ON THE DEGREE POLYNOMIAL OF GRAPHS

by

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Abstract

Graph polynomials encode graph theoretic information in the form of polynomials. These polynomials have been of interest for decades, for both graph theoretic insight and their mathematical properties. This thesis discusses one particular graph polynomial: the degree polynomial. This graph polynomial encodes the degree sequence of a graph, and has only recently appeared in the literature. We begin by examining basic properties of the degree polynomial, including some special evaluations. Much of our focus is on roots of the degree polynomial: we explore how these roots are related to the roots of polynomials with non-negative integer coefficients, their density, and how they are impacted by restricting certain graph parameters. The latter leads us to bounds on the roots in terms of graph order. We also study the roots of degree polynomials for a few families of graphs, as we are able to say much more about their roots. Some possible extensions or generalizations of the degree polynomial are also briefly discussed.

List of Abbreviations and Symbols Used

Notation	Description
$V(G)$	The vertex set of a graph G.
$E(G)$	The edge set of a graph G.
$G-e$	The graph G with an edge e removed, that is $V(G-e)=V(G)$ and $E(G-e)=E(G)-\{e\}$.
G^c	The graph complement of G.
$\overline{z}$	The complex conjugate of a complex number z.
$\text{Re}(z)$	The real component of a complex number z.
$\text{Im}(z)$	The imaginary component of a complex number z.
$\mathbb{Z}$	The domain of integers.
$\mathbb{Q}$	The field of rational numbers.
$\mathbb{R}$	The field of real numbers.
$\mathbb{C}$	The field of complex numbers.
$\mathbb{Z}_{\geq a}$	The integers greater than or equal to an integer a.
$\mathbb{Z}_{\geq a}[x]$	The polynomials with coefficients that are integers greater than or equal to an integer a.
$Z(\mathcal{F})$	The set of all roots of polynomials in a family of polynomials $\mathcal{F}$.
$D(G;x)$	The Degree Polynomial of a graph G, with indeterminate x.
$\mathcal{D}(\mathcal{G})$	The set of all degree polynomials for graphs in $\mathcal{G}$, a family of graphs.
$\mathcal{D}$	The set of all degree polynomials for (simple) graphs.
$\mathcal{D}_{multi}$	The set of all degree polynomials for multigraphs.
$\mathcal{S}_n$	The set of all simple graphs of order n.

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Chapter 1

Introduction

1.1 Basic Graph Theory

We begin simply by defining a graph and its variations, as the terminology for graphs is not universal across the literature.

Definition 1.1. A *multigraph* G is an ordered pair (V, E) where $V = V(G)$ is a (possibly empty) finite set whose elements are called *vertices*, and $E = E(G)$ is a finite multiset of sets of vertices of size two, called *edges*. G is called *simple* if an edge appears in E at most once, ie. if E is a set. The term *graph* by default will refer to a simple multigraph, unless specified otherwise.

A graph with n vertices and m edges (i.e. *order* n and *size* m) may be referred to as an (n, m)-graph. Unless otherwise indicated, n and m will stand for the number of vertices and edges of a graph under discussion. We shall follow [38] for further graph theory definitions.

Now we list some well known families of graphs. These graphs will repeatedly appear and prove useful for certain examples, so it is worth defining them here.

- The empty graph $O_n = ([n], \emptyset)$ has vertices $[n] = \{0, 1, 2, ..., n-1\}$ and no edges. In other words, O_n consists of n *isolated* vertices. See Figure 1.1.

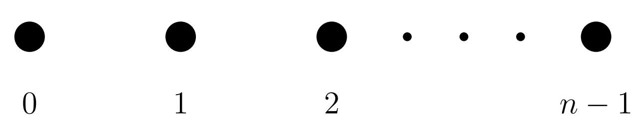


Figure 1.1: The empty graph with vertices $0, 1, ..., n-1$.

- The complete graphs $K_n = ([n], E)$ where $E = \{\{i, j\} \ : \ i < j\}$. These simple graphs have n vertices, and every possible edge. See Figure 1.2.

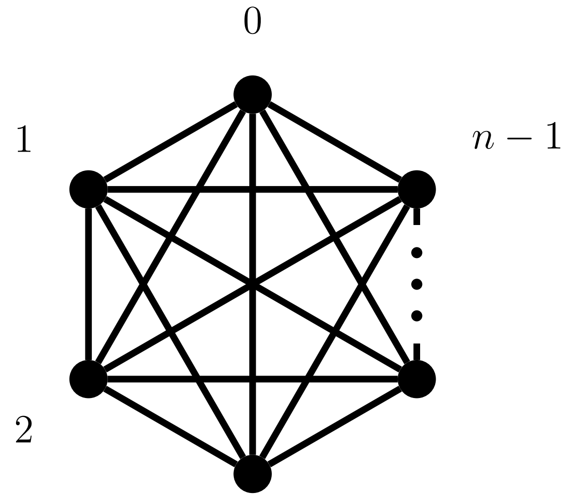


Figure 1.2: The complete graph with vertices $0, 1, ..., n-1$.

- The path graphs $P_n = ([n], E)$, where $E = \{\{i, i+1\} \ : \ 0 \leq i \leq n-2\}$. See Figure 1.3.

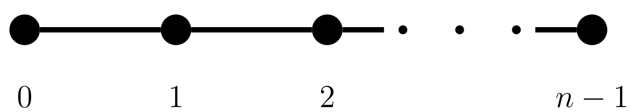


Figure 1.3: The path graph with vertices $0, 1, ..., n-1$.

- The cycle graphs $C_n = ([n], E)$, where $E = \{\{i, i+1\} \ : \ i \in [n]\}$. The addition is taken to be modulo n. See Figure 1.4.

- The star graphs $S_n = ([n], E)$, where $E = \{\{0, i\} \ : \ 1 \leq i \leq n-1\}$. See Figure 1.5.

- The complete bipartite graphs $K_{n,m} = ([n+m], E)$, having edge set $E = \{\{i, j\} \ : \ 0 \leq i \leq n-1, \ n \leq j \leq n+m-1\}$. See Figure 1.6.

Let G, H be graphs. Unless otherwise stated, we refer to [38] for the following graph operations. If $V(G)$ and $V(H)$ are disjoint, the *graph union* $G \cup H$ is the graph with vertex and edge sets

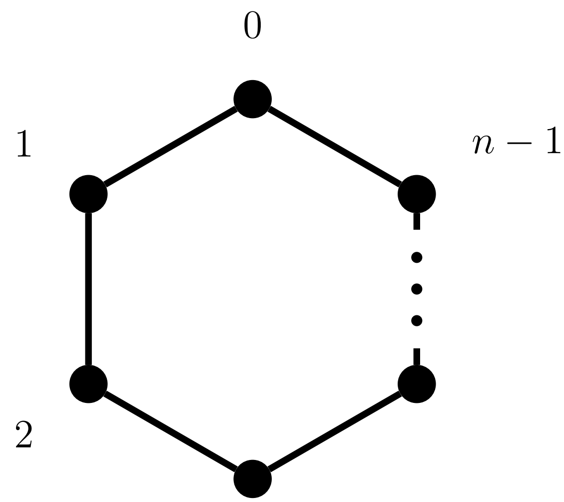


Figure 1.4: The cycle graph with vertices $0, 1, ..., n-1$.

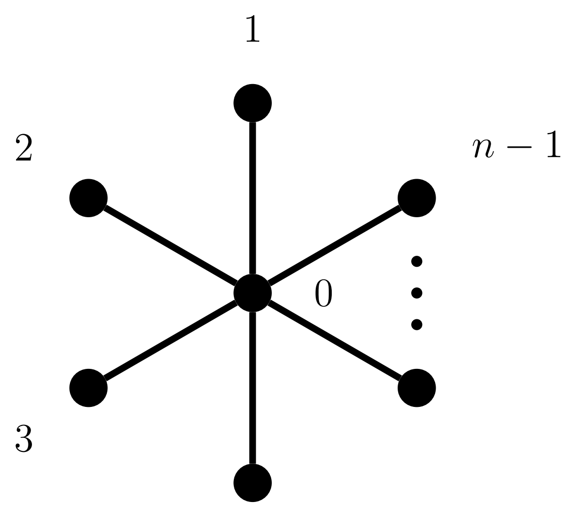


Figure 1.5: The star graph with vertices $0, 1, ..., n-1$.

$$V(G \cup H) = V(G) \cup V(H), \quad E(G \cup H) = E(G) \cup E(H).$$

If $V(G)$ and $V(H)$ are not disjoint, simply take disjoint isomorphic copies of G, H. As an example, consider the path graphs $G = P_3$: $G = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}\})$ and $H = P_2$: $H = (\{4, 5\}, \{\{4, 5\}\})$. Then $G \cup H$ is the graph with vertices $\{1, 2, 3, 4, 5\}$ and edges $\{\{1, 2\}, \{2, 3\}, \{4, 5\}\}$. If $V(G)$ and $V(H)$ are disjoint, then the *graph join* $G + H$ has vertices and edges given by

$$V(G + H) = V(G) \cup V(H),$$

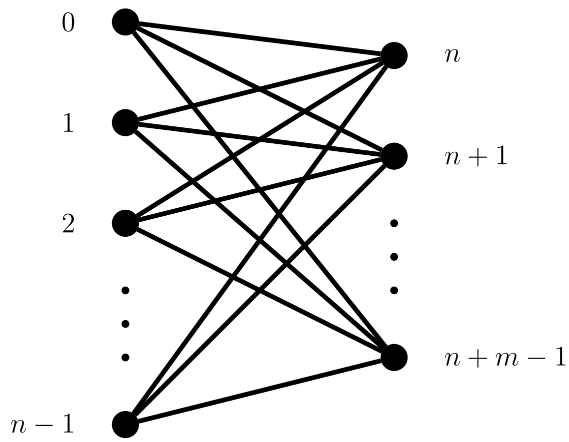


Figure 1.6: The complete bipartite graph $K_{n,m}$.

$$E(G+H) = E(G) \cup E(H) \cup \{\{g,h\} \ : \ g \in V(G), \ h \in V(H)\}.$$

If $V(G)$ and $V(H)$ are not disjoint, take disjoint copies as was done for the graph union. The *corona* [21] $G \odot H$ is formed in the following way: first, take $|V(G)|$ disjoint copies H_v of H. Join each vertex v of G to H_v, and the resulting graph is $G \odot H$. The *lexicographic product* $G[H]$ is the graph where

$$V(G[H]) = V(G) \times V(H),$$

and

$$E(G[H]) = \{\{(g,h),(g',h')\} \ : \ \{g,g'\} \in E(G), \text{ or } g = g' \text{ and } \{h,h'\} \in E(H)\}.$$

The *cartesian product* $G \,\square\, H$ has vertices $V(G) \times V(H)$ and edges

$$E(G \,\square\, H) = \left\{ \{(g,h),(g',h')\} \ : \ \begin{array}{l} \{g,g'\} \in E(G) \text{ and } h = h', \text{ or} \\ g = g' \text{ and } \{h,h'\} \in E(H) \end{array} \right\}.$$

The *tensor product* $G \times H$ has vertices $V(G) \times V(H)$ and edges

$$E(G \times H) = \{\{(g,h),(g',h')\} \ : \ \{g,g'\} \in E(G) \text{ and } \{h,h'\} \in E(H)\}.$$

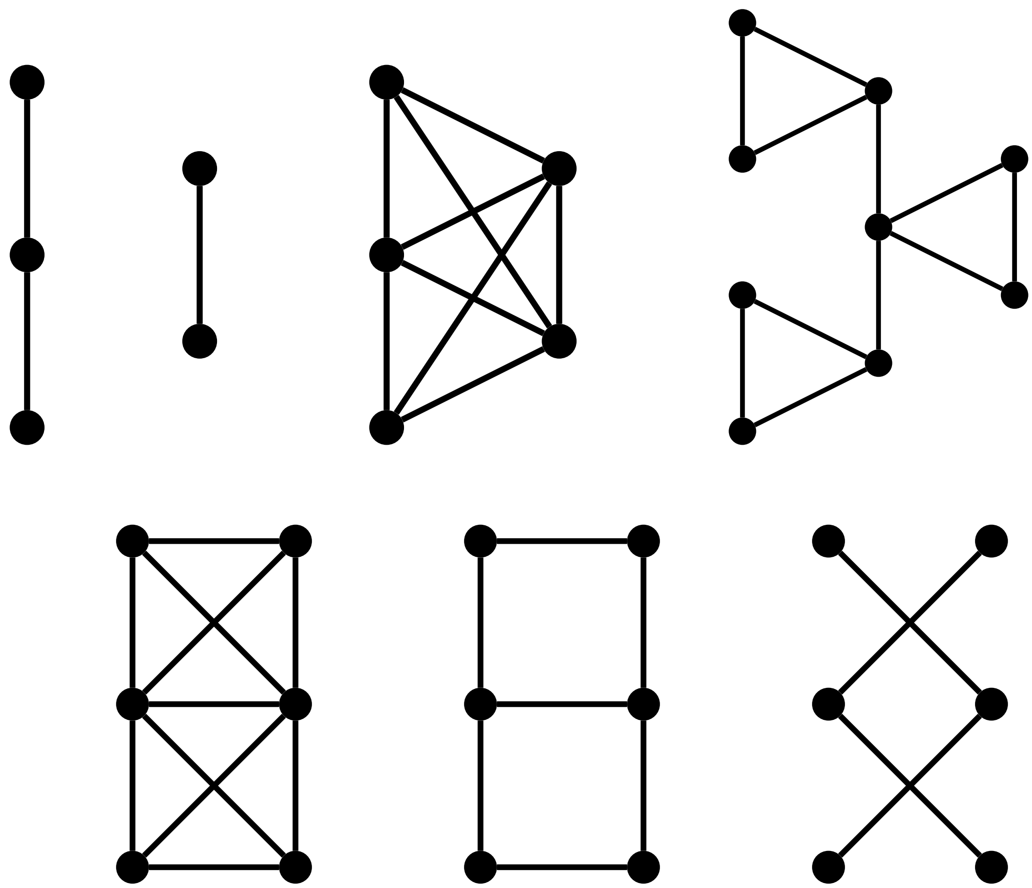


Figure 1.7: Examples for the above graph operations, taking $G = P_3$ and $H = P_2$. Top row, left to right: $G \cup H$, $G + H$, $G \odot H$. Bottom row, left to right: $G[H]$, $G \,\square\, H$, $G \times H$.

See Figure 1.7 for an example of these graph products.

Each operation just outlined is a binary operation of graphs. An important unary operator of graphs is the *graph complement*, or simply *complement*. For a graph G, its complement, denoted G^c, is the graph with the same vertices as G and with edges $e \in E(G^c)$ if and only if $e \notin E(G)$. In other words, two distinct vertices are adjacent in G^c if and only if they are not adjacent in G.

For a vertex v of a graph G, the *neighbourhood set* $N_G(v)$ is the set of vertices

$$N_G(v) = \{u \ : \ \{v, u\} \in E(G)\}.$$

That is, $N_G(v)$ is the set of all vertices adjacent to v in G. The *degree* of v, denoted

$\deg_G(v)$ (or $\deg(v)$ if G is clear by the context), is the number of edges to which v belongs. If G is simple, then $\deg_G(v) = |N_G(v)|$. The maximum degree across all vertices we shall denote by $\Delta(G)$, or simply Δ. Similarly, the minimum degree will be denoted by $\delta(G)$ or δ. Suppose $\{v_1, ..., v_n\}$ are the vertices of G, and that $d_i = \deg(v_i)$. If, without loss of generality, the vertices have been labelled such that $d_1 \geq d_2 \geq \cdots \geq d_n$, then the sequence $d_1, d_2, ..., d_n$ is called the *degree sequence* of G. For example, consider the graph in Figure 1.8. Vertex a has a degree of 4, while vertex f has degree 0. Furthermore, we have $\Delta = 4$, $\delta = 0$, and the degree sequence of the graph is $4, 3, 2, 2, 1, 0$.

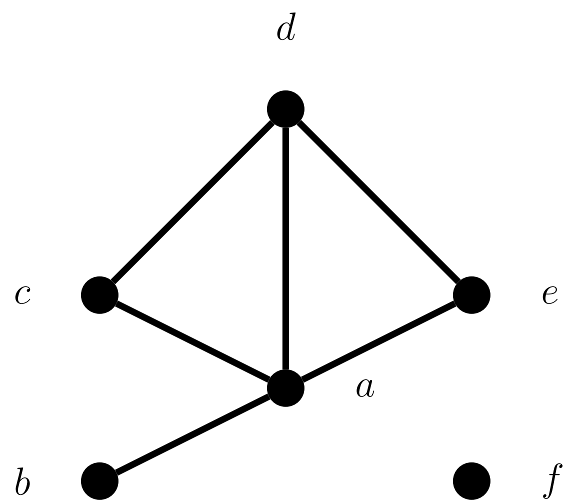


Figure 1.8: An example graph to illustrate various degree related concepts. This graph has $\Delta = 4$, $\delta = 0$, and degree sequence $4, 3, 2, 2, 1, 0$.

1.2 Graph Polynomials

Those who study graph polynomials are concerned with special types of polynomials that encode graph theoretic information about a graph. Some graph polynomials were motivated by applications of graph theory, and others are simply derived from a generating function definition. Let us now give a brief overview of some well known graph polynomials.

A *proper vertex colouring* of a graph G with x colours is a function $c : V(G) \to \{1, 2, ..., x\}$ such that if $\{u, v\} \in E(G)$, then $c(u) \neq c(v)$. The *chromatic polynomial* $\pi(G, x)$ of a graph G is a polynomial in x that counts the number proper vertex colourings of G with x colours [18]. If G is an (n, m)-graph, then $\pi(G, x)$ is a monic polynomial of degree n, with alternating integer coefficients. Due to a theorem of Whitney [39], the coefficients of $\pi(G, x)$ can be expressed in terms of the broken cycles of G: suppose that $b : E(G) \to [m]$ is a fixed bijective map, ie. an ordering of the edges of G. For any cycle C of G, let e be the edge of C such that $b(e) > b(f)$ for all other edges f of C. Then $C - e$ is a *broken cycle* with respect to b [18]. Setting h_i to be the number of spanning subgraphs of G with exactly $n - i$ edges, and having no broken cycles with respect b, the Broken Cycle Theorem [18] says that

$$\pi(G, x) = \sum_{i=1}^{n} (-1)^{n-i} h_i x^i.$$

In particular, the coefficient on x^{n-1} is $-m$, and the coefficient on x^{n-2} is $\binom{m}{2} - n(C_3)$, where $n(C_3)$ is the number of subgraphs of G isomorphic to C_3. More information about G is encoded in the roots of $\pi(G, x)$: the multiplicity of 0 as a root is the number of components of G, and the multiplicity of 1 as a root is the number of blocks [18]. The least positive integer that is not a root is the *chromatic number* of G.

Example 1.1. Consider the graph C_4, labelled as in Figure 1.9. Before computing the chromatic polynomial of C_4, we define an ordering of the edges: $ab < bc < cd < da$. C_4 is a cycle, and thus contains only one broken cycle (with respect to our ordering), namely the subgraph with edges ab, bc, cd. We can now find the coefficients h_i for the chromatic polynomial: there are three spanning subgraphs with three edges that do not contain the broken cycle (the other spanning subgraph with three edges is precisely the broken cycle), namely the subgraphs with edge sets $\{bc, cd, da\}$, $\{cd, da, ab\}$, and $\{da, ab, bc\}$. Hence, $h_1 = 3$. Since the only broken cycle of C_4 has three edges, we need not worry about spanning subgraphs containing a broken cycle for h_2, h_3, or h_4 as for these coefficients we count subgraphs with fewer than three edges. It is therefore easy to see that $h_2 = 6$ (every pair of edges forms a spanning subgraph), $h_3 = 4$, and $h_4 = 1$. Putting this all together, we have

$$\begin{aligned}\pi(C_4, x) &= (-1)^{4-4} h_4 x^4 + (-1)^{4-3} h_3 x^3 + (-1)^{4-2} h_2 x^2 + (-1)^{4-1} h_1 x \\ &= x^4 - 4x^3 + 6x^2 - 3x.\end{aligned}$$

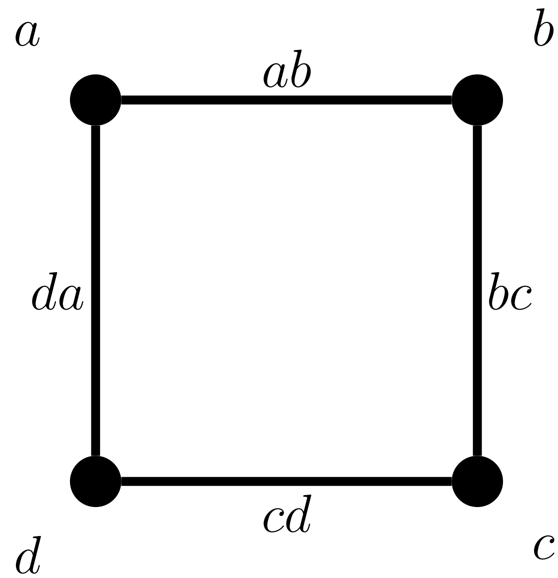


Figure 1.9: The graph C_4 with labelled vertices and edges.

The *reliability polynomial* of a graph is useful when studying a graph as a network of communication. For this brief discussion on reliability, we refer to [15]. In this model vertices are considered to be agents that communicate with each other through edges, and each edge is independently operational with probability $p \in [0, 1]$. Only operational edges can pass information between agents, and agents are assumed to be always operational (ie. they cannot fail). There are various notions of graph reliability, such as *two-terminal reliability*: for two specified agents a, b, one is concerned with whether or not a and b can communicate, ie. if there is a path between a and b consisting of all operational edges. This can be naturally extended for k specified agents. The type of reliability we consider here is *all-terminal* reliability, where we wish for all pairs of agents to be able to communicate. That is, the spanning subgraph with the operational edges contains a spanning tree.

A subset S of the edge set is called a *state*. Thus a state allows all pairs of agents to communicate if and only if it contains a spanning tree of the graph, and such a state is called an *operational state*. Let N_i be the number of operational states with i edges. The reliability polynomial is a polynomial in p giving the probability that a graph has all-terminal communication, and is calculated from the N_i:

$$Rel(G,p) = \sum_{i=0}^{m} N_i p^i (1-p)^{m-i}.$$

Of particular interest are methods for computing the reliability polynomial (see, for example, [28], [5], [23]), or properties of its roots [11].

Example 1.2. Consider again the graph C_4 from Figure 1.9. C_4 has one edge subset of cardinality 4, namely all the edges, which indeed contains a spanning tree. Thus $N_4 = 1$. We have $N_3 = 4$, since any three edges of C_4 form a spanning tree. Any subgraph of C_4 with two edges is disconnected, so $N_2 = 0$, and similarly $N_1 = N_0 = 0$. Therefore,

$$\begin{aligned} Rel(C_4,p) &= N_3 p^3 (1-p)^{4-3} + N_4 p^4 (1-p)^{4-4} \\ &= 4p^3(1-p) + p^4 \\ &= 4p^3 - 3p^4. \end{aligned}$$

The *independence polynomial* (or *independent set polynomial*) $I(G,x)$ of a graph is formed by counting the number of independent sets of each possible size ([25]):

$$I(G,x) = \sum_{k=0}^{n} b_k(G) x^k.$$

The coefficients $b_k(G)$ are the number of independent sets of G of cardinality k (b_0 is taken to be 1). Thus $b_1 = n$, $b_2 = |E(G^c)| = \binom{n}{2} - m$, and all non-zero coefficients are positive integers. The degree of $I(G,x)$ is β, the independence number of G. Closely related to the independence polynomial is the *clique polynomial* $C(G,x)$: if $a_k(G)$ is the number of cliques of G with order k, then the clique polynomial is

$$C(G,x) = \sum_{k=0}^{n} a_k(G) x^k.$$

Since $b_k(G) = a_k(G^c)$, it is easily seen that $I(G,x) = C(G^c,x)$. As before, the roots of the independent polynomial are of interest. Such roots are called *independence roots*. In [12] it is shown that the modulus of roots of independence polynomials of graphs of order n and independence number β is bounded by $(n/\beta)^{\beta-1} + O(n^{\beta-2})$.

A graph is said to be *well-covered* if all of its maximal independent sets have the same cardinality [32]. If G is well-covered, then the bound on the moduli of the independence roots is improved to simply be β [8]. Furthermore, the real roots of independence polynomials are dense in the set of non-positive real numbers, and the complex roots are dense in $\mathbb{C}$ [9].

Example 1.3. We return to C_4 to compute its independence polynomial. It is easily seen that $b_4 = b_3 = 0$, since any subgraph induced by three or four vertices will contain an edge. For the other coefficients, we have $b_2 = \binom{n}{2} - m = 6 - 4 = 2$, $b_1 = n = 4$, and $b_0 = 1$ by definition. Thus the independence polynomial of C_4 is

$$\begin{aligned} I(C_4, x) &= b_2x^2 + b_1x + b_0 \\ &= 2x^2 + 4x + 1. \end{aligned}$$

Lastly, we mention two other graph polynomials that are generating functions. First is the *domination polynomial*: a set of vertices D of a graph G is called a *dominating set* if, for any vertex $v \in V(G)$, either $v \in D$ or v is adjacent to some $u \in D$. In other words, a dominating set is a set of vertices who, collectively, are adjacent to every other vertex in the graph [24]. Consider the sequence $(s_0(G), s_1(G), ..., s_n(G))$ where $s_i(G)$ is the number of dominating sets of G with i vertices. The domination polynomial $Dom(G, x)$ is the generating function of this sequence ([2]):

$$Dom(G, x) = \sum_{i=0}^{n} s_i(G)x^i.$$

The lowest degree of a non-zero term of $Dom(G, x)$ has exponent $\gamma(G)$, the domination number of G, which is the order of the smallest dominating set of G. Hence $\gamma(G)$ is the multiplicity of the root 0 of $Dom(G, x)$. It is conjectured that the coefficients of $Dom(G, x)$ form a unimodal sequence for all graphs [2], and several families of graphs are known for which this holds [2, 4]. Recently it was shown that this holds for almost all graphs [4], and a counterexample has yet to be found.

Example 1.4. For the cycle C_4, it is clear that $s_0 = 0$ and $s_1 = 0$ as there is no universal vertex. We also see that any subset of vertices with at least two vertices forms a dominating set, hence $s_2 = 6$, $s_3 = 4$, and $s_4 = 1$. Hence,

$$\begin{aligned} Dom(C_4, x) &= s_4x^4 + s_3x^3 + s_2x^2 \\ &= x^4 + 4x^3 + 6x^2. \end{aligned}$$

Another generating function type polynomial is the *Wiener polynomial* [34]:

$$W(G, x) = \sum x^{\text{dist}_G(u,v)} = \sum_{i=1}^{\text{diam}(G)} c_i(G)x^i.$$

Here, $c_i(G)$ is the number of pairs of vertices in G that are distance i apart, and the first sum is taken over all subsets $\{u, v\}$ of V of size two. In particular, $c_1 = m$. Hence $W(G, x)$ is the generating function of the sequence $(c_1(G), c_2(G), ..., c_{\text{diam}(G)}(G))$. This polynomial is closely related to the Wiener index $W(G)$, first defined by Harry Wiener [40] for use in chemistry. If $\mathbf{D}$ is the *distance matrix* of G (the matrix whose (i, j)-*th* element is the distance from vertex i to vertex j), the Wiener index is defined to be $W(G) = (1/2)\sum_{i\neq j} \mathbf{D}_{ij}$ [31]. If an alkane (hydrocarbon) molecule is represented with a graph by having vertices and edges represent atoms and chemical bonds respectively, then the Wiener index gives insight to the boiling point of that molecule [31]. Two notable evaluations [34] of the Wiener polynomial are

$$W(G, 1) = \sum 1 = \binom{n}{2},$$

and

$$W'(G, 1) = \sum \text{dist}_G(u, v) = W(G).$$

Example 1.5. Once again, we will use C_4 to illustrate the Wiener polynomial. The pairs of vertices which are adjacent are $\{a, b\}, \{b, c\}, \{c, d\}$, and $\{d, a\}$. Hence, $c_1 = 4$. The only remaining pairs of vertices are the corners of C_4, or $\{a, c\}$ and $\{b, d\}$, both of which are distance two apart. Thus $c_2 = 2$, and all other coefficients are zero. Therefore,

$$\begin{aligned} W(C_4, x) &= c_2x^2 + c_1x \\ &= 2x^2 + 4x. \end{aligned}$$

1.3 The Roots of Polynomials

In this section we highlight some results concerning the roots of polynomials. This is a rich area of study, so we will focus on what will be useful for us later on. A complex number $z \in \mathbb{C}$ is called a *root* (or *zero*) of a polynomial $f(x)$ if $f(z) = 0$. Every polynomial we will encounter will be a *real polynomial*: a polynomial whose coefficients are all real. Thus it is worth recalling that if z is a non-real root of a real polynomial f, then the complex conjugate $\overline{z}$ is also a root of f.

A useful result for counting the number of positive (or negative) roots of a polynomial is Descartes' Rule of Signs (see, for example, [3]):

Theorem 1.1 (Rule of Signs)**.** *If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a real polynomial, then the number of positive roots (counting multiplicity) of $f(x)$ is equal to the number of sign changes in consecutive (non-zero) coefficients of $f(x)$, or less than that amount by an even number.*

A corollary is that the number of negative roots of $f(x)$ is found by counting the sign changes of $f(-x)$. Let us demonstrate this with a few examples.

Example 1.6. Consider the polynomial $f(x) = 2x^5 - 3x^4 - 19x^3 + 33x^2 + 17x - 30$, shown in Figure 1.10. Quick inspection reveals three changes in sign of the coefficients: $2 \to -3$, $-19 \to 33$, and $17 \to -30$. Therefore by the Rule of Signs, $f(x)$ has three positive roots or just one. In fact $f(x)$ has three positive roots: $x = 1, 2, 5/2$, substantiated by Figure 1.10. For the negative roots of $f(x)$, we examine

$$\begin{aligned} f(-x) &= 2(-x)^5 - 3(-x)^4 - 19(-x)^3 + 33(-x)^2 + 17(-x) - 30 \\ &= -2x^5 - 3x^4 + 19x^3 + 33x^2 - 17x - 30. \end{aligned}$$

There are two changes of sign in the coefficients of $f(-x)$: $-3 \to 19$, and $33 \to -17$. Thus $f(x)$ has either two negative roots or none. Figure 1.10 reveals that there are two negative roots: $x = -1, -3$.

Example 1.7. Now consider the polynomial $g(x) = x^4 - 6x^3 + 9x^2 - 6x + 8$, as in Figure 1.11. There are four sign changes in the coefficients: $1 \to -6$, $-6 \to 9$, $9 \to -6$, and $-6 \to 8$, so by the Rule of Signs $g(x)$ either has four, two, or zero positive roots.

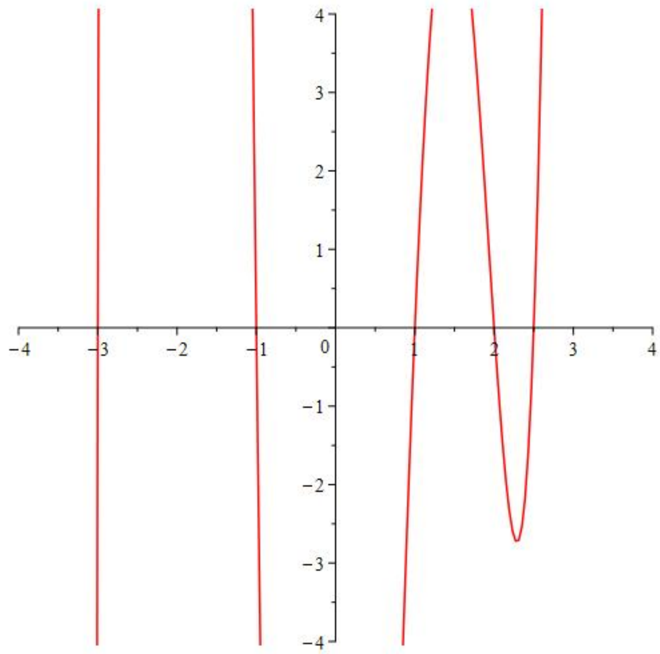


Figure 1.10: A portion of the polynomial $f(x) = 2x^5 - 3x^4 - 19x^3 + 33x^2 + 17x - 30$. In the case of this polynomial, the number of positive (negative) roots of $f(x)$ is precisely the number of sign changes in the coefficients of $f(x)$ ($f(-x)$).

