An Introduction to Totally Cocomplete Categories

by

Michael Albert Wendt

©

Submitted in partial fulfillment of the requirements
for the degree of Master of Science
at
Dalhousie University
Halifax, Nova Scotia
August, 1985

Dalhousie University

Department of

Mathematics, Statistics and Computing Science

The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "An Introduction to Totally Cocomplete Categories" by Michael Albert Wendt in partial fulfillment of the requirements for the degree of Master of Science.

Dated: 19 Sept 1985

Supervisor: RJWood

Readers:

K. Johnson

DALHOUSIE UNIVERSITY

Date Sept 20 1985

Author Michael A. Wendt

Title An Introduction to Totally Cocomplete Categories

Department or School Mathematics, Statistics and Computing Science

Degree M.Sc. Convocation Fall Year 1985

Permission is herewith granted to Dalhousie University to circulate and to have copied for non-commercial purposes, at its discretion, the above title upon the request of individuals or institutions.

Michael A. Wendt

Signature of Author

To my parents.

Table of Contents

Abstract vi

Notation vii

Acknowledgements viii

Introduction 1

§1–Preliminaries 5

§2–Cocompleteness in a poset 12

§3–Totality 16

§4–Examples and closure properties 21

§5–Total adjoint functor theorem 28

§6–Duality 32

§7–Topological examples 43

§8–Cartesian closedness 48

§9–Lex total categories 51

§10–n–total categories 61

Bibliography 67

Abstract

A total category is defined as a locally small category whose Yoneda embedding, Y, has a left adjoint, L. Totality implies cocompleteness (and completeness). The converse is not true. However, many familiar cocomplete categories are total. In fact, total categories enjoy good closure properties.

In the total setting, arguments are more conceptual than for merely cocomplete categories; often expressed in terms of adjointness situations. For example, one may specialize total categories by considering lex total categories, total categories whose L is lex. Such categories are closely related to topoi.

Two interesting conjectures are introduced. Attempts to characterize $\mathbf{set}^{\mathbf{A}^{op}}$ (for small $\mathbf{A}$) and $\mathbf{set}$, via adjoints left of Yoneda, are made.

Notation

Most of the notation we use is standard except where, for typographical reasons, conventions are not followed.

Categories such as **set**, **grp**, and **top** are denoted by boldface lower case. Small categories are denoted by bold face upper case ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, ...). Large categories are denoted by upper case calligraphic letters ($\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ...).

"$\longrightarrow$" is used for morphism, functor, or natural transformation. Composition of morphisms is written in the "functional" way. Generally, arguments are written on the right. However, in the name of a functor, the argument is sometimes written as a blank space or a "–". "$\hookrightarrow$" denotes a monomorphism. An identity arrow is denoted by " $A \xrightarrow{A} A$ " or "$A \xrightarrow{1_A} A$."

Set theoretic notation is standard with the exception that $\subseteq$ means inclusion and $\subset$ means *strict* inclusion.

Other notation is defined in the text.

Acknowledgements

With utmost gratitude, I would like to thank my advisor, Professor R.J. Wood, for his patience, guidance, and encouragement.

I would also like to thank Dalhousie University for providing financial support during my tenure as a Master's student.

Thanks go to Fran Ornstein, for help in typesetting the manuscript.

Introduction

One may think of a category, $\mathcal{B}$, as a generalized poset, $\mathbf{P}:=(P, \leq)$, in the sense that a poset is a category with hom sets equal to 0 or 1. For a poset, $\mathbf{P}$, it is well known that sup–completeness, existence of $\bigvee_{i\in I} a_i$, for all families $(a_i)_{i\in I}$ in $\mathbf{P}$, generalizes to small cocompleteness for a category; existence of $\varinjlim_{i\in\mathbf{I}} F(i)$ for all small diagrams $F:\mathbf{I}\longrightarrow\mathcal{B}$.

In the poset case, we can phrase our discussion of sup–completeness in terms of sups of families, $(a_i)_{i\in I}$, which form a downclosed subset of P. Generalizing this form of sup–completeness to categories gives a notion of cocompleteness which is, in general, strictly stronger than small cocompleteness. We make this more precise.

Sup–completeness, as defined by sups of families, may also be expressed by asserting the existence of left adjoints to the diagonals $\mathbf{P}\longrightarrow\mathbf{P}^I$ for each (small) set I. In a similar manner, for small cocompleteness of a category, $\mathcal{B}$, we may ask for left adjoints to the diagonals $\mathcal{B}\longrightarrow\mathcal{B}^{\mathbf{I}}$ for each small category $\mathbf{I}$.

Now (in the poset case), as suggested above, existence of left adjoints to the diagonals may be replaced by "existence of a single left adjoint to $\downarrow:\mathbf{P}\longrightarrow\mathcal{D}\mathbf{P}$" where $\mathcal{D}\mathbf{P}$ denotes the set of downclosed subsets of P ordered by inclusion and $\downarrow(a):=\{b\in P|b\leq a\}$ for $a\in P$. $\mathcal{D}\mathbf{P}$ is equivalent to $2^{\mathbf{P}^{op}}$. Moreover, $\downarrow:\mathbf{P}\longrightarrow 2^{\mathbf{P}^{op}}$ corresponds, via exponential adjointness, to the order relation, $\mathbf{P}^{op}\times\mathbf{P}\longrightarrow 2$, on $\mathbf{P}$.

The "order relation" for a locally small category, $\mathcal{B}$, is the hom functor,

$\mathcal{B}(-,-) : \mathcal{B}^{op} \times \mathcal{B} \longrightarrow \mathbf{set}$, where $\mathcal{B}(A, B)$ denotes the set of "reasons" for $A \leq B$. Indeed, a poset is just a category whose hom functor factors through the inclusion, $2 \longrightarrow \mathbf{set}$. The transpose, through exponential adjointness, of the hom functor for $\mathcal{B}$ is called the Yoneda functor, $Y : \mathcal{B} \longrightarrow \mathbf{set}^{\mathcal{B}^{op}}$, and posets are exactly those categories whose Yoneda functor factors through $2^{\mathbf{P}^{op}} \longrightarrow \mathbf{set}^{\mathbf{P}^{op}}$.

Note that $2 \longrightarrow \mathbf{set}$ has a left adjoint so $\downarrow : \mathbf{P} \longrightarrow 2^{\mathbf{P}^{op}}$ has a left adjoint iff $\mathbf{P} \longrightarrow \mathbf{set}^{\mathbf{P}^{op}}$ does. Thus, sup–completeness in a poset is equivalent to the existence of a left adjoint to $\mathbf{P} \longrightarrow \mathbf{set}^{\mathbf{P}^{op}}$. This idea leads us to ask for the existence of a left adjoint to Yoneda for an arbitrary category, $\mathcal{B}$. Following Street and Walters, we call such a category ***totally cocomplete*** or, simply, ***total***.

Total categories were first introduced, in published form, in "Yoneda Structures in 2–categories" [S&W1]. Street and Walters defined the notions of total arrow and total object in a 2–category setting. Subsequent works, such as [RJW1], [Th] and [Wal], explored many properties of total categories some of which are presented here. Later papers in the subject dealt with special types of total categories. One example is [St1], in which Street gives some relationships between topoi and lex total categories; total categories for which the left adjoint to Yoneda is left exact.

A total category is small complete and small cocomplete. The converse is not true. Despite the fact that not all cocomplete categories are total, a wealth of examples exists. Many familiar categories, such as **set**, **grp**, and **top**, are total.

One motivation for studying total categories, apart from the unification of certain properties of the examples, is the total adjoint functor theorem which eliminates, for total categories, the solution set condition of Freyd's adjoint functor theorem.

Section 1 consists of some of the necessary set theoretic preliminaries such as the distinction between large and small sets. Large, small and locally small categories are defined. We also reproduce Street's interesting characterization of small categories. In sections 2 and 3, we make precise the ideas about posets discussed above. We said that many familiar categories are total. Section 4 is a listing of some "closure properties" of total categories. That is, it gives some methods of determining whether a category is total. Other examples are discussed in section 7.

The total adjoint functor theorem is proved in section 5. As well, a "homomorphism-like" property of cocontinuous functors between total categories is proved as a consequence of the total adjoint functor theorem. Other consequences are introduced in later sections.

The dual of total is *cototal*. A category is said to be cototally complete if it is locally small and its co–Yoneda embedding has a right adjoint. Totality and cototality are not equivalent as is shown in section 6. However, they may be related (other than by dualization). A category which is both total and cototal shares many of the properties of a sup–complete poset. **set** and, more generally, Grothendieck topoi are categories which are both total and cototal.

In section 8, we prove another property of total categories; a relationship between cartesian closed and total categories. Explicitly, if B is total, then B is cartesian closed iff the left adjoint (to Yoneda) preserves binary products. We also define a lex total category in section 8.

Some examples of lex totals are listed in section 9 and some of their elementary properties are given. A lex total category has many of the exactness properties of topoi. In fact, lex totals are "very nearly" topoi and in section 9, we state some of the results of Street's comparison of lex totals and topoi. [St1]

As a special case of lex totals, one may consider n–totals $n \in \mathbf{N}$. That is, locally small categories for which there is a string of n adjunctions left of Yoneda. Examples and elementary properties of such are discussed in section 10.

1–Preliminaries

One has an intuitive notion of **SET**, the category of sets and functions. Within **SET**, we distinguish certain sets which we call small (see, for example, [Mac]). Recall the notion of *strongly inaccessible cardinal* . That is, a cardinal $\mathcal{N}$, with the following two properties:

1. For every set, I, with $|I| < \mathcal{N}$ and for every family of sets $(X_i)_{i \in I}$ with $|X_i| < \mathcal{N}, \forall i \in I;\ |\bigcup_{i \in I} X_i| < \mathcal{N}$.
2. For every set, X, with $|X| < \mathcal{N}$, $|\mathcal{P}(X)| < \mathcal{N}$ where $\mathcal{P}(X)$ denotes the set of subsets of X.

We assume the existence of such a cardinal with $\mathcal{N} > |\mathbf{N}|$, the cardinality of the natural numbers. If $\mathcal{U}$ is such that $|\mathcal{U}| = \mathcal{N}$, $\mathcal{U}$ is called a ***universe.*** Using these assumptions, $\{X |\, |X| < \mathcal{N}\}$ provides a model for Zermelo-Frankel set theory.

Definition 1–1: We say "X is a small set" to indicate $|X| < \mathcal{N}$. **set** denotes the category of all small sets and functions. We say "X is a large set" to indicate $|X| \geq \mathcal{N}$. □

The small sets are the usual building blocks of mathematics. They include, for example, $\mathbf{N}$, $\mathbf{R}$, $\mathcal{P}(\mathbf{R})$.

We denote by **cat** := **cat(set)** the (2-)category of category objects in **set**. The objects of **cat** are called small categories. Notice that among the objects of **CAT** := **cat(SET)** are small categories. A category which is not small is said to be large. Note also that **set**$\notin$ **cat** but **set**$\in$ **CAT** , the set of all small sets being large.

Definition 1–2: $\mathcal{C}$ in **CAT** is said to be locally small (or to have small hom sets) if $\mathcal{C}(A, B) \in \mathbf{set}\ \forall A, B \in \mathcal{C}$. □

Certainly a small category is locally small. Street [St2] and Freyd have given a characterization of small categories in terms of local smallness.

Proposition 1–3: (i)–If **C** is small, then **C** and $\widehat{\mathbf{C}} := \mathbf{set}^{\mathbf{C}^{op}}$ are both locally small.

(ii)– If **C** and $\widehat{\mathbf{C}}$ are locally small, then **C** is essentially small (i.e. equivalent to a small category)

Proof: (i)– If **C** is small, it is locally small. Furthermore, for $\Phi, \Psi \in \widehat{\mathbf{C}}$,

$$\widehat{\mathbf{C}}(\Phi, \Psi) \simeq \int_{C \in \mathbf{C}} \mathbf{set}(\Phi(C), \Psi(C))$$

Now, $\mathbf{set}(\Phi(C), \Psi(C))$ is small (**set** is locally small) so the above is a small end of small sets. Hence $\widehat{\mathbf{C}}(\Phi, \Psi) \in \mathbf{set}$ and$\widehat{\mathbf{C}}$ is locally small.

(ii)–(Street) Assume $\mathbf{C}$ is skeletal. We show that $\mathbf{C}$ is small and hence that in the general case $\mathbf{C}$ is essentially small. It suffices to show that $Ob(\mathbf{C})$ is small, for $\mathbf{C}$ is locally small and

$$Mor(\mathbf{C}) = \sum_{A,B \in Ob(\mathbf{C})} \mathbf{C}(A,B)$$

We proced to show $\exists T$ such that $Ob(\mathbf{C}) \hookrightarrow \widehat{\mathbf{C}}(T,T)$; whence $\mathbf{C}$ is small.

Define $T\colon \mathbf{C}^{op} \longrightarrow \mathbf{set}$ by $T(C) = \{c\colon C \to B | c \text{ is a split epi } \} + \{0\}$ and for $a\colon C \to C'$ in $\mathbf{C}$, $T(a)\colon T(C') \to T(C)$ by

$$T(a)(c') = \begin{cases} c'a, & \text{if } c'a \text{ is a split epi;} \\ 0, & \text{if not.} \end{cases}$$

and $T(a)(0) = 0$.

We must check that T is well defined. That is, that $T(C) \in \mathbf{set}$ and T is functorial.

Consider $\bar{T}(C) := \{(c,b) | C \underset{c}{\overset{b}{\leftrightarrows}} B, c \in T(c) \text{ and } cb = B\}$

We have $\bar{T}(C) \longrightarrow T(C)$ ($(c,b) \mapsto c$) ; an onto function.

Define $\bar{T}(C) \xrightarrow{h} \mathbf{C}(C,C)$ by $h(c,b) = bc$. Now, bc is an idempotent since $(bc)(bc) = b(cb)c = bc$. Suppose $h(c,b) = h(c',b')$ where $B \xrightarrow{b} C \xleftarrow{b'} B'$ Then $(cb')(c'b) = c(b'c')b = c(bc)b = (cb)(cb) = BB = B$ and $(c'b)(cb') = c'(bc)b' = c'(b'c')b' = (c'b')(c'b') = B'B' = B'$; whence $B \simeq B'$. But $\mathbf{C}$ is skeletal, so $B = B'$.

$\bar{T}(C) = \sum_{e \in \mathbf{C}(C,C)} h^{-1}(e)$; a small sum. We proceed to show $h^{-1}(e) \in \mathbf{set}$; whence $\bar{T}(C)$ is small.

Now, $h^{-1}(e) = \{(c,b) | bc = e\}$

$$= \begin{cases} \emptyset, & \text{e not idem} \\ \{(c,b) | C \underset{c}{\overset{b}{\leftrightarrows}} B \text{ such that } bc = e \text{ for unique } B\} & \text{otherwise} \end{cases}$$

So $h^{-1}(e) \subseteq \mathbf{C}(C,B) \times \mathbf{C}(B,C)$ for a single B; and so $h^{-1}(e) \in$ **set.**

We have $\bar{T}(C) \longrightarrow T(C)$ onto so $T(C) \in$ **set** (a quotient of a small set is a small set).

T is functorial:

Identity condition: For $C \xrightarrow{C} C$, $T(C) \xrightarrow{T(C)} T(C)$ is given by $T(C)(c) = Cc = c$ and $T(C)(0) = 0$.

Composition: Suppose $C'' \xrightarrow{a'} C' \xrightarrow{a} C$ are in $\mathbf{C}$. Consider $T(C) \xrightarrow{T(aa')} T(C'')$ and $T(C) \xrightarrow{T(a)} T(C') \xrightarrow{T(a')} T(C'')$. We have $T(a'a)(0) = 0 = T(a')T(a)(0)$.

For $C \xrightarrow{c} B$ a split epi, we must consider cases:

1: ca split epi, caa' split epi : $T(aa') = caa'$ and $T(a')T(a)(c) = T(a')(ca) = caa'$.

2: ca split epi, caa' not a split epi: $T(aa') = 0$ and $T(a')T(a)(c) = T(a')(ca) = 0$.

3: ca not a split epi: $T(aa')(c) = 0$, since if ca is not a split epi then neither is caa', and $T(a')T(a)(c) = T(a')(0) = 0$.

