TOWARDS A NEW MATHEMATICAL PARADIGM FOR THE DEVELOPMENT OF ECONOMIC GROWTH THEORY

by

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To my parents.

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Abstract

We contribute to the development of the growth theory in economics, using mathematical and statistical tools. In particular, we employ various techniques rooted in the theory of Hamiltonian systems on Poisson manifolds, jet bundles theory, calculus of variation, and statistical data analysis to study the properties of the Cobb-Douglas production function as an invariant of the one-parameter Lie group action determined by exponential growth in factors (capital and labor) and production. This approach is extended to more general models determined by logistic growth and the Lotka-Volterra type interactions between factors. The resulting new production functions are shown with the aid of statistical methods to provide a good fit to the current economic data.

List of Abbreviations and Symbols Used

Abbreviations and Symbols Used in Chapter 2

ODE	Ordinary differential equation
E	Open subset
$\mathbb{R}^n$	n-dimensional real space
$\mathbf{x}_0$	Initial conditions
X	Vector field
$I(\mathbf{x}_0)$	Maximal interval of existence
$\phi_t(\mathbf{x}_0)$	Flow of a corresponding differential equation
Γ	Orbit
M	Smooth manifold
Ψ	(Local) group of transformations
G	Lie group
TM	Tangent bundle of M
T_pM	Tangent space of M
T^*M	Cotangent bundle of M
T_p^*M	Cotangent space of M
$[\cdot,\cdot]$	Schouten bracket or Lie bracket if all components are vector fields
R_g	Right multiplication map on a Lie group
dR_g	Induced map of R_g
$\mathfrak{g}$	Lie algebra
E	Total manifold
M	Base manifold
π	Projection map
(E, π, M)	Fibred manifold (Bundle)

Abbreviations and Symbols Used in Chapter 2 (Continued)

(TM, τ_M, M)	Tangent bundle
Δ	Distribution
$j^1_p\phi$	One-jet of a section ϕ
$(J^1\pi, \pi_1, M)$	Jet bundle
D_i	Cartan structure
$\mathcal{C}$	Cartan distribution
$\{\cdot, \cdot\}$	Poisson bracket
$\frac{\partial}{\partial x}$	Basic vector field
dx	Basic one-form
ω	Differential form
π	Poisson bivector
H	Hamiltonian function
$\mathfrak{v}$, $\mathfrak{y}$	Multi-vector field
$C^\infty(M)$	Space of smooth functions on M
RSS	Residual sum of squares
ϵ	Residual

Abbreviations and Symbols Used in Chapter 3

iff	If and only if
$\mathbb{R}^2_+$	Two-dimensional non-negative real space
G	Lie group
$T(G, \mathbb{R}^2_+)$ or $(G, \mathbb{R}^2_+)$	One-parameter group of transformations
$U,\ X$	Vector field
$\mathcal{L}$	Lie derivative
$[\cdot, \cdot]$	Lie bracket
d	Exterior derivative
$\oplus$	Direct sum
$\wedge$	Wedge product
K	Capital
L	Labor
Y	Production
N_i	Carrying capacity of each quantity
H	Hamiltonian function
$\mathrm{pr}^{(1)}\mathbf{u}$	First prolongation of a vector field **u**
I_i	Fundamental differential invariants

Abbreviations and Symbols Used in Chapter 4

TEG	Transformed exponential growth
TLG	Transformed logistic growth
TCELG	Transformed combined exponential and logistic growth
TCLK	Transformed combined Lotka-Volterra
π or π_i	Poisson bivector
$\dfrac{\partial}{\partial x} \wedge \dfrac{\partial}{\partial y}$	Basic bivector
H	Hamiltonian function
D	Debt

Abbreviations and Symbols Used in Chapter 6

$\mathrm{int}\mathbb{R}^4_+$	Interior of four-dimensional non-negative real space
V	Vector field
π	Poisson bivector
Ω	Four-form
d	Exterior derivative
Tr	Trace of a matrix
Det	Determinant of a matrix

Abbreviations and Symbols Used in Chapter 7

J	Social welfare
Y	Gross production
C	Consumption
I	Investment
X	Export
M	Import
D	Debt
T	Taxation
Π	Revenue
U	Utility
c	Consumption per capita
k	Capital per capita
u	Utility per capita
$\Gamma_{k\alpha}$	Components of Poisson bivector

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Chapter 1

Introduction

As is well known, production functions are commonly used in both macroeconomics and microeconomics models because they have a number of convenient and, as is widely believed, realistic properties. By definition, they relate the quantities representing physical inputs (*e.g.*, land, capital, labor) to the quantities representing output of goods. In fact, many models are largely determined by the mathematical properties and parameters of the production functions involved. For example, the elasticity of substitution between capital and labor is one such a parameter that is derived from the form of a particular production function.

Recall that in 1928 Charles Cobb and Paul Douglas published their seminal paper [27] dedicated to the study of the data for the US manufacturing sector for 1899-1922. Their ultimate goal was to determine how the variations of the elastisities of labor and capital affected the distribution of income (see Douglas [31] and Samuelson [103] for more details and references). The authors plotted the statistical series for the labor force (L), the stock capital (K) and the resulting product (Y) on a logarithmic scale and concluded that a function of the form

$$Y = f(L, K) = AL^k K^{1-k} \tag{1.0.1}$$

could be fitted to this data. Using statistical methods, Cobb and Douglas found the coefficients k and A in (1.0.1) to determine the following production function

$$Y = f(L, K) = 1.01 L^{\cdot 75} K^{\cdot 25} \tag{1.0.2}$$

that fitted to the data very well. See Samuelson [103] to learn about the use and derivation of the function (1.0.1) by Wicksell, Wicksteed, and Wilcox prior to 1928.

Roughly speaking, the Cobb-Douglas function (1.0.1) can be easily derived under the assumptions that there is no production if either capital or labor vanishes, the marginal productivity of capital is proportional to the amount of production per unit of capital (*i.e.*, $\frac{\partial Y}{\partial K} = \alpha \frac{Y}{K}$), and the marginal productivity of labor is proportional to the amount of production per unit of labor (*i.e.*, $\frac{\partial Y}{\partial L} = \beta \frac{Y}{L}$).

Later, Ruzyo Sato [107] developed the theory of technical change and economic invariance, in which a production function was an output obtained within the framework of a model. In particular, the author and his collaborators have derived the Cobb-Douglas production function as a consequence of the exponential growth in factors (capital and labor) and production.

In this thesis we continue the development of Sato's theory by changing the assumptions about the Lie group theoretical properties of the technical progress representing the growth in factors.

The first goal is to use the existing model to develop a new mathematical paradigm that can be used to study the current state of economy. Accordingly, in what follows we will modify the economic growth models described by Sato within the framework of the Lie group theory according to the present economic realities [8]. More specifically, we will replace in a neoclassical growth model in the sense of Sato $(G, \mathbb{R}^2_+)$, where $\mathbb{R}^2_+ = \{(K, L) | K, L \in \mathbb{R}_+\}$, a group G representing an exponential growth with another one-parameter Lie group that describes a *logistic growth*:

$$G : \text{exponential growth} \to \text{logistic growth}.$$

This idea is currently being exploited and developed from different perspectives and in different directions quite extensively in the literature by economists and mathematicians alike (see, for example, [1, 2, 18, 19, 21, 23]), which is quite natural, given that the resources on our planet are limited. We will show that this approach can be used to derive other production functions whose properties are determined, for example, by logistic growth in factors, the presense of additional contributing factors (say, debt), nonlinear interraction, *etc.*

Our second goal is to extend the applicability of the method used to derive the production functions in Chapter 3. More specifically, we will enlarge the set of available tools that can be employed to derive production functions, ranging from data analysis [27, 31] to symmetry and Lie group theory methods [107, 110], by incorporating a Hamiltonian formalism into the theory. In this view, the use of the Hamiltonian methods appears to be a natural next step and it is our contention that the theory can be further developed at this point by recasting its setting within a Hamiltonian framework. More specifically, we will redefine the existing models presented in Chapter 3 and introduce a new one by presenting them as a special case of the general n-dimensional Lotka-Volterra model in population dynamics (see, for example, Kerner [65] and the relevant references therein). This model is given by the following formula:

$$\dot{x}_i = x_i \left(b_i + \sum_{j=1}^{n} a_{ij} x_j \right), \quad i = 1, \ldots, n, \tag{1.0.3}$$

where the linear terms describe the Malthusian growth (or decay) of the species in question $x_1, \ldots, x_n$ in the absence of interaction (*i.e.*, when the parameters a_{ij} all vanish), while the quadratic terms tell us about the binary interaction between the species, assuming spatial homogeneity. More specifically, $a_{ij} = \frac{1}{\beta_i}\alpha_{ij}$, where β_i is Volterra's "equivalent number" parameter that has the meaning of mean effective biomass of the individuals in the ith species, while α_{ij} is normally assumed to be a skew-symmetric matrix representing the interaction strength of species i with species j [65]. We recall that the Lotka-Volterra systems with vanishing linear terms (*i.e.*, when $b_i = 0$, $i = 1, \ldots, n$ in (1.0.3)), as well as their integrable perturbations are an important and well-studied topic in the field of mathematical physics, in particular, they appear as discretizations of the KdV equation (see, for example, Bogoyavlenskij *et al.* [13] and Damianou *et al.* [122] for more details and references). Furthermore, Plank [95, 94] (see also Kerner [66]) studied general n-dimensional Lotra-Volterra systems from the Hamiltonian viewpoint and found bi-Hamiltonian formulations for the 3-dimensional model (1.0.3).

Our next goal in this thesis is to revisit the Cobb-Douglas production function controversy described in Chapter 5 and discuss its legidimacy from a mathematical viewpoint that extends the approach to the growth theory established by Sato [107] (see also Sato and Ramachadran [110]). Specifically, we will review the data studied by Cobb and Douglas in [27] from the viewpoint of a mathematical model based on the assumption of exponential growth in factors and the corresponding output that originated in Sato [107] to explain its relevance and the properties of the corresponding Cobb-Douglas production function (1.0.2). Our main conclusion in this respect is that Cobb and Douglas in [27] did derive “a production function”, but not “the production function”. In fact, one can determine a whole class of production functions of the Cobb-Douglas class (3.0.1) that can be fitted to the 1928 data investigated by the authors. Based on our findings, we certainly agree with Samuelson [103] who expressed serious doubts that the choice of the form of the production function (1.0.1) was uniquely determined by the specific data studied by Cobb and Douglas in [27]. At the same time we are convinced that the function (1.0.1) with the coefficients specified in (1.0.2) is a legitimate production function (*i.e.*, a function that relates the quantity of factor inputs of labor and capital to the amount of output in manufacturing) that can be fitted with good accuracy to the data for the US manufacturing sector for 1899-1922 used by Cobb and Douglas in [27].

A production function also plays an important role in various economic growth models. An example of the application of a production function is the celebrated Ramsey-Cass-Koopmans model [109, 99, 25, 70, 93] initially introduced by the British mathematician Frank P. Ramsey [99] in 1928 to investigate the optimal savings of a country. He aimed to determine the consumption level at which the country can attain the maximal social welfare. His contribution did not receive much attention until the 1950s. In 1956 Samuelson and Solow [104] extended the Ramsey model and considered it from viewpoints of a different mathematical formalism. The model was further modified and completed by Cass [25] and Koopmans [70], at which point it was named the Ramsey-Cass-Koopmans model.

At the core of the Ramsey-Cass-Koopmans model and its generalizations is a production function $Y = f(K, L)$, normally of the Cobb-Douglas type (3.0.1), where the factors K and L represent capital and labor respectively. The function Y is required to satisfy the so-called Inada conditions [60]. From a mathematical standpoint, the Ramsey-Cass-Koopmans model and its generalizations, for example, the Solow-Swan economic growth model [120, 126], are governed by a single nonlinear differential equation or a system of such equations that describe the evolution of per capita capital stock, consumption, *etc.* We formulate a new variational problem based on the Ramsey-Cass-Koopmans model by considering the production function derived in Section 3.3 and incorporating a new factor, debt D.

The thesis is organized as follows. In Chapter 2 we review the requisite theoretical background. In Chapter 3 we lay the groundwork for the introduction of a new growth model and derivation of new production functions. Specifically, we review the Lie group approach introduced in [107] and employ it to rederive the Cobb-Douglas function (1.0.1). We depart from the growth model described by Sato based on exponential growth and introduce instead a new one — based on the assumption that factors grow logistically and derive a new production function (3.3.14) within the framework of the growth model (3.2.7). We also explain, using mathematical reasonings and the results obtained in preceeding sections, why Bowley's law [15, 16] no longer holds true in post-1960 data. In the process we also derive another production function (3.6.23) and a new modified wage share (3.6.22). We use statistical analysis to investigate how estimations of the new production function (3.3.14) compare to economic data. Some of the results presented in Chapter 3 have already been published in [118].

Chapter 4 is devoted to the Hamiltonian formalism of the economic growth model. We consider special cases of the Lotka-Volterra model that characterize the evolution of capital, labor, production as well as debt, the Hamiltonian formulations of which via corresponding Poisson structures are given. The Hamiltonian function in each model can be used as a production function. We employ the bi-Hamiltonian formalism to relax Sato's condition of simultaneous holotheticity, based on which we derive the production functions satisfying the condition $\alpha + \beta = 1$.

In Chapter 5 we demonstrate the validity of the concept of a production function from both mathematical and statistical perspectives . We attempt to resolve some of the controversies surrounding the Cobb-Douglas function. The Cobb-Douglas function and the new production function (3.3.14) are reviewed within the framework of invariants of corresponding one-parameter Lie group. The invariant conditions are given, with which a production function is a time invariant along the flow defined by its corresponding growth model. We show that the condition $\alpha + \beta = 1$ is not necessary for a production function by comparing each production function to the US economic data in different periods using the R programming language. An algorithm of fitting a production function to data is presented. Some of the results presented in Chapters 4 and 5 have been published in [116, 117].

Chapter 6 deals with the qualitative analysis of the four-dimensional economic growth model involving debt. We continue the discussion concerning the four-dimensional model shown in Chapter 4. We investigate the divergence as well as the Hamiltonian formalism of the corresponding dynamical system. The stability of the equilibrium of the model is analyzed. We also discuss the production function given by the corresponding Hamiltonian function of the model. Chapter 6 is necessarily incomplete. A more detailed numerical analysis of the model will be completed in forthcoming research.

In Chapter 7 we consider new variational problems. Firstly, we derive the Ramsey golden rule of accumulation employing the Euler-Lagrange method. We consider a new variational problem based on the Ramsey-Cass-Koopmans model, in which the integral of the social welfare is subject to logistic growth of capital and labor as well as a different growth path of consumption related to the new production function (3.3.14). We also extend the model of the maximum of profits of a company proposed by Nerlove [84] by incorporating the new production function (3.3.14).

Concluding remarks are the subject of Chapter 8.

Chapter 2

Requisite theoretical background

In this chapter we will briefly review the necessary theoretical background for the thesis. The requisite material comes from Perko [91], Olver [88], Saunders [111], Gelfand and Fomin [47] as well as Fernandes [39]. We follow and adopt their notations. The chapter is also based on the material presented in [9, 14, 74, 83, 80, 115]. We use statistical tools when fitting our models to data. The required statistical techniques and methods based on [82, 123] are reviewed in Section 2.6.

2.1 Dynamical systems

The evolution of an economic, physical or biological model can be described by a dynamical system. Throughout the section we deal only with autonomous dynamical systems. A common example of a dynamical system can be given by a system of first-order ordinary differential equations (ODEs):

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),\ \mathbf{x} \in U \subset \mathbb{R}^n, \tag{2.1.1}$$

where U is an open subset on $\mathbb{R}^n$.

Let us consider an initial value problem given by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}), \\ \mathbf{x}(0) &= \mathbf{x}_0, \end{aligned} \tag{2.1.2}$$

where $\mathbf{x}_0 \in U$ is an initial value.

Remark 2.1.1. $\mathbf{x}$ *denotes a vector* $(x_1, \ldots, x_n) \in U \subset R^n$ *and* $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}, \ldots, f_n(\mathbf{x}))$ *denotes a smooth vector function.*

Suppose (2.1.1) admits a family of solutions $\mathbf{x}(t)$ where t is defined in some finite interval I, where $0 \in I$. Given an initial value $\mathbf{x}(0) = \mathbf{x}_0$, we can obtain a unique solution $\mathbf{x}(t, \mathbf{x}_0)$ through determining the value of the constant C in a solution. This is guaranteed by *the existence and uniqueness theorem* (See pp. 70-76 in [91]). Suppose $\mathbf{x}(t)$ defines a family of smooth curves on U. Then we can view $\mathbf{x}(t)$ as parameterized curves on U. More significantly, $\mathbf{x}(t, \mathbf{x}_0)$ uniquely defines the curve passing through the point $\mathbf{x}(0) = \mathbf{x}_0$. In this framework, $\dot{\mathbf{x}}$ describes the tangent vector at each point to a curve. A vector field assigns each point to a tangent vector on U. Hence, the assignment $\mathbf{f}$ in (2.1.1) naturally defines a vector field on U and we denote the vector field by $X = (f_1, f_2, \ldots, f_n)$. Then the solution $\mathbf{x}(t)$ is also called *an integral curve* of the vector field X. Among all integral curves, we want to find the unique maximal integral curve, which is determined by the unique solution $\mathbf{x}(t, \mathbf{x}_0)$ defined on the maximal interval. The maximal interval for the unique solution is called *the maximal interval of existence*, denoted by $I(\mathbf{x}_0) = (\alpha, \beta)$ since α and β generally depend on $\mathbf{x}_0$. We denote the unique solution defined on $I(\mathbf{x}_0)$ (the unique maximal integral curve) by $\phi(t, \mathbf{x}_0)$. Then for $t \in I(\mathbf{x}_0)$, the set of mappings $\phi_t(\mathbf{x}_0)$ defined by $\phi_t(\mathbf{x}_0) = \phi(t, \mathbf{x}_0)$ is called *the flow of the differential equation* (2.1.1) or *the flow of the vector field* X. The mapping satisfies the following properties

(1) $\phi_0(\mathbf{x}_0) = \mathbf{x}_0$ for all $\mathbf{x}_0 \in U$,

(2) $\phi_t \circ \phi_s(\mathbf{x}_0) = \phi_{t+s}(\mathbf{x}_0)$ for all $t, s \in \mathbb{R}$ and $\mathbf{x}_0 \in U$.

Suppose $\mathbf{x}_0$ is a fixed point, then the flow $\phi(t, \mathbf{x}_0) : I(x_0) \longrightarrow U$ defines a *trajectory* of (2.1.1) through the point $\mathbf{x}_0$. The corresponding set $\Gamma = \{\phi(t, x_0) : t \in I(\mathbf{x}_0)\}$ is called *an orbit* through $\mathbf{x}_0$. If we move the initial point on U and choose different values of $\mathbf{x}_0$, namely, treat $\mathbf{x}_0$ as a variable $\mathbf{x}$, then the mapping $\phi(t, \mathbf{x})$ gives rise to a subset $\Omega = \{(t, \mathbf{x}) \in R \times U : t \in I(\mathbf{x}_0)\}$, in which the system (2.1.1) evolves.

Following Perko [91], one can in turn employ the idea of a flow to define a dynamical system. To complete this, we need to extend the maximal interval of existence $I(\mathbf{x}_0)$ to $\mathbb{R}$, *i.e.*, for all $\mathbf{x}_0 \in U$, the flow is defined on $\mathbb{R}$. First, let us review the following general

Definition 2.1.2. A dynamical system is a triple $(\mathcal{S}, \mathcal{T}, \phi)$, where $\mathcal{S}$ is the state space, $\mathcal{T}$ is the parameter space and $\phi : \mathcal{T} \times \mathcal{S} \longrightarrow \mathcal{S}$ is the evolution.

More specifically, we restrict the evolution to Euclidean space. Suppose the maximal interval of existence $I(\mathbf{x}_0)$ can be extended to an infinite interval $\mathbb{R}$, namely, for all $\mathbf{x}_0 \in U$, the flow $\phi(t, \mathbf{x}_0)$ admits the maximal interval of existence $I(\mathbf{x_0}) = \mathbb{R}$, then we can define the flow $\phi(t, \mathbf{x})$ for all $t \in \mathbb{R}$ and identify $\Omega = \mathbb{R} \times U$. In what follows, we write $\mathbf{x}$ rather than $\mathbf{x}_0$ since we vary the value of $\mathbf{x}_0$. Thus, it follows

Definition 2.1.3. A dynamical system is given by a smooth flow

$$\phi : \mathbb{R} \times U \longrightarrow U \tag{2.1.3}$$

where U is an open subset of $\mathbb{R}^n$ and $\phi(t, \mathbf{x}) = \phi_t(\mathbf{x})$ satisfies

1) $\phi_0(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in U$,

2) $\phi_t \circ \phi_s(\mathbf{x}) = \phi_{t+s}(\mathbf{x})$ for all $t, s \in \mathbb{R}$ and $\mathbf{x} \in U$.

Suppose (2.1.1) has a maximal interval of existence $\mathbb{R}$, we can say that its flow $\phi(t, \mathbf{x})$ is a dynamical system on U defined by (2.1.1), namely, it is associated with the vector field X

$$X = \frac{d}{dt}\phi(t, \mathbf{x})\bigg|_{t=0}, \quad t \in \mathbb{R}, \ \mathbf{x} \in U, \tag{2.1.4}$$

where $X = (f_1, \ldots, f_n)$ or $X = f_1 \dfrac{\partial}{\partial x_1} + f_2 \dfrac{\partial}{\partial x_2} + \cdots + f_n \dfrac{\partial}{\partial x_n}$.

2.2 Lie group theory

Throughout the thesis we base our model on smooth manifolds. Let us consider the following

Definition 2.2.1. A smooth manifold or, simply, manifold M is a topological space with a family of pairs $\{(U_i, \ \phi_i)\}$, where

(1) $\{U_i\}$ is a family of open subsets covering M, namely, $\bigcup_i U_i = M$,

(2) ϕ is a homeomorphism from U_i onto an open subset $V_i \subset \mathbb{R}^m$, which is called *the coordinate function*, or *the coordinate*,

(3) given U_i and U_j such that $U_i \bigcap U_i \neq \emptyset$, the map $\psi_{ij} = \phi_i \circ \phi_j^{-1}$ from $\phi_j(U_i \bigcap U_j)$ to $\phi_i(U_i \bigcap U_j)$ is infinitely differentiable.

The pair $\{(U_i, \phi_i)\}$ is called *a coordinate chart* and the family of the pair is called *an atlas*. Roughly speaking, a manifold is a topological space which locally looks like Euclidean space, for example, $M = \mathbb{R}^n$ is a trivial manifold. Correspondingly, we define a submanifold $N \subset M$ as follows:

Definition 2.2.2. Let M be a smooth manifold. Then a submanifold is a subset $N \subset M$ with an *embedding* $f : \tilde{N} \longrightarrow N \subset M$, where $\tilde{N}$ is a different manifold and N is the image of f. The dimension of N is same as $\tilde{N}$, and does not exceed the dimension of M.

Remark 2.2.3. *An embedding $f : \tilde{N} \longrightarrow N$ is* an injection *and* an immersion. *An injection is a one-to-one function. The map f is called* an immersion *if the induced map $f_* : T_pM \longrightarrow T_{f(p)}N$ is an injection, that is, rank $f_* =$ rank $\tilde{N}$, where T_pM and $T_{f(p)}M$ denote tangent spaces of M and N, respectively. Roughly speaking, the immersion means that the first derivative of f never vanishes considering $\tilde{N}$ is a parameter space.*

An important example of a manifold is a Lie group, which is a group with a manifold structure.

Definition 2.2.4. An r-dimensional Lie group G is a group admitting a structure of an r-dimensional manifold, that is, the group operation

$$m : G \times G \longrightarrow G, \quad m(g,h) = g \cdot h, \quad g, h \in G, \tag{2.2.1}$$

and the inversion

$$i : G \times G, \quad i(g) = g^{-1}, \quad g \in G, \tag{2.2.2}$$

define smooth maps between manifolds.

An r-dimensional Lie group is often referred to as an r-parameter group. In practice, a Lie group is associated with a specific group of transformations. In some cases, one may only consider a local group.

Definition 2.2.5. Let M be a smooth manifold. *A local group of transformations* acting on M is given by a (local) Lie group G and a smooth map

$$\Psi : \mathcal{U} \longrightarrow M, \tag{2.2.3}$$

where $\mathcal{U}$ is an open subset of $G \times M$, which must include the Lie group identity e, satisfying the group properties:

(1) *Associativity.* $\Psi(g, \Psi(h, x)) = \Psi(g \cdot h, x)$, where $(h, x), (g, \Psi(h, x)), (g \cdot h, x) \in \mathcal{U}$.

(2) *Identity.* $\Psi(e, x) = x$ for all $x \in M$.

(3) *Inverse.* $\Psi(g^{-1}, \Psi(g, x)) = x$, where $(g, x), (g^{-1}, \Psi(g, x)) \in \mathcal{U}$.

Note when $\mathcal{U} = G \times M$, then Ψ is called *a global group of transformations*. For our convenience, we can denote $\Psi(g, x)$ by $g \cdot x$. Hence, we denote a group of transformations by either Ψ or G. We can also check that for each $x \in M$, g forms a local Lie group $G_x = \{g \in G : (g, x) \in \mathcal{U}\}$. We only investigate a *connected* group of transformations in this thesis. It is connected in the sense that G, M, $\mathcal{U}$ and G_x are all connected, namely, they cannot be represented by a union of two or more disjoint non-empty subsets.

Consider a smooth manifold M. *The tangent space* of M is the collection of all tangent vectors to all possible curves passing through a given point $p \in M$. *The tangent bundle* TM of M is the collection of all tangent spaces corresponding to all points p in M, that is,

$$TM = \bigcup_{p \in M} T_p M. \tag{2.2.4}$$

Then it gives rise to a bundle (TM, τ_M, M), where $\tau_M : TM \longrightarrow M$ is a projection map. A vector field is a section of the bundle (TM, τ_M, M), namely, a vector field on M is given by a smooth map $X : M \longrightarrow TM$. We can see that for each $p \in M$,

$X(p) \in T_pM$. Hence, a vector field is a smooth assignment of each point in M to the tangent vector $X(p)$ or, commonly denoted, X_p in T_pM. In local coordinates $(x_1, \ldots, x_n)$, where $n = \dim M$, a vector field X, which we have seen in Section 2.1, can be represented by

$$X = f_1(\mathbf{x})\frac{\partial}{\partial x_1} + f_2(\mathbf{x})\frac{\partial}{\partial x_2} + \cdots + f_n(\mathbf{x})\frac{\partial}{\partial x_n}, \tag{2.2.5}$$

where $f_i(\mathbf{x})$, $i = 1, \ldots, n$ are smooth functions of $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. The dual space of T_pM at the point p on M is called a cotangent space, denoted by T_p^*M. A cotangent bundle is a collection of all cotangent spaces at each point on M, that is,

$$T^*M = \bigcup_{p \in M} T_p^*M. \tag{2.2.6}$$

The element $\omega : T_pM \longrightarrow \mathbb{R}$ on T_p^*M, which is a linear functional on T_pM, is called *a one-form*. In local coordinates, a one-form can be presented in the following form:

$$\omega = f_1(\mathbf{x})dx_1 + f_2(\mathbf{x})dx_2 + \cdots + f_n(\mathbf{x})dx_n, \tag{2.2.7}$$

where $f_i(\mathbf{x})$, $i = 1, \ldots, n$ are smooth functions defined on M. We can define an inner product between a one-form and a vector field in local coordinates as follows

$$< dx_i, \frac{\partial}{\partial x_j} >= \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j, \end{cases} \quad i, j = 1, \ldots, n, \tag{2.2.8}$$

where δ_{ij} is the Kronecker delta.

The flow of a vector field has been discussed in Section 2.1. We often call a flow $\phi(t, \mathbf{x})$ a *one-parameter group of transformations*. Then the vector field X is called *the infinitesimal generator* of the action defined by the equation (2.1.4). The flow generated by X is identical to the given local action of $\mathbb{R}$ on M guaranteed by the uniqueness of solutions to (2.1.2). Hence, the local one-parameter group of transformations and its infinitesimal generator are uniquely related.

Let us briefly review the operations defined on vector fields. Suppose X, Y are vector fields on M, *a Lie bracket* is an operator assigning the two vector fields to a vector field $[X, Y]$ on M given by

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \tag{2.2.9}$$

where f is a smooth function on M.

The Lie bracket has the following properties:

(1) *Bilinearity*

$$\begin{aligned} [\lambda X + \mu Y, Z] &= \lambda[X, Z] + \mu[Y, Z], \\ [X, \lambda Y + \mu Z] &= \lambda[X, Y] + \mu[X, Z], \end{aligned} \tag{2.2.10}$$

where λ, μ are constants and X, Y, Z are vector fields on M.

(2) *Skew-symmetry*

$$[X, Y] = -[Y, X], \tag{2.2.11}$$

(3) *Jacobi Identity*

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0. \tag{2.2.12}$$

Lastly, let us briefly discuss the concept of a Lie algebra. Algebraically, let us consider *the right multiplication map* of a Lie group G. For any group element $g \in G$, *the right multiplication map* $R_g : G \longrightarrow G$ defined by $R_g(h) = h \cdot g$, $h \in G$, is a diffeomorphism, a bijective differential map on G whose inverse is also differentiable, with inverse $R_{g^{-1}} = (R_g)^{-1}$. Then consider all *right-invariant* vector fields X on G. They are right-invariant in the sense that

$$dR_g(X|_h) = X|_{R_g(h)} = X|_{hg}, \tag{2.2.13}$$

where $dR_g : T_gG \longrightarrow T_{R(g)G}$ is an induced map, T_gG and $T_{R(g)}$ are tangent spaces of G at the point g and $R(g)$, respectively. To have a better understanding, let us consider the following

Example 2.2.6. Consider a simple Lie group $G = \mathbb{R}$. Let us define a right-multiplication map, *i.e.,* a translation $R_a(x) = x + a$, where a is a constant. Obviously, $X = \frac{\partial}{\partial x}$ is right-invariant since

$$dR_a(X) = X(R_a(x))\frac{\partial}{\partial x} = \left(\frac{\partial(x+a)}{\partial x}\right)\frac{\partial}{\partial x} = \frac{\partial}{\partial x} = X. \tag{2.2.14}$$

Remark 2.2.7. *Although R_g is commonly called a right-multiplication map, it denotes a general right group operation.*

Hence, the set of all right-invariant vectors form a vector space since it satisfies

(1) *Identity.* A zero vector field $\mathbf{0}$ must be right-invariant since $dR_g(\mathbf{0}|_h) = \mathbf{0}|_{hg}$.

(2) *Closure under Addition and Scalar Multiplication.*

$$dR_g((aX + bY)|_h) = aX|_{hg} + bY|_{hg}, \tag{2.2.15}$$

where a, b are constants and X and Y are vector fields on G.

Thus, it follows

Definition 2.2.8. *The Lie algebra* of a Lie group G is the vector space of all right invariant vector fields on G, conventionally denoted by $\mathfrak{g}$.

An algebra is a vector space with a bilinear operation. Hence, we can also describe a Lie algebra by considering only finite-dimensional vector spaces as follows

Definition 2.2.9. A Lie algebra is a vector space $\mathfrak{g}$ with a Lie bracket

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}, \tag{2.2.16}$$

which satisfies the above three properties (2.2.10), (2.2.11) and (2.2.12).

Geometrically, a Lie algebra is tangent to its Lie group at the identity and characterizes a Lie group locally, *e.g.*, the infinitesimal generator of the one-parameter group is an element in the corresponding Lie algebra. Nevertheless, the Lie algebra is a powerful tool, which, for example, enables us to consider a linear infinitesimal condition rather than complicated conditions of invariance under the corresponding group actions.

2.3 Jet bundles

We start by reviewing the basic terminology of the theory of jet bundles.

Let us consider a bundle (E, π, M) where E and M are smooth manifolds with dim $E = m + n$, dim $M = m$ and π is a surjective submersion. A submersion is a map between two manifolds whose differential map is a surjective linear map. As a shorthand notation, we denote (E, π, M) by π. Then a map $\phi : M \longrightarrow E$ is a *section* of π if $\phi \circ \pi = id_M$ where id_M denotes an identity map. One can also define a local section of a open submanifold of M. If $p \in M$ then the set of all local sections of π whose domains contain p is denoted by $\Gamma_p(\pi)$. $\phi(p)$ or $\pi^{-1}(p)$ is called a fibre of π over p.

Let $u : U \longrightarrow \mathbb{R}^{n+m}$ be a coordinate system on the open set $U \subset E$. The coordinate system u is called *an adapted coordinate system*, if $a, b \in U$ and $\pi(a) = \pi(b) = p$, then $pr_1(u(a)) = pr_1(u(b))$, where $pr_1 : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^m$. Thus (U, u), where one can choose $u = (x^i, u^j)$, is called an adapted coordinate chart on E. Presenting this in terms of a commutative diagram, we have

$$\begin{array}{ccc} U & \xrightarrow{\;u\;} & \mathbb{R}^{m+n} \\ {\scriptstyle \pi|_U}\big\downarrow & & \big\downarrow{\scriptstyle pr_1} \\ \pi(U) & \xrightarrow{\;x\;} & \mathbb{R}^m \end{array},$$

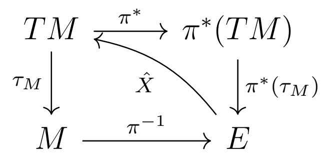
where $U \subset E$ and $\pi(U) \subset M$.

Let (H, ρ, N) be another bundle, then *a bundle morphism* from π to ρ is a pair $(f, \bar{f})$ where $f : E \longrightarrow H$, $\bar{f} : M \longrightarrow N$ and $\rho \circ f = \bar{f} \circ \pi$. The map $\bar{f}$ is called *the projection* of f. If a vector field X on E is also the bundle morphism, then the vector field is called *a projectable vector field* to π.

Let (V, v) be an adapted coordinate system on ρ where $v = (y^\alpha, v^\beta)$. Then the coordinate representation of the morphism f is given by the pair (f^α, f^β) where

$$f^\alpha = y^\alpha \circ f, \;\; f^\beta = y^\beta \circ f. \tag{2.3.1}$$

Now consider a tangent bundle (TM, τ_M, M) and the corresponding pullback bundle by π is $(\pi^*(TM), \pi^*(\tau_M), E)$. Thus, the pair (π^*, π^{-1}) is a natural bundle morphism, which can be presented in the following commutative diagram



where $\tau_M = \pi \circ \pi^*(\tau_M) \circ \pi^*$. One can see that a section of the bundle $\pi^*(\tau_M)$ is a map $X : E \longrightarrow \pi^*(TM)$. However, we note that we will consider the map $\hat{X} : E \longrightarrow TM$ defined by $\hat{X} = \tau_M^*(\pi) \circ X$ as a section of $\pi^*(\tau_M)$ instead of X. We will call the map $\hat{X}$ *a vector field along* π, and denote the set of all such vector fields along π by $\mathcal{X}(\pi)$.

Consider a tangent bundle (TE, τ_E, E). The triple $(V\pi, \tau_E|_{V_\pi}, E)$ is a subbundle of τ_E and is called *the vertical bundle* to π, where

$$V\pi = \left\{\xi \in TE : \pi_*(\xi) = 0 \in T_{\tau_M(\pi_*(\xi))}M\right\} \tag{2.3.2}$$

is a submanifold of TE.

A distribution Δ on an m-dimensional smooth manifold M is a vector subbundle of the the tangent bundle satisfying certain conditions, that is, $\Delta : p \longrightarrow \Delta_p$ where $p \in M$ and $\Delta_p \subset T_pM$ is a subspace of the tangent space. An ℓ-dimensional distribution Δ can be spanned by a set of independent vector fields $X_1, X_2, \ldots, X_\ell$; equivalently, it can also be determined by a set of independent differential 1-forms $\omega^1, \omega^2, \ldots, \omega^{n-\ell}$ such that any $X_i \in \Delta$ satisfies $\omega^j(X_i) = 0$, where $i = 1, \ldots, \ell$ and $j = 1, \ldots, n - \ell$. A distribution is involutive if $[X_i, X_j] \in \{X_1, X_2, \ldots, X_\ell\}$, for all $i, j = 1, 2, \ldots, \ell$, where $[\cdot, \cdot]$ denotes the Lie bracket.

First-order jet bundles

Let (E, π, M) be a bundle and let $p \in M$. Two sections $\phi, \psi \in \Gamma_p(\pi)$ are said to be *one-equivalent* at p if their graphs are tangent to each other at the point $\phi(p) = \psi(p) \in E$, that is, in some adapted coordinate system (x^i, u^j),

$$\begin{aligned} \phi(p) &= \psi(p), \\ \frac{\partial \phi^j}{\partial x^i}(p) &= \frac{\partial \psi^j}{\partial x^i}(p). \end{aligned} \tag{2.3.3}$$

The equivalence class containing ϕ is called *the one-jet of* ϕ at p and is denoted by $j^1_p\phi$.

The set of the one-jets has a natural structure of a smooth manifold. Let us review the following

Definition 2.3.1. [111] *The first jet manifold* of π is the set

$$\left\{ j^1_p\phi : p \in M, \phi \in \Gamma_p(\pi) \right\} \tag{2.3.4}$$

and is denoted by $J^1\pi$.

Moreover, the maps

$$\begin{aligned} \pi_1 : \quad J^1\pi &\longrightarrow M, \\ j^1_p\phi &\longrightarrow p, \end{aligned} \tag{2.3.5}$$

and

$$\begin{aligned} \pi_{1,0} : \quad J^1\pi &\longrightarrow E, \\ j^1_p &\longrightarrow \phi(p), \end{aligned} \tag{2.3.6}$$

are called *source and target projections* respectively. The triples $(J^1\pi, \pi_1, M)$ and $(J^1\pi, \pi_{1,0}, E)$ are bundles, where π_1 is called the *first jet bundle of* π. For any section $\phi \in \Gamma_p(\pi)$ the map

$$\begin{aligned} j^1(\phi) : \quad M &\longrightarrow J^1\pi, \\ p &\longrightarrow j^1_p(\phi), \end{aligned} \tag{2.3.7}$$

is a section of π_1 and is called the one-jet of ϕ. The fibre $\pi_1^{-1}(p) = j^1_p(\phi)$ (locally) is denoted by $J^1_p\pi$, which is a submanifold of $J^1\pi$. One needs to note that the one-jet is also *the first prolongation* of ϕ.

As a conclusion, we present the following commutative diagram

$$\begin{array}{ccc} J^1\pi & \xrightarrow{\pi_{1,0}} & E \\ {\scriptstyle \pi_1}\downarrow\uparrow{\scriptstyle j^1(\phi)} & & {\scriptstyle \pi}\downarrow\uparrow{\scriptstyle \phi} \\ M & \xrightarrow{id} & M \end{array}$$

where $j^1(\phi) \circ \pi_1 = id_M$ and $\phi \circ \pi = id_M$.

Let $p \in M$ and let (U, u) be an adapted coordinate chart on E where $u = (x^i, u^j)$. Then *the induced coordinate chart* is given by

$$\begin{aligned} U^1 &= \left\{ j^1_p : \phi(p) \in U \right\}, \\ u^1 &= (x^i, u^j, u^j_i), \end{aligned} \tag{2.3.8}$$

where $x^i(j^1_p\phi) = x^i(p)$, $u^j(j^1_p\phi) = u^j(\phi(p))$ and the new function

$$u^j_i : U^1 \longrightarrow \mathbb{R} \tag{2.3.9}$$

denotes the partial differentiation that is

$$u^j_i(j^1_p\phi) = \frac{\partial \phi^j}{\partial x^i}(p) \tag{2.3.10}$$

known as *the derivative coordinates*. Thus the 1-jet j^1_p in local coordinates is given by

$$\left(\phi^j, \frac{\partial \phi^j}{\partial x^i} \right). \tag{2.3.11}$$

Total derivatives

More generally, let $(J^1\rho, \rho_1, H)$ be another first jet bundle and the bundle morphism from π_1 to ρ_1 be given by the first prolongation of the pair $(f, \bar{f})$, namely, the map $j^1(f, \bar{f}) : J^1\pi \longrightarrow J^1\rho$ defined by

$$j^1(f, \bar{f})(j^1_p\phi) = j^1_{\bar{f}(p)}(\tilde{f}(\phi)), \tag{2.3.12}$$

where $\tilde{f}(\phi) = f \circ \phi \circ \bar{f}^{-1}|_U$ for $U \subset M$ is an open subset, and is abbreviated as $j^1 f$ if causing no confusion.

Both maps

$$(j^1 f, f) : \pi_{1,0} \longrightarrow \rho_{1,0} \tag{2.3.13}$$

and

$$(j^1 f, \bar{f}) : \pi_1 \longrightarrow \rho_1 \tag{2.3.14}$$

are bundle morphisms.

It follows that the commutative diagram

$$\begin{array}{ccc} J^1\pi & \xrightarrow{j^1 f} & J^1\rho \\ {\scriptstyle \pi_{1,0}}\downarrow & & \downarrow{\scriptstyle \rho_{1,0}} \\ E & \xrightarrow{f} & H \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \rho} \\ M & \xrightarrow{\bar{f}} & N \end{array}$$

where $f \circ \pi_{1,0} = \rho_{1,0} \circ j^1 f$ and $\bar{f} \circ \pi = \rho \circ f$.

Let (V, v) be an adapted coordinate chart on $J^1\rho$ where $v = (y^\alpha, v^\beta, v^\beta_\alpha)$. The coordinate representation of $j^1 f$ is given by

$$y^\alpha \circ j^1 f = f^\alpha, \tag{2.3.15}$$

$$v^\beta \circ j^1 f = f^\beta, \tag{2.3.16}$$

and

$$v^\beta_\alpha \circ j^1 f = \left(\frac{f^\beta}{\partial x^i} + u^j_i \frac{\partial f^\beta}{\partial u^j} \right) \left(\frac{\partial (\bar{f}^{-1})^i}{\partial y^\alpha} \circ \bar{f} \right), \tag{2.3.17}$$

where x^i, u^j, u^j_i are adapted coordinate functions on $J^1\pi$.

The expression in the first pair of parentheses in (2.3.17) is called a *total derivative* and is denoted by

$$D_i f^\beta = \frac{\partial f^\beta}{\partial x^i} + u^j_i \frac{\partial f^\beta}{\partial u^j}. \tag{2.3.18}$$

The operator

$$D_i = \frac{\partial}{\partial x^i} + u_i^j \frac{\partial}{\partial u^j} \tag{2.3.19}$$

is related to the concept of *the Cartan structure* [74] and can also be viewed as a vector field, details of which will be given in what follows.

Cartan distributions

The Cartan distribution $\mathcal{C}$ on $J^1\pi$ is a n-dimensional vector subbundle of the tangent bundle $TJ^1\pi$ and $\mathcal{C} : \theta^1 \longrightarrow \mathcal{C}_{\theta^1}$, where $\theta^1 \in J^1\pi$ and $\mathcal{C}_{\theta^1}$ is a subspace of $T_pJ^1\pi$.

Let us consider a pullback bundle $(\pi_{1,0}^*(TE), \pi_{1,0}^*(\tau_E), J^1\pi)$ and denote the vector fields along $\pi_{1,0}$ by $\mathcal{X}(\pi_{1,0})$. The bundle $\pi_{1,0}^*(\tau_E)$ admits a unique decomposition of two subbundles

$$(\pi_{1,0}^*(V\pi) \oplus \mathcal{C}, \pi_{1,0}^*(\tau_E), J^1\pi), \tag{2.3.20}$$

where $\pi_{1,0}^*(V\pi)$ denotes the vertical subbundle and $\mathcal{C}$ is *the Cartan distribution* on $J^1\pi$.

It follows that any vector field $Z \in \mathcal{X}(\pi_{1,0})$ has a canonical decomposition into its vertical and horizontal components

$$Z = Z^v + Z^h, \tag{2.3.21}$$

where $Z^v \in \pi_{1,0}^*(V\pi)$ and Z^h lies in the Cartan distribution.

Therefore, the module $\mathcal{X}(\pi_{1,0})$ can be written into two submodules

$$\mathcal{X}(\pi_{1,0}) = \mathcal{X}^v(\pi_{1,0}) \oplus \mathcal{X}^h(\pi_{1,0}), \tag{2.3.22}$$

where $\mathcal{X}^v(\pi_{1,0})$ consists of vertical vectors to $\pi_{1,o}$ and $\mathcal{X}^h(\pi_{1,0})$ consists of vectors of the Cartan distribution.

The module of differential forms $\wedge_0^1\pi_{1,0}$ dual to (2.3.22) can be correspondingly written as

$$\wedge_0^1\pi_{1,0} = \wedge_0^1\pi_1 \oplus \ \wedge_C^1\pi_{1,0}, \tag{2.3.23}$$

where $\wedge_0^1 \pi_1$ is comprised of one-forms dual to the vertical vector and $\wedge_C^1$ consists of one-forms annihilating Cartan distribution called Cartan form or contact form.

Let (x^i, u^j, u_i^j) be adapted coordinate functions on $J^1\pi$. Any vector field on M can be mapped to $J^1\pi$ in a way analogous to what we have seen in Section 2.3. The Cartan distribution $\mathcal{C}$ on $J^1\pi$ is thus spanned by

$$D_i = \frac{\partial}{\partial x^i} + u_i^j \frac{\partial}{\partial u^j}. \tag{2.3.24}$$

where $i = 1, \dots, n$ and $j = 1, \dots, n$.

The contact form dual to (2.3.24) in local coordinates enjoys the following form:

$$\omega_i^j = du^j - u_i^j dx^i. \tag{2.3.25}$$

Prolongations of vector fields

The Cartan distribution $\mathcal{C}$ on $J^1\pi$ is the main structure for us to study the first prolongation of a vector field on E, a *symmetry* of which is a diffeomorphism f of $J^1\pi$ preserving $\mathcal{C}$, that is, if $D \in \mathcal{C}$, then $f_*(D) \in \mathcal{C}$, or by duality, preserving the space of Cartan forms, namely, if ω is a Cartan form, then $f^*\omega$ is also a contact form so that f is called *a contact transformation*.

The infinitesimal symmetry or *infinitesimal contact transformation* of $\mathcal{C}$ is a vector field X^1 on $J^1\pi$ with property that if the vector field D belongs to $\mathcal{C}$, then so does the vector field $[X^1, D]$, or, according to the duality, if the one-form ω is in $\wedge_C^1 \pi_{1,0}$, so is the one-form $\mathcal{L}_{X^1}\omega$.

The following theorem [111] gives conditions for a vector field X^1 on $J^1\pi$ to be the first prolongation of a vector field X on E.

