AN ELEMENTARY ACCOUNT OF FLAT 2-FUNCTORS

by

Michael Lambert

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

at

Dalhousie University
Halifax, Nova Scotia
April 2019

© Copyright by Michael Lambert, 2019

Table of Contents

Abstract

A set-valued functor is “flat” if its tensor product extension is finite-limit preserving. Such a functor is flat if, and only if, its category of elements is filtered. Analogously, a category-valued 2-functor on a 2-category is defined to be flat in terms of a finite-limit preserving property. The characterization in the work of M. E. Descotte, E. J. Dubuc, and M. Szyld is that a 2-functor is flat if, and only if, its 2-category of elements is appropriately 2-filtered. The goal of the present work is to prove a generalization in the internal 2-category theory of a suitable 1-category. This follows the pattern of R. Diaconescu’s generalized account of the theory 1-dimensional flatness in the internal category theory of a 1-topos. The 1-topos is here replaced by the 2-category of internal categories of an exact 1-category.

This work follows a novel approach. The first step is in computing, for a category-valued pseudo-functor, a tensor product extension. This is done as a category of fractions. Supposing this extension is finite-limit preserving, 2-filteredness conditions are obtained related to those of Descotte, Dubuc and Szyld. The converse result, namely, that our 2-filteredness conditions imply finite-limit preservation, is approached using the right calculus of fractions. That is, under the assumption of 2-filteredness, the tensor product is formed through a right calculus of fractions. This gives a tractable characterization of the morphisms of the tensor product, from which follows an “elementary” proof that filteredness implies limit-preservation.

For the internal generalization, the right calculus of fractions is described in internal category theory. The internal 2-filteredness conditions imply that an internal tensor-product construction is formed through the internal right calculus of fractions. Finally, it is seen that internal 2-filteredness implies that the internal tensor product is finite-limit preserving. Partly this is achieved by showing that the internal tensor product reduces to Diaconescu’s internal colimit construction. For this reason, exactness of the internal tensor product partly reduces to known cases. The remaining case is that of certain cotensors, which are shown to be preserved using an elementary argument.

List of Abbreviations and Symbols Used

Notation	Description
$M \otimes_R N$	The tensor product of right- and left-modules M and N over a ring R.
Ab	The category of abelian groups.
Mod-R	The category of right R-modules and R-module homomorphisms.
$\square$	End of proof.
Set	The category of small sets and functions.
$Q \otimes_{\mathscr{C}} P$	The tensor product of functors $Q\colon \mathscr{C} \to \mathbf{Set}$ and $P\colon \mathscr{C}^{op} \to \mathbf{Set}$.
$[\mathscr{C}, \mathbf{Set}]$	The category of functors $\mathscr{C} \to \mathbf{Set}$ and natural transformations.
$\int_{\mathscr{C}} Q$	The category of elements of a functor $Q\colon \mathscr{C} \to \mathbf{Set}$.
$\mathbf{Geom}(\mathscr{F}, \mathscr{E})$	The category of geometric morphisms between toposes $\mathscr{F}$ and $\mathscr{E}$ and geometric transformations.
$\mathbf{Flat}(\mathscr{C}, \mathbf{Set})$	The category of flat set-valued functors on $\mathscr{C}$ and natural transformations.
$\mathbf{DFib}(\mathbb{C})$	The category of discrete fibrations on a category $\mathbb{C}$ internal to a finitely-complete category $\mathscr{E}$ and internal functors over $\mathbb{C}$.
$\mathbf{DOpf}(\mathbb{C})$	The category of discrete opfibrations on a category $\mathbb{C}$ internal to a finitely-complete category $\mathscr{E}$ and internal functors over $\mathbb{C}$.
$\mathfrak{Toposes}$	The 2-category of toposes, geometric morphisms and geometric transformations.
$\mathfrak{Cat}$	The 2-category of small categories, functors, and natural transformations.
$\mathfrak{Cat}(\mathscr{E})$	The 2-category of categories internal to a finitely-complete category $\mathscr{E}$, internal functors, and internal natural transformations.
$[\mathcal{A}, \mathcal{V}]$	The enriched functor category for a $\mathcal{V}$-category $\mathcal{A}$.
Cat	The 1-category of categories and functors.
$\int_{\mathfrak{C}} E$	The 2-category of elements of a pseudo-functor $E\colon \mathfrak{C} \to \mathfrak{Cat}$.
$\mathfrak{Hom}(\mathfrak{C}, \mathfrak{D})$	The 2-category of pseudo-functors $\mathfrak{C} \to \mathfrak{D}$ between 2-categories, pseudo-natural transformations, and modifications.
$E \star W$	The pseudo-colimit of a pseudo-functor $E\colon \mathfrak{C} \to \mathfrak{K}$ weighted by a pseudo-functor $W\colon \mathfrak{C}^{op} \to \mathfrak{Cat}$.

Notation	Description
$E \otimes_{\mathfrak{C}} F$	The tensor product of category-valued pseudo-functors $E\colon \mathfrak{C} \to \mathfrak{Cat}$ and $F\colon \mathfrak{C}^{op} \to \mathfrak{Cat}$ on a 2-category $\mathfrak{C}$.
$\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F}$	The internal tensor product of a discrete 2-opfibration $e\colon \mathcal{E} \to \mathcal{C}$ and a discrete 2-fibration $f\colon \mathcal{F} \to \mathcal{C}$ internal to a finitely-complete category $\mathscr{E}$.
$\mathfrak{K}(A, B)$	The category of morphims $A \to B$ and 2-cells between them in the 2-category $\mathfrak{K}$.
$\mathfrak{A}_0$	The underlying 1-category of a 2-category $\mathfrak{A}$.
$\mathfrak{K}/A$	The 2-slice of a 2-category $\mathfrak{K}$ by an object A.
$\mathfrak{K} \mathbin{/\!/} A$	The lax-slice of a 2-category $\mathfrak{K}$ by an object A.
$[\mathfrak{A}, \mathfrak{B}]$	The 2-category of 2-functors between 2-categories $\mathfrak{A}$ and $\mathfrak{B}$, 2-natural transformations, and modifications.
2-$\mathfrak{Cat}$	The 3-category of small 2-categories, 2-functors, 2-natural transformations and modifications.
$\pi_0\mathscr{C}$	The connected components of a small 1-category $\mathscr{C}$.
$\pi_0\mathfrak{A}$	The connected components of a 2-category $\mathfrak{A}$.
$\mathfrak{Alg}(T)$	The 2-category of pseudo-algebras for a 2-monad T, algebra homomorphisms, and their transformations.
$\mathbf{DFib}(\mathscr{C})$	The category of discrete fibrations on a small category $\mathscr{C}$ and functors over $\mathscr{C}$.
$\mathbf{DOpf}(\mathscr{C})$	The category of discrete opfibrations on a small category $\mathscr{C}$ and functors over $\mathscr{C}$.
$\mathfrak{cFib}(\mathscr{C})$	The 2-category of cloven fibrations over a small category $\mathscr{C}$, cartesian morphism-preserving functors and transformations with vertical components.
$\mathfrak{sFib}(\mathscr{C})$	The 2-category of split fibrations over a small category $\mathscr{C}$, cartesian morphism-preserving functors and transformations with vertical components.
$\mathfrak{cOpf}(\mathscr{C})$	The 2-category of opcloven opfibrations over a small category $\mathscr{C}$, opcartesian morphism-preserving functors and transformations with vertical components.

Notation	Description
$\mathfrak{sOpf}(\mathscr{C})$	The 2-category of split opfibrations over a small category $\mathscr{C}$, opcartesian morphism-preserving functors and transformations with vertical components.
$\mathbb{C}^{\mathbf{2}}$	The internal arrow category of a small category $\mathbb{C}$ internal to a finitely-complete category $\mathscr{E}$.
$\mathbf{Cat}(\mathscr{E})$	The 1-category of categories internal to a finitely-complete category $\mathscr{E}$ and internal functors.
$\mathbf{Iso}(\mathbb{D})$	The object of isomorphims of an internal category $\mathbb{D}$.
$\mathscr{E}^{\mathbb{C}}$	The category of internal diagrams on $\mathbb{C}$ with action-preserving morphisms.
$\lim\limits_{\to \mathbb{C}}$	The internal colimit functor.
$\mathbb{C}^*$	The constant diagram functor.
$\mathfrak{DOpf}(\mathfrak{C})$	The 2-category of discrete 2-opfibrations over $\mathfrak{C}$, cartesian-morphism-preserving functors over $\mathfrak{C}$ and transformations with vertical components.
$\mathfrak{DFib}(\mathfrak{C})$	The 2-category of discrete 2-fibrations over $\mathfrak{C}$.
$\mathcal{K}^{\mathbf{2}}$	The internal 2-arrow category of a 2-category $\mathcal{K}$ internal to a finitely-complete category $\mathscr{E}$.
$\mathcal{K}(a, b)$	The internal 1-category of internal morphisms and internal 2-cells.
$2\text{-}\mathfrak{Cat}(\mathscr{E})$	The 2-category of 2-categories internal to $\mathscr{E}$, internal 2-functors, and internal 2-natural transformations.
$\pi_0\mathcal{K}$	The internal connected components of an internal 2-category $\mathcal{K}$.
$\mathfrak{DFib}(\mathcal{C})$	The 2-category of discrete 2-fibrations over $\mathcal{C}$ internal to $\mathscr{E}$, internal cleavage-preserving functors, and internal 2-natural transformations.
$\mathfrak{DOpf}(\mathcal{C})$	The 2-category of discrete 2-opfibrations over $\mathcal{C}$ internal to $\mathscr{E}$, internal opcleavage-preserving functors, and internal 2-natural transformations.
$\{P, Q\}_s$	The 2-limit of a 2-functor $Q\colon \mathfrak{J} \to \mathfrak{K}$ between 2-categories $\mathfrak{J}$ and $\mathfrak{K}$ weighted by a 2-functor $P\colon \mathfrak{J} \to \mathfrak{Cat}$.
f/g	The comma object of morphisms $f\colon A \to C$ and $g\colon B \to C$ in a 2-category $\mathfrak{K}$.
$\mathscr{A} \pitchfork A$	The cotensor of A in a 2-category $\mathfrak{K}$ with a category $\mathscr{A}$.

Notation	Description
$\{P, Q\}$	The pseudo-limit of a pseudo-functor $Q\colon \mathfrak{J} \to \mathfrak{K}$ weighted by a pseudo-functor $P\colon \mathfrak{J} \to \mathfrak{Cat}$.
$E \star_s W$	The 2-colimit of a 2-functor $E\colon \mathfrak{C} \to \mathfrak{K}$ weighted by a 2-functor $W\colon \mathfrak{C}^{op} \to \mathfrak{Cat}$.
$E \star W$	The pseudo-colimit of a 2-functor $E\colon \mathfrak{C} \to \mathfrak{K}$ weighted by a 2-functor $W\colon \mathfrak{C}^{op} \to \mathfrak{Cat}$.
$\Delta(E, W)$	The diagonal 2-category of elements of category-valued pseudo-functors $E\colon \mathfrak{C} \to \mathfrak{Cat}$ and $W\colon \mathfrak{C}^{op} \to \mathfrak{Cat}$ on a 2-category $\mathfrak{C}$.
$\mathscr{C}[\Sigma^{-1}]$	The category of fractions of $\mathscr{C}$ with respect to a subset of arrows Σ.
$\mathbb{C}[\Sigma^{-1}]$	The internal category of fractions of an internal category $\mathbb{C}$ with respect to Σ.

Acknowledgements

First and foremost, the author would like to thank his supervisor, Dr. Dorette Pronk, for over four years of generous, ongoing support and encouragement during the research and writing that went into this thesis. The author would like to thank Dr. Pieter Hofstra, Dr. Robert Paré and Dr. Peter Selinger for serving as examiners on the thesis committee. In particular, Dr. Paré and Dr. Selinger provided extremely helpful comments and suggestions on an earlier draft of the document.

The author would also like to thank Dr. Richard Foote at the University of Vermont and Dr. Calin Chindris at the University of Missouri-Columbia, each of whom provided much needed support in the early stages of the author's graduate studies.

The research of this thesis was supported in part by a Nova Scotia Graduate Scholarship (NSGS) and by the Natural Sciences and Engineering Research Council of Canada (NSERC) Discovery Grant of Dr. Pronk and by the U.S. Air Force Office of Scientific Research, Air Force Material Command, USAF under Award No. FA9550-15-1-0331, held by Dr. Selinger.

Chapter 1

Introduction

The present thesis is directed toward a generalization of a characterization of flat category-valued 2-functors due to M. E. Descotte, E. Dubuc, and M. Szyld in [DDS18b] in terms of 2-dimensional filteredness conditions. The generalization undertaken here is motivated by the way in which an account of the theory of flat presheaves was given in the internal category theory of a topos by R. Diaconescu in [Dia73].

1.1 Flat Functors and 2-Functors

1.1.1 Modules

The concept of flatness has its origin in the theory of modules over a ring.

If M is a right module and N is a left module over a ring R with identity 1, the tensor product of M and N is defined by a universal property as in §IV.5 of [Hun74]. It is an abelian group T admitting a so-called "middle-linear" map $M \times N \to T$ that is universal among all such middle-linear maps. There is a canonical construction of the tensor product as a quotient of the free abelian group on $M \times N$. That is, the tensor, denoted $M \otimes_R N$, is given explicitly as the quotient of the free abelian group on $M \times N$ by the subgroup D generated by the middle-linearity expressions

$$(x + y, z) - (x, z) - (y, z)$$

$$(x, w + z) - (x, w) - (x, z)$$

$$(xr, w) - (x, rw)$$

where $x, y \in M$, $w, z \in N$ and $r \in R$. The map $M \times N \to M \otimes_R N$ sending $(m, n) \mapsto m \otimes n$ is middle-linear. There is a one-to-one correspondence between middle-linear maps $M \times N \to P$ and homomorphisms of abelian groups $M \otimes_R N \to P$, for any abelian group P, as described in Theorem IV.5.2 of [Hun74]. This is given by composition with the canonical middle linear map $M \times N \to M \otimes_R N$.

The tensor-hom adjunction is the statement that homomorphisms $M \otimes_R N \to P$ into an abelian group P are in one-to-one correspondence with homomorphisms $M \to \mathbf{Ab}(N, P)$, in the sense that there is an isomorphism of abelian groups

$$\mathbf{Ab}(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathbf{Ab}(N, P)). \tag{1.1.1}$$

as in Theorem IV.5.10 of [Hun74].

The induced functor $- \otimes_R N \colon \mathbf{Mod}\text{-}R \to \mathbf{Ab}$ preserves short exact sequences on the right, in the sense that, if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of right R-modules, then the tensored sequence

$$M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$$

is still exact. Accordingly, $- \otimes_R N$ is said to be right exact. But in general exactness on the left fails. The example that shows this is the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

which has multiplication by 2 as the injective map on one side and projection to the quotient on the other. The functor $- \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ takes the above sequence to

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{Id} \mathbb{Z}/2\mathbb{Z} \to 0$$

which again is exact on the right. The rightmost nonzero map is the identity map. The sequence is exact in the middle. The leftmost map is the zero map and thus not injective. Thus, the whole sequence fails to be exact on the left.

Definition 1.1.1. *A left R-module N is defined to be flat if $- \otimes_R N \colon \mathbf{Mod}\text{-}R \to \mathbf{Ab}$ preserves short exact sequences on the left. In other words, an R-module N is flat if $- \otimes_R N$ is left exact.*

An analogous development and corresponding definitions can be given for the right R-module M. Any free or projective left R-module is flat. And in fact flat modules are characterized in the following way.

Theorem 1.1.2 (Lazard's Criterion)**.** *An R-module N is flat if, and only if, it is a filtered colimit of finitely-generated free modules.*

Proof. See [Laz64]. □

1.1.2 Presheaves

Let $\mathscr{C}$ denote a small category. The notation $\mathscr{C}_0$ indicates its set of objects; and $\mathscr{C}_1$, the set of arrows. By a presheaf is meant a functor $P\colon \mathscr{C}^{op} \to \mathbf{Set}$. A copresheaf is one $Q\colon \mathscr{C} \to \mathbf{Set}$. Throughout these are viewed as set-valued representations of $\mathscr{C}$. This viewpoint generalizes the case of modules over a ring R which are certain abelian group-valued additive functors.

Any presheaf P and copresheaf Q admit a tensor-product like construction analogous to that for ordinary modules given above. Let

$$\pi_P\colon \int_{\mathscr{C}} P \to \mathscr{C}$$

denote the usual category of elements of P and its projection to $\mathscr{C}$ as in, for example, §III.7 of [Mac98]. The tensor product of P and Q can be defined as the colimit

$$Q \otimes_{\mathscr{C}} P := \varinjlim Q \circ \pi_P$$

of the composite diagram $Q \circ \pi_P$ taken in the category of sets.

Now, the tensor product $M \otimes_R N$ of modules over a ring is generated by so-called "simple tensors" of the form $m \otimes n$, which really are equivalence classes of pairs (m, n) under the relation generated by the middle linearity expressions. In particular, this means that $m \otimes rn = mr \otimes n$ holds for all $m \in M$, $n \in N$ and $r \in R$. Now, the development of §VII.2 of [MLM92] shows that the tensor product of set-valued functors fits into a coequalizer of sets

$$\coprod_{C,C'} QC' \times \mathscr{C}(C', C) \times PC \underset{\rho}{\overset{\mu}{\rightrightarrows}} \coprod_C QC \times PC \dashrightarrow Q \otimes_{\mathscr{C}} P$$

where the actions of the parallel maps are

$$\mu(x, f, y) = (Qf(x), y) \qquad \rho(x, f, y) = (x, Pf(y)).$$

Write $u \otimes v$ for the image of $(u, v) \in QC \times PC$ in the tensor. Write xf and fy, respectively, for the actions $Qf(x)$ and $Pf(y)$ whenever (x, f, y) is an element of the coproduct on the left above. In this notation there is thus the analogous equation $x \otimes fy = xf \otimes y$ between simple tensors for any such (x, f, y). Thus, the elements of the tensor $Q \otimes_{\mathscr{C}} P$ behave somewhat like those of the tensor $M \otimes_R N$ of modules but without the additivity.

And indeed, by the universal property of the colimit, there results a tensor functor

$$Q \otimes_{\mathscr{C}} -\colon [\mathscr{C}^{op}, \mathbf{Set}] \to \mathbf{Set}$$

where $[\mathscr{C}^{op}, \mathbf{Set}]$ is the 1-category of ordinary presheaves. Let $\mathbf{Set}(Q, -)$ denote the functor

$$\mathbf{Set}(Q, -)\colon \mathbf{Set} \to [\mathscr{C}^{op}, \mathbf{Set}]$$

given by assigning to each set X the functor

$$\mathbf{Set}(Q(-), X) \colon \mathscr{C}^{op} \to \mathbf{Set} \tag{1.1.2}$$

which takes an object $C \in \mathscr{C}_0$ to the hom-set $\mathbf{Set}(QC, X)$. The arrow assignment is given by composition. The functor of Display 1.1.2 will be denoted by '$\mathbf{Set}(Q, X)$' to cut down on notational clutter.

Proposition 1.1.3. *The tensor functor $Q \otimes_{\mathscr{C}} - \colon [\mathscr{C}^{op}, \mathbf{Set}] \to \mathbf{Set}$ has the following properties.*

1. *It fits into a tensor-hom adjunction, that is, a system of isomorphisms*

$$\mathbf{Set}(Q \otimes_{\mathscr{C}} P, X) \cong [\mathscr{C}^{op}, \mathbf{Set}](P, \mathbf{Set}(Q, X)),$$

one for each set X, natural in P and X.

2. *It fits into a diagram*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "Q"] \arrow[d, "\mathbf{y}"'] \arrow[dr, phantom, "\cong"] & \mathbf{Set} \\ {[\mathscr{C}^{op}, \mathbf{Set}]} \arrow[ur, dashed, "Q \otimes_{\mathscr{C}} -"'] & \end{tikzcd}$$

making $Q \otimes_{\mathscr{C}} -$ the left Kan extension of Q along the Yoneda embedding.

Proof. See Theorem I.5.2 of [MLM92] for the first statement. See §X.3 of [Mac98] for Kan extensions and Corollary I.5.4 of [MLM92] for the proof of the second statement. □

Remark 1.1.4. The second condition shows that the Yoneda embedding is a unit for the tensor functor.

Definition 1.1.5. *A copresheaf $Q \colon \mathscr{C} \to \mathbf{Set}$ is flat if the tensor product extension*

$$Q \otimes_{\mathscr{C}} - \colon [\mathscr{C}^{op}, \mathbf{Set}] \longrightarrow \mathbf{Set}$$

is left exact in the sense that it preserves, up to isomorphism, finite limits. Let $\mathbf{Flat}(\mathscr{C}, \mathbf{Set})$ denote the category of flat copresheaves.

Of course this definition, while elegant in its abstraction, is not a very concrete characterization of the phenomenon. Something more tractable is given in the following development.

Definition 1.1.6. *A category $\mathscr{X}$ is filtered if*

1. *it has an object;*

2. *any two objects $X, Y \in \mathscr{X}$ fit into a span $X \leftarrow Z \rightarrow Y$;*

3. *any parallel arrows $f, g\colon X \rightrightarrows Y$ are equalized by an arrow $h\colon Z \to X$, in that $fh = gh$.*

Remark 1.1.7. The terminology in Definition 1.1.6 is consistent with the usage of §VII.6 of [MLM92], whereas to be consistent with §IX.1 of [Mac98], it would have to be "cofiltered" instead. The choice of the former convention is made on aesthetic grounds; for if "filtered" is characterized by the presence of certain spans and equalizing arrows, then "cofiltered" is axiomatized as the presence of certain cospans and coequalizing arrows. In any event, the result characterizing flatness is the following.

Theorem 1.1.8. *A copresheaf $Q\colon \mathscr{C} \to \mathbf{Set}$ is flat if, and only if, either of the following equivalent conditions hold.*

1. *Its category of elements $\int_{\mathscr{C}} Q$ is filtered in the sense of Definition 1.1.6.*

2. *The copresheaf $Q\colon \mathscr{C} \to \mathbf{Set}$ is canonically a filtered colimit of representable functors.*

Proof. For the fist condition, see Theorem VII.6.3 of [MLM92] and its proof. As part of that of the second, note that by Theorem III.7.1 of [Mac98], the functor Q is always colimit over its category of elements. □

Remark 1.1.9. Since the Yoneda embedding is the unit of the tensor product (i.e. representable functors are the "free modules" over $\mathscr{C}$), the second condition of the theorem is the copresheaf analogue of Lazard's Criterion in Theorem 1.1.2.

Partly the interest in flat copresheaves $Q\colon \mathscr{C} \to \mathbf{Set}$ is the following classification result. For this, recall that a geometric morphism between toposes, denoted $f\colon \mathscr{F} \to \mathscr{E}$ is an adjoint pair of functors $f^*\colon \mathscr{E} \rightleftarrows \mathscr{F} \colon f_*$ with f^*, the left adjoint, a finite-limit preserving functor. Call f^* the inverse image and f_* the direct image. A transformation of geometric morphisms is a natural transformation between inverse images. Geometric morphisms and their transformations form a category $\mathbf{Geom}(\mathscr{F}, \mathscr{E})$. If Q is a flat copresheaf, then the tensor-hom adjunction

$$Q \otimes_{\mathscr{C}} - \dashv \mathbf{Set}(Q, -)$$

is thus an example of a geometric morphism $\mathbf{Set} \to [\mathscr{C}^{op}, \mathbf{Set}]$. In general, a point of a topos $\mathscr{E}$ is a geometric morphism $g\colon \mathbf{Set} \to \mathscr{E}$. These are classified in the following way.

Theorem 1.1.10. *There is an equivalence of categories*

$$\mathbf{Flat}(\mathscr{C}, \mathbf{Set}) \simeq \mathbf{Geom}(\mathbf{Set}, [\mathscr{C}^{op}, \mathbf{Set}])$$

sending a flat functor $Q\colon \mathscr{C} \to \mathbf{Set}$ to the geometric morphism determined by its tensor product extension $Q \otimes_{\mathscr{C}} -$.

Proof. See Theorem VII.5.2 of [MLM92] and its proof. □

Remark 1.1.11. The real interest of the theorem is that every point of the presheaf topos appears, up to isomorphism, as a tensor-hom adjunction associated to a flat copresheaf.

Remark 1.1.12. The foregoing development can be redone in the event that **Set** is replaced by a cocomplete topos $\mathscr{E}$. That is, a functor $Q\colon \mathscr{C} \to \mathscr{E}$ admits a tensor product extension along the Yoneda embedding and the definition is that Q is flat if the resulting extension is finite-limit preserving. A generalization of Theorem 1.1.8 is then given in §VII.9 of [MLM92]. The generalization of Theorem 1.1.10 is then given in Theorem VII.7.2 of [MLM92].

1.1.3 Internalization I

R. Diaconescu's thesis [Dia73] and subsequent paper [Dia75] gave a generalization of the foregoing results in the internal category theory of an ambient topos $\mathscr{E}$ replacing **Set**. These results are also summarized over the course of Chapter 2 of [Joh14].

As set up, replace the ambient category of sets by an elementary topos $\mathscr{E}$ and work in the 2-category of internal categories $\mathfrak{Cat}(\mathscr{E})$. Fix an internal category $\mathbb{C}$. Set-valued presheaves and copresheaves are replaced by certain "internal diagrams" which will be seen to be equivalent to internal discrete fibrations and opfibrations over $\mathbb{C}$. A tensor product of an internal discrete opfibration $E\colon \mathbb{E} \to \mathbb{C}$ and an internal discrete fibration $F\colon \mathbb{F} \to \mathbb{C}$ is given as a certain coequalizer $E \otimes_{\mathbb{C}} F$ in $\mathscr{E}$.

Definition 1.1.13. *An internal discrete opfibration $E\colon \mathbb{E} \to \mathbb{C}$ is said to be flat if the induced tensor functor $E \otimes_{\mathbb{C}} -\colon \mathbf{DFib}(\mathbb{C}) \to \mathscr{E}$ on the category of internal discrete fibrations over $\mathbb{C}$ is finite-limit preserving.*

The main result, generalizing Theorem 1.1.8, is the following.

Theorem 1.1.14. *An internal discrete opfibration $E\colon \mathbb{E} \to \mathbb{C}$ is flat if, and only if, $\mathbb{E}$ is (suitably internally) filtered.*

Proof. See, for example, §2.5 and §4.3 of [Joh14]. □

Remark 1.1.15. The proof proceeds by reducing to the exactness of an internal colimit functor

$$\varinjlim_{\mathbb{C}} \colon \mathbf{DFib}(\mathbb{C}) \to \mathscr{E}$$

and showing that $\varinjlim_{\mathbb{C}}$ is left exact if, and only if, $\mathbb{C}$ is suitably internally filtered.

The theorem is a crucial ingredient in the elementary generalization of the classification result, Theorem 1.1.10. Recall that an $\mathscr{E}$-topos is a topos $\mathscr{F}$ equipped with a geometric morphism $f\colon \mathscr{F} \to \mathscr{E}$. The 2-category of $\mathscr{E}$-toposes is denoted by $\mathfrak{Topos}/\mathscr{E}$. Theorem 2.32 of [Joh14] shows that $\mathbf{DFib}(\mathbb{C})$ (denoted by $\mathscr{E}^{\mathbb{C}^{op}}$ in the reference) is an $\mathscr{E}$-topos.

Theorem 1.1.16. *Let $f\colon \mathscr{F} \to \mathscr{E}$ denote a geometric morphism. There is an equivalence of categories*

$$\mathbf{Flat}(f^*\mathbb{C}, \mathscr{F}) \simeq \mathfrak{Topos}/\mathscr{E}(\mathscr{F}, \mathbf{DFib}(\mathbb{C}))$$

natural in $\mathscr{F}$.

Proof. See Theorem 4.34 of [Joh14] or Theorem B3.2.7 of [Joh01]. $\square$

Remark 1.1.17. A crucial step in the proof is that of showing that a certain Yoneda profunctor is flat. Theorem 1.1.14 is used for this purpose.

The ultimate goal of the research in the present thesis is a fully 2-categorical version of Theorem 1.1.16. The first step, of course, is understanding the 2-dimensional analogues of the components of the 1-dimensional result. For example, what is meant by a 2-copresheaf and what it should mean for such a thing to be flat both need to be understood. To this end, as in the classical case, the work should begin with the nicest base 2-category, namely, $\mathfrak{Cat}$ itself, in the place of **Set**. Thus, the thesis firstly studies what should be meant by a flat 2- or pseudo-functor $E\colon \mathfrak{C} \to \mathfrak{Cat}$ on a 2-category $\mathfrak{C}$. On the basis of these results, and the manner of their presentation, a more generic 2-categorical version can be pursued. The setting for the 2-categorical generalization will be the 2-category $\mathfrak{K} = \mathfrak{Cat}(\mathscr{E})$ for an exact category $\mathscr{E}$.

1.2 2-Dimensional Flatness

A good deal is known about flat functors $E\colon \mathfrak{C} \to \mathfrak{Cat}$. For example, in the context of $\mathcal{V}$-categories, flatness seems first to have been studied in §6 of Kelly's [Kel82b] where a base-valued $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{V}$ is defined to be flat if the induced weighted colimit functor $F \star -\colon [\mathcal{A}^{op}, \mathcal{V}] \to \mathcal{V}$ is left exact. Thus, it was recognized at this point that the induced internal colimit functor is a kind of tensor product. This is further confirmed for the case of $\mathcal{V} = \mathbf{Cat}$ in the computations of §4.2.2 of the present work.

Kelly's paper, referenced above, includes as Theorem 6.11 a kind of enriched analogue of Theorem 1.1.8 above where the domain is a finitely-complete $\mathcal{V}$-category $\mathcal{A}$. However, the general connection between flatness and filteredness for the case of $\mathcal{V} = \mathbf{Cat}$ emerged only recently in the paper of M. E. Descotte, E. Dubuc, and M. Szyld, namely, [DDS18b]. Approaches to 2-dimensional filteredness were studied, for example, in Dubuc and Street's [DS06] and in Kennison's [Ken92]. The following filteredness definition of [DDS18b] is meant to be a generalization of Kennison's.

Definition 1.2.1. *Let $\mathfrak{C}$ denote a 2-category and Σ a 1-subcategory of $\mathfrak{C}$ containing all the objects of $\mathfrak{C}$. It is said that $\mathfrak{C}$ is Σ-filtered, or filtered with respect to Σ, if $\mathfrak{C}$ has an object and*

σ**F0** *any two objects $X, Y \in \mathfrak{C}$ fit into a span $X \leftarrow \cdot \rightarrow Y$ with both arrows in Σ;*

σ**F1** *given arrows $f, g\colon X \rightrightarrows Y$ with $g \in \Sigma$, there is $h \in \Sigma$ with $h\colon Z \to X$ and a 2-cell $\alpha\colon hf \Rightarrow hg$; if $f \in \Sigma$ too, then α can be taken to be invertible;*

σ**F2** *given 2-cells $\alpha, \beta\colon f \Rightarrow g$ with $f, g\colon X \to Y$ and $g \in \Sigma$, there is a morphism $h\colon Z \to X$ with $\alpha * h = \beta * h$.*

Remark 1.2.2. This follows the pattern of Definition 1.1.6 with a nonemptiness condition, a spanning condition, an equalizing condition, and a uniqueness condition on 2-cells.

To state the main result of [DDS18b], recall that there is a 2-category of elements construction for any 2- or pseudo-functor $E\colon \mathfrak{C} \to \mathfrak{Cat}$, detailed in §1.4 of [Bir84] and in §I,2.5 of [Gra74]. The main result of [DDS18b], namely, Theorem 4.2.7, is then the following.

Theorem 1.2.3. *A 2-functor $E\colon \mathfrak{C} \to \mathfrak{Cat}$ is flat if, and only if, either of the following equivalent conditions hold.*

1. *The 2-category of elements $\int_{\mathfrak{C}} E$ is filtered in the sense of Definition 1.2.1 with respect to the subcategory of opcartesian arrows.*

2. *The 2-functor $E\colon \mathfrak{C} \to \mathfrak{Cat}$ is a Σ-filtered colimit of representable 2-functors where Σ is a subcategory of opcartesian arrows.*

1.3 Overview of the Thesis

The ultimate goal of the research in the present thesis is a purely elementary phrasing and proof of a flatness result of the form of Theorem 1.1.14 but in a general 2-categorical setting where a base topos $\mathscr{E}$ is replaced by something like a base 2-topos $\mathfrak{K}$ in the sense of M.

Weber's paper [Web07]. The model for this development is the elementary version of the topos-theoretic results summarized in the previous section, namely, §1.1.3. This result is not achieved completely here. Rather a version is given in the case where $\mathfrak{K}$ is a 2-category $\mathfrak{Cat}(\mathscr{E})$ for an exact category $\mathscr{E}$ in the sense of M. Barr [Bar71].

The work of the thesis starts with the case of $\mathscr{E} = \mathbf{Set}$, essentially obtaining the results presented in [DDS18b]. But the approach of the present work is somewhat different. The approach here is geared toward presenting an elementary account that is generalizable in the internal category theory of such a category $\mathscr{E}$. Summarized below are the points of interest. Briefly put, Chapters 2 and 3 of the thesis for the most part present background for the rest of the work; Chapter 4 covers colimits and tensor products; Chapters 5 and 6 are directed toward the elementary generalizations for the case of $\mathfrak{K} = \mathfrak{Cat}(\mathscr{E})$.

1.3.1 Fibrations

The internal version of the set-theoretic results summarized in §1.1.3 view set-valued functors as discrete fibrations and opfibrations, since the latter admit of elementary generalization, whereas the idea of a set-valued functor does not. Chapter 2 will isolate the notion of a discrete 2-fibration for 2-dimensional generalization. This will be a 2-functor $\Pi\colon \mathfrak{F} \to \mathfrak{C}$ whose underlying 1-functor is a fibration and that is locally a discrete opfibration. Some argument is given that this is the correct notion by exhibiting a category of elements construction that is part of an equivalence between functors $F\colon \mathfrak{C}^{op} \to \mathfrak{Cat}$ and discrete 2-fibrations $\Pi\colon \mathfrak{F} \to \mathfrak{C}$. The elementary study of this notion begins in the end of Chapter 3.

1.3.2 Colimits

The main object of the entire work is to describe as explicitly as possible a tensor product extension

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "E"] \arrow[d, "\mathbf{y}"'] \arrow[dr, phantom, "\simeq", near start] & \mathfrak{Cat}. \\ \mathfrak{Hom}(\mathfrak{C}^{op}, \mathfrak{Cat}) \arrow[ur, dashed, "E \otimes_{\mathscr{C}} -"'] & \end{tikzcd}$$

This is done is §4.2 with the explicit construction appearing in Display 4.2.2. Roughly speaking, this is done in the following way. On the basis of the colimit computations of §6.4.0 of [AGV72], the weighted colimit of any pseudo-functor $E\colon \mathfrak{C} \to \mathfrak{Cat}$ is constructed as a category of fractions. If $F\colon \mathfrak{C}^{op} \to \mathfrak{Cat}$ denotes the weight, the universal property of the weighted colimit $E \star F$ is expressed as the existence of natural isomorphisms

$$\mathfrak{Cat}(E \star W, \mathscr{X}) \cong \mathfrak{Hom}(\mathfrak{C}^{op}, \mathfrak{Cat})(W, \mathfrak{Cat}(E, \mathscr{X}))$$

which is formally a 2-dimensional tensor-hom adjunction analogous to that in Proposition 1.1.3. The main result of the present work, Theorem 4.2.11, shows that the proposed computation yields such an isomorphism. For this reason and the fact that $E \star F$ turns out to be a coinverter and a codescent object, the notation $E \otimes_{\mathfrak{C}} F = E \star F$ is adopted. The required extension property of the first display is shown to hold for this construction in Corollaries 4.2.15 and 4.2.16. Now, make the following definition.

Definition 1.3.1. *A pseudo-functor $E\colon \mathfrak{C} \to \mathfrak{Cat}$ is flat if the induced tensor 2-functor*

$$E \otimes_{\mathfrak{C}} -\colon \mathfrak{Hom}(\mathfrak{C}^{op}, \mathfrak{Cat}) \to \mathfrak{Cat}$$

preserves up to equivalence all finite weighted limits.

1.3.3 Flatness

In §4.4, filteredness conditions slightly refining those of Definition 1.2.1 are obtained from the assumption that $E \otimes_{\mathfrak{C}} -$ as above is left exact. These conditions are axiomatized in Definition 4.4.8. It is also seen that the obtained conditions imply those of Definition 1.2.1.