We will use elementary arguments from calculus to deduce the number of positive roots of $g(x)$, rather than relying on the plot of $g(x)$. First, we eliminate the choice of having zero positive roots:

$$\begin{aligned} g(1) &= (1)^4 - 6(1)^3 + 9(1)^2 - 6(1) + 8 \\ &= 6 \end{aligned}$$

and

$$\begin{aligned} g(3) &= (3)^4 - 6(3)^3 + 9(3)^2 - 6(3) + 8 \\ &= -10. \end{aligned}$$

Therefore, by the Intermediate Value Theorem (IVT), $g(x)$ must have a root in the interval $(1, 3)$. Furthermore, we now know $g(x)$ has either four or two positive roots. Consider $g'(x)$ and $g''(x)$:

$$g'(x) = 4x^3 - 18x^2 + 18x - 6,$$

$$g''(x) = 12x^2 - 36x + 18.$$

By the Rule of Signs, $g'(x)$ has one or three positive roots. Solving $g''(x) = 0$ gives the two critical points of $g'(x)$:

$$x_1 = \frac{3+\sqrt{3}}{2}, \quad x_2 = \frac{3-\sqrt{3}}{2}.$$

Simple calculation reveals that the value of $g'(x)$ at these critical points is less than zero:

$$\begin{aligned}
g'(x_1) &= 4\left(\frac{3+\sqrt{3}}{2}\right)^3 - 18\left(\frac{3+\sqrt{3}}{2}\right)^2 + 18\left(\frac{3+\sqrt{3}}{2}\right) - 6 \\
&= \frac{1}{2}(54 - 30\sqrt{3}) - \frac{9}{2}(12 - 6\sqrt{3}) + 27 - 9\sqrt{3} - 6 \\
&= 3\sqrt{3} - 6 \\
&< 0,
\end{aligned}$$

and

$$\begin{aligned}
g'(x_2) &= 4\left(\frac{3-\sqrt{3}}{2}\right)^3 - 18\left(\frac{3-\sqrt{3}}{2}\right)^2 + 18\left(\frac{3-\sqrt{3}}{2}\right) - 6 \\
&= \frac{1}{2}(54 + 30\sqrt{3}) - \frac{9}{2}(12 + 6\sqrt{3}) + 27 + 9\sqrt{3} - 6 \\
&= -3\sqrt{3} - 6 \\
&< 0.
\end{aligned}$$

Since each critical point of $g'(x)$ is located below the x-axis, any local maximum of $g'(x)$ must be below the x-axis. But $\deg(g'(x)) = 3$ and $g'(x)$ has positive leading coefficient, so we conclude that $g'(x)$ can only have one real, and thus one positive, root (the only way to have three real roots would be to have a local maximum on or above the x-axis). Consequently, $g(x)$ has only one critical point for positive x.

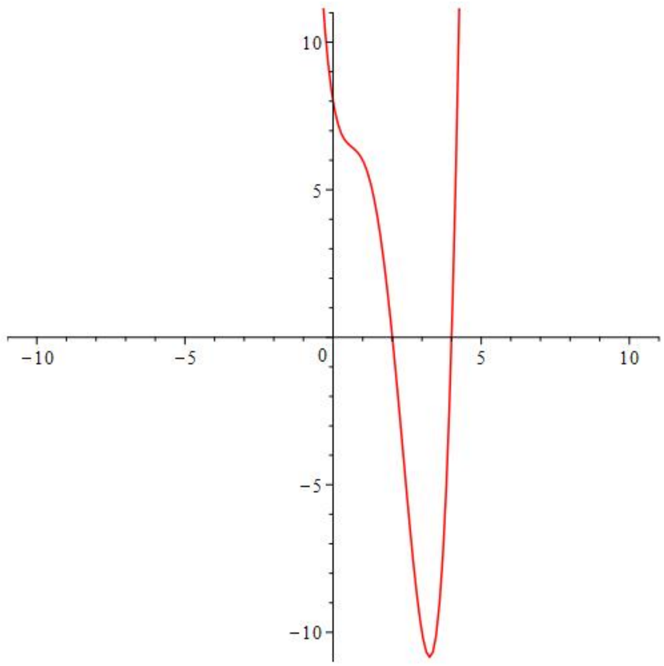


Figure 1.11: A portion of the polynomial $g(x) = x^4 - 6x^3 + 9x^2 - 6x + 8$. While there are four sign changes in the coefficients of $g(x)$, there are only two positive roots.

But $g(x)$ has positive leading coefficient, so $g(x) > 0$ for all sufficiently large x. Since $g(1) > 0$ and $g(3) < 0$, as x increases along the positive x-axis $g(x)$ initially decreases, but then must eventually increase without bound. Thus the positive critical point of $g(x)$ must be a local minimum located at some $x > 1$. Furthermore, the minimum must be below the x-axis, as if not we could not have $g(x)$ being negative for any $x > 1$. Hence, $g(x)$ has at least two positive roots: one in the interval $(1, 3)$, and the other in the interval $(3, \infty)$. However, there cannot be any more positive roots since there is only one positive critical point (local minimum). Any other roots would require $g(x)$ to "bend" in order to touch the x-axis, resulting in additional critical points. Therefore, $g(x)$ has exactly two positive roots.

Another kind of root counting theorem is Rouché's Theorem [29], which relates the number of zeros of two analytic functions inside a region of the complex plane.

Theorem 1.2 (Rouché, [29]). *Let $P(z)$ and $Q(z)$ be functions that are analytic on the interior of a simple, closed Jordan curve C. If $P(z)$ and $Q(z)$ are continuous on C and $|P(z)| < |Q(z)|$ on C, then $P(z) + Q(z)$ and $Q(z)$ have the same number of zeros in the interior of C.*

In many cases we may want to bound the roots of a polynomial. The following theorems give circular bounds to all roots of a polynomial, that need not be a real polynomial.

Theorem 1.3 (Cauchy, [29]). *Let* $g(z) = c_0 + c_1 z + \ldots + c_{n-1}z^{n-1} + c_n z^n$ *be a polynomial with complex coefficients,* $c_n \neq 0$*, and set* $h(z) = |c_n|z^n - |c_{n-1}|z^{n-1} - \cdots - |c_1|z - |c_0|$*. Then all the zeros of* $g(z)$ *lie in the circle* $|z| \leq r$*, where* r *is the unique positive root of the equation* $h(z) = 0$*.*

Example 1.8. Consider the complex coefficient polynomial

$$g(z) = 8z^6 + 8iz^4 - 11z^3 + 2z^2 + (2 + 2\sqrt{3}i)z - 3.$$

Finding all roots of this polynomial would prove difficult, but Cauchy's bound restricts the roots to a small domain: let $h(z)$ be the polynomial

$$\begin{aligned} h(z) &= |8|z^6 - |8i|z^4 - |-11|z^3 - |2|z^2 - |(2 + 2\sqrt{3}i)|z - |-3| \\ &= 8z^6 - 8z^4 - 11z^3 - 2z^2 - 4z - 3, \end{aligned}$$

which admits the factorization

$$h(z) = (2z - 3)(4z^5 + 6z^4 + 5z^3 + 2z^2 + 2z + 1).$$

Thus $h(z)$ has the positive root $r = \frac{3}{2}$, and the modulus of any root of $g(z)$ cannot exceed $\frac{3}{2}$. See Figure 1.12 for the roots of $g(z)$ and this bound.

Sometimes it may not be ideal to try and find the positive root r of $h(z)$, as even that could be difficult. However, since the root is unique and $h(z)$ has positive leading coefficient, if we find some positive value x_0 such that $h(x_0) > 0$, then all roots of $g(z)$ satisfy $|z| < x_0$. Finding such a value x_0 only requires testing some reasonable points, so a strict bound like this can be much easier to obtain.

The following theorem provides a bound which is much easier to compute than that from Theorem 1.3. However, it is not always as useful, as illustrated in Example 1.9.

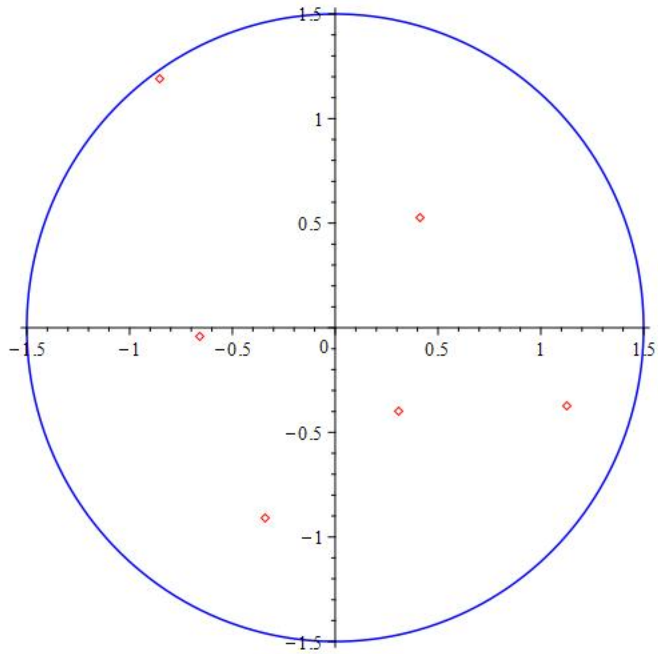


Figure 1.12: The roots of $g(z) = 8z^6 + 8iz^4 - 11z^3 + 2z^2 + (2 + 2\sqrt{3}i)z - 3$ (red), and the modulus bound $|z| \leq \frac{3}{2}$ on these roots found with Theorem 1.3.

Theorem 1.4 (Cauchy, [29]). *Let $g(z) = c_0 + c_1 z + ... + c_{n-1}z^{n-1} + c_n z^n$ be a polynomial with complex coefficients, $c_n \neq 0$. Then any root z of $g(z)$ satisfies*

$$|z| < 1 + \max_{k \leq n-1}\{|c_k/c_n|\}.$$

Example 1.9. Recall the polynomial $g(z)$ from Example 1.8. It is not difficult to see that $\max_{k \leq n-1}\{|a_k/a_n|\} = |-11/8| = 11/8$. Thus all the roots of $g(z)$ lie in the circle of radius $1 + 11/8 = 19/8$. While true, the bound from Theorem 1.3 gives a much tighter bound on the modulus of the roots.

An advantage of the previous theorems is that they apply to any polynomial with complex coefficients. However, for specific types of polynomials it may not give a tight or interesting bound. For polynomials that have real and positive coefficients, there is the following result.

Theorem 1.5 (Enestrom-Kakeya, [19], [27]). *Consider the following real polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where each a_i is positive. Let $q_k = a_{k-1}/a_k$, for $1 \leq k \leq n$. Then any root z of $p(x)$ satisfies*

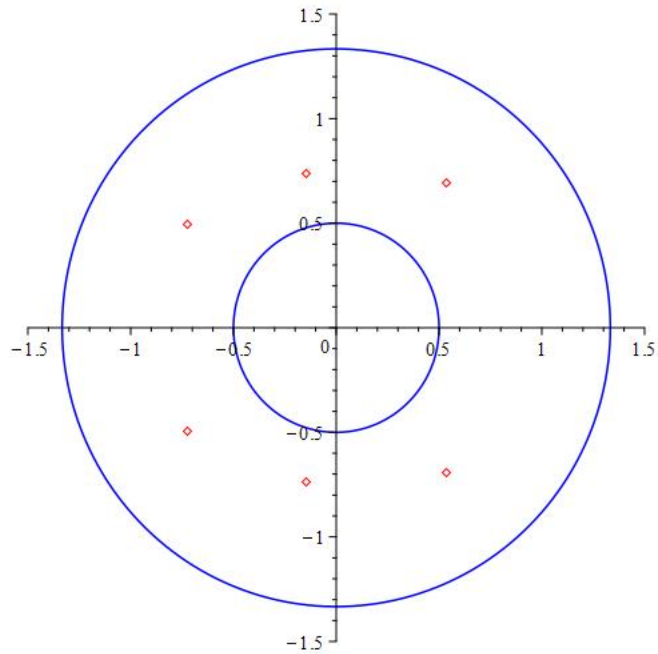


Figure 1.13: The roots of the polynomial $p(x)$ (red), and the annulus that bounds them as determined from Theorem 1.5. The inner radius of the annulus is 1/2, while the outer radius is 4/3.

$$\min_k\{q_k\} \leq |z| \leq \max_k\{q_k\}.$$

Example 1.10. Consider the polynomial $p(x) = 3x^6 + 2x^5 + 2x^4 + \frac{3}{2}x^3 + 2x^2 + x + 1$. Then the q_k's are 2/3, 1, 3/4, 4/3, 1/2, and 1. Therefore the roots z of $p(x)$ satisfy

$$\frac{1}{2} \leq |z| \leq \frac{4}{3},$$

as verified in Figure 1.13.

The next result concerns limits of the roots to polynomials in certain polynomial sequences. If $p_1(x), p_2(x), ...,$ or $\{p_t(x)\}$, is a sequence of polynomials, then z is a *limit of the roots of* $\{p_t(x)\}$ if there is a sequence $\{z_t\}$ such that $p_t(z_t) = 0$ for each t and $z_t \to z$.

Theorem 1.6 (Beraha-Kahane-Weiss, [6])**.** *Suppose* $\{P_t(x) : t \in \mathbb{N}\}$ *is a sequence of polynomials having the form*

$$P_t(x) = \sum_{j=1}^{s} \alpha_j(x)\lambda_j(x)^t$$

for some functions α_j and λ_j, satisfying the following non-degeneracy condition: there is no constant ω, with $|\omega| = 1$, such that $\lambda_i = \omega\lambda_j$ for some $i \neq j$. Then z is a limit of zeros for $\{P_t(x)\}$ if and only if the α_j's, λ_j's can be reordered such that at least one of the following holds:

1. $|\lambda_1(z)| > |\lambda_j(z)|, \;\; 2 \leq j \leq s, \;\; \textit{and} \;\; \alpha_1(z) = 0$
2. $|\lambda_1(z)| = |\lambda_2(z)| = \cdots = |\lambda_l(z)| > |\lambda_j(z)|, \;\; l+1 \leq j \leq s, \;\; \textit{for some} \;\; l \geq 2.$

Example 1.11. Consider the sequence of polynomials $\{P_t(x)\}$, where

$$P_t(x) = (x-1)x^t + x(x-1)^t + x^2.$$

These polynomials are quickly seen to be of the correct form to apply Theorem 1.6, the BKW Theorem. Letting $\alpha_1(x) = x - 1$, $\alpha_2(x) = x$, $\alpha_3(x) = x^2$, and $\lambda_1(x) = x$, $\lambda_2(x) = x - 1$, $\lambda_3(x) = 1$, we have

$$P_t(x) = \alpha_1(x)\lambda_1(x)^t + \alpha_2(x)\lambda_2(x)^t + \alpha_3(x)\lambda_3(x)^t.$$

Furthermore, there is no ω with $|\omega| = 1$ such that $\lambda_i = \omega\lambda_j$ for $i \neq j$. It is acceptable to permute the indices of the α_j's and λ_j's, as our labelling was arbitrary. To find the limits of the roots of these polynomials, we check conditions 1) and 2) from the BKW Theorem (1.6) across all possible index permutations.

Condition 1.

- $|z| > |z-1|$, $|z| > 1$, and $z - 1 = 0$. The last equality gives $z = 1$, which contradicts the inequality $|z| > 1$. Therefore no complex z satisfies these conditions.
- $|z-1| > |z|$, $|z-1| > 1$, and $z = 0$. Since $z = 0$ contradicts $|z-1| > 1$, we again find no root limits.
- $1 > |z|$, $1 > |z-1|$, and $z^2 = 0$. But $z^2 = 0$ implies $z = 0$, contradicting $1 > |z-1|$. So once again, we obtain no root limits.

Condition 2.

- $|z| = |z-1| > 1$. The equality may be interpreted as being all z that are equidistant from 0 and 1. That is, all z with real part equal to $\frac{1}{2}$. Therefore the solutions to these constraints are precisely elements of the set

$$S_1 = \left\{ z \in \mathbb{C} \ : \ \mathrm{Re}(z) = \frac{1}{2}, \ |z| > 1 \right\}.$$

- $|z-1| = 1 > |z|$. The equality describes the points of a circle in the complex plane of radius 1, centered at 1. The inequalities $1 > |z|$ and $|z-1| > |z|$ restrict solutions to be on the interior of the unit circle, and with real component less than $\frac{1}{2}$, respectively. Formally, $|z-1| = 1$ implies $z = 1 + e^{i\theta}$, and z lying inside the unit circle is equivalent to z having real part equal to $\frac{1}{2}$, which is equivalent to $\theta \in (\frac{2\pi}{3}, \frac{4\pi}{3})$. Therefore all z satisfying these constraints are

$$S_2 = \left\{ z \in \mathbb{C} \ : \ z = 1 + e^{i\theta}, \ \theta \in \left(\frac{2\pi}{3}, \frac{4\pi}{3} \right) \right\}.$$

- $1 = |z| > |z-1|$. These constraints are similar to the previous, giving the following solutions: z must lie on the unit circle, and be interior to the circle of radius 1 centred at 1 (equivalent to having a real part greater than $\frac{1}{2}$). In other words, the solutions are

$$S_3 = \left\{ z \in \mathbb{C} \ : \ z = e^{i\theta}, \ \theta \in \left(\frac{-\pi}{3}, \frac{\pi}{3} \right) \right\}.$$

- $|z-1| = |z| = 1$. There are only two points satisfying this constraint: the points where the unit circle and circle of radius 1 centered at 1 intersect. Precisely, these are the points $e^{i\pi/3}$ and $e^{-i\pi/3}$. In fact, these are the points that were missed by S_2 and S_3, in the sense that the union $S_2 \cup S_3 \cup \{e^{i\pi/3}, e^{-i\pi/3}\}$ forms a continuous closed curve in the complex plane.

Therefore the limits of the roots of $\{P_t(x)\}$ are the points $S_1 \cup S_2 \cup S_3 \cup \{e^{i\pi/3}, e^{-i\pi/3}\}$ (see Figure 1.14).

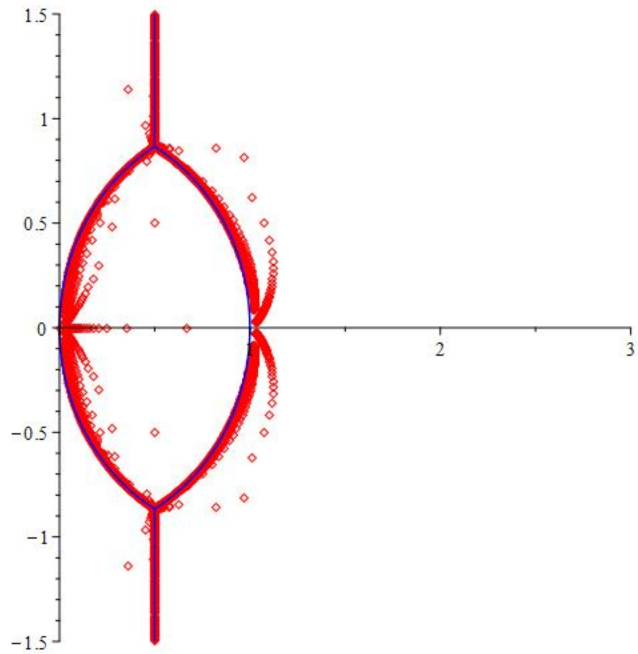


Figure 1.14: Roots of the polynomials $P_t(x) = (x-1)x^t + x(x-1)^t + x^2$, for $1 \leq t \leq 100$ (red). In blue are the limits of these roots found using the BKW Theorem (Theorem 1.6): $S_1 \cup S_2 \cup S_3 \cup \{e^{i\pi/3}, e^{-i\pi/3}\}$.

While powerful, the BKW Theorem suffers limitations. For example, it does not apply to polynomials $P_t(x)$ whose coefficients depend on t. Polynomials like this frequently appear when studying graph polynomials, prompting the following extension of the BKW Theorem to handle the root limits of some of these sequences.

Theorem 1.7 ([10])**.** *Suppose $\{P_t(x)\}$ is a sequence of analytic functions where*

$$P_t(x) = \alpha_1(t;x)\lambda_1(x)^t + \alpha_2(t;x)\lambda_2(x)^t$$

are such that the λ_i are analytic, non-zero, and $\lambda_1 \neq \omega\lambda_2$ for some unit constant $\omega \in \mathbb{C}$, and the α_i have the form

$$\alpha_i(t;x) = t^{d_i} q_{i,d_i}(x) + t^{d_i-1} q_{i,d_i-1}(x) + \cdots + t q_{i,1}(x) + q_{i,0}(x).$$

The functions $q_{i,j}$ are assumed to be analytic, and q_{i,d_i} non-zero. Then $z \in \mathbb{C}$ is a limit of the zeros of $\{P_t(x)\}$ if either of the following hold:

1. $|\lambda_1(z)| > |\lambda_2(z)|$ *and* $q_{1,d_1}(z) = 0$, *or* $|\lambda_2(z)| > |\lambda_1(z)|$ *and* $q_{2,d_2}(z) = 0$.
2. $|\lambda_1(z)| = |\lambda_2(z)| > 0$ *and at least one of* $q_{1,d_1}(z)$, $q_{2,d_2}(z)$ *is non-zero.*

Let us give a simple example of an application of this theorem.

Example 1.12. Consider the polynomial sequence $\{P_t(x)\}$ where

$$P_t(x) = (tx + 1)x^t + t^2(x - 1)^t.$$

Writing $P_t(x)$ in the required form to apply Theorem 1.7, we have $\lambda_1(x) = x$, $\alpha_1(t; x) = tx+1$, $\lambda_2(x) = x - 1$, and $\alpha_2(t; x) = t^2$. Thus $d_1 = 1$ and $q_{1,1}(x) = x$, while $d_2 = 2$ and $q_{2,2}(x) = 1$. Now it is straightforward to find the limits of the roots, using Theorem 1.7:

Condition 1.

- $|z| > |z - 1|$ and $q_{1,1}(z) = 0$. Thus $z = 0$, contradicting the inequality.
- $|z - 1| > |z|$ and $q_{2,2}(z) = 0$. This would imply $1 = 0$, a clear contradiction.

Condition 2.

- $|z| = |z - 1| > 0$, and at least one of $q_{1,1}(z)$ and $q_{2,2}(z)$ is non-zero. This is easily seen to be satisfied by the points $z = \frac{1}{2} + ib$, $b \in \mathbb{R}$.

Therefore, the limits of the roots of $\{P_t(x)\}$ is the line of points $\{\frac{1}{2} + ib \ : \ b \in \mathbb{R}\}$ (see Figure 1.15).

Next, we discuss the Hermite-Biehler Theorem (see, for example, [37]). This theorem concerns polynomials which have all their roots lying in the closed left-half-plane (LHP). First, some definitions are needed. A polynomial is considered *Hurwitz quasi-stable*, or simply *stable*, if each of its roots lies in the closed LHP. That is, if $\text{Re}(z) \leq 0$ for each root z. A polynomial is called *standard* if it is identically zero, or its leading coefficient is positive. Let $f(x)$, $g(x)$ be real polynomials only having real roots, where $u_1 \leq \cdots \leq u_n$ are the roots of $f(x)$ and $v_1 \leq \cdots \leq v_m$ are the roots of

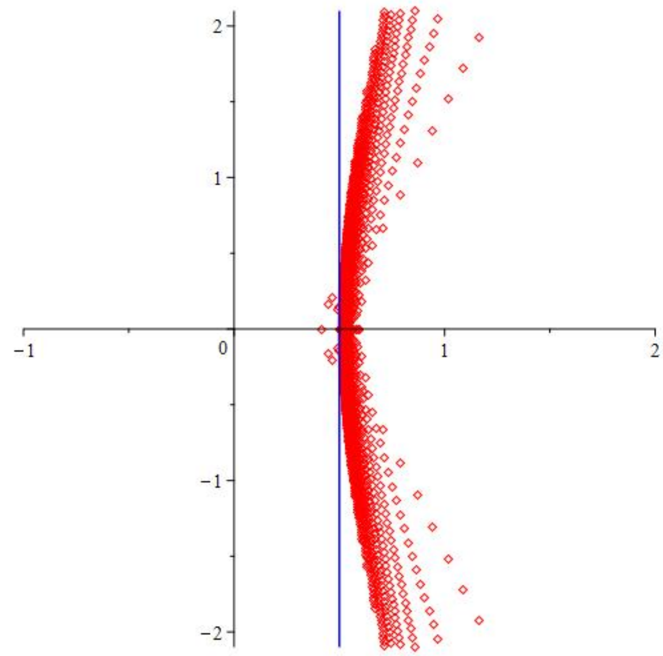


Figure 1.15: Roots of the polynomials $P_t(x) = (tx+1)x^t + t^2(x-1)^t$, for $1 \leq t \leq 80$ (red). In blue are the limits of these roots found using the extension of the BKW Theorem (Theorem 1.7).

$g(x)$. We say f *interlaces* g if $m = n+1$ and $v_1 \leq u_1 \leq v_2 \leq \cdots \leq u_n \leq v_{n+1}$, or that f *alternates left of* g if $m = n$ and $u_1 \leq v_1 \leq u_2 \leq \cdots \leq u_n \leq v_n$. In either case, we say $f \prec g$. The following theorem gives necessary and sufficient conditions for a polynomial to be stable.

Theorem 1.8 (Hermite-Biehler)**.** *Let $p(x) = p_e(x^2) + xp_o(x^2)$ be a real, standard polynomial. Then $p(x)$ is stable if and only if $p_e(x), p_o(x)$ are standard, have only non-positive roots, and $p_o(x) \prec p_e(x)$.*

The following examples show how this theorem may be used.

Example 1.13. Consider the polynomial $p(x) = x^4 + 2x^3 + 3x^2 + 5x + 1$. Does $p(x)$ have a root in the open right-half-plane (RHP)? First, let us write $p(x) = p_e(x^2) + xp_o(x^2)$, so that $p_e(x) = x^2 + 3x + 1$ and $p_o(x) = 2x + 5$. We immediately see that both $p_e(x)$ and $p_o(x)$ are standard as they have positive leading coefficient. Letting $v_{1,2}$ be the roots of $p_e(x)$ and u_1 be the root of $p_o(x)$, we have

$$v_1 = \frac{-3-\sqrt{5}}{2}, \quad v_2 = \frac{-3+\sqrt{5}}{2},$$

and $u_1 = -5/2$. Each root is non-positive, and $v_1 \leq u_1 \leq v_2$. Hence, $p_o(x) \prec p_e(x)$. Therefore, by the Hermite-Biehler Theorem, all roots of $p(x)$ are located in the closed LHP, and thus $p(x)$ has no roots in the open RHP. See Figure 1.16 for the location of the roots.

Example 1.14. Consider the polynomial $p(x) = x^4 + 2x^3 + 2x^2 + 7x + 6$. Writing $p(x) = p_e(x^2) + xp_o(x^2)$, we find $p_e(x) = x^2 + 2x + 6$ and $p_o(x) = 2x + 7$. The only root of $p_o(x)$ is non-positive and real, but the roots of $p_e(x)$ are $-1 \pm i\sqrt{5}$. Since these roots are non-real, we cannot say that $p_o(x) \prec p_e(x)$. Therefore, by the Hermite-Biehler Theorem, $p(x)$ has roots in the open RHP. Solving for the roots of $p(x)$ reveals it has the roots $(1 \pm i\sqrt{11})/2$, which lie in the open RHP. This can be seen in Figure 1.16.

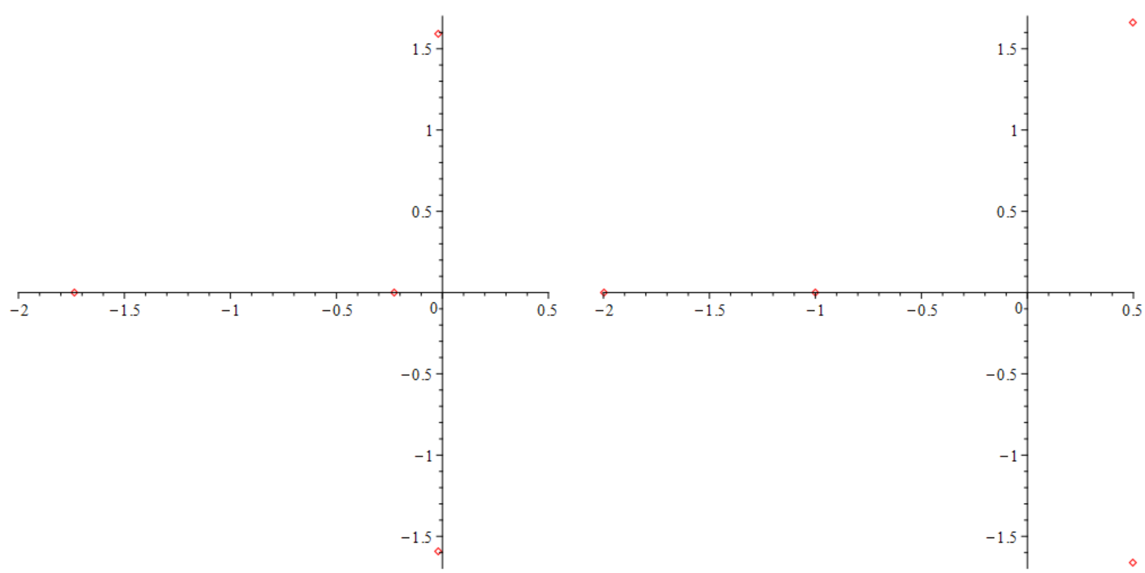


Figure 1.16: Left: the roots of the polynomial $p(x) = x^4 + 2x^3 + 3x^2 + 5x + 1$. Each root lies to the left of the origin, a fact shown with the Hermite-Biehler Theorem. Right: the roots of the polynomial $p(x) = x^4 + 2x^3 + 2x^2 + 7x + 6$. This polynomial has roots to the right of the origin, as shown with the Hermite-Biehler Theorem.

Chapter 2

The Degree Polynomial

2.1 Definition

We now define the degree polynomial of a graph. Unless otherwise stated, all graphs considered will be simple. Given an (n, m)-graph G, let $a_k(G)$ (or a_k) denote the number of vertices of degree k in G. Furthermore, let δ and Δ denote the minimum and maximum degrees of G, respectively. We define the *degree polynomial* $D(G; x)$ of G to be the generating function for the sequence $(a_0(G), a_1(G), ..., a_{n-1}(G))$. In other words,

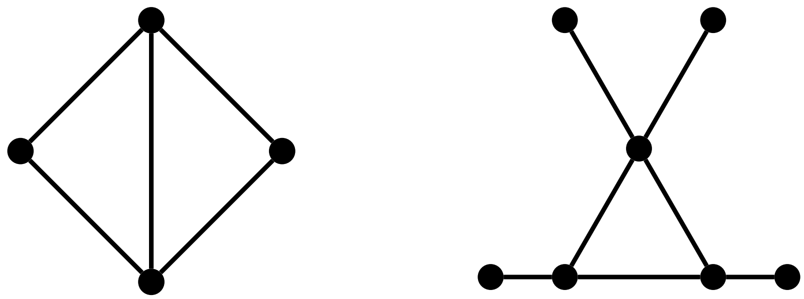
$$D(G; x) \equiv \sum_{k=0}^{n-1} a_k x^k = \sum_{k=\delta}^{\Delta} a_k x^k = \sum_{v \in V} x^{\deg(v)}$$

Such a graph polynomial has been previously defined independently by a number of researchers ([13], [26]). By $\mathcal{D}$ and $\mathcal{D}_{multi}$ we denote the set of degree polynomials for simple graphs and multigraphs, respectively. If $\mathcal{F}$ is a family of graphs, then we use $\mathcal{D}(\mathcal{F})$ to denote the set of all degree polynomials for graphs in $\mathcal{F}$ with indeterminate x. Let us illustrate the definition of this polynomial with some examples.

Example 2.1. In Figure 2.1 we show two graphs, G_1 and G_2, and their degree polynomials. G_1 contains two vertices each of degree two and three, so its degree polynomial is $D(G_1; x) = 2x^3 + 2x^2$. G_2 has a vertex of degree four, two vertices of degree three, and four leaves, and thus $D(G_2; x) = x^4 + 2x^3 + 4x$.

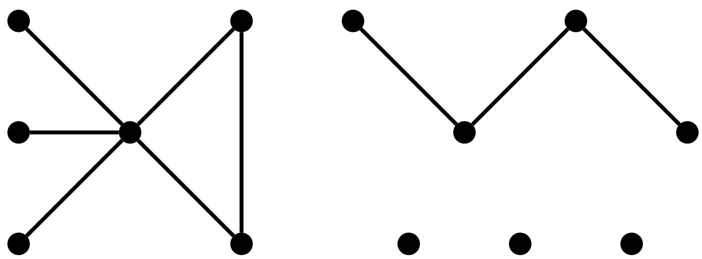
Example 2.2. There is no additional difficulty even if the graph is disconnected. As in Figure 2.2, we form the degree polynomial simply by counting degrees across each component to obtain $D(G; x) = x^5 + 4x^2 + 5x + 3$.

Example 2.3. Sometimes a family of graphs has enough structure to form a general expression for their degree polynomials. Consider the following family of graphs: let



$D(G_1; x) = 2x^3 + 2x^2$ $\qquad$ $D(G_2; x) = x^4 + 2x^3 + 4x$

Figure 2.1: The graphs G_1, G_2, and their degree polynomials.



$$D(G; x) = x^5 + 4x^2 + 5x + 3$$

Figure 2.2: A disconnected graph and its degree polynomial.