Thus, $T \in \widehat{\mathbf{C}}$ as claimed.

Finally, define $m : Ob(\mathbf{C}) \longrightarrow \widehat{\mathbf{C}}(T,T)$. For $X \in Ob(\mathbf{C})$, mX is the natural transformation $T \xrightarrow{mX} T$ whose C^{th} component $T(C) \xrightarrow{mXC} T(C)$ is given by $mXC(0) = 0$ and for $C \xrightarrow{c} B$,

$$mXC(c) = \begin{cases} 0 & \text{if } B \neq X \\ c & \text{if } B = X \end{cases}$$

By considering cases, as with T above, we see that m is well defined.

We claim that m is one-to-one: Suppose $X \neq Y$ and consider $1_X \in T(X)$ ($X \xrightarrow{1_x} X$ being a split epi). Now, $mXX(1_X) = 1_X$ and $mYX(1_X) = 0$ so $mX \neq mY$.

Thus, $Ob(\mathbf{C})$ is small as required. This completes the proof. ∎

"small" is used as an adjective in appropriate situations. For example,

Definition 1–4: $\mathcal{C}$ in **CAT** is (small) complete (respectively (small) cocomplete) if $\forall \mathbf{D} \in \mathbf{cat}$ and all functors $F : \mathbf{D} \longrightarrow \mathcal{C}$, $\varprojlim F$ (respectively $\varinjlim F$) exists. $\square$

For small categories, small completeness is trivial:

Proposition 1–5:(Freyd): A small category, **C**, which is small complete is simply a preorder which has an inf for each small set of its elements.

Proof: $\Leftarrow$: categorical product in a preorder is inf and a preorder has equalizers: $(S \xrightarrow{\preceq} S \overset{\preceq}{\underset{\preceq}{\rightrightarrows}} T)$.

$\Rightarrow$: [Mac] p.110: Suppose **C** is not a preorder. Then there are objects S, $T \in \mathbf{C}$ and arrows $S \overset{f}{\underset{g}{\rightrightarrows}} T$ with $f \neq g$. Let $I \in \mathbf{set}$ and form $\prod_{i \in I} T$. There are at least 2^I arrows $S \longrightarrow \prod T$ since each arrow is a family $(S \longrightarrow T)_{i \in I}$ and each member of the family can be either f or g. By taking $|I| \geq |Mor(\mathbf{C})|$, $I \subset 2^I \subseteq \mathbf{C}(S, \prod t) \subseteq Mor(\mathbf{C})$; yields a contradiction. ■

Note that $\varinjlim$ and $\varprojlim$, considered as functors, occur in the adjointness situation:

$$\begin{tikzcd}
\mathcal{C}^{\mathbf{D}} \arrow[r, shift left=6, "\varinjlim"] \arrow[r, shift right=6, "\varprojlim"'] & \mathcal{C} \arrow[l, "\Delta\ \perp"', "\perp"]
\end{tikzcd}$$

where $\Delta : \mathcal{C} \longrightarrow \mathcal{C}^{\mathbf{D}}, \Delta(C)(D) = C$ for $C \in \mathcal{C}$, $D \in \mathbf{D}$.

Familiar categories such as **set**, **grp**, and **top** are both small complete and small cocomplete. A category that is small complete is not necessarily small cocomplete (and vice-versa). Adámek [Ad] gives an example.

Example 1–6: Let $P_\star\colon \mathbf{set} \longrightarrow \mathbf{set}$ be given by $P_\star(A) = \{A_0 \subseteq A | A_0 \neq \emptyset\}$ and for $A \xrightarrow{f} B$, $P_\star(A) \xrightarrow{P_\star(f)=f(\ )} P_\star(B)$.

Consider the category $(P_\star; \mathbf{set})$ whose objects are pairs (A, a) where $P_\star(A) \xrightarrow{a} A$ and whose morphisms are functions $A \xrightarrow{f} B$ such that

$$\begin{tikzcd}
P_\star(A) \arrow[r, "f(\ )"] \arrow[d, "a"'] & P_\star(B) \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

commutes.

$(P_\star; \mathbf{set})$ is complete. Indeed, $U\colon (P_\star; \mathbf{set}) \longrightarrow \mathbf{set}$ ($(A, a) \mapsto A$) creates all limits.

We claim that $(P_\star; \mathbf{set})$ is not cocomplete.

Let (A, a) and (B, b) be nonempty. We proceed to show that they have no coproduct.

Let X be any set containing A and B disjointly and define $x\colon P_\star(X) \longrightarrow X$ by

$$xX_0 = \begin{cases} aX_0, & \text{if } X_0 \subseteq A; \\ bX_0, & \text{if } X_0 \subseteq B; \\ \text{arbitrarily}, & \text{if } X_0 = X; \\ \text{any} x \notin X_0, & \text{otherwise.} \end{cases}$$

Assume that (A, a) and (B, b) have a coproduct (C, c). The inclusions $A \hookrightarrow X$ and $B \hookrightarrow X$ induce morphisms $(A, a) \longrightarrow (X, x)$ and $(B, b) \longrightarrow (X, x)$.

Since (C, c) is a coproduct, we have a morphism $(C, c) \xrightarrow{f} (X, x)$:

$$\begin{tikzcd}
(A, a) \arrow[dr, "i"'] \arrow[drr] & & \\
& (C, c) \arrow[r, "f"] & (X, x) \\
(B, b) \arrow[ur, "j"] \arrow[urr] & &
\end{tikzcd}$$

where i and j are the injections.

The image of f, $P_\star(f(C)) \xrightarrow{d} f(C)$, where d is the restriction of x, is a subobject of (X, x) which contains (A, a) and (B, b). Call it $(D, d) \hookrightarrow (X, x)$.

$D \not\subseteq A$ and $D \not\subseteq B$ since D contains both. Now, since (D, d) is a subobject, $xD \in D$ so we must have $D = X$ (by definition of x).

Thus, f is onto. But X was arbitrarily large giving an arbitrarily large coproduct; a contradiction. ∎

There *are* theorems relating completeness and cocompleteness. For example, Grillet [Gri] has shown " regular, complete and regularly cowellpowered " implies " cocomplete ".

2–Cocompleteness in a poset

We would like a class of categories in which completeness and cocompleteness both hold. The motivation for such a notion comes from the case of a partially ordered set. This section is devoted to a study of cocompleteness in a poset.

Theorem 2–1: Let **P**:= $(P, \preceq)$ be a poset. The following are equivalent:

(1)– $\bigvee S$ exists for all downclosed subsets, S, of P

(2)– $\bigvee T$ exists for all subsets, T, of P

(3)– $\forall \mathcal{X} \in \mathbf{CAT}$, $\forall \mathcal{X} \xrightarrow{p} \mathbf{P}$ [$\varinjlim p$ exists.].

Proof: (2) $\Rightarrow$ (1) is trivial.

(1)$\Rightarrow$(2): Let $L(T) = \{x \in P | \ \exists t \in T$ such that $x \preceq t\}$ denote the downclosure of a subset, T, of P. We claim that $\bigvee T = \bigvee(L(T))$, which exists by (i).

If T is empty, it is certainly downclosed, so $\bigvee \emptyset$ exists. Now,

$\bigvee(L(T))$ is an upper bound for T:

$x \preceq \bigvee(L(T)) \ \forall x \in L(T)$ by definition of sup. Hence, $x \preceq \bigvee(L(T)) \ \forall x \in T$ since $T \subseteq L(T)$.

$\bigvee(L(T))$ is a least upper bound for T:

Suppose $w \in P$ is an upper bound for T. That is, $x \preceq w \ \forall x \in T$. We wish to show that $\bigvee(L(T)) \preceq w$. Now, $x \preceq w \ \forall x \in T$ implies $t \preceq w \ \forall t \in L(T)$ since $t \preceq x$ for some $x \in T$, by definition. And so $\bigvee(L(T)) \preceq w$.

(2)⇒ (3): For a poset, $\bigvee$ is coproduct:

$$\begin{array}{c}
\bigvee T \longrightarrow w \\
\hline
\bigvee T \preceq w \\
\hline
t \preceq w \ \forall t \in T \\
\hline
t \longrightarrow w \ \forall t \in T
\end{array}$$

Thus, if **P** has all sups, it has all coproducts (including the empty one which is $\bigvee \emptyset$) **P** has all coequalizers (for $x \overset{\preceq}{\underset{\preceq}{\rightrightarrows}} w$, its coequalizer is the identity $w \xrightarrow{\preceq} w$). Thus, **P** has all colimits.

(3)⇒(2): We prove a slightly stronger result. $\varinjlim \dashv \Delta$ so

$$\begin{array}{c}
\varinjlim p \preceq q \\
\hline
\forall x \in \mathcal{X} \ [p(x) \preceq q] \\
\hline
\bigvee_{x \in \mathcal{X}} p(x) \preceq q
\end{array}$$

Thus, by Yoneda, $\varinjlim p \simeq \bigvee_{x \in \mathcal{X}} p(x)$ and so $\varinjlim p$ exists iff $\bigvee_{x \in \mathcal{X}} p(x)$ exists. In particular, we may consider, $i \colon T \longrightarrow P$, the inclusion. Given that $\varinjlim i$ exists (T considered as a discrete category and i as a functor), $\bigvee_{t \in T} i(t) = \bigvee_{t \in T} t = \bigvee T$ exists. ∎

Remarks 2–2:

1. Note that, in particular, $\bigvee \emptyset$, if it exists, is the initial object of **P**.

2. We may replace "poset" and "**CAT**" by "small poset" and "**cat**" in the above theorem. □

Theorem 2–3: **P** has all sups iff it has all infs. More precisely, $\bigvee S$ exists for all $S \subseteq P$ iff $\bigwedge T$ exists for all $T \subseteq P$.

Proof: We need only prove "$\Rightarrow$". Without loss of generality, assume S is upclosed. Let S^- denote the set of lower bounds for S. We claim that $\bigwedge S = \bigvee S^-$.

$\bigvee S^-$ is a lower bound for S:

$t \preceq s \ \forall s \in S \ \forall t \in S^-$ by definition of lower bound. Thus, $\bigvee S^- \preceq s \ \forall s \in S$.

$\bigvee S^-$ is the greatest lower bound:

Suppose $t \preceq s \ \forall s \in S$. Then $t \in S^-$ and hence $t \preceq \bigvee S^-$. ∎

A poset has equalizers. Thus, an inf-complete (small) poset is (small) complete in the categorical sense as defined in 1–4. Theorem 2–3 may be translated as " **P** is (small) complete iff **P** is (small) cocomplete."

There is a "nice" adjoint functor theorem for posets. We will study the general case in section 5.

Theorem 2–4: Let $\mathbf{P} \xrightarrow{f} \mathbf{Q}$ be an order preserving function between posets $\mathbf{P} := (P, \preceq)$ and $\mathbf{Q} := (Q, \leq)$. If f preserves all sups, then $\exists u \vdash f$.

Proof: Define u by $uq = \bigvee\{x \in P | fx \leq q\}$. Then $u \vdash f$:

Let $p \in P$, $q \in Q$ and suppose $fp \leq q$. Then $p \in A = \{x | fx \leq q\}$ which implies $p \preceq \bigvee A \Rightarrow \ p \preceq uq$.

Conversely, suppose $p \preceq uq$. Then $fp \leq fuq$ since f is order preserving. But $fuq = f(\bigvee\{x|fx \leq q\} = \bigvee\{fx|fx \leq q\}$ since f preserves sups.

Now, $\bigvee\{fx|fx \leq q\} \leq q$ so $fp \leq q$. ▮

3–Totality

Theorem 2–1 characterizes cocompleteness of a poset in terms of its sup completeness. One may also characterize cocompleteness in terms of a left adjoint to the Yoneda embedding.

We first note that $2^{\mathbf{P}^{op}} \simeq$ the set of down-closed subsets of $\mathbf{P}$ via

$$\begin{tikzcd} \mathbf{P}^{op} \arrow[d, "f"'] \\ \mathbf{2} \end{tikzcd} \longmapsto f^{-1}(1)$$

and

$$\chi_S \longmapsfrom S$$

where χ_S denotes the characteristic function.

Theorem 3–1: The statements of theorem 2–1 are equivalent to:

(4)– $\downarrow$ has a left adjoint, $\downarrow(x) = \{w \in P | w \preceq x\}$ for $x \in P$.

Proof: We show that (4) is equivalent to (1) (i.e. $\bigvee S$ exists for down-closed subsets, S, of P.)

Assume (1). We wish to show, for S down-closed and $x \in P$:

$$\frac{\bigvee S \longrightarrow x}{S \longrightarrow \downarrow x}$$

Now,

$$\underline{\bigvee S \longrightarrow x}$$

$$\begin{array}{c} \bigvee S \preceq x \\ \hline w \preceq x \ \forall w \in S \\ \hline w \in \downarrow x \ \forall w \in S \\ \hline S \subseteq \downarrow x \\ \hline S \longrightarrow \downarrow x \end{array}$$

Conversely, suppose we have $m \dashv \downarrow$. We proceed to show that m must be $\bigvee$:

mS is an upper bound for S:

$\mathbf{P}(mS, mS) \simeq 2^{\mathbf{P}^{op}}(S, \downarrow mS)$. Thus, $S \subseteq \downarrow mS$. And so, mS is an upper bound.

mS is the least upper bound:

Suppose $x \leq b$ for all $x \in S$. Then $S \subseteq \downarrow b$. Hence $mS \leq b$ by adjointness.

Hence, $mS \simeq \bigvee S$. ∎

The inclusion $2 \xrightarrow{i} \mathbf{set}$ and its left adjoint, $2 \xleftarrow{f} \mathbf{set}$, given by

$$fX = \begin{cases} 1 & \text{if } X \neq \emptyset \\ 0 & \text{if } X = \emptyset \end{cases}$$

induce an adjointness situation:

$$\begin{tikzcd} 2^{\mathbf{P}^{op}} \arrow[r, shift right=1.5, "i^{\mathbf{P}^{op}}"'] \arrow[r, phantom, "\perp"] & \mathbf{set}^{\mathbf{P}^{op}} \arrow[l, shift right=1.5, "f^{\mathbf{P}^{op}}"'] \end{tikzcd}$$

Adjoints compose, so we have

$$\begin{tikzcd} \mathbf{P} \arrow[r, shift right=1.5, "\downarrow"'] \arrow[r, phantom, "\perp"] & 2^{\mathbf{P}^{op}} \arrow[l, shift right=1.5, "\vee"'] \arrow[r, shift right=1.5, "i^{\mathbf{P}^{op}}"'] \arrow[r, phantom, "\perp"] & \mathbf{set}^{\mathbf{P}^{op}} \arrow[l, shift right=1.5, "f^{\mathbf{P}^{op}}"'] \end{tikzcd} ;$$

$\bigvee f^{\mathbf{P}^{op}} \dashv i^{\mathbf{P}^{op}} \downarrow$. In fact, $i^{\mathbf{P}^{op}} \downarrow$ is the Yoneda embedding for **P**:

$$\mathbf{P} \xrightarrow{Y} \mathbf{set}^{\mathbf{P}^{op}} \quad (\, x \longmapsto \mathbf{P}(-, x) \,)$$

Thus, we may characterize cocompleteness in a poset as follows:

Theorem 3–2: **P** is cocomplete iff $\mathbf{P} \xrightarrow{Y} \mathbf{set}^{\mathbf{P}^{op}}$ has a left adjoint, L.

Proof: $\bigvee$ is L applied to functors $\Phi \in \mathbf{set}^{\mathbf{P}^{op}}$ which factor through 2. By theorem 3–1, **P** is cocomplete. ∎

Theorem 3–2 motivates the definition of a *total* category.

Definition 3–3: $\mathcal{B}$ in **CAT** is said to be *totally cocomplete* (or *total*) if it is locally small and the Yoneda embedding

$$\mathcal{B} \xrightarrow{Y} \mathbf{set}^{\mathcal{B}^{op}}$$

$$(\, B \longmapsto \mathcal{B}(-, B) \,)$$

has a left adjoint. ☐

Remarks 3–4:

1. We reserve 'Y' for the Yoneda embedding and 'L' for its left adjoint when it exists. When necessary, such functors are subscripted with the appropriate category (for example, $L_{\mathcal{B}}$).