Now, in fact a converse for this necessity result is true. That is, it can be seen that if the 2-category of elements construction is filtered in the sense of our Definition 4.4.8, then the tensor $E \otimes_{\mathfrak{C}} -$ is left exact. By the limit-construction result of R. Street in [Str76], this can be seen by showing that the tensor preserves the terminal object, binary products, equalizers, and cotensors with **2**. This converse result is proved in the internal category theory of an exact category $\mathscr{E}$ over the course of Chapters 5 and 6. But it is worth pointing out here that it can be seen for the case of $\mathscr{E} = \mathbf{Set}$ by mimicking the elementary proofs for the case of cotensors given in §6.3. This involves some tedious cone-building to show that certain diagrams commute in various tensor products. This cone-building is avoided, at least for conical limits, by a technical contrivance in the internalization of Chapters 5 and 6.

1.3.4 Internalization II: Internal Calculus of Fractions

The elementary cone-building mentioned in the previous subsection requires knowing what the morphisms of various tensor products $E \otimes_{\mathfrak{C}} F$ look like. For the tensor is a category of fractions formed by inverting pairs of cartesian morphisms of the total 2-categories $\mathfrak{E}$ and $\mathfrak{F}$. Thus, without further assumptions, all that is known about the arrows of the tensor is that they are certain formal sequences of arrows modulo the necessary equations. Seemingly fortuitously, the 2-filteredness conditions axiomatized in Definition 4.4.8 are shown in Chapter 5 to imply that the category of fractions giving the tensor product $E \otimes_{\mathfrak{C}} F$ is formed via a right calculus of fractions. This is proved in Theorem 5.1.2.

The more abstract characterization given in §4.3.1 is that the tensor product $E \otimes_{\mathfrak{C}} F$ arises as the reflexive coinverter of the 2-cell induced from the opcleavage for E and the cleavage for F. On the basis of this result, a tensor product in the internal case can be defined to be a reflexive coinverter of 2-cells arising from internalized cleavages. Indeed this is the approach that is taken in §4.3.1. Whether the tensor exists is then the natural question.

There are two parts to our approach to existence, basically suggested by the comments in the first paragraph above. First is to describe the process of forming a localization through a right calculus of fractions in internal category theory; second is to show that under our elementary 2-filteredness conditions, the reflexive coinverter of an appropriate 2-cell can be constructed through a right calculus of fractions. The first part is solved in §5.2. The second is solved in §5.3, where it is seen that a suitable internalization of the 2-filteredness axioms of Definition 4.4.8 implies that the 2-cell coming from the cleavages for the internal discrete 2-fibrations admits a right calculus of fractions in the internal sense of Definition 5.2.1.

1.3.5 Internalization III: Limit Preservation

The internal category of fractions construction has the following consequence that allows circumvention of all tedious cone-constructions in the case of conical limits. It was noted above that the arrows of the internal localization are obtained as a certain coequalizer in a slice of $\mathscr{E}$. The parallel arrows coequalized turn out to be domain and codomain morphisms of an internal groupoid under the 2-filteredness hypothesis in Definition 5.3.1. Thus, the consequence, summarized in Theorem 6.1.7, is that the arrow object of the internal tensor product $\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F}$ is equal to the internal colimit functor from §1.1.3 for a certain internal groupoid. This will be shown in §6.1.

It turns out, additionally, that 2-filteredness in the sense of Definition 5.3.1 implies that the groupoid is filtered in the ordinary internal sense presented, for example, in §2.5 of [Joh14]. This filteredness is equivalent to exactness of the internal colimit functor (Theorem 2.58 and Theorem 2.59 in the reference). And as a result, it will be seen in §6.2 and ultimately Theorem 6.2.6, that the tensor $\mathcal{E} \otimes_{\mathcal{C}} -$ preserves all finite conical limits.

It will be left to see that the tensor preserves cotensors with **2**. The following sections, namely, §6.3 and §6.4 give first the elementary proof in the case of $\mathscr{E} = \mathbf{Set}$; and then a proof that cotensors are preserved in the case of $\mathscr{E}$ exact and $\mathfrak{K} = \mathfrak{Cat}(\mathscr{E})$. The stress in the final section is on the proof that the canonical internal functor is suitably internally essentially surjective. The proof of internally fully faithful follows a similar pattern.

1.4 Application: Classification of Principal 2-Bundles

Throughout let G denote a topological group. As in §VIII.1 of [MLM92], a principal G-bundle over a space X is a continuous map $p\colon E \to X$ equipped with a fiber-wise continuous (left) action of G that is "locally trivial" in the sense that X admits an open cover $\{U_i\}$ and a system of appropriately compatible isomorphisms $\phi_i\colon G \times U_i \cong p^{-1}(U_i)$ for each i. Connections on principal bundles model particle trajectories along 1-dimensional paths.

When G is discrete, a principal G-bundle is equivalently a so-called "torsor," namely, a étale morphism of spaces $p\colon E \to X$ equipped with a fiber-wise continuous (left) action of G that is free and transitive in each fiber. Phrased in terms of sheaves on a space, a torsor is thus a sheaf F on X, such that $F \to 1$ is an epimorphism, together with an action $\mu\colon \Delta G \times F \to F$ that is free and transitive in the sense that

$$\langle \mu, \pi_2 \rangle\colon \Delta G \times F \xrightarrow{\cong} F \times F$$

is an isomorphism. This motivates the following.

Definition 1.4.1. *A G-torsor in a topos $\mathscr{E}$ over* **Set** *is an object $T \in \mathscr{E}$ equipped with an action $\mu\colon g^*G \times T \to T$ such that $T \to 1$ is an epimorphism and for which*

$$\langle \mu, \pi_2 \rangle\colon g^*G \times T \xrightarrow{\cong} T \times T$$

is an isomorphism where g is the geometric morphism $g\colon \mathscr{E} \to \mathbf{Set}$. Let $\mathbf{Tor}(\mathscr{E}, G)$ denote the category of G-torsors in $\mathscr{E}$ and suitably equivariant maps between them.

The characterization is that the topos $\mathbf{B}G$ of right G-sets classifies torsors, hence principal bundles, in the following sense.

Theorem 1.4.2. *For any discrete group G and topos $g\colon \mathscr{E} \to$* **Set** *over sets, there is an equivalence of categories*

$$\mathbf{Tor}(\mathscr{E}, G) \simeq \mathbf{Geom}(\mathscr{E}, \mathbf{B}G)$$

where $\mathbf{B}G$ denotes the topos of right G-sets.

Proof. See Theorem VIII.2.7 of [MLM92]. The generalization of Theorem 1.1.10 shows that geometric morphisms correspond to flat functors. Thus, the point of the proof consists in showing that flat functors $G \to \mathscr{E}$ correspond to G-torsors in $\mathscr{E}$. For a more elementary line of development, see §8.3 of [Joh14]. □

The categorification of gauge theory in the work of J. Baez and coauthors (for example, [BS07], [BL04], [BC04]) is an area of potential application of the results of the thesis. In this research program, the intended use of higher connections on higher principal bundles is to model trajectories of strings along surfaces. First recall that a (strict) 2-group $\mathscr{G}$ is a group object in **Cat**. A topological 2-group $\mathscr{G}$ is a group object in category objects in a nice category of spaces. As part of the higher gauge theory program, T. Bartels [Bar04] developed the notion of a principal 2-$\mathscr{G}$-bundle for a topological 2-group $\mathscr{G}$. This involves the definition of 2-space and what it means for a 2-group action to be "locally trivial." The idea of 2-space was further pursued in U. Schreiber's thesis [Sch05] as a category object in a certain category of smooth spaces, and principal 2-$\mathscr{G}$-bundles were developed there in that context. The goal of our application is to show that principal 2-bundles, at least for discrete topological 2-groups, are essentially the same as flat functors on $\mathscr{G}$ valued in some 2-topos-like 2-category of spaces, as in the proof of Theorem 1.4.1 above. The idea is that this result would facilitate showing that some 2-category of (potentially category-valued) representations of $\mathscr{G}$ is a classifying geometric 2-topos for principal 2-$\mathscr{G}$-bundles.

Chapter 2

Background and Notation

The present chapter summarizes the needed background on 2-categories, fibrations, and exact 1-categories that will be used throughout. Some original material appears in the section on fibrations, where the notion of a "discrete 2-fibration" is isolated in Definition 2.2.15 as one of the central definitions of the thesis.

2.1 2-Categories

Roughly, the assumed background in the theory of 2-categories corresponds to Chapters I,2 and I,3 of Gray's [Gra74]. Other references are Chapter 7 of [Bor94], Chapter B1 of [Joh01], and the overview paper of [KS74]. The material on 2-monads can be found in, for example, §1 of [Lac02]. Some notation and terminology will differ, so here will be summarized notions used throughout the paper.

A 2-category $\mathfrak{K}$ consists of objects, 1-cells, and transformations satisfying well-known axioms (see §7.1 of [Bor94] for example). Vertical composition of 2-cells will be denoted by juxtapostion '$\beta\alpha$'; while horizontal composition is denoted by '$*$' as in $\gamma * \delta$. When horizontally composing a 2-cell with a vertical identity morphism write, for example, '$\alpha * f$' or '$g * \beta$'. In general $\mathfrak{K}(A, B)$ denotes the vertical category of morphisms $A \to B$ of $\mathfrak{K}$ and 2-cells between them. Any 2-category is a bicategory in the sense of [Bén67] with strict unit and associativity.

The notation '$\mathfrak{K}^{op}$' indicates the 1-dimensional dual of $\mathfrak{K}$; and '$\mathfrak{K}^{co}$' denotes the 2-dimensional dual with 2-cells formally reversed; and '$\mathfrak{K}^{coop}$' indicates the 2-category with both 1- and 2-cells formally reversed.

The basic example is the 2-category $\mathfrak{Cat}$ of small categories relative to a fixed category of sets **Set**, functors between them, and their natural transformations. The notation $\mathfrak{CAT}$ will be used for an enlarged 2-category of categories containing a 1-category **Set** of sets as an object. The notation **Cat** is used for the 1-category of small categories and functors between them, without considering the 2-dimensional structure. Generally, any 2-category $\mathfrak{A}$ has an underlying 1-category $\mathfrak{A}_0$ obtained by forgetting the 2-cells. Thus, $\mathfrak{Cat}_0$ and **Cat** are notation for the same category.

Example 2.1.1. *Every 1-category is a "locally discrete" 2-category whose 2-cells are identities.*

Example 2.1.2. *Given a 2-category $\mathfrak{K}$, the 2-slice over $A \in \mathfrak{K}$ is the 2-category whose objects are morphisms $x\colon X \to A$. A morphism from $x\colon X \to A$ to $y\colon Y \to A$ is a morphism $f\colon X \to Y$ of $\mathfrak{K}$ such that $yf = x$ holds. A 2-cell between such morphisms f and g is one of $\mathfrak{K}$ of the form $\alpha\colon f \Rightarrow g$ such that $y * \alpha = x$ holds. Denote the 2-slice over A by $\mathfrak{K}/A$, as usual.*

Example 2.1.3. *The lax slice of a 2-category $\mathfrak{K}$ is the same as the 2-slice above, with the difference that a morphism from $x\colon X \to A$ to $y\colon Y \to A$ is a morphism $f\colon X \to Y$ and a 2-cell $\alpha\colon x \Rightarrow yf$. The 2-cells then satisfy an analogous commutativity condition. Denote the lax slice by $\mathfrak{K}\,/\!/\,A$.*

Example 2.1.4. *Given a 2-category $\mathfrak{K}$, the 2-arrow category $\mathfrak{K}^{\mathbf{2}}$ has as its objects morphisms of $\mathfrak{K}$, as its arrows those pairs of arrows of $\mathfrak{K}$ making commutative squares in $\mathfrak{K}$, and as its 2-cells those cells making two composites yielding an equality of 2-cells*

$$\begin{tikzcd} X \arrow[r] \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\Rightarrow"] \arrow[dr, phantom, "="] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "="] & Y \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\Rightarrow"] \\ Z \arrow[r] & W \end{tikzcd}$$

For a discrete 2-category, this definition reduces to that of the usual arrow category.

Definition 2.1.5. *A pseudo-functor, or "homomorphism" as in [Bén67], between 2-categories $F\colon \mathfrak{K} \to \mathfrak{L}$ assigns to each object $A \in \mathfrak{K}$ an object FA of $\mathfrak{L}$; to each arrow f of $\mathfrak{K}$ an arrow Ff of $\mathfrak{L}$; and to each 2-cell α of $\mathfrak{K}$ a 2-cell $F\alpha$ of $\mathfrak{L}$; and includes coherence isomorphisms $\phi_{f,g}\colon FgFf \Rightarrow F(gf)$ and $\phi_A\colon 1_{FA} \to F1_A$ for each object $A \in \mathfrak{K}$ and composable pair of arrows f and g all satisfying the axioms of Definition B1.1.2 on p.238 of [Joh01]. Among these is the statement that $F(\beta\alpha) = F\beta F\alpha$ holds for any vertically composable 2-cells α and β. There is also the requirement that for horizontally composable 2-cells*

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & C \end{tikzcd}$$

the relationship between the images under F is described by the equation

$$\phi_{g,k}(F\beta * F\alpha) = F(\beta * \alpha)\phi_{f,h}. \tag{2.1.1}$$

A pseudo-functor is "normalized" if the ϕ_A as above are identities. Pseudo-functors will always be assumed to be normalized in the present work. A pseudo-functor is called a 2-functor if all of the cells $\phi_{f,g}$ are identities.

Definition 2.1.6. *A lax-natural transformation* $\alpha\colon F \to G$ *of pseudo-functors* $F, G\colon \mathfrak{K} \rightrightarrows \mathfrak{L}$ *consists of a family of arrows* $\alpha_A\colon FA \to GA$ *of* $\mathfrak{L}$ *indexed over the objects* $A \in \mathfrak{K}$ *together with, for each arrow* f *of* $\mathfrak{K}$, *a 2-cell*

$$\begin{tikzcd}
FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\alpha_f \Rightarrow"] & GA \arrow[d, "Gf"] \\
FB \arrow[r, "\alpha_B"'] & GB
\end{tikzcd}$$

satisfying the following two compatibility conditions.

1. *For any composable arrows* f *and* g *of* $\mathfrak{K}$, *there is an equality of 2-cells*

$$\begin{tikzcd}
FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Rightarrow"] \arrow[dd, bend right=60, "F(gf)"', ""{name=L}] & GA \arrow[d, "Gf"] \\
FB \arrow[r, "\alpha_B"'] \arrow[d, "Fg"'] \arrow[dr, phantom, "\Rightarrow"] \arrow[phantom, to=L, "\cong"] & GB \arrow[d, "Gg"] \\
FC \arrow[r, "\alpha_C"'] & GC
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\alpha_A"] \arrow[d, "F(gf)"'] \arrow[dr, phantom, "\Rightarrow"] & GA \arrow[d, "G(gf)"] \arrow[dr, bend left, "Gf"] \arrow[r, phantom, "\cong", shift right=4ex] & \\
FC \arrow[r, "\alpha_C"'] & GC & GB. \arrow[l, bend left, "Gg"]
\end{tikzcd}$$

2. *For any 2-cell* $\theta\colon f \Rightarrow g$ *of* $\mathfrak{K}$, *there is an equality of 2-cells as depicted in the diagram*

$$\begin{tikzcd}
FA \arrow[r, "\alpha_A"] \arrow[d, bend right, "Ff"'] \arrow[d, bend left, "Fg"] \arrow[d, phantom, "\overset{F\theta}{\Rightarrow}"] \arrow[dr, phantom, "\Rightarrow"] & GA \arrow[d, "Gg"] \\
FB \arrow[r, "\alpha_B"'] & GB
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Rightarrow"] & GA \arrow[d, bend right, "Gf"'] \arrow[d, bend left, "Gg"] \arrow[d, phantom, "\overset{G\theta}{\Rightarrow}"] \\
FB \arrow[r, "\alpha_B"'] & GB.
\end{tikzcd}$$

A lax-natural transformation is "pseudo natural" if the cells α_f *are invertible. If they are identities, the transformation is "2-natural."*

Remark 2.1.7. Pseudo-naturality is the basic concept in the present work. Lax natural transformations would, in the language of §I,2.4 of [Gra74], be called "quasi-natural."

Definition 2.1.8. *A modification* $m\colon \alpha \to \beta$ *of lax-natural transformations* $\alpha, \beta\colon F \rightrightarrows G$ *consists of a family of 2-cells* $m_A\colon \alpha_A \Rightarrow \beta_A$ *of* $\mathfrak{L}$ *satisfying the following condition.*

1. *For an arrow $f\colon A \to B$ of $\mathfrak{K}$, there is required an equality of 2-cells*

$$
\begin{tikzcd}
FA \arrow[rr, bend left, "\beta_A"] \arrow[rr, bend right, "\alpha_A"'] \arrow[rr, phantom, "\Uparrow m_A"] \arrow[d, "Ff"'] & & GA \arrow[d, "Gf"] \\
FB \arrow[rr, bend right, "\alpha_B"'] \arrow[urr, phantom, "\Rightarrow \alpha_f"] & & GB
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FA \arrow[rr, bend left, "\beta_A"] \arrow[d, "Ff"'] \arrow[drr, phantom, "\Rightarrow \beta_f"] & & GA \arrow[d, "Gf"] \\
FB \arrow[rr, bend left, "\beta_B"] \arrow[rr, bend right, "\alpha_B"'] \arrow[rr, phantom, "\Uparrow m_B"] & & GB.
\end{tikzcd}
$$

Example 2.1.9. *The 2-functors between 2-categories $F\colon \mathfrak{A} \to \mathfrak{B}$, with 2-natural transformations and modifications, form a 2-category, denoted using the "internal hom" notation $[\mathfrak{A}, \mathfrak{B}]$. In particular, $[\mathfrak{C}, \mathfrak{Cat}]$ denotes the 2-category of category-valued 2-functors, 2-natural transformations, and modifications. Pseudo-functors $\mathfrak{K} \to \mathfrak{L}$, together with pseudo-natural transformations and modifications between them, form a 2-category, denoted $\mathfrak{Hom}(\mathfrak{K}, \mathfrak{L})$. Thus, in particular, $\mathfrak{Hom}(\mathfrak{K}, \mathfrak{Cat})$ denotes the 2-category of category-valued pseudo-functors, pseudo-natural transformations, and modifications.*

Remark 2.1.10. The '$\mathfrak{Hom}$' notation will be used since pseudo-functors are also called "homomorphisms" in [Bén67]. In general this notation will always indicate "pseudo" whereas the brackets '$[-,-]$' will always mean taking everything as strict as possible. When dealing with 1-categories there is no distinction, so the brackets will be used. Since strict "2-structure" is always pseudo, there is an inclusion

$$[\mathfrak{A}, \mathfrak{B}] \to \mathfrak{Hom}(\mathfrak{A}, \mathfrak{B}).$$

If $\mathscr{C}$ is a 1-category, viewed as a locally discrete 2-category, both $[\mathscr{C}^{op}, \mathfrak{Cat}]$ and $\mathfrak{Hom}(\mathscr{C}^{op}, \mathfrak{Cat})$ are potential 2-categorical analogues of the category of ordinary presheaves $[\mathscr{C}^{op}, \mathbf{Set}]$.

Example 2.1.11. *Small 2-categories, 2-functors, 2-natural transformations, and modifications form a 3-category in the sense of §7.3 of [Bor94]. Roughly speaking, a 3-category is in a suitable sense "enriched in 2-categories." The "hom objects" are precisely the strict 2-categories $[\mathfrak{A}, \mathfrak{B}]$ in the bracket notation above.*

Every set is a category whose objects and arrows are just the members of the set. Such a category is "discrete" and there is a "discrete category" functor $disc\colon \mathbf{Set} \to \mathbf{Cat}$ of ordinary 1-categories. The discrete category functor is right adjoint to the "connected components" functor $\pi_0\colon \mathbf{Cat} \to \mathbf{Set}$ given by sending a category $\mathscr{C}$ to the set of its connected components. In other words, $\pi_0\mathscr{C}$ is given as a coequalizer

$$\begin{tikzcd} \mathscr{C}_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \mathscr{C}_0 \arrow[r] & \pi_0\mathscr{C} \end{tikzcd}$$

of the domain and codomain functions coming with the category structure. As observed, for example, in §I,2.3 of [Gra74], there is a similar situation in dimension 2. For 1-categories can be viewed as "locally discrete" 2-categories in the sense that all 2-cells are identities. This extends to a 2-functor $disc\colon \mathfrak{Cat} \to 2\text{-}\mathfrak{Cat}$. Again $disc$ has a left adjoint, a "connected components" functor, given by taking a 2-category $\mathfrak{A}$ to the 1-category $\pi_0\mathfrak{A}$, having the same objects and whose morphisms between say $A, B \in \mathfrak{A}$ are given by taking the connected components of the hom-category

$$(\pi_0\mathfrak{A})(A,B) := \pi_0\mathfrak{A}(A,B).$$

In other words, $\pi_0\mathfrak{A}$ is given by taking connected components locally. This construction also makes sense for bicategories. In particular, it is discussed in §7.1 of [Bén67] where it is called the "Poincaré category" of the bicategory.

2.1.1 2-Monads and their Algebras

Definition 2.1.12. *A 2-monad on a 2-category $\mathfrak{K}$ is a 2-functor $T\colon \mathfrak{K} \to \mathfrak{K}$ with 2-natural transformations $\eta\colon 1 \to T$ and $\mu\colon TT \to T$ for which the following diagrams commute:*

$$\begin{tikzcd} TTT \arrow[r, "T\mu"] \arrow[d, "\mu T"'] \arrow[dr, phantom, "="] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "1"'] & TT \arrow[d, "\mu"] & T. \arrow[l, "\eta T"'] \arrow[dl, "1"] \\ & T & \end{tikzcd}$$

Definition 2.1.13. *A lax algebra for $T\colon \mathfrak{K} \to \mathfrak{K}$ is an object A with an arrow $a\colon TA \to A$ and 2-cells*

$$\begin{tikzcd} TTA \arrow[r, "\mu_A"] \arrow[d, "Ta"'] \arrow[dr, phantom, "\tau \Rightarrow"] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"'] & TA \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=3, "\iota \Rightarrow"] & TA \arrow[d, "a"] \\ & A \end{tikzcd}$$

satisfying the following conditions.

1. An associativity condition, namely, that there is an equality of 2-cells as in

$$
\begin{tikzcd}
TTTA \arrow[r, "\mu_{TA}"] \arrow[d, "TTa"'] \arrow[dr, "T\mu_A"] & TTA \arrow[dr, "\mu_A"] & \\
TTA \arrow[r, phantom, "T\tau \Rightarrow"] \arrow[dr, "Ta"'] & TTA \arrow[r, "\mu_A"] \arrow[d, "Ta"'] \arrow[dr, phantom, "\tau \Rightarrow"] & TA \arrow[d, "a"] \\
& TA \arrow[r, "a"'] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
TTTA \arrow[r, "\mu_{TA}"] \arrow[d, "TTa"'] & TTA \arrow[d, "Ta"] \arrow[dr, "\mu_A"] & \\
TTA \arrow[r, "\mu_A"'] \arrow[dr, "Ta"'] & TA \arrow[r, phantom, "\tau \Rightarrow"] \arrow[dr, "a"] \arrow[d, phantom, "\tau \Rightarrow"] & TA \arrow[d, "a"] \\
& TA \arrow[r, "a"'] & A.
\end{tikzcd}
$$

2. A unit condition asserting that each of the composite 2-cells is equal to the identity on a:

$$
\begin{tikzcd}
& & TA \arrow[dr, "a"] & \\
TA \arrow[urr, bend left, "1"] \arrow[r, "T\eta_A"'] \arrow[drr, bend right, "1"'] \arrow[ur, phantom, "T\iota \Downarrow"] & TTA \arrow[ur, "Ta"] \arrow[dr, "\mu_A"'] \arrow[rr, phantom, "\tau \Downarrow"] & & A \\
& & TA \arrow[ur, "a"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
TA \arrow[r, "\eta_{TA}"] \arrow[d, "a"'] & TTA \arrow[r, "\mu_A"] \arrow[d, "Ta"'] \arrow[dr, phantom, "\tau \Rightarrow"] & TA \arrow[d, "a"] \\
A \arrow[r, "\eta_A"'] \arrow[rr, bend right=60, "1"'] \arrow[rr, phantom, bend right=30, "\iota \Uparrow"] & TA \arrow[r, "a"'] & A
\end{tikzcd}
$$

A lax algebra is a pseudo-algebra if τ and ι are invertible; and is a strict 2-algebra if they are identity cells. In general an algebra is "normalized" if ι is an identity whether or not τ is one.

Proposition 2.1.14. *Pseudo-algebras, their homomorphisms, and transformations between them comprise a 2-category, denote by $\mathfrak{Alg}(T)$.*

Proof. The definitions of homomorphism and transformation are stated in [Lac02]. □

2.2 Fibrations and Category of Elements Constructions

Throughout let $\mathscr{C}$ denote a small category. Recall the following standard definition.

Definition 2.2.1. *A discrete fibration over $\mathscr{C}$ is a functor $F\colon \mathscr{F} \to \mathscr{C}$ such that for each morphism $f\colon C \to FX$ with $X \in \mathscr{F}$, there is a unique morphism $Y \to X$ of $\mathscr{F}$ above f. A functor $E\colon \mathscr{E} \to \mathscr{C}$ is a discrete opfibration if E^{op} is a discrete fibration. Let* $\mathbf{DFib}(\mathscr{C})$ *denote the category of discrete fibrations over $\mathscr{C}$ and* $\mathbf{DOpf}(\mathscr{C})$ *denote the category of discrete opfibrations over $\mathscr{C}$.*

Remark 2.2.2. Notice that F as above is a discrete fibration if, and only if, the square

$$\begin{tikzcd}
\mathscr{F}_1 \arrow[r, "F_1"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{C}_1 \arrow[d, "d_1"] \\
\mathscr{F}_0 \arrow[r, "F_0"'] & \mathscr{C}_0
\end{tikzcd}$$

is a pullback in **Set**. A functor E as in the definition is a discrete opfibration if, and only if, an analogous square with domain arrows replacing the codomain arrows is a pullback.

For each set-valued functor $E\colon \mathscr{C} \to \mathbf{Set}$, there is an associated category of elements, or "Grothendieck semi-direct product," detailed for example in §II.6 and §III.7 of [Mac98], yielding a discrete opfibration

$$\pi_E\colon \int_{\mathscr{C}} E \to \mathscr{C}.$$

The source category has as objects pairs (C, x) with $C \in \mathscr{C}_0$ and $x \in EC$ and as morphisms $(C, x) \to (D, y)$ those morphisms $f\colon C \to D$ of $\mathscr{C}$ with $Ef(x) = y$.

Theorem 2.2.3. *The category of elements construction is half of an equivalence of categories*

$$\mathbf{DOpf}(\mathscr{C}) \simeq [\mathscr{C}, \mathbf{Set}].$$

*The pseudo-inverse sends a discrete opfibration $e\colon \mathscr{E} \to \mathscr{C}$ to the functor $\mathscr{C} \to$ **Set** whose action on $C \in \mathscr{C}$ is to take the fiber of E above C.*

Definition 2.2.4. *A functor $F\colon \mathscr{F} \to \mathscr{C}$ is a fibration if for each $x\colon X \to FA$ there is an $f\colon B \to A$ of $\mathscr{F}$ having the property that whenever $h\colon C \to A$ makes a commutative triangle $xu = Fh$ as below there is a unique F-lift $C \to B$ over u making a commutative triangle in $\mathscr{F}$ as indicated in the following picture*

$$\begin{tikzcd}
 & C \arrow[dr, dashed, "!"'] \arrow[drr, "h"] & & \\
\mathscr{F} \arrow[dd, "F"'] & & B \arrow[r, "f"'] & A \\
 & FC \arrow[dr, "u"'] \arrow[drr, "Fh"] & & \\
\mathscr{C} & & X \arrow[r, "x"'] & FA.
\end{tikzcd}$$

Such a morphism f is "cartesian" over x. A morphism of $\mathscr{F}$ is F-vertical if its image under F is an identity. The fiber of F over an object $C \in \mathscr{F}$ is the subcategory of $\mathscr{F}$ of objects and vertical morphisms over C via F. A functor $E\colon \mathscr{E} \to \mathscr{C}$ is an opfibration if E^{op} is a fibration; in this case the morphisms of $\mathscr{E}$ having the special lifting property are called "opcartesian."

A cleavage σ for a fibration specifies a cartesian morphism in $\mathscr{F}$ for each such $x\colon X \to FA$ in $\mathscr{C}$. Denote the chosen cartesian morphism by $\sigma(x, A)$. A fibration with a cleavage is said to be "cloven." Notice that each discrete fibration is a cloven fibration. An opfibration with chosen opcartesian morphisms is said to be "opcloven" or to be equipped with an "opcleavage."

Remark 2.2.5. In general a cleavage σ for a fibration $F\colon \mathscr{F} \to \mathscr{C}$ need not be functorial. That is, given composable arrows $f\colon X \to Y$ and $g\colon Y \to FB$ of $\mathscr{C}$, there is a diagram of chosen cartesian arrows in $\mathscr{F}$ of the form

$$\begin{tikzcd}
f^*g^*B \arrow[r, "\sigma(f, g^*B)"] \arrow[d, dashed, "\cong"'] \arrow[dr, phantom, "="] & g^*B \arrow[d, "\sigma(g, B)"] \\
(gf)^*B \arrow[r, "\sigma(gf, B)"'] & B.
\end{tikzcd}$$

The dashed arrow exists since a composition of cartesian morphisms is again cartesian. It is an isomorphism by the uniqueness aspect of the definition. But in general this isomorphism is not an identity. When every such isomorphism is an identity, the fibration $F\colon \mathscr{F} \to \mathscr{C}$ is said to be split. The difference between cloven and split fibrations is precisely the difference between category-valued pseudo-functors and 2-functors, as will be seen presently.

Let $\mathfrak{cFib}(\mathscr{C})$ denote the 2-category of cloven fibrations over $\mathscr{C}$, whose arrows are functors over $\mathscr{C}$ that preserve cartesian morphisms (but not necessarily the cleavage), and whose 2-cells are those transformations between such functors whose components are vertical. Let $\mathfrak{sFib}(\mathscr{C})$ denote the full sub-2-category of split fibrations over $\mathscr{C}$. Dually, $\mathfrak{cOpf}(\mathscr{C})$ denotes the 2-category of opcloven opfibrations over $\mathscr{C}$ and $\mathfrak{sOpf}(\mathscr{C})$ the 2-category of split opfibrations over $\mathscr{C}$.

As set-up for the next result, consider the 2-monad in the sense of Definition 2.1.12 on $\mathfrak{Cat}/\mathscr{C}$ given by sending a functor $H\colon \mathscr{X} \to \mathscr{C}$ to the pullback d_1^*H as in

$$\begin{tikzcd}
\mathscr{C}^{\mathbf{2}} \times_{\mathscr{C}} \mathscr{X} \arrow[r, "\pi_2"] \arrow[d, "d_1^*H"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{X} \arrow[d, "H"] \\
\mathscr{C}^{\mathbf{2}} \arrow[r, "d_1"'] & \mathscr{C}
\end{tikzcd}$$

composed with the domain functor $d_0 \colon \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$. Let T denote this 2-monad.

Theorem 2.2.6. *Cloven fibrations over $\mathscr{C}$ are precisely the normalized pseudo-T-algebras as in Definition 2.1.13 for T as above. Additionally, a cleavage for a fibration is, equivalently, a natural transformation*

$$\begin{tikzcd} \mathscr{C}^{\mathbf{2}} \times_{\mathscr{C}} \mathscr{F} \arrow[rr, bend left, "m"] \arrow[rr, bend right, "\pi_{\mathscr{F}}"'] \arrow[rr, phantom, "\sigma \Downarrow"] & & \mathscr{F} \end{tikzcd}$$

where m denotes the action coming with the pseudo-algebra structure. Split fibrations are the strict 2-algebras for the same 2-monad. Dually, cloven/split opfibrations over $\mathscr{C}$ are precisely the normalized pseudo/strict 2-algebras for the 2-monad on $\mathfrak{Cat}/\mathscr{C}$ given by pulling back along $d_0 \colon \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ and then composing with $d_1 \colon \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$.

Proof. The correspondence is discussed in §I,3.5 of [Gra74]. A detailed account is in [Gra66]. The correspondence led to the definition of a fibration in a 2-category as a certain pseudo-algebra in §2 of [Str74]. This approach will be followed in Definition 3.3.1 below. □

Now, start with a pseudo-functor $E \colon \mathscr{C} \to \mathfrak{Cat}$. Denote the image of $f \colon C \to D$ in $\mathscr{C}$ by $f_! \colon EC \to ED$. As in the discrete case, there is an associated opfibration arising as a category of elements construction

$$\pi_E \colon \int_{\mathscr{C}} E \to \mathscr{C}.$$

The source category has objects pairs (C, X) with $X \in EC$ and as morphisms $(C, X) \to (D, Y)$ pairs (f, u) where $f \colon C \to D$ and $u \colon f_! X \to Y$ is a morphism of ED. The units and composition are described for example in §B1.3 of [Joh01].

Theorem 2.2.7. *The category of elements construction is one-half of an equivalence of 2-categories*

$$\mathfrak{cOpf}(\mathscr{C}) \simeq \mathfrak{Hom}(\mathscr{C}, \mathfrak{Cat}).$$

Again the pseudo-inverse sends a cloven opfibration E to the pseudo-functor that associates to each $C \in \mathscr{C}$ the fiber of E over it. Moreover, this equivalence restricts to one

$$\mathfrak{sOpf}(\mathscr{C}) \simeq [\mathscr{C}, \mathfrak{Cat}]$$

between split opfibrations and strict category-valued 2-functors.

Proof. See Theorem B1.3.5 of [Joh01] for example. □

Several sources, namely, §I,2.9 of [Gra74] and more recently §2.1.6 of [Buc14], boost up both the domain and codomain of the given representation to a 2-functor $E\colon \mathfrak{C} \to 2\text{-}\mathfrak{Cat}$ on an honest 2-category $\mathfrak{C}$ and then give an associated 2-category of elements construction

$$\pi_E\colon \int_{\mathfrak{C}} E \to \mathfrak{C}.$$

Each source give a definition of a 2-(op)fibration and show a correspondence between 2-category-valued functors and 2-fibrations, one direction of which is the 2-category of elements construction. This should be seen as a 2-dimensional analogue of the correspondence between category-valued functors on a 1-category and cloven opfibrations as in Theorem 2.2.7 above.

Now, there is an evident gap in the sense that there ought to be an analogue of Theorem 2.2.3 for the 2-dimensional case. That is, just as set-valued functors are the discrete objects relative to 1-categorical opfibrations, there should be a concept of discrete 2-fibration giving the discrete objects relative to 2-fibrations. The insight is that objects of **Set** are discrete relative to objects of $\mathfrak{Cat}$; analogously, objects of $\mathfrak{Cat}$ are discrete relative to objects of 2-$\mathfrak{Cat}$. Thus, the representation to be considered is a 2-functor $E\colon \mathfrak{C} \to \mathfrak{Cat}$. The goal is to find the discrete 2-fibration concept corresponding to this under a category of elements construction.

Now, the development will be more precise. For the pseudo-functor $E\colon \mathfrak{C} \to \mathfrak{Cat}$ and any arrow $f\colon C \to D$ of $\mathfrak{C}$, denote the corresponding transition functor by $f_!\colon EC \to ED$. Similarly, let $\alpha_!\colon f_! \Rightarrow g_!$ denote the transformation associated to a 2-cell $\alpha\colon f \Rightarrow g$ of $\mathfrak{C}$. To avoid cluttering notation, subscripts on components of α may be dropped. For the following, compare §1,2.5 of [Gra74].

Definition 2.2.8. *The 2-category of elements of E is the 2-category whose*

1. *objects are pairs (C, X) with $C \in \mathfrak{C}$ and $X \in EC$;*
2. *arrows are pairs $(f, u)\colon (C, X) \to (D, Y)$ with $f\colon C \to D$ in $\mathfrak{C}$ and $u\colon f_!X \to Y$ in the fiber ED;*
3. *and whose 2-cells $\alpha\colon (f, u) \Rightarrow (g, v)$ are those $\alpha\colon f \Rightarrow g$ in $\mathfrak{C}$ for which there is a commutative triangle*

$$\begin{tikzcd} f_!X \arrow[dr, "u"] \arrow[dd, "(\alpha_!)_X"'] & \\ & Y \\ g_!X \arrow[ur, "v"'] & \end{tikzcd}$$

of arrows in the category ED.

Denote this 2-category by $\int_{\mathfrak{C}} E$*. There is an evident projection 2-functor* $\Pi\colon \int_{\mathfrak{C}} E \to \mathfrak{C}$.