$G_0 = S_5$. Let G_1 be the graph made from adding a leaf to each leaf of G_0. In general, let G_t be the graph formed by adding a leaf to each leaf of G_{t-1}. Thus G_t has a single vertex of degree four connected to four distinct paths, each ending in a vertex of degree one. Each of the four paths contributes t vertices of degree two, and therefore we find $D(G_t; x) = x^4 + 4tx^2 + 4x$.

2.2 Observations

Graph polynomials encode graph theoretic information in the form of a polynomial. For example, the chromatic polynomial encodes information about the number of vertices and edges, cycles, block number, colourability, etc. [18]. The degree polynomial encodes precisely the degree sequence of a graph, and hence the same information as the degree sequence. If $d_1 \geq d_2 \geq \cdots \geq d_n$ is the degree sequence of a graph G, then

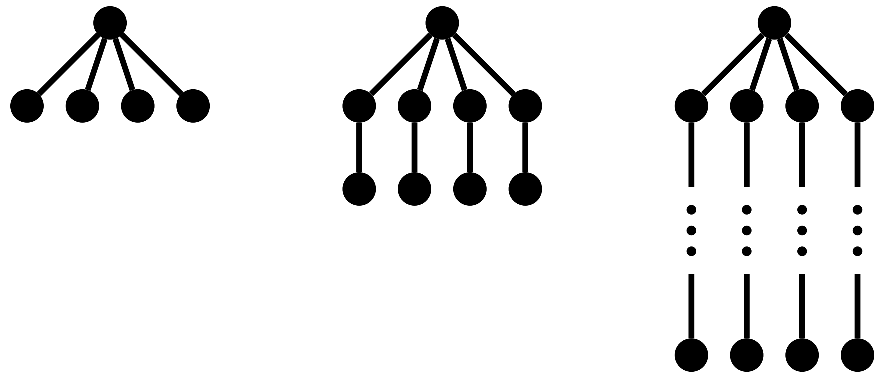


Figure 2.3: The graphs G_0, G_1, and the generalization G_t form the family $\{G_t\}_{t=0}^{\infty}$. The degree polynomial of G_t is easily found to be $D(G_t;x) = x^4 + 4tx^2 + 4x$.

$D(G;x) = \sum_{i=1}^{n} x^{d_i}$. Conversely, if G has degree polynomial $D(G;x) = \sum_{k=\delta}^{\Delta} a_k x^k$ then

$$\underbrace{\Delta, ..., \Delta}_{a_\Delta}, ..., \underbrace{\delta+1, ..., \delta+1}_{a_{\delta+1}}, \underbrace{\delta, ..., \delta}_{a_\delta}$$

is the degree sequence of G. In the following example we illustrate the lack of graph theoretic information encoded by degree polynomials/degree sequences.

Example 2.4. The two graphs shown in Figure 2.4 have the same degree polynomial $3x^3 + 2x^2 + 5x$, yet bear different structural properties and are non-isomorphic. The graph on the left is connected, has a diameter of 5, and is a tree, thus is bipartite and has no cycles. In contrast, the graph on the right is disconnected, thus has infinite diameter, has a cycle C_3, and has chromatic number 3. Hence, degree polynomials cannot encode such graph theoretic information.

For the question of whether or not a given polynomial is *degree-graphic*, that is if it is the degree polynomial of some graph G, there of course is the necessary condition that all its coefficients are non-negative integers. Given such a polynomial, we can extract a sequence of non-negative integers from the coefficients and exponents as described above. Viewing this sequence as a possible degree sequence, we recall the Erdös-Gallai theorem which gives the answer to this problem:

Theorem 2.1 (Erdös-Gallai, [14]). *A sequence of non-negative integers* $d_1 \geq \cdots \geq d_n$

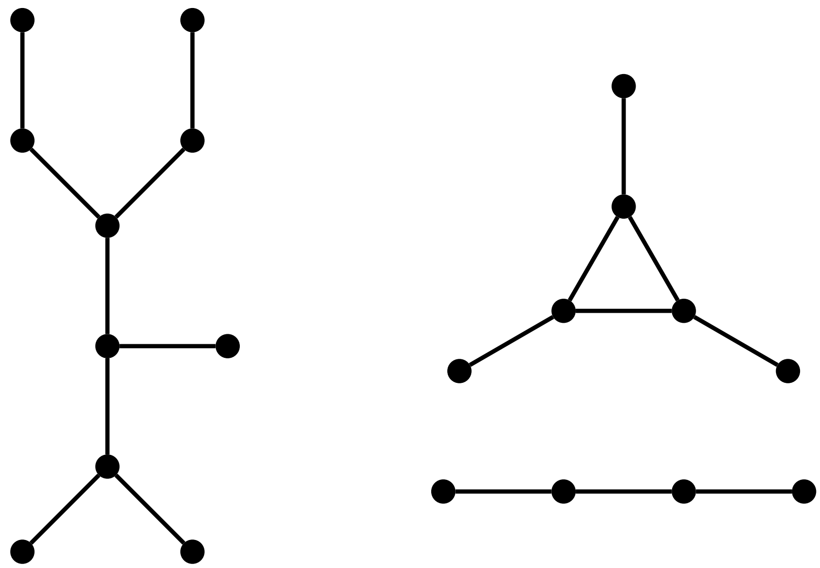


Figure 2.4: Two graphs having the same degree polynomial $3x^3+2x^2+5x$, and hence the same degree sequence, that have very different structures. They do not share the same diameter, chromatic number, cycles, or connectedness.

is graphic (the degree sequence of a finite, simple graph) if and only if the following hold:

1. $d_1 + \cdots + d_n$ *is even,*
2. $\sum_{i=1}^{j} d_i \leq j(j-1) + \sum_{i=j+1}^{n} \min(d_i, j),$ *for all* $1 \leq j \leq n.$

If we expand our consideration to (multi)graphs, the same question has an answer due to Hakimi:

Theorem 2.2 (Hakimi, [22])**.** *A sequence of non-negative integers* $d_1 \geq \cdots \geq d_n$ *is the degree sequence of a multigraph if and only if*

1. $d_1 + \cdots + d_n$ *is even,*
2. $d_1 \leq \sum_{i=2}^{n} d_i,$ *or equivalently* $d_1 \leq \frac{1}{2}\sum_{i=1}^{n} d_i.$

Below we list some more observations on the degree polynomial. For each of the following, G is a (n, m)-graph, and H_1, H_2 are graphs of order n_1, n_2, respectively.

Observation 2.1. Some facts concerning degree polynomials.

1. $D(G; 1) = \sum_{k=\delta}^{\Delta} a_k = n$.

2. $D(G; -1) = \sum_{k \text{ even}}^{\Delta} a_k - \sum_{k \text{ odd}}^{\Delta} a_k$.

3. $D^{(r)}(G; 0) = r!a_r$, for $0 \le r \le \Delta$.

4. $D'(G; 1) = \sum_{k=1}^{\Delta} ka_k = 2m$.

5. The multiplicity of the root 0 of $D(G; x)$ is δ.

6. $D(G^c; x) = x^{n-1} D(G; 1/x)$.

7. If G' is the graph resulting from adding a universal vertex to G, then

$$D(G'; x) = x^n + xD(G; x).$$

8. If H_1, H_2 are vertex disjoint graphs, then

$$D(H_1 \cup H_2; x) = D(H_1; x) + D(H_2; x).$$

What follows are some results about degree polynomials under graph operations that deserve more attention than just a quick mention.

Proposition 2.3 ([13])**.** *If H_1 and H_2 are vertex disjoint, then the degree polynomial of the join $H_1 + H_2$ of graphs H_1 and H_2 is*

$$D(H_1 + H_2; x) = x^{n_2} D(H_1; x) + x^{n_1} D(H_2; x).$$

Proof. Recall that $V(H_1+H_2) = V(H_1) \cup V(H_2)$. Furthermore if u is a vertex of H_1 and v is a vertex of H_2, then $\deg_{H_1+H_2}(u) = n_2 + \deg_{H_1}(u)$ and similarly $\deg_{H_1+H_2}(v) = n_1 + \deg_{H_2}(v)$. Thus,

$$\begin{aligned}
D(H_1+H_2;x) &= \sum_{w \in V(H_1+H_2)} x^{\deg_{H_1+H_2}(w)} \\
&= \sum_{w \in V(H_1)} x^{\deg_{H_1+H_2}(w)} + \sum_{w \in V(H_2)} x^{\deg_{H_1+H_2}(w)} \\
&= \sum_{w \in V(H_1)} x^{n_2+\deg_{H_1}(w)} + \sum_{w \in V(H_2)} x^{n_1+\deg_{H_2}(w)} \\
&= x^{n_2} \sum_{w \in V(H_1)} x^{\deg_{H_1}(w)} + x^{n_1} \sum_{w \in V(H_2)} x^{\deg_{H_2}(w)} \\
&= x^{n_2} D(H_1;x) + x^{n_1} D(H_2;x).
\end{aligned}$$

□

Proposition 2.4 ([13]). *The degree polynomial for the corona $H_1 \odot H_2$ of H_1 and H_2 is*

$$D(H_1 \odot H_2;x) = x^{n_2} D(H_1;x) + n_1 x \cdot D(H_2;x).$$

Proof. If u is a vertex of H_1, then $\deg_{H_1 \odot H_2}(u) = \deg_{H_1}(u) + n_2$. If v is a vertex belonging to one of the copies of H_2 in $H_1 \odot H_2$, then $\deg_{H_1 \odot H_2}(v) = \deg_{H_2}(v) + 1$ since each copy of H_2 is joined to a single vertex of H_1. Hence,

$$\begin{aligned}
D(H_1 \odot H_2;x) &= \sum_{w \in V(H_1 \odot H_2)} x^{\deg_{H_1 \odot H_2}(w)} \\
&= \sum_{w \in V(H_1)} x^{\deg_{H_1 \odot H_2}(w)} + \sum_{w \in V(H_1 \odot H_2) - V(H_1)} x^{\deg_{H_1 \odot H_2}(w)} \\
&= \sum_{w \in V(H_1)} x^{\deg_{H_1}(w)+n_2} + \sum_{w \in V(H_1 \odot H_2) - V(H_1)} x^{\deg_{H_2}(v)+1} \\
&= x^{n_2} \sum_{w \in V(H_1)} x^{\deg_{H_1}(w)} + n_1 \sum_{w \in V(H_2)} x^{\deg_{H_2}(v)+1} \\
&= x^{n_2} D(H_1;x) + n_1 x \cdot D(H_2;x).
\end{aligned}$$

□

Proposition 2.5 ([13]). *The degree polynomial of the lexicographic product $H_1[H_2]$ is*

$$D(H_1[H_2];x) = D(H_1;x^{n_2})D(H_2;x).$$

Proof. Recall that two vertices (u,v) and (u',v'), where $u,u' \in V(H_1)$ and $v,v' \in V(H_2)$, of $H_1[H_2]$ are adjacent if and only if $\{u,u'\} \in E(H_1)$ or $u = u'$ and $\{v,v'\} \in E(H_2)$. Therefore $\deg_{H_1[H_2]}((u,v)) = n_2\deg_{H_1}(u) + \deg_{H_2}(v)$. Computing the degree polynomial,

$$\begin{aligned}
D(H_1[H_2];x) &= \sum_{(u,v)\in V(H_1[H_2])} x^{\deg_{H_1[H_2]}((u,v))} \\
&= \sum_{(u,v)\in V(H_1[H_2])} x^{n_2\deg_{H_1}(u)+\deg_{H_2}(v)} \\
&= \sum_{(u,v)\in V(H_1[H_2])} x^{n_2\deg_{H_1}(u)} x^{\deg_{H_2}(v)} \\
&= \sum_{u\in V(H_1)} x^{n_2\deg_{H_1}(u)} \sum_{v\in V(H_2)} x^{\deg_{H_2}(v)} \\
&= D(H_1;x^{n_2})D(H_2;x).
\end{aligned}$$

□

Proposition 2.6 ([13]). *If H_1 and H_2 are connected, then the degree polynomial of their cartesian product is*

$$D(H_1 \square H_2;x) = D(H_1;x)D(H_2;x).$$

Proof. Recall that two vertices (u,v) and (u',v'), where $u,u' \in V(H_1)$ and $v,v' \in V(H_2)$, of $H_1 \square H_2$ are adjacent if and only if $\{u,u'\} \in E(H_1)$ and $v = v'$ or $u = u'$ and $\{v,v'\} \in E(H_2)$. Thus $\deg_{H_1 \square H_2}((u,v)) = \deg_{H_1}(u) + \deg_{H_2}(v)$. Computing the degree polynomial,

$$\begin{aligned}
D(H_1 \,\square\, H_2; x) &= \sum_{(u,v)\in V(H_1 \,\square\, H_2)} x^{\deg_{H_1 \,\square\, H_2}((u,v))} \\
&= \sum_{(u,v)\in V(H_1 \,\square\, H_2)} x^{\deg_{H_1}(u)+\deg_{H_2}(v)} \\
&= \sum_{(u,v)\in V(H_1 \,\square\, H_2)} x^{\deg_{H_1}(u)} x^{\deg_{H_2}(v)} \\
&= \sum_{u\in V(H_1)} x^{\deg_{H_1}(u)} \sum_{v\in V(H_2)} x^{\deg_{H_2}(v)} \\
&= D(H_1; x) D(H_2; x).
\end{aligned}$$

□

Proposition 2.7 ([13]). *If H_1 and H_2 are connected, then the degree polynomial of their tensor product is*

$$D(H_1 \times H_2; x) = \sum_{u\in V(H_1)} D(H_2; x^{deg_{H_1}(u)}) = \sum_{v\in V(H_2)} D(H_1; x^{deg_{H_2}(v)}).$$

Proof. Recall that the edges of $H_1 \times H_2$ are

$$E(H_1 \times H_2) = \{\{(u,v),(u',v')\} \;:\; \{u,u'\} \in E(H_1) \text{ and } \{v,v'\} \in E(H_2)\}.$$

Thus $\deg_{H_1\times H_2}((u,v)) = \deg_{H_1}(u)\deg_{H_2}(v)$, since there are $\deg_{H_1}(u)$ choices for u' adjacent to u, and $\deg_{H_2}(v)$ choices for v' adjacent to v. Therefore the degree polynomial is

$$\begin{aligned}
D(H_1 \times H_2; x) &= \sum_{(u,v)\in V(H_1\times H_2)} x^{\deg_{H_1\times H_2}((u,v))} \\
&= \sum_{(u,v)\in V(H_1\times H_2)} x^{\deg_{H_1}(u)\deg_{H_2}(v)} \\
&= \sum_{u\in V(H_1)} \sum_{v\in V(H_2)} (x^{\deg_{H_1}(u)})^{\deg_{H_2}(v)} \\
&= \sum_{u\in V(H_1)} D(H_2; x^{\deg_{H_1}(u)}),
\end{aligned}$$

and equivalently

$$\begin{aligned}D(H_1 \times H_2; x) &= \sum_{(u,v)\in V(H_1\times H_2)} x^{\deg_{H_1}(u)\deg_{H_2}(v)} \\ &= \sum_{v\in V(H_2)} \sum_{u\in V(H_1)} (x^{\deg_{H_2}(v)})^{\deg_{H_1}(u)} \\ &= \sum_{v\in V(H_2)} D(H_1; x^{\deg_{H_2}(v)}).\end{aligned}$$

□

In addition to the observations made above, we can obtain an alternate statement of Theorem 2.2. Using the method previously described to extract a sequence of non-negative integers from a polynomial $p(x) \in \mathbb{Z}_{\geq 0}[x]$ and the result of Observation 4, we can re-express Theorem 2.2 in non-graph theoretic terms and give a characterization of $\mathcal{D}_{multi}$.

Theorem 2.8 (Hakimi, re-expressed)**.** *A polynomial $p(x) \in \mathbb{Z}_{\geq 0}[x]$ is the degree polynomial of a multigraph if and only if the following hold:*

1. $p'(1)$ *is even,*
2. *deg*$(p(x)) \leq p'(1)/2$.

In other words, we have

$$\mathcal{D}_{multi} = \{p(x) \in \mathbb{Z}_{\geq 0}[x] \ : \ p'(1) \textit{ is even, and } \textit{deg}(p(x)) \leq p'(1)/2\}.$$

2.3 Degree Polynomials of Some Families of Graphs

We begin this section by listing general formulas for the degree polynomials of common graph families.

G	$D(G;x)$
O_n	n
K_n	nx^{n-1}
k-regular, order n	nx^k
P_n	$(n-2)x^2+2x$
C_n	nx^2
$S_n = K_{1,n-1}$	$x^{n-1}+(n-1)x$
$K_{a,n-a}$	$ax^{n-a}+(n-a)x^a$

Table 2.1: Some degree polynomials of graphs.

The above families of graphs present fairly simple degree polynomials, requiring very little derivation. Below we attempt to show the variety of degree polynomials by describing particular families of graphs.

Example 2.5. We define the family of graphs $\{A_t\}$ recursively as follows: $A_1 = P_2$, $A_2 = P_3$, and for A_{t+1}, $t \geq 2$, connect t leaves to one of the leaf neighbours of the vertex of highest degree in A_t (see Figure 2.5). The degree polynomials for these graphs are thus also recursive:

$$D(A_{t+1};x) = D(A_t;x) + x^{t+1} + (t-1)x.$$

Explicitly,

$$\begin{aligned} D(A_{t+1};x) &= x^{t+1} + (t-1)x + x^t + (t-2)x + \cdots + x^3 + x + x^2 + 2x \\ &= \sum_{j=2}^{t+1} x^j + l_{t+1}x, \end{aligned}$$

where l_{t+1} is the number of leaves of A_{t+1}. For $t \geq 2$, A_{t+1} has $t-1$ more leaves than A_t. Thus $l_{t+1} = l_t + t - 1$, with $l_1 = l_2 = 2$. Therefore,

$$\begin{aligned} l_{t+1} &= (t-1) + (t-2) + \cdots + 2 + 1 + 2 \\ &= t(t-1)/2 + 2, \end{aligned}$$

or

$$l_t = t(t-3)/2 + 3.$$

Part of what makes these graphs interesting is that their degree polynomials have no gaps (in the exponents). Additionally, the polynomial $D(A_t; x) = \sum_{j=2}^{t} + l_t x$ has t terms. Thus by choosing large enough t we can find a polynomial that has as many terms as desired.

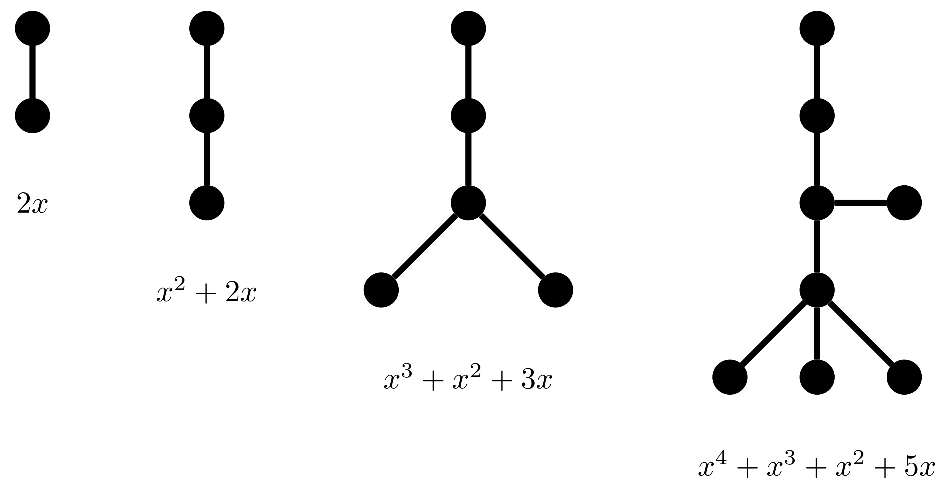
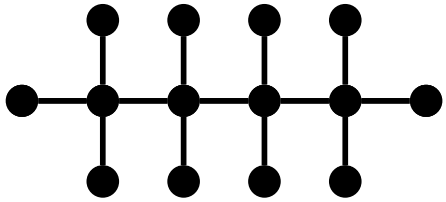


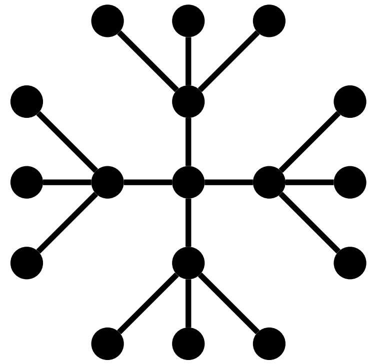
Figure 2.5: The first four graphs of the family $\{A_t\}$, from $t = 1$ to $t = 4$, and their degree polynomials.

Example 2.6. Hydrocarbons (alkanes): a tree consisting of degree four vertices (carbon) and leaves (hydrogen) (see Figure 2.6). With k carbons and n total atoms (vertices), a hydrocarbon HC_k has degree polynomial $D(HC_k; x) = kx^4 + (n-k)x$. Summing the degrees, and knowing there are $n-1$ edges, we have $2(n-1) = 4k+n-k \implies n = 3k + 2$. Thus $D(HC_k; x) = kx^4 + (2k + 2)x$. These polynomials are binomials, and no matter the graph there are only terms of degree one and four.

Example 2.7. Starting with C_{2k} ($k \geq 2$), construct a *half-wheel* by adding a new vertex, and connect this new vertex to every other vertex of the cycle. The resulting degree polynomial is $D(G; x) = x^k + kx^3 + kx^2$. As k increases, these polynomials have arbitrarily large gaps in their terms. If $k \geq 4$ then $D(G; x)$ will be a trinomial, and a binomial otherwise.

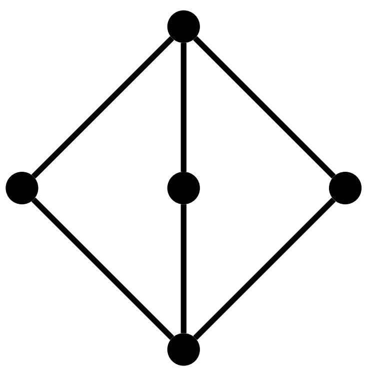


$4x^4 + 10x$

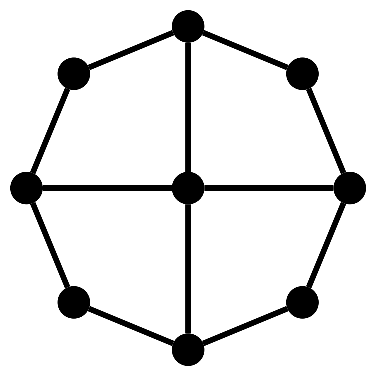


$5x^4 + 12x$

Figure 2.6: Two examples of hydrocarbon/alkane graphs. The left hydrocarbon has four carbon vertices (ten hydrogen), and the rightmost has five (and 12 hydrogen).



$2x^3 + 3x^2$



$x^4 + 4x^3 + 4x^2$

Figure 2.7: Two example half-wheels, using cycles C_4 and C_8.

Example 2.8. For complete p-partite graphs $K_{a_1,\ldots,a_p}$, set $n = a_1 + \cdots + a_p$. Then the degree polynomial is $D(K_{a_1,\ldots,a_p}; x) = \sum_{i=1}^{p} a_i x^{n-a_i}$. These degree polynomials have as many terms as unique a_i, and can have arbitrary gaps in the terms, that is, they can be *lacunary*.

Example 2.9. Consider the family of graphs $\{Y_n\}$, $n \geq 4$ constructed in the following manner: take a complete graph K_{n-4}, and join to it O_2. The current degree polynomial is $(n-4)x^{n-3} + 2x^{n-4}$. Next, connect a new vertex to the two just added, changing the polynomial to $(n-2)x^{n-3} + x^2$. Finally, add a universal vertex. We now

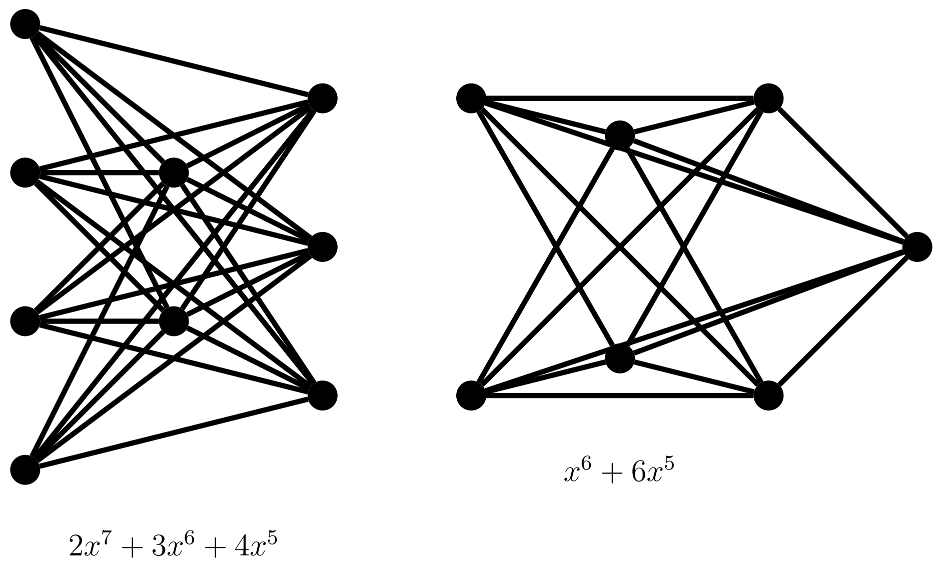


Figure 2.8: Two complete p-partite graphs. Left: the complete 3-partite graph $K_{4,2,3}$. Right: the complete 4-partite graph $K_{2,2,2,1}$.

have a graph on n vertices, with degree polynomial $D(Y_n; x) = x^{n-1}+(n-2)x^{n-2}+x^3$. These polynomials will also have an arbitrarily large gap, as n increases.

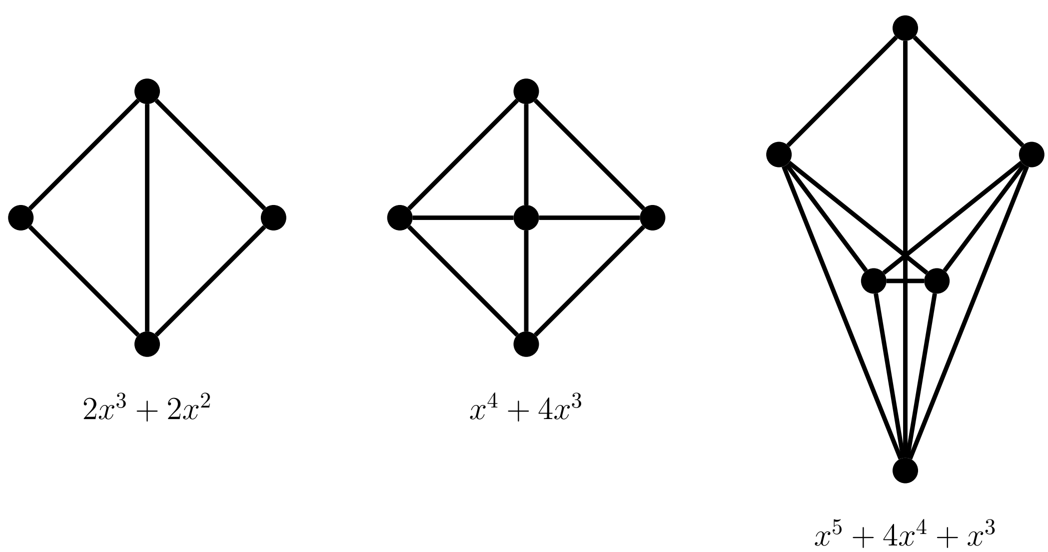


Figure 2.9: The first three graphs from the family $\{Y_n\}$: Y_4, Y_5, Y_6.

Example 2.10. For $n \geq 2$, attach a leaf to a vertex of K_{n-1}. This graph, which we will call CL_n, has degree polynomial $D(CL_n; x) = x^{n-1}+(n-2)x^{n-2}+x$. This example is very similar to the previous, the only difference between the degree polynomials is the replacement of the x^3 term with x.

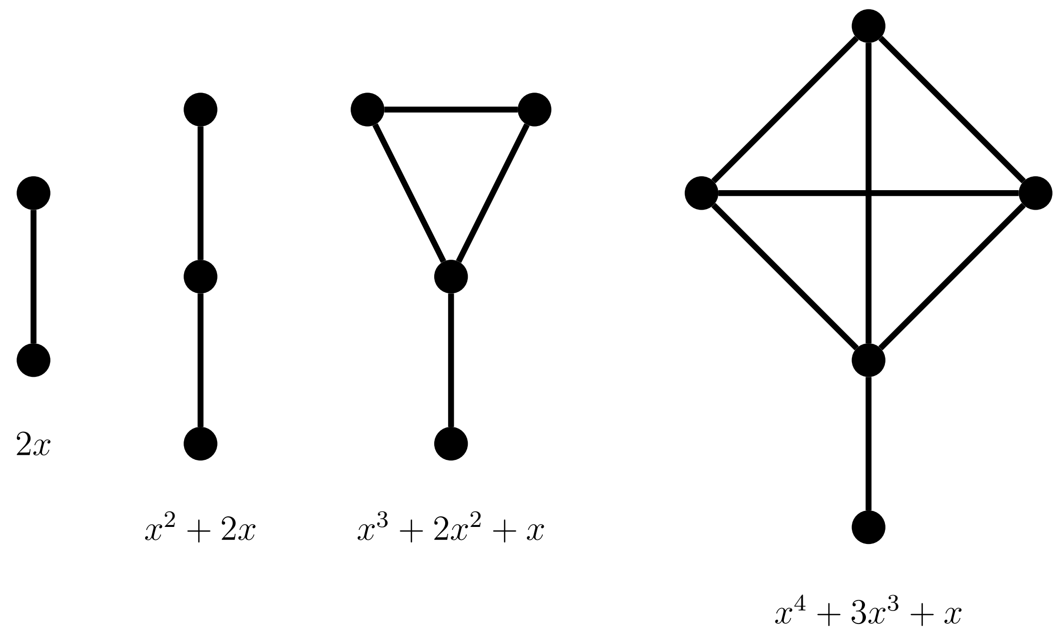


Figure 2.10: The four smallest CL_n graphs, constructed by attaching a leaf to complete graphs.

Example 2.11. An interesting family of graphs are the *anti-regular* graphs [1], also known as quasi-perfect, maximally non-regular, degree anti-regular, or half-complete ([1], [20], [17]): those with n vertices and having $n-1$ distinct degrees. Indeed there cannot be more than $n-1$ distinct degrees: if there were n distinct degrees, then each of $0, 1, ..., n-1$ must appear as the degree of exactly one vertex. Thus there would simultaneously be a vertex adjacent to all others (degree $n-1$) and a vertex not adjacent to any (degree 0), which is a contradiction.

For a given $n \geq 2$, there are precisely two graphs (up to isomorphism) with $n-1$ distinct degrees (we refer to [17] for the following): first, the graph H_n, with degrees $1, 2, ..., n-1$. Every degree appears once in the degree sequence, except for $\lfloor n/2 \rfloor$, which appears twice. H_n can be formed by taking vertices $v_1, ..., v_n$, and adding all edges of the form $\{v_i, v_j\}$ such that $i + j \geq n + 1$. The other graph has degrees $0, 1, ..., n-2$, and is the graph complement H_n^c of H_n. The degree which appears twice in the degree sequence in this case is $n - 1 - \lfloor n/2 \rfloor$, or $\lfloor (n-1)/2 \rfloor$:

- $n = 2k : n - 1 - \lfloor n/2 \rfloor = 2k - 1 - k = k - 1 = n/2 - 1 = \lfloor (n-1)/2 \rfloor$
- $n = 2k + 1 : n - 1 - \lfloor n/2 \rfloor = 2k - \lfloor k + 1/2 \rfloor = k = (n-1)/2 = \lfloor (n-1)/2 \rfloor$.

Furthermore, H_n^c can be formed by taking vertices $v_1, ..., v_n$ and adding edges $\{v_i, v_j\}$ such that $i + j > n + 1$. Thus we can easily write the degree polynomials for these graphs:

$$\begin{aligned} D(H_n; x) &= \sum_{1}^{n-1} x^i + x^{\lfloor n/2 \rfloor} \\ &= \frac{x(x^{n-1} - 1)}{x - 1} + x^{\lfloor n/2 \rfloor}, \end{aligned}$$

and

$$\begin{aligned} D(H_n^c; x) &= x^{n-1} D(H_n; 1/x) \\ &= \sum_{0}^{n-2} x^i + x^{\lfloor (n-1)/2 \rfloor} \\ &= \frac{x^{n-1} - 1}{x - 1} + x^{\lfloor (n-1)/2 \rfloor}. \end{aligned}$$

These polynomials have no gaps, and have only a single term with coefficient greater than one. This contrasts with many other families of graphs, who have arbitrarily many vertices with a certain degree. See Figure 2.11 for some examples of anti-regular graphs and their degree polynomials.

