FAST PREPROCESSING FOR OPTIMAL ORTHOGONAL RANGE REPORTING AND RANGE SUCCESSOR WITH APPLICATIONS TO TEXT INDEXING

by

Younan Gao

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Abstract

Under the word RAM model, we design three data structures that can be constructed in $O(n\sqrt{\lg n})$ time over n points in an $n \times n$ grid. The first data structure is an $O(n \lg^{\epsilon} n)$-word structure supporting orthogonal range reporting in $O(\lg\lg n + \mathsf{occ})$ time, where occ denotes output size and ϵ is an arbitrarily small constant. The second is an $O(n \lg\lg n)$-word structure supporting orthogonal range successor in $O(\lg\lg n)$ time, while the third is an $O(n \lg^{\epsilon} n)$-word structure supporting sorted range reporting in $O(\lg\lg n + \mathsf{occ})$ time. The query times of these data structures are optimal when the space costs must be within $O(n \operatorname{polylog} n)$ words. Their exact space bounds match those of the best known results achieving the same query times, and the $O(n\sqrt{\lg n})$ construction time beats the previous bounds on preprocessing. We also apply our results to improve the construction time of text indexes.

List of Abbreviations and Symbols Used

$C(s..f)$	A function that returns the bits between and including the s- and f-th most significant bits of C, where C can be an integer or a character
$[\sigma]$	A symbol denoting the universe $\{0, 1, \cdots, \sigma - 1\}$
Sp	Skipping pointers
$\texttt{lca}(a, b)$	A function that returns the lowest common ancestor of nodes a and b.
`occ`	The output size
polylog	Poly-logarithmic function
`pred`	A predecessor query
`rMq`	A range maximum query
`rmq`	A range minimum query
`succ`	A successor query

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Chapter 1

Introduction

Two dimensional orthogonal range search problems have been studied intensively in the communities of computational geometry, data structures and databases. The goal of these problems is to maintain a set, N, of points on the plane in a data structure such that one can efficiently compute aggregate information about the points contained in an axis-aligned query rectangle Q. Among these problems, *orthogonal range counting* and *orthogonal range reporting* are perhaps the most fundamental; the former counts the number of points contained in $N \cap Q$ while the latter reports them. Another well-known problem is *orthogonal range successor*, which asks for the point in $N \cap Q$ with the smallest x- or y-coordinate. Range counting, reporting and successor have many applications including text indexing [27, 9, 7, 30], Lempel-Ziv factorization [4] and consensus trees in phylogenetics [22], to name a few. See [26] for a survey on the connection between text indexing and various range searching techniques.

Most work on orthogonal range searching [14, 21, 12, 32, 37] focuses on achieving the best tradeoffs between query time and space, and preprocessing time is often neglected. However, the preprocessing time of a data structure matters when it is used as a building block of an algorithm processing plain data, as the total running time includes that needed to build the structure. Furthermore, an orthogonal range search structures with fast construction time are preferred when preprocessing huge amounts of data, e.g., when used as components of text indexes built upon large data sets from search engines and bioinformatics applications. The work of Chan and Pătraşcu [11] is the first that improves the $\Omega(n \lg n)$ bound on the construction time of 2d orthogonal range counting structures; they designed an $O(n)$-word structure with $O(\lg n/\lg\lg n)$ query time that can be built in $O(n\sqrt{\lg n})$ time. Their ideas were further extended to design an $O(n \lg\sigma/\sqrt{\lg n})$-time algorithm to build a binary

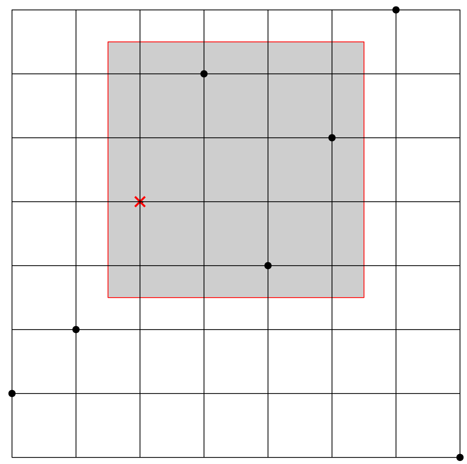


Figure 1.1: 8 points in an 8×8 grid with $Q = [2, 5] \times [3, 6]$. There are four points in $Q \cap N$, which are $(2, 4)$, $(3, 6)$, $(4, 3)$, and $(5, 5)$. The point marked in red is the leftmost point.

wavelet trees over a string of length n drawn from $[\sigma]$ [29, 3][1], which is a key data structure used in succinct text indexes. More recently, Belazzougui and Puglisi [4] showed how to construct, in $O(n\sqrt{\lg n})$ time, an $O(n)$-word data structure supporting range successor in $O(\lg^{\epsilon} n)$ time, and applied this algorithm to achieve new results on Lempel-Ziv parsing.

The previous work on constructing orthogonal range search structures in $O(n\sqrt{\lg n})$ time focuses on linear space data structures. To achieve optimal query time for 2d orthogonal range reporting and range successor using near-linear space, however, the best tradeoffs under the word RAM model require superlinear space [12, 37]. The increased space costs are needed to encode more information, posing new challenges to fast construction. We thus investigate the problem of designing data structures with optimal query times for range reporting and range successor that can be built in $O(n\sqrt{\lg n})$ time, while matching the space costs of the best known solutions. We also consider a closely related problem called *sorted range reporting* [32] to achieve similar goals. In this problem, we report all the points in $N \cap Q$ in a sorted order along either x- or y-axis. The query time should depend on the number of points actually reported even if the procedure is ended early by user.

[1] In this thesis, $[\sigma]$ denotes $\{0, 1, \ldots, \sigma - 1\}$.

1.1 Previous Work

The research on 2d orthogonal range reporting has a long history [35, 14, 2, 21, 31, 23, 10, 12]. Researchers have achieved three best tradeoffs between query time and space costs under the word RAM model; we follow the state of the art and assume that the input points are in rank space. The solution with optimal query time of $O(\lg\lg n + \mathsf{occ})$ and space cost of $O(n \lg^\epsilon n)$ words is due to Alstrup et al. [2], while the best linear-space solution is designed by Chan et al. [12] which answers a query in $O((1+\mathsf{occ})\lg^\epsilon n)$ time, where occ is the output size and ϵ is an arbitrarily small constant. Chan et al. also proposed an $O(\lg\lg n)$-word structure with $O((1+\mathsf{occ})\lg\lg n)$ query time and another tradeoff matching that of Alstrup et al. [2].

The 2d orthogonal range successor problem was also studied extensively. After a series of work [25, 24, 15, 16, 36], Nekrich and Navarro [32] gave two solutions to this problem; the first uses $O(n)$ words and answers a query in $O(\lg^\epsilon n)$ time, while the second uses $O(n \lg\lg n)$ words to answer a query in $O((\lg\lg n)^2)$ time. Zhou [37] decreased the query time of the latter to $O(\lg\lg n)$ without increasing space costs. By definition, a solution to orthogonal range successor can be used to answer sorted range reporting queries. Furthermore, Nekrich and Navarro [32] also designed a data structure using $O(n \lg^\epsilon n)$ words to support sorted range reporting in $O(\lg\lg n + \mathsf{occ})$ time. Hence, the best three time-space tradeoffs for the original 2d orthogonal range reporting problem have also been achieved for the sorted version. The optimality of the $O(\lg\lg n + \mathsf{occ})$ query time for orthogonal range reporting and the $O(\lg\lg n)$ query time for orthogonal range successor when no more than $O(n \operatorname{polylog} n)$ space can be used is established by a lower bound on range emptiness [33].

Alstrup et al. [2] claimed that their structure for optimal orthogonal range reporting can be constructed in $O(n \lg n)$ expected time. Even though preprocessing times are not given in [12, 32, 37], straightforward analyses reveal that the other data structures we surveyed here can be built in $O(n \lg n)$ worst-case time (Bille and Gørtz [7] also claimed that the preprocessing time of the $O(n \lg\lg n)$-word structure of Chan et al. [12] is $O(n \lg n)$). Hence, when faster preprocessing time is needed in their solution to Lempel-Ziv decomposition, Belazzougui and Puglisi [4] had to design a new linear-space data structure for orthogonal range successor with $O(n\sqrt{\lg n})$ preprocessing time and $O(\lg^\epsilon n)$ query time. No attempts have been published to achieve similar preprocessing times for other tradeoffs.

1.2 Our Results

Under the word RAM model, we design the following three data structures that can be constructed in $O(n\sqrt{\lg n})$ time over n points in an $n \times n$ grid:

- An $O(n \lg^{\epsilon} n)$-word structure supporting orthogonal range reporting in $O(\lg \lg n + \texttt{occ})$ time, where $\texttt{occ}$ denotes the output size and ϵ is an arbitrarily small constant;
- An $O(n \lg \lg n)$-word structure supporting orthogonal range successor in $O(\lg \lg n)$ time;
- An $O(n \lg^{\epsilon} n)$-word structure supporting sorted range reporting in $O(\lg \lg n + \texttt{occ})$ time.

The query times of these structures are optimal when space costs must be within $O(n \operatorname{polylog} n)$ words. Their exact space bounds match those of the best known results achieving the same query times, and the $O(n\sqrt{\lg n})$ construction time beats the previous bounds on preprocessing. Note that even though our third result implies the first one, our data structure for the first is much simpler. In addition, our results can be used to improve the construction time of text indexes. For a text string T of length n over alphabet $[\sigma]$, we design

- A text index of $O(n \lg \sigma \lg^{\epsilon} n)$ bits that can be constructed in $O(n \lg \sigma / \sqrt{\lg n})$ time and can report the $\texttt{occ}$ occurrences of a pattern of length p in time $O(p/\log_{\sigma} n + \log_{\sigma} n \lg \lg n + \texttt{occ})$, where ϵ is any small positive constant. This improves one result of Munro et al. [30] who designed the first text indexes with both sublinear construction time and query time for small σ; for the same time-space tradeoff, their preprocessing time is $O(n \lg \sigma \lg^{\epsilon} n)$.
- A text index of $O(n \lg^{1+\epsilon} n)$ bits for any constant $\epsilon > 0$ built in $O(n\sqrt{\lg n})$ time that supports position-restricted substring search [27] in $O(p/\log_{\sigma} n + \lg p + \lg \lg \sigma + \texttt{occ})$ time. Previous indexes with similar query performance require $O(n \lg n)$ construction time.

The thesis is based on joint work with Meng He and Yakov Nekrich [19].

1.3 Overview of Our Approach

We first discuss why some obvious approaches will not work. The modern approach of Chan et al. [12] for orthogonal range reporting is based on a problem called ball inheritance which they defined over range trees. This solution is well-known for its simplicity, and by choosing different parameters in their approach to ball inheritance, they obtain all three best known tradeoffs. One natural idea is to redesign the structures stored at range tree nodes to use bit packing to speed up construction. However, even though we have achieved construction time matching the state of the art for these structures, it is still not enough to construct the data structures for the tradeoffs of ball inheritance that we need quickly enough. Another idea is to tune the parameters in the approach of Belazzougui and Puglisi [4], hoping to obtain the tradeoffs that we aim for, as they already showed how to construct in $O(n\sqrt{\lg n})$ time a linear space, $O((k+1)\lg^{\epsilon} n)$ query time structure for orthogonal range reporting. Their solution uses many trees grouped into $O(\lg^{\epsilon} n)$ levels of granularity. If we borrow ideas from [12] and set parameters to achieve different tradeoffs, we would use $O(1/\epsilon)$ or $O(\lg\lg n)$ levels of granularity. However, to return a point in the answer, their query algorithm would perform operations requiring $O(\lg\lg n)$ time at each level of granularity. Thus, at best, the former would give an $O(n\lg^{\epsilon} n)$-word structure with $O((k+1)\lg\lg n)$ query time and the latter an $O(n\lg\lg n)$-word structure with $O((k+1)(\lg\lg n)^2)$ query time. Either solution is inferior to the best known tradeoffs. This however is fine in the original solution, as the total cost of spending $O(\lg\lg n)$ time at each of the $O(\lg^{\epsilon} n)$ levels is bounded by $O(\lg^{\epsilon'} n)$ for any $\epsilon' > \epsilon$.

We thus design new approaches. For optimal orthogonal range reporting, our overall strategy is to perform two levels of reductions, making it sufficient to solve ball inheritance in special cases with fast preprocessing time. More specifically, we first use a generalized wavelet tree and range minimum/maximum structures to reduce the problem in the general case to the special case in which the points are from a $2^{\sqrt{\lg n}} \times n'$ (narrow) grid, where $n' \leq n$. In this reduction, we need only support ball inheritance over a wavelet tree with high fanout. We further reduce the problem over points in a narrow grid to that over a (small) grid of size at most $2^{\sqrt{\lg n}} \times 2^{2\sqrt{\lg n}}$. This is done by grouping points and selecting representatives from each group, so that previous results with slower preprocessing time can be used over a smaller set of representatives. Finally, over the small grid, we solve ball inheritance when the coordinates of each point can be encoded in $O(\sqrt{\lg n})$ bits. The ball inheritance structures in both special cases can be built quickly by redesigning components with fast preprocessing, though the second case requires a twist to the approach of Chan et al. [12]. Our solutions to optimal range successor and sorted range reporting are based on similar strategies, though we preform more levels of reductions.

1.4 Road Map

The rest of the thesis is organized as follows. In Chapter 2, we describe the previous results that are used in our solution. In addition, we also introduce the model of computation that will be adopted in this thesis. In Chapter 3, we present a data structure to support $\texttt{rank}'$ queries in constant time that can be constructed fast, which will be later used for constructing the ball inheritance structure. Given a sequence A and an index i, a $\texttt{rank}'(A, i)$ operation computes the number of elements equal to $A[i]$ in the subarray $A[0..i]$. Chapter 4 addresses the problem of quickly building data structures for ball inheritance in some useful special cases. Those special cases are sufficient to solve orthogonal range search problems including orthogonal range reporting, orthogonal range successor, and orthogonal sorted range reporting. In Chapters 5, 6 and 7, we design data structures with fast construction time that support orthogonal range reporting, range successor queries and sorted range reporting in optimal time, respectively. In Chapter 8, we apply our range search structures to the text indexing problem and improve the construction time of text indexes. Finally, Chapter 9 presents concluding remarks and lists some open problems.

Chapter 2

Preliminaries

In this chapter, we first introduce the word RAM model that is adopted by this thesis. Then, we describe and sometimes extend the previous results that are used in our solution. Especially, by combining the bit parallelism, we achieve fast construction for building a generalized wavelet tree, and data structures for range minimum/maximum queries, `rank` and `count` queries over sequences from small alphabets, and predecessor/successor queries.

2.1 Notation

We adopt the word RAM model with word size $w = \Theta(\lg n)$ bits, where n denotes the size of the given data. Our complete solutions use several sets of homogeneous components. We present a lemma to bound the costs of each different type of components, which is then applied over the entire set of these components to calculate the total cost. The size, n', of the data that each component represents may be less than n which is the input size of the entire problem, so when the cost of constructing the component is bounded by a function of the form $f(n')/\operatorname{polylog}(n)$ to take advantage of the word size, we keep both n' and n in the lemma statement, as commonly done in previous work on similar topics. In this case, the construction algorithm usually uses a universal table of $o(n)$ bits, whose content solely depends on the value of n, and hence can be constructed once in $o(n)$ time and used for all data structure components of the same type. Thus unless otherwise stated, these lemmas assume the existence of such a table without stating so explicitly in the lemma statements, and we define and analyze the table in the proof. This also applies to algorithms that manipulate sequences of size n'. Occasionally the query algorithms of a data structure may need a universal table as well, and we explicitly state it if this is the case.

2.2 Algorithms on Packed Sequences

We say a sequence $A \in [\sigma]^n$ is in *packed* form if the bits of its elements are concatenated and stored in as few words as possible. Thus, when packed, A occupies $\lceil n \lceil \lg \sigma \rceil / w \rceil$ words. Throughout the thesis, we define $C(s..f)$ to be the bits between and including the s- and f-th most significant bits of C, where C can be an integer or a character. With the bit parallelism technique, we can extract bits at specified positions from multiple elements packed into a word in constant time.

Lemma 1 *Let $C[0..n'-1]$ be a packed sequence of c-bit elements, where $n' \leq n$. Given a pair of parameters s and f satisfying that $0 \leq s \leq f \leq c-1$, a packed sequence $A[0..n'-1]$ of $(f-s+1)$-bit elements in which $A[i] = C[i](s..f)$ for each entry $i \in [0..n'-1]$ can be constructed in $O(n'c/\lg n + 1)$ time.*

Proof. Let δ denote the block size $\lfloor \frac{\lg n}{2 \times c} \rfloor$. We construct a universal lookup table U. For each possible δ-element packed sequence S_1 drawn from alphabet $[2^c]$, and each different range $[s, f]$ where $0 \leq s \leq f \leq c-1$, U stores a packed sequence S_2 in which $S_2[i] = S_1[i](s..f)$ for each $i \in [0..\delta - 1]$. As there are $2^{\delta \times c} \times c^2 \leq \sqrt{n} \times c^2$ entries in U, and each entry stores a result of $\delta \times (f-s+1)$ bits, table U occupies $O(\sqrt{n} \times c^2 \times \delta \times (f-s+1)) = o(n)$ bits of space. Given a pair of parameters s and f, we can apply table U to extract the bits from δ consecutive elements $C[i], C[i+1], \ldots, C[i+\delta-1]$ for each $i \in [0, n'-1-\delta]$ in constant time. Therefore, the overall processing time is $O(n'c/\lg n + 1)$. ■

2.3 Generalized Wavelet Trees

Given a sequence $A[0..n-1]$ drawn from alphabet $[\sigma]$, a d-ary wavelet tree [28] T_d over A is a balanced tree in which each internal node has d children. Each node of T_d then represents a range of alphabet symbols defined as follows: At the leaf level, the i-th leaf from the left represents the integer range $[i, i]$ for each $i \in [0..\sigma-1]$. The range represented by an internal node is the union of the ranges represented by its children. Hence the root represents $[0, \sigma-1]$, and T_d is a complete tree having $\log_d \sigma + 1$ levels. Each node u is further associated with a subsequence, $A(u)$, of A, in

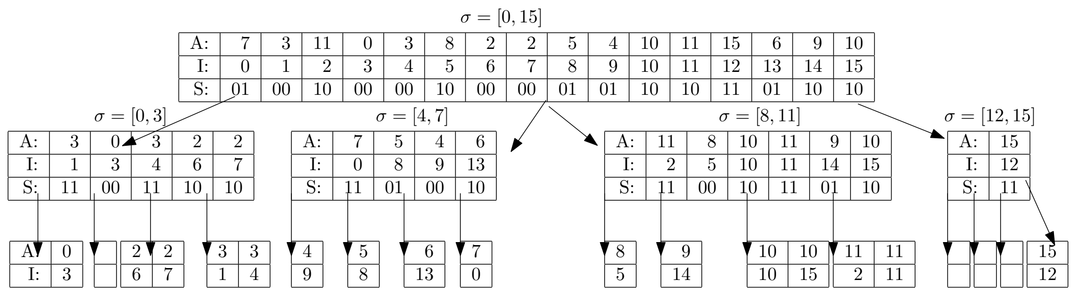


Figure 2.1: A wavelet tree with degree equals to 4. Each internal node v of the tree is associated with a value array $A(u)$, an index array $I(v)$, and the array $S(v)$.

which $A(u)[i]$ stores the i-th entry in A that is in the range represented by u. Thus the root is associated with the entire sequence A. To save storage, $A[u]$ is not stored explicitly in [28]. Instead, each internal node u stores a sequence $S(u)$ of integers in $[d]$, where $S(u)[i] = j$ if $A(u)[i]$ is within the range represented by the jth child of u. All the $S(u)$'s built for internal nodes occupy $O(n \lg \sigma)$ bits in total.

Generalized wavelet trees share fundamental ideas with range trees but are more suitable for compact data structures over sequences which may contain duplicate values. When we use them in this thesis, we sometimes explicitly store $A(u)$ for each node u, and may even associate with u an additional array $I(u)$ in which $I(u)[i]$ stores the index of $A(u)[i]$ in the original sequence A. We call $A(u)$ the *value array* of u, and $I(u)$ the *index array*. See Figure 2.1 for an example of a generalized wavelet tree. In this thesis, if we construct value and/or index arrays for each node, we explicitly state so. If not, it implies that we build a wavelet tree in which each node u is associated with $S(u)$ only. Furthermore, unless otherwise specified, we apply the standard pointer-based implementation to represent the tree structure of a wavelet tree, which is preprocessed in time linear to the number of tree nodes such that the lowest common ancestor of any two nodes can be located in $O(1)$ time [6]. We also number the levels of the tree incrementally starting from the root level, which is level 0. We have the following two lemmas on constructing wavelet trees:

Lemma 2 *Let $A[0..n'-1]$ be a packed sequence drawn from alphabet $[\sigma]$ and let $I[0..n'-1]$ be a packed sequence in which $I[i] = i$ for each $i \in [0..n'-1]$, where $n' \leq n$ and $\sigma \leq 2^{O(\sqrt{\lg n})}$. Given A and I as input, a d-ary wavelet tree over A with value and index arrays in packed form can be constructed in $O(n' \lg \sigma (\lg n' + \lg \sigma)/\lg n + \sigma)$ time, where d is an arbitrary power of 2 with $2 \leq d \leq \sigma$. If index arrays are not* *constructed, the construction time can be lowered to* $O(n' \lg^2 \sigma / \lg n + \sigma)$*; this bound still applies when neither value nor index arrays are built.*

Proof. We only prove the result when value and index arrays are required; the other results in the lemma follow by removing the steps of constructing them. The construction consists of two steps: we first build a binary wavelet tree T_2 and then convert it to a d-ary wavelet tree T_d.

To construct T_2, let r denote its root node, and we have $A(r) = A$ and $I(r) = I$. We then create the left child, r_0, and the right child, r_1, of r, and perform a linear scan of $A(r)$ and $I(r)$. During the scan, for each $i \in [0, |A(r)| - 1]$, if the highest bit of $A(r)[i]$ is 0, then $A(r)[i]$ is appended to $A(r_0)$, and $I(r)[i]$ is appended to $I(r_0)$. Otherwise, they are appended to $A(r_1)$ and $I(r_1)$. Afterwards, we recursively process the child node r_0 and r_1 in the same manner, but we examine the second highest bit of each element of $A(r_0)$ and $A(r_1)$. In general, when generating the sequences for the child nodes of an internal node u at tree level l where $l \in [0, \lg \sigma - 1]$, we append $A(u)[i]$ and $I(u)[i]$ to $A(u_0)$ and $I(u_0)$, respectively, if the l-th highest bit of $A(u)[i]$ is 0. Otherwise. they are appended to $A(u_1)$ and $I(u_1)$. If $A(v)$ for some node v is empty but v is above the leaf level, then we keep v as an empty node, and at next phase we create empty children v_0 and v_1 under v. T_2 have been constructed completely after processing all the $\lceil \lg \sigma \rceil$ bits of each element of A.

To speed up this process, we use a universal table U. Let $b = \lfloor \frac{\lg n}{2t} \rfloor$, where $t = \lceil \lg n' \rceil + \lceil \lg \sigma \rceil$. This table U has an entry for each possible triple (D, E, c), where D is a sequence of length b drawn from universe $[\sigma]$, E is a sequence of length b drawn from universe $[n']$, and c is an integer in $[0, \lceil \lg \sigma \rceil - 1]$. This entry $U[D, E, c]$ stores four packed sequences D_0, D_1, E_0 and E_1 defined as follows: $D_0[i]$ or $D_1[i]$ stores the ith element in D whose c-th most significant bit is 0 or 1, respectively, while $E_0[i]$ or $E_1[i]$ stores $E[j]$ if $D[j]$ is the $E[j]$-th element in A whose c-th most significant bit is 0 or 1, respectively. Similar to the table U in the proof of Lemma 1, U uses $o(n)$ bits. By performing table lookups with U, we can process b consecutive elements in $A(u)$ and $I(u)$ in constant time, and hence we spend $O(|A(u)|/b + 1)$ time on each internal node u. The sum of the lengths of all the value arrays for the nodes at the same level of T_2 is n'. As T_2 has $\lceil \lg \sigma \rceil + 1$ levels and $O(\sigma)$ nodes, the total time required to construct T_2 is $O(n' \lg\sigma(\lg n' + \lg\sigma)/\lg n + \sigma)$.

We then transform T_2 into a d-ary tree T_d. We first remove the nodes of T_2 whose levels are not multiples of $\lg d$, and add edges between each remaining node u and its descendants at the next remaining level. We then visit each internal node u of T_d and associate it with a packed sequence $S(u)$ storing $A(u)[i](l \lg d..(l+1)\lg d - 1)$ for all $i \in [0, |A(u)| - 1]$, where l is the level of u in T_d. It remains to analyze the time needed to transform T_2 into T_d. For each internal node $u \in T_d$, it takes $O(|A(u)| \lg\sigma/\lg n + 1)$ time to construct $S(u)$. The sum of the lengths of all the value arrays for the nodes at the same level of T_d is n'. As T_d has $\lg\sigma/\lg d + 1$ levels and $O(\sigma)$ nodes, the time required to construct $S(u)$'s for all the internal node of T_d is $O(n' \lg^2\sigma/(\lg n \times \lg d) + \sigma)$. Therefore, the two steps of our construction algorithm use $O(n' \lg\sigma(\lg n' + \lg\sigma)/\lg n + \sigma)$ time in total. ■

Next, we show how to construct a d-ary wavelet tree efficiently when both σ and d are large. In this case, we store the sequences $A(u)$, $I(u)$ and $S(u)$ in an unpacked form; this does not matter as we do not save space asymptotically by packing bits when σ and d are sufficiently large.

