H,$$

then Right responds with the 'mirror' move in H to

$$\mathrm{CP}[\alpha_1, \ominus, (\alpha_2-1), (c-1), (2n-1), (\beta-1), \gamma] - \mathrm{CP}[\alpha_1, \ominus, (\alpha_2-1), (2n-1), (\beta-1), \gamma]$$

which is zero by induction since $(2n-1) \leq (c-1)$ with $c-1$ even.

(c) If Right moves in H past $2n$, then Left can respond with the 'mirror' move in G to some

$$\mathrm{CP}[\alpha_1, \ominus, \alpha_2, (c-1), (2n-1), (\beta-1), \gamma] - \mathrm{CP}[\alpha_1, \ominus, \alpha_2, (2n-1), (\beta-1), \gamma]$$

which Left wins by induction since $2n-1 \leq c-1$ with $c-1$ even.

□

The following gives an example of how we can even out positions.

Example 4.4.16. *Even it out:*

$\mathrm{CP}[1, 1, 3, 2, \ominus, 1, 4, 5, 6, 3, 1, 2]$
$= \mathrm{CP}[1, 1, 2, \ominus, 1, 4, 3, 1, 2]$
$= \mathrm{CP}[1, 1, \ominus, 1, 3, 1, 2]$
$= \mathrm{CP}[1, 1, \ominus, 1, 3, 1]$

Low Points Are Big

Our final necessary reduction is based on the fact that since we cannot cross over zeros, any larger integers outside of a smaller one from the roller do not 'act' any larger and so we can, in fact, cut the tops off all bumps outside of the smaller integer that are higher than it. That is, the number of times the roller crosses the smaller inside integer is what determines the possible number of times the outside larger integer can be crossed before it is pruned.

Lemma 4.4.17. *[Low Point reduction] Let α, β, γ be (possibly empty) sequences of nonnegative integers and let $c, d \in \mathbb{Z}^{>0}$ such that c and d are both odd with $c < d$. Then,*

$$\mathrm{CP}[\alpha, d, c, \beta, \ominus, \gamma] = \mathrm{CP}[\alpha, c, c, \beta, \ominus, \gamma],$$

and

$$\mathrm{CP}[\alpha, \ominus, \beta, c, d, \gamma] = \mathrm{CP}[\alpha, \ominus, \beta, c, c, \gamma].$$

Proof. Let $G = \mathrm{CP}[\alpha, d, c, \beta, \ominus, \gamma]$ and $H = \mathrm{CP}[\alpha, c, c, \beta, \ominus, \gamma]$. We consider the game $G - H$ and claim that it is a second player win.

If Left plays in a component, Right can always respond with the 'mirror' move in the other component. After such play, they finally reach a position

$$\mathrm{CP}[\alpha', d', 1, \beta', \ominus, \gamma'] - \mathrm{CP}[\alpha', 1, 1, \beta', \ominus, \gamma']$$

with $d' > 1$ odd and Left to play. Again, Right can always mirror Left's move in the other component, until he crosses d' to some position

$$\mathrm{CP}[\alpha_1, \ominus, (\alpha_2 - 1), (d' - 1)] - \mathrm{CP}[\alpha_1, \ominus, (\alpha_2 - 1)] \leq 0.$$

□

The following gives an example of how we can cut the tops off and keep it low.

Example 4.4.18. *Keep it low:*

$\mathrm{CP}[1, 1\ominus, 3, 5, 1, 3, 3, 0, 7, 9]$
$= \mathrm{CP}[1, 1\ominus, 3, 5, 1, 3, 3, 0]$
$= \mathrm{CP}[1, 1\ominus, 3, 5, 1, 3, 3]$
$= \mathrm{CP}[1, 1\ominus, 3, 5, 1, 1, 1]$
$= \mathrm{CP}[1, 1\ominus, 3, 3, 1, 1, 1]$

Main Theorem

Finally, all of this comes together to give us our game values. We notice that from repeated application we have equivalent games that are made up of odd, non-increasing strings of integers as we move out from the roller. We also note that because of the reduction of Lemma 4.4.9, the game is either $\mathrm{CP}[\ominus]$ or there exists at least one 1 on the outside of the roller.