2.4 Outline of the Thesis

So far we have defined the degree polynomial, investigated some of its properties, and given some examples of computing it. The remainder of the thesis will focus on roots of degree polynomials, which is the subject of Chapter 3. We will begin by making some observations on these roots, and provide a characterization of them independent of graph theoretic constraints. This will allow us to make statements concerning the density of these roots. We then investigate how roots of degree polynomials depend on certain graph parameters. In particular, we will give bounds on the roots that

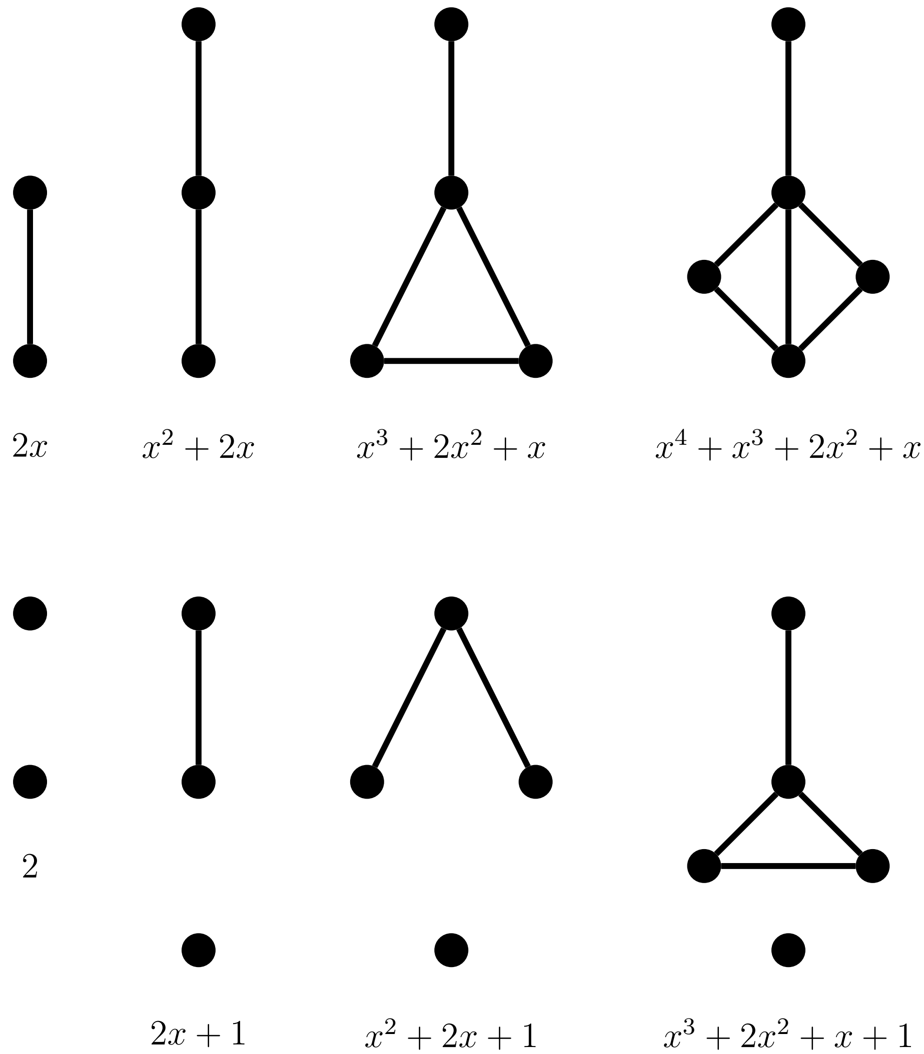


Figure 2.11: Examples of anti-regular graphs and their degree polynomials. Top, left to right: H_2, H_3, H_4, H_5. Bottom, left to right: $H_2^c, H_3^c, H_4^c, H_5^c$.

depend on graph order, and locate them to certain regions of the complex plane. Chapter 3 concludes with an exploration of roots for some families of graphs. By narrowing in on families for which their degree polynomials can be explicitly written, we are able to prove results on the roots specific to those families.

In Chapter 4 we discuss future directions for the study of the degree polynomial. We mention open problems, conjectures, and ways to generalize the degree polynomial. In particular, a two variable degree polynomial for directed graphs and a multivariate generalization based on vertex labellings are presented.

Chapter 3

Roots of the Degree Polynomial

3.1 Degree Roots

Possibly the most natural problem concerning polynomials is that of the location of their roots. The study of roots for degree polynomials is absent in the literature, so we will explore this topic beginning with the following definition:

Definition 3.1. A complex number z is a *degree root* if it is the root of a degree polynomial for some multigraph, that is, if there exists a multigraph G such that $D(G; z) = 0$. In this case, we also say that z is a *degree root of* G. If $\mathcal{F}$ is a family of graphs, the set of all degree roots for graphs in $\mathcal{F}$ is $Z(\mathcal{D}(\mathcal{F}))$, as per our notation.

To help motivate conjectures and future results, let us examine the behaviour of degree roots for graphs of low order. Figure 3.1 shows all degree roots for (simple) graphs from orders two to ten, from which we make the following observations.

- Focusing on the real axis, degree roots appear to be filling up the negative real axis as the order of graphs increases. Thus we conjecture that degree roots are dense on the negative real axis.

- Since the coefficients of a degree polynomial are non-negative, there cannot be a positive root of a degree polynomial. Thus we see no positive roots in any plot.

- When the order n of graphs is odd, the largest negative root seems to be located at $-(n-1)$. When n is even, the largest negative root seems to be near $-(n-2)$.

- For orders $n \geq 5$, the root of largest modulus seems to be real and never surpasses a modulus of $n-1$. Furthermore, roots on the imaginary axis appear to never exceed a modulus of $\sqrt{n}$.

- The region surrounding the origin (excluding the negative real axis) that extends a small distance to the right seems to be filling in very slow. Only seeing roots up to order ten, it is not clear whether this region will actually fill with roots or not. On the same note, the real parts of the non-real roots lie within a small band, only being the approximate interval $(-2, 1.5)$ for graphs of order ten.

Recall that $\mathcal{D}$ and $\mathcal{D}_{multi}$ represent, respectively, the set of degree polynomials for (simple) graphs and multigraphs. While $\mathcal{D} \subseteq \mathcal{D}_{multi}$, how are their roots related? The previous inclusion implies $Z(\mathcal{D}) \subset Z(\mathcal{D}_{multi})$, but is this inclusion strict? Furthermore, since both $\mathcal{D}$ and $\mathcal{D}_{multi}$ are contained in $\mathbb{Z}_{\geq 0}[x]$, the set of polynomials with non-negative integer coefficients, how does $Z(\mathbb{Z}_{\geq 0}[x])$ fit into this picture? We will now answer these questions, beginning with the following folklore lemma.

Lemma 3.1. *There exists an r-regular (simple) graph on n vertices if and only if nr is even, and $n \geq r+1$. It follows that for any $r \geq 1$, there are infinitely many r-regular (simple) graphs.*

Proof. ($\Longrightarrow$) This direction is clear since $nr = 2m$, where m is the number of edges, and $r \leq n-1$ as the graph is simple.
($\Longleftarrow$) We will show the existence of such a desired regular (simple) graph with the Erdös-Gallai Theorem. The sequence in question is $r, r, ..., r$ which is of length n. By the assumption that nr is even, the theorem's first condition is met. The second condition requires that

$$kr \leq k(k-1) + (n-k) \cdot \min(k, r)$$

hold true for all $1 \leq k \leq n$. We can consider two cases:
Case 1: $k \leq r$. Therefore $\min(k, r) = k$, and we have

$$\begin{aligned} & kr \leq k(k-1) + (n-k)k \\ \iff & kr \leq nk - k \\ \iff & \ r \leq n-1, \end{aligned}$$

which is true by assumption.

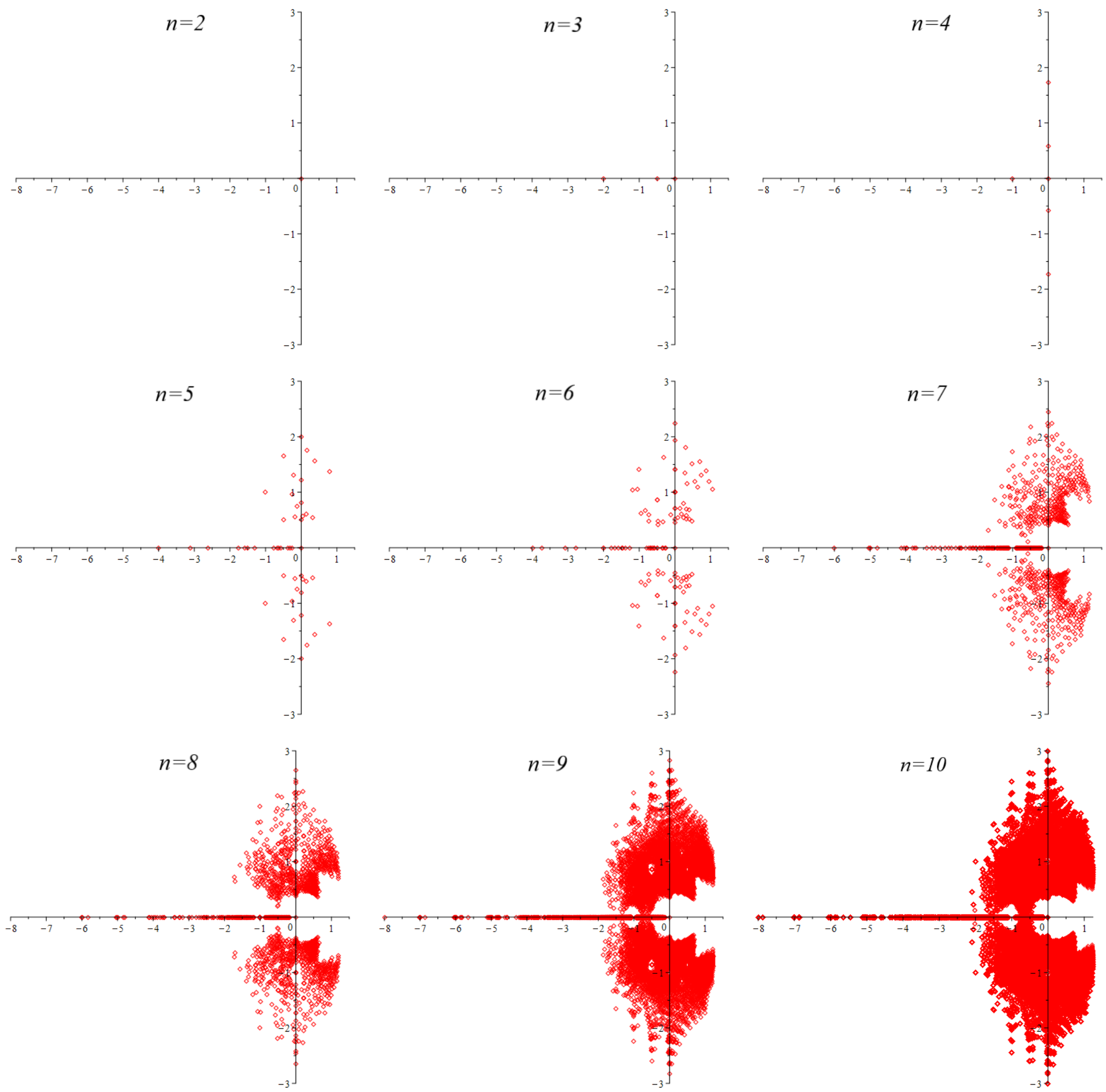


Figure 3.1: Degree roots for graphs from orders two to ten. Beginning with order two in the top left, each plot shows all degree roots for graphs of order n, ending with order ten in the bottom right.

Case 2: $r < k$. This is equivalent to $r \leq k-1$. Since $n-k \geq 0$, and $r \leq k-1 \implies kr \leq k(k-1)$, we indeed have $kr \leq k(k-1) + (n-k)r$.

Thus the second condition is met, and it follows that there exists a (simple) graph with degree sequence $r, ..., r$, (of length n), ie. there exists a (simple) r-regular graph on n vertices. □

This lemma allows us to show that given a multigraph, there is a process which creates a (simple) graph while preserving degree roots.

Proposition 3.2. *Let M be a multigraph. Then there exists a (simple) graph G for which $D(M;x)$ and $D(G;x)$ have precisely the same roots, including multiplicities.*

Proof. Let $e_1, ..., e_k$ be the pairs of vertices of M that are joined with multiple edges. That is, $e_i = \{v_{i,1}, v_{i,2}\}$ is an unordered pair of vertices being the endpoints of $b_i \geq 2$ parallel edges. Suppose the pairs are indexed so that $b_1 \leq \cdots \leq b_k$. Let $M_1, ..., M_{b_k}$ be disjoint isomorphic copies of M, and set $G = M_1 \cup \cdots \cup M_{b_k}$. Thus $D(G;x) = b_k D(M;x)$. Furthermore, let e_i^j be the copy of the pair e_i belonging to M_j. We shall modify G so that it turns into a simple graph through a sequence of degree-preserving steps:

1. Take the subgraph of G induced by $e_1^1, e_1^2, ..., e_1^{b_k}$, and observe that each vertex has an induced degree of b_1. Remove all edges of this subgraph, and add new edges in the vertices of the subgraph so that the subgraph is simple and b_1-regular (this can be achieved by Lemma 3.1 since there are $2b_k$ vertices in the subgraph, and $b_1 + 1 \leq 2b_k$). Each vertex still has an induced degree of b_1.

2. Repeat Step 1. with the vertices from $e_2^1, ..., e_2^{b_k}$, and making the induced subgraph b_2-regular.

   . . .

k. Repeat Step 1. with the vertices from $e_k^1, ..., e_k^{b_k}$, and making the induced subgraph b_k-regular.

Since in each step the induced degree of every vertex is preserved, these rewiring procedures preserve all vertex degrees of G, and hence the degree polynomial $D(G;x)$. Furthermore, we have eliminated all multi-edges by creating the b_i-regular subgraphs and thus G is now simple. Since $D(G;x) = b_k D(M;x)$, G and M have precisely the same degree roots and we are done. □

Let us illustrate this proof with an example.

Example 3.1. Consider the multigraph M on the left of Figure 3.2, which has degree polynomial $D(M;x) = x^4 + x^3 + x$. M has a single edge bundle: there are 3 edges between vertices u and v. First, we form a graph G that is the disjoint union of 3 copies of M. Let u_j, v_j be the copies of vertices u, v in the j'th copy of M ($j = 1, 2, 3$). We then modify G by removing all edges in the subgraph induced by $\{u_1, v_1, u_2, v_2, u_3, v_3\}$. This step removes all bundles of edges. Lastly, we add edges back into this subgraph so that it induces a 3-regular (simple) subgraph. This can be done in several ways, but here we choose to add edges $\{u_j, v_j\}$ ($j = 1, 2, 3$), $\{u_j, u_{j'}\}$ ($j \neq j'$), and $\{v_j, v_{j'}\}$ ($j \neq j'$). The resulting graph G is simple, and has degree polynomial $D(G;x) = 3x^4 + 3x^3 + 3x = 3D(M;x)$. See the right of Figure 3.2 for this final graph G.

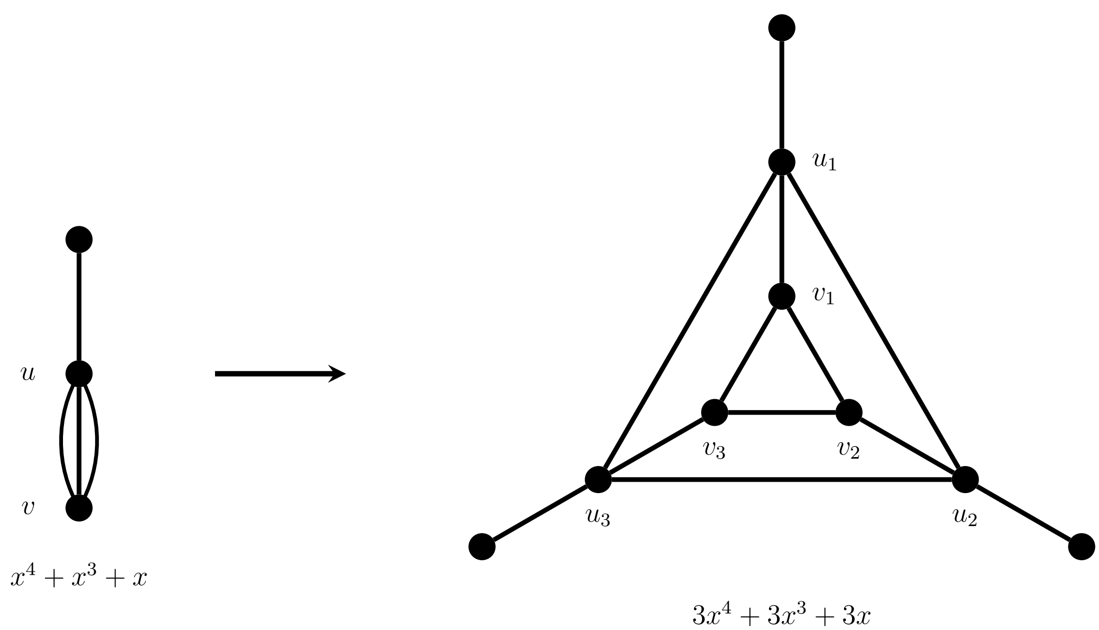


Figure 3.2: Left: a multigraph M with $D(M;x) = x^4 + x^3 + x$. Right: a simple graph G with degree polynomial $D(G;x) = 3D(M;x)$, formed through the process described in Proposition 3.2.

As a corollary, we are able to relate the degree roots of multigraphs to those of (simple) graphs.

Corollary 3.3. $Z(\mathcal{D}) = Z(\mathcal{D}_{multi})$.

Proof. The inclusion $Z(\mathcal{D}_{multi}) \subseteq Z(\mathcal{D})$ follows from Proposition 3.2. □

Interestingly, the expansion to multigraphs adds no new degree roots. We will shortly see how the roots $Z(\mathbb{Z}_{\geq 0}[x])$ are related. First, however, we need the following corollary due to our re-expressed version of Hakimi's Theorem (Theorem 2.8). Recall that this theorem asserts that a polynomial $p(x) \in \mathbb{Z}_{\geq 0}[x]$ is the degree polynomial of a multigraph if and only if $p'(1)$ is even and $deg(p(x)) \leq p'(1)/2$.

Corollary 3.4. *If* $f(x) \in \mathbb{Z}_{\geq 0}[x]$*, then* $2f(x) \in \mathcal{D}_{multi}$.

Proof. Let $g(x) = 2f(x)$. Then as $g'(x) = 2f'(x)$ and $f'(x) \in \mathbb{Z}_{\geq 0}[x]$ it is easy to see that $2 \mid g'(1)$. Furthermore, we can write $g(x) = 2ax^{\Delta} + \cdots$, where $a \geq 1$, and thus

$$\begin{aligned} g'(1)/2 &= a\Delta + \cdots \\ &\geq \Delta \\ &= \deg(g(x)). \end{aligned}$$

Therefore by Theorem 2.8, $g(x) \in \mathcal{D}_{multi}$. □

The above result implies that the roots of polynomials with non-negative integer coefficients are degree roots for some multigraph, which are degree roots for a (simple) graph. Thus we also have the following.

Corollary 3.5. $Z(\mathcal{D}) = Z(\mathbb{Z}_{\geq 0}[x])$.

Proof. The forward inclusion $Z(\mathcal{D}) \subseteq Z(\mathbb{Z}_{\geq 0}[x])$ is true by definition, so we need only to show the other direction. Suppose $p(x) \in \mathbb{Z}_{\geq 0}[x]$, and that $p(z) = 0$ (so $z \in Z(\mathbb{Z}_{\geq 0}[x])$). Then $2p(z) = 0$, and since $2p(x) \in \mathcal{D}_{multi}$ by Corollary 3.4, we have $z \in Z(\mathcal{D}_{multi}) = Z(\mathcal{D})$, and we are done. □

While $Z(\mathcal{D}) = Z(\mathcal{D}_{multi}) = Z(\mathbb{Z}_{\geq 0}[x])$ may make it seem that degree roots are not deserving of investigation beyond examining the roots of $Z(\mathbb{Z}_{\geq 0}[x])$, the restriction of certain graph parameters can restrict degree roots. This problem is the concern of a later section.

3.2 Density of Degree Roots

In this section we investigate the distribution of degree roots in the complex plane. In particular, we are interested in the density of degree roots. The density of roots for other graph polynomials has been studied. For example, see [35] for the density of chromatic roots. Our approach will make use of the results from the previous section, in particular Corollary 3.5.

To begin, consider the following set of complex numbers, which contains all a^{th} roots of all negative rational numbers:

$$\mathcal{R} = \left\{ \omega_a \left(\frac{p}{q}\right)^{1/a} \ : \ a, p, q \in \mathbb{Z}_{\geq 1}, \ \gcd(p,q) = 1, \ (\omega_a)^a = -1 \right\}.$$

Observe that an element $z = \omega_a \left(p/q\right)^{1/a}$ of $\mathcal{R}$ is a root of the polynomial $qx^{a+s} + px^s \in \mathbb{Z}_{\geq 0}[x]$. Therefore, $z \in Z(\mathcal{D}) = Z(\mathbb{Z}_{\geq 0}[x])$ and hence $\mathcal{R} \subseteq Z(\mathcal{D})$. A useful subset of $\mathcal{R}$ for showing density of degree roots is

$$\mathcal{A} = \left\{ \omega_a \left(\frac{p}{q}\right) \ : \ a, p, q \in \mathbb{Z}_{\geq 1}, \ \gcd(p,q) = 1, \ (\omega_a)^a = -1 \right\},$$

which is obtained by taking a^{th} roots of negative rational powers $-(p/q)^a$. Before using $\mathcal{A}$ to give a result on the density of degree roots, let us describe how to form graphs with degree roots belonging to $\mathcal{A}$.

Let $s \geq 2$, $a \geq 1$ be integers. Take any t graphs G_i that are each s-regular of the same order n_0, and set $G_i' = G_i - e_i$ for any edge e_i of G_i. Let $v_{i,0}$ and $v_{i,1}$ be the endpoints of e_i, so they now have degree $s-1$ in G_i'. Similarly, take r graphs H_i that are $(s+a)$-regular of the same order n_a, and set $H_i' = H_i - e_i'$ (e_i' any edge of H_i). Let $u_{i,0}$ and $u_{i,1}$ be the endpoints of e_i', so they have degree $s+a-1$ in H_i'. Next, form a new graph F in the following way: start by adding an edge between $v_{1,1}$ and $v_{2,0}$. Then add an edge between $v_{2,1}$ and $v_{3,0}$. Continue adding edges between vertices $v_{i-1,1}$ and $v_{i,0}$, up to $i = t$. Repeat this process for $u_{i-1,1}$ and $u_{i,0}$, for $2 \leq i \leq r$. Finally, add an edge between $v_{1,0}$ and $u_{1,0}$, and between $v_{t,1}$ and $u_{r,1}$. Figure 3.3 gives a sketch of this process. The resulting graph F will have order $tn_0 + rn_a$. Furthermore, we began with tn_0 vertices of degree s from the G_i graphs, and rn_a vertices of degree $s+a$ from the H_i graphs. After removing edges, we had $t(n_0-2)$ vertices of degree s, $2t$ vertices of degree $s-1$, $r(n_a-2)$ vertices of degree $s+a$, and $2r$ vertices of degree

$s+a-1$. However, adding edges between vertices $v_{i-1,1}$ and $v_{i,0}$, $u_{i-1,1}$ and $u_{i,0}$, $v_{1,0}$ and $u_{1,0}$, and $v_{t,1}$ and $u_{r,1}$ increased their degrees back up by 1. Therefore F has tn_0 vertices of degree s and rn_a vertices of degree $s+a$, thus its degree polynomial is $D(F;x) = rn_a x^{s+a} + tn_0 x^s$.

If p/q is any positive (and without loss of generality, fully reduced) rational number, observe that if we set $t = Lp^a/n_0$ and $r = Lq^a/n_a$, where $L = \mathrm{LCM}(n_0, n_a)$, then the degree polynomial of F is $D(F;x) = Lq^a x^{s+a} + Lp^a x^s$. This polynomial has s roots at 0, and the remaining roots are the a^{th} roots of $-p^a/q^a$, that is, roots of the form $\omega_a p/q$ where again ω_a is any a^{th} root of -1. Therefore, the non-zero degree roots of F belong to $\mathcal{A}$.

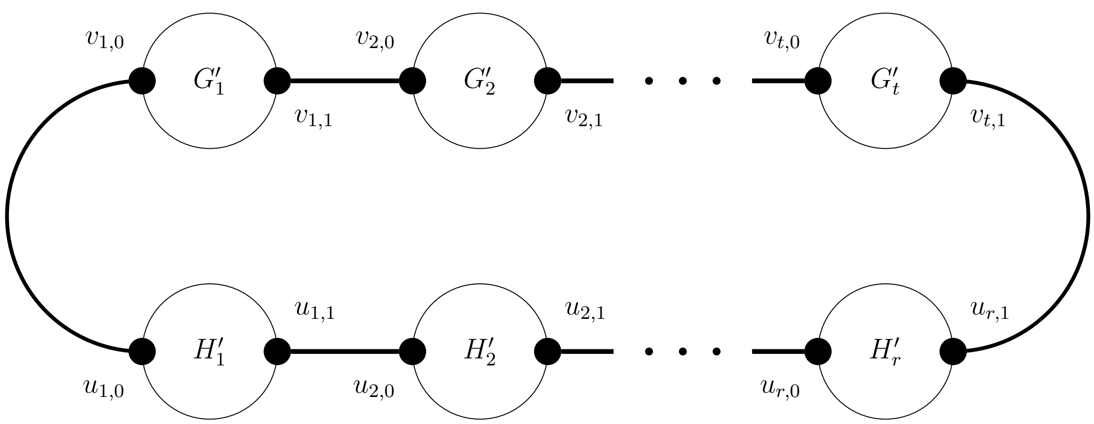


Figure 3.3: Schematic of the construction of the graph F.

We remark that in the construction of F it did not matter which $v_{i,j}$ or $u_{k,l}$ were connected to each other. As long as each only received one new edge, the degree polynomial would remain $Lq^a x^{s+a} + Lp^a x^s$. It is the roots of these polynomials that gives us the following theorem concerning the density of degree roots.

Theorem 3.6. *Degree roots are dense in the complex numbers* $\mathbb{C}$*, in the non-positive real axis* $(-\infty, 0]$*, and in the imaginary axis* $i\mathbb{R}$*.*

Proof. To begin, we claim that the set $\mathbb{Q}_{odd}$ of odd-numerator rational numbers is dense in $\mathbb{R}$. We are interested in $\mathbb{Q}_{odd}$ for the following reason: consider an a^{th} root of -1, ω_a, as in the set $\mathcal{A}$ above. Then the argument of ω_a has the form $\frac{\pi}{a} + \frac{2\pi j}{a} = \frac{(2j+1)\pi}{a}$, which is precisely π times an element of $\mathbb{Q}_{odd}$. Conversely, if $h = \frac{2j+1}{a} \in \mathbb{Q}_{odd}$, then πh is the argument of an a^{th} root of -1 since

$$\left(e^{i\pi h}\right)^a = e^{i\pi(2j+1)}$$
$$= e^{i\pi}$$
$$= -1.$$

To prove our claim, we shall show that between any two real numbers there exists an element of $\mathbb{Q}_{odd}$. Let $x, y \in \mathbb{R}$ such that $x < y$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational number $\frac{n}{m}$ in reduced form such that $x < \frac{n}{m} < y$. If n is odd, then we are done. Suppose n is even, so $n = 2k$. Then m must be odd (otherwise $\frac{n}{m}$ would not be reduced). Then observe $0 < y - \frac{2k}{m}$. As the sequence $\{\frac{1}{t}\}_{t\to\infty}$ converges to zero, there exists $t \in \mathbb{N}$ such that $0 < \frac{1}{t} < y - \frac{2k}{m}$, implying $x < \frac{2k}{m} < \frac{2k}{m} + \frac{1}{t} < y \implies x < \frac{2kt+m}{mt} < y$. But since m is odd, it follows that $\frac{2kt+m}{mt} \in \mathbb{Q}_{odd}$. Hence we have shown there is an odd-numerator rational number between any two real numbers, which means $\mathbb{Q}_{odd}$ is dense in $\mathbb{R}$.

Now we may continue with the proof that degree roots are dense in $\mathbb{C}$. Let $z = re^{i\theta} \in \mathbb{C}$, $\theta \in [0, 2\pi)$. We will show there are degree roots arbitrarily close to z. Let $\epsilon > 0$ be given. We may assume that $r > 0$, as we have already seen that 0 is a degree root (so of course all balls around 0 contain a degree root). We can also assume $\epsilon < r$. Choose $r' \in \mathbb{Q}$ so that $|r - r'| < \epsilon$, which is possible since $\mathbb{Q}$ is dense in $\mathbb{R}$. Furthermore, let $h \in \mathbb{Q}_{odd}$ be such that $|\theta/\pi - h| < \epsilon/\pi$, which can be done since $\mathbb{Q}_{odd}$ is dense in $\mathbb{R}$ by our argument above. Notice that $\theta' = \pi h$ has the form $\pi(2j+1)/a$, for some a and j. Hence, $|\theta - \theta'| < \epsilon$ and θ' is the argument of an a^{th} root of -1 (as shown above). Therefore, $z' = r'e^{i\theta'} \in \mathcal{A}$ has modulus and argument within ϵ of the modulus and argument of z. Thus we can find degree roots belonging to the set $\mathcal{A}$ arbitrarily close to z, implying $\mathcal{A}$, and consequently the set of all degree roots, is dense in $\mathbb{C}$.

For the case of density in $(-\infty, 0]$, the complex number z would have argument $\theta = \pi$. Therefore, to ensure z' is within ϵ of z we need only to set $h = 1$, or $\theta' = \pi$, and choose r' to be within ϵ of r.

For the case density in the imaginary axis $i\mathbb{R}$, we proceed as in the real case except setting $h = 1/2$ or $3/2$ (and hence $\theta' = \pi/2$ or $3\pi/2$), depending on if z is in the upper or lower half-plane (as these would also be the values of θ). □

3.3 Degree Roots and Graph Parameters

In this section we investigate how restraining certain graph parameters impacts degree roots, or how degree roots may depend on certain graph parameters. So far we have seen that, as a whole, degree roots for all graphs are the same as roots for polynomials with non-negative integer coefficients, and that these roots are dense in $\mathbb{C}$, $(-\infty, 0]$, and $i\mathbb{R}$. We wish to see how these facts may change with the addition of graph theoretic constraints, and discover any dependence of degree roots on graph parameters.

3.3.1 Order of Graphs

The simplest of graph parameters is the order, or the number of vertices, n. If n is fixed, there are only finitely many graphs and thus only finitely many degree roots which are, of course, bounded. Furthermore, the degrees are naturally bounded since no degree may exceed $n-1$. As early as Figure 3.1 the influence of n on degree roots could be seen, as the roots were restricted to a finite region that grew as n increased. We look to better understand this influence and any other dependence of degree roots on n. Let us start with the following simple result about when -1 is a degree root.

Proposition 3.7. *If -1 is a degree root of a graph G with order n, then n is a multiple of four.*

Proof. Recall our observation from Section 2.2 that -1 is a degree root of G if and only if G has an equal number of even and odd degree vertices. Suppose there are k of each, so that $n = 2k$. But we also know that G must have an even number of odd degree vertices, hence $k = 2t$ for some t. Therefore, $n = 4t$ and we are done. □

Conversely, if $n = 4t$ there exists a graph for which -1 is a degree root. One such graph is formed by taking the disjoint union of t copies of P_4, which has degree polynomial $2x^2 + 2x$, though this is not the only construction.

Let us now see how restricting n creates a difference between degree roots for (simple) graphs, multigraphs, and the roots of polynomials with non-negative integer coefficients. In Figure 3.4, we show a comparison of these roots for $n = 4$. The left plot shows degree roots for (simple) graphs for $n = 4$, the middle shows degree roots of multigraphs for $n = 4$ and a maximum edge bundle size of 3, and the rightmost plot shows the roots of all polynomials of $\mathbb{Z}_{\geq 0}[x]$ with sum of coefficients equal to $n = 4$ and degree at most 9. Degree polynomials for multigraphs were calculated from their adjacency matrices found with *nauty* [30].

While the roots in each plot are distributed in similar shapes, these plots are strikingly different. The main reason for this is that each plot has a visibly different number of roots. Unsurprisingly, degree roots for (simple) graphs has the least dense plot: we are looking at roots to polynomials of $\mathbb{Z}_{\geq 0}[x]$ having a sum of coefficients equal to 4, degree 3, and also satisfying restraints of being degree-graphic. Having more roots, the plot for multigraphs shows roots for polynomials in $\mathbb{Z}_{\geq 0}[x]$ with sum of coefficients equal to 4, degree at most 9 (a vertex has at most 3 neighbours, each of which having an edge bundle of at most 3 edges), and satisfying weaker degree-graphic restraints. Removing these weaker graph theoretic restraints produces the final plot.