Theorem 2.3.2. *If $X^1 \in \mathcal{X}(J^1\pi)$ is projectable onto E, then X is an infinitesimal symmetry of the Cartan distribution if and only if X^1 is the prolongation of a vector field on E. If $n > 1$(n=dim E-dim M), then every infinitesimal symmetry of the Cartan distribution is necessarily projectable onto E.*

Proof. See pp. 144-145 in [111]. □

Remark 2.3.3. *If $n = 1$ but X is projectable, then the result also holds.*

Therefore, the first prolongation of a vector field X on E is a vector field X^1 on $J^1\pi$ where the components of X^1 can be determined by the contact form since X_1 is an infinitesimal symmetry, *i.e.*, $\mathcal{L}_{X^1}(\omega) \in \wedge^1_C \pi_{1,0}$ where $\omega \in \wedge^1_C \pi_{1,0}$.

2.4 Calculus of variations

We investigate the optimized welfare in Chapter 7. A classical example of a variational problem can be presented as follows. Suppose $F(x, y, y')$ is a smooth function, we want to find the necessary condition for the extremum of the following functional

$$J[y] = \int_a^b F(x, y, y')dx, \tag{2.4.1}$$

where $y(x)$, $x \in E \subset \mathbb{R}$ is a smooth function satisfying

$$y(a) = A, \quad y(b) = B. \tag{2.4.2}$$

The necessary condition is given by *the Euler-Lagrange equation*, namely,

$$F_y - \frac{d}{dx}F_{y'} = 0. \tag{2.4.3}$$

Throughout the thesis, we focus on the necessary condition for an extremum of a functional. Before we review the general condition, let us consider the following

Lemma 2.4.1. *[47] If $\alpha(x)$ and $\beta(x)$ are continuous in a finite interval $[a, b]$, and if*

$$\int_a^b [\alpha(x)h(x) + \beta(x)h'(x)]dx = 0 \tag{2.4.4}$$

for every smooth function $h(x)$ defined on $[a, b]$ such that $h(a) = h(b) = 0$, then $\beta(x)$ is differentiable and $\beta'(x) = \alpha(x)$ for all x in $[a, b]$.

Proof. Using *the fundamental theorem of calculus*, we can write

$$A(x) = \int_a^x \alpha(t)dt. \tag{2.4.5}$$

Integrating (2.4.4) by parts, we get

$$\int_a^b \alpha(x)h(x)dx = \int_a^b h(x)dA(x) = A(x)h(x)|_a^b - \int_a^b A(x)h'(x)dx. \tag{2.4.6}$$

Taking into account the boundary conditions $h(a) = h(b) = 0$, we obtain

$$\int_a^b \alpha(x)h(x)dx = -\int_a^b A(x)h'(x)dx. \tag{2.4.7}$$

Then (2.4.4) becomes

$$\int_a^b [-A(x) + \beta(x)]h'(x)dx = 0. \tag{2.4.8}$$

We want to show $-A(x) + \beta(x) = c$, where c is constant.

Let us construct

$$h(x) = \int_a^x [-A(t) + \beta(t) - c]dt, \tag{2.4.9}$$

then we can show

$$\begin{aligned} \int_a^b [-A(x) + \beta(x) - c]h'(x)dx &= \int_a^b [-A(x) + \beta(x)]h'(x)dx \\ &\quad -c(h(b) - h(a)) = 0. \end{aligned} \tag{2.4.10}$$

On the other hand, we have

$$\int_a^b [-A(x) + \beta(x) - c]h'(x)dx = \int_a^b [-A(x) + \beta(x) - c]^2 dx. \tag{2.4.11}$$

Hence, we must obtain

$$\beta(x) - A(x) = c, \tag{2.4.12}$$

differentiating gives

$$\beta'(x) = \alpha(x). \tag{2.4.13}$$

The differentiability follows from *the fundamental theorem of calculus*, namely, the structure of $A(x)$. □

Let us consider the following functional

$$J[y_1,\ldots,y_n]=\int_a^b F(x,y_1,\ldots,y_n,y_1',\ldots,y_n')dx, \tag{2.4.14}$$

where $y_i(x)$, $i=1,\ldots,n$ are smooth functions, satisfying the boundary conditions

$$y_i(a)=A_i,\quad y_i(b)=B_i,\ i=1,2,\ldots,n. \tag{2.4.15}$$

We want to find the necessary conditions of the extremum of $J[y_1,\ldots,y_n]$. For brevity, let us denote $\mathbf{y}=(y_1,\ldots,y_n)$ and $\mathbf{y}'=(y_1',\ldots,y_n')$. Consider two points $J[\mathbf{y}+\mathbf{h}]$ and $J[\mathbf{y}]$ on the functional that differ by $\mathbf{h}(x)=(h_1(x),h_2(x),\ldots,h_n(x))$, where $\mathbf{h}(x)$ is a smooth function. Thus, the increment is given by

$$\begin{aligned}\Delta J &= J[\mathbf{y}+\mathbf{h}]-J[\mathbf{y}]\\ &=\int_a^b \left[F(x,\mathbf{y}+\mathbf{h},\mathbf{y}'+\mathbf{h}')-F(x,\mathbf{y},\mathbf{y}')\right]dx,\end{aligned} \tag{2.4.16}$$

The Taylor expansion of $F(x,\mathbf{y}+\mathbf{h},\mathbf{y}'+\mathbf{h}')$ at the point $(x,y_1,\ldots,y_n,y_1',\ldots,y_n')$ yields

$$\begin{aligned}\Delta J &= \int_a^b \Bigg[F(x,\mathbf{y},\mathbf{y}')+\sum_{i=1}^n((y_i+h_i-y_i)F_{y_i})+\sum_{i=1}^n((y_i'+h_i'-y_i')F_{y_i'})\\ &\qquad -F(x,\mathbf{y},\mathbf{y}')+\ldots\Big]dx\\ &=\int_a^b\left[\sum_{i=1}^n\left(F_{y_i}h_i+F_{y_i'}h_i'\right)+\ldots\right]dx,\end{aligned} \tag{2.4.17}$$

where $\ldots$ denotes terms with higher degrees in h. The necessary condition for the extremum is

$$\delta J=\int_a^b\sum_{i=1}^n\left(F_{y_i}h_i+F_{y_i'}h_i'\right)dx=0, \tag{2.4.18}$$

where δJ is called *the variation* of J.

Remark 2.4.2. *The variation $\delta J=0$ is called* the principle of least action.

Since $\mathbf{h}$ is an independent function, we can choose, for example, $h_1 \neq 0$ and $h_2, \ldots, h_n = 0$, then choose $h_2 \neq 0$ and $h_1, h_3, \ldots, h_n = 0$, *etc.*, until we choose $h_n \neq 0$ and $h_1, h_2, \ldots, h_{n-1} = 0$. Under each of the assumptions, we reduce the problem to the classical example at the beginning of the section and derive the corresponding Euler-Lagrange equation. Therefore, (2.4.18) implies that

$$\int_a^b \left(F_{y_i} h_i + F_{y_i'} h_i' \right) dx = 0, \quad i = 1, \ldots, n, \tag{2.4.19}$$

Using Lemma 2.4.1, we obtain the following system of Euler-Lagrange equations

$$F_{y_i} - \frac{d}{dx} F_{y_i'} = 0, \quad i = 1, \ldots, n, \tag{2.4.20}$$

which are the necessary condition for the extremum of (2.4.14).

Next, we review the variations with subsidiary conditions, which is given in the following

Theorem 2.4.3. *[47] Given the functional*

$$J[y, z] = \int_a^b F(x, y, z, y', z') dx. \tag{2.4.21}$$

Suppose the admissible curves $y(x), z(x)$, where x is in the open subset $E \subset \mathbb{R}$, lie on the surface

$$g(x, y, z) = 0 \tag{2.4.22}$$

and satisfy the boundary conditions

$$y(a) = A_1, \quad y(b) = B_1, \tag{2.4.23}$$

$$z(a) = A_2, \quad z(b) = B_2. \tag{2.4.24}$$

Suppose we allow $J[y]$ to attain the extremum along the curves

$$y = y(x), \quad z = z(x). \tag{2.4.25}$$

If g_y and g_z do not vanish simultaneously on the surface defined by $g(x,y,z) = 0$, then there exists a function $\lambda(x)$ such that (2.4.25) *is an extremal of the functional*

$$\int_a^b [F + \lambda g]dx \tag{2.4.26}$$

satisfying the differential equations

$$\begin{aligned} F_y + \lambda g_y - \frac{d}{dx}F_{y'} = 0, \\ F_z + \lambda g_z - \frac{d}{dx}F_{z'} = 0. \end{aligned} \tag{2.4.27}$$

The proof is similar to the derivation of the necessary condition for extremum of (2.4.14). Details can be found on pp. 46-47 in [47]. Note that the function $F(x,y,z,y',z')$ may only depend on y or z. It can be further extended by considering a surface $g(x,y,z,y',z') = 0$, which is shown in the following

Remark 2.4.4. *Let us consider the admissible curves in Theorem 2.4.3 defined on a smooth space given by*

$$g(x,y,z,y',z') = 0. \tag{2.4.28}$$

If the functional has an extremum along a curve γ, subject to the surface (2.4.28)*, assuming g_z and g_y do not vanish simultaneously along γ, then there exists a function $\lambda(x)$ such that* (2.4.26) *attains the extremum along the curve γ determined by the following system*

$$\Phi_y - \frac{d}{dx}\Phi_{y'} = 0, \quad \Phi_z - \frac{d}{dx}\Phi_{z'} = 0, \tag{2.4.29}$$

where $\Phi = F + \lambda G$.

2.5 Hamiltonian formalism via Poisson geometry

All models that we discuss in Chapter 4 will be studied within the framework of the Hamiltonian systems defined on Poisson manifolds. We denote the space of smooth functions on a manifold M by $C^\infty(M)$. Recall the following

Definition 2.5.1. *A Poisson structure* on a manifold M is a bilinear bracket

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M), \tag{2.5.1}$$

satisfying the following properties

(1) *Skew-Symmetry*

$$\{f, g\} = -\{g, f\}, \tag{2.5.2}$$

(2) *Leibniz Rule*

$$\{f, gh\} = \{f, g\}h + g\{f, h\}, \tag{2.5.3}$$

(3) *Jacobi Identity*

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0, \tag{2.5.4}$$

where $f, g, h \in C^\infty(M)$.

In canonical coordinates (q_i, p_i), $i = 1, \ldots, n$, we define the Poisson bracket as follows:

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right), \tag{2.5.5}$$

where f, g are smooth functions.

The pair of a manifold M and a Poisson structure $\{\cdot,\cdot\}$ defined on M, is called a *Poisson manifold.* Next, let $(M, \{\cdot,\cdot\})$ be a Poisson manifold, then the vector field X_H given by

$$X_H = \{\cdot, H\}$$

is called the *Hamiltonian vector field* determined by the *Hamiltonian function* H. Note that the value of $\{f, g\}$ at any point $p \in M$ depends linearly on the differentials $\mathrm{d}f, \mathrm{d}g$ at $p \in M$. We denote the space of all bivectors on M by $\mathfrak{X}^2(M)$. Let us consider $\Lambda^2 TM$, which is the second order exterior derivative of the tangent bundle TM. By analogy with the definition of a vector field, a bivector field is a section of a bundle $(\Lambda^2 TM, \tau_{TM}, TM)$. The set of all sections in the bundle denoted by $\Gamma(\Lambda^2 TM)$

is equivalent to $\mathfrak{X}^2 M$. In this view, the bracket $\{\cdot,\cdot\}$ gives rise to a Poisson bivector field $\pi \in \mathfrak{X}^2(M) = \Gamma(\Lambda^2 TM)$ such that

$$\pi(\mathrm{d}f, \mathrm{d}g) = \{f, g\},$$

for all $f, g \in C^\infty(M)$. Conversely, given a Poisson bivector $\pi \in \Gamma(\Lambda^2 TM)$, then π defines the corresponding Poisson bracket satisfying the properties specified above. In local coordinates $(x_1, \ldots, x_n)$, where $n = \dim M$, a bivector π can be presented in the following form

$$\pi = \sum_{i,j=1}^{n} \pi_{ij}(\mathbf{x}) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \tag{2.5.6}$$

where $\pi_{ij}(\mathbf{x}) = \{x_i, x_j\}$ is a smooth function depending on $\mathbf{x} = (x_1, \ldots, x_n)$.

In what follows we will refer to a Poisson manifold as a pair (M, π), which gives rise to the following definition of a Hamiltonian vector field:

$$X_H = \pi dH, \tag{2.5.7}$$

or, in terms of local coordinates $(x_1, \ldots, x_n)$ in a neighbourhood of $p \in M$ using Einstein summation convention,

$$X_H = \pi^{i\ell} \frac{\partial H}{\partial x_\ell} \frac{\partial}{\partial x_i}, \tag{2.5.8}$$

where $\pi^{i\ell}$ are components of π. See, for example, Fernandes and Mărcut [39] for more details. Note that in [39] the authors defined the canonical coordinates as (p_i, q_i), $i = 1, \ldots, n$. Hence, all the structures in [39] have p_i and q_i switched. We note, however, that most of the above formulas can be represented in a uniform way via the Schouten bracket $[\cdot,\cdot]$ [72, 39] that can be defined as follows

Definition 2.5.2. Let $\mathfrak{v} \in \mathfrak{X}^k(M)$ and $\mathfrak{y} \in \mathfrak{X}^l(M)$ be multi-vector fields. *The Schouten bracket* of $\mathfrak{v}$ and $\mathfrak{y}$ is the multi-vector field $[\mathfrak{v}, \mathfrak{y}] \in \mathfrak{X}^{k+l-1}(M)$ defined by

$$[\mathfrak{v}, \mathfrak{y}] = \mathfrak{v} \circ \mathfrak{y} - (-1)^{(k-1)(l-1)} \mathfrak{y} \circ \mathfrak{v}, \tag{2.5.9}$$

where $\mathfrak{v} = v_1 \wedge \ldots \wedge v_k$, $v_i \in \mathfrak{X}(M)$, $i = 1, \ldots, k$ and $\mathfrak{y} = y_1 \wedge \ldots \wedge y_l$, $y_i \in \mathfrak{X}(M)$, $i =$

$1, \ldots, l$.

We can see that the Schouten bracket is a natural generalization of the Lie bracket defined on multi-vector fields. Thus, for instance, the Jacobi identity condition for a Poisson bracket $\{\cdot, \cdot\}$ defined by a Poisson bivector π is simply equivalent to the condition $[\pi, \pi] = 0$. A Hamiltonian vector field X_f defined on a Poisson manifold (M, π) can be now determined as

$$X_f = [\pi, H].$$

Similarly, the Poisson bracket of any functions $f, g \in C^\infty(M)$ defined on a Poisson manifold (M, π) may be defined via the Schouten bracket as

$$\{f, g\} = [[\pi, f], g]$$

and so on (see, for example, [115] for more details).

2.6 Statistical tools

We employ the curve fitting techniques in what follows. All regressions in the thesis are conducted using the R programming language. Let us briefly review the definition of "*a best-fitting curve*". Suppose we have a dataset (x_i, y_i), $i = 1, \ldots, n$, and we want to fit a curve C to the given dataset. Let us denote the estimated values by $(x_i, \hat{y}_i)$, $i = 1, \ldots, n$. Note data arranged according to time are called *time series*. The difference $\hat{y}_i - y_i$ is referred to as a *residual*, *error* or *deviation*. A measure of the *goodness of fit* of a curve C to the given data is determined by the *residual sum of squares* $\sum\limits_{i=1}^{n}(\hat{y}_i - y_i)^2$, which is commonly abbreviated by RSS. Hence, we have the following

Definition 2.6.1. Of all curves approximating a given set of data points, the curve having the minimum RSS is called *a best-fitting curve*.

The method of fitting a curve to a given set of data having the minimum value of RSS is called *the method of least squares*. A derived curve using the method is called *a* *least-squares curve* or *a regression curve*. The method can be applied to a multilinear regression model as follows

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n + \epsilon = \beta_0 + \sum_{i=1}^{n} \beta_i x_i + \epsilon, \tag{2.6.1}$$

where parameters β_i, $i = 0, 1, \ldots, n$, are called *regression coefficients* and ϵ denotes the residual.

Suppose we have a given dataset (x_i, y_i), $i = 1, \ldots, n$. Furthermore, the fitting of the model (2.6.1) to the data can be expressed as

$$y_i = \beta_0 + \sum_{j=1}^{n} \beta_j x_{ij} + \epsilon_i, \; i = 1, \ldots, n, \tag{2.6.2}$$

where x_{ij} denotes the value of the corresponding variable x_j.

Then, the RSS is given by

$$RSS(\beta_0, \beta_1, \ldots, \beta_n) = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{n} \beta_j x_{ij} \right)^2 . \tag{2.6.3}$$

The values of regression coefficients $\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_n$ of the best-fitting curve are determined by the following system of linear equations

$$\frac{\partial RSS}{\partial \beta_0} = -2 \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{n} \beta_j x_{ij} \right) \tag{2.6.4}$$

and

$$\frac{\partial RSS}{\partial \beta_j} = -2 x_{ij} \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{n} \beta_j x_{ij} \right), \; j = 1, \ldots, n. \tag{2.6.5}$$

A more general regression approach can be stated in terms of the Gauss-Markov theorem, which gives the values of regression coefficients of a best-fitting linear model based on the unbiased requirement. More details about the theorem can be found in [78].

Let us briefly discuss the unbiased requirement. We illustrate it with the following linear model

$$y = \beta_0 + \beta_1 x. \tag{2.6.6}$$

Suppose the observed values of regression coefficients are $\hat{\beta}_0$ and $\hat{\beta}_1$. Then, the results are unbiased provided that $E(\hat{\beta}_0) = \beta_0$ and $E(\hat{\beta}_1) = \beta_1$, where $E(\cdot)$ denotes the expected value. It can be shown by using (2.6.4) and (2.6.5).

Note that the method of least squares can be also applied to a non-linear model. Some non-linear models can be turned into linear models using proper transformations. For example, an exponential model $y = Aa^x$ can be transformed into a linear model $\ln y = \ln A + x \ln a$ by taking the logarithm. We need to mention that the parameters minimizing the RSS of transformed model may not necessarily minimize the residuals of the original model. One can employ the Gauss-Newton method to compare a non-linear model to a given set of data using the method of least squares.

Suppose we have a non-linear model

$$y = f(\mathbf{x}; \beta) + \epsilon, \tag{2.6.7}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is the vector of variables and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ denotes the vector of parameters.

The Taylor expansion of (2.6.7) around the value $\beta_0 = (\beta_{01}, \beta_{02}, \ldots, \beta_{0n})$ is given by

$$p(\mathbf{x}; \beta) = f(\mathbf{x}; \beta_0) + \sum_{i=1}^{n} f_{\beta_i}(\mathbf{x}; \beta_0)(\beta_i - \beta_{0i}) + \epsilon, \tag{2.6.8}$$

where $f_{\beta_i}(\mathbf{x}; \beta_0) = \dfrac{\partial f(\mathbf{x}; \beta_0)}{\partial \beta_i}$.

Then the sum of squared residuals of a given dataset is given by

$$RSS(\beta_1, \ldots, \beta_n) = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - f(\mathbf{x}_{ij}; \beta_0) - \sum_{j=1}^{n} f_{\beta_j}(\mathbf{x}_{ij}; \beta_0)(\beta_j - \beta_{0j}) \right)^2, \tag{2.6.9}$$

where $\mathbf{x}_{ij} = (x_{1j}, x_{2j}, \ldots, x_{nj})$, $j = 1, \ldots, n$, denotes the data points.

Then the regression coefficients of the best-fitting curve are attained at

$$\frac{\partial RSS}{\partial \beta_j} = -2 f_{\beta_j}(\mathbf{x}_{ij};\beta_0) \sum_{i=1}^{n} \left(y_i - f(\mathbf{x}_{ij};\beta_0) - \sum_{j=1}^{n} f_{\beta_j}(\mathbf{x}_{ij};\beta_0)(\beta_j - \beta_{0j}) \right), \quad (2.6.10)$$

where $j = 1, \ldots, n$.

There are many different non-linear regression methods, more discussions of which can be found in [78, 61, 100] and the relevant references therein.

Chapter 3

In search of a new economic growth model determined by logistic growth

Empirical estimates of an aggregate production function play a pivotal role in any economic growth model. Normally, such a function relates the output and inputs and can be used either to estimate the output of a model by studying the input factors or to study the dynamics of other quantities.

A two-dimensional economic growth model can be described by a generic production function $Y = f(K, L)$, where $K = K(t)$ is a time-dependent capital function, $L = L(t)$ is a time-dependent labor function and Y is the production function. Recall the Cobb-Douglas production function is given by

$$Y = f(K, L) = AK^{\alpha}L^{\beta}, \tag{3.0.1}$$

where A is the total factor productivity while α and $\beta \geq 0$ are the output elasticity of capital and labor, respectively. It is said to have constant returns to scale when

$$\alpha + \beta = 1. \tag{3.0.2}$$

The production function was derived by Charles Cobb and Paul Douglas [27] by studying the American economic data during the period of 1899-1922. Solow [120] and Stigler [125] studied production of an economic growth model to observe that the increase of output is not proportional to the growth of capital and labor and thus realized that the technical progress also contributed to the production. This phenomenon is called the "Solow-Stigler controversy". More recently, Sato [107, 108] (see also Sato and Ramachandran [110]), while resolving the "Solow-Stigler controversy", developed a Lie group theoretical framework to study technical progress and production functions. Sato [108] showed that we can treat technical progress as economies of scale and identify the corresponding exogeneous technical progress with the action of a one-parameter Lie group that acts in $C^2(\mathbb{R}_+^2)$, where $\mathbb{R}_+^2 = \{(K, L)|K, L \in \mathbb{R}_+\}$. Then, a production function can be derived as an invariant under the action of a one-parameter group. For instance, within this framework the Cobb-Douglas production function (3.0.1) can be recovered as an invariant of the one-parameter Lie group action [28] that afford exponential growth in both K and L in the first quadrant of the two-dimensional Euclidean space $\mathbb{R}_+^2$. The principle of invariance, which can be traced back to Emma Noether, who in 1918 demonstrated the fundamental invariance principle known as Neother's theorem, allows one to study the more general invariance of a dynamical system [1]. This principle has been employed also by other mathematical economists. As an illustration, Samuelson [102] introduced the conservation law of the aggregate capital-output ratio in a neoclassical von Neumann economic model in 1970. Then a new conservation law of ratio between the national wealth and income was established in 1981 by Sato [108]. Sato and his collaborator's work that utilizes Lie group theoretical methods is very fruitful, but it mainly focuses on the case where both labor and capital grow exponentially. We extend their work by presenting a new economics growth model under the assumption that labor and capital follow a logistical growth in [118]. The new economics model is characterized by a new production function and we have shown that it compares reasonably well against US economic data for the period 1947-2016. We believe the new growth model can be used to describe some aspects of the current economy. We have also observed within the framework of the new model that the Bowley's law, a stylized fact of economics stating the constant of wage share among different countries, no longer holds true in the post-1960 data. In addition, we have a fairly rigorous mathematical explanation of the phenomenon by using a projective logistic transformation group. This transformation group leads to a new economic invariant, which we believe can be viewed as a new notion of wage share.

3.1 A Lie group approach to the study of holothetic production functions

In this section we will briefly review the Lie group theoretical approach developed by Sato to study holothetic production functions and employ it to derive the Cobb-Douglas production function (3.0.1).

In order to show that the increases in efficiency of inputs due to technical progress can be explained by economies of scale, Sato interpreted technical progress as the action of a one-parameter Lie group of transformations. Accordingly, one can introduce the following

Definition 3.1.1. *A Lie type of technical progress* in an economics growth model acting on $\mathbb{R}^2_+$ is a one-parameter group of transformations given by a Lie group G

$$T : G \times \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+ \tag{3.1.1}$$

where T is a smooth map satisfying all group properties.

Remark 3.1.2. *It is important to clarify that Sato defined the technical progress in [108] in a more general way, that is, T may not have a group structure and be a one-parameter group transformation. But we are, from the mathematical viewpoint, particularly interested in the group-structured technical progress, that is, we mainly consider the technical progress that can be identified as a continuous one-parameter group. Hence, the technical progress in the following context mostly refers to the Lie type of technical progress defined in Definition 3.1.1.*

Remark 3.1.3. *For our convenience, we denote $T(G, \mathbb{R}^2_+)$ by $(G, \mathbb{R}^2_+)$. Note that $G = \mathbb{R}$ in most cases. We will also use $(G, \mathbb{R}^2_+)$ to denote an economic growth model.*

Suppose that a technical progress T is defined by the functions ϕ and ψ such that

$$T_t : \quad \bar{K} = \phi(K, L, t), \quad \bar{L} = \psi(K, L, t), \tag{3.1.2}$$

where t is the technical progress parameter and the functions ϕ, ψ are analytic and functionally independent.

Sato [110] defined the holothetical functions as follows:

Definition 3.1.4. When the technical progress T acting on a production function f can be represented by some strictly monotone transformation F, then the production is said to be *holothetic* to the technical progress T, *i.e.*,

$$f(T_t(K, L)) = f(\bar{K}, \bar{L}) = F_t(f). \tag{3.1.3}$$

If the technical progress T_t is in Definition 3.1.1, we can have the following

Lemma 3.1.5 (Fundamental Lemma on Holotheticity)**.** *A production function f is holothetic to a technical progress T_t iff the production function f is invariant under a group action.*

Proof. According to Definition 3.1.4, if T_t is a group of transformations, then f is invariant under the T_t, and vice versa. □

More specifically, T_t preserves the isoquant map of Y (or, in mathematical terms, level curves of Y), that is, we can interpret the action as the mapping of one level curve (representing a production level) to another defined by T_t. Hence, the technical progress has the same effect as an economy of scale. As a result, a production function is an invaraint under T_t.

More specifically, let capital and labor affected by technical progress and measured in the efficiency units, $\bar{K}$ and $\bar{L}$, be given by

$$\bar{K} = \lambda_1 K, \quad \bar{L} = \lambda_2 L, \tag{3.1.4}$$

where λ_1 and λ_2 represent the effect of the exogenous technical progress. Following Sato and Ramachardan [110], let us remark that if $\lambda_1 = \lambda_2$ the change generated by technical progress is Hicks-neutral. If technical progress is factor augmenting and biased, then $\lambda_1 \neq \lambda_2$. The functions λ_i, $i = 1, 2$ may depend on t only, or they may be functions of K/L, which would imply that the rate of technical progress on different rays are different, but the rate is constant on each of them. More generally, the functions λ_i, $i = 1, 2$ can be functions of the form $\lambda_i(K, L, t)$, which would entail that the rate of technical progress will also vary along a ray. In what follows, we will also require that the technical progress functions λ_i, $i = 1, 2$ represent the action of a one-parameter Lie group.

Consider now the case when both $\lambda_i = \lambda_i(t)$, $i = 1, 2$. Moreover, let $\lambda_1(t) = e^{\alpha t}$, $\lambda_2(t) = e^{\beta t}$, $\alpha, \beta \geq 0$. Note, if $\alpha = \beta$ the change generated by such technical progress is Hicks-neutral. Clearly, the corresponding transformations

$$\bar{K} = e^{\alpha t} K, \quad \bar{L} = e^{\beta t} L \tag{3.1.5}$$

form a continuous one-parameter Lie group, which follows from the fact, for example, that transformation (3.1.5) determines the flow

$$\sigma(t, (K, L)) = \begin{bmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{bmatrix} \begin{bmatrix} K \\ L \end{bmatrix} \tag{3.1.6}$$

generated by the following vector field

$$U = \alpha K \frac{\partial}{\partial K} + \beta L \frac{\partial}{\partial L}, \tag{3.1.7}$$

which generates the Lie algebra of the one-parameter Lie group $G = \{g \,|\, g = \sigma_t, t \in \mathbb{R}\}$, where $\sigma_t : \mathbb{R}^2 \to \mathbb{R}^2$ is determined by (3.1.6) for each fixed $t \in \mathbb{R}$.

Let us also suppose the family of transformations T_t (3.1.2) forms a one-parameter Lie group (as per Definition 3.1.1). Then the infinitesimal generator of T_t is given by

$$U = \xi(K, L) \frac{\partial}{\partial K} + \eta(K, L) \frac{\partial}{\partial L}, \tag{3.1.8}$$

where $\xi(K, L) = \left(\frac{\partial \phi}{\partial t}\right)_{t=0}$, $\eta(K, L) = \left(\frac{\partial \psi}{\partial t}\right)_{t=0}$.

Recall that Sato formulated the following theorem [108]:

Theorem 3.1.6. *A production function f is holothetic under a continuous one-parameter Lie group of transformatoins (3.1.2) iff*

$$Uf = \xi(K,L)\frac{\partial f}{\partial K} + \eta(K,L)\frac{\partial f}{\partial L} = H(f), \tag{3.1.9}$$

where $\xi(K,L) = \left(\frac{\partial \phi}{\partial t}\right)_{t=0}$, $\eta(K,L) = \left(\frac{\partial \psi}{\partial t}\right)_{t=0}$.

Proof. The proof is done by equating principal parts of functions on both sides. We elaborate on Sato's proof and introduce modern terminology.

Using Lemma 3.1.5, we obtain

$$f(T_t(K,L)) = f(\bar{K},\bar{L}) = F_t(f), \tag{3.1.10}$$

where T_t is given by the transformation group (3.1.2) and F_t is a strictly monotone function.

Applying Taylor's theorem to the transformation T_t, we obtain

$$\begin{aligned} f(T_t(K,L)) &= f(\bar{K},\bar{L}) \\ &= f(\phi(K,L,t),\psi(K,L,t)) \\ &= f\left(\phi(K,L,0) + \left(\left.\frac{\partial \phi}{\partial t}\right|_{t=0}\right)t + O(t^2), \psi(K,L,0) + \left(\left.\frac{\partial \psi}{\partial t}\right|_{t=0}\right)t + O(t^2)\right) \end{aligned} \tag{3.1.11}$$

Let $\xi(K,L) = \left(\frac{\partial \phi}{\partial t}\right)_{t=0}$, $\eta(K,L) = \left(\frac{\partial \psi}{\partial t}\right)_{t=0}$, $\phi(K,L,0) = \phi(K,L)$ and $\psi(K,L,0) = \psi(K,L)$. Then, it follows from Talyor's theorem applied to f that

$$f(T_t(K,L)) = f(\phi(K,L),\psi(K,L)) + \left(\left.\frac{\partial f}{\partial t}\right|_{t=0}\right)t + O(t^2), \tag{3.1.12}$$

where $\left(\left.\frac{\partial f}{\partial t}\right|_{t=0}\right) = \xi(K,L)\frac{\partial f}{\partial K} + \eta(K,L)\frac{\partial f}{\partial L}$.

Working on the $F_t(f)$, we obtain

$$F_t(f) = f + H(f)t + O(t^2), \tag{3.1.13}$$

where $H(f) = \left(\frac{\partial F}{\partial t} \Big|_{t=0} \right)$ is a function of f.

Equating (3.1.12) and (3.1.13) yields

$$Uf = \xi(K, L)\frac{\partial f}{\partial K} + \eta(K, L)\frac{\partial f}{\partial L} = H(f). \tag{3.1.14}$$

□

The condition of holotheticity is crucial from the economic standpoint, because it assures that the isoquant map (*i.e.*, the family of level curves of f) is invariant under the transformation (3.1.2) representing the technical change, which in turn means that under T isoquants are mapped onto isoquants and the techinical change in this case is transformed into a scale effect.

Using Theorem 3.1.6, we can derive a family of production functions.

Example 3.1.7. If $\xi = \alpha K$ and $\eta = \beta L$ in (3.1.9), $\alpha \neq \beta$, $\alpha, \beta > 0$, which means $\lambda_1 = e^{\alpha t}$, $\lambda_2 = e^{\beta t}$ in (3.1.4), $H(f) \neq 0$. It is a straigforward calculation, using the method of characteristics, that the general solution to the partial differential equation (3.1.9) is given by [110] (see also [106])

$$Y = f\left[K^{1/\alpha} Q\left(\frac{L^\alpha}{K^\beta}\right)\right], \tag{3.1.15}$$

where $Q(\cdot)$ is an arbitrary function.

The converse problem was also considered by Sato. Specifically, he established necessary and sufficient conditions for the existence of a technical progress that affords holotheticity of a given production function (see Lemma 4 in [108] on p. 34).

Now let us derive the Cobb-Douglas function (3.0.1) within the framework of the model $(G, \mathbb{R}^2_+)$, where the one-parameter Lie group of transformations G determines the exponential growth (3.1.5). Consider the partial differential equation (3.1.9) with the coefficients ξ and η determined by (3.1.5) for $\bar{K} = e^{at}K$, $\bar{L} = e^{bt}L$, $a, b \geqslant 0$. Clearly, we can determine a particular production function (3.1.15) by specifying the function $H(f) \neq 0$ in (3.1.5). Since G in this case defines an exponential growth, it is natural to impose the corresponding condition on $H(f)$ — so that it is also subject to an exponential growth. Indeed, let $H(f) = cf$, $c \geqslant 0$. Therefore we have

$$Uf = aK\frac{\partial f}{\partial K} + bL\frac{\partial f}{\partial L} = cf, \tag{3.1.16}$$

or, alternatively, we can solve instead the following partial differential equation as a lift of the equation (3.1.16)

$$X\varphi = aK\frac{\partial \varphi}{\partial K} + bL\frac{\partial \varphi}{\partial L} + cf\frac{\partial \varphi}{\partial f} = 0, \tag{3.1.17}$$

where $\varphi(K, L, f) = 0$, $\partial\varphi/\partial f \not\equiv 0$ is a solution to (3.1.17), while f is a solution to (3.1.16) and an invariant as such. Solving the corresponding system of ordinary differential equations

$$\frac{dK}{aK} = \frac{dL}{bL} = \frac{df}{cf}, \tag{3.1.18}$$

using the method of characteristics, yields the function (3.0.1), where $\alpha = \alpha(a, b, c), \beta = \beta(a, b, c)$.

Unfortunately, the elasticity elements in this case do not attain economically meaningful values like (3.0.2). To overcome this problem Sato in [108] adjusted the model accordingly. Specifically, he introduced the notion of the simultaneous holothenticity, which implies that a production function is holothetic under more than one type of technical change simultaneously. Mathematically, it means that a production function is an invariant of an integrable distribution of vector fields Δ [4], each representing a technical change as per the formula (3.1.2).

Let us introduce the definition of compatible types of technical progress as follows

Definition 3.1.8. Two technical progress T_1 and T_2 are called *compatible if their corresponding vector fields X_1 and X_2 are in involution.*

Therefore, a production function f is simultaneously holothetic under two compatible technical progress, using Theorem 3.1.6, the following conditions hold true

$$\begin{aligned} X_1 f &= \xi_1(K, L)\frac{\partial f}{\partial K} + \eta_1(K, L)\frac{\partial f}{\partial L} = H_1(f), \\ X_2 f &= \xi_2(K, L)\frac{\partial f}{\partial K} + \eta_2(K, L)\frac{\partial f}{\partial L} = H_2(f) \end{aligned} \tag{3.1.19}$$

It is more convenient to employ the lift of the two vector fields, namely, use the following equations instead

$$\begin{aligned} X_1\varphi &= \xi_1(K,L)\frac{\partial\varphi}{\partial K} + \eta_1(K,L)\frac{\partial\varphi}{\partial L} + H_1(f)\frac{\partial\varphi}{\partial f} = 0, \\ X_2\varphi &= \xi_2(K,L)\frac{\partial\varphi}{\partial K} + \eta_2(K,L)\frac{\partial\varphi}{\partial L} + H_2(f)\frac{\partial\varphi}{\partial f} = 0, \end{aligned} \tag{3.1.20}$$

for which a function $\varphi = \varphi(K, L, f)$ is an invariant.

Solving the system (3.1.20), Sato derived

$$\frac{\partial\varphi}{\partial K}\bigg/\frac{\partial\varphi}{\partial f} = \frac{H_2\eta_1 - H_1\eta_2}{\xi_1\eta_2 - \xi_2\eta_1} \tag{3.1.21}$$

and

$$\frac{\partial\varphi}{\partial L}\bigg/\frac{\partial\varphi}{\partial f} = \frac{H_1\xi_2 - H_2\xi_1}{\xi_1\eta_2 - \xi_2\eta_1}, \tag{3.1.22}$$

where $\xi_1\eta_2 - \xi_2\eta_1 \neq 0$.

The total differential equation corresponding to $\phi(K, L, f) = \text{const}$ is

$$d\varphi = \frac{\partial\varphi}{\partial K}dK + \frac{\partial\varphi}{\partial L}dL + \frac{\partial\varphi}{\partial f}df = 0, \tag{3.1.23}$$

substituting (3.1.21) and (3.1.22) into which, Sato, by assuming $\dfrac{\partial\varphi}{\partial f} \neq 0$, arrived at

$$(H_2\eta_1 - H_1\eta_2)dK + (H_1\xi_2 - H_2\xi_1)dL + (\xi_1\eta_2 - \xi_2\eta_1)df = 0. \tag{3.1.24}$$

The total differential equation (3.1.24), if solvable, leads to the function $f = f(K, L)$.

Sato commented that the total differential equation (3.1.24), noting $P = H_2\eta_1 - H_1\eta_2,\ Q = H_1\eta_2 - H_2\eta_1,\ R = \xi_1\eta_2 - \xi_2\eta_1 \neq 0$, followed the condition of integrability

$$P\left(\frac{\partial Q}{\partial f} - \frac{\partial R}{\partial L}\right) + Q\left(\frac{\partial R}{\partial K} - \frac{\partial P}{\partial f}\right) + R\left(\frac{\partial P}{\partial L} - \frac{\partial Q}{\partial K}\right) = 0, \tag{3.1.25}$$

which is also a necessary and sufficient condition for this problem of finding a function f simultaneously holothetic under two technical progress.

In modern terminology, the integrability condition (3.1.25) is reasonably obvious as we have seen it in Definition 3.1.8, namely, the two vector fields X_1 and X_2 are in involution, and $\{X_1, X_2\}$ form an integrable distribution.

Agricola and Forrest, as a special case of the Frobenius' theorem [4], considered the case of an $(m-1)$-dimensional $\mathcal{E}^{m-1}$ on an m-dimensional manifold. They showed that if $\mathcal{E}^{m-1}$ is defined by one nowhere vanishing one-form ω, the integrability of the distribution reduces to the condition that the three-form $d\omega \wedge \omega$ vanishes, namely, $d\omega \wedge \omega = 0$. Alternatively, we can reform the above problem, finding a function $\phi = const$ holothetic under two compatible technical progress, in terms of differential forms. Consider a 2-dimensional submanifold of $\mathbb{R}^3_+$

$$M = \left\{(K, L, f) \in \mathbb{R}^3_+ : \varphi(K, L, f) = const\right\}, \tag{3.1.26}$$

linearizing which by passing to the tangent bundle, we obtain

$$\omega(X_1) = 0 \text{ and } \omega(X_2) = 0, \tag{3.1.27}$$

where ω is a one-form associated with the function ϕ, *i.e.*, $\omega = d\phi$.

Therefore, the problem can be expressed in the following equivalent form

$$X_1(\varphi) = 0,\ X_2(\varphi) = 0,\ [X_1, X_2] \in \{X_1, X_2\}. \tag{3.1.28}$$

$$\Updownarrow$$

$$\omega(X_1) = 0,\ \omega(X_2) = 0,\ d\omega \wedge \omega = 0. \tag{3.1.29}$$

The equivalence of $X_i(\varphi) = 0$ and $\omega(X_i) = 0$, $i = 1, 2$, can be verified by using the formula $\mathcal{L}_X(\varphi) = i_X(d\varphi)$, where the interior derivative i_X maps an m-form to an $m-1$-form, while $d\omega \wedge \omega = 0$ states the integrability, which is same as the involution of the distribution $\{X_1, X_2\}$. Specifically, the one-form ω can be expressed as

$$\omega = P(K, L, f)dK + Q(K, L, f)dL + R(K, L, f)df, \tag{3.1.30}$$

the contraction of which with X_1 and X_2 leads to

$$P\xi_1 + Q\eta_1 + RH_1 = 0, \quad P\xi_2 + Q\eta_2 + RH_2 = 0. \tag{3.1.31}$$

Expressing P and Q in terms of R (assuming $R \neq 0$), we obtain

$$P = R\frac{H_2\eta_1 - H_1\eta_2}{\xi_1\eta_2 - \xi_2\eta_1} \tag{3.1.32}$$

and

$$Q = R\frac{H_1\xi_2 - H_2\xi_1}{\xi_1\eta_2 - \xi_2\eta_1} \tag{3.1.33}$$

substituting both of which into (3.1.30), we have the one-form ω in the following form

$$\omega = R\frac{H_2\eta_1 - H_1\eta_2}{\xi_1\eta_2 - \xi_2\eta_1}dK + R\frac{H_1\xi_2 - H_2\xi_1}{\xi_1\eta_2 - \xi_2\eta_1}dL + Rdf. \tag{3.1.34}$$

The one-form ω is indeed nowhere vanishing since three components are not equal to zero simultaneously.

We have the Pfaffian equation $\omega = 0$ along the submanifold M since ω is associated with $\phi(K, L, f) = const$, that is,

$$R\frac{H_2\eta_1 - H_1\eta_2}{\xi_1\eta_2 - \xi_2\eta_1}dK + R\frac{H_1\xi_2 - H_2\xi_1}{\xi_1\eta_2 - \xi_2\eta_1}dL + Rdf = 0, \tag{3.1.35}$$

or

$$(H_2\eta_1 - H_1\eta_2)dK + (H_1\xi_2 - H_2\xi_1)dL + (\xi_1\eta_2 - \xi_2\eta_1)df = 0, \tag{3.1.36}$$

which shows that we have recovered (3.1.24) in terms of a one-form.

Moreover, it follows from $\omega = d\varphi$ that the integrability condition $d\omega \wedge \omega = 0$ holds because ω is an exact form. To have a detailed view of the specific problem and recover (3.1.25), let us work on the condition of integrability. Upon substituting (3.1.21) and (3.1.22) a non-trivial calculation shows that

$$d\omega \wedge \omega = \left[P\left(\frac{\partial Q}{\partial f} - \frac{\partial R}{\partial L}\right) + Q\left(\frac{\partial R}{\partial K} - \frac{\partial P}{\partial f}\right) + R\left(\frac{\partial P}{\partial L} - \frac{\partial Q}{\partial K}\right)\right] dK \wedge dL \wedge df = 0. \tag{3.1.37}$$

Thus, we recover the condition of integrability (3.1.25):

$$P\left(\frac{\partial Q}{\partial f}-\frac{\partial R}{\partial L}\right)+Q\left(\frac{\partial R}{\partial K}-\frac{\partial P}{\partial f}\right)+R\left(\frac{\partial P}{\partial L}-\frac{\partial Q}{\partial K}\right)=0. \tag{3.1.38}$$

To derive the exact form of the Cobb-Douglas function, let us consider the following two vector fields, for which a function $\varphi(K,L,f)$ is an invariant:

$$X_1\varphi = K\frac{\partial\varphi}{\partial K}+L\frac{\partial\varphi}{\partial L}+f\frac{\partial\varphi}{\partial f}=0,$$

$$X_2\varphi = aK\frac{\partial\varphi}{\partial K}+bL\frac{\partial\varphi}{\partial L}+f\frac{\partial\varphi}{\partial f}=0. \tag{3.1.39}$$

Clearly, the vector fields X_1, X_2 form a two-dimensional integrable distribution: $[X_1,X_2]=\rho_1X_1+\rho_2X_2$, where $\rho_1=\rho_2=0$. The corresponding total differential equation is given by (see Chapter VII, Sato [108] for more details)

$$(fL-bfL)dK+(afK-fK)dL+(bKL-aKL)df=0, \tag{3.1.40}$$

or,

$$(1-b)\frac{dK}{K}+(a-1)\frac{dL}{L}+(b-a)\frac{df}{f}=0. \tag{3.1.41}$$

Remark 3.1.9. *Alternatively, the problem can be formulated in terms of differential forms. Indeed, let us consider a submanifold*

$$M_1=\left\{(K,L,f)\in\mathbb{R}^3_+ : \varphi(K,L,f)=const\right\}. \tag{3.1.42}$$

Next, consider a differential form

$$\omega_1=P_1dK+Q_1dL+R_1df, \tag{3.1.43}$$

where $P_1=P_1(K,L,f)$, $Q_1=Q_1(K,L,f)$ and $R_1=R_1(K,L,f)\neq 0$ are smooth functions and $\omega_1=d\varphi$. To derive the Cobb-Douglas function, we employ the condition (3.1.39)*, which is dual to the simultaneous holotheticity, that is*

$$\omega_1(X_1)=0 \text{ and } \omega_1(X_2)=0, \tag{3.1.44}$$

where X_1 *and* X_2 *are vector fields in* (3.1.39).

Hence, we obtain

$$KP_1 + LQ_1 + fR_1 = 0, \tag{3.1.45}$$

$$aKP_1 + bLQ_1 + fR_1 = 0, \tag{3.1.46}$$

which yields

$$P_1 = \frac{(1-b)f}{(b-a)K}R_1, \tag{3.1.47}$$

$$Q_1 = \frac{(a-1)f}{(b-a)L}R_1. \tag{3.1.48}$$

The equation (3.1.47) *leads to*

$$\omega_1 = \frac{(1-b)f}{(b-a)K}R_1 dK + \frac{(a-1)f}{(b-a)L}R_1 dL + R_1 df. \tag{3.1.49}$$

ω_1 *gives rise to a Pfaffian equation on the submanifold* M*, namely,* $\omega_1 = 0$ *along* M*.*

Therefore, we have recovered (3.1.41) *as follows*

$$\frac{(1-b)f}{(b-a)K}R_1 dK + \frac{(a-1)f}{(b-a)L}R_1 dL + R_1 df = 0, \tag{3.1.50}$$

or

$$(1-b)Lf dK + (a-1)Kf dL + (b-a)KL df = 0. \tag{3.1.51}$$

Integrating (3.1.41), we arrive at a Cobb-Douglas function of the form (3.0.1), where the elasticity coefficients

$$\alpha = \frac{1-b}{a-b}, \quad \beta = \frac{a-1}{a-b} \tag{3.1.52}$$

satisfy the condition of constant return to scale (3.0.2).

Remark 3.1.10. *Note that, in principle, we could have used only one vector field generating a partial differential equation of the type (3.1.16). However, the resulting Cobb-Douglas function would have had the elasticity of holotheticity satisfying* *the condition $\alpha\beta < 0$ (see (3.0.1)). The latter constraint on the parameters α and β in (3.0.1) is incompatible with the economic growth theory main postulates. We suppose that exactly for this reason Sato [108] introduced the concept of simultaneous holotheticity. This arrangement, in particular, allows us to generate two-input Cobb-Douglas functions of the type (3.0.1) depending on a wide range of parameters α and β, which we can, for instance, make to satify the condition $\alpha + \beta = 1$, so that the function (3.0.1) displays constant returns to scale as in the example above.*

These considerations lead to a very important conclusion. Namely, the Cobb-Douglas function, derived within the framework of the growth model $(G, \mathbb{R}^2_+)$, where the Lie group G is determined by the exponential growth (3.1.5), is precisely a manifestation of this exponential growth, or, more succinctly, we have

$$\text{exponential growth} \Rightarrow \text{the Cobb-Douglas function},$$

which means that the Cobb-Douglas function (3.0.1) is a consequence of exponential growth representing technical change.