This leads us to our main theorem which gives us the value of the game as the ordinal sum of the game based on those outside ones and all the other bits in the middle.

Theorem 4.4.19 (Ordinal Sum Approach)**.** *Let $\alpha = \langle a_i \rangle_{i=1}^{m}$ and $\beta = \langle b_i \rangle_{i=1}^{n}$ be (possibly empty) sequences of odd integers such that $\forall i < j, 1 < a_i \leq a_j$ and $b_i \geq b_j > 1$.*

Let $a, b \in \mathbb{Z}^{\geq 0}$. Then,

$$\mathrm{CP}[1^a, \alpha, \circleddash, \beta, 1^b] = \mathrm{CP}[1^a, \circleddash, 1^b]\mathbf{:}\mathrm{CP}[\alpha, \circleddash, \beta] = \{a-1\,|1-b\}\mathbf{:}\mathrm{CP}[\alpha, \circleddash, \beta].$$

Proof. Let $G = \mathrm{CP}[1^a, \alpha, \circleddash, \beta, 1^b]$ and $H = \mathrm{CP}[1^a, \circleddash, 1^b]\mathbf{:}\mathrm{CP}[\alpha, \circleddash, \beta]$. We claim that $G - H$ is a second player win.

If Left plays in G to $\mathrm{CP}[1^k, \circleddash]$ where $0 \leq k < a$, then Right can move to zero by playing to the $\mathrm{CP}[1^k, \circleddash]$ in the H component.

If Left plays in G to some $\mathrm{CP}[1^a, \alpha_1, \circleddash, (\alpha_2 - 1), \beta, 1^b]$, then Right can respond by playing in H to $\mathrm{CP}[1^a, \circleddash, 1^b]\mathbf{:}\mathrm{CP}[\alpha_1, \circleddash, (\alpha_2 - 1), \beta]$, leaving position

$$\mathrm{CP}[1^a, \alpha_1, \circleddash, (\alpha_2 - 1), \beta, 1^b] - \mathrm{CP}[1^a, \circleddash, 1^b]\mathbf{:}\mathrm{CP}[\alpha_1, \circleddash, (\alpha_2 - 1), \beta]$$

which is zero by induction.

Similarly, if Left plays in H to $\mathrm{CP}[\circleddash, 1^k]$ where $0 \leq k < b$, then Right can move to zero by playing to $\mathrm{CP}[\circleddash, 1^k]$ in the G component.

If Left plays in H to $\mathrm{CP}[1^a, \circleddash, 1^b]\mathbf{:}\mathrm{CP}[\alpha, (\beta_1 - 1), \circleddash, \beta_2]$, then Right can respond by playing in G to $\mathrm{CP}[1^a, \alpha, (\beta_1 - 1), \circleddash, \beta_2, 1^b]$, leaving position

$$\mathrm{CP}[1^a, \alpha_1, \circleddash, (\alpha_2 - 1), \beta, 1^b] - \mathrm{CP}[1^a, \circleddash, 1^b]\mathbf{:}\mathrm{CP}[\alpha_1, \circleddash, (\alpha_2 - 1), \beta]$$

which is zero by induction.

□

The following gives an example of the application of this ordinal sum approach.

Example 4.4.20. *Ordinal sum approach:*

$\mathrm{CP}[1, 1, 5, \circleddash, 7, 3, 1]$
$= \mathrm{CP}[1, 1, \circleddash, 1]\mathbf{:}\mathrm{CP}[5, \circleddash, 7, 3]$
$= \{1\,|0\}\mathbf{:}\mathrm{CP}[3, \circleddash, 5, 1]$
$= \{1\,|0\}\mathbf{:}(\mathrm{CP}[\circleddash, 1]\mathbf{:}\mathrm{CP}[3, \circleddash, 5])$
$= \{1\,|0\}\mathbf{:}(-1\mathbf{:}\mathrm{CP}[1, \circleddash, 3])$
$= \{1\,|0\}\mathbf{:}(-1\mathbf{:}(1\mathbf{:}-1))$
$= \{1\,|0\}\mathbf{:}\left(-1\mathbf{:}\frac{1}{2}\right)$
$= \{1\,|0\}\mathbf{:}\frac{3}{4}$
$= \{1, \{1\,|0, \{1\,|0\}\}\,|0, \{1, \{1\,|0, \{1\,|0\}\}\,|0, \{1\,|0\}\}\}$