Remark 2.2.9 (Dualization). Some care must be exercised in forming the category of elements of a contravariant pseudo-functor $F\colon \mathfrak{C}^{op} \to \mathfrak{Cat}$ on a 2-category $\mathfrak{C}$. Denote the image of an arrow $f\colon C \to D$ under F by $f^*\colon FD \to DC$. Now, objects of the 2-category of elements are again pairs (C, X) where $X \in FC$. But morphisms $(C, X) \to (D, Y)$ are pairs (f, u) where $u\colon X \to f^*Y$ is a morphism of FC. Note the difference that u has the action of the transition functor on Y as its codomain. Consequently the commutative triangles in the definition of a 2-cell will be of the form

$$\begin{tikzcd} & f^*Y \arrow[dd, "\alpha^*_Y"] \\ X \arrow[ur, "u"] \arrow[dr, "v"'] & \\ & g^*Y \end{tikzcd}$$

This is consistent with forming the 1-categorical dual of the construction in Definition 2.2.8. Moreover, the indicated direction of the morphisms u is required to prove associativity of composition.

Proposition 2.2.10. *Let* $E\colon \mathfrak{C} \to \mathfrak{Cat}$ *denote a pseudo-functor. The 2-functor* $\Pi\colon \int_{\mathfrak{C}} E \to \mathfrak{C}$ *from the 2-category of elements has the following fibration properties.*

1. *The ordinary functor* Π_0 *of underlying 1-categories is a cloven opfibration.*

2. *Locally* Π *is a discrete fibration.*

Additionally, if E is in fact a 2-functor, then Π_0 is a split opfibration.

Proof. Since at the level of 1-categories, the 2-category of elements is the same as the ordinary 1-category of elements, the first point has been established. The final comments also follows for the same reason.

Now, for the discrete fibration claim, start with a morphism $(g, v)\colon (C, X) \to (D, Y)$ and a cell $\alpha\colon f \Rightarrow g\colon C \rightrightarrows D$ to the codomain of (g, v) under Π in $\mathfrak{C}$. The required lift is the cell

$$\begin{tikzcd} (C, X) \arrow[rr, bend left, dashed, "{(f, u)}"] \arrow[rr, bend right, "{(g, v)}"'] \arrow[rr, phantom, "\Downarrow \alpha"] & & (D, Y) \end{tikzcd}$$

where u is defined as the composite $u := v(\alpha_!)_X$. This is defined to give the correct commutative triangle as in Definition 2.2.8 and evidently is over α via the projection Π. $\square$

Remark 2.2.11. As a result of Remark 2.2.9 above, the 2-category of elements construction for a pseudo-functor $F\colon \mathfrak{C}^{op} \to \mathfrak{Cat}$ will be a cloven fibration at the level of underlying 1-categories and a discrete *op*fibration locally.

Proposition 2.2.12. *In the notation of the proof of Proposition 2.2.10, the lifts coming with the opfibration and discrete fibration properties of* Π *are compatible in the following sense.*

1. *For* α *and its lift as in the proof of Proposition 2.2.10 and a vertical morphism* $u\colon X \to Y$*, the composite 2-cells*

$$\begin{tikzcd}
(C,X) \arrow[rr, dashed, bend left, "{(f,(\alpha_!)_X)}"] \arrow[rr, bend right, "{(g,1)}"'] \arrow[rr, phantom, "\Downarrow\alpha"] \arrow[d, "{(1,u)}"'] & & (D,g_!X) \arrow[d, "{(1,g_!u)}"] \\
(C,Y) \arrow[rr, "{(g,1)}"'] & & (D,g_!Y)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
(C,X) \arrow[rr, "{(f,(\alpha_!)_X)}"] \arrow[d, "{(1,u)}"'] & & (D,g_!X) \arrow[d, "{(1,g_!u)}"] \\
(C,Y) \arrow[rr, dashed, bend left, "{(f,(\alpha_!)_Y)}"] \arrow[rr, bend right, "{(g,1)}"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & (D,g_!Y).
\end{tikzcd}$$

   are equal.

Proof. The condition follows by the naturality of α. $\square$

For a 2-functor $E\colon \mathfrak{E} \to \mathfrak{C}$, let $\mathfrak{E}_C$ denote the fiber of E over C. What follows in the next two results is a sort of inverse to the 2-category of elements construction above, specific to the discrete case. Compare the proofs here to the material of §2.2.3 and §2.2.4 in [Buc14].

Proposition 2.2.13. *Let* $E\colon \mathfrak{E} \to \mathfrak{C}$ *denote a 2-functor such that*

1. *the functor* $E_0\colon \mathfrak{E}_0 \to \mathfrak{C}_0$ *of underlying 1-categories is an opfibration with opcleavage* ρ*;*

2. E *is locally a discrete fibration;*

It then follows, conversely, that E *determines a pseudo-functor* $\tilde{E}\colon \mathfrak{C}_0 \to \mathfrak{Cat}$.

Proof. For $C \in \mathfrak{C}$, take the object assignment to be the fiber $\mathfrak{E}_C$. For a morphism $f\colon C \to D$, a transition functor $f_!\colon \mathfrak{E}_C \to \mathfrak{E}_D$ is given in the following way. On an object X of the fiber over C, take $f_!X$ to be the codomain of the opcartesian morphism $\rho(X,f)\colon X \to f_!X$ specified by the opcleavage. Thus, for a morphism $u\colon X \to Y$ of the fiber $\mathfrak{E}_C$, the value $f_!u$ is the unique lift of identity in the square

$$\begin{tikzcd}
X \arrow[r, "\rho"] \arrow[d, "u"'] \arrow[dr, phantom, "="] & f_!X \arrow[d, dashed, "f_!u"] \\
Y \arrow[r, "\rho"'] & f_!Y.
\end{tikzcd}$$

This is plainly functorial by uniquesness. $\square$

Proposition 2.2.14. *Suppose that $E\colon \mathfrak{E} \to \mathfrak{C}$ satisfies the hypotheses of the last result, Proposition 2.2.13. It then follows that $E\colon \mathfrak{E} \to \mathfrak{C}$ satisfies the following compatibility condition:*

1. *Let $\alpha\colon f \Rightarrow g\colon C \rightrightarrows D$ denote a 2-cell of $\mathfrak{C}$ and $u\colon X \to Y$ an arrow of the fiber $\mathfrak{E}_C$. Since E is locally a discrete fibration, there are unique 2-cells*

$$\begin{tikzcd}
X \arrow[r, bend left, dashed] \arrow[r, bend right, "\rho(X,g)"'] \arrow[r, phantom, "\tilde{\alpha}_X \Downarrow"] & g_!X
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, bend left, dashed] \arrow[r, bend right, "\rho(Y,g)"'] \arrow[r, phantom, "\tilde{\alpha}_Y \Downarrow"] & g_!Y
\end{tikzcd}$$

each over α. It follows that the composite 2-cells

$$\begin{tikzcd}
X \arrow[r, bend left, dashed] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \tilde{\alpha}_X"] \arrow[d, "u"'] & g_!X \arrow[d, "g_!u"] \\
Y \arrow[r, "\rho(Y,g)"'] & g_!Y
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "\rho(X,g)"] \arrow[d, "u"'] & g_!X \arrow[d, "g_!u"] \\
Y \arrow[r, bend left, dashed] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \tilde{\alpha}_Y"] & g_!Y.
\end{tikzcd}$$

are equal, that is, in equations, that $g_!u * \tilde{\alpha}_X = \tilde{\alpha}_Y * u$.

Consequently, the pseudo-functor $\tilde{E}\colon \mathfrak{C}_0 \to \mathfrak{Cat}$ in Proposition 2.2.13 extends to one $\mathfrak{C} \to \mathfrak{Cat}$ making the same underlying assignments.

Proof. The compatibility condition follows since E is locally a discrete fibration. For a given 2-cell $\alpha\colon f \Rightarrow g\colon C \rightrightarrows D$, define the component of a purported natural transformation $\alpha_!\colon f_! \Rightarrow g_!$ in the following way. Since locally E is a discrete fibration, there is a unique cell

$$\begin{tikzcd}
X \arrow[r, "\rho"] \arrow[d, "1"'] \arrow[dr, dashed, bend left] \arrow[dr, phantom, "\tilde{\alpha}_X \Downarrow"] & f_!X \arrow[d, dashed, "(\alpha_!)_X"] \\
X \arrow[r, "\rho"'] & g_!X
\end{tikzcd}$$

over α. In particular the domain of the lift is over f. Thus, the desired component of $\alpha_!$ then occurs as a lift of identity making a commutative triangle as above. Naturality of $\alpha_!$ now follows by the compatibility condition. For the condition says precisely that the two ways around the naturality square

$$\begin{tikzcd} f_!X \arrow[r, "f_!u"] \arrow[d, "(\alpha_!)_X"'] \arrow[dr, phantom, "="] & f_!Y \arrow[d, "(\alpha_!)_Y"] \\ g_!X \arrow[r, "g_!u"'] & g_!Y \end{tikzcd}$$

solve the same lifting problem and thus are identical by uniqueness. □

The foregoing development now justifies the following definition.

Definition 2.2.15. *A discrete 2-opfibration is a 2-functor $E\colon \mathfrak{E} \to \mathfrak{C}$ such that*

1. *the underlying functor $E_0\colon \mathfrak{E}_0 \to \mathfrak{C}_0$ is an opcloven opfibration;*

2. *E itself is locally a discrete fibration, in that each functor $E\colon \mathfrak{E}(X,Y) \to \mathfrak{C}(EX,EY)$ is a discrete fibration.*

A discrete 2-fibration is a 2-functor $F\colon \mathfrak{F} \to \mathfrak{C}$ whose underlying functor of 1-categories is a cloven fibration and which is locally a discrete opfibration.

Remark 2.2.16. Notice that in the definition of a discrete 2-fibration it is required that F_0 be a fibration and that F be locally a discrete *op*fibration. This "mixed variance" is a result of the formation of the 2-category of elements as summarized in the Dualization Remark 2.2.9. That is, the category of elements construction for contravariant pseudo-functors establishes a correspondence with discrete 2-fibrations, as defined above, as a result of the definition of morphisms in the construction.

Let $\mathfrak{DOpf}(\mathfrak{C})$ denote the 2-category of discrete 2-opfibrations over $\mathfrak{C}$, cartesian-morphism-preserving functors over $\mathfrak{C}$ and transformations with vertical components. Similarly, let $\mathfrak{DFib}(\mathfrak{C})$ denote the 2-category of discrete 2-fibrations over $\mathfrak{C}$.

2.3 Regular and Exact Categories

A morphism of a 1-category is a regular epimorphism if it is a coequalizer of some pair of arrows of the category. Every regular epimorphism is an epimorphism. A strong epimorphism of $\mathscr{X}$ is an epimorphism $e\colon A \to B$ such that for any monomorphism m of $\mathscr{X}$ fitting into a square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "e"'] & C \arrow[d, tail, "m"] \\ B \arrow[r, "g"'] \arrow[ur, dashed] & D \end{tikzcd}$$

there is a lift as depicted by the dashed arrow, making two commutative triangles. The composition of strong epimorphisms is again a strong epimorphism.

Example 2.3.1. *Every regular epimorphism is strong. Every split epi is regular.*

Proof. For the first statement, see, for example, Theorem 2.6 of [Bar71]. ☐

Lemma 2.3.2. *If $e\colon A \to B$ is a regular epi and factors as $e = fg$ for a regular epimorphism g, then f is a regular epimorphism too.*

Proof. By the assumption e is the coequalizer of some pair of morphisms. It follows that f is thus the coequalizer of the same pair postcomposed with g. ☐

Lemma 2.3.3. *A morphism that is regular epi and a monomorphism is also an isomorphism.*

Proof. See Corollary 2.7 of [Bar71]. ☐

Recall that the kernel pair of a morphism is its pullback along itself, if it exists.

Definition 2.3.4. *A category $\mathscr{X}$ is regular if it possesses all finite limits and*

1. *regular epimorphisms are stable under pullback; and*

2. *every kernel pair has a coequalizer.*

An image factorization for a morphism $f\colon X \to Y$ in a regular category $\mathscr{X}$ is a commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e"'] & & Y \\ & I \arrow[ur, tail, "m"'] & \end{tikzcd}$$

in $\mathscr{X}$ where e is a regular epimorphism and m is a monomorphism. This is required to be universal in that any other such factorization $f = m'e'$ admits a unique arrow $I' \to I$ making two commutative triangles.

Lemma 2.3.5. *Every morphism of a regular category has a pullback-stable image factorization.*

Proof. This is proved in Theorem 2.3 of [Bar71]. □

Proposition 2.3.6. *A finitely-complete category $\mathscr{X}$ is regular if every arrow of $\mathscr{X}$ has a pullback-stable factorization as a regular epimorphism followed by a monic.*

Proof. Any regular epimorphism is a factorization of itself as a regular epimorphism followed by a monic. Therefore, regular epimorphisms are stable under pullback. So, let $f\colon X \to Y$ denote any morphism with $d_0, d_1\colon Z \rightrightarrows X$ denoting its kernel pair. By the assumption f has a factorization $f = me$ where e is regular epi and m is monic. Note that $ed_0 = ed_1$ since m is monic. Now, e is the coequalizer of, say, $p, q\colon W \rightrightarrows X$. Since the kernel pair of f is a pullback, there is a unique arrow $h\colon W \to Z$ making $hd_0 = p$ and $hd_1 = q$ as in the diagram

$$\begin{tikzcd}
& W \arrow[d, shift right, "p"'] \arrow[d, shift left, "q"] \arrow[dl, dashed, bend right, "h"'] & & \\
Z \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X \arrow[rr, "f"] \arrow[dr, "e"'] & & Y \\
& & I \arrow[ur, "m"'] &
\end{tikzcd}$$

Thus, if r is any morphism coequalizing d_0 and d_1, then r also coequalizes p and q, yielding a unique morphism t such that $te = r$. Thus, e is the coequalizer of d_0 and d_1. □

Thus, a regular category is equivalently a finitely-complete category with a pullback-stable image factorization for each morphism.

Remark 2.3.7. In fact a bit more is true. In the notation of the proof above, $d_0, d_1\colon Z \rightrightarrows X$ is actually the kernel pair of e as well since m is monic. And conversely in the presence of image factorizations (for example, in a regular category), that e and f have the same kernel pair will imply that m is monic.

Example 2.3.8. *Any 1-topos is regular. This is proved in §IV.6 and §IV.7 of [MLM92].*

Example 2.3.9. *As a 1-category,* **Cat** *is not regular. A proof is sketched in A1.5 of [Joh01] on p.48. Let* $\mathbf{2}+\mathbf{2}$ *denote the coproduct of* $\mathbf{2}$ *with itself. There is a functor* $\mathbf{2}+\mathbf{2} \to \mathbf{3}$*, sending the first summand to* $\{0 \leq 1\}$ *and the second to* $\{1 \leq 2\}$*. This is an epimorphism in* **Cat** *and is the coequalizer of two injections* $\mathbf{1} \rightrightarrows \mathbf{2}+\mathbf{2}$*. This epimorphism can be pulled back along the functor* $\mathbf{2} \to \mathbf{3}$ *whose image is* $\{0 \leq 2\}$*. The resulting morphism from the pullback to* $\mathbf{2}$ *is not an epimorphism.*

2.3.1 Pullback-Image Lemma

The following lemma makes precise the idea that a square that is "almost a pullback" becomes one when passing to certain images. It does not seem to appear in the literature and may be folklore. In any event, as it will be of vital importance in the proof of Lemma 6.1.4, a complete proof is given here.

As set up, consider, in a regular category $\mathscr{E}$, a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Each of the horizontal morphisms f and g has an image factorization, denoted respectively, by I and J. In each case, it is obtained as the coequalizer of the kernel of the morphism in question. There results a morphism $\tilde{h}$ between the images, as in the diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[rr, bend left, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "="] & I \arrow[r, "m"] \arrow[d, "\tilde{h}"] \arrow[dr, phantom, "="] & B \arrow[d, "k"] \\ C \arrow[r, "v"'] \arrow[rr, bend right, "g"'] & J \arrow[r, "n"'] & D \end{tikzcd}$$

making two commutative squares.

Lemma 2.3.10 (Pullback-Image Lemma)**.** *Let $\mathscr{E}$ denote a regular category and use the notation established immediately above. Suppose that the commutative square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

has the existence but not necessarily the uniqueness aspect of the universal property of a pullback *square. The induced commutative square arising from the image factorizations, namely,*

$$\begin{tikzcd} I \arrow[r, "m"] \arrow[d, "\tilde{h}"'] & B \arrow[d, "k"] \\ J \arrow[r, "n"'] & D \end{tikzcd}$$

is then a pullback.

Proof. Let X denote any object of $\mathscr{E}$ admitting two maps $x\colon X \to J$ and $y\colon X \to B$ satisfying the equation $nx = ky$. Form the pullback

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "x"] \\ C \arrow[r, "v"'] & J \end{tikzcd}$$

Note that π_2 is a regular epimorphism because v is one. It is a straightforward computation that the equation $ky\pi_2 = g\pi_1$ holds. Thus, by the assumption on the square in the first display of the statement of the lemma, there is a morphism $w\colon P \to A$ for which it is true that $fw = x\pi_2$ and $hw = \pi_1$. Now, since π_2 is a regular, hence strong, epi, there is a (unique) lift as in the diagram

$$\begin{tikzcd} P \arrow[r, "uw"] \arrow[d, "\pi_2"'] & I \arrow[d, tail, "m"] \\ X \arrow[r, "y"'] \arrow[ur, dashed, "\tilde{y}"] & B \end{tikzcd}$$

making two commutative triangles. By construction $\tilde{y}$ is the required arrow $X \to I$ verifying the universal property of a pullback. For on the one hand

$$n\tilde{h}\tilde{y} = km\tilde{y} = ky = nx$$

so that since n is monic, $\tilde{h}\tilde{y} = x$ holds; and on the other hand $m\tilde{y} = y$ is true by construction. Uniqueness follows now since any morphism satisfying these last two equations would also be a lift of y as above. $\square$

Remark 2.3.11. Notice that the hypotheses of the lemma can be weakened. For the fact that $hw = \pi_1$ holds was not used in the course of the proof. Hence it need only be required that the square in the assumption of the lemma yields a morphism to A making a commutative triangle with f as one side, without any further statement about a commutative triangle involving h.

2.3.2 Exact Categories

Exact categories were introduced by M. Barr in [Bar71]. As stated in the introduction to that paper, the intention was to axiomatize the features of those categories that are somehow abelian but not necessarily additive. Exact categories are in particular regular, as above, but additionally have the property that each internal equivalence relation is a kernel.

Definition 2.3.12. *A pair of arrows $d_0, d_1 \colon R \rightrightarrows X$ in a finitely-complete category $\mathscr{E}$ is an equivalence relation on $X \in \mathscr{E}$ if*

1. *the morphisms d_0 and d_1 are jointly monic;*
2. *reflexivity holds in the sense that the diagonal factors through $\langle d_0, d_1 \rangle$ as in*

$$\begin{tikzcd} R \arrow[rr, "\langle d_0, d_1 \rangle"] & & X \times X \\ & X \arrow[ul, dashed, "i"] \arrow[ur, "\Delta"'] & \end{tikzcd}$$

   making a commutative triangle;

3. *symmetry holds in the sense that there is a twist morphism $(-)^{-1}$ as in*

$$\begin{tikzcd} R \arrow[rr, "\langle d_0, d_1 \rangle"] & & X \times X \\ & R \arrow[ul, dashed, "(-)^{-1}"] \arrow[ur, "\langle d_1, d_0 \rangle"'] & \end{tikzcd}$$

   making a commutative triangle;

4. *the corner object of the pullback*

$$\begin{tikzcd} T \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "d_0"] \\ R \arrow[r, "d_1"'] & X \end{tikzcd}$$

   factors through $\langle d_0, d_1 \rangle$ as in

$$\begin{tikzcd} R \arrow[rr, "\langle d_0, d_1 \rangle"] & & X \times X \\ & T \arrow[ul, dashed, "\circ"] \arrow[ur, "\langle d_0\pi_1, d_1\pi_2 \rangle"'] & \end{tikzcd}$$

making a commutative triangle.

Definition 2.3.13. *A regular category $\mathscr{E}$ is exact if every equivalence relation is the kernel of some morphism.*

Example 2.3.14. *Any 1-topos is an exact category. See, for example, §1.5 of [Joh14].*

Example 2.3.15. *The category of torsion-free abelian groups is regular but not exact. See §A1.3 on p.24 of [Joh01].*

Lemma 2.3.16. *In an exact category $\mathscr{E}$, every equivalence relation is the kernel of its coequalizer.*

Proof. By exactness every equivalence relation is the kernel of some arrow. But by regularity, every kernel has a coequalizer. It follows then that the equivalence relation is also the kernel of its coequalizer. □

Lemma 2.3.17. *The slice of any regular or exact category is again regular or exact, as the case may be.*

Proof. See Theorem 5.4 of [Bar71]. □

Chapter 3

Internal Category Theory

3.1 Internal 1-Categories

Let $\mathscr{E}$ denote a regular category as in Definition 2.3.4. Most of the following is standard material found in any reference on category theory or topos theory, for example, Chapter 8 of [Bor94], Chapter XII of [Mac98], or Chapter V of [MLM92], or Chapter 2 of [Joh14].

Definition 3.1.1. *A 1-category $\mathbb{C}$ internal to $\mathscr{E}$ consists of the data of objects and arrows of $\mathscr{E}$, displayed as*

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, shift left=2, "\pi_1"] \arrow[r, "\circ" description] \arrow[r, shift right=2, "\pi_2"'] & C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & C_0 \arrow[l, "i" description] \end{tikzcd}$$

subject to the requirements that

1. $d_0 i = d_1 i = 1$*;*

2. $d_0 \circ = d_0 \pi_1$ *and* $d_1 \circ = d_1 \pi_2$*;*

3. $\circ\langle 1, id_1\rangle = 1_{C_1}$ *and* $\circ\langle id_0, 1\rangle = 1_{C_1}$*;*

4. $\circ(\circ \times 1) = \circ(1 \times \circ)$.

Display the data as a tuple $\mathbb{C} = (C_0, C_1, d_0, d_1, i, \circ)$. Given such $\mathbb{C}$, the internal opposite category, denote by $\mathbb{C}^{op}$, is formed from the data of $\mathbb{C}$ but with the roles of d_0 and d_1 interchanged.

Remark 3.1.2. In the first display of Definition 3.1.1, the object $C_1 \times_{C_0} C_1$ is formed by pulling back d_0 along d_1 as in

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "d_0"] \\ C_1 \arrow[r, "d_1"'] & C_0 \end{tikzcd}$$

Thus this object of composable pairs is written in the "diagrammatic order." This will always be the convention when dealing with internal 1-categories.

Remark 3.1.3. The third condition of Definition 3.1.1 is a unit condition requiring that the triangles

$$\begin{tikzcd} C_1 \arrow[r, "\langle 1, id_1\rangle"] \arrow[dr, "1"'] & C_1 \times_{C_0} C_1 \arrow[d, "\circ"] \\ & C_1 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "\langle id_0, 1\rangle"] \arrow[dr, "1"'] & C_1 \times_{C_0} C_1 \arrow[d, "\circ"] \\ & C_1 \end{tikzcd}$$

each commute. Notationally, the expressions $\circ\langle 1, id_1\rangle$ and $\circ\langle id_0, 1\rangle$ in the equations, and others of the same form involving composition and angle brackets '$\langle -, -\rangle$', will be written

$$\circ\langle id_0, 1\rangle =: id_0 \circ 1 \qquad \circ\,\langle 1, id_1\rangle =: 1 \circ id_1$$

treating $\circ$ as though it has two arguments $- \circ -$ taking generalized elements of C_1 as values.

Example 3.1.4. *Any object $X \in \mathscr{E}$ can be viewed as a "discrete" internal category $\mathbb{X}$ with $X_0 = X_1 = X$ and all required morphisms identities.*

Example 3.1.5. *For any internal category $\mathbb{C}$, the internal arrow category $\mathbb{C}^{\mathbf{2}}$ is given in the following way. The object of objects is C_1. The object of arrows is given as the corner object of the pullback*

$$\begin{tikzcd} (\mathbb{C}^{\mathbf{2}})_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \times_{C_0} C_1 \arrow[d, "- \circ -"] \\ C_1 \times_{C_0} C_1 \arrow[r, "- \circ -"'] & C_1 \end{tikzcd}$$

The domain arrow is $\pi_1\pi_2 \colon (\mathbb{C}^{\mathbf{2}})_1 \to C_1$ and the codomain arrow is $\pi_2\pi_1 \colon (\mathbb{C}^{\mathbf{2}})_1 \to C_1$. The composition is induced from that of $\mathbb{C}$.

Definition 3.1.6. *An internal groupoid is an internal category $\mathbb{G} = (G_0, G_1, d_0, d_1, i, \circ)$, as in Definition 3.1.1, equipped with an additional morphism $(-)^{-1} \colon G_1 \to G_1$ for which the equations*

1. $d_0(-)^{-1} = d_1$ *and* $d_1(-)^{-1} = d_0$

2. $1_{G_1} \circ (-)^{-1} = id_0$

3. $(-)^{-1} \circ 1_{G_1} = id_1$

are satisfied.

Remark 3.1.7. The last two conditions express that the morphism $(-)^{-1}\colon G_1 \to G_1$ provides each "arrow" of $\mathbb{G}$ with an inverse under internal composition. These equations express the commutativity of the two squares

$$
\begin{tikzcd}
G_1 \arrow[r, "{\langle 1, (-)^{-1} \rangle}"] \arrow[d, "d_0"'] \arrow[dr, phantom, "="] & G_1 \times_{G_0} G_1 \arrow[d, "\circ"] \\
G_0 \arrow[r, "i"'] & G_1
\end{tikzcd}
\qquad
\begin{tikzcd}
G_1 \arrow[r, "{\langle (-)^{-1}, 1 \rangle}"] \arrow[d, "d_1"'] \arrow[dr, phantom, "="] & G_1 \times_{G_0} G_1 \arrow[d, "\circ"] \\
G_0 \arrow[r, "i"'] & G_1.
\end{tikzcd}
$$

Definition 3.1.8. *A generalized arrow $f\colon X \to D_1$ of $\mathbb{D}$ is an (internal) isomorphism if there is an arrow $g\colon X \to D_1$ such that*

1. $d_1 f = d_0 g$ *and* $d_0 f = d_1 g$
2. $f \circ g = id_0 f$
3. $g \circ f = id_0 g$

are each valid equations. A pair of generalized objects $x, y\colon X \rightrightarrows D_0$ are (internally) isomorphic if there is a regular epimorphism $p\colon Z \to X$ and an internal isomorphism $f\colon Z \to D_1$ between them, in the sense that $d_0 f = xp$ and $d_1 f = yp$ each hold.

Lemma 3.1.9. *Every generalized morphism of an internal groupoid $\mathbb{G}$ is an isomorphism in the sense of Definition 3.1.8 above.*

Example 3.1.10. *Given an object $X \in \mathscr{E}$, the chaotic internal groupoid on X is given by the simplicial data*

$$
\begin{tikzcd}
X \times X \times X \arrow[r, shift left=2, "\pi_{1,2}"] \arrow[r, "\pi_{1,3}" description] \arrow[r, shift right=2, "\pi_{2,3}"'] & X \times X \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] & X \arrow[l, "\Delta" description]
\end{tikzcd}
$$

It will be seen in Lemma 3.1.21 that any such groupoid on an "inhabited" object X is suitably "weakly equivalent" to the terminal internal category.

Proposition 3.1.11. *An equivalence relation $d_0, d_1\colon R \rightrightarrows X$ as in Definition 2.3.12 determines a groupoid internal to $\mathscr{E}$ as above in Definition 3.1.6.*

Proof. The three morphisms i, $(-)^{-1}$, and $\circ$ given in Definition 2.3.12 above give the required morphisms for the category and groupoid structure. The four conditions in Definition 2.3.12 give all the category and groupoid axioms except the associativity, unit and inverse laws. Proofs of these follow a similar pattern. The diagrams expressing each law commute up to post-composition with the arrow $\langle d_0, d_1 \rangle\colon R \to X \times X$, which cancels since it is monic. $\square$

Definition 3.1.12. *A functor of internal categories $f\colon \mathbb{C} \to \mathbb{D}$ consists of arrows $f_0\colon C_0 \to D_0$ and $f_1\colon C_1 \to D_1$ satisfying the functoriality conditions*

1. $f_0 d_0 = d_0 f_1$
2. $f_0 d_1 = d_1 f_1$
3. $f_1 \circ f_1 = f_1(- \circ -)$
4. $f_1 i = i f_0$.

Let $\mathbf{Cat}(\mathscr{E})$ *denote the 1-category of internal categories and internal functors.*

Example 3.1.13. *The discrete category as in Example 3.1.4 on a terminal object of $\mathscr{E}$ is a terminal object of* $\mathbf{Cat}(\mathscr{E})$.

Definition 3.1.14 (Internally Fully Faithful)**.** *A functor of internal categories $f\colon \mathbb{C} \to \mathbb{D}$ is internally fully faithful if the commutative square*

$$\begin{tikzcd} C_1 \arrow[r, "f_1"] \arrow[d, "\langle d_0, d_1\rangle"'] & D_1 \arrow[d, "\langle d_0, d_1\rangle"] \\ C_0 \times C_0 \arrow[r, "f_0 \times f_0"'] & D_0 \times D_0 \end{tikzcd}$$

is a pullback.

Construction 3.1.1 (Object of Isomorphisms)**.** *Let $\mathbb{D}$ denote an internal category. Construct the object of isomorphisms in $\mathbb{D}$. First form the pullback*

$$\begin{tikzcd} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D_0 \arrow[d, "i"] \\ D_1 \times_{D_0} D_1 \arrow[r, "\circ"'] & D_1 \end{tikzcd}$$

whose elements are interpreted as pairs of morphisms composing to identity. Denote the images of the two projections from B to D_1 by I and J, respectively. The object $\mathbf{Iso}(\mathbb{D})$ *is then the pullback*

$$\begin{tikzcd} \mathbf{Iso}(\mathbb{D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & J \arrow[d, tail, "n"] \\ I \arrow[r, tail, "m"'] & D_1 \end{tikzcd}$$

Let $d_0, d_1\colon$ $\mathbf{Iso}(\mathbb{D}) \rightrightarrows D_0$ *denote the associated domain and codomain morphisms.*

Lemma 3.1.15. *Any internal isomorphism of $\mathbb{D}$ determines a generalized object of $\mathbf{Iso}(\mathbb{D})$.*

Proof. Let $f\colon X \to D_1$ denote an internal isomorphism with inverse $g\colon X \to D_1$ as in Definition 3.1.8. Consider the canonical morphisms to B induced by its universal property, as in the diagrams

$$\begin{tikzcd} X \arrow[dr, dashed, "x"] \arrow[drr, bend left, "d_0 f"] \arrow[ddr, bend right, "\langle f,g\rangle"'] & & \\ & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D_0 \arrow[d, "i"] \\ & D_1 \times_{D_0} D_1 \arrow[r, "\circ"'] & D_1 \end{tikzcd} \qquad \begin{tikzcd} X \arrow[dr, dashed, "y"] \arrow[drr, bend left, "d_0 g"] \arrow[ddr, bend right, "\langle g,f\rangle"'] & & \\ & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D_0 \arrow[d, "i"] \\ & D_1 \times_{D_0} D_1 \arrow[r, "\circ"'] & D_1. \end{tikzcd}$$

Denote the projections $B \to I$ and $B \to J$ by p and q, respectively. By construction, the equations $f = \pi_1\pi_1 x = mpx$ and $f = \pi_2\pi_1 y = nqy$ hold. Thus, by the universal property of $\mathbf{Iso}(\mathbb{D})$ as constructed above there is a universal map $z\colon X \to \mathbf{Iso}(\mathbb{D})$ with $\pi_1 z = px$ and $\pi_2 z = qy$, as required. $\square$

Now, compare the following definition to Proposition 1.5 on p. 376 of [BP79].

Definition 3.1.16. *An internal functor $f\colon \mathbb{C} \to \mathbb{D}$ is essentially surjective on objects if the composite $d_1 d_0^*(f_0)$ in the diagram*

$$\begin{tikzcd} P \arrow[r, "d_0^*(f_0)"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbf{Iso}(\mathbb{D}) \arrow[r, "d_1"] \arrow[d, "d_0"] & D_0 \\ C_0 \arrow[r, "f_0"'] & D_0 & \end{tikzcd}$$

is a regular epimorphism.

Definition 3.1.17. *An internal functor $f\colon \mathbb{C} \to \mathbb{D}$ is surjective-on-objects if the object part $f_0\colon C_0 \to D_0$ is a regular epimorphism.*

Example 3.1.18. *In a regular category, $\mathscr{E}$, any surjective-on-objects internal functor is internally essentially surjective. For d_0 splits and thus is regular epi; and $d_0^*(f_0)$ is a pullback of a regular epi. The composition of regular epis is again a regular epi.*

Definition 3.1.19. *A functor $f\colon \mathbb{C} \to \mathbb{D}$ of internal categories is a weak equivalence if f is internally essentially-surjective and internally fully-faithful.*

Definition 3.1.20. *An object $A \in \mathscr{E}$ is inhabited if the canonical map $A \to 1$ is a regular epimorphism.*

Lemma 3.1.21. *The chaotic category from Example 3.1.10 on an inhabited object $A \in \mathscr{E}$ is weakly equivalent to the terminal object* 1 *in* $\mathfrak{K}$.

Proof. By the example, above, the unique functor $A \to 1$ in $\mathfrak{K}$ is essentially surjective since it is inhabited. Additionally, the square

$$\begin{tikzcd} A \times A \arrow[r] \arrow[d, "1 = \langle \pi_1, \pi_2 \rangle"'] & 1 \arrow[d] \\ A \times A \arrow[r] & 1 \end{tikzcd}$$

is evidently a pullback, showing that $A \to 1$ is fully-faithful as in Definition 3.1.14. □

Definition 3.1.22. *An internal natural transformation $\theta\colon f \Rightarrow g$ between internal functors $f, g\colon \mathbb{C} \to \mathbb{D}$ is an arrow $\theta\colon C_0 \to D_1$ satisfying the conditions*

1. $d_0\theta = f_0$;
2. $d_1\theta = g_0$;
3. $\theta d_0 \circ g_1 = f_1 \circ \theta d_1$.

Proposition 3.1.23. *Internal 1-categories, functors, and natural transformations form a 2-category $\mathfrak{Cat}(\mathscr{E})$. Additionally, $\mathfrak{Cat}(\mathscr{E})$ is finitely complete and is cartesian closed if $\mathscr{E}$ is.*

Proof. Finite limits are constructed, for example, in §7.2 of [Jac99]. Exponentials are given, for example, in §B2.3 of [Joh01]. □

3.2 Internal Diagrams and Colimits

Much of the following material is summarized in Chapter 2 of [Joh14]. These results, and the main one, Theorem 3.2.7, originate in [Dia73] and the subsequent paper [Dia75].

Throughout let $\mathbb{C}$ denote an internal category in $\mathscr{E}$, an exact category. Think of $\mathscr{E}$ as the base category replacing **Set**. As motivation for the following definition, note that a copresheaf $E\colon \mathscr{C} \to$ **Set** on a small category is also called a set-valued diagram on $\mathscr{C}$. Insofar as **Set** is the "base-category" of category theory, such E could be called a "base-valued diagram on $\mathscr{C}$." Notice that such a diagram yields a coproduct

$$\coprod_{C \in \mathscr{C}_0} EC$$

admitting a certain action of $\mathscr{C}_1$ that respects the fibers of the projection of the coproduct to $\mathscr{C}_0$. In fact to give a base-valued diagram on $\mathscr{C}$ is essentially the same as giving a set function $e\colon E \to \mathscr{C}_0$ admitting a suitable action of $\mathscr{C}_1$. This correspondence is discussed in more detail in §V.7 of [MLM92]. The point is that the latter description admits of an internal formulation when the base category **Set** is replaced by $\mathscr{E}$ whereas the notion of an $\mathscr{E}$-valued functor on an internal category does not even make sense.