2. We always have $\mathcal{B} \xrightarrow{Y} \mathbf{SET}^{\mathcal{B}^{op}}$ which is not in **CAT** . If $\mathcal{B}$ is locally small, then $\mathcal{B} \xrightarrow{Y} \mathbf{set}^{\mathcal{B}^{op}}$.
3. In general, $\widehat{\mathcal{B}} := \mathbf{set}^{\mathcal{B}^{op}}$ is a large category; in a sense, much larger than $\mathcal{B}$. It is not necessarily locally small. □

If $\mathcal{B}$ is total, it is a full reflective subcategory of $\mathbf{set}^{\mathcal{B}^{op}}$, a complete and cocomplete category. Thus, a total category is both complete and cocomplete. It is not true, in general, that a cocomplete category is total for then every cocomplete category would be complete, contradicting 1–6. In fact, as we shall see in example 6–3, a cocomplete and complete category need not be total. However, we have the following result. Recall the definition of *dense*.

Definition 3–5: Let $\mathcal{A}$ and $\mathcal{B}$ be locally small. A functor $\mathcal{A} \xrightarrow{D} \mathcal{B}$ is said to be *dense* if $\mathcal{B} \xrightarrow{(D,-)} \mathbf{set}^{\mathcal{A}^{op}}$ ($B \mapsto \mathcal{B}(D-, B)$) is fully faithful.

Proposition 3–6: If $\mathbf{A}$ is in **cat** , $\mathbf{A} \xrightarrow{D} \mathcal{B}$ dense and $\mathcal{B}$ small cocomplete, then $\mathcal{B}$ is total.

Proof: see corollary 2, theorem 4–4. ∎

We conclude this section with a characterization of total categories in terms of discrete fibrations. Recall the category of elements.

Definition 3–7: Let $\Phi \in \widehat{\mathcal{A}}$. Φ_{el} denotes the category whose objects are pairs, (A, α), where $A \in \mathcal{A}$ and $\alpha \in \Phi(A)$, and a morphism, $(A, \alpha) \xrightarrow{f} (B, \beta)$, is

$A \xrightarrow{f} B$ in $\mathcal{A}$ such that $\alpha = \Phi f(\beta)$. □

For $\Phi \in \widehat{\mathcal{A}}$, $\mathcal{A}$ total, we have $L_{\mathcal{A}}\Phi \simeq \varinjlim(\Phi_{el} \xrightarrow{p} \mathcal{A})$ where $p : (A, \alpha) \mapsto A$, (see [S&W1]). Recall, [S&W2], that any functor $\mathcal{G} \xrightarrow{D} \mathcal{A}$ admits a factorization:

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "E"] \arrow[dr, "D"'] & \mathcal{E} \arrow[d, "M"] \\
 & \mathcal{A}
\end{tikzcd}$$

with E final and M a discrete fibration (for a definition of discrete fibration, see, for example [PTJ]). It follows that, for $\mathcal{A}$ with small hom sets, $\mathcal{A}$ is total iff every diagram in $\mathcal{A}$, whose associated discrete fibration has small fibres, has a colimit.

It is not easy to realize colimits in $\mathcal{B}$ as $L\Phi$ for $\Phi \in \widehat{\mathcal{B}}$. However, we may consider, as an easy example, $A + B \in \mathcal{B}$ and search for a $\Phi \in \widehat{\mathcal{B}}$ such that $L\Phi \simeq A + B$. $L(\mathcal{B}(-, A)) \simeq A$ and $L(\mathcal{B}(-, B) \simeq B$ since Y is fully faithful. L is cocontinuous so $A + B \simeq L(\mathcal{B}(-, A)) + L(\mathcal{B}(-, B)) \simeq L(\mathcal{B}(-, A) + \mathcal{B}(-, B))$. Thus, $\Phi \simeq \mathcal{B}(-, A) + \mathcal{B}(-, B)$.

4– Examples and closure properties

In the previous section, we saw that a sup-complete poset is total. The results in this section show that the class of total categories contains many familiar ones.

Theorem 4–1 (Street and Walters): For $\mathbf{A} \in \mathbf{cat}$, $\mathbf{set}^{\mathbf{A}^{op}}$ is total.

Proof: [S&W1]: Recall that $\widehat{\mathbf{A}} := \mathbf{set}^{\mathbf{A}^{op}}$. For $f : \mathbf{A} \longrightarrow \mathcal{B}$, we have an induced functor, $\widehat{f} : \widehat{\mathcal{B}} \longrightarrow \widehat{\mathbf{A}}$, namely $\mathbf{set}^{f^{op}}$; precomposition with f^{op}.

In particular, $\mathbf{A} \xrightarrow{Y_{\mathbf{A}}} \widehat{\mathbf{A}}$ induces $\widehat{\mathbf{A}} \xleftarrow{\widehat{Y_{\mathbf{A}}}} \widehat{\widehat{\mathbf{A}}}$ ($Y_{\widehat{\mathbf{A}}}$ makes sense since $\mathbf{A}$ is small.).

Explicitly, $\widehat{\mathbf{A}} \underset{Y_{\widehat{\mathbf{A}}}}{\overset{\widehat{Y_{\mathbf{A}}}}{\leftrightarrows}} \widehat{\widehat{\mathbf{A}}}$: $Y_{\widehat{\mathbf{A}}}(\Phi) = \widehat{\mathbf{A}}(-,\Phi)$ for $\Phi \in \widehat{\mathbf{A}}$ and , for $\psi \in \widehat{\widehat{\mathbf{A}}}$,

$$\begin{tikzcd}
\widehat{Y_{\mathbf{A}}} : \widehat{\mathbf{A}}^{op} \arrow[d, "\psi"] \arrow[r, mapsto] & \mathbf{A}^{op} \arrow[r, "Y^{op}"] \arrow[dr] & \widehat{\mathbf{A}}^{op} \arrow[d, "\psi"] \\
\mathbf{set} & & \mathbf{set}
\end{tikzcd}$$

We claim that $\widehat{Y_{\mathbf{A}}} \dashv Y_{\widehat{\mathbf{A}}}$:

We wish to show, for $\Phi \in \widehat{\mathbf{A}}$, $\psi \in \widehat{\widehat{\mathbf{A}}}$, $\widehat{\mathbf{A}}(\widehat{Y_{\mathbf{A}}}\psi, \Phi) \simeq \widehat{\widehat{\mathbf{A}}}(\psi, \widehat{\mathbf{A}}(-,\Phi))$.

$\widehat{\mathbf{A}}(\widehat{Y_{\mathbf{A}}}\psi, \Phi) \simeq \int_{A \in \mathbf{A}} \mathbf{set}(\widehat{Y_{\mathbf{A}}}\psi A, \Phi A) \simeq \int_{A \in \mathbf{A}} \mathbf{set}(\psi(\mathbf{A}(-,A)), \Phi A))$ by definition of $\widehat{Y_{\mathbf{A}}}$. Now, $\psi \in \widehat{\widehat{\mathbf{A}}}$ so, by the yoneda density lemma, we may write it as a coend:

$$\psi \simeq \int^{\Gamma \in \widehat{\mathbf{A}}} \psi\Gamma . \widehat{\mathbf{A}}(-,\Gamma)$$

$$\text{In particular, } \psi(\mathbf{A}(-,A)) \simeq \int^{\Gamma\in\widehat{\mathbf{A}}} \psi\Gamma . \widehat{\mathbf{A}}(\mathbf{A}(-,A),\Gamma)$$
$$\simeq \int^{\Gamma\in\widehat{\mathbf{A}}} \psi\Gamma \times \widehat{\mathbf{A}}(\mathbf{A}(-,A),\Gamma))$$
$$\simeq \int^{\Gamma\in\widehat{\mathbf{A}}} \psi\Gamma \times \Gamma A \quad \text{(by the Yoneda lemma)}$$

$$\text{Thus, } \widehat{\mathbf{A}}(\widehat{Y_{\mathbf{A}}}\psi,\Phi) \simeq \int_{A\in\mathbf{A}} \mathbf{set}(\int^{\Gamma\in\widehat{\mathbf{A}}} \psi\Gamma \times \Gamma A, \Phi A)$$
$$\simeq \int_{A\in\mathbf{A}} \int_{\Gamma\in\widehat{\mathbf{A}}} \mathbf{set}(\psi\Gamma \times \Gamma A, \Phi A) \qquad \text{(coend out becomes end)}$$
$$\simeq \int_{A\in\mathbf{A}} \int_{\Gamma\in\widehat{\mathbf{A}}} \mathbf{set}(\psi\Gamma, \mathbf{set}(\Gamma A, \Phi A)) \qquad (\, - \times \Gamma A \dashv \mathbf{set}(\Gamma A, -) \,)$$
$$\simeq \int_{\Gamma\in\widehat{\mathbf{A}}} \int_{A\in\mathbf{A}} \mathbf{set}(\psi\Gamma, \mathbf{set}(\Gamma A, \Phi A)) \qquad \text{(``Fubini'')}$$
$$\simeq \int_{\Gamma\in\widehat{\mathbf{A}}} \mathbf{set}(\psi\Gamma, \int_{A\in\mathbf{A}} \mathbf{set}(\Gamma A, \Phi A)) \qquad \text{(take end in)}$$
$$\simeq \int_{\Gamma\in\widehat{\mathbf{A}}} \mathbf{set}(\psi\Gamma, \widehat{\mathbf{A}}(\Gamma,\Phi))$$
$$\simeq \widehat{\widehat{\mathbf{A}}}(\psi, \widehat{\mathbf{A}}(-,\Phi)). \ ■$$

In particular,

Corollary 1: **set** is total.

Proof: $\mathbf{set} \simeq \mathbf{set}^{\mathbf{1}^{op}} \xleftarrow{\widehat{Y_1}} \mathbf{set}^{\mathbf{set}^{op}}$ is given by $\mathbf{set} \xleftarrow{evat(1)} \mathbf{set}^{\mathbf{set}^{op}}$ where

$evat(1)(f) = f(1)$ since

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "Y_1"] \arrow[dr, "\ulcorner 1 \urcorner"'] & \mathbf{set}^{\mathbf{1}^{op}} \arrow[d, "\wr"] \\
& \mathbf{set}
\end{tikzcd} \quad ■$$

Remark 4–2: *evat*(1) may be constructed as the left adjoint to $Y_{\mathbf{set}}$ via proposition 3–5 using the facts that **set** is small cocomplete and $1 \xrightarrow{i} \mathbf{set}$ is dense (every small set is a small sum of one point sets). □

More generally, we have the following closure property for total categories.

Theorem 4–3: If $\mathcal{B}$ is total and $\mathbf{A}$ is small, then $\mathcal{B}^{\mathbf{A}}$ is total.

Proof: [RJW3]: Recall, for each $A \in \mathbf{A}$, we have the evaluation map $\mathcal{B}^{\mathbf{A}} \xrightarrow{\epsilon_A} \mathcal{B}$ ($H \mapsto H(A)$) which has left and right adjoints:

$$\begin{tikzcd}
\mathcal{B}^{\mathbf{A}} \arrow[r, "\epsilon_A\ \perp", "\perp"'] \arrow[r, leftarrow, shift left=4, "A_L"] \arrow[r, leftarrow, shift right=4, "A_R"'] & \mathcal{B}
\end{tikzcd}$$

where $(A_L B)(X) = \mathbf{A}(A, X).B$ and $(A_R B)(X) = \{\mathbf{A}(X, A), B\}$ for $X \in \mathbf{A}, B \in \mathcal{B}$

$A_L \dashv \epsilon_A$: Let $H \in \mathcal{B}^{\mathbf{A}}$, $B \in \mathcal{B}$. We wish to show, $\mathcal{B}^{\mathbf{A}}(A_L B, H) \simeq \mathcal{B}(B, H(A))$

$$\begin{aligned}
\mathcal{B}^{\mathbf{A}}(A_L B, H) &\simeq \int_{X \in \mathbf{A}} \mathcal{B}((A_L B)X, HX) & \\
&\simeq \int_{X \in \mathbf{A}} \mathcal{B}(\mathbf{A}(A, X).B, HX)) & \text{(definition } A_L) \\
&\simeq \int_{X \in \mathbf{A}} \mathbf{set}(\mathbf{A}(A, X), \mathcal{B}(B, HX)) & \text{(universal property)} \\
&\simeq \mathbf{set}^{\mathbf{A}}(\mathbf{A}(A, -), \mathcal{B}(B, H-)) & \\
&\simeq \mathcal{B}(B, HA) & \text{(Yoneda)}
\end{aligned}$$

$\epsilon \dashv A_R$: Let $B \in \mathcal{B}$, $H \in \mathcal{B}^{\mathbf{A}}$. We wish to show $\mathcal{B}(HA, B) \simeq \mathcal{B}^{\mathbf{A}}(H, A_R B)$.

$$\begin{aligned}
\mathcal{B}^{\mathbf{A}}(H, A_R B) &\simeq \int_{X \in \mathbf{A}} \mathcal{B}(HX, (A_R B)(X)) \\
&\simeq \int_{X \in \mathbf{A}} \mathcal{B}(HX, \{\mathbf{A}(X, A), B\}) && \text{(definition } A_R \text{)} \\
&\simeq \int_{X \in \mathbf{A}} \mathbf{set}(\mathbf{A}(X, A), \mathcal{B}(HX, B)) && \text{(universal property)} \\
&\simeq \widehat{\mathbf{A}}(\mathbf{A}(-, A), \mathcal{B}(H-, B)) \\
&\simeq \mathcal{B}(HA, B) && \text{(Yoneda)}
\end{aligned}$$

So, $A_L{}^{op} \vdash \epsilon_A{}^{op}$.

Let $\Phi \in (\widehat{\mathcal{B}^{\mathbf{A}}})$, then $\Phi A_L{}^{op} \in \widehat{\mathcal{B}}$:

$$\begin{tikzcd}
(\mathcal{B}^{\mathbf{A}})^{op} \arrow[d, "\Phi"'] & & \mathcal{B}^{op} \arrow[ll, "A_L{}^{op}"'] \arrow[dll] \\
\mathbf{set} & &
\end{tikzcd}$$

We claim that $\mathcal{B}^{\mathbf{A}} \xleftarrow{\quad L \quad} (\widehat{\mathcal{B}^{\mathbf{A}}})$ is given by $(L\Phi)(A) \simeq L_{\mathcal{B}}(\Phi A_L{}^{op})$.

We wish to show, for $F \in \mathcal{B}^{\mathbf{A}}$, $\mathcal{B}^{\mathbf{A}}(L\Phi, F) \simeq (\widehat{\mathcal{B}^{\mathbf{A}}})(\Phi, \mathcal{B}^{\mathbf{A}}(-, F))$.

$$\begin{aligned}
\mathcal{B}^{\mathbf{A}}(L\Phi, F) &\simeq \int_{A\in\mathbf{A}} \mathcal{B}(L\Phi A, FA) \\
&\simeq \int_{A\in\mathbf{A}} \mathcal{B}(L_{\mathcal{B}}(\Phi A_L{}^{op}), FA) && \text{(definition)} \\
&\simeq \int_{A\in\mathbf{A}} \widehat{\mathcal{B}}(\Phi A_L{}^{op}, \mathcal{B}(-, FA)) && (L_{\mathcal{B}} \dashv Y_{\mathcal{B}}) \\
&\simeq \int_{A\in\mathbf{A}} \widehat{\mathcal{B}^{\mathbf{A}}}(\Phi, \mathcal{B}(-, FA)\epsilon_A{}^{op}) && (A_L{}^{op} \vdash \epsilon_A{}^{op}) \\
&\simeq \int_{A\in\mathbf{A}} \int_{G\in\mathcal{B}^{\mathbf{A}}} \mathbf{set}(\Phi G, \mathcal{B}(-, FA)\epsilon_A{}^{op}G) \\
&\simeq \int_{A\in\mathbf{A}} \int_{G\in\mathcal{B}^{\mathbf{A}}} \mathbf{set}(\Phi G, \mathcal{B}(-, FA)GA) && (\text{definition } \epsilon_A{}^{op}) \\
&\simeq \int_{A\in\mathbf{A}} \int_{G\in\mathcal{B}^{\mathbf{A}}} \mathbf{set}(\Phi G, \mathcal{B}(GA, FA)) \\
&\simeq \int_{G\in\mathcal{B}^{\mathbf{A}}} \int_{A\in\mathbf{A}} \mathbf{set}(\Phi G, \mathcal{B}(GA, FA)) && \text{(``Fubini'')} \\
&\simeq \int_{G\in\mathcal{B}^{\mathbf{A}}} \mathbf{set}(\Phi G, \int_{A\in\mathbf{A}} \mathcal{B}(GA, FA)) && \text{(take end in)} \\
&\simeq \int_{G\in\mathcal{B}^{\mathbf{A}}} \mathbf{set}(\Phi G, \mathcal{B}^{\mathbf{A}}(G, F)) \\
&\simeq (\widehat{\mathcal{B}^{\mathbf{A}}})(\Phi, \mathcal{B}^{\mathbf{A}}(-, F)). \blacksquare
\end{aligned}$$

Another closure property of total categories is the following:

Theorem 4–4: (Street and Walters): If $\mathcal{B}$ is total and $\mathcal{A}$ is a full reflective subcategory of $\mathcal{B}$, i.e. $\mathcal{A} \underset{F}{\overset{T}{\rightleftarrows}} \mathcal{B}$ (with $\perp$) with F fully faithful, then $\mathcal{A}$ is total.