Finally, we answer a question posed by Nowakowski and Ottaway in [23]. They simply ask who wins in the game

$$\mathrm{CP}[1,2,3,\ominus,4,1,3]+\mathrm{CP}[1,1,3,2,\ominus,1,3,1]+\mathrm{CP}[3,3,2,1,2,\ominus,2,2,1]+\mathrm{CP}[1,2,\ominus,3,1,2].$$

We make use of all of our strategies to arrive at the solution.

$$\begin{aligned}
&\mathrm{CP}[1,2,3,\ominus,4,1,3]+\mathrm{CP}[1,1,3,2,\ominus,1,3,1]+\mathrm{CP}[3,3,2,1,2,\ominus,2,2,1]+\mathrm{CP}[1,2,\ominus,3,1,2]\\
&=\mathrm{CP}[1,3,\ominus,1,3]+\mathrm{CP}[1,1,\ominus,1,3,1]+\mathrm{CP}[1,\ominus,1]+\mathrm{CP}[1,\ominus,3,1] \text{ (Even it out)}\\
&=\mathrm{CP}[1,3,\ominus,1,1]+\mathrm{CP}[1,1,\ominus,1,1,1]+\mathrm{CP}[1,\ominus,1]+\mathrm{CP}[1,\ominus,3,1] \text{ (Low points rule)}\\
&=\{0\,|{-1}\}\mathbin{:}1 \quad+\quad \{1\,|{-2}\} \quad+\quad \{0\,|\,0\} \quad+\quad \{0\,|\,0\}\mathbin{:}{-1} \qquad \text{(Main Theorem)}\\
&=\{0,\{0\,|{-1}\}\,|{-1}\} \;+\; \{1\,|{-2}\} \;+\; * \;+\; {\downarrow}*\\
&=\{\{1\downarrow,\{1\downarrow|\downarrow\}\,|\downarrow\} \;\mid\; \{-2\downarrow,\{-2\downarrow|-3\downarrow\}\,|-3\downarrow\}\}\\
&<0\\
&\implies \text{Right wins}
\end{aligned}$$

Thus, through the ordinal sum approach, we are able to provide them with the actual value of the game. To have simply answered the question “who wins?”, we could have stopped at the point right after we utilized the main theorem and noted that the reduced canonical form of this simplifies everything quite nicely. In fact, we see that

$$\operatorname{rcf}(G)=\{\{1\,|\,0\}\,|\{-2\,|{-3}\}\}<0.$$

Chapter 5

Conclusion

5.1 $\mathcal{P}$-Positions: Where We Will End

"You learn from a conglomeration of the incredible past:
whatever experience gotten in any way whatsoever."
- Bob Dylan

Understanding the underlying structure of a game can give rise to new solutions and perspectives on how to approach a game. Once the hidden structure is known, the analysis of a game can then sometimes follow rather simply. However, spotting this often-camouflaged structure is the true art. Exploration of various interactions, and further understanding of functions that describe these interactions, will help to fill the combinatorial game theorists' toolkit.

Once the interactions are spotted, understanding of the functions describing them can be valuable. A better comprehension of how and when these functions can be of use to us will allow us to better make use of these tools meant to simplify a game. We have made use of these tools in order to obtain results for several games. We were able to apply ordinal sums to give a solution to the game of LENRES played on zeros and ones and were also able to give an alternate solution to the game of SHOVE. Through the application of side-sums, we were able to identify a solution to the game of RESTRICTED TOPPLING DOMINOES.

In terms of the overarching structure of posets of games, this too can serve a valuable purpose. Once we have a better understanding of the forms of games that can exist within a classification of games, such as that of option-closed games, we can gain insight into the restrictions that may be placed on options within that set.

