Vibration Analysis of Pole-vaulting via Mathematical Modeling

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Abstract

This project studied the factors including the weight of athlete (m) and stiffness of the pole (k) that can influence the efficiency of the energy conversion. The efficiency refers to the percentage of initial kinetic energy transferred into potential energy stored in a pole. A mathematical model is built based on Hamilton's method to simulate the pole motion. The efficiency can be optimized if m and k satisfy a linear expression that has been derived in the project.

Introduction

In pole vault, the work done by athlete and the gravitational potential energy of the athlete is converted into potential energy stored in the pole.

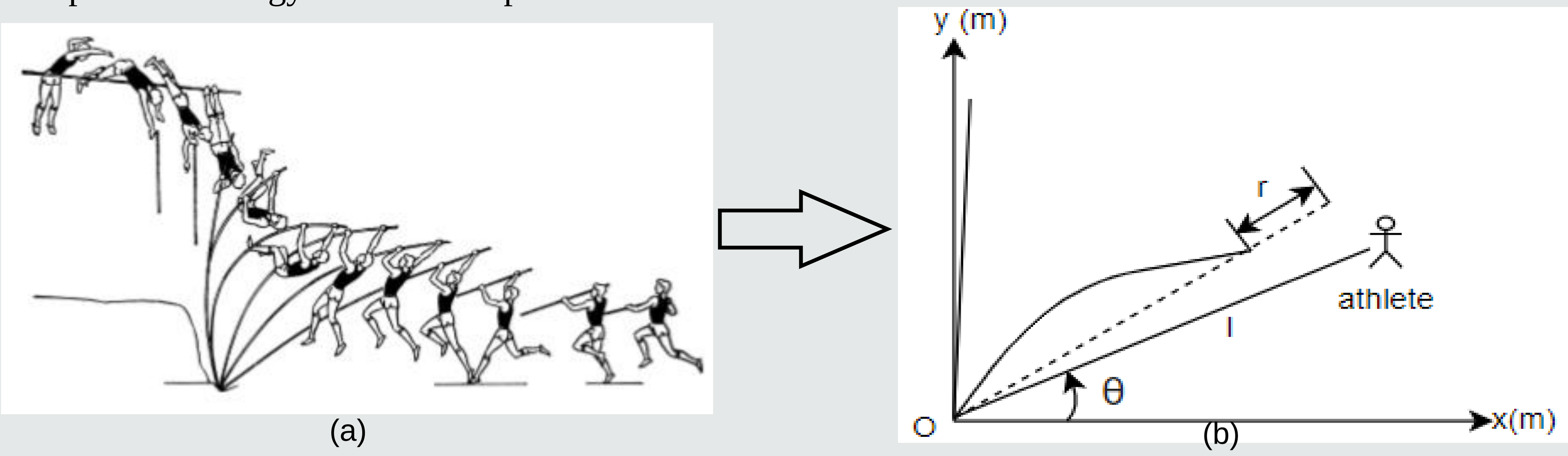

Figure 2 (a) Real-World Pole Vaulting (Linthorne, 2000). (b) Schematic of the Pole-vaulting Model

Linthorne (2000) indicates that a flexible pole can store more energy by comparing a fiberglass pole model with a bamboo pole model.

In this project, a mathematical model is built to simulate the motion of the pole. The relation between stiffness of the pole, weight of athlete and energy storage in pole-vaulting is studied. The model is built with the following assumptions:

- The model is a linear, 2-DOF system with respect to the radial and the angular displacement of the pole.
- The athlete is a mass point at pole tip. The pole is a massless spring system.

Figure 1 (a) is the real pole vault. Figure 1(b) is the graphical representation of the mathematical model.

Method

In Hamilton's Method, Hamiltonian (H) is defined by a general formula:

$$H = \sum_{i=1}^{n} v_i p_i - L \tag{1}$$

where L is the Lagrangian function with respect to kinetic energy T and potential energy U:

$$L = T - U \tag{2}$$

The summation of the product of momentum p and speed v equals to two times of T. Therefore, the Hamiltonian ends up to be the summation of U and T:

$$H = T(p_r, L_a) + U(r, \theta) \tag{3}$$

The momentum-based T and displacement-based U can be expressed by:

$$T(p_r, L_a) = \frac{{p_r}^2}{2m} + \frac{{L_a}^2}{(l-r)^2} \tag{4}$$

$$U(r,\theta) = mg[(l-r)\sin(\theta)] + \frac{1}{2}kr^2 \tag{5}$$

where L_a is angular momentum, p_r is linear momentum along tangential direction.