Lemma 3 *Let $A[0..n-1]$ be a sequence drawn from alphabet $[\sigma]$. A d-ary wavelet tree over A with value and index arrays can be built in $O(n \lg\sigma/\lg d)$ time, where $2 \le d \le \sigma$.*

Proof. We use $O(n)$ time to create a sequence $I[0..n-1]$ in which $I[i] = i$ for each $i \in [0..n-1]$. At the root node r of the wavelet tree, set $A(r) = A$ and $I(r) = I$. We then create an empty sequence $S(r)$, and, for each $i \in [0, n-1]$, we append $A(r)[i](0..\lg d - 1)$ to $S(r)$. At the second level, there are d children of r. We linearly scan $A(r)$, $I(r)$ and $S(r)$, appending $A(r)[i]$ or $I(r)[i]$ to $A(r_\alpha)$ or $I(r_\alpha)$, respectively, where $\alpha = S(r)[i]$ and r_α represents the α-th child node of r. Next, we construct $S(v)$ for each node v at the second tree level by appending $A(v)[i](\lg d..2\lg d - 1)$ to $S(v)[i]$ for each $i \in [0, |A(v)| - 1]$.

This process continues at each successive level: in general, when generating $S(u)$ for a node u at a level ℓ where $\ell \in [0, \frac{\lg\sigma}{\lg d} - 1]$, we append $A(u)[i](\ell \times \lg d..(\ell+1) \times \lg d - 1)$ to $S(u)[i]$ for each $i \in [0, |A(u)| - 1]$. If u is an internal node, we append $A(u)[i]$ or

$I(u)[i]$ to the sequence $A(u_\alpha)$ or $I(u_\alpha)$, respectively, where $\alpha = S(u)[i]$. After reaching the leaf level, $\frac{\lg \sigma}{\lg d} + 1$ levels have been created on T_d. As it uses $O(n)$ time for the non-empty nodes at each tree level and there are in total at most $O(\sigma)$ empty nodes, overall the construction time is $O(n \times \lg \sigma / \lg d + \sigma) = O(n \lg \sigma / \lg d)$, as $\sigma \leq n$. ■

2.4 The Ball Inheritance Problem

A sequence $A[0..n-1]$ drawn from $[\sigma]$ can be viewed as a point set $N = \{(A[i], i) | 0 \leq i \leq n-1\}$. Let T be a d-ary wavelet tree constructed over A. Then *ball inheritance* [12] can be defined over T which asks for the support of these operations:

- `point`(v, i), which returns the point $(A(v)[i], I(v)[i])$ in N for an arbitrary node v in T and an integer i; and
- `noderange`(c, d, v), which, given a range $[c, d]$ and a node v of T, finds the range $[c_v, d_v]$ such that $I(v)[i] \in [c, d]$ iff $i \in [c_v, d_v]$.

If we store the value and index arrays explicitly, it is trivial to support these operations, but the space cost is high. To save space, we only store $S(v)$ for each node v and design auxiliary structures. The following lemma presents previous results:

Lemma 4 ([12, Theorem 2.1], [13, Lemma 2.3]) *A generalized wavelet tree over a sequence $A[0..n-1]$ drawn from $[\sigma]$ can be augmented with a ball inheritance data structure in $O(n \lg n f(\sigma))$ bits to support* `point` *in $O(g(\sigma))$ time and* `noderange` *in $O(g(\sigma) + \lg \lg n)$ time, such that,*

(a) $f(\sigma) = O(1)$ and $g(\sigma) = O(\lg^\epsilon \sigma)$;

(b) $f(\sigma) = O(\lg \lg \sigma)$ and $g(\sigma) = O(\lg \lg \sigma)$;

(c) $f(\sigma) = O(\lg^\epsilon \sigma)$ and $g(\sigma) = O(1)$.

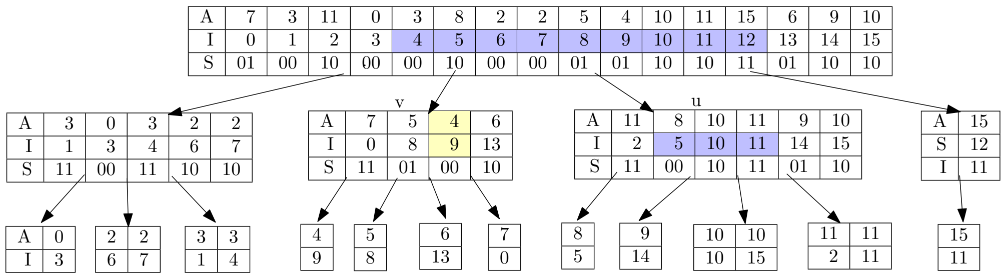


Figure 2.2: A wavelet tree with degree equals to 4; On the tree, the value array and index array associated with each node are explicitly stored. The operation $\texttt{point}(v, 2)$ returns the coordinates $(4, 9)$. Meanwhile, $\texttt{noderange}(u, 4, 12)$ returns the range $(1, 3)$.

2.5 Range Minimum/Maximum Queries

Given a sequence A of n integers, a range minimum/maximum query $\texttt{rmq}(i, j)/\texttt{rMq}(i, j)$ with $i \leq j$ returns the position of a minimum/maximum element in the subsequence $A[i..j]$, i.e., $\texttt{rmq}(i, j) = \text{argmin}_{i \leq k \leq j}\{A[k]\}$ and $\texttt{rMq}(i, j) = \text{argmax}_{i \leq k \leq j}\{A[k]\}$.

Fischer and Heun [17] considered $\texttt{rmq}(i, j)/\texttt{rMq}(i, j)$ queries without consulting the input array after the construction of the data structure. The result is summarized as the following lemma.

Lemma 5 ([17, Theorem 5.8]) *Given an array A of n integers, a data structure of $O(n)$ bits can be constructed in $O(n)$ time, which answers $\texttt{rmq}(i, j)/\texttt{rMq}(i, j)$ queries in constant time without accessing A.*

Belazzougui and Puglisi [4] provided a *systematic* scheme with efficient construction for range minimum/maximum queries over an input sequence from small alphabets. A data structure is called systematic if it stores the input sequence verbatim along with the additional information for answering the queries. Their result is presented as follows.

Lemma 6 ([4, Lemma D.1]) *Let $A[0..n' - 1]$ be a packed sequence drawn from alphabet $[\sigma]$, where $n' \leq n$ and $\sigma \leq 2^{\sqrt{\lg n}}$. There is a systematic data structure using $O(n' \lg \sigma / \lg n)$ extra bits constructed in $O(n' \lg \sigma / \lg n)$ time, which answers $\texttt{rmq}(i, j)/\texttt{rMq}(i, j)$ queries in constant time. The query procedure uses a universal table of $o(n)$ bits.*

We further build an auxiliary data structure upon a packed input sequence A from a small alphabet under the *indexing model*: after the construction of the data structure, each query operation needs to access certain elements in A. A itself need not be stored verbatim; it suffices to provide an operator supporting access to an arbitrary element of A.

Lemma 7 *Let $A[0..n'-1]$ be a packed sequence drawn from alphabet $[\sigma]$, where $\sigma \leq 2^{\sqrt{\lg n}}$ and $n' \leq n$. There is a data structure using $O(n' \lg\lg n)$ extra bits constructed in $O(n' \lg \sigma / \lg n)$ time, which answers $\texttt{rmq}(i,j)/\texttt{rMq}(i,j)$ queries in $O(1)$ time and $O(1)$ accesses to the elements of A. The query procedure uses a universal table of $o(n)$ bits.*

Proof. If $\sigma \leq \lg n$, each element in A can be encoded with $O(\lg\lg n)$ bits, so explicitly storing elements of A requires $O(n' \lg\lg n)$ bits, which is affordable. We then apply Lemma 6 for $\texttt{rmq}/\texttt{rMq}$ queries over A, which achieves the efficient construction time and the constant query time. Therefore, for the rest of the proof, it suffices to assume $\sigma > \lg n$.

We only show the proof for range minimum as the support for range maximum is similar. Let b denote the block size $\lfloor \lg n/(2\lg\lceil\sigma\rceil) \rfloor$. The elements of A are conceptually divided into blocks of b elements each. With a universal lookup table U, we can retrieve the minimum value of a block of elements in constant time. For each possible b elements drawn from $[\sigma]$, U stores the minimum element value of these b elements. Similar to the table U in the proof of Lemma 1, U uses $o(n)$ bits.

Next, we store the minimum values of the blocks in a sequence A' ordered by their original position in A. The sequence A' occupies $O(n'/b \times \lg \sigma) = O(n')$ bits. Over A' we build a data structure DS_1 of $O(n'/b)$ bits in $O(n'/b)$ time by Lemma 5.

To save storage, we do not keep the original element values in a block. Instead, each element value e is replaced with its rank, i.e., the number of elements in the block that are smaller than e. As each block has b elements, the rank value can be encoded with $O(\lg b) = O(\lg\lg n)$ bits. The transformation from element values to their corresponding rank values for each block can be processed in constant time by applying a universal lookup table U'. For each possible b elements drawn from $\{0, 1, ..., \sigma - 1\}$, we store the rank values in $O(b \lg\lg n)$ bits. Similar to the table U

in the proof of Lemma 1, U' uses $o(n)$ bits. With table U', we spend $O(\frac{n'}{b})$ time on transferring n' elements into their ranks.

At last, we construct a universal lookup table U'' in which for each possible b elements drawn from $[b]$ and for each different query range $[q_1, q_2]$ where $0 \leq q_1 \leq q_2 \leq b-1$, we store the in-block index of the minimum value in the range $[q_1, q_2]$. Similar to the table U in the proof of Lemma 1, U'' uses $o(n)$ bits. The universal table U'' will be used amid the querying procedure only.

All rank values occupy $O(n' \lg\lg n)$ bits. In addition to the $O(n' + \frac{n'}{b})$-bit space usage of A' and DS_1, the overall space cost is $O(n' \lg\lg n)$ bits. As shown above, computing the rank value for all n' elements uses $O(\frac{n'}{b})$ time. Constructing the sequence A' and building the data structure DS_1 over A' takes $O(\frac{n'}{b})$ time. Overall the construction time is bounded by $O(n'/b) = O(n' \lg \sigma / \lg n)$.

Now we show how to answer the range minimum query given a query range $[i, j]$. Let B_s and B_t denote the block containing i and j, respectively, where $s = \lfloor \frac{i}{b} \rfloor$ and $t = \lfloor \frac{j}{b} \rfloor$. We only consider the case when $s < t$; the remaining case in which s is equal to t can be handled similarly. Let m_1 denote the minimum value in $B_s[i \texttt{ mod } b, b-1]$, m_2 denote the minimum value among blocks B_{s+1}, $B_{s+2}, \ldots,$ B_{t-1}, and m_3 denote the minimum value in $B_t[0, j \texttt{ mod } b]$. The answer is clearly $\min(m_1, m_2, m_3)$.

The value m_2 can be retrieved in constant time as follows: We search the data structure DS_1 for the index τ of the minimum value in the query range $[s+1, t-1]$ and complete with accessing $A'[\tau]$. The values m_1 and m_3 both can be answered in a similar way, and we take how to retrieve m_1 as an example. Given the pattern of block B_s and the query range $[i \texttt{ mod } b, b-1]$, we can apply U'' to retrieve the in-block index τ of the minimum value e in constant time, compute the original index τ' of e in A, where $\tau' = b \times s + \tau$, and retrieve m_1 by accessing $A[\tau']$. Overall, the query requires $O(1)$ time and $O(1)$ accesses to the elements of A. ■

2.6 Data Structures for `rank` and `select`

Given a sequence $A[0..n'-1]$ drawn from alphabet $[\sigma]$, a $\texttt{rank}_c(A, i)$ operation computes the number of elements equal to c in $A[0..i]$. An even more powerful operation $\texttt{count}_c(A, j)$ operation computes the number of elements less than or equal to c in

$A[0..j]$. Clearly the support for `count` implies that for `rank`. The following lemma shows how to construct a data structure supporting `count` when $\sigma < \lg^{1/3} n$.

Lemma 8 ([3, Lemma 2.3]) *Let $A[0..n'-1]$ be a packed sequence drawn from alphabet $[\sigma]$, where $n' \leq n$ and $\sigma < \lg^{1/3} n$. A systematic data structure occupying $o(n')$ extra bits supporting* `count` *in $O(1)$ time can be constructed in $O(n' \lg \sigma / \lg n)$ time.*

We then consider slightly larger alphabet with $\sigma = O(\text{polylog}\, n)$.

Lemma 9 *Let $A[0..n'-1]$ be a packed sequence drawn from alphabet $[\sigma]$, where $n' \leq n$ and $\sigma = O(\text{polylog}\, n)$. A data structure of $n'\lceil \lg \sigma \rceil + o(n' \lg \sigma)$ bits supporting* `count` *and* `rank` *in $O(1)$ time can be constructed in $O(n' \lg^2 \sigma / \lg n + \sigma)$ time. The query procedure requires access to a universal table of size $o(n)$ bits.*

Proof. Lemma 8 already subsumes this lemma when $\sigma \leq 2^{\lceil 1/4 \lg \lg n \rceil}$, so it suffices to assume that $2^{\lceil 1/4 \lg \lg n \rceil} < \sigma$ in the rest of the proof.

Let $d = 2^{\lceil 1/4 \lg \lg n \rceil}$. We first build a d-ary wavelet tree T in $O(n' \lg^2 \sigma / \lg n + \sigma)$ time by Lemma 2. Then the height of the tree is $h = O(\lg \sigma / (1/4 \lg \lg n)) = O(1)$. For each level l of T except the leaf level, we construct a packed sequence S_l by concatenating all the $S(v)$'s for the nodes at this level from left to right. As S_l is drawn from alphabet $[d]$ and there are at most σ nodes in total, it takes $O(|S_l| \lg d / \lg n + \sigma) = O(n' \lg \sigma / \lg n + \sigma)$ time to construct S_l. We then build a data structure C_l with constant-time support for `count` over S_l; by Lemma 8, this data structure occupies $o(n' \lg \sigma)$ extra bits, and it can be constructed in $O(n' \lg d / \lg n) = O(n' \lg \sigma / \lg n)$ time. At last, we discard all sequences $S(v)$ and the tree T to save space. As T has a constant number of levels, all the S_l's and C_l's occupy $n' \lceil \lg \sigma \rceil + o(n' \lg \sigma)$ bits in total, and their construction time, including that of T which we discard later, is $O(n' \lg^2 \sigma / \lg n + \sigma)$. The set of S_l's and C_l's is the data structures Bose et al. [9, Theorem 4] designed, which support `count` operations in constant time over a sequence drawn from an alphabet of size $O(\text{polylog}(n))$. This implies the support for `rank`. ■

Another widely-studied operation over sequences is $\texttt{select}_c(A, i)$, which returns the index of the entry of A containing the i-th occurrence of c. The following lemma addresses the problem of efficiently constructing data structures supporting `rank` and `select` queries.

Lemma 10 ([3, Lemma 2.1]) *Given a packed bit sequence* $B[0..n-1]$*, a systematic data structure occupying* $o(n)$ *extra bits can be constructed in* $O(n/\lg n)$ *time, which supports* `rank` *and* `select` *in constant time.*

Lemma 11 ([4, Lemma C.3]) *Let* $A[0..n'-1]$ *be a packed sequence drawn from alphabet* $[\sigma]$*, where* $n' \leq n$*. A data structure of* $O(n' \lg \sigma)$ *bits supporting* `select` *in* $O(1)$ *time can be constructed in* $O(n' \lg^2 \sigma / \lg n + \sigma)$ *time.*

There is a restricted version of `rank` called *partial rank*; a partial rank operation, $\texttt{rank}'(A, i)$, computes the number of elements equal to $A[j]$ in $A[0..j]$. The following lemma presents a solution to supporting $\texttt{rank}'$.

Lemma 12 *Given a sequence* $A[0..n-1]$ *drawn from alphabet* $[\sigma]$*, a data structure of* $O(n \lg \sigma)$ *bits can be constructed in* $O(n + \sigma)$ *time, which supports* $\texttt{rank}'$ *in constant time.*

Proof. Belazzougui et al. [5, Lemma 3.5] already proved this lemma for the case in which $\sigma \leq n$. When $\sigma > n$, then the data structure we construct is simply an array $A'[0..n-1]$ storing all the answers, i.e., $A'[i] = \texttt{rank}'(A, i)$ for any $i \in [0, n-1]$. A' occupies $O(n \lg n) = O(n \lg \sigma)$ bits. To construct A', it is enough to perform a linear scan of A, and during the scan, we maintain an array $C[0..\sigma-1]$ in which $C[j]$, for any $j \in [0, \sigma-1]$, stores how many times symbol j occurs in the portion of A that we have scanned so far. This uses $O(n + \sigma)$ time. ■

2.7 Fast Construction of Predecessor Query Structures

Let $A[0..n-1]$ be a sequence of integers sorted in the increasing order. Given a query integer x, we define operations $\texttt{pred}(x)$ and $\texttt{succ}(x)$:

$$\texttt{pred}(x) = \max\{j \mid A[j] \leq x, 0 \leq j \leq n-1\}$$

$$\texttt{succ}(x) = \min\{j \mid x \leq A[j], 0 \leq j \leq n-1\}$$

Belazzougui et al. [5] show a data structure with deterministic linear preprocessing time for predecessor/successor queries. Their result shown as follows will be used later in our methods.

Lemma 13 ([5, Lemma 3.6]) *Given a sorted sequence A of integers from universe $[0, u-1]$, a data structure of $O(n \lg u)$ bits can be constructed in linear time, which answers a* `pred` *or* `succ` *query in $O(\lg\lg u)$ time.*

We also need a solution under the indexing model over packed sequences. The following result can be achieved by combining an approach of Grossi et al. [20] with Lemma 13, while applying universal tables.

Lemma 14 *Given a sorted packed sequence A of n' distinct integers from $[\sigma]$, where $n' \leq n$ and $\sigma \leq 2^{c\sqrt{\lg n}}$ for any arbitrary positive constant c, a data structure using $O(n' \lg\lg\sigma)$ extra bits of space can be constructed in $O(n'/\sqrt{\lg n})$ time, which answers a* `pred`(x) *or* `succ`(x) *query in $O(\lg\lg\sigma)$ time and $O(1)$ accesses to the elements of A. The query procedure requires access to a universal table of size $o(n)$ bits.*

Proof. Let $b = \lfloor\sqrt{\lg n}/(2c)\rfloor$. We divide A into blocks of length b each. We retrieve the last element of each block and construct a predecessor/successor data structure R over these elements using Lemma 13. R uses $O((n'/b) \times \lg\sigma) = O(n')$ bits of space and can be constructed in $O(n'/b)$ time. Then, over each block, we regard each integer in it as a binary string of length $\lg\sigma$ and construct a Patricia trie over the integers in this block, as done by Grossi et al. [20, Lemma 3.3]. This is a compressed bitwise trie with a skip value of $O(\lg\lg\sigma)$ bits stored at each node. It stores elements at the leaves in sorted order. As the trie has b leaves and $b-1$ internal nodes, its tree structure can be encoded in $O(b)$ bits. With skip values, each trie can be encoded in $O(b \lg\lg\sigma)$ bits, without encoding the b elements at its leaves. To construct such a trie fast, we use a universal U which has an entry for any possible packed sequence S of b elements drawn from $[\sigma]$. This entry stores the Patricia trie (without the elements stored at leaves) of $O(b \lg\lg\sigma)$ bits constructed upon S. As there are at most $2^{(\lg\sigma)\times b} \leq \sqrt{n}$ different entries in U, U uses $O(\sqrt{n} \times b \lg\lg\sigma) = o(n)$ bits. With U, a Patricia trie over any block of b elements can be constructed in constant time. As there are $\lceil n'/b\rceil$ blocks in total, the overall space usage of the tries and R is $O((n'/b) \times b \lg\lg\sigma + n') = O(n' \lg\lg\sigma)$ bits, and and the overall processing time is $O(n'/\sqrt{\lg n})$.

To answer a query, given an integer y, where $y \in [0, \sigma-1]$, we first perform a predecessor or successor query over R to find the block B containing `pred`(y) or

$\texttt{succ}(y)$ in $O(\lg\lg\sigma)$ time. Then, with the help of an $o(n)$-bit universal table, we query over the trie built upon B using the query algorithm by Grossi et al. [20] in $O(1)$ time and $O(1)$ accesses of elements of A to retrieve $\texttt{pred}(y)$ or $\texttt{succ}(y)$. Therefore, the overall query cost is $O(\lg\lg\sigma)$ time and $O(1)$ accesses to elements of A. ■

We then extend Lemma 14 for a sequence of integers that are not necessarily distinct.

Lemma 15 *Given a sorted packed sequence A of n' integers from $[\sigma]$, where $n' \leq \sigma \leq 2^{c\sqrt{\lg n}}$ for any arbitrary positive constant c, a data structure using $O(n'\lg\lg\sigma)$ extra bits of space can be constructed in $O(n'/\sqrt{\lg n})$ time, which answers a $\texttt{pred}(x)$ or $\texttt{succ}(x)$ query in $O(\lg\lg\sigma)$ time and $O(1)$ accesses to the elements of A. The query procedure requires access to a universal table of size $o(n)$ bits.*

Proof. We create a bitvector $B[0..n']$ in which $B[i] = 1$ if $i = 0$ or $A[i] > A[i-1]$, and represent it by Lemma 10 to support $\texttt{rank}$ and $\texttt{select}$. Thus B records the position of the first occurrence of each distinct integer in A. We also define a sequence $A'[0..t-1]$, in which $A'[i] = A[\texttt{select}_1(B,i)]$, where t is the number of distinct elements in A. $A'[0..t-1]$ then stores the distinct elements of A. We construct a predecessor/successor structure over A' using Lemma 14. A' is needed during construction but is discarded at the end of preprocessing, as each element, $A'[i]$, can be accessed by retrieving $A[\texttt{select}_1(B,i)]$. By Lemmas 10 and 14, B and the predecessor/successor structure over A' occupy $O(n'\lg\lg\sigma)$ bits in total.

Next, we give a query algorithm for $\texttt{succ}(x)$ over A; the support for $\texttt{pred}(x)$ is similar. We perform a successor query over A' to retrieve the successor, x', in A', which uses $O(\lg\lg\sigma)$ time and $O(1)$ accesses of elements of A. This will also give the position, i, of x' in A'. Then, we find $\texttt{succ}(x)$ over A in constant time, which is $\texttt{select}_1(B,i)$. Overall, we require $O(\lg\lg\sigma)$ time and $O(1)$ accesses of elements of A.

To build these data structures, we first show how to compute B and A'. Let $b = \lfloor\sqrt{\lg n}/(2c)\rfloor$. We use a universal table U to generate b elements of B and A' in constant time. U has an entry for each possible packed sequence $S[0..b-1]$ drawn from $[\sigma]$ and each possible flag $f \in \{0,1\}$, which stores a bitvector $V[0..b-1]$ and a

Index	0	1	2	3	4	5	6	7	8	9
A	2	2	4	4	4	5	5	7	7	7
B	1	0	1	0	0	1	0	1	0	0
A'	2	4	5	7						

Figure 2.3: An example of a `succ` queries over a sequence of non-distinct elements, e.g. $\texttt{succ}(A, 6) = \texttt{select}_1(B, \texttt{succ}(A', 6) + 1) = \texttt{select}_1(B, 4) = 7$.

packed sequence S' of length at most b defined as follows: If $f = 1$, we set $V[0] = 1$ and $V[0] = 0$ otherwise. For each $i \in [1, b-1]$, if $S[i-1] = S[i]$, then $V[i]$ is set to 0. Otherwise, $V[i]$ is set to 1. Then, the length of S' is equal to the number of 1's in V, and $S'[i] = S[\texttt{rank}_1(V, i)]$. As U has $O(n^{1/2})$ entries each using $O(\text{polylog}(n))$ bits, U occupies $o(n)$ bits. With U, we can generate b bits in B and b entries of A' in one table lookup, and thus the content of B and A' can be computed in $O(n/b)$ time. Adding the construction time needed to build query structures over them, the overall preprocessing time is $O(n/b) = O(n'/\sqrt{\lg n})$. ■

For general integer sequences, we will use the $\texttt{pred}(x)/\texttt{succ}(x)$ data structure by Chan et al. [12], which is summarized in the following lemma (even though Chan et al. did not analyze the construction time, it follows directly from previous results on the data structure components used):

Lemma 16 ([12, Section 2]) *Given an increasingly sorted sequence A of n' distinct integers from universe $[n]$ where $n' \leq n$, a data structure using extra $O(n' \lg\lg n)$ bits of space can be constructed in linear time, which answers a $\texttt{pred}(x)$ or $\texttt{succ}(x)$ query in $O(\lg\lg n)$ time and $O(1)$ accesses to elements of A. The query algorithm requires access to a universal table of $o(n)$ bits.*

Chapter 3

The $\texttt{rank}'$ Query Structures with Fast Preprocessing

In this section we focus on how to quickly construct data structures for $\texttt{rank}'$ queries over a sequence $A[0..n'-1]$ drawn from alphabet $[\sigma]$, where $n' \leq n$ and $\sigma \leq 2^{\sqrt{\lg n}}$. This is needed to solve ball inheritance in a special case. Lemma 9 already solves this problem when $\sigma \leq \lg n$, so we assume $\lg n < \sigma \leq 2^{\sqrt{\lg n}}$ in the rest of this section.