Definition 3.2.1. *An internal base-valued diagram, or an internal diagram, is a morphism* $e\colon E \to C_0$ *of* $\mathscr{E}$ *equipped with an action morphism* $m\colon E \times_{C_0} C_1 \to E$*, where* $E \times_{C_0} C_1$ *denotes the pullback of* d_0 *along* e*, such that the equations*

1. $em = d_1\pi_{C_1}$
2. $m\langle 1, ie\rangle = 1$
3. $m(1 \times m) = m(m \times 1)$

are satisfied. A morphism of internal diagrams is one $g\colon E \to E'$ *with* $e'g = e$ *that commutes with the given actions. Let* $\mathscr{E}^{\mathbb{C}}$ *denote the category of such diagrams. Analogously,* $\mathscr{E}^{\mathbb{C}^{op}}$ *is the category of contravariant diagrams on* $\mathbb{C}$*.*

Definition 3.2.2. *An internal functor* $e\colon \mathbb{E} \to \mathbb{C}$ *is an internal discrete opfibration if, as in remark 2.2.2, the square*

$$\begin{tikzcd} E_1 \arrow[r, "e_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "d_0"] \\ E_0 \arrow[r, "e_0"'] & C_0 \end{tikzcd}$$

is a pullback. An internal functor is a discrete fibration if the analogous square with codomain arrows instead is a pullback. A morphism of discrete opfibrations $e\colon \mathbb{E} \to \mathbb{C}$ *and* $g\colon \mathbb{G} \to \mathbb{C}$ *is a internal functor* $h\colon \mathbb{E} \to \mathbb{G}$ *such that* $gh = e$*. Morphisms of internal discrete fibrations are analogous. Denote these categories by* $\mathbf{DOpf}(\mathbb{C})$ *and* $\mathbf{DFib}(\mathbb{C})$*, respectively.*

Theorem 3.2.3. *An internal category of elements construction, as for example in §2.1 of [Joh14], gives an equivalence of categories*

$$\mathscr{E}^{\mathbb{C}} \simeq \mathbf{DOpf}(\mathbb{C})$$

between internal diagrams and discrete opfibrations. An analogous result holds for contravariant internal diagrams and discrete fibrations.

Proof. The proof is discussed more explicitly in Proposition B2.5.3 and its proof of [Joh01]. □

Lemma 3.2.4. *The category* $\mathscr{E}^{\mathbb{C}} \simeq \mathbf{DOpf}(\mathbb{C})$ *has finite limits.*

Proof. The category $\mathbf{Cat}(\mathscr{E})$ is finitely-complete since $\mathscr{E}$ is; thus the slice $\mathfrak{Cat}(\mathscr{E})/\mathbb{C}$ is finitely-complete too. The forgetful functor $\mathbf{DOpf}(\mathbb{C}) \to \mathbf{Cat}(\mathscr{E})/\mathbb{C}$ creates finite limits. Explicitly, the terminal object is the identity $1\colon \mathbb{C} \to \mathbb{C}$. Products are given by taking a pullback in $\mathbf{Cat}(\mathscr{E})$. Finally equalizers are given by taking equalizers in $\mathbf{Cat}(\mathscr{E})$ as well. □

Recall that a reflexive pair in $\mathscr{E}$ is a pair of parallel arrows $f, g\colon A \rightrightarrows B$ with a common splitting, that is, an arrow $p\colon B \to A$ with $fp = 1 = gp$. Now, assume that $\mathscr{E}$ has coequalizers of reflexive pairs. Let $\pi_0\colon \mathbf{Cat}(\mathscr{E}) \to \mathscr{E}$ denote the functor induced by taking the coequalizer

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & C_0 \arrow[r] & \pi_0(\mathbb{C}). \end{tikzcd}$$

This is a connected components functor, left adjoint to the "discrete category" functor as in the case of $\mathscr{E} = \mathbf{Set}$ discussed in §2.1. Now, let $\varinjlim_{\mathbb{C}}$ denote the functor $\mathbf{DOpf}(\mathbb{C}) \to \mathscr{E}$ induced by declaring

$$\varinjlim_{\mathbb{C}} e := \pi_0(\mathbb{E}). \tag{3.2.1}$$

Let $\mathbb{C}^*\colon \mathscr{E} \to \mathbf{DOpf}(\mathbb{C})$ denote the "constant diagram" functor taking an object $X \in \mathscr{E}_0$ to the discrete opfibration $X \times \mathbb{C} \to \mathbb{C}$ given by projection where $(X \times \mathbb{C})_0 = X \times C_0$ and $(X \times \mathbb{C})_1 = X \times C_1$. Since these functors are adjoint $\varinjlim_{\mathbb{C}} \dashv \mathbb{C}^*$, the colimit notation is appropriate. Accordingly, $\varinjlim_{\mathbb{C}}$ will be referred to as an "internal colimit functor." Notice that $\mathscr{E}$ is thus "internally cocomplete" if, for example, it has coequalizers of reflexive pairs.

Definition 3.2.5. *The internal category* $\mathbb{C}$ *is cofiltered if*

1. *the arrow* $C_0 \to 1$ *is a regular epimorphism;*
2. *for any two generalized objects* $c, d\colon U \rightrightarrows C_0$*, there is a regular epimorphism* $p\colon V \to U$ *and arrows* $f, g\colon V \to C_1$ *with* $d_1 f = d_1 g$ *such that* $d_0 f = cp$ *and* $d_0 g = dp$*;*
3. *for any two* $f, g\colon U \rightrightarrows C_1$ *with* $d_0 f = d_0 g$ *and* $d_1 f = d_1 g$*, there is a regular epimorphism* $p\colon V \to U$ *and an* $h\colon V \to C_1$ *with* $d_1 h = d_0 fp = d_0 gp$ *and* $h \circ fp = h \circ gp$*.*

Remark 3.2.6. Definition 3.2.5 is an "elementary" version of the original definition, phrased in terms of the existence of certain regular epimorphisms in [Dia73] and [Dia75].

Theorem 3.2.7. *The internal colimit functor*

$$\varinjlim_{\mathbb{C}} \colon \mathbf{DOpf}(\mathbb{C}) \longrightarrow \mathscr{E}$$

is finite-limit preserving if, and only if, $\mathbb{C}$ is a cofiltered internal category in the sense of Definition 3.2.5.

Proof. See, for example, the developments of §2.5 of [Joh14]. □

3.3 Internal Fibrations, 2-Fibrations, and Discreteness

Following [Str74] and the development summarized above, a fibration in a 2-category $\mathfrak{K}$ is a pseudo-algebra for a certain 2-monad on a slice of $\mathfrak{K}$. Here the definitions are specialized to the case of $\mathfrak{K} = \mathfrak{Cat}(\mathscr{E})$ for finitely-complete $\mathscr{E}$. Throughout fix $\mathbb{C}$ an internal category.

Definition 3.3.1. *An internal cloven opfibration $e\colon \mathbb{E} \to \mathbb{C}$ is a normalized pseudo-algebra for the 2-monad*

$$T\colon \mathfrak{K}/\mathbb{C} \longrightarrow \mathfrak{K}/\mathbb{C}$$

given by pulling back along $d_0\colon \mathbb{C}^{\mathbf{2}} \to \mathbb{C}$ and then composing with $d_1\colon \mathbb{C}^{\mathbf{2}} \to \mathbb{C}$. Such an opfibration is understood to be split if it is a strict 2-algebra and not merely pseudo. Denote the corresponding 2-categories by $\mathfrak{cOpf}(\mathbb{C})$ and $\mathfrak{sOpf}(\mathbb{C})$. The duals are internal cloven fibrations and internal split fibrations, namely, normalized pseudo- and strict-algebras for the 2-monad on $\mathfrak{K}/\mathbb{C}$ given by pulling back along $d_1\colon \mathbb{C}^{\mathbf{2}} \to \mathbb{C}$ and then composing with $d_0\colon \mathbb{C}^{\mathbf{2}} \to \mathbb{C}$. The corresponding 2-categories are denoted by $\mathfrak{cFib}(\mathbb{C})$ and $\mathfrak{sFib}(\mathbb{C})$, respectively.

The abstract definition can be reconciled with the internal version of the following ordinary notion.

Definition 3.3.2. *Let $e\colon \mathbb{E} \to \mathbb{C}$ denote an internal functor. A generalized morphism $g\colon X \to E_1$ is e-opcartesian, or just opcartesian, if given any morphism $f\colon X \to E_1$ with $d_0 f = d_0 g$ for which there exists a fill $k\colon X \to C_1$ with $e_1 g \circ k = e_1 f$ in $\mathbb{C}$, there then exists a unique lift of k, say, $\tilde{k}\colon X \to E_1$ over k in that $e_1\tilde{k} = k$ and making a commutative triangle $g \circ \tilde{k} = f$ in $\mathbb{E}$.*

Lemma 3.3.3. *The composite of any two (op)cartesian morphisms is (op)cartesian.*

Proof. This is just a translation of the usual set-theoretic argument into elementary terms. □

Proposition 3.3.4. *For an internal opcloven opfibration $e\colon \mathbb{E} \to \mathbb{C}$ with action morphism $m\colon \mathbb{E} \times_{\mathbb{C}} \mathbb{C}^{\mathbf{2}} \to \mathbb{E}$,*

1. *there is an internal natural transformation* $\rho\colon \pi_{\mathbb{E}} \Rightarrow m$ *where* $\pi_{\mathbb{E}}$ *is the projection to* $\mathbb{E}$*;*

2. *for each morphism* $\langle x, g\rangle\colon X \to E_0 \times_{C_0} C_1$*, the composite* $\rho\langle x, g\rangle$ *is opcartesian over* g*; and* $e_1\rho\langle x, g\rangle = x$ *holds;*

3. ρ *is normalized in the sense that* $\rho\langle x, ie_0x\rangle = ie_0$*.*

Remark 3.3.5. Street [Str74] has this result in full generality, in the sense that the paper shows that the 2-monad is lax idempotent.

Proof. 1. Let $\ker(\circ)$ denote the kernel of the composition in $\mathbb{C}$, obtained as the pullback of $\circ$ along itself. Let q denote the canonical map $C_1 \to \ker(\circ)$ arising by the universal property in the following diagram

$$\begin{tikzcd}
C_1 \arrow[dr, dashed] \arrow[drr, bend left, "\langle id_0, 1\rangle"] \arrow[ddr, bend right, "\langle id_0, 1\rangle"'] & & \\
& \ker(\circ) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_1 \times_{C_0} C_1 \arrow[d, "\circ"] \\
& C_1 \times_{C_0} C_1 \arrow[r, "\circ"'] & C_1.
\end{tikzcd}$$

Now, the composite $m_1(i \times q)$ determines the required natural transformation $\rho\colon \pi \Rightarrow m$. That the two identities

$$d_0m_1(i \times q) = \pi \qquad\qquad d_1m_1(i \times q) = m_0$$

hold follows readily; the first because i splits d_0; and the second because m is an internal functor. Naturality is the requirement that $m_1d_0 \circ m_1 = \pi \circ m_1d_1$ holds. That this is true, set-theoretically speaking, follows essentially by equality of composed squares

$$\begin{tikzcd}
\cdot \arrow[d, "1"'] & \cdot \arrow[r, "1"] \arrow[d, "1"'] & \cdot \arrow[d, "f"] & & \cdot \arrow[d, "u"'] & \cdot \arrow[r, "1"] \arrow[d, "h"'] & \cdot \arrow[d, "h"] \\
\cdot \arrow[d, "u"'] & \cdot \arrow[r, "f"] \arrow[d, "h"'] & \cdot \arrow[d, "k"] & = & \cdot \arrow[d, "1"'] & \cdot \arrow[r, "1"] \arrow[d, "1"'] & \cdot \arrow[d, "g"] \\
\cdot & \cdot \arrow[r, "g"'] & \cdot & & \cdot & \cdot \arrow[r, "g"'] & \cdot
\end{tikzcd}$$

which can easily be translated into the language of projection morphisms of $\mathbb{E}$.

2. The action satisfies the commutativity condition expressed by the diagram

$$\begin{tikzcd}
\mathbb{E} \times_{\mathbb{C}} \mathbb{C}^{\mathbf{2}} \arrow[r, "m"] \arrow[d] \arrow[dr, phantom, "="] & \mathbb{E} \arrow[d, "e"] \\
\mathbb{C}^{\mathbf{2}} \arrow[r, "d_1"'] & \mathbb{C}
\end{tikzcd}$$

which ensures that the condition $e_1\rho\langle x, g\rangle = g$ holds. In this sense $\rho\langle x, g\rangle$ is "over g." That $\rho\langle x, g\rangle$ is opcartesian now follows by the functoriality of the action m. For let $f\colon X \to E_1$ denote a morphism with $d_0 f = x$ and let $k\colon X \to C_1$ denote a fill in $\mathbb{C}$ with $k \circ g = e_1 f$. Now, m applied to each side of the internal analogue of the situation

$$\begin{tikzcd}
\cdot \arrow[d, "1"'] & \cdot \arrow[r, "1"] \arrow[d, "1"'] & \cdot \arrow[d, "g"] & & \cdot \arrow[dd, "f"'] & \cdot \arrow[r, "1"] \arrow[dd, "e_1 f"'] & \cdot \arrow[dd, "e_1 f"] \\
\cdot \arrow[d, "f"'] & \cdot \arrow[r, "g"] \arrow[d, "e_1 f"'] & \cdot \arrow[d, "k"] & = & & & \\
\cdot & \cdot \arrow[r, "1"'] & \cdot & & \cdot & \cdot \arrow[r, "1"'] & \cdot
\end{tikzcd}$$

in $\mathbb{E} \times_{\mathbb{C}} \mathbb{C}^{\mathbf{2}}$ yields the required commutative square in $\mathbb{E}$ by functoriality and the unit conditions from the algebra axioms.

3. Normalization follows from the unit laws for the algebra. □

Remark 3.3.6. Thus, the result says that the internal natural transformation ρ is an internal normalized opcleavage for e. Dually, an internal cloven fibration $f\colon \mathbb{F} \to \mathbb{C}$ with action n yields an internal normalized cleavage σ as a natural transformation $\sigma\colon n \Rightarrow \pi$ with the right properties.

Lemma 3.3.7. *If $e\colon \mathbb{E} \to \mathbb{C}$ is a split opfibration, then the opcleavage is functorial in the sense that the equation*

$$\rho\langle f \circ g, x\rangle = \rho\langle f, x\rangle \circ \rho\langle g, m_0 x\rangle$$

holds for any generalized morphisms f and g, and generalized object x.

Proof. This follows from the fact that e is a strict algebra as in Definition 3.3.1 and from the associativity condition in Definition 2.1.13. □

Lemma 3.3.8. *The opcleavage $\rho\colon E_0 \times_{C_0} C_1 \to E_1$ coming with an internal opcloven opfibration $e\colon \mathbb{E} \to \mathbb{C}$ is monic. Similarly for a cleavage $\sigma\colon C_1 \times_{C_0} F_0 \to F_1$ for an internal fibration.*

Proof. Supposing that $\rho\langle x, f\rangle = \rho\langle y, g\rangle$ holds, it follows that $f = g$ since each morphism is over f or g, respectively, via e. Additionally, $x = y$ holds since each is domain of the same opcartesian morphism. $\square$

3.4 Internal 2-Categories

The present section culminates in an elementary axiomatization of the notion of a discrete 2-fibration from Definition 2.2.15. First some set-up and generalities.

Let $\mathscr{E}$ denote a finitely-complete 1-category. Assume as give objects K_0, K_1 and K_2 of $\mathscr{E}$ with certain morphisms between them, displayed as

$$\begin{tikzcd} K_2 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & K_1 \arrow[l, "\iota" description] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & K_0 \arrow[l, "i" description] \end{tikzcd}$$

Think of K_0 as an object of objects; K_1 as an object of 1-cells, or morphims; and K_2 as an object of 2-cells, or transformations. Form three pullbacks

$$\begin{tikzcd} K_1 \times_{K_0} K_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & K_1 \arrow[d, "d_1"] \\ K_1 \arrow[r, "d_0"'] & K_0 \end{tikzcd} \qquad \begin{tikzcd} K_2 \times_{K_1} K_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & K_2 \arrow[d, "s"] \\ K_2 \arrow[r, "t"'] & K_1 \end{tikzcd} \qquad \begin{tikzcd} K_2 \times_{K_0} K_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & K_2 \arrow[d, "d_1 s"] \\ K_2 \arrow[r, "d_0 t"'] & K_0 \end{tikzcd}$$

The leftmost pullback is an object of pairs of composable morphisms. The middle is an object of pairs of vertically composable 2-cells; and the rightmost object is one of horizontally composable cells. Now, the corner objects of the following pullbacks

$$\begin{tikzcd} M \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & K_2 \times_{K_0} K_2 \arrow[d, "\langle s, s\rangle"] \\ K_2 \times_{K_0} K_2 \arrow[r, "\langle t, t\rangle"'] & K_1 \times_{K_0} K_1 \end{tikzcd} \qquad \begin{tikzcd} N \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & K_2 \times_{K_1} K_2 \arrow[d, "d_0 t \pi_1"] \\ K_2 \times_{K_0} K_2 \arrow[r, "d_1 t \pi_1"'] & K_0 \end{tikzcd}$$

are isomorphic. Each is interpreted as an objects of elements consisting of four 2-cells with two pairs to be composed horizontally, and two pairs to be composed vertically. The object M sets up to do the horizontal compositions first and then the vertical one; while N sets up to do the vertical first and then the horizontal. For the following compare the "global" definition of a bicategory in §1.3 of [Bén67].

Definition 3.4.1. *In the notation of the discussion above, a 2-category internal to $\mathscr{E}$ is given by the data of objects and maps displayed as*

$$\begin{tikzcd}
K_2 \times_{K_0} K_2 \arrow[d, "\odot"] & K_1 \times_{K_0} K_1 \arrow[d, "\circ"] & \\
K_2 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & K_1 \arrow[l, "\iota" description] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & K_0 \arrow[l, "i" description] \\
K_2 \times_{K_1} K_2 \arrow[u, "*"] & &
\end{tikzcd}$$

subject to the following axioms:

1. *splitting:*
   (a) $d_0 i = d_1 i = 1$ *and* $s\iota = t\iota = 1$*;*
   (b) $d_0 s = d_0 t$ *and* $d_1 s = d_1 t$*;*
2. *domain/codomain of compositions:*
   (a) $d_0 \circ = d_0 \pi_2$ *and* $d_1 \circ = d_1 \pi_2$*;*
   (b) $s* = s\pi_2$ *and* $t* = t\pi_1$*;*
   (c) $s\odot = \circ(s \times s)$ *and* $t\odot = \circ(t \times t)$*;*
3. *identities:*
   (a) $id_1 \circ 1 = 1$ *and* $1 \circ id_0 = 1$*;*
   (b) $\iota s * 1 = 1$ *and* $1 * \iota t = 1$*;*
   (c) $\iota id_0 s \odot 1 = 1$ *and* $1 \odot \iota id_1 s = 1$*;*
4. *associativity:*
   (a) $\circ(\circ \times 1) = \circ(1 \times \circ)$*;*
   (b) $*(* \times 1) = *(1 \times *)$
   (c) $\odot(\odot \times 1) = \odot(1 \times \odot)$

5. the interchange law holds, as in the commutativity of the diagram

$$\begin{tikzcd} M \cong N \arrow[r, "\langle *, * \rangle"] \arrow[d, "\langle \odot, \odot \rangle"'] & K_2 \times_{K_1} K_2 \arrow[d, "\odot"] \\ K_2 \times_{K_0} K_2 \arrow[r, "*"'] & K_2 \end{tikzcd}$$

6. compatibility of identities, in the sense that

$$\begin{tikzcd} K_1 \times_{K_0} K_1 \arrow[r, "\langle \iota, \iota \rangle"] \arrow[d, "\circ"'] & K_2 \times_{K_0} K_2 \arrow[d, "\odot"] \\ K_1 \arrow[r, "i"'] & K_2 \end{tikzcd}$$

commutes.

Remark 3.4.2. In the definition, K_0 is the object of objects; K_1 is the object of arrows or 1-cells; and K_2 is the object of transformations, or 2-cells. The morphism $\circ$ is the composition of 1-cells, while $*$ is the vertical composition of 2-cells and $\odot$ is the horizontal composition of 2-cells. Think of $\odot$ as an "external" composition; hence it is written here in diagrammatic order. Now, the first equations just mean that i and ι are simultaneous splittings for the domain/source and codomain/target maps, respectively. The next four pairs of equations say that sources and domains of the various composites are what they should be. Identity, associativity, and interchange are more-or-less self-explanatory.

Example 3.4.3. *Let $\mathcal{K}$ denote an internal 2-category as in Definition 3.4.1. What follows is an elementary version of the 2-arrow category of Example 2.1.4. The internal 2-arrow category of $\mathcal{K}$, denoted by $\mathcal{K}^{\mathbf{2}}$, has as its object of objects K_1, the object of arrows of $\mathcal{K}$. The object of arrows is the corner object of the pullback*

$$\begin{tikzcd} (\mathcal{K}^{\mathbf{2}})_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & K_1 \times_{K_0} K_1 \arrow[d, "- \circ -"] \\ K_1 \times_{K_0} K_1 \arrow[r, "- \circ -"'] & K_1 \end{tikzcd}$$

that is, the object of commutative squares of $\mathcal{K}$. The domain and codomain morphisms are $d_0 = \pi_1\pi_2$ and $d_1 = \pi_2\pi_1$, respectively. The object of 2-cells is then given as as a limit in $\mathscr{E}$ in the following way. First form the limit of the diagram

$$\begin{tikzcd}
 & & & P \arrow[dl, dashed, "\pi_1"'] \arrow[dr, dashed, "\pi_2"] & & & \\
 & & K_2 \times_{K_1} (\mathcal{K}^{\mathbf{2}})_1 \arrow[dl, dashed, "\pi_1"'] \arrow[dr, dashed, "\pi_2"] & & (\mathcal{K}^{\mathbf{2}})_1 \times_{K_1} K_2 \arrow[dl, dashed, "\pi_1"'] \arrow[dr, dashed, "\pi_2"] & & \\
 & K_2 \arrow[dr, "t"'] & & (\mathcal{K}^{\mathbf{2}})_1 \arrow[dl, "\pi_1\pi_1"] \arrow[dr, "\pi_2\pi_2"'] & & K_2 \arrow[dl, "s"] & \\
 & & K_1 & & K_1 & &
\end{tikzcd}$$

as indicated by the dashed arrows. Let w_1 and w_2 denote the composites

$$w_1 := \pi_1 \odot d_1\pi_2 \colon K_2 \times_{K_1} (\mathcal{K}^{\mathbf{2}})_1 \to K_2$$
$$w_2 := d_0\pi_1 \odot \pi_2 \colon (\mathcal{K}^{\mathbf{2}})_1 \times_{K_1} K_2 \to K_2.$$

The object of 2-cells is then the equalizer

$$\begin{tikzcd}
(\mathcal{K}^{\mathbf{2}})_2 \arrow[r, dashed] & P \arrow[r, shift left, "w_1\pi_1"] \arrow[r, shift right, "w_2\pi_2"'] & K_2.
\end{tikzcd}$$

With induced compositions, $\mathcal{K}^{\mathbf{2}}$ is an internal 2-category.

Lemma 3.4.4. *If $\mathcal{K}$ is a 2-category internal to $\mathscr{E}$, then the data*

$$\begin{tikzcd}
K_2 \times_{K_1} K_2 \arrow[r, shift left=2, "\pi_1"] \arrow[r, "*" description] \arrow[r, shift right=2, "\pi_2"'] & K_2 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & K_1 \arrow[l, "\iota" description]
\end{tikzcd}$$

determines a category object of $\mathscr{E}$, called the "vertical category" of $\mathcal{K}$.

Proof. That the axioms for an internal category from §3.1 are satisfied follows from the axioms for $\mathcal{K}$ in Defintion 3.4.1. ☐

Definition 3.4.5. *Let $\mathcal{K}$ denote an internal 2-category and $a, b \colon X \rightrightarrows K_0$ any pair of objects. By $\mathcal{K}(a, b)$ denote the internal category with*

$$\begin{tikzcd}
\mathcal{K}(a,b)_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "b"] \\
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & K_1 \arrow[r, "d_1"'] \arrow[d, "d_0"] & K_0 \\
X \arrow[r, "a"'] & K_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{K}(a,b)_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "b"] \\
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & K_2 \arrow[r, "d_1 s"'] \arrow[d, "d_0 s"] & K_0 \\
X \arrow[r, "a"'] & K_0. &
\end{tikzcd}$$

Definition 3.4.6. *Let $\mathcal{A}$ and $\mathcal{B}$ denote 2-categories internal to a finitely-complete 1-category $\mathscr{E}$. An internal 2-functor $l\colon \mathcal{A} \to \mathcal{B}$ consists of three arrows*

$$l_0\colon A_0 \to B_0 \qquad l_1\colon A_1 \to B_1 \qquad l_2\colon A_2 \to B_2$$

such that

1. *l_0 and l_1 give an internal functor $\mathcal{A}_0 \to \mathcal{B}_0$ of underlying internal 1-categories;*
2. *l_2 satisfies the functoriality conditions*
   (a) *$l_2 \odot l_2 = l_2(- \odot -)$;*
   (b) *$l_2 * l_2 = l_2(- * -)$;*
   (c) *$l_2\iota = \iota l_1$.*

Lemma 3.4.7. *Any internal 2-functor $l\colon \mathcal{A} \to \mathcal{B}$ in the above sense determines, for each $a, b\colon X \rightrightarrows A_0$, an internal functor $l_{a,b}\colon \mathcal{A}(a,b) \to \mathcal{B}(l_0a, l_0b)$ of internal hom categories as in Definition 3.4.5.*

Proof. By the construction of $\mathcal{A}(a,b)_0$ and $\mathcal{B}(l_0a, l_0b)_0$, the object-part of the functor $(l_{a,b})_0$ can be induced from universal properties using l_0 and l_1 and assumed functoriality. Similarly for the arrow-part $(l_{a,b})_1$. Functoriality follows by the functoriality assumed in Definition 3.4.6. $\square$

Remark 3.4.8. Let P, informally speaking, stand for some property of internal 1-functors (such as being internally fully faithful, or internally eso et cetera). An internal 2-functor is said to be locally P if each induced functor as in Lemma 3.4.7 has property P.

Definition 3.4.9. *An internal 2-natural transformation $\theta\colon k \Rightarrow l$ between internal 2-functors $k, l\colon \mathcal{A} \rightrightarrows \mathcal{B}$ is a natural transformation θ of the underlying functors $k, l\colon \mathcal{A}_0 \rightrightarrows \mathcal{B}_0$ satisfying the compatibility condition*

$$k_2 * \iota\theta d_1 t = \iota\theta d_0 s * l_2.$$

Remark 3.4.10. The further compatibility condition, for ordinary 2-categories in the case that $\mathscr{E} = \mathbf{Set}$, is exactly the requirement that there is an equality of composite 2-cells

$$\begin{tikzcd} KA \arrow[r, "\alpha_A"] \arrow[d, bend right=60, "Kf"'] \arrow[d, bend left=60, "Kg"] \arrow[d, phantom, "\overset{K\alpha}{\Rightarrow}"] \arrow[dr, phantom, "="] & LA \arrow[d, "Lg"] \\ KB \arrow[r, "\alpha_B"'] & LB \end{tikzcd} \quad = \quad \begin{tikzcd} KA \arrow[r, "\alpha_A"] \arrow[d, "Kf"'] \arrow[dr, phantom, "="] & LA \arrow[d, bend right=60, "Lf"'] \arrow[d, bend left=60, "Lg"] \arrow[d, phantom, "\overset{L\alpha}{\Rightarrow}"] \\ KB \arrow[r, "\alpha_B"'] & LB. \end{tikzcd}$$

where $\alpha\colon f \Rightarrow g\colon A \rightrightarrows B$ is a 2-cell of $\mathcal{A}$. Let 2-$\mathfrak{Cat}(\mathscr{E})$ denote the 2-category of 2-categories internal to $\mathscr{E}$ with internal 2-functors and internal 2-transformations.

3.4.1 Internal Connected Components

Consider now the following way of looking at the connected components construction for 2-categories that was summarized in §2.1 in the case of $\mathscr{E} = \mathbf{Set}$. In particular note that the collection of morphisms of $\pi_0\mathfrak{A}$, for an ordinary 2-category $\mathfrak{A}$, occurs as the coequalizer, taken in $\mathbf{Set}/A_0 \times A_0$ of the source and target maps as in

$$\begin{tikzcd} A_2 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A_1 \arrow[r, dashed] \arrow[d, "\langle d_0, d_1\rangle"'] & (\pi_0\mathfrak{A})_1 \arrow[dl, dashed] \\ & A_0 \times A_0 & \end{tikzcd}$$

Of course this makes sense because the slice of $\mathbf{Set}$ is cocomplete. And indeed the fibers of the resulting map over $A_0 \times A_0$ are precisely the sets $\pi_0\mathfrak{A}(A, B)$ as the objects A, B vary over $A_0 \times A_0$. This shows, then, how to give an elementary version of the connected components construction for internal 2-categories.

For let $\mathcal{K}$ denote an internal 2-category as in Definition 3.4.1 where $\mathscr{E}$ is an exact category with pullback-stable coequalizers of reflexive pairs. Let $\pi_0\mathcal{K}$ denote what will be an internal 1-category whose objects are those of $\mathcal{K}$ and whose object of arrows is the coequalizer

$$\begin{tikzcd} K_2 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & K_1 \arrow[r, dashed, "q"] \arrow[d, "\langle d_0, d_1\rangle"'] & (\pi_0\mathcal{K})_1 \arrow[dl, dashed] \\ & K_0 \times K_0 & \end{tikzcd}$$

taken in the slice $\mathscr{E}/K_0 \times K_0$.

Proposition 3.4.11. *The construction $\pi_0\mathcal{K}$ defines an internal 1-category. Moreover π_0 extends to a 2-functor $\pi_0\colon$ 2-$\mathfrak{Cat}(\mathscr{E}) \to \mathfrak{Cat}(\mathscr{E})$, left adjoint to the discrete 2-category functor disc: $\mathfrak{Cat}(\mathscr{E}) \to$ 2-$\mathfrak{Cat}(\mathscr{E})$.*

Proof. Composition for $\pi_0\mathcal{K}$ is induced from the universal property of the coequalizers. This requires that $q \times q$ is a coequalizer of $s \times s$ and $t \times t$. This statement follows by pullback stability and the "3 x 3 Lemma" of §0.17 in [Joh14]. That π_0 is a functor is immediate and the adjunction is a routine verification. $\square$

3.4.2 Internal Discrete 2-Fibrations

Now let $e\colon \mathcal{E} \to \mathcal{C}$ denote an internal 2-functor of internal 2-categories. Following Definition 2.2.15, the internal version is now the following.

Definition 3.4.12. *The 2-functor e is an internal discrete 2-opfibration if*

1. *the underlying internal 1-functor* $e_0\colon \mathcal{E}_0 \to \mathcal{C}_0$ *is a split internal opfibration;*
2. *locally* E *is an internal discrete fibration.*

The dual notion is that of an internal discrete 2-fibration $f\colon \mathcal{F} \to \mathcal{C}$ *which is an internal split fibration at the level of its under 1-functor and should be locally an internal discrete opfibration.*

Definition 3.4.13. *A morphism of internal discrete 2-fibrations* $f\colon \mathcal{F} \to \mathcal{C}$ *and* $g\colon \mathcal{G} \to \mathcal{C}$ *with cleavages* σ *and* τ*, respectively, is an internal 2-functor* $h\colon \mathcal{F} \to \mathcal{G}$ *over* $\mathcal{C}$ *in that* $f = gh$ *holds strictly and for which the cleavage-preservation condition*

$$f_1\sigma = \tau(1 \times f_0) \tag{3.4.1}$$

holds. A transformation of such morphisms $\theta\colon h \Rightarrow k$ *is an internal 2-natural transformation as in Definition 3.4.9 vertical over* $\mathcal{C}$*. Denote the 2-category of internal discrete 2-fibration by* $\mathfrak{DFib}(\mathcal{C})$*. Dually,* $\mathfrak{DOpf}(\mathcal{C})$ *denotes the 2-category of internal discrete 2-opfibrations.*

Theorem 3.4.14. *The 2-categories* $\mathfrak{DFib}(\mathcal{C})$ *and* $\mathfrak{DOpf}(\mathcal{C})$ *have all finite conical limits.*

Proof. These are inherited from 2-$\mathfrak{Cat}(\mathscr{E})$ in a manner similar to the proof of Lemma 3.2.4. $\square$

Chapter 4

Limits and Colimits

The present chapter gives further background on 2-categorical limits and colimits. The main original result is Theorem 4.2.11 which shows how to compute the weighted pseudo-colimit of any category-valued pseudo-functor on a small 2-category. Some further arguments are given that this weighted pseudo-colimit ought to be seen as a tensor product of pseudo-functors.

4.1 Limits

A standard reference for 2-categorical limits and colimits is Kelly's [Kel89]. More generally Chapter 3 of [Kel82a] describes the theory of enriched limits of which the 2-limits in $\mathfrak{Cat}$ are but an instance.

Let $Q\colon \mathfrak{J} \to \mathfrak{K}$ denote a 2-functor on a 2-category. Treat this as a diagram of shape $\mathfrak{J}$ in $\mathfrak{K}$. For each $A \in \mathfrak{K}$ there is a canonical functor $\mathfrak{K}(A, Q(-))\colon \mathfrak{K} \to \mathfrak{Cat}$. Denote this by $\mathfrak{K}(A, Q)$. Let $P\colon \mathfrak{J} \to \mathfrak{Cat}$ denote another 2-functor called the "weight." A 2-cone is a 2-natural transformation $P \to \mathfrak{K}(A, Q)$.

Definition 4.1.1 (Weighted 2-Limit)**.** *In the notation above, the 2-limit of Q weighted by P is an object $\{P, Q\}_s$ of $\mathfrak{K}$ together with a unit $\zeta\colon P \to \mathfrak{K}(\{P, Q\}_s, Q)$ making an isomorphism of 1-categories*

$$\mathfrak{K}(A, \{P, Q\}_s) \cong [\mathfrak{J}, \mathfrak{Cat}](P, \mathfrak{K}(A, Q)). \tag{4.1.1}$$

where $[\mathfrak{J}, \mathfrak{Cat}]$, as in the Example 2.1.9, denotes the 2-category of category-valued 2-functors on $\mathfrak{J}$, 2-natural transformations, and modifications. A 2-limit is called conical if the weight 2-functor $P\colon \mathfrak{J} \to \mathfrak{Cat}$ is constant at the value $\mathbf{1}$. A 2-limit is finite if $\mathfrak{J}$ is a finite 2-category and each $P(J)$ is finitely-presentable.

Example 4.1.2. *In* **Cat***, the usual finite products and equalizers are again finite 2-products and 2-equalizers in $\mathfrak{Cat}$, since these can be seen to satisfy automatically the 2-dimensional aspect of the universal property in 4.1.1.*

One important 2-categorical limit is the comma object associated to morphisms $f\colon A \to C$ and $g\colon B \to C$ of $\mathfrak{K}$. As a weighted limit, take $\mathfrak{J}$ to consist of a generic corner $\cdot \to \cdot \leftarrow \cdot$.

Take Q as the diagram that takes $\mathfrak{J}$ to the cospan with legs f and g; additionally, P to be the diagram on $\mathfrak{J}$ with values

$$\mathbf{1} \xrightarrow{0} \mathbf{2} \xleftarrow{1} \mathbf{1}.$$

The 2-limit is an object f/g with two morphisms $p\colon f/g \to A$ and $q\colon f/g \to B$ and a cell $\phi\colon fp \Rightarrow gq$ that is universal in the following sense: given any arrows $s\colon D \to A$ and $t\colon D \to B$ and a cell $\psi\colon fs \Rightarrow gt$, there is a unique $r\colon D \to f/g$ making two commutative triangles as on the left diagram of the figure

$$\begin{tikzcd}
D \arrow[dr, dashed, "r"'] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & \\
& f/g \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\phi \Rightarrow"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
D \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & \\
& \psi \Rightarrow & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C.
\end{tikzcd}$$

The further 2-dimensional aspect of the universal property is discussed in §1 of [Str74].

Comma objects can be constructed from cotensors and pullbacks. The cotensor of $A \in \mathfrak{K}$ with some category $\mathscr{A}$ is a finite weighted 2-limit on the indexing category $\mathbf{1}$. It consists of an object $\mathscr{A} \pitchfork A$ of $\mathfrak{K}$ inducing an isomorphism

$$\mathfrak{K}(B, \mathscr{A} \pitchfork A) \cong [\mathscr{A}, \mathfrak{K}(B, A)]$$

for any $B \in \mathfrak{K}$. Now, cotensors with $\mathscr{A} = \mathbf{2}$ together with pullbacks gives a construction of comma squares. That is, given arrows f and g, the comma object f/g is the vertex in the diagram of pullbacks

$$\begin{tikzcd}
f/g \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "g"] \\
S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{2} \pitchfork B \arrow[r, "d_1"'] \arrow[d, "d_0"] & B \\
A \arrow[r, "f"'] & B. &
\end{tikzcd}$$

For an object A, the identity 2-cell $1_A \Rightarrow 1_A$ induces a morphism $i\colon A \to A^{\mathbf{2}}$ from the universal property for the cotensor; additionally, the composite cell arising from

$$\begin{tikzcd}
B^{\mathbf{2}} \times_B B^{\mathbf{2}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B^{\mathbf{2}} \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d, "1"] \\
B^{\mathbf{2}} \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[r, "1"'] \arrow[d, "1"] & B \\
B \arrow[r, "1"'] & B &
\end{tikzcd}$$

yields a morphism

$$c\colon B^{\mathbf{2}} \times_B B^{\mathbf{2}} \longrightarrow B^{\mathbf{2}}.$$

Propositions 2 and 8 in §1 and §2 of [Str74] indicate that these morphisms make $B^{\mathbf{2}} \rightrightarrows B$ into a category object in $\mathfrak{K}$. The same result also shows that any morphism $f\colon A \to B$ extends to a functor $A^{\mathbf{2}} \to B^{\mathbf{2}}$.