Proof: Consider the diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "Y"] \arrow[d, shift left=2, "F"] \arrow[d, phantom, "\dashv"] & \hat{\mathcal{A}} \arrow[d, shift right=2, "F_1"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{B} \arrow[u, shift left=2, "T"] \arrow[r, shift right=2, "Y_{\mathcal{B}}"'] \arrow[r, phantom, "\perp"] & \hat{\mathcal{B}} \arrow[l, shift right=2, "L_{\mathcal{B}}"'] \arrow[u, shift right=2, "\hat{F}"']
\end{tikzcd}$$

where F is the fully faithful "inclusion", T is the reflector and $F_!$ is left Kan extension; for $\Phi \in \widehat{\mathcal{A}}$,

$$\begin{tikzcd}
\mathcal{A}^{op} \arrow[r, "F^{op}"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\xrightarrow{\sim}"] & \mathcal{B}^{op} \arrow[dl, "F_!(\Phi)"] \\
\mathbf{set} &
\end{tikzcd}$$

$F_!(\Phi) = Lan_{F^{op}}(\Phi)$. Recall, for $T \dashv F$, we have $\widehat{T} \dashv \widehat{F}$ so $F_!$ is isomorphic to $\widehat{L}$.

We claim that $TL_{\mathcal{B}}F_! \dashv Y_{\mathcal{A}}$: We wish to show $\mathcal{A}(TL_{\mathcal{B}}F_!\Phi, A) \simeq \widehat{\mathcal{A}}(\Phi, \mathcal{A}(-, A))$ for $A \in \mathcal{A}$, $\Phi \in \widehat{\mathcal{A}}$.

$$\begin{aligned}
\mathcal{A}(TL_{\mathcal{B}}F_!\Phi, A) &\simeq \mathcal{B}(L_{\mathcal{B}}F_!\Phi, FA) && (\ T \dashv F\ ) \\
&\simeq \widehat{\mathcal{B}}(F_!\Phi, \mathcal{B}(-, FA))) && (\ L_{\mathcal{B}} \dashv Y_{\mathcal{B}}\ ) \\
&\simeq \widehat{\mathcal{A}}(\Phi, \widehat{F}(\mathcal{B}(-, FA))) && (\ F_! \dashv \widehat{F}\ )
\end{aligned}$$

Now, $\widehat{F}(\mathcal{B}(-, FA)) \simeq \mathcal{B}(F-, FA)$ and $\mathcal{B}(F-, FA) \simeq \mathcal{A}(-, A)$ since F is fully faithful. Substituting, we have $\widehat{\mathcal{A}}(\Phi, \widehat{F}(\mathcal{B}(-, FA))) \simeq \widehat{\mathcal{A}}(\Phi, \mathcal{A}(-, A))$. ∎

The above theorem tells us how to compute colimits in a full reflective subcategory of a total category. To compute the colimit of a diagram in $\mathcal{A}$, notation as above, one first considers it as an object of $\widehat{\mathcal{A}}$, then as an object of $\widehat{\mathcal{B}}$ by applying $F_!$, calculates its colimit in $\mathcal{B}$ ($\mathcal{B}$ is total) using $L_{\mathcal{B}}$, and reflects the result into $\mathcal{A}$ via T.

Recall that a theory, $\mathbf{T}$, is said to have rank if the arrity of the operations is bounded. More precisely,

Definition 4–5: [Man] p. 56: Let $\mathbf{T} = (T, \eta, \circ, \mu)$ be an algebraic theory in **set**. We say that $\mathbf{T}$ has rank if there exists a cardinal $\mathcal{N}$ such that $\forall X \in \mathbf{set}$ $\forall \omega \in T(X)$, *arrity*$(\omega) < \mathcal{N}$. □

Recall also that the category of **T**–algebras for a theory, **T**, with rank is a full reflective subcategory of $\widehat{\mathbf{A}}$ for some small **A**. Hence, from theorems 4–1 and 4–4, we have:

Corollary 1: The category of **T**–algebras for a theory, **T**, with rank is total. ∎

This corollary provides a wealth of examples. **grp, ab, rng** and such are total categories. As another corollary, we are able to prove proposition 3–6.

Corollary 2: If **A** is in **cat** , $\mathbf{A} \xrightarrow{D} \mathcal{B}$ dense and $\mathcal{B}$ small cocomplete, then $\mathcal{B}$ is total.

Proof: [RJW1]: By definition (D dense), $(D,-) : \mathcal{B} \longrightarrow \widehat{\mathbf{A}}$ ($B \longmapsto \mathcal{B}(D-,B)$) is fully faithful.

We claim that $F : \widehat{\mathbf{A}} \longrightarrow \mathcal{B}$ ($\Gamma \longrightarrow \int^{A\in\mathbf{A}} \Gamma A.DA$) is left adjoint to $(D,-)$.

We wish to show $\mathcal{B}(F\Gamma, B) \simeq \widehat{\mathbf{A}}(\Gamma, \mathcal{B}(D,B))$ $\forall B \in \mathcal{B},\ \Gamma \in \widehat{\mathbf{A}}$.

$$\begin{aligned}
\mathcal{B}(F\Gamma, B) &\simeq \mathcal{B}\Big(\int^{A\in\mathbf{A}} \Gamma A.DA, B\Big) \\
&\simeq \int_{A\in\mathbf{A}} \mathcal{B}(\Gamma A.DA, B) && \text{(coend out becomes end)} \\
&\simeq \int_{A\in\mathbf{A}} \mathbf{set}(\Gamma A, \mathcal{B}(D-,B)) && \text{(universal property)} \\
&\simeq \widehat{\mathbf{A}}(\Gamma, \mathcal{B}(D-,B)).
\end{aligned}$$

Thus, $\mathcal{B}$ is a full reflective subcategory of $\widehat{\mathbf{A}}$ which is total by theorem 4–1. ∎

5– Total adjoint functor theorem

A simple application showing the power of the total category setting is the total adjoint functor theorem. Recall Freyd's adjoint functor theorem.

Theorem 5–1: (Freyd): Suppose $\mathcal{A}$ is (small) complete and locally small. A functor $\mathcal{A} \xrightarrow{U} \mathcal{C}$ has a left adjoint iff it preserves all small limits and satisfies the solution set condition:

For each $C \in \mathcal{C}$, there is a small set I and an I-indexed family $f_i: C \longrightarrow UA_i$ in $\mathcal{C}$ such that every $h: C \longrightarrow UA$ in $\mathcal{C}$ can be written as a composite $h = Ua \circ f_i$ for some index i and some $a: A_i \longrightarrow A$.

Proof: see, for example, [Mac] p. 117. ∎

There is a "nice" adjoint functor theorem in the total category setting. We first define *admissible functor*.

Definition 5–2: A functor $F: \mathcal{B} \longrightarrow \mathcal{X}$ is said to be *admissible* if $\mathcal{X}(FB, X) \in$ **set** $\forall B \in \mathcal{B}$, $X \in \mathcal{X}$. □

Note, in particular, if $\mathcal{X}$ is locally small, then any such F is admissible.

Theorem 5–3: (Street and Walters): If $\mathcal{B}$ is total and $F: \mathcal{B} \longrightarrow \mathcal{X}$ is admissible, then F has a right adjoint iff F is cocontinuous, i.e. preserves colimits.

Proof: $\Rightarrow$: (well known): For $F \dashv R$:

$$\begin{array}{c} F(\varinjlim W) \longrightarrow T \\ \hline \varinjlim W \longrightarrow RT \\ \hline W \longrightarrow RT \text{ cocones} \\ \hline FW \longrightarrow T \text{ cocones} \\ \hline \varinjlim FW \longrightarrow T \end{array}$$

$\Leftarrow$: For fixed $X \in \mathcal{X}$, $\mathcal{X}(F-,X) : \mathcal{B}^{op} \longrightarrow \mathbf{set}$ since F is admissible. Let $G : \mathcal{X} \longrightarrow \widehat{\mathcal{B}}$ $(X \mapsto \mathcal{X}(F-,X))$ and consider the diagram:

$$\begin{tikzcd}
\mathcal{B} \arrow[rr, "F"] \arrow[d, shift left=2, "Y"] \arrow[d, phantom, "\dashv"] & & \mathcal{X} \arrow[dll, "G"] \\
\widehat{\mathcal{B}} \arrow[u, shift left=2, "L"] & &
\end{tikzcd}$$

Then $F \dashv LG$. ∎

It is interesting to note that this theorem generalizes theorem 2–4 , the adjoint functor theorem for posets. As an immediate consequence of the total adjoint functor theorem, we have:

Corollary 1: If $\mathcal{B}$ is total, $\mathcal{C}$ locally small , then any cocontinuous $F: \mathcal{B} \longrightarrow \mathcal{C}$ has a right adjoint. ∎

We conclude this section with an interesting property of cocontinuous functors between total categories.

Theorem 5–4: Suppose $\mathcal{A}$ and $\mathcal{B}$ are total and $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is cocontinuous, hence has a right adjoint, U, say. Suppose further that $F_!$ ($\dashv \widehat{F}$) exists. Then both

$$\begin{tikzcd}
\widehat{\mathcal{A}} \arrow[r, "F_!"] \arrow[d, shift right=2, "L"'] \arrow[d, phantom, "\dashv"] & \widehat{\mathcal{B}} \arrow[d, shift right=2, "L"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{A} \arrow[r, "F"'] \arrow[u, shift right=2, "Y"'] & \mathcal{B} \arrow[u, shift right=2, "Y"']
\end{tikzcd}$$

commute.

Proof:[RJW3]: The " Y – square " always commutes.

Let $\Phi \in \widehat{\mathcal{A}}$, $B \in \mathcal{B}$. Then $\mathcal{B}(FL_{\mathcal{A}}\Phi, B) \simeq \mathcal{A}(L_{\mathcal{A}}\Phi, UB)$, since $F \dashv U$,

$\simeq \widehat{\mathcal{A}}(\Phi, \mathcal{A}(-, UB)) = (P)$, since ($L_{\mathcal{A}} \dashv Y_{\mathcal{A}}$).

Now, $\mathcal{A}(-, UB)) \simeq \mathcal{B}(F-, B))$, since ($F \dashv U$). So,

$$\begin{aligned}
(P) &\simeq \widehat{\mathcal{A}}(\Phi, \mathcal{B}(F-, B)) \\
&\simeq \widehat{\mathcal{A}}(\Phi, \widehat{F}(\mathcal{B}(-, B))) && \text{(by Yoneda)} \\
&\simeq \widehat{\mathcal{A}}(\Phi, \widehat{F}(\mathcal{B}(-, B))) && \text{(definition of } \widehat{F} \text{)} \\
&\simeq \widehat{\mathcal{B}}(F_!\Phi, \mathcal{B}(-, B) && (\ F_! \dashv \widehat{F}\ ) \\
&\simeq \mathcal{B}(L_{\mathcal{B}}F_!\Phi, B) && (\ L_{\mathcal{B}} \dashv Y_{\mathcal{B}}\ )
\end{aligned}$$

Thus, $FL_{\mathcal{A}}\Phi \simeq L_{\mathcal{B}}F_!\Phi$. ∎

The "L – square" of the above theorem is reminiscent of the homomorphism condition for a function between groups. Recall that a function $G \xrightarrow{f} H$ between groups G and H is said to be a homomorphism if

$$\begin{tikzcd}
G \times G \arrow[r, "f \times f"] \arrow[d, "*"'] & H \times H \arrow[d, "*"] \\
G \arrow[r, "f"'] & H
\end{tikzcd}$$

commutes.

In this context, one may think of a cocontinuous functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$, as above, as a "homomorphism" between total categories in that it preserves the "sup operation" $L : \widehat{\mathcal{A}} \longrightarrow \mathcal{A}$.

6–Duality

The dual notion of totality is *cototality*.

Definition 6–1: **B** in **CAT** is said to be *cototally complete* (or *cototal*) if it is locally small and

$$\mathcal{B} \xrightarrow{Z} \check{\mathcal{B}} := (\mathbf{set}^{\mathcal{B}})^{op}$$

$$(B \longmapsto \mathcal{B}(B,-))$$

has a right adjoint. □

Remarks 6–2:

1. We denote this right adjoint, when it exists, by $R_{\mathcal{B}}$
2. $(\ )^{op}: \mathbf{CAT}^{co} \longrightarrow \mathbf{CAT}$ (That is, op-ing reverses the direction of natural transformations and hence the sense of adjunctions) so, since $(\check{\mathcal{B}})^{op} = (\widehat{\mathcal{B}^{op}})$, $\mathcal{B}$ is cototal iff $\mathcal{B}^{op}$ is total.
3. Cototal categories are complete and cocomplete (being full coreflective subcategories of $(\mathbf{set}^{\mathcal{B}})^{op}$). □

The example below, due to Paré, shows that totality and cototality are not equivalent.

Example 6–3: [RJW1]: **grp** is total as was seen in section 4. We proceed to show that it is not cototal.

For every infinite cardinal, α, let S_α be a simple group of cardinality α. Note that $A = \{\alpha | \alpha \text{ infinite cardinal in } \mathbf{set}\}$ is in **SET** but not in **set**.

Let 1 denote the trivial group and write $S_\alpha \xrightarrow{!_\alpha} 1$ for the unique homomorphism. Consider the diagram in $\mathbf{set}^{\mathbf{grp}}$:

$$\begin{tikzcd}
 & \mathbf{grp}(S_\alpha, -) \\
(\star) & \vdots \\
\mathbf{grp}(1, -) \arrow[uur, "\mathbf{grp}(!_\alpha, -)"] & & \forall \alpha \in A
\end{tikzcd}$$

We claim that it has a colimit in $\mathbf{set}^{\mathbf{grp}}$.

Let G be any group and "evaluate" $(\star)$ at G:

$$\begin{tikzcd}
 & \mathbf{grp}(S_\alpha, G) \\
(\star(G)) & \vdots \\
\mathbf{grp}(1, G) \arrow[uur, "\mathbf{grp}(!_\alpha, G)"] & & \forall \alpha \in A
\end{tikzcd}$$

which is a large diagram of sets.

The group, G, has cardinality, β, say. Since G is a small group, $\bar{\beta} = \{\alpha \in A | \alpha \leq \beta\}$ is small.

If $\alpha \in A - \bar{\beta}$, then $\mathbf{grp}(S_\alpha, G) = 1$ since $\alpha > \beta$.

Now,

$$\varinjlim \star(G) \simeq \sum_{\alpha \in A} \mathbf{grp}(S_\alpha, G) \; / \mathcal{E} \quad = (P)$$

where $\mathcal{E}$ is the equivalence relation generated by identification of all the 0 maps: $S_\alpha \xrightarrow{!} 1 \xrightarrow{!} G$. But, except for all but a small set, indexed by $\bar{\beta}$, the sets

$\mathbf{grp}(S_\alpha, G)$ consist only of the 0 map. Thus,

$$(P) \simeq (\sum_{\alpha\in\bar{\beta}} (\mathbf{grp}(S_\alpha, G) \ \setminus \{S_\alpha \xrightarrow{0} G\}) + 1$$

which is in **set**.