The Hamiltonian can be derived by substituting Eqn.4 and 5 into Eqn.3:

$$H = \frac{{p_r}^2}{2m} + \frac{{L_a}^2}{(l-r)^2} + mg[(l-r)\sin(\theta)] + \frac{1}{2}kr^2 \tag{6}$$

By taking the derivative of H with respect to p_r, L_a, r and θ respectively, the Hamiltonian equation of motion is obtained:

$$Hamiltonan\ equation\ of\ motion \begin{cases} \dot{r} = \frac{\partial H}{\partial p_r} \\ \dot{\theta} = \frac{\partial H}{\partial L_a} \\ \dot{p_r} = \frac{\partial H}{\partial r} \\ \dot{L} = \frac{\partial H}{\partial \theta} \end{cases} \Longrightarrow \begin{cases} \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} & (4) \\ \dot{\theta} = \frac{\partial H}{\partial L_a} = \frac{L_a}{m(l-r)^2} & (5) \\ \dot{p_r} = \frac{\partial H}{\partial r} = \frac{-{L_a}^2}{m(l-r)^3} - mg\sin\theta + kr & (6) \\ \dot{L} = \frac{\partial H}{\partial \theta} = \frac{-{L_a}^2}{m(l-r)^2} + (l-r)mg\cos\theta & (7) \end{cases}$$

With initial condition:

$$r(0) = 0\,m; \theta(0) = 30°; V_{x,athlete}(0) = 10\,m/s; V_{y,athlete}(0) = 3\,m/s$$