Our focus from here on is (simple) graphs. We will now look toward bounding degree roots in terms of n, since there are only finitely many of them. To find a modulus bound on degree roots of graphs with order n, the following lemma will be useful.

Lemma 3.8. *Consider the weighted sum* $B = \sum_{i=1}^{N} b_i y_i$ *where* $y_1 \geq y_2 \geq \ldots \geq y_N$ *are real numbers and each* b_i *is a non-negative real number with* $\sum_{i=1}^{N} b_i = k$*. Then* $B \leq k y_1$*.*

Proof. Rewrite B in the following way:

$$\begin{aligned} B &= \sum_{i=1}^{N} b_i y_i \\ &= (k - b_2 - \cdots - b_N) y_1 + b_2 y_2 + \cdots + b_N y_N \\ &= k y_1 + b_2 (y_2 - y_1) + \cdots + b_N (y_N - y_1). \end{aligned}$$

Since $y_i - y_1 \leq 0$ for each i, and the b_i are non-negative, it is immediately seen that $B \leq k y_1$. □

We recall Cauchy's bound (Theorem 1.3) on the modulus of roots of a polynomial with complex coefficients: every root of a polynomial $g(x) = c_n x^n + \cdots + c_1 x + c_0$ of

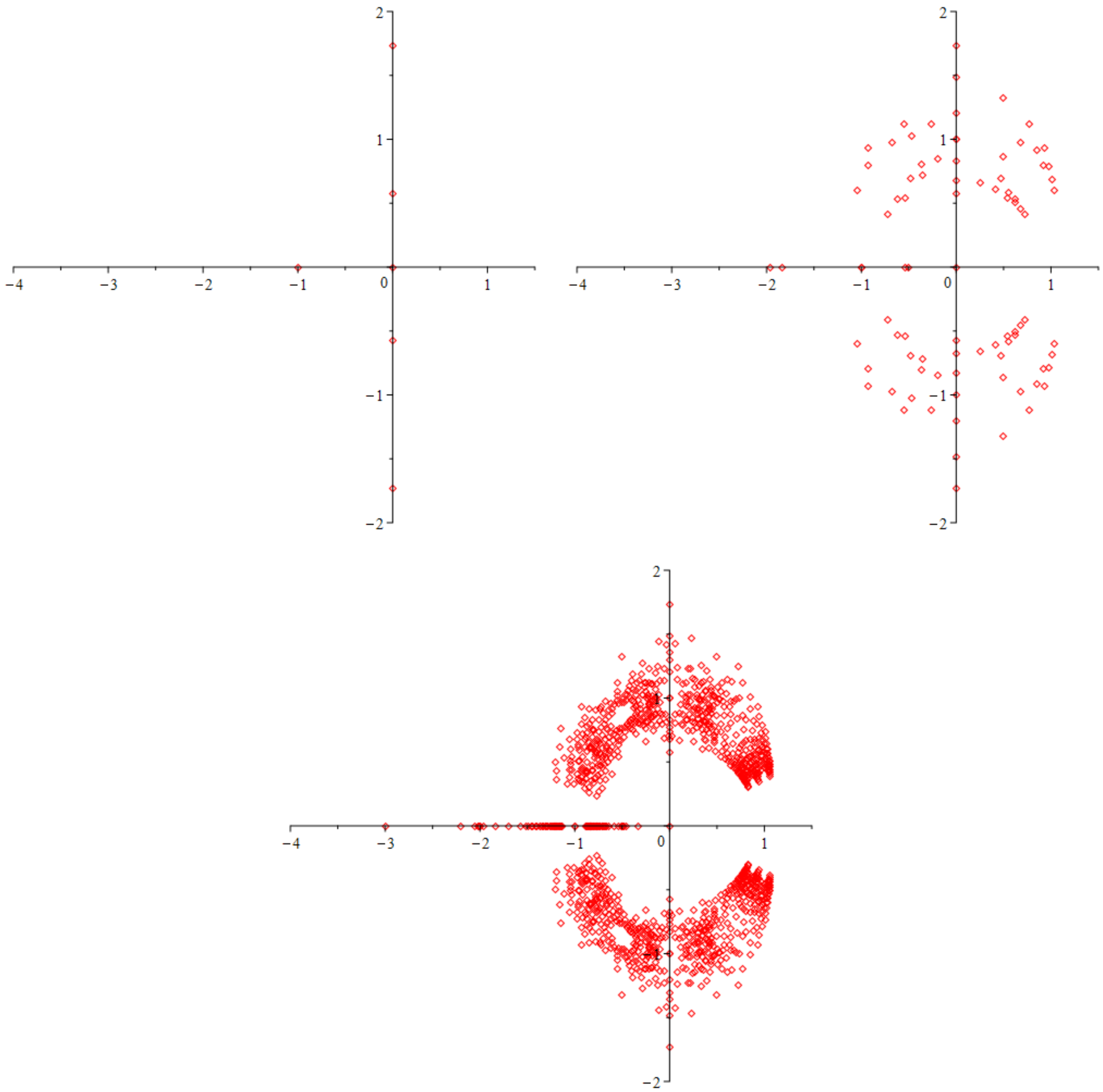


Figure 3.4: Top left: degree roots of (simple) graphs for $n = 4$. Top right: degree roots of multigraphs of order $n = 4$, allowing a maximum edge bundle size of 3. Bottom: roots of polynomials belonging to $\mathbb{Z}_{\geq 0}[x]$ whose sum of coefficients is 4, and whose degrees are at most 9.

degree $n \geq 1$ with $c_0 \neq 0$ has modulus at most r, where r is the unique positive root of the real polynomial $h(x) = |c_n|x^n - \cdots - |c_1|x - |c_0|$. We are now able to prove a modulus and argument bound on the roots of polynomials with non-negative integer coefficients.

Proposition 3.9. *Let* $p(x) = a_\Delta x^\Delta + \cdots + a_\delta x^\delta \in \mathbb{Z}_{\geq 0}[x]$ *where* $\Delta > \delta$ *and with* $a_\Delta, a_\delta \geq 1$. *Suppose that* $p(1) = a_\Delta + \cdots + a_\delta = n$. *If* z *is a root of* $p(x)$, *then*

$$|z| \leq \max\left\{\frac{n - a_\Delta}{a_\Delta}, \frac{a_\Delta}{n - a_\Delta}\right\}.$$

Proof. Let $c = \max\left\{\frac{n-a_\Delta}{a_\Delta}, \frac{a_\Delta}{n-a_\Delta}\right\}$, and observe that c takes on the value of whichever of $(n - a_\Delta)/a_\Delta$, $a_\Delta/(n - a_\Delta)$ is greater than or equal to one. Hence in any case $c \geq 1$. By Theorem 1.3 we have $|z| \leq R$, where R is the positive root of

$$h(x) = a_\Delta x^{\Delta-\delta} - a_{\Delta-1}x^{\Delta-\delta-1} - \cdots - a_\delta.$$

Notice that $h(0) = -a_\delta < 0$. We now consider two cases:

Case 1: $c = (n - a_\Delta)/a_\Delta$.

Observe

$$\begin{aligned} \frac{h(c)}{c^{\Delta-\delta}} &= a_\Delta - \frac{a_{\Delta-1}}{c} - \cdots - \frac{a_\delta}{c^{\Delta-\delta}} \\ &= a_\Delta - S, \end{aligned}$$

where $S = \sum_{j=1}^{\Delta-\delta} a_{\Delta-j}/c^j$. Since $c \geq 1$, we have $1/c \geq 1/c^2 \geq \cdots \geq 1/c^{\Delta-\delta}$. By Lemma 3.8 where we have $y_i = 1/c^i$,

$$S \leq \frac{n - a_\Delta}{c} = a_\Delta.$$

Therefore $h(c) = a_\Delta - S \geq 0$, and since $h(0) < 0$, by the IVT and Theorem 1.3, we have $R \in (0, c]$.

Case 2: $c = a_\Delta/(n - a_\Delta)$.

We proceed in the same manner as in Case 1, and find

$$S \leq \frac{n - a_\Delta}{c} = \frac{(n - a_\Delta)^2}{a_\Delta}.$$

However,

$$\begin{aligned} \frac{(n-a_\Delta)^2}{a_\Delta} &\leq a_\Delta \\ \iff n^2 - 2na_\Delta &\leq 0 \\ \iff \frac{n}{2} &\leq a_\Delta. \end{aligned}$$

Since we have $c = a_\Delta/(n - a_\Delta) \geq 1$, the last inequality is indeed true. Thus $S \leq a_\Delta$ and again we have $h(c) \geq 0$, so $R \in (0, c]$.

In any case on c, we have found $R \leq c$. Therefore by Theorem 1.3 $|z| \leq c$, as desired. □

Equality is met for this bound when $p(x) = a_\Delta x^\Delta + a_{\Delta-1}x^{\Delta-1}$. Thus we can use this bound to give another for the degree roots of graphs with fixed order. Recall that in Figure 3.1 we observed that the degree roots for graphs of fixed order n seemed to never exceed a modulus of $n-1$, and roots that had such a modulus were real. This modulus bound is in fact true, as stated in the following Corollary which is due to considering the extreme of Proposition 3.9. For simplicity, we let $\mathcal{S}_n$ be the family of all (simple) graphs of order n.

Corollary 3.10. *If $z \in Z(\mathcal{D}(\mathcal{S}_n))$ is non-zero, then*

$$\frac{1}{n-1} \leq |z| \leq n-1.$$

Proof. The upper bound follows directly from Proposition 3.9, when $a_\Delta = 1$ or $n-1$. The lower bound follows since $1/z$ also belongs to $Z(\mathcal{D}(\mathcal{S}_n))$ (as a degree root to the graph complement of a graph for which z was a degree root). Thus $1/|z| \leq n-1$, and the lower bound is obtained from rearranging. □

Let us now consider the extreme of Corollary 3.10, when there is a degree root of modulus $n-1$ (considering graphs of order n). Figure 3.1 seems to suggest that the only roots of such modulus are real, and only appear when n is odd. The following propositions address these observations for $n \geq 4$, since all degree roots for $n = 2$ or $n = 3$ are already real.

Proposition 3.11. *Let G be a graph of order $n \geq 4$, and suppose $D(G;x)$ has a degree root z, where $|z| = n-1$. Then $D(G;x)$ has the form $D(G;x) = x^{\Delta} + (n-1)x^{\Delta-1}$, and in particular $z = -(n-1)$.*

Proof. Let $D(G;x) = a_{\Delta}x^{\Delta} + \cdots + a_{\delta}x^{\delta}$ ($a_{\Delta} > 0$) be the degree polynomial of G which has the root z of modulus $n-1$. Recall Cauchy's theorem (Theorem 1.4) which states that all roots of $D(G;x)$ have modulus strictly less than

$$1 + \max_{k \neq \Delta}\left\{\left|\frac{a_k}{a_{\Delta}}\right|\right\} = 1 + \frac{\max_{k\neq\Delta}\{a_k\}}{a_{\Delta}}.$$

Since this applies to the root z with modulus $n-1$, we must have

$$n - 2 < \frac{\max_{k\neq\Delta}\{a_k\}}{a_{\Delta}}.$$

This inequality is only satisfied when $a_{\Delta} = 1$ and $\max_{k\neq\Delta}\{a_k\} = n-1$. If $\max_{k\neq\Delta}\{a_k\}$ were any smaller value, the RHS of the above inequality would be at most $n-2$ and thus gives a contradiction. We must also ensure that the sum of the a_k's is equal to n, so no single coefficient may be greater than $n-1$ (if $a_k = n$ for some k, then all other coefficients are zero and therefore $D(G;x)$ is a monomial, having no non-zero roots). Thus we have $a_{\Delta} = 1$, $a_k = n-1$ for some $k < \Delta$, and all other coefficients are zero. This gives $D(G;x)$ the form $D(G;x) = x^{\Delta} + (n-1)x^{k}$.

Furthermore, since $D(G;z) = 0$ we must have $z^{\Delta-k} = -(n-1)$. As $|z| = n-1$, it follows that $\Delta - k = 1$, or $k = \Delta - 1$. Thus $D(G;x) = x^{\Delta} + (n-1)x^{\Delta-1}$, and we also conclude that $z = -(n-1)$. □

Polynomials of the form $x^{\Delta} + (n-1)x^{\Delta-1}$ are not degree-graphic for all values of n and Δ, however. The next proposition tells us precisely when they are.

Proposition 3.12. *A polynomial of the form $x^{\Delta} + (n-1)x^{\Delta-1}$ with $n \geq 4$, $\Delta \leq n-1$, is degree-graphic if and only if n is odd and Δ is even.*

Proof. ($\Longrightarrow$) We first prove the forward direction. Since $x^{\Delta} + (n-1)x^{\Delta-1}$ is degree-graphic, we know that the sum of the degrees must be even (the degree-sum formula). Hence, $\Delta + (n-1)(\Delta-1)$ is even, implying Δ and $(n-1)(\Delta-1)$ are of the same parity. If both are odd, then $n-1$ and $\Delta-1$ must also be odd, so that their product is odd. But here we have a contradiction since Δ and $\Delta-1$ are odd. In the case where Δ and $(n-1)(\Delta-1)$ are both even, we know $\Delta - 1$ is odd, and hence $n-1$ must be even (to ensure $(n-1)(\Delta-1)$ is even). In other words, n is odd. Therefore, we conclude that Δ is even and n is odd.

($\Longleftarrow$) Suppose n is odd, so $n = 2k+1$ for some $k \geq 2$, and also that Δ is even, so $\Delta = 2d$ for some $1 \leq d \leq k$. The potential degree sequence from the polynomial $x^{\Delta} + (n-1)x^{\Delta-1} = x^{2d} + 2kx^{2d-1}$ is $2d, 2d-1, ..., 2d-1$. We will show this sequence is graphic using the Erdös-Gallai Theorem. First, observe that the sum of the sequence is $2d + 2k(2d-1) = 2(d + k(2d-1))$, which is even $(2d-1 \geq 1)$. Thus the first condition is satisfied. The second condition may be expressed as

$$2d + (j-1)(2d-1) \leq j(j-1) + (2k+1-j) \cdot \min(2d-1, j),$$

which simplifies to

$$2dj + 1 \leq j^2 + (2k+1-j) \cdot \min(2d-1, j)$$

for each $j \in 1, ..., 2k+1$. We show this condition holds by considering three cases on j, in terms of d.

Case 1: $j \leq 2d-1$. Then

$$\begin{aligned} & 2dj{+}1 \leq j^2 + (2k+1-j) \cdot \min(2d-1, j) \\ \Longleftrightarrow\ & 2dj{+}1 \leq 2kj + j \\ \Longleftrightarrow\ & \qquad 1 \leq j(2(k-d)+1), \end{aligned}$$

which is indeed true since $k - d \geq 0$ and $j \geq 1$.

Case 2: $j = 2d$. In this case the second condition becomes

$$\begin{aligned} & 2d(2d){+}1 \leq (2d)^2 + (2k+1-2d)(2d-1) \\ \Longleftrightarrow\ & \qquad 1 \leq (2k+1-2d)(2d-1), \end{aligned}$$

which is true since $2d-1 \geq 1$, and $d \leq k$ so $2k - 2d + 1 \geq 1$.

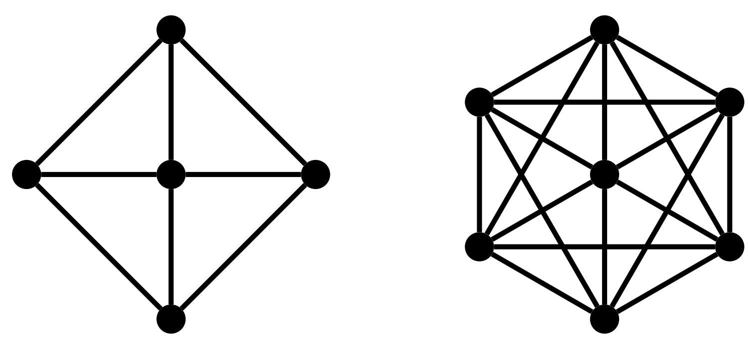
Case 3: $j \geq 2d+1$. The second condition is then

$$2dj{+}1 \leq j^2 + (2k+1-j)\cdot\min(2d-1, j)$$
$$\iff \qquad 1 \leq j^2 - 2dj + (2k+1-j)(2d-1).$$

But $j^2 - 2dj \geq j \geq 1$, $2k+1-j \geq 0$, and $2d-1 \geq 1$, so the final inequality is true. This exhausts all cases for j, and we are done. □

Propositions 3.11 and 3.12 confirm our observations of Figure 3.1 previously mentioned. In the following example we show some types of graphs that have a degree root at $-(n-1)$, for small values of n.

Example 3.2. Let $n = 2k+1$, $k \geq 1$. One construction of graphs having a degree root at $-(n-1)$ is the following: remove a perfect matching from K_{2k}, creating a graph with degree polynomial $2kx^{2k-2}$, and then add a universal vertex. The resulting graph has degree polynomial $x^{2k}+2kx^{2k-1}$, with a root at $-2k = -(n-1)$. See Figure 3.5 for some of these graphs.



$x^2 + 2x$ $\qquad$ $x^4 + 4x^3$ $\qquad$ $x^6 + 6x^5$

Figure 3.5: Three examples of graphs formed from adding a universal vertex to K_{2k} with a perfect matching removed. These graphs have degree roots at $-(n-1)$, where $n = 2k+1$.

There is also the following family of disconnected graphs: take the disjoint union of P_3 with $(n-3)/2$ copies of P_2 (see Figure 3.6). These graphs have degree polynomial $x^2 + 2x + 2x(n-3)/2 = x^2 + (n-1)x$, again with a root at $-(n-1)$.

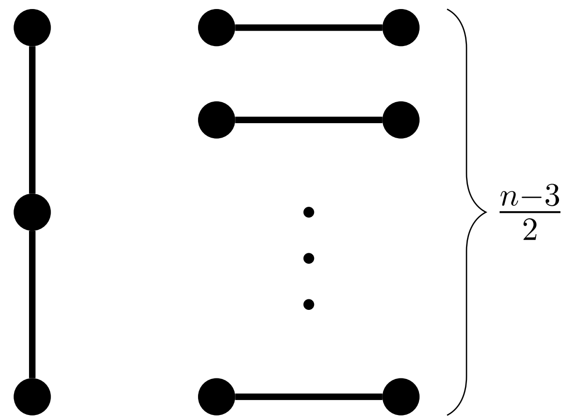


Figure 3.6: Construction of disconnected graph with root at $-(n-1)$, made by taking the union of P_3 and copies of P_2.

In addition to these constructions, when $n = 7$ there are four distinct graphs with degree polynomial x^4+6x^3 (see Figure 3.7), and when $n = 9$ there are 28 graphs with degree polynomial $x^4 + 8x^3$ and 20 graphs with polynomial $x^6 + 8x^5$ (see Figure 3.8 for examples). All of these graphs have a real degree root at $-(n-1)$.

We have seen there are real degree roots for graphs of order n with modulus as large as $n-1$. Now we look toward the imaginary axis. It was observed in Figure 3.1 that purely imaginary roots seemed to never exceed a modulus of $\sqrt{n}$. The next proposition confirms and improves on this observation.

Proposition 3.13. *Let $z = ir$ $(r \in \mathbb{R})$ be a purely imaginary root of a degree polynomial $D(G;x)$, where G has order n. Then $|z| \leq \sqrt{n-1}$.*

Proof. Let us write $D(G;x) = a_\Delta x^\Delta + \cdots + a_\delta x^\delta$. $D(G;ir) = 0$ can be written as

$$i^\Delta \left(a_\Delta r^\Delta - a_{\Delta-2}r^{\Delta-2} + \cdots\right) + i^{\Delta-1}\left(a_{\Delta-1}r^{\Delta-1} - a_{\Delta-3}r^{\Delta-3} + \cdots\right) = 0,$$

or simply $i^\Delta A + i^{\Delta-1}B = 0$. Therefore, both A and B must be equal to zero. Since $a_\Delta \geq 1$, there must be another coefficient (a_k, for some k) in A that is non-zero. Let us now consider two cases on the parity of Δ.

Case 1: $\Delta = 2k$. In this case, we may write $A = 0$ as

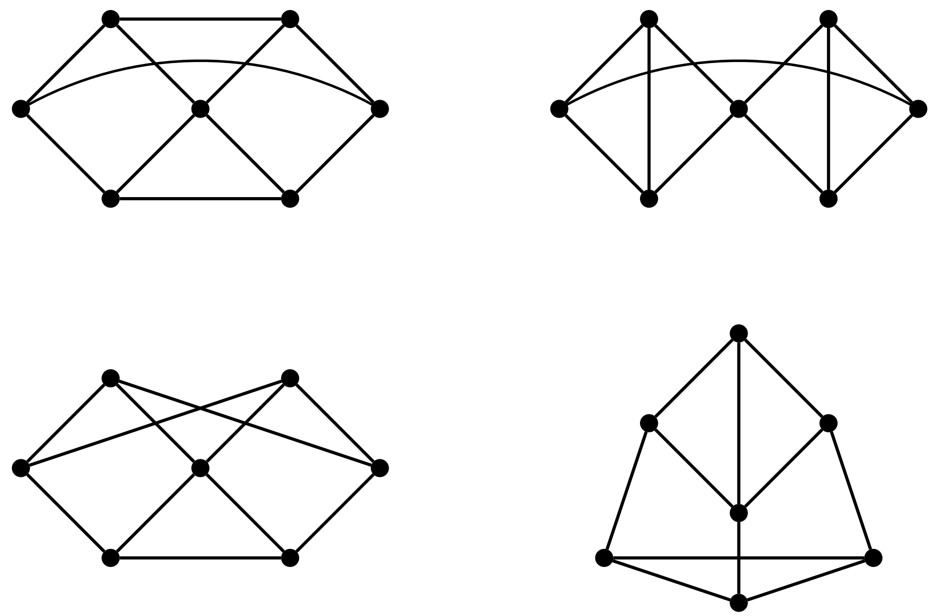


Figure 3.7: The four graphs of order $n = 7$ with degree polynomial $x^4 + 6x^3$.

$$a_{2k}r^{2k} - a_{2k-2}r^{2k-2} + \cdots = 0.$$

Setting $s = r^2$ we have

$$a_{2k}s^{k} - a_{2k-2}s^{k-1} + \cdots = 0,$$

and thus $-s$ is a root of $f(x) = a_{2k}x^k + a_{2k-2}x^{k-1} + \cdots$. Since $f(x)$ has only non-negative integer coefficients, we apply Proposition 3.9: $f(1) \leq n$ and $1 \leq a_{2k} \leq n-1$, so

$$\begin{aligned} |s| &= |-s| \\ &\leq \max\left\{\frac{n-a_{2k}}{a_{2k}}, \frac{a_{2k}}{n-a_{2k}}\right\} \\ &\leq n-1. \end{aligned}$$

Therefore, $|z| = |r| \leq \sqrt{n-1}$.

Case 2: $\Delta - 2k + 1$. In this case, we may write $A = 0$ as

$$a_{2k+1}r^{2k+1} - a_{2k-1}r^{2k-1} + \cdots = 0.$$

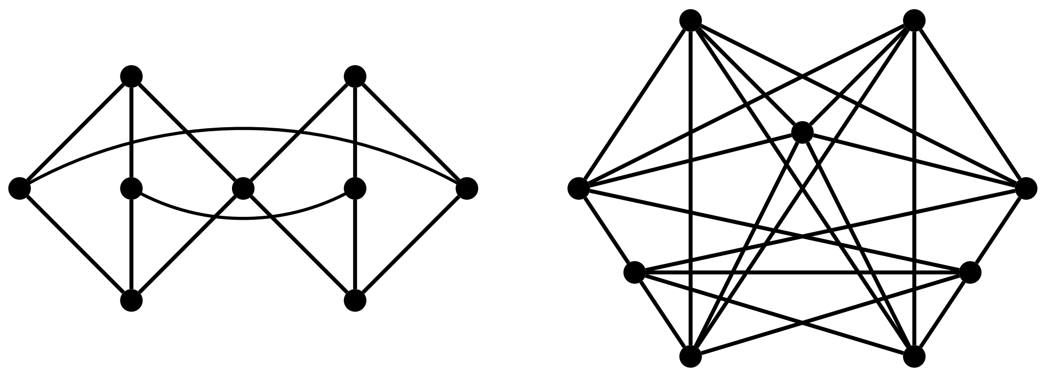


Figure 3.8: Left: one of the 28 graphs of order $n = 9$ with degree polynomial $x^4 + 8x^3$. Right: one of the 20 graphs of order $n = 9$ with degree polynomial $x^6 + 8x^5$.

Dividing by r and again setting $s = r^2$ we have

$$a_{2k+1}s^k - a_{2k-1}s^{k-1} + \cdots = 0,$$

so $-s$ is a root of $g(x) = a_{2k+1}x^k + a_{2k-1}x^{k-1} + \cdots$. As above, we can apply Proposition 3.9 to obtain $|z| \leq \sqrt{n-1}$.

Therefore, in any case, we have $|z| \leq \sqrt{n-1}$. Note that it is possible that B is equal to zero because each of its coefficients c_k are zero. If B has some non-zero coefficients, we may proceed as above to reach the same conclusion. The only difference in this instance is that $a_{\Delta-1}$ may not be the leading coefficient of B, so we would simply write B starting from its leading coefficient. □

The following example describes some graphs that have imaginary roots with largest possible modulus, that is, with modulus $\sqrt{n-1}$.

Example 3.3. For $n \leq 3$, all degree roots are real. Starting at $n = 4$, we begin to see purely imaginary degree roots with largest possible modulus: the complete bipartite graph $K_{1,3}$ has degree polynomial $D(K_{1,3}; x) = x^3 + 3x$. Thus its degree roots are 0 and $\pm i\sqrt{3}$. For $n = 5$, consider the graph formed by intersecting two copies of K_3 on a single vertex. This graph has degree polynomial $x^4 + 4x^2$, and thus has degree roots 0 (with multiplicity two) and $\pm 2i$. For $n = 6$, make a graph by adding a universal vertex to C_5. The resulting graph has degree polynomial $x^5 + 5x^3$, having degree roots at 0 (with multiplicity three) and $\pm i\sqrt{5}$. Each of these graphs, shown in Figure 3.9, share a common construction. To make a graph of order $n \geq 4$ with degree roots at

$\pm i\sqrt{n-1}$, begin with an $(n-4)$-regular graph on $n-1$ vertices (this can always be done since $n-4 \leq n-1$ and $(n-4)(n-1)$ is always even). Then, add a universal vertex. The graph resulting from this process has a single vertex of degree $n-1$, and $n-1$ vertices of degree $n-3$. Therefore its degree polynomial is $x^{n-1} + (n-1)x^{n-3}$, having roots at $\pm i\sqrt{n-1}$.

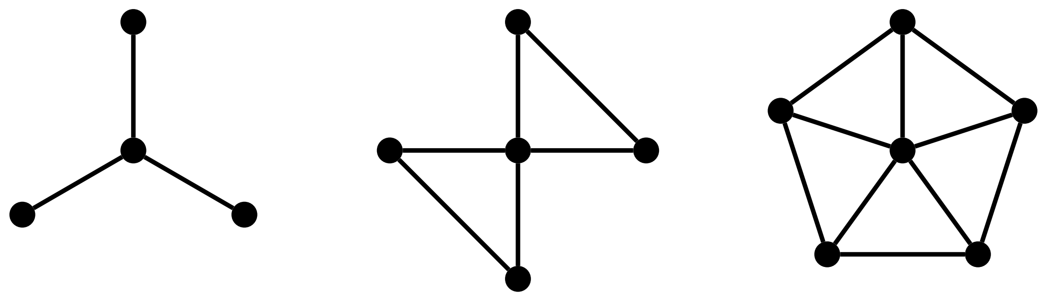


Figure 3.9: Three graphs having imaginary degree roots with largest possible modulus.

Another way to form graphs with degree roots at $\pm i\sqrt{n-1}$ is the following, which works for $n \geq 5$. Take two disjoint cycles C_s and C_t, and intersect them on a single vertex. The resulting graph, as shown in Figure 3.10, will have $n = s + t - 1$ vertices ($n \geq 5$ since $s, t \geq 3$), one of which has degree 4 while the remaining vertices have degree 2. Hence, it has degree polynomial $x^4 + (n-1)x^2$ and has the desired degree roots. Notice that when $s = t = 3$ this graph is the same as the graph of order 5 from the previous construction (Figure 3.9, middle).

Each of the graphs described above had a degree polynomial of a particular form: $x^{\Delta} + (n-1)x^{\Delta-2}$. This is, in fact, the only form such degree polynomials can have, as shown in the following proposition.

Proposition 3.14. *Let G be a graph of order n that has an imaginary degree root with modulus $\sqrt{n-1}$. Then $D(G;x) = x^{\Delta} + (n-1)x^{\Delta-2}$, for some Δ.*

Proof. Suppose $D(G;x) = a_{\Delta}x^{\Delta} + \cdots + a_{\delta}x^{\delta}$, so $a_{\Delta}, a_{\delta} \geq 1$. Since there is a root at $i\sqrt{n-1}$, there is also a root at $-i\sqrt{n-1}$. Thus we can factor $D(G;x)$ as:

$$a_{\Delta}x^{\Delta} + \cdots + a_{\delta}x^{\delta} = (x^2 + (n-1))(b_{\Delta-2}x^{\Delta-2} + \cdots + b_{\delta}x^{\delta}).$$

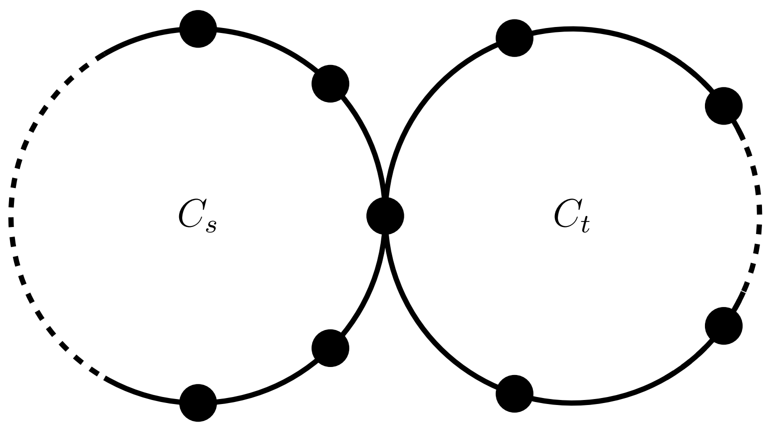


Figure 3.10: Two cycles C_s and C_t intersecting on a single vertex. Setting $n = s+t-1$, this graph has degree polynomial $x^4 + (n-1)x^2$.

Expanding the product on the right and equating coefficients, we obtain the following relations:

$$\begin{aligned} b_{\Delta-2} &= a_{\Delta}, \\ b_{\Delta-3} &= a_{\Delta-1}, \\ b_{\Delta-k-2} + (n-1)b_{\Delta-k} &= a_{\Delta-k}, \quad 2 \le k \le \Delta - \delta - 2, \\ (n-1)b_{\delta+1} &= a_{\delta+1}, \\ (n-1)b_{\delta} &= a_{\delta}. \end{aligned}$$

The first two of these relations tell us $b_{\Delta-2}, b_{\Delta-3} \in \mathbb{Z}_{\geq 0}$, and $b_{\Delta-2} \geq 1$. Since we have $b_{\Delta-4} + (n-1)b_{\Delta-2} = a_{\Delta-2}$, it follows that $b_{\Delta-4} \in \mathbb{Z}$. Similarly, $b_{\Delta-5} \in \mathbb{Z}$ as $b_{\Delta-5} + (n-1)b_{\Delta-3} = a_{\Delta-3}$. This pattern continues for decreasing indices, so from $b_{\delta} + (n-1)b_{\delta+2} = a_{\delta+2}$ we can conclude $b_{\delta} \in \mathbb{Z}$. The last relation gives us the inequality $1 \leq (n-1)b_{\delta} \leq n-1$, and therefore $b_{\delta} = 1$. Hence, $a_{\delta} = n-1$, and the only other non-zero coefficient of $D(G;x)$ is $a_{\Delta} = 1$ (as the coefficients must sum to n). Thus $D(G;x) = x^{\Delta} + (n-1)x^{\delta}$, and since there are roots at $\pm i\sqrt{n-1}$, it must be that $\delta = \Delta - 2$. $\square$

3.3.2 Maximum Degree of Graphs

Another graph parameter we may focus on is the maximum degree, or Δ. Fixing Δ does not limit the number of graphs to a finite amount (for example, even for $\Delta = 2$ there are infinitely many path graphs), nor bound degree roots: the graphs that are a disjoint union of C_4 with k copies of P_2 all have $\Delta = 2$, yet their degree polynomials are $4x^2 + 2kx$ which have a root at $-k/2$ that becomes arbitrarily large in absolute value as k increases.

Let us begin with a lemma that will allow us to find a bound on the argument of roots of non-negative real coefficient polynomials, and consequently degree roots.