3.2 From exponential to logistic growth models

In this section we depart from the assumption that the input factors K and L follow an exponential growth in order to extend Sato's growth model $(G, \mathbb{R}^2_+)$. In the following context we will assume capital and labor grow logistically. There is already a substantial literature on logistic growth on population and labor in mathematics, statistics and economics.

Recall that Verhulst [128] introduced the idea of logistic growth in population dynamics. He obtained the logistic equation by adjusting the exponential equation while studying population growth, that is,

$$\begin{array}{ccc} \dot{x} = rx & \longrightarrow & \dot{x} = rx\left(1 - \dfrac{x}{N}\right), \\ \text{(exponential growth)} & \longrightarrow & \text{(logistic growth)}, \end{array} \tag{3.2.1}$$

where x is population size, r is the growth rate and N is the carrying capacity.

By solving the logistic equation (3.2.1) Verhulst [128] derived the logistic function of the following form

$$x(t) = \frac{x(0)e^{rt}}{1 + x(0)(e^{rt} - 1)/N}, \tag{3.2.2}$$

where $x(0) > 0$ is the initial population.

Verhulst compared his logistic function against available demographic data (more details can be found in [10]).

The 1920s were an important period for the development of the logistic growth model. The logistic model has become accepted by mathematicians, statisticians, economists and biologists thanks to the promotion efforts by Pearl and Leed [90, 89], who extended the logistic model and proved the population data of US from 1790 to 1910 fits well to the logistic function. More importantly, Pearl [68] concluded the law of logistical growth of population based on the accumulated empirical evidence.

Meanwhile, Lotka [79] and Volterra [129] independently introduced the predator-prey equations given by (also known as the Lotka-Volterra equations)

$$\begin{aligned} \dot{x} &= \alpha x - \beta xy, \\ \dot{y} &= \delta xy - \gamma y, \end{aligned} \tag{3.2.3}$$

where x and y are population size or population density of different species and α, β, δ, γ represent different growth or decay rates. The model can be used in (but not limited within) describing the population dynamics of ecological species or interactions of chemicals. The application of Lotka-Volterra model in distinct branches of mathematics and other disciplines is quite fruitful. The model can, by considering logistic growth, be generalized to

$$\begin{aligned} \dot{x} &= \alpha \left(x - \frac{x}{N_1} \right) - \beta xy, \\ \dot{y} &= \delta xy - \gamma \left(y - \frac{y}{N_2} \right), \end{aligned} \tag{3.2.4}$$

where N_1 and N_2 are carrying capacities. The system (3.2.4) describes an interaction between logistic growth and decay of two different species. From this point of view, we can comment that the Lotka-Volterra model is an extension of Verhulst's logistic model. The early history of the development of the logistic model can be found in [68]. We make use of the model (3.2.4), as well as its generalization in different dimensions, in the study of economic dynamics of inputs and output and derivation of production functions, the details of which will be discussed in Chapters 4 and 5.

Subsequently, in 1959 Holling [58] introduced the Holling-type interaction based on the predator-prey model of Gause (see in [56]), who also made distinct contributions to the development of the logistic growth model, by considering the ecological saturation effect, *i.e.*, the high density of the predator decreases the possibility of catching the prey. The mathematical model is as follows

$$\begin{aligned} \dot{x} &= \alpha x \left(1 - \frac{x}{N}\right) - y p(x), \\ \dot{y} &= y \left(\delta p(x) - \gamma\right), \end{aligned} \tag{3.2.5}$$

where $p(x)$ is called the response function or functional response.

There are three types of functional responses, Type I, Type II and Type III. In particular, we noted that the Type III functional response is characterized by a sigmoid function as follows

$$p(x) = \frac{cx^n}{a + x^n}, \tag{3.2.6}$$

where a is a parameter controlling the growth rate and c is the carrying capacity.

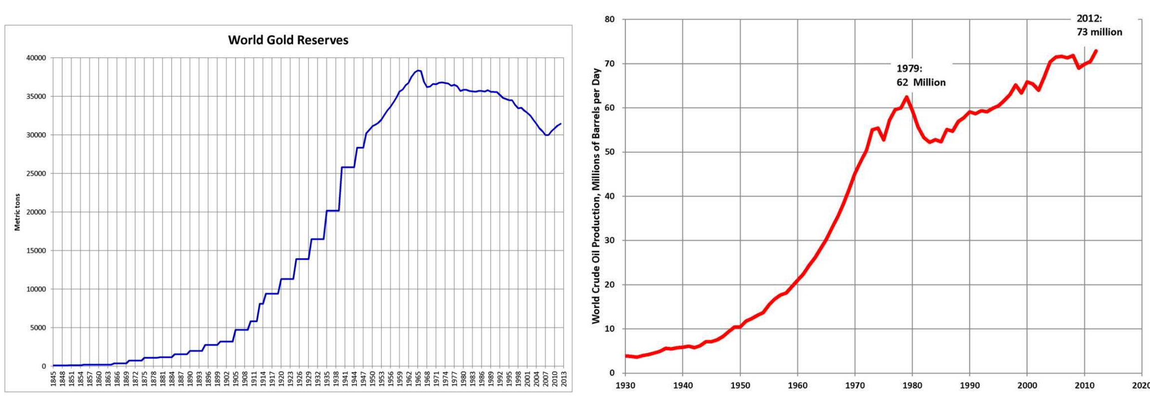
As we have mentioned, the response function corresponds to the ecological saturation effect. As we can see, $p(x) \to c$ as $x \to \infty$. Then the interaction of the predator and prey in (3.2.5) is determined by the density of the prey. This is the mathematical interpretation of the effect. We are interested in the Type III functional response because the same "S-shaped" function is also employed in the study of econometric dynamics, in which some mathematicians and economists suggest that we should consider a production function characterized by a sigmoid function. We will discuss this in Section 3.3.

More recently, the law of logistic growth of population was confirmed by more evidence. See, for example, Brass [17], Leach [76], Oliver [86], in which the authors validated the law via studying population of the US, Scotland and Great Britain using modern statistical tools. The idea of logistic growth has also been accepted and adopted by economists since labor force is naturally subject to population, for example, Accinelli and Brida [2, 1, 19]. extended the Solow's model by considering labor force is affected by the logistic growth.

The same assumption can be made about the growth in capital, if, using natural resources such as crude oil and gold as proxies for energy and money, respectively, we can see, for example, in Figure 3.1 that the accumulation of gold reserves and oil production are almost subject to logistic growth rather than exponential growth.

We note that from the mathematical viewpoint it is also evident that there cannot be unbounded, continuous exponential growth, whether in terms of production, capital, or population, on a planet with limited resources as per the following well-known theorem [101]:

Theorem 3.2.1 (Extreme value theorem)**.** *If S is a compact set and $f : S \to \mathbb{R}$ is a continuous function, then f is bounded and there exist $p, q \in S$ such that $f(p) = \sup_{x \in S} f(x)$ and $f(q) = \inf_{x \in S} f(x)$.*



(a) World Gold Reserves from 1845 to 2013, in metric tonnes (Wikipedia [132]).

(b) World crude oil production 1930 to 2012 (Wikipedia [133]).

Figure 3.1: Logistic growth in basic factors of production (gold and oil).

In view of the above, we propose the following growth model based on the assumption that both capital K and labor L are affected by logistic growth, namely

$$(G_1, \mathbb{R}^2_+), \quad G_1 : \bar{K} = \frac{N_K K}{K + (N_K - K)\, e^{-\alpha t}}, \quad \bar{L} = \frac{N_L L}{L + (N_L - L)\, e^{-\beta t}}, \qquad (3.2.7)$$

where $\alpha, \beta > 0$ and N_K, N_L are the respective carrying capacities.

We verify that G_1 satisfies the group properties presented in Section 2.2. Let us illustrate using the transformation $\bar{K} = \phi_1(K,t)$ in G_1, that is,

(a) Associativity

$$\begin{aligned} & \phi_1(\phi_1(K,t),s) \\ = \; & \phi_1\left(\frac{N_K K}{K+(N_K-K)e^{-\alpha t}}, s\right) \\ = \; & \frac{N_K K}{K+[K+(N_K-K)e^{-\alpha t}-K]e^{-\alpha s}} \\ = \; & \frac{N_K K}{K+(N_K-K)e^{-\alpha(t+s)}} \\ = \; & \phi_1(K,t+s), \end{aligned} \tag{3.2.8}$$

(b) The identity for $\phi_1(K,t)$ is $\phi(K,0) = \dfrac{N_K K}{K+N_K-K} = K$,

(c) The inverse of $\phi_1(K,t)$ is

$$\phi_1(K,-t) = \frac{N_K K}{K+(N_K-K)\,e^{\alpha t}} \tag{3.2.9}$$

such that

$$\begin{aligned} & \phi_1(\phi_1(K,-t),t) \\ = \; & \phi_1\left(\frac{N_K K}{K+(N_K-K)e^{-\alpha t}}, t\right) \\ = \; & \frac{N_K K}{K+[K+(N_K-K)e^{\alpha t}-K]e^{-\alpha t}} = K. \end{aligned} \tag{3.2.10}$$

Hence, $\phi_1(K,-t)$ is indeed the inverse.

Therefore, G_1 is a one-parameter Lie group, acting in $\mathbb{R}^2_+$, whose flow is generated by the vector field

$$U_1 = \alpha K\left(1-\frac{K}{N_K}\right)\frac{\partial}{\partial K} + \beta L\left(1-\frac{L}{N_L}\right)\frac{\partial}{\partial L}. \tag{3.2.11}$$

Remark 3.2.2. *It is also natural to consider the growth models $(G_2, \mathbb{R}^2_+)$ and $(G_3, \mathbb{R}^2_+)$ determined by the assumption that only one of the two variables grow logistically, while* *the other is affected by exponential growth, that is*

$$(G_2, \mathbb{R}_+^2), \quad G_2 : \bar{K} = \frac{N_K K}{K + (N_K - K)\, e^{-\alpha t}}, \quad \bar{L} = e^{\beta t} L, \tag{3.2.12}$$

or,

$$(G_3, \mathbb{R}_+^2), \quad G_3 : \bar{K} = e^{\alpha t} K, \quad \bar{L} = \frac{N_L L}{L + (N_L - L)\, e^{-\beta t}}. \tag{3.2.13}$$

Following the approach developed by Sato in [108], we can now determine the corresponding family of production functions by solving the partial differential equation determined by the vector field U_1 (3.2.11):

$$U_1 f = \alpha K \left(1 - \frac{K}{N_K}\right) \frac{\partial f}{\partial K} + \beta L \left(1 - \frac{L}{N_L}\right) \frac{\partial f}{\partial L} = H(f), \tag{3.2.14}$$

where $H(f)$ is an arbitrary function of f. Employing the method of characteristics, we arrive at the following family of functions:

$$Y = f_1 \left\{ \left(\frac{K}{|N_K - K|} \right)^{1/\alpha} Q \left[\left(\frac{L}{|N_L - L|} \right)^{\alpha} \left(\frac{|N_K - K|}{K} \right)^{\beta} \right] \right\}, \tag{3.2.15}$$

where $Q(\cdot)$ is an arbitrary function. We note that for $N_K = N_L = 1$ and $K, L \ll 1$ the family of functions given by (3.2.15) $f_1 \sim f$, where f is given by (3.1.15). Therefore, we arrive at the following

Proposition 3.2.3. *The most general family of production functions holothetic within the growth model (3.2.7) is given by (3.2.15).*

Remark 3.2.4. *The same argument applied to the "partially" logistic neoclassical growth models (3.2.12) and (3.2.13) yields the families of functions*

$$Y = f_2 \left\{ \left(\frac{K}{|N_K - K|} \right)^{1/\alpha} Q \left[L^{\alpha} \left(\frac{|N_K - K|}{K} \right)^{\beta} \right] \right\} \tag{3.2.16}$$

and

$$Y = f_3 \left\{ K^{1/\alpha} Q \left[\left(\frac{L}{|N_L - L|} \right)^{\alpha} K^{-\beta} \right] \right\}, \tag{3.2.17}$$

respectively.

Our next goal is to derive a new production function under the assumption of logistic growth in both capital K and labor L. Since the Cobb-Douglas function (3.0.1) has been shown above to be a member of the family of production functions (3.1.15) determined within the neoclassical growth model $(G, \mathbb{R}^2_+)$, where the Lie group G is given by (3.1.5), it is natural to seek a new production function compatible with the logistic growth determined by the action of the Lie group G_1 (3.2.7) within the growth model $(G_1, \mathbb{R}^2_+)$. This is the subject of the considerations that follow.

3.3 From logistic growth to a new production function

In Section 3.1 we saw how the Cobb-Douglas production function could be derived as an element of the family of production functions (3.1.15) within the framework of the growth model $(G, \mathbb{R}^2_+)$, where the Lie group G was defined by (3.1.5). Now let us consider the new growth model $(G_1, \mathbb{R}^2_+)$, where the Lie group G_1 was given by (3.2.7). By analogy, we are supposed to derive a new type of production function based on the model $(G_1, \mathbb{R}^2_+)$ by using the holotheticity, that is, we formulate the following

Conjecture 3.3.1. *The growth model $(G_1, \mathbb{R}^2_+)$ leads to a production function of a new type holothetic to G_1*

$$\textit{logistic growth} \Rightarrow \textit{production function of a new type.} \tag{3.3.1}$$

Before we formally derive the corresponding production function as an element of the family of production functions (3.2.15), following the procedure outlined above, let us first give a reasonable justification for the calculations that we will present below.

Harrod [53] and Domar [30], when studying the long-run production and accumulation of wealth, independently introduced the following Harrod-Domar model

$$\begin{aligned} Y &= f(K;t), \\ \frac{dY}{dK} &= \frac{Y}{K} = c, \\ f(0) &= 0, s > 0 \\ sY &= S = I, \\ \Delta K &= I - \delta K, \end{aligned} \tag{3.3.2}$$

where C and I represent consumption and investment (savings) respectively, while c, s and δ denotes marginal rate of production, rate of saving and depreciation of capital. The core of the dynamic model (3.3.2) is the behaviour of Y, that is,

$$\frac{\dot{Y}}{Y} = sc - \delta, \tag{3.3.3}$$

which shows that rate of production is affected by the marginal rate of production, rate of saving as well as depreciation of capital and production follows growth of an exponential type. But the model has been criticized by Solow and some neoclassical economists that the authors employ a short-run model to analyze the long-run growth, *i.e.*, the production is assumed only to be affected by capital, and the model has an unstable equilibrium provided that the economy grows.

Solow [120] and Swan [126] extended the Harrod-Domar model by considering that production is affected by capital and labor, in which labor force follows an exponential growth

$$\begin{aligned} Y &= f(K, L; t), \\ Y &= C + I, \\ I &= sY, s > 0 \\ \dot{K} &= I - \delta K,\ K_0,\ \delta \geq 0, \\ L &= L_0 e^{\alpha t},\ L > 0,\ \alpha \geq 0, \end{aligned} \tag{3.3.4}$$

where L is labor and α is the growth rate of labor. It is noted economists prefer the model presented in projective coordinates, namely, $y = \frac{Y}{L}$ and $k = \frac{K}{L}$, when doing mathematical analysis, Solow commented that his model (3.3.5) (known as the Solow-Swan model or simply Solow model) in comparison to the Harrod-Domar model was "less sensitive", that is, the model (3.3.5) described dynamics of capital rather than of production. The model is well accepted by neoclassical economists (see, for example, Jones and Scrimgeour [62], a model with decay in produced capital was studied in Cheviakov and Hartwick [26]).

Another neoclassical model of economic growth is the Ramsey model [99], in which Ramsey considered the problem of optimal social welfare, namely, utility of consumers, subject to dynamics of accumulation of capital. We extend the Ramsey model in Chapter 5 by considering our new production function, which is derived in what follows, and logistic growth of capital and labor.

It is assumed in all above models that production function f satisfies the Inada conditions [60]:

1. $f_K, f_L > 0$, this condition accounts for growth in both K and L,
2. $f_{KK}, f_{LL} < 0$, that implies diminishing marginal returns also in both K and L,
3. f has constant returns to scale, that is $f(\lambda K, \lambda L) = \lambda f(K, L)$ for all $\lambda > 0$,
4. f satisfies the following properties:

$$\lim_{K\to 0} f_K = \infty, \lim_{K\to\infty} f_K = 0,$$
$$\lim_{L\to 0} f_L = \infty, \lim_{L\to\infty} f_L = 0.$$

For example, the Cobb-Douglas function (3.0.1) satisfies the above assumptions, provided the condition (3.0.2) holds. Many important examples of endogenous growth support this assumption (see, for example, Cobb and Douglas [27]). Nevertheless, there are situations when growth cannot be described by a strictly concave production function. Skiba [114] indicated that it is very realistic to apply a narrow class of production functions to different economic models, for example, in both developed and developing countries. He considered a model of the Ramsey type based on the non-concave production function. Economically, for instance, the business cycle of a company is a notable counterexample. Indeed, at a microeconomic level a company may develop a product based on an original idea, such a product initially can be sold unrestricted in the absence of competition, generating increasing marginal returns. After a while, a competition may become a factor (*e.g.*, other companies may introduce similar products) affecting the sales of the original product, whose market share may shrink. In turn, this situation in a long-run will manifest itself in decreasing marginal returns. Mathematically, the corresponding production function will no longer be strictly concave. Capasso *et al.* [24] gave a different motivation for the introduction of a (globally) nonconcave production function based on the idea of "poverty traps".

To address the issue Capasso *et al.* [24] (see also Engbers *et al.* [34], La Torre *et al.* [75], Anita *et al.* [5, 6, 7]) employed a sigmoid function (3.2.6) of the Type III functional response in the following form

$$Y = f_4(K, L) = \frac{\alpha_1 K^p L^{1-p}}{1 + \alpha_2 K^p L^{1-p}} \tag{3.3.5}$$

reducible to the Cobb-Douglas function (3.0.1) and enjoying an "S-shaped" (concave-convex) behavior for $p \geq 2$. Clearly, the functions of the class (3.3.5) have a horizontal asymptote as $(K, L) \to (\infty, \infty)$ when $\alpha_2 \neq 0$ and are compatible with logistic growth. As mentioned in Section 3.2, mathematical biologists introduced the functional response in the study of the saturation phenomenon. We can base the application of the new production (3.3.5) on the same reason, a saturation of economic growth, which we have discussed in Section 3.2. These functions were used by the authors as a cornerstone for building a new, highly non-trivial generalization of the Solow model with spacial component in which they did not make assumptions about logistic growth for L. It is worth mentioning at this point that La Torre *et al.* [1, 2, 19, 23], while generalizing the Ramsey models of economic growth, assumed logistic growth in L, but kept the Cobb-Douglas function (3.0.1) intact.

The introduction of the family of production functions (3.3.5) is in agreement with a big step in the right direction, nevertheless these functions cannot account for all possible examples of growth (and decay). For example, a production function can exhibit growth, followed by a period of stabilization and then decay (see, for example, [22]). Another option is growth followed by a period of stabilization, which is followed by growth again. In this view our next goal is to derive a more general production function that can be used to describe a wider range of economic growth models, including the situations outlined above. We shall employ the Lie group theoretical method developed by Sato [108] and briefly described in Section 3.1.

Indeed, consider the growth model $(G_1, \mathbb{R}^2_+)$ given by (3.2.7). Next, we are going to identify a member of the family (3.2.15) compatible with logistic growth given by (3.2.7) by imposing the corresponding constraints on the RHS of the equation (3.2.14). By analogy with the case of the Cobb-Douglas function derived by Sato [108] within the framework of the growth model $(G, \mathbb{R}^2_+)$, where the action of the Lie group G is determined by (3.1.5), let us consider the following partial differential equation determined by the vector field U_1 given by (3.2.11):

$$U_1 f = aK\left(1 - \frac{K}{N_K}\right)\frac{\partial f}{\partial K} + bL\left(1 - \frac{L}{N_L}\right)\frac{\partial f}{\partial L} = cf\left(1 - \frac{f}{N_f}\right), \qquad (3.3.6)$$

or, in other words, let us specify the function $H(f)$ in (3.2.14) to be $cf\left(1 - \frac{f}{N_f}\right)$ that implies logistic growth in the production function as well. Compare (3.3.6) with the equation (3.1.16).

Remark 3.3.2. *We note that the choice for the RHS of (3.3.6) is not arbitrary. It turns out that in order to obtain a meaningful solution one needs to assure that the properties of the function $H(f)$ in (3.2.11) are compatible with the logistic growth determined by (3.2.7). For example, if we set $H(f) = f$ in (3.2.11), which would imply that the growth in both K and L is logistic, while f grows exponentially, the resulting production function would have singularities (see the equation (3.8.1)). Therefore the above equation reflects the fact that the growth determined by (3.3.6) is consistent for all quantities involved, that is for K, L and f.*

Next, we employ the same reasoning that Sato in [108] based his derivation of the

Cobb-Douglas function (3.0.1) upon (see also Section 3.1). Let us consider two *compatible* technical progress, under which the production function is holothetic, and solve (again) the corresponding *simultaneous holotheticity problem*. Let us consider the following two vector fields acting on a function $\varphi(K, L, f)$:

$$\begin{aligned} X_3\varphi &= K\left(1-\frac{K}{N_K}\right)\frac{\partial\varphi}{\partial K}+L\left(1-\frac{L}{N_L}\right)\frac{\partial\varphi}{\partial L}+f\left(1-\frac{f}{N_f}\right)\frac{\partial\varphi}{\partial f}=0,\\ X_4\varphi &= aK\left(1-\frac{K}{N_K}\right)\frac{\partial\varphi}{\partial K}+bL\left(1-\frac{L}{N_L}\right)\frac{\partial\varphi}{\partial L}+cf\left(1-\frac{f}{N_f}\right)\frac{\partial\varphi}{\partial f}=0. \end{aligned} \tag{3.3.7}$$

Clearly, the vector fields X_3 and X_4 form an integrable distribution Δ, because $[X_3, X_4] = \rho_3 X_3 + \rho_4 X_4$, where $\rho_3 = \rho_4 = 0$. Then the corresponding total differential equation which has $\varphi(K, L, f) = \text{const}$ for a solution assumes the following form:

$$\begin{aligned} \left[(c-b)f\left(1-\frac{f}{N_f}\right)L\left(1-\frac{L}{N_L}\right)\right]dK &+\\ \left[(a-c)f\left(1-\frac{f}{N_f}\right)K\left(1-\frac{K}{N_K}\right)\right]dL &+\\ \left[(b-a)f\left(1-\frac{K}{N_K}\right)L\left(1-\frac{L}{N_L}\right)\right]df &= 0, \end{aligned} \tag{3.3.8}$$

or,

$$(c-b)\frac{dK}{K\left(1-\frac{K}{N_K}\right)}+(a-c)\frac{dL}{L\left(1-\frac{L}{N_L}\right)}+(b-a)\frac{df}{f\left(1-\frac{f}{N_f}\right)}=0. \tag{3.3.9}$$

Remark 3.3.3. *By analogy with Remark 3.1.9, we can also derive* (3.3.9) *using a differential form. Consider a differential form*

$$\omega_2 = P_2dK + Q_2dL + R_2df,\ R_2 \neq 0, \tag{3.3.10}$$

where $\omega_2 = d\varphi$.

Employing the condition (3.1.27)*, we have*

$$\omega_2(X_2) = 0 \text{ and } \omega_2(X_3) = 0, \tag{3.3.11}$$

where X_3 *and* X_4 *are vector fields in* (3.3.7).

It follows that

$$K\left(1-\frac{K}{N_K}\right)P_2+L\left(1-\frac{L}{N_L}\right)Q_2+f\left(1-\frac{f}{N_f}\right)R_2=0,$$
$$aK\left(1-\frac{K}{N_K}\right)P_2+bL\left(1-\frac{L}{N_L}\right)Q_2+cf\left(1-\frac{f}{N_f}\right)R_2=0. \tag{3.3.12}$$

Expressing P_2 and Q_2 in terms of R_2, we obtain, by substituting into (3.3.10)*, the following expression*

$$\omega_2=\frac{(c-b)f\left(1-\frac{f}{N_f}\right)}{(b-a)K\left(1-\frac{K}{N_K}\right)}R_2dK+\frac{(a-c)f\left(1-\frac{f}{N_f}\right)}{(b-a)L\left(1-\frac{L}{N_L}\right)}R_2dL+R_2df. \tag{3.3.13}$$

We note that we see $\omega_2\wedge d\omega_2=0$ by identifying $\omega_2=d\phi$, where $\phi(K,L,f)=const$. $\omega_2=0$ along the submanifold for $\phi(K,L,f)=const$ (see (3.1.35)*). Therefore, we have recovered* (3.3.9)*.*

Integrating the differential equation (3.3.9) (compare it with (3.1.41)), we arrive at a solution of the form $\varphi(K,L,f)=0$ defined in the open domain

$$D=]0,N_K[\times]0,N_L[\times]0,N_f[\subset\mathbb{R}^3$$

and satisfying the condition $\frac{\partial\varphi}{\partial f}\not\equiv 0$. Solving for f by the implicit function theorem, we arrive at the following hypersurface in $\mathbb{R}^3$:

$$Y=f_5(K,L)=\frac{N_{f_5}K^\alpha L^\beta}{C\left|N_K-K\right|^\alpha\left|N_L-L\right|^\beta+K^\alpha L^\beta},\ (K,L)\in\mathbb{R}^2_+, \tag{3.3.14}$$

where $C\in\mathbb{R}$ is the constant of integration, $\alpha=\frac{c-b}{a-b}$, $\beta=\frac{a-c}{a-b}$. Note $\alpha+\beta=1$. Note that in view of the symmetry of the differential equation (3.3.9), we could have solved the equation $\varphi(K,L,f)=0$ for K and L as well. The function $Y=f_5(K,L)$ given by (3.3.14) whose range is $]0,N_f[$ coinsides with the function $\varphi(K,L,f)=0$ on D.

Furthermore, we note that in the subset $D'=]0,N_K[\times]0,N_L[\subset\mathbb{R}^2_+$ of the domain of the function $Y=f_5(K,L)$ its growth is governed by the logistic growth in the factors K and L. Note that in this region the growth of the production function f_5 is "S-shaped", which agrees with the assumptions that led to the introduction of the production function (3.3.5). However, the production function (3.3.14) is also defined outside of the region D', which impies in turn that its shape in the subset $\mathbb{R}_+^2 \setminus D' = [N_K, \infty[\times[N_L, \infty[$ is determined by the growth in K and L that goes beyond the respective carrying capacities N_K and N_L. We will elaborate on this matter without loss of generality while dealing with the corresponding one-input analog of the new two-input production function (3.3.14) below.

We conclude, therefore, that by analogy with the algorithm based on the Lie group theory methods devised by Sato and applied in [108] to generate the Cobb-Douglas function (3.0.1), we have used it, after some modifications, to generate a *new production function (3.3.14)*. More succinctly, we have shown that

logistic growth $\Rightarrow$ the new production function (3.3.14).

Remark 3.3.4. *Taking the limit as $K, L \to \infty$ (even though K and L cannot grow beyond a certain "horizon" - see below), we obtain*

$$\lim_{\substack{K\to\infty \\ L\to\infty}} f_5(K,L) = \lim_{\substack{K\to\infty \\ L\to\infty}} \frac{N_{f_5} K^\alpha L^\beta}{C\,|N_K - K|^\alpha\,|N_L - L|^\beta + K^\alpha L^\beta} \tag{3.3.15}$$

$$= \lim_{\substack{K\to\infty \\ L\to\infty}} \frac{N_{f_5}}{C\,\left|\frac{N_K}{K} - 1\right|^\alpha \left|\frac{N_L}{L} - 1\right|^\beta + 1} \tag{3.3.16}$$

$$= \frac{N_{f_5}}{C+1}. \tag{3.3.17}$$

The quantity

$$S_{f_5} = \frac{N_{f_5}}{C+1} \tag{3.3.18}$$

is the steady state *of the new production function f_5 given by (3.3.14). Note that by changing the constant C in (3.3.18) we can regulate the steady state S_{f_5}.*

Remark 3.3.5. *See Remark 3.1.10.*

Remark 3.3.6. *We observe that the new production function f_5 (3.3.14) is reducible to the production function (3.3.5) proposed by Capasso* et al. *[24] when K and $L \ll N_K$ and N_L respectively, $N_L, N_K \approx 1$, $C = 1$ in (3.3.14) and $\alpha_1 = N_{f_5}$, $\alpha_2 = 1$ in*

(3.3.5) .

Remark 3.3.7. *Figure 3.2 presents the surface of a two-input production function of the type (3.3.14) for* $N_f = 120$, $\alpha = \beta = 3$, $N_K = 113$, $N_L = 115$, $C = 1.18$ *without singularities (see Remark 3.3.8).*

Remark 3.3.8. *Employing the same procedure, we can determine now in a fairly straightforward manner the corresponding one-input analogue of the new two-input production function (3.3.14). Thus, let us derive a new production function* $Y = f(x)$ *whose growth is governed by the growth in* x *which we assume to be logistic. Hence, we can formulate the following problem within the framework of the growth model* $(G_2, \mathbb{R}_+)$*:*

$$(G_2, \mathbb{R}_+), \quad G_2 : \bar{x} = \frac{N_x x}{x + (N_x - x)\, e^{-at}},\; a > 0, x \in \mathbb{R}_+, \tag{3.3.19}$$

$$U_2 f = ax\left(1 - \frac{x}{N_x}\right)\frac{df}{dx} = bf\left(1 - \frac{f}{N_f}\right), \tag{3.3.20}$$

where the vector field $U_2 = ax\left(1 - \frac{x}{N_x}\right)\frac{\partial}{\partial x}$ *represents the infinitesimal action defined by the Lie group* G_2 *(3.3.19). Separating the variables and integrating the differential equation (3.3.20) yields the following solution (production function):*

$$Y = f_6(x) = \frac{N_{f_6} x^{\alpha}}{C|N_x - x|^{\alpha} + x^{\alpha}}, \tag{3.3.21}$$

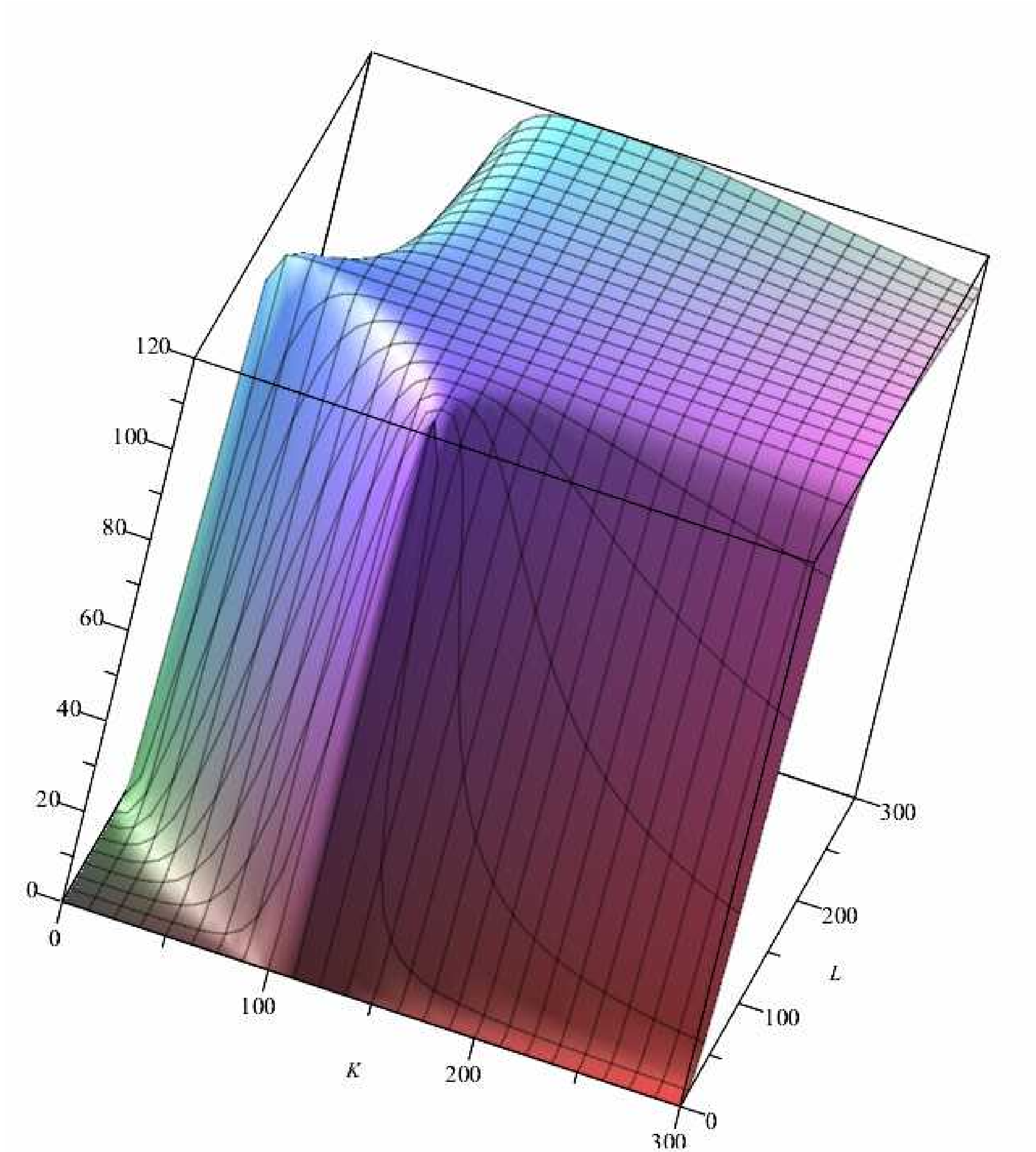
where $C \in \mathbb{R}$ *is the constant of integration and* $\alpha = b/a$ *with the corresponding steady state given by*

$$S_{f_6} = \frac{N_{f_6}}{C + 1}. \tag{3.3.22}$$

Note that in this case as well the new production function (3.3.21) exhibits first an "S-shaped" growth in the region $]0, N_x[$*, followed by a decline for* $x > N_x$*. Let us investigate this case from the economics point of view in more detail.*

Let us recover the corresponding group action that affects the input $x(t)$*, so that this action could be viewed as growth which entails the condition* $\dot{x}(t) > 0$*. Indeed, consider the infinitesimal action* $\tilde{U}$ *given by* $\tilde{U} = \tilde{U}_1\frac{\partial}{\partial x} + \tilde{U}_2\frac{\partial}{\partial y}$ *so that* $\tilde{U} f_6 = 0$*. Solving the last*

Figure 3.2: A two-input production function of the type (3.3.14) with isoquants.



equation, we arrive at the following solutions:

$$\begin{aligned} U_1 &= a\frac{x(N_x - x)}{N_x}, \\ U_2 &= b\frac{y(N_y - y)}{N_y} \end{aligned} \tag{3.3.23}$$

and

$$\begin{aligned} U_1 &= a\frac{x(x - N_x)}{N_x}, \\ U_2 &= b\frac{y(y - N_y)}{N_y}. \end{aligned} \tag{3.3.24}$$

In view of the fact that $x(t)$, $y(t) > 0$, *it follows from* (3.3.23) *and* (3.3.24) *that*

$$\begin{aligned} \dot{x} &= a\frac{x(N_x - x)}{N_x}, \quad 0 < x < N_x, \\ \dot{y} &= b\frac{y(N_y - y)}{N_y}, \quad 0 < y < N_y \end{aligned} \tag{3.3.25}$$

and

$$\begin{aligned} \dot{x} &= a\frac{x(x - N_x)}{N_x}, \quad x > N_x, \\ \dot{y} &= b\frac{y(y - N_y)}{N_y}, \quad y > N_y, \end{aligned} \tag{3.3.26}$$

so that both $x(t)$ *and* $y(t)$ *represent growth. Solving the above equations, we obtain*

$$x(t) = \begin{cases} \dfrac{N_x}{1 + C_1 e^{-at}}, & 0 < x(t) < N_x, \\ \dfrac{N_x}{1 + C_2 e^{at}}, & x(t) > N_x, \end{cases} \tag{3.3.27}$$

where $C_1 > 0$ *and* $C_2 > 0$ *are constants of integration. Next, we determine the time interval corresponding to growth in* $x(t)$. *It follows from (3.3.27) that* $t > 0$ *for* $0 < \frac{N_x}{1+C_1 e^{-at}} < N_x$ *and* $0 < t < \frac{1}{a}\ln\frac{1}{C_2}$ *for* $\frac{N_x}{1+C_2 e^{at}} > N_x$. *Substituting the equation (3.3.27) into (3.3.21), we arrive at the following function:*

$$y(t) = \begin{cases} \dfrac{N_{f_6}}{C(C_1 e^{-at})^{\alpha} + 1}, & 0 < t < t_1, \\ \dfrac{N_{f_6}}{C(C_2 e^{at})^{\alpha} + 1}, & t_1 < t < \frac{1}{a}\ln\frac{1}{C_2}, \end{cases} \tag{3.3.28}$$

where t_1 is the time at which the function shifts from the logistic to a different growth type. Let us assume α to be a positive integer. Furthermore, we note that $y(t)$ increases or decreases depending on whether α is odd or even respectively. To assure that (3.3.28) is compatible with (3.3.21) we assume that α is an even integer (see below). Next, rewrite the production function given by (3.3.28) as follows:

$$y = (H_0(t) - H_{t_1}(t))y_1(t) + H_{t_1}y_2(t), \tag{3.3.29}$$

where $H_c(t)$ is the Heaviside (unit) step function,

$$y_1(t) = \frac{N_{f_6}}{C(C_1 e^{-at})^\alpha + 1}, \quad y_2(t) = \frac{N_{f_6}}{C(C_2 e^{at})^\alpha + 1}.$$

In this view the function (3.3.29) may interpreted as an impulse response function. Indeed, a sudden change in the input at $t = t_1$ causes a jump in the output from $y_1(t)$ to $y_2(t)$. From the economic viewpoint we can identify this phenomenon as a "shock" [113], which means that a sudden change in exogenous factors yields the corresponding sudden change in production (see [69, 92, 54] for more details and references). The gap between $y_1(t)$ and $y_2(t)$ caused by a sudden change in $x(t)$ at $t = t_1$ is given by

$$d_{(y_1,y_2)}(t_1) = \frac{CN_{f_6}(C_2^a e^{bt_1} - C_1^a e^{-bt_1})}{(C(C_1 e^{-at_1})^a + 1)(C(C_2 e^{at_1})^a + 1)}, \tag{3.3.30}$$

where $d_{(y_1,y_2)}(t_1)$ denotes the distance between the two curves at $t = t_1$. Next, we note that

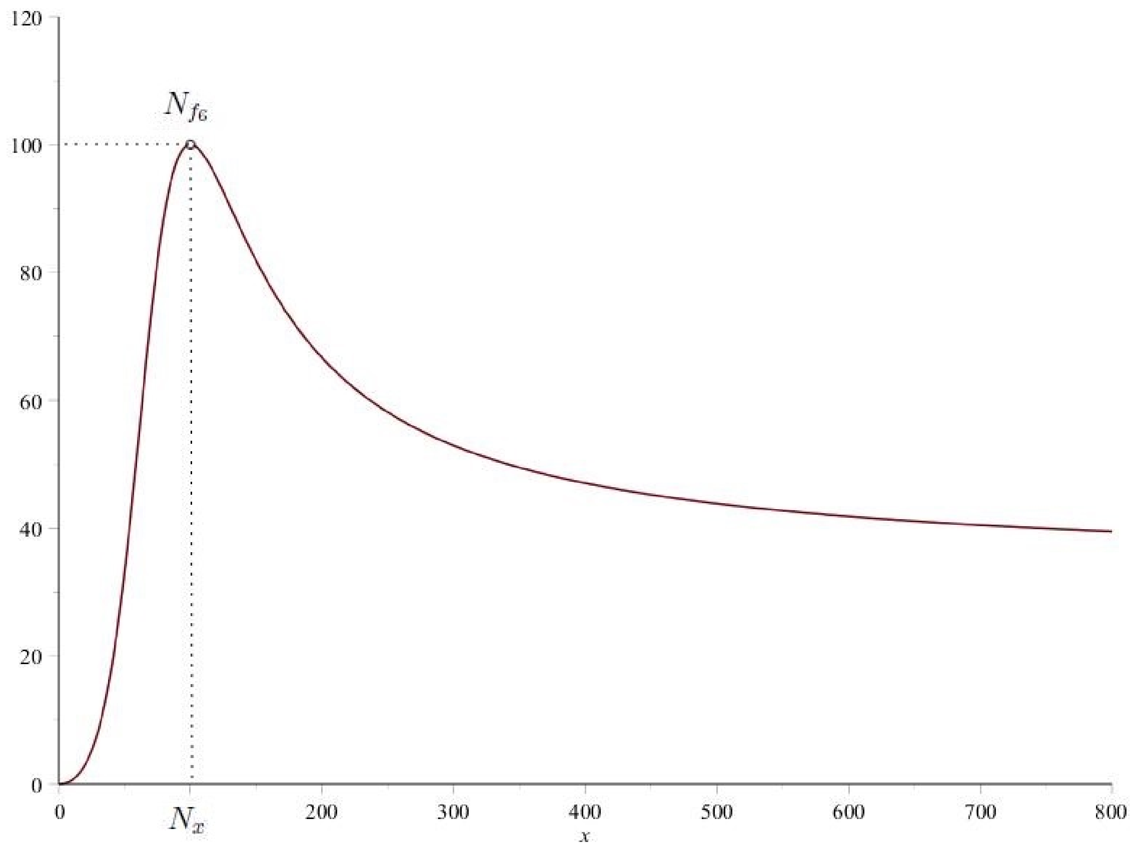
$$y(t) \to \frac{N_{f_6}}{C+1}, \quad \text{as} \quad t \to \frac{1}{a}\ln\frac{1}{C_2}. \tag{3.3.31}$$

Note that if α is an even number, the RHS of (3.3.31) is precisely the steady state (3.3.22).

Figure 3.3 presents the graph of a one-input production function of the type (3.3.21) generated for $N_{f_6} = 100$, $\alpha = 2$ and $C = 2$. Note the function given by (3.3.21) defines an invariant $I(K,L)$ of the infinitesimal action determined by vector field U_1 (3.2.11) for $f_6 = K$ (or, L) and $x = L$ (or, K), namely $U_1 I = 0$, where

$$I(K,L) = \frac{L^\alpha}{|N_L - L|^\alpha} \cdot \frac{N_K - K}{K}.$$

Figure 3.3: A one-input production function of the type (3.3.21).



Remark 3.3.9. *Repeating the above calculation within the frameworks of the growth models (3.2.12) and (3.2.13), we arrive at the production functions*

$$Y = f_7(K, L) = \frac{N_{f_7} K^\alpha L^\beta}{C\,|N_K - K|^\alpha + K^\alpha L^\beta} \tag{3.3.32}$$

and

$$Y = f_8(K, L) = \frac{N_{f_8} K^\alpha L^\beta}{C\,|N_L - L|^\beta + K^\alpha L^\beta}, \tag{3.3.33}$$

respectively, where the parameters α and β are the same as in (3.3.14).

We also note that the functions (3.3.32) and (3.3.33) are elements of the families (3.2.16) and (3.2.17) respectively, as expected.

3.4 Modeling economic bubbles using the new production function defined by logistic growth

Economic bubbles have occurred repeatedly throughout history. Recent examples are the IT bubble of the late 1990s, the US housing bubble of the early 2000s and the global financial crisis of 2007-2008. The review of history of economic bubbles can be found in [42, 45]. Economic bubbles have some common characteristics. In [85] the authors described an economic bubble occurs when an asset has a market price exceeding the price that a rational person would compensate. Another feature of an economic bubble is, as mentioned in [55, 45, 97], that speculation occurs with new, or perceived to be new technological enhancements.

The study of causes of economic bubbles is of intrinsic interest in economics while the mathematical study emphasizes on the dynamics of asset prices. Roughly speaking, the development of an economic bubble can be characterized by an escalation of an asset price followed by a sudden contraction. Mathematicians are interested in identifying if a crisis is happening or not through behaviour of prices. Based on the martingale theory (a martingale is a sequence of random variables for which the next expectation conditional on all previous variables is equal to the present expectation), in [59, 85, 97, 121] authors modeled the asset price as a solution of stochastic differential equations and tested the model against real economic data, by doing which, they were able to describe the bubble mathematically. For example, Sornette *et al.* [121, 122] concluded that the asset price in a bubble phase follows a faster-than-exponential growth with oscillations following the log-periodic power law. The log-periodic power law is characterized by a time-dependent function

$$y(t) = A + B(t_c - t)^z + C(t_c - t)^z \cos(\omega \log(t_c - t) + \Phi), \tag{3.4.1}$$

where y denotes an asset price, t is the time variable, t_c represents the most probable time of crash, z is the growth parameter and A, B, C, Φ are constant. Protter *et al.* [85] showed that stock prices in a financial crisis followed a gamma distribution and estimated parameters in the distribution using real data.

Alternatively, mathematicians and statisticians have also attempted to model the process of the crash in economy using a non-stochastic approach. For example, Watanabe *et al.* [130] fitted the trend of NASDAQ data in the Internet bubble to a discrete exponential growth and decay model. Herzog [55] modeled financial crisis using the shocking wave models, namely, the following equation

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial p} = 0, \tag{3.4.2}$$

where $\rho = \rho(p,t)$ denotes the number of trades within a certain price range, $q = q(p,t)$ represents the total trading price of an asset, that is, a product of the number of trades ρ and the buy or sell price $u = u(p,t)$ while the asset price $p = p(f,t)$ depends on the fundamental price f (a reasonable market price for an asset) and time t. Korobeinikov [71] employed a disease infection model to describe the global financial crisis of 2007-2008. By considering the healthy agents $x(t)$, who follow financial regulations strictly, and the activated agents $y(t)$, who are unable to fulfill their financial obligations, in an economy, he viewed the economy as the population of agents and described the economic bubble as the process of infection of healthy agents by activated agents, that is,

$$\begin{aligned} \dot{x} &= -\beta x y^{\alpha}, \\ \dot{y} &= \beta x y^{\alpha} - \frac{1}{\sigma} y, \end{aligned} \tag{3.4.3}$$

where $x(t)$ and $y(t)$ are the size of agents.