In our solution, we conceptually divide sequence A into chunks of length σ. For simplicity, assume that n' is a multiple of σ. Let A_k denote the kth chunk, where $0 \leq k \leq n'/\sigma - 1$. For each $c \in [0, \sigma-1]$, we define the following data structures:

- A bitvector $B_c = 1^{\texttt{rank}_c(A_0,\sigma)}01^{\texttt{rank}_c(A_1,\sigma)}0\ldots1^{\texttt{rank}_c(A_{n'/\sigma-1},\sigma)}0$, which encodes the number of occurrences of symbol c in each chunk in unary. B_c is represented using Lemma 10 to support $\texttt{rank}$ and $\texttt{select}$ in constant time.

- A sequence $P_c[0..n'/\sigma-1]$, in which $P_c[i] = \texttt{rank}'(A_i, c)$ for each $i \in [0, n'/\sigma-1]$, i.e., $P_c[i]$ stores the answer to a partial rank query performed locally within A_i at position c.

Note that we have one B_c for each alphabet symbol c, while we have one P_c for each relative position c in the chunks of A. See Figure 3.1 for an example. We have the following lemma on supporting queries using these data structures, with a space analysis.

Lemma 17 *The data structures in this section occupy $n' \lg \sigma + o(n' \lg \sigma)$ extra bits and support $\texttt{rank}'$ in $O(1)$ time and $O(1)$ accesses to elements of A.*

Proof. In B_c, each 1 bit corresponds to an occurrence of symbol c in A, while each 0 corresponds to a chunk. Thus, these bit vectors have n' 1s and $n'/\sigma \times \sigma = n'$ 0s in total. Therefore, the lengths of all these bit vectors sum up to $2n'$. By Lemma 10, $o(n')$ bits are needed to augment them to support $\texttt{rank}$ and $\texttt{select}$. As each chunk has σ elements, encoding an entry of each P_c requires $\lceil \lg \sigma \rceil$ bits. Thus P_0, ..., $P_{\sigma-1}$

	A_0				A_1				A_2			
A:	a	a	b	c	d	d	**c**	b	a	b	b	c

B_a:110010 B_b:1010110 B_c: **10**1010 B_d:01100

Chunk Index:	0	1	2
P_0:	1	1	1

Chunk Index:	0	1	2
P_1:	2	2	1

Chunk Index:	0	**1**	2
P_2:	1	**1**	2

Chunk Index:	0	1	2
P_3:	1	1	1

Figure 3.1: A sequence A of length $n = 12$ over an alphabet of $\sigma = 4$ symbols, split into $n/\sigma = 3$ chunks A_0, A_1 and A_2. Given a query $\texttt{rank}'(A, 6)$, we have $\texttt{rank}'(A, 6) = \texttt{select}(B_c, \lfloor 6/4 \rfloor) - (\lfloor 6/4 \rfloor - 1) + P_{(6 \texttt{ mod } 4)}[\lfloor 6/4 \rfloor] = 2$.

occupy $n'\lceil \lg \sigma \rceil$ bits in total. The total space usage of all the data structures in this section is therefore $2n' + o(n') + n'\lceil \lg \sigma \rceil$ bits, which is $n' \lg \sigma + o(n' \lg \sigma)$ when $\sigma > \lg n$.

A query $\texttt{rank}'(A, j)$ can be answered as follows:

$$\texttt{rank}'(A, j) = select_0(B_c, t) - (t-1) + P_\tau[t], where\ \tau = j \texttt{ mod } \sigma, t = \lfloor \frac{j}{\sigma} \rfloor, and\ c = A[j]$$

As the $\texttt{select}$ query over B_c takes constant time, answering $\texttt{rank}'(A, j)$ requires $O(1)$ time and a single access to A. ■

Next, we consider how to construct the sequences B_c's efficiently.

Lemma 18 *Bitvectors $B_0, B_1, \ldots, B_{\sigma-1}$ can be constructed in $O(n' \lg^2 \sigma / \lg n + \sigma)$ time.*

Proof. We first construct a sequence $M[0..n' + n'/\sigma - 1]$ in which each element is encoded in $\lceil \lg \sigma \rceil + 1$ bits. In M, n' elements are *regular elements*, and the rest are *boundary elements* each of which is an integer whose binary expression simply consists of $\lceil \lg \sigma \rceil + 1$ 0-bits. M is divided into n'/σ chunks, and each chunks contains σ regular elements followed by a boundary element. The subsequence of the σ regular elements in the i-th chunk can be obtained by appending a 1-bit to the end of the binary expression of each element in A_k.

Next we show how to create M efficiently with the help of a universal table U. This table has an entry for each possible pair (D, t), where D is a sequence of length $b = \lfloor \frac{\lg n}{2\lceil \lg \sigma \rceil} \rfloor$ drawn from $[\sigma]$ and t is an integer in $[0, b]$. If $t = 0$, this entry stores a sequence of length b which is obtained by appending a 1-bit to the end of the binary expression of each element in D. Otherwise, this entry stores a sequence of length $b+1$ consisting of three sections: the first section is obtained by appending a 1-bit to the end of the binary expression of each of the first t elements in D, the second section is a boundary element, and the third section is obtained by appending a 1-bit to the end of the binary expression of each of the last $b-t$ elements in D. As there are at most $n^{1/2}$ possible sequences of length b drawn from σ and t has $b+1$ possible values, U has at most $n^{1/2}(b+1)$ entries. Since each entry is encoded in at most $(b+1)(\lceil \lg \sigma \rceil + 1) = O(\text{polylog}(n))$ bits, U uses $o(n)$ bits. With U, we can scan A and process b of its elements in constant time; whether or where a boundary element should be created when processing these b elements can be inferred by keeping track of the number of elements that we have scanned so far. Note that at most one boundary element will be created when reading b elements from A, as $b < \lg n < \sigma$. The time needed to create M is hence $O(n'/b) = O(n' \lg \sigma / \lg n)$.

From M we determine the content of $B_0, B_1, \ldots, B_{\sigma-1}$ by constructing a tree T over M similar to a large extent to a binary wavelet tree and associating each node u of T with a sequence $M(u)$. At the root node r of T, we set $M(r) = M$, and we perform the following recursive procedure at any node u at level l of T where $l \in [0, \lceil \lg \sigma \rceil - 1]$: We create the left child, u_0, and the right child, u_1, of u, and perform a linear scan of $M(u)$. During the scan, for each $i \in [0, |M(u) - 1|]$, if $M(u)[i]$ is a boundary element, it is appended to both $M(u_0)$ and $M(u_1)$. If $M(u)[i]$ is not a boundary element and its lth most significant bit is 0, $M(u)[i]$ is appended to $M(u_0)$. If its lth significant bit is 1, it is appended to $M(u_1)$. After generating the sequences $M(u_0)$ and $M(u_1)$, we discard the sequence $M(u)$. We finish recursion after we create $\lceil \lg \sigma \rceil$ levels, i.e., we only examine the first $\lceil \lg \sigma \rceil$ bits of each element of M to determine the tree structure. Thus, this tree has σ leaves, and the sequences associated with the leaves from left to right are named M_0, M_1, $\ldots$, $M_{\sigma-1}$. They form a partition of M.

To speed up this process, we use a universal table U'. Recall that $b = \lfloor \frac{\lg n}{2\lceil \lg \sigma \rceil} \rfloor$. U' has an entry for each possible pair (E, c), where E is a sequence of length b drawn from universe $[2\sigma]$ and c is an integer in $[0, \lceil \lg \sigma \rceil - 1]$. This entry stores a pair of packed sequences E_0 and E_1 defined as follows: E_0 or E_1 stores the boundary

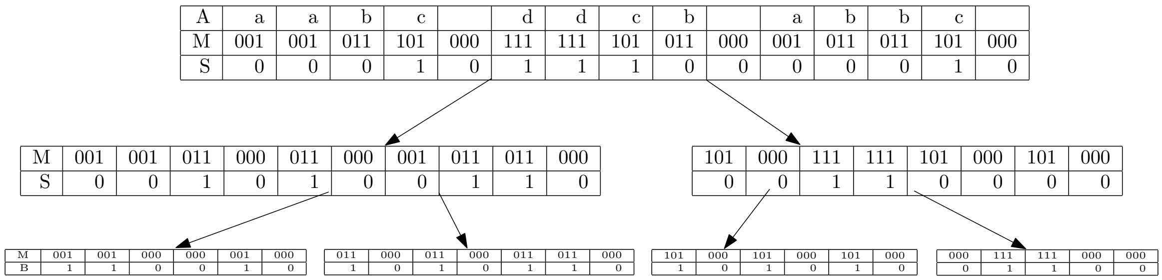


Figure 3.2: An example of constructing all B_c's. The input sequence A is divided into chunks of length 4. Each boundary element is encode by "000".

elements in E and the regular elements in E whose c-th most significant bit is 0 or 1, respectively. The elements in E_0 retain their relative order in E, and the same is true with E_1. As U' has $2^{b\times(\lceil\lg\sigma\rceil+1)} \times \lceil\lg\sigma\rceil$ entries and each entry stores a pair of packed sequences occupying $O(b\lceil\lg\sigma\rceil)$ bits in total, U' uses $o(n)$ bits. By performing table lookups in U', we can process $M(u)$ in $O(|M(u)|\lg\sigma/\lg n + 1)$ time. Note that we assign n' regular and $2^l \times \frac{n'}{\sigma}$ boundary elements to the nodes at tree level l. Summing over all $O(\sigma)$ nodes of the tree, the total time required to construct this tree is $O(\sum_{l=0}^{\lceil\lg\sigma\rceil-1}((n' + 2^l \times \frac{n'}{\sigma})\lg\sigma/\lg n) + \sigma) = O(n'\lg^2\sigma/\lg n + \sigma)$.

To construct bitvectors B_c for any $0 \le c \le \sigma - 1$, a crucial observation is that the i-th bit in B_c is the same as the least significant bits of the i-th elements of M_c. Thus it takes $O(|B_c|(\lg\sigma + 1)/\lg n + 1)$ time to compute the content of B_c using bit packing. B_c can then be represented in $O(|B_c|/\lg n + 1)$ time to support `rank` and `select` by Lemma 10. Summing over all σ bitvectors, the time required to construct $B_0, B_1, \ldots, B_{\sigma-1}$ from $M_0, M_1, \ldots, M_{\sigma-1}$ is $O(n'\lg\sigma/\lg n + \sigma)$.

Overall, given A, the construction time of these bit vectors is

$$O(n'\lg\sigma/\lg n + (n'\lg^2\sigma/\lg n + \sigma) + (n'\lg\sigma/\lg n + \sigma)) = O(n'\lg^2\sigma/\lg n + \sigma).$$

■

It remains to show how to build all sequences $P_0, P_1, \ldots, P_{\sigma-1}$ efficiently.

Lemma 19 *Sequences $P_0, P_1, \ldots, P_{\sigma-1}$ can be constructed in $O(n'\lg^2\sigma/\lg n + \sigma)$ time.*

Proof. The construction consists of two phases. In the first phase, we compute the set of pairs $R_k = \{(i, \texttt{rank}'(A_k, i)) | 0 \le i \le \sigma - 1\}$ for each chunk A_k. Even though

$P_i[k] = \texttt{rank}'(A_k, i)$ and thus the entries of all the P_i's have been computed in this phase, the pairs themselves generated for A_k are not in any order that allows us to directly assign values from these pairs to entries of P_i's quickly enough. Thus, in the second phase, we reorganize all n' pairs computed from all the chunks, to construct $P_0, P_1, \ldots, P_{\sigma-1}$ efficiently.

We first show how to compute the pair set R_k for each A_k efficiently. Let $I[0, \sigma-1]$ denote a packed sequence such that $I[i] = i$ for each $i \in [0, \sigma - 1]$. Note that I can be constructed once in $O(\sigma)$ time and shared with all chunks. By Lemma 2, a binary wavelet tree, in which node u is associated with $A(u)$ and $I(u)$ as defined before, over A_k could be constructed in $O(\sigma \lg^2 \sigma / \lg n + \sigma)$ time. However, the second term $O(\sigma)$, when summed over all n'/σ chunks, is too expensive to afford. Thus, we modify the structure of a wavelet tree to decrease this term. In the modified tree, when a node v satisfies $|A(v)| \leq b = \lfloor \frac{\lg n}{2\lceil \lg \sigma \rceil} \rfloor$, we make v a leaf node without any descendants. With this modification, we observe the following two properties. First, if a leaf node l satisfies $|A(l)| > b$, then the tree level of l must be $\lg \sigma$ and all entries of $A(l)$ store the same symbol. Second, as there are at most $\lceil \sigma/b \rceil$ nodes at each level, the modified tree has $O(\sigma/b \times \lg \sigma) = O(\sigma \lg^2 \sigma / \lg n)$ nodes. The $O(\sigma)$ term in construction time in Lemma 2 follows from the fact that a wavelet tree has $O(\sigma)$ leaves. With fewer leaves, the modified tree can be constructed in $O(\sigma \lg^2 \sigma / \lg n)$ time. After this tree is constructed, we only keep the sequences $A(l)$ and $I(l)$ for each leaf node l and call them *leaf sequences*. We discard the rest of the tree.

To further compute R_k using these leaf sequences, observe that, for any symbol α, there exists one leaf l such that $A(l)$ contains all the occurrences of α in A. Thus $(I(l)[i], \texttt{rank}'(A_k, I(l)[i])) = (I(l)[i], \texttt{rank}'(A(l), i))$ holds, which we can use to reduce the problem of computing the pairs in R_k to the problem of computing the answer to a partial rank query at each position of $A(l)$ for each leaf l. Hence for each leaf l, we define a packed sequence $Q(l)[0..|A(l)| - 1]$ in which $Q(l)[i] = \texttt{rank}'(A(l), i)$ to store these answers. To construct $Q(l)$ efficiently, we consider two cases. When $|A(l)| \leq b$, we apply a universal table U'' to generate $Q(l)$ in constant time. U'' has an entry for each possible pair (F, x), where F is a sequence of length b drawn from universe $[\sigma]$, and x is an integer in $[0, b]$. This entry stores a packed sequence $G[0..x]$ in which $G[i] = \texttt{rank}'(F, i)$. Similar to U in the proof of Lemma 18, U''

uses $o(n)$ bits. When $|A(l)| > b$, all entries of $A(l)$ store the same symbol. Thus, we have $Q(l)[i] = i$ for each $i \in [0, |A(l)| - 1]$, and hence we can create $Q(l)$ by copying the first $|A(l)|$ elements from the sequence I which we created before. In either case, $Q(l)$ can be constructed in $O(|A(l)| \lg \sigma / \lg n + 1)$ time. Let l_i denote the $(i+1)$-st leaf visited in a preorder traversal of the tree, and f the number of leaves. Since $\sum_{i=0}^{f} |Q(l_i)| = \sigma$ and $f = O(\sigma \lg^2 \sigma / \lg n)$, the total time required to build $Q(l_0), Q(l_1), \ldots, Q(l_{f-1})$ is $O(\sigma \lg^2 \sigma / \lg n)$. Then we construct the concatenated packed sequence $I_k = I(l_0)I(l_1)\ldots I(l_{f-1})$ and $Q_k = Q(l_0)Q(l_1)\ldots Q(l_{f-1})$. It requires $O(\sigma \lg^2 \sigma / \lg n)$ time to concatenate these sequences if we process $\Theta(\lg n)$ bits, i.e., $O(1)$ words, in constant time by performing bit operations. Since for any $i \in [0, \sigma - 1]$, $(I_k[i], Q_k[i])$ is a distinct pair in R_k, I_k and Q_k store all the pairs in R_k. We perform the steps in this and the previous paragraphs for all the chunks in A, and the total time spent in this phase is $O(n' \lg^2 \sigma / \lg n + \sigma)$.

Next we construct $P_0, P_1, \ldots, P_{\sigma-1}$ efficiently using the pairs computed in the previous phase. We first build in $O(n' \lg^2 \sigma / \lg n)$ time two concatenated packed sequences each of length n': $I' = I_0 I_1 \ldots I_{n'/\sigma-1}$ and $Q = Q_0 Q_1 \ldots Q_{n'/\sigma-1}$. Then we construct a binary wavelet tree over I'. Each node, u, of the wavelet tree is associated with two sequences, $I'(v)$ which contains all the elements of I' whose values are within the range represented by v, retaining their relative order in I', and $Q(v)$ in which $Q(v)[i]$ is the element in Q corresponding to $I'(v)[i]$. The wavelet tree construction algorithm of Lemma 2 can be modified easily to construct this wavelet tree in $O(n' \lg^2 \sigma / \lg n + \sigma)$ time. Let l'_i denote the $(i+1)$st leaf of this wavelet tree in preorder. Observe that all the entries in $I'(l'_i)$ store i, and $I'(l'_i)[j]$ initially came from A_j, i.e., $I'(l'_i)[j]$ corresponds to the ith position in chunk A_j. Therefore, $Q(l'_i)[j] = P_i[j]$, and we have $P_i = Q(l'_i)$. The processing time required for this phase is also $O(n' \lg^2 \sigma / \lg n + \sigma)$, which is the same as the bound for the first phase. Therefore, the total time required to construct all sequences $P_0, P_1, \ldots, P_{\sigma-1}$ is $O(n' \lg^2 \sigma / \lg n + \sigma)$. ∎

Combining Lemmas 9, 17, 18 and 19, we have the following result:

Lemma 20 *Let $A[0..n'-1]$ be a packed sequence drawn from alphabet $[\sigma]$, where $n' \leq n$ and $\sigma = O(2^{O(\sqrt{\lg n})})$. A data structure using $n' \lceil \lg \sigma \rceil + o(n' \lg \sigma)$ extra bits*

	A_0				A_1				A_2			
Index:	0	1	2	3	4	5	6	7	8	9	10	11
A:	a	a	b	c	d	d	c	b	a	b	b	c

In-chunk index (j)	0	1	2	3	3	2	0	1	0	1	2	3
$rank'(A_t, j)$	1	2	1	1	1	1	1	2	1	1	2	1

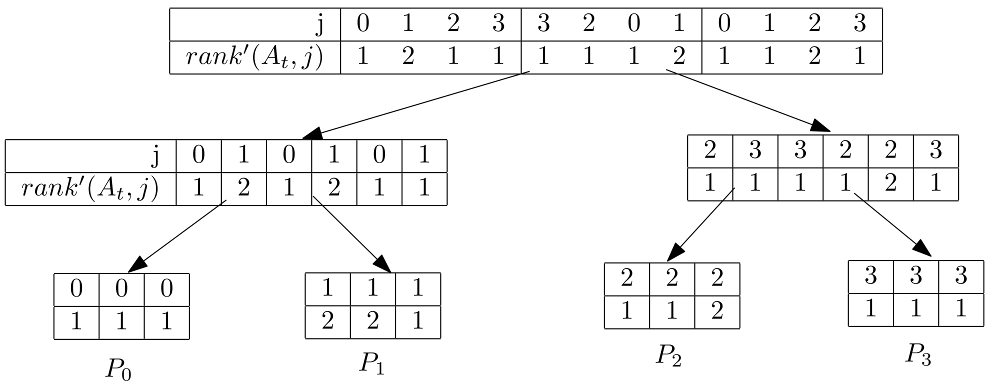


Figure 3.3: An example of Phase II; The first table shows the input sequence A is divided into chunks of length 4. The second table in the middle shows the output of Phase I; The binary tree at the bottom presents how to construct the arrays P_i for all $i \in [0, \sigma - 1]$. And all arrays P_i's are stored at the leaf level.

can be constructed in $O(n' \lg^2 \sigma / \lg n + \sigma)$ *time to support* `rank'` *queries in* $O(1)$ *time and* $O(1)$ *accesses to elements of* A.

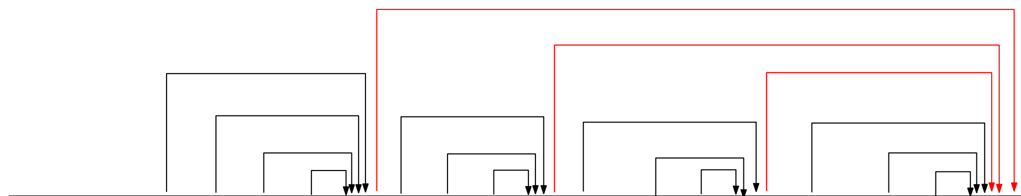
Chapter 4

The Ball Inheritance Structure with Fast Preprocessing

We now address the problem of efficiently building data structures for ball inheritance whose time and space bounds match those in parts (b) and (c) of Lemma 4. We first discuss, in Section 4.1, how to construct the ball inheritance structures of Chan et al. [12] efficiently over generalized wavelet trees, by replacing some of their data structure components with those we designed in previous sections to achieve faster construction time. This strategy however can not solve the problem in the general case, even though the data structures from previous sections match the state of the art, e.g., the construction time of our data structures for $\texttt{rank}'$ in Lemma 20 is on par with other data structures for sequences so that we can not hope for better. Instead, we only use this approach to solve some useful special cases. In Section 4.2, we further consider a special case, where the point coordinates can be encoded in $O(\sqrt{\lg n})$ bits, and we aim at achieving part (c) of Lemma 4. Our solution in this case then requires a twist to the approach of Chan et al. [12] to take advantage of the smaller grid size.

4.1 Ball Inheritance Based on the Approach of Chan et al. [12]

When used to represent the point set N, each node, u, of wavelet tree T is conceptually associated with an ordered list, $N(u)$, of points whose x-coordinates are within the range represented by u, and these points are ordered by y-coordinate. To save space, Chan et al. [12] do not encode each ordered point list explicitly. Instead, they define a sequence, $Sp(u)$, of *skipping pointers* for u, in which $Sp(u)[i]$ stores, at a certain number of levels below u, which descendant of u has $N(u)[i]$ in its ordered list of points; different choices of the distance between u and its descendant give different time-space tradeoffs. Then, since both $N(u)$ and $N(Sp(u)[i])$ order points by y-coordinate, the result of a $\texttt{rank}'(Sp(u), i)$ query is the position of the point $N(u)[i]$ in $N(Sp(u)[i])$. Thus, to compute $\texttt{point}(v, i)$, we can follow these skip pointers starting from v and perform $\texttt{rank}'$ queries along the way, until we reach the leaf level of T. As



Tree Level	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Color 0	0	1	2	3		5	6	7		9	10	11		13	14	15	
Color 1					4				8				12				
Color 2																	16

Figure 4.1: An example of coloring the tree levels over a tree whose height is 16, where τ is set to 4. The arrows on the top show the skipping pointers such that nodes on levels of the tree that are a multiple of 4^i store pointers to the next level multiple of 4^{i+1}.

Chan et al. store the coordinates of each point in the ordered lists associated with the leaves, this process will answer $\texttt{point}(v, i)$.

We now describe the details of these skipping pointers. Let τ be a parameter to be set later and h denote $\log_d \sigma$. Assume for simplicity that h is a power of τ. We assign a color to each level of T as follows: Level 0 is assigned color 0, while any other Level l is assigned color $\max\{c \mid \tau^c \text{ divides } l \text{ and } 0 \le c \le \log_\tau h\}$. Hence there are $\log_\tau h + 1$ different colors, and only the leaf level is assigned color $\log_\tau h$. For each leaf node l of T, we store the coordinates of the points in $N(u)$ explicitly. For any other node v, let color c be the color assigned to its level, l, where $c \in [0, \log_\tau h - 1]$. We do not store $N(v)$ explicitly, and instead, for each $i \in [0, |N(v)|]$, we store a skipping pointer $Sp(v)[i]$. This pointer stores, at the closest level l' satisfying $l' > l$ and l' is a multiple of τ^{c+1}, the descendant of v at level l' containing point $N(v)[i]$ in its ordered list of points. This descendant is encoded by its rank among all the descendants of v at level l' in left-to-right order. If each skipping pointer in $Sp(v)$ can be encoded in $O(\sqrt{\lg n})$ bits, we use Lemma 20 to build auxiliary data structures to support $O(1)$-time $\texttt{rank}'$ over $Sp(v)$ for each node v. Otherwise, we use Lemma 12. Finally, since T is a complete d-ary tree, we can store its structure implicitly by laying out its nodes in an array of σ entries in breadth-first order. The entry of this array corresponding to node v stores a pointer to a block of memory encoding the depth of v and either $N(v)$ if v is a leaf node or $Sp(v)$ otherwise. We also store the color of each level.

The following lemma gives analysis of this approach, in which the analysis of preprocessing time is restricted to the special cases that we need later.