Result and Discussion

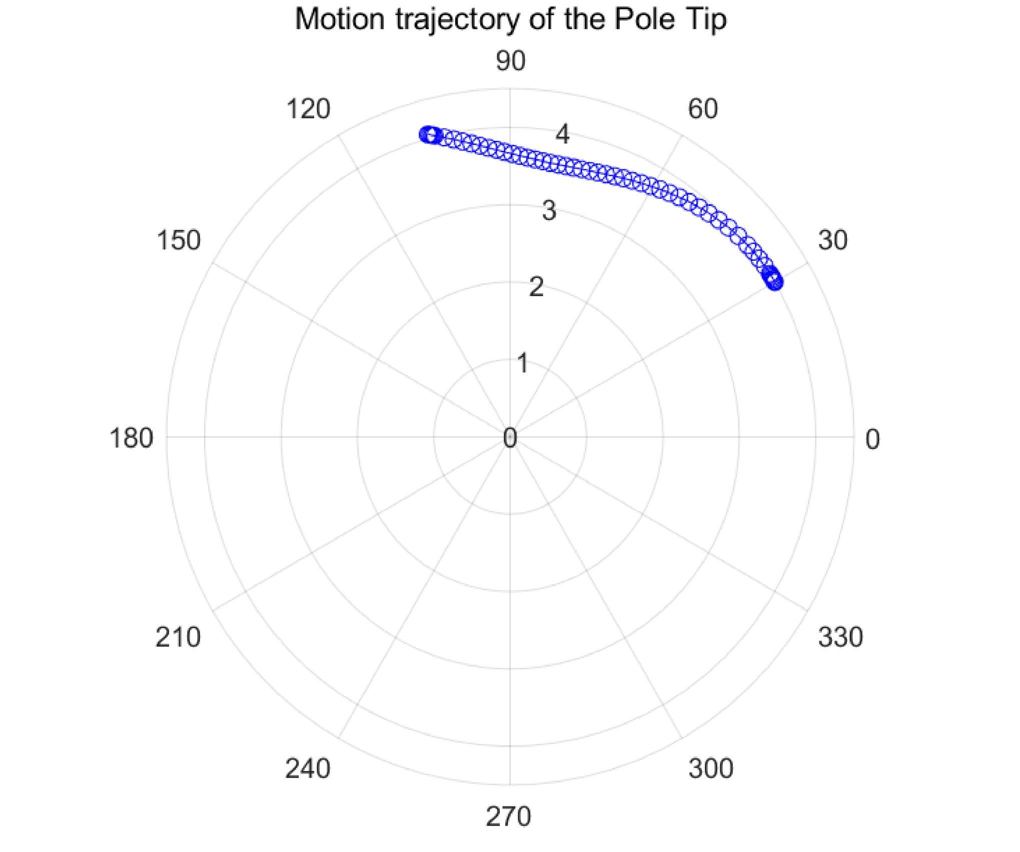

Figure 2 Motion Trajectory of the Tip of the Pole when k=300N/m, m = 70kg

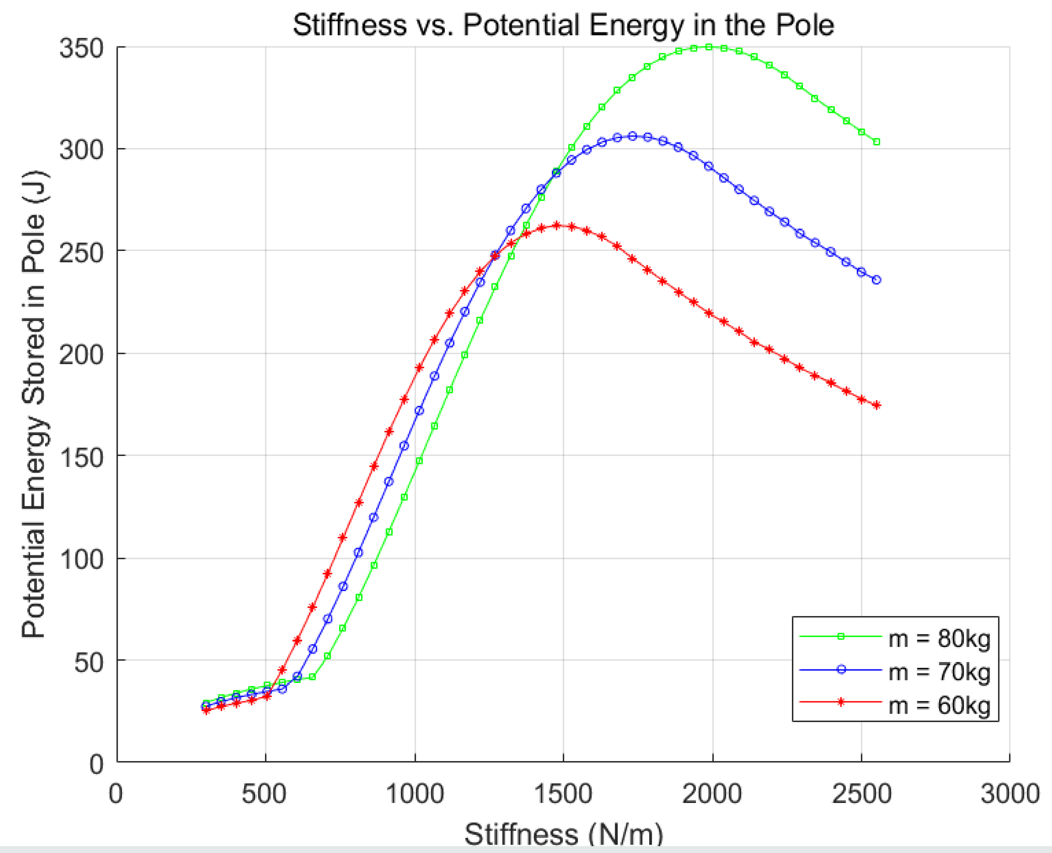

Figure 3 Change of energy storage in a pole with respect to different stiffness and athletes' weight

On Figure 2, the pole motion when θ is larger than 90° will not be discussed due to the discrepancy between the simulation and real pole-vaulting. In the real-world scenario, athletes push the pole back to avoid hitting the cross bar. Meanwhile, athletes release the pole after the pole reaches the peak (i.e. load is removed).

This project focuses on the maximum radial deflection and the corresponding maximum potential energy ($U_{p,max}$) stored in the pole. According to Figure 2, the maximum deflection happens before θ reaches 90°. Figure 3 shows the change of energy storage with respect to k and m. $U_{p,max}$ increases if the weight of athlete is larger. If k keeps increasing after reaching $U_{p,max}$, the energy storage starts decreasing.