Suppose we have $L \dashv Y_{\mathbf{grp}^{op}}$ (i.e. $\mathbf{grp}^{op}$ total). Then the $\varinjlim$ of $(\star)$ is preserved by L (since L is a left adjoint) inducing a $\varprojlim$ diagram in **grp**. But $LY \simeq 1$ since Y is fully faithful. So, if L exists,

$$L(\varinjlim(\star)) = \prod_{\alpha\in A} S_\alpha \quad ;$$

a contradiction since $\prod_{\alpha\in A} S_\alpha$ does not exist. □

The above example also illustrates a category which is complete and cocomplete ($\mathbf{grp}^{op}$; since **grp** is cocomplete and complete) but not total.

Totality and cototality may be related:

Theorem 6–4: If $\mathcal{B}$ is cocomplete and has a small set of cogenerators, then $\mathcal{B}$ is total..

Proof: [RJW1]: Z preserves all colimits so, by the total adjoint functor theorem, we need only show that it is admissible.

Let $\mathcal{C}$ denote the small full subcategory of $\mathcal{B}$ determined by a small set of cogenerators. For all $X \in \mathcal{B}$ there is a canonical morphism:

$$(6-5) \qquad X \longrightarrow \int_{C\in\mathcal{C}} \{\mathcal{B}(X, C), C\} \ :$$

We seek a $\beta \in \mathcal{B}(X, \int_{C\in\mathcal{C}}\{\mathcal{B}(X,C),C\})$.

$$\begin{aligned}
\mathcal{B}(X, \int_{C\in\mathcal{C}}\{\mathcal{B}(X,C),C\}) &\simeq \int_{C\in\mathcal{C}} \mathcal{B}(X,\{\mathcal{B}(X,C),C\}) && \text{(take end out)}\\
&\simeq \int_{C\in\mathcal{C}} \mathbf{set}(\mathcal{B}(X,C),\mathcal{B}(X,C)) && \text{(universal property)}\\
&\simeq \mathbf{set}^{\mathcal{C}}(\mathcal{B}(X,-),\mathcal{B}(X,-)) = (P)
\end{aligned}$$

One element of (P) is the identity natural transformation $\mathcal{B}(X,-)\xrightarrow{1}\mathcal{B}(X,-)$ which induces a canonical morphism in (6–5).

By the definition of generators, the map in (6–5) is monic. $\forall B\in\mathcal{B}$, $\psi\in\check{\mathcal{B}}$,

$\check{\mathcal{B}}(ZB,\psi) \simeq \mathbf{set}^{\mathcal{B}}(\psi, ZB) \simeq \mathbf{set}^{\mathcal{B}}(\psi,\mathcal{B}(B,-)) \hookrightarrow \mathbf{set}^{\mathcal{B}}(\psi,\mathcal{B}(B,\int_{C\in\mathcal{C}}\{\mathcal{B}(-,C),C\}))$ $= (Q)$ since (6–5) is mono and $\mathcal{B}(B,-)$ and $\mathbf{set}^{\mathcal{B}}(\psi,-)$ preserve monos.

We have,

$$\begin{aligned}
(Q) &\simeq \mathbf{set}^{\mathcal{B}}(\psi, \int_{C\in\mathcal{C}} \mathcal{B}(B,\{\mathcal{B}(-,C),C\})) && \text{(take end out)}\\
&\simeq \mathbf{set}^{\mathcal{B}}(\psi, \int_{C\in\mathcal{C}} \mathbf{set}(\mathcal{B}(-,C),\mathcal{B}(B,C))) && \text{(universal property)}\\
&\simeq \int_{C\in\mathcal{C}} \mathbf{set}^{\mathcal{B}}(\psi, \mathbf{set}(\mathcal{B}(-,C),\mathcal{B}(B,C))) && \text{(take end out)}\\
&\simeq \int_{C\in\mathcal{C}}\int_{D\in\mathcal{B}} \mathbf{set}(\psi D, \mathbf{set}(\mathcal{B}(D,C),\mathcal{B}(B,C)))\\
&\simeq \int_{C\in\mathcal{C}}\int_{D\in\mathcal{B}} \mathbf{set}(\mathcal{B}(D,C), \mathbf{set}(\psi D,\mathcal{B}(B,C))) && \text{(set is cartesian closed)}\\
&\simeq \int_{C\in\mathcal{C}} \widehat{\mathcal{B}}(\mathcal{B}(-,C), \mathbf{set}(\psi -,\mathcal{B}(B,C)))\\
&\simeq \int_{C\in\mathcal{C}} \mathbf{set}(\psi C,\mathcal{B}(B,C)) = (R)
\end{aligned}$$

(R) is a small end of small sets so $(R)\in\mathbf{set}$. Thus $\check{\mathcal{B}}(ZB,\psi)\in\mathbf{set}$ $\forall B\in\mathcal{B}, \psi\in\check{\mathcal{B}}$ whence Z is admissible. ∎

A Grothendieck topos has a small set of cogenerators (if $(G_i)_{i \in I}$ generate, then $((\Omega)^{G_i})_{i \in I}$ cogenerate). Furthermore, a Grothendieck topos, being a full reflective subcategory of $\widehat{\mathbf{A}}$, for small $\mathbf{A}$, is total. Thus, Grothendieck topoi are both total and cototal.

We continue our study of duality by first considering the case of a poset. Recall that a poset has all sups iff it has all infs. We have

$$\begin{tikzcd}
 & 2^{\mathbf{P}^{op}} \arrow[dd, shift right, "(\ )^+"'] & \simeq \{\text{downclosed subs}, \ \subseteq\} \\
\mathbf{P} \arrow[ur, "\downarrow -"] \arrow[dr, "\uparrow -"'] & & \\
 & (2^{\mathbf{P}})^{op} \arrow[uu, shift right, "(\ )^-"'] & \simeq \{\text{upclosed subs}, \ \supseteq\}
\end{tikzcd}$$

where S^- denotes the set of all lower bounds for an upclosed subset, S, of P and T^+ denotes the set of all upper bounds for a downclosed subset, T, of P.

Proposition 6–6: $(\ )^+ \dashv (\ )^-$.

Proof: We wish to show $T^+ \supseteq S$ iff $T \subseteq S^-$.

$$\begin{array}{c}
T^+ \supseteq S \\
\hline
\forall s \in S[\forall t \in T[t \leq s]] \\
\hline
\forall t \in T[\forall s \in S[t \leq s]] \\
\hline
\forall t \in T[t \in S^-] \\
\hline
T \subseteq S^-
\end{array}$$

Note that $(\downarrow x)^+ = \uparrow x$ and $(\uparrow x)^- = \downarrow x$ for all $x \in P$. So, $(\ )^+$ and $(\ )^-$ commute with the Yoneda embeddings. A sup-complete poset is total and cototal. In fact, the diagram

$$\begin{tikzcd}
& 2^{\mathbf{P}^{op}} \arrow[dl, shift right=1.5, "\bigvee"'] \arrow[dd, shift right=1.5, "(\ )^+"'] \arrow[dd, phantom, "\dashv"] \\
\mathbf{P} \arrow[ur, shift right=1.5, "\downarrow -"'] \arrow[ur, phantom, "\perp" sloped] \arrow[dr, shift left=1.5, "\uparrow -"] \arrow[dr, phantom, "\perp" sloped] & \\
& (2^{\mathbf{P}})^{op} \arrow[ul, shift left=1.5, "\bigwedge"] \arrow[uu, shift right=1.5, "(\ )^-"']
\end{tikzcd}$$

commutes; $\bigvee(\ )^- = \bigwedge$ and $\bigwedge(\ )^+ = \bigvee$ by theorem 2–2; $\uparrow \bigvee = (\ )^+$ and $\downarrow \bigwedge = (\ )^-$ following immediately from their definitions.

This situation does not generalize to **CAT** . However, in the total category setting, it does.

Definition 6–7: Let $\mathbf{A} \in$ **cat** . The *Isbell conjugation functors* : $\widehat{\mathbf{A}} \underset{(\ )^+}{\overset{(\ )^-}{\leftrightarrows}} \check{\mathbf{A}}$ are defined pointwise by $\Phi^+(A) = \widehat{\mathbf{A}}(\Phi, \mathbf{A}(-, A))$ and $\Psi^-(A) = \mathbf{set}^{\mathbf{A}}(\Psi, \mathbf{A}(A, -))$ for $A \in \mathbf{A}, \Phi \in \widehat{\mathbf{A}}, \psi \in \check{\mathbf{A}}$. □

For $\mathcal{A} \in$ **CAT** , these functors do not necessarily exist. We need $Y_{\mathcal{A}}$ to be coadmissible (i.e. $\widehat{\mathcal{A}}(\Phi, \mathcal{A}(-, A)) \in \mathbf{set}\ \forall A \in \mathcal{A}, \Phi \in \widehat{\mathcal{A}}$) in order to define $(\ )^+$. Similarly, we need Z to be admissible in order to define $(\ )^-$. If $\mathcal{B}$ is total, then $\widehat{\mathcal{B}}(\Phi, \mathcal{B}(-, B)) \simeq \mathcal{B}(L\Phi, B) \in \mathbf{set}$ since $\mathcal{B}$ is locally small and so $(\ )^+$ is defined. Similarly, if $\mathcal{B}$ is cototal, $(\ )^-$ exists.

$(\ )^+$ and $(\ )^-$ commute with the appropriate Yoneda embeddings. In addition, they are adjoint:

Proposition 6–8: $(\ )^+ \dashv (\ )^-$

Proof: let Φ, Ψ be as above. We want to show $(\mathbf{set}^{\mathcal{A}})^{op}(\Phi^+,\Psi) \simeq \mathbf{set}^{\mathcal{A}^{op}}(\Phi,\Psi^-)$.

$$\begin{aligned}
(\mathbf{set}^{\mathcal{A}})^{op}(\Phi^+,\Psi) &\simeq \mathbf{set}^{\mathcal{A}}(\Psi,\Phi^+) && (\text{op} - \text{ing})\\
&\simeq \int_{A\in\mathcal{A}} \mathbf{set}(\Psi A, \Phi^+ A) && \\
&\simeq \int_{A\in\mathcal{A}} \mathbf{set}(\Psi A, \mathbf{set}^{\mathcal{A}^{op}}(\Phi, \mathcal{A}(-,A))) && (\text{definition } (\ )^+\ )\\
&\simeq \int_{A\in\mathcal{A}} \mathbf{set}(\Psi A, \int_{B\in\mathcal{A}} \mathbf{set}(\Phi B, \mathcal{A}(B,A))) && \\
&\simeq \int_{A\in\mathcal{A}} \int_{B\in\mathcal{A}} \mathbf{set}(\Psi A, \mathbf{set}(\Phi B, \mathcal{A}(B,A))) && (\text{take end out})\\
&\simeq \int_{A\in\mathcal{A}} \int_{B\in\mathcal{A}} \mathbf{set}(\Phi B, \mathbf{set}(\Psi A, \mathcal{A}(B,A))) && (\text{set cartesian closed})\\
&\simeq \int_{B\in\mathcal{A}} \int_{A\in\mathcal{A}} \mathbf{set}(\Phi B, \mathbf{set}(\Psi A, \mathcal{A}(B,A))) && (\text{“Fubini”})\\
&\simeq \int_{B\in\mathcal{A}} \mathbf{set}(\Phi B, \int_{A\in\mathcal{A}} \mathbf{set}(\Psi A, \mathcal{A}(B,A))) && (\text{take end in})\\
&\simeq \int_{B\in\mathcal{A}} \mathbf{set}(\Phi B, \mathbf{set}^{\mathcal{A}}(\Psi, \mathcal{A}(B,-))) && \\
&\simeq \int_{B\in\mathcal{A}} \mathbf{set}(\Phi B, \Psi^- B) && (\text{definition } (\ )^-)\\
&\simeq \mathbf{set}^{\mathcal{A}^{op}}(\Phi,\Psi^-). \ \blacksquare
\end{aligned}$$

For $\mathcal{B}$ total and cototal, we have

(6 − 9)

$$\begin{tikzcd}
 & \hat{\mathcal{B}} \arrow[dl, bend right, "L"'] \arrow[dd, shift right=2, "(\ )^+"'] \arrow[dd, phantom, "\dashv"] \\
\mathcal{B} \arrow[ur, "Y"] \arrow[ur, phantom, "\perp"'] \arrow[dr, "Z"'] \arrow[dr, phantom, "\perp"] & \\
 & \check{\mathcal{B}} \arrow[ul, bend left, "R"] \arrow[uu, shift right=2, "(\ )^-"']
\end{tikzcd}$$

We proceed to show that this diagram commutes. The adjointness situation of proposition 6–8 is $\check{\mathcal{B}}(\Phi^+,\Psi) \simeq \hat{\mathcal{B}}(\Phi,\Psi^-)$. In addition, the following relations hold:

Theorem 6–10: Suppose($)^+$ and ($)^-$ exist. Then for $\Phi \in \hat{\mathcal{B}}, \Psi \in \check{\mathcal{B}}$,

(i)– If $\mathcal{B}$ is total, $\check{\mathcal{B}}(\Psi,\Phi^+) \simeq \Psi(L\Phi)$

(ii)– If $\mathcal{B}$ is cototal,$\hat{\mathcal{B}}(\Psi^-,\Phi) \simeq \Phi(R\Psi)$.

Proof: [RJW1]:

$$\begin{aligned}
(i) - \ (\mathbf{set}^{\mathcal{B}})^{op}(\Psi,\Phi^+) &\simeq \mathbf{set}^{\mathcal{B}}(\Phi^+,\Psi) && \text{(op – ing)}\\
&\simeq \int_{X\in\mathcal{B}} \mathbf{set}(\Phi^+X,\Psi X)\\
&\simeq \int_{X\in\mathcal{B}} \mathbf{set}(\mathbf{set}^{\mathcal{B}^{op}}(\Phi,\mathcal{B}(-,X)),\Psi X) && \text{(definition ( )}^+\text{)}\\
&\simeq \int_{X\in\mathcal{B}} \mathbf{set}(\mathcal{B}((L\Phi,X),\Psi X) && (\ L \dashv Y\ )\\
&\simeq \mathbf{set}^{\mathcal{B}}(\mathcal{B}(L\Phi,-),\Psi)\\
&\simeq \Psi(L\Phi) && \text{(Yoneda)}
\end{aligned}$$

$$\begin{aligned}
(ii) - \ \mathbf{set}^{\mathcal{B}}(\Psi^-,\Phi) &\simeq \int_{X\in\mathcal{B}} \mathbf{set}(\Psi^-X,\Phi X)\\
&\simeq \int_{X\in\mathcal{B}} \mathbf{set}(\mathbf{set}^{\mathcal{B}}(\Psi,\mathcal{B}(X,-)),\Phi X) && \text{(definition ( )}^-\text{)}\\
&\simeq \int_{X\in\mathcal{B}} \mathbf{set}((\mathbf{set}^{\mathcal{B}})^{op}(\mathcal{B}(X,-),\Psi),\Phi X) && \text{(op – ing)}\\
&\simeq \int_{X\in\mathcal{B}} \mathbf{set}(\mathcal{B}(X,R\Psi),\Phi X) && (\ Z \dashv R\ )\\
&\simeq \mathbf{set}^{\mathcal{B}^{op}}(\mathcal{B}(-,R\Psi),\Phi)\\
&\simeq \Phi(R\Psi). && \text{(Yoneda)} \ \blacksquare
\end{aligned}$$

Theorem 6–11: If $\mathcal{B}$ is total and cototal, for $\Phi \in \hat{\mathcal{B}}, \Psi \in \check{\mathcal{B}}$,

$$\begin{tikzcd}
\check{\mathcal{B}}(\Phi^+,\Psi) \arrow[rr, "\sim"] \arrow[d, "\wr"'] & & \hat{\mathcal{B}}(\Phi,\Psi^-) \arrow[d, "\wr"] \\
\Psi^-(L\Phi) \arrow[dr, "\sim"'] & & \Phi^+(R\Psi) \arrow[dl, "\sim"] \\
& \mathcal{B}(L\Phi,R\Psi) &
\end{tikzcd}$$

Proof: [RJW1]: (i) $\check{\mathcal{B}}(\Phi^+,\Psi) \simeq \hat{\mathcal{B}}(\Phi,\Psi^-)$ is the adjointness relation $(\ )+ \dashv (\ )^-$.