Lemma 21 *Let $X[0..n'-1]$ be a sequence drawn from alphabet $[\sigma]$ and $Y[0..n'-1]$ be a sequence in which $Y[i]=i$ for each $i \in [0..n'-1]$, where $max(\sigma, n') \leq n$. A d-ary wavelet tree over X, where d is a power of 2 upper bounded by σ, can be represented using $O(n'\tau(\lg\sigma)\log_\tau(\log_d\sigma)+n'\lg n'+\sigma w)$ bits to support* `point` *in $O(\log_\tau(\log_d\sigma))$ time. Given X and Y as input, this tree can be constructed in $O(n'\tau\lg^2\sigma/\lg n + n'\lg n'\lg\sigma/\lg n+\sigma\log_d\sigma)$ time if $\sigma = O(2^{O(\sqrt{\lg n})})$. If $d \geq 2^{\sqrt{\lg n}}$, the construction time is $O((n'+\sigma)\log_d\sigma)$.*

Proof. Each point in N appears in the ordered point list associated with a node u at each level, l, of T. At each internal node, a skipping pointer is created, which encodes the rank of the descendant of u among all the descendants of u at level $l' = \tau^{c+1}\lceil l/\tau^{c+1}\rceil$. As u has $d^{l'-l} \leq d^{\tau^{c+1}}$ descendants at level l', $Sp(v)[i]$ can be encoded using at most $\tau^{c+1}\lg d$ bits. Since there are at most h/τ^c levels with color c, the skipping pointers created for this point across all levels of T occupy at most $\sum_{c=0}^{\log_\tau h-1}\frac{h}{\tau^c}\times\tau^{c+1}\times\lg d = O(\tau\lg\sigma\log_\tau h)$ bits. As there are n' points in N, the space usage of all skipping pointers is $O(n'\tau\lg\sigma\log_\tau h) = O(n'\tau(\lg\sigma)\log_\tau(\log_d\sigma))$ bits. By either Lemma 20 or Lemma 12, the extra space cost needed to build data structures to support `rank`$'$ in sequences of skip pointers is also $O(n'\tau(\lg\sigma)\log_\tau(\log_d\sigma))$ bits. We know that there are in total n' points at the leaf level and the coordinates of each point can be encoded in $O(\lg\sigma+\lg n')$ bits, so the cost of storing point coordinates at the leaf level is $O(n'(\lg\sigma+\lg n'))$ bits. As T has $O(\sigma)$ nodes, the implicit representation of T, with color and depth information, occupies $O(\sigma w)$ bits. Overall, the space usage of these structures is $O(n'\tau(\lg\sigma)\log_\tau(\log_d\sigma)+n'\lg n'+\sigma w)$ bits.

Now we analyze the query time of `point`. We retrieve the depth, l, of v to get the color, c, assigned to level l. If v is a leaf node, i.e., $c = \log_\tau h$, then $N(v)$ is stored explicitly, and we return $N(v)[i]$ as the answer. Otherwise, let $l' = \tau^{c+1}\lceil l/\tau^{c+1}\rceil$. Then the point p that we will eventually return as the answer to the query is also distributed to the ordered point list associated with the $Sp(v)[i]$-th descendant, u, of v at level l'. Node u can be located in constant time using the implicit representation of T as a complete d-ary tree. Furthermore, p is at position $j =$ `rank`$'(Sp(v), i)$ of $N(u)$. We then perform the query `point`(u, j) recursively to compute the answer. To bound the running time, observe that this process is terminated once we reach a leaf level. Hence, the process is applied recursively $O(\log_\tau h)$ times, each with a cost of $O(1)$. Therefore, it requires $O(\log_\tau h)$ time to support $\texttt{point}(v, i)$.

To construct these data structures, we first build T as a d-ary wavelet tree T over X with value and index arrays. T can be built in $O(n' \lg \sigma(\lg n' + \lg \sigma)/\lg n + \sigma)$ time by Lemma 2, if $\sigma = O(2^{\sqrt{\lg n}})$. Otherwise, it takes $O(n' \log_d \sigma + \sigma)$ time using the algorithm shown in Lemma 3. Computing the depth of each node of T and storing T implicitly use $O(\sigma)$ time. We then assign colors to its levels as follows: We first assign color 0 to the root level. Then we assign color $\log_\tau h$ to the leaf level. Among the remaining levels, we assign color $\log_\tau h - 1$ to those that are multiples of $\tau^{(\log_\tau h - 1)} = h/\tau$, and so on. During this process, an array of flags is used to mark those levels that have been assigned colors. As we use $O(1)$ time for each level, this requires $O(\log_d \sigma)$ time. Observe that the value and index arrays of each node u of T encode the x- and y-coordinates of the points in $N(u)$, respectively. Therefore, at the leaf level, we keep its value and index arrays as the encoding of $N(u)$. Otherwise, let c be the color assigned to level l. Then, for any $i \in [0, |N(u)| - 1]$, $Sp(u)[i]$ needs to store the rank of the descendant of u at level $l' = \tau^{c+1} \lceil l/\tau^{c+1} \rceil)$, which can be computed as $A(u)[i](l \lg d..l' \lg d)$; recall that $A(u)$ is the value array of u storing the x-coordinates of the points in $N(u)$. By Lemma 1, all elements of $Sp(u)$ can be generated in $O(|N(u)| \lg \sigma / \lg n + 1)$ time. The overall time needed to generate all the skipping pointers across the entire tree T is thus bounded by $O(n' \lg^2 \sigma/(\lg n \times \lg d) + \sigma)$, which is subsumed by the time cost spending on building T. We discard the value and index arrays of u after $Sp(u)$ has been built.

Next, we show how to build the data structure for $\texttt{rank}'$ queries upon $Sp(u)$. We first consider the case in which $\sigma = O(2^{\sqrt{\lg n}})$, in which the alphabet size of $Sp(u)$ is at most $\sigma = O(2^{\sqrt{\lg n}})$, so we apply Lemma 20 to build a $\texttt{rank}'$ structure over $S(u)$. $Sp(u)$ is drawn from alphabet $d^{l'-l}$, and since $l' - l \le \tau^{c+1}$, this structure can be built in $O(|Sp(u)|(l' - l)^2 \lg^2 d/ \lg n + d^{l'-l}) = O(|Sp(u)| \tau^{2c+2} \lg^2 d/ \lg n + d^{l'-l})$ time. Over all nodes at level l, observe that the term, $|Sp(u)| \tau^{2c+2} \lg^2 d/ \lg n$, sums up to $n' \tau^{2c+2} \lg^2 d/ \lg n$, while the term, $d^{l'-l}$, sums up to $f \times d^{l'-l}$, where f is the number of nodes at level l. To bound f, observe that, as each node at level l has $d^{l'-l}$ descendants, there are $f \times d^{l'-l}$ nodes at level l' and we have $f \times d^{l'-l} \le \sigma$. Thus the sum of the term, $d^{l'-l}$, over nodes at level l is bounded by σ. Hence the total time required to build auxiliary data structures for $\texttt{rank}'$ for nodes at a level with color c is $O(n'\tau^{2c+2}\lg^2 d/\lg n+\sigma)$. As there are at most h/τ^c levels with color c, the total construction time over all levels of T, including the time spent building and coloring T itself, is $O(n'\lg\sigma(\lg n'+\lg\sigma)/\lg n+\sigma+\sum_{c=0}^{(\log_\tau h)-1}(h/\tau^c)\times O(n'\tau^{2c+2}\lg^2 d/\lg n+\sigma) = O(n'\tau\lg^2\sigma/\lg n+n'\lg n'\lg\sigma/\lg n+\sigma\log_d\sigma)$. Finally, we consider the case in which $d\geq 2^{\sqrt{\lg n}}$. In this case, all $\texttt{rank}'$ structures are built using Lemma 12, so constructing overall takes $O(n'\log_d\sigma+\sigma)+\sum_{c=0}^{(\log_\tau h)-1}O(n'+\sigma) = O(n'\log_d\sigma+\sigma\log_d\sigma)$ time. ■

Another operation of ball inheritance is $\texttt{noderange}(c,d,v)$. Recall that given a range $[c,d]$ and a node v of T, $\texttt{noderange}(c,d,v)$ finds the range $[c_v,d_v]$ such that $I(v)[i]\in[c,d]$ iff $i\in[c_v,d_v]$. Obviously, c_v or d_v is equal to the positions of $\texttt{succ}(c)$ or $\texttt{pred}(d)$ in $I(v)$, respectively. Hence by constructing predecessor/successor data structures over $I(v)$, we can support $\texttt{noderange}$. The following lemmas addressing special cases of ball inheritance, in which either the wavelet tree has high fanout or the coordinates can be encoded in $O(\sqrt{\lg n})$ bits, can thus be obtained by choosing appropriate values for τ in Lemma 21 and applying different data structures for $\texttt{pred}/\texttt{succ}$ operations.

Lemma 22 *Let $X[0,n-1]$ be a sequence drawn from alphabet $[\sigma]$ denoting the point set $N=\{(X[i],i)|0\leq i\leq n-1\}$, where $2^{\sqrt{\lg n}}\leq\sigma\leq n$. A $2^{\sqrt{\lg n}}$-ary wavelet tree over X using $O(n\lg\sigma\cdot f(\sigma)+n\lg n)$ bits of space can be constructed in $O(n\lg\sigma/\sqrt{\lg n})$ time to support* `point` *in $O(g(\sigma))$ time and* `noderange` *in $O(\lg\lg n+g(\sigma))$ time, where (a) $f(\sigma)=O(\lg(\lg\sigma/\sqrt{\lg n}))$ and $g(\sigma)=O(\lg(\lg\sigma/\sqrt{\lg n}))$; or (b) $f(\sigma)=O(\lg^\epsilon\sigma)$ and $g(\sigma)=O(1)$ for any constant $\epsilon>0$. The* `noderange` *query requires a universal table of $o(n)$ bits.*

Proof. Consider the case in Lemma 21 for $d\geq 2^{\sqrt{\lg n}}$. In this case, set $n'=n$ and $d=2^{\sqrt{\lg n}}$. By further setting $\tau=2$ or $\tau=\lg^\epsilon\sigma$, we have the result (a) or (b), respectively, apart from the support of $\texttt{noderange}$. Next, we show the data structure supporting $\texttt{noderange}$, whose space cost and construction time are both subsumed by the data structure supporting $\texttt{point}$. We apply Lemma 16 to construct a data structure supporting $\texttt{pred}/\texttt{succ}$ over $I(v)$ at each node v, which answers $\texttt{noderange}$ in $O(\lg\lg n)$ time and $O(1)$ calls to $\texttt{point}$ without requiring $I(v)$ to be stored explicitly. This structure occupies $O(|I(v)|\lg\lg n)$ extra bits of space can be built upon $I(v)$

in $O(|I(v)|)$ time. As T has $\lg\sigma/\sqrt{\lg n}+1$ levels and there are n elements at each level, the overall time needed to construct it over all nodes is $O(n\lg\sigma/\sqrt{\lg n})$, and the overall extra space cost is $O(n\lg\lg n \times \lg\sigma/\sqrt{\lg n}) = o(n\lg\sigma)$ bits. ■

Lemma 23 *Let $X[0..n'-1]$ be a packed sequence drawn from alphabet $[\sigma]$ and $Y[0..n'-1]$ be a packed sequence in which $Y[i]=i$ for each $i\in[0..n'-1]$, where $\sigma=O(2^{O(\sqrt{\lg n})})$ and $n'=O(\sigma^{O(1)})$. Given X and Y as input, a d-ary wavelet tree over X using $O(n'\lg\sigma\lg(\lg\sigma/\lg d)+\sigma w)$ bits of space can be constructed in $O(n'\lg^2\sigma/\lg n+\sigma\log_d\sigma)$ time to support* `point` *in $O(\lg(\lg\sigma/\lg d))$ time and* `noderange` *in $O(\lg\lg\sigma)$ time, where d is a power of 2 upper bounded by $min(\sigma, 2^{\sqrt{\lg n}})$. The* `noderange` *query requires a universal table of $o(n)$ bits.*

Proof. Consider the case in Lemma 21 for $\sigma=O(2^{O(\sqrt{\lg n})})$. By setting $\tau=2$ and applying $n'=O(\sigma^{O(1)})$, we can obtain the construction time, the space cost and the query time needed to support `point`, which match the bounds shown in this lemma. It remains to show the support of `noderange`. As n' is bounded by $O(\sigma^{O(1)})$, each element of the index array $I(u)$ of each node u can be encoded with $O(\lg\sigma)=O(\sqrt{\lg n})$ bits. We then build the predecessor/successor data structure over $I(u)$ using Lemma 15. Given that `point` takes $O(\lg(\lg\sigma/\lg d))$ time, `noderange` can be answered in $O(\lg\lg\sigma+\lg(\lg\sigma/\lg d))=O(\lg\lg\sigma)$ time without explicitly storing $I(u)$. By Lemma 15, this data structure occupies $O(|I(u)|\lg\lg\sigma)$ extra bits of space and can be built upon $I(u)$ in $O(|I(u)|/\sqrt{\lg n}+1)$ time. As T has σ nodes and $h+1$ levels, the overall time needed to construct it over all nodes is $\sum_u O(|I(u)|/\sqrt{\lg n}+1)=O(n'\lg\sigma/(\sqrt{\lg n}\times\lg d)+\sigma)$ bounded by $O(n'\lg^2\sigma/\lg n+\sigma\log_d\sigma)$. Similarly, the overall extra space cost is $O(n'\lg\lg\sigma\log_d\sigma)$ bits. Therefore, the overall space cost required by the data structure designed is $O(n'\lg\sigma\lg(\lg\sigma/\lg d)+n'\lg\lg\sigma\log_d\sigma+\sigma w)=O(n'\lg\sigma\lg(\lg\sigma/\lg d)+\sigma w)$ bits. ■

4.2 Ball Inheritance in a Small Grid with Optimal Query Time

We now discuss how to efficiently build ball inheritance structures whose time and space bounds match those in part (c) of Lemma 4, when $\sigma=O(2^{O(\sqrt{\lg n})})$ and

$n' = O(\sigma^{O(1)})$. One may attempt to achieve this by setting τ to $(\log_d \sigma)^\epsilon$ in Lemma 21 to achieve constant-time support for `point`, but then the construction time is $O(n'\tau \lg^2 \sigma / \lg n + \sigma \log_d \sigma)$, in which the first term is not small enough. This term shows the time spent on building the auxiliary data structures for `rank`′. To remove the τ factor in it, we design a variant of the solution by Chan et al. [12] by storing point coordinates at a subset of levels of T instead of only at the leaf level. This twist allows us to build `rank`′ structures at fewer tree levels to decrease the construction time, and we still achieve the query time and space bounds that match those in part (c) of Lemma 4.

The details are as follows. Assume for simplicity that σ is a power of d, and that both $1/\epsilon$ and $\tau = \log_d^\epsilon \sigma$ are integers. We assign a color to each level of T as follows: Level 0 is assigned color 0, while any other Level l is assigned color $\max\{c \mid \tau^c \text{ divides } l \text{ and } 0 \le c \le 1/\epsilon - 1\}$. Hence there are $1/\epsilon$ different colors. Note that due to our simplifying assumptions, the leaf level is always assigned color $1/\epsilon - 1$ since $\log_d \sigma$ is divisible by $\tau^{1/\epsilon - 1}$; if these simplifying assumptions do not hold, we can manually assign color $1/\epsilon - 1$ to the leaf level and our proof can go through with trivial modifications. For each node u of T at a level assigned with color $1/\epsilon - 1$, we store the coordinates of the points in $N(u)$ explicitly. For any other node v, if the level l of v is assigned color c where $c \in [0, 1/\epsilon - 2]$, we do not store $N(v)$ explicitly, and instead, for each $i \in [0, |N(v)|]$, we store a skipping pointer $Sp(v)[i]$ as defined before. The other auxiliary data structures constructed including the one supporting `rank`′ over $Sp(v)$ and the implicit representation of T are all the same as in Lemma 21.

Lemma 24 *All skipping pointers $Sp(u)$ defined so far occupy $O(n' \lg \sigma \log_d^\epsilon \sigma + \sigma w)$ bits.*

Proof. Each point in N appears in the ordered point list associated with a node u at each level, l, of T. When the color, c, of l is not $1/\epsilon - 1$, a skipping pointer is created for this node, which encodes the rank of the descendant of u among all the descendants of u at level $l' = \tau^{c+1}\lceil l/\tau^{c+1} \rceil$. As u has $d^{l'-l} \le d^{\tau^{c+1}}$ descendants at level l', $Sp(v)[j]$ can be encoded using at most $\tau^{c+1} \lg d$ bits. Since there are at most $\log_d \sigma / \tau^c$ levels with color c, the skipping pointers created for this point across all levels of T occupy at most $\sum_{c=0}^{1/\epsilon-2} \frac{\log_d \sigma}{\tau^c} \times \tau^{c+1} \times \lg d = O(\tau \lg \sigma)$ bits. As there are n' points in N, the space usage of all skipping pointers is $O(n' \lg \sigma \log_d^\epsilon \sigma)$ bits. The space costs of all other data structures are similar to those in Lemma 21. ■

Lemma 25 *The data structures defined so far can support* $\texttt{point}(v, i)$ *in* $O(1)$ *time.*

Proof. We retrieve the depth, l, of v to get the color, c, assigned to level l. If $c = 1/\epsilon - 1$, then $N(v)$ is stored explicitly, and we return $N(v)[i]$ as the answer. Otherwise, let $l' = \tau^{c+1}\lceil l/\tau^{c+1}\rceil$. Then the point p that we will eventually return as the answer to the query is also distributed to the ordered point list associated with the $Sp(v)[i]$-th descendant, u, of v at level l'. Node u can be located in constant time using the implicit representation of T as a complete d-ary tree. Furthermore, p is at position $j = \texttt{rank}'(Sp(v), i)$ of $N(u)$. We then perform the query $\texttt{point}(u, j)$ recursively to compute the answer. To bound the running time, observe that this process is terminated once we reach a level with color $1/\epsilon - 1$, and one out of every $\tau^{1/\epsilon-1}$ levels of T is assigned this color. Hence, the process is applied recursively $O(\log_\tau \tau^{1/\epsilon-1}) = O(1)$ times, each with a cost of $O(1)$. Therefore, it requires constant time to support $\texttt{point}(v, i)$. ■

Lemma 26 *The data structures defined so far can be constructed in* $O(n'\lg^2 \sigma/\lg n + \sigma \log_d \sigma)$ *time.*

Proof. We build T and the skipping pointer sequences for its nodes as in the proof of Lemma 21, which uses $O(n'\lg^2 \sigma/\lg n + \sigma)$ time. We then apply Lemma 20 to build the data structure for $\texttt{rank}'$ queries upon $Sp(u)$. As $Sp(u)$ is drawn from alphabet $d^{l'-l}$, this requires $O(|Sp(u)|(l'-l)^2 \lg^2 d/\lg n + d^{l'-l})$ time, which is bounded by $O(|Sp(u)|\tau^{2c+2} \lg^2 d/\lg n + d^{l'-l})$ as $l' - l \leq \tau^{c+1}$. Hence the total time required to build auxiliary data structures for $\texttt{rank}'$ for nodes at a level with color c is $O(n'\tau^{2c+2} \lg^2 d/\lg n + \sigma)$. As there are at most $\log_d \sigma/\tau^c$ levels with color c, the total construction time over all levels of T is $\sum_{c=0}^{1/\epsilon-2}(\log_d \sigma/\tau^c) \times O(n'\tau^{2c+2} \lg^2 d/\lg n + \sigma) =$

$O(n'\lg^2\sigma/\lg n + \sigma\log_d\sigma)$, which dominates the construction time of all data structures. ■

To support `noderange`, we use the same method as shown in the proof of Lemma 23. Combining Lemmas 24, 25 and 26, we have the following result on ball inheritance:

Lemma 27 *Let $X[0..n'-1]$ be a packed sequence drawn from alphabet $[\sigma]$ and $Y[0..n'-1]$ be a packed sequence in which $Y[i] = i$ for each $i \in [0..n'-1]$, where $\sigma = O(2^{O(\sqrt{\lg n})})$ and $n' = O(\sigma^{O(1)})$. Given X and Y as input, a d-ary wavelet tree over X using $O(n'\lg\sigma\log_d^{\epsilon}\sigma + \sigma w)$ bits of space for any positive constant ϵ can be constructed in $O(n'\lg^2\sigma/\lg n + \sigma\log_d\sigma)$ time to support* `point` *in $O(1)$ time and* `noderange` *in $O(\lg\lg\sigma)$ time, where d is a power of 2 upper bounded by $min(\sigma, 2^{\sqrt{\lg n}})$. The* `noderange` *query requires a universal table of $o(n)$ bits.*

Chapter 5

Optimal Orthogonal Range Reporting with Fast Preprocessing

We now design data structures that support orthogonal range reporting in optimal time and can be constructed fast. Previously, with a solution to ball inheritance, Chan et al. [12] was able to design a relatively simple approach achieving three current best tradeoffs for orthogonal range reporting. However, we have only designed alternative solutions to ball inheritance with fast construction time in special cases. Therefore, we design a different data structure with optimal query time for orthogonal range reporting. The strategy is to use a generalized wavelet tree and our solution to range minimum/maximum (Lemma 7) to reduce the orthogonal range reporting problem in the general case to the special case in which the points are from a $2^{\sqrt{\lg n}} \times n'$ (narrow) grid. In this reduction, we need only support ball-inheritance over a wavelet tree with high fanout which is solved by part (b) of Lemma 22. We further reduce the range reporting problem over points in a narrow grid to this problem over a (small) grid of size at most $2^{\sqrt{\lg n}} \times 2^{2\sqrt{\lg n}}$, to which we can apply Lemma 27 for ball inheritance. Hence we describe our solutions over a small, narrow and general grid in this order, as the solution to the next case uses that to the previous.

5.1 Orthogonal Range Reporting in a Small Grid

We first show how to support orthogonal range reporting in a small grid:

Lemma 28 *Let N be a set of δ points with distinct y-coordinates in a $2^{\sqrt{\lg n}} \times \delta$ grid where $\delta \le 2^{2\sqrt{\lg n}}$. Given packed sequences X and Y respectively encoding the x- and y-coordinates of these points where $Y[i] = i$ for any $i \in [0, \delta - 1]$, a data structure occupying $O(\delta \lg^{1/2+\epsilon} n + w \cdot 2^{\sqrt{\lg n}})$ bits can be constructed in $O(\delta + \sqrt{\lg n} \cdot 2^{\sqrt{\lg n}})$ time to support orthogonal range reporting over N in $O(\lg \lg n + \mathsf{occ})$ time, where ϵ is an arbitrary positive constant and occ is the number of reported points.*

Proof. We build a binary wavelet tree T over X augmented with support for ball inheritance. By Lemma 27, T occupies $O(\delta \lg^{1/2+\epsilon} n + w \cdot 2^{\sqrt{\lg n}})$ bits and can be built in $O(\delta + \sqrt{\lg n} \cdot 2^{\sqrt{\lg n}})$ time. It also supports `point` in $O(1)$ time and `noderange` in $O(\lg\lg n)$ time. For any internal node v of T, its value array $A(v)$ is built at some point when augmenting T to solve ball inheritance, though $A(v)$ may be discarded eventually. When $A(v)$ was available, we built a data structure $M(v)$ to support range minimum and maximum queries over $A(v)$ using Lemma 7. As T has $\lceil \sqrt{\lg n} \rceil$ non-leaf levels and the total length of the value arrays of the nodes at each tree level is δ, over all internal nodes, these structures use $O(\delta\sqrt{\lg n}\lg\lg n)$ bits in total and the overall construction time is $\sum_v O(|A(v)|/\sqrt{\lg n} + 1) = O(\delta + 2^{\sqrt{\lg n}})$. These costs are subsumed in the storage and construction costs of T. Recall that $A(v)$ stores the x-coordinates of the set, $N(v)$, of points from N whose x-coordinates are within the range represented by v, and the entries of $A(v)$ are ordered by the corresponding y-coordinates of these points. Thus any entry of $A(v)$ can be retrieved by `point` in constant time. Therefore, even after $A(v)$ is discarded, $M(v)$ can still support `rmq`/`rMq` over $A(v)$ in $O(1)$ time.