In this section, we want to present a model of an economic bubble involving the new production function $f_6(x)$ (3.3.21). More specifically, we want to exploit the shape of $f_6(x)$ to roughly characterize dynamics of a bubble. According to the *greater fool theory*, bubbles are driven by the behaviour of irrational market participants who are willing to buy an overvalued asset in order to sell the asset to the next speculator at a higher price. We propose that overvalued asset price is determined by the number of market participants. Let us consider the excessive price of an asset as a function of the number of market participants. We assume that the number of speculators buying or investing in an asset grows logistically. Obviously, the volume of the number of all buyers in an economy is fixed at a time, that is, there cannot be infinitely many market participants. Based on this, we introduce the following DE

$$\dot{x} = \beta x \left(1 - \frac{x}{N_x}\right), \tag{3.4.4}$$

where x represents the number of market participants.

Next, we consider the behaviour of an asset price. Recall that the function $f_6(x)$ (in (3.3.21)) derived in Section 3.2

$$Y = f_6(x) = \frac{N_{f_6} x^\alpha}{C|N_x - x|^\alpha + x^\alpha}. \tag{3.4.5}$$

The function (3.3.21) describes the relation of inputs (for example, labor, capital and *etc.*) and output (production), the graph of which exhibits first an "S-shaped" growth and follows a decline with the input beyond the carrying capacity. We derive the function using the holotheticity given by the differential equation (3.3.19).

Let us define the following non-autonomous first-order differential equation based on the form of the equation (3.3.20)

$$\frac{dp}{dt} = \alpha \frac{p\left(1 - \frac{p}{N_p}\right)}{t\left(1 - \frac{t}{N_t}\right)}, \tag{3.4.6}$$

where p denotes the asset price, t is the time variable, N_t represents the most probable time of crash, N_p is the carrying capacity and α is a constant parameter.

Remark 3.4.1. *N_t does not represent the carrying capacity of time since it is not reasonable to discuss the carrying capacity of time t. By analogy with t_c in* (3.4.1)*, we assume N_t represents the most probable time of crash.*

Remark 3.4.2. *The equation* (3.4.6) *is not derived from the holotheticity or from a variational principle. We modify the equation* (3.3.19) *to obtain a new equation. By imposing the condition that the asset price following this dynamics, we can qualitatively recover an asset price in a crisis phase in some sense, namely, a rapid expansion followed by a sudden contraction.*

Hence, the dynamics of an economic bubble can be characterized by the following system

$$\begin{cases} \dfrac{dp}{dt} &= \alpha \dfrac{p\left(1-\dfrac{p}{N_p}\right)}{t\left(1-\dfrac{t}{N_t}\right)}, \\ \dfrac{dx}{dt} &= \beta x\left(1-\dfrac{x}{N_x}\right). \end{cases} \tag{3.4.7}$$

Solving equations (3.4.7) yields

$$p(t) = \frac{N_p t^\alpha}{C_1|N_t - t|^\alpha + t^\alpha}, \quad C_1 \in \mathbb{R} \tag{3.4.8}$$

and

$$x(t) = \frac{N_x}{1 + C_2 e^{-\beta t}}, \quad C_2 \in \mathbb{R}. \tag{3.4.9}$$

The equation (3.4.9) gives

$$t = \frac{1}{\beta} \ln\left(\frac{x}{C_2(N_x - x)}\right). \tag{3.4.10}$$

In view of the asset price as a function of market participants, we substitute (3.4.10) into (3.4.7) and derive

$$p(x) = \frac{N_p}{C_1\left|\dfrac{N_t}{\dfrac{1}{\beta}\ln\left(\dfrac{x}{C_2(N_x-x)}\right)} - 1\right|^\alpha + 1}. \tag{3.4.11}$$

In fact, when less speculators participate in the market, it becomes difficult for the asset holders in the market to sell their assets. Namely, it can be argued, roughly, when the growth rate of speculators slows down, the asset price declines. We hypothesize that the downfall of the asset price happens at the stationary point of (3.4.9), that is,

$$N_t = \frac{\ln C_2}{\beta}. \tag{3.4.12}$$

By assuming $N_t = \frac{\ln C_2}{\beta}$, our model of an economic bubble is characterized by the following function

$$p(x) = \frac{N_p \left| \ln\left(\frac{x}{C_2(N_x - x)}\right) \right|^{\alpha}}{C_1 \left| \ln C_2 - \ln\left(\frac{x}{C_2(N_x - x)}\right) \right|^{\alpha} + \left| \ln\left(\frac{x}{C_2(N_x - x)}\right) \right|^{\alpha}}, \quad C_2 > 0. \quad (3.4.13)$$

We generate Figure 3.4 using the symbolic algebra programming Maple for $C_1 = 100$, $C_2 = 2$, $N_p = 1000$, $N_x = 100$ and $\alpha = 2$.

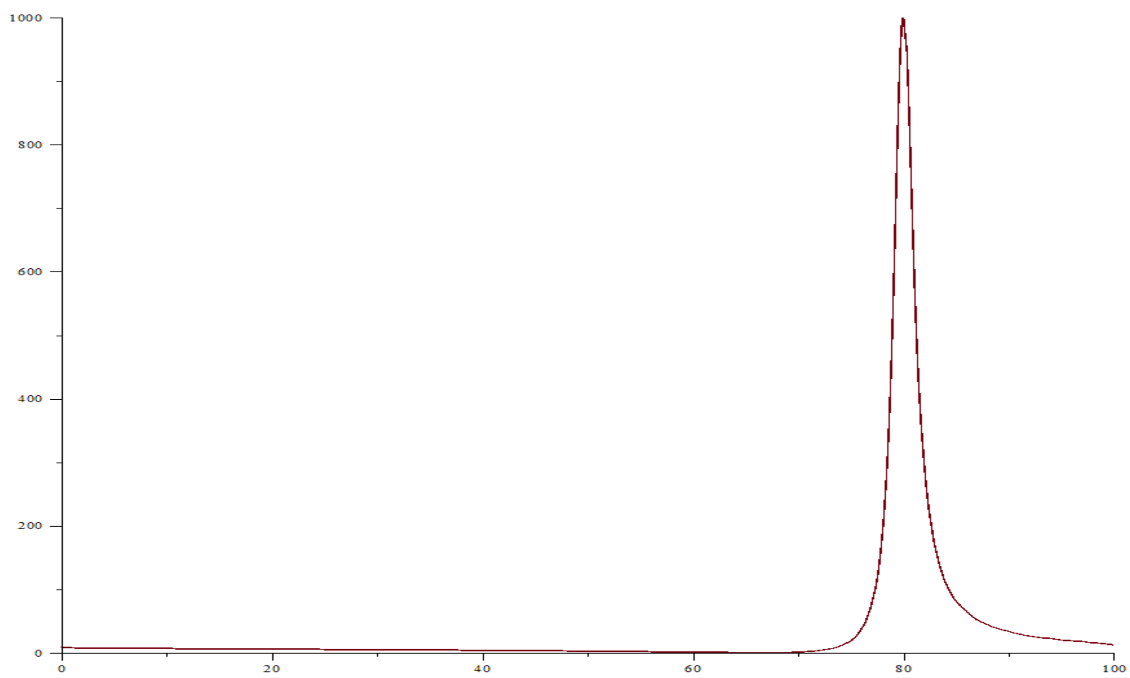


Figure 3.4: An asset price versus the number of market participants.

Figure 3.4 illustrates the dynamics of an economic model. At the beginning, the price of an asset stays at a level. After a while, the speculation involves more participants and the price grows rapidly. When the market is nearly saturated, it is difficult for asset holders to sell their assets and the asset price starts to plummet. It is notable that the most probable time of crash is given by $N_t = \frac{\ln C_2}{\beta}$, which is determined by the logistic growth of market participants. This allows us to monitor a bubble and somehow predict the crash time of a bubble through monitoring the population model describing the number of speculators at a given time.

We attempted to model economic crisis based on *the greater fool theory* by relating the asset price to the size of market participants, in which we assume the asset price p is affected by the growth type of the function $f_6(x)$ and the number of speculators x follows a logistic growth. We admit that using the model we have some difficulties in explaining the asset price prior to the escalation since it shows that the price may decline with increasing participants. One may argue that it may be normal since the supply is greater than the demand, or, the asset has not become an investment good. We admit that it is not enough to model an economic bubble accurately by focusing only on the number of participants. However, the model is robust in some sense. For example, it reflects certain features of an economic bubble, that is, the asset price increases rapidly followed by a crash. What is even more important is that the crash time of a bubble is predictable in this model, namely, the crash time is related in the model to the number of participants. It must also be kept in consideration that, as mentioned in [55, 71, 97], the economic bubble is a chaotic and complex process depending on a large amount of parameters and affected by a variety of factors, thus a simple model is not able to fully explain or describe all details of the process. Considering this complexity, a precise quantitative description of the economic bubble is not to be expected or even not possible at all.

3.5 The problem of maximization of profit under conditions of perfect competition

In 1947 Paul Duglas gave his presidential address to the American Economics Association in which he referred to a coherent assembly of the statistical evidence accumulated in the course of the previous 20 years while he and other people were studying various economic data that confirmed the validity of the Cobb-Douglas production function. It is safe to assume that this event marked the beginning of its universal acceptance by the mainstream economic science. He wrote in [31]: "... the Cobb-Douglas function was being widely used, and that a host of younger scholars led by my former student, Paul Samuelson, his colleague Solow and Marc Nerlove, the son of my friend and former colleague, Samuel Nerlove, were all pushing forward into new and more sophisticated fields." In fact, Marc Nerlove gave a series of lectures at the

Econometric Workshop held at the University of Minnesota in 1957, which were subsequently published a few years later in a book [84]. One of the problem considered by the author was the problem of maximization of profit of a firm under conditions of perfect competition in both factors and product markets under the assumption that the revenue of the firm from sales was determined by the Cobb-Douglas production function. In what follows we shall solve the problem using the same arguments *mutatis mutandis* as in [84] by assuming that the revenue of the firm from sales is now determined by the new production function (3.3.14).

Consider an individual firm functioning under conditions of perfect competition in both factors and product markets. It attempts to maximize its profits by employing optimal quantities of inputs and producing an optimal quantity of output. At the same time its purchases of factors and supply of output do not affect the prices of the factors involved and the final product. Therefore the said prices are assumed to be given, while the profits are to be maximized. Let Π, p_0, p_1, p_2 be the profit, the price of the final product, the cost of using one unit of capital, and the wage of labor respectively. Hence, we have

$$\Pi = p_0 Y - p_1 K - p_2 L. \tag{3.5.1}$$

Traditionally, in problems like this the output Y is assumed to be related to the inputs K (capital) and L (labor) by the Cobb-Douglas production function (3.0.1). Instead, suppose now Y is related to K and L via the new production function f_5 (3.3.14). Next, let us solve the problem of maximization of the profit Π given by (3.5.1) subject to the constraint implied by (3.3.14). The corresponding Lagrangian function $\mathcal{L}$ is readily found to be

$$\mathcal{L}(Y, K, L, \lambda) = \Pi - \lambda \left(Y - \frac{N_{f_5} K^{\alpha} L^{\beta}}{C \left| N_K - K \right|^{\alpha} \left| N_L - L \right|^{\beta} + K^{\alpha} L^{\beta}} \right), \tag{3.5.2}$$

where λ is a Lagrange multiplier. For the profit to be a maximal, we must have

$$d\mathcal{L}(Y, K, L, \lambda) = d(\Pi - \lambda g) = 0, \tag{3.5.3}$$

where

$$g = Y - \frac{N_{f_5} K^\alpha L^\beta}{C\,|N_K - K|^\alpha\,|N_L - L|^\beta + K^\alpha L^\beta}. \tag{3.5.4}$$

The condition (3.5.3) yields

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \lambda} &= -Y + \frac{N_{f_5} K^\alpha L^\beta}{C\,|N_K - K|^\alpha\,|N_L - L|^\beta + K^\alpha L^\beta} = 0,\\
\frac{\partial \mathcal{L}}{\partial K} &= -p_1 + p_0 \frac{\beta N_{f_5} C (N_K - K)^\alpha K^\alpha (L^{\beta-1}(N_L - L)^\beta + (N_L - L)^{\beta-1} L^\beta)}{(C(N_K - K)^\alpha (N_L - L)^\beta + K^\alpha L^\beta)^2} = 0,\\
\frac{\partial \mathcal{L}}{\partial L} &= -p_2 + p_0 \frac{\alpha N_{f_5} C (N_L - L)^\beta L^\beta (K^{\alpha-1}(N_K - K)^\alpha + (N_K - K)^{\alpha-1} K^\alpha)}{(C(N_K - K)^\alpha (N_L - L)^\beta + K^\alpha L^\beta)^2} = 0,\\
\frac{\partial \mathcal{L}}{\partial Y} &= p_0 - \lambda = 0.
\end{aligned} \tag{3.5.5}$$

The equations (3.5.5) give us necessary conditions for maximum profit. Solving (3.5.5) with the aid of the computer algebra system Maple, we get

$$\begin{aligned}
Y &= \frac{N_{f_5} K^\alpha L^\beta}{C\,|N_K - K|^\alpha\,|N_L - L|^\beta + K^\alpha L^\beta},\\
\alpha &= \frac{p_2 N_{f_5} K (N_K - K)}{p_0 N_K Y (N_{f_5} - Y)},\\
\beta &= \frac{p_0 N_K Y (N_{f_5} - Y) \ln \frac{|N_{f_5} - Y|}{CY} - p_2 N_{f_5} K (N_K - K) \ln \frac{|N_K - K|}{K}}{p_0 N_K Y (N_f - Y) \ln \frac{|N_L - L|}{L}}.
\end{aligned} \tag{3.5.6}$$

The resulting equations (3.5.6) are sufficient to determine the variables Y, K and L. The corresponding sufficient conditions for maximum profit are provided by the necessary conditions established above supplemented by the following second-order condition:

$$d^2\mathcal{L} < 0,$$

or, given the fact that Π in (3.5.2) is linear in Y, K and L (see (3.5.1)) and $\lambda = p_0$ by (3.5.5), we have

$$d^2\tilde{g} > 0, \tag{3.5.7}$$

where

$$\tilde{g}(K,L) = \frac{p_0 N_{f_5} K^\alpha L^\beta}{C\,|N_K - K|^\alpha\,|N_L - L|^\beta + K^\alpha L^\beta}.$$

Solving (3.5.7), using Maple, we arrive at the following set of inequalities:

$$\begin{aligned}
&\alpha(\alpha-1) < 0,\\
&\beta(\beta-1) < 0,\\
&(2K-N_K)(2L-N_L) + N_L(2K-N_K)\beta + N_K(2L-N_L)\alpha > 0,\\
&(2K-N_K)(2L-N_L) - N_L(2K-N_K)\beta - N_K(2L-N_L)\alpha > 0.
\end{aligned} \tag{3.5.8}$$

The first two inequalities entail that $0 < \alpha, \beta < 1$. The second two inequalities imply that $K > N_K/2$ and $L > N_L/2$. Hence, we arrive at the following conditions that assure maximum profit:

$$\begin{aligned}
&0 < \alpha, \beta < 1, \quad K > N_K/2, \quad L > N_L/2,\\
&(2K-N_K)(2L-N_L) + N_L(2K-N_K)\beta + N_K(2L-N_L)\alpha > 0,\\
&(2K-N_K)(2L-N_L) - N_L(2K-N_K)\beta - N_K(2L-N_L)\alpha > 0.
\end{aligned} \tag{3.5.9}$$

Next, we observe that since $\lim_{t\to\infty} K(t) = N_K$ and $\lim_{t\to\infty} L(t) = N_L$, the last inequality in (3.5.9) implies that

$$0 < \alpha + \beta < 1, \tag{3.5.10}$$

which in turn implies that the assumption of perfect competition and maximization of profit are inconsistent in the case when

$$\alpha + \beta \geq 1.$$

Finally, we conclude that the equations and inequalities (3.5.6), (3.5.9) and (3.5.10) constitute sufficient conditions for maximum profit of a firm in the environment of perfect competition. The equations (3.5.6) determine the output a firm will deliver and the inputs of factors it will employ once the prices of the product and factors are established. Therefore the conclusions are pretty much the same as in the case when the revenue is determined by the Cobb-Douglas production function (3.0.1) considered in Nerlove [84]. The case of imperfect competition in both factor and production markets will be considered in future research.

Note that all of the calculations above have been carried out under the assumption that $C > 0$. If $C < 0$ the condition (3.5.10) changes to $\alpha + \beta > 1$.

3.6 The wage share and logistic growth

The labor share is the fraction of national income, or the income of a particular economic sector, defined as the share which is paid out to employees. Therefore it is often also called the wage share. As is well-known, the wage share in the economic growth models governed by the Cobb-Douglas production function (3.0.1) is a constant. More specifically, its constant value can be derived directly from the Cobb-Douglas function and expressed in terms of the output elasticity of capital in a simple and elegant way when the Cobb-Douglas function, say, enjoys constant return to scale (see, for example, Rabbani [98]). The invariance of the wage share is subject to Bowley's law [15, 16] or the law of the constant wage share, which states that the share of national income that is paid out to the employees as compensation for their work (normally, in the form of wages), remains unchanged (invariant) over time [67, 73, 112]. Economic data collected in different countries till about 1980 gave rise to and most strongly supported this law, which was widely accepted by the economics community at the time. However, this is no longer the case on both counts (see, for example, Schneider [112] for more details and references).

In view of the mathematical models presented above, it should not be viewed as a surprise. Indeed, the ivariance of wage share is linked to the Cobb-Douglas production function, which in turn is a consequence of exponential growth, as shown by Sato [107]. Next, since one of the the main points of this research project is the idea that we must depart from the exponential growth model and accept the logistic one, let us investigate how this transition affects the wage share.

In what follows we shall propose a new formula for the wage share compatible with logistic growth and support our claim by a rigorous mathematical analysis.

First, let us recover the formula for the wage share as an invariant of a prolonged infinitesimal group action given in terms of the corresponding projective coordinates defined as the output-capital ration $Y/K = y$ and the labor-capital output $L/K = x$. The terminology and notations that we will use are compatible with those adopted by Olver [88, 87] and Saunders [111]. Consider a general production function

$$Y = f(K, L; t) \tag{3.6.1}$$

under the assumption that the dependent and independent variables K, L and Y grow exponentially:

$$\bar{K} = Ke^{\alpha t}, \quad \bar{L} = Le^{\beta t}, \quad \bar{Y} = Ye^{\epsilon t}, \ \alpha, \beta, \epsilon \geqslant 0. \tag{3.6.2}$$

In view of the material presented in Section 3.1 we know that the production function (3.6.1) is bound to be of the Cobb-Douglas type (3.0.1). In terms of the projective coordinates it assumes the following form:

$$y = f(x; t), \tag{3.6.3}$$

where x and y are projective variables.

Clearly, the one-parameter Lie group of transformations (3.6.2) induces the corresponding action on the projective coordinates, which is also exponential:

$$\bar{y} = ye^{\gamma t}, \quad \bar{x} = xe^{\lambda t}, \ \gamma, \lambda \geqslant 0 \tag{3.6.4}$$

with the corresponding infinitesimal action given by the vector field $\mathbf{u}$ (compare it with (3.1.7)) given by

$$\mathbf{u} = \lambda x \frac{\partial}{\partial x} + \gamma y \frac{\partial}{\partial y}. \tag{3.6.5}$$

Following terminologies introduced in Section 2.3, let us suppose that $(\mathbb{R}^2, \pi, \mathbb{R})$ is a trivial bundle so that $\pi = pr_1$ and (x, y) are adapted coordinates. Then the corresponding jet bundles are $(J^1\pi, \pi_1, \mathbb{R})$ and $(J^1\pi, \pi_{1,0}, \mathbb{R}^2)$, as per the commutative diagram (3.6.7), where the first-jet manifold of π is given by

$$J^1\pi = \left\{ j^1_p\phi : p \in \mathbb{R}, \phi \in \Gamma_p(\pi) \right\} \tag{3.6.6}$$

in terms of adapted coordinates (x, y, y_x).

$$\begin{array}{ccc} J^1\pi & \xrightarrow{\pi_{1,0}} & \mathbb{R}^2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ \mathbb{R} & \xrightarrow{id} & \mathbb{R} \end{array} \tag{3.6.7}$$

Here $\pi_1 = \pi \circ \pi_{1,0}$.

Next, the first prolongation of $\mathbf{u}$ on $\mathbb{R}^2$ is the following vector field $\mathrm{pr}^{(1)}\mathbf{u} = \mathbf{u}^{(1)}$, which, using Theorem 2.3.2, has to be a symmetry of the Cartan distribution on $J^1\pi$, that is the vector field

$$\mathrm{pr}^{(1)}\mathbf{u} = \mathbf{u}^{(1)} = \lambda x \frac{\partial}{\partial x} + \gamma y \frac{\partial}{\partial y} + \xi(x, y, y_x) \frac{\partial}{\partial y_x} \tag{3.6.8}$$

is required to be a symmetry of the Cartan distribution on $J^1\pi$. Indeed, consider a basic contact form $\omega = dy - y_x dx$. Next, in view of the above, we require the one-form $\mathcal{L}_{\mathbf{u}^{(1)}}(\omega)$ to be a contact form, where $\mathcal{L}$ denotes the Lie derivative. Thus, we compute

$$\begin{aligned} \mathcal{L}_{\mathbf{u}^{(1)}}(\omega) &= \mathcal{L}_{\mathbf{u}^{(1)}}(dy - y_x dx) \\ &= \mathcal{L}_{\mathbf{u}^{(1)}}(dy) - (\mathcal{L}_{\mathbf{u}^{(1)}} y_x) dx - y_x(\mathcal{L}_{\mathbf{u}^{(1)}}(dx)) \\ &= \mathrm{d}(\mathbf{u}^{(1)})(y)) - (\mathbf{u}^{(1)}(y_x))\mathrm{d}x - y_x \mathrm{d}(\mathbf{u}^{(1)}(x)) \\ &= \gamma \mathrm{d}y - \xi(x, y, y_x)\mathrm{d}x - \lambda y_x \mathrm{d}x \\ &= \gamma(\omega + y_x \mathrm{d}x) - \xi(x, y, y_x) - \lambda y_x \mathrm{d}x \\ &= \gamma\omega + (\gamma y_x - \xi(x, y, y_x) - \lambda y_x)\mathrm{d}x. \end{aligned} \tag{3.6.9}$$

The last line of (3.6.9) implies that the expression in the parentheses above vanishes, which entails that $\xi(x, y, y_x) = (\gamma - \lambda)y_x$. Therefore the first prolongation $\mathbf{u}^{(1)}$ of $\mathbf{u}$ is found to be

$$\mathbf{u}^{(1)} = \lambda x \frac{\partial}{\partial x} + \gamma y \frac{\partial}{\partial y} + (\gamma - \lambda) y_x \frac{\partial}{\partial y_x}. \tag{3.6.10}$$

The vector field (3.6.10) represents an infinitesimal action of a one-parameter Lie group of transformations in a three-dimensional (prolonged) space. Hence, we expect to obtain $3-1=2$ fundamental differential invariants. Indeed, solving the corresponding partial differential equation by the method of characteristics, we arrive at the following set of two fundamenal differential invariants:

$$I_1 = yx^{-\frac{\gamma}{\lambda}}, \quad I_2 = y_x x^{\frac{\lambda-\gamma}{\lambda}}, \tag{3.6.11}$$

as expected, which means that any other differential invariant of the prolonged infinitesimal group action defined by (3.6.10) if a function of I_1 and I_2. Now, combining the fundamental differential invariants (3.6.11) in such a way that the parameters λ and γ disappear, we arrive at the following differential invariant:

$$\mathcal{I}(I_1, I_2) = \frac{xy_x}{y}, \tag{3.6.12}$$

which we immediately recognize to be precisely the wage share s_L (see, for example, Rabbani [98] and Schneider [112] for more details).

Therefore we conclude that not only the Cobb-Douglas production function (3.0.1), but also the wage share $s_L = \mathcal{I}$ given by (3.6.12) is a consequence of the exponential growth in K and L as a differential invariant obtained within the framework of the growth model $(G, \mathbb{R}^2_+)$, where the action of the Lie group G is given by (3.1.5), that is

exponential growth $\Rightarrow$ the wage share function (3.6.12).

Now let us redo the above calculations for the growth model $(G_1, \mathbb{R}^2_+)$, where the action of G_1 is given by (3.2.7) and thus give a solution to the seemingly unresolved problem of the determination of why Bowley's law [15, 16] does not hold true anymore in post-1960s data [12, 33, 52, 64].

First, we observe in the example considered above the exponential growth in K and L induced the corresponding exponential growth in the projective coordinates $x = L/K$

and $y = Y/K$. However, the logistic growth in K and L given by (3.2.7) does not translate into the same type of transformations for the projective coordinates x and y. Therefore, let us assume that the growth in K is suppressed by, say, excessive debt and so it does not affect logistic growth in L and Y. Hence, both projective coordinates x and y grow logistically, that is we have

$$\bar{x} = \frac{1}{1 + (\frac{1}{x} - 1)e^{-\lambda t}}, \quad \bar{y} = \frac{1}{1 + (\frac{1}{y} - 1)e^{-\gamma t}}, \ \lambda, \gamma \geqslant 0, \tag{3.6.13}$$

where we assumed without loss of generality that both carrying capacities were equal to one. The corresponding infinitesimal action of the Lie group G_1 is given by the vector field

$$\mathbf{u}_1 = \lambda x(1 - x)\frac{\partial}{\partial x} + \gamma y(1 - y)\frac{\partial}{\partial y}. \tag{3.6.14}$$

To determine its first prolongation $\mathbf{u}_1^{(1)} = \mathrm{pr}^{(1)}\mathbf{u}_1$ we proceed as above within the same framework as in the previous case (see the commutative diagram (3.6.7)). We note first that the vector field $\mathbf{u}_1^{(1)}$ on $J^1\pi$ is projectable, since the bundle $(T\mathbb{R}^2, \tau, \mathbb{R}^2)$ is endowed with a vector structure (see Saunders [111], Chapter 2 for more details). Next, define

$$\mathbf{u}_1^{(1)} = \lambda x(1 - x)\frac{\partial}{\partial x} + \gamma y(1 - y)\frac{\partial}{\partial y} + \xi(x, y, y_x)\frac{\partial}{\partial y_x} \tag{3.6.15}$$

and require the vector field (3.6.15) to be a symmetry of the Cartan distribution, which will assure that (3.6.15) is the first prolongation of (3.6.14). Indeed, consider again a basic contact form $\omega = \mathrm{d}y - y_x\mathrm{d}x$. Then again, $\mathcal{L}_{\mathbf{u}_1^{(1)}}(\omega)$ is a contact form iff $\mathbf{u}_1^{(1)}$ is a symmetry of the Cartan distribution on $J^1\pi$, which in turn assures that (3.6.15) is indeed the first prolongation of (3.6.14), where $\mathcal{L}$ as before denotes the Lie derivative. Thus, we compute

$$\begin{aligned}
\mathcal{L}_{\mathbf{u}_1^{(1)}}(\omega) &= \mathcal{L}_{\mathbf{u}_1^{(1)}}(\mathrm{d}y - y_x\mathrm{d}x) \\
&= \mathcal{L}_{\mathbf{u}_1^{(1)}}(\mathrm{d}y) - (\mathcal{L}_{\mathbf{u}_1^{(1)}}(y_x)\mathrm{d}x - y_x(\mathcal{L}_{\mathbf{u}_1^{(1)}}(\mathrm{d}x)) \\
&= \mathrm{d}(\mathbf{u}_1^{(1)}(y)) - (\mathbf{u}_1^{(1)}(y_x))\mathrm{d}x - y_x\mathrm{d}(\mathbf{u}_1^{(1)}(x)) \\
&= \gamma(1-2y)\mathrm{d}y - \xi(x,y,y_x)\mathrm{d}x - \lambda(y_x\mathrm{d}x - 2xy_x\mathrm{d}x) \\
&= \gamma(1-2y)(\omega + y_x\mathrm{d}x) - (\xi(x,y,y_x) + \lambda y_x - 2\lambda x y_x)\mathrm{d}x \\
&= \gamma(1-2y)\omega + (\gamma y_x - 2\gamma y y_x - \xi(x,y,y_x) - \lambda y_x + 2\lambda x y_x)\mathrm{d}x.
\end{aligned} \tag{3.6.16}$$

In view of the above, $\mathcal{L}_{\mathbf{u}_1^{(1)}}(\omega)$ is again a contact form, provided the expression in the parenthesis that appears in the last line of (3.6.16) vanishes. Hence, we have

$$\gamma y_x - 2\gamma y y_x - \xi(x,y,y_x) - \lambda y_x + 2\lambda x y_x = 0, \tag{3.6.17}$$

or,

$$\xi(x,y,y_x) = (\gamma - \lambda + 2\lambda x - 2\gamma y)y_x. \tag{3.6.18}$$

We conclude therefore that the first prolongation of the vector field $\mathbf{u}_1$ given by (3.6.14) is the following fector field:

$$\mathbf{u}_1^{(1)} = \lambda x(1-x)\frac{\partial}{\partial x} + \gamma y(1-y)\frac{\partial}{\partial y} + (\gamma - \lambda + 2\lambda x - 2\gamma y)y_x\frac{\partial}{\partial y_x}, \tag{3.6.19}$$

whose infinitesimal action brings about the following two fundamental differential invariants:

$$I_1 = -\left(\frac{y-1}{y}\right)\left(\frac{x}{x-1}\right)^{\frac{\gamma}{\lambda}}, \quad I_2 = (2\gamma x)^2\left(\frac{y_x}{(y-1)^2}\right)\left(\frac{1-x}{x}\right)^{\frac{\gamma+\lambda}{\lambda}}. \tag{3.6.20}$$

In order to eliminate the parameters λ and γ let us consider the following combination:

$$\mathcal{I}(I_1, I_2) = I_1 \cdot \frac{I_2}{(2\gamma)^2} = x|x-1|\left(\frac{y_x}{y|y-1|}\right). \tag{3.6.21}$$

Definition 3.6.1. The differential invariant $\mathcal{I}$ given by (3.6.21) is called a *modified* *wage share* $s'_L = \mathcal{I}$, so that

$$s'_L = \frac{|x-1|}{|y-1|} s_L = \text{const}, \tag{3.6.22}$$

where s_L is the classical wage share given by (3.6.12).

Remark 3.6.2. *The modified wage share s'_L given by (3.6.22) is a differential invariant of the growth model $(G_1, \mathbb{R}^2_+)$, where the action of the Lie group G_1 is given by (3.2.7), while the classical wage share s_L given by (3.6.12)* is not. *That is a reason why s_L has been in decline: it may be attributed to the fact that post-1960 economic data has been generated within the framework of the growth model $(G_1, \mathbb{R}^2_+)$, rather than $(G, \mathbb{R}^2_+)$. More specifically, it follows that the decline in s_L is due to the relation $\gamma > \lambda$ (see (3.6.22)). Indeed, if the output-to-capital ratio y grows logistically faster than the labor-to-capital ratio x under the condition of supressed capital (*e.g., *by excessive debt), that is if $\gamma > \lambda$ the ratio $\frac{|x-1|}{|y-1|}$ in (3.6.22) clearly contributes to decline in s_L, since s'_L is a constant. Simply put, more wealth (real or perceived) distributed among fewer people implies a marked decrease in the classical wage share s_L and so Bowley's law [15, 16] no longer holds in the economic environment of the logistic growth model $(G_1, \mathbb{R}^2_+)$.*

Remark 3.6.3. *The corresponding production function compatible with the infinitesimal action generated by the vector field $\mathbf{u}_1$ (3.6.14) is readily found to be*

$$Y = f_9(K, L; t) = \frac{KL^{C_3}}{L^{C_3} + C_4|L-K|^{C_3}}, \quad C_3 \in (0,1), C_4 \in \mathbb{R}, \tag{3.6.23}$$

which we derived by integrating the equation $\mathcal{I} = const$, where $\mathcal{I}$ is given by (3.6.21) and rewriting the solution in terms of K and L.

Now, let us analyse the second new production function (3.6.23). The partial derivatives of the production function f_9 (3.6.23), called in economic literature marginal productivities*, are found to be*

$$MP_K = \frac{1}{1 + C_4|1-\frac{K}{L}|^{C_3}} + C_3C_4\frac{K}{L-K}\frac{|1-\frac{K}{L}|^{C_3}}{(1+C_4|1-\frac{K}{L}|^{C_3})^2}, \tag{3.6.24}$$

$$MP_L = C_3C_4 \frac{K^2}{L(L-K)} \frac{|1-\frac{K}{L}|^{C_3}}{(1+C_4|1-\frac{K}{L}|^{C_3})^2}. \tag{3.6.25}$$

Next, the slope of an isoquant is the marginal rate of technical substitution *(MRTS), or* technical rate of substitution *(TRS). Thus,* $MRTS = \frac{MP_K}{MP_L}$ *so that in our case*

$$MRTS(K,L) = \frac{1}{C_3C_4} \frac{L(L-K)}{K^2} \frac{1+C_4|1-\frac{K}{L}|^{C_3}}{(1-\frac{K}{L})^{C_3}} + \frac{L}{K}, \tag{3.6.26}$$

which decreases when L *grows and* K *declines. We conclude, therefore, that* (3.6.26) *has concave up isoquants when* L *increases and* K *decreases, that is if the labor-capital ratio is less than approximately* $\frac{1+C_3}{2}$*, in which case* $MRTS$ *increases, while otherwise the isoquants are concave down, since* $MRTS$ *decreases.*

Recall that the new productoin function (3.3.14) does not enjoy constant return to scale. Now let us examine the function (3.6.23) from this viewpoint. Indeed, for a factor $r > 1$*, the substitution* $(K,L) \to (rK, rL)$ *in (3.6.23) yields*

$$\begin{aligned} f_9(rK, rL) &= \frac{rK(rL)^{C_3}}{(rL)^{C_3} + C_4|(rL)-(rK)|^{C_3}} \\ &= \frac{rKL^{C_3}}{L^{C_3} + C_4|L-K|^{C_3}}. \end{aligned} \tag{3.6.27}$$

which means that the new production function (3.6.23) *has* constant returns to scale*, since it is a homogeneous function of degree one. Therefore we conclude that it satisfies* the law of diminishing marginal returns and has constant return to scale*, which means it has a great potential for playing a pivotal role in various economic growth models.*

Finally, let us investigate the behavior of the new production function (3.6.23) as $t \to 0$ *and* $t \to \infty$ *under the assumption that both* $K(t)$ *and* $L(t)$ *grow logistically according to the one-parameter Lie group transformations defined by (3.2.7). To understand its behaviour when* K *and* L *are small, we employ economic reasoning. Thus, at the beginning of a production cycle a company, say, invests much of its resources into fixed assets (*e.g.*, infrastructure, materials, land, etc) and so when* t *is* *small it is safe to assume that* $K \gg L$*, which implies that*

$$f_9(t) \sim \frac{1}{C_4}(K(t))^{1-C_3}(L(t))^{C_3}, \tag{3.6.28}$$

that is the production function Y *enjoys a similar behaviour to that of the Cobb-Douglas production function (3.0.1) that has constant returns to scale. When* $t \to \infty$ *both* K *and* L *grow logistically and so we have by (3.6.23)*

$$\lim_{t\to\infty} f_9(K, L; t) = const.$$

3.7 The new production function f_5 vis-à-vis economic data

In this section we present a similar analysis to the one conducted by Cobb and Doublas [27], namely we compare the new production function with some available US economic data from 1947-2016. We make use of the data from the period 1947-2016 that is provided by the Federal Reserve Bank of St. Louis (https://fred.stlouisfed.org), employing the FRED tool. The variables are as follows: K — capital services of nonfarm business sector [36], L — compensations of employees of nonfarm business sector [35], Y — real output of nonfarm business sector [37]. The values of all variables are dimensionless, they are index values with the values at 2009 taken as 100. To estimate the new production function (3.3.14), we have used the R Programming language [63], employing the method of least squares, and assuming the corresponding carrying capacities to be of the following values: $N_{f_5} = 120$, $N_L = 150$. We have also assumed that $\alpha + \beta = 1$.

The resulting production function of the type (3.3.14) is found to be

$$Y = \frac{120K^{(0.4063544)}L^{(0.5936456)}}{(0.3118901)|150 - K|^{(0.4063544)}|150 - L|^{(0.5936456)} + K^{(0.4063544)}L^{(0.5936456)}}, \tag{3.7.1}$$

where $C = 0.3118901$, $\alpha = 0.4063544$ and $\beta = 0.5936456$ (see Figure 3.5).

The elasticity of substitution σ_1 (see Sato [105]) of the new production function

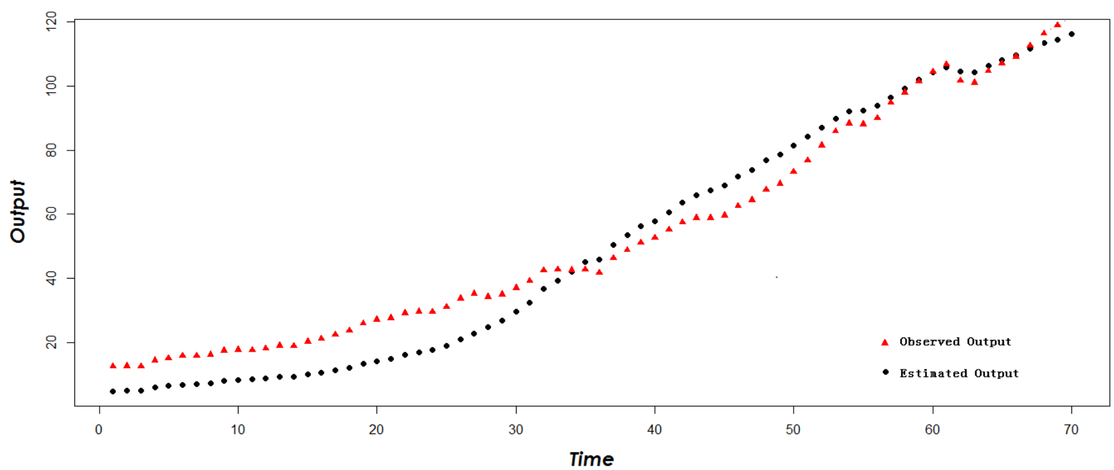


Figure 3.5: Observed output vs estimated output using the new production function (3.3.14).

(3.3.14) in this case assumes the following form:

$$\sigma_1 = \frac{\frac{\dot{L}}{L} - \frac{\dot{K}}{K}}{\frac{\dot{L}}{L} - \frac{\dot{K}}{K} - \frac{\dot{K}}{K-1} - \frac{\dot{L}}{L-1}}, \tag{3.7.2}$$

where $K = \frac{N_K C_1}{C_1 + (N_K - C_1)e^{-at}}$, $L = \frac{N_L C_2}{C_2 + (N_L - C_2)e^{-bt}}$, while C_1 and C_2 are constants. The variable σ_1, giving the best estimate when $C_1 = 0.203$, $a = 0.129$, $C_2 = 0.432$ and $b = 0.118$, ranges approximately from -0.0151724079 to 0.4982041724.

Whether the function f_5, derived using the Lie group theoretical methods, can accurately predict the future still remains to be seen, but it looks like the function f_5 can "predict" the past. More specifically, while running our simulations, we have noticed that the negative value of $\sigma_1 = -0.0151724079$ occurs in the year of 1958 - exactly the year of a sharp economic downturn [44], see Figure 3.6.

We conclude from the above that the time series from the period 1947-2016 that compares the observed and estimated outputs (see Figure 3.7) reveals that our model fits quite well the data with the adjusted R-squared value of 97.65%. On the other hand, the Cobb-Douglas function (3.0.1) with a constant elasticity of substitutions, *i.e.*, $\sigma = 1$, does not provide satisfactory results in terms of the values of parameters

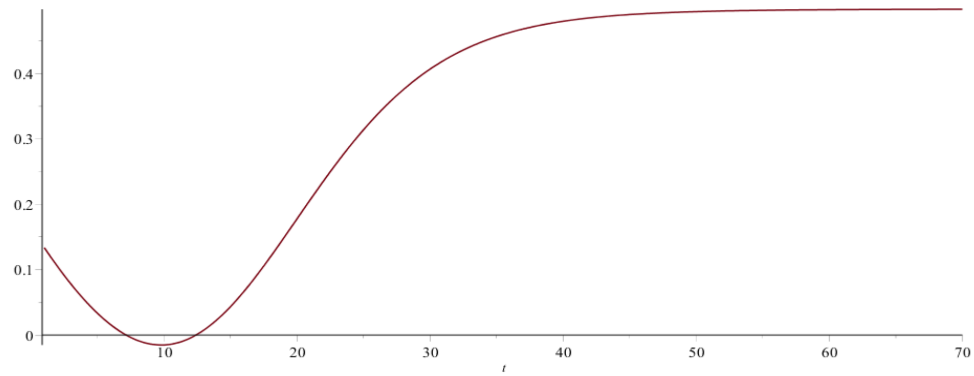


Figure 3.6: The elasticity of substitution of the new production function from 1947 to 2016.

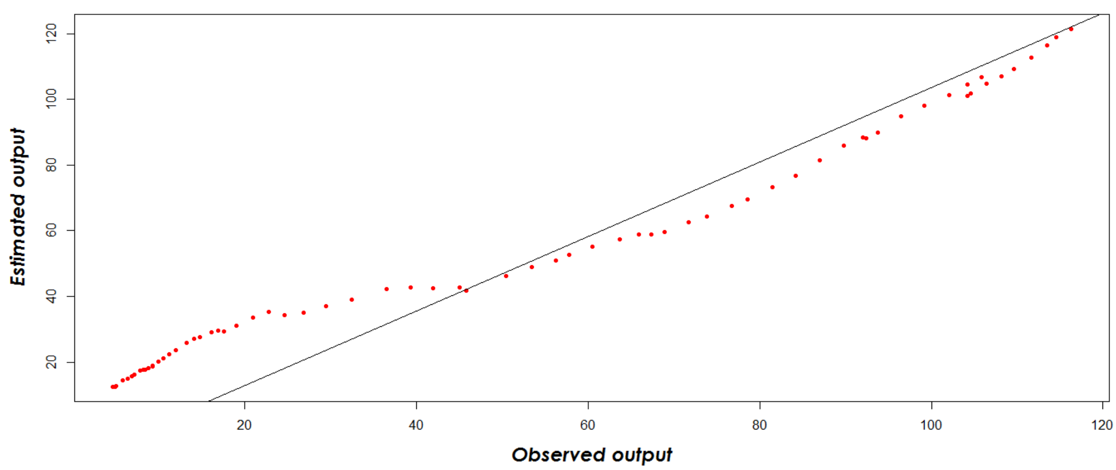


Figure 3.7: The linear regression of the observed and estimated outputs of the period from 1947 to 2016.

α and β. The best estimation of the Cobb-Douglas function that we managed to have obtained, using the same method, is as follows:

$$Y = (0.2464455)K^{(1.6612365)}L^{(-0.6612365)}, \tag{3.7.3}$$

where $C = 0.2464455$, $\alpha = 1.6612365$ and $\beta = -0.6612365$. We see that this (negative!) value of the parameter β is not compatible with the definition of the Cobb-Douglas production function given by the formula (3.0.1).

3.8 Concluding remarks

Our research has also demonstrated that *there can not be exponential growth of production while factors grow logistically.* We are inclined to believe that this is the most important consequence of our studies. Indeed, if one "forces" the production function to grow exponentially (*i.e.*, by setting $H(f) = cf$ in (3.3.6)), while the factors K and L grow logistically as in (3.2.7), the resulting production function will be of the form

$$Y = f_{10}(K, L; t) = C_1 \left(\frac{K}{|1-K|} \right)^{C_2} \left(\frac{L}{|1-L|} \right)^{C_3}, \tag{3.8.1}$$

where we assumed without loss of generality that $N_K = N_L = 1$. The production function f_{10} (3.8.1) blows up very quickly near the singularities at $K = 1$ and $L = 1$. Similarly unsatisfactory result can by obtained by enforcing logistic growth in the production function, while the factors K and L grow exponentially, that is by setting $H(f) = cf(1-f)$ in (3.1.16): the resulting production function will not even grow.

When we were starting this project, our original goal was to only extend the theoretical framework based on the Lie group theory developed by Sato, we did not expect that the resulting production functions would perform so well. Therefore the results obtained in this chapter have exceeded our expectations.

We see many applications in both economic theory of growth and applied mathematics where the new production functions (3.3.14) and (3.6.23), as well as the new modified wage share (3.6.22) can be used essentially *mutatis mutandis* by simply replacing the Cobb-Douglas function or its generalizations (like the CES function, for example) and wage share with them as appropriate.

We have argued in Section 3.2 that the system of Lotka-Volterra equations can be viewed as an extension of the exponential or logistic growth of two species. Indeed, if certain coefficients equal to zero, then systems (3.2.3) and (3.2.5) become exponential and logistic growth or decay, for instance, the system (3.2.3), by choosing $\beta = \delta = 0$, represents exponential growth or decay, namely,

$$\begin{aligned} \dot{x} &= \alpha x, \\ \dot{y} &= -\gamma y. \end{aligned} \tag{3.8.2}$$

From this viewpoint, we employ the Lotka-Volterra model in the study of econometric dynamics, that is, modeling behaviour of capital, labor and production using the Lotka-Volterra model. We will discuss in what follows.

Chapter 4

The Hamiltonian approach to the problem of derivation of production functions in economic growth theory

In Chapter 3 we have reviewed the Lie theoretical approach to the study of holothetic productions and the simultaneous holotheticity to recover the exact form of the Cobb-Douglas function, *i.e.*, $Y = f(K, L) = AK^{\alpha}L^{\beta}$ with $\alpha + \beta = 1$. We are able to derive the new production function $f_5(x)$ (3.3.14) employing the simultaneous holotheticity based upon assuming that capital and labor are affected by logistic growth. Following this approach, we investigate a four-dimensional model involving capital K, labor L, production Y and debt D, namely,

$$\begin{aligned}
\frac{dK}{dt} &= K(b_1 + a_{11}K + a_{12}D), \\
\frac{dD}{dt} &= D(b_2 + a_{21}K + a_{22}D), \\
\frac{dL}{dt} &= b_3L(1 - \frac{L}{N_L}), \\
\frac{dY}{dt} &= b_4Y(1 - \frac{Y}{N_f}),
\end{aligned} \tag{4.0.1}$$

where parameters a_{ij} and b_i ($i = 1, \ldots, 4$, $j = 1, 2$) satisfy certain conditions, and N_L, N_f are carrying capacities. More details about the economic growth model involving debt are given in Chapter 5. We want to derive a function holothetic to (4.0.1). The holotheticity condition is given by a vector field representing an infinitesimal action of a Lie transformation group in a four-dimensional space. Thus, we expect to obtain $4 - 1 = 3$ fundamental invariants. However, it is difficult to determine an economically meaningful combination of the three fundamental invariants. Since the Lie theoretical approach does not perform well in this special case, we will employ a different method.