Lemma 3.15 ([29]). *Suppose that w_j, $1 \leq j \leq p$ are non-zero complex numbers such that $\gamma \leq arg\,(w_j) < \gamma + \pi$ for some real constant γ. Then $\sum_1^p w_j \neq 0$.*

The next result is stated in [16] as being true for polynomials with strictly positive coefficients. Here, we allow for coefficients to simply be non-negative and give a simple proof.

Lemma 3.16. *Let $p(x) = b_d x^d + \cdots + b_1 x + b_0$ be a polynomial with non-negative, real coefficients, and without loss of generality $b_d, b_0 \neq 0$. If z is a root of $p(x)$, then*

$$|arg\,(z)| \geq \frac{\pi}{d}.$$

Proof. Let $z = re^{i\theta}$, $r > 0$, $\theta \in (-\pi, \pi]$. Clearly $\theta \neq 0$, since $p(x)$ has non-negative coefficients. It suffices to prove the result for $0 < \theta \leq \pi$, due to the symmetry $p(z) = 0 \iff p(\overline{z}) = 0$, and $\arg(\overline{z}) = -\theta$. For all j such that $b_j \neq 0$, let $w_j = b_j r^j e^{i\theta j}$. Then $p(z) = p(re^{i\theta}) = \sum w_j = 0$. Toward a contradiction, suppose that $\theta < \pi/d$, so then $0 < \theta d < \pi$. But $\arg(w_j) = \theta j$, so for all j such that $b_j \neq 0$, we have

$$0 \leq \arg(w_j) \leq \theta d < \pi.$$

Thus by Lemma 3.15, $\sum w_j \neq 0$, which is in contradiction to $p(z) = 0$. Therefore $\theta \geq \pi/d$, and we are done. ∎

As a corollary, we have a lower bound to the argument of non-zero degree roots.

Corollary 3.17. *Let G be a graph with degree polynomial $D(G;x) = a_\Delta x^\Delta + \cdots + a_\delta x^\delta$ $(\Delta > \delta)$. If z is a non-zero degree root of G, then*

$$|\arg(z)| \geq \frac{\pi}{\Delta - \delta}.$$

Proof. This follows directly from applying Lemma 3.16 to $D(G;x)/x^\delta$. □

Furthermore, we can give a lower bound to the arguments of all non-zero degree roots for graphs of order n.

Corollary 3.18. *If $z \in Z(\mathcal{D}(\mathcal{S}_n))$ is non-zero, then*

$$|\arg(z)| \geq \frac{\pi}{n-2}.$$

Proof. This follows directly from Corollary 3.17, observing that $\Delta - \delta \leq n-2$ since G cannot have both a universal vertex (which has degree $n-1$) and an isolated vertex. □

We can prove a simple result on degree root density for certain families of graphs. We can show that not all families of graphs have degree roots that are dense in $\mathbb{C}$, as a necessary condition is that the maximum degrees of graphs in a family is unbounded.

Corollary 3.19. *Suppose $\mathcal{F}$ is a family of graphs. If $\sup\{\Delta(G) : G \in \mathcal{F}\} < \infty$, then $Z(\mathcal{D}(\mathcal{F}))$ is not dense in $\mathbb{C}$.*

Proof. Let $M = \sup\{\Delta(G) : G \in \mathcal{F}\}$, so that M is finite. By Lemma 3.16, any non-zero $z \in Z(\mathcal{D}(\mathcal{F}))$ satisfies $|\arg(z)| \geq \pi/M$. Therefore the region

$$\{z \in \mathbb{C} : |z| > 0, |\arg(z)| < \pi/M\} \subset \mathbb{C}$$

will be free from elements of $Z(\mathcal{D}(\mathcal{F}))$, and thus $Z(\mathcal{D}(\mathcal{F}))$ cannot be dense in $\mathbb{C}$. □

3.4 Degree Roots for Some Families of Graphs

Here we shall focus on some families of graphs for which we can say much about their degree roots. We have so far considered degree polynomials more generally, which leaves many patterns and properties obscured. Dealing with graph families for which we can write more explicit degree polynomials, we expect to see simpler pictures of degree roots with identifiable behaviour and properties.

3.4.1 Trees

By a *tree*, we mean a connected and acyclic graph. While being a large family of graphs, trees have distinctive structure and properties. It is natural, then, to wonder how these properties influence the location of degree roots. The degree roots for trees of orders three through eighteen are shown in Figure 3.11. The roots appear to be filling in the negative real line, extend along near vertical lines, yet do not extend far into the right-half-plane (RHP). We also note the apparent lack of non-real roots inside the unit circle, which is our first point of investigation into the degree roots for trees.

With evidence of a root-free region of the complex plane, we make the following conjecture:

Conjecture 1. *Trees have no non-real roots inside the unit circle. Consequently, the degree roots for trees are not dense in* $\mathbb{C}$*.*

Both a proof or a counterexample to this conjecture have eluded us. However, we are able to provide a partial answer. The Lemma below will be useful.

Lemma 3.20. *Let* T *be a tree with* a_k *vertices of degree* k, $1 \leq k \leq \Delta$*. Then* $a_1 = 2 + \sum_{k=3}^{\Delta}(k-2)a_k$.

Proof. If $n = \sum_{k=1}^{\Delta} a_k$ is the order of T, then the degree-sum formula states the sum of the degrees of T is equal to $2(n-1)$. We arrive at our result through some simple rearranging:

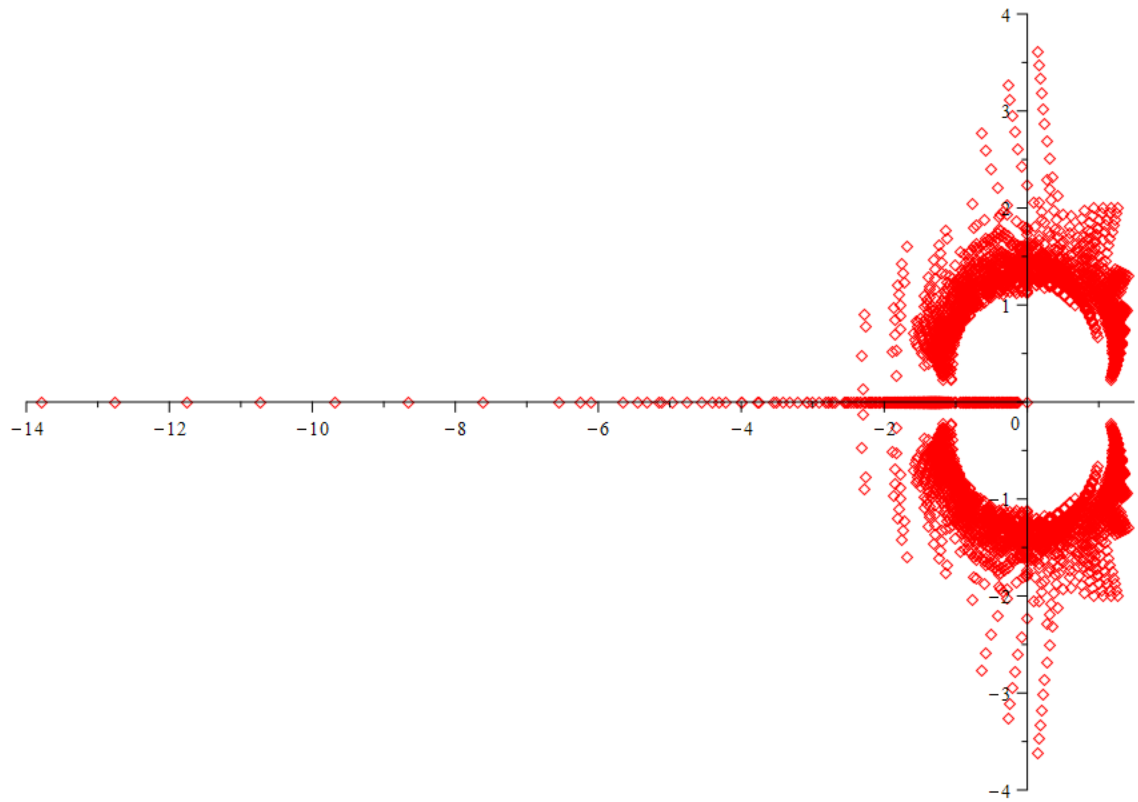


Figure 3.11: Degree roots for trees of orders three through eighteen.

$$\sum_{k=1}^{\Delta} ka_k = 2\left(\sum_{k=1}^{\Delta} a_k - 1\right)$$
$$\sum_{k=2}^{\Delta} ka_k + a_1 = 2\sum_{k=2}^{\Delta} a_k + 2a_1 - 2$$
$$\sum_{k=2}^{\Delta}(k-2)a_k + 2 = a_1$$
$$\sum_{k=3}^{\Delta}(k-2)a_k + 2 = a_1.$$

□

Corollary 3.21. *For a tree T with a_k vertices of degree k, $1 \leq k \leq \Delta$, it follows that* $a_1 > \sum_{k=3}^{\Delta} a_k$.

Lemma 3.20 shows us that we may change the number of vertices of degree two in a tree (in fact, in any graph) without changing the number of vertices of any other degree. Thus for a tree with fixed $a_1, a_3, ..., a_\Delta$, we can allow a_2 to be any value we want. Graphically, this corresponds to subdividing the edges of a tree arbitrarily many times.

The proposition below gives a partial answer to Conjecture 1, stating that trees with sufficiently few or sufficiently many subdivisions have no non-real degree roots inside the unit circle. It is still unknown if this is true for *any* number of subdivisions.

Proposition 3.22. *Let T be a tree with degree polynomial $D(T;x) = \sum_{k=1}^{\Delta} a_k x^k$. If $a_2 < a_1 - \sum_{k=3}^{\Delta} a_k$ or $a_1 + \sum_{k=3}^{\Delta} a_k < a_2$, then $D(T;x)$ has no non-real roots inside the complex unit circle.*

Proof. Our proof will make use of Rouché's Theorem (Theorem 1.2), taking the simple closed Jordan curve to be the unit circle $C = \{z \ : \ |z| = 1\}$. Let us first consider the case when $a_2 < a_1 - \sum_{k=3}^{\Delta} a_k$, or when $\sum_{k=2}^{\Delta} a_k < a_1$. This inequality holds for at least one value of a_2 ($a_2 = 0$) due to Corollary 3.21. Define the polynomials $P(z) = D(T;z) - a_1 z$, and $Q(z) = a_1 z$. Then for $z \in C$, we have

$$\begin{aligned} |P(z)| &= |D(T;z) - a_1 z| \\ &= |\sum_{k=2}^{\Delta} a_k z^k| \\ &\leq \sum_{k=2}^{\Delta} a_k |z|^k \\ &= \sum_{k=2}^{\Delta} a_k. \end{aligned}$$

Furthermore, $|Q(z)| = |a_1 z| = a_1$ for $z \in C$. Thus for $z \in C$, $|P(z)| < |Q(z)|$ and so by Rouché's Theorem, $Q(z)$ and $Q(z) + P(z)$ have the same roots inside the unit circle. But $Q(z)$ has only one root, $z = 0$. Thus $P(z) + Q(z) = D(T;z)$ has exactly one root inside the unit circle. It is easily seen that this root is also $z = 0$.

In the second case, we have $a_1 + \sum_{k=3}^{\Delta} a_k < a_2$. Now define $P(z) = D(T;z) - a_2 z^2$ and $Q(z) = a_2 z^2$. For $z \in C$,

$$
\begin{aligned}
|P(z)| &= |D(T;z) - a_2 z^2| \\
&= |\sum_{k=3}^{\Delta} a_k z^k + a_1 z| \\
&\le \sum_{k=3}^{\Delta} a_k |z|^k + a_1 |z| \\
&= \sum_{k=3}^{\Delta} a_k + a_1,
\end{aligned}
$$

while $|Q(z)| = |a_2 z^2| = a_2$. Therefore $|P(z)| < |Q(z)|$ on C. By Rouché's Theorem, $Q(z) + P(z) = D(T;z)$ has the same number of roots inside the unit circle as $Q(z)$, which is exactly two ($Q(z)$ has two roots at $z = 0$). Since we know $D(T;z)$ has one root at $z = 0$, the other root inside the unit circle must be real. Otherwise, it would have a conjugate root also inside the unit circle. □

Now we turn our attention to the real parts of degree roots for trees, asking if the real parts can be bounded. It is not difficult to show that degree roots for trees are unbounded along the negative real line. Consider trees with $\Delta = 3$, that is, trees which have degree polynomial $D(T;x) = a_3 x^3 + a_2 x^2 + (a_3 + 2)x$ (there are $a_3 + 2$ leaves as a result of Lemma 3.20). The roots of these polynomials are simply

$$
0, \quad \frac{-a_2 \pm \sqrt{a_2^2 - 4a_3(a_3+2)}}{2a_3}.
$$

For sufficiently large a_2, these roots are all real. Since

$$
\begin{aligned}
\lim_{a_2 \to \infty} \left(\frac{-a_2 - \sqrt{a_2^2 - 4a_3(a_3+2)}}{2a_3} \right) &= \lim_{a_2 \to \infty} \frac{-a_2}{a_3} \\
&= -\infty
\end{aligned}
$$

it is clear that these degree roots are unbounded on the negative real line.

While not having positive real roots, there are degree roots for trees with positive real part (as seen in Figure 3.11). Thus we might wonder if there is a bound to the positive real parts of degree roots for trees. To answer this, we make use of the following Lemma.

Lemma 3.23 ([7])**.** *Let $R > 0$ and $f(x) \in \mathbb{R}[x]$ be a polynomial of degree d with positive coefficients. Then*

1. *if $d \geq 4$, then for m sufficiently large $f(x)+mx$ has a root with real part greater than R, and*

2. *if $d \geq 3$, then for l sufficiently large $f(x)+l$ has a root with real part greater than R.*

The original statement of the lemma assumes $f(x)$ has positive coefficients. However, the result still holds if $f(x)$ is only assumed to have non-negative coefficients. Thus we are able to apply Lemma 3.23 to degree polynomials and obtain the following.

Proposition 3.24. *For any $R > 0$, there exists a tree with a degree root having real part greater than R.*

Proof. Let $R > 0$ be given, and fix a tree T with $\Delta \geq 5$. Recall that vertices of degree two can be inserted by subdividing edges without impacting the degrees of any other vertices. Thus if T' is the tree resulting from adding a_2 edge subdivisions of T, we have $D(T';x) = D(T;x) + a_2x^2$. Define

$$\begin{aligned} g(x) &= \frac{D(T';x)}{x} \\ &= \frac{D(T;x)}{x} + a_2x, \end{aligned}$$

which is a polynomial since the lowest degree term of $D(T;x)$ has exponent equal to one. Since $\deg(D(T;x)/x) \geq 4$, from Lemma 3.23 it follows that for sufficiently large a_2, $g(x)$ and thus $D(T';x)$ has a root with real part greater than R. □

The above proposition used trees with $\Delta \geq 5$ to find roots with arbitrarily large real part. Plotting the degree roots for trees up to order $n = 18$ that have $\Delta \leq 4$, we observe that they do not extend far into the RHP (see Figure 3.12). In fact, the real parts of degree roots for trees with $\Delta \leq 4$ do not exceed a value of 1 (and therefore are bounded to the right) as we shall now argue. For $\Delta = 1$, the only tree is the path P_2 with degree polynomial $2x$, and thus the only degree root is at 0. For $\Delta = 2$, the only trees are paths P_n, for $n \geq 3$. With degree polynomials $D(P_n; x) = (n-2)x^2 + 2x$, the degree roots are 0 and $-2/(n-2)$, which of course lie in the closed LHP. In the case of $\Delta = 3$, recall from above that these trees have the following degree roots: 0, and $(-a_2 \pm \sqrt{a_2^2 - 4a_3(a_3+2)})/2a_3$, where a_k is the number of vertices of degree k. Other than 0, these roots will have negative real part: either the roots are real and negative, or are non-real and have real part $-a_2/2a_3$. Hence, all of these roots also lie in the (closed) LHP. For the case of $\Delta = 4$, a tree T has degree polynomial $D(T;x) = a_4x^4 + a_3x^3 + a_2x^2 + (2a_4 + a_3 + 2)x$ (there are $2a_4 + a_3 + 2$ leaves due to Lemma 3.20). Since there is always a root at 0, we shall remove it for simplicity and instead consider the polynomial $P(x)$:

$$\begin{aligned} P(x) &= \frac{D(T;x)}{x} \\ &= a_4x^3 + a_3x^2 + a_2x + (2a_4 + a_3 + 2). \end{aligned}$$

Notice that $P(x)$ has a root $a + ib$, where $a > 1$, if and only if the polynomial $Q(x) = P(x+1)$ has a root in the RHP. Recall that a polynomial is considered stable if all of its roots lie in the closed LHP, and the Hermite-Biehler Theorem (Theorem 1.8) which states a polynomial $f(x) = f_{even}(x^2) + xf_{odd}(x^2)$ is stable if and only if $f_{even}(x)$ and $f_{odd}(x)$ are standard, have only non-positive roots, and $f_{odd}(x) \prec f_{even}(x)$. Therefore, after setting $Q(x) = Q_{even}(x^2) + xQ_{odd}(x^2)$, we see that $P(x)$ has a root with real part greater than 1 if and only if at least one of $Q_{even}(x)$, $Q_{odd}(x)$ are non-standard or have some non-positive root, or $Q_{odd}(x) \not\prec Q_{even}(x)$. First, we must compute $Q(x)$, $Q_{even}(x)$, and $Q_{odd}(x)$:

$$\begin{aligned} Q(x) &= a_4(x+1)^3 + a_3(x+1)^2 + a_2(x+1) + (2a_4 + a_3 + 2) \\ &= a_4x^3 + (3a_4 + a_3)x^2 + (3a_4 + 2a_3 + a_2)x + (3a_4 + 2a_3 + a_2 + 2), \end{aligned}$$

and thus

$$Q_{even}(x) = (3a_4 + a_3)x + (3a_4 + 2a_3 + a_2 + 2)$$

and

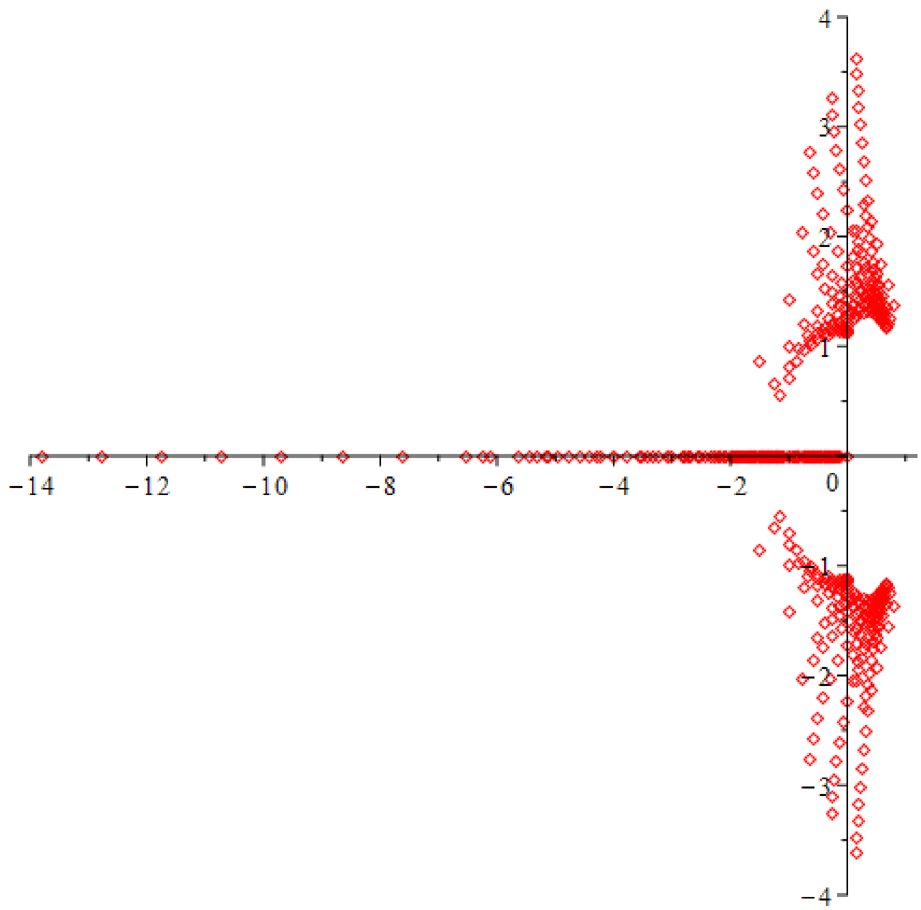


Figure 3.12: Degree roots for trees of orders three through eighteen that have $\Delta \leq 4$.

$$Q_{odd}(x) = a_4 x + (3a_4 + 2a_3 + a_2).$$

We quickly see that both $Q_{even}(x)$ and $Q_{odd}(x)$ are standard, and have only non-positive roots. If z_e is the root of $Q_{even}(x)$ and z_o is the root of $Q_{odd}(x)$, then

$$z_e = -\frac{3a_4 + 2a_3 + a_2 + 2}{3a_4 + a_3}$$

and

$$z_o = -\frac{3a_4 + 2a_3 + a_2}{a_4}.$$

$Q_{odd}(x) \not\prec Q_{even}(x)$ is now equivalent to the inequality $z_o > z_e$. We will now show that this inequality does not hold for any values of a_2, a_3, a_4. By rearranging the inequality we have the following:

$$\begin{aligned} & z_o > z_e \\ \iff & -\frac{3a_4 + 2a_3 + a_2}{a_4} > -\frac{3a_4 + 2a_3 + a_2 + 2}{3a_4 + a_3} \\ \iff & (3a_4 + a_3)(3a_4 + 2a_3 + a_2) < a_4(3a_4 + 2a_3 + a_2 + 2), \end{aligned}$$

which after expanding and collecting terms becomes

$$6a_4^2 + 7a_4a_3 + 2a_4a_2 + 2a_3^2 + a_3a_2 < 2a_4.$$

This inequality implies (since $a_2, a_3 \geq 0$) $6a_4^2 < 2a_4$, which is a contradiction as $a_4 \geq 1$. Therefore there are no values of a_2, a_3, a_4 for which $z_o > z_e$. Hence, there is no $P(x)$ with a root having real part greater than 1, i.e. no tree with $\Delta = 4$ with degree root having real part greater than 1.

To conclude this discussion of trees, let us consider trees which have only two degrees present. This is a subfamily of trees for which we can exactly solve for the degree roots. For a tree of order at least two that has only two degrees, one of them must be 1 (ie. the tree must have leaves). The other degree, call it k, takes values in the set $\{2, 3, ..., n-1\}$. We let $\mathcal{T}_{k,n}$ denote the set of all trees on n vertices that only have degrees 1 and k (up to isomorphism). The hydrocarbon graphs, or alkane graphs from Example 2.6, constitute $\mathcal{T}_{4,n}$ while path graphs make up $\mathcal{T}_{2,n}$. It is quickly seen that for some pairs (k, n), $\mathcal{T}_{k,n} = \emptyset$. For example, there is no tree with 6 vertices having degrees 1 and 4. The star $K_{1,4}$ has 5 vertices, and having an additional vertex would introduce a degree other than 1 or 4. In [33] all admissible (k, n) pairs, that is, pairs for which $\mathcal{T}_{k,n} \neq \emptyset$, are characterized:

Theorem 3.25 ([33])**.** *There exists a tree on n vertices whose degrees are only 1 and $k \geq 2$ (ie. $\mathcal{T}_{k,n} \neq \emptyset$) if and only if $k-1$ divides $n-2$.*

Therefore every pair (k, n) for which $\mathcal{T}_{k,n} \neq \emptyset$ has $n-2 = (k-1)t$, or $n = (k-1)t+2$, for some integer $t \geq 1$. The form for a degree polynomial of a tree T belonging to $\mathcal{T}_{k,n}$ is $D(T; x) = (n-l)x^k + lx$. Summing the degrees, and knowing that T has $n-1$ edges, we obtain

$$2(n-1) = k(n-l) + l,$$

and simplifying yields

$$l = \frac{kn - 2n + 2}{k - 1}.$$

Since we must have $n = (k-1)t + 2$ for some $t \geq 1$, this further becomes

$$\begin{aligned} l &= \frac{((k-1)t+2)k - 2((k-1)t+2) + 2}{k-1} \\ &= kt - 2(t-1). \end{aligned}$$

We can then also find

$$\begin{aligned} n - l &= (k-1)t + 2 - (kt - 2(t-1)) \\ &= t. \end{aligned}$$

Thus t is the number of vertices of degree k. Putting this all together, the degree polynomial for a tree T with t vertices of degree k and the remaining vertices being leaves is

$$D(T;x) = tx^k + [kt - 2(t-1)]x.$$

Finding the roots of these polynomials is not difficult: there is a root at $x = 0$, and the remaining roots are the $(k-1)^{th}$ roots of $-(k - 2(t-1)/t) = -l/t$. Figure 3.13 shows these roots for $1 \leq t \leq 10$, and $2 \leq k \leq 40$. This picture obscures two kinds of behaviour going on. The first kind is when t is fixed, and k increases ($k \to \infty$). In this case, the roots have modulus $(k - 2(t-1)/t)^{\frac{1}{k-1}}$ which has the following limit:

$$\lim_{k\to\infty} \left(k - 2\frac{t-1}{t}\right)^{\frac{1}{k-1}} = e^{\lim_{k\to\infty} \frac{\ln\left(k - 2\frac{t-1}{t}\right)}{k-1}}.$$

Since by L'Hôpital's Rule,

$$\begin{aligned} \lim_{k\to\infty} \frac{\ln\left(k - 2\frac{t-1}{t}\right)}{k-1} &= \lim_{k\to\infty} \frac{\frac{1}{k - 2\frac{t-1}{t}} \cdot 1}{1} \\ &= 0, \end{aligned}$$

we have

$$\lim_{k\to\infty}\left(k-2\frac{t-1}{t}\right)^{\frac{1}{k-1}} = e^0$$
$$= 1.$$

Thus as k increases the modulus of the roots approach 1, and since there are $k-1$ roots with arguments being those of the $(k-1)^{th}$ roots of -1, the roots of $D(T;x)$ approach the entire unit circle. Figure 3.14 shows this for $t=4$, and $2 \leq k \leq 40$.

The second kind of behaviour occurs when k is fixed, and t increases ($t \to \infty$). In this case, we see the limit of the modulus of the roots is

$$\lim_{t\to\infty}\left(k-2\frac{t-1}{t}\right)^{\frac{1}{k-1}} = \left(k-2+\lim_{t\to\infty}\frac{2}{t}\right)^{\frac{1}{k-1}}$$
$$= (k-2)^{\frac{1}{k-1}}.$$

Since k is fixed, we can say that as $t \to \infty$ the roots approach the points which lie on the same rays as the $(k-1)^{th}$ roots of -1, and have a modulus of $(k-2)^{1/(k-1)}$. We give an example of this in Figure 3.15, for $k=6$. There are five points which the roots are approaching (other than the always present root at the origin). These points have modulus $(6-2)^{1/(6-1)} = \sqrt[5]{4}$, and lie on the same rays as the 5^{th} roots of -1. We also see that the roots approach these points along the rays, and not from another direction. This is a consequence of k being fixed.

3.4.2 Complete Graphs with a Leaf

Here we investigate the degree roots of the CL_n graphs from Example 2.10. The graph CL_n is constructed by attaching a leaf onto any vertex of K_{n-1}, and has degree polynomial $D(CL_n;x) = x^{n-1} + (n-2)x^{n-2} + x$. While complete graphs (and any regular graph) have uninteresting degree polynomials, the simple addition of a leaf creates degree polynomials with non-trivial roots. We will study these roots in order to explain the behaviour observed in Figure 3.16, Figure 3.17, and Figure 3.18. Figure 3.16 shows all roots for $D(CL_n;x)$, $2 \leq n \leq 50$. We can observe two things: there are roots which appear to be spaced out along the negative real line, and there are roots

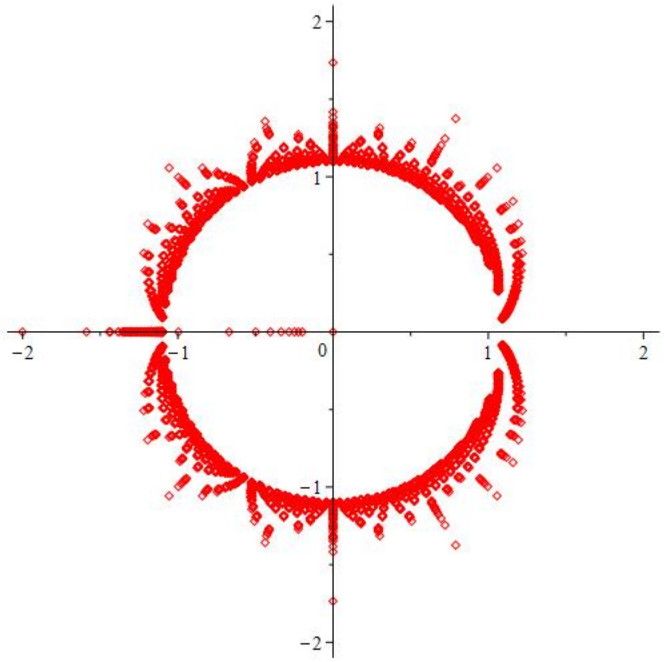


Figure 3.13: Roots of $D(T;x)$ for trees T with t vertices of degree k, and the rest being leaves, where $1 \leq t \leq 10$ and $2 \leq k \leq 40$.

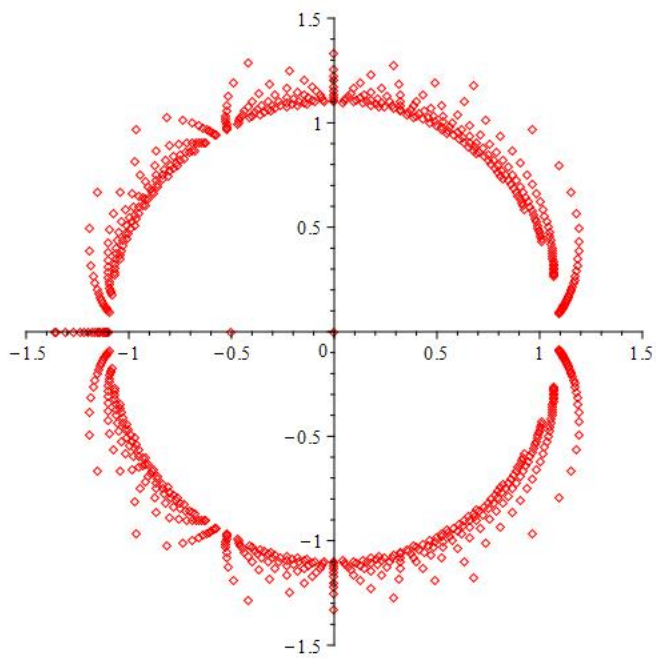


Figure 3.14: Roots of $D(T;x)$ for trees T with $t = 4$ vertices of degree k, $2 \leq k \leq 40$, and the remaining vertices being leaves.

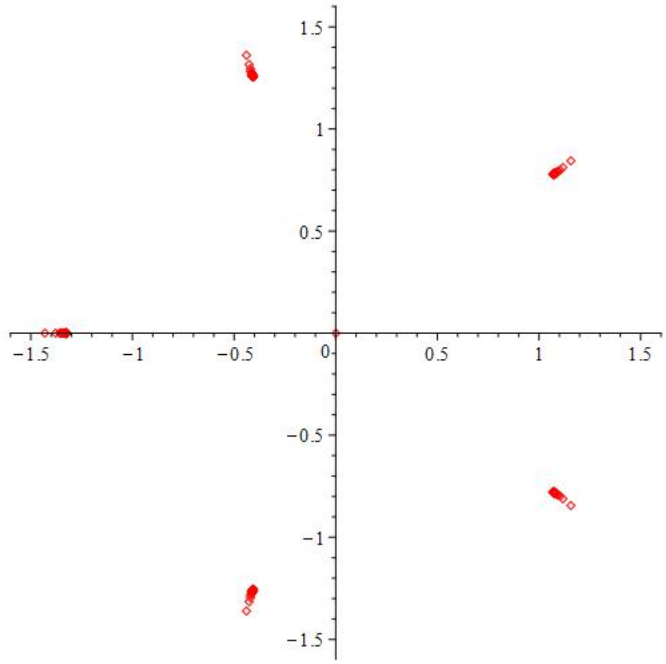


Figure 3.15: Roots of $D(T;x)$ for trees T with t vertices of degree $k=6$, for $1 \leq t \leq 40$. There are five non-zero limit points of the roots, which lie on and are approached along the rays that pass through the 5^{th} roots of -1.

which have modulus close to 1. Figure 3.17 focuses on those roots that are within the unit circle. It appears that the roots are approaching the entire unit circle from the inside, save for the roots at the origin and -1. Figure 3.18 focuses on the negative real line. The roots shown here all seem to be real, and are located near the negative integers.