Given a query range $Q = [a, b] \times [c, d]$, we first locate the lowest common ancestor u of l_a and l_b in constant time, where l_a and l_b denote the a-th and b-th leftmost leaves of T, respectively. Let u_l and u_r denote the left and right children of u, respectively, $[c_l, d_l] =$ `noderange`(c, d, u_l) and $[c_r, d_r] =$ `noderange`(c, d, u_r). Then $Q \cap N = (([a, +\infty) \times [c_l, d_l]) \cap N(u_l)) \cup (([0, b] \times [c_r, d_r]) \cap N(u_r))$. In this way, we reduce a 2-d 4-sided range reporting in N to 2-d 3-sided range reporting in $N(u_l)$ and $N(u_r)$. To report points in $([a, +\infty) \times [c_l, d_l]) \cap N(u_l)$, we need only report the points in $N(u_l)[c_l, d_l]$ whose x-coordinates are at least a. This can be done by performing range maximum queries over $A(u_l)$ recursively as follows. We perform `rMq`(c_l, d_l) to get the index m of the point p that has the maximum x-coordinate in $N(u_l)[c_l, d_l]$, and retrieve its coordinates $(p.x, p.y)$ by `point`(u_l, m). If $p.x \geq a$, we report p and perform the same process recursively in $N(u_l)[c_l, m-1]$ and $N(u_l)[m+1, d_l]$. Otherwise we stop. The points in $([0, b] \times [c_r, d_r]) \cap N(u_r)]$ can be reported in a similar way. To analyze the query time, observe that we perform `noderange` twice in $O(\lg\lg n)$ time. The recursive procedure is called $O($`occ`$)$ times, and each time it is performed, it uses $O(1)$ time. All other steps require $O(1)$ time. Therefore, the overall query time is

$O(\lg\lg n + \texttt{occc})$. ■

5.2 Orthogonal Range Reporting in a Narrow Grid

Our solution for points in a $2^{\sqrt{\lg n}} \times n'$ grid for any $n' \leq n$ uses the following previous result:

Lemma 29 ([12, Section 2],[7, Lemma 5]) *Given a set, N, of n points in $[u] \times [u]$, a data structure of $O(n \lg^{1+\epsilon} n)$ bits can be constructed in $O(n \lg n)$ time, which supports orthogonal range reporting over N in $O(\lg\lg u + \texttt{occ})$ time, where* occ *is the number of reported points.*

The following lemma presents our solution for a narrow grid:

Lemma 30 *Let N be a set of n' points with distinct y-coordinates in a $2^{\sqrt{\lg n}} \times n'$ grid where $n' \leq n$. Given packed sequences X and Y respectively encoding the x- and y-coordinates of these points where $Y[i] = i$ for any $i \in [0, n'-1]$, a data structure occupying $O(n' \lg^{1/2+\epsilon} n + w \cdot 2^{\sqrt{\lg n}} + n'w/2^{\sqrt{\lg n}})$ bits can be constructed in $O(n' + \sqrt{\lg n} \cdot 2^{\sqrt{\lg n}})$ time to support orthogonal range reporting over N in $O(\lg\lg n + \texttt{occ})$ time, where ϵ is an arbitrary positive constant and* occ *is the number of reported points.*

Proof. Let $b = 2^{2\sqrt{\lg n}}$. We need only consider the case in which $n' > b$ as Lemma 28 applies otherwise. Assume for simplicity that n' is divisible by b. We divide N into n'/b subsets, and for each $i \in [0, n'/b - 1]$, the ith subset, N_i, contains points in N whose y-coordinates are in $[ib, (i+1)b - 1]$. Let p be a point in N_i. We call its coordinates $(p.x, p.y)$ *global coordinates*, while $(p.x', p.y') = (p.x, p.y \bmod b)$ its *local coordinates* in N_i; the conversion between global and local coordinates can be done in constant time. Hence the points in N_i with their local coordinates can be viewed as a point set in a $2^{\sqrt{\lg n}} \times 2^{2\sqrt{\lg n}}$ grid, and we apply Lemma 28 to construct an orthogonal range search structure over N_i.

We also define a point set $\hat{N}$ in a $2^{\sqrt{\lg n}} \times n'/b$ grid. For each set N_i where $i \in [0, n'/b - 1]$ and each $j \in [0, 2^{\sqrt{\lg n}} - 1]$, we store a point (j, i) in $\hat{N}$ iff there exists at least one point in N_i whose x-coordinate is j. Thus the number of points in $\hat{N}$ is at most $n'/b \times 2^{\sqrt{\lg n}} = n'/2^{\sqrt{\lg n}}$. We apply Lemma 29 to construct an orthogonal range search structure over $\hat{N}$. In addition, for each $i \in [0, n'/b-1]$ and $j \in [0, 2^{\sqrt{\lg n}}-1]$, we store a list $P_{i,j}$ storing the local y-coordinates of the points in N_i whose x-coordinates are equal to j.

Given a query range $Q = [x_1, x_2] \times [y_1, y_2]$, we first check if $\lfloor y_1/b \rfloor$ is equal to $\lfloor y_2/b \rfloor$. If it is, then the points in the answer to the query reside in the same subset $N_{\lfloor y_1/b \rfloor}$, and we can retrieve these points by performing an orthogonal range query in $N_{\lfloor y_1/b \rfloor}$, which requires $O(\lg\lg n + \mathtt{occ})$ time by Lemma 28. Otherwise, we decompose Q into three subranges $Q_1 = [x_1, x_2] \times [y_1, b(\lfloor y_1/b \rfloor + 1) - 1]$, $Q_2 = [x_1, x_2] \times [b(\lfloor y_1/b \rfloor + 1), b\lfloor y_2/b \rfloor - 1]$ and $Q_3 = [x_1, x_2] \times [b\lfloor y_2/b \rfloor, y_2]$. The points in $N \cap Q_1$ and $N \cap Q_3$ are in $N_{\lfloor y_1/b \rfloor}$ and $N_{\lfloor y_2/b \rfloor}$, respectively, and by Lemma 28, they can be reported in $O(\lg\lg n + \mathtt{occ}_1)$ and $O(\lg\lg n + \mathtt{occ}_3)$ time, respectively, where $\mathtt{occ}_1 = |N \cap Q_1|$ and $\mathtt{occ}_3 = |N \cap Q_3|$. The points in $N \cap Q_2$ are in $N_{\lfloor y_1/b \rfloor + 1}, N_{\lfloor y_1/b \rfloor + 2}, \ldots, N_{\lfloor y_2/b \rfloor - 1}$. To retrieve them, we first perform an orthogonal range query in $\hat{N}$ with query range $\hat{Q} = [x_1, x_2] \times [\lfloor y_1/b \rfloor + 1, \lfloor y_2/b \rfloor - 1]$. Let (x, y) be a point in $\hat{N} \cap \hat{Q}$. The existence of this point means that is at least one point in $N_y \cap Q_2$ whose x-coordinates are equal to x; the local y-coordinates of these points are stored in $P_{y,x}$ which we retrieve and convert to global coordinates. After examining all the points in $\hat{N} \cap \hat{Q}$ and retrieving their corresponding points in $N \cap Q_2$ in this way, we have computed all the points in $N \cap Q_2$ in $O(\lg\lg n + \mathtt{occ}_2)$ time where $\mathtt{occ}_2 = |N \cap Q_2|$. The overall query processing time is thus $O(\lg\lg n + \mathtt{occ})$.

To bound the storage costs, by Lemma 28, the orthogonal range reporting structure over each N_i uses $O(2^{2\sqrt{\lg n}} \lg^{1/2+\epsilon} n + w \cdot 2^{\sqrt{\lg n}})$ bits. Thus, the range reporting structures over $N_0, N_1, \ldots, N_{n/b-1}$ occupy $O((n'/b) \times (2^{2\sqrt{\lg n}} \lg^{1/2+\epsilon} n + w \cdot 2^{\sqrt{\lg n}})) = O(n' \lg^{1/2+\epsilon} n + n'w/2^{\sqrt{\lg n}})$ bits. As there are at most $n'/2^{\sqrt{\lg n}}$ points in $\hat{N}$, by Lemma 29, the range reporting structure for $\hat{N}$ occupies $O(n' \lg^{1+\epsilon} n / 2^{\sqrt{\lg n}}) = o(n')$ bits. There are n' points in all $P_{i,j}$'s and each of their local y-coordinates can be encoded in $\lg b = 2\sqrt{\lg n}$ bits. In addition, each $P_{i,j}$ requires a pointer to encode its memory location, so $n'/b \times 2^{\sqrt{\lg n}} = n'/2^{\sqrt{\lg n}}$ pointers are needed. Therefore, the total storage cost of all $P_{i,j}$'s is $O(n'w/2^{\sqrt{\lg n}} + n'\sqrt{\lg n})$. Thus the space costs of all structures add up to $O(n' \lg^{1/2+\epsilon} n + n'w/2^{\sqrt{\lg n}})$ bits. Note that the above analysis assumes $n' > b$. Otherwise, $O(n' \lg^{1/2+\epsilon} n + w \cdot 2^{\sqrt{\lg n}})$ bits are needed, so we use

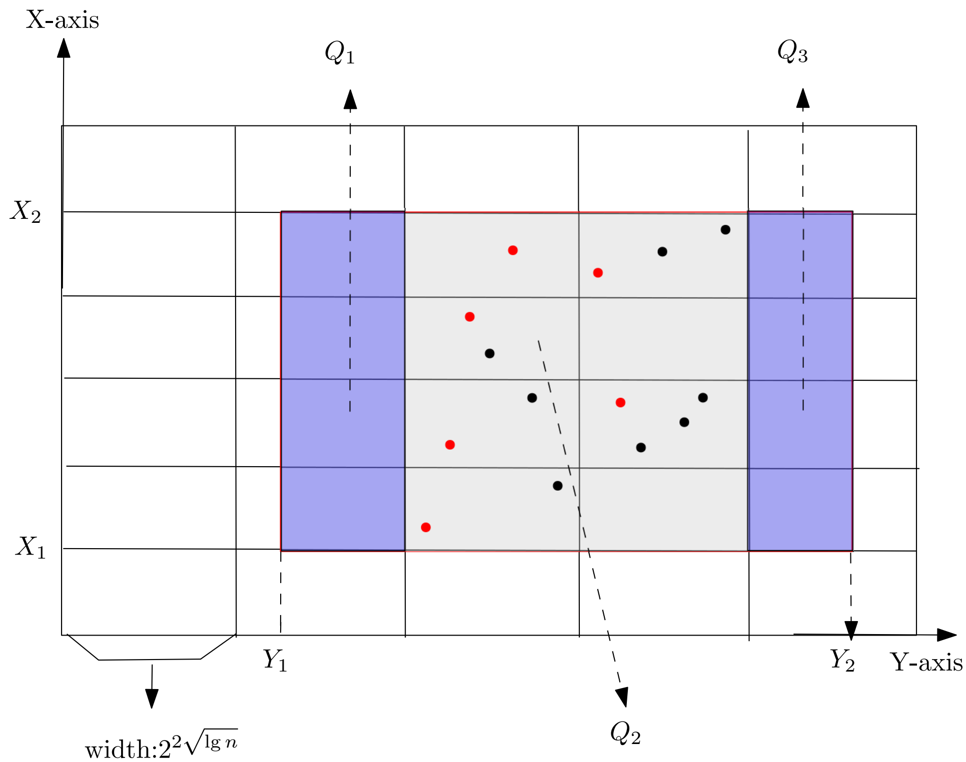


Figure 5.1: The n' points are divided into $2^{\sqrt{\lg n}} \times n'/2^{2\sqrt{\lg n}}$ cells. If the cell (j, i) has points, we store (j, i) in $\hat{N}$. Given a query range Q, it is split into Q_1, Q_2, and Q_3 three non-overlapping parts.

$O(n' \lg^{1/2+\epsilon} n + w \cdot 2^{\sqrt{\lg n}} + n'w/2^{\sqrt{\lg n}})$ as the space bound on both cases.

Regarding construction time, observe that the point sets $N_0, N_1, \ldots, N_{n'/b-1}$ and $\hat{N}$, as well as the sequences $P[i, j]$ for $i = 0, 1, \ldots, n'/b - 1$ and $j = 0, 1, \ldots, 2^{\sqrt{\lg n}} - 1$, can be computed in $O(n')$ time, when $n' > b$. By Lemma 29, the range reporting structure for $\hat{N}$ can be built in $O(n'/b \times \lg n) = o(n')$ time. Finally, the total construction time of the range reporting structures for $N_0, N_1, \ldots, N_{n/b-1}$ is $O(n'/2^{2\sqrt{\lg n}} \times (2^{2\sqrt{\lg n}} + \sqrt{\lg n} \times 2^{\sqrt{\lg n}})) = O(n')$, which dominates the total preprocessing time of all our data structures. When $n' \leq b$, the construction time is $O(n' + \sqrt{\lg n} \cdot 2^{\sqrt{\lg n}})$ by Lemma 28, so we use $O(n' + \sqrt{\lg n} \cdot 2^{\sqrt{\lg n}})$ as the upper bound on construction time in both cases. ■

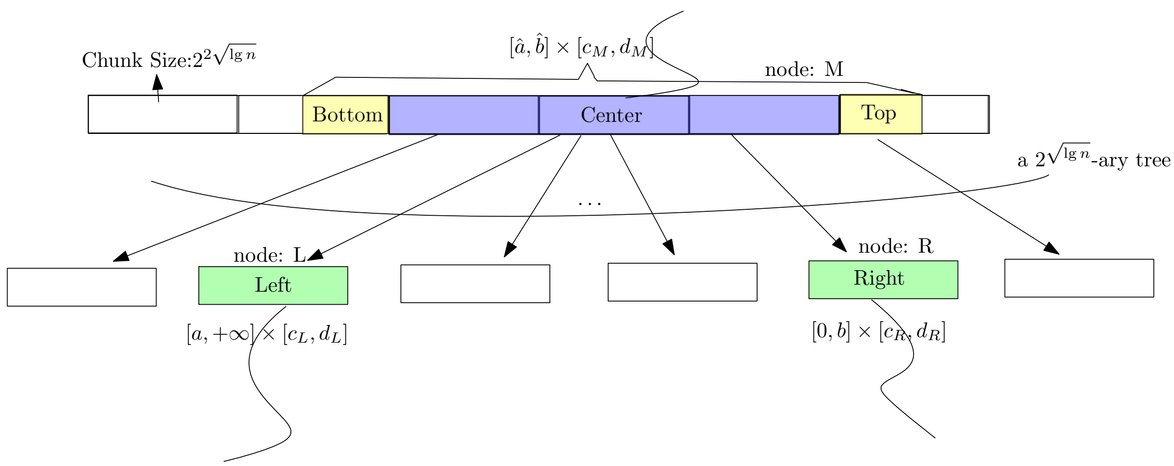
5.3 Orthogonal Range Reporting in an $n \times n$ Grid

We first describe a solution that is slightly more general, which requires the grid to be of size $\sigma \times n$ with $2^{\sqrt{\lg n}} \leq \sigma \leq n$, as it will be needed for some applications to be described later.

Lemma 31 *Given a sequence $X[0, n-1]$ drawn from alphabet $[\sigma]$ denoting the point set $N = \{(X[i], i) | 0 \leq i \leq n-1\}$, a data structure of $O(n \lg^{1+\epsilon} \sigma + n \lg n)$ bits for any constant $\epsilon > 0$ can be constructed in $O(n \lg \sigma / \sqrt{\lg n})$ time to support orthogonal range reporting over N in $O(\lg \lg n + \texttt{occ})$ time, where $2^{\sqrt{\lg n}} \leq \sigma \leq n$ and $\texttt{occ}$ is the number of reported points.*

Proof. We build a $2^{\sqrt{\lg n}}$-ary wavelet tree T upon $X[0, n-1]$ with support for ball inheritance using part (b) of Lemma 22. As in the proof of Lemma 28, for each internal node $v \in T$, we build a data structure $M(v)$ to support range minimum and maximum queries over its value array $A(v)$ in constant time using Lemma 7, even $A(v)$ is not explicitly stored. Recall that $A(v)$ stores the x-coordinates of the ordered list, $N(v)$, of points from N whose x-coordinates are within the range represented by v, and these points are ordered by y-coordinate. Furthermore, v is associated with another sequence $S(v)$ drawn from alphabet $[2^{\sqrt{\lg n}}]$, in which $S(v)[i]$ encodes the rank of the child of v that contains $N(v)[i]$ in its ordered list. Let $\hat{S}(v)$ denote the point set $\{(S(v)[i], i) | 0 \leq i \leq |S(v)| - 1\}$, and we use Lemma 30 to build a structure supporting orthogonal range reporting over $\hat{S}(v)$.

Given a query range $Q = [a, b] \times [c, d]$, we first locate the lowest common ancestor u of l_a and l_b in constant time, where l_a and l_b denote the a-th and b-th leftmost leaves of T, respectively. Let u_i denote the ith child of u, for any $i \in [0, 2^{\sqrt{\lg n}} - 1]$. We first locate two children, $u_{a'}$ and $u_{b'}$, of u that are ancestors of l_a and l_b, respectively. They can be found in constant time by simple arithmetic as each child of u represents a range of equal size. Then the answer, $Q \cap N$, to the query can be partitioned into three point sets $A_1 = Q \cap N(v_{a'})$, $A_2 = Q \cap (N(v_{a'+1}) \cup N(v_{a'+2}) \cup \ldots N(v_{b'-1}))$ and $A_3 = Q \cap N(v_{b'})$. With $O(\lg \lg n)$-time support for `noderange` and constant-time support for `point` and `rmq`/`rMq`, we can use the algorithm in the proof of Lemma 28 to perform 3-sided range queries over $N(v'_a)$ and $N(v'_b)$ to compute $A_1 \cup A_3$ in



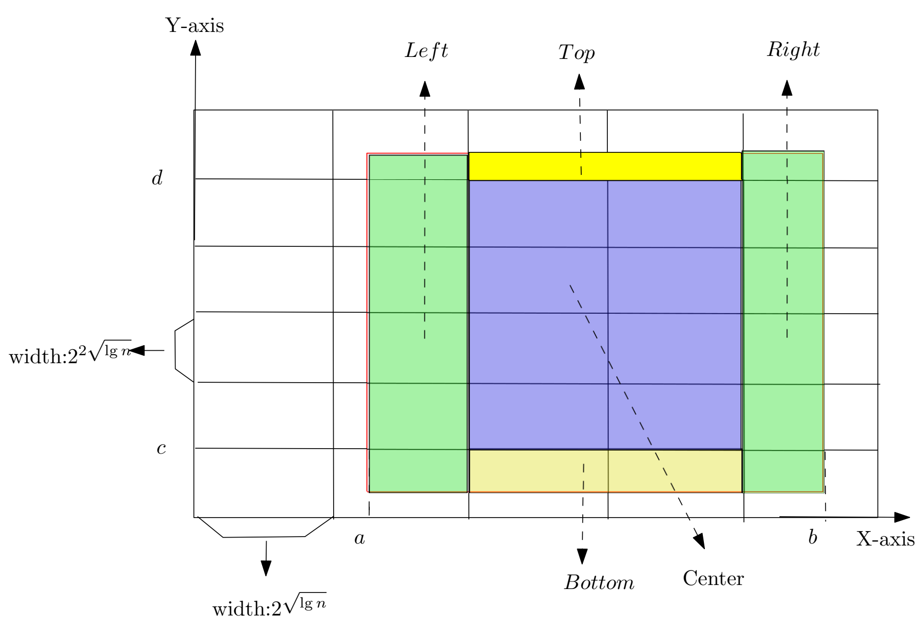


Figure 5.2: On a $2^{\sqrt{\lg n}}$-ary wavelet tree, the query range $[a, b] \times [c, d]$ is divided into three parts: Left, middle, and right. Furthermore, the elements in the middle part is split into chunks of length $2^{2\sqrt{\lg n}}$.

$O(\lg\lg n + |A_1| + |A_3|)$ time. To compute A_2, observe that any entry, $\hat{S}(v)[i]$, can be obtained by replacing the x-coordinate of point $N(v)[i]$ with the rank of the child whose ordered list contains $N(v)[i]$. Hence, by performing range reporting over $\hat{S}$ to compute $S \cap ([a'+1, b'-1] \times [c_v, d_v])$, where $[c_v, d_v] = \texttt{noderange}(c, d, v)$, we can find the set of points in $\hat{S}(v)$ corresponding to the points in A_2. For each point returned, we use `point` to find its original coordinates in N and return it as part of A_2. This process uses $O(\lg\lg n + |A_2|)$ time. Hence we can compute $Q \cap N$ as $A_1 \cup A_2 \cup A_3$ in $O(\lg\lg n + \texttt{occ})$ time.

Now we analyze the space costs. The wavelet tree T with support for ball inheritance uses $O(n \lg^{1+\epsilon} \sigma + n \lg n)$ bits for any positive ϵ. For each internal node v, since $w = \Theta(\lg n)$, the data structure for range reporting over $\hat{S}$ uses $O(|S(u)| \lg^{1/2+\epsilon'} n + 2^{\sqrt{\lg n}} \lg n + |S(u)| \lg n / 2^{\sqrt{\lg n}})$ bits for any positive ϵ'. This subsumes the cost of storing $M(u)$ which is $O(|S(u)| \lg\lg n)$ bits. As T has $O(\sigma / 2^{\sqrt{\lg n}})$ internal nodes, the total cost of storing these structures at all internal nodes is $\sum_u O(|S(u)| \lg^{1/2+\epsilon'} n + 2^{\sqrt{\lg n}} \lg n + |S(u)| \lg n / 2^{\sqrt{\lg n}}) = O(n \lg \sigma / \sqrt{\lg n} \times \lg^{1/2+\epsilon'} n + \sigma \lg n) = O(n \lg \sigma \lg^{\epsilon'} n + \sigma \lg n)$ bits. As $\lg n \leq \lg^2 \sigma$ and $\sigma \leq n$, this is bounded by $O(n \lg^{1+2\epsilon'} \sigma)$. Setting $\epsilon' = \epsilon/2$, the space bound turns out to be $O(n \lg^{1+\epsilon} \sigma)$ bits. Overall, the data structures occupy $O(n \lg^{1+\epsilon} \sigma + n \lg n)$ bits.

Finally, we analyze the construction time. As shown in Lemma 22, T with support for ball inheritance can be constructed in $O(n \lg \sigma / \sqrt{\lg n})$ time. For each internal node u of T, constructing $M(u)$ and the range reporting structure over $\hat{S}(v)$ requires $O(|S(u)| + \sqrt{\lg n} \cdot 2^{\sqrt{\lg n}})$ time. As T has $O(\sigma / 2^{\sqrt{\lg n}})$ internal nodes, these structures over all internal nodes can be built in $\sum_u O(|S(u)| + \sqrt{\lg n} \times 2^{\sqrt{\lg n}}) = O(n \lg \sigma / \sqrt{\lg n} + \sigma \sqrt{\lg n}) = O(n \lg \sigma / \sqrt{\lg n})$ time as $\sigma \leq n$. The preprocessing time of all data structures is hence $O(n \lg \sigma / \sqrt{\lg n})$. ■

Our result on points over an $n \times n$ gird immediately follows.

Theorem 1 *Given a set, N, of n points in rank space, a data structure of $O(n \lg^{\epsilon} n)$ words for any constant $\epsilon > 0$ can be constructed in $O(n\sqrt{\lg n})$ time to support orthogonal range reporting in $O(\lg\lg n + \texttt{occ})$ time, where `occ` is the number of reported points.*

Chapter 6

Optimal Orthogonal Range Successor with Fast Preprocessing

We now design data structures over n points in 2d rank space that support an orthogonal range successor query in optimal time and can be constructed fast. Previously, using a solution to the three-sided next point problem defined in Section 6.1 and ball inheritance, Zhou [37] solved the orthogonal range successor problem within optimal query time. As their solution relies on auxiliary structures on a binary wavelet tree, the preprocessing time requires $O(n \lg n)$. Our data structure is constructed upon a $2^{\sqrt{\lg n}}$-ary wavelet tree to reduce the problem in the general case to the three-sided next point query problem and the orthogonal range successor problem in the special case in which the points are from a $2^{\sqrt{\lg n}} \times n'$ *medium narrow* grid. Then our solutions to ball inheritance upon a generalized wavelet tree with high fanout can apply, which reduces the processing time from $O(n \lg n)$ to $O(n\sqrt{\lg n})$. We further design data structures with fast construction time supporting the three-sided next point problem and the reduced orthogonal range successor problem in the special cases. Hence, we describe our solutions in this order: in Section 6.1, we introduce the methods to solve the three sided next point query, and in Section 6.2, we describe our solutions to the orthogonal range successor problem over a medium narrow and general grid.

6.1 Fast Construction of the Three-Sided Next Point Structures

In this section, we show how to efficiently construct data structures for *three-sided next point* queries, defined as follows. Given a set of points, N, of n points in the rank space, a three-sided next point query is the problem of retrieving the point with the smallest y-coordinate among all the points in $N \cap Q$ where $Q = [a, +\infty] \times [c, d]$.

The methods shown in Lemmas 34 and 36 are under the indexing model: after the construction of the data structure, each query operation needs to access some points and report them. The point set N itself need not be stored explicitly; it suffices to provide an operator supporting the access to an arbitrary point of N. The operator is implemented by `point`(v, i) from ball inheritance. Our solutions will use the previous results as follows:

Lemma 32 ([32, Lemma 5]) *There exists a data structure of $O(n \lg^3 n)$-bit space constructed upon a set of n points in rank space in $O(n \lg^2 n)$ time, which supports three-sided next point query in $O(\lg \lg n)$ time.*

Both Lemmas 33 and 34 are originally designed by Zhou [37]. But they did not mention the construction time before. Here, we only give the analysis of the construction time.