Oslo games are a tractable subset of all loopy games. Understanding of their form and structure may lead to insight into that of other loopy games. We have seen that the addition of a Left pass can sometimes serve to simplify otherwise difficult games. Through addition of the Left pass, we were able to fully solve the Oslo variants of WYTHOFF'S GAME, the known killer OCTAL .007, and GRUNDY'S GAME. We would like to think that better understanding of these variants might give insight into the non-Oslo variants of these games.

In general, answers always lead to more questions. As such, we now air the laundry list of questions that have arisen during the work that has led us to this end.

5.2 $\mathcal{N}$-Positions: Where We Want To Go Next

"We go in there and we work on altering those ideas
and in many cases go in different directions."
- Les Paul

With respect to juxtapositions of games, part of the challenge is finding that internal structure. How can you readily see/know when these structures exist? The structures that we were able to describe in the analyzed games that made use of both ordinal sums and side-sum was at first hidden, lurking within the nuances of the games themselves. However, once "found", clean solutions presented themselves. How do we view these games in a different light so that we might illuminate that which had first seemed to be camouflaged?

Much remains to be understood regarding the application of ordinal sums and what their use is in simplifying a game.

Question 5.2.1. *For games G and H, What can be said about the bounds of $G{:}H-G$ as a function of base G? As a function of both base and branch?*

From the Branch Outcome theorem (2.2.11), we know that if a game H is confused with zero, then $G{:}H$ is confused with G.

Question 5.2.2. *If starting with a game G', can we find a base G and some $H \parallel 0$ such that $G' = G{:}H$?*

Question 5.2.3. *Can every nonzero game be represented as an ordinal sum (with nonzero base and branch)?*

With respect to Oslo games, much is left to be done. Consideration of Oslo games is still in its infancy. More questions remain open than have been answered. For now, from the long list of possible topics to be explored, we briefly list those that might be of the most relevance moving forward.

With respect to the structure of the lattice of Oslo games, many questions remain open. For instance, while ON will be the maximal element for all Oslo lattices, we think that there will be a maximal element, specific to the day, sitting below ON. If this is so, what is the description of this maximal element? One could also look to dimension of this lattice. What is the largest antichain formed from the set of join-irreducibles? One could also explore sublattices; what cover-preserving sublattices, if any, appear forever?

Yet another realm to explore is the possible applications of the side-out function. What if we were to look at other forms of $H \odot G$. For instance, what about $\text{OVER} \odot G$. Since OVER kills all-small games, how will $\text{OVER} \odot G$ affect all-small games G? What of $\text{UNDER} \odot G$?

In terms of game application, further understanding of other "hard" games with a Left pass would be interesting to look at. For instance, what can be said of the Oslo version of other octal games or subtraction games?

Lastly, one final area that would certainly be worth pushing forward, would be the application and understanding of uponic weight. What values can be obtained? The surface of this concept has only just been scratched. However, it is clear that the game $\text{UPON}*$ plays a leading role in Oslo games. Uponic weight is introduced as a possible tool in better understanding that role. What if the values in the lattice of Oslo games were changed to their uponic weight? How would this affect the structure? What can be said of the underlying game based on the uponic weight of its Oslo version? Currently our definition of uponic weight relates the uponic weight of an Oslo game to the atomic weight of the passified version of the underlying game.

Question 5.2.4. *What can be said of the relationship between a game and its passified version?*

Question 5.2.5. *For a game G, can we find a bijection between the uponic weight of* $\operatorname{oslo}(G)$ *and the atomic weight of the underlying game G?*

In combinatorial game theory, we are always considering where to go next. As it seems that answers breed questions faster than questions breed answers, we are sure to have an unlimited supply of enticing work ahead and new directions to explore!

Appendix A

Rulesets

"You have to learn the rules of the game.
Then you have to play better than anyone else."
- Albert Einstein

The following is a compilation of the rulesets of all games referenced in the thesis.