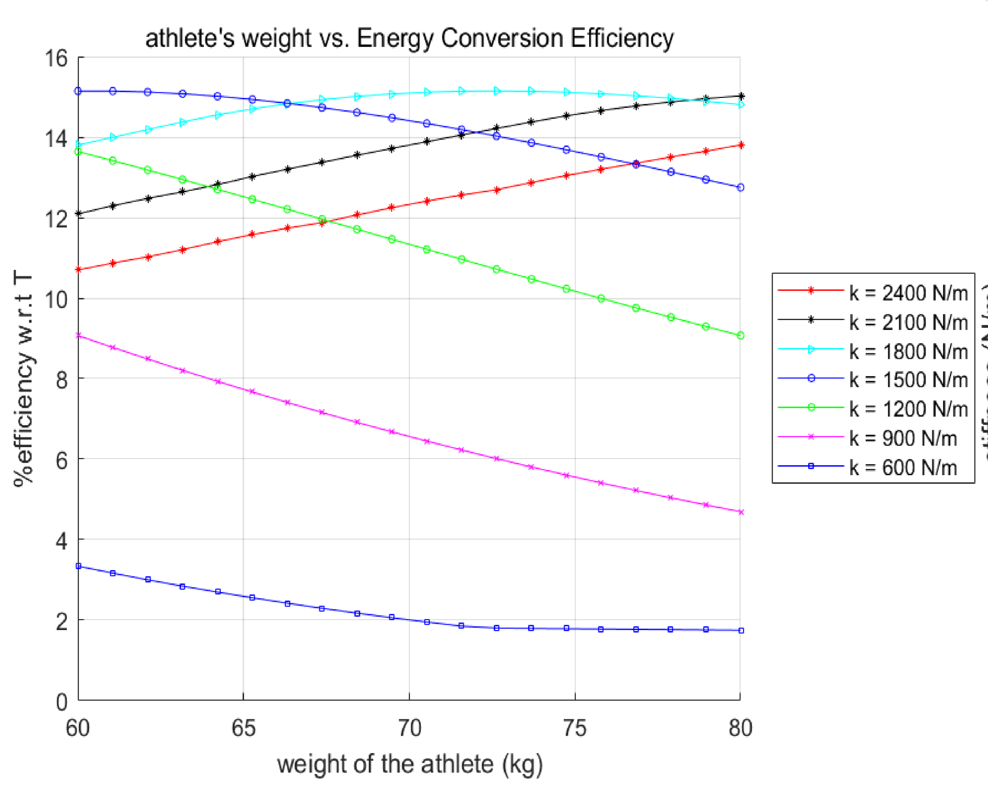

Figure 4 Efficiency of energy conversion for various athlete weight and pole stiffness

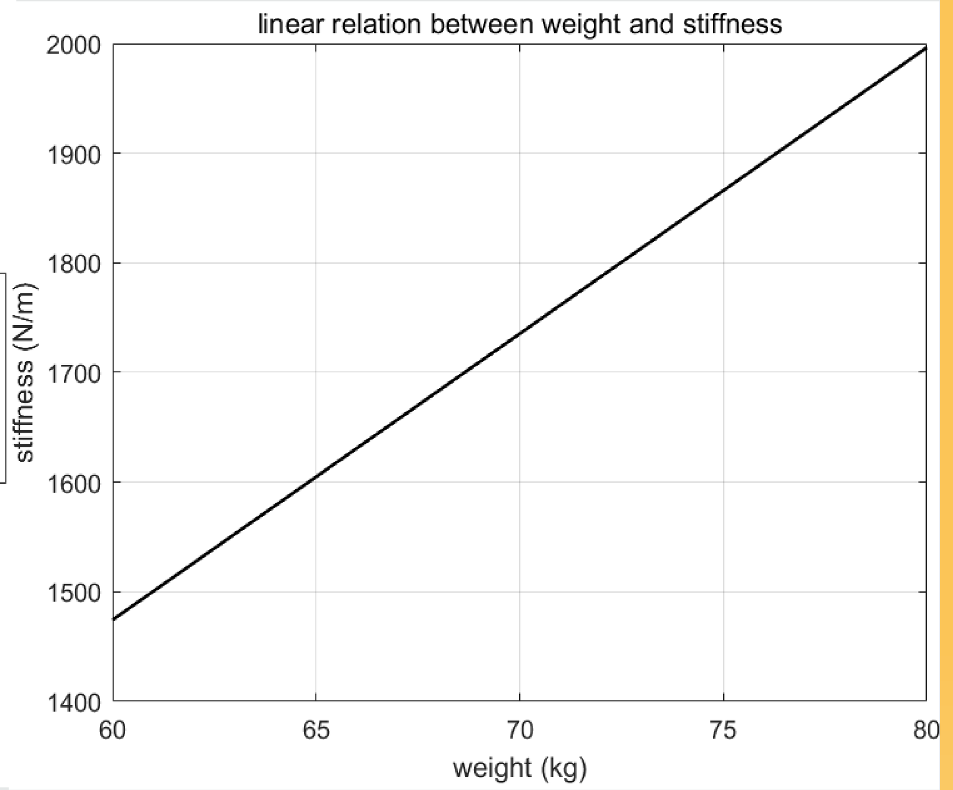

Figure 5 Result of linear regression for the optimized stiffness and weight combination

Figure 4 shows the change of efficiency of energy conversion with respect k and m. In order to maximize the efficiency, a larger k is preferred when the weight of the athlete is larger. However, when k exceeds a 'threshold', the efficiency and energy storage start decreasing.

A linear relationship between m and k that can give optimized efficiency is derived by linear regression based on m and k that can produce a $U_{p,max}$ on Figure 2:

$$k = 26.1m - 91.67 \tag{8}$$

The plot of the linear expression is shown on Figure 5.

Conclusion and Recommendation

In conclusion, the model analysis studied the optimization of energy conversion in pole vaulting by selecting optimal combination of m and k. The combination of m and k that satisfy Eqn.8 can provide an optimal efficiency of energy conversion. However, some further improvement can be made. For example, Euler's Buckling theory can be applied in the model instead of treating the pole as a spring system. The real-world scenario of pole-vaulting can be studied. The timing when the athlete releases the pole can be considered.

Reference

Linthorne, N.P. (2000) Energy loss in the pole vault take-off and the advantage of the flexible pole. *Sports Engineering*, 3(4), 205 – 218.