(ii)- $\hat{\mathcal{B}}(\Phi,\Psi^-) \simeq \Phi^+(R\Psi)$:

$$\begin{aligned}
\mathbf{set}^{\mathcal{B}^{op}}(\Phi,\Psi^-) &\simeq \int_{X\in\mathcal{B}} \mathbf{set}(\Phi X,\Psi^- X) \\
&\simeq \int_{X\in\mathcal{B}} \mathbf{set}(\Phi X,\mathbf{set}^{\mathcal{B}}(\Psi,\mathcal{B}(X,-))) && \text{(definition ( )}^-\text{)} \\
&\simeq \int_{X\in\mathcal{B}} \mathbf{set}(\Phi X,(\mathbf{set}^{\mathcal{B}})^{op}(\mathcal{B}(X,-),\Psi)) && \text{(op – ing)} \\
&\simeq \int_{X\in\mathcal{B}} \mathbf{set}(\Phi X,\mathcal{B}(X,R\Psi)) && (\ Z \dashv R\ ) \\
&\simeq \mathbf{set}^{\mathcal{B}^{op}}(\Phi,\mathcal{B}(-,R\Psi)) && \text{.....(1)} \\
&\simeq \Phi^+(R\Psi) && \text{(definition ( )}^+\text{)}
\end{aligned}$$

(iii)- $\Phi^+(R\Psi) \simeq \mathcal{B}(L\Phi,R\Psi)$:

$\mathcal{B}(L\Phi,R\Psi) \simeq \hat{\mathcal{B}}(\Phi,\mathcal{B}(-,R\Psi))$ which is (1) above, so the result is proved in (ii).

(iv)- $\check{\mathcal{B}}(\Phi^+,\Psi) \simeq \Psi^-(L\Phi)$:

$$\begin{aligned}
(\mathbf{set}^{\mathcal{B}})^{op}(\Phi^+,\Psi) &\simeq \mathbf{set}^{\mathcal{B}}(\Psi,\Phi^+) && \text{(op – ing)} \\
&\simeq \int_{X\in\mathcal{B}} \mathbf{set}(\Psi X,\Phi^+ X) \\
&\simeq \int_{X\in\mathcal{B}} \mathbf{set}(\Psi X,\mathbf{set}^{\mathcal{B}^{op}}(\Phi,\mathcal{B}(-,X))) && \text{(definition ( )}^+\text{)} \\
&\simeq \int_{X\in\mathcal{B}} \mathbf{set}(\Psi X,\mathcal{B}(L\Phi,X)) && (\ L \dashv Y\ ) \\
&\simeq \mathbf{set}^{\mathcal{B}}(\Psi,\mathcal{B}(L\Phi,-)) && \text{.....(2)} \\
&\simeq \Psi^-(L\Phi).
\end{aligned}$$

(v)- $\Psi^{-}(L\Phi) \simeq \mathcal{B}(L\Phi, R\Psi)$:

$\mathcal{B}(L\Phi, R\Psi) \simeq \check{\mathcal{B}}(\mathcal{B}(L\Phi, -), \Psi)$ since $Z \dashv R$. But $\check{\mathcal{B}}(\mathcal{B}(L\Phi, -), \Psi)$

$\simeq \mathbf{set}^{\mathcal{B}}(\Psi, \mathcal{B}(L\Phi, -))$ which is (2) above, so the result is proved in (iv). ∎

As a generalization of the poset case, we may relate R and L via the Isbell conjugation functors.

Theorem 6–12: If $\mathcal{B}$ is total and cototal,

(i)- $R \simeq L(\ )^{-}$

(ii)- $L \simeq R(\ )^{+}$

Proof: [RJW1]: (i) We wish to show, for $\Psi \in \widehat{\mathcal{B}}, B \in \mathcal{B}$, $\mathcal{B}(R\Psi, B) \simeq \mathcal{B}(L\Psi^{-}, B)$.

$$\begin{aligned}
\mathcal{B}(R\Psi, B) &\simeq \mathbf{set}^{\mathcal{B}^{op}}(\mathcal{B}(-, R\Psi), \mathcal{B}(-, B)) && (Z \text{ is fully faithful}) \\
&\simeq \int_{X \in \mathcal{B}} \mathbf{set}(\mathcal{B}(X, R\Psi), \mathcal{B}(X, B)) && \\
&\simeq \int_{X \in \mathcal{B}} \mathbf{set}((\mathbf{set}^{\mathcal{B}})^{op}(\mathcal{B}(X, -), \Psi), \mathcal{B}(X, B)) && (\ Z \dashv R\ ) \\
&\simeq \int_{X \in \mathcal{B}} \mathbf{set}(\mathbf{set}^{\mathcal{B}}(\Psi, \mathcal{B}(X, -)), \mathcal{B}(X, B)) && (\text{op} - \text{ing}) \\
&\simeq \int_{X \in \mathcal{B}} \mathbf{set}(\Psi^{-}X, \mathcal{B}(X, B)) && (\text{definition } (\ )^{-}\ ) \\
&\simeq \mathbf{set}^{\mathcal{B}^{op}}(\Psi^{-}, \mathcal{B}(-, B)) && \\
&\simeq \mathcal{B}(L\Psi^{-}, B) && (L \dashv Y)
\end{aligned}$$

(ii) We wish to show, for $\Phi \in \widehat{\mathcal{B}}, B \in \mathcal{B}$, $\mathcal{B}(B, L\Phi) \simeq \mathcal{B}(B, R\Phi^+)$.

$$\begin{aligned}
\text{Now, } \mathcal{B}(B, L\Phi) &\simeq (\mathbf{set}^{\mathcal{B}})^{op}(\mathcal{B}(B,-), \mathcal{B}(L\Phi,-)) && (Z \text{ is fully faithful}) \\
&\simeq \mathbf{set}^{\mathcal{B}}(\mathcal{B}(L\Phi,-), \mathcal{B}(B,-)) && (\text{op} - \text{ing}) \\
&\simeq \int_{X\in\mathcal{B}} \mathbf{set}(\mathcal{B}(L\Phi, X), \mathcal{B}(B, X)) \\
&\simeq \int_{X\in\mathcal{B}} \mathbf{set}(\mathbf{set}^{\mathcal{B}^{op}}(\Phi, \mathcal{B}(-,X), \mathcal{B}(B, X)) && (\ L \dashv Y\ ) \\
&\simeq \int_{X\in\mathcal{B}} \mathbf{set}(\Phi^+ X, \mathcal{B}(X, B)) && (\text{definition } (\ )^+\ ) \\
&\simeq \mathbf{set}^{\mathcal{B}}(\Phi^+, \mathcal{B}(B,-)) \\
&\simeq \check{\mathcal{B}}(\mathcal{B}(B,-), \Phi^+) && (\text{op} - \text{ing}) \\
&\simeq \mathcal{B}(B, R\Phi^+) && (\ Z \dashv R\ ) \blacksquare
\end{aligned}$$

To see that diagram (6–9) commutes, it remains to be shown that $ZL \simeq (\ )^+$:

Lemma 6–13: $ZL \simeq (\ )^+$.

Proof: Let $\Phi \in \widehat{\mathcal{B}}$, $\Psi \in \check{\mathcal{B}}$. Then $\check{\mathcal{B}}(ZL\Phi, \Psi) \simeq \mathcal{B}(L\Phi, R\Psi)$, since $(Z \dashv R)$, $\simeq \check{\mathcal{B}}(\Phi^+, \Psi)$ (by theorem 6–11). $\blacksquare$

Finally, we note that $(\ )^{-+} \simeq ZL(\ )^- \simeq ZR$. Since Z is fully faithful, it is cotripleable. And so, in this context, L is the canonical comparison functor:

$$\begin{tikzcd}
\widehat{\mathcal{B}} \arrow[rr, "L"] \arrow[dr, shift left=2, "(\ )^+"] \arrow[dr, phantom, "\dashv"'] & & \mathcal{B} \arrow[dl, shift right=2, "Z"'] \arrow[dl, phantom, "\dashv"] \\
& \check{\mathcal{B}} \arrow[ul, shift left=2, "(\ )^-"] \arrow[ur, shift right=2, "R"'] &
\end{tikzcd}$$

7–Topological examples

We continue to construct examples of total categories. In this ection, we show that **top** and like categories are total. First, recall how colimits are constructed in **top**. Let $\mathcal{E} \xrightarrow{M}$ **top** . To form $\varinjlim M$, one composes with the forgetful functor

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "M"] & \textbf{top} \arrow[d, "U"] \\
 & \textbf{set}
\end{tikzcd}$$

forms $\ell\colon UM \longrightarrow \varinjlim UM$ as a colimit in **set**, and equips $\varinjlim UM$ with the finest topology for which all components of ℓ are morphisms of **top** (i.e. continuous functions). This situation is abstracted via the notion of *total opfibration.*

Definition 7–1: A functor $U\colon \mathcal{T} \longrightarrow \mathcal{B}$ is said to be a *total opfibration* if $\mathcal{T}$ is locally small, $\widehat{U}\colon \widehat{\mathcal{B}} \longrightarrow \widehat{\mathcal{T}}$ has a left adjoint, $U_!$, and U satisfies the lifting condition: $D\colon \mathcal{T} \longrightarrow U_!/Y_{\mathcal{T}}$ defined by

(7 − 2)

$$\begin{tikzcd}
 & \mathcal{T} \arrow[ddl, "Y"'] \arrow[d, "D"] \arrow[ddr, "U"] & \\
 & U_!/Y \arrow[dl, "P"] \arrow[dr, "Q"] & \\
\widehat{\mathcal{T}} \arrow[rr, phantom, "\longrightarrow"] \arrow[dr, "U_!"'] & & \mathcal{B} \arrow[dl, "Y"] \\
 & \widehat{\mathcal{B}} \arrow[loop below, "\sim"] &
\end{tikzcd}$$

has a left adjoint over $\mathcal{B}$. □

Remarks 7–3:

1. left kan extension commutes with Yoneda. So, explicitly, D is given by

$$D(T) =< \mathcal{T}(-,T), \eta\colon U_! \mathcal{T}(-,T) \xrightarrow{\sim} \mathbf{B}(-,UT), UT > \text{ for } T \in \mathcal{T}$$

2. Recall that $\Phi \in \hat{\mathcal{T}}$ corresponds to a discrete fibration $M\colon \mathcal{E} \longrightarrow \mathcal{T}$ with small fibres. $U_!\Phi$ corresponds to the diagram, in $\mathcal{B}$, obtained by factoring UM. Furthermore, an object $< \Phi, \gamma\colon U_!\Phi \rightarrow \mathcal{B}(-,B), B >\in U_!/Y$ corresponds to a cone from the diagram to B. If D has a left adjoint, then it and its unit provide, for such an object of $U_!/Y$, a $T \in$ T and a "cone" $\Phi \longrightarrow \mathcal{T}(-,T)$ which is a best lifting of γ. □

The motivating example of a total opfibration is the forgetful functor $\mathbf{top} \xrightarrow{U} \mathbf{set}$.

Example 7–4: $U\colon \mathbf{top} \longrightarrow \mathbf{set}$ is a total opfibration: $U : \mathbf{top} \longrightarrow \mathbf{set}$ has a left adjoint (discrete topology functor) which induces a left adjoint to $\hat{U}$.
Now, let $S \in \mathbf{set}$. $U_!\Phi \longrightarrow \mathbf{set}(-,S)$ effectively gives a cone to S. The left adjoint to D sends $< \Phi, U_!\Phi \xrightarrow{\eta} \mathbf{set}(-,S), S >$ to S equipped with the finest topology for which the components of η are continuous. □

Theorem 7–5: If $U\colon \mathcal{T} \longrightarrow \mathcal{B}$ is a total opfibration and $\mathcal{B}$ is total, then $\mathcal{T}$ is total and U is cocontinuous.

Before we prove this theorem, we require a lemma from general category theory.

Lemma 7–6: Given a comma category

$$\begin{tikzcd}
 & G/U \arrow[dl, "P"'] \arrow[dr, "Q"] & \\
\mathcal{W} \arrow[dr, "G"'] \arrow[rr, phantom, "\longrightarrow"] & & \mathcal{A} \arrow[dl, "U"] \\
 & \mathcal{X} &
\end{tikzcd}$$

and $F \dashv U$ then there exists $T \dashv P$ and $\mathcal{W} \xrightarrow{T} G/U \xrightarrow{P} \mathcal{W}$ is the identity. (i.e. T is fully faithful).

Proof: We have

$$\begin{tikzcd}
 & \mathcal{W} \arrow[dl, "\mathcal{W}"'] \arrow[dr, "FG"] \arrow[ddl, bend right=60, "G"'] \arrow[ddr, bend left=60, "UFG"] & \\
\mathcal{W} \arrow[dr, "G"'] \arrow[rr, phantom, "\xrightarrow{\eta_G}"] & & \mathcal{A} \arrow[dl, "U"] \\
 & \mathcal{X} &
\end{tikzcd}$$

where ηG is the unit $\eta \mathcal{A} \longrightarrow UF$ (of $F \dashv U$) applied to G. define T by $T(W) = \langle W, GW \xrightarrow{\eta_G W = \eta G_W} UFGW, FGW \rangle$. Now, let $\langle W', GW' \xrightarrow{x} UA, A \rangle$ be an object of G/U. Then

$$\begin{array}{c}
T(W) = \langle W, GW \xrightarrow{\eta_G W} UFGW, FGW \rangle \\
\Big\downarrow x \\
\langle W', GW' \xrightarrow{x} UA, A \rangle \\
\hline
W \xrightarrow{w} W', FGW \xrightarrow{a} A \text{ such that}
\end{array}$$

$$\begin{tikzcd}
GW \arrow[r, "\eta_G w"] \arrow[d, "Gw"'] & UFGW \arrow[d, "Ua"] \\
GW' \arrow[r, "x"'] \arrow[ur, phantom, "//"] & UA
\end{tikzcd}$$

$$\begin{array}{c}
\hline
(*)
\end{array}$$

But this says that $FGW \xrightarrow{a} A$ is the transpose $(F \dashv U)$ of $GW \xrightarrow{xGw} UA$.

Thus a map $T(W) \longrightarrow \; < W', x, A >$ is the same as a map $W \longrightarrow W'$.

Summarizing:

$$\frac{\overline{\quad (*) \quad}}{\dfrac{W \longrightarrow W'}{W \to P < W', x, A >}}$$

Hence $T \dashv P$. ∎

Proof:(of theorem 7–5):[RJW2]: For reference recall diagram (7–2)

$$\begin{tikzcd}
 & \mathcal{T} \arrow[d, "D"] \arrow[ddl, "Y"'] \arrow[ddr, "U"] & \\
 & U_!/Y \arrow[dl, "P"'] \arrow[dr, "Q"] & \\
\hat{\mathcal{T}} \arrow[dr, "U_!"'] \arrow[rr, phantom, "\longrightarrow"] & & \mathcal{B} \arrow[dl, "Y"] \\
 & \hat{\mathcal{B}} \arrow[loop below, no head, "\sim"] &
\end{tikzcd}$$

$\mathcal{B}$ is total so $Y_{\mathcal{B}}$ has a left adjoint. Hence, by the lemma, $P : U_!/Y \longrightarrow \hat{\mathcal{T}}$ has a left adjoint. Explicitly, it sends $\Phi \in \hat{\mathcal{T}}$ to $< \Phi, U_!\Phi \xrightarrow{\eta} \mathcal{B}(-, LU_!\Phi), LU_!\Phi >$.

By hypothesis, D has a left adjoint and adjoints compose so $Y_{\mathcal{T}}$ has a left adjoint. The left adjoint of D is over $\mathcal{B}$ so U is cocontinuous. ∎

In the above proof, the construction of $L_{\mathcal{T}}$, through $U_!/Y$, is a generalization of the construction of colimits in **top** mentioned at the beginning of this section.

The dual notion of total opfibration is *cototal fibration*.