Example 4.1.3. *Let $F\colon \mathfrak{F} \to \mathfrak{C}$ denote a discrete 2-fibration as in Definition 2.2.15. The cotensor of F with $\mathbf{2} = \{0 \leq 1\}$ is given in the following way. The objects are vertical maps $u\colon X \to Y$ of the total 2-category $\mathfrak{F}$. The arrows and 2-cells are those of the 2-arrow category as in Example 2.1.4. There is an evident forgetful 2-functor to $\mathfrak{C}$. Denote the total 2-category and forgetful 2-functor by $\Pi\colon \mathbf{2} \pitchfork F \to \mathfrak{C}$. That this is the cotensor in $\mathfrak{DFib}(\mathfrak{C})$ is easy to check.*

Example 4.1.4. *For a category $\mathbb{C}$ internal to a finitely-complete category $\mathscr{E}$, the internal arrow category $\mathbf{2} \pitchfork \mathbb{C} \cong \mathbb{C}^{\mathbf{2}}$ from Example 3.1.5 is the cotensor with $\mathbf{2}$ in the 2-category $\mathfrak{K} = \mathfrak{Cat}(\mathscr{E})$.*

Example 4.1.5. *Let $f\colon \mathcal{F} \to \mathcal{C}$ denote an internal discrete 2-fibration as in Definition 3.4.12. The cotensor $\mathbf{2} \pitchfork f$ in the 2-category $\mathfrak{DFib}(\mathcal{C})$ has the following description. The object of objects is given as the corner of the pullback*

$$\begin{tikzcd}
(\mathbf{2} \pitchfork f)_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_1 \arrow[d, "f_1"] \\
C_0 \arrow[r, "i"'] & C_1
\end{tikzcd}$$

that is, as the object of the maps of $\mathfrak{F}$ that sit over identity morphisms of $\mathfrak{C}$ via f. The object of arrows is that object of commutative squares with domain and codomain in $(\mathbf{2} \pitchfork f)_0$. That *is, the object of arrows is given as the corner object of the pullback*

$$\begin{tikzcd}
(\mathbf{2} \pitchfork f)_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathbf{2} \pitchfork f)_0 \times_{F_0} F_1 \arrow[d, "- \circ -"] \\
F_1 \times_{F_0} (\mathbf{2} \pitchfork f)_0 \arrow[r, "- \circ -"'] & F_1
\end{tikzcd}$$

viewing the composition law of $\mathcal{F}$ as restricted to $(\mathbf{2} \pitchfork f)_0 \to F_1$. The 2-cells are given in a manner analogous to that of the internal 2-arrow category from Example 3.4.3. First take the limit of the diagram

$$\begin{tikzcd}
F_2 \arrow[dr, "t"'] & & (\mathbf{2} \pitchfork f)_1 \arrow[dl, "\pi_1\pi_1"] \arrow[dr, "\pi_2\pi_2"'] & & F_2 \arrow[dl, "s"] \\
& F_1 & & F_1 s &
\end{tikzcd}$$

and then take the equalizer in $\mathscr{E}$ of the analogous pair of arrows. Note that there is an internal inclusion 2-functor $\mathbf{2} \pitchfork f \to \mathcal{F}^{\mathbf{2}}$ and an internal projection 2-functor $\Pi\colon \mathbf{2} \pitchfork f \to \mathcal{C}$. And Π is an internal discrete 2-fibration since f is assumed to be one.

Just as ordinary 1-categorical limits and colimits are constructed canonically from certain basic limit shapes, arbitrary finite pseudo-limits can be constructed from simpler ones. The following result of R. Street gives the precise sense in which this is the case.

Theorem 4.1.6 (Limit Construction)**.** *In a 2-category $\mathfrak{K}$, finite weighted 2-limits can be constructed from a terminal object, binary products, equalizers, and cotensors with* $\mathbf{2}$.

Proof. The argument on p. 106 of [Kel89] is that every weighted 2-limit is obtained as the equalizer of a certain parallel pair of morphisms between products of cotensors with the categories indexed by P. Cotensors with categories can be constructed from cotensors with $\mathbf{2}$. □

The notion of "2-limit" is that of enriched category theory with $\mathcal{V} = \mathbf{Cat}$. There are variations obtained by weakening either the notion of weighted cone or the universal property. For example, let $Q\colon \mathfrak{J} \to \mathfrak{K}$ denote a pseudo-functor on a 2-category. For each object A of $\mathfrak{K}$ there is a canonical functor $\mathfrak{K}(A, Q)$. Let $P\colon \mathfrak{J}^{op} \to \mathfrak{Cat}$, a pseudo-functor, denote the weight.

Definition 4.1.7 (Weighted Pseudo-Limit)**.** *In the notation above, the pseudo-limit of Q weighted by P is an object $\{P, Q\}$ of $\mathfrak{K}$ together with a unit $\zeta\colon P \to \mathfrak{K}(\{P, Q\}, Q)$ making an isomorphism of 1-categories*

$$\mathfrak{K}(A, \{P, Q\}) \cong \mathfrak{Hom}(\mathfrak{J}, \mathfrak{Cat})(P, \mathfrak{K}(A, Q)). \tag{4.1.2}$$

where $\mathfrak{Hom}(\mathfrak{J}, \mathfrak{Cat})$*, as in the Example 2.1.9, denotes the 2-category of category-valued pseudo-functors on* $\mathfrak{J}$*, pseudo-natural transformations, and modifications. A conical pseudo-limit is one weighted by the constant functor taking* $\mathbf{1}$ *as its only value.*

Example 4.1.8. *The pseudo-equalizer in* $\mathfrak{Cat}$ *of parallel functors* $F, G\colon \mathscr{C} \rightrightarrows \mathscr{D}$ *has as its objects those pairs* (C, ϕ) *where* $\phi\colon FC \cong GC$ *is an isomorphism in* $\mathscr{D}$*.*

Remark 4.1.9. The isomorphism in Display 4.1.2 expresses the universal property of the pseudo-limit as in §1.14 of [Str80]. This isomorphism could be weakened to require only an equivalence of categories, in which case would be given the definition of the pseudo-bilimit of Q weighted by P as in §1.13 of [Str80]. In general bilimits and bicolimits will not be considered in the present work. That is, limits and colimits will always have universal properties expressed by isomorphisms of categories such as that above.

Remark 4.1.10. The "pseudo" in pseudo-limit refers to the fact that the cones on the right side of Display 4.1.2 are pseudo-natural transformations $P \to \mathfrak{K}(A, Q)$. There are analogous limit-concepts in the cases that pseudo-natural transformations are replaced by lax- or oplax-natural transformations. Each also admits of a weakened universal property as a bilimit. Thus, considering all the various combinations, one might study oplax-bilimits, or 2-bilimits, or any other combination that makes sense. In the present work, however, only 2-(co)limits and pseudo-(co)limits will be studied.

4.2 Weighted Colimits of Category-Valued Functors

Let $\mathfrak{C}$ denote a 2-category. Let $E\colon \mathfrak{C} \to \mathfrak{Cat}$ and $W\colon \mathfrak{C}^{op} \to \mathfrak{Cat}$ denote pseudo-functors. There is a "hom" 2-functor

$$\mathfrak{Cat}(E, -)\colon \mathfrak{Cat} \longrightarrow [\mathfrak{C}^{op}, \mathfrak{Cat}]$$

given by sending a small category $\mathscr{X}$ to the pseudo-functor

$$\mathfrak{Cat}(E, \mathscr{X})\colon \mathfrak{C}^{op} \to \mathfrak{Cat}$$

given on objects by taking each C of $\mathfrak{C}^{op}$ to the 1-category of functors and natural transformation $\mathfrak{Cat}(EC, \mathscr{X})$. The 2-functor $\mathfrak{Cat}(E, -)$ could also be viewed as taking its values in $\mathfrak{Hom}(\mathfrak{C}^{op}, \mathfrak{Cat})$ since every 2-functor is pseudo. In general a separate notation will not be used to indicate this change of target. A pseudo-cocone on E weighted by W is a pseudo-natural transformation $W \to \mathfrak{Cat}(E, \mathscr{X})$.

The present section is concerned primarily with pseudo-colimits. For the ensuing computations are more involved in the pseudo-case. And so in this section "pseudo" is taken as the primary notion. The development here is specialized to $\mathfrak{K} = \mathfrak{Cat}$.

Definition 4.2.1 (Weighted Pseudo-Colimit)**.** *The pseudo-colimit of E weighted by W is a category $E \star W$ together with a cocone $\xi\colon W \to \mathfrak{Cat}(E, E \star W)$ inducing into an isomorphism of categories*

$$\mathfrak{Cat}(E \star W, \mathscr{X}) \cong \mathfrak{Hom}(\mathfrak{C}^{op}, \mathfrak{Cat})(W, \mathfrak{Cat}(E, \mathscr{X})) \tag{4.2.1}$$

for any small category $\mathscr{X}$. A pseudo-colimit is conical if W has $\mathbf{1}$ as its only value. It is finite if $\mathfrak{C}$ is a finite 2-category and each WC is finitely-presentable.

The strict version is also of interest. For completeness it is recalled here.

Definition 4.2.2 (Weighted 2-Colimit)**.** *Suppose that E and W are in fact 2-functors. The 2-colimit of E weighted by W is a category $E \star_s W$ together with a cocone $\xi\colon W \to \mathfrak{Cat}(E, E \star_s W)$ inducing an isomorphism of categories*

$$\mathfrak{Cat}(E \star_s W, \mathscr{X}) \cong [\mathfrak{C}^{op}, \mathfrak{Cat}](W, \mathfrak{Cat}(E, \mathscr{X}))$$

for any small category $\mathscr{X}$. A 2-colimit is conical if W has $\mathbf{1}$ as its only value.

Example 4.2.3. *The coinverter of a 2-cell is a 1-morphism that universally inverts the 2-cell by horizontal composition. That is, let $s, t\colon S \rightrightarrows C$ denote arrows admitting a 2-cell $\alpha\colon s \Rightarrow t$. The coinverter of α is an arrow $q\colon C \to Q$ such that $q * \alpha$ is invertible and such that composition with q induces an isomorphism of categories*

$$\mathfrak{K}(Q, X) \cong \mathfrak{K}(C, X)_\alpha$$

where $\mathfrak{K}(C, X)_\alpha$ is the full subcategory of arrows $C \to X$ inverting α by horizontal composition. Following the slight abuse of language in [KLW93], a coinverter will be called "reflexive" if the 2-cell α admits a morphism $i\colon A \to S$ with $\alpha * i = 1$.

The main result of the section, Theorem 4.2.11, is a computation of the weighted pseudo-colimit in the case $\mathfrak{K} = \mathfrak{Cat}$ and a direct verification of the universal property as in 4.2.1. Theorem 4.2.11 should be seen as a weighted and genuinely 2-categorical generalization of the computation of §6.4.0 in [AGV72], where the pseudo-colimit of a pseudo-functor on a 1-category $\mathscr{C} \to \mathfrak{Cat}$ is computed as a category of fractions. Colimit computations have been of some interest recently. In §3.2 of F. Lawler's thesis [Law13], there is a computation of conical pseudo-colimits indexed by bicategories similar to the one subsequently presented here. From this, Lawler computes weighted bicolimits using certain descent diagrams. The paper [DDS18a] of Descotte, Dubuc, and Szyld shows how to compute certain σ-filtered σ-colimits in $\mathfrak{Cat}$.

The insight leading to the present construction in §4.2.1 is the observation that the category of fractions technique can be carried out for both a category-valued functor and its weight by using a "diagonal category" $\Delta(E, W)$ that carries out each functor's category of elements construction simultaneously. (It is worth pointing out that this would not just be the 2-category of elements of the product bifunctor given by E and W.) Of course 2-cells must be added in to account for the indexing 2-category, but passing to connected components makes the candidate sufficiently 2-categorically discrete not only to be a 1-category but also to satisfy all the necessary 2-dimensional coherence conditions. Passing to connected components seems first to have featured in the conical colimit computations of §I,7.11 of [Gra74].

4.2.1 Candidate for Colimit

Let $\Delta(E, W)$ denote the category with objects triples (C, X, Y) with $C \in \mathfrak{C}$ and $X \in EC$ and $Y \in WC$; and with arrows $(C, X, Y) \to (D, A, B)$ those triples (f, u, v) with $f\colon C \to D$ and $u\colon f_!X \to A$ and $v\colon Y \to f^*B$. Call a morphism (f, u, v) "cartesian" if both u and v are invertible. Composition and identities in $\Delta(E, W)$ are as in the 2-category of elements of category-valued pseudo-functors. Boost $\Delta(E, W)$ up to a 2-category as follows. Declare a 2-cell $(f, u, v) \Rightarrow (g, x, y)$ to one $\alpha\colon f \Rightarrow g$ of $\mathfrak{C}$ for which there are commutative triangles

$$\begin{tikzcd} f_!X \arrow[d, "(\alpha_!)_X"'] \arrow[dr, "u"] & \\ g_!X \arrow[r, "x"'] & U \end{tikzcd} \qquad \begin{tikzcd} & f^*Y \arrow[d, "(\alpha^*)_Y"] \\ V \arrow[ur, "v"] \arrow[r, "y"'] & g^*Y \end{tikzcd}$$

in the respective fibers. Notice that this construction basically combines the 2-category of elements constructions of Definition 2.2.8 for each pseudo-functor "along the diagonal."

Now, recall from §2.1 that there is a "connected components" functor $\pi_0\colon 2\text{-}\mathfrak{Cat} \to \mathfrak{Cat}$ taking a 2-category $\mathfrak{A}$ to its 1-category of connected components, given by taking the π_0 in the usual sense of each hom-category $\mathfrak{A}(X, Y)$ for $X, Y \in \mathfrak{A}$. Now, declare as notation

$$E \star W := \pi_0\Delta(E, W)[\Sigma^{-1}] \tag{4.2.2}$$

by first taking the connected components of the 2-category $\Delta(E, W)$ and then inverting Σ, the set of images of cartesian morphisms in the resulting 1-category. The notation may seem somewhat prejudicial, but Theorem 4.2.11 – perhaps the central result of the present work – shows that this construction is in fact a computation of the weighted pseudo-colimit of E. For the computations, note that there is a canonical map $L\colon \Delta(E, W) \to E \star W$ viewing a morphism (f, u, v) as a span with left leg identity.

Remark 4.2.4. Recall that category-valued pseudo-functors on 2-categories correspond, roughly speaking, to discrete 2-fibrations, axiomatized in Definition 2.2.15. Thus, bracketing, temporarily, the question of the correctness of the colimit computation in 4.2.2 above, let us notice that an analogous construction can be carried out for a discrete 2-fibration $F\colon \mathfrak{F} \to \mathfrak{C}$ and a discrete 2-opfibration $E\colon \mathfrak{E} \to \mathfrak{C}$. Start by taking the ordinary pullback of the total 2-categories, namely, $\mathfrak{E} \times_{\mathfrak{C}} \mathfrak{F}$. Then apply the connected components functor π_0 and pass to the category of fractions

$$E \otimes_{\mathfrak{C}} F := \pi_0(\mathfrak{E} \times_{\mathfrak{C}} \mathfrak{F})[\Sigma^{-1}] \tag{4.2.3}$$

where Σ is the set of images of arrows of the pullback whose components are (op)cartesian. The use of the tensor notation is tendentious, but it will be justified in Corollary 4.2.13 below.

4.2.2 Assignments and Universal Property

Let $E\colon \mathfrak{C} \to \mathfrak{Cat}$ and $W\colon \mathfrak{C}^{op} \to \mathfrak{Cat}$ denote pseudo-functors on a small 2-category $\mathfrak{C}$.

Now, begin to define a correspondence

$$\Phi\colon \mathfrak{Cat}(E \star W, \mathscr{X}) \longrightarrow \mathfrak{Hom}(\mathfrak{C}^{op}, \mathfrak{Cat})(W, \mathfrak{Cat}(E, \mathscr{X})) \tag{4.2.4}$$

Start with a functor $F\colon E \star W \to \mathscr{X}$. The image under Φ should be a pseudo-natural transformation $\Phi(F)$ whose components over $C \in \mathfrak{C}$ should be functors

$$\Phi(F)_C\colon WC \to \mathfrak{Cat}(EC, \mathscr{X}). \tag{4.2.5}$$

To define such $\Phi(F)_C$, fix an object $Y \in WC$. The image should be a functor $EC \to \mathscr{X}$. For an object $X \in EC$, declare

$$\Phi(F)_C(Y)(X) := F(C, X, Y). \tag{4.2.6}$$

And for an arrow $u\colon X \to Z$ of EC, the image under $\Phi(F)_C(Y)$ is taken to be the image under F of $(1, u, 1)$ viewed as a span in $E \star W$ with left leg identity, i.e., as the image of $(1, u, 1_Y)$ of $\Delta(E, W)$ viewed in the category of fractions under the canonical map L above. Of course this means that $\Phi(F)_C(Y)\colon EC \to \mathscr{X}$ is a functor since F is one.

Now, finish the assignment of 4.2.5. For an arrow $v\colon Y \to Z$ of WC, declare $\Phi(F)_C(v)$ to be the natural transformation $\Phi(F)_C(Y) \Rightarrow \Phi(F)_C(Z)$ whose components $\Phi(F)_C(v)_X$ are the images of the morphisms $(1, 1_X, v)$ viewed as a span with left leg identity. Naturality in $X \in EC$ and that $\Phi(F)_C$ is a functor both follow because F is a functor.

Now, the components $\Phi(F)_C$ as in 4.2.5 indexed over $C \in \mathfrak{C}$ comprise a pseudo-natural transformation. To see this, required are invertible cells

$$\begin{tikzcd} WD \arrow[r] \arrow[d, "f^*"'] \arrow[dr, phantom, "\cong"] & \mathfrak{Cat}(ED, \mathscr{X}) \arrow[d, "(f_!)^*"] \\ WC \arrow[r] & \mathfrak{Cat}(EC, \mathscr{X}) \end{tikzcd}$$

for each $f\colon C \to D$ of $\mathfrak{C}$. Such a cell should be a natural isomorphism with components indexed over $Y \in WD$. For such Y, the component of the coherence isomorphism should be a natural isomorphism $\Phi(F)_C(f^*Y) \Rightarrow (f_!)^*\Phi(F)_D(Y)$ of functors $EC \to \mathscr{X}$. For $X \in EC$, a component will be the image under F of the arrow in $E \star W$ given by the span

$$(C, X, f^*Y) \xleftarrow{\;1\;} (C, X, f^*Y) \xrightarrow{(f,1,1)} (D, f_!X, Y).$$

Note that the image of the span above upon passing to the category of fractions $E \star W$ is an isomorphism. That these arrows amount to a natural isomorphism results from the fact that F and the canonical morphisms L are functors. Now, the proposed components of the purported isomorphism in the square above should be natural in $Y \in WD$. For $v\colon Y \to Z$ in WD, the naturality square commutes because L and F are functors.

Lemma 4.2.5. *The components $\Phi(F)_C$ over $C \in \mathfrak{C}$ as in 4.2.5, with coherence isos as above, are a pseudo-natural transformation. Thus, the object assigment for Φ as in 4.2.4 is well-defined.*

Proof. Condition 1 of the pseudo-natural transformation axioms in 2.1.6 can be seen to hold in the following way. Let $f\colon B \to C$ and $g\colon C \to D$ denote two arrows of $\mathfrak{C}$. The equality of the corresponding 2-cells of the form of the first part of the condition then follows from the commutativity of the figure

$$\begin{tikzcd} (C, f_!X, g^*Y) \arrow[rr, "{(g,1,1)}"] & & (D, g_!f_!X, Y) \arrow[d, "{(1,\cong,1)}"] \\ (B, X, f^*g^*Y) \arrow[u, "{(f,1,1)}"] \arrow[dr, "{(1,1,\cong)}"'] & & (D, (gf)_!X, Y) \\ & (B, X, (gf)^*Y) \arrow[ur, "{(gf,1,1)}"'] & \end{tikzcd}$$

and the fact that the canonical map L and the given F are functors.

The second pseudo-naturality condition of 2.1.6 is verified in the following way. Start with a 2-cell $\alpha\colon f \Rightarrow g$ between arrows $f, g\colon C \rightrightarrows D$ of $\mathfrak{C}$. The equality of 2-cells in the condition boils down to the commutativity of the square

$$\begin{tikzcd}
(C, X, f^*Y) \arrow[r, "{(1,1,\alpha)}"] \arrow[d, "{(f,1,1)}"'] & (C, X, g^*Y) \arrow[d, "{(g,1,1)}"] \\
(D, f_!X, Y) \arrow[r, "{(1,\alpha,1)}"'] & (D, g_!X, Y)
\end{tikzcd}$$

when reduced to path-classes and subsequently to the category of fractions $E \star W$. But this can be seen by exhibiting a path between the composite sides of the square, namely, $(f, \alpha, 1)$ and $(g, 1, \alpha)$. The path is a 2-cell of $\Delta(E, W)$ between these two arrows. Take α itself. The commutative triangles

$$\begin{tikzcd}
f_!X \arrow[dr, "(\alpha_!)_X"] \arrow[dd, "(\alpha_!)_X"'] & \\
& g_!X \\
g_!X \arrow[ur, "1"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& f^*Y \arrow[dd, "(\alpha^*)_Y"] \\
f^*Y \arrow[ur, "1"] \arrow[dr, "(\alpha^*)_Y"'] & \\
& g^*Y
\end{tikzcd}$$

show precisely that $\alpha\colon (f, \alpha, 1) \Rightarrow (g, 1, \alpha)$ is such a 2-cell, hence a path in the localization $E \star W$, meaning that the two arrows in the commutative square reduce to the same class in the localization. Thus, the images of these classes under F are equal, proving the condition. $\square$

Now, continue the assignments for 4.2.4. In particular, take a natural transformation $\alpha\colon F \Rightarrow G$ for functors $F, G\colon E \star W \rightrightarrows \mathscr{X}$. The image under Φ should be a modification $\Phi(\alpha)$ with components

$$\Phi(\alpha)_C\colon \Phi(F)_C \to \Phi(G)_C \tag{4.2.7}$$

indexed over $C \in \mathfrak{C}$. Each such component should be a natural transformation with components

$$\Phi(\alpha)_{C,Y}\colon \Phi(F)_C(Y) \to \Phi(G)_C(Y) \tag{4.2.8}$$

indexed by $Y \in WC$. Further each such component should be a natural transformation

$$\Phi(\alpha)_{C,Y,X}\colon \Phi(F)_C(Y)(X) \to \Phi(G)_C(Y)(X) \tag{4.2.9}$$

indexed over $X \in EC$. Unpacking the last condition from the definitions, this means that $\Phi(\alpha)_{C,Y,X}$ ought to be an arrow of $\mathscr{X}$ of the form $F(C, X, Y) \to G(C, X, Y)$. Thus, make the definition

$$\Phi(\alpha)_{C,Y,X} := \alpha_{C,X,Y}\colon \Phi(F)_C(Y)(X) \to \Phi(G)_C(Y)(X). \tag{4.2.10}$$

That the collections indicated by the displays 4.2.8 and 4.2.9 are natural in their proper variables follows from the definition in 4.2.10 by the naturality of α. What remains to check is that the components of 4.2.7 comprise a modification.

Lemma 4.2.6. *The arrow assignment for Φ with components $\Phi(\alpha)_C$ over $C \in \mathfrak{C}$ as in 4.2.7 is a modification. In particular, the arrow assignment for Φ of 4.2.4 is well-defined.*

Proof. Let $f\colon C \to D$ denote an arrow of $\mathfrak{C}$. The modification condition in Definition 2.1.8 requires equality of two composite 2-cells making two sides of a cylindrical figure. Chasing $Y \in WD$ around each composite reveals that the equality will follow from commutativity of the square

$$\begin{tikzcd} F(C, X, f^*Y) \arrow[r, "\alpha_{C,X,f^*Y}"] \arrow[d, "F(f,1,1)"'] & F(C, X, f^*Y) \arrow[d, "F(f,1,1)"] \\ F(D, f_!X, Y) \arrow[r, "\alpha_{D,f_!X,Y}"'] & F(D, f_!X, Y) \end{tikzcd}$$

But this is commutative in $\mathscr{X}$ because it is a naturality square for α at the morphism $(f, 1, 1)$. $\square$

Lemma 4.2.7. *The assignments giving Φ of 4.2.4 are functorial.*

Proof. This follows by the definition of composition of natural transformations on the one hand and of modifications on the other. $\square$

Now, begin assignments for a reverse correspondence, namely, what will be a functor

$$\Psi\colon \mathfrak{Hom}(\mathfrak{C}^{op}, \mathfrak{Cat})(W, \mathfrak{Cat}(E, \mathscr{X})) \longrightarrow \mathfrak{Cat}(E \star W, \mathscr{X}). \tag{4.2.11}$$

Start with a pseudo-natural transformation $\theta\colon W \to \mathfrak{Cat}(E, \mathscr{X})$ of the domain. The image $\Psi(\theta)$ will be a functor; it can be induced from the underlying category $\pi_0\Delta(E, W)$ of $E \star W$ using the universality of the category of fractions construction. To this end, define

$$\Psi(\theta)\colon \pi_0\Delta(E, W) \longrightarrow \mathscr{X} \tag{4.2.12}$$

in the following way. On an object (C, X, Y) of the domain, take

$$\Psi(\theta)(C, X, Y) := \theta_C(Y)(X). \tag{4.2.13}$$

Now, for an arrow assignment, observe first that since θ is pseudo-natural, it comes with coherence isomorphisms for each arrow $f\colon C \to D$ of $\mathfrak{C}$ of the form

$$\begin{tikzcd} WD \arrow[r, "\theta_D"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\cong"] & \mathfrak{Cat}(ED, \mathscr{X}) \arrow[d, "f_!"] \\ WC \arrow[r, "\theta_C"'] & \mathfrak{Cat}(EC, \mathscr{X}) \end{tikzcd}$$

Denote such a coherence isomorphism by θ_f. Thus, for a morphism (f, u, v) of $\Delta(E, W)$ with morphisms $u\colon f_!X \to U$ and $v\colon Y \to f^*V$ of the appropriate fibers, take $\Psi(\theta)(f, u, v)$ to be the composite morphism

$$\theta_C(Y)(X) \xrightarrow{\theta_C(v)_X} \theta_C(f^*V)(X) \xrightarrow{\theta_{f,V,X}} \theta_D(V)(f_!X) \xrightarrow{\theta_D(V)(u)} \theta_D(V)(U)$$

of $\mathscr{X}$. It must be shown that this induces a well-defined assigment when passing to path-classes.

Lemma 4.2.8. *The arrow assignment immediately above is independent of representative of path-class. Additionally, the induced assignment on $\pi_0\Delta(E, W)$ gives a functor $\Psi(\theta)$ as in 4.2.12.*

Proof. The first statement reduces to the case where $\alpha\colon (f, u, v) \Rightarrow (g, x, y)$ is a 2-cell of $\Delta(E, W)$ between arrows $(C, X, Y) \rightrightarrows (D, U, V)$. The claim is that the top and bottom sides of the outside of the following figure are equal.

$$\begin{tikzcd} & \theta_C(f^*V)(X) \arrow[r, "\theta_f"] \arrow[d, dashed, "\theta_C(\alpha)_X"'] & \theta_D(V)(f_!X) \arrow[dr, "\theta_D(V)(u)"] \arrow[d, dashed, "\theta_D(V)(\alpha)"] & \\ \theta_C(Y)(X) \arrow[ur, "\theta_C(v)_X"] \arrow[dr, "\theta_C(y)_X"'] & & & \theta_D(V)(U) \\ & \theta_C(g^*V)(X) \arrow[r, "\theta_g"'] & \theta_D(V)(g_!X) \arrow[ur, "\theta_D(V)(x)"'] & \end{tikzcd}$$

But this is immediate. For the dashed vertical arrows give a square in the center that commutes by the second coherence condition for θ_f and θ_g in 2.1.6 and the two triangles are the images of the commutative triangles coming with the 2-cell α under θ_C and under $\theta_D(V)$, respectively. Thus any two such arrows connected by such a 2-cell α are in the same path class. Since an arbitrary path is just alternating 2-cells of this form, this special case proves the first claim.

Therefore, the assignments for Ψ induce assignments on $\pi_0\Delta(E, W)$. That the arrow assignment is functorial also follows. The unit condition is trivial. That the assignment respects composition is involved but ultimately straightforward. One sets up a triangular figure each of whose sides is a three-fold composite of morphisms arising as in the arrow assignment. The claim is that one side of the triangle is equal to the composite of the other two. This can be seen by filling in the figure with the various naturality and coherence conditions, a tedious but straightforward task. □

Corollary 4.2.9. *The functor $\Psi(\theta)\colon \pi_0\Delta(E,W) \to \mathscr{X}$ inverts the images of cartesian morphisms, hence induces a functor on the category of fractions, also denoted by $\Psi(\theta)\colon E\star W \to \mathscr{X}$. In particular the object assignment of Ψ above in 4.2.11 is well-defined.*

Proof. The main claim basically follows from the definition of the arrow assignment for Ψ. For if (f,u,v) is cartesian, then u and v are invertible and so are $\theta_C(v)$ and $\theta_D(V)(u)$. Of course the components of θ_f are invertible. Thus, $\Psi(\theta)(f,u,v)$ for such (f,u,v) is invertible in $\mathscr{X}$. □

For an arrow assignment for Ψ, begin with a modification $m\colon \theta \to \gamma$ of two given pseudo-natural transformations $\theta, \gamma\colon W \rightrightarrows \mathfrak{Cat}(E,\mathscr{X})$. It suffices to induce the required natural transformation from the underlying category $\Delta(E,W)$. Take an object (C,X,Y). The evident definition of the required $\Psi(m)\colon \Psi(\theta) \Rightarrow \Psi(\gamma)$ is just

$$\Psi(m)_{C,X,Y} := m_{C,Y,X}\colon \theta_C(Y)(X) \to \theta_C(Y)(X) \tag{4.2.14}$$

that is, the X-component of the Y-component of the C-component of the modification m.

Lemma 4.2.10. *The definition of 4.2.14 defines a natural transformation. Thus, in particular, the arrow assignment of Ψ from 4.2.11 is well-defined. Additionally, Ψ, so defined, is a functor.*

Proof. That the required naturality square commutes is just a result of the modification condition 2.1.8 satisfied by m. That Ψ is a functor again follows by the definitions of the assignments and the definitions of composition of modifications and of natural transformations. □

Theorem 4.2.11 (Colimit Computation)**.** *The functors Φ and Ψ of 4.2.4 and 4.2.11 are mutually inverse. In particular, for pseudo-functors $E\colon \mathfrak{C} \to \mathfrak{Cat}$ and $W\colon \mathfrak{C}^{op} \to \mathfrak{Cat}$, the category $E \star W$ is the pseudo-colimit of E weighted by W in the sense that Φ and Ψ thus provide an isomorphism*

$$\mathfrak{Cat}(E\star W, \mathscr{X}) \cong \mathfrak{Hom}(\mathfrak{C}^{op}, \mathfrak{Cat})(W, \mathfrak{Cat}(E,\mathscr{X}))$$

of categories for any small category $\mathscr{X}$.

Proof. That Φ and Ψ are mutually inverse follows by computation from the definitions given over the preceding development. That $E \star W$ is the pseudo-colimit follows by definition. □

Remark 4.2.12. The theorem is the 2-dimensional analogue of the presheaf tensor-hom adjunction from Proposition 1.1.3.

4.2.3 Consequences of Theorem 4.2.11

Notice that for E and W, co- and contravariant pseudo-functors on a 2-category $\mathfrak{C}$ as in the previous subsection, the pseudo-colimit extends to a 2-functor $E\star -\colon \mathfrak{Hom}(\mathfrak{C}^{op},\mathfrak{Cat})\to\mathfrak{Cat}$. The assignments on arrows and on 2-cells are the ones suggested by the construction of $E\star W$.

Corollary 4.2.13. *The induced 2-functor $E\star -$ is left 2-adjoint to the 2-functor $\mathfrak{Cat}(E,-)$.*

Proof. Theorem 4.2.11 almost proves this. The isomorphism in the conclusion of the statement is also natural in $\mathscr{X}$ and in W, as can be seen from the definitions of the morphisms giving the isomorphism. □

Remark 4.2.14. The 2-adjunction of Corollary 4.2.13 above is, formally speaking, a 2-categorical "tensor-hom adjunction" analogous to the 1-categorical case reviewed in the introduction. Thus, to emphasize the analogy, use the notation

$$E\otimes_{\mathfrak{C}} W := E\star W \tag{4.2.15}$$

and call this the tensor product of the pseudo-functors E and W over $\mathfrak{C}$.

Now, if $C\in\mathfrak{C}$ is an object, then consider the colimit weighted by the canonical representable 2-functor $\mathbf{y}C\colon\mathfrak{C}^{op}\to\mathfrak{Cat}$. The computation underlying Theorem 4.2.11 shows explicitly that $E\otimes_{\mathfrak{C}}\mathbf{y}C$ is equivalent to the fiber EC. For indeed on the one hand there is a functor

$$F\colon EC\to\pi_0\Delta(E,\mathbf{y}C)$$

given by

$$F(X)=(C,X,1)\qquad F(u)=(1,u,1) \tag{4.2.16}$$

where the latter arrow is viewed reduced modulo its path class in the target. The assignments for F are completed by then passing to the category of fractions. Denote the composite again by F. This is plainly a functor. On the other hand, there is a functor $G\colon\Delta(E,\mathbf{y}C)\to EC$ given in the following way. On an object (B,X,f) with $f\colon B\to C$, take the image under G to be the image of X under the transition functor $f_!$, namely,

$$G(B,X,f):=f_!X. \tag{4.2.17}$$

Now, fix a morphism $(B, X, f) \to (D, Y, g)$ given by (h, u, θ) with $u\colon h_!X \to Y$ in ED and $\theta\colon f \Rightarrow gh$ a 2-cell of $\mathfrak{C}$. The image under G is defined to be the composite

$$f_!X \xrightarrow{(\theta_!)_X} g_!h_!X \xrightarrow{g_!u} g_!Y$$

where of course $\theta_!$ is the image under E of the 2-cell θ. That G is a functor follows by the naturality of the images of the various 2-cells under E. But $\Delta(E, \mathbf{y}C)$ is also a 2-category. The assignments for G are well-defined on paths in $\Delta(E, \mathbf{y}C)$. For let $\alpha\colon (h, u, \theta) \Rightarrow (k, v, \gamma)$ denote such a 2-cell. In particular, the 2-cells α, γ, and θ satisfy the relationship

$$\begin{tikzcd} B \arrow[dd, "k"'] \arrow[dr, "f"] & \\ \arrow[r, phantom, "\Downarrow\gamma"] & C \\ D \arrow[ur, "g"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} B \arrow[dd, bend right, "k"'] \arrow[dd, bend left, "h"] \arrow[dd, phantom, "\overset{\alpha}{\Leftarrow}"] \arrow[dr, "f"] & \\ \arrow[r, phantom, "\Downarrow\theta"] & C. \\ D \arrow[ur, "g"'] & \end{tikzcd}$$

And so, the images under G of the two 1-cells of $\Delta(E, \mathbf{y}C)$ above are the left and right sides of the diamond in the following figure.

$$\begin{tikzcd} & f_!X \arrow[dl, "(\theta_!)_X"'] \arrow[dr, "(\gamma_!)_X"] \arrow[d, phantom, "(I)"] & \\ g_!h_!X \arrow[rr, dashed] \arrow[dr, "g_!u"'] & \arrow[d, phantom, "(II)"] & g_!k_!X \arrow[dl, "g_!v"] \\ & g_!Y & \end{tikzcd}$$

The dashed arrow is the image under the transition functor $g_!$ of the component $(\alpha_!)_X$. The triangle (I) commutes by the condition on the 2-cells α, γ, and θ mentioned above. The triangle (II) commutes since it is the image under the transition functor $g_!$ of the commutative triangle

$$\begin{tikzcd} h_!X \arrow[dd, "(\alpha_!)_X"'] \arrow[dr, "u"] & \\ & Y \\ k_!X \arrow[ur, "v"'] & \end{tikzcd}$$

coming by definition with the 2-cell α. In particular, the discussion shows that G extends to a functor on the 1-category of connected components, also denoted by $G\colon \pi_0\Delta(E, \mathbf{y}C) \to EC$, since every path is constructed from such 2-cells.