Recall that Plank [95] showed the first two equations in (4.0.1), namely,

$$\begin{aligned} \dot{x}_1 &= x_1(b_1 + a_{11}x_1 + a_{12}x_2), \\ \dot{x}_2 &= x_2(b_2 + a_{21}x_1 + a_{22}x_2), \end{aligned} \tag{4.0.2}$$

can be interpreted as a Hamiltonian system determined by the following Poisson structure

$$\pi = x_1^{1-\ell_1} x_2^{1-\ell_2} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}, \tag{4.0.3}$$

where ℓ_1, ℓ_2 are constant under certain conditions. From the perspective of the Lyapounov stability theory, Plank introduced an integrating factor $x_1^{1-\ell_1} x_2^{1-\ell_2}$ to the system so that the system becomes Lyapounov stable. The phase flow is a closed orbit. It is proved (see more details in [56]) that a two-dimensional Lotka-Volterra system does not admit an isolated orbit. Hence, the phase flow of the system (4.0.2) is given by periodic orbits. Then it must be a Hamiltonian system. We will continue this discussion in Chapter 5.

Considering different algebraic conditions, for example, varying the values of ℓ_1 and ℓ_2, Plank discussed all possible Poisson structures and corresponding Hamiltonian functions. His approach is quite general. Kerner [66] commented that some special Lotka-Volterra systems can be written as a Hamiltonian system using a change of variables and proved the new system indeed admits a Hamiltonian structure by employing the Lie-Koenigs theorem (we will discuss the theorem in Section 7.3). For example, he considered the following system

$$\begin{aligned} \dot{N}_1 &= \epsilon_1 N_1 + \frac{a_{12}}{\beta_1} N_1 N_2, \\ \dot{N}_2 &= \epsilon_2 N_2 + \frac{a_{21}}{\beta_2} N_1 N_2, \end{aligned} \tag{4.0.4}$$

where $N_i,\ i = 1, 2$ represent population of species and $\epsilon_i,\ a_{ij},\ \beta_i,\ i, j = 1, 2$ are coefficients.

The stationary points of the system (4.0.4), considering $N_i > 0$, are given by

$$N_1^* = -\frac{\epsilon_2 \beta_2}{a_{21}},\ N_2^* = -\frac{\epsilon_1 \beta_1}{a_{12}}. \tag{4.0.5}$$

Introducing the following change of variables

$$q = \ln\left(\frac{N_1}{N_1^*}\right), \; p = \ln\left(\frac{N_2}{N_2^*}\right), \tag{4.0.6}$$

we rewrite as follows

$$\begin{aligned} \dot{q} &= \epsilon_1 - \epsilon_1 e^p = \frac{\partial H}{\partial p}, \\ \dot{p} &= \epsilon_2 - \epsilon_2 e^q = -\frac{\partial H}{\partial q}. \end{aligned} \tag{4.0.7}$$

Hence, (4.0.7) is a Hamiltonian system admitting the Hamiltonian function

$$H = \epsilon_1 p - \epsilon_1 e^p - \epsilon_2 q + \epsilon_2 e^q \tag{4.0.8}$$

determined by the canonical symplectic structure $dp \wedge dq$.

Following their approaches, we identify the whole system (4.0.4) as a Hamiltonian system with a Poisson structure. For instance, we, assuming $a_{11} = a_{22} = 0$, have the Hamiltonian function (4.3.14). Then, we propose the Hamiltonian function can be used as a production function. Thus, we see that the economic dynamics can be described by a special case of a Lotka-Volterra system and the Hamiltonian function of the model, provided that the system has a Hamiltonian structure, represents a corresponding production function.

Let us review an n-dimensional Lotka-Volterra model

$$\dot{x}_i = b_i x_i + \sum_{j=1}^{n} a_{ij} x_i x_j, \quad (i = 1, \ldots, n), \tag{4.0.9}$$

where x_i represents the population of a species and parameters b_i, a_{ij} satisfy certain conditions.

Fernandes and Oliva [40] discussed the Hamiltonian structure of the Lotka-Volterra model based on the symplectic realization from $\mathbb{R}^{2n} \to \mathbb{R}^n$ by introducing new canonical coordinates. Tsuchida *et al.* [127] considered the tri-Hamiltonian structure of the Lotka-Volterra system through linking it to the Toda lattice model via introducing a change of variables. More recent treatment of the Arnold-Liouville integrability of the system can be found in [122]. We remark the study of Arnold-Liouville integrability of the Lotka-Volterra model is normally based on the assumption that the coefficients $b_i = 0$ (in (4.0.9)) and the skew-symmetry of matrix of the coefficients in quadratic parts, namely, a_{ij} in (4.0.9). The assumption is not essential in Plank's and Kerner's approach, but plays a vital role in [40, 122, 127] for investigating the complete integrability in the sense of Arnold-Liouville.

In what follows we consider special cases of the Lotka-Volterra model, in which we assume $b_i \neq 0$ and do not necessarily require the skew symmetry of matrix of a_{ij}, and show that they are Hamiltonian systems determined by the corresponding Poisson structures and Hamiltonian functions that can be considered to be production functions.

The Lie theoretical method has been proved to be a powerful technique in the production theory. It is our contention that the theory can be further developed at this point by recasting its setting within a Hamiltonian framework. In this chapter we want to demonstrate the following

Conjecture 4.0.1.

$$\begin{gathered}\textit{a holothetic production function}\\ \Updownarrow\\ \textit{an invariant of a transformation group}\\ \Updownarrow\\ \textit{a Hamiltonian function of the corresponding dynamical system.}\end{gathered} \tag{4.0.10}$$

Note the above equivalences hold in a sense that a dynamical system has a Hamiltonian structure, namely, the vector field giving rise to the dynamical system preserves a Poisson structure.

In Sections 4.1, 4.2 and 4.3, we will discuss low-dimensional models and their Hamiltonian formalism, in which we will present the Hamiltonian functions and Poisson bivectors. A bivector, which is intrinsically not a matrix, can be represented by a skew-symmetric matrix at each point and the matrix presentation is more intuitive and convenient in some applications, particularly in low-dimensional models. We remark that we employ the matrix representation in the proof associated with linear algebra when necessary. The n-dimensional generalization is considered in Section 4.4. Hamiltonian functions (4.2.4) and (4.2.23) induce the Cobb-Douglas function and the new production function, respectively, but elasticity elements do not attain economically meaningful values. By analogy with Sato's *simultaneous holotheticity* (see Section 3.3), we introduce a bi-Hamiltonian approach to recovering the exact form of the Cobb-Douglas function (3.0.1) and the new production function (3.3.14). The bi-Hamiltonian approach can be used to describe the topological features of real statistical data in some sense and we derive the values of coefficients in (3.0.1) through the bi-Hamiltonian approach using the economic data from 1899-1922, which is employed by Cobb and Douglas in [27] to derive the Cobb-Douglas function (3.0.1).

4.1 Two-dimensional Hamiltonian systems

We realize that the dynamical system

$$\begin{aligned} \dot{x}_1 &= b_1 x_1, \\ \dot{x}_2 &= b_2 x_2, \end{aligned} \tag{4.1.1}$$

which gives rise to the exponential transformation group, is a special case of the system (4.0.9), where $n = 2$ and $a_{ij} = 0$. We want to show (4.1.1) is a Hamiltonian system and the Hamiltonian is a production function.

Following Kerner [66], let us introduce the transformation

$$v_i = \ln x_i, \tag{4.1.2}$$

the system (4.1.1) becomes

$$\dot{v}_i = b_i, \quad i = 1, 2. \tag{4.1.3}$$

Considering the following

Lemma 4.1.1. *A two dimensional system of separable ordinary differential equations*

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{y} &= g(y) \end{aligned} \tag{4.1.4}$$

defined on an open subset $E \subset \mathbb{R}^2$ *can be written as a Hamiltonian system with a two dimensional Poisson structure,* i.e.,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -f(x)g(y) \\ f(x)g(y) & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{bmatrix} \tag{4.1.5}$$

with a Hamiltonian function

$$H = \int \frac{1}{f(x)}\, dx - \int \frac{1}{g(y)}\, dy \tag{4.1.6}$$

if $\frac{1}{f(x)}$ *and* $\frac{1}{g(y)}$ *are well-defined.*

Proof. The Hamiltonian function $H = \int \frac{1}{f(x)}\, dx - \int \frac{1}{g(y)}\, dy$ is clearly smooth on E.

The matrix $J(x, y)$ is skew-symmetric since $J_{ij} = -J_{ji},\ i, j = 1, 2$ and we only need to check it satisfies the Jacobi identity

$$\sum_{l=1}^{2} \left(J_{il} \frac{\partial J_{mk}}{\partial x_l} + J_{kl} \frac{\partial J_{im}}{\partial x_l} + J_{ml} \frac{\partial J_{ki}}{\partial x_l} \right) = 0,\ \ i, m, k = 1, 2, \tag{4.1.7}$$

where $x_1 = x$ and $x_2 = y$.

Note the equation (4.1.7) involving trivial entries $J_{11} = J_{22} = 0$ naturally yields zeros. Let us consider the non-trivial equations given by

$$J_{11} \frac{\partial J_{22}}{\partial x} + J_{21} \frac{\partial J_{12}}{\partial x} + J_{21} \frac{J_{21}}{\partial x} = J_{21} \left(\frac{\partial J_{12}}{\partial x} - \frac{\partial J_{12}}{\partial x} \right) = 0 \tag{4.1.8}$$

and

$$J_{22} \frac{\partial J_{11}}{\partial y} + J_{12} \frac{\partial J_{21}}{\partial y} + J_{12} \frac{J_{12}}{\partial y} = J_{12} \left(\frac{\partial J_{21}}{\partial y} - \frac{\partial J_{21}}{\partial y} \right) = 0. \tag{4.1.9}$$

Hence, the matrix $J(x)$ representing the bivector $-f(x)g(y)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ determines a Poisson structure. Note $-f(x)g(y)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ is a tensor since it is the wedge product of two vector fields given by differential equations in (4.1.4), respectively. □

Using Lemma 4.1.1, the system (4.1.3) is a Hamiltonian system with a Hamiltonian function

$$H_1 = \frac{1}{b_1}v_1 - \frac{1}{b_2}v_2 \tag{4.1.10}$$

and the following Poisson bivector

$$\pi_1 = b_1 b_2 \frac{\partial}{\partial v_j} \wedge \frac{\partial}{\partial v_i}, \;\; i,j = 1,2, \tag{4.1.11}$$

where $\frac{\partial}{\partial v_j} \wedge \frac{\partial}{\partial v_i} = -\frac{\partial}{\partial v_i} \wedge \frac{\partial}{\partial v_j}$.

In original coordinates, the system (4.1.1) is a Hamiltonian system with a Hamiltonian function

$$H_1^* = \frac{1}{b_1}\ln x_1 - \frac{1}{b_2}\ln x_2 \tag{4.1.12}$$

and the Poisson bivector given by

$$\pi_1^* = b_1 b_2 x_1 x_2 \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_i}, \;\; i,j = 1,2. \tag{4.1.13}$$

Let us consider the other special case of the system (4.0.9), where $a_{ij} = -\frac{b_i}{N_i}$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$, namely,

$$\begin{aligned} \dot{x}_1 &= b_1 x_1 \left(1 - \tfrac{x_1}{N_1}\right), \\ \dot{x}_2 &= b_2 x_2 \left(1 - \tfrac{x_2}{N_2}\right), \end{aligned} \tag{4.1.14}$$

where N_i, $i = 1,2$ are carrying capacities.

Applying the following transformation to the system (4.1.14)

$$v_i = \ln \frac{x_i}{N_i}, \tag{4.1.15}$$

we obtain

$$\dot{v}_i = b_1(1 - e^{v_i}), \;\; i = 1,2, \tag{4.1.16}$$

which is a Hamiltonian system with a Hamiltonian function

$$H_2 = \frac{1}{b_1}v_1 - \frac{1}{b_1}\ln(1 - e^{v_1}) - \frac{1}{b_2}v_2 + \frac{1}{b_2}\ln(1 - e^{v_2}) \tag{4.1.17}$$

and a Poisson bivecor

$$\pi_2 = b_1 b_2 (1 - e^{v_1})(1 - e^{v_2}) \frac{\partial}{\partial v_j} \wedge \frac{\partial}{\partial v_i}, \; i, j = 1, 2. \tag{4.1.18}$$

4.2 Three-dimensional Hamiltonian systems

The following dynamical system

$$\begin{aligned} \dot{x}_1 &= b_1 x_1, \\ \dot{x}_2 &= b_2 x_2, \\ \dot{x}_3 &= b_3 x_3, \end{aligned} \tag{4.2.1}$$

is a special case of the system (4.0.9), where $n = 3$ and $a_{ij} = 0$. Let us discuss the system from the point of view of the Hamiltonian formalism.

Plank [94] has shown the existence of the Hamiltonian structure of a three-dimensional Lotka-Volterra model under certain conditions. He viewed a three-dimensional system of differential equations $\dot{x}_i = f(x_i)$, $i = 1, 2, 3$, as the cross product of the gradients of two constants of motion, $K(x_1, x_2, x_3)$ and $H(x_1, x_2, x_3)$. He further introduced a bi-Hamiltonian structure and proved the system to be integrable. A similar approach to constructing a Poisson structure associated with a three-dimensional dynamical system by employing the cross product can be also found in [51]. We realize that the Hamiltonian structure of a econometric dynamical system naturally gives rise to a production function. Let us present the Hamiltonian structure of the system (4.2.1).

Following Plank's convention, we assume that

$$K(x_1, x_2, x_3) = \gamma x_1 x_2 x_3, \quad \gamma \neq 0 \in \mathbb{R}, \tag{4.2.2}$$

then the system (4.2.1) becomes

$$\begin{bmatrix} 0 & -\gamma x_1 x_2 & -\gamma x_1 x_3 \\ \gamma x_1 x_2 & 0 & -\gamma x_2 x_3 \\ \gamma x_1 x_3 & \gamma x_2 x_3 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \\ \frac{\partial H}{\partial x_3} \end{bmatrix} = \begin{bmatrix} b_1 x_1 \\ b_2 x_2 \\ b_3 x_3 \end{bmatrix}. \tag{4.2.3}$$

We know that the Hamiltonian of (4.2.1) (that is the Cobb-Douglas function!) must be of the following form

$$H_3 = \alpha_1 \ln x_1 + \alpha_2 \ln x_2 + \alpha_3 \ln x_3, \tag{4.2.4}$$

where parameters α_i, $i = 1, 2, 3$, will be determined later.

Substituting H_3 into (4.2.3) yields a system of algebraic equations, *i.e.*,

$$\begin{aligned} -\gamma\alpha_2 x_1 - \gamma\alpha_3 x_1 &= b_1 x_1, \\ \gamma\alpha_1 x_2 - \gamma\alpha_3 x_2 &= b_2 x_2, \\ \gamma\alpha_1 x_3 + \gamma\alpha_2 x_3 &= b_3 x_3, \end{aligned} \tag{4.2.5}$$

which determines parameters α_i, $i = 1, 2, 3$.

The equations (4.2.5) (assuming $x_i \neq 0$) gives

$$\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \frac{b_1}{\gamma} \\ \frac{b_2}{\gamma} \\ \frac{b_3}{\gamma} \end{bmatrix}. \tag{4.2.6}$$

We see that the rank of the 3×3 coefficient matrix in (4.2.6) is 2 and the system of equations has solutions iff

$$\frac{b_1 + b_3}{\gamma} = \frac{b_2}{\gamma}, \tag{4.2.7}$$

or, simply,

$$b_1 + b_3 = b_2. \tag{4.2.8}$$

Indeed, (4.2.4) is a first integral of the system (4.2.1), namely, a constant along the flow generated by (4.2.1) . Consider the vector field corresponding to (4.2.1)

$$X = b_1 x_1 \frac{\partial}{\partial x_1} + b_2 x_2 \frac{\partial}{\partial x_2} + b_3 x_3 \frac{\partial}{\partial x_3}. \tag{4.2.9}$$

Then, the Hamiltonian function H_3 is preserved along the flow generated by (4.2.1), *i.e.*, $\mathcal{L}_X(H_3) = 0$. In more details, we have

$$\mathcal{L}_X(H_3) = X(H_3) = b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3, \tag{4.2.10}$$

it follows from (4.2.5) that we arrive at

$$\mathcal{L}_X(H_3) = -\gamma\alpha_1(\alpha_2 + \alpha_3) + \gamma\alpha_2(\alpha_1 - \alpha_3) + \gamma\alpha_3(\alpha_1 + \alpha_2) = 0. \tag{4.2.11}$$

Therefore, we state that the system (4.2.1) has a Hamiltonian structure iff (4.2.8) holds true. The Hamiltonian function is not unique. Consider the following

Example 4.2.1. Let us assume $\alpha_3 = -1$ and, according to the equation (4.2.6), we obtain $\alpha_1 = \dfrac{-\gamma + b_2}{\gamma}$ and $\alpha_2 = \dfrac{\gamma - b_1}{\gamma}$.

Then (4.2.1) is a Hamiltonian system with the following Hamiltonian function

$$H = \frac{-\gamma + b_2}{\gamma} \ln x_1 + \frac{\gamma - b_1}{\gamma} \ln x_2 - \ln x_3, \tag{4.2.12}$$

where we require $\dfrac{-\gamma + b_2}{\gamma} > 0$ and $\dfrac{\gamma - b_1}{\gamma} > 0$,

and the following Poisson bivector

$$\pi = \gamma x_i x_j \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^i},\ \ i, j = 1, 2, 3. \tag{4.2.13}$$

In the new coordinates $v_i = \ln x_i$, $i = 1, 2, 3$, the system (4.2.1) becomes

$$\dot{v}_i = b_i,\ \ i = 1, 2, 3, \tag{4.2.14}$$

which is a Hamiltonian system under the same condition that (4.2.8) holds true.

Remark 4.2.2. *We introduce the parameter γ in the Poisson bivector* (4.2.13) *to obtain the economically meaningful coefficients,* i.e., *the elasticity condition $\alpha + \beta = 1$ in* (3.0.1). *However, this attempt does not work out. We address the issue in Section 4.5. Nevertheless, we employ the Poisson bivector* (4.2.16) *in what follows.*

We have a freedom to choose the values of α_i and the corresponding Poisson structure. For our convenience, we present the simplest possible form of the Poisson structure

(4.2.16). The system (4.2.1) admits a Hamiltonian function given by

$$H_3 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \tag{4.2.15}$$

corresponding to the following Poisson bivector

$$\pi_3 = \frac{\partial}{\partial v_j} \wedge \frac{\partial}{\partial v_i}, \;\; i, j = 1, 2, 3. \tag{4.2.16}$$

Remark 4.2.3. *Identifying* $x_1 = L,\ x_2 = K,\ x_3 = f,\ A = exp\left(\frac{H_3}{\alpha_3}\right),\ \alpha = -\frac{\alpha_2}{\alpha_3}$, $\beta = -\frac{\alpha_1}{\alpha_3}$ *we arrive at the Cobb-Douglas function* (3.0.1).

Remark 4.2.4. *We have also considered the Poisson bivector of the following form*

$$\pi_3^* = \begin{bmatrix} 0 & -C_1 & -C_2 \\ C_1 & 0 & -C_3 \\ C_2 & C_3 & 0 \end{bmatrix}, \tag{4.2.17}$$

where C_i, $i = 1, 2, 3$ *are constant.*

Note if π^* *is applied, then the system* (4.2.14) *has a Hamiltonian structure under the condition* $b_1 C_3 + b_3 C_1 = b_2 C_2$. *For this reason, let us call the Poisson bivector of the form of* (4.2.16) *a standard form.*

Let us consider the following dynamical system giving rise to the transformation group of logistic growth

$$\begin{aligned} \dot{x}_1 &= b_1 x_1 (1 - \tfrac{x_1}{N_1}), \\ \dot{x}_2 &= b_2 x_2 (1 - \tfrac{x_2}{N_2}), \\ \dot{x}_3 &= b_3 x_3 (1 - \tfrac{x_3}{N_3}), \end{aligned} \tag{4.2.18}$$

where N_i, $i = 1, 2, 3$ are carrying capacities.

We present the new production function (3.3.14) in the new coordinates as follows

$$x_3 = f(x_1, x_2) = \frac{N_3 x_1^{\alpha} x_2^{\beta}}{C|N_1 - x_1|^{\alpha} |N_2 - x_2|^{\beta} + x_1^{\alpha} x_2^{\beta}}, \tag{4.2.19}$$

where α, β and C are constant.

We can also identify the above system (4.2.18) as a special case of a three-dimensional Lotka-Volterra equation determined by the conditions $a_{ij} = -\frac{b_i}{N_i}$ when $i = j$, and $a_{ij} = 0$, when $i \neq j$.

According to the production function (4.2.19), we can, using the hypothesis that the production function is equivalent to a Hamiltonian function of a dynamical system, make the following ansatz of the Hamiltonian of the system (4.2.18)

$$H_4 = \alpha_1 \ln \frac{x_1}{|N_1 - x_1|} + \alpha_2 \ln \frac{x_2}{|N_2 - x_2|} + \alpha_3 \ln \frac{x_3}{|N_3 - x_3|}, \tag{4.2.20}$$

where α_i, $i = 1, 2, 3$, are constant.

By analogy with (4.2.1), the system (4.2.18) becomes

$$\dot{v_i} = b_i(1 - e^{v_i}), \quad i = 1, 2, 3, \tag{4.2.21}$$

via employing the following transformation

$$x_i = N_i e^{v_i}, \tag{4.2.22}$$

and the Hamiltonian (4.2.20) becomes

$$H_4 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 - \alpha_1 \ln(1 - e^{v_1}) - \alpha_2 \ln(1 - e^{v_2}) - \alpha_3 \ln(1 - e^{v_3}). \tag{4.2.23}$$

We are looking for a Poisson bivector π_4 given by the matrix in the coordinates (v_1, v_2, v_3),

$$\begin{bmatrix} 0 & -K_1 & -K_2 \\ K_1 & 0 & -K_3 \\ K_2 & K_3 & 0 \end{bmatrix} \tag{4.2.24}$$

such that

$$\pi_4 \begin{bmatrix} \frac{A}{1-e^{v_1}} \\ \frac{B}{1-e^{v_2}} \\ \frac{C}{1-e^{v_3}} \end{bmatrix} = \begin{bmatrix} b_1(1 - e^{v_1}) \\ b_2(1 - e^{v_2}) \\ b_3(1 - e^{v_3}) \end{bmatrix}, \tag{4.2.25}$$

solving which, we obtain

$$\pi_4 = \begin{bmatrix} 0 & -(1-e^{v_1})(1-e^{v_2}) & -(1-e^{v_1})(1-e^{v_3}) \\ (1-e^{v_1})(1-e^{v_2}) & 0 & -(1-e^{v_2})(1-e^{v_3}) \\ (1-e^{v_1})(1-e^{v_3}) & (1-e^{v_2})(1-e^{v_3}) & 0 \end{bmatrix}, \tag{4.2.26}$$

or in terms of the local coordinates (v_1, v_2, v_3):

$$\pi_4 = (1-e^{v_i})(1-e^{v_j})\frac{\partial}{\partial v_j} \wedge \frac{\partial}{\partial v_i}, \ i,j = 1,2,3. \tag{4.2.27}$$

It can be proved that π_4 is skew-symmetric and satisfies the Jacobian identity.

Next, we need to equate all coefficients. By analogy with (4.2.6), the system (4.2.25) yields

$$\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \tag{4.2.28}$$

which means (4.2.21) is a Hamiltonian system if $b_1 + b_3 = b_3$.

Then, (4.2.18) in original coordinates is a Hamiltonian system defined by the Poisson bivector

$$\pi_4 = x_i x_j \left(1 - \frac{x_i}{N_i}\right)\left(1 - \frac{x_j}{N_j}\right)\frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_i}, \ i,j = 1,2,3, \tag{4.2.29}$$

and the Hamiltonian function

$$H_4 = \alpha_1 \ln \frac{x_1}{|N_1 - x_1|} + \alpha_2 \ln \frac{x_2}{|N_2 - x_2|} + \alpha_3 \ln \frac{x_3}{|N_3 - x_3|} \tag{4.2.30}$$

under the condition

$$b_1 + b_3 = b_2. \tag{4.2.31}$$

Remark 4.2.5. *Identifying $x_1 = L$, $x_2 = K$, $x_3 = f$, $N_1 = N_L$, $N_2 = N_K$, $N_3 = N_f$, $-\frac{\alpha_2}{\alpha_3} = \alpha$, $-\frac{\alpha_1}{\alpha_3} = \beta$, $e^{-H_2/\alpha_3} = C$, we obtain the production function* (3.3.14).

Remark 4.2.6. *In Section 3.3, we assume that $b_i > 0$, $i = 1, 2, 3$. We consider* *the Poisson vector of the aforementioned form, and, notably find a new condition $b_1 + b_3 = b_2$, which is consistent with the corresponding assumption made in Section 3.3.*

4.3 Four-dimensional Hamiltonian systems

We have shown in previous sections that the examples of evolution of input factors and the output, which can be viewed as a special case of a Lotka-Volterra model, are Hamiltonian systems and the Hamiltonian functions H_i, $i = 1, 2, 3, 4$, can be used as production functions.

Note that treating debt as an independent variable has recently become an acceptable practice in economic modeling (see [3, 11, 50, 49], for example). The model involving debt is discussed in greater detail in Chapter 6. Let us introduce the new economic factor debt D in this section, which we will denote by x_2.

Consider the following four-dimensional Lotka-Volterra model,

$$\dot{x}_i = b_i x_i + \sum_{j=1}^{4} a_{ij} x_i x_j, \quad (i = 1, 2, 3, 4), \tag{4.3.1}$$

assuming x_1, x_2 grow exponentially ($a_{ij} = 0$, $i = 1, 2$) and x_3, x_4 grow logistically ($a_{ij} = -\frac{b_i}{N_i}$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$, $i = 3, 4$), the equation (4.3.1) becomes

$$\begin{aligned} \dot{x}_1 &= b_1 x_1, \\ \dot{x}_2 &= b_2 x_2, \\ \dot{x}_3 &= b_3 x_3 \left(1 - \frac{x_3}{N_3}\right), \\ \dot{x}_4 &= b_4 x_4 \left(1 - \frac{x_4}{N_4}\right), \end{aligned} \tag{4.3.2}$$

where N_i, $i = 3, 4$ are carrying capacities.

Applying the following transformations

$$\begin{aligned} v_i &= \ln x_i, \ i = 1, 2 \\ v_i &= \ln \frac{x_i}{N_i}, \ i = 3, 4, \end{aligned} \tag{4.3.3}$$

the system becomes

$$\begin{aligned}\dot{v_1} &= b_1,\\ \dot{v_2} &= b_2,\\ \dot{v_3} &= b_3(1-e^{v_3}),\\ \dot{v_4} &= b_4(1-e^{v_4}).\end{aligned} \tag{4.3.4}$$

It follows from the results presented in Section 4.1 that the system (4.3.4) is a Hamiltonian system admitting the following Hamiltonian function

$$H_5 = \frac{1}{b_1}v_1 - \frac{1}{b_2}v_2 + \frac{1}{b_3}v_3 - \frac{1}{b_4}v_4 - \frac{1}{b_3}\ln(1-e^{v_3}) + \frac{1}{b_4}\ln(1-e^{v_4}) \tag{4.3.5}$$

and the following Poisson bivector

$$\pi_5 = \pi^{ij}\frac{\partial}{\partial v_j}\wedge\frac{\partial}{\partial v_i},\ \ i,j = 1,2,3,4, \tag{4.3.6}$$

where

$$\pi^{ij} = \begin{cases} b_1 b_2, & i,j = 1,2,\\ b_3 b_4(1-e^{v_3})(1-e^{v_4}), & i,j = 3,4,\end{cases} \tag{4.3.7}$$

which is skew-symmetric and satisfies the Jacobian identity.

Identifying $x_1 = K$, $x_2 = D$, $x_3 = L$ and $x_4 = Y$, we obtain a new production function given by

$$Y = f(K,D,L) = \frac{CN_Y G(K,D,L)}{1+CG(K,D,L)}, \tag{4.3.8}$$

where $C = \exp(-H_5 b_4)$ and $G(K,D,L) = K^{\frac{b_4}{b_1}} D^{-\frac{b_4}{b_2}} \left(\frac{L}{|N_L - L|}\right)^{-\frac{b_4}{b_3}}$.

The production function (4.3.8) characterizing the dynamics (4.3.4) is a sigmoid function of the type III functional response (3.2.6). We tend to consider a more realistic model in what follows.

We have stated in Section 3.2 that we live in "a compact world" and any continuous growth in this world must be bounded. Hence, we consider a more realistic model, in which K and D interact in the following manner, *i.e.*,

$$\begin{aligned} \dot{x}_1 &= x_1(b_1 + a_{21}x_2), \\ \dot{x}_2 &= x_2(b_2 + a_{12}x_1), \\ \dot{x}_3 &= b_3x_3\left(1 - \tfrac{x_3}{N_3}\right), \\ \dot{x}_4 &= b_4x_4\left(1 - \tfrac{x_4}{N_4}\right). \end{aligned} \tag{4.3.9}$$

The Hamiltonian structures of the first two equations in (4.3.9) has been given in (4.0.7). Employing the transformation

$$v_i = \ln\left(-\frac{a_{ji}}{b_i}x_i\right), \ j = 1 \text{ or } 2, \ i = 1, 2, \tag{4.3.10}$$

we obtain

$$\begin{aligned} \dot{v_1} &= b_1(1 - e^{v_2}), \\ \dot{v_2} &= b_2(1 - e^{v_1}), \end{aligned} \tag{4.3.11}$$

which gives rise to the canonical Hamiltonian structure given by the Hamiltonian function

$$H = b_1(v_2 - e^{v_2}) - b_2(v_1 - e^{v_1}) \tag{4.3.12}$$

and the canonical Poisson bivector

$$\pi = \frac{\partial}{\partial v^j} \wedge \frac{\partial}{\partial v^i}, \ i, j = 1, 2. \tag{4.3.13}$$

Employing π_4 (4.2.27) for the last two equations in (4.3.9), we derive a Hamiltonian structure for the system (4.3.9) with the Hamiltonian function given by

$$H_6 = b_1v_2 - b_2v_1 + \frac{1}{b_3}v_3 - \frac{1}{b_4}v_4 + b_2e^{v_1} - b_1e^{v_2} - \frac{1}{b_3}\ln(1 - e^{v_3}) + \frac{1}{b_4}\ln(1 - e^{v_4}) \tag{4.3.14}$$

and the Poisson bivector

$$\pi_6 = p^{ij}\frac{\partial}{\partial v_j} \wedge \frac{\partial}{\partial v_i}, \ i, j = 1, 2, 3, 4, \tag{4.3.15}$$

where

$$p^{ij} = \begin{cases} 1, & i, j = 1, 2, \\ b_3b_4(1 - e^{v_3})(1 - e^{v_4}), & i, j = 3, 4, \end{cases} . \tag{4.3.16}$$

Let us assume $x_1 = K$, $x_2 = D$, $x_3 = f$, $x_4 = L$, $N_3 = N_f$, $N_4 = N_L$. Solving for f, we arrive at a new production function of the following form

$$Y = f(L, K, D) = \frac{N_f e^{b_3 G(L,K,D)}}{1 + e^{b_3 G(L,K,D)}}, \tag{4.3.17}$$

where the fuction G is given by

$$\begin{aligned} G &= C - b_1 \left[\ln \left(-\frac{a_{21}}{b_2} D \right) + \frac{a_{21}}{b_2} D \right] + b_2 \left[\ln \left(-\frac{a_{12}}{b_1} K \right) + \frac{a_{12}}{b_1} K \right] + \\ & \frac{1}{b_4} \ln \frac{L}{N_L - L}, \quad C \in \mathbb{R}. \end{aligned} \tag{4.3.18}$$

4.4 N-dimensional Hamiltonian systems

In previous sections we have seen how a production function can be derived as a Hamiltonian function of a special case of Lotka-Volterra model characterizing the economic growth.

In this section, we will consider dynamical systems evolving in the $\mathbb{R}^n_+$ space, the Hamiltonian formalism of which will be summarized in the following theorems. The proofs of the theorems essentially are based on the existence of the Poisson structures, i.e, it amounts to checking the skew-symmetry and Jacobian identity of the corresponding Poisson structure as follows from the proof of Lemma 4.1.1.

4.4.1 The generalized exponential growth model

Let us consider the special case of the model where $a_{ij} = 0$, namely,

$$\dot{x}_i = b_i x_i, \; b_i \neq 0, \; i = 1, \ldots, n, \tag{4.4.1}$$

employing the transformation

$$v_i = \ln x_i, \tag{4.4.2}$$

the system (4.4.1) becomes

$$\dot{v}_i = b_i,\ i = 1, \ldots, n. \tag{4.4.3}$$

Definition 4.4.1. The n-dimensional dynamical system

$$\dot{v}_i = b_i,\ i = 1, \ldots, n, \tag{4.4.4}$$

is called a *transformed exponential growth* (TEG) model.

Using the results in Sections 4.1 and 4.2, we can formulate the following

Theorem 4.4.2. *The n-dimensional ($n = 2k$, $k \in \mathbb{N}^+$) TEG model is a Hamiltonian system admitting the following Poisson bivector*

$$\pi_{2k} = b_{2i-1}b_{2i}\frac{\partial}{\partial v_{2i}} \wedge \frac{\partial}{\partial v_{2i-1}},\ 1 \leq i \leq k \tag{4.4.5}$$

and the corresponding Hamiltonian function

$$H = \sum_{i=1}^{k}\left(\frac{1}{b_{2i-1}}v_{2i-1} - \frac{1}{b_{2i}}v_{2i}\right). \tag{4.4.6}$$

In order to present a Hamiltonian structure for odd dimensional models, we use the result of the following

Proposition 4.4.3. *The n-dimensional ($n = 2k+1$, $k \in \mathbb{N}^+$) system of linear equations*

$$\begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ 1 & 0 & -1 & \cdots & -1 \\ 1 & 1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} \tag{4.4.7}$$

has solutions iff

$$\sum_{i=1}^{k} b_{2i+1} = \sum_{i=1}^{k} b_{2i},\ b_i \neq 0. \tag{4.4.8}$$

Proof. Note the rank of the $n \times n$ skew-symmetric matrix is $n-1$, which suggests the system has no solutions or infinitely many solutions. Using the rank-nullity theorem, the system has solutions iff $\sum_{i=1}^{k} b_{2i+1} = \sum_{i=1}^{k} b_{2i}$. □

Proposition 4.4.3 can be used to prove the following

Theorem 4.4.4. *The n-dimensional ($n = 2k+1$, $k \in \mathbb{N}^+$) TEG model is a Hamiltonian system if $\sum_{i=1}^{k} b_{2i+1} = \sum_{i=1}^{k} b_{2i}$. The Poisson bivector is as follows*

$$\pi_{2k+1} = \frac{\partial}{\partial v_j} \wedge \frac{\partial}{\partial v_i}, \ 1 \leq i, j \leq 2k+1 \tag{4.4.9}$$

and the Hamiltonian function is

$$H = \sum_{i=1}^{n} \alpha_i v_i, \tag{4.4.10}$$

where $\alpha_i = \alpha_i(b_1, \ldots, b_n)$ are constants related to values of b_i, where $i = 1, \ldots, n$.

Suppose we treat the last variable v_n as the production and introduce the original coordinates, the Hamiltonian functions (4.4.6) and (4.4.10) are the Cobb-Douglas production function as follows

$$x_n = C_1 x_1^{\beta_1} x_2^{\beta_2} \cdots x_{n-1}^{\beta_{n-1}}, \tag{4.4.11}$$

where C_1 and β_i, $i = 1, \ldots, n$, are positive constants.

4.4.2 The generalized logistic growth model

Let $a_{ij} = -\frac{b_i}{N_i}$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$, the model (4.0.9) becomes an n-dimensional logistic growth model, namely,

$$\dot{x}_i = b_i x_i \left(1 - \frac{x_i}{N_i}\right), \ b_i \neq 0, \ i = 1, \ldots, n, \tag{4.4.12}$$

which, under the transformation

$$v_i = \ln\left(\frac{x_i}{N_i}\right), \tag{4.4.13}$$

becomes

$$\dot{v_i} = b_i(1 - e^{v_i}), \ i = 1, \ldots, n. \tag{4.4.14}$$

Definition 4.4.5. The n-dimensional dynamical system

$$\dot{v_i} = b_i(1 - e^{v_i}), \ i = 1, \ldots, n, \tag{4.4.15}$$

is called a *transformed logistic growth* (TLG) model.

It follows from conclusions in Sections 4.1 and 4.2, the Hamiltonian structure of the TLG model is presented in the following

Theorem 4.4.6. *The n-dimensional ($n = 2k$, $k \in \mathbb{N}^+$) TLG model is a Hamiltonian system with the following Poisson bivector*

$$\pi_{2k} = b_{2i-1}b_{2i}(1 - e^{v_{2i-1}})(1 - e^{v_{2i}})\frac{\partial}{\partial v_{2i}} \wedge \frac{\partial}{\partial v_{2i-1}}, \ 1 \leq i \leq k \tag{4.4.16}$$

and the corresponding Hamiltonian function

$$H = \sum_{i=1}^{k}\left(\frac{1}{b_{2i-1}}v_{2i-1} - \frac{1}{b_{2i}}v_{2i} + \frac{1}{b_{2i-1}}\ln(1 - e^{v_{2i-1}}) - \frac{1}{b_{2i}}\ln(1 - e^{v_{2i}})\right). \tag{4.4.17}$$

Theorem 4.4.7. *The n-dimensional ($n = 2k + 1$, $k \in \mathbb{N}^+$) TLG model is a Hamiltonian system provided $\sum\limits_{i=1}^{k} b_{2i+1} = \sum\limits_{i=1}^{k} b_{2i}$. The Poisson bivector is given by*

$$\pi_{2k+1} = (1 - e^{v_i})(1 - e^{v_j})\frac{\partial}{\partial v_j} \wedge \frac{\partial}{\partial v_i}, \ 1 \leq i, j \leq 2k + 1 \tag{4.4.18}$$

and the Hamiltonian function is

$$H = \sum_{i=1}^{n} \alpha_i v_i - \sum_{i=1}^{n} \alpha_i \ln(1 - e^{v_i}), \tag{4.4.19}$$

where $\alpha_i = \alpha_i(b_1, \ldots, b_n)$ *are constants related to values of* b_i*, where* $i = 1, \ldots, n$*.*

The Hamiltonian functions (4.4.17) and (4.4.19) correspond to an n-dimensional production function of the type f_5 in Section 3.3.

4.4.3 The generalized model of combined exponential and logistic growth

We have studied in Section 4.3 models of combined exponential and logistic growth. We will investigate the generalized n-dimensional model in this section. Let us consider the combination of the form $k + \ell = n$, where k and ℓ represent dimensions of each growth model. We see k and ℓ are not necessarily identical.

Let us consider special values of coefficients of the Lotka-Volterra model (4.0.9) as follows

- if $1 \leq i, j \leq k$, $a_{ij} = 0$,
- if $k+1 \leq i, j \leq k+\ell$, $a_{ij} = -\frac{b_i}{N_i}$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$.

Thus, we arrive at the following system

$$\dot{x}_i = \begin{cases} b_i x_i, & 1 \leq i \leq k, \\ b_i x_i \left(1 - \frac{x_i}{N_i}\right), & k+1 \leq i \leq k+\ell, \end{cases} \quad b_i \neq 0. \tag{4.4.20}$$

Applying the transformations

$$\begin{aligned} v_i &= \ln x_i, \\ v_i &= \ln \tfrac{x_i}{N_i}, \end{aligned} \tag{4.4.21}$$

the equation (4.4.20) reduces to

$$\dot{v}_i = \begin{cases} b_i, & 1 \leq i \leq k, \\ b_i \left(1 - e^{v_i}\right), & k+1 \leq i \leq k+\ell. \end{cases} \tag{4.4.22}$$

Definition 4.4.8. The n-dimensional dynamical system

$$\dot{v_i} = \begin{cases} b_i, & 1 \leq i \leq k, \\ b_i\left(1 - e^{v_i}\right), & k+1 \leq i \leq k+\ell. \end{cases} \tag{4.4.23}$$

is called a *transformed combined exponential and logistic growth* (TCELG) model.

In Subsection 4.4.1 and 4.4.2, we have seen the form of the Poisson bivector depends on the parity of the dimension of the growth model, which follows four cases

- Case 1: k even and ℓ even,
- Case 2: k even and ℓ odd,
- Case 3: k odd and ℓ even,
- Case 4: k odd and ℓ odd.

Applying Theorems 4.4.2, 4.4.4, 4.4.6 and 4.4.7, we summarize the Hamiltonian formalism for the TCELG model of each case in the following

Theorem 4.4.9. *The n-dimensional ($n = k + \ell$) TCELG model is a Hamiltonian system with the following Poisson bivector and Hamiltonian function.*

- *If k and ℓ are even, then the Poisson bivector is*

$$\pi_{k+\ell} = p^i \frac{\partial}{\partial v_{2i}} \wedge \frac{\partial}{\partial v_{2i-1}}, \tag{4.4.24}$$

where

$$p^i = \begin{cases} b_{2i-1}b_{2i}, & 1 \leq i \leq \frac{k}{2}, \\ b_{2i-1}b_{2i}(1-e^{v_{2i-1}})(1-e^{v_{2i}}), & \frac{k}{2}+1 \leq i \leq \frac{k+\ell}{2}, \end{cases} \tag{4.4.25}$$

and the Hamiltonian is

$$\begin{aligned} H_{k+\ell} &= \sum_{i=1}^{k/2}\left(\frac{1}{b_{2i-1}}v_{2i-1} - \frac{1}{b_{2i}}v_{2i}\right) + \sum_{i=\frac{k}{2}+1}^{(k+\ell)/2}\left(\frac{1}{b_{2i-1}}v_{2i-1} - \frac{1}{b_{2i}}v_{2i}\right. \\ &\quad \left. + \frac{1}{b_{2i-1}}\ln(1-e^{v_{2i-1}}) - \frac{1}{b_{2i}}\ln(1-e^{v_{2i}})\right). \end{aligned} \tag{4.4.26}$$

- *If k is even and ℓ is odd, assuming the condition that* $\sum_{i=\frac{k}{2}+1}^{\frac{k+\ell-1}{2}} b_{2i+1} = \sum_{i=\frac{k}{2}+1}^{\frac{k+\ell-1}{2}} b_{2i}$, *then the Poisson bivector is*

$$\pi_{k+\ell} = b_{2i-1}b_{2i}\frac{\partial}{\partial v_{2i}} \wedge \frac{\partial}{\partial v_{2i-1}} + b_{j_1}b_{j_2}(1-e^{v_{j_1}})(1-e^{v_{j_2}})\frac{\partial}{\partial v_{j_2}} \wedge \frac{\partial}{\partial v_{j_1}}, \quad (4.4.27)$$

where $1 \leq i \leq \frac{k}{2}$ *and* $k+1 \leq j_1, j_2 \leq k+\ell$,

and the Hamiltonian is

$$H_{k+\ell} = \sum_{i=1}^{k/2}\left(\frac{1}{b_{2i-1}}v_{2i-1} - \frac{1}{b_{2i}}v_{2i}\right) + \sum_{i=k+1}^{k+\ell}\alpha_i(v_i - \ln(1-e^{v_i})), \quad (4.4.28)$$

where $\alpha_i = \alpha_i(b_{k+1}, \ldots, b_{k+\ell})$ *are constants related to values of* b_i, *where* $i = k+1, \ldots, k+\ell$.

- *If k is odd and ℓ is even, assuming the condition that* $\sum_{i=1}^{(k-1)/2} b_{2i+1} = \sum_{i=1}^{(k-1)/2} b_{2i}$, *then the Poisson bivector is*

$$\pi_{k+\ell} = \frac{\partial}{\partial v_{j_2}} \wedge \frac{\partial}{\partial v_{j_1}} + b_{2i-1}b_{2i}(1-e^{v_{2i}})(1-e^{v_{2i+1}})\frac{\partial}{\partial v_{2i+1}} \wedge \frac{\partial}{\partial v_{2i}}, \quad (4.4.29)$$

where $1 \leq j_1, j_2 \leq k$ *and* $\frac{k+1}{2} \leq i \leq \frac{k+\ell-1}{2}$,

and the Hamiltonian is

$$\begin{aligned} H_{k+\ell} = & \sum_{i=1}^{n}\alpha_i v_i + \sum_{i=\frac{k+1}{2}}^{\frac{k+\ell-1}{2}}\left(\frac{1}{b_{2i}}v_{2i} - \frac{1}{b_{2i+1}}v_{2i+1} + \frac{1}{b_{2i}}\ln(1-e^{v_{2i}})\right. \\ & \left. - \frac{1}{b_{2i+1}}\ln(1-e^{v_{2i+1}})\right), \end{aligned} \quad (4.4.30)$$

where $\alpha_i = \alpha_i(b_1, \ldots, b_k)$ *are constants related to the values of* b_i, *where* $i = k+1, \ldots, k+\ell$.

- *If k and ℓ are odd, assuming the condition that* $\sum_{i=1}^{(k+\ell)/2} b_{2i+1} = \sum_{i=1}^{(k+\ell)/2} b_{2i}$, *then the Poisson bivector is*

$$\pi_{k+\ell} = p^{ij}\frac{\partial}{\partial v_j} \wedge \frac{\partial}{\partial v_i}, \quad (4.4.31)$$

where

$$p^{ij} = \begin{cases} 1, & 1 \le i, j \le k, \\ b_i b_j (1 - e^{v_i})(1 - e^{v_j}), & k+1 \le i, j \le k + \ell, \end{cases} \tag{4.4.32}$$

and the Hamiltonian is

$$H_{k+\ell} = \sum_{i=1}^{k} \alpha_i v_i + \sum_{i=k+1}^{k+l} \alpha(v_i - \ln(1 - e^{v_i})), \tag{4.4.33}$$

where $\alpha_i = \alpha_i(b_1, \ldots, b_{k+\ell})$ *are the constants related to values of* b_i*, where* $i = 1, \ldots, k + \ell$.

We note that each Hamiltonian function $H_{k+\ell}$ corresponds to a production function of a certain type.

4.4.4 The generalized model of the combined non-linear dynamics and logistic growth

In this subsection, we present an n-dimensional model of a combination of an m-dimensional (for m even) Lotka-Volterra dynamics and an ℓ-dimensional logistic growth model, which can also be viewed as a special case of the equation (4.0.9) by taking the following coefficients:

- if $1 \le i \le \frac{m}{2}$, $a_{(2i-1)(2i)},\ a_{(2i)(2i-1)} \neq 0$ and values of the rest of coefficients a_{ij} equal 0,
- if $m + 1 \le i, j \le m + \ell$, $a_{ij} = -\frac{b_i}{N_i}$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$.

Note that we are interested in a Lotka-Volterra model of even dimensions and each pair of variables x_{2i-1} and x_{2i} interacts in the manner of a Lotka-Volterra model, namely, the matrix of the coefficients a_{ij} when $1 \le i, j \le k$ is as follows:

$$\begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & a_{34} \\ a_{43} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & a_{(m-1)(m)} \\ a_{(m)(m-1)} & 0 \end{bmatrix}. \tag{4.4.34}$$

The logistic growth model can be viewed as a degenerate Lotka-Volterra model when there are no nonlinear terms.