Lemma 33 ([37, Lemma 3.2]) *Let N be a set of $\lg^3 n$ points in rank space. Given packed sequences X and Y respectively encoding the x- and y-coordinates of these points where $Y[i] = i$ for any $i \in [0, \lg^3 n - 1]$, a data structure using $O(\lg^3 n \lg \lg n)$ bits of space constructed over N in $o(\lg^3 n/\sqrt{\lg n})$ time that answers the three-sided next point query in $O(\lg \lg n)$ time. The query procedure requires access to a universal table of $o(n)$ bits.*

Proof. We divide each consecutive $\lg^{3/4} n$ points along y-axis of N into blocks. The dividing operation can be done in $O(\lg^3 n/\lg^{3/4} n) = o(\lg^3 n/\sqrt{\lg n})$ time by bit-wise operations. As each point requires $6 \lg \lg n$ bits of space, each block uses $6 \lg \lg n \times \lg^{3/4} n$ bits, which are less than a word. From each block, we apply a universal table U of $o(n)$ bits to retrieve the point with maximum x-coordinate in constant time. U has an entry for each possible triple (α, β, γ), where α or β is a packed sequence of length at most $\lg^{3/4} n$ drawn from $[\lg^3 n]$ denoting the x- or y-coordinates of the points, respectively, and γ is an integer $\in [0..(\lg^{3/4} n) - 1]$ denoting the number of points. This entry stores the point with the maximum x-coordinate among the point set denoted by α and β. As U has $O(2^{(\lg^{3/4} n) \times (6 \lg \lg n)} \times \lg^{3/4} n)$ entries and each entry stores a point of $6 \lg \lg n$ bits, U uses $o(n)$ bits of space. Let $\hat{N}$ denote the set of the selected points and $|\hat{N}| = \lceil \lg^3 n/\lg^{3/4} n \rceil$. We use Lemma 32 to build a data structure $DS(\hat{N})$ over $\hat{N}$ for the the three-sided next point query. The data structure $DS(\hat{N})$ using $O(|\hat{N}| \lg^3 |\hat{N}|) = o(\lg^3 n)$ bits of space can be built in $O(|\hat{N}| \lg^2 |\hat{N}|) = O(|\hat{N}| \lg^2 \lg n)$ time bounded by $o(\lg^3 n/\sqrt{\lg n})$. Overall, the data structure uses $O(\lg^3 n \lg\lg n + o(\lg^3 n)) = O(\lg^3 n \lg\lg n)$ bits of space and can be constructed in $o(\lg^3 n/\sqrt{\lg n})$ time. ■

Lemma 34 ([37]) *Let the sequence $A[0..n'-1]$ of distinct elements drawn from $[n]$ denote a point set $N = \{(A[i], i) | 0 \le i \le n'-1\}$, where $n' \le n$. There exists a data structure using $O(n' \lg\lg n)$ bits of extra space constructed over N in $O(n')$ time that answers three-sided next point query in $O(\lg\lg n)$ time and $O(1)$ access to A. The query procedure requires access to a universal table of $o(n)$ bits.*

Proof. We divide N into $n'/\lg^3 n$ blocks, and for each $i \in [0, n'/\lg^3 n - 1]$, the i-th block, N_i, contains points in N whose y-coordinates are in $[i\lg^3 n, (i+1)\lg^3 n - 1]$. Assume for simplicity that n' is divisible by $\lg^3 n$. We linearly scan the points of each block and retrieve the one with maximum x-coordinate from each block. Let $\hat{N}$ denote the selected points and $|\hat{N}| = n'/\lg^3 n$. We apply Lemma 32 to build the data structure $DS(\hat{N})$ over $\hat{N}$ for three-sided next point queries. As shown in Lemma 32, the data structure $DS(\hat{N})$ uses $O(\hat{N}\lg^3 \hat{N}) = O(n')$ bits of space and can be built in $O(\hat{N}\lg^2 \hat{N}) = O(n'/\lg n)$ time.

We apply the general rank reduction technique [34] to reduce the points of each block to the rank space, which can be accomplished by sorting the points once with respect to each of x- and y-coordinate. As there are only polylog(n) points within each block, it is well-known that an atomic heap [18] can be used to sort them in linear time. As each point can be encoded in $O(\lg\lg n)$ bits after rank reduction, we take linear time to store the x- and y- coordinates of points of each block N_i in packed sequences $X'(N_i)$ and $Y'(N_i)$, respectively. Note that the y-coordinates of points in $Y'(N_i)$ denote the in-block indexes. Afterwards, we build data structure $TS(N_i)$ over $X'(N_i)$ and $Y'(N_i)$ of each block N_i by Lemma 33 in $o(\lg^3 n/\sqrt{\lg n})$ time for three-sided next point query within a block. As each point in N has a distinct y-coordinate represented by its index j in $A[0..n'-1]$, we can use the block index i and in-block index i' to compute j, i.e., $j = i \times \lg^3 n + i'$, and then apply $\texttt{point}(v, j)$ to retrieve the original x-coordinate of that point. Thus, we do not need to store the coordinates of points in N to save space.

As defined above, let Q denote the query range $[a, +\infty] \times [c, d]$. When a query happens upon some block N_i, the query range $[a, +\infty]$ along x-axis need to be reduced to $[\hat{a}, +\infty]$ in rank space. Before discarding the x-coordinates of points in N_i, we sort them in linear time using an atomic heap [18]. Let $S(N_i)$ denote the sequence storing all sorted x-coordinates. If $S(N_i)$ is available at the querying procedure, we can apply $\texttt{succ}(a)$ over $S(N_i)$ to find $\hat{a}$. However, storing $S(N_i)$ will overflow the total space usage. Instead, all the points of N_i are still sorted by x-coordinate and each point e after sorting is specified by its in-block index i' using $O(\lg\lg n)$ bits of space. As all x-coordinates are distinct in N, we can use Lemma 16 to build the predecessor/successor data structure $PS(N_i)$ of $O(\lg^3 n \lg\lg n)$ bits in linear time over $S(N_i)$. Afterwards $S(N_i)$ can be discarded. The $\texttt{succ}(a)$ query can be retrieved in $O(\lg\lg n)$ time and $O(1)$ calls to $\texttt{point}$ without storing the sequence $S(N_i)$. For all $n'/\lg^3 n$ blocks, the data structure can be constructed in $O(n' + n'/\lg n + (n'/\lg^3 n) \times \lg^3 n + (n'/\lg^3 n) \times o(\lg^3 n/\sqrt{\lg n}) + (n'/\lg^3 n) \times \lg^3 n) = O(n')$ time. ■

More interestingly, when the x- and y-coordinates of the points are stored in the packed form, we can solve the three-side queries with a data structure built in sublinear time. Our method requires a fast sorting algorithm for performing rank reduction over a small set of points. When a sequence of n' integers from $[\sigma]$ is bit packed into $O(n' \lg\sigma/\lg n)$ words, it can be sorted using a bit-packed version of mergesort:

Lemma 35 ([1]) *A packed sequence $A[0..n'-1]$ from alphabet $[\sigma]$, where $max(\sigma, n') \le n$, can be sorted in $O(n' \lg n' \lg\sigma/\lg n)$ time with the help of a universal tables of $o(n)$ bits.*

Note one difference between Lemmas 34 and 36: Lemma 36 allows multiple points with the same x-coordinate.

Lemma 36 *Let N be a set of n' points with distinct y-coordinates in a $2^{\sqrt{\lg n}} \times n'$ grid, where $n' = O(2^{c\sqrt{\lg n}})$ for any constant integer c. Given packed sequences X and Y respectively encoding the x- and y-coordinates of these points where $Y[i] = i$ for any $i \in [0, n'-1]$, a data structure using $O(n' \lg\lg n)$ bits of extra space constructed over N in $O(n' \lg\lg n/\sqrt{\lg n})$ time that answers three-sided next point query in $O(\lg\lg n)$ time and $O(1)$ access to A. The query procedure requires access to a universal table of $o(n)$ bits.*

Proof. As shown in the proof of Lemma 34, the linear construction time is bounded by the rank reduction operation and building the predecessor/successor index data structure upon sorted x-coordinates of points for each block. As the x- and y-coordinates of each point are encoded with $O(\sqrt{\lg n})$ bits and the coordinates of points are stored in packed sequences, we can sort a block of $\lg^3 n$ points in $O((\lg^3 n) \times (\sqrt{\lg n}/\lg n) \times \lg\lg n)$ time by applying Lemma 35. Meanwhile, the predecessor and successor data structure for each block N_i can be constructed in $O(\lg^3 n/\sqrt{\lg n})$ time by applying Lemma 15. Overall, the whole data structure over N can be built in $O(n' \lg\lg n/\sqrt{\lg n})$ time. ■

6.2 Fast Construction of the Orthogonal Range Successor Structures

Now we consider the solution of the orthogonal range successor problem with optimal time. We describe our solution first for a small narrow grid of size $\lg^{1/4} \times n'$, then for a medium narrow grid of size $2^{\sqrt{\lg n}} \times n'$, and finally for an $n \times n$ grid. Every step in our construction relies on the previous one.

6.2.1 Orthogonal Range Successor Queries in a Small Narrow Grid

First, we consider a special case such that the number of points is less than $\lg n$.

Lemma 37 *Let N be a set of n' points with distinct y-coordinates in a $\lg^{1/4} n \times n'$ grid where $n' < \lg n$. Given packed sequences X and Y respectively encoding the x- and y-coordinates of these points where $Y[i] = i$ for any $i \in [0, n'-1]$, a data structure of $O(n' \lg\lg n)$ bits can be built in $o(n'/\sqrt{\lg n})$ time over N to answer orthogonal range successor query in $O(1)$ time. The query procedure requires access to a universal table of $o(n)$ bits.*

Proof. When $n' \leq \lg^{3/4} n$, we can apply a universal table U of $o(n)$ bits to retrieve in constant time the point with the smallest y-coordinate in the query range. U has an entry for each possible set $(\alpha, \beta, \gamma, a', b', c', d')$, where α (or β, respectively) is a packed sequence of length at most $\lg^{3/4} n$ drawn from $[\lg^{1/4} n]$ (or $[\lg^{3/4} n]$, respectively) denoting the x-coordinate (or y-coordinate, respectively), γ is an integer in $[0..(\lg^{3/4} n) - 1]$ denoting the number of points, and a', b', c', d' each is an integer in $[0, (\lg n) - 1]$ and all together denotes the query range. This entry stores the point with the smallest y-coordinate in the point set denoted by α and β. As U has $O(2^{(\lg^{3/4} n) \times (\lg\lg n)} \times \lg^{3/4} n \times \lg^4 n)$ entries and each entry stores a point of at most $\lg\lg n$ bits, U uses $o(n)$ bits of space.

Assume for simplicity that n' is divisible by $\lg^{3/4} n$. We divide N into $n'/\lg^{3/4} n$ subsets, and for each $i \in [0, n'/\lg^{3/4} n - 1]$, the i-th subset, N_i, contains points in N whose y-coordinates are in $[i \lg^{3/4} n, ((i+1)\lg^{3/4} n) - 1]$. The division of N into N_i can be implemented in $O(n'/\lg^{3/4} n)$ time using bitwise operations. We also define a point set $\hat{N}$ in a $\lg^{1/4} n \times n'$ grid. For each set N_i where $i \in [0, n'/\lg^{3/4} n - 1]$ and each $j \in [0, \lg^{1/4} n - 1]$, if there exists at least one point in N_i whose x-coordinate is j, we store the one with the smallest y-coordinate among them in $\hat{N}$. Thus the number of points in $\hat{N}$ is at most $n'/\lg^{3/4} n \times \lg^{1/4} n = n'/\sqrt{\lg n} < \sqrt{\lg n}$. As each block of points occupies in total $(1/4 \lg\lg n + \lg n') \times \lg^{3/4} n$ bits, creating points for $\hat{N}$ from each block can be implemented in $O(1)$ time with a universal table U' of $o(n)$ bits. U' has an entry for each possible triple (α, β, γ), where α (or β, respectively) is a packed sequence of length at most $\lg^{3/4} n$ drawn from $[\lg^{1/4} n]$ (or $[(\lg n) - 1]$, respectively) denoting the x-coordinates (or y-coordinates, respectively), and γ is an integer $\in [0..(\lg^{3/4} n)]$ denoting the number of points. This entry stores a packed sequence of at most $\lg^{1/4} n$ points for $\hat{N}$ occupying at most $(\lg^{1/4} n) \times (1/4 \lg\lg n + \lg\lg n)$ bits. Similar to the universal table U, U' uses $o(n)$ bits. Therefore, constructing $\hat{N}$ takes $O(n'/\lg^{3/4} n)$ time. Obviously, storing all the points in N and $\hat{N}$ occupies $O(n' \lg\lg n)$ bits of space in total.

Let $Q = [a, b] \times [c, d]$ denote the query range and $N_i, ..., N_j$ denote the blocks intersecting the range $[c, d]$ such that $i = \lfloor c/\lg^{3/4} n \rfloor$ and $j = \lfloor d/\lg^{3/4} n \rfloor$. If $i = j$, then the query range Q is within a single block and we can apply U to retrieve the answer in constant time. Otherwise, we sequentially check $B_i \cap Q$, $(B_{i+1} \cup B_{i+2} \cup ... \cup B_{j-1}) \cap Q$, $B_j \cap Q$, and stop querying once the lowest point is retrieved. The second case can also be answered in constant time by querying over U with the range $\hat{N} \cap [a, b] \times [i \times b + b, j \times b]$. Overall, the query time is $O(1)$. ∎

Next, we consider the orthogonal range successor problem over a larger number of points.

Lemma 38 *Given packed sequence* $X[0..n'-1]$ *drawn from* $[\lg^{1/4} n]$ *denote a point set* $N = \{(A[i], i) | 0 \leq i \leq n'-1\}$, *where* $n' \leq n$, *a data structure of* $O(n' \lg^2 \lg n + w \times \lg^{1/4} n)$ *bits can be built in* $O(n'/\sqrt{\lg n})$ *time over* N *to answer an orthogonal range successor query in* $O(\lg \lg n)$ *time. The query procedure requires access to a universal table of* $o(n)$ *bits.*

Proof. Lemma 37 already achieves this result for $n' < \lg n$, so it suffices to consider the case $n' \geq \lg n$ in the rest of the proof.

We construct a binary wavelet tree T upon $X[0..n'-1]$ by Lemma 2 together with the value array $A(v)$ in packed form at each node v and the bit sequence $S(v)$ if v is an internal node. Recall that $A(v)$ stores the x-coordinates of the ordered list, $N(v)$, of points from N whose x-coordinates are within the range represented by v, and these points are ordered by y-coordinate. The tree T has $\lceil 1/4 \lg \lg n \rceil + 1$ levels and $\lg^{1/4} n$ nodes. Over the sequences associated with each internal node u, we build the following data structures:

- $RK_{ds}(u)$ supports $O(1)$-time `rank` queries over $A(u)$ by Lemma 8;
- $SL_{ds}(u)$ supports $O(1)$-time `select` queries over $A(u)$ by Lemma 11;
- $B_{ds}(u)$ supports $O(1)$-time `rank` queries over $S(u)$ by Lemma 10.

As shown in Lemma 2, T uses $O(n' \lg^2 \lg n + w \times \lg^{1/4} n)$ bits of space and can be constructed in $O(n' \lg^2 \lg n / \lg n + \lg^{1/4} n) = o(n'/\sqrt{\lg n})$ time as $n' \geq \lg n$. Both $RK_{ds}(u)$ and $SL_{ds}(u)$ use $O(|A(u)| \lg \lg n)$ bits of space, while $B_{ds}(u)$ only requires $o(|S(u)|)$ bits of space. As there are $\lceil 1/4 \lg \lg n \rceil$ non-leaf levels in T and n' elements across each level, all data structures $RK_{ds}(u)$, $SL_{ds}(u)$ and $B_{ds}(u)$ use $O(n' \lg^2 \lg n)$ bits. Constructing $RK_{ds}(u)$ takes $O(|A(u)| \lg \lg n / \lg n + 1)$ time, $SL_{ds}(u)$ uses $O(|A(u)| \lg^2 \lg n / \lg n + \lg^{1/4} n)$ time to build, and it takes $O(|S(u)| / \lg n + 1)$ time to build $B_{ds}(u)$. As T has less than $\lg^{1/4} n$ internal nodes, the overall construction time for these data structures is $\sum_u (O(|A(u)| \lg \lg n / \lg n + 1) + O(|A(u)| \lg^2 \lg n / \lg n + \lg^{1/4} n) + O(|S(u)| / \lg n + 1)) = O(n' \lg^3 \lg n / \lg n + \sqrt{\lg n}) = O(n'/\sqrt{\lg n})$ as $n' \geq \lg n$. Therefore, this data structure requires $O(n' \lg^2 \lg n + w \times \lg^{1/4} n)$ bits of space and takes $O(n'/\sqrt{\lg n})$ time to construct. With $RK_{ds}(u)$ and $SL_{ds}(u)$, we can implement the operation $\texttt{point}(u, i)$ in constant time. Let r denote the root node and we have,

$$\texttt{point}(u, i) = (A(u)[i], \texttt{select}_{A(u)[i]}(A(r), \texttt{rank}_{A(u)[i]}(A(u), i)))$$

Given a query range $Q = [a, b] \times [c, d]$, we first locate the lowest common ancestor v of l_a and l_b in constant time, where l_a and l_b denote the a-th and b-th leftmost leaves of T, respectively. Let π_a and π_b denote the paths from v to the a-th leaf and from a to the b-th leaf respectively. For each node u on π_a we mark the right child of u if it exists and is not on the path π_a. For each node u on π_b we mark the left child of u if it exists and is not on the path π_b. In addition, we mark the a-th and b-th leaves. The points on the marked node have the x-coordinate in the range $[a, b]$. As the height of T is $O(\lg \lg n)$, there are in total $O(\lg \lg n)$ marked nodes.

The points at all marked nodes within the query range Q can be identified in total $O(\lg \lg n)$ time. Let $[c_v, d_v]$ denote the range such that $I(v)[c_v..d_v]$ within $[c, d]$. Recall that $I(v)$ is the index array that is not explicitly stored in our data structure. Clearly, the range $[c_v, d_v]$ can be retrieved by answering $\texttt{rank}$ query over $S(u)$ where u is the parent of v, i.e., $[c_v, d_v] = [\texttt{rank}_0(S(u), c_u), \texttt{rank}_0(S(u), d_u)]$ if v is the left child of u. Otherwise, $[c_v, d_v] = [\texttt{rank}_1(S(u), c_u), \texttt{rank}_1(S(u), d_u)]$. As we move down the path from the root node to the a-th leaf (b-th leaf, respectively), we answer $\texttt{rank}$ queries at the visited nodes. If a marked node v is identified, we can find the index range $[c_v, d_v]$ by $\texttt{rank}$ queries over the bit sequence $S(u)$ where u is the parent of v. Obviously, within each marked node v the point represented by $(A(v)[c_v], I(v)[c_v])$ carries the "locally" smallest y-coordinate in Q, where

$$I(v)[c_v] = \texttt{select}_{A(v)[c_v]}(A(r), \texttt{rank}_{A(v)[c_v]}(A(v), c_v))$$

Therefore, the lowest point in Q can be retrieved by comparing the $O(\lg \lg n)$ locally lowest points at all marked nodes. Overall, the query time is $O(\lg \lg n)$. ■

6.2.2 Orthogonal Range Successor Queries in a Medium Narrow Grid

Our solution for points in a $2^{\sqrt{\lg n}} \times n'$ grid for any $2^{\sqrt{\lg n}} \leq n' \leq n$ uses the following previous result:

Lemma 39 ([37, Theorem 3.3]) *There exists a data structure of $O(n \lg n \lg\lg n)$ bits constructed upon a set of n points in rank space in $O(n \lg n)$ time that answers orthogonal range successor queries in $O(\lg\lg n)$ time.*

The following lemma presents our solution for a medium narrow grid.

Lemma 40 *Let N be a set of n' points with distinct y-coordinates in a $2^{\sqrt{\lg n}} \times n'$ grid where $2^{\sqrt{\lg n}} \le n' \le 2^{2\sqrt{\lg n}}$. Given packed sequences X and Y respectively encoding the x- and y-coordinates of these points where $Y[i] = i$ for any $i \in [0, n'-1]$, a data structure of $O(n'\sqrt{\lg n}\lg\lg n + w \times 2^{\sqrt{\lg n}})$ bits can be built over N in $O(n' + 2^{\sqrt{\lg n}} \times \sqrt{\lg n}/\lg\lg n)$ time to answer an orthogonal range successor query in $O(\lg\lg n)$ time. The query procedure requires access to a universal table of $o(n)$ bits.*

Proof. We build a $\lg^{1/4}$-ary wavelet tree T upon $X[0, n'-1]$ and $Y[0, n'-1]$ with support for ball inheritance using Lemma 23. Recall that each node u of T is associated with (but does not explicitly store) the value array $A(u)$ and the index array $I(u)$, in which $A(u)$ and $I(u)$ store the x- and y-coordinates of the ordered list, $N(u)$, of points from N whose x-coordinates are within the range represented by u, and these points are ordered by y-coordinate. Furthermore, u is associated with another sequence $S(u)$ drawn from alphabet $[\lg^{1/4} n]$, in which $S(u)[i]$ encodes the rank of the child of u that contains $N(u)[i]$ in its ordered list. Let $\hat{S}(u)$ denote the point set $\{(S(u)[i], i) | 0 \le i \le |S(u)| - 1\}$, and we use Lemma 38 to build a structure $RS_{ds}(u)$ supporting orthogonal range successor queries over $\hat{S}(u)$. Let $\hat{N}(u)$ denote the point set $\hat{N}(u) = \{(A(u)[i], i) | 0 \le i \le |A(u)| - 1\}$, and we use Lemma 36 to build a structure $TS_{ds}(u)$ supporting three sided next point queries over $\hat{N}(u)$. Note that as shown in Lemma 23, both $\texttt{point}(v, i)$ and $\texttt{noderange}(c, d, v)$ can be answered in $O(\lg\lg n)$ time on T.

Given a query range $Q = [a, b] \times [c, d]$, we first locate the lowest common ancestor v of l_a and l_b in constant time, where l_a and l_b denote the a-th and b-th leftmost leaves of T, respectively. Let v_i denote the i-th child of v, for any $i \in [0, \lg^{1/4} n - 1]$. We first locate two children, $v_{a'}$ and $v_{b'}$, of v that are ancestors of l_a and l_b, respectively. They can be found in constant time by simple arithmetic as each child of v represents a range of equal size. Then the answer, $Q \cap N$, to the query can be reduced to retrieving the lowest point among three point sets $A_1 = Q \cap N(v_{a'})$, $A_2 = Q \cap (N(v_{a'+1}) \cup N(v_{a'+2}) \cup$

$\ldots N(v_{b'-1}))$ and $A_3 = Q \cap N(v_{b'})$. To find the lowest point in A_1, we need only retrieve the point p'_1 with the smallest y-coordinate in $[a, +\infty] \times [c_{v_{a'}}, d_{v_{a'}}]$ where $[c_{v_{a'}}, d_{v_{a'}}] =$ $\texttt{noderange}(c, d, v_{a'})$ and then use $\texttt{point}(v_{a'}, p'_1.y)$ to find its original coordinates p_1 in N. The point p'_1 can be found by querying over $TS_{ds}(u)$ in $O(\lg\lg n)$ time using the algorithm shown in the proof of Lemma 36. With $O(\lg\lg n)$-time support for noderange and point, p_1 can be retrieved in $O(\lg\lg n)$ time. Similarly, we can find the lowest point p_3 in A_3 in $O(\lg\lg n)$ time. To compute A_2, observe that any entry, $\hat{S}(v)[i]$, can be obtained by replacing the x-coordinate of point $N(v)[i]$ with the rank of the child whose ordered list contains $N(v)[i]$. Hence, by performing an orthogonal range successor query over $RS_{ds}(v)$ to compute $\hat{S}(v) \cap ([a'+1, b'-1] \times [c_v, d_v])$, where $[c_v, d_v] = \texttt{noderange}(c, d, v)$, we can find in $O(\lg\lg n)$ time the lowest point p'_2 in $\hat{S}(v) \cap ([a'+1, b'-1] \times [c_v, d_v])$. Again, we use point to find its original coordinates p_2 in N. Obviously, the lowest point in $Q \cap N$ is the point with the smallest y-coordinate among p_1, p_2, and p_3. Therefore, the overall query time required is $O(\lg\lg n)$.

Now we analyze the space costs. The wavelet tree T with support for ball inheritance uses $O(n'\sqrt{\lg n}\lg\lg n + w \times 2^{\sqrt{\lg n}})$ bits by Lemma 23. For each internal node u, $RS_{ds}(u)$ over $\hat{S}(u)$ uses $O(|S(u)| \lg^2 \lg n + w \times \lg^{1/4} n)$ bits of space as shown in Lemma 38. This subsumes the cost of storing $TS_{ds}(u)$ over $\hat{N}(u)$, which is $O(|S(u)| \lg\lg n)$ bits. As T has $O(2^{\sqrt{\lg n}}/\lg^{1/4} n)$ internal nodes and $4\sqrt{\lg n}/\lg\lg n$ tree levels, the total cost of storing these structures at all internal nodes is $\sum_u O(|S(u)| \lg^2 \lg n + w \times \lg^{1/4} n) = O(n'\sqrt{\lg n}\lg\lg n + w \times 2^{\sqrt{\lg n}})$ bits of space. Therefore, all the data structures occupy $O(n'\sqrt{\lg n}\lg\lg n + w \times 2^{\sqrt{\lg n}})$ bits of space.