Definition 7–7: A functor $V : \mathcal{T} \longrightarrow \mathcal{B}$ is a *cototal fibration* if $\mathcal{T}$ is locally small, $\check{V} : \check{\mathcal{B}} \longrightarrow \check{\mathcal{T}}$ has a right adjoint V_* and E defined by

$$\begin{tikzcd}
 & \mathcal{T} \arrow[d, "E"] \arrow[ddl, "V"'] \arrow[ddr, "Z"] & \\
 & Z/V_* \arrow[dl, "P"'] \arrow[dr, "Q"] & \\
\mathcal{B} \arrow[rr] \arrow[dr, "Z"'] & & \check{\mathcal{T}} \arrow[dl, "V_*"] \\
 & \check{\mathcal{B}} \arrow[loop below, "\sim"] &
\end{tikzcd}$$

has a right adjoint over $\mathcal{B}$. ☐

Dualizing lemma 7–6 gives a right adjoint to Q so the dual of theorem 7–5 is:

Theorem 7–8: If $\mathcal{T} \xrightarrow{V} \mathcal{B}$ is a cototal fibration and $\mathcal{B}$ is cototal, then $\mathcal{T}$ is also cototal. ■

We may conclude that **top** is also cototal since the indiscrete topology functor is right adjoint to the forgetful functor; which induces $\check{I} \vdash \check{U}$. Another example is the following:

Example 7–9: Comphaus, the category of compact Hausdorff spaces, is both total and cototal: **Comphaus** is total since it is a full reflective subcategory of **top** ($S \dashv I$: **Comphaus** $\longrightarrow$ **top**, where S is the Stone–Čech compactification of a space). [0,1] is a cogenerator for **Comphaus** so, by theorem 6–4, **Comphaus** is also cototal. ☐

8– Cartesian Closedness

In this section, we give a simple application of theorem 5–3; a relationship between total and cartesian closed categories. Recall the definition of a cartesian closed category.

Definition 8–1: $\mathcal{B}$ in **CAT** is said to be cartesian closed if it has finite products and for each $B \in \mathcal{B}$,

$$(8-2) \qquad B \times - : \mathcal{B} \longrightarrow \mathcal{B}$$

has a right adjoint. ☐

If $\mathcal{B}$ is total, then $\mathcal{B}$ has finite products and the functors in (8–2) are admissible. To show that $\mathcal{B}$ is cartesian closed, we need only have $B \times -$ cocontinuous by the total adjoint functor theorem. In fact, we have the following characterization:

Theorem 8–3: If $\mathcal{B}$ is total, then $\mathcal{B}$ is cartesian closed iff L preserves binary products.

Proof: Consider the diagram

$$\begin{tikzcd}
\hat{\mathcal{B}} \arrow[r, "(A\times -)_!"] \arrow[d, "L"'] & \hat{\mathcal{B}} \arrow[d, "L"] \\
\mathcal{B} \arrow[r, "(A\times -)"'] & \mathcal{B}
\end{tikzcd}$$

A direct calculation shows that $(A \times -)_!$ is given by $\mathcal{B}(-, A) \times -$.

$(A \times -)$ is cocontinuous iff the diagram commutes up to isomrphism. That is, iff $L(\mathcal{B}(-,A) \times \Phi) \simeq A \times L\Phi \ \forall \Phi \in \widehat{\mathcal{B}}$.

If L preserves binary products, then $L(\mathcal{B}(-,A) \times \Phi) \simeq L(\mathcal{B}(-,A)) \times L\Phi \simeq A \times L\Phi$.

Conversely, Suppose $L(\mathcal{B}(-,A) \times \Phi) \simeq A \times L\Phi$. Then

$$\begin{aligned}
L(\Psi \times \Phi) &\simeq L((\int^{A \in \mathcal{B}} \Psi A.\mathcal{B}(-,A)) \times \Phi) && \text{(Yoneda density lemma)}\\
&\simeq L(\int^{A \in \mathcal{B}} \Psi A.(\mathcal{B}(-,A) \times \Phi)) && (- \times \Phi \text{ cocontinuous})\\
&\simeq \int^{A \in \mathcal{B}} \Psi A.L(\mathcal{B}(-,A) \times \Phi) && (L \text{ cocontinuous})\\
&\simeq \int^{A \in \mathcal{B}} \Psi A.(A \times L\Phi) && \text{(by hypothesis)}\\
&\simeq (\int^{A \in \mathcal{B}} \Psi A.A) \times L\Phi && (- \times L\Phi \text{ cocontinuous})\\
&\simeq L(\Psi) \times L(\Phi) && (L \text{ as a coend}). \blacksquare
\end{aligned}$$

Definition 8–4: A total category, $\mathcal{B}$, is said to be lex total if L preserves finite limits. □

Corollary 1: If $\mathcal{B}$ is lex total, then $\mathcal{B}$ is cartesian closed.

Proof: obvious. ■

Corollary 2: Let $\mathcal{B}$ be total and suppose L preserves equalizers. Then $\mathcal{B}$ is lex total iff $\mathcal{B}$ is cartesian closed.

Proof: limits by products and equalizers. ■

Theorem 8–3 provides an example of a total category which is total but not lex total.

Example 8–5: **grp** is not lex total since **grp** is not cartesian closed. □

It should be noted that $\hat{\mathcal{B}}$ is not, in general, cartesian closed. However, it is well known that for small $\mathbf{B}$, $\hat{\mathbf{B}}$ is cartesian closed.

9– Lex total categories

In the previous section, we defined a lex total category as a total category whose L is lex. As with total categories, it is useful to begin with lex total posets. Recall that a locale is a sup complete, cartesian closed poset. If $\mathbf{P}$ is a lex total poset, $\bigvee \dashv (\downarrow -)$ and $\bigvee$ is lex. Thus, by corollary 2, theorem 8–3, $\mathbf{P}$ is a lex total poset iff it is a locale.

The primary example of a locale, and hence of a lex total poset, is the lattice of opens of a topological space with $\bigvee$ given by union and finite $\bigwedge$ given by finite intersection. We now return to general, lex total categories. An important example is the following:

Example 9–1: $\hat{\mathbf{A}}$, for $\mathbf{A}$ in **cat** , is lex total: In fact we have

$$\begin{tikzcd}
\hat{\mathbf{A}} \arrow[r, shift left=5] \arrow[r, phantom, shift left=2.5, "\perp"] \arrow[r, leftarrow] \arrow[r, phantom, shift right=2.5, "\perp"] \arrow[r, shift right=5, "Y_{\mathbf{A}}"'] & \hat{\hat{\mathbf{A}}}
\end{tikzcd}$$

We saw, in section 4, that $\widehat{Y_{\mathbf{A}}} \dashv Y_{\hat{\mathbf{A}}}$. Now, $\widehat{Y_{\mathbf{A}}}$ has a left adjoint given by $(Y_{\mathbf{A}})_!$; left Kan extension. □

Many of the closure properties of section 4 hold in a lex total setting with appropriate modifications. For example,

Theorem 9–2: Let $\mathcal{A}$ be a lex reflective subcategory of a lex total category $\mathcal{B}$. Then $\mathcal{A}$ is lex total.

Proof: Recall the diagram of theorem 4–4:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "Y_{\mathcal{A}}"] \arrow[d, shift left=2, "F"] \arrow[d, phantom, "\dashv"] & \widehat{\mathcal{A}} \arrow[d, shift right=2, "F_!"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{B} \arrow[u, shift left=2, "T"] \arrow[r, shift right=2, "Y_{\mathcal{B}}"'] \arrow[r, phantom, "\perp"] & \widehat{\mathcal{B}} \arrow[l, shift right=2, "L_{\mathcal{B}}"'] \arrow[u, shift right=2, "\widehat{F}"']
\end{tikzcd}$$

where F is the "inclusion".

In theorem 4–4, it was proved that $TL_{\mathcal{B}}F_! \dashv Y_{\mathcal{A}}$. If T is lex (i.e. $\mathcal{A}$ lex reflective subcategory of $\mathcal{B}$) then each of the factors of $L_{\mathcal{A}}$, as constructed above, is lex. Indeed, $F_! = \widehat{L}$ so it preserves limits and colimits. And so, $L_{\mathcal{A}}$ is lex. ■

A grothendieck topos is a lex reflective subcategory of $\widehat{\mathbf{A}}$, for small $\mathbf{A}$ and so Grothendieck topoi are lex total. In fact, as we shall see later in this section, lex totals are "very nearly" topoi. Lex totals share many properties with topoi. An example is given by theorem 9–5 below.

Definition 9–3: ([Ba] p.4): An epimorphism is said to be regular if it is the coequalizer of its kernel pair. A category, $\mathcal{B}$, is said to be regular if the pullback, in $\mathcal{B}$, of a regular epi is regular epi. □

Remark 9–4: We denote a regular epi by $\xrightarrow{\quad\quad}\!\!\!\!\rightarrow$. □

Theorem 9–5: A lex total category is regular.

Proof: Given $C \xrightarrow{g} B \overset{f}{\twoheadleftarrow} A$ in $\mathcal{A}$, form its pullback ($\mathcal{A}$ is complete):

$$\begin{tikzcd}
P \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\times"] & A \arrow[d, two heads, "f"] \\
C \arrow[r, "g"'] & B
\end{tikzcd}$$

f is a regular epi so suppose its kernel pair is h, k. Applying Y gives a diagram in $\widehat{\mathcal{A}}$:

$$\begin{tikzcd}
 & {} \arrow[d, shift right, "Yh"'] \arrow[d, shift left, "Yk"] \\
YP \arrow[r, "Yv"] \arrow[d, "Yu"'] \arrow[dr, phantom, "\times"] & YA \arrow[d, "Yf"] \\
YC \arrow[r, "Yg"'] & YB
\end{tikzcd}$$

The diagram is still a pullback since Y is continuous. However, Y is not cocontinuous (in fact, Y preserves only absolute colimits) so Yf is not (necessarily) a regular epi.

We may factor Yf into a regular epi followed by a mono:

$$\begin{tikzcd}
YA \arrow[r, two heads, "\delta"'] \arrow[rr, bend left, "Yf"] & \Phi \arrow[r, "\gamma"'] & YB
\end{tikzcd}$$

Consider the diagram:

$$\begin{tikzcd}
 & {} \arrow[d, shift right, "Yh"'] \arrow[d, shift left, "Yk"] \\
YP \arrow[r, "Yv"] \arrow[d, two heads, "\alpha"'] \arrow[dr, phantom, "\times"] & YA \arrow[d, two heads, "\delta"] \arrow[dd, bend left=50, "Yf"] \\
\Psi \arrow[r] \arrow[d, "\beta"'] \arrow[dr, phantom, "\times"] & \Phi \arrow[d, "\gamma"] \\
YC \arrow[r, "Yg"'] & YB
\end{tikzcd}$$

By the pullback lemma, $Yu = \beta\alpha$. Furthermore, α is a regular epi since $\widehat{\mathcal{A}}$ is regular.

Applying L gives a diagram:

$$\begin{tikzcd}
 & {} \arrow[d, shift right, "h"'] \arrow[d, shift left, "k"] \\
P \arrow[r] \arrow[d, two heads, "L\alpha"'] \arrow[dr, phantom, "\times"] & A \arrow[d, two heads, "L\delta"] \arrow[dd, bend left=50, "f"] \\
L\Psi \arrow[r] \arrow[d, "L\beta"'] \arrow[dr, phantom, "\times"] & L\Phi \arrow[d, "L\gamma"] \\
C \arrow[r, "g"'] & B
\end{tikzcd}$$

The two squares are pullbacks since L is lex. Also $L(\delta)$ and $L(\alpha)$ are regular epis since L is a left adjoint. Now, f and $L(\delta)$ are coequalizers of h, k. But coequalizers are unique up to isomrophism so $L(\gamma)$ is an iso. Hence $L(\beta)$ is an iso.

And so, we have

$$\begin{tikzcd}
P \arrow[r, "v"] \arrow[d, two heads, "u"'] \arrow[dr, phantom, "\times"] & A \arrow[d, two heads, "f"] \\
C \arrow[r, "g"'] & B
\end{tikzcd}$$

making $\mathcal{A}$ regular. ▮

Theorem 9–6: Let $\mathcal{A}$ be lex total. Equivalence relations in $\mathcal{A}$ are effective.

Proof: Let $R \overset{a}{\underset{b}{\rightrightarrows}} X$ be an equivalence relation in $\mathcal{A}$. Applying Y gives: $YR \overset{Ya}{\underset{Yb}{\rightrightarrows}} YX$, an equivalence relation in $\widehat{\mathcal{A}}$.

Equivalence relations in $\widehat{\mathcal{A}}$ are effective so $\exists$ F, such that

$$\begin{tikzcd}
YR \arrow[r, "Ya"] \arrow[d, "Yb"'] \arrow[dr, phantom, "\times"] & YX \arrow[d] \\
YX \arrow[r] & F
\end{tikzcd}$$

L preserves pullbacks and hence kernel pairs, so

$$\begin{tikzcd}
R \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\times"] & X \arrow[d] \\
X \arrow[r] & LF
\end{tikzcd}$$

and equivalence relations in $\mathcal{A}$ are effective. ∎

Corollary:(Walters): A lex total is exact. ∎

Exactness may be regarded, in a sense, as a relationship between limits and colimits. The proofs of the above two theorems suggest a method for discovering such relationships in a lex total category:

1: send the diagram to $\widehat{\mathcal{A}}$ via Yoneda.

2: use the exactness properties of $\widehat{\mathcal{A}}$.

3: send the diagram back using the facts that L is lex and a left adjoint.

The corollary above provides another "proof" that **grp** is total but not lex total, since **grp** is not exact (see [He] p. 295).

We said above that lex totals are "very nearly" topoi. In the remainder of this section, we make the term "very nearly" more precise.

Theorem 9–8: (Street): Let $\mathcal{E}$ be a locally small category with a small generating set of objects. The following are equivalent:

1- $\mathcal{E}$ is a Grothendieck topos.

2- $\mathcal{E}$ is lex total.

3- Every **set**-valued canonical sheaf on $\mathcal{E}$ is representable and $\mathcal{E}$ has all small colimits.

4- $\mathcal{E}$ is an elementary topos with all small colimits.

5- $\mathcal{E}$ is a pretopos with all small coproducts, and these are universal (i.e. preserved by pullback).

Remarks 9–9:

1. Recall that the canonical topology on a category, $\mathcal{E}$, is the largest topology for which the representables are sheaves; canonical sheaves are sheaves for this topology.
2. A category, $\mathcal{E}$, is an elementary topos if $\mathcal{E}$ has finite limits, is cartesian closed, and has a subobject classifier.
3. Also, recall ([PTJ] p. 238) that a category, $\mathcal{E}$, is said to be a pretopos if it has finite limits, has finite coproducts which are disjoint and universal, has coequalizers of equivalence relations which are universal, and every equivalence relation in $\mathcal{E}$ is effective and every epi in $\mathcal{E}$ is a coequalizer. ☐

The proof of theorem 9–8 uses Giraud's theorem.

Theorem 9–10: (Giraud): The following are equivalent:

(i) $\mathcal{E}$ is a Grothendieck topos.

(ii) $\mathcal{E}$ satisfies:

- (a) $\mathcal{E}$ has finite limits.
- (b) $\mathcal{E}$ has all **set**–indexed coproducts, and they are disjoint and universal.
- (c) Equivalence relations in $\mathcal{E}$ have universal coequalizers.
- (d) Equivalence relations in $\mathcal{E}$ are effective, and every epimorphism is a coequalizer.
- (e) $\mathcal{E}$ is locally small.
- (f) $\mathcal{E}$ has a small set of generators.

Proof: see, for example, [PTJ] pp. 17–18. ∎

Proof:(of theorem 9–8): (1) $\iff$ (3) $\iff$ (4) $\iff$ (5) follow from Giraud's theorem.

(1) $\Rightarrow$ (2): was noted above. Recall that a Grothendieck topos is a lex reflective subcategory of $\widehat{\mathbf{A}}$ and, hence, is lex total.

(2) $\Rightarrow$ (1): We proceed to verify (a)–(f) of theorem 9–10. (a) follows from the fact that $\mathcal{E}$ is total. The exactness conditions (b)–(d) are (partially) proved above in theorems 9–5 and 9–6. Other exactness properties follow from the methods described after those theorems. Finally, (e) and (f) are hypotheses. ∎

In fact, relationships between some of the five conditions of theorem 9–8 may be proved using fewer assumptions. An extensive list appears in [St1]. We reproduce only one of Street's results, the proof of which is straightforward. We first introduce a "size" (for sets) between small and large.