The equation (4.0.9) becomes

$$\begin{bmatrix} \dot{x}_{2i-1} \\ \dot{x}_{2i} \end{bmatrix} = \begin{bmatrix} b_{2i-1} & a_{(2i)(2i-1)}x_{2i-1} \\ a_{(2i-1)(2i)}x_{2i} & b_{2i} \end{bmatrix} \begin{bmatrix} x_{2i-1} \\ x_{2i} \end{bmatrix}, b_{2i-1}, b_{2i} \neq 0, \ 1 \leq i \leq \tfrac{m}{2}, \tag{4.4.35}$$

and

$$\dot{x}_i = b_i x_i \left(1 - \frac{x_i}{N_i}\right), \ b_i \neq 0, \ m+1 \leq i \leq m+\ell, \tag{4.4.36}$$

which, under the transformations

$$v_i = \ln\left(-\frac{a_{ji}}{b_i} x_i\right), \ j = 1 \text{ or } \dots \text{ or } m, \ i = 1, \dots, m \tag{4.4.37}$$

and

$$v_i = \ln\left(\frac{x_i}{N_i}\right), \ i = m+1, \dots, m+\ell, \tag{4.4.38}$$

becomes

$$\begin{bmatrix} \dot{v}_{2i-1} \\ \dot{v}_{2i} \end{bmatrix} = \begin{bmatrix} b_{2i-1}(1 - e^{v_{2i}}) \\ b_{2i}(1 - e^{v_{2i-1}}) \end{bmatrix}, \quad 1 \leq i \leq \tfrac{m}{2} \tag{4.4.39}$$

and

$$\dot{v}_i = b_i(1 - e^{v_i}), \ m+1 \leq i \leq m+\ell, \tag{4.4.40}$$

which is defined in the following

Definition 4.4.10. The n-dimensional $(m + \ell = n)$ dynamical system

$$\begin{bmatrix} \dot{v}_{2i-1} \\ \dot{v}_{2i} \end{bmatrix} = \begin{bmatrix} b_{2i-1}(1 - e^{v_{2i}}) \\ b_{2i}(1 - e^{v_{2i-1}}) \end{bmatrix}, \quad 1 \leq i \leq \tfrac{m}{2} \tag{4.4.41}$$

and

$$\dot{v}_i = b_i(1 - e^{v_i}), \ m+1 \leq i \leq m+\ell, \tag{4.4.42}$$

is called a *transformed combined Lotka-Volterra* (TCLV) model.

Using results in Sections 4.3 and Subseciton 4.4.3, we introduce the Hamiltonian structure of the TCLV model in the following

Theorem 4.4.11. *The n-dimensional ($n = m + \ell$) TCLV model is a Hamiltonian system defined by the following Poisson bivectors and Hamiltonian functions.*

- *If ℓ is even, then the Poisson bivector is*

$$\pi_{m+\ell} = p^i \frac{\partial}{\partial v_{2i}} \wedge \frac{\partial}{\partial v_{2i-1}}, \; 1 \leq i \leq \frac{m+\ell}{2}, \tag{4.4.43}$$

where

$$p^i = \begin{cases} 1, & 1 \leq i \leq \frac{m}{2}, \\ b_{2i-1} b_{2i} (1 - e^{v_{2i-1}})(1 - e^{v_{2i}}), & \frac{m}{2} + 1 \leq i \leq \frac{m+\ell}{2}, \end{cases} \tag{4.4.44}$$

and the Hamiltonian is

$$\begin{aligned} H_{m+\ell} = \; & \sum_{i=1}^{\frac{m}{2}} (b_{2i-1}(1 - e^{v_{2i}}) - b_{2i}(1 - e^{v_{2i-1}})) + \sum_{i=\frac{m}{2}+1}^{(m+\ell)/2} \Bigg(\frac{1}{b_{2i-1}} v_{2i-1} \\ & - \frac{1}{b_{2i}} v_{2i} + \frac{1}{b_{2i-1}} \ln(1 - e^{v_{2i-1}}) - \frac{1}{b_{2i}} \ln(1 - e^{v_{2i}}) \Bigg). \end{aligned} \tag{4.4.45}$$

- *If ℓ is odd, under the condition that $\sum_{i=\frac{m}{2}+1}^{\frac{m+\ell-1}{2}} b_{2i+1} = \sum_{i=\frac{m}{2}+1}^{\frac{m+\ell-1}{2}} b_{2i}$, then the Poisson bivector is*

$$\pi_{m+\ell} = \frac{\partial}{\partial v_{2i}} \wedge \frac{\partial}{\partial v_{2i-1}} + b_{j_1} b_{j_2} (1 - e^{v_{j_1}})(1 - e^{v_{j_2}}) \frac{\partial}{\partial v_{j_2}} \wedge \frac{\partial}{\partial v_{j_1}}, \tag{4.4.46}$$

where $1 \leq i \leq \frac{m}{2}$ and $m + 1 \leq j_1, j_2 \leq m + \ell$,

and the Hamiltonian is

$$\begin{aligned} H_{m+l} = \; & \sum_{i=1}^{m/2} (b_{2i-1}(1 - e^{v_{2i}}) - b_{2i}(1 - e^{v_{2i-1}})) \\ & + \sum_{i=m+1}^{m+\ell} \alpha_i (v_i - \ln(1 - e^{v_i})), \end{aligned} \tag{4.4.47}$$

where $\alpha_i = \alpha_i(b_{m+1}, \ldots, b_{m+\ell})$ are constant related to values of b_i, $i = m + 1, \ldots, m + \ell$.

The Hamiltonian function $H_{m+\ell}$ is of the type of the production function (3.3.14).

4.5 Bi-Hamiltonian structures for three-dimensional systems

We have recovered the Cobb-Douglas function (3.0.1) and the production function $f_5(x)$ (3.3.14) via the Hamiltonian formalism. As mentioned in Remark 4.2.2, we did not obtain the desired economically meaningful results, namely, $\alpha+\beta=1$. This issue happened in Sato's Lie theoretical approach. To address this issue, as discussed in Section 3.3, Sato introduced *the simultaneous holotheticity*, *i.e.*, the condition that a function is holothetic under more than one technical progress. Mathematically, he considered a production function as an invaraint of an involutive distribution of two vector fields. He succeeded in recovering the Cobb-Douglas function with constant returns to scale.

We attempted to solve this issue via the bi-Hamiltonian approach. Introduce the following bi-Hamiltonian structure for the dynamical system (4.2.1):

$$\dot{x}_i = X_{H_1,H_2} = \pi_1 dH_1 = \pi_1 dH_2,\ \ i = 1, 2, 3, \tag{4.5.1}$$

where the Hamiltonian functions H_1 and H_2 are given by

$$H_1 = b \ln x_1 + \ln x_2 + a \ln x_3, \quad H_2 = \ln x_1 + a \ln x_2 + b \ln x_3, \tag{4.5.2}$$

corresponding to the Poisson bivectors π_1 and π_2 given by

$$\pi_1 = a_{ij} x_i x_j \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_i}, \quad \pi_2 = b_{ij} x_i x_j \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_i}, \quad i, j = 1, 2, 3 \tag{4.5.3}$$

respectively under the conditions

$$\begin{cases} bb_1 + b_2 + ab_3 &= 0, \\ b_1 + ab_2 + b_3 b &= 0. \end{cases} \tag{4.5.4}$$

Remark 4.5.1. *Note notations π_1, π_2, H_1 and H_2 used in the bi-Hamiltonian approach do not refer to the Poisson bivectors and Hamiltonian functions presented in previous sections.*

Remark 4.5.2. *Let us briefly discuss the algebraic conditions* (4.5.4)*. It slightly differs from* (4.2.6)*, in which we want to determine the value of coefficients* α_i *in the Hamiltonian* H_3*. The conditions* (4.5.4) *are derived from determining coefficients of the Poisson bivectors* a_{ij} *and* b_{ij}*, (which are not identical) the conditions (4.5.4) (compare them to (4.2.8)) assuring that* π_1 *and* π_2 *are indeed Poisson bivectors compatible with the dynamics of (4.2.1) and corresponding to the Hamiltonians* H_1 *and* H_2 *given by (4.5.2). Let us take* a_{ij} *for illustration. The Hamiltonian structure leads to a system of equations given by the following matrix*

$$\left[\begin{array}{ccc|c} -1 & -a & 0 & b_1 \\ b & 0 & -a & b_2 \\ 0 & b & 1 & b_3 \end{array}\right]. \tag{4.5.5}$$

The system has solutions iff $bb_1 + b_2 + ab_3 = 0$.

Analogously, it follows $b_1 + ab_2 + b_3 b = 0$ *for* b_{ij}*.*

Remark 4.5.3. *Let* $\mathcal{H} = \mathcal{H}(H_1, H_2)$ *denotes the function depending on* H_1 *and* H_2*. The Jacobian of* $\mathcal{H}$ *is given by*

$$\begin{bmatrix} \dfrac{b}{x_1} & \dfrac{1}{x_2} & \dfrac{a}{x_3} \\ \dfrac{1}{x_1} & \dfrac{a}{x_2} & \dfrac{b}{x_3}, \end{bmatrix} \tag{4.5.6}$$

which is a matrix of rank 2.

Thus, H_1 *and* H_2 *are indeed functionally independent (*$dH_1 \wedge dH_2 \neq 0$*).*

Solving the linear system (4.5.4) for a and b under the additional condition $b_1 b_2 - b_3^2 \neq 0$, we arrive at

$$\alpha = \frac{1-b}{a-b} = \frac{b_3 - b_1}{b_2 - b_1}, \quad \beta = \frac{a-1}{a-b} = \frac{b_3 - b_2}{b_1 - b_2}. \tag{4.5.7}$$

Consider now the first integral H_3 given by

$$H_3 = H_1 - H_2 = (b-1)\ln x_1 + (1-a)\ln x_2 + (a-b)\ln x_3. \tag{4.5.8}$$

Solving the equation $H_3 = \text{const}$ determined by (4.5.8) for x_3, we arrive at the Cobb-Douglas function (3.0.1) with the elastisities of substitution α and β given by

$$\alpha = \frac{1-b}{a-b}, \quad \beta = \frac{a-1}{a-b}, \tag{4.5.9}$$

where a and b are given by (4.5.7). Note $\alpha + \beta = 1$, as expected. Also, $\alpha, \beta > 0$ under the additonal condition

$$b_2 > b_3 > b_1, \tag{4.5.10}$$

which implies by (4.2.1) that capital ($x_2 = K$) grows faster than production ($x_3 = f$), which, in turn, grows faster than labor ($x_1 = L$).

Remark 4.5.4. *Note,* $\alpha = \frac{b_3-b_1}{b_2-b_1} > 0$ *and* $\beta = \frac{b_3-b_2}{b_1-b_2} > 0$ *imply* $b_2 > b_3 > b_1$ *or* $b_1 > b_3 > b_2$. *We choose* (4.5.10) *in view of the linear condition* $b_1 + b_3 = b_2$.

We have also determined the corresponding formula for total factor productivity A, that is,

$$A = \exp\left(\frac{H_3}{a-b}\right), \tag{4.5.11}$$

where H_3 is a constant along the flow (4.2.1) as linear combination of the two Hamiltonians H_1 and H_2 given by (4.5.2).

In addition, we introduce the following bi-Hamiltonian structure to the dynamical system (4.2.18)

$$\dot{x}_i = X_{H_4,H_5} = \pi_4 dH_4 = \pi_5 dH_5, \; i = 1, 2, 3, \tag{4.5.12}$$

where the Hamiltonian functions H_4 and H_5 are

$$\begin{aligned} H_4 &= b \ln \frac{x_1}{|N_1 - x_1|} + \ln \frac{x_2}{|N_2 - x_2|} + a \ln \frac{x_3}{|N_3 - x_3|}, \\ H_5 &= \ln \frac{x_1}{|N_1 - x_1|} + a \ln \frac{x_2}{|N_2 - x_2|} + b \ln \frac{x_3}{|N_3 - x_3|}, \end{aligned} \tag{4.5.13}$$

with the corresponding Poisson bivectors π_4 and π_5

$$\begin{aligned} \pi_4 &= a_{ij} x_i x_j (1 - \frac{x_i}{N_i})(1 - \frac{x_j}{N_j}) \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_i}, \; i, j = 1, 2, 3, \\ \pi_5 &= b_{ij} x_i x_j (1 - \frac{x_i}{N_i})(1 - \frac{x_j}{N_j}) \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_i}, \; i, j = 1, 2, 3. \end{aligned} \tag{4.5.14}$$

under the same conditions (4.5.4).

Note H_4 and H_5 are functionally independent, a and b, assuming $b_1b_2 - b_3^2 \neq 0$, are given by (4.5.7).

A new first integral H_6 is in the following form

$$\begin{aligned} H_6 = H_4 - H_5 = (b-1)\ln\frac{x_1}{|N_1 - x_1|} \quad &+ (1-a)\ln\frac{x_2}{|N_2 - x_2|} \\ &+ (a-b)\ln\frac{x_3}{|N_3 - x_3|}, \end{aligned} \tag{4.5.15}$$

solving which for x_3, we recover the production function (3.3.14) satisfying the condition $\alpha + \beta = 1$, provided $b_2 > b_3 > b_1$ and the constant $C = \pm\exp\left(\frac{H_6}{b-a}\right)$.

4.6 The Cobb-Douglas function revisited via the bi-Hamiltonian approach

We have shown that the production function can be recovered from a Hamiltonian structure, that is, the production function can be represented by the Hamiltonian function of the dynamical system of the output and inputs, including capital, labor, *etc.* In this section, we consider the Cobb-Douglas function to show that the formulas obtained via the bi-Hamiltonian approach are compatible with data employed by Cobb and Douglas in [27].

It is important to note that Cobb and Douglas [27], employing the US economic data from 1899 to 1922, numerically determined the relation between production, labor and capital based on a pure statistical analysis, *i.e.*, their function was of the form

$$Y = 1.01K^{\frac{1}{4}}L^{\frac{3}{4}}, \tag{4.6.1}$$

where Y, L, K represented production, labor and capital, respectively, and generalized the relation (4.6.1) to the well-known Cobb-Douglas function by introducing the parameters $A = 1.01$, $\alpha = \frac{1}{4}$ and $\beta = \frac{3}{4}$.

Remark 4.6.1. *Cobb and Douglas used the notation P' and C in lieu of Y and K respectively in [27], which denote the estimated values for production and capital,* *respectively. We use* Y *and* K *for our convenience.*

The Hamiltonian formalism has proven to be a powerful tool in deriving a production function. We have recovered the exact form of the Cobb-Douglas function via the bi-Hamiltonian approach in Section 4.5. We believe the Hamiltonian approach is compatible with the statistical analysis conducted by Cobb and Douglas, that is, intuitively, we can show $\alpha = \frac{1}{4}$ and $\beta = \frac{3}{4}$ by employing (4.5.7).

Solving the equation (4.2.1) yields

$$x_i = c_i \exp(b_i t), \quad i = 1, 2, 3, \tag{4.6.2}$$

where $c_i \in \mathbb{R}^+$ and b_i are to be determined by the following statistical analysis.

The model after the logarithmic transformation assumes the form:

$$\ln x_i = C_i + b_i t, \quad i = 1, 2, 3, \tag{4.6.3}$$

where $C_i = \ln c_i$.

Note Table 4.1 shows US economic dimensionless data 1899-1922 from [27] after the logarithmic transformation.

To estimate the model (4.6.3), we have used the R Programming language, employing the method of least squares, and obtained the following estimates (see Figure 4.1, 4.2 and 4.3 for references)

- Estimations for labor

$$b_1 = 0.02549605, \ C_1 = 4.66953290; \tag{4.6.4}$$

- Estimations for capital

$$b_2 = 0.06472564, \ C_2 = 4.61213588; \tag{4.6.5}$$

Year	Output	Capital	Labor
1899	4.605170	4.605170	4.605170
1900	4.615121	4.672829	4.653960
1901	4.718499	4.736198	4.700480
1902	4.804021	4.804021	4.770685
1903	4.820282	4.875197	4.812184
1904	4.804021	4.927254	4.753590
1905	4.962845	5.003946	4.828314
1906	5.023881	5.093750	4.890349
1907	5.017280	5.170484	4.927254
1908	4.836282	5.220356	4.795791
1909	5.043425	5.288267	4.941642
1910	5.068904	5.337538	4.969813
1911	5.030438	5.375278	4.976734
1912	5.176150	5.420535	5.023881
1913	5.214936	5.463832	5.036953
1914	5.129899	5.497168	5.003946
1915	5.241747	5.583469	5.036953
1916	5.416100	5.697093	5.204007
1917	5.424950	5.814131	5.278115
1918	5.407172	5.902633	5.298317
1919	5.384495	5.958425	5.262690
1920	5.442418	6.008813	5.262690
1921	5.187386	6.033086	4.990433
1922	5.480639	6.066108	5.081404

Table 4.1: The time series data used by Cobb and Douglas in [27]

- Estimations for production

$$b_3 = 0.03592651,\ C_3 = 4.66415363. \tag{4.6.6}$$

The model (4.6.3) has become a straight line after the logarithmic transformation. We can see that the errors (represented by $value in Figure 4.1, 4.2 and 4.3) are less than 1, which suggests that the linear regression performed quite well. Let us take the estimation of capital for example. The observed capital and estimated capital are illustrated by Figure 4.4. The linear regression (see Figure 4.5) test using the R Programming language shows the adjusted R-squared value of the model is 0.9934, which is very close to 1 (representing perfection). Note that we have 0.02549605 +

```
>
> myfun=function(par,data){
+ l = data$labour
+ t = data$year
+ func=sum((l-par[2]-(par[1]*(t-1899)))^2)
+ return(func)
+ }
> optim(myfun,par=c(0.1,4.60517018599),data=mydata)
$par
[1] 0.02549605 4.66953290

$value
[1] 0.1827943

$counts
function gradient
      59       NA

$convergence
[1] 0

$message
NULL
```

Figure 4.1: Labor fitting.

```
> myfun=function(par,data){
+ k = data$capital
+ t = data$year
+ func=sum((k-par[2]-(par[1]*(t-1899)))^2)
+ return(func)
+ }
> optim(myfun,par=c(0.1,4.60517018599),data=mydata)
$par
[1] 0.06472564 4.61213588

$value
[1] 0.03065574

$counts
function gradient
      65       NA

$convergence
[1] 0

$message
NULL
```

Figure 4.2: Capital fitting.

```
> myfun=function(par,data){
+ p = data$output
+ t = data$year
+ func=sum((p-par[2]-(par[1]*(t-1899)))^2)
+ return(func)
+ }
> optim(myfun,par=c(0.1,4.60517018599),data=mydata)
$par
[1] 0.03592651 4.66415363

$value
[1] 0.1825852

$counts
function gradient
      63       NA

$convergence
[1] 0

$message
NULL
```

Figure 4.3: Production fitting.

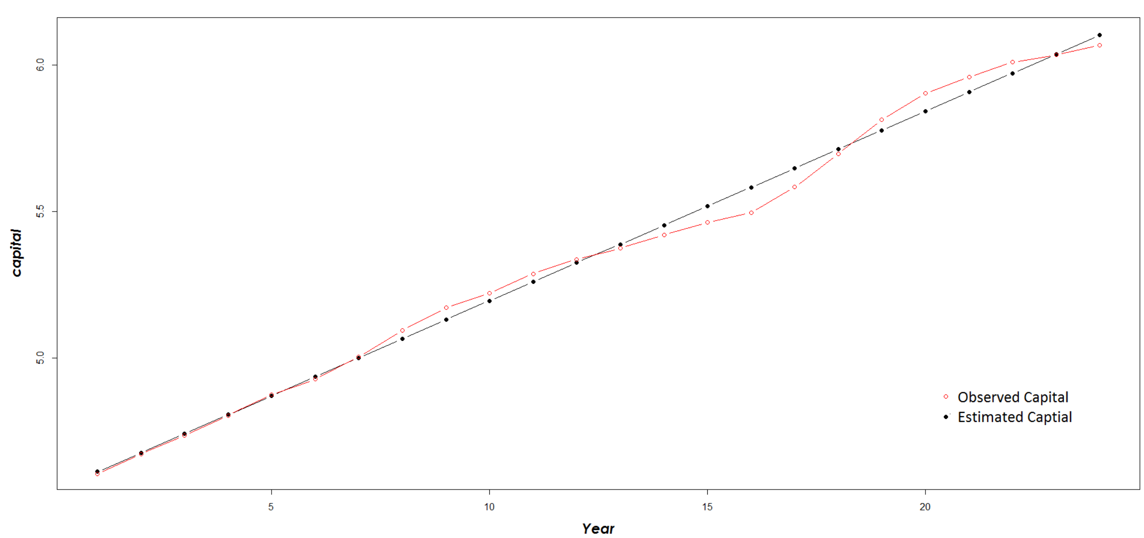


Figure 4.4: Observed capital versus estimated capital during the period 1899-1922.

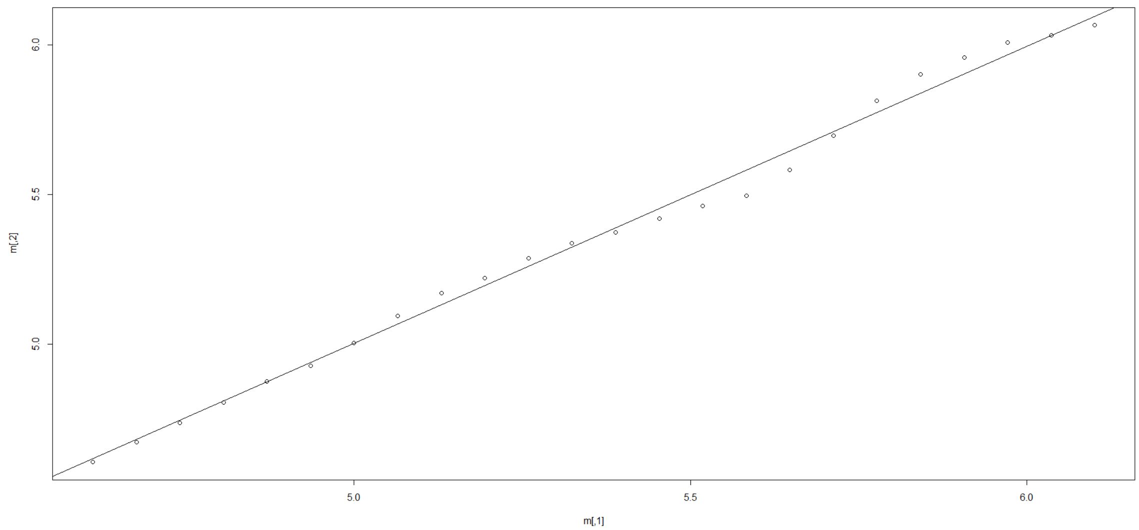


Figure 4.5: The linear regression of the observed and estimated capital from 1899 to 1922.

$0.03592651 \approx 0.06472564$ from the dataset in Table 4.1, which implies that the model describing the US economy has a Poisson structure of the standard form.

We also note the values of coefficients are constrained by the inequality $b_2 > b_3 > b_1$, which satisfies our requirement. The numerical values help us to identify $x_1 = L$ and $x_2 = K$. Substituting values of parameters b_i into the equation (4.5.4), we obtain

$$a = 4.659691804, \quad b = -9.104630098, \tag{4.6.7}$$

which, in turn, determine the values of α and β as follows

$$\alpha = 0.2658824627, \quad \beta = 0.7341175376. \tag{4.6.8}$$

Next, using the data from Table 4.1 and the formula (4.5.11) we employ the R programming language to evaluate the value of H_3, arriving at the following result: the variance of the resulting distribution of values of H_3 is 0.5923171 and the mean of the distribution is 0.1365228. By letting $H_3 = 0.1365228$ and using (4.5.11), the value of A is found to be $A = 1.00996795211 \approx 1.01$ (compare with (4.6.1)).

Considering admissible errors, we have exactly recovered the values of elasticity of substitution for capital and labor as well as total productivity factor, which Cobb and Douglas obtained from a statistical analysis. Mathematically, we believe this work demonstrates that Sato's assumption about exponential growth in production and its factors (3.1.5) is compatible with the results by Cobb and Douglas based on the statistical analysis of the data from the US manufacturing studied in [27].

4.7 Concluding remarks

We have reduced several problems of derivation of a production function to the corresponding algebraic problems by employing the Hamiltonian approach to and describing the dynamics in econometrics in each case as a special case of the Lotka-Volterra model (4.0.9). We have extended Sato's Lie theoretical approach to the problem of the determination of a production function by employing the Hamiltonian formalism, that is, we showed the Hamiltonian structures of considered models are preserved along the flow given by each corresponding technical progress. We believe that the Hamiltonian function in each case represents the state of an economy. The Hamiltonian approach to the function production of considered special cases is classified and summarized in previous theorems.

To derive the Cobb-Douglas function and the production function (3.3.14) with desired algebraic condition, we introduce the bi-Hamiltonian structure to the system (4.2.1) and (4.2.18). The advantage of the bi-Hamiltonian approach is that one does not have to consider two sectors of an economy, that is, we can derive the production function with constant returns to scale based on one vector field. The statistical analysis shows the formulas (4.5.4) derived via the bi-Hamiltonian approach match the US manufacturing data (4.1) utilized by Cobb and Douglas. Mathematically, it shows that the Cobb-Douglas function can be in some appropriate sense viewed as a conservation law of the economy in early 20th century since it is a Hamiltonian of the dynamical system relatively adequately characterizing the economic growth.

We have considered a four-dimensional economic model involving debt in Section 4.3. The model leads to a new production of capital, debt and labor. We want to gain an insight of relations between debt and output, which is considered as a complex dynamics in mathematical economics, with the help of the model (4.3.9). More details of analysis of (4.3.9) are given in Chapter 5.

Chapter 5

On the validity of the concept of a production function in economics: A mathematical perspective

In this chapter we revisit some of the controversies around the derivation of the Cobb-Douglas production function in [27] and discuss its legitimacy from a mathematical viewpoint that extends the approach to the growth theory established by Sato which we have reviewed in Section 3.3. Let us start with the function (4.6.1) in Section 4.6

$$Y = 1.01K^{\frac{1}{4}}L^{\frac{3}{4}}. \tag{5.0.1}$$

The results came from the statistical analysis of time series of capital, labor and production conducted by Cobb and Douglas [27] using the following production function

$$Y = AL^{k}K^{1-k}, \tag{5.0.2}$$

where k is parameter. The function (5.0.1) turned out to be a good fit for the US economic data from 1899-1922 used in [27]. This calculation led the authors and other scientists to belive that the coefficients α and A in (5.0.2) were determined empirically from the given dataset and the relation given by (5.0.2) was a general time-invariant property relating the output product to the corresponding inputs (capital and labor). In fact, the function (5.0.2) was successfully fitted to many other datasets with invariably good results. For example, Douglas [31] showed that the Cobb-Douglas function with constant return to scale ($\alpha + \beta = 1$) fit well to the economic data of Canada, Australia, New Zealand and South Africa during around the 1950s and 1960s, and Leser [77] concluded that the British economic data 40 years preceding the First World War can be described by a Cobb-Douglas function with homogeneity of degree one while the data after the war do not support this. We need to note that Leser actually described the time series of capital, labor and production using the exponential growth model. Those efforts further promoted the widespread acceptance of the Cobb-Douglas function in the scientific community as a viable example of a production function that could be used in various growth models. As we have discussed in previous sections, in the decades since 1928 the Cobb-Douglas function and its generalizations have become a common feature of many works dedicated to the development of the growth theory in economics. It must be noted that the importance of the Cobb-Douglas production function goes beyond merely the growth theory in economics. For example, it manifests itself most prominently also in the theory of aggregate demand (see Giraud & Quah [48] for more details). It is very telling that by now the function (3.0.1), due to its interesting properties (*e.g.*, diminishing marginal returns in both factors), numerous applications and simplicity has become a standard feature not only of textbooks in economics (see Felipe & McCombie [38] and the relevant references therein), but in mathematics [124] as well. At the same time, some conclusions drawn by Cobb and Douglas from their results obtained in [27] have been met with strong and justifiable criticisms [20, 38, 103], or even outright rejected [81]. Thus, Mendershausen [81] indicated that no empirical evidence supports the constant return to scale for the Cobb-Douglas function and one can fit the Cobb-Douglas function to the data used in [27] without assuming $\alpha + \beta = 1$. A similar critical argument can be also found in Brown [20], Samuelson [103], Felipe & McCombie [38]–among others. In recent years some authors have stated that the Cobb-Douglas function no longer can be fitted to the relevant ecnomic data coming from today's economy (see Antràs [8], Gechet *et al* [46]). In fact, the question of whether the function (3.0.1) is a viable production function has been the subject of debate practically from the time of its inception, while there is no doubt it can be fitted to various datasets, which has been demonstrated by Cobb and Douglas from the get-go in 1928, as well as many others in the years to come. In particular, as early as 1938 Mendershausen [81] (see also Brown [20]) had used some statistical tools based on the notion of multicollinearity to argue that (5.0.1) was not the only function that could be fitted to the dataset studied by Cobb and Douglas in [27], at the same time confirming that the function (5.0.1) could be fitted to the data for the US manufacturing sector for 1899-1922 with good accuracy.

In what follows we demonstrate that $\alpha+\beta=1$ is not a necessary condition for a production function relatively rigorously from the viewpoint of mathematics by treating a production function as an invariant of the corresponding group of transformations under a certain condition that affords the possibility to fit a production function to the corresponding dataset.

5.1 The invariant condition for the Cobb-Douglas function

According to Lemma 3.1.5, we know that a production function is an invariant of a certain group of transformations. Following notations in Chapter 4, we present the group of transformations (3.6.2) in the following form

$$(G_4, \mathbb{R}^3_+), \quad G_4: \ \bar{x}_i = x_i e^{b_i t}, \quad b_i \geqslant 0, \quad i = 1, 2, 3. \tag{5.1.1}$$

We know that the Cobb-Douglas function (3.0.1) is holothetic to the above group of transformations based on the discussion in Section 3.2. We want to show that a holothetic production function is the Cobb-Douglas function iff G_4 holds. We have shown that G_4 leads to the Cobb-Douglas function in Chapter 3. Let us show the following converse problem: suppose capital x_2 and labor x_1 grow exponentially, given a Cobb-Douglas production function holothetic to G_4, production x_3 must follow an exponential growth. Assume the following transformation on f

$$\bar{x}_3 = \xi(x_3, t), \tag{5.1.2}$$

applying Taylor's theorem, we get

$$\bar{x}_3 = x_3 + H(x_3)t + O(t^2), \tag{5.1.3}$$

where $x_3 = \xi(x_3, 0)$ and $H(x_3) = \left(\dfrac{\partial \xi}{\partial t}\right)_{t=0}$.

Now the holothetic condition upon assuming x_1 and x_2 are affected by exponential growth in G_4 becomes

$$b_1 x_1 \frac{\partial f}{\partial x_1} + b_2 x_2 \frac{\partial f}{\partial x_2} = H(x_3), \tag{5.1.4}$$

where $x_3 = f$ is given by a Cobb-Douglas function.

Substituting $f = Ax_1^{\beta}x_2^{\alpha}$ into (5.1.4) gives

$$(b_1\beta + b_2\alpha)f = H(x_3), \tag{5.1.5}$$

Since f can not always be zero, we obtain

$$H(x_3) = (b_1\beta + b_2\alpha)f = b_3x_3, \tag{5.1.6}$$

where b_3 is a constant. Then the corresponding vector field of (5.1.2) is

$$U = H(x_3)\frac{\partial}{\partial x_3} = b_3x_3\frac{\partial}{\partial x_3}. \tag{5.1.7}$$

Solving the associated ODE

$$\dot{x}_3 = b_3x_3, \tag{5.1.8}$$

we arrive at

$$x_3 = x_3^0e^{b_3t},\ x_3^0 \in \mathbb{R}_+, \tag{5.1.9}$$

or

$$\bar{x}_3 = x_3e^{b_3t}, \tag{5.1.10}$$

which proves that x_3 is affected by an exponential growth.

Next, let us prove that the family of Cobb-Douglas function is indeed an invariant under G_4. We present the function as follows

$$\varphi(x_1, x_2, x_3) = x_1^{\beta}x_2^{\alpha}x_3^{\gamma} = A, \tag{5.1.11}$$

where $\alpha,\ \beta \geq 0$ and $\gamma \in \mathbb{R}$.

The function $\varphi(x_1, x_2, x_3)$ is an invariant under G_4 iff

$$b_1x_1\frac{\partial\varphi}{\partial x_1} + b_2x_2\frac{\partial\varphi}{\partial x_2} + b_3x_3\frac{\partial\varphi}{\partial x_3} = 0, \tag{5.1.12}$$

substituting $\varphi(x_1, x_2, x_3) = x_1^{\beta} x_2^{\alpha} x_3^{\gamma}$, we get

$$(b_1\beta + b_2\alpha + b_3\gamma)x_3 = 0. \tag{5.1.13}$$

Since x_3 is not always zero, we arrive at

$$b_1\beta + b_2\alpha + b_3\gamma = 0, \tag{5.1.14}$$

under which the function φ is invariant of the group transformation G_4.

Now we are clear that the function φ is an invariant of G_4 under (5.1.14). G_4 is a one-parameter group defined by the parameter t, hence, the above proof equivalently shows that

$$\frac{d\varphi}{dt} = 0 \quad \text{iff} \quad b_1\beta + b_2\alpha + b_3\gamma = 0. \tag{5.1.15}$$

Note (5.1.1) can be presented as follows

$$x_i = c_i e^{b_i t}, \quad i = 1, 2, 3, \tag{5.1.16}$$

Let us give more details about (5.1.15). Expressing φ as the following combination, in view of (5.1.16), we arrive at

$$\varphi = (c_1)^{\alpha}(c_2)^{\beta}(c_3)^{\gamma} e^{(\alpha b_1 + \beta b_2 + \gamma b_3)t}. \tag{5.1.17}$$

Therefore, we conclude that in view of (5.1.16), the function is constant along the flow generated by (5.1.16) iff the condition (5.1.14) holds. Since t denotes time, we can see that the function φ is a time invariant under the condition (5.1.14).

More specifically, we arrive at the Cobb-Douglas production function (3.0.1), provided (and that is the key!)

$$b_1\beta + b_2\alpha - b_3 = 0, \tag{5.1.18}$$

where b_i, $i = 1, 2, 3$ are determined by the exponential growth in input factors and production given by (5.1.16). We conclude, therefore, that the one-parameter Lie group action (5.1.1) admits a family of the Cobb-Douglas functions given by (3.0.1) iff the elasticities of substitution α and β are constrained by the linear relation (5.1.14). Moreover, if additionally the parameters b_i, $i = 1, 2, 3$ satisfy the inequality (4.5.10), we can always use the formulas (4.5.7) to pick among the functions given by (3.0.1), the Cobb-Douglas function (5.0.2) enjoying constant returns to scale. Note (5.1.14) is compatible with (4.2.8).

Based upon the statistical analysis in Section 4.6, we can see the values of α, β, b_1, b_2 and b_3 satisfy (5.1.14) as follows:

$$\begin{aligned} &0.02549605 \cdot 0.7341175376 + 0.06472564 \cdot 0.2658824627 - 0.03592651 \\ &= 7.5601053 \times 10^{-12} \approx 0. \end{aligned} \tag{5.1.19}$$

We comment that (5.1.19) is the condition assuring that the Cobb-Douglas function (5.0.1) is a time invariant for the dataset used in [27]. In fact, it explains some of the controversy surrounding the Cobb-Douglas function. For instance, Felipe [38] commented that the use of the Cobb-Douglas function appeals to the authority and it is not enough for the existence of the production function simply because the function compares well against the real data. Now, according to our analysis, the existence of the Cobb-Douglas function (5.0.1) is due to the exponential growth of factors as well as production and the satisfied linear condition (5.1.19). However, we also confirm the criticism of Mandershaunsen [81], Samuelson [103], *etc.*, that Cobb and Douglas [27] did not derive "the production function" but "a production function", namely, $\alpha + \beta = 1$ is not a necessary condition for the existence of the Cobb-Douglas function. There are other values of α and β satisfying the linear condition (5.1.14) for the values of b_1, b_2 and b_3 given by (4.6.4), (4.6.5) and (4.6.6). For example, setting $\alpha = 1$, we find, via (5.1.14) and using the values given by (4.1), the corresponding value for β:

$$\beta = \frac{b_3 - b_1}{b_2} = 0.16114881212. \tag{5.1.20}$$

Note that the values $\alpha = 1$ and $\beta = 0.16114881212$ in this case no longer add up to one, while the function

$$Y = ALK^{0.16114881212} \tag{5.1.21}$$

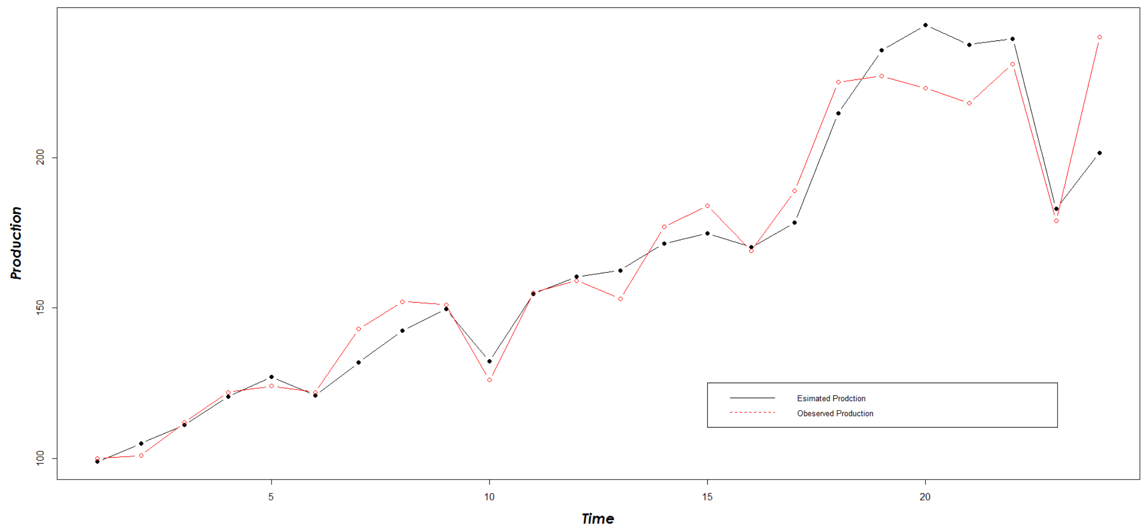


Figure 5.1: The function (5.1.21) vs the index values for the production studied by Cobb and Douglas in [27].

is a legitimate Cobb-Douglas function compatible with the data studied in [27]. Using the R programming language, we demonstrate that the new production function $Y = 0.4710156LK^{0.16114881212}$ fits to the data very well (see Figure 5.1). Indeed, we obtain a set of values of the parameters α and β, $P_{\alpha,\beta} = \{\alpha, \beta \in (0, \infty) : 0.02549605\beta + 0.06472564\alpha = 0.03592651\}$ (see Figure 5.2), for which the Cobb-Douglas function is valid for the data during the period 1899-1922.

Note that in Figure 5.2 we find $\alpha > \beta$ where the parameter line $0.02549605\beta + 0.06472564\alpha = 0.03592651$ is below the identity line $\beta = \alpha$ while $\alpha < \beta$ where the parameter line lies above the identity line. The intersection of the parameter line and the line $\alpha + \beta = 1$ determines the value of coefficients in (5.0.1).

5.2 The invariant condition for the new production function f_5

We know from Section 3.2 that the new production function f_5 (3.3.14) is derived from the following one-parameter Lie group action given by

$$(G_5, \mathbb{R}^3_+), \quad G_5 : \bar{x}_i = \frac{N_i x_i}{x_i + (N_i - x_i)e^{-b_i t}}, \quad b_i \geq 0, \quad i = 1, 2, 3, \tag{5.2.1}$$

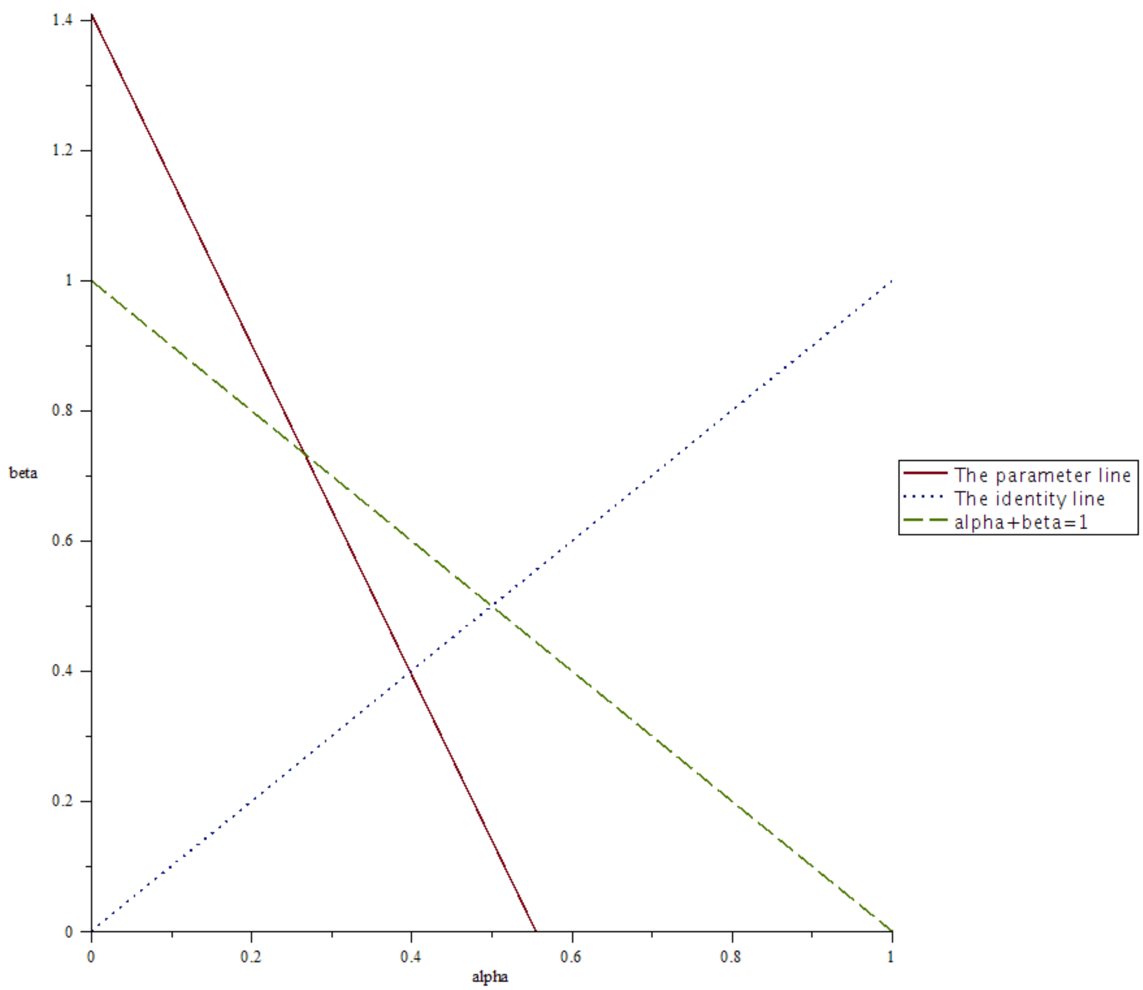


Figure 5.2: The line of parameters of the Cobb-Douglas function that fits well to the data used in [27].

which is generated by the model (4.2.18).

We have shown in Chapter 3 that there can not be exponential growth for production if all factors grow logistically. By analogy with the proof in Section 5.1, we can also show that the new production function f_5 also implies the logistic growth in x_3 given that x_i, $i = 1, 2$ are affected by logistic growth.

It follows from (5.2.1) that

$$\frac{x_i(N_i - x_i^0)}{x_i^0(N_i - x_i)} = e^{b_i t}, \quad i = 1, 2, 3. \tag{5.2.2}$$

Next, we obtain

$$\prod_{i=1}^{3} \left[\frac{x_i(N_i - x_i^0)}{x_i^0(N_i - x_i)} \right]^{\alpha_i} = e^{\left(\sum_{i=1}^{3} \alpha_i b_i\right)t}, \tag{5.2.3}$$

where α_i, $i = 1, 2, 3$ are some paremters. We see that the LHS of the equation (3.3.21) is an invariant of the one-parameter group action generated by (4.2.18) iff the parameters α_i, $i = 1, 2, 3$ satisfy the linear relation

$$\alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 = 0, \tag{5.2.4}$$

for the fixed values of b_i, $i = 1, 2, 3$ determined by (4.2.18).

In terms of the notations in the new production function f_5 (3.3.14), the linear condition (5.1.14) becomes

$$b_1 \beta + b_2 \alpha - b_3 = 0. \tag{5.2.5}$$

Note that our new production function is an invariant along the flow generated by (4.2.18) under the linear condition (5.2.5). It implies that the condition $\alpha + \beta = 1$ is not necessary for the new production. We confirm the condition (5.2.5) using the values in Chapter 3. Recall that we compared the new production function to the US economic data from 1947 to 2016, assuming $\alpha+\beta = 1$, which gave the following values of parameters $C = 0.3118901$, $\alpha = 0.4063544$ and $\beta = 0.5936456$. Next we fit the logistic model to the time series of capital, labor and production used in Chapter 3. The results of the regression are as follows: Assuming $N_{f_5} = 120$ and $N_K = N_L = 150$, we fit, to the time series of capital, labor and production, the following logistic curve

$$x_i = \frac{N_i x_i^0}{x_i^0 + (N_i - x_i^0)e^{-b_i t}}, \; i = 1, 2, 3, \tag{5.2.6}$$

where x_i^0 denotes the initial condition, N_i represents the carrying capacity and b_i is the growth rate.

The curve fitting, using the R programming language, shows the following results:

- Estimations for labor ($N_L = 150$)

$$b_1 = 0.09244029, \; x_1^0 = 1.21720215, \; RSS_1 = 2310.192; \tag{5.2.7}$$

- Estimations for capital ($N_K = 150$)

$$b_2 = 0.1037214, \; x_2^0 = 0.5875279, \; RSS_2 = 398.5569; \tag{5.2.8}$$

- Estimations for production ($N_Y = 120$)

$$b_3 = 0.0701327, \; x_3^0 = 6.8962901, \; RSS_3 = 1028.567, \tag{5.2.9}$$

where RSS_i, $i = 1, 2, 3$ is the residual sum of squares used to indicate the goodness-of-fit of a model. A small RSS indicates a good fit of the model to the data.