Addressing the observation of the real roots, let us first count the negative real roots. Consider the polynomial $D(CL_n;-x)$:

$$D(CL_n;-x) = (-1)^{n-1}x^{n-1} + (-1)^{n-2}(n-2)x^{n-2} - x.$$

If n is odd, the coefficients have exactly one sign change. Thus $D(CL_n;x)$ has exactly one negative root by the Rule of Signs (Theorem 1.1). If n is even, there are two sign changes in the coefficients. Thus $D(CL_n;x)$ has zero or two negative roots, also by the Rule of Signs. We shall see there are in fact two negative roots for even n, except for $n=2$ when the degree polynomial is $D(CL_2;x) = 2x$. For $n \geq 4$, which is when $D(CL_n;x)$ is a trinomial (and not a binomial), we can locate a *large* negative root within an error that vanishes as $n \to \infty$.

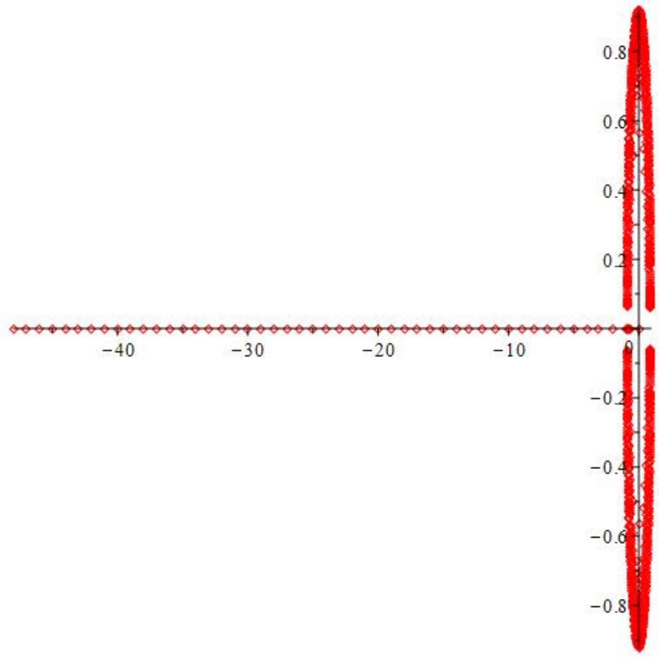


Figure 3.16: All roots of $D(CL_n; x)$ for $2 \leq n \leq 50$.

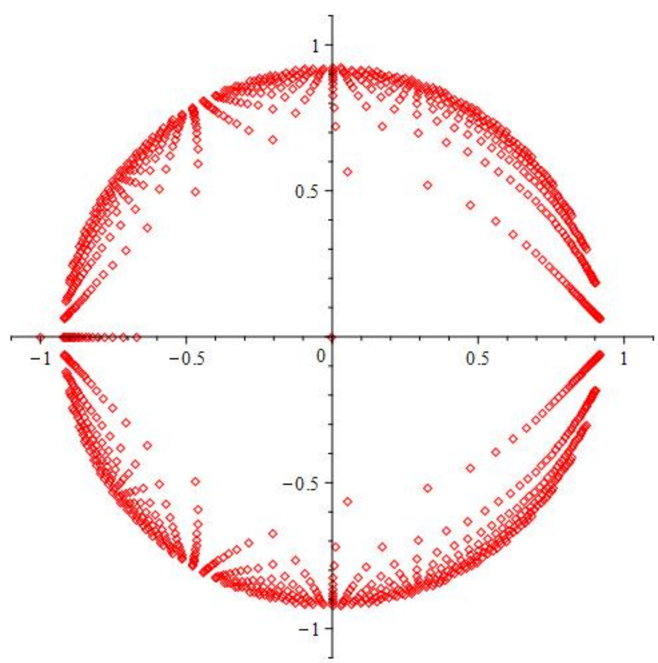


Figure 3.17: The roots of $D(CL_n; x)$ for $2 \leq n \leq 50$ that are contained in the unit circle. The roots appear to be converging outward to the unit circle as n increases.

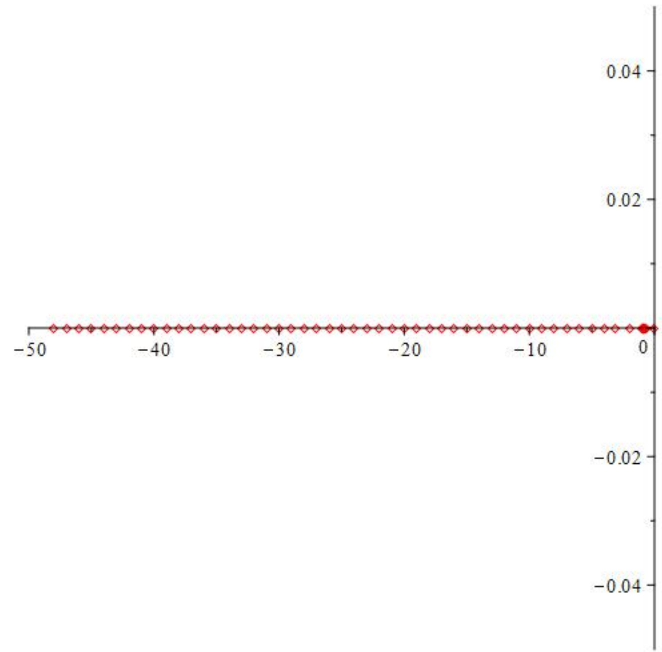


Figure 3.18: The roots of $D(CL_n;x)$ for $2 \leq n \leq 50$ on the real line (or at least seeming to be real).

Proposition 3.26. *Consider the graphs CL_n, $n \geq 4$. For odd n, $D(CL_n;x)$ has a real root in the interval $(-(n-2)-\epsilon_o(n), -(n-2))$ where*

$$\epsilon_o(n) = \frac{1}{(n-2)^{n-3}}.$$

For even n, $D(CL_n;x)$ has a real root in the interval $(-(n-2), -(n-2)+\epsilon_e(n)]$, where

$$\epsilon_e(n) = \frac{1}{(n-3)^{n-3}}.$$

Proof. To simplify some calculations, make the change of variables $x = (n-2)y$, and consider the polynomial

$$\begin{aligned} f(y) &= \frac{1}{(n-2)^{n-1}} D(CL_n;(n-2)y) \\ &= y^{n-1} + y^{n-2} + \frac{y}{(n-2)^{n-2}}. \end{aligned}$$

The roots of $f(y)$ and $D(CL_n;x)$ are in one-to-one correspondence via the change of variables. We first consider when n is odd. We will evaluate $f(y)$ at two points that give values with opposite sign, and apply the IVT. The first point is $y=-1$:

$$\begin{aligned} f(-1) &= (-1)^{n-1} + (-1)^{n-2} + \frac{-1}{(n-2)^{n-2}} \\ &= \frac{-1}{(n-2)^{n-2}} \\ &< 0. \end{aligned}$$

The next point is $y = -1 - 1/(n-2)^{n-2}$:

$$\begin{aligned} f\left(-1 - \frac{1}{(n-2)^{n-2}}\right) &= \left(-1 - \frac{1}{(n-2)^{n-2}}\right)^{n-1} + \left(-1 - \frac{1}{(n-2)^{n-2}}\right)^{n-2} \\ &\quad + \frac{-1 - \frac{1}{(n-2)^{n-2}}}{(n-2)^{n-2}} \\ &= (-1)^{n-1}\left(1 + \frac{1}{(n-2)^{n-2}}\right)^{n-1} \\ &\quad + (-1)^{n-2}\left(1 + \frac{1}{(n-2)^{n-2}}\right)^{n-2} - \frac{1 + \frac{1}{(n-2)^{n-2}}}{(n-2)^{n-2}} \\ &= \left(1 + \frac{1}{(n-2)^{n-2}}\right)^{n-2}\left[\left(1 + \frac{1}{(n-2)^{n-2}}\right) - 1\right] \\ &\quad - \frac{1 + \frac{1}{(n-2)^{n-2}}}{(n-2)^{n-2}} \\ &= \frac{1 + \frac{1}{(n-2)^{n-2}}}{(n-2)^{n-2}}\left[\left(1 + \frac{1}{(n-2)^{n-2}}\right)^{n-3} - 1\right] \\ &> 0 \end{aligned}$$

since $(1 + 1/(n-2)^{n-2})^{n-3} > 1$. Thus by the IVT, $f(y)$ has a root in the interval $(-1 - 1/(n-2)^{n-2}, -1)$. Through the change of variables $x = (n-2)y$, it is clear that $D(CL_n;x)$ has a root in the interval $(-(n-2) - 1/(n-2)^{n-3}, -(n-2))$, or $(-(n-2) - \epsilon_o(n), -(n-2))$.

Similarly, suppose that n is even. We still have that $f(-1) = -1/(n-2)^{n-2} < 0$. Let us evaluate $f(y)$ at another point, namely $y = -1 + 1/(n-2)(n-3)^{n-3}$:

$$
\begin{aligned}
f(y) &= f\left(-1+\frac{1}{(n-2)(n-3)^{n-3}}\right)\\
&= \left(-1+\frac{1}{(n-2)(n-3)^{n-3}}\right)^{n-1}+\left(-1+\frac{1}{(n-2)(n-3)^{n-3}}\right)^{n-2}\\
&\quad+\frac{-1+\frac{1}{(n-2)(n-3)^{n-3}}}{(n-2)^{n-2}}\\
&= -\left(1-\frac{1}{(n-2)(n-3)^{n-3}}\right)^{n-1}+\left(1-\frac{1}{(n-2)(n-3)^{n-3}}\right)^{n-2}\\
&\quad-\frac{1-\frac{1}{(n-2)(n-3)^{n-3}}}{(n-2)^{n-2}}\\
&= \left(1-\frac{1}{(n-2)(n-3)^{n-3}}\right)^{n-2}\left[-\left(1-\frac{1}{(n-2)(n-3)^{n-3}}\right)+1\right]\\
&\quad-\frac{1-\frac{1}{(n-2)(n-3)^{n-3}}}{(n-2)^{n-2}}\\
&= \frac{\left(1-\frac{1}{(n-2)(n-3)^{n-3}}\right)^{n-2}}{(n-2)(n-3)^{n-3}}-\frac{1-\frac{1}{(n-2)(n-3)^{n-3}}}{(n-2)^{n-2}}.
\end{aligned}
$$

This quantity is non-negative, as

$$
\begin{aligned}
& f\left(-1+\frac{1}{(n-2)(n-3)^{n-3}}\right)\geq 0\\
\iff\quad & \frac{\left(1-\frac{1}{(n-2)(n-3)^{n-3}}\right)^{n-2}}{(n-2)(n-3)^{n-3}}\geq\frac{1-\frac{1}{(n-2)(n-3)^{n-3}}}{(n-2)^{n-2}}\\
\iff\quad & \left(1-\frac{1}{(n-2)(n-3)^{n-3}}\right)^{n-3}\geq\frac{(n-3)^{n-3}}{(n-2)^{n-3}}\\
\iff\quad & 1-\frac{1}{(n-2)(n-3)^{n-3}}\geq\frac{n-3}{n-2}\\
\iff\quad & (n-2)(n-3)^{n-3}-1\geq(n-3)^{n-2}\\
\iff\quad & (n-2)(n-3)^{n-3}-1\geq(n-2)(n-3)^{n-3}-(n-3)^{n-3}\\
\iff\quad & 1\leq(n-3)^{n-3},
\end{aligned}
$$

and this last inequality is indeed true since $n \geq 4$. Furthermore, there is equality if and only if $n = 4$. Applying the IVT, we conclude that $f(y)$ has a root in the interval $(-1, -1 + 1/(n-2)(n-3)^{n-3})$. Thus $D(CL_n; x)$ has a root in the interval $(-(n-2), -(n-2) + 1/(n-3)^{n-3})$, or $(-(n-2), -(n-2) + \epsilon_e(n))$.

□

Since we have shown there is at least one negative root when n is even, there in fact must be two negative roots by what we found above with the Rule of Signs. Using the IVT, we can quickly find that this root is in the interval $[-1, 0)$. Evaluating the polynomial $g(x) = D(CL_n; x)/x$ (just removing the known root at 0) at these endpoints, we find

$$\begin{aligned} g(0) &= (0)^{n-2} + (n-2)(0)^{n-3} + 1 \\ &= 1 \\ &> 0, \end{aligned}$$

and

$$\begin{aligned} g(-1) &= (-1)^{n-2} + (n-2)(-1)^{n-3} + 1 \\ &= (-1)^{n-3}(-1 + n - 2) + 1 \\ &= -(n-3) + 1 \\ &\leq 0, \end{aligned}$$

as $n \geq 4$. Therefore $D(CL_n; x)$ has a root in $[-1, 0)$ when n is even. In fact, this last inequality is equality if and only if $n = 4$, when $D(CL_4; x) = x^3 + 2x^2 + x = x(x+1)^2$. In this case there is a double root at -1, which is why the half-closed interval is needed in Proposition 3.26.

We can address the observation of roots converging to the unit circle, from Figure 3.17, with the extended BKW Theorem (Theorem 1.7). Let us examine the limits of the roots of $D(CL_n; x)$, as $n \to \infty$. Since there is always a root at $x = 0$, we can just consider the polynomial

$$\begin{aligned} g_{n-3}(x) &= \frac{D(CL_n; x)}{x} \\ &= x^{n-2} + (n-2)x^{n-3} + 1 \\ &= x^{n-3}(x + n - 2) + 1. \end{aligned}$$

With a substitution of $N = n-3$, $g_N(x) = x^N(x+N+1)+1$ is in the form to apply Theorem 1.7 if we let $\lambda_1(x) = x$, $\lambda_2(x) = 1$, $\alpha_1(N;x) = x+N+1$, and $\alpha_2(N;x) = 1$. Furthermore, we have $p_{1,1}(x) = 1$ as the coefficient polynomial on N in $\alpha_1(N;x)$, and $p_{2,0}(x) = 1$ as the coefficient polynomial on N in $\alpha_2(N;x)$. Since both $p_{1,1}(x)$ and $p_{2,0}(x)$ are non-zero, we can rule out using the first condition of Theorem 1.7 to find the limits. The second condition immediately gives that the limits of $g_N(x)$ are the points z where $|\lambda_1(z)| = |\lambda_2(z)|$, or where $|z| = 1$, i.e. the unit circle. Thus the limits of the roots of $D(CL_n;x)$, as $n \to \infty$, are 0 and the unit circle $|z| = 1$.

We have now verified part of our observation that there are roots of $D(CL_n;x)$ which approach the unit circle from its interior. What we will now show is that for $n \geq 5$, all the roots of $D(CL_n;x)$ except for the real root located near $-(n-2)$ from Proposition 3.26 are contained within the unit circle. When we say the root located near $-(n-2)$, we mean the root inside the interval $(-(n-2)-\epsilon_o(n), -(n-2))$ if n is odd, or inside the interval $(-(n-2), -(n-2)+\epsilon_e(n))$ if n is even. Furthermore, the root in this interval is unique. When n is odd, $D(CL_n;x)$ has only one negative root, and thus it must be in the respective interval. When n is even, there are two negative roots: one is contained in the interval $(-(n-2), -(n-2)+\epsilon_e(n))$, and the other is contained in the disjoint interval $(-1, 0)$. Thus there is no confusion when saying the root located near $-(n-2)$. Let $-r$ be this root, so that $r \approx n-2$. Then $x+r$ is a factor of $D(CL_n;x)$ and we can write

$$\begin{aligned} D(CL_n;x) &= (x+r)f(x) \\ &= (x+r)(b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + \cdots + b_1 x), \end{aligned}$$

where $f(x) = b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + \cdots + b_1 x$. Expanding the product on the right and equating the coefficients with $D(CL_n;x) = x^{n-1} + (n-2)x^{n-2} + x$ gives the following relations:

$$\begin{aligned} b_{n-2} &= 1, \\ b_{n-3} + rb_{n-2} &= n-2, \\ b_k + rb_{k+1} &= 0, \quad 1 \leq k \leq n-4, \\ rb_1 &= 1. \end{aligned}$$

Thus from these equations we obtain $b_{n-2} = 1$, $b_{n-3} = n-2-r$, and $b_k = (-1)^{k+1} 1/r^k$ for $1 \le k \le n-4$. Hence,

$$f(x) = x^{n-2} + (n-2-r)x^{n-3} + \sum_{k=1}^{n-4} (-1)^{k+1} \frac{1}{r^k} x^k.$$

We will show $f(x)$ has all its roots interior to the unit circle using Cauchy's bound (Theorem 1.3). First, let us remove the root at 0 by dividing by x:

$$\begin{aligned} f_0(x) &= \frac{f(x)}{x} \\ &= x^{n-3} + (n-2-r)x^{n-4} + \sum_{k=1}^{n-4} (-1)^{k+1} \frac{1}{r^k} x^{k-1}. \end{aligned}$$

To apply Cauchy's bound, we construct the following polynomial from $f_0(x)$ and look for its (unique) positive root:

$$g(x) = x^{n-3} - |n-2-r| x^{n-4} - \sum_{k=1}^{n-4} \frac{1}{r^k} x^{k-1}.$$

Evaluating at $x = 0$ gives $g(0) = -1/r < 0$. Evaluating at $x = 1$ gives

$$g(1) = 1 - |n-2-r| - \sum_{k=1}^{n-4} \frac{1}{r^k},$$

and we claim that $g(1) > 0$. Recall from Proposition 3.26 that $|n-2-r| < \epsilon_o(n)$ for odd n, where $\epsilon_o(n) = 1/(n-2)^{n-3}$, and for even n, $|n-2-r| < \epsilon_e(n)$ where $\epsilon_e(n) = 1/(n-3)^{n-3}$. In any case on the parity of n, we have $|n-2-r| < 1/(n-3)^{n-3}$. Furthermore, since either $-r < -(n-2)$ or $-r < -(n-2) + \epsilon_e(n)$, we also have $1/r < 1/(n-2-\epsilon_e(n))$. These inequalities immediately show

$$|n-2-r| + \sum_{k=1}^{n-4} \frac{1}{r^k} < \frac{1}{(n-3)^{n-3}} + \sum_{k=1}^{n-4} \left(\frac{1}{n-2-\epsilon_e(n)} \right)^k. \tag{3.1}$$

The sum on the RHS can be rewritten as it is a geometric series:

$$\sum_{k=1}^{n-4} \left(\frac{1}{n-2-\epsilon_e(n)} \right)^k = \frac{\left(\frac{1}{n-2-\epsilon_e(n)} \right)^{n-3} - 1}{\frac{1}{n-2-\epsilon_e(n)} - 1} - 1,$$

and at this point we make a substitution of $N = n - 3$ (where $N \geq 2$ since $n \geq 5$) to simplify our expressions. Therefore the RHS of (3.1) becomes

$$\begin{aligned}
\frac{1}{N^N} + \frac{\left(\frac{1}{N+1-\frac{1}{N^N}}\right)^N - 1}{\frac{1}{N+1-\frac{1}{N^N}} - 1} - 1 &= \frac{1}{N^N} + \frac{\left(\frac{N^N}{N^N(N+1)-1}\right)^N - 1}{\frac{N^N}{N^N(N+1)-1} - 1} - 1 \\
&= \frac{1}{N^N} + \frac{\frac{N^{N^2}}{(N^N(N+1)-1)^N} - 1}{\frac{N^N - N^N(N+1)+1}{N^N(N+1)-1}} - 1 \\
&= \frac{1}{N^N} + \frac{N^{N^2} - (N^N(N+1)-1)^N}{(1-N^{N+1})(N^N(N+1)-1)^{N-1}} - 1 \\
&= \frac{1}{N^N} + \frac{(N^N(N+1)-1)^N - N^{N^2}}{(N^{N+1}-1)(N^N(N+1)-1)^{N-1}} - 1 \\
&= \frac{1}{N^N} + \frac{N^N(N+1)-1}{N^{N+1}-1} \\
&\quad - \frac{N^{N^2}}{(N^{N+1}-1)(N^N(N+1)-1)^{N-1}} - 1 \\
&= \frac{1}{N^N} + \frac{N^N}{N^{N+1}-1} \\
&\quad - \frac{N^{N^2}}{(N^{N+1}-1)(N^N(N+1)-1)^{N-1}} \\
&= \frac{1}{N^N} + \frac{1}{N - \frac{1}{N^N}} - \frac{N^{N^2}}{(N^{N+1}-1)(N^N(N+1)-1)^{N-1}} \\
&< \frac{1}{N^N} + \frac{1}{N - \frac{1}{N^N}}.
\end{aligned}$$

Furthermore, each term in the final inequality above is strictly decreasing as N increases. Therefore, since $N \geq 2$ we have

$$\begin{aligned}
\frac{1}{N^N} + \frac{\left(\frac{1}{N+1-\frac{1}{N^N}}\right)^N - 1}{\frac{1}{N+1-\frac{1}{N^N}} - 1} - 1 &< \frac{1}{N^N} + \frac{1}{N - \frac{1}{N^N}} \\
&\leq \frac{1}{2^2} + \frac{1}{2 - \frac{1}{2^2}} \\
&= \frac{23}{28} \\
&< 1.
\end{aligned}$$

By inequality (3.1) it follows that

$$|n-2-r| + \sum_{k=1}^{n-4} \frac{1}{r^k} < 1,$$

thus proving $g(1) > 0$. Since $g(0) < 0$ and $g(1) > 0$, by the IVT it follows that $g(x)$ has its positive root in the interval $(0,1)$. By Cauchy's bound, all the roots of $f_0(x)$ (and also $f(x)$) have modulus less than 1. This completes our argument: that for $n \geq 5$, all roots of $D(CL_n; x)$ except for the real root near $-(n-2)$ is located in the interior of the unit circle.

3.4.3 Anti-Regular Graphs

Recall the anti-regular graphs from Example 2.11. This family was partitioned into two sets of graphs: connected and disconnected graphs, which were complementary to one another (in the graph theoretic sense). Figure 3.19 shows the roots of these graphs up to order $n = 50$, separating roots for the connected graphs (top) and disconnected graphs (bottom). Furthermore, we identify roots for even n with red and those for odd n with blue. Some immediate observations are: when n is even the (non-zero) roots appear to be on the unit circle, and not so for odd n. However, the roots for odd n surround the unit circle and possibly converge to it. No root seems to exceed a modulus of 2, which occurs for a real root. By examining closely, we can also notice the reciprocal relationship between the (non-zero) roots of the two plots, since the graphs giving these degree roots are complements of each other.

We will start investigating these roots by considering the connected graphs. For a graph of order $n \geq 2$, the connected anti-regular graph H_n has degree polynomial $D(H_n; x) = \sum_{j=1}^{n-1} x^j + x^{\lfloor n/2 \rfloor}$. However, we may write $D(H_n; x)$ in the form

$$\begin{aligned} D(H_n; x) &= x \sum_{j=0}^{n-2} x^j + x^{\lfloor n/2 \rfloor} \\ &= x \frac{1 - x^{n-1}}{1 - x} + x^{\lfloor n/2 \rfloor}, \end{aligned}$$

which holds for all $x \neq 1$. Thus the roots of $D(H_n; x)$ are the solutions to the equation

$$x - x^n + x^{\lfloor n/2 \rfloor} - x^{\lfloor n/2 \rfloor + 1} = 0 \tag{3.2}$$

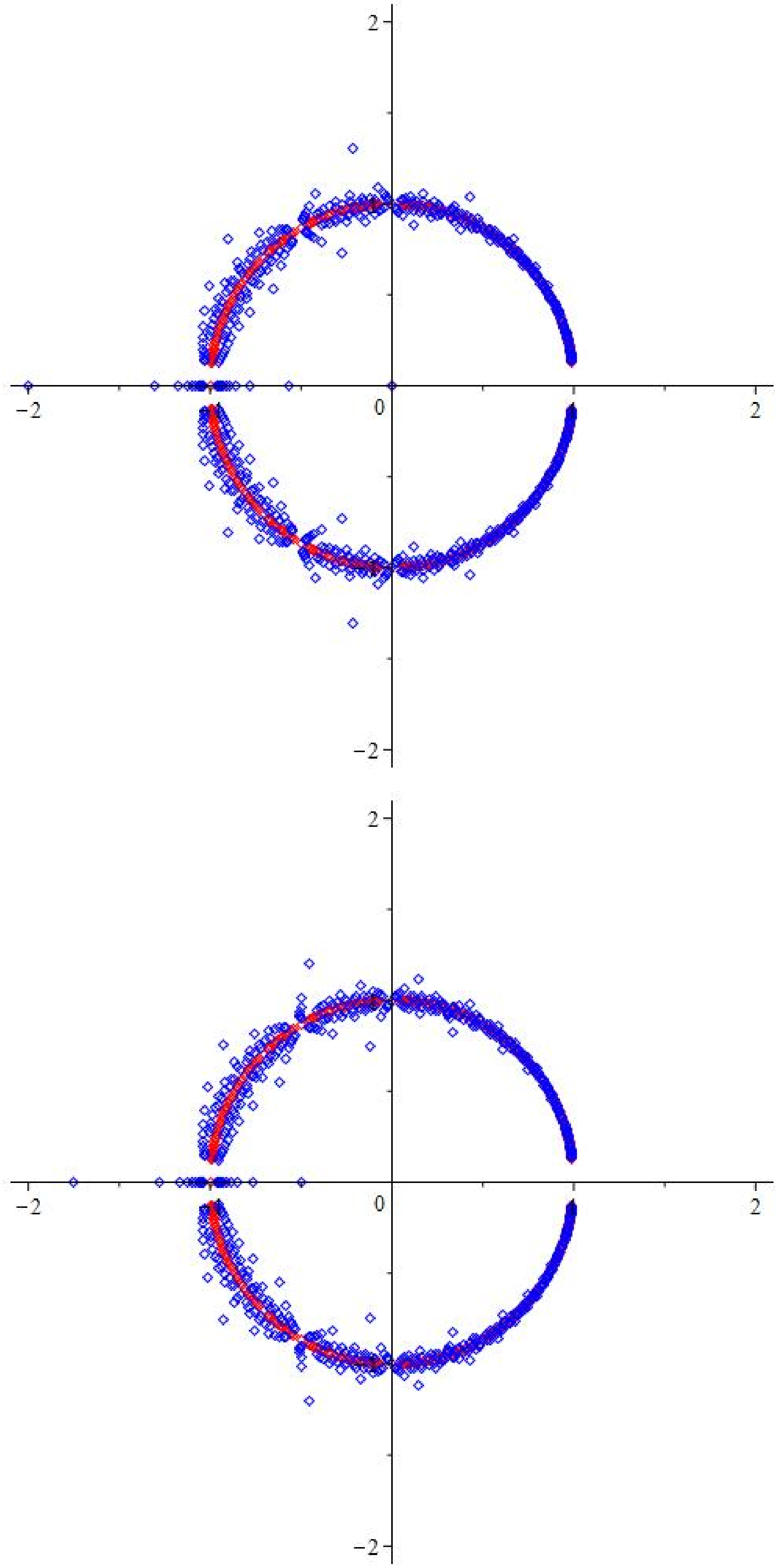


Figure 3.19: Degree roots for anti-regular graphs, up to order $n = 50$. (Top) All roots of $D(H_n; x)$. (Bottom) All roots of $D(H_n^c; x)$. In each plot, roots for even n are shown in red, and roots for odd n are shown in blue. Note the property that each root of the bottom plot is a reciprocal of a non-zero root of the top plot, and vice versa.

except $x = 1$. We now examine the solutions to (3.2) via two cases on n.

Case 1: $n = 2k$, $k \geq 1$. Here, (3.2) simplifies to

$$x^{2k} + x^{k+1} - x^k - x = 0$$

or

$$x(x^k - 1)(1 + x^{k-1}) = 0.$$

Thus the roots of $D(H_{2k}; x)$ are $x = 0$, the k^{th} roots of unity (except for 1 itself), and the $(k-1)^{th}$ roots of -1.

Case 2: $n = 2k + 1$, $k \geq 1$. In this case, (3.2) becomes

$$x^{2k+1} + x^{k+1} - x^k - x = 0. \tag{3.3}$$

These polynomials require numerical techniques to find their solutions. However, we can deduce some information about them. Of course, there is a root at 0. Recall Theorem 1.1, The Rule of Signs. Since there is exactly one sign change in the coefficients, there is exactly one positive solution to (3.3). A quick check verifies that this solution is $x = 1$, which is exactly the point we are excluding. This simply confirms that $D(H_{2k+1}; x)$ has no positive roots, which we know to be true. If we substitute $x \to -x$, we obtain

$$-x^{2k+1} + (-1)^{k+1}x^{k+1} + (-1)^{k+1}x^k + x = 0.$$

Regardless if k is even or odd, this equation has exactly one sign change and thus has exactly one positive root. Therefore, equation (3.3), and hence $D(H_{2k+1}; x)$, has exactly one negative root. A quick check rules out -1 from being this root. However, we can bound this negative root to the interval $[-2, -1/2)$ using the Enestrom-Kakeya Theorem (Theorem 1.5) and some algebra. Since $D(H_n; x) = x\sum_{j=0}^{n-2} x^j + x^{\lfloor n/2 \rfloor}$, $D(H_n; x)/x$ has all positive coefficients. Furthermore, all of these coefficients is 1 except for the coefficient on $x^{\lfloor n/2 \rfloor - 1}$, which is 2. Thus the minimum ratio of consecutive coefficients is $1/2$, while the maximum ratio is 2. By Theorem 1.5, every root of $D(H_n; x)/x$ (i.e. the non-zero roots of $D(H_n; x)$) has modulus in the interval $[1/2, 2]$. This is true of the real and non-real roots, and also holds for even n. For the real root in the case of odd n (which we were originally interested in), we can slightly improve this interval. Evaluating the left hand side of (3.3) at $x = -1/2$, we obtain

$$\begin{aligned} x^{2k+1} + x^{k+1} - x^k - x &= \left(\frac{-1}{2}\right)^{2k+1} + \left(\frac{-1}{2}\right)^{k+1} - \left(\frac{-1}{2}\right)^{k} + \frac{1}{2} \\ &= -\left(\frac{1}{2}\right)^{2k+1} + (-1)^{k+1}\left(\left(\frac{1}{2}\right)^{k+1} + \left(\frac{1}{2}\right)^{k}\right) + \frac{1}{2} \\ &= \frac{-1 + (-1)^{k+1}(2^k + 2^{k+1}) + 2^{2k}}{2^{2k+1}}. \end{aligned}$$

If k is odd, then this quantity is clearly positive. When k is even, observe

$$\begin{aligned} -1 + (-1)^{k+1}(2^k + 2^{k+1}) + 2^{2k} &= -1 - (2^k + 2^{k+1}) + 2^{2k} \\ &= -1 + 2^k(2^k - 3) \\ &> 0, \end{aligned}$$

since $2^k \geq 4$ and $2^k - 3 \geq 1$, so $2^k(2^k - 3) \geq 4 > 1$. Thus in any case, the left hand side of (3.3) is positive when $x = -1/2$. Therefore the negative root of equation (3.3) is actually in the interval $[-2, -1/2)$.

We can also study the limits of degree roots for H_n, as $n \to \infty$, using the BKW Theorem (Theorem 1.6). The first step in applying the BKW Theorem is writing the polynomials in the right form. Recall that we can write

$$(1-x)D(H_n; x) = x(1 - x^{n-1}) + (1-x)x^{\lfloor n/2 \rfloor}.$$

As before, we shall consider cases on the parity of n.

Case 1: $n = 2k$, $k \geq 1$. Define the polynomial

$$\begin{aligned} f_k(z) &\equiv (1-z)D(H_{2k}; z) \\ &= z(1 - z^{2k-1}) + (1-z)z^k \\ &= -z^{2k} + (1-z)z^k + z. \end{aligned}$$

Other than the root of f_k at $z = 1$, $f_k(z)$ and $D(H_{2k}; z)$ have the same roots. Thus aside from 1, $f_k(z)$ will have the same root limits as $D(H_{2k}; z)$. Observe that $f_k(z)$ is readily in the form to apply the BKW Theorem by setting $\alpha_1(z) = -1$, $\alpha_2(z) = 1-z$, $\alpha_3(z) = z$, and $\lambda_1(z) = z^2$, $\lambda_2(z) = z$, and $\lambda_3(z) = 1$. Thus we can now check the conditions of the theorem:

Condition 1:

- $|z^2| > |z|$, $|z^2| > 1$, and $-1 = 0$. The equality immediately gives a contradiction, so we move on.
- $|z| > |z^2|$, $|z| > 1$, and $1 - z = 0$. The equality gives $z = 1$, contradicting the inequalities. So again, we move on.
- $1 > |z^2|$, $1 > |z|$, and $z = 0$. The point $z = 0$ satisfies the inequalities, and thus it is a limit of the roots.

Condition 2:

- $|z^2| = |z| > 1$. No z satisfy these constraints, since the equality implies $|z| = 1$ or $z = 0$, contradicting $|z| > 1$.
- $|z| = 1 > |z^2|$. There are also no z that satisfy these constraints.
- $|z^2| = 1 > |z|$. Again, no z are possible.
- $|z^2| = |z| = 1$. These equalities define precisely the unit circle. Thus every point on the unit circle is a limit of the roots.