Corollary 4.2.15. *For each $C \in \mathfrak{C}$, the functors F and G in the discussion above induce an equivalence of categories $E \otimes_{\mathfrak{C}} \mathbf{y}C \simeq EC$.*

Proof. In fact, it follows immediately from the definitions that $GF = 1$. On the other hand, it is straightforward, again from the definitions, to construct a natural system of maps $1 \Rightarrow FG$, each component of which is a cartesian arrow in $\Delta(E, \mathbf{y}C)$, hence invertible when passing to the category of fractions, and thus yielding the rest of the equivalence. □

Corollary 4.2.16. *The equivalence $E \otimes_{\mathfrak{C}} \mathbf{y}C \simeq EC$ of Corollary 4.2.15 is pseudo-natural in C, yielding a pseudo-natural equivalence $E \otimes_{\mathfrak{C}} \mathbf{y} \simeq E$. In this sense, Yoneda is a unit for the tensor 2-functor $E \otimes_{\mathfrak{C}} -$.*

Proof. For an arrow $f\colon C \to D$ of $\mathfrak{C}$, the required coherence cell

$$\begin{tikzcd} EC \arrow[r, "F_C"] \arrow[d, "f_!"'] \arrow[dr, phantom, "\phi_f \cong"] & E \otimes_{\mathfrak{C}} \mathbf{y}C \arrow[d, "1 \star \mathbf{y}f"] \\ ED \arrow[r, "F_D"'] & E \otimes_{\mathfrak{C}} \mathbf{y}D \end{tikzcd}$$

has as its X-component for $X \in EC$, the arrow

$$(f, 1, 1)\colon (C, X, f) \to (D, f_! X, 1)$$

which is plainly cartesian, hence invertible in $E \otimes_{\mathfrak{C}} \mathbf{y}D$. Naturality in X follows straight from the definition. The two pseudo-naturality conditions of Definition 2.1.6 follow by the construction of the colimit. □

Corollary 4.2.17. *Every category-valued pseudo-functor $E\colon \mathfrak{C} \to \mathfrak{Cat}$ has a "strictification."*

Proof. The previous corollary shows that E is pseudo-naturally equivalent to a strict 2-functor. □

Remark 4.2.18. Corollaries 4.2.15 and 4.2.16 admit another, but substantially less informative, proof from Theorem 4.2.11. That is, in the following display, the theorem provides the following left-most isomorphism, while the equivalence on the right is the pseudo-Yoneda lemma:

$$\mathfrak{Cat}(E \otimes_{\mathfrak{C}} \mathbf{y}C, \mathscr{X}) \cong \mathfrak{Hom}(\mathscr{C}^{op}, \mathfrak{CAT})(\mathbf{y}C, \mathfrak{Cat}(E, \mathscr{X})) \simeq \mathfrak{Cat}(EC, \mathscr{X}).$$

These hold pseudo-naturally in C and $\mathscr{X}$, which yields an equivalence $E \simeq E \otimes_{\mathfrak{C}} \mathbf{y}C$. That this is pseudo-natural in C is a further consequence of the pseudo-naturality of the equivalences.

4.3 The Tensor Product as a Coinverter

For the moment, let $P\colon \mathscr{C}^{op} \to \mathbf{Set}$ and $Q\colon \mathscr{C} \to \mathbf{Set}$ denote ordinary functors. There is another characterization of their tensor product, based on the construction of colimits from coproducts and coequalizers. For this, recall that P and Q can be viewed as functions $\mathscr{P} \to \mathscr{C}_0$ and $\mathscr{Q} \to \mathscr{C}_0$ where $\mathscr{P}$ and $\mathscr{Q}$ are the sets formed by taking the disjoint unions of the sets PC and QC over all $C \in \mathscr{C}_0$. The arrows of $\mathscr{C}$ act on $\mathscr{P}$ and $\mathscr{Q}$. For example, $n\colon \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{P} \to \mathscr{P}$ is the action $n(f,p) = Pf(p)$. The action m on $\mathscr{Q}$ is given analogously. And the tensor product $Q \otimes_{\mathscr{C}} P$ is then the coequalizer of these actions as in the diagram

$$\begin{tikzcd} \mathscr{Q} \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{P} \arrow[r, shift left, "1\times n"] \arrow[r, shift right, "m\times 1"'] & \mathscr{Q} \times_{\mathscr{C}_0} \mathscr{P} \arrow[r, dashed] & Q \otimes_{\mathscr{C}} P. \end{tikzcd}$$

This is described in VII.5.(3) on p.379 of [MLM92]. Notice that it is the formation of these actions that suggests viewing Q as a right $\mathscr{C}$-module and P as a left $\mathscr{C}$-module.

What follows is a 2-dimensional version of the coequalizer condition above for the tensor product of discrete 2-fibrations constructed in the last subsection.

Let $F\colon \mathfrak{F} \to \mathfrak{C}$ denote a discrete 2-fibration with cleavage σ; and let $E\colon \mathfrak{F} \to \mathfrak{C}$ denote a discrete 2-opfibration; each as in Definition 2.2.15. The first lemma gives part of the proof of the omnibus fibration theorem from Chapter 2, namely, Theorem 2.2.6.

Lemma 4.3.1. *In the notation above, the cleavage and opcleavage determine a 2-cell*

$$\begin{tikzcd} \mathfrak{E}_0 \times_{\mathfrak{C}_0} (\mathfrak{C}_0)^{\mathbf{2}} \times_{\mathfrak{C}_0} \mathfrak{F}_0 \arrow[rr, bend left, "1\times n"] \arrow[rr, bend right, "m\times 1"'] \arrow[rr, phantom, "\Downarrow \rho\times\sigma"] & & \pi_0(\mathfrak{E} \times_{\mathfrak{C}} \mathfrak{F}) \end{tikzcd}$$

between the action functors coming with the algebra structure.

Proof. Given a morphism $(X, f, Y) \to (Z, g, W)$ in the domain category on the right with components (u, h, k, v) represented by

$$\begin{tikzcd} X \arrow[d, "u"'] & C \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "="] & D \arrow[d, "k"] & Y \arrow[d, "v"] \\ Y & A \arrow[r, "g"'] & B & W \end{tikzcd}$$

the naturality square takes the following form. The naturality square is represented by the diagram

$$\begin{tikzcd}
(X, f^*Y) \arrow[r, "(\rho,\sigma)"] \arrow[d, "(u,!)"'] \arrow[dr, phantom, "="] & (f_!X, Y) \arrow[d, "(!,v)"] \\
(Z, g^*W) \arrow[r, "(\rho,\sigma)"'] & (g_!Z, W)
\end{tikzcd}$$

This square evidently commutes by definition of the unique lifts. $\square$

Theorem 4.3.2. *The tensor product $E \otimes_{\mathfrak{C}} F$, constructed as in Equation 4.2.3 with its universal map L as in*

$$\begin{tikzcd}
\mathfrak{E}_0 \times_{\mathfrak{C}_0} (\mathfrak{C}_0)^{\mathbf{2}} \times_{\mathfrak{C}_0} \mathfrak{F}_0 \arrow[rr, bend left, "1 \times n"] \arrow[rr, bend right, "m \times 1"'] \arrow[rr, phantom, "\Downarrow \rho \times \sigma"] & & \pi_0(\mathfrak{E} \times_{\mathfrak{C}} \mathfrak{F}) \arrow[r, dashed, "L"] & E \otimes_{\mathfrak{C}} F
\end{tikzcd}$$

is the reflexive coinverter of the cell $\rho \times \sigma$ as in Example 4.2.3.

Proof. Since the cleavage and opcleavage are assumed to be normalized, the 2-cell $\rho \times \sigma$ is reflexive. And indeed the canonical morphism $L\colon \pi_0(\mathfrak{E} \times_{\mathfrak{C}} \mathfrak{F}) \to E \otimes_{\mathfrak{C}} F$ inverts the images of cartesian morphisms. The task is to show that it does so suitably universally.

To see this, start with a functor $K\colon \pi_0(\mathfrak{E} \times_{\mathfrak{C}} \mathfrak{F}) \to \mathscr{X}$ with a reflexive cell $\zeta\colon K(1 \times n) \cong K(m \times 1)$. The required induced functor

$$\widetilde{K}\colon E \otimes_{\mathfrak{C}} F \to \mathscr{X}$$

arises in the following way. The point is that K inverts the images of cartesian morphisms. But it suffices to show that K inverts the cell $\rho \times \sigma$ since any cartesian morphism is isomorphic to one specified by $\rho \times \sigma$. To this end, consider the morphism

$$\begin{tikzcd}
X \arrow[d, "\rho(X,f)"'] & C \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & D \arrow[d, "1"] & Y \arrow[d, "1"] \\
f_!X & D \arrow[r, "1"'] & D & Y.
\end{tikzcd}$$

Now, the naturality square corresponding to this morphism under the given normalized isomorphism $\zeta\colon K(1\times n)\cong K(m\times 1)$ takes the form

$$\begin{tikzcd} K(X,f^*Y) \arrow[r, "\zeta_{X,f,Y}"] \arrow[d, "K(\rho,\sigma)"'] \arrow[dr, phantom, "="] & (f_!X,Y) \arrow[d, "K(1,1)"] \\ (f_!X,Y) \arrow[r, "\zeta_{f_!X,1,Y}"'] & (f_!X,Y) \end{tikzcd}$$

showing that $K(\rho,\sigma)=\zeta_{X,f,Y}$, an isomorphism. Thus, there is a functor $\widetilde{K}\colon E\otimes_{\mathfrak{C}}F\to\mathscr{X}$ induced by the universal property of the category of fractions making an appropriate commutative triangle. The 2-dimensional aspect of the universal property of a reflexive coinverter is similarly established. □

4.3.1 Elementary Construction of Tensor Product

The last theorem motivates an internal definition of the tensor product in the case of $\mathfrak{K}=\mathfrak{Cat}(\mathscr{E})$ for suitable $\mathscr{E}$. Let $e\colon\mathcal{E}\to\mathcal{C}$ denote a discrete 2-opfibration with underlying opcleavage ρ; and let $f\colon\mathcal{F}\to\mathcal{C}$ denote a discrete 2-fibration with underlying cleavage σ, each as in Definition 3.4.12. The cleavage and opcleavage determine an internal natural transformation of underlying internal 1-functors as displayed in the following diagram; the tensor product is defined to be the coinverter, as in Example 4.2.3, of the reflexive 2-cell

$$\begin{tikzcd} \mathcal{E}_0\times_{\mathcal{C}_0}(\mathcal{C}_0)^{\mathbf{2}}\times_{\mathcal{C}_0}\mathcal{F}_0 \arrow[r, bend left, "1\times n"] \arrow[r, bend right, "m\times 1"'] \arrow[r, phantom, "\Downarrow\rho\times\sigma"] & \pi_0(\mathcal{E}\times_{\mathcal{C}}\mathcal{F}) \arrow[r, dashed, "L"] & \mathcal{E}\otimes_{\mathcal{C}}\mathcal{F} \end{tikzcd}$$

appearing as the dashed arrow, if it exists. Recall that the 'π_0' indicates the internal connected components construction of §3.4.1. Provided that the tensor always exists, it will define a 2-functor

$$\mathcal{E}\otimes_{\mathcal{C}}-\colon\mathfrak{DFib}(\mathcal{C})\longrightarrow\mathfrak{K}=\mathfrak{Cat}(\mathscr{E}).$$

The following section extracts necessary filteredness conditions under which the tensor product can be seen to arise through a right calculus of fractions, both in the classical case of ordinary categories and internally in the elementary case of $\mathfrak{K}=\mathfrak{Cat}(\mathscr{E})$ for sufficiently nice $\mathscr{E}$, as described in Chapter 5.

4.4 Extraction of Filteredness Conditions

What follows are necessary "intrinsic" conditions following from the assumption that the tensor 2-functor $E \otimes_{\mathfrak{C}} - \colon \mathfrak{Hom}(\mathfrak{C}^{op}, \mathfrak{Cat}) \to \mathfrak{Cat}$ preserves finite weighted limits. Throughout use the result of Theorem 4.2.11 that there is an equivalence $E \otimes_{\mathfrak{C}} \mathbf{y}C \simeq EC$ for any $C \in \mathfrak{C}$.

Definition 4.4.1. *A pseudo-functor $E \colon \mathfrak{C} \to \mathfrak{Cat}$ is 2-filtered if*

1. *some fiber EC has an object;*
2. *for any objects $X \in EC$ and $Y \in ED$, there is a span in $\mathfrak{C}$ with legs $f \colon B \to C$ and $g \colon B \to D$ and an object $Z \in EB$ such that $f_!Z \cong X$ and $g_!Z \cong Y$ in the respective fibers;*
3. *for any parallel $f, g \colon C \rightrightarrows D$ of $\mathfrak{C}$ and an object $X \in EC$ with $f_!X \cong g_!X$, there is an arrow $h \colon B \to C$ and an object $Z \in EB$ such that*
   (a) *$fh = gh$ holds;*
   (b) *$h_!Z \cong X$ holds; and*
   (c) *the coherence condition*

$$\begin{tikzcd} g_!h_!Z \arrow[r, "g_!w"] \arrow[d, "\cong"'] \arrow[dr, phantom, "="] & g_!X \arrow[d, "\cong"] \\ f_!h_!Z \arrow[r, "f_!w"'] & f_!X \end{tikzcd}$$

   holds;
4. *for each arrow $u \colon X \to Y$ of any fiber EC, there is a 2-cell*

$$\begin{tikzcd} B \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \alpha"] & C \end{tikzcd}$$

   and an object $Z \in EB$ yielding between u and $(\alpha_!)_X$ an isomorphism

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\cong"'] \arrow[dr, phantom, "="] & Y \arrow[d, "\cong"] \\ f_!Z \arrow[r, "(\alpha_!)_Z"'] & g_!Z \end{tikzcd}$$

   in the arrow category $E(C)^{\mathbf{2}}$.

Remark 4.4.2. The above definition is justified in the following proposition. Notice first how it recalls the standard definitions of filteredness in a 1-categorical case, namely, that of Moerdijk's "principal $\mathscr{C}$-bundle" in Definition 2.2 of [Moe95]. For this reason, here in Definition 4.4.1, the first condition is a non-emptiness, or non-triviality condition. The second is a spanning, or transitivity condition. The third is a freeness condition. The significance of the last condition is explained partly below.

Example 4.4.3. *Let $\mathfrak{C}$ denote a 2-category. Any representable 2-functor*

$$\mathbf{y}C = \mathfrak{C}(C, -) \colon \mathfrak{C} \to \mathfrak{Cat}$$

is 2-filtered as above. This is essentially the analogue of a free module over a ring R being flat.

Now, Definition 4.4.1 is justified by the following result. The pattern of the proof follows that of the necessity direction of Theorem VII.6.3 of [MLM92], showing that left-exactness of the set-theoretic tensor product implies the usual 1-categorical notion of filteredness.

Proposition 4.4.4. *Let $E \colon \mathfrak{C} \to \mathfrak{Cat}$ denote a pseudo-functor. If the tensor 2-functor*

$$E \otimes_{\mathfrak{C}} - \colon \mathfrak{Hom}(\mathfrak{C}^{op}, \mathfrak{Cat}) \to \mathfrak{Cat}$$

preserves finite weighted pseudo-limits up to equivalence, then E is 2-filtered in the sense of Definition 4.4.1.

Proof. Since $E \otimes_{\mathfrak{C}} 1$ is weakly equivalent to the terminal category $\mathbf{1}$, there is some fiber of E with an object, which verifies the non-emptiness condition.

Let $\mathbf{y}A$ and $\mathbf{y}B$ denote two representables at A and B in $\mathfrak{C}$. By the preservation hypothesis, there is a sequence of equivalences

$$E \otimes_{\mathfrak{C}} (\mathbf{y}A \times \mathbf{y}B) \simeq (E \otimes_{\mathfrak{C}} \mathbf{y}A) \times (E \otimes_{\mathfrak{C}} \mathbf{y}A) \simeq EA \times EB$$

the left being weak and the rightmost being the equivalence as a consequence of 4.2.11. In any event, since the composite is essentially surjective, given two objects $X \in EC$ and $Y \in ED$, there is in particular a span $f \colon B \to C$ and $g \colon B \to D$ in $\mathfrak{C}$ and an object Z in EC such that, by definition of the functors making the equivalence, it follows that there are isomorphisms $f_!Z \cong X$ and $g_!Z \cong Y$, as required.

For the equalizing condition, suppose that there are morphisms $f, g \colon C \rightrightarrows D$ of $\mathfrak{C}$ and an object $X \in EC$ with $f_!X \cong g_!X$. Let $\epsilon \colon Q \to \mathbf{y}C$ denote the pseudo-equalizer in $[\mathfrak{C}^{op}, \mathfrak{Cat}]$ of the induced arrows $f_*, g_* \colon \mathbf{y}C \rightrightarrows \mathbf{y}D$. Now, since E is a 2-functor, the squares on the right in the following diagram commute and thus there is an induced dashed arrow

$$\begin{tikzcd}
E \otimes_{\mathfrak{C}} Q \arrow[r, "E \otimes \epsilon"] \arrow[d, dashed, "\exists !"'] & E \otimes_{\mathfrak{C}} \mathbf{y}C \arrow[r, shift left, "E \otimes f_*"] \arrow[r, shift right, "E \otimes g_*"'] \arrow[d, "\simeq"'] & E \otimes_{\mathfrak{C}} \mathbf{y}D \arrow[d, "\simeq"] \\
\mathscr{E} \arrow[r, "K"'] & EC \arrow[r, shift left, "f_!"] \arrow[r, shift right, "g_!"'] & ED
\end{tikzcd}$$

where $K \colon \mathscr{E} \to EC$ is the pseudo-equalizer of $f_!, g_! \colon EC \rightrightarrows ED$ in $\mathfrak{Cat}$. The dashed arrow is in particular essentially surjective by the preservation hypothesis; and this yields the arrow h and object Z with the desired properties. The coherence condition follows from the fact that the squares on the right side of the diagram commute up to isomorphism.

Finally, by the preservation hypothesis, there is a sequence of equivalences

$$E \otimes_{\mathfrak{C}} (\mathbf{2} \pitchfork \mathbf{y}C) \simeq \mathbf{2} \pitchfork (E \otimes_{\mathfrak{C}} \mathbf{y}C) \simeq (EC)^{\mathbf{2}}$$

the rightmost being a weak equivalence and the leftmost coming from the corollary to Theorem 4.2.11. Since in particular the composite is essentially surjective, there is an object (B, Z, α) of the domain whose image is isomorphic to $u \colon X \to Y$ in the target. The definitions of the object correspondences in the equivalences show that this yields the required isomorphism in the statement of the condition. □

Remark 4.4.5. In fact, a converse to Proposition 4.4.4 holds. The proof again is by considering the various finite-limit shapes and can be executed, technically speaking, by building cones on the required diagrams in the manner of the proofs of the lemmas leading to Theorem 6.3.6. But in any event, the elementary results of Chapter 6 prove this converse in greater generality.

If $\mathfrak{C}$ is a 1-category, then requiring 2-filteredness of a category-valued pseudo-functor E on $\mathfrak{C}$ essentially forces E to take sets as values. Thus, such E is basically a discrete opfibration.

Corollary 4.4.6. *Each category EC, for a 2-filtered pseudo-functor $E \colon \mathscr{C} \to \mathfrak{Cat}$ on a 1-category $\mathscr{C}$, is a connected preordered groupoid, thus equivalent to a set.*

Proof. The proof of Proposition 4.4.4 shows that any morphism $u \colon X \to Y$ in a given category EC is isomorphic in the arrow category $E(C)^{\mathbf{2}}$ to a component of the natural transformation of the image of 2-cell of $\mathscr{C}$ under E. However, as $\mathscr{C}$ is 2-categorically discrete, such a transformation can only be an identity morphism, meaning that u is isomorphic in the arrow category to an identity, making it invertible itself. The remaining filteredness conditions now imply that between any two objects of each category EC there is precisely one morphism. Thus, each EC is equivalent to a set. $\square$

Remark 4.4.7. Now, consider the 2-category of elements construction of $E\colon \mathfrak{C} \to \mathfrak{Cat}$ from Definition 2.2.8, denoted in the usual fashion by

$$\Pi\colon \int_{\mathfrak{C}} E \longrightarrow \mathfrak{C}.$$

Recall from Proposition 2.2.10 that Π is a discrete 2-fibration in the sense that Π_0 is an opcloven opfibration and locally Π is a discrete fibration. The 2-filteredness conditions of Definition 4.4.1, stated in terms of the existence of certain arrows in the completion, take the following form.

1. There is an object (C, X) of the category of elements construction.
2. For any two objects (C, X) and (D, Y), there is a span with legs $(f, v)\colon (B, Z) \to (C, X)$ and $(g, v)\colon (D, Y)$ with u and v invertible.
3. For any parallel arrows $(f, u), (g, v)\colon (C, X) \rightrightarrows (D, Y)$ with u and v invertible, there is an arrow $(h, w)\colon (B, Z) \to (C, X)$ with w invertible, equalizing the given parallel pair.
4. Each arrow $(1, u)\colon (C, X) \to (C, Y)$ fits into a 2-cell

$$\begin{tikzcd} & (C,X) \arrow[d, "{(1,u)}"] \\ (B,Z) \arrow[ur, "{(f,u)}"] \arrow[dr, "{(g,v)}"'] \arrow[r, phantom, "\Downarrow\alpha"] & (C,Y) \end{tikzcd}$$

   with u and v invertible.

It follows that E is filtered in the sense of Definition 4.4.1 if, and only if, the conditions immediately above are satisfied. Recalling that the morphisms (f, u) with u invertible are precisely the opcartesian morphisms for the underlying opfibration Π_0, this discussion justifies the following definition.

Definition 4.4.8. *A discrete 2-opfibration $E\colon \mathfrak{E} \to \mathfrak{C}$ with opcleavage ρ as in Definition 2.2.15 is understood to be 2-filtered with respect to the opcartesian morphisms of the underlying opfibration E_0 if*

1. *the 2-category $\mathfrak{E}$ has an object;*
2. *for any two objects $A, B \in \mathfrak{E}$, there is a span $A \leftarrow Z \to B$ with both arrows opcartesian;*

3. *for any parallel opcartesian arrows $A \rightrightarrows B$ of $\mathfrak{E}$, there is a further opcartesian arrow $D \to A$ that equalizes the given parallel pair;*

4. *each vertical arrow $u\colon A \to B$ of $\mathfrak{E}$ fits into a 2-cell*

$$\begin{tikzcd}
 & A \arrow[dd, "u"] \\
Z \arrow[ur] \arrow[dr] \arrow[r, phantom, "\Downarrow\alpha"] & \\
 & B
\end{tikzcd}$$

with the two unlabeled arrows opcartesian.

Remark 4.4.9. The conditions of Definition 4.4.8 differ from those of the notion of "bifiltered," given in Definition 3.2 of [Ken92]. Here no equifying condition on parallel 2-cells is required since E is already locally discrete. For the same reason, and for the reason that there are no lax cells under consideration here, Definition 4.4.8 also differs from the more recent Definition 3.1.1 of [DDS18b]. Here follow technical results that will be needed in subsequent developments.

Lemma 4.4.10. *Let $E\colon \mathfrak{E} \to \mathfrak{C}$ denote a discrete 2-opfibration. Assume that E is filtered with respect to opcartesian morphisms as in Definition 4.4.8. Then for any arrows $f, g\colon A \rightrightarrows B$ of $\mathfrak{E}$ with g opcartesian, there is opcartesian $h\colon Z \to A$ and a 2-cell $\alpha\colon fh \Rightarrow gh$ of $\mathfrak{E}$.*

Remark 4.4.11. This shows that for any discrete 2-opfibration, filtered in the present sense, the condition 'σ**F1**' of Definition 3.1.2 in [DDS18b] is also satisfied. It will be used in the proof of Theorem 5.1.2.

Proof. The given arrow f factors as $f = vk$ an opcartesian followed by a vertical morphism. This factorization and the rest of the proof is contained in the following diagram.

$$\begin{tikzcd}
 & & & \cdot \arrow[dr] \arrow[ddrr, bend left] & & \\
 & & & & \cdot \arrow[dr, "v"] \arrow[r, phantom, "\Rightarrow"] & \\
Z \arrow[r, "w"'] & \cdot \arrow[r, "u"'] & \cdot \arrow[uur] \arrow[r] & A \arrow[ur, "k"] \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & B
\end{tikzcd}$$

The 2-cell arises from the fact that the vertical arrow v fits into such a cell by the fourth condition of Definition 4.4.8. Each unmarked arrow is opcartesian. The span of arrows making the square is the "spanning" condition of the same definition; the arrow u making a commutative square arises from the "freeness" condition; finally the arrow w equalizes the topmost and bottommost composites of opcartesian arrows, yielding the desired 2-cell. In particular, h is the composite of the rightmost three opcartesian arrows $Z \to A$. $\square$

Lemma 4.4.12. *Let $E\colon \mathfrak{E} \to \mathfrak{C}$ denote a discrete 2-opfibration as in Definition 2.2.15. If E is filtered in the sense of Definition 4.4.8, then, for any morphism $f\colon A \to Z$ of the total 2-category $\mathfrak{E}$, there are opcartesian morphisms w and r and a 2-cell $fw \Rightarrow r$ of $\mathfrak{E}$.*

Remark 4.4.13. This result will play a crucial role in the proof of Lemma 6.3.1.

Proof. Again f factors as $f = vk$ for k opcartesian and v vertical. By the assumed filteredness conditions, f with its factorization fits into a diagram

$$\begin{tikzcd}
& & A \arrow[rr, "f"] \arrow[dr, "k"'] & & Z \\
\cdot \arrow[r] & \cdot \arrow[ur] \arrow[dr] & & \cdot \arrow[ur, "v"'] \arrow[r, phantom, "\Rightarrow"] & {} \\
& & \cdot \arrow[ur] \arrow[uurr, bend right=60] & &
\end{tikzcd}$$

with all unlabeled arrows opcartesian; the 2-cell exists since v is vertical; the rightmost horizontal arrow equalizes the two sides of the diamond figure construced by the spanning condition. This shows that there are opcartesian arrows w and r and a 2-cell $\theta\colon fw \Rightarrow r$. $\square$

Chapter 5

Localization of Internal Categories

5.1 A Calculus of Fractions

The colimit computation $\varinjlim F = \mathscr{F}[\Sigma^{-1}]$, reviewed in §4.2, under certain filteredness conditions on the base category, admits a right calculus of fractions. This is proved in §6.4.0 of [AGV72]. This is a desirable situation. For the ordinary category of fractions has as its morphisms only certain formal "zig-zags" of alternating arrows coming from the free category construction. The right calculus of fractions gives a more tractable characterization of these morphisms as equivalence classes of certain spans. The point of the computation is that filteredness should imply the existence of a right calculus of fractions. Such a result ought to extend to the colimit construction of 4.2.2 in the present work under the filteredness conditions of Definition 4.4.1 or Definition 4.4.8. That it does is the content of the present subsection. Let us recall the definition which originated with [GZ67].

Definition 5.1.1. *A set of arrows Σ of a category $\mathscr{C}$ admits a right calculus of fractions if*

1. *Σ has all identities and is closed under composition;*
2. *any corner diagram with horizontal arrow in Σ can be completed to a commutative square*

$$\begin{tikzcd} \cdot \arrow[r, dashed, "\tau"] \arrow[d, dashed] & \cdot \arrow[d] \\ \cdot \arrow[r, "\sigma"'] & \cdot \end{tikzcd}$$

   with τ also in Σ;

3. *any parallel arrows coequalized by one in Σ are also equalized by one in Σ as in the diagram*

$$\begin{tikzcd} \cdot \arrow[r, dashed, "\tau"] & \cdot \arrow[r, shift left] \arrow[r, shift right] & \cdot \arrow[r, "\sigma"] & \cdot \end{tikzcd}$$

   again with τ in Σ.

The description of the resulting category $\mathscr{C}[\Sigma^{-1}]$ is set out in detail over the course of §5.2 of [Bor94]. The objects are just the objects of $\mathscr{C}$. And for such Σ, the morphisms are given by equivalence classes of spans $\cdot \leftarrow \cdot \rightarrow \cdot$ whose left leg is in Σ and whose right leg is an arbitrary arrow of $\mathscr{C}$. Two such spans are considered to be equivalent if there is a further span indicated by the dashed arrows in

$$\begin{tikzcd} & \cdot \arrow[dl, "\sigma'"'] \arrow[dr] & \\ \cdot & \cdot \arrow[u, dashed] \arrow[d, dashed] & \cdot \\ & \cdot \arrow[ul, "\sigma"] \arrow[ur] & \end{tikzcd}$$

making two commutative squares with each side of the leftmost square composing to an arrow of Σ. The process of forming a category of fractions from a set admitting a calculus of fraction will be referred to as "localization."

Theorem 5.1.2. *Let $E\colon \mathfrak{E} \to \mathfrak{C}$ denote a discrete 2-opfibration as in Definition 2.2.15. If E is 2-filtered as in Definition 4.4.8, then for any discrete 2-fibration, $F\colon \mathfrak{F} \to \mathfrak{C}$, the set Σ of images of cartesian morphisms, inverted to form the tensor product $E \otimes_{\mathfrak{C}} F$ as in Equation 4.2.3, admits a right calculus of fractions as described in Definition 5.1.1.*

Proof. The set Σ of images of cartesian morphisms of $\mathfrak{E} \times_{\mathfrak{C}} \mathfrak{F}$ contains all identities and is closed under composition. Thus, for the second condition, assume given a corner diagram of the form

$$\begin{tikzcd} & (A,B) \arrow[d, "{(e,f)}"] \\ (X,Y) \arrow[r, "{(s,t)}"'] & (Z,W) \end{tikzcd}$$

with s opcartesian and t cartesian. The arrows e and s determine a corner in $\mathfrak{E}$ that, by the spanning condition of the hypothesis and the extra filteredness condition of Lemma 4.4.10, can be completed to a cell by cartesian arrows u and v as at left below. The image in $\mathfrak{C}$ under E is on the right.

$$\begin{tikzcd} C \arrow[r, dashed, "v"] \arrow[d, dashed, "u"'] \arrow[dr, phantom, "\Leftarrow"] & A \arrow[d, "e"] \\ X \arrow[r, "s"'] & Z \end{tikzcd} \qquad \begin{tikzcd} EC \arrow[r, dashed, "Ev"] \arrow[d, dashed, "Eu"'] \arrow[dr, phantom, "\Leftarrow"] & EA \arrow[d, "Ee"] \\ EX \arrow[r, "Es"'] & EZ \end{tikzcd}$$

The objects B and Y of $\mathfrak{F}$ are over EA and EX, respectively. Since F is in particular a fibration, there are chosen cartesian arrows

$$\sigma(Ev, B)\colon E(v)^*B \to B \qquad \sigma(Eu, Y)\colon E(u)^*Y \to Y$$

of $\mathfrak{F}$ over Ev and Eu, respectively. In the following diagram, the 2-cell arises because F is locally a discrete fibration and the image in $\mathfrak{C}$ of the constructed 2-cell in $\mathfrak{E}$ thus lifts to a 2-cell of $\mathfrak{F}$ whose target is over $E(s)E(u)$.

$$\begin{tikzcd}
E(v)^*B \arrow[rr, "\sigma"] \arrow[d, dashed, "h"'] \arrow[drr, dashed] & \arrow[d, phantom, "\Downarrow"] & B \arrow[d, "f"] \\
E(u)^*Y \arrow[r, "\sigma"'] & Y \arrow[r, "t"'] & W
\end{tikzcd}$$

Additionally, since $t\sigma(Eu, Y)$ is cartesian over $E(s)E(u)$, there is a unique $h\colon E(u)^*Y \to E(v)^*B$ arising as a vertical lift of identity on EC making the depicted triangle commute in $\mathfrak{F}$. Thus, the initially given corner diagram can be completed to a 2-cell of $\mathfrak{E} \times_{\mathfrak{C}} \mathfrak{F}$ by the arrows u and v of $\mathfrak{E}$ and the arrows σh and σ of $\mathfrak{F}$ as in

$$\begin{tikzcd}
(C, E(v)^*B) \arrow[r, "{(v,\sigma)}"] \arrow[d, "{(u,\sigma h)}"'] \arrow[dr, phantom, "\Leftarrow"] & (A, B) \arrow[d, "{(e,f)}"] \\
(X, Y) \arrow[r, "{(s,t)}"'] & (Z, W)
\end{tikzcd}$$

which of course becomes a commutative square upon passing to path-classes in the reduction $\pi_0(\mathfrak{E} \times_{\mathfrak{C}} \mathfrak{F})$. This verifies the second condition.

Finally, a statement stronger than the third condition is true. Let $e, g\colon A \rightrightarrows X$ denote parallel arrows of $\mathfrak{E}$. The diagram below is constructed in the following way. The fourth condition of the 2-filteredness definition guarantees that e and g each fit into the depicted 2-cells with opcartesian morphisms. The commutative square is formed using the spanning and equalizing conditions. And finally r equalizes the two outside legs of the triangle with codomain A.

$$\begin{tikzcd}
 & & & A \arrow[dd, "e"] \\
 & & \cdot \arrow[ur] \arrow[dr] \arrow[r, phantom, "\Downarrow"] & {} \\
\cdot \arrow[r, "r"] & \cdot \arrow[ur] \arrow[dr] \arrow[r, phantom, "="] & {} & X \\
 & & \cdot \arrow[ur] \arrow[dr] \arrow[r, phantom, "\Uparrow"] & {} \\
 & & & A \arrow[uu, "g"']
\end{tikzcd}$$

Now, all the morphisms in the diagram beside possibly e and g are opcartesian. Therefore, the diagram shows that there is an opcartesian morphism $l\colon D \to A$ and a path of 2-cells between el and gl. Now, since $F\colon \mathfrak{F} \to \mathfrak{C}$ is a fibration and locally a discrete opfibration, for any $f, h\colon B \rightrightarrows Y$ of $\mathfrak{F}$ over Ee and Fh, respectively, there is a cartesian morphism $k\colon C \to B$ over El and a path between fk and hk in $\mathfrak{F}$ over the image in $\mathfrak{C}$ of the path in $\mathfrak{E}$ under E. Thus, for any parallel pair of morphisms of $\mathfrak{E} \times_{\mathfrak{C}} \mathfrak{F}$ as in

$$\begin{tikzcd}(D,C) \arrow[r, dashed, "{(l,k)}"] & (A,B) \arrow[r, shift left, "{(e,f)}"] \arrow[r, shift right, "{(g,h)}"'] & (X,Y)\end{tikzcd}$$

a dashed cartesian arrow exists admitting a path in $\mathfrak{E} \times_{\mathfrak{C}} \mathfrak{F}$ between the compositions. This is certainly still true, if, as in the hypothesis of the final condition for a right calculus of fractions, the image of the parallel pair modulo connected components is coequalized by the image of a cartesian morphism. $\square$

5.2 Localization, Internally

Throughout let $\mathscr{E}$ denote an exact category in the sense of §2.3.2. Let $\mathfrak{K}$ denote $\mathfrak{Cat}(\mathscr{E})$, viewed as a 2-category. Fix throughout $\mathbb{C}$, a category internal to $\mathscr{E}$ displayed as the tuple

$$\mathbb{C} = (C_0, C_1, d_0, d_1, i, \circ)$$

and satisfying the axioms of §3.1.

The subsequent sections are directed toward reproducing in $\mathfrak{K}$ the calculus of fractions constructions as summarized in §5.1.1. To this end, let $s\colon \Sigma \to \mathbb{C}_1$ denote a monomorphism. The internalized version of Definition 5.1.1 is now the following.