A.1 Cricket Pitch

The game of CRICKET PITCH uses a row of nonnegative integers (or bumps) and a roller that is placed between any two bumps or at either end. Left (Right) moves the roller to the left (right), flattening each bump it passes over by 1. Once a bump has been reduced to zero, the roller may not cross it again. At least one bump must decrease in size at each move.

The game can be represented as a path graph with weighted edges. In this visual representation, the bumps are edge-weights in the graph and the roller is located on one of the vertices. On a move, the roller is moved to the left by Left (right by Right) to another vertex. In doing so, the edge-weight of all edges traversed having positive edge-weight is decreased by one. Edges having weight zero may not be crossed.

The game of CRICKET PITCH was introduced by Nowakowski and Ottaway [23] as an example of an option-closed game.

Variant: ROLL THE LAWN

A.2 Grundy's Game

In GRUNDY'S GAME, the only legal move is to split a single heap of tokens into two smaller heaps of different sizes. The winner is the the player who is last able to split a heap. For instance, a heap of size 4 can be split into two heaps, one of size 1 and the other of size 3. However, it could not have been split into two heaps both of size 2. A heap of size 3 can be split into heaps of size 1 and 2. Heaps of size 1 and 2 can no longer be split.

Variant: OSLO GRUNDY=GRUNDY'S GAME with a pass for left.

A.3 Hackenbush

A position in the game of HACKENBUSH consists of a edges colored black, white or gray. The *ground* will be shown as a horizontal line. On a move, Left is allowed to cut either black or gray lines. Right is allowed to cut white or gray lines. That edge, along with any portion of the edges no longer connected to the ground is then removed.

The game of HACKENBUSH was first introduced by Berlekamp, Conway and Guy in *Winning Ways* [3].

A.4 Independence Game

In the INDEPENDENCE GAME, we start with a graph G and an independent set $I \subseteq V(G)$ that is initally set to $I = \emptyset$. On her turn, Right chooses a vertex and adds it to the set I such that I remains an independent set. On his turn, Left may swap 2 adjacent vertices, provided that I remains independent. The game ends when I is a maximal independent set.

A.5 LenRes

The game of lenres is played on a sequence of integers. On his turn, Left is allowed to move any integer from its current position, to replace (cover) any other integer to the right (east) of it that is larger than, or up (north) from, it. Right is allowed to move to cover to the right (east) any integer that is less than, or down (south) from, it. Thus, the name comes from an acronym of the ruleset: Left-East-North, Right-East-South.

The game of lenres was created by Richard Nowakowski in a fourth-year Introduction to Game Theory class. It was originally considered by Richard Nowakowski and Paul Ottaway who looked at it within a class of one-dimensional games that they were interested in. They were able to analyze certain simple positions, yet no meaningful conjectures were formed[22].

A.6 Maze

The game of MAZE is played on a board with a token starting in the top-left position. Solid edges on the board are walls that may not be crossed. On a move, Left is allowed to move the token downward any distance and Right is allowed to move the token any number of squares to the right, neither moving it over a sold wall.

This game can be considered using multiple tokens on the same board. In this case, the tokens do not affect one another, and so their positions can be thought of as the disjunctive sum.

The game of MAZE was introduced in *Lessons in Play*[2] and considered by Nowakowski and Ottaway in [23] and later by McKay, Nowakowski and Siegel in [21].

A.7 Nim

In the game of NIM, players take turns removing stones from distinct piles, called nim-heaps. On a turn, a player may remove any number of stones, provided he removes at least one, from any one pile. NIM is the quintessential example of an impartial game.

NIM was first introduced by Charles Bouton[6].

Variants: SUBTRACTION GAMES, WYTHOFF'S GAME

A.8 Octal Game .007

The octal game OCTAL .007 is played on heaps of tokens. On a move, a player is allowed to take 3 tokens from any one pile at which point he may split that pile into

2 separate heaps, if he would like. Thus, from heaps of size 0, 1 and 2, there are no allowed moves. Thus, heaps of size 1 and 2 have value 0. From a heap of size 3, each player has a move to 0 and so this has value $*$. From a heap of size 6, a choice exists to split the remaining pile. Each player can move to either a single heap of size 3 or to two heaps of sizes 1 and 2. Hence, this has value $*2$.