Therefore the limits of the roots for $f_k(z)$, and also $D(H_{2k}; z)$ are the unit circle and the origin, or

$$C_0 = \{z \in \mathbb{C} \ : \ z = 0 \ \text{ or } \ |z| = 1\}.$$

Case 2: $n = 2k + 1$, $k \geq 1$. Similar to the first case, define

$$\begin{aligned} g_k(z) &\equiv (1-z)D(H_{2k+1}; z) \\ &= z(1 - z^{2k}) + (1-z)z^k \\ &= -z^{2k+1} + (1-z)z^k + z. \end{aligned}$$

Observe that $g_k(z)$ is also immediately in the form to apply the BKW Theorem, by setting $\alpha_1(z) = -z$, $\alpha_2(z) = 1 - z$, $\alpha_3(z) = z$, and $\lambda_1(z) = z^2$, $\lambda_2(z) = z$, $\lambda_3(z) = 1$. The only difference between $g_k(z)$ and $f_k(z)$ is α_1; thus the limits of the roots will be the same except for possibly those from the constraint $|\lambda_1(z)| > |\lambda_2(z)|$, $|\lambda_1(z)| > |\lambda_3(z)|$, and $\alpha_1(z) = 0$. For $f_k(z)$ this gave a contradiction, and for $g_k(z)$ this also gives a contradiction: $|z^2| > |z|$, $|z^2| > 1$, and $-z = 0$ which implies $z = 0$, contradicting the inequalities. Thus the limits of the roots of $g_k(z)$ are the same as $f_k(z)$, which overall gives that the limits of the roots of $D(H_n; z)$, as $n \to \infty$, are simply C_0.

For the disconnected anti-regular graphs, we recall that they are precisely the complements of the connected graphs H_n. Using the fact that $D(G^c; x) = x^{n-1}D(G; 1/x)$ for any graph of order n, we immediately obtain the degree roots of H_n^c as the reciprocals of the non-zero degree roots of H_n. In fact, when $n = 2k$, $k \geq 1$, the non-zero degree roots of H_{2k} and H_{2k}^c are the same: the non-zero roots of $D(H_{2k}; x)$ are the k^{th} roots of unity (except for 1), and the $(k-1)^{th}$ roots of -1. Since the k^{th} roots of unity have the form $z = e^{2i\pi j/k}$, where $0 \leq j \leq k - 1$, we have $z^{-1} = e^{-2i\pi j/k} = e^{2i\pi(k-j)/k}$, which is simply another k^{th} root of unity. Thus the reciprocals of the k^{th} roots of unity are exactly the k^{th} roots of unity, simply with different arguments. This is similarly the case for the $(k - 1)^{th}$ roots of -1. When $n = 2k + 1$, $k \geq 1$, however, H_{2k+1} and H_{2k+1}^c have different degree roots. This is immediate simply from the fact that H_{2k+1} has a root with modulus not equal to 1. The negative root of $D(H_{2k+1}; x)$, which we found to lie in the interval $[-2, -1/2)$, tells us that $D(H_{2k+1}^c; x)$ has a negative root in the interval $(-2, -1/2]$. The limits of the roots of $D(H_n^c; x)$, as $n \to \infty$ are the reciprocals of the limits for $D(H_n; x)$, and thus are easily seen to be the unit circle $|z| = 1$.

3.4.4 Complete p-Partite Graphs

Complete p-partite graphs have degree polynomials that are easy to compute. Recall from Example 2.8 that the complete p-partite graph $K_{a_1,\ldots,a_p}$ has degree polynomial

$$D(K_{a_1,\ldots,a_p}; x) = \sum_{j=1}^{p} a_j x^{n-a_j}$$

where $n = \sum_1^p a_j$. When $p = 2$ (i.e. for complete bipartite graphs), the degree polynomial is simply $D(K_{a_1,a_2}; x) = a_1 x^{a_2} + a_2 x^{a_1}$. Assuming, without loss of generality, that $a_1 < a_2$, then the roots of this polynomial are 0 (with multiplicity a_1) and the $(a_2 - a_1)^{th}$-roots of $-a_2/a_1$. Thus, from here on we assume that $p \geq 3$. In this section we consider the case where the a_j's have the form $a_j = sj$, for some integer $s \geq 1$. In this situation, the degree roots have a particular symmetry as seen in Figure 3.20. This figure shows the degree roots for four such p-partite graphs, with varying p and s. The upper left plot shows roots for $p = 10$ and $s = 1$. We can observe nine non-zero roots that are approximately evenly distributed in both directions around a negative root. That is, there are four roots in both the counterclockwise and counter-clockwise directions from a negative root, and each root appears to be located after traversing some fixed angle θ. These roots are also similar in modulus; it is not clear whether they are actually equal or not. The upper right plot shows the roots for $p = 6$ and $s = 2$. In this case, there are five roots distributed evenly (in the same sense as above) around two distinct points: one on the imaginary axis in the upper half-plane, and one on the imaginary axis in the lower half-plane. At this point we can already sense a pattern: that there are s clusters of roots, where each cluster contains $p - 1$ roots distributed approximately evenly (in the angular sense) and have similar modulus, and the angular centres of these clusters have the same argument as one of the s^{th} roots of -1. This pattern holds for the bottom plots, where in the left plot $p = 8$ and $s = 3$, and on the right $p = s = 4$.

Let us now address these observations. Having $a_j = sj$, we can write the degree polynomial of $K_{a_1,\ldots,a_p}$ as follows:

$$\begin{aligned} D(K_{a_1,\ldots,a_p}; x) &= \sum_{j=1}^{p} sjx^{n-sj} \\ &= sx^{n-ps} \sum_{j=1}^{p} jx^{(p-j)s}. \end{aligned}$$

Hence there are $n - ps$ roots at zero. Let $g(x) = \sum_{j=1}^{p} jx^{(p-j)s}$ be the polynomial which has all of the non-zero roots of $D(K_{a_1,\ldots,a_p}; x)$. We can perform a substitution $y = x^s$ to obtain the polynomial

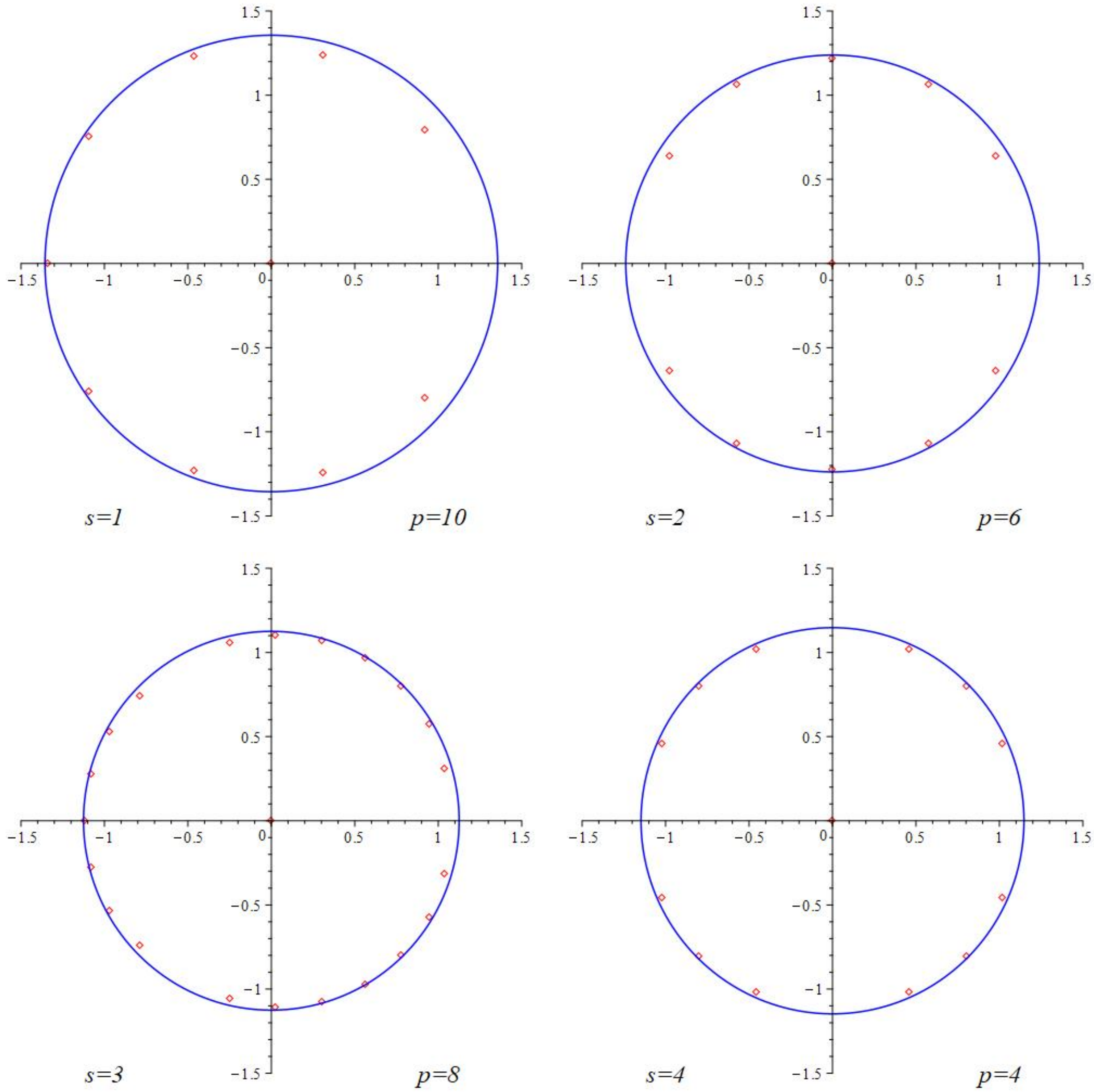


Figure 3.20: Degree roots for some complete p-partite graphs which have the form $a_j = sj$ for $s \geq 1$. Top left: the roots for $p = 10$, $s = 1$. Top right: the roots for $p = 6$, $s = 2$. Bottom left: the roots for $p = 8$, $s = 3$. Bottom right: the roots for $p = s = 4$. In each plot, the blue circle is the circle $|x| = (2p+1)^{\frac{1}{ps}}$, which is a bound on the modulus of the roots that will be discussed later.

$$h(y) = \sum_{j=1}^{p} j y^{p-j}$$
$$= y^{p-1} + 2y^{p-2} + \cdots + (p-1)y + p.$$

Thus we are close to confirming one of our observations: for each root z of $h(y)$, we obtain s roots for $D(K_{a_1,\ldots,a_p}; x)$ which are the s^{th}-roots of z. Thus each of the $p-1$ roots of $h(y)$ yields s roots of $D(K_{a_1,\ldots,a_p}; x)$ at equal angles around the origin, and overall give s "clusters" of roots for the degree polynomial.

To better understand the roots of $h(y)$, let us rewrite $h(y)$ using the geometric series:

$$yh(y) = \sum_{j=1}^{p} j y^{p-j+1}$$
$$= y^p + 2y^{p-1} + \cdots + (p-1)y^2 + py$$

and thus

$$(y-1)h(y) = y^p + y^{p-1} + \cdots + y - p$$
$$= y\frac{y^p - 1}{y-1} - p,$$

as long as $y \neq 1$. Therefore we have, for all $y \neq 1$,

$$h(y) = \frac{y(y^p - 1) - p(y-1)}{(y-1)^2}.$$

Hence the roots of $h(y)$ are the roots of

$$w(y) = y(y^p - 1) - p(y-1)$$
$$= y^{p+1} - (p+1)y + p \qquad (3.4)$$

different from 1. While the roots of $w(y)$ must be found numerically, we can bound their modulus using the following interesting theorem inspired by physics.

Theorem 3.27 ([36])**.** *The equation $Az^{N+M} + Bz^M + C = 0$, where*

$$A = |A|e^{i\alpha}, \quad B = |B|e^{i\beta}, \quad C = |C|e^{i\gamma},$$

defines two regular polygons S_{N+M} and S_N concentric in the complex plane. The $(N+M)$-gon S_{N+M} has vertices

$$\left(\frac{2N+M}{N}\left|\frac{C}{A}\right|\right)^{\frac{1}{N+M}} e^{i\frac{\gamma-\alpha+(2\lambda+1)\pi}{N+M}}, \quad \lambda = 1, ..., N+M.$$

The N-gon S_N has vertices

$$\left(\frac{2N+M}{N+M}\left|\frac{B}{A}\right|\right)^{\frac{1}{N}} e^{i\frac{\beta-\alpha+(2v+1)\pi}{N}}, \quad v = 1, ..., N.$$

If at the vertices of these polygons unit masses are placed, and these masses determine a force field that is inversely proportional to distance, then the solutions to the trinomial equation are precisely the equilibrium points of these force fields.

Consider the equilibrium points of the above theorem. No equilibrium point may be further from the origin than any of the polygon vertices, for at such a location there would be a non-zero net force. Thus we have the following corollary.

Corollary 3.28. *If z is a solution to the equation $Az^{N+M} + Bz^M + C = 0$, then*

$$|z| \leq \max\left\{\left(\frac{2N+M}{N}\left|\frac{C}{A}\right|\right)^{\frac{1}{N+M}}, \left(\frac{2N+M}{N+M}\left|\frac{B}{A}\right|\right)^{\frac{1}{N}}\right\}.$$

Applying this to $w(y)$ (3.4), we have $M = 1$, $N = p$, $A = 1$, $B = -(p+1)$, and $C = p$. Substituting, we get

$$\left(\frac{2N+M}{N}\left|\frac{C}{A}\right|\right)^{\frac{1}{N+M}} = (2p+1)^{\frac{1}{p+1}}$$

and

$$\left(\frac{2N+M}{N+M}\left|\frac{B}{A}\right|\right)^{\frac{1}{N}} = (2p+1)^{\frac{1}{p}}.$$

Therefore, any root y of $w(y)$ satisfies

$$|y| \leq (2p+1)^{\frac{1}{p}}.$$

Since the non-zero roots x of $D(K_{a_1,\ldots,a_p};x)$ were s^{th} roots of such roots $y \neq 1$, it follows that the non-zero roots x of $D(K_{a_1,\ldots,a_p};x)$ satisfy

$$|x| \leq (2p+1)^{\frac{1}{ps}}.$$

In each plot of Figure 3.20, the circle $|x| = (2p+1)^{\frac{1}{ps}}$ is shown in blue for the corresponding values of p and s.

We may also examine the limits of the roots of $D(K_{a_1,\ldots,a_p};x)$ as $p \to \infty$ by applying the extended BKW theorem (Theorem 1.7) to $w(y)$. First, we rewrite $w(y)$ and relabel it as $w_p(y)$:

$$w_p(y) = y \cdot y^p + (p(1-y) - y).$$

Labelling each part of $w_p(y)$ in the notation of Theorem 1.7, we have: $\lambda_1(y) = y$, $\alpha_1(p;y) = y$, so $d_1 = 0$ and $q_{1,d_1}(y) = q_{1,0} = y$. Similarly, $\lambda_2(y) = 1$, $\alpha_2(p;y) = p(1-y) - y$, so $d_2 = 1$ and $q_{2,d_2}(y) = q_{2,1}(y) = 1 - y$. Indeed, $\lambda_1 \neq \omega\lambda_2$ for any complex ω such that $|\omega| = 1$, so we are able to apply Theorem 1.7:

Condition 1:

- $|\lambda_1(y)| > |\lambda_2(y)|$ and $q_{1,d_1}(y) = 0$: the inequality implies $|y| > 1$ while $q_{1,d_1}(y) = 0$ gives $y = 0$, which is a contradiction.

- $|\lambda_1(y)| < |\lambda_2(y)|$ and $q_{2,d_2}(y) = 0$: the inequality implies $|y| < 1$ while $q_{2,d_2}(y) = 0$ gives $y = 1$, which again gives a contradiction.

Condition 2:

- $|\lambda_1(y)| = |\lambda_2(y)| > 0$, and at least one of $q_{1,d_1}(y)$, $q_{2,d_2}(y)$ are non-zero: this gives $|y| = 1$, and indeed $q_{1,d_1}(y) = y$ is non-zero for such y.

From Condition 2, we can conclude that the limits of the roots of $w_p(y)$ (or $w(y)$) as $p \to \infty$ are the points $\{z \in \mathbb{C} \ : \ |z| = 1\}$, or simply the complex unit circle. Consequently, since $y = x^s$, the non-zero roots of $D(K_{a_1,\ldots,a_p};x)$ have the same limits as $p \to \infty$.

Chapter 4

Conclusion

4.1 Open Problems

Many questions about degree roots remain unanswered. For instance, there is our conjecture concerning the degree roots of trees which proposes there are no non-real degree roots of trees inside the unit circle. In this section we discuss a few open problems of degree roots.

We did not focus on the degree roots of multigraphs, save only to briefly compare or contrast them with degree roots of (simple) graphs. Hence, there is much left to wonder about the degree roots for multigraphs. For example, how do multigraph degree roots behave when imposing restrictions to n and Δ as we did for (simple) graphs? For fixed n, what are some degree roots for multigraphs that aren't degree roots for (simple) graphs?

Returning to the degree roots of (simple) graphs, in Theorem 3.6 we saw that degree roots were dense in the negative real axis. In particular, we observed that every negative rational number is a degree root of some graph. One question we may ask is how rational degree roots depend on graph order n. That is, for graphs of some order n what rational degree roots may we see? Table 4.1 lists all rational degree roots that appear for some small values of n. Other than seeing some expected patterns, such as -1 only appearing when n is a multiple of 4 and the appearance of $-(n-1)$ as a degree root for odd n, we can make some observations: first, it appears that every rational degree root $-\frac{p}{q}$ (in fully reduced form) satisfies $(p+q)|n$. We also notice that for odd n, all rational numbers $-\frac{n-k}{k}$, for $1 \leq k \leq n$ seem to be degree roots. For even n, it appears there are fewer rational roots, and that a (fully reduced) root $-\frac{p}{q}$ satisfies $p+q=\frac{n}{2}$.

The rational root theorem tells us that a degree polynomial $D(G;x) = a_{\Delta}x^{\Delta} + \cdots + a_{\delta}x^{\delta}$ with rational root $-\frac{p}{q}$ satisfies $q|a_{\Delta}$ and $p|a_{\delta}$. Therefore, $(p+q) \leq a_{\Delta} + a_{\delta} \leq n$, but the divisibility is yet to be proven (or disproven). Our observation for odd n and

n	Rational Degree Roots
2	0
3	0, $-\frac{1}{2}$, -2
4	0, -1
5	0, $-\frac{1}{4}$, $-\frac{2}{3}$, $-\frac{3}{2}$, -4
6	0, $-\frac{1}{2}$, -2
7	0, $-\frac{1}{6}$, $-\frac{2}{5}$, $-\frac{3}{4}$, $-\frac{4}{3}$, $-\frac{5}{2}$, -6
8	0, $-\frac{1}{3}$, -1, -3

Table 4.1: Rational degree roots for graphs of small order.

the roots of the form $-\frac{n-k}{k}$, however, can be confirmed. For $k = 1$, we already know the existence of the degree root $-(n-1)$. For odd $k \geq 3$, construct the following graph: take the union of a copy of C_k and $(n-k)/2$ copies of P_2. This graph has degree polynomial $kx^2 + (n-k)x$, having a root at $-\frac{n-k}{k}$. For even k, take the graph that is the union of $k/2$ copies of P_2 and $n-k$ isolated vertices. This graph has degree polynomial $kx + n - k$, and thus has a root at $-\frac{n-k}{k}$. Thus for all k $(1 \leq k \leq n)$, $-\frac{n-k}{k}$ is a degree root.

Another area deserving of further study is the bounding of degree roots. Corollary 3.10 bounds the moduli of degree roots for graphs of order n to be at most $n-1$. However, only real roots meet this bound and non-real roots seem to fall quite short of it (see Figure 4.1). Furthermore, there seems to be a relationship between the argument of a degree root and its maximum possible modulus: roots with smaller argument $\theta \in (0, \pi]$ seem to have smaller moduli. Here we conjecture a modified bound to the moduli of degree roots that takes into account this behaviour.

Conjecture 2. *If z is a non-zero degree root of a graph of order n with argument* $\arg(z) = \theta \in (-\pi, \pi]$, *then*

$$|z| \leq (n-1)^{\left|\frac{\theta}{\pi}\right|}.$$

This bound agrees with Corollary 3.10 for real roots $(\theta = \pi)$ and Proposition 3.13 for imaginary roots $(\theta = \pm\pi/2)$. Figure 4.1 shows the curve $|z| = (n-1)^{\left|\frac{\theta}{\pi}\right|}$ along with the circular bound $|z| = n-1$ on plots of degree roots for some small values of n. This conjecture has been verified for $n \leq 9$.

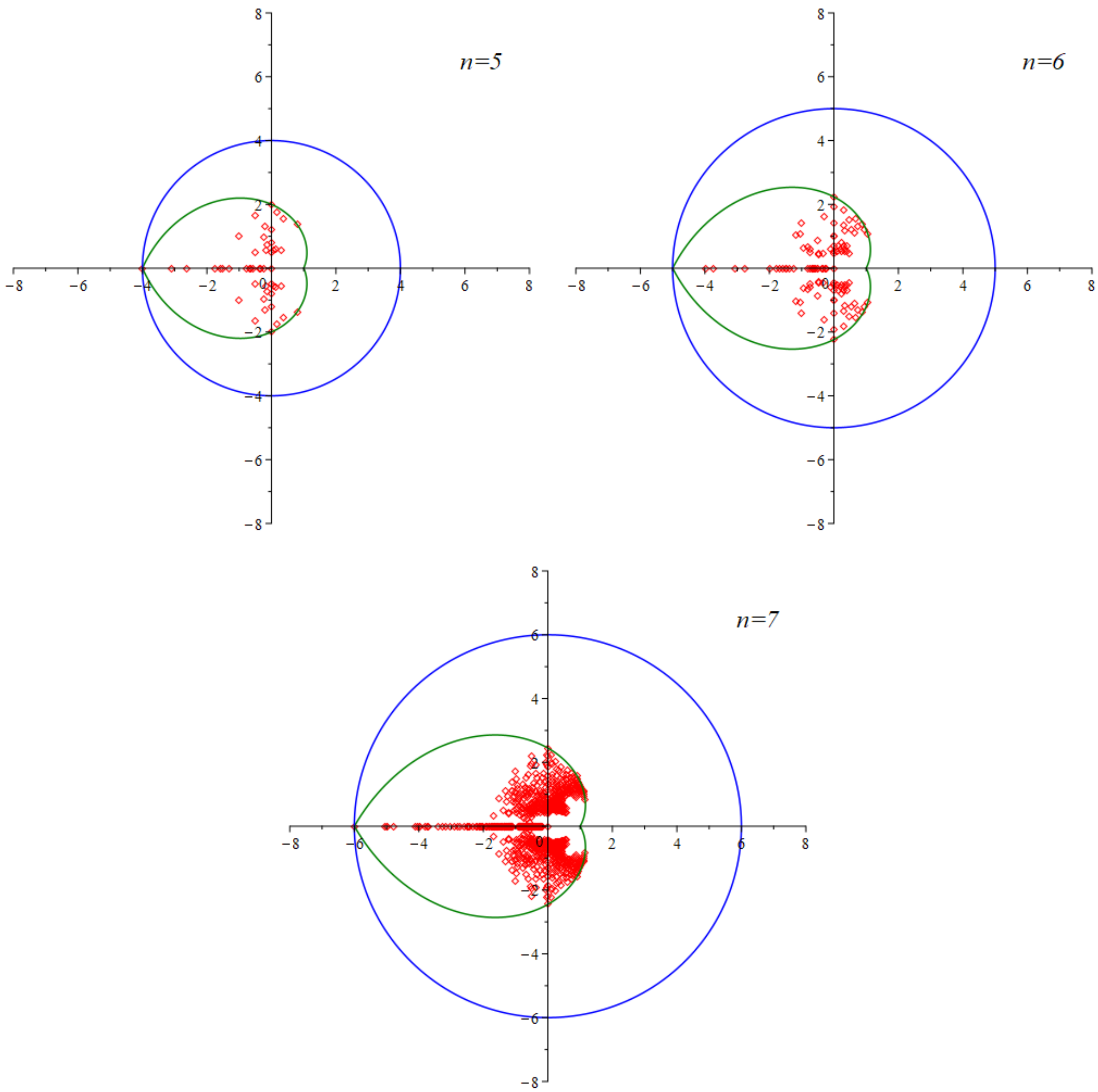


Figure 4.1: Degree roots for some small values of n (red). The blue curves show $|z| = n - 1$, while the green curves are $|z| = (n-1)^{\left|\frac{\theta}{\pi}\right|}$.

4.2 Generalizations

There are many ways in which to generalize the degree polynomial. Here we present a few possible generalizations and extensions with some supporting examples that may motivate future directions of study.

4.2.1 Directed Graphs

Here we give some extensions for directed graphs. A *directed graph* is an ordered pair $\overrightarrow{G} = (V, \overrightarrow{E})$ where V is a set of vertices and $\overrightarrow{E}$ is a set of ordered pairs of vertices, called (directed) edges. A (directed) edge $(u, v) \in \overrightarrow{E}$ represents a directional adjacency from vertex u to vertex v (but not vice-versa). If both (u, v) and (v, u) are present, we may simply consider there to be an undirected edge between u and v with the understanding that there is adjacency in both directions. See Figure 4.2 for an example of a directed graph. For a vertex $v \in V$, its *out-degree* $\mathrm{odeg}_{\overrightarrow{G}}(v)$ is the number of edges for which v is the first coordinate. Similarly, its *in-degree* $\mathrm{ideg}_{\overrightarrow{G}}(v)$ is the number of edges for which v is the second coordinate.

The first way we could extend the degree polynomial is with the following *directed degree polynomial*:

$$D_{Dir}(\overrightarrow{G}; x, y) = \sum_{v \in V} x^{\mathrm{odeg}_{\overrightarrow{G}}(v)} y^{\mathrm{ideg}_{\overrightarrow{G}}(v)}.$$

This directed degree polynomial generalizes the ordinary degree polynomial, since if G is the underlying undirected graph of $\overrightarrow{G}$ (formed by replacing directed edges with undirected edges) then

$$D(G; x) = D_{Dir}(\overrightarrow{G}; x, x),$$

as $\deg_G(v) = \mathrm{odeg}_{\overrightarrow{G}}(v) + \mathrm{ideg}_{\overrightarrow{G}}(v)$.

As an example, consider the directed graph $\overrightarrow{H}$ from Figure 4.2. Vertex 0 has an out-degree of 3 and in-degree of 0, so it contributes a term x^3. Vertex 2, having an out-degree and in-degree of 1, contributes a term xy. The terms for the remaining vertices are found similarly, and thus we find the directed degree polynomial of $\overrightarrow{H}$ to be $D_{Dir}(\overrightarrow{H}; x, y) = x^3 + y^2 + x^2y + xy$.

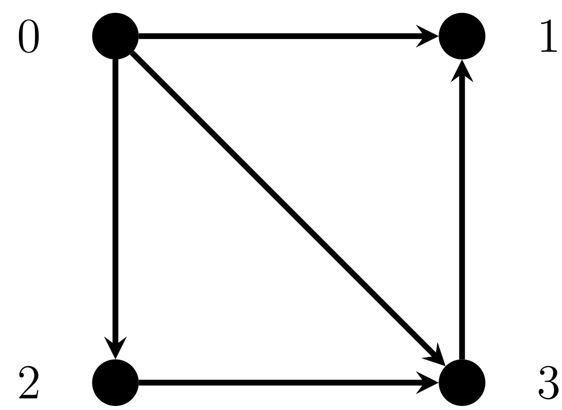


Figure 4.2: A directed graph $\overrightarrow{H}$.

Another approach to a degree polynomial for directed graphs is based on Laurent polynomials. A Laurent polynomial with real coefficients $\mathcal{L}(x)$ is an element of the ring $\mathbb{R}[x, x^{-1}]$. That is, $\mathcal{L}(x)$ has the form

$$\mathcal{L}(x) = \sum_{k=-a}^{b} c_k x^k$$

where a, b are positive integers and $c_k \in \mathbb{R}$. We define the following *Laurent-degree polynomial* for directed graphs:

$$D_{Lau}(\overrightarrow{G}; x) = \sum_{v \in V} \left(x^{\text{odeg}_{\overrightarrow{G}}(v)} + x^{-\text{ideg}_{\overrightarrow{G}}(v)} \right).$$

Observe that this polynomial does not extend the ordinary degree polynomial in the same sense that the previous does. We cannot obtain the ordinary degree polynomial of the underlying undirected graph from this polynomial.

Consider again the directed graph in Figure 4.2. Vertex 0 contributes $x^3 + 1$ to the polynomial, while vertex 2 contributes $x + 1/x$. Summing the contributions of each vertex, we find $D_{Lau}(\overrightarrow{H}; x) = x^3 + 2x + 1/x + 2/x^2 + 2$.

4.2.2 Multivariable Polynomials

We can also form multivariable versions of the degree polynomial using labellings of vertices and edges. The extensions in this section will be defined for multigraphs to allow a more general treatment of graphs.

Suppose $M = (V, E)$ is a multigraph where $|V| = n$, $|E| = m$, and let $w(u, v)$ be the number of edges between vertices u and v. If we associate a variable x_k to vertex v_k ($1 \leq k \leq n$), then we can form a *vertex-labelled degree polynomial*:

$$D_{vl}(M; x_1, ..., x_n) = \sum_{j=1}^{n} \prod_{k=1}^{n} x_k^{w(v_j, v_k)}.$$

Notice that this polynomial generalizes the ordinary degree polynomial since we have $D(M; x) = D_{vl}(M; x, ..., x)$. Similarly, if we associate a variable y_k to edge e_k, we obtain an *edge-labelled degree polynomial*:

$$D_{el}(M; y_1, ..., y_m) = \sum_{j=1}^{n} \prod_{\substack{e_k \in E \\ e_k \ni v_j}} x_k.$$

This polynomial generalizes the ordinary degree polynomial in the same sense as the previous: $D(M; x) = D_{el}(M; x, ..., x)$.

Let us illustrate these polynomials with an example, beginning with the vertex-labelled degree polynomial. Consider the multigraph H in Figure 4.3. Vertex 1 has neighbours $2, 3, 4$, though there are two edges to vertex 3. Thus it contributes a term $x_2 x_3^2 x_4$. Vertex 2 contributes a term $x_1 x_4$. Computing the other terms similarly, we have

$$D_{vl}(H; x_1, x_2, x_3, x_4) = x_2 x_3^2 x_4 + x_1 x_4 + x_1^2 x_4 + x_1 x_2 x_3.$$

For the edge-labelled degree polynomial, we see that vertex 1 belongs to edges a, b, c, and d. Thus this vertex contributes a term $y_a y_b y_c y_d$ to the polynomial. Vertex 2 contributes a term $y_a y_e$, and we can compute the other terms to obtain

$$D_{el}(H; y_a, y_b, y_c, y_d, y_e, y_f) = y_a y_b y_c y_d + y_a y_e + y_c y_d y_f + y_b y_e y_f.$$

4.3 Concluding Remarks

This thesis has presented a number of topics related to the degree polynomial of graphs. Our goal was to contribute to the new and small body of work regarding this graph polynomial, and generate interest for future research. Some properties and

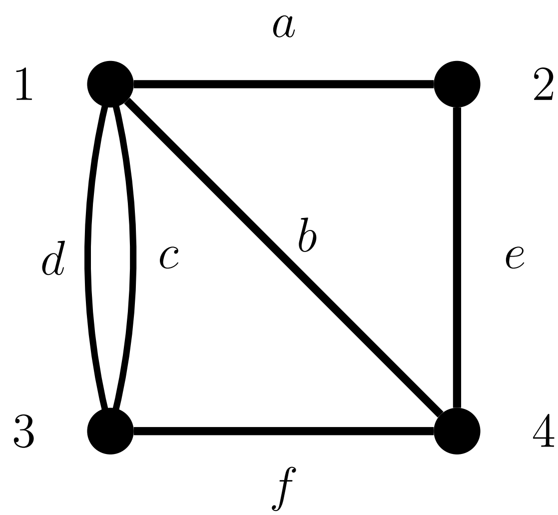


Figure 4.3: A multigraph H with labelled vertices and edges.

special evaluations of the degree polynomial were discussed, notably that the degree polynomial precisely encodes the degree sequence of a graph. Much of our focus was on degree roots, or roots of the degree polynomial. We explored the connection between degree roots for (simple) graphs, multigraphs, and the roots of polynomials with non-negative integer coefficients. In particular, it was found that the sets of these roots are equivalent. Inspired by research into the density of roots for other graph polynomials, we showed that degree roots are dense in the complex plane. Our investigation into the effect of restricting certain graph parameters on degree roots showed a distinction between the degree roots of (simple) graphs, multigraphs, and the roots of polynomials with non-negative integer coefficients for fixed graph order. Furthermore, these restrictions allowed for degree roots to be bounded in modulus. We also examined the degree roots for some families of graphs. Dealing with more explicit degree polynomials, we were able to prove more precise statements about the degree roots, detect their interesting behaviour, and even calculate limits of the roots. Future research on the degree polynomial may involve further investigation into the topics we have tackled thus far, specifically to answer problems that remain open, or might involve generalizations/extensions of the degree polynomial.

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