Definition 5.2.1 (Internal Right Calculus of Fractions)**.** *The morphism $s\colon \Sigma \to C_1$ admits a right calculus of fractions if*

1. *given $x\colon X \to C_0$, the composite $ix\colon X \to C_1$ factors through $s\colon \Sigma \to C_1$;*
2. *given $f, g\colon X \rightrightarrows \Sigma$, the $\mathbb{C}$-composite $sf \circ sg\colon X \to C_1$ factors through $s\colon \Sigma \to C_1$;*
3. *given generalized morphisms $f\colon X \to C_1$ and $g\colon X \to \Sigma$ with $d_1 f = d_1 sg$, there exists a regular epimorphism $p\colon Z \to X$ and generalized morphisms $h\colon Z \to C_1$ and $k\colon Z \to \Sigma$ for which the equation $sk \circ fp = h \circ sgp$ holds;*
4. *and finally given $f, g\colon X \rightrightarrows C_1$ and $h\colon X \to \Sigma$ such that the equations*

(a) $d_0 f = d_0 g$

(b) $d_1 f = d_1 g$

(c) $d_0 sh = d_1 f = d_1 g$

(d) $f \circ sh = g \circ sh$

all hold, it follows that there exist a regular epimorphism $p\colon Z \to X$ *and a generalized morphism* $k\colon Z \to \Sigma$ *such that* $sk \circ fp = sk \circ gp$*.*

Remark 5.2.2. Think of the third condition as a sort of (pseudo) "spanning" condition; and of the fourth condition as a sort of "freeness" condition.

Remark 5.2.3. The axioms above are given in an "elementary" form. However, the conditions can be stated in terms of the existence of certain regular epimorphisms. For example, one such condition is implied by the spanning condition above. For this, let X and Y denote the corner objects of the pullbacks

$$\begin{tikzcd} Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma \arrow[d, "d_1 s"] \\ C_1 \arrow[r, "d_1"'] & C_0 \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \times_{C_0} \Sigma \arrow[d, "\circ(1 \times s)"] \\ \Sigma \times_{C_0} C_1 \arrow[r, "\circ(s \times 1)"'] & C_1 \end{tikzcd}$$

There is induced a canonical morphism $X \to Y$ by the universal property of Y.

Lemma 5.2.4. *If* $s\colon \Sigma \to C_1$ *admits a right calculus of fractions as in Definition 5.2.1, then the canonically induced morphism* $r\colon X \to Y$ *as above is a regular epimorphism.*

Proof. By the spanning axiom for the right calculus of fractions there is are regular epimorphisms $p\colon Z \to Y$ and $q\colon Z \to X$. These can be taken to have the same domain by taking pullbacks. By uniqueness these satisfy $rq = p$. Hence by Lemma 2.3.2, the induced map $r\colon X \to Y$ is a regular epimorphism. ☐

5.2.1 Arrows of Localization

From the classical construction, the object of objects of a category of fractions for $s\colon \Sigma \to \mathscr{C}_1$ ought to be nothing other than C_0 itself. Thus, the non-trivial task is to give an object of arrows. This is constructed in the present subsection as a certain coequalizer. Throughout, denote by S the corner object of the pullback

$$\begin{tikzcd} S \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "d_0"] \\ \Sigma \arrow[r, "d_0 s"'] & C_0 \end{tikzcd}$$

This is the object of spans of $\mathbb{C}$ whose left leg is in Σ. Think of $d_1 s\pi_1 \colon S \to C_0$ as giving the domain of a span; and of $d_1\pi_2 \colon S \to C_0$ as giving the codomain. Denote by $S \times S$ the total object of the product of $\langle sd_1\pi_1, d_1\pi_2\rangle \colon S \to C_0 \times C_0$ with itself in the slice $\mathscr{E}/C_0 \times C_0$. This is formed as a pullback in $\mathscr{E}$.

Remark 5.2.5. It is worth noting that the exactness hypothesis on $\mathscr{E}$ will be essential in the following development. The sequence resulting from Theorem 5.2.7 below will be a kernel by exactness of $\mathscr{E}$. That the sequence is thus a coequalizer and a kernel pair is used in the proof that the composition morphism as defined in Construction 5.2.2 is well-defined.

Construction 5.2.1. *Let P denote the corner object of the pullback on the left and Q the limit object of the diagram on the right*

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \times_{C_0} C_1 \arrow[d, "\circ"] \\ C_1 \times_{C_0} C_1 \arrow[r, "\circ"'] & C_1 \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[rr, dashed, "\pi_2"] \arrow[dr, dashed, "\pi_3"] \arrow[d, dashed, "\pi_1"'] & & C_1 \times_{C_0} \Sigma \arrow[d, "\circ(1\times s)"] \\ C_1 \times_{C_0} \Sigma \arrow[rr, "\circ(1\times s)"'] & \Sigma \arrow[r, "s"] & C_1. \end{tikzcd}$$

These are the objects of commutative squares and commutative squares with two bottom sides in Σ and whose sides compose to an element Σ, respectively. Let R denote the corner object of the pullback

$$\begin{tikzcd} R \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "\langle \pi_1\pi_1, \pi_1\pi_2\rangle"] \\ Q \arrow[r, "\langle \pi_1\pi_1, \pi_1\pi_2\rangle"'] & C_1 \times C_1. \end{tikzcd}$$

An element of R is a pair of spans related in the manner depicted in the remark immediately below. Note that by the universal property of the pullback giving S, there are two canonical *maps*

$$\begin{tikzcd}
R \arrow[dr, dashed, "\delta_0"] \arrow[drr, bend left, "\pi_2\pi_1\pi_2"] \arrow[ddr, bend right, "\pi_2\pi_1\pi_1"'] & & \\
& S \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "d_0"] \\
& \Sigma \arrow[r, "d_0 s"'] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[dr, dashed, "\delta_1"] \arrow[drr, bend left, "\pi_2\pi_2\pi_2"] \arrow[ddr, bend right, "\pi_2\pi_2\pi_1"'] & & \\
& S \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "d_0"] \\
& \Sigma \arrow[r, "d_0 s"'] & C_0
\end{tikzcd}$$

since the outside squares commute. Let R_0 denote the object of the image factorization

$$\begin{tikzcd}
R \arrow[rr, "\langle \delta_0, \delta_1 \rangle"] \arrow[dr] & & S \times S \\
& R_0 \arrow[ur] &
\end{tikzcd}$$

taken in $\mathscr{E}/C_0 \times C_0$. Denote the components of the monomorphism $R_0 \to S \times S$ by ∂_0 and ∂_1. Interpret R_0 as the set of pairs of elements of S related by a span in the manner of R above.

Remark 5.2.6. Under set-theoretic interpretation in the case that $\mathscr{E} = \mathbf{Set}$, an element of R, viewed as a set, is a figure of the form

$$\begin{tikzcd}
& \cdot \arrow[dl, "\sigma'"'] \arrow[dr] & \\
\cdot & {} \arrow[u, dashed] \arrow[d, dashed] & \cdot \\
& \cdot \arrow[ul, "\sigma"] \arrow[ur] &
\end{tikzcd}$$

with the left square composing again to an element of Σ. The right square comes from the set P and the left from Q. Note that a chosen dashed span comes with each such element of R whereas for any element of R_0 the two outside spans are related by some such dashed span, but a particular one is not given.

Theorem 5.2.7. *The image $R_0 \to S \times S$ is an equivalence relation in the slice $\mathscr{E}/C_0 \times C_0$.*

Proof. The reflexivity and symmetry conditions of 2.3.12 have straightforward elementary constructions using the conditions of Definition 5.2.1. As set-up for transitivity, take three generalized spans ϕ,ψ,χ viewed as arrows $X \to S$ over $C_0 \times C_0$. Suppose further that, on the one hand, there is $\alpha\colon X \to R_0$ for which $\partial_0\alpha = \phi$ and $\partial_1\alpha = \psi$ hold; and, on the other hand, that there is $\beta\colon X \to R_0$ such that $\partial_0\beta = \psi$ and $\partial_1\beta = \chi$. Refer to this as the "given diagram."

Using the conditions of Definition 5.2.1, it is possible to build a cone on the given diagram, that is, a regular epimorphism $N \to X$ fitting into a commutative square

$$\begin{tikzcd} N \arrow[r] \arrow[d] & R_0 \arrow[d, "\langle \partial_0, \partial_1 \rangle"] \\ X \arrow[r, "\langle \partial_0\alpha, \partial_1\beta \rangle"'] \arrow[ur, dashed] & S \times S \end{tikzcd}$$

viewed in the slice category over $C_0 \times C_0$. Since the right vertical arrow is a monomorphism, and since each regular epimorphism is strong by Example 2.3.1, the dashed arrow making two commutative triangles exists. This is the required arrow for the transitivity condition. $\square$

Definition 5.2.8. *Let $u\colon S \to \mathbb{C}[\Sigma^{-1}]_1$ denote the coequalizer in $\mathscr{E}/C_0 \times C_0$ of the equivalence relation $R \rightrightarrows S$ from Theorem 5.2.7. Notice that $\mathbb{C}[\Sigma^{-1}]_1$ thus comes with domain and codomain arrows $d_0, d_1\colon \mathbb{C}[\Sigma^{-1}]_1 \rightrightarrows C_0$ induced from $\langle sd_1\pi_1, d_1\pi_2\rangle\colon S \to C_0 \times C_0$.*

Corollary 5.2.9. *The pair $R_0 \rightrightarrows S$ of Construction 5.2.1 is the kernel of u in $\mathscr{E}/C_0 \times C_0$.*

Proof. This follows by Theorem 5.2.7 and exactness, in particular, Lemma 2.3.16. $\square$

Corollary 5.2.10. *The parallel pair $R_0 \rightrightarrows S$ of Construction 5.2.1 determines a groupoid internal to $\mathscr{E}/C_0 \times C_0$.*

Proof. This follows by Theorem 5.2.7 and Theorem 3.1.11. $\square$

5.2.2 The Composition Arrow

Suppose that $s\colon \Sigma \to C_1$ admits a right calculus of fractions as in Definition 5.2.1. Use $u\colon S \to \mathbb{C}[\Sigma^{-1}]_1$ to denote the coequalizer as in Definition 5.2.8.

Construction 5.2.2. *Let V denote the corner object of the pullback*

$$\begin{tikzcd} V \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \times_{C_0} S \arrow[d, "- \circ s"] \\ \Sigma \times_{C_0} S \arrow[r, "s \circ -"'] & C_1 \end{tikzcd}$$

taken in $\mathscr{E}$. The corner object $\Sigma \times_{C_0} S$ is the pullback of $d_0\pi_2$ along $d_1 s$; the other corner is the pullback of $d_0\pi_2$ along d_1. Thus, V is the object of compositions of two composable spans *in S. In the set-theoretic case, an element is depicted in the display of the remark below. In the general case, there are two useful equations, namely,*

$$d_1 s\pi_1\pi_1 = d_0\pi_2\pi_2\pi_1 = d_0 s\pi_1\pi_2\pi_1 \tag{5.2.1}$$

and

$$d_1\pi_1\pi_2 = d_0 s\pi_1\pi_2\pi_2 = d_0\pi_2\pi_2\pi_2 \tag{5.2.2}$$

which show that the codomains of the arrows of the top span do match with the appropriate domains of the arrows of the two bottom spans. By the construction of V above, there are morphisms $V \to \Sigma$ and $V \to C_1$ arising by equations 5.2.1 and 5.2.2. These appear in the following diagram:

$$\begin{tikzcd}
V \arrow[dr, dashed, "c"'] \arrow[drr, bend left, "\pi_1\pi_2 \circ \pi_2\pi_2\pi_2"] \arrow[ddr, bend right, "\pi_1\pi_1 \circ \pi_1\pi_2\pi_1"'] & & \\
& S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "d_0"] \\
& \Sigma \arrow[r, "d_1"'] & C_0
\end{tikzcd}$$

Since the outside commutes, the dashed arrow $V \to S$ exists by the universal property of S. In the set theoretic interpretation, the effect of this induced morphism is to compose the spans making an element of V and to send the result to the outside span.

Remark 5.2.11. Set-theoretically speaking, an element of such a set V would be a figure of the form

$$\begin{tikzcd}
& & \cdot \arrow[dl] \arrow[dr] & & \\
& \cdot \arrow[dl] \arrow[dr] & & \cdot \arrow[dl] \arrow[dr] & \\
\cdot & & \cdot & & \cdot
\end{tikzcd}$$

with all southwest-pointing arrows in Σ. So, an element of V is a pair of composable spans of S with a chosen span composing them. The arrow $c\colon V \to S$ of Construction 5.2.2 composes all the arrows of the above figure and sends the result to the corresponding span. In the general case, this morphism will induce the required composition arrow for $\mathbb{C}[\Sigma^{-1}]_1$ provided that the assignment is well-defined on equivalence classes. That this is the case will be shown in Proposition 5.2.16. The rest of this section provides constructions and set-up required for this result and those in subsequent sections.

The object of composable pairs of elements of S is the corner object of the pullback

$$\begin{tikzcd}
S \times_{C_0} S \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & S \arrow[d, "d_1 s \pi_1"] \\
S \arrow[r, "d_1 \pi_2"'] & C_0
\end{tikzcd}$$

taken in $\mathscr{E}$. An arrow $q\colon V \to S \times_{C_0} S$ is given by the universal property of $S \times_{C_0} S$ as in the diagram

$$\begin{tikzcd}
V \arrow[dr, dashed, "q"] \arrow[drr, bend left, "\pi_2\pi_2"] \arrow[ddr, bend right, "\pi_2\pi_1"'] & & \\
& S \times_{C_0} S \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & S \arrow[d, "d_0 s \pi_1"] \\
& S \arrow[r, "d_1 \pi_2"'] & C_0.
\end{tikzcd}$$

as the outside square commutes by the category axioms for $\mathbb{C}$.

Lemma 5.2.12. *The arrow $q\colon V \to S \times_{C_0} S$ immediately above is regular epi.*

Proof. The claim is that q is a pullback of a regular epimorphism. To see this, note that $S \times_{C_0} S$ and V admit canonical arrows to the objects Y and X from remark 5.2.3, respectively, as in the diagrams

$$\begin{tikzcd}
S \times_{C_0} S \arrow[dr, dashed] \arrow[drr, bend left, "\pi_1\pi_2"] \arrow[ddr, bend right, "\pi_2\pi_1"'] & & \\
& Y \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \Sigma \arrow[d] \\
& C_1 \arrow[r] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[dr, dashed] \arrow[drr, bend left, "\langle \pi_1\pi_2, \pi_1\pi_2\pi_2 \rangle"] \arrow[ddr, bend right, "\langle \pi_1\pi_1, \pi_2\pi_2\pi_1 \rangle"'] & & \\
& X \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & C_1 \times_{C_0} \Sigma \arrow[d] \\
& \Sigma \times_{C_0} C_1 \arrow[r] & C_0.
\end{tikzcd}$$

The resulting square

$$\begin{tikzcd}
V \arrow[r] \arrow[d, "q"'] & X \arrow[d] \\
S \times_{C_0} S \arrow[r] & Y
\end{tikzcd}$$

is a pullback in $\mathscr{E}$. Commutativity follows from the uniqueness clause of the universal property for Y. Universality follows from universality for V. $\square$

The object of composable classes of spans is the corner object of the pullback

$$\begin{tikzcd} \mathbb{C}[\Sigma^{-1}]_1 \times_{C_0} \mathbb{C}[\Sigma^{-1}]_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}[\Sigma^{-1}]_1 \arrow[d, "d_0"] \\ \mathbb{C}[\Sigma^{-1}]_1 \arrow[r, "d_1"'] & C_0. \end{tikzcd}$$

Notice that there is a canonical map

$$u \times u \colon S \times_{C_0} S \longrightarrow \mathbb{C}[\Sigma^{-1}]_1 \times_{C_0} \mathbb{C}[\Sigma^{-1}]_1$$

given by universal properties.

Lemma 5.2.13. *The arrow $u \times u \colon S \times_{C_0} S \to \mathbb{C}[\Sigma^{-1}]_1 \times_{C_0} \mathbb{C}[\Sigma^{-1}]_1$ is regular epi. Hence $v = (u \times u)q$ is regular epi.*

Proof. Factor $u \times u$ as the composition $(1 \times u)(u \times 1)$. The maps $1 \times u$ and $u \times 1$ are regular epis because they are pullbacks of u. The second statement follows since regular epis are stable under composition. $\square$

Construction 5.2.3. *The object K_0 is given in the following way. First let I_0 and J_0 denote the corner objects of the two pullbacks*

$$\begin{tikzcd} I_0 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "\pi_2\pi_1"] \\ R_0 \arrow[r, "\partial_1"'] & S \end{tikzcd} \qquad \begin{tikzcd} J_0 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & R_0 \arrow[d, "\partial_0"] \\ V \arrow[r, "\pi_2\pi_2"'] & S \end{tikzcd}$$

with V as above and R_0 with kernel arrows ∂_0 and ∂_1 as in Construction 5.2.1. The object K_0 then denotes the corner object of the pullback square

$$\begin{tikzcd} K_0 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & J_0 \arrow[d, "\langle \pi_2\pi_1\pi_1, \partial_1\pi_2 \rangle"] \\ I_0 \arrow[r, "\langle \partial_0\pi_1, \pi_2\pi_2\pi_2 \rangle"'] & S \times S \end{tikzcd}$$

Intuitively, an object of K_0 consists of two elements of V whose first spans are related under R_0 and whose second spans are related by R_0.

Lemma 5.2.14. *The object K_0 is the kernel of v. As a consequence, v is the coequalizer of canonical morphisms $K_0 \rightrightarrows V$.*

Proof. By Lemmas 5.2.12 and 5.2.13, the map v is a regular epi, hence the quotient of its kernel, whatever it is. Now, to prove the first statement, note that it is a direct calculation that the square

$$\begin{tikzcd} K_0 \arrow[r, "\pi_1\pi_2"] \arrow[d, "\pi_2\pi_1"'] & V \arrow[d, "v"] \\ V \arrow[r, "v"'] & \mathbb{C}[\Sigma^{-1}]_1 \times_{C_0} \mathbb{C}[\Sigma^{-1}]_1 \end{tikzcd}$$

is a pullback. That the square commutes follows by the uniqueness clause of the universal property for the pullback object in the lower righthand corner. That the universal property is satisfied uses twice that $R \rightrightarrows S \to \mathbb{C}[\Sigma^{-1}]_1$ is exact, hence in particular a kernel on the left side by Corollary 5.2.9. Regularity means that every regular epi is the coequalizer of its kernel. □

Construction 5.2.4. *There is a further object K that is to K_0 as R is to R_0, at least in the sense that an element of K is essentially one of K_0, but with specified structures under which the elements are related. It is constructed in the following way. First let I and J denote the corner objects of the two pullbacks*

$$\begin{tikzcd} I \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "\pi_2\pi_1"] \\ R \arrow[r, "\delta_1"'] & S \end{tikzcd} \qquad \begin{tikzcd} J \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "\delta_0"] \\ V \arrow[r, "\pi_2\pi_2"'] & S \end{tikzcd}$$

with V as above and R with kernel arrows δ_0 and δ_1 as in §5.2.1. The object K is then denotes the corner object of the pullback square

$$\begin{tikzcd} K \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & J \arrow[d, "\langle \pi_2\pi_1\pi_1, \delta_1\pi_2 \rangle"] \\ I \arrow[r, "\langle \delta_0\pi_1, \pi_2\pi_2\pi_2 \rangle"'] & S \times S \end{tikzcd}$$

Intuitively, an object of K consists of two elements of V whose first spans are related under R and whose second spans are related by R.

There are canonical maps $I \to I_0$ and $J \to J_0$ arising by universal properties from the map $e\colon R \to R_0$. Accordingly, K admits a canonical morphism to K_0 as in

$$\begin{tikzcd}
K \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& K_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & J_0 \arrow[d] \\
& I_0 \arrow[r] & S \times S.
\end{tikzcd}$$

by the construction of K_0 in §5.2.3 above.

Corollary 5.2.15. *The canonical map $K \to K_0$ is regular epi.*

Proof. The square

$$\begin{tikzcd}
K \arrow[r, "\langle \pi_1\pi_1, \pi_2\pi_2 \rangle"] \arrow[d] & R \times_{C_0} R \arrow[d, "e \times e"] \\
K_0 \arrow[r, "\langle \pi_1\pi_1, \pi_2\pi_2 \rangle"'] & R_0 \times_{C_0} R_0
\end{tikzcd}$$

is a pullback in $\mathscr{E}$. ☐

Proposition 5.2.16. *Composition is well-defined on equivalence classes in the sense that there is an induced morphism as in the diagram*

$$\begin{tikzcd}
V \arrow[r, "c"] \arrow[d, "v"'] & S \arrow[d, "u"] \\
\mathbb{C}[\Sigma^{-1}]_1 \times_{C_0} \mathbb{C}[\Sigma^{-1}]_1 \arrow[r, dashed, "c"'] & \mathbb{C}[\Sigma^{-1}]_1
\end{tikzcd}$$

where c is as in Construction 5.2.2.

Proof. The calculus of fractions conditions in Definition 5.2.1 suffice for the purpose of building a cone on elements of K from Construction 5.2.4 above in the form of a regular epimorphism $E \to K$. Under set-theoretic interpretation an element of K has four open corners that can be closed successively using the spanning and freeness conditions of the mentioned definition. Now, an element of each such cone thus relates the two V-sides of an elements of K; and so E

admits a map $r\colon E \to R$ making the two top squares of the following diagram commute:

$$\begin{tikzcd}
E \arrow[rr, "r"] \arrow[d, "\epsilon"'] & & R \arrow[d, "e"'] \\
K_0 \arrow[d, shift right, "\pi_2\pi_1"'] \arrow[d, shift left, "\pi_1\pi_2"] & & R_0 \arrow[d, shift right, "\partial_0"'] \arrow[d, shift left, "\partial_1"] \\
V \arrow[rr, "c"] \arrow[d, "v"'] & & S \arrow[d, "u"'] \\
\mathbb{C}[\Sigma^{-1}]_1 \times_{C_0} \mathbb{C}[\Sigma^{-1}]_1 \arrow[rr, dashed, "c"'] & & \mathbb{C}[\Sigma^{-1}]_1
\end{tikzcd}$$

by definition of $\delta_0 = \partial_0 e$ and $\delta_1 = \partial_1 e$ as in Construction 5.2.1. Therefore, since $u\delta_0 = u\delta_1$ holds by construction, it follows that

$$u\delta_0 r = uc\pi_2\pi_1\epsilon = uc\pi_1\pi_2\epsilon = u\delta_1 r.$$

Since the regular epimorphism ϵ cancels, this means that the morphism uc coequalizes $\pi_2\pi_1$ and $\pi_1\pi_2$. Since v is the coequalizer of $\pi_2\pi_1$ and $\pi_1\pi_2$ by Lemma 5.2.14, the dashed morphism in the display above exists. $\square$

5.2.3 Composition is Associative

It will be seen in Proposition 5.2.21 that the associativity condition holds up to precomposition with a certain regular epimorphism morphism. Recall that V from Construction 5.2.2 denotes the object of compositions of composable elements of S.

Construction 5.2.5. *Form the pullback*

$$\begin{tikzcd}
V \times_S V \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "\pi_2\pi_1"] \\
V \arrow[r, "\pi_2\pi_2"'] & S.
\end{tikzcd}$$

Thus, set-theoretically, an element of $V \times_S V$ would be a pair of elements of V having one of their respective bottom spans in common. By the universal property of the three-fold pullback on the right below, there is a canonical map

$$\langle \pi_2\pi_1\pi_1, \xi, \pi_2\pi_2\pi_2\rangle\colon V \times_S V \to S \times_{C_0} S \times_{C_0} S$$

where ξ denotes either map of the pullback defining the object $V \times_S V$ above. Its composite with three instances of the projection u appears on the left side of the square

$$\begin{tikzcd}
V \times_S V \arrow[r, "c \times \pi_2"] \arrow[d] & S \times_{C_0} S \arrow[d, "u \times u"] \\
\mathbb{C}[\Sigma^{-1}]_1 \times_{C_0} \mathbb{C}[\Sigma^{-1}]_1 \times_{C_0} \mathbb{C}[\Sigma^{-1}]_1 \arrow[r, "c \times 1"'] & \mathbb{C}[\Sigma^{-1}]_1 \times_{C_0} \mathbb{C}[\Sigma^{-1}]_1
\end{tikzcd}$$

which commutes by the universality of the pullback in the lower right corner. The unlabeled morphism on the left side is a regular epimorphism.

Construction 5.2.6. *An object W of three-fold compositions of elements of S can be constructed from $V \times_S V$. First note that $V \times_S V$ admits a morphism to Y of Construction 5.2.3 as in the diagram*

$$\begin{tikzcd}
V \times_S V \arrow[dr, dashed] \arrow[drr, bend left, "\pi_1\pi_1\pi_2"] \arrow[ddr, bend right, "\pi_1\pi_2\pi_1"'] & & \\
& Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma \arrow[d, "d_1 s"] \\
& C_1 \arrow[r, "d_1"'] & C_0
\end{tikzcd}$$

by the commutativity of the outside square. Define W to be the corner object of the pullback

$$\begin{tikzcd}
W \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
V \times_S V \arrow[r] & Y
\end{tikzcd}$$

with $V \times_S V \to Y$ as above and $X \to Y$ as in Construction 5.2.3. Set-theoretically, an element of W consists of two elements of V with one overlapping span and an element of S capping the open corner between the two V-elements. Note that π_1 is regular epi.

Remark 5.2.17. Set-theoretically, an element of the W is a figure of the form

$$\begin{tikzcd}[column sep=small]
 & & & \cdot \arrow[dl] \arrow[dr] & & & \\
 & & \cdot \arrow[dl] \arrow[dr] & & \cdot \arrow[dl] \arrow[dr] & & \\
 & \cdot \arrow[dl] \arrow[dr] & & \cdot \arrow[dl] \arrow[dr] & & \cdot \arrow[dl] \arrow[dr] & \\
\cdot & & \cdot & & \cdot & & \cdot
\end{tikzcd}$$

all of whose southwest-facing arrows are elements of Σ. This is why, in general, W should be viewed as an object of thrice-fold compositions of spans of the form of elements of S. The map $\pi_1 \colon W \to V \times_S V$ projects to the figure without the topmost span.

Construction 5.2.7. *Let p denote the composite of the projection $\pi_1 \colon W \to V \times_S V$ with the morphism on the lefthand side of the last square in Construction 5.2.5. Notice that p is a regular epimorphism. The object W admits two canonical morphisms to the corner objects of the diagram of which V is a pullback as in*

$$\begin{tikzcd}
W \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "c\pi_1\pi_1"] \arrow[ddr, bend right, "\pi_1\pi_1\pi_2"'] & & \\
 & \Sigma \times_{C_0} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "d_0\pi_2"] \\
 & \Sigma \arrow[r, "d_1 s"'] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[dr, dashed, "\psi"] \arrow[drr, bend left, "\pi_2\pi_2\pi_2\pi_1"] \arrow[ddr, bend right, "l"'] & & \\
 & C_1 \times_{C_0} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "d_0\pi_2"] \\
 & C_1 \arrow[r, "d_1"'] & C_0
\end{tikzcd}$$

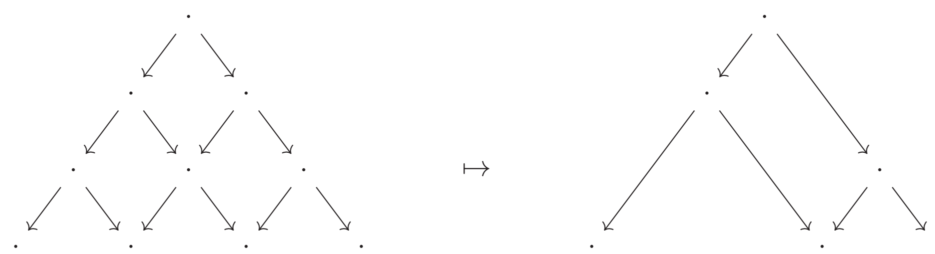
where l denotes the $\mathbb{C}$-composite $\pi_1\pi_2\pi_2 \circ \pi_1\pi_2\pi_2\pi_1$.

The morphism λ is the canonical one arising by the universal property of V as in the diagram

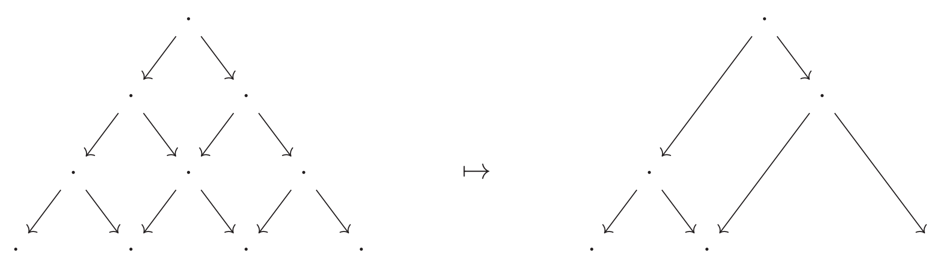
$$\begin{tikzcd}
W \arrow[dr, dashed, "\lambda"] \arrow[drr, bend left, "\psi"] \arrow[ddr, bend right, "\phi"'] & & \\
 & V \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \times_{C_0} S \arrow[d, "d_1 s"] \\
 & \Sigma \times_{C_0} S \arrow[r, "d_1"'] & C_1.
\end{tikzcd}$$

That the outside of the square does indeed commute is a moderatly involved computation from the definitions using the following result.

Remark 5.2.18. The set-theoretic interpretation is that the morphism λ has the following effect. It sends a figure as at the left below to the figure at the right by making the obvious compositions:



Another map $\rho\colon W \to V$ can be built along the lines of Construction 5.2.7 but sending a given element of W as at the left below to the figure on the right:



The details of the construction are left to the reader. The point is that λ and ρ each do one or the other of the first two possible compositions given three consecutive composable spans. And in any case, the pictures provide intuition for the next lemma and following remark.

Lemma 5.2.19. *The square*

$$\begin{tikzcd} W \arrow[r, "\lambda"] \arrow[d, "\pi_1"'] & V \arrow[d, "q"] \\ V \times_S V \arrow[r, "c \times \pi_2"'] & S \times_{C_0} S \end{tikzcd}$$

with q as in Lemma 5.2.12 is commutative.

Proof. This follows by the uniqueness clause of the universal property of $S \times_{C_0} S$ by checking on the projections. $\square$

Remark 5.2.20. A result analogous to Lemma 5.2.19 holds for $\rho\colon W \to V$ from Construction 5.2.18 in relation to the induced map $\pi_1 \times c$. By construction of u, v and p, these lemmas therefore imply that the equations

$$c(c \times 1)p = uc\lambda \qquad\qquad c(1 \times c)p = uc\rho$$

both hold.

Proposition 5.2.21. *The induced morphism*

$$c\colon \mathbb{C}[\Sigma^{-1}]_1 \times_{C_0} \mathbb{C}[\Sigma^{-1}]_1 \longrightarrow \mathbb{C}[\Sigma^{-1}]_1$$

of Construction 5.2.2 satisfies the associativity law.

Proof. Consider the diagram

$$\begin{tikzcd}
 & & \mathbb{C}[\Sigma^{-1}]_1 \times_{C_0} \mathbb{C}[\Sigma^{-1}]_1 \arrow[dr, "c"] & \\
W \arrow[r, "p"] & \mathbb{C}[\Sigma^{-1}]_1 \times_{C_0} \mathbb{C}[\Sigma^{-1}]_1 \times_{C_0} \mathbb{C}[\Sigma^{-1}]_1 \arrow[ur, "1 \times c"] \arrow[dr, "c \times 1"'] & & \mathbb{C}[\Sigma^{-1}]_1 \\
 & & \mathbb{C}[\Sigma^{-1}]_1 \times_{C_0} \mathbb{C}[\Sigma^{-1}]_1 \arrow[ur, "c"'] &
\end{tikzcd}$$

with p as above in Construction 5.2.6. The equation $c\lambda = c\rho$ holds by the construction of λ and ρ above and by the uniqueness aspect of the universal property of $S \times_{C_0} S$. Therefore, by Lemma 5.2.19 and Remark 5.2.20 the diagram commutes. The statement of the proposition now follows since p is an epimorphism, hence right cancelable. $\square$

5.2.4 An Identity Morphism

Consider the canonical morphism arising from the universal property of S as in the diagram

$$\begin{tikzcd}
C_1 \arrow[dr, dashed] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "jd_0"'] & & \\
 & S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "d_0"] \\
 & \Sigma \arrow[r, "d_0 s"'] & C_0
\end{tikzcd}$$

where $j\colon C_0 \to \Sigma$ factors i through s as in Definition 5.2.1. Denote this canonical map by $\langle jd_0, 1\rangle$; it is the elementary equivalent of the canonical reduction map that views an arrow of $\mathscr{C}$ as an arrow of $\mathscr{C}[\Sigma^{-1}]$ in the set-theoretic case. Now, let $\iota\colon C_0 \to \mathbb{C}[\Sigma^{-1}]_1$ denote the composite

$$\iota := u\langle jd_0, 1\rangle i \tag{5.2.3}$$

where u is the coequalizer of Definition 5.2.8.

Lemma 5.2.22. *The map ι in Equation 5.2.3 splits $d_0, d_1\colon \mathbb{C}[\Sigma^{-1}]_1 \to C_0$ given in Definition 5.2.8.*

Proof. The two computations are straightforward. For example, there is the computation

$$\begin{aligned}
d_0\iota &= d_0u\langle jd_0, 1\rangle i && \text{(definition of } \iota \text{ above)}\\
&= d_1s\pi_1\langle jd_0, 1\rangle i && \text{(construction of } d_0 \text{ and } d_1 \text{ in Definition 5.2.8)}\\
&= d_1sjd_0i &&\\
&= d_1id_0i && \text{(hypothesis of Definition 5.2.1)}\\
&= 1 &&
\end{aligned}$$

The other is similar. □

5.2.5 Universal Property

Definition 5.2.23. *An internal functor $f\colon \mathbb{C} \to \mathbb{D}$ inverts a generalized morphism $s\colon \Sigma \to C_1$ of $\mathbb{C}$ if there is a morphism*

$$f_1(s)^{-1}\colon \Sigma \to D_1$$

for which the equations

1. $d_0f_1(s)^{-1} = d_1f_1s$
2. $d_1f_1(s)^{-1} = d_0f_1s$
3. $f_1(s) \circ f_1(s)^{-1} = if_0d_0s$
4. $f_1(s)^{-1} \circ f_1s = if_0d_1s$

all hold. Let $\mathfrak{K}(\mathbb{C}, \mathbb{D})_\Sigma$ denote the full subcategory of $\mathfrak{K}(\mathbb{C}, \mathbb{D})$ of such internal functors.

Remark 5.2.24. Put another way, the generalized arrow $f_1(s)\colon \Sigma \to D_1$ of the internal category $\mathbb{D}$ is an isomorphism with inverse $f_1(s)^{-1}\colon \Sigma \to D_1$ in the sense of Definition 3.1.8. As a consequence, for such $f\colon \mathbb{C} \to \mathbb{D}$, there is an induced generalized element $\Sigma \to \mathbf{Iso}(\mathbb{D})$ as in Lemma 3.1.15 and its proof.

Definition 5.2.25. *A category of fractions for a monomorphism* $s\colon \Sigma \to C_1$ *is an internal category* $\mathbb{F}$ *admitting a functor* $l\colon \mathbb{C} \to \mathbb{F}$ *inverting* $s\colon \Sigma \to C_1$ *and that is universal in the sense that there is an isomorphism of categories*

$$\mathfrak{K}(\mathbb{F}, \mathbb{D}) \cong \mathfrak{K}(\mathbb{C}, \mathbb{D})_\Sigma$$

induced by composition with L.

For the rest of the subsection, let $f\colon \mathbb{C} \to \mathbb{D}$ denote a functor of internal categories that inverts $s\colon \Sigma \to C_1$. The subsequent development shows that $\mathbb{C}[\Sigma^{-1}]$ as in the next result, Theorem 5.2.26, is the category of fractions for $s\colon \Sigma \to C_1$. First a few necessary preliminaries.

Theorem 5.2.26. *The tuple*

$$\mathbb{C}[\Sigma^{-1}] = (C_0, \mathbb{C}[\Sigma^{-1}]_1, d_0, d_1, i, c)$$

defines a category object in $\mathscr{E}$.