Variant: OSLO(OCTAL .007)=OCTAL .007 with a pass for left.

A.9 Push

The game of PUSH is played on finite strips of squares. Each square can either be empty or occupied by a black or white piece. Left moves by choosing a black piece and moving it one square to the left, Right by choosing a white piece and doing the same.

At most one piece can occupy a square at a time. Any pieces immediately adjacent and to the left of that which is being moved are also pushed one square to the left. Once a piece is pushed past the left end of the strip, it is removed from play.

The game of PUSH was introduced in *Lessons in Play* as a tool for demonstrating various properties of games[2].

A.10 Restricted Toppling Dominoes

The game of X/Y-RESTRICTED TOPPLING DOMINOES is played on a row of black, white or gray dominoes. The X and Y in its name are placeholders for either "E" or "W" and depict which direction Left and Right, respectively, are allowed to topple their dominoes. The letter E, for east, implies toppling to the right and the letter W, for west, implies toppling to the left. On his move, Left may choose either a black or gray domino and topple it in the designated X direction. Every domino in that direction also topples over. All toppled dominoes are removed from the game. Right maybe topple either a white or gray domino in the designated Y direction, with similar effect.

Variant: TOPPLING DOMINOES

A.11 Roll The Lawn

The game of ROLL THE LAWN uses a row of nonnegative integers (or bumps) and a roller that is placed between any two bumps or at either end. Left (Right) moves the roller to the left (right), flattening each bump it passes over by 1 unless the bump has already been completely flattened to zero. At least one bump must decrease in size at each move.

The game can be represented as a path graph with weighted edges. In this visual representation, the bumps are edge-weights in the graph and the roller is located on one of the vertices. On a move, the roller is moved to the left by Left (right by Right) to another vertex. In doing so, the edge-weight of all edges traversed having positive edge-weight is decreased by one.

The game of ROLL THE LAWN was introduced by Nowakowski and Ottaway [23] as an example of an option-closed game. Ottaway looked at this game under Misère Play in his Ph.D. thesis [24].

Variant: CRICKET PITCH

A.12 Shove

The game of SHOVE is played on finite strips of squares. Each square can either be empty or be occupied by a black or white piece. Left moves by choosing a black piece and moving it one square to the left, Right by choosing a white piece and doing the same.

When a piece is moved to the left one square, all pieces to the left of it on the board are also moved over one square. Once a piece is pushed passed the left end of the strip, it is removed from play.

The game of SHOVE was introduced in *Lessons in Play* as a tool for demonstrating various properties of games [2].

Variant: PUSH

A.13 Subtraction Games

A SUBTRACTION GAME is played on a pile of stones. Each player is assigned a subtraction set. On his turn, a player may remove any number of stones so long as that number belongs to his subtraction set and provided that he removes at least one stone from the pile.

Variant: NIM

A.14 Toppling Dominoes

The game of TOPPLING DOMINOES is played on a row of black, white or gray dominoes. On his move, Left may choose either a black or gray domino and topple it either left or right. Every domino in that direction also topples over. All toppled dominoes are removed from the game. Right maybe topple either a white or gray domino, with similar effect.

Variant: RESTRICTED TOPPLING DOMINOES

A.15 Wythoff's Game

WYTHOFF'S GAME was introduced in 1907 by Willem A. Wythoff [33].

In WYTHOFF'S GAME, play begins with two heaps of tokens. On a move, a player may take any number of tokens from one pile or an equal number of tokens from both piles, provided at least one token is removed on a turn. The player taking the last token wins.

For example, starting from a set of heaps of size x and $x+y$ $(x, y \geq 0)$, a position we will denote as $W(x, x+y)$, legal moves are to any (i) $W(i, x+y)$ for $0 \leq i < x$; (ii) $W(x, j)$ for $0 \leq j < x+y$; or $W(k, k+y)$ where $0 \leq k < x$.

Variants: NIM, OSLO WYTHOFF=WYTHOFF'S GAME with a pass for left.

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