Note the values of b_i, $i = 1, 2, 3$, are not precisely consistent with the inequality (4.5.10). This implies that the condition $\alpha + \beta = 1$ is not suitable for the fitting of the new production function to the dataset used in Chapter 3, or the fitting restricted by $\alpha + \beta = 1$ does not generate the best estimation. Indeed, we have $RSS = 4336.974$ when we assume $\alpha + \beta = 1$ while $RSS = 1447.294$ without the condition. If we drop the condition, then the best fitting given by the R programming language is $C = 0.3549321$, $\alpha = 1.1882808$, $\beta = -0.4668962$, which matches our argument in some sense. Hence, we conclude that $\alpha + \beta = 1$ is not necessary for the new production function with the dataset used in Chapter 3 although we can still obtain a relatively good fit assuming $\alpha + \beta = 1$ using the R programming language.

Remark 5.2.1. *We also determine the goodness-of-fit using the adjusted R-values. The* $RSS = \sum(y - \hat{y})^2$*, where* y *is the observed value while* $\hat{y}$ *is the estimated value, is a good indicator, but the disadvantage is that it also depends on the measure of the data. For example, the* RSS *of a dataset consisting of values between* $(0, 10)$ *can be much smaller than the value of a dataset consisting of values between* $(1000, 10000)$*. Within the same dataset, a small* RSS *definitely means a good fit.*

Remark 5.2.2. *One can check that* $b_1 + b_3 \neq b_2$ *in this case, which implies that the corresponding Poisson structure is not in the standard form.*

Then the linear condition for this model becomes $\alpha b_3 + \beta b_2 - b_1 = 0$. Using the results $\alpha = 1.1882808$, $\beta = -0.4668962$, we can see that the linear condition holds since

$$\begin{aligned} -0.4668962 \cdot 0.09244029 + 1.1882808 \cdot 0.1037214 - 0.0701327 &= 0.00995742804 \\ &\approx 0. \end{aligned} \tag{5.2.10}$$

However, the results $C = 0.3549321$, $\alpha = 1.1882808$, $\beta = -0.4668962$ are not satisfactory since we require positive parameters. We promote the approach in what follows.

5.3 A new algorithm for the fitting of a production function to empirical data

Next, we modify our approach by treating all N_i, x_i^0 and b_i as predictors in the logistic model. Using R, we obtain the following values of regression coefficients using the method of least squares

- Estimations for labor

$$b_1 = 0.07842367, \; x_1^0 = 2.092004, \; N_1 = 175.97, \; RSS_4 = 508.0948. \tag{5.3.1}$$

- Estimations for capital

$$b_2 = 0.07793777,\ x_2^0 = 1.575667,\ N_2 = 230.26,\ RSS_5 = 299.7033; \quad (5.3.2)$$

- Estimations for production

$$b_3 = 0.04619786,\ x_3^0 = 11.312991,\ N_3 = 211.30,\ RSS_6 = 419.7767. \quad (5.3.3)$$

In contrast to RSS_i, $i = 1, 2, 3$ the new fitting yields much better results (see plot of the three time series in Figures 5.3, 5.4 and 5.5).

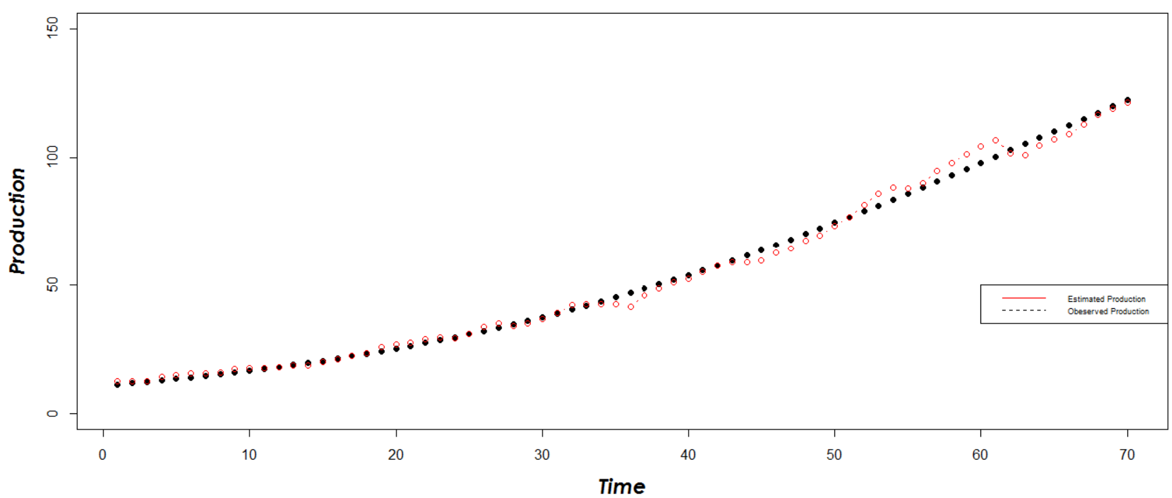


Figure 5.3: Time series of observed and estimated production from 1947 to 2016.

Choosing the new carrying capacities, we compare the new production function against the data used in Chapter 3 without assuming $\alpha + \beta = 1$ to obtain

$$\alpha = 0.46780229,\ \beta = 0.05955408,\ C = 1.59899336,\ RSS = 428.27. \quad (5.3.4)$$

In contrast to $RSS = 4336.975$ in the model fitting in Chapter 3, the new approach yields a better result. More importantly, we no longer need to assume the value of the carrying capacities in the new production function. The values of carrying capacities are obtained through the statistical analysis of the time series of capital, labor and

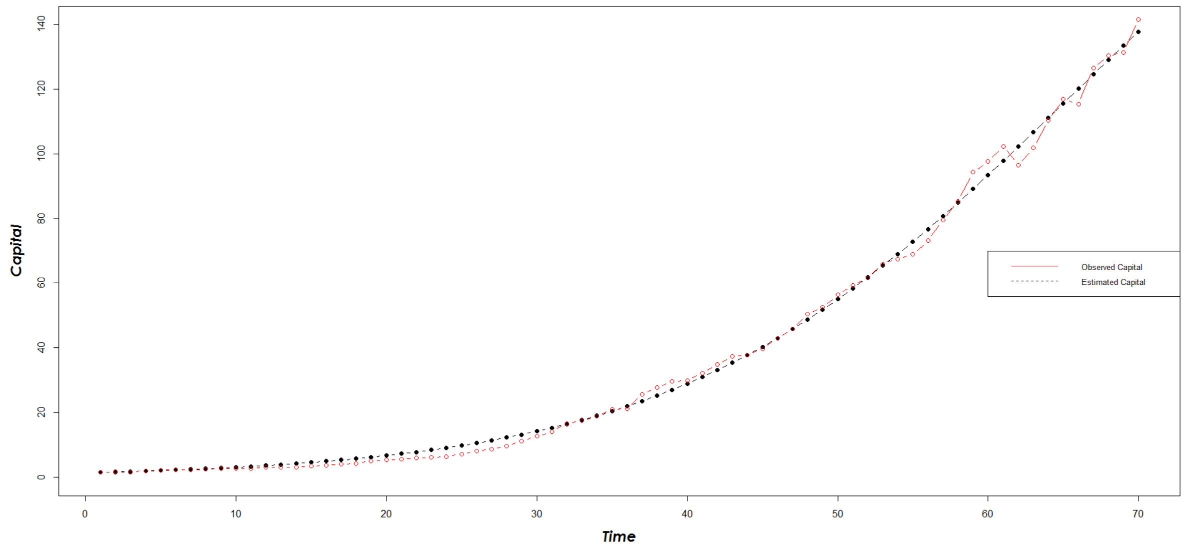


Figure 5.4: Time series of observed and estimated capital from 1947 to 2016.

production using the new approach.

Note this supports the linear condition (5.1.14), since

$$\begin{aligned} 0.46780229 \cdot 0.07842367 + 0.05955408 \cdot \quad & 0.07793777 - 0.04619786 \\ & = -0.00486957539 \approx 0, \end{aligned} \tag{5.3.5}$$

which is closer to 0.

On the other hand, we also compare the Cobb-Douglas function against the data without $\alpha + \beta = 1$. We start by fitting the exponential growth model in logarithmic form

$$\ln y_i = b_i \ln x_i + c_i \tag{5.3.6}$$

to the data and arrive at the following values

- Labor

$$b_1 = 0.06983731, \ c_1 = 0.45741448, \tag{5.3.7}$$

- Capital

$$b_2 = 0.065705809, \ c_2 = 0.75835155, \tag{5.3.8}$$

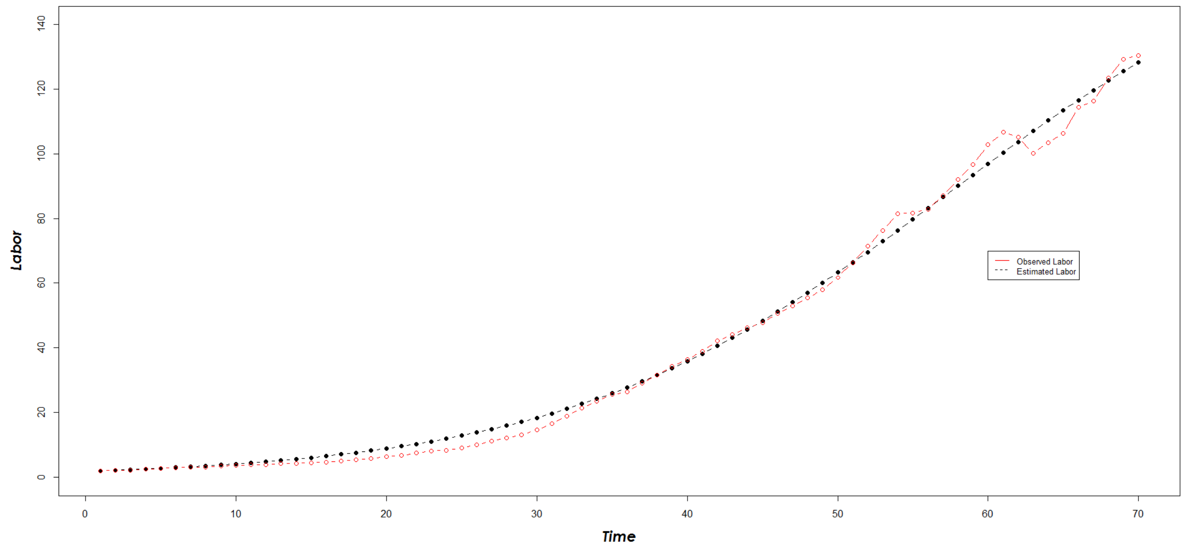


Figure 5.5: Time series of observed and estimated labor from 1947 to 2016.

- Production

$$b_3 = 0.03421333, \;\; c_3 = 2.58402362. \tag{5.3.9}$$

Remark 5.3.1. *Note we no longer require the inequality* (4.5.10).

We also compute the values of RSS for each fitting in the form of $x_i = x_i^0 e^{b_i t}$, $i = 1, 2, 3$, which are

- Labor $RSS_7 = 18421.53$;
- Capital $RSS_8 = 13566.78$;
- Production $RSS_9 = 1991.283$.

The above results demonstrate that the value of RSS is not preserved by the transformation as stated in Section 2.6. The RSS for (5.3.6) is significantly small, which shows it is a good fitting. However, the value of RSS in the original form without the logarithmic transformation are very large. Considering the magnitude of number in the dataset and the insignificant fitting of the exponential model to the data during the last period, the values are expected to be reasonably large.

The new fitting for Cobb-Douglas function to the data of the period 1947-2016 without $\alpha + \beta = 1$ gives

$$\alpha = 0.05018686,\ \beta = 0.45529695,\ A = 9.89921606,\ RSS = 584.4616, \tag{5.3.10}$$

which also supports the linear condition.

We note that the RSS for the Cobb-Douglas function is 584.4616 while the RSS for the new production function is 428.27, which shows that the production function is a better estimation for the data from 1947-2016. We believe that this is due to the fact that the logistic model better describes the time series of capital, labor and production.

Remark 5.3.2. *We do not use the adjusted R-values as indicators in the contrast since the values for two cases do not differ significantly, namely, the value for the new production function is* 0.9945 *while the one for the Cobb-Douglas function is* 0.9926*.*

Therefore, we have formally formulated an algorithm of fitting the Cobb-Douglas or the new production function to given empirical economic data. First, we compare a growth model to the given time series of factors and production. Next, given an exponential growth in factors and production, we fit the Cobb-Douglas function to the data. For the logistic growth, we can obtain the values of carrying capacity in the first step. Then, we use the new production function. In the last step, we check that all obtained values b_i, $i = 1, 2, 3$, α and β satisfy the linear condition (5.1.14) or (5.2.5). One can see that the best fitting is obtained along the approach.

5.4 Concluding remarks

In this section we have extended Sato's Lie theoretical approach to the derivation of a production function. In particular, we have shown that a production function is an invariant of a Lie one-parameter group characterizing an economic model under the condition (5.1.14). We reasonably rigorously showed the validity of a production function from a mathematical perspective. More specifically, we have shown that production, capital and labor in economy are related via the production function iff growth rates in each quantity and elasticity of substitution satisfy a certain condition. However, conditions (5.1.14) and (5.2.5) also prove that the imposed condition $\alpha+\beta = 1$ is not essential for a production function. We have composed an algorithm of fitting a production function to empirical data. In fact, the fitting can be expressed by an optimization problem of the values of the elasticity of substitution α and β for the minimized RSS restricted by the corresponding linear condition. Taking the Cobb-Douglas production function $y = Ax_1^{\beta}x_2^{\alpha}$, $\alpha, \beta > 0$ for example, we want to

$$\text{minimize} \quad RSS(\alpha, \beta, \tilde{A}) = \sum_{i=1}^{n}(y_i - (\tilde{A} + \beta \ln x_i^1 + \alpha \ln x_i^2))^2, \tag{5.4.1}$$

where $\tilde{A} = \ln A$ and x^i denote x_i, $i = 1, 2$,

subject to

$$b_1\beta + b_2\alpha - b_3 = 0. \tag{5.4.2}$$

Using techniques of the Lagrange multipliers, we want to minimize the following function

$$\mathcal{L} = RSS(\alpha, \beta, \tilde{A}) - \lambda(b_1\beta + b_2\alpha - b_3), \tag{5.4.3}$$

where λ is a constant.

For brevity, let us use the following notations:

$$X_1 = \sum_{i=1}^{n} \ln x_i^1,\ X_2 = \sum_{i=1}^{n} \ln x_i^2,\ X_3 = \sum_{i=1}^{n}(\ln x_i^1)^2,\ X_4 = \sum_{i=1}^{n}(\ln x_i^2)^2, \tag{5.4.4}$$

$$Y_1 = \sum_{i=1}^{n} y_i,\ Y_2 = \sum_{i=1}^{n} y_i^2, \tag{5.4.5}$$

$$S_1 = \sum_{i=1}^{n} y_i \ln x_i^1,\ S_2 = \sum_{i=1}^{n} y_i \ln x_i^2,\ S_3 = \sum_{i=1}^{n}(\ln x_i^1)(\ln x_i^2). \tag{5.4.6}$$

Hence, given a specific dataset, the best fitting is given by

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \alpha} &= 2X_3\alpha + S_3\beta + 2X_1\tilde{A} - 2S_1 - \lambda b_2 = 0, \\
\frac{\partial \mathcal{L}}{\partial \beta} &= S_3\alpha + 2X_4\beta + 2X_2\tilde{A} - 2S_2 - \lambda b_1 = 0, \\
\frac{\partial \mathcal{L}}{\partial \tilde{A}} &= 2X_1\alpha + 2X_2\beta + 2n\tilde{A} - 2Y_1 = 0, \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= b_1\beta + b_2\alpha - b_3 = 0.
\end{aligned} \tag{5.4.7}$$

But we need to note that the parameters derived with the aid of the above approach may not minimize the value of RSS of the original form.

Analogously, we can formulate the optimal fitting problem for the new production function f_5 using the Gauss-Newton method. Note that we conduct all regression in the section using the R programming language, which, as discussed above, yields the best fitting in each case.

Chapter 6

Dynamics of the four-dimensional economics growth model involving debt

6.1 A four-dimensional Lotka-Volterra model

Continuing the discussions of Section 4.3, let us consider a four-dimensional Lotka-Volterra model given by

$$\begin{aligned} \dot{x}_1 &= x_1(b_1 + a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4), \\ \dot{x}_2 &= x_2(b_2 + a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4), \\ \dot{x}_3 &= x_3(b_3 + a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4), \\ \dot{x}_4 &= x_4(b_3 + a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4), \end{aligned} \tag{6.1.1}$$

where b_i, a_{ij} $(i, j = 1, 2, 3, 4)$ are arbitrary parameters.

We have investigated special cases of the four dimensional model (4.3.2) and (4.3.9) in Section 4.3. In this thesis, we will consider a new model based on (4.3.9) and impose the additional conditions of $a_{11} = -\frac{b_1}{N_1}$ and $a_{22} = -\frac{b_2}{N_2}$, that is,

$$\begin{aligned} \dot{x}_1 &= x_1(b_1 - \tfrac{b_1}{N_1}x_1 - a_{12}x_2), \\ \dot{x}_2 &= x_2(b_2 - \tfrac{b_2}{N_2}x_2 - a_{21}x_1), \\ \dot{x}_3 &= x_3(b_3 - \tfrac{b_3}{N_3}x_3), \\ \dot{x}_4 &= x_4(b_3 - \tfrac{b_4}{N_4}x_4), \end{aligned} \tag{6.1.2}$$

where $b_i > 0$, $N_i > 0$, $a_{12} > 0$ and $a_{21} > 0$ $(i = 1, 2, 3, 4)$.

The variables x_i $(i = 1, 2)$ represent capital, debt, production, and labor, respectively. It is natural to restrict the analysis of the equation to the space $\mathbb{R}^4_+$. By analogy with the analysis to a two-dimensional model in [57], we observe the following five solutions to the system (6.1.2):

(1) $x_1(t) = x_2(t) = x_3(t) = x_4(t) = 0$,

(2) $x_1(t) = \frac{N_1C_1}{C_1+(N_1-C_1)e^{-b_1t}}$ $(C_1 > 0)$, $x_2(t) = 0$, $x_3(t) = 0$ and $x_4(t) = 0$,

(3) $x_1(t) = 0$, $x_2(t) = \frac{N_2C_2}{C_2+(N_2-C_2)e^{-b_2t}}$ $(C_2 > 0)$, $x_3(t) = 0$ and $x_4(t) = 0$,

(4) $x_1(t) = 0$, $x_2(t) = 0$, $x_3(t) = \frac{N_3C_3}{C_3+(N_3-C_3)e^{-b_3t}}$ $(C_3 > 0)$ and $x_4(t) = 0$,

(5) $x_1(t) = 0$, $x_2(t) = 0$, $x_3(t) = 0$ and $x_4(t) = \frac{N_4C_4}{C_4+(N_4-C_4)e^{-b_4t}}$ $(C_4 > 0)$,

which correspond to five orbits: (1) the origin, which is an equilibrium,(2) the positive x_1-axis, (3) the positive x_2-axis, (4) the positive x_3-axis, (5) the positive x_4-axis. Together the five orbits form the boundary of the space $\mathbb{R}^4_+$.

The set is invariant in the sense that any solution which starts in it remains there for all time for which it is defined. Indeed, we can see the boundary of $\mathbb{R}^4_+$ is invariant. Since orbits can not cross, the interior

$$\text{int}\mathbb{R}^4_+ = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 > 0,\ x_2 > 0,\ x_3 > 0,\ x_4 > 0\right\}. \tag{6.1.3}$$

is also invariant.

We can thus restrict our analysis to the interior of the space denoted by $\text{int}\mathbb{R}^4_+$.

6.2 A Hamiltonian system for the four-dimensional Lotka-Volterra model

The dynamics of a two-dimensional Lotka-Volterra model has been completely studied in [57]. In conclusion, the two-dimensional model admits no isolated periodic orbit. The algebraic property of a two-dimensional Lotka-Volterra model will be preserved in a higher dimensional model.

Note that the system (6.1.2) is a separable system, *i.e.*, the vector field $V = V_1(x_1, x_2)+ V_2(x_3, x_4)$, which makes the system (6.1.2) separable. Note V_1 is determined by the first two equations while V_2 corresponds to the last two equations.

The Hamiltonian system of a two-dimensional Lotka-Volterra model has been extensively investigated by Plank [95] and Kerner [66], where Plank introduced a Poisson bivector showing that any two-dimensional Lotka-Volterra system can be viewed as a Hamiltonian system and Kerner introduced a transformation for a special case of Lotka-Volterra system and reviewed the Lie-Koenig's theorem that states any dynamical system can be locally redefined as a Hamiltonian system.

We can employ Plank's approach to study the Hamiltonian structure of the first two equations of the system (6.1.2). Let us introduce the Poisson bivector π_1 given by

$$\pi_1 = -x_1^{1-\ell_1} x_2^{1-\ell_2} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \quad i,j = 1,2. \tag{6.2.1}$$

Then, the first two equations form the Hamiltonian system given by

$$\dot{x}_i = \pi_1^{i\ell} \frac{\partial H_1}{\partial x_\ell}, \ i = 1,2 \tag{6.2.2}$$

associated with the following Hamiltonian function

$$\begin{aligned} H_1(x_1,x_2) &= \int -x_1^{\ell_1} x_2^{\ell_2 - 1} \left(b_1 - \frac{b_1}{N_1} x_1 - a_{12} x_2 \right) dx_2 \\ &= \int x_1^{\ell_1 - 1} x_2^{\ell_2} \left(b_2 - \frac{b_2}{N_2} x_2 - a_{21} x_1 \right) dx_1 \end{aligned}. \tag{6.2.3}$$

As stated, all coefficients in the system (6.1.2) are positive. Let us assume $\ell_1, \ell_2 \neq 0, -1$. Integrating, we get

$$H_1(x_1,x_2) = x^{\ell_1} x^{\ell_2} \left(-\frac{b_1}{l_2} + \frac{b_1}{\ell_2 N_1} x_1 + \frac{a_{12}}{\ell_2 + 1} x_2 \right), \tag{6.2.4}$$

or

$$H_1(x_1,x_2) = x^{\ell_1} x^{\ell_2} \left(\frac{b_2}{\ell_1} - \frac{b_2}{\ell_1 N_2} x_2 - \frac{a_{21}}{\ell_1 + 1} x_1 \right). \tag{6.2.5}$$

Note the above two forms are equivalent under the following conditions

$$b_1 \ell_1 + b_2 \ell_2 = 0, \quad b_1(\ell_1 + 1) + a_{21} N_1 \ell_2 = 0, \quad b_2(\ell_2 + 1) + a_{12} N_2 \ell_1 = 0. \tag{6.2.6}$$

The last two equations define the following Hamiltonian system given by

$$\dot{x}_i = \pi_2^{i\ell} \frac{\partial H_2}{\partial x_\ell}, \ \ i = 3, 4, \tag{6.2.7}$$

defined by the following Poisson bivector

$$\pi_2 = -b_3 b_4 x_3 x_4 \left(1 - \frac{x_3}{N_3}\right)\left(1 - \frac{x_4}{N_4}\right) \tag{6.2.8}$$

and the Hamiltonian function H_2 given by

$$H_2(x_3, x_4) = \frac{1}{b_3} \ln \frac{x_3}{|N_3 - x_3|} - \frac{1}{b_4} \ln \frac{x_4}{|N_4 - x_4|}. \tag{6.2.9}$$

Therefore, the system (6.1.2) is a Hamiltonian system

$$\dot{x}_i = \pi^{il} \frac{\partial H}{\partial x_l}, \ \ i = 1, 2, 3, 4, \tag{6.2.10}$$

where π is the Poisson bivector determined by the components

$$\pi^{ij} = \begin{cases} -x_1^{1-\ell_1} x_2^{1-\ell_2}, & i, j = 1, 2, \\ -b_3 b_4 x_3 x_4 \left(1 - \dfrac{x_3}{N_3}\right)\left(1 - \dfrac{x_4}{N_4}\right), & i, j = 3, 4, \end{cases} \tag{6.2.11}$$

and the Hamiltonian function H is given by

$$\begin{aligned} H(x_1, x_2, x_3, x_4) = &\ x^{\ell_1} x^{\ell_2} \left(\frac{b_2}{\ell_1} - \frac{b_2}{\ell_1 N_2} x_2 - \frac{a_{21}}{\ell_1 + 1} x_1\right) \\ &+ \frac{1}{b_3} \ln \frac{x_3}{|N_3 - x_3|} - \frac{1}{b_4} \ln \frac{x_4}{|N_4 - x_4|} \end{aligned} \tag{6.2.12}$$

under the conditions

$$\begin{gathered} \ell_1, \ell_2 \neq 0, -1, \quad b_1 \ell_1 + b_2 \ell_2 = 0, \\ b_1(\ell_1 + 1) + a_{21} N_1 \ell_2 = 0, \quad b_2(\ell_2 + 1) + a_{12} N_2 \ell_1 = 0. \end{gathered} \tag{6.2.13}$$

The Hamiltonian function (6.2.12) yields, when we identify production $x_3 = f$, capital $x_1 = K$, debt $x_2 = D$ and labor $x_4 = L$, a new production function of the following form

$$Y = f(K, L, D) = \frac{N_f e^{b_3 G(K,D,L)}}{1 + e^{b_3 G(K,D,L)}}, \tag{6.2.14}$$

where the function G is given by

$$G = C + K^{\ell_1} D^{\ell_2} \left(-\frac{b_2}{\ell_1} + \frac{b_2}{\ell_1 N_D} D + \frac{a_{21}}{\ell_1 + 1} K \right) + \frac{1}{b_4} \ln \frac{L}{|N_L - L|}. \tag{6.2.15}$$

We will analyze the system (6.1.2) in what follows.

6.3 Divergence and a volume form

Let us introduce the notion of divergence into the growth model, in which we want to use the idea of the divergence and volume to describe the expansion and contraction of an economy. Mathematically speaking, we want to characterize the status of an economy with the volume of the corresponding economics model.

The following vector field gives rise to the system (6.1.2):

$$\begin{aligned} V = \quad & x_1 \left(b_1 - \tfrac{b_1}{N_1} x_1 - a_{12} x_2 \right) \tfrac{\partial}{\partial x_1} + x_2 \left(b_2 - \tfrac{b_2}{N_2} x_2 - a_{21} x_1 \right) \tfrac{\partial}{\partial x_2}, \\ & + x_3 \left(b_3 - \tfrac{b_3}{N_3} x_3 \right) \tfrac{\partial}{\partial x_3} + x_4 \left(b_4 - \tfrac{b_4}{N_4} x_4 \right) \tfrac{\partial}{\partial x_4}. \end{aligned} \tag{6.3.1}$$

Then, the divergence of the vector field V is given by

$$\begin{aligned} \operatorname{div}(V) \quad &= \nabla \cdot V \\ &= \left(\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_2}, \tfrac{\partial}{\partial x_3}, \tfrac{\partial}{\partial x_4} \right) \cdot \left(x_1 \left(b_1 - \tfrac{b_1}{N_1} x_1 - a_{12} x_2 \right), x_2 \left(b_2 - \tfrac{b_2}{N_2} x_2 - a_{21} x_1 \right), \right. \\ &\quad \left. x_3 \left(b_3 - \tfrac{b_3}{N_3} x_3 \right), x_4 \left(b_4 - \tfrac{b_4}{N_4} x_4 \right) \right) \\ &= b_1 - 2\tfrac{b_1}{N_1} x_1 - a_{12} x_2 + b_2 - 2\tfrac{b_2}{N_2} x_2 - a_{21} x_1 + b_3 - 2\tfrac{b_3}{N_3} x_3 + b_4 - 2\tfrac{b_4}{N_4} x_4 \\ &= (b_1 + b_2 + b_3 + b_4) - \left(2\tfrac{b_1}{N_1} + a_{21} \right) x_1 - \left(2\tfrac{b_2}{N_2} + a_{12} \right) x_2 - 2\tfrac{b_3}{N_3} x_3 - 2\tfrac{b_4}{N_4} x_4. \end{aligned} \tag{6.3.2}$$

Note $\operatorname{div}(V)$ does not equal to 0 everywhere for $\mathbf{x} \in \operatorname{int}\mathbb{R}^{\mathbf{4}}$. For example, when $x_i = \frac{N_i}{3}$, $i = 1 \ldots 4$, respectively, the divergence becomes

$$\operatorname{div}(V) = \frac{1}{3} (b_1 + b_2 + b_3 + b_4) - \frac{1}{3} (N_1 a_{21} + N_2 a_{12}). \tag{6.3.3}$$

More importantly, we want to explore the dynamics of an economy using the notion of the divergence, *i.e.*, the economy characterized by the equation (6.1.2) shrinks when $\mathrm{div}(V) < 0$ and expands when $\mathrm{div}(V) > 0$.

The equation (6.1.2) is associated with the following differential four-form

$$\Omega = (dx_1 - X_1 dt) \wedge (dx_2 - X_2 dt) \wedge (dx_3 - X_3 dt) \wedge (dx_4 - X_4 dt)\,, \tag{6.3.4}$$

where X_i, $i = 1, 2, 3, 4$ denotes the RHS of equations in the system (6.1.2).

Rewriting the four-form (6.3.4), we obtain

$$\begin{aligned} \Omega = \;& dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 - X_1 dt \wedge dx_2 \wedge dx_3 \wedge dx_4 + X_2 dt \wedge dx_1 \wedge dx_3 \wedge dx_4 \\ & - X_3 dt \wedge dx_1 \wedge dx_2 \wedge dx_4 + X_4 dt \wedge dx_1 \wedge dx_2 \wedge dx_3, \end{aligned} \tag{6.3.5}$$

the exterior derivative of which yields

$$\begin{aligned} d\Omega &= -\frac{\partial X_1}{\partial x_1} dx_1 \wedge dt \wedge dx_2 \wedge dx_3 \wedge dx_4 + \frac{\partial X_2}{\partial x_2} dx_2 \wedge dt \wedge dx_1 \wedge dx_3 \wedge dx_4 \\ & -\frac{\partial X_3}{\partial x_3} dx_3 \wedge dt \wedge dx_1 \wedge dx_2 \wedge dx_4 + \frac{\partial X_4}{\partial x_4} dx_4 \wedge dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \\ &= \left(\sum_{i=1}^{4} \frac{\partial X_i}{\partial x_i} \right) dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \\ &= \mathrm{div}(V) dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4. \end{aligned} \tag{6.3.6}$$

If follows from above that $d\Omega$ is not always zero. Hence, Ω is not a closed form on $\mathrm{int}\mathbb{R}^4$, which corresponds to the non-canonical Poisson bivector derived in Section 6.2.

6.4 Analysis of a four-dimensional Lotka-Volterra model

We have restricted our attention to the set $\text{int}\mathbb{R}^4_+$. Next, we, following similar procedures in [56], reduce the equations (6.1.2) to the following system

$$\begin{aligned} \frac{\dot{x}_1}{x_1} &= b_1 - \tfrac{b_1}{N_1}x_1 - a_{12}x_2, \\ \frac{\dot{x}_2}{x_2} &= b_2 - \tfrac{b_2}{N_2}x_2 - a_{21}x_1, \\ \frac{\dot{x}_3}{x_3} &= b_3 - \tfrac{b_3}{N_3}x_3, \\ \frac{\dot{x}_4}{x_4} &= b_3 - \tfrac{b_4}{N_4}x_4, \end{aligned} \tag{6.4.1}$$

The four null-lines defined by Eq. (6.1.2) are then given by

$$\begin{aligned} & b_1 - \tfrac{b_1}{N_1}x_1 - a_{12}x_2 = 0, \\ & b_2 - \tfrac{b_2}{N_2}x_2 - a_{21}x_1 = 0, \\ & b_3 - \tfrac{b_3}{N_3}x_3 = 0, \\ & b_3 - \tfrac{b_4}{N_4}x_4 = 0. \end{aligned} \tag{6.4.2}$$

Suppose that the first two null-lines are nonparallel

$$\frac{b_1 b_2}{N_1 N_2} - a_{12}a_{21} \neq 0, \tag{6.4.3}$$

then the interior equilibrium $\mathbf{p} = (p_1, p_2, p_3, p_4)$ is determined by the intersection of the first two null lines and the zeros of the last two null lines, that is,

$$p_1 = \frac{-N_1 b_1 b_2 - N_1 N_2 b_2 a_{12}}{b_1 b_2 - N_1 N_2 a_{12} a_{21}},\ p_2 = \frac{-N_2 b_1 b_2 - N_1 N_2 b_1 a_{21}}{b_1 b_2 - N_1 N_2 a_{12} a_{21}},\ p_3 = N_3,\ p_4 = N_4. \tag{6.4.4}$$

Note that we require the equilibrium lives in the region $\text{int}\mathbb{R}^4_+$, which entails that $p_i > 0$, $i = 1, 2, 3, 4$. Hence, we obtain the following condition from $p_1, p_2 > 0$

$$\frac{b_1 b_2}{N_1 N_2} - a_{12}a_{21} < 0. \tag{6.4.5}$$

The Jacobian matrix of the system (6.1.2) is given by

$$DJ = \begin{bmatrix} b_1 - \frac{2b_1}{N_1}x_1 - a_{12}x_2 & -a_{12}x_1 & 0 & 0 \\ -a_{21}x_2 & b_2 - \frac{2b_2}{N_2}x_2 - a_{21}x_1 & 0 & 0 \\ 0 & 0 & b_3 - \frac{2b_3}{N_3}x_3 & 0 \\ 0 & 0 & 0 & b_4 - \frac{2b_4}{N_4}x_4 \end{bmatrix}. \qquad (6.4.6)$$

Linearising the equation (6.1.2) around $\mathbf{p}$, we obtain the matrix

$$M = DJ|_{\mathbf{p}} = \begin{bmatrix} -\frac{b_1}{N_1}p_1 & -a_{12}p_1 & 0 & 0 \\ -a_{21}p_2 & -\frac{b_2}{N_2}p_2 & 0 & 0 \\ 0 & 0 & -b_3 & 0 \\ 0 & 0 & 0 & -b_4 \end{bmatrix}, \qquad (6.4.7)$$

which is a direct sum of the following matrices $M = M_1 \oplus M_2$, where $M_1 = \begin{bmatrix} -\frac{b_1}{N_1}p_1 & -a_{12}p_1 \\ -a_{21}p_2 & -\frac{b_2}{N_2}p_2 \end{bmatrix}$ and $M_2 = \begin{bmatrix} -b_3 & 0 \\ 0 & -b_4 \end{bmatrix}$.

Hence, the eigenvalues of M are given by the eigenvalues of M_1 and M_2, that is,

$$\lambda_{1,2} = \frac{\mathrm{Tr}(M_1) \pm \sqrt{(\mathrm{Tr}(M_1))^2 - 4\mathrm{Det}(M_1)}}{2}, \quad \lambda_3 = -b_3, \quad \lambda_4 = -b_4. \qquad (6.4.8)$$

where $\mathrm{Det}(M_1) = p_1p_2\left(\frac{b_1b_2}{N_1N_2} - a_{12}a_{21}\right)$, $\mathrm{Tr}(M_1) = -\frac{b_1}{N_1}p_1 - \frac{b_2}{N_2}p_2$.

Note $\mathrm{Det}(M_1) < 0$ and $\mathrm{Tr}(M_1) < 0$, then $\lambda_1\lambda_2 < 0$. Therefore, the equilibrium $\mathbf{p}$ is unstable.

6.5 Concluding remarks

Let us express the equation (6.2.14) in the following form

$$Y = f(K, D, L) = \frac{N_f L^{\frac{1}{b_4}}}{L^{\frac{1}{4}} + C(N_L - L)^{\frac{1}{b_4}} G_1(K, D)}, \qquad (6.5.1)$$

where $C \in \mathbb{R}_+$ and $G_1(K, D) = \exp\left(\frac{b_2}{l_1} K^{l_1} D^{l_2} \left(\frac{D}{N_D} + \frac{K}{N_K} - 1\right)\right)$.

Let us check that the function Y is well-defined from the economic point of view, namely, we first need to check that the function satisfies the following three conditions and then analyze the properties of the function.

1. *The function is not monotone decreasing.*

   It is sufficient to state that the function is not monotone decreasing. Let us choose the direction where K and D are constant. It can be shown that Y becomes the type of the function f_6 in Section 3.3. It follows from the shape of f_6, which has been analyzed in Section 3.3, that the production function Y is not monotone decreasing.

2. *The boundedness.*

   Let us assume the domain of L is $S_L = [0, N_L]$. Including N_L can be viewed as an analytic continuation since the function Y is well-defined at the point N_L. We can see that $[0, N_L]$ is a bounded interval. The domain of K and L is also a bounded set. Let us denote the domains of K and L by S_K and S_L, respectively. It follows that $S_K \times S_L \times S_D$ is a bounded set. We can conclude from the continuity of Y on the given domain, using the bounded value theorem, that Y is bounded, where the continuation preserves the boundedness from the domain to the image.

3. *The function has an absolute maximum.*

   It follows from the boundedness that Y attains an absolute maximum N_L at the surface $L = N_L$.

The function Y is of the type of the function f_6 in Section 3.3. Let us focus on the following function

$$G_1(K, D) = \exp\left(\frac{b_2}{\ell_1} K^{\ell_1} D^{\ell_2} \left(\frac{D}{N_D} + \frac{K}{N_K} - 1\right)\right). \tag{6.5.2}$$

We know that

$$-1 < \frac{D}{N_D} + \frac{K}{N_K} - 1 < 1. \tag{6.5.3}$$

Assuming $\ell_1, \ell_2 > 0$, we know that $\mathbf{p}$ is an unstable equilibrium, which implies that either capital or debt dies out in the competition. In the end behaviour ($K \to N_K,\ D \to 0$ or $K \to 0,\ D \to N_D$), $\frac{D}{N_D} + \frac{K}{N_K} - 1$ approaches zero. The function $G_1(K, D)$ is close to 1 and the production function (6.5.1), roughly speaking, becomes the function f_6. Economically speaking, capital and debt do not contribute to production. For example, in the aftermath of the European debt crisis of 2010, new fiscal policies, including increasing the government spending, issuing new treasure bills and bonds, *etc.*, did not stimulate the economy in the countries already with a large amount of debt. From the perspective of our model, it is clear that more debt can not stimulate production since $D \to N_D$.

In Chapter 6 we have briefly investigated the stability of the four-dimensional economic growth model. The chapter is necessarily incomplete. We have realized that, since the dimension of the model is greater than 2, the model potentially has interesting properties, *e.g.,* chaotic behaviour, the Hamiltonian may lie around a potential attractor, *etc.* In future research we plan to employ a more detailed numerical analysis of the model.

Chapter 7

Optimization problems

We investigate variational problems involving the new production function f_5 (3.3.14) in this chapter, which is a natural next step in the study of the function (3.3.14). Let us start by reviewing the Ramsey model. Recall the income identity in macroeconomics is given by

$$Y = C + I + G + X - M, \tag{7.0.1}$$

where Y is the gross production, C is consumption, I represents investment, G is government spending, X and M are export and import, respectively.

The Ramsey-Cass-Koopmans model [109, 99, 25, 70, 93] studies the optimal savings in one country, in which they aim to find the maximum of social welfare under a certain consumption level, *i.e.*, an optimum problem of the following functional

$$J(k) = \int_0^\infty e^{-\rho t} u(c(t)) dt, \tag{7.0.2}$$

where we denote $F(c, k, t) = e^{-\rho t} u(c(t))$ and $\rho > 0$ represents the discount rate reflecting time preference. The time preference [43] refers to the current relative valuation placed on receiving a good at an earlier date and receiving it at a later date. Agents in an economy with high time preference emphasize substantially on their welfare at present and in the immediate future while those with low time preference place their focus in the distant future.

The authors restrict the problem of the social welfare in a closed economy, namely, such a problem does not involve import and export, hence, the income identity (7.0.1), ignoring the government spending, becomes

$$Y = C + I, \tag{7.0.3}$$

and they assume that there is no capital depreciation such that

$$\dot{K} = I. \tag{7.0.4}$$

Therefore, we arrive at the following dynamical system involving capital,

$$Y = C + \dot{K}. \tag{7.0.5}$$

They postulate that the general production Y is a homogeneous function $Y = f(K, L)$. By introducing the projective coordinates, $y = \frac{Y}{L}$, $k = \frac{K}{L}$ and $c = \frac{C}{L}$, the equation (7.0.5) becomes

$$y = f(k) = c + \dot{k} + nk, \tag{7.0.6}$$

where $\dot{k} = \frac{\dot{K}}{L} - k\frac{\dot{L}}{L}$ represents the accumulation of capital per labor and $n = \frac{\dot{L}}{L}$ is the growth rate of population and assumed constant.

Note, it follows from the equation (7.0.6) that the consumption c is also a function of k, that is, $c(k(t)) = f(k) - \dot{k} - nk$. We still write $c(k(t))$ as $c(t)$ for our convenience in the following context.

Then the variational problem of the optimal social welfare (7.0.2) can be presented as follows

$$J(k) = \int_0^\infty e^{-\rho t} u(c(t))dt, \tag{7.0.7}$$

subject to

$$g(c, k, \dot{k}) = f(k) - c(t) - \dot{k} - nk = 0. \tag{7.0.8}$$

An optimal problem of a functional with finite subsidiary conditions as (7.0.7) can be either solved using Pontryagin's maximum principle, which was developed by Lev Pontryagin [96], *i.e.*, by deriving the Hamiltonian function of optimal control theory from the above functional, details of which can be found in [93], or applying the Euler-Lagrange method. We will illustrate the latter approach in what follows.

According to Theorem 2.4.3 and Remark 2.4.4, since $g_{\dot{k}}$ does not vanish, there exists a function $\lambda(t)$ such that the maximal social welfare is attained along the integral curve of the system

$$\Phi_c - \frac{d}{dt}\Phi_{\dot{c}} = 0 \tag{7.0.9}$$

and

$$\Phi_k - \frac{d}{dt}\Phi_{\dot{k}} = 0, \tag{7.0.10}$$

where $\Phi = F + \lambda g$.

The Ramsey problem can be described in terms of the following functional

$$\int_0^\infty e^{-\rho t} u(c(t)) + \lambda(t) g(c, k, \dot{k}) dt. \tag{7.0.11}$$

Then the corresponding Euler-Lagrange equations are

$$\begin{aligned} &e^{-\rho t}(f'(k) - n)u_c + (f'(k) - n)\lambda - \frac{d}{dt}(-e^{-\rho t}u_c - \lambda) = 0, \\ &e^{-\rho t}u_c - \lambda = 0, \end{aligned} \tag{7.0.12}$$

or, simply,

$$\begin{aligned} &e^{-\rho t}(f'(k) - n)u_c + (f'(k) - n)\lambda - (\rho e^{-\rho t}u_c - e^{-\rho t}\frac{du_c}{dt} - \dot{\lambda}) = 0, \\ &e^{-\rho t}u_c - \lambda = 0. \end{aligned} \tag{7.0.13}$$

It follows, $\lambda = e^{-\rho t}u_c$, differentiating this equation with respect to t yields

$$\dot{\lambda} = \rho e^{-\rho t}u_c - e^{-\rho t}\frac{du_c}{dt}. \tag{7.0.14}$$

Eliminating λ and $\dot{\lambda}$ in the first equation of the equation (7.0.13), we obtain

$$2e^{-\rho t}(f'(k) - n)u_c - 2\rho e^{-\rho t}u_c + 2e^{-\rho t}\frac{du_c}{dt} = 0, \tag{7.0.15}$$

rearranging which gives us the Ramsey golden rule of accumulation, that is,

$$\frac{du_c}{dt} = (\rho + n - f'(k))u_c. \tag{7.0.16}$$

7.1 A new macroeconomical model

Let us consider the government spending in a closed economy, *i.e.,* we only ignore import and export. Thus, the income identity (7.0.1) becomes

$$Y = C + I + G, \tag{7.1.1}$$

The accumulation of capital $\dot{K}$ is the difference between the total investment and the capital depreciation δK, that is,

$$\dot{K} = I - \delta K. \tag{7.1.2}$$

We can view government spending as debt and taxation in a country (see [1, 41] for more details) and the government budget constraint is given by the following difference equation

$$D_t = (1 + r)D_{t-1} + G_t - T_t, \tag{7.1.3}$$

where r is the interest rate, D_t is the debt at the current time, D_{t-1} is the debt at the previous time, G_t is the government spending at the current time, T_t is the taxation at the current time.

Remark 7.1.1. *Some applied mathematicians and economists write B_t, the budget in the current time, in the government budget constraint. We assume the government is in debt and consider the debt repayment in our model. Thus, we use D_t in the constraint.*

Moving D_{t-1} to the LHS, the equation (7.1.3) assumes the following form

$$\Delta D_t = rD_{t-1} + G_t - T_t, \tag{7.1.4}$$

and expressing D_{t-1} in terms of D_t, we obtain

$$\Delta D_t = \frac{1}{1+r} D_t + \frac{r}{1+r} G_t - \frac{r}{1+r} T_t. \tag{7.1.5}$$

Then, it follows from considering a continuous model of the equation (7.1.5) that the continuous dynamical model of the government budget constraint is as follows:

$$G = \frac{1+r}{r}\dot{D} - \frac{1}{r}D - T(t). \tag{7.1.6}$$

Let us assume the taxation is a linear and homogeneous function of capital K, that is,

$$T(t) = F(rK(t)), \tag{7.1.7}$$

where r is the interest rate.

It follows from equations (7.1.2) and (7.1.6) that the income identity becomes a dynamical system of K and D, that is

$$Y(t) = C(t) + \dot{K}(t) + \delta K(t) + \frac{1+r}{r}\dot{D}(t) - \frac{1}{r}D(t) - F(rK(t)). \tag{7.1.8}$$

The model (7.1.1) is mainly used by neoclassical economists to describe the capital accumulation. The production function $Y = f(K, L)$ is most commonly assumed to be of the Cobb-Douglas type in previous studies by economists and applied mathematicians [109, 1, 41].

Let us recall the new production function

$$Y = f_5(K, L) = \frac{N_{f_5} K^\alpha L^\beta}{C_5 |N_K - K|^\alpha |N_L - L|^\beta + K^\alpha L^\beta}, \tag{7.1.9}$$

where $\alpha > 0$, $\beta > 0$, $f_5(K, L)$ represents production, K is capital, L is labor, $N_{f_5} = N_Y$ is the maximum value of production, N_K and N_L are the steady states of the function K and L respectively, and C_5 is the integrating factor.

We consider an optimization problem of the social welfare with the production function (7.1.9). Let us assume K and L grow logistically. Hence, we aim to find a consumption function $C(t)$ and a debt function $D(t)$ along which the social welfare is maximized, that is, the optimization of the functional

$$J(K, L) \quad = \int_0^\infty e^{-\rho t} U(C(t)) dt, \tag{7.1.10}$$

subject to the constraints

$$C + aK(1 - \frac{K}{N_K}) + \delta K + \frac{1+r}{r}\dot{D} - \frac{1}{r}D - F(rK) - f_5(K, L) = 0, \tag{7.1.11}$$

$$\dot{K} - aK(1 - \frac{K}{N_K}) = 0 \tag{7.1.12}$$

and

$$\dot{L} - bL(1 - \frac{L}{N_L}) = 0, \tag{7.1.13}$$

where a and b are positive constants and N_K and N_L are carrying capacities of K and L, respectively,

with the boundary conditions

$$C(0) = C_0, \quad \lim_{t\to\infty} \lambda_1(t)C(t) = 0, \tag{7.1.14}$$

$$D(0) = D_0, \quad \lim_{t\to\infty} \lambda_1(t)D(t) = 0, \tag{7.1.15}$$

where C_0 and D_0 are constant and the end point condition is called the transversality condition.