Finally, we analyze the construction time. As shown in Lemma 23, T with support for ball inheritance can be constructed in $O(n' + 2^{\sqrt{\lg n}} \times \sqrt{\lg n}/\lg\lg n)$ time. At each internal node u of T, constructing $TS_{ds}(u)$ requires $O(|A(u)|/\sqrt{\lg n} \times \lg\lg n + 1)$ time using the algorithm in the proof of Lemma 36 and $RS_{ds}(v)$ requires $O(|S(u)|/\sqrt{\lg n} + 1)$ time by Lemma 38. As T has $\sqrt{\lg n}/(1/4\lg\lg n)$ non-leaf levels and $O(2^{\sqrt{\lg n}}/\lg^{1/4} n)$ internal nodes, these structures over all internal nodes can be built in $\sum_u O(|A(u)|/\sqrt{\lg n} \times \lg\lg n + 1) = O(n')$ time. Therefore, the overall construction time is $O(n' + 2^{\sqrt{\lg n}} \times \sqrt{\lg n}/\lg\lg n)$. ■

Lemma 41 *Let N be a set of n' points with distinct y-coordinates in a $2^{\sqrt{\lg n}} \times n'$ grid where $2^{\sqrt{\lg n}} \le n' \le n$. Given packed sequences X and Y respectively encoding* *the x- and y-coordinates of these points where $Y[i] = i$ for any $i \in [0, n'-1]$, a data structure of $O(n'\sqrt{\lg n}\lg\lg n + w(n'/2^{\sqrt{\lg n}} + 2^{\sqrt{\lg n}}))$ bits can be built over N in $O(n' + \sqrt{\lg n} \cdot 2^{\sqrt{\lg n}}/\lg\lg n)$ time to answer an orthogonal range successor query in $O(\lg\lg n)$ time. The query procedure requires access to a universal table of $o(n)$ bits.*

Proof. Let b denote $2^{2\sqrt{\lg n}}$. We need only consider the case in which $n' > b$ as Lemma 40 applies otherwise. Assume for simplicity that n' is divisible by b. We divide N into n'/b subsets, and for each $i \in [0, n'/b-1]$, the ith subset, N_i, contains the points in N whose y-coordinates are in $[ib, (i+1)b-1]$. The dividing procedure can be performed in linear time. Let p be a point in N_i. We call its coordinates $(p.x, p.y)$ *global coordinates*, while $(p.x', p.y') = (p.x, p.y \texttt{ mod } b)$ its *local coordinates* in N_i; the conversion between global and local coordinates can be done in constant time. Hence the points in N_i with their local coordinates can be viewed as a point set in a $2^{\sqrt{\lg n}} \times 2^{2\sqrt{\lg n}}$ grid, and we apply Lemma 40 to construct an orthogonal range search structure $RS(N_i)$ over N_i. We also define a point set $\hat{N}$ in a $2^{\sqrt{\lg n}} \times n'$ grid. For each set N_i where $i \in [0, n'/b-1]$ and each $j \in [0, 2^{\sqrt{\lg n}}-1]$, if there exists at least one point in N_i whose x-coordinate is j, we store the one among them with the smallest y-coordinate in $\hat{N}$. Thus the number of points in $\hat{N}$ is at most $2^{\sqrt{\lg n}} \times n'/b = n'/2^{\sqrt{\lg n}}$. Finally, we build the data structure $\hat{RS}_{ds}$ for orthogonal range successor over $\hat{N}$ by Lemma 39.

Given a query range $Q = [x_1, x_2] \times [y_1, y_2]$, we first check if $\lfloor y_1/b \rfloor$ is equal to $\lfloor y_2/b \rfloor$. If it is, then the points in the answer to the query reside in the same subset $N_{\lfloor y_1/b \rfloor}$, and we can retrieve the lowest point e by performing an orthogonal range successor query in $N_{\lfloor y_1/b \rfloor} \cap Q$, which requires $O(\lg\lg n)$ time by Lemma 40. Then we retrieve its original coordinates in N, which is $(e.x, b\lfloor y_1/b \rfloor + e.y)$. Otherwise, let $N_s, \ldots, N_e$ denote the blocks interacting $[y_1, y_2]$, where $s = \lfloor y_1/b \rfloor$ and $e = \lfloor y_2/b \rfloor$. We sequentially look for the lowest point in $A_1 = N_s \cap [x_1, x_2] \times [y_1 \mod b, +\infty]$, $A_2 = (N_{s+1} \cup \cdots \cup N_{e-1}) \cap [x_1, x_2] \times [-\infty, +\infty]$, and $A_3 = N_e \cap [x_1, x_2] \times [0, y_2 \mod b]$. Once a point e is returned, we retrieve the original coordinates of e in N and terminate the the query procedure. The lowest point in A_1 or A_3 can be answered in $O(\lg\lg n)$ time by Lemma 40. It remains to find the lowest point in A_2, which can be implemented by querying in $O(\lg\lg n)$ time over $\hat{N}$ for the lowest point in Q using

Lemma 39. Overall, the query procedure requires $O(\lg\lg n)$ time.

To bound the storage costs, by Lemma 40, the orthogonal range successor structure over each N_i uses $O(2^{2\sqrt{\lg n}}\lg\lg n + w\cdot 2^{\sqrt{\lg n}})$ bits. Thus, the orthogonal range successor structures over $N_0, N_1, \ldots, N_{n'/b-1}$ occupy $O((n'/b)\times(2^{2\sqrt{\lg n}}\sqrt{\lg n}\lg\lg n + w\cdot 2^{\sqrt{\lg n}})) = O(n'\sqrt{\lg n}\lg\lg n + n'w/2^{\sqrt{\lg n}})$ bits. As there are at most $n'/2^{\sqrt{\lg n}}$ points in $\hat{N}$, by Lemma 39, the range successor structure for $\hat{N}$ occupies $O(n'\lg\lg n\lg n/2^{\sqrt{\lg n}}) = o(n')$ bits. Thus the space costs of all structures add up to $O(n'\sqrt{\lg n}\lg\lg n + n'w/2^{\sqrt{\lg n}})$ bits. Note that the above analysis assumes $n' > b$. Otherwise, the data structure uses $O(n'\sqrt{\lg n}\lg\lg n + w2^{\sqrt{\lg n}})$ bits, so we use $O(n'\sqrt{\lg n}\lg\lg n + w(n'/2^{\sqrt{\lg n}} + 2^{\sqrt{\lg n}}))$ bits as the space bound in both cases.

Regarding construction time, observe that the point sets $N_0, N_1, \ldots, N_{n'/b-1}$ and $\hat{N}$, can be computed in $O(n')$ time. By Lemma 39, The range successor structure for $\hat{N}$ can be built in $O(n'/b\times\lg n') = o(n')$ time. Finally, the total construction time of the range successor structures for $N_0, N_1, \ldots, N_{n/b-1}$ is $O(n'/2^{2\sqrt{\lg n}}\times(2^{2\sqrt{\lg n}} + \sqrt{\lg n}\times 2^{\sqrt{\lg n}}/\lg\lg n)) = O(n')$, which dominates the total preprocessing time of all our data structures. When $n' \leq b$, the construction time is $O(n' + \sqrt{\lg n}\cdot 2^{\sqrt{\lg n}}/\lg\lg n)$ by Lemma 40, so we use $O(n' + \sqrt{\lg n}\cdot 2^{\sqrt{\lg n}}/\lg\lg n)$ as the upper bound on construction time in both cases. ■

6.2.3 Orthogonal Range Successor Queries in an $n \times n$ Grid

The following theorem presents our result on fast construction of structures for optimal range successor.

Theorem 2 *Given n points in rank space, a data structure of $O(n\lg\lg n)$ words can be constructed in $O(n\sqrt{\lg n})$ time to support orthogonal range successor in $O(\lg\lg n)$ time. The query procedure requires access to a universal table of $o(n)$ bits.*

Proof. Let the sequence $X[0, n-1]$ denote the point set $N = \{(X[i], i) | 0 \leq i \leq n-1\}$. We build a $2^{\sqrt{\lg n}}$-ary wavelet tree T upon $X[0, n-1]$ with support for ball inheritance using part (a) of Lemma 22. Recall that each node u of T is associated with the value array $A(u)$ and the index array $I(u)$ (these arrays are not stored explicitly); $A(u)$ and $I(u)$ contain the x- and y-coordinates of $N(u)$, where $N(u)$ is the list of points from N whose x-coordinates are within the range of u, and points in $N(u)$ are ordered by their y-coordinates. Furthermore, u is associated with another sequence $S(u)$ drawn from alphabet $[2^{\sqrt{\lg n}}]$, in which $S(u)[i]$ encodes the rank of the child of u that contains $N(u)[i]$ in its ordered list. Let $\hat{S}(u)$ denote the point set $\{(S(u)[i], i) | 0 \leq i \leq |S(u)| - 1\}$, and we use Lemma 41 to build a structure $RS_{ds}(u)$ supporting orthogonal range successor queries over $\hat{S}(u)$. Let $\hat{N}(u)$ denote the point set $\hat{N}(u) = \{(A(u)[i], i) | 0 \leq i \leq |A(u)| - 1\}$, and we use Lemma 34 to build a structure $TS_{ds}(u)$ supporting three sided next point queries over $\hat{N}(u)$. The query procedure is exactly the same as in the proof of Lemma 40 and requires $O(\lg \lg n)$ time.

Now we analyze the space usage. The wavelet tree T with support for ball inheritance uses $O(n \lg n \lg \lg n)$ bits. For each internal node u, since $w = \Theta(\lg n)$, $RS_{ds}(u)$ over $\hat{S}(u)$ uses space in bits:

$$O(|S(u)|\sqrt{\lg n} \lg \lg n + (|S(u)|/2^{\sqrt{\lg n}} + 2^{\sqrt{\lg n}}) \lg n) = O(|S(u)|\sqrt{\lg n} \lg \lg n + 2^{\sqrt{\lg n}} \lg n)$$

This subsumes the cost of storing $TS_{ds}(u)$ over $\hat{N}(u)$, which is $O(|S(u)| \lg \lg n)$ bits. As T has $O(n/2^{\sqrt{\lg n}})$ internal nodes, the total cost of storing these structures at all internal nodes is $\sum_u O(|S(u)|\sqrt{\lg n} \lg \lg n + \lg n \times 2^{\sqrt{\lg n}}) = O(n \lg n \lg \lg n + n \lg n) = O(n \lg n \lg \lg n)$ bits of space. Therefore, all the data structures occupy $O(n \lg n \lg \lg n)$ bits of space.

Finally, we analyze the construction time. As shown in part (a) of Lemma 22, the tree T with ball inheritance structures can be constructed in $O(n\sqrt{\lg n})$ time. For each internal node u of T, $TS_{ds}(u)$ can be constructed in linear time and $RS_{ds}(v)$ can be constructed in $O(|S(v)| + 2^{\sqrt{\lg n}} \times \sqrt{\lg n}/\lg \lg n)$ time. As T has $O(n/2^{\sqrt{\lg n}})$ internal nodes, these structures over all internal nodes can be built in $\sum_u O(|S(v)| + 2^{\sqrt{\lg n}} \times \sqrt{\lg n}/\lg \lg n) = O(n\sqrt{\lg n} + n\sqrt{\lg n}/\lg \lg n) = O(n\sqrt{\lg n})$ time. The preprocessing time of all data structures is thus $O(n\sqrt{\lg n})$. ■

Chapter 7

Optimal Orthogonal Sorted Range Reporting with Fast Preprocessing

In this section we study the orthogonal sorted range reporting problem over n points in 2d rank space. In our methods for three-sided sorted reporting and orthogonal sorted range reporting problems, we adopt the same strategy as shown in Chapter 6 which is to reduce a big point set N into blocks of small point sets and sample several special points from each block. Both three-sided sorted reporting and orthogonal sorted range reporting queries over a block will take $O(\lg\lg n + \mathtt{occ})$ time. However, the points in the query range are possibly distributed among different blocks. As the $\lg\lg n$-term might subsumes the number of reported points from some block, we can not afford the $\lg\lg n$-term in the query time unless there are at least $\lg\lg n$ points reported from that block. In this way, the $\lg\lg n$-term can be dismissed.

For the three-sided next point problem with the query range $[a, +\infty] \times [c, d]$, we sample the point with the maximum x-coordinate of each block. Then for its counterpart problem three-sided sorted reporting, we need to sample $\lg\lg n$ points with largest x-coordinates from each block. Similarly, for the orthogonal range successor problem, we sample the points from each block with the smallest y-coordinate for each distinct x-coordinate. Then for its counterpart problem, we need to sample the points from each block with $\lg\lg n$ smallest y-coordinate for each distinct x-coordinates. This sample strategy makes sure that if the number of points reported from the sampled point set that belongs to the same block B is less than $\lg\lg n$, all the points in $B \cap Q$ have been reported from the query over the sampled point set, where Q denote the query range. Otherwise, there are at least $\lg\lg n$ points in $B \cap Q$, and we can afford to query over the data structure built upon B.

Given the same query range, an answer to the orthogonal range successor is always the first point reported among the reported points from the orthogonal sorted range reporting query. Our methods between the orthogonal range successor and orthogonal sorted range reporting are almost the same, apart from the sampling strategy described above. In addition, our data structures can work in an online fashion: points within the query range Q are reported in ascending order of x- or y-coordinates until the query procedure is terminated or all the points in Q are reported.

7.1 Fast Construction of the Three-Sided Sorted Reporting Structures

Now, we show how to efficiently construct data structures for three-sided sorted reporting. Let N be a set of n points in 2d rank space. Given a query range $Q = [a, +\infty] \times [c, d]$, we define three-sided sorted reporting query to be the problem of reporting points in $N \cap Q$ in increasing order of y-coordinates. The methods to be shown in Lemmas 44 and 45 are under the indexing model. The following previous results will be adopted in our method:

Lemma 42 ([32, Lemma 5]) *There exists a data structure of $O(n \lg^3 n)$-bit space constructed upon a set of n points in 2d rank space in $O(n \lg^2 n)$ time, which supports three-sided reporting query in $O(\lg \lg n + \mathsf{occ})$ time, where occ denotes the number of reported points.*

Lemma 43 ([32]) *Let N be a set of $\lg^3 n$ points in rank space. Given packed sequences X and Y respectively encoding the x- and y-coordinates of these points where $Y[i] = i$ for any $i \in [0, \lg^3 n - 1]$, a data structure using $O(\lg^3 n \lg \lg n)$ bits of space constructed over N in $o(\lg^3 n/\sqrt{\lg n})$ time that answers the three-sided sorted reporting query in $O(\lg \lg n + \mathsf{occ})$ time. The query algorithm requires access to a universal table of $o(n)$ bits.*

Proof. The proof is similar to the one shown in Lemma 33. We construct almost the same data structure, apart from that upon the sampled point $\hat{N}$ from each block, we build data structure $DS(\hat{N})$ for three-sided sorted reporting by Lemma 42. Let $Q = [a, +\infty] \times [c, d]$ denote the query range, and N_s and N_e denote the blocks containing c and d, where $s = \lfloor c/ \lg^{3/4} n \rfloor$ and $e = \lfloor d/ \lg^{3/4} n \rfloor$. If e is equal to s, then the points in the answer to the query reside in the same subset N_s, and we can retrieve the target points in constant time by performing lookups with a universal table U

of $o(n)$ bits. U has an entry for each possible set $(\alpha, \beta, \gamma, a', b', c', d')$, where α or β is a packed sequence of length at most $\lg^{3/4} n$ drawn from $[\lg^3 n]$ denoting the x- or y-coordinates of the points, γ is an integer $\in [0..(\lg^{3/4} n) - 1]$ denoting the number of points, and a', b', c', d' each is an integer $\in [0..\lg^3 n - 1]$ such that all a', b', c', d' together denote the query range. This entry stores a sorted point set of γ points in the range $[a', b'] \times [c', d']$. As U has $O(2^{(\lg^{3/4} n) \times (6 \lg \lg n)} \times \lg^{3\times 4} n \times \lg^{3/4} n)$ entries and each entry stores a point set of at most $(6 \lg \lg n) \times \lg^{3/4} n$ bits, U uses $o(n)$ bits of space. If $s < e$, we sequentially check $A_1 = N_s \cap Q$, $A_2 = (N_{s+1} \cup \cdots \cup N_{e-1}) \cap [a, +\infty] \times [-\infty, +\infty]$, and $A_3 = N_e \cap Q$, and report points in increasing order of y-coordinates in each of the three cases. Among them, points in A_1 and A_3 can be reported in constant time by performing lookups with U. It remains to report point in A_2. We query over $DS(\hat{N})$ in $O(\lg \lg n + \mathsf{occ}')$ time and retrieve all the blocks that each contains at least one point in Q, where occ' is the number of blocks reported and $\mathsf{occ}' \leq \mathsf{occ}$. For each reported block B, we report points by performing lookups with U. Overall, the query time is $O(\lg \lg n + \mathsf{occ})$. ■

Lemma 44 ([32]) *Let the sequence $A[0..n' - 1]$ of distinct elements drawn from $[n]$ denote a point set $N = \{(A[i], i) | 0 \leq i \leq n' - 1\}$, where $n' \leq n$. There exists a data structure using $O(n' \lg \lg n)$ bits of extra space constructed over N in $O(n')$ time that answers a three-sided sorted reporting query in $O(\lg \lg n + t \times \mathsf{occ})$ time, given that reporting the x/y-coordinate of a certain point of A takes $O(t)$ time after the construction of the data structure. The query procedure requires access to a universal table of $o(n)$ bits.*

Proof. The proof is similar to the one shown in Lemma 34. We divide the points along y-axis of N into blocks of length $\lg^3 n$ each. Within each block, we retrieve the $\lceil \lg \lg n \rceil$ points with largest x-coordinates into the point set $\hat{N}$. The capacity of $\hat{N}$ is $\lceil n' / \lg^3 n \rceil \times \lceil \lg \lg n \rceil$. We build in $O(|\hat{N}| \lg^2 |\hat{N}|) = O(n' \lg^2 n' / \lg^3 n \times \lg \lg n) = o(n')$ time the data structure $DS(\hat{N})$ of $O(|\hat{N}| \lg^3 |\hat{N}|) = o(n')$ bits for three-sided sorted reporting by Lemma 42. Over each block N_i of points in rank space, we build in $o(\lg^3 n / \sqrt{\lg n})$ time the data structure $TS(N_i)$ of $O(\lg^3 n \lg \lg n)$ bits of space for three-sided reporting by Lemma 43. The remaining data structures to be built are all the same as the proof of Lemma 34.

Let $Q = [a, +\infty] \times [c, d]$ denote the query range, and N_s and N_e denote the blocks containing c and d, respectively. If s is equal to e, we perform $\texttt{succ}(a)$ over the index data structure of sorted x-coordinates of points from N_s to retrieve $\hat{a}$ in rank space and perform a three-sided sorted reporting query in $N_s \times [\hat{a}, +\infty] \times [c \texttt{ mod } \lg^3 n, d \texttt{ mod } \lg^3 n]$, which requires $O(\lg\lg n + t \cdot \texttt{occ})$ time by Lemma 43. Note that once a point e is reported from a block, we can compute its original y-coordinate by $i \times \lg^3 n + e.y$, where i denotes the block index. Then, we can retrieve its original x-coordinate by the computed y-coordinate. We dismiss the details here, but assume that the original x- and y-coordinates of e can be retrieved in $O(t)$ time. Otherwise, we sequentially report points from $A_1 = N_s \cap [a, +\infty] \times [c \mod \lg^3 n, +\infty]$, $A_2 = (N_{s+1} \cup \cdots \cup N_{e-1}) \cap [a, +\infty] \times [-\infty, +\infty]$, and $A_3 = N_s \cap [a, +\infty] \times [0, d \mod \lg^3 n]$. We first take $O(\lg\lg n + t \cdot \texttt{occ}_1)$ time to report points in A_1 following the similar way as we did when $s = e$, where $\texttt{occ}_1 = |A_1|$. Then, we query over $DS(\hat{N})$ for points in A_2. If there are consecutive $\lceil \lg\lg n \rceil$ points reported from the same block N_i, it means that there are at least $\lceil \lg\lg n \rceil$ points in $N_i \cap Q'$, where $Q' = N_i \cap [a, +\infty] \times [-\infty, +\infty]$ and $s < i < e$. Then we query over the data structure $TS(N_i)$ for points in $N_i \cap Q'$ in $O(\lg\lg n + t \cdot \texttt{occ}_i) = O(t \cdot \texttt{occ}_i)$ time, where $\texttt{occ}_i$ denotes the number of points in $N_i \cap Q'$. If some block N_i contains less than $\lceil \lg\lg n \rceil$ points in $N_i \cap Q$, then all the points in $N_i \cap Q$ are reported when performing queries over $DS(\hat{N})$ and we do not need to check $TS(N_i)$. Thus reporting points in A_2 requires $O(\lg\lg n + t \cdot \texttt{occ}_2)$ time, where $\texttt{occ}_2 = |A_2|$. Finally, we query over $TS(N_e)$ for points in A_3 using $O(\lg\lg n + t \cdot \texttt{occ}_3)$ time. Overall, the points in $N \cap Q$ can be reported in increasing order of y-coordinates in $O(\lg\lg n + t \cdot \texttt{occ})$ time. ■

More interestingly, when the x- and y-coordinates of the points are stored in the packed form, we can solve the three-sided sorted reporting with a data structure built in sublinear time. Here, we allow the point set N to have duplicated x-coordinates.

Lemma 45 *Let N be a set of n' points with distinct y-coordinates in a $2^{\sqrt{\lg n}} \times n'$ grid, where $n' = O(2^{c\sqrt{\lg n}})$ for any constant integer c. Given packed sequences X and Y respectively encoding the x- and y-coordinates of these points where $Y[i] = i$ for any $i \in [0, n'-1]$, a data structure using $O(n' \lg\lg n)$ bits of extra space constructed over N in $O(n'/\sqrt{\lg n} \times \lg\lg n)$ time that answers a three-sided sorted reporting query in $O(\lg\lg n + t \times \texttt{occ})$ time, given that reporting x/y-coordinate of a certain point of A*

takes $O(t)$ time after construction. The query procedure requires access to a universal table of $o(n)$ bits.

Proof. The proof is similar to the one shown in Lemma 36. ■

7.2 Fast Construction of Orthogonal Sorted Range Reporting Structures

Let N be a set of n points in 2d rank space. Given a query range $Q = [a, b] \times [c, d]$, we define the orthogonal sorted range reporting to be the problem of reporting points in $N \cap Q$ in increasing order of y-coordinates. In this subsection, we consider the orthogonal range queries in three different cases: on a $\lg^{1/4} \times n'$ small narrow grid, on a $2^{\sqrt{\lg n}} \times n'$ medium narrow grid, and eventually on an $n \times n$ grid.

7.2.1 Orthogonal Sorted Range Reporting on a Small Narrow Grid

First, we consider a special case such that the number of points is less than $\lg n$.

Lemma 46 *Let N be a set of n' points with distinct y-coordinates in a $\lg^{1/4} n \times n'$ grid, where $n' < \lg n$. Given packed sequences X and Y respectively encoding the x- and y-coordinates of these points where $Y[i] = i$ for any $i \in [0, n'-1]$, a data structure of $O(n' \lg\lg n)$ bits can be built in $o(n'/\sqrt{\lg n})$ time over N to answer orthogonal sorted range reporting in $O(\mathtt{occ})$ time, where $\mathtt{occ}$ is the number of the reported points. The query procedure requires access to a universal table of $o(n)$ bits.*

Proof. The data structure for orthogonal range successor queries shown in Lemma 37 can also be used for sorted range reporting queries. Therefore, we only show the query algorithm. Let $Q = [a, b] \times [c, d]$ denotes the query range, and N_s and N_e denote the block contain c and d, respectively. We sequentially check $A_1 = N_s \cap Q$, $A_2 = (N_{s+1} \cup \cdots \cup N_{e-1}) \cap Q$ and $A_3 = N_e \cap Q$ and report points in increasing order of y-coordinates. Both points in A_1 and A_3 can be reported in constant time with a universal table U of $o(n)$ bits, similar to U in the proof of Lemma 43. As $\hat{N}$ has at most $\sqrt{\lg n}$ of points, querying over $\hat{N}$ can be also achieved by performing lookups with U. To report points in A_2, we query over $\hat{N}$ to find all blocks that contains at least one point in the query range. Then we iterate each reported block from left to right, and report the target points in increasing order of y-coordinates. Overall, the query time is $O(\texttt{occc})$. ■

Next, we give the method for a point set with any number of points. As the data structure show in the proof of Lemma 38 can be used for both orthogonal range successor and sorted range reporting, we only show the query algorithm for orthogonal sorted range reporting in the proof of the following Lemma:

Lemma 47 *Let the packed sequence $A[0..n'-1]$ drawn from $[\lg^{1/4} n]$ denote a point set $N = \{(A[i], i) | 0 \leq i \leq n'-1\}$, where $n' \leq n$. A data structure of $O(n' \lg^2 \lg n + w \times \lg^{1/4} n)$ bits can be built in $O(n'/\sqrt{\lg n})$ time over N to answer orthogonal sorted range reporting in $O(\lg\lg n + \texttt{occ})$ time, where $\texttt{occ}$ is the number of the reported points. The query procedure requires access to a universal table of $o(n)$ bits.*

Proof. Lemma 46 already subsumes this lemma when $n' < \lg n$, so it suffices to assume that $n' \geq \lg n$ in the rest of the proof.

Let $Q = [a, b] \times [c, d]$ denote the query range and π_a/π_b denote the path from $\texttt{lca}(a, b)$ to a/b-th leaf, where $\texttt{lca}(a, b)$ denotes the lowest common ancestor of the a-th and b-th leaves. For each node u on π_a we mark the right child of u if it exists and it does not stay on the path π_a. For each node u on π_b we mark the left child of u if it exists and it does not stay on the path π_b. In addition, we mark the a-th and b-th leaves. The points on the marked node have the x-coordinate in the range $[a, b]$. As the height of T is $O(\lg\lg n)$, there are in total $O(\lg\lg n)$ marked nodes.

The points at all marked nodes within the query range Q can be identified in $O(\lg\lg n)$ time in total. Let $[c_v, d_v]$ denote the range such that $I(v)[c_v..d_v]$ is within $[c, d]$. Recall that $I(v)$ is an index sequence associated with (but not explicitly stored at) each node v. Clearly, the range $[c_v, d_v]$ can be retrieved by operating $\texttt{rank}$ query over the sequence $S(u)$ where u is the parent of v, i.e., $[c_v, d_v] = [\texttt{rank}_0(S(u), c_u), \texttt{rank}_0(S(u), d_u)]$ if v is the left child of u. Otherwise, $[c_v, d_v] = [\texttt{rank}_1(S(u), c_u), \texttt{rank}_1(S(u), d_u)]$. As traversing each internal node v on the path from the root node to the a-th leaf (b-th leaf, respectively), we keep operating $\texttt{rank}$ queries. And if a marked node v is identified, we can find the range $[c_v, d_v]$ by $\texttt{rank}$ queries over $S(u)$ where u is the parent of v.