Definition 9–11: Let $\mathcal{N}$ be the strongly inaccessible cardinal used to define **set**. We say "X is a moderate set" to indicate $|X| \leq \mathcal{N}$. ☐

Theorem 9–12: (Freyd and Street): Suppose $\mathcal{E}$ is a total category satisfying:

(a) if a pushout of a mono is an iso, then the mono is an iso.

(b) there is a moderate set M of objects of $\mathcal{E}$ such that, for each $E \in \mathcal{E}$, there is an extremal epi $D \longrightarrow E$ with D in M.

Then $\mathcal{E}$ has a strongly generating small set of objects.

Remarks 9–13:

1. If $\mathcal{E}$ is lex total and

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "+"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

in $\mathcal{E}$, then the square is also a pullback:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\times"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

Hence, condition (a) holds for lex totals.

2. If $\mathcal{E}$ has a moderate set of iso classes, then condition (b) is satisfied. ☐

Proof: [ST1]: Suppose that $\mathcal{E}$ has no small strongly generating set of objects. We proceed to show a contradiction.

Observe that M is not small. Assume $\mathcal{E}$ is skeletal and well-order M so that $\{D \in M | D \leq E\} =: \downarrow E$ is small $\forall E \in M$.

For each $E \in M$, the set $\downarrow E$ cannot strongly generate $\mathcal{E}$ so

(*) there are objects, C_E, $D_E \in \mathcal{E}$, and a monomorphism, $m_E : C_E \longrightarrow D_E$ which is not an isomorphism and yet $\mathcal{E}(B, m_E) : \mathcal{E}(B, C_E) \longrightarrow \mathcal{E}(B, D_E)$ is an isomorphism for all $B \leq E$.

In $\mathbf{SET}^{\mathcal{E}^{op}}$, we may form the colimit:

$$\begin{tikzcd}
YC_E \arrow[r, "Ym_E"] \arrow[d] & YD_E \arrow[d, "g_E{,}\ E \in M"] \\
1 \arrow[r, "w"'] & P
\end{tikzcd}$$

$P : \mathcal{E}^{op} \longrightarrow \mathbf{SET}$, at $A \in \mathcal{E}$, is given by the sum of 1 and, for each $F \in M$, the set of arrows $A \to D_F$ which do not factor through m_F.

For each $A \in \mathcal{E}$, (b) gives us an extremal epi $D \longrightarrow A$ with $D \in M$. Thus, $PA \longrightarrow PD$ is a monomorphism. Now, $\mathcal{E}(D, m_G)$ is an isomorphism for all $G \geq D$. Thus, PD is small. But, $PA \hookrightarrow PD$ so P factors through **set** and, considered as a colimit, is in $\mathbf{set}^{\mathcal{E}^{op}}$.

Applying $L_{\mathcal{E}}$, (note that $LY = 1$ since Y is fully faithful) yields a colimit (L is a left adjoint) in $\mathcal{E}$:

$$\begin{tikzcd}
C_E \arrow[r, "m_E"] \arrow[d] & D_E \arrow[d, "h_E{,}\ E \in M"] \\
1 \arrow[r, "Lw"'] & LP
\end{tikzcd}$$

For each $X \in M$, define $k_X : LP \longrightarrow LP$ by:

$$k_X = \begin{cases} k_X(LF) = LF, & \\ k_X(h_E) = h_E, & \text{when } E = X\ ; \\ k_X(h_E) = D_E \longrightarrow 1 \xrightarrow[Lw]{} LP, & \text{when } E \neq X. \end{cases}$$

Now, $M \xrightarrow{f} \mathcal{E}(LP, LP)$ ($X \mapsto k_X$) is a monomorphism for suppose $k_X = k_{X'}$ and $X \neq X'$. In particular, $k_X h_X = k_{X'} h_X$. So h_X factors through Lw

Consider the following diagram:

$$\begin{tikzcd} C_X \arrow[r, "m_X"] \arrow[d] & D_X \arrow[d] \\ 1 \arrow[r] & 1 \end{tikzcd}$$

Since LP is a colimit, this diagram is a pushout and, by (a), m_X must be an isomorphism; contradicting $(*)$ above. Thus, f, above is a monomorphism. But, $\mathcal{E}(LP, LP)$ is small since $\mathcal{E}$ is locally small. And so, this contradicts the fact that M is not small. ∎

The above theorem shows us that we may replace "small generating set" in "lex total + small generating set = Grothendieck topos" by "moderate generating set"; a slightly weaker hypotheses.

10– n–total categories

We may specialize total categories by considering a left adjoint to L. More generally,

Definition 10–1 A locally small category, $\mathcal{A}$, is said to be n–total ($n \in \mathbf{N}$) if there are functors $F_1,\ F_2, \ldots,\ F_n$ in the adjointness situation:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, shift right=12, "Y"'] \arrow[r, phantom, shift right=8, "\perp"] & \hat{\mathcal{A}} \arrow[l, shift right=4, "F_1 \perp"'] \arrow[l, shift left=4, "F_2 \perp"', leftarrow] \arrow[l, phantom, shift left=10, "\vdots"] \arrow[l, shift left=16, "F_n"']
\end{tikzcd}$$

Remarks 10–2:

1. Note that, in particular, a total category is 1–total. We call a 2–total category "essential total"; terminology derived from essential geometric morphism.
2. One may define $\mathcal{N}$–total, for $\mathcal{N}$=$\mathbf{N}$, by considering a countable string of adjunctions. It is not clear, however, how to define α–total for an arbitrary cardinal $\mathcal{N}$.
3. One may use a sequence of n adjunctions on the right of Z as a definition of n–cototal. Of course, $\mathcal{B}$ is n–cototal iff $\mathcal{B}^{op}$ is n–total.
4. If $\mathcal{B}$ is n–total ($n \geq 2$) it is, in particular, lex total so it is cartesian closed. Now, if $\mathcal{B}$ is also n–cototal, then $\mathcal{B}^{op}$ is cartesian closed. Examples of such seem to be rare. ☐

As we saw, there are many examples of total categories. For larger n, examples are rarer. Indeed, it is not known whether there are n–total categories for $n \geq 5$. However, the following theorem shows that there are posets which have an arbitrarily long (finite) string of adjunctions left of $\downarrow$.

Theorem 10–3: For each $n \in \mathbf{N}$, there is a poset which has a string of n adjoints left of $\downarrow$.

Proof: We proceed to show that $\mathbf{n}= \{0, 1, \ldots n-1\}$ is $2n$–total (in the poset context).

$2^{\mathbf{n}^{op}} \simeq \mathbf{n}+1$ and we have a sequence of adjunctions; "face" and "degeneracy":

$$\begin{tikzcd}
\mathbf{n} \arrow[r, shift left=30, "\perp" description, phantom] \arrow[from=1-2, to=1-1, shift right=36, "s_{n-1}"'] \arrow[r, shift left=24, "m_{n-1}"'] \arrow[r, shift left=14, "\vdots" description, phantom] \arrow[from=1-2, to=1-1, shift right=6, "s_1 \perp"'] \arrow[r, shift left=3, "\perp" description, phantom] \arrow[r, "m_1"'] \arrow[r, shift right=3, "\perp" description, phantom] \arrow[from=1-2, to=1-1, shift left=6, "s_0"] \arrow[r, shift right=9, "\perp" description, phantom] \arrow[r, shift right=12, "m_0"'] & \mathbf{n}+1
\end{tikzcd}$$

where $m_i : \mathbf{n} \longrightarrow \mathbf{n}+1$ acts as the identity on $\{0, 1, \ldots, i-1\}$ and sends $i \mapsto i+1$, $i+1 \mapsto i+2$, etc. and $s_i : \mathbf{n}+1 \longrightarrow \mathbf{n}$ and sends i, $i+1 \mapsto i$ and sends $i+2 \mapsto i+1$, $i+3 \mapsto i+2$, etc.

Finally, note that m_0 is $\downarrow$ so $\downarrow$ has a string of $2n$ adjunctions on the left. ∎

It should be noted that $\mathbf{n}$ is "$2n$–total" in the poset context only. There is not a string of $2n$–adjoints left of $2 \longrightarrow$ **set**.

Our "chief" example of a lex total category was $\widehat{\mathbf{A}}$ for small $\mathbf{A}$. We saw that it was an example of an essential total category. Indeed, this particular category leads to an interesting conjecture. In [MB], Bunge proved that a category, $\mathcal{X}$, is equivalent to $\widehat{\mathbf{A}}$ iff $\mathcal{X}$ is complete, wellpowered, cowellpowered, coregular, and has a generating set of abstractly unary projectives. K, abstractly unary projective, in her context, means $\mathcal{X}(-,K)$ is cocontinuous.

Suppose $\mathcal{B}$ is essential total with a small generating set of objects. In particular, $\mathcal{B}$ is lex total so, by theorem 9–8, it is a Grothendieck topos and, as such, satisfies the first four hypotheses of Bunge's characterization.

For an object, K, of $\mathcal{B}$, $\mathcal{B}(-,K) : \mathcal{B}^{op} \longrightarrow \mathbf{set}$. **set** is locally small so $\mathcal{B}(-,K)$ is admissible. Now, $\mathcal{B}$ is a Grothendieck topos, so $\mathcal{B}^{op}$ is total. By the total adjoint functor theorem, $\mathcal{B}(-,K)$ is cocontinuous iff it has a right adjoint. In particular, this is true for generating objects, K. Thus, K, abstractly unary projective, in our context, means $\mathcal{B}(-,K)$ has a right adjoint. The question remains whether an essential total category with a small generating set contains a generating set of objects, K, for which $\mathcal{B}(-,K)$ has a right adjoint. In other words, is an essential total "very close" to $\widehat{\mathbf{A}}$ in the same sense as a lex total is almost a topos?

Wood has suggested a way in which to attack this problem. Suppose we have:

$$\begin{tikzcd}[column sep=huge]
\mathcal{B} \arrow[r, shift left=4, "T"] \arrow[r, shift right=4, "Y"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathbf{set}^{\mathcal{B}^{op}} \arrow[l]
\end{tikzcd}$$

Transpose Y and T (exponential adjunction) to H and S: $\mathcal{B}^{op} \times \mathcal{B} \overset{S}{\underset{H}{\rightrightarrows}} \mathbf{set}$. Consider the full subcategory of $\mathcal{B}$ determined by those $B \in \mathcal{B}$ for which $S(B,B) \simeq H(B,B)$ as sets. Let $\mathcal{A}$ denote the cauchy completion of this subcategory. If we can show

(i)– $\mathcal{A}$ is essentially small

(ii)– $\mathcal{B} \simeq \mathbf{set}^{\mathcal{A}^{op}}$

then we have $\mathcal{A} \simeq \widehat{\mathbf{A}}$ as above.

In fact, item (ii) implies item (i). Indeed, $\mathcal{B}$ is assumed to be locally small, so $\mathbf{set}^{\mathcal{A}^{op}}$ is locally small, assuming (ii). Furthermore, $\mathcal{A}$ is locally small since it is a subcategory of $\mathcal{B}$ (recall that $\mathcal{B}$ is complete). From theorem 1–3, it follows that $\mathcal{A}$ is essentially small. Thus, our question about essential totals is reduced to showing (ii). Cauchy completeness of $\mathcal{A}$ is apparently needed to construct the equivalence for (ii).

When $\mathbf{A} = 1$ (i.e. $\widehat{\mathbf{A}} = \mathbf{set}$), the string of adjunctions is somewhat longer:

Theorem 10–4: set is 4–total.

Proof: In fact,

(10 – 5)
$$\begin{tikzcd}[column sep=huge, row sep=huge]
\check{\mathbf{set}} \arrow[r, phantom, "\perp"] \arrow[r, shift right=3, "R"'] & \mathbf{set} \arrow[l, shift right=3, "z"'] \arrow[r, shift left=12, "K"] \arrow[r, shift left=3, "\Delta\ \perp"] \arrow[r, shift right=12, "\perp", "Y"'] & \widehat{\mathbf{set}} \arrow[l, shift right=7, "\mathit{evat}(\emptyset)\ \perp"'] \arrow[l, shift left=2, "\mathit{evat}(1)\ \perp"]
\end{tikzcd}$$

where Δ is the diagonal and $K(X) = X.\mathbf{set}(-,\emptyset)$ for $X \in \mathbf{set}$.

(i)– **set** is cototal as a special case of the fact that Grothendieck topoi are cototal.

(ii)– $evat(1) \dashv Y$ was shown in corollary 1 to theorem 4–1.

(iii)– $evat(\emptyset) \dashv \Delta \dashv evat(1)$ since $\emptyset$ is the terminal object and 1 is the initial object of $\mathbf{set}^{op}$.

(iv)– $K \dashv evat(\emptyset)$ for $X \in \mathbf{set}$, $\Phi \in \widehat{\mathbf{set}}$,

$$\begin{array}{c} X.\mathbf{set}(-,\emptyset) \longrightarrow \Phi \\ \hline X \longrightarrow \widehat{\mathbf{set}}(\mathbf{set}(-,\emptyset),\Phi) \\ \hline X \longrightarrow \Phi(\emptyset). \end{array} \qquad \blacksquare$$

Remarks 10–6:

1. It is interesting to note that K acts as a "characteristic function":

$$K(X)(W) = \begin{cases} X & \text{if } W = \emptyset \\ \emptyset & \text{if } W \neq \emptyset \end{cases}$$

2. K does not have a left adjoint since it does not preserve the terminal object. It is interesting to note, however, that K preserves products indexed by nonempty sets and all equalizers.

3. $\mathbf{set}^{op}$ is not cartesian closed so, by theorem 8–3, R does not have a right adjoint. □

It is not known whether the adjunctions of (10-5) characterize **set**. It appears that the answer to this question swings on the discussion above about characterizing

$\hat{\mathbf{A}}$. Another question is raised: Do the extra adjoints, K and $evat(\emptyset)$, guarantee that $\mathbf{A} = 1$?

Finally, we may (partially) summarize section 10 via a series of nested inclusions:

locally small categories

$\cup$

totals

$\cup$

cartesian closed totals

$\cup$

topoi $\supset$ Grothendieck topoi $\subset$ lex totals

$\cup$

$\hat{\mathbf{A}} \subset$ essential totals

$\cup$

set $\subset$ 4–totals

$\cup$

n–totals $n \geq 4$

Bibliography

[Ad] Adámek, Jiri, "Colimits of algebras revistited", Bull. Aust. Math. Soc. **17** (1977), 433–450.

[Ba] Barr, Michael, "Exact categories", Lecture Notes in Math. **236**, Springer (Berlin, 1971).

[MB] Bunge, Marta,"Relative functor categories and categories of algebras", J. Algebra **11** (1969), 64–101.

[Gri] Grillet, Pierre, "Regular categories", Lecture Notes in Math. **236** (Springer, Berlin, 1971).

[He] Herrlich, Horst; Strecker, George, **Category Theory**, Allyn and Bacon, Inc., (Boston, 1973).

[PTJ] Johnstone, P.T., **Topos Theory**, L.M.S. Monographs **10**, Acadenic Press (London, 1977).

[Mac] MacLane, Saunders, **Categories for the Working Mathematician**, Graduate Texts in Math. **5**, (Springer, New York, 1977).

[Man] Manes, Ernest, **Algebraic Theories**, Graduate Texts in Math. **26**, (Springer, New York, 1976).

[St1] Street, Ross, "Notions of topos", Bull. Aust. Math. Soc. **23** (1981), 199–208.

[St2] Street, Ross, unpublished material.

[S&W1] Street, Ross; Walters, Robert, "Yoneda Structures on 2–categories", J. Algebra **50** (1978), 350–379.

[S&W2] Street, Ross; Walters, Robert, "The comprehensive factorization of a functor", Bull. Amer. Math. Soc. **79** (1973), 936–941.

[Th] Tholen, Walter, "Note on total categories", Bull. Aust. Math. Soc. **21** (1980), 169–173.

[Wal] Walters, Robert, "Total cocompleteness", The Isle of Thorns Category Theory Meeting, 1976 [Abstract].

[RJW1] Wood, R.J., "Some remarks on total categories", J. Algebra **75** (1982), 538–545.

[RJW2] Wood, R.J., "Proarrows II", to appear in Cahiers Topologie Géom. Differentielle.

[RJW3] Wood, R.J., unpublished material.