Remark 7.1.2. *The utility function $U(C(t))$ in our functional represents the total utility of the society.*

This is also an optimization problem with finite subsidiary conditions. Let us denote

$$\begin{aligned}\Phi = \quad & e^{-\rho t}U(C) + \lambda_1 \left(C + \dot{K} + \delta K + \frac{1+r}{r}\dot{D} - \frac{1}{r}D - F(rK) - f_5(K, L) \right) \\ & + \lambda_2 \left(\dot{K} - aK \left(1 - \frac{K}{N_K} \right) \right) + \lambda_3 \left(\dot{L} - bL \left(1 - \frac{L}{N_L} \right) \right),\end{aligned} \tag{7.1.16}$$

where $\lambda_i = \lambda_i(t)$, $i = 1, 2, 3$, is a function of t.

The Euler-Lagrange equations associated with the functional (7.1.10) are given by

$$\Phi_C = 0, \quad \Phi_K - \frac{d}{dt}\Phi_{\dot{K}} = 0, \quad \Phi_L - \frac{d}{dt}\Phi_{\dot{L}} = 0, \quad \Phi_D - \frac{d}{dt}\Phi_{\dot{D}} = 0, \tag{7.1.17}$$

or,

$$e^{-\rho t}U_C + \lambda_1 = 0, \tag{7.1.18}$$

$$\lambda_1\left(\delta - r\frac{\partial F}{\partial K} - \frac{\partial f_5}{\partial L}\right) + \lambda_2\left(-a + a\frac{2K}{N_K}\right) - \frac{d}{dt}(\lambda_1 + \lambda_2) = 0, \tag{7.1.19}$$

$$\lambda_1\left(-\frac{\partial f_5}{\partial L}\right) + \lambda_3\left(-b + b\frac{2L}{N_L}\right) - \frac{d}{dt}\lambda_3 = 0, \tag{7.1.20}$$

$$\lambda_1(-\frac{1}{r}) - \frac{d}{dt}\left(\frac{1+r}{r}\lambda_1\right) = 0. \tag{7.1.21}$$

The equation (7.1.18) yields

$$\lambda_1 = -e^{-\rho t}U_C, \tag{7.1.22}$$

substituting which into the equation (7.1.21), we obtain

$$e^{-\rho t}U_C + \frac{d}{dt}((1+r)e^{-\rho t}U_C) = 0, \tag{7.1.23}$$

or,

$$(1+r)\left(\frac{dU_C}{dt} - \rho U_C\right) + U_C = 0. \tag{7.1.24}$$

Remark 7.1.3. *The equations* (7.1.19) *and* (7.1.20) *are first-order ODEs of* λ_2 *and* λ_3 *and solving the equations yields the corresponding form of Lagrange multipliers. The existence of solutions in* (7.1.19) *and* (7.1.20) *proves the validity of the optimal problem.*

Remark 7.1.4. *Due to the complicated form of the partial differentiation of* f_5*, we present the calculation in what follows and denote them by corresponding partial derivative notations,*

$$\frac{\partial f_5(K,L)}{\partial K} = \frac{\alpha N_Y N_K C_5 \left|\frac{N_K}{K} - 1\right|^\alpha \left|\frac{N_L}{L} - 1\right|^\beta}{K(N_K - K)\left(C_5\left|\frac{N_K}{K} - 1\right|^\alpha \left|\frac{N_L}{L} - 1\right|^\beta + 1\right)^2}, \tag{7.1.25}$$

$$\frac{\partial f_5(K,L)}{\partial L} = \frac{\beta N_Y N_L C_5 \left|\frac{N_K}{K}-1\right|^{\alpha}\left|\frac{N_L}{L}-1\right|^{\beta}}{L(N_L-L)\left(C_5\left|\frac{N_K}{K}-1\right|^{\alpha}\left|\frac{N_L}{L}-1\right|^{\beta}+1\right)^2}. \tag{7.1.26}$$

We notice that

$$\frac{dU_C}{dt} = \frac{dU_C}{dC}\frac{dC}{dt} = U_{CC}\frac{dC}{dt}, \tag{7.1.27}$$

and the equation (7.1.24) becomes

$$(1+r)U_{CC}\frac{dC}{dt} + (1-\rho(1+r))U_C = 0, \tag{7.1.28}$$

which is a first-order ODE of the consumption C. By identifying the utility function $U(C)$ with a specific function, we can determine the exact form of the consumption function $C = C(t)$ and solve for $D(t)$ explicitly.

Therefore, the optimal social welfare is determined by the following system of differential equations

$$(1+r)U_{CC}\dot{C}(t) + (1-\rho(1+r))U_C = 0, \tag{7.1.29}$$

$$\frac{1+r}{r}\dot{D} = \frac{1}{r}D + F(rK) + f_5(K,L) - C - aK(1-\frac{K}{N_K}) - \delta K, \tag{7.1.30}$$

$$\dot{K} = aK(1-\frac{K}{N_K}), \tag{7.1.31}$$

$$\dot{L} = bL(1-\frac{L}{N_L}). \tag{7.1.32}$$

Several classes of utility functions can be used in our model. Let us illustrate with the constant intertemporal elasticity of substitution utility function, *i.e.*,

$$U(C(t)) = \frac{C(t)^{1-\theta}-1}{1-\theta}, \tag{7.1.33}$$

where $\theta > 0$ and $\dfrac{1}{\theta}$ represents the elasticity. The utility function means that each household is more willing to change the consumption style for a smaller value of θ, and *vice versa*. We notice that

$$\frac{U_{CC}C}{U_C} = \frac{-\theta C^{-\theta-1}C}{C^{-\theta}} = -\theta. \tag{7.1.34}$$

Then the equation (7.1.29) becomes

$$\frac{U_{CC}C}{U_C}\dot{C}(t) = \frac{\rho(1+r)-1}{1+r}C, \tag{7.1.35}$$

or

$$-\theta\dot{C}(t) = \frac{\rho(1+r)-1}{1+r}C. \tag{7.1.36}$$

Hence, the consumption function is

$$C(t) = C_0 e^{\left(\frac{1}{\theta(1+r)} - \frac{\rho}{\theta}\right)t}, \tag{7.1.37}$$

where $C_0 \in \mathbb{R}_+$ is an integration constant. Considering the transversality condition in (7.1.14), we also have $\rho > \frac{\theta - 1}{1+r}$. Note, θ is normally assumed to be $[0, 1]$. Hence, the transversality condition yields $\rho > 0$.

It is clear that $\frac{1}{\theta} > 0$, hence, the consumption function is affected by the interest rate r and the discount rate ρ, that is,

- if $\frac{1}{1+r} > \rho$, the consumption function $C(t)$ increases,
- if $\frac{1}{1+r} < \rho$, the consumption function $C(t)$ decreases.

Note we relate the consumption in our model with r and ρ. The economic meaning of $\frac{1}{r+1}$ can be related to the discount rate of a saving model (which is not the effective discount rate d). Suppose one deposits s dollars into a saving account, then after one period there are $s(1+r)$ dollars in the account. Conversely, suppose one has s dollars at the end of the period, then he must save $\frac{s}{1+r}$ at the beginning. Hence, $v = \frac{1}{1+r}$ is the discount rate reflecting the value of money at the initial time. We can measure the time preference using the discount rate. Let us assume the time preference of the average is represented by v. When the interest rate r is high, consumers tend to save money in the bank rather than spend money. Otherwise, when r is low, people tend to consume more.

The equation (7.1.30) can be reduced to a linear non-homogeneous ODE as follows

$$\dot{D} - \frac{1}{1+r}D = \frac{r}{1+r}\left((r-\delta-a)K(t) + \frac{a}{N_K}K^2(t) + Y(t) - C(t)\right). \qquad (7.1.38)$$

solving (7.1.38) yields,

$$\begin{aligned} D(t) &= \frac{r}{1+r}e^{\frac{t}{1+r}}\int e^{-\frac{t}{1+r}}\left((r-\delta-a)\frac{N_K K_0}{K_0+(N_K-K_0)e^{-at}}\right. \\ &\quad \left. +\frac{a}{N_K}\left(\frac{N_K K_0}{K_0+(N_K-K_0)e^{-at}}\right)^2 + \frac{N_Y Y_0}{Y_0+(N_Y-Y_0)e^{-ct}} - C_0 e^{\left(\frac{1}{\theta(1+r)}-\frac{\rho}{\theta}\right)t}\right) dt \\ &\quad + D_0 e^{\frac{t}{1+r}}, \end{aligned} \qquad (7.1.39)$$

where a, b and c are constant and K_0, C_0 and D_0 are initial conditions.

Therefore, assuming a constant intertemporal elasticity of substitution utility function, the optimal social welfare (7.1.10) is achieved along the curves determined by (7.1.38) and (7.1.39).

7.2 A new microeconomical model

Let us consider a firm in an industry of a certain type. The optimal profit of a firm is of our interest. Nerlove [84] assumed the output of the firm following the form of the Cobb-Douglas function, and then investigated the discounted profit maximization from now to the economic horizon (the time when the firm no longer exists).

We suppose the output of the firm follows the form of the production function $f_5(K, L)$, where capital $K = K(t)$, labor $L = L(t)$. The profit of the firm at a time can be presented as

$$\Pi^*(t) = \Pi(t) - p_0 Y - p_1 K - p_2 L - Q_1(\dot{K}) - Q_2(\dot{L}), \qquad (7.2.1)$$

where $\Pi(t)$ is revenue, p_i, $i = 0, 1, 2$, are prices of each factor, Q_i, $i = 1, 2$, are costs of changing the level of each input and assumed to be positive for all values of $\dot{K}$ and $\dot{L}$.

The total of discounted profits from the present time to the economic horizon can be expressed as

$$\int_{p}^{T} e^{-\rho t}\Pi^{*}(t)dt, \tag{7.2.2}$$

where $e^{-\rho t}$ is the discount factor, ρ is the discount rate, p and T represent the current time and the economic horizon, respectively.

Remark 7.2.1. *The reason for introducing the discount factor is that the values of money vary at different times due to the interest rate and we use the discount factor to calculate the present value of profit. The discount rate can be also viewed as the interest rate.*

We want to maximize the following functional

$$J(K,L) = \int_{p}^{T} e^{-\rho t}\Pi^{*}(t)dt, \tag{7.2.3}$$

subject to

$$g_3(Y,K,L) = Y - \frac{N_{f_5}K^{\alpha}L^{\beta}}{C_5|N_K - K|^{\alpha}|N_L - L|^{\beta} + K^{\alpha}L^{\beta}} = 0. \tag{7.2.4}$$

Let us denote

$$\Phi = e^{-\rho t}\Pi^{*}(t) + \lambda g_3(Y,K,L). \tag{7.2.5}$$

The following Euler-Lagrange equations lead to the necessary conditions of the above optimization problem

$$-p_0 e^{-\rho t} + \lambda = 0, \tag{7.2.6}$$

$$-p_1 e^{-\rho t} - \lambda(f_5)_K + \frac{d}{dt}\left(e^{-\rho t}\frac{dQ_1}{d\dot{K}}\right) = 0, \tag{7.2.7}$$

$$-p_2 e^{-\rho t} - \lambda(f_5)_L + \frac{d}{dt}\left(e^{-\rho t}\frac{dQ_2}{d\dot{L}}\right) = 0. \tag{7.2.8}$$

The equation (7.2.6) yields

$$\lambda = p_0 e^{-\rho t}, \tag{7.2.9}$$

substituting (7.2.9) into the equation (7.2.7), we obtain

$$-p_1 e^{-\rho t} - p_0 e^{-\rho t}(f_5)_K + \frac{d}{dt}\left(e^{-\rho t}\frac{dQ_1}{d\dot{K}}\right) = 0. \tag{7.2.10}$$

It follows from

$$\frac{d}{dt}\left(e^{-\rho t}\frac{dQ_1}{d\dot{K}}\right) = -\rho e^{-\rho t}\frac{dQ_1}{d\dot{K}} + e^{-\rho t}\frac{d}{dt}\left(\frac{dQ_1}{d\dot{K}}\right) \tag{7.2.11}$$

that the equation (7.2.10) becomes

$$-\rho e^{-\rho t}\frac{dQ_1}{d\dot{K}} + e^{-\rho t}\frac{d}{dt}\left(\frac{dQ_1}{d\dot{K}}\right) = p_1 e^{-\rho t} + p_0 e^{-\rho t}(f_5)_K, \tag{7.2.12}$$

or

$$-\rho\frac{dQ_1}{d\dot{K}} + \frac{d^2Q_1}{d\dot{K}^2}\ddot{K} = p_1 + p_0(f_5)_K. \tag{7.2.13}$$

Similarly, we can obtain a differential equation in terms of L by substituting the equation (7.2.6) into the equation (7.2.8), that is,

$$-\rho\frac{dQ_2}{d\dot{L}} + \frac{d^2Q_2}{d\dot{L}^2}\ddot{K} = p_2 + p_0(f_5)_L. \tag{7.2.14}$$

Therefore, the maximal profit of the functional (7.2.3) subject to (7.2.4) is determined by the following system of differential equations

$$-\rho\frac{dQ_1}{d\dot{K}} + \frac{d^2Q_1}{d\dot{K}^2}\ddot{K} = p_1 + p_0(f_5)_K \tag{7.2.15}$$

and

$$-\rho\frac{dQ_2}{d\dot{L}} + \frac{d^2Q_2}{d\dot{L}^2}\ddot{L} = p_2 + p_0(f_5)_L. \tag{7.2.16}$$

We assume the functions Q_i, $i = 1, 2$, are positive for any values of $\dot{K}$ or $\dot{L}$. The simplest possible form of Q_i is of the following quadratic form [84]:

$$Q_1(\dot{K}) = \frac{m_1}{2}(\dot{K})^2 \quad \text{and} \quad Q_2(\dot{L}) = \frac{m_2}{2}(\dot{L})^2, \tag{7.2.17}$$

where m_1 and m_2 are positive constants.

This entails that the equations (7.2.15) and (7.2.16) become the following system of second order ordinary differential equations

$$m_1\ddot{K} - \rho m_1\dot{K} - p_0\frac{\alpha N_Y N_K C_5\left|\frac{N_K}{K}-1\right|^{\alpha}\left|\frac{N_L}{L}-1\right|^{\beta}}{K(N_K-K)\left(C_5\left|\frac{N_K}{K}-1\right|^{\alpha}\left|\frac{N_L}{L}-1\right|^{\beta}+1\right)^2} = p_1 \tag{7.2.18}$$

and

$$m_2\ddot{L} - \rho m_2\dot{L} - p_0\frac{\beta N_Y N_L C_5\left|\frac{N_K}{K}-1\right|^{\alpha}\left|\frac{N_L}{L}-1\right|^{\beta}}{L(N_L-L)\left(C_5\left|\frac{N_K}{K}-1\right|^{\alpha}\left|\frac{N_L}{L}-1\right|^{\beta}+1\right)^2} = p_2, \tag{7.2.19}$$

the solution of which represents the optimal path of K and L.

We can see that solving the system of equations (7.2.18) and (7.2.19) analytically is a highly non-trivial matter due to the complicated forms of the partial derivatives of the production function Y.

7.2.1 A Hamiltonian approach

We attempt to identify the above system of differential equations as a Hamiltonian system in what follows.

1. A Hamiltonian system in canonical coordinates

If we assume $\rho = 0$, then the system becomes

$$m_1\ddot{K} - p_0\frac{\partial Y}{\partial K} = p_1 \tag{7.2.20}$$

and

$$m_2\ddot{L} - p_0\frac{\partial Y}{\partial L} = p_2. \tag{7.2.21}$$

Let us introduce the momentum coordinates x_1 and x_2 so that

$$x_1 = m_1\dot{K} \tag{7.2.22}$$

and

$$x_2 = m_2 \dot{L}. \tag{7.2.23}$$

The system can be written as a Hamiltonian system defined by the Hamiltonian function H given by

$$H = \frac{1}{2}\left(\frac{x_1^2}{m_1} + \frac{x_2^2}{m_2}\right) - (p_0 Y + p_1 K + p_2 L) \tag{7.2.24}$$

on the phase space with the canonical coordinates (K, x_1, L, x_2), so that

$$\begin{aligned} \frac{\partial H}{\partial K} &= -\dot{x}_1 = -p_0 \frac{\partial Y}{\partial K} - p_1, \\ \frac{\partial H}{\partial x_1} &= \dot{K} = \frac{x_1}{m_1}, \end{aligned} \tag{7.2.25}$$

and

$$\begin{aligned} \frac{\partial H}{\partial L} &= -\dot{x}_2 = -p_0 \frac{\partial Y}{\partial L} - p_2, \\ \frac{\partial H}{\partial x_2} &= \dot{L} = \frac{x_2}{m_2}. \end{aligned} \tag{7.2.26}$$

We have obtained a Hamiltonian structure of the system. Our next goal is to prove this is a completely integrable system, that is, the system of ODEs has solutions. The system can be viewed as a four-dimensional canonical Hamiltonian system, namely, it has two degrees of freedom. The Arnold-Liouville integrability requires n first integrals for a $2n$- dimensional canonical Hamiltonian system, namely, we need one additional first integral to show the system is completely integrable.

In search of an additional first integral

As is known, if f is a first integral of the system, then $\{f, H\} = 0$, or in terms of a Hamiltonian vector field $X_H(f) = 0$. We want f to be an additional first integral, namely, $X_H(f) = 0$ and $df \wedge dH \neq 0$, *i.e.*, f and H are functionally independent.

The Hamiltonian vector field of the system is given by

$$
\begin{aligned}
X_H &= \frac{\partial H}{\partial x_1}\frac{\partial}{\partial K} - \frac{\partial H}{\partial K}\frac{\partial}{\partial x_1} + \frac{\partial H}{\partial x_2}\frac{\partial}{\partial L} - \frac{\partial H}{\partial L}\frac{\partial}{\partial x_2} \\
&= \frac{x_1}{m_1}\frac{\partial}{\partial K} + (p_0Y_K + p_1)\frac{\partial}{\partial x_1} + \frac{x_2}{m_2}\frac{\partial}{\partial L} + (p_0Y_L + p_2)\frac{\partial}{\partial x_2},
\end{aligned} \tag{7.2.27}
$$

hence, we arrive at the following partial differential equation

$$
\frac{x_1}{m_1}\frac{\partial f}{\partial K} + (p_0Y_K + p_1)\frac{\partial f}{\partial x_1} + \frac{x_2}{m_2}\frac{\partial f}{\partial L} + (p_0Y_L + p_2)\frac{\partial f}{\partial x_2} = 0. \tag{7.2.28}
$$

1. Applying the method of characteristics

Initially, we attempted to solve the PDE via the method of characteristics, that is, by solving the DE

$$
\frac{dK}{\frac{x_1}{m_1}} = \frac{dx_1}{p_0Y_K + p_1} = \frac{dL}{\frac{x_2}{m_2}} = \frac{dx_2}{p_0Y_L + p_2}, \tag{7.2.29}
$$

it seems that the only reasonable combination of the above ordinary equation is

$$
p_0Y_K + p_1dK = \frac{x_1}{m_1}dx_1, \quad p_0Y_L + p_2dL = \frac{x_2}{m_2}dx_2, \tag{7.2.30}
$$

which yields

$$
\frac{x_1^2}{2m_1} - (p_0Y + p_1K) = C_1 \tag{7.2.31}
$$

and

$$
\frac{x_2^2}{2m_2} - (p_0Y + p_2L) = C_2. \tag{7.2.32}
$$

A proper arrangement of the two invariants C_1 and C_2 leads to

$$
f = \frac{1}{2}\left(\frac{x_1^2}{m_1} + \frac{x_2^2}{m_2}\right) - (p_0Y + p_1K + p_2L), \tag{7.2.33}
$$

which is identical to the Hamiltonian function.

We have also tried other combinations, for example,

$$
\frac{dK}{\frac{x_1}{m_1}} = \frac{dx_2}{p_0Y_L + p_2}, \quad \frac{dL}{\frac{x_2}{m_2}} = \frac{dx_1}{p_0Y_K + p_1}, \tag{7.2.34}
$$

but none of which gives us a proper first integral. We can see that the method of characteristics does not yield an ideal result.

2. A new first integral whose momentum is a polynomial

Let us review the Liouville-Arnold integratbility. The $2n$ dynamical system is completely integrable if it has n independent first integrals $P_1, \ldots, P_n$ such that $\{P_k, P_l\} = 0$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket.

Note the system has a canonical Hamiltonian structure, namely, we are considering canonical one-forms and polynomials in the momentum, that is, the case of two Schouten bracket commuting Killing tensors [32].

Let us confine our considerations to the case of first integrals with polynomials in the momentum. We will use the following ansatz for the additional first integral

$$f = g(x_1, x_2) - (p_0 Y + p_1 K + p_2 L), \tag{7.2.35}$$

where $g = g(x_1, x_2)$ is a polynomial of momentum variables x_1 and x_2.

Substituting the equation (7.2.28) into the PDE (7.2.35), we obtain

$$\frac{x_1}{m_1}(-p_0 Y_K - p_1) + (p_0 Y_K + p_1)\frac{\partial g}{\partial x_1} + \frac{x_2}{m_2}(-p_0 Y_L - p_2) + (p_0 Y_L + p_2)\frac{\partial g}{\partial x_2} = 0, \tag{7.2.36}$$

We note that the solution to the equation (7.2.36) is a quadratic polynomial. A polynomial with a different degree simply does not satisfy the equation (7.2.36), *e.g.*, suppose g is a cubic polynomial, then the terms

$$(p_0 Y_K + p_1)\frac{\partial g}{\partial x_1} \quad \text{and} \quad (p_0 Y_L + p_2)\frac{\partial g}{\partial x_2} \tag{7.2.37}$$

are of degree 2 in variables x_1 and x_2, but the terms

$$\frac{x_1}{m_1}(-p_0 Y_K - p_1) \quad \text{and} \quad \frac{x_2}{m_2}(-p_0 Y_L - p_2) \tag{7.2.38}$$

are linear in variables x_1 and x_2, which is obviously inconsistent.

Thus, we obtain

$$g = \frac{1}{2}\left(\frac{x_1^2}{m_1} + \frac{x_2^2}{m_2}\right), \tag{7.2.39}$$

and

$$f = \frac{1}{2}\left(\frac{x_1^2}{m_1} + \frac{x_2^2}{m_2}\right) - (p_0 Y + p_1 K + p_2 L), \tag{7.2.40}$$

which is again identical to the Hamiltonian.

We can conclude the system does not have an independent first integral whose momentum is a polynomial.

7.2.2 An economic approach

The solution to the system determines curves of K and L, along which the profit of the firm is maximized. We will tend to analyze the problem from the viewpoint of economics. $\Pi^*(t)$ in the equation (7.2.1) represents the economic profit in the economic sense. A firm can not obtain the optimal profit in perfect competition and enjoy equilibrium in the long run. We are discussing a firm of any type. A firm is able to achieve an optimal profit in non-perfect competition, namely, the system of the equations (7.2.18) and (7.2.19) with the given boundary conditions has analytic solutions, which determine the optimal path of $K(t)$ and $L(t)$. Along the optimal path the profit attains its maximum.

7.3 The Lie-Koenigs theorem

We have established the Hamiltonian formalism of a special case of the system of equations (7.2.18) and (7.2.19), in which we assume $\rho = 0$, in the previous section. This implies a potential Hamiltonian structure for the general case ($\rho \neq 0$) of the system. We want to further study the system and discuss the possibility of the Hamiltonian formalism of the system in this section.

Whittaker [131] followed Lie and Koenigs in the study of the inverse problem of Hamiltonian formalization, namely, identifying a system of first-order differential equations as a Hamiltonian system using the variational principle (see pp. 275-276 in [131]) and introduced the Lie-Koenigs theorem, which shows that any system of first order differential equations may be viewed as a Hamiltonian system [66]. Whittaker proved the theorem, but the details of the theorem (especially in the sense of applications) was extended by Kerner [66]. Let us briefly illustrate the Lie-Koenigs theorem based on Kerner's version, which is written in a more modern manner and includes more details.

Suppose we have the following system of first order differential equations

$$\dot{x}_i = X_i(x),\ i = 1, \ldots, m \tag{7.3.1}$$

which Kerner coupled with the following variational principle (Whittaker called it an integral invariant of the system)

$$\delta \int [U_\alpha(x)\dot{x}_\alpha - U_0]\, dt = 0. \tag{7.3.2}$$

and wrote the Euler-Lagrange equation in the following manner

$$\left(\frac{\partial U_k}{\partial x_\alpha} - \frac{\partial U_\alpha}{\partial x_k}\right)\dot{x}_\alpha = -\frac{\partial U_0}{\partial x_k},\ k = 1, \ldots, m, \tag{7.3.3}$$

or,

$$\Gamma_{k\alpha}\dot{x}_\alpha = -\frac{\partial U_0}{\partial x_k},\ k = 1, \ldots, m, \tag{7.3.4}$$

by letting

$$\Gamma_{k\alpha} = \left(\frac{\partial U_k}{\partial x_\alpha} - \frac{\partial U_\alpha}{\partial x_k}\right). \tag{7.3.5}$$

Kerner showed $\Gamma_{k\alpha}$ was a component of a Poisson bivector. We can see that the equation can have a Hamiltonian structure if U_0 is a Hamiltonian function.

Let us introduce the following ansatz

$$U_0 = U_\alpha X_\alpha + W_0, \tag{7.3.6}$$

where W_0 is an arbitrary function.

Remark 7.3.1. *The freedom provided by W_0 plays an important role in finding the*

Hamiltonian function U_0 in applications.

Substituting the equation (7.3.1) into the equation (7.3.3) yields

$$\left(\frac{\partial U_k}{\partial x_\alpha} - \frac{\partial U_\alpha}{\partial x_k}\right) X_\alpha = -\frac{\partial U_0}{\partial x_k},\ k = 1, \ldots, m, \tag{7.3.7}$$

using the ansatz, we obtain

$$X_\alpha \frac{\partial U_k}{\partial x_\alpha} = -U_\alpha \frac{\partial X_\alpha}{\partial x_k} - \frac{\partial W_0}{\partial x_k},\ k = 1, \ldots, m, \tag{7.3.8}$$

in which solving U_k and W_0 yields the Hamiltonian function U_0.

Kerner commented that the equation (7.3.8) is a system of Cauchy-Kowalevskaya type. We realize, according to the Cauchy-Kowalevskaya theorem, the system whose coefficients are analytic functions has a unique and analytic solution around the proper initial condition, namely, the equation (7.3.8) can always have a local analytic solution. Kerner proved that any system of first-order differential equations could be viewed, at least locally, as a Hamiltonian system.

He completed the proof by introducing a canonical transformation to the Hamiltonian structure (7.3.5) to show that

$$H(Q, P) = U_0(x(Q, P)), \tag{7.3.9}$$

where (Q, P) are the generalized canonical coordinates.

7.3.1 The case of a harmonic oscillator

The Lie-Koenigs theorem presents a systematic way of finding a Hamiltonian structure for a system, namely, it reduces the problem of finding a Hamiltonian to the corresponding problem of solving a system of PDEs. We will first consider a two-dimensional harmonic oscillator as an example.

Let us consider the following harmonic oscillator

$$\begin{aligned} \dot{p} &= -q, \\ \dot{q} &= p, \end{aligned} \tag{7.3.10}$$

where (q, p) is the generalized coordinates.

The system (7.3.10) is a Hamiltonian system with a canonical symplectic structure

$$\omega = dq \wedge dp \tag{7.3.11}$$

and a Hamiltonian function

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2. \tag{7.3.12}$$

Indeed, let us apply the Lie-Koenigs theorem. As we stated, the core of the Lie-Koenigs theorem in applications is to solve the system (7.3.8). Assume the Hamiltonian $U_0 = -qU_1 + pU_2 + W_0$ and the equation (7.3.8) in this case becomes

$$\begin{aligned} -q\frac{\partial U_1}{\partial p} + p\frac{\partial U_1}{\partial q} &= -U_2 - \frac{\partial W_0}{\partial p}, \\ -q\frac{\partial U_2}{\partial p} + p\frac{\partial U_2}{\partial q} &= U_1 - \frac{\partial W_0}{\partial q}. \end{aligned} \tag{7.3.13}$$

We can see that $U_1 = -\frac{1}{2}q$, $U_2 = \frac{1}{2}p$ and $W_0 = C$, $C \in \mathbb{R}$, is a solution to the above system. This leads to the Hamiltonian of the form:

$$U_0 = \frac{1}{2}q^2 + \frac{1}{2}p^2 + C. \tag{7.3.14}$$

7.3.2 The special case revisited

The Lie-Koenigs approach can also be applied to our special case in Section 7.2, which gives rise to the following system of PDEs

$$(p_0Y_K + p_1)\frac{\partial U_1}{\partial x_1} + \frac{x_1}{m_1}\frac{\partial U_1}{\partial K} + (p_0Y_L + p_2)\frac{\partial U_1}{\partial x_2} + \frac{x_2}{m_2}\frac{\partial U_1}{\partial L} = -\frac{1}{m_1}U_2 - \frac{\partial W_0}{\partial x_1}, \tag{7.3.15}$$

$$(p_0Y_K+p_1)\frac{\partial U_2}{\partial x_1}+\frac{x_1}{m_1}\frac{\partial U_2}{\partial K}+(p_0Y_L+p_2)\frac{\partial U_2}{\partial x_2}+\frac{x_2}{m_2}\frac{\partial U_2}{\partial L}=-(p_0Y_{KK})U_1-(p_0Y_{KL})U_3-\frac{\partial W_0}{\partial K}, \tag{7.3.16}$$

$$(p_0Y_K+p_1)\frac{\partial U_3}{\partial x_1}+\frac{x_1}{m_1}\frac{\partial U_3}{\partial K}+(p_0Y_L+p_2)\frac{\partial U_3}{\partial x_2}+\frac{x_2}{m_2}\frac{\partial U_3}{\partial L}=-\frac{1}{m_2}U_4-\frac{\partial W_0}{\partial x_2}, \tag{7.3.17}$$

$$(p_0Y_K+p_1)\frac{\partial U_4}{\partial x_1}+\frac{x_1}{m_1}\frac{\partial U_4}{\partial K}+(p_0Y_L+p_2)\frac{\partial U_4}{\partial x_2}+\frac{x_2}{m_2}\frac{\partial U_4}{\partial L}=-(p_0Y_{KL})U_1-(p_0Y_{LL})U_3-\frac{\partial W_0}{\partial L}, \tag{7.3.18}$$

with the Hamiltonian function of the form

$$U_0=(p_0Y_K+p_1)U_1+\frac{x_1}{m_1}U_2+(p_0Y_L+p_2)U_3+\frac{x_2}{m_2}U_4+W_0. \tag{7.3.19}$$

According to the form of the Hamiltonian, we can assume that

$$U_2=x_1,\ U_4=x_2. \tag{7.3.20}$$

Substituting U_2 and U_4 into equations (7.3.16) and (7.3.18) yields

$$-(p_0Y_{KK})U_1-(p_0Y_{KL})U_3-\frac{\partial W_0}{\partial K}=p_0Y_K+p_1, \tag{7.3.21}$$

$$-(p_0Y_{KL})U_1-(p_0Y_{LL})U_3-\frac{\partial W_0}{\partial L}=p_0Y_L+p_2. \tag{7.3.22}$$

The different order of derivatives of the production function Y leads us to the consideration that

$$U_1=U_3=0. \tag{7.3.23}$$

Then the system reduces to

$$\begin{aligned}\frac{\partial W_0}{\partial x_1} &= -\frac{x_1}{m_1},\\ \frac{\partial W_0}{\partial K} &= -p_0 Y_K - p_1,\\ \frac{\partial W_0}{\partial x_2} &= -\frac{x_2}{m_2},\\ \frac{\partial W_0}{\partial L} &= -p_0 Y_L - p_2,\end{aligned} \tag{7.3.24}$$

which gives rise to

$$W_0 = -\frac{1}{2}\left(\frac{x_1^2}{m_1} + \frac{x_2^2}{m_2}\right) - (p_0 Y + p_1 K + p_2 L). \tag{7.3.25}$$

Eventually, we arrive at the following Hamiltonian

$$\begin{aligned}U_0 &= \frac{x_1^2}{m_1} + \frac{x_2^2}{m_2} + W_0\\ &= \frac{1}{2}\left(\frac{x_1^2}{m_1} + \frac{x_2^2}{m_2}\right) - (p_0 Y + p_1 K + p_2 L).\end{aligned} \tag{7.3.26}$$

7.4 The microeconomics model revisited

We proposed an optimal problem of profit of a firm in Section 7.2. We followed Nerlove and considered the profit (7.2.1), in which we assumed all prices were constant. It is true that the prices are stable in the equilibrium, but the model we considered may not be the equilibrium. Thus, we think we can also view prices as functions of time.

Then Lagrangian (7.2.3) in Section 7.2 becomes

$$J(K, L) = \int_p^T e^{\rho(\tilde{t})} \Pi^*(\tilde{t}) d\tilde{t}, \tag{7.4.1}$$

subject to

$$g_3(Y, K, L) = Y - \frac{N_{f_5} K^\alpha L^\beta}{C_5 |N_K - K|^\alpha |N_L - L|^\beta + K^\alpha L^\beta} = 0, \tag{7.4.2}$$

where

$$\Pi^*(\tilde{t}) = \Pi(\tilde{t}) - \tilde{p}_0(\tilde{t})Y - \tilde{p}_1(\tilde{t})K - \tilde{p}_2(\tilde{t})L - Q_1(K') - Q_2(L'), \tag{7.4.3}$$

$$K = K(\tilde{t}),\ L = L(\tilde{t}) \text{ and } Y = Y(\tilde{t}), \tag{7.4.4}$$

$$K' = \frac{dK}{d\tilde{t}} \text{ and } L' = \frac{dL}{d\tilde{t}}. \tag{7.4.5}$$

We also generalize the discount factor

$$e^{-\rho\tilde{t}} \longrightarrow e^{\rho(\tilde{t})}. \tag{7.4.6}$$

The Euler-Lagrange equations of the above Lagrangian give rise to

$$-\rho(\tilde{t})\frac{dQ_1}{dK'} + \frac{d^2Q_1}{dK'^2}K'' = \tilde{p}_1 + \tilde{p}_0(f_5)_K \tag{7.4.7}$$

and

$$-\rho(\tilde{t})\frac{dQ_2}{dL'} + \frac{d^2Q_2}{dL'^2}L'' = \tilde{p}_2 + \tilde{p}_0(f_5)_L, \tag{7.4.8}$$

which determine the optimal path of K and L.

Similarly, let us apply the equation (7.2.17) and the system becomes

$$m_1K'' - \rho(\tilde{t})m_1K' - \tilde{p}_0\frac{\alpha N_Y N_K C_5 \left|\frac{N_K}{K} - 1\right|^\alpha \left|\frac{N_L}{L} - 1\right|^\beta}{K(N_K - K)\left(C_5\left|\frac{N_K}{K} - 1\right|^\alpha \left|\frac{N_L}{L} - 1\right|^\beta + 1\right)^2} = \tilde{p}_1 \tag{7.4.9}$$

and

$$m_2L'' - \rho(\tilde{t})m_2L' - \tilde{p}_0\frac{\beta N_Y N_L C_5 \left|\frac{N_K}{K} - 1\right|^\alpha \left|\frac{N_L}{L} - 1\right|^\beta}{L(N_L - L)\left(C_5\left|\frac{N_K}{K} - 1\right|^\alpha \left|\frac{N_L}{L} - 1\right|^\beta + 1\right)^2} = \tilde{p}_2. \tag{7.4.10}$$

The new system gives us more freedom, since we can assume different functions for discount and price factors.

Let us introduce the following functions

$$\rho(\tilde{t}) = -\frac{1}{\tilde{t}},\ \tilde{p}_0 = \frac{p_0}{\tilde{t}^2},\ \tilde{p}_1 = \frac{p_1}{\tilde{t}^2} \text{ and } \tilde{p}_2 = \frac{p_2}{\tilde{t}^2}. \tag{7.4.11}$$

The value of $e^{-\frac{1}{t}}$ is in $(0, 1)$ and is a proper discount factor.

We propose that price is inversely proportional to time, which makes sense that the firm may take much money in hiring employees and spend much money in production materials in the pioneering stage, since it may lack bargaining power.

Then the equations (7.4.9) and (7.4.10) become

$$m_1K'' + \frac{1}{\tilde{t}}m_1K' - \frac{p_0}{\tilde{t}^2}\frac{\alpha N_Y N_K C_5 \left|\frac{N_K}{K} - 1\right|^\alpha \left|\frac{N_L}{L} - 1\right|^\beta}{K(N_K - K)\left(C_5 \left|\frac{N_K}{K} - 1\right|^\alpha \left|\frac{N_L}{L} - 1\right|^\beta + 1\right)^2} = \frac{p_1}{\tilde{t}^2} \tag{7.4.12}$$

and

$$m_2L'' + \frac{1}{\tilde{t}}m_2L' - \frac{p_0}{\tilde{t}^2}\frac{\beta N_Y N_L C_5 \left|\frac{N_K}{K} - 1\right|^\alpha \left|\frac{N_L}{L} - 1\right|^\beta}{L(N_L - L)\left(C_5 \left|\frac{N_K}{K} - 1\right|^\alpha \left|\frac{N_L}{L} - 1\right|^\beta + 1\right)^2} = \frac{p_2}{\tilde{t}^2}, \tag{7.4.13}$$

or

$$m_1\tilde{t}^2K'' + m_1\tilde{t}K' - p_0Y_K = p_1 \tag{7.4.14}$$

and

$$m_2\tilde{t}^2L'' + m_2\tilde{t}L' - p_0Y_L = p_2. \tag{7.4.15}$$

Recall that Yatsun [134] investigated an integrable model of the Yang-Mill theory and quasi-instantons. Under the $O(4)$-symmetry, the model becomes a system of second-order ODEs, which Yatsun showed, under a certain transformation, was a Hamiltonian system. We noticed that the transformation can be also applied to our model.

Let us introduce

$$t = \ln \tilde{t}, \tag{7.4.16}$$

then

$$\tilde{t}K' = \frac{dK}{dt} = \dot{K} \tag{7.4.17}$$

and

$$\tilde{t}^2 K'' = -\dot{K} + \ddot{K}. \tag{7.4.18}$$

It follows from the above transformation that the equations (7.4.14) and (7.4.15) become

$$m_1 \ddot{K} = p_0 Y_K + p_1 \tag{7.4.19}$$

and

$$m_2 \ddot{L} = p_0 Y_L + p_2, \tag{7.4.20}$$

Referring to the result in Section 7.2, we can introduce

$$x_1 = m_1 \dot{K} \tag{7.4.21}$$

and

$$x_2 = m_2 \dot{L}. \tag{7.4.22}$$

Then we obtain a four-dimensional Hamiltonian system with a canonical Poisson bivector and a Hamiltonian function

$$H = \frac{1}{2}\left(\frac{x_1^2}{m_1} + \frac{x_2^2}{m_2}\right) - (p_0 Y + p_1 K + p_2 L). \tag{7.4.23}$$

7.5 Concluding remarks

In this chapter, we have briefly reviewed and rederived, using the Euler-Lagrange method, the Ramsey golden rule of accumulation. A new macroeconomics model of social welfare has been considered, in which we find the optimal path of consumption and debt. We have extended the model of the optimal profit of a firm given by Nerlove through employing a new production Y of the bounded growth type. A system of ODEs determines the optimal path of capital and labor. We have identified a special case of the system as a Hamiltonian system, in which a Hamiltonian function is derived. We have discussed the existence of a potential solution from the viewpoint of economics.

Chapter 8

Conclusions

In the following we summarize the results obtained in Chapters 3-7. See also [119].

1. *The validity of the Cobb-Douglas function.* We have demonstrated that the functions of the Cobb-Douglas type (3.0.1) arise naturally as invariants of the one-parameter Lie group action (5.1.1). That is the Cobb-Douglas function (3.0.1) is a consequence of the exponential growth in factors and production determined by the corresponding parameters b_1, b_2, and b_3 in (4.2.1), provided the elastisities of substitution α and β in (3.0.1) satisfy the linearity condition (5.1.18). The latter explains why various authors (see, for example, Mendershausen [81] and Doll [29]) have observed the property of multicollinearity while studying data compatible with the Cobb-Douglas production function, using statistical methods. In view of this observation, we modify the definition of the Cobb-Douglas function as follows.

   Definition 8.0.1 (Cobb-Douglas function)**.** Given the one-parameter group action

   $$x_i = x_i^0 e^{b_i t}, \quad x_i^0, b_i > 0, \quad i = 1, \ldots, n \tag{8.0.1}$$

   in $\mathbb{R}_+^n$. Then the *Cobb-Douglas function* is defined as an element of the following family of invariants of the action (8.0.1):

   $$\prod_{i=1}^{n} x_i^0 x_i^{\alpha_i} = C, \quad \alpha_i > 0, \quad i = 1, \ldots, n, \tag{8.0.2}$$

   where $C \in \mathbb{R}$ is an arbitrary constant and x_i^0, $i = 1, \ldots, n$ are the corresponding initial conditions, if the linearity condition

$$\sum_{i=1}^{n} \alpha_i b_i = 0 \tag{8.0.3}$$

holds true.

We note that the parameters α_i, $i = 1, \ldots, n$ may satisfy an additional linearity condition

$$\sum_{i=1}^{n} \alpha_i a_i = a, \quad a_i, a \in \mathbb{R}, \quad i = 1, \ldots, n, \tag{8.0.4}$$

provided the lines (8.0.3) and (8.0.4) intersect in $\mathbb{R}_+$.

Importantly, the results presented above put in evidence that Cobb and Douglas in [27] did derive a production function, as a function that related physical output of a production process to physical inputs or factors of production, compatible with the data studied by the authors. However, it was not the only production function of the type (3.0.1) compatible with it. We hope that our analysis will help to clarify the many controversies surrounding the question of derivation, applicability and properties of the Cobb-Douglas production function (3.0.1).

2. *The logistic production function.* The assumption about exponential growth in production and factors of production that led to the introduction of the Cobb-Douglas function given by (8.0.2) can naturally be modified under the assumption that any factors of production, as well as production grow logistically, rather than exponentially. Thus, we give the following

Definition 8.0.2 (Logistic production function)**.** Given the following one-parameter group action

$$x_i = \frac{N_i x_i^0}{x_i^0 + (N_i - x_i^0)e^{-b_i t}}, \quad x_i^0, b_i, N_i > 0, \quad i = 1, \ldots, n \tag{8.0.5}$$

in $\mathbb{R}_+^n$. Then the *logistic production function* is defined as an element of the following family of invariants of the action (5.2.1) :

$$\prod_{i=1}^{n} \left[\frac{x_i(N_i - x_i^0)}{x_i^0(N_i - x_i)} \right]^{\alpha_i} = C, \quad \alpha_i, N_i > 0 \quad i = 1, \ldots, n, \tag{8.0.6}$$

where $C \in \mathbb{R}$ is an arbitrary constant and x_i^0, $i = 1, \ldots, n$ are the corresponding initial conditions, if the linearity condition

$$\sum_{i=1}^{n} \alpha_i b_i = 0 \tag{8.0.7}$$

holds true.

We note that the parameters α_i, $i = 1, \ldots, n$ in (8.0.6) may satisfy an additional linearity condition (8.0.4), provided the lines (8.0.7) and (8.0.4) intersect in $\mathbb{R}_+$.

3. *The Hamiltonian approach.* Mathematicians often say that “a mathematical problem is essentially solved when it is reduced to an algebraic problem.” In this thesis we have reduced several problems of the derivation of a production function to the corresponding algebraic problems by employing the Hamiltonian approach and describing the dynamics in question in each case as a special case of the Lotka-Volterra model (1.0.3). In particular, we have rederived the celebrated Cobb-Douglas production function (3.0.1) with economically acceptable elasticities of substitution as a linear combination of two Hamiltonians of the bi-Hamiltonian structure (4.5.1) defined by two quadratic (degenerate) Poisson bivectors. In Chapter 3 we derived a new production function (3.3.14) by assuming logistic rather than exponential growth in factors. In this case too, we identified the corresponding dynamical system as a special case of the Lotka-Volterra model (1.0.3) and a Hamiltonian system as such, which enabled us to derive the corresponding production function (3.3.14) as a Hamiltonian. The last model presented Chapter 6 is new — we have introduced an additional variable (debt) and described the dynamics built around the “predator-prey” type interaction between capital and debt also as a special case of the Lotka-Volterra model (1.0.3), which ultimately led to the derivation of a new production function (3.3.21).

4. *The algorithm.* Based on Definitions 8.0.1 and 8.0.2, the data analysis presented by Cobb and Douglas in [27] and in this paper, we derive the following algorithm that can be used to fit the production functions (3.0.1) and (3.3.14) (which are special cases of the functions (8.0.2) and (8.0.6) respectively) to real data.

   Given data representing production, labor, and capital.

   - First, employ R to fit the functions (5.1.16), or (5.2.6) to the given data, recovering in the process the values of parameters x_i^0 and b_i, or x_i^0, b_i, and N_i respectively for $i = 1, 2, 3$. More specifically, we choose either (5.1.16) or (5.2.6) to fit to the data, depending on the corresponding values of RSS's (*i.e.*, whichever more accurate).
   - Second, use the values of the parameters b_1, b_2, and b_3 determined in the preceding step to form the linearity condition (5.1.18) for the parameters α and β.
   - Next, if the parameters b_1, b_2, and b_3 satisfy either the inequality (4.5.10), we can choose the values of the parameters α and β in (3.0.1) or (8.0.6) so that $\alpha + \beta = 1$, using the formulas (4.5.9).
   - Finally, use the values of the parameters of α and β, satisfying the linearity condition (5.1.18), that afford the best fit for the data that represents production by the function (3.0.1) (if the data was approximated by exponential formulas (5.1.16) in the first step), or the production function (3.3.14) (if the data is compatible with logistic growth given by (5.2.6)) to define by these parameters either the production (3.0.1), or (3.3.14).

5. *The four-dimensional dynamical model and the new variational problems* We have shown in Chapter 6 that the four-dimensional dynamical system has an unstable equilibrium. The Hamiltonian function of the model can be used as a production function. We have formulated a new variational problem involving debt based on the Ramsey-Cass-Koopmans model with our new production function (3.3.14) in Chapter 7. The maximal social welfare is attained along the path of consumption and debt given by (7.1.38) and (7.1.39), respectively.

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