Note that the points associated with each node of T are increasingly sorted by y-coordinate. Once the marked nodes and the index range at each of them are identified, we can use the $O(\lg\lg n)$-way mergesort algorithm to merge all the points in $N \cap Q$ in increasing order of y-coordinates. However, the merging algorithm takes $O(\lg\lg n \times \mathsf{occ})$ time. To speed up the querying efficiency, we can apply the Q-heap data structure [18] which supports to find the minimum among $O(\lg\lg n)$ elements in constant time. By combining the Q-heap with the $O(\lg\lg n)$-way mergesort algorithm, reporting target points at marked nodes in increasing order of y-coordinates takes $O(\lg\lg n + \mathsf{occ})$ time. Note that the $\lg\lg n$-term spends on filling elements into Q-heap. Overall, the orthogonal sorted range reporting can be answered in $O(\lg\lg n + \lg\lg n + \mathsf{occ}) = O(\lg\lg n + \mathsf{occ})$ time. ■

7.2.2 Orthogonal Sorted Range Reporting on a Medium Narrow Grid

In this subsection, we solve the orthogonal sorted range reporting over a point set N on a $2^{\sqrt{\lg n}} \times n'$ grid with fast processing data structures. We consider two cases depending on whether $2^{\sqrt{\lg n}} \leq n' \leq 2^{2\sqrt{\lg n}}$ or $2^{2\sqrt{\lg n}} < n'$.

Lemma 48 *Let N be a set of n' points with distinct y-coordinates in a $2^{\sqrt{\lg n}} \times n'$ grid, where $2^{\sqrt{\lg n}} \leq n' \leq 2^{2\sqrt{\lg n}}$. Given packed sequences X and Y respectively encoding the x- and y-coordinates of these points where $Y[i] = i$ for any $i \in [0, n'-1]$, a data structure of $O(n'(\lg n)^{(1+\epsilon)/2} + w \times 2^{\sqrt{\lg n}})$ bits can be built over N in $O(n' + 2^{\sqrt{\lg n}} \times \sqrt{\lg n}/\lg\lg n)$ time to answer orthogonal sorted range reporting in $O(\lg\lg n + \mathsf{occ})$ time, where ϵ is any positive constant and occ is the number of the reported points. The query procedure requires access to a universal table of $o(n)$ bits.*

Proof. The proof is similar to the one shown in Lemma 40. Here, we only discuss the different data structures required to build. We build a $\lg^{1/4}$-ary wavelet tree T upon $X[0, n'-1]$ and $Y[0, n'-1]$ with support for ball inheritance using Lemma 27. We construct the data structures over the sequences associated with each internal node u as follows:

- $TS_{ds}(u)$ over $\hat{N}(u)$ by Lemma 45 for three-sided sorted reporting;

- $RS_{ds}(u)$ over $\hat{S}(u)$ by Lemma 47 for orthogonal sorted range reporting;

The overall space usage and the construction time are both dominated by the data structure for ball inheritance, which is $O(n'(\lg n)^{(1+\epsilon)/2} + w \times 2^{\sqrt{\lg n}})$ bits of space and $O(n' + 2^{\sqrt{\lg n}} \times \sqrt{\lg n}/\lg\lg n)$ time, respectively.

Let Q denote the query range $[a, b] \times [c, d]$. Similarly, we retrieve the lowest common ancestor v of the a- and b-th leaf. Let v_s and v_e each denote the child of v on the path from v to the a- and b-th leaf. Note that s and e denote the child indexes, and $s \leq e$. At node v_s and v_e, we query over $TS_{ds}(v_s)$ and $TS_{ds}(v_e)$ for reporting points in $\hat{N}(v_s) \cap [a, +\infty] \times [c_{v_s}, d_{v_s}]$ and $\hat{N}(v_e) \cap [0, b] \times [c_{v_e}, d_{v_e}]$ in increasing order of y-coordinates, respectively, where $[c_{v_s}, d_{v_s}] = \texttt{noderange}(c, d, v_s)$ and $[c_{v_e}, d_{v_e}] = \texttt{noderange}(c, d, v_e)$. At node v, we query over $RS_{ds}(v)$ for reporting points in $\hat{N}(u) \cap [s+1, e-1] \times [c_v, d_v]$ in increasing order of y-coordinates, where $[c_v, d_v] = \texttt{noderange}(c, d, v)$. Finally, we use the 3-way mergesort algorithm to merge three sorted point lists reported into a single sorted list. As $\texttt{point}(v, i)$ takes $O(1)$ time and $\texttt{noderange}(c, d, v)$ takes $O(\lg\lg n)$ time, the overall query time is $O(\lg\lg n + \texttt{occ})$. ■

Next, we provide a method when the capacity of the point set is more than $2^{2\sqrt{\lg n}}$. Our method will use the following result:

Lemma 49 ([32, Theorem 2]) *There exists a data structure of $O(n \lg^{1+\epsilon} n)$ bits constructed upon a set of n points in 2d rank space in $O(n \lg n)$ time that supports sorted range reporting queries in $O(\lg\lg n + \texttt{occ})$ time, where occ is the number of the reported points.*

Lemma 50 *Let N be a set of n' points with distinct y-coordinates in a $2^{\sqrt{\lg n}} \times n'$ grid, where $2^{2\sqrt{\lg n}} < n'$. Given packed sequences X and Y respectively encoding the x- and y-coordinates of these points where $Y[i] = i$ for any $i \in [0, n'-1]$, a data structure of $O(n'(\lg n)^{(1+\epsilon)/2} + w \times n'/2^{\sqrt{\lg n}})$ bits can be built over N in $O(n')$ time to answer orthogonal sorted range reporting in $O(\lg\lg n + \texttt{occ})$ time, where ϵ is any positive constant and $\texttt{occ}$ is the number of reported points. The query procedure requires access to a universal table of $o(n)$ bits.*

Proof. The proof is similar to the one shown in Lemma 41. Let b denote $2^{2\sqrt{\lg n}}$. We divide N into n'/b subsets, and for each $i \in [0, n'/b-1]$, the i-th subset, N_i, contains points in N whose y-coordinates are in $[ib, (i+1)b-1]$. Let p be a point in N_i. We call its coordinates $(p.x, p.y)$ *global coordinates*, while $(p.x', p.y') = (p.x, p.y \texttt{ mod } b)$ its *local coordinates* in N_i, where $p.x' = p.x$; the conversion between global and local coordinates can be done in constant time. Hence the points in N_i with their local coordinates can be viewed as a point set in a $2^{\sqrt{\lg n}} \times 2^{2\sqrt{\lg n}}$ grid, and we apply Lemma 48 to construct an orthogonal sorted range reporting structure $RS(N_i)$ over N_i. We also define a point set $\hat{N}$ in a $2^{\sqrt{\lg n}} \times n'$ grid. For each set N_i where $i \in [0, n'/b-1]$ and each $j \in [0, 2^{\sqrt{\lg n}}-1]$, if there exists $\geq \lceil \lg\lg n \rceil$ points in N_i whose x-coordinate is j, we store $\lceil \lg\lg n \rceil$ points among them with smallest y-coordinates into $\hat{N}$. Otherwise, we store all the points in N_i whose x-coordinate is j into $\hat{N}$. Thus the number of points in $\hat{N}$ is at most $\lceil \lg\lg n \rceil \times 2^{\sqrt{\lg n}} \times n'/b = \lceil \lg\lg n \rceil n'/2^{\sqrt{\lg n}}$. Finally, we build the data structure $\hat{RS}_{ds}(\hat{N})$ for orthogonal sorted range reporting over $\hat{N}$ by Lemma 49.

Given a query range $Q = [x_1, x_2] \times [y_1, y_2]$, we first check if $\lfloor y_1/b \rfloor$ is equal to $\lfloor y_2/b \rfloor$. If it is, then the points in the answer to the query reside in the same subset $N_{\lfloor y_1/b \rfloor}$, and we can report points in $N_{\lfloor y_1/b \rfloor} \cap Q$ in increasing order of y-coordinates by querying over $RS(N_{\lfloor y_1/b \rfloor})$, which requires $O(\lg\lg n + \texttt{occ})$ time by Lemma 48. Otherwise, let $N_s, \ldots, N_e$ denote the blocks intersecting $[y_1, y_2]$, where $s = \lfloor y_1/b \rfloor$ and $e = \lfloor y_2/b \rfloor$. We sequentially look for points in $A_1 = N_s \cap [x_1, x_2] \times [y_1 \texttt{ mod } b, +\infty]$, $A_2 = (N_{s+1} \cup \cdots \cup N_{e-1}) \cap [x_1, x_2] \times [-\infty, +\infty]$, and $A_3 = N_e \cap [x_1, x_2] \times [0, y_2 \texttt{ mod } b]$. Both the cases A_1 and A_3 can be answered in $O(\lg\lg n + \texttt{occ}_1)$ and $O(\lg\lg n + \texttt{occ}_3)$ time by Lemma 48, where $\texttt{occ}_1 = |A_1|$ and $\texttt{occ}_2 = |A_2|$, respectively. It remains to show how to report points in A_2. We query over $\hat{RS}_{ds}(\hat{N})$ to report points in $Q \cap \hat{N}$ by Lemma 49. If consecutive $\lceil \lg\lg n \rceil$ points reported by querying over $\hat{RS}_{ds}(\hat{N})$ are from a same bock N_i, then we immediately query over $RS_{ds}(N_i)$ for the remaining points in N_i, where $s < i < e$. It requires $O(\lg\lg n + \texttt{occ}_i) = O(\texttt{occ}_i)$ time, as $\lceil \lg\lg n \rceil \leq \texttt{occ}_i$, where $\texttt{occ}_i$ is the number of points in $N_i \cap Q$. Otherwise, if some block N_i contains less than $\lceil \lg\lg n \rceil$ points in $N_i \cap Q$, then all the points in $N_i \cap Q$ has been reported by querying over $\hat{RS}_{ds}(\hat{N})$. Overall, the query procedure requires $O(\lg\lg n + \texttt{occ})$ time.

To bound the storage costs, by Lemma 48, the orthogonal range reporting structure over each N_i uses $O(2^{2\sqrt{\lg n}} \lg^{1/2+\epsilon} n + w \cdot 2^{\sqrt{\lg n}})$ bits. Thus, the range reporting structures over $N_0, N_1, \ldots, N_{n/b-1}$ occupy $O((n'/b) \times (2^{2\sqrt{\lg n}} \lg^{1/2+\epsilon} n + w \cdot 2^{\sqrt{\lg n}})) = O(n' \lg^{1/2+\epsilon} n + n'w/2^{\sqrt{\lg n}})$ bits. As there are at most $\lceil \lg\lg n \rceil n'/2^{\sqrt{\lg n}}$ points in $\hat{N}$, by Lemma 49, the sorted reporting structure for $\hat{N}$ occupies $O((\lg\lg n)n' \lg^{1+\epsilon} n/2^{\sqrt{\lg n}}) = o(n')$ bits. Thus the space costs of all structures add up to $O(n' \lg^{1/2+\epsilon} n + n'w/2^{\sqrt{\lg n}})$ bits.

Regarding construction time, observe that the point sets $N_0, N_1, \ldots, N_{n'/b-1}$ and $\hat{N}$ can be computed in $O(n')$ time. By Lemma 49, the sorted range reporting structure for $\hat{N}$ can be built in $O(n'/b \times \lg n) = o(n')$ time. Finally, the total construction time of the sorted range reporting structures for $N_0, N_1, \ldots, N_{n/b-1}$ is $O(n'/2^{2\sqrt{\lg n}} \times (2^{2\sqrt{\lg n}} + \sqrt{\lg n} \times 2^{\sqrt{\lg n}}/\lg\lg n)) = O(n')$, which dominates the total preprocessing time of all our data structures. ■

Combining Lemma 50 and Lemma 48, we have the following result on the orthogonal sorted range reporting:

Lemma 51 *Let N be a set of n' points with distinct y-coordinates in a $2^{\sqrt{\lg n}} \times n'$ grid, where $2^{\sqrt{\lg n}} \leq n' \leq n$. Given packed sequences X and Y respectively encoding the x- and y-coordinates of these points where $Y[i] = i$ for any $i \in [0, n'-1]$, a data structure of $O(n'(\lg n)^{(1+\epsilon)/2} + w \times (n'/2^{\sqrt{\lg n}} + 2^{\sqrt{\lg n}}))$ bits can be built over N in $O(n' + 2^{\sqrt{\lg n}} \times \sqrt{\lg n}/\lg\lg n)$ time to answer orthogonal sorted range reporting in $O(\lg\lg n + \mathsf{occ})$ time, where ϵ is any positive constant and occ is the number of reported points. The query procedure requires access to a universal table of $o(n)$ bits.*

7.2.3 Orthogonal Sorted Range Reporting on an $n \times n$ Grid

Finally, we show the data structure with the fast construction algorithm for n points in 2d rank space.

Theorem 3 *Let N denote a set of n points in 2d rank space. A data structure of $O(n \lg^{1+\epsilon} n)$ bits of space can be built over N in $O(n\sqrt{\lg n})$ time to answer orthogonal sorted range reporting in $O(\lg\lg n + \mathsf{occ})$ time, where occ is the number of reported points and ϵ is any small positive constant. The query procedure requires access to a universal table of $o(n)$ bits.*

Proof. Let sequence $X[0, n-1]$ denote the point set N such that $N = \{(X[i], i) | 0 \leq i \leq n-1\}$. We build a $2^{\sqrt{\lg n}}$-ary wavelet tree T upon $X[0, n-1]$ with support for ball inheritance using part (b) of Lemma 22. Recall that $A(v)$ stores the x-coordinates of the ordered list, $N(v)$, of points from N whose x-coordinates are within the range represented by v, and these points are ordered by y-coordinate. Furthermore, v is associated with another sequence $S(v)$ drawn from alphabet $[2^{\sqrt{\lg n}}]$, in which $S(v)[i]$ encodes the rank of the child of v that contains $N(v)[i]$ in its ordered list. We regard $A(u)$ at each internal u as a point set $\hat{N}(u) = \{(A(u)[i], i) | 0 \leq i \leq |A(u)| - 1\}$ and construct the data structure $TS_{ds}(u)$ over $\hat{N}(u)$ for three-sided sorted reporting by Lemma 44. We regard $S(u)$ at each internal u as a set $\hat{S}(u)) = \{(S(u)[i], i) | 0 \leq i \leq |S(u)| - 1\}$ and construct the data structure $RS_{ds}(u)$ over $\hat{S}(u)$ for orthogonal sorted range reporting by Lemma 51.

Given a query range $Q = [a, b] \times [c, d]$, we first locate the lowest common ancestor u of l_a and l_b in constant time, where l_a and l_b denote the a-th and b-th leftmost leaves of T, respectively. Let u_i denote the i-th child of u, for any $i \in [0, 2^{\sqrt{\lg n}} - 1]$. We first locate two children, $u_{a'}$ and $u_{b'}$, of u that are ancestors of l_a and l_b, respectively. They can be found in constant time by simple arithmetic as each child of u represents a range of equal size. Then the answer, $Q \cap N$, to the query can be partitioned into three point sets $A_1 = Q \cap N(u_{a'})$, $A_2 = Q \cap (N(u_{a'+1}) \cup N(u_{a'+2}) \cup \ldots N(u_{b'-1}))$ and $A_3 = Q \cap N(u_{b'})$. At node $u_{a'}$, we query over $TS_{ds}(u_{a'})$ to report all the points in $[a, +\infty] \times [c_{u_{a'}}, d_{u_{a'}}]$ in increasing order of y-coordinates in $O(\lg \lg n + \mathtt{occ}_0)$ time, where $\mathtt{occ}_0$ is the number of the reported points and $[c_{u_{a'}}, d_{u_{a'}}] = \mathtt{noderange}(c, d, u_{a'})$. At node $u_{b'}$, we query over $TS_{ds}(u_{b'})$ to report all the points in $[0, b] \times [c_{u_{b'}}, d_{u_{b'}}]$ in increasing order of y-coordinates in $O(\lg \lg n + \mathtt{occ}_1)$ time, where $\mathtt{occ}_1$ is the number of the reported points and $[c_{u_{b'}}, d_{u_{b'}}] = \mathtt{noderange}(c, d, u_{b'})$. At node u, we query over $RS_{ds}(u)$ to report all the points in $[a' + 1, b' - 1] \times [c_u, d_u]$ in increasing order of y-coordinates in $O(\lg \lg n + \mathtt{occ}_2)$ time, where $\mathtt{occ}_2$ is the number of the reported points and $[c_u, d_u] = \mathtt{noderange}(c, d, u)$. With constant-time support for $\mathtt{point}$, we can retrieve the original x- and y-coordinates of each reported point in constant time. Finally, we use the 3-way mergesort algorithm to merge three sorted list into one sorted list in $O(\mathtt{occ}_0 + \mathtt{occ}_1 + \mathtt{occ}_2) = O(\mathtt{occ})$ time. Therefore, the overall query time for orthogonal sorted range reporting is $O(\lg \lg n + \mathtt{occ})$ time.

Now we analyze the space costs. The wavelet tree T with support for ball inheritance uses $O(n\lg^{1+\epsilon'} n + n\lg n) = O(n\lg^{1+\epsilon'} n)$ bits for any positive ϵ'. For each internal node v, since $w = \Theta(\lg n)$, the data structure $RS_{ds}(u)$ for orthogonal sorted range reporting over $\hat{S}(v)$ uses $O(|S(u)|\lg^{1/2+\epsilon''} n + 2^{\sqrt{\lg n}}\lg n + |S(u)|\lg n/2^{\sqrt{\lg n}})$ bits for any positive ϵ''. This subsumes the cost of the data structure $TS_{ds}(v)$ for three-sided sorted reporting over $\hat{N}(v)$ which is $O(|A(v)|\lg\lg n)$ bits. As T has $O(n/2^{\sqrt{\lg n}})$ internal nodes, the total cost of storing these structures at all internal nodes is $\sum_u O(|S(u)|\lg^{1/2+\epsilon''} n + 2^{\sqrt{\lg n}}\lg n + |S(u)|\lg n/2^{\sqrt{\lg n}}) = O(n\lg n/\sqrt{\lg n} \times \lg^{1/2+\epsilon''} n + n\lg n) = O(n\lg n\lg^{\epsilon''} n)$. Setting $\epsilon = max(\epsilon', \epsilon'')$, the total space bound turns out to be $O(n\lg^{1+\epsilon} n)$ bits. Overall, the data structures occupy $O(n\lg^{1+\epsilon} n)$ bits.

Finally, we analyze the construction time. As shown in Lemma 22, T with support for ball inheritance can be constructed in $O(n\sqrt{\lg n})$ time. For each internal node v of T, constructing $TS_{ds}(v)$ over $\hat{N}(v)$ and the orthogonal sorted range reporting structure $RS_{ds}(v)$ over $\hat{S}(v)$ requires $O(|A(u)| + |S(u)| + \sqrt{\lg n} \cdot 2^{\sqrt{\lg n}}/\lg\lg n) = O(|S(u)| + \sqrt{\lg n} \cdot 2^{\sqrt{\lg n}}/\lg\lg n)$ time. As T has $O(n/2^{\sqrt{\lg n}})$ internal nodes, these structures over all internal nodes can be built in $\sum_u O(|S(u)| + \sqrt{\lg n} \times 2^{\sqrt{\lg n}}) = O(n\lg n/\sqrt{\lg n} + n\sqrt{\lg n}/\lg\lg n) = O(n\sqrt{\lg n})$ time. Therefore, the preprocessing time of all data structures is hence $O(n\sqrt{\lg n})$. ■

Chapter 8

Applications

We now apply our range search structures to the text indexing problem, in which we preprocess a text string $T \in [\sigma]^n$, where $\sigma \leq n$. Given a pattern string $P[0..p-1]$, a *counting query* computes the number of occurrences of P in T and a *listing query* reports these occurrences.

8.1 Text indexing and searching in sublinear time

When both T and P are given in packed form, a text index of Munro et al. [30] occupies $O(n \lg \sigma)$ bits, can be built in $O(n \lg \sigma / \sqrt{\lg n})$ time and supports counting queries in $O(p/\log_\sigma n + \lg n \log_\sigma n)$ time (there are other tradeoffs, but this is their main result). Thus for small alphabet size, which is common in practice, they achieve both $o(n)$ construction time and $o(p)$ query time, while previous results achieve at most one of these bounds. To support listing queries, however, they need to increase the space cost to $O(n \lg \sigma \lg^\epsilon n)$ bits and construction time to $O(n \lg \sigma \lg^\epsilon n)$, and then a listing query can be answered in $O(p/\log_\sigma n + \log_\sigma n \lg\lg n + \texttt{occ})$ time. The increase in storage and construction costs stems from one component they used which is an orthogonal range reporting structure over $t = O(n/r)$ points in a $\sigma^{O(r)} \times t$ grid, for $r = c \log_\sigma n$ for any constant $c < 1/4$. We can apply Lemma 31 over this point set

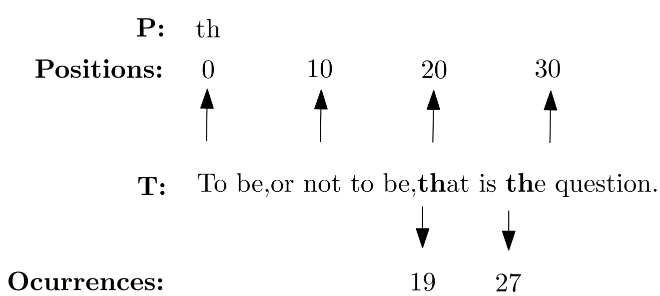


Figure 8.1: An example of listing and counting queries. The pattern "th" appears twice in the text string at the positions of 19 and 27. Therefore, the counting query returns 2, and the listing query returns 19 and 27.

to decrease the construction time of their index for listing queries to match that for counting queries:

Theorem 4 *Given a packed text string T of length n over an alphabet of size σ, an index of $O(n \lg \sigma \lg^{\epsilon} n)$ bits can be built in $O(n \lg \sigma / \sqrt{\lg n})$ time for any positive constant ϵ. Given a packed pattern string P of length p, this index supports listing queries in $O(p/\log_{\sigma} n + \log_{\sigma} n \lg \lg n + \mathsf{occ})$ time where occ is the number of occurrences of P in T.*

8.2 Position-restricted substring search

In a position-restricted substring search [27], we are given both a pattern P and two indices $0 \leq l \leq r \leq n-1$, and we report all occurrences of P in $T[l..r]$. Mäkinen and Navarro [27] solved this problem using an index for the original text indexing problem and a two-dimensional orthogonal range reporting structure. Different text indexes and range reporting structures yield different tradeoffs. The tradeoff with the fastest query time supports position-restricted substring search in $O(p + \lg \lg n + \mathsf{occ})$ time, where occ is the output size, and it uses $O(n \lg^{1+\epsilon} n)$ bits and can be constructed in $O(n \lg n)$ time. Again, the construction time of the range reporting structure is the bottleneck, which can be improved by Theorem 1. We can also use a new text index by Bille et al. [8] to achieve speedup when P is given as a packed sequence. We have:

Theorem 5 *Given a text T of length n over an alphabet of size σ, an index of $O(n \lg^{1+\epsilon} n)$ bits can be built in $O(n\sqrt{\lg n})$ time for any constant $0 < \epsilon < 1/2$. Given a packed pattern string P of length p, this index supports position-restricted substring search in $O(p/\log_{\sigma} n + \lg p + \lg \lg \sigma + \mathsf{occ})$ time, where occ in the size of the output.*

Chapter 9

Conclusion

In this thesis, we have presented data structures for three fundamental geometry problems that improve upon the preprocessing cost of the previous optimal solutions. These three fundamental problems are orthogonal range reporting, orthogonal range successor, and sorted range reporting. For n points in 2-dimension rank space, constructing our data structures only takes $O(n\sqrt{\lg n})$ time, significantly improving the previous bound of $O(n \lg n)$. Furthermore, our data structure has the same space costs and query time of the previous best tradeoffs.

Our results are particularly relevant to text indexing, where the input string is often very long, as orthogonal range search structures are often used as building blocks of text indexes. Thus, we apply our range search structures to the text indexing problem. For listing queries, our index achieved the same space cost and query time as the ones by Munro at al. [30]. Meanwhile, we significantly improved the preprocessing time from $O(n \lg \sigma \lg^{\epsilon} n)$ to $O(n \lg \sigma / \sqrt{\lg n})$. Our solutions can also be adopted for position-restricted substring queries.

When designing the ball inheritance structure, the fast processing is only achieved over either a wavelet tree with high fanout, or the points whose coordinates have to be encoded in $O(\sqrt{\lg n})$ bits. One interesting open problem is how to quickly construct the ball inheritance data structure without those limitations, as the ball inheritance data structure itself has been applied to many different research problems.

Other contributions of the thesis include various preliminary results such as those for `rank′`, `rmq/rMq`, and `succ/pred` queries. These components of our data structures might be useful in other more complex solutions in the future.

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