Proof. It remains only to verify the domain and codomain equations. For the unit equations were verified in Lemma 5.2.22 and associativity is Proposition 5.2.21. But the domain equation is a straightforward computation:

$$\begin{aligned} d_0cv &= d_0uc \\ &= d_1s\pi_1c && \text{(definition of } d_0 \text{ in Definition 5.2.8)} \\ &= d_1s(\pi_1\pi_1 \circ \pi_1\pi_2\pi_1) && \text{(construction of } c \text{ in Construction 5.2.2)} \\ &= d_1s\pi_1\pi_2\pi_1 \\ &= d_0u\pi_1q && \text{(construction of } q \text{ as before Lemma 5.2.12)} \\ &= d_0\pi_1v && \text{(construction of } v \text{ in Lemma 5.2.13)} \end{aligned}$$

Since v is regular epi, it cancels so that $d_0c = d_0\pi_1$ holds, as required. The computation for the codomain arrow is similar. □

Lemma 5.2.27. *A functor of internal categories* $l\colon \mathbb{C} \to \mathbb{C}[\Sigma^{-1}]$ *is given by*

$$l_0 = 1\colon C_0 \to C_0 \qquad\qquad l_1 = u\langle jd_0, 1\rangle\colon C_1 \to \mathbb{C}[\Sigma^{-1}]_1$$

with $\langle jd_0, 1\rangle$ *as above and* u *as in Definition 5.2.8.*

Proof. That the axioms of Definition 3.1.12 hold follows from the universal properties of the given constructions. □

Lemma 5.2.28. *The internal functor l of Lemma 5.2.27 above inverts $s\colon \Sigma \to C_1$.*

Proof. The inverse is the composite $u\langle 1, id_0 s\rangle\colon \Sigma \to \mathbb{C}[\Sigma^{-1}]_1$, where $\langle 1, id_0 s\rangle$ is the canonical map arising in the diagram

$$\begin{tikzcd}
\Sigma \arrow[dr, dashed] \arrow[drr, bend left, "id_0 s"] \arrow[ddr, bend right, "1"'] & & \\
& S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C_1 \arrow[d, "d_0"] \\
& \Sigma \arrow[r, "d_0 s"'] & C_0
\end{tikzcd}$$

That the required equations do hold is an exercise in cone-building. $\square$

For $f\colon \mathbb{C} \to \mathbb{D}$ consider $f_1(s)^{-1} \circ f_1\colon S \to D_1$. This is well-defined on equivalence classes of spans in the sense that it coequalizes $\partial_0, \partial_1\colon R \rightrightarrows S$ from Construction 5.2.1. Essentially, the construction of R allows the computation on p. 187 of [Bor94] to be reproduced using projection morphisms:

$$\begin{aligned}
(f_1(s)^{-1} \circ f_1)\delta_0 &= f_1(s)^{-1}\pi_1\delta_0 \circ f_1\pi_2\delta_0 \\
&= f_1(s)^{-1}\pi_2\pi_1\pi_1 \circ f_1\pi_2\pi_1\pi_2 \\
&= f_1(s)^{-1}(\pi_1\pi_1\pi_1 \circ \pi_2\pi_1\pi_1) \circ f_1(\pi_1\pi_1\pi_1 \circ \pi_2\pi_1\pi_1) \circ f_1(s)^{-1}\pi_2\pi_1\pi_1 \circ f_1\pi_2\pi_1\pi_2 \\
&= f_1(s)^{-1}(\pi_1\pi_1\pi_1 \circ \pi_2\pi_1\pi_1) \circ f_1\pi_1\pi_1\pi_1 \circ f_1\pi_2\pi_1\pi_2 \\
&= f_1(s)^{-1}(\pi_1\pi_1\pi_1 \circ \pi_2\pi_1\pi_1) \circ f_1\pi_1\pi_1\pi_1 \circ f_1\pi_2\pi_2\pi_1 \circ f_1(s)^{-1}\pi_2\pi_2\pi_1 \circ f_1\pi_2\pi_1\pi_2 \\
&= f_1(s)^{-1}\pi_2\pi_2\pi_1 \circ f_1\pi_2\pi_2\pi_2 \\
&= (f_1(s)^{-1} \circ f_1)\delta_2
\end{aligned}$$

In particular, it follows that

$$(f_1(s)^{-1} \circ f_1)\partial_0 = (f_1(s)^{-1} \circ f_1)\partial_1$$

holds since the morphism $R \to R_0$ is regular epi, hence cancelable on the right. Therefore, there exists a morphism $\mathbb{C}[\Sigma^{-1}]_1 \to D_1$ making a commutative triangle, as in

$$\begin{tikzcd}
S \arrow[rr, "u"] \arrow[dr, "f_1(s)^{-1} \circ f_1"'] & & \mathbb{C}[\Sigma^{-1}]_1. \arrow[dl, dashed, "(\widetilde{f})_1"] \\
& D_1 &
\end{tikzcd}$$

For the morphism u is a coequalizer in $\mathscr{E}/C_0 \times C_0$, hence in $\mathscr{E}$, since the forgetful functor $\mathscr{E}/C_0 \times C_0 \to \mathscr{E}$ is a left adjoint.

Lemma 5.2.29. *The choices $(\widetilde{f})_0 := f_0$ and $(\widetilde{f})_1$ as in the discussion immediately above yield an internal functor $\widetilde{f} \colon \mathbb{C}[\Sigma^{-1}] \to \mathbb{D}$.*

Proof. The identity law is very easily proved. Recall that the identity arrow was defined in Equation 5.2.3. Now, compute that

$$(\widetilde{f})_1 u\langle jd_0, 1\rangle i = (f_1(s)^{-1} \circ f_1)\langle jd_0, 1\rangle i = f_1 i = i$$

since f is an internal functor. And indeed $\widetilde{f}$ respects composition as well; for the usual square expressing this fact is equalized by the morphism v from Lemma 5.2.13, a regular epi, hence a right-cancelable morphism. □

Theorem 5.2.30. *The category object $\mathbb{C}[\Sigma^{-1}]$ is, up to isomorphism, the category of fractions associated to Σ in the sense of Definition 5.2.25.*

Proof. With $\widetilde{f}$ as in Lemma 5.2.29, the diagram of internal functors

$$\begin{tikzcd} \mathbb{C} \arrow[rr, "l"] \arrow[dr, "f"'] & & \mathbb{C}[\Sigma^{-1}] \arrow[dl, "\widetilde{f}"] \\ & \mathbb{D} & \end{tikzcd}$$

commutes. At the level of objects, this is immediate. At the level of arrows, compute that

$$(\widetilde{f})_1 u\langle jd_0, 1\rangle = (f_1(s)^{-1}\pi_1 \circ f_1\pi_2)\langle jd_0, 1\rangle = f_1(s)^{-1}jd_0 \circ f_1 = f_1$$

as required. By construction $\widetilde{f}$ is unique. The 2-dimensional aspect of the universal property is trivial. For a natural transformation $\theta \colon f \Rightarrow g$ of internal functors that invert $s \colon \Sigma \to C_1$ is really a morphism $\theta \colon C_0 \to D_1$ satisfying the conditions of Definition 3.1.22. Thus, by construction of the localization, for the required lift $\widetilde{\theta} \colon \widetilde{f} \Rightarrow \widetilde{g}$, just take θ itself. □

The last point of the general development is to show that the localization is a reflexive coinverter in the sense of Example 4.2.3. Maintain the hypothesis that the monomorphism $s \colon \Sigma \to C_1$ admits a right calculus of fractions as in Definition 5.2.1. Denote by $(\Sigma^{\mathbf{2}})_1$ the corner object of the pullback

$$\begin{tikzcd} (\Sigma^{\mathbf{2}})_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma \times_{C_0} \Sigma \arrow[d, "\circ"] \\ \Sigma \times_{C_0} \Sigma \arrow[r, "\circ"'] & \Sigma. \end{tikzcd}$$

Let $d_0 := \pi_1\pi_2 \colon (\Sigma^{\mathbf{2}})_1 \to \Sigma$ and $d_1 := \pi_2\pi_1 \colon (\Sigma^{\mathbf{2}})_1 \to \Sigma$. Evidently, then take $(\Sigma^{\mathbf{2}})_0 := \Sigma$.

Lemma 5.2.31. *If $s \colon \Sigma \to C_1$ admits an internal right calculus of fractions, then $\Sigma^{\mathbf{2}}$ is an internal category. And functors* dom, cod: $\Sigma^{\mathbf{2}} \rightrightarrows \mathbb{C}$ *are given with object-level assignments*

$$\mathrm{dom}_0 := d_0 s \colon \Sigma \to C_0 \qquad \mathrm{cod}_0 := d_1 s \colon \Sigma \to C_0$$

respectively; and arrow-level assignments

$$\mathrm{dom}_1 := d_0 s \colon (\Sigma^{\mathbf{2}})_1 \to C_1 \qquad \mathrm{cod}_1 := d_1 s \colon (\Sigma^{\mathbf{2}})_1 \to C_1$$

with in this case d_0 and d_1 as above in the discussion.

Proof. Straightforward verification. □

Lemma 5.2.32. *The monomorphism $s \colon \Sigma \to C_1$ determines an internal natural transformation $\sigma \colon$* dom ⇒ cod. *Additionally, as a 2-cell σ is reflexive.*

Proof. The equations specified in Definition 3.1.22 are all satisfied by the construction of the internal functors dom and cod. □

Theorem 5.2.33. *The internal category of fractions construction $l \colon \mathbb{C} \to \mathbb{C}[\Sigma^{-1}]$ is the reflexive coinverter of $\sigma \colon$* dom ⇒ cod.

Proof. This is entirely a restatement of Theorem 5.2.30 in light of Example 4.2.3. □

5.3 Elementary 2-Filteredness

Throughout let $\mathscr{E}$ denote an exact category with pullback-stable coequalizers of reflexive pairs; and $\mathfrak{K}$, the 2-category of internal categories $\mathfrak{Cat}(\mathscr{E})$. Let $\mathcal{C}$ denote an internal 2-category as in Definition 3.4.1. The following is perhaps the central definition of the present work, axiomatizing the condition extracted from the exactness assumption in the special case of $\mathscr{E} = \mathbf{Set}$ in Theorem 4.4.4 and enshrined in Definition 4.4.8.

Definition 5.3.1. *A discrete 2-opfibration $e \colon \mathcal{E} \to \mathcal{C}$ is 2-filtered with respect to opcartesian morphisms if the following conditions are satisfied.*

1. *The canonical map $E_0 \to 1$ is a regular epimorphism.*

2. *Given two generalized objects $x, y \colon X \rightrightarrows E_0$, there is a regular epimorphism $p \colon Z \to X$ and opcartesian generalized morphisms $f, g \colon Z \rightrightarrows E_1$ such that the following hold*

(a) $d_0 f = d_0 g$

(b) $d_1 f = xp$

(c) $d_1 g = yp$.

3. *For generalized opcartesian morphisms $f, g\colon X \rightrightarrows E_1$ with $d_0 f = d_0$ and $d_1 f = d_1 g$, there is a regular epimorphism $p\colon Z \to X$ and a generalized opcartesian morphism $h\colon Z \to E_1$ such that the following equations hold:*

   (a) $d_1 h = d_0 f p = d_0 g p$

   (b) $h \circ fp = h \circ gp$.

4. *For any vertical generalized morphism $u\colon X \to E_1$, there is a regular epi $p\colon Z \to X$, opcartesian morphisms $f, g\colon Z \rightrightarrows E_1$, and a generalized 2-cell $\alpha\colon Z \to E_2$ for which the following equations are satisfied*

   (a) $d_0 f = d_0 g$

   (b) $d_1 f = d_0 up$

   (c) $d_1 g = d_1 up$

   (d) $s\alpha = f \circ up$

   (e) $t\alpha = g$.

Remark 5.3.2. Definition 5.3.1 is an internal version of Definition 4.4.8 for an ordinary discrete 2-opfibration. Basically it says in purely elementary language of internal 2-categories that e is non-trivial; that any two objects are connected by an opcartesian span (transitivity); that any two opcartesian arrows are equalized by a third (freeness); and finally that any vertical arrow of the total category pulls back by an opcartesian arrow to an opcartesian arrow. The main result will be that if e is 2-filtered with respect to opcartesian morphisms, then the induced tensor product $\mathcal{E} \otimes_{\mathcal{C}} -$ as in 4.3.1 has certain expected exactness properties.

Lemma 5.3.3. *Let $e\colon \mathcal{E} \to \mathcal{C}$ denote an internal discrete 2-opfibration that is 2-filtered in the sense of Definition 5.3.1 above. For any parallel generalized morphisms $h, k\colon X \to E_1$ with k internally opcartesian, there is a regular epimorphism $p\colon Z \to X$, a opcartesian generalized morphism $w\colon Z \to E_1$, and a generalized 2-cell $\alpha\colon w \circ hp \Rightarrow w \circ kp$.*

Proof. The proof of Lemma 4.4.10 can be rewritten in the internal category theory of $\mathscr{E}$. □

Lemma 5.3.4. *Let $e\colon \mathcal{E} \to \mathcal{C}$ denote a internal discrete 2-fibration. Suppose that e is 2-filtered in the sense of Definition 5.3.1. It follows that for each generalized morphism $j\colon X \to E_1$,* *there is a regular epimorphism* $p\colon Z \to X$*, two generalized opcartesian morphisms* $l, r\colon Z \to E_1$ *and a generalized 2-cell* $\theta\colon Z \to E_2$ *such that the equations*

1. $d_0 l = d_0 r$
2. $d_1 l = d_0 j$
3. $d_1 r = d_1 j$
4. $s\theta = l \circ jp$
5. $t\theta = r$

all hold.

Proof. The given filteredness conditions allow the elementary construction of a cell of the form of that in the proof of Lemma 4.4.12. □

Remark 5.3.5. The proof of Theorem 5.1.2 can now be internalized. In particular, all but the third condition of Definition 5.2.1 are trivially rewritten in the elementary internal category theory of $\mathscr{E}$. The following gives an outline of the elementary proof of the third condition.

Construction 5.3.1. *Let* $e\colon \mathcal{E} \to \mathcal{C}$ *denote an internal discrete 2-opfibration and* $f\colon \mathcal{F} \to \mathcal{C}$ *an internal discrete 2-fibration as in Definition 2.2.15. Denote by* $\Sigma_{e,f}$ *the corner object of the successive pullbacks*

$$\begin{tikzcd}
\Sigma_{e,f} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \times_{C_0} F_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_0 \arrow[d, "f_0"] \\
E_0 \times_{C_0} C_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[r, "d_1"'] \arrow[d, "d_0"] & C_0 \\
E_0 \arrow[r, "e_0"'] & C_0.
\end{tikzcd}$$

By construction, there is then a morphism $\rho \times \sigma\colon \Sigma \longrightarrow E_1 \times_{C_1} F_1$ *commuting with the appropriate projections. Now, as in the opening of §5.2, form the object of distinguished spans in* $\mathcal{E} \times_{\mathcal{C}} \mathcal{F}$ *by taking the pullback*

$$\begin{tikzcd}
S_{e,f} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_1 \times_{C_1} F_1 \arrow[d, "d_0 \times d_0"] \\
\Sigma_{e,f} \arrow[r, "\rho \times \sigma"'] & E_0 \times_{C_0} F_0.
\end{tikzcd}$$

In particular, this whole process applies to the identity fibration over $\mathcal{C}$ and yields the object of spans in $\mathcal{E}$ as a pullback

$$\begin{tikzcd}
S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_1 \arrow[d, "d_0"] \\
\Sigma \arrow[r, "\rho"'] & E_0
\end{tikzcd}$$

together with a canonical morphism $\pi\colon S \to S$ induced by the projections.

Theorem 5.3.6. *If an internal discrete 2-opfibration $e\colon \mathcal{E} \to \mathcal{C}$ is 2-filtered in the sense of Definition 5.3.1, then for any internal discrete 2-fibration $f\colon \mathcal{F} \to \mathcal{C}$, the morphism*

$$\rho \times \sigma\colon \Sigma_{e,f} \to E_1 \times_{C_1} F_1$$

admits an internal right calculus of fractions as in Definition 5.2.1.

Proof. Lemma 3.3.8 shows that $\rho \times \sigma$ is a monomorphism.

The fact that $e\colon \mathcal{E} \to \mathcal{C}$ is a split internal opfibration means that the opcleavage is functorial, hence closed under composition as in Lemma 3.3.7.

Let $z\colon X \to \Sigma$ and $\langle h, k\rangle\colon X \to E_1 \times_{C_1} F_1$ denote morphisms with $d_1 h = d_1 \rho z$ and $d_1 k = d_1 \sigma z$. Use the notation

$$d_0 h =: a, \quad d_0 k =: b, \quad d_0 \rho z =: x, \quad d_0 \sigma z =: y.$$

This gives the elementary analogue of the corner diagram that starts the proof in the case $\mathscr{E} = \mathbf{Set}$. Now, in the $\mathcal{E}$-component, the filteredness axioms and Lemma 5.3.3 give a regular epimorphism $p\colon Z \to X$, suitably composable opcartesian morphisms $u, v\colon Z \to E_1$, and finally a 2-cell $\alpha\colon Z \to E_2$ with $\alpha\colon hp \circ v \Rightarrow xp \circ u$. Viewed in $\mathcal{C}$ via $e\colon \mathcal{E} \to \mathcal{C}$, this gives a cell $e_2\alpha\colon e_1 v \circ e_1 h \Rightarrow e_1 u \circ e_1 i$. Now, since $f\colon \mathcal{F} \to \mathcal{C}$ is an internal cloven fibration, there are internally cartesian generalized morphisms

$$\sigma\langle e_1 v, bp\rangle\colon Z \to F_1$$

$$\sigma\langle e_1 u, yp\rangle\colon Z \to F_1$$

over $e_1 v$ and $e_1 u$, respectively, in the sense that

$$f_1 \sigma\langle e_1 v, bp\rangle = e_1 v$$

$$f_1 \sigma\langle e_1 u, yp\rangle = e_1 u.$$

Thus, since, additionally, $e_1h = f_1k$ holds and $f\colon \mathcal{F} \to \mathcal{C}$ is locally an internal discrete opfibration, there is a unique lifted 2-cell

$$\widetilde{e\alpha}\colon \sigma\langle e_1v, bp\rangle \circ k \Rightarrow l$$

over α for some generalized arrow $l\colon Z \to F_1$ over $e_1u \circ e_1i$ in that $f_1l = e_1u \circ e_1i$. Now, since $e_1i = f_1j$ and $f\colon \mathcal{F} \to \mathcal{C}$ is an internal fibration, there is a unique lift of identity $w\colon Z \to F_1$ such that

$$w\colon d_0l \to d_0\sigma\langle e_1u, yp\rangle$$

$$w \circ \sigma\langle e_1u, yp\rangle \circ jp = l.$$

Thus, α and the lift $\widetilde{e\alpha}$ yield a 2-cell of $\mathcal{E} \times_{\mathcal{C}} \mathcal{F}$ that when reduced modulo internal connected components as in §3.4.1 via the coequalizer

$$\mathcal{E} \times_{\mathcal{C}} \mathcal{F} \to \pi_0(\mathcal{E} \times_{\mathcal{C}} \mathcal{F})$$

completes the given corner diagram to a commutative square, as required. $\square$

Corollary 5.3.7. *Under the same hypotheses, the tensor product $\mathcal{E} \otimes_{\mathscr{C}} \mathcal{F}$ arises through a right calculus of fractions as in Definition 5.2.1. It defines a 2-functor*

$$\mathcal{E} \otimes_{\mathcal{C}} -\colon \mathfrak{DFib}(\mathcal{C}) \to \mathfrak{K}$$

by the universal property of the coinverter.

Proof. This is a restatement of the above result using the interpretation of Theorem 5.2.33 and the internalized definition of the tensor product in §4.3.1. $\square$

Chapter 6

Elementary Account of Flatness

Now that it is assured that a tensor product exists under 2-filteredness conditions, its exactness properties can be studied. The present chapter culminates in an internalized version of the result that a discrete 2-fibration $E\colon \mathfrak{E} \to \mathfrak{C}$, 2-filtered in the sense of Definition 4.4.8, is flat in that the tensor $E \otimes_{\mathfrak{C}} -\colon \mathfrak{DFib}(\mathfrak{C}) \to \mathfrak{Cat}$ has several exactness properties.

6.1 Conical Limits Reduce to the Internal Colimit

Work in $\mathfrak{K} = \mathfrak{Cat}(\mathscr{E})$ for a exact 1-category $\mathscr{E}$ with pullback-stable coequalizers of reflexive pairs. Fix $e\colon \mathcal{E} \to \mathcal{C}$, an internal discrete 2-opfibration and let $f\colon \mathcal{F} \to \mathcal{C}$ denote an internal discrete 2-fibration, as in Definition 2.2.15. Think of f as variable. The opcleavage for the underlying opfibration of e is a morphism $\rho\colon E_0 \times_{C_0} C_1 \to E_1$ making a natural transformation $\rho\colon \pi \Rightarrow m$, where m is the action of $\mathbf{2} \pitchfork \mathcal{C}_0$ on $\mathcal{E}_0$ as described in §3.3. Similarly, let σ denote the cleavage for f.

Construction 6.1.1. *Form the objects P_e, Q_e and $P_{e,f}$, $Q_{e,f}$ for the internal categories $\mathcal{E}$ and $\mathcal{E} \times_{\mathcal{C}} \mathcal{F}$ as in Construction 5.2.1, respectively. Then the relations objects R_e and $R_{e,f}$ are formed as pullbacks*

$$\begin{tikzcd} R_e \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_e \arrow[d, "\langle \pi_1\pi_1, \pi_1\pi_2\rangle"] \\ Q_e \arrow[r, "\langle \pi_1\pi_1, \pi_1\pi_2\rangle"'] & E_1 \times E_1 \end{tikzcd} \qquad \begin{tikzcd} R_{e,f} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_{e,f} \arrow[d, "\langle \pi_1\pi_1, \pi_1\pi_2\rangle"] \\ Q_{e,f} \arrow[r, "\langle \pi_1\pi_1, \pi_1\pi_2\rangle"'] & (E_1 \times_{C_1} F_1) \times (E_1 \times_{C_1} F_1). \end{tikzcd}$$

From the evident projection morphisms $\pi\colon P_{e,f} \to P_e$ and $\pi\colon Q_{e,f} \to Q_e$, there is a canonical projection morphism induced $\pi\colon R_{e,f} \to R_e$ making the appropriate commutative diagram. Henceforth R_e and $R_{e,f}$ will be confused with the object of their respective image factorizations, unless for some reason in either case the image and the original object need to be distinguished; in the former case use ∂_0, ∂_1 and in the latter case use δ_0, δ_1. Thus, as in Construction 5.2.1, each relation object comes with parallel arrows $R_e \rightrightarrows S_e$ and $R_{e,f} \rightrightarrows S_{e,f}$ jointly monic in the appropriate slice category.

Theorem 6.1.1. *If $e\colon \mathcal{E} \to \mathcal{C}$ is 2-filtered in the sense of Definition 4.4.8, then the morphisms $\partial_0, \partial_1\colon R_{e,f} \rightrightarrows S_{e,f}$ define a groupoid internal to $\mathscr{E}/E_0 \times E_0$. Denote this groupoid by $\mathbb{S}_{e,f}$.*

Proof. From Theorem 5.2.7 and Theorem 5.3.6, it follows that $R_{e,f} \rightrightarrows S_{e,f}$ determines an equivalence relation, hence an internal groupoid, in the slice over the product $(E_0 \times_{C_0} F_0) \times (E_0 \times_{C_0} F_0)$. However, that it is an equivalence relation over $E_0 \times E_0$ follows as well. The proofs of reflexivity and symmetry are essentially the same. For transitivity, note that the pullback T taken over $E_0 \times E_0$ in the set-up for the condition can also be viewed as a pullback over $(E_0 \times_{C_0} F_0) \times (E_0 \times_{C_0} F_0)$. Thus, the arrow required for transitivity over $E_0 \times E_0$ arises by using transitivity over $(E_0 \times_{C_0} F_0) \times (E_0 \times_{C_0} F_0)$. The result now follows by the proposition. $\square$

Corollary 6.1.2. *If $e\colon \mathcal{E} \to \mathcal{C}$ is 2-filtered as in Definition 4.4.8, then $R_e \rightrightarrows S_e$ is a groupoid in $\mathscr{E}/E_0 \times E_0$, denoted by $\mathbb{S}_e$.*

Proof. Take the identity fibration on $\mathcal{C}$ in the previous theorem. $\square$

Corollary 6.1.3. *If $e\colon \mathcal{E} \to \mathcal{C}$ is 2-filtered as in Definition 4.4.8, the projection $\pi\colon \mathcal{E} \times_{\mathcal{C}} \mathcal{F} \to \mathcal{E}$ determines an internal functor of groupoids $\pi\colon \mathbb{S}_{e,f} \to \mathbb{S}_e$.*

Proof. Evidently, the components of π are the projections $\pi\colon S_{e,f} \to S_e$ and $\pi\colon R_{e,f} \to R_e$ from the discussion above. For example, the two squares in

$$\begin{tikzcd}
R_{e,f} \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] \arrow[d, "\pi"'] & S_{e,f} \arrow[d, "\pi"] \\
R_e \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & S_e
\end{tikzcd}$$

commute by the uniqueness clause of the pullback S_e. The other conditions for an internal functor as in Definition 3.1.12 are similarly verified. $\square$

Lemma 6.1.4. *The commutative square*

$$\begin{tikzcd}
R_{e,f} \arrow[r, "\delta_1"] \arrow[d, "\pi"'] & S_{e,f} \arrow[d, "\pi"] \\
R_e \arrow[r, "\delta_1"'] & S_e
\end{tikzcd}$$

satisfies the modified hypotheses of Lemma 2.3.10 given in Remark 2.3.11. Consequently, the commutative square of images

$$\begin{tikzcd}
R_f \arrow[r, tail, "\partial_1"] \arrow[d] & S_{e,f} \arrow[d] \\
R_{e,f} \arrow[r, tail, "\partial_1"'] & S_e
\end{tikzcd}$$

is a pullback.

Remark 6.1.5. The proof below in the internal category theory of $\mathscr{E}$ is technical and uninformative. Set-theoretically, the proof is an exercise in cone-building. For $\mathcal{E} = \mathbf{Set}$, the point is that assumed as given is a figure of the form

$$\begin{tikzcd}
& \cdot \arrow[dl, "s"'] \arrow[dr, "f"] & \\
\cdot & \cdot \arrow[u, dashed, "x"] \arrow[d, dashed, "y"'] & \cdot \\
& \cdot \arrow[ul, "t"] \arrow[ur, "g"'] &
\end{tikzcd}$$

of arrows of $\mathcal{E}$, relating by x and y two of the special spans with left legs s and t opcartesian; as well as a span $\cdot \leftarrow \cdot \rightarrow \cdot$ of $\mathcal{E} \times_{\mathcal{C}} \mathcal{F}$, namely,

$$\cdot \xleftarrow{(t,r)} \cdot \xrightarrow{(g,l)} \cdot$$

projecting to its $\mathcal{E}$-components t and g above. The condition of Lemma 2.3.10 requires construction of a diagram of arrows in $\mathcal{E} \times_{\mathcal{C}} \mathcal{F}$ of the same form, namely,

$$\begin{tikzcd}
& \cdot \arrow[dl] \arrow[dr] & \\
\cdot & \cdot \arrow[u, dashed] \arrow[d, dashed] & \cdot \\
& \cdot \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

one side of which projects to the given span $\cdot \leftarrow \cdot \rightarrow \cdot$ of $\mathcal{E} \times_{\mathcal{C}} \mathcal{F}$, but does not necessarily project to the given figure in $\mathcal{E}$, above. Such a figure can be constructed using the fact that $F \colon \mathcal{F} \to \mathcal{C}$ is a cloven fibration. Indeed take the chosen cartesian morphisms of $\mathcal{F}$ over sx and tx and over y with appropriate codomains, denoted, respectively, by u, v and w. These almost work, but the required squares do not necessarily commute. So, take p and q, lifts of identities such that $up = rw$ and $vq = lw$. Then the $\mathcal{E}$- and $\mathcal{F}$-components of the required diagram in

$\mathcal{E} \times_{\mathcal{C}} \mathcal{F}$ are represented by

$$\begin{tikzcd}
& \cdot \arrow[dl, "sx"'] \arrow[dr, "fx"] & \\
\cdot & \cdot \arrow[u, dashed, "1"] \arrow[d, dashed, "y"] & \cdot \\
& \cdot \arrow[ul, "t"] \arrow[ur, "g"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& \cdot \arrow[dl, "up"'] \arrow[dr, "vq"] & \\
\cdot & \cdot \arrow[u, dashed, "1"] \arrow[d, dashed, "w"] & \cdot \\
& \cdot \arrow[ul, "r"] \arrow[ur, "l"'] &
\end{tikzcd}$$

respectively. Evidently, this will project to the given span of $\mathcal{E} \times_{\mathcal{C}} \mathcal{F}$. Now, the following proof rephrases these remark in the internal category theory of $\mathscr{E}$.

Proof. Let X denote an object with two morphisms $h\colon X \to R_e$ and $k\colon X \to S_{e,f}$ satisfying the equation $\pi k = \delta_1 h$ as in

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "h"'] & & \\
& R_{e,f} \arrow[r, "\delta_1"] \arrow[d, "\pi"'] & S_{e,f} \arrow[d, "\pi"] \\
& R_e \arrow[r, "\delta_1"'] & S_e
\end{tikzcd}$$

The goal is to produce the dashed arrow $X \to R_{e,f}$ making the top triangle commute. Use the fact that f is a cloven fibration to build a cone. Indeed there are the following three cartesian morphisms:

1. $u := \sigma\langle e_1\pi_1\pi_1\pi_1 h \circ e_1\pi_2\pi_1\pi_1 h, \pi_2\pi_2\pi_1 k\rangle\colon X \to F_1$

2. $v := \sigma\langle e_1\pi_1\pi_1\pi_2 h \circ e_1\pi_2\pi_1\pi_2 h, d_1\pi_2\pi_2 k\rangle\colon X \to F_1$

3. $w := \sigma\langle e_1\pi_1\pi_2 h, d_0\pi_2\pi_2 k\rangle\colon X \to F_1$.

That is, for example, u is the cartesian morphism picked by the cleavage for $f\colon \mathcal{F} \to \mathcal{C}$ over the composite $e_1\pi_1\pi_1\pi_1 h \circ e_1\pi_2\pi_1\pi_1 h$ and having codomain $\pi_2\pi_2\pi_1 k$. Now, since, in particular, u and v are cartesian, there are lifts of identity, denoted by $p, q\colon X \rightrightarrows F_1$ satisfying

$$d_0 p = d_0 w = d_0 q \qquad d_1 p = d_0 v \qquad d_1 q = d_0 u$$

and, most importantly, making commutative squares in $\mathcal{F}$, given in equations by

$$p \circ v = w \circ \pi_2 k \qquad q \circ u = w \circ \pi_1 k.$$

The required morphism $X \to R_{e,f}$ arises as in the following diagram, essentially by inserting an identity morphism:

$$\begin{tikzcd}
X \arrow[dr, dashed, "r"] \arrow[rrd, bend left, "\langle\langle id_0 p, p\circ v\rangle, \langle w, \pi_2 k\rangle\rangle"] \arrow[dd, bend right, swap, "\langle\langle id_0 q, q\circ u\rangle, \langle w, \pi_1 k\rangle\rangle"] & & \\
 & R_{e,f} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_{e,f} \arrow[d, "\langle \pi_1\pi_1, \pi_1\pi_2\rangle"] \\
 & Q_{e,f} \arrow[r, swap, "\langle \pi_1\pi_1, \pi_1\pi_2\rangle"] & (E_1 \times_{C_0} F_1) \times (E_1 \times_{C_0} F_1)
\end{tikzcd}$$

That the outside does commute is an easy computation. It now follows that $\delta_1 r = k$ holds in the first diagram of the proof, as can be seen by checking on components. □

Corollary 6.1.6. *If $e \colon \mathcal{E} \to \mathcal{C}$ is 2-filtered as in Definition 4.4.8, then the internal functor $\pi \colon \mathbb{S}_{e,f} \to \mathbb{S}_e$ is a discrete fibration internal to $\mathscr{E}/E_0 \times E_0$.*

Proof. This is just an interpretation of Lemma 6.1.4 in light of Definition 3.2.2. □

Corollary 6.1.7. *If $e \colon \mathcal{E} \to \mathcal{C}$ is 2-filtered as in Definition 4.4.8, then the object of arrows $(\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_1$ of the tensor product is the internal colimit of Equation 3.2.1, in that there is an equality*

$$(\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_1 = \varinjlim_{\mathbb{S}_e^{op}} \pi_{e,f}$$

with $\pi_{e,f}$ the internal discrete fibration of Corollary 6.1.3.

Proof. By Lemma 6.1.4, $(\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_1$ is formed through the right calculus of fractions, hence as a certain coequalizer in a slice of $\mathscr{E}$. Since the forgetful functor from the slice preserves colimits, the two objects thus have the same definition. Since a choice of colimits is assumed, the values are literally equal. □

Lemma 6.1.8. *If $e \colon \mathcal{E} \to \mathcal{C}$ is 2-filtered as in Definition 4.4.8, then the groupoid $\mathbb{S}_e$ is internally filtered as in Definition 3.2.5.*

Proof. The non-emptiness condition is trivial. Suppose that given are two elements of S_e over $E_0 \times E_0$ as depicted in the diagram

$$\begin{tikzcd}
U \arrow[rr, shift left, "x"] \arrow[rr, shift right, swap, "y"] \arrow[dr] & & S_e. \arrow[dl] \\
 & E_0 \times E_0 &
\end{tikzcd}$$

Now, set theoretically, this is just to give two spans with the same endpoints, as depicted in the square

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \\ \cdot & \cdot \arrow[l] \arrow[u] \end{tikzcd}$$

A cone relating the two spans can be built using the filteredness assumptions on E at the level of 1-cells. The equalizing condition is trivial. □

Theorem 6.1.9. *If the discrete 2-opfibration $e\colon \mathcal{E} \to \mathcal{C}$ is 2-filtered as in Definition 4.4.8, then the internal colimit functor*

$$\varinjlim_{\mathbb{S}_e^{op}} : \mathbf{DFib}(\mathbb{S}_e) \longrightarrow \mathscr{E}/E_0 \times E_0$$

is left exact.

Proof. By the Lemma 6.1.8 above and Lemma 3.2.7. □

6.2 Preservation of Conical Limits

As a result of the last corollary, it can now be seen that $\mathcal{E} \otimes_{\mathcal{C}} -$ has the required left-exactness properties if $e\colon \mathcal{E} \to \mathcal{C}$ is 2-filtered as in Definition 4.4.8. Recall that by Corollary 5.3.7, if $e\colon \mathcal{E} \to \mathcal{C}$ is 2-filtered, then the tensor is a 2-functor

$$\mathcal{E} \otimes_{\mathcal{C}} - \colon \mathfrak{DFib}(\mathcal{C}) \longrightarrow \mathfrak{K}$$

where $\mathfrak{K} = \mathfrak{Cat}(\mathscr{E})$ for $\mathscr{E}$ an exact category with pullback-stable coequalizers of reflexive pairs. In particular, $\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F}$ is given as the reflexive coinverter of the cleavage and opcleavage coming with e and f. If e is 2-filtered as in Definition 4.4.8, then the tensor always exists, given as a right calculus of fractions. For the first part of the proof, recall that the terminal object of $\mathfrak{DFib}(\mathcal{C})$ is the identity fibration $1\colon \mathcal{C} \to \mathcal{C}$.

Lemma 6.2.1. *If $e\colon \mathcal{E} \to \mathcal{C}$ is 2-filtered as in Definition 4.4.8, then $\mathcal{E} \otimes_{\mathcal{C}} -$ preserves the terminal object.*

Proof. Since e is 2-filtered, the tensor arises through a right calculus of fractions. Thus, by construction, the object of objects of the tensor is

$$(\mathcal{E} \otimes_{\mathcal{C}} 1)_0 = E_0 \times_{C_0} C_0 \cong E_0.$$

On the other hand, by Corollary 6.1.7, the object of arrows is given by the coequalizer in the definition of the internal colimit

$$(\mathcal{E} \otimes_{\mathcal{C}} 1)_1 \cong \varinjlim_{\mathbb{S}_e^{op}} 1$$

since $\pi_1 = 1$. Now, since the groupoid $\mathbb{S}_e$ is filtered, this internal colimit is isomorphic to the terminal object in the slice over $E_0 \times E_0$. That is, the object of arrows is, up to isomorphism, $E_0 \times E_0$. Therefore, the tensor $\mathcal{E} \otimes_{\mathbb{C}} 1$, up to isomorphism, as an internal category, is the chaotic category on E_0, which is weakly equivalent to 1 in $\mathfrak{K}$ by Lemma 3.1.21. $\square$

The product of two internal discrete 2-fibrations $f\colon \mathcal{F} \to \mathcal{C}$ and $g\colon \mathcal{G} \to \mathcal{C}$ with cleavages σ and τ, respectively, is given by their pullback, namely,

$$\begin{tikzcd}
\mathcal{F} \times_{\mathcal{C}} \mathcal{G} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d, "g"] \\
\mathcal{F} \arrow[r, "f"'] & \mathcal{C}
\end{tikzcd}$$

taken in 2-$\mathfrak{Cat}(\mathscr{E})$. In particular, the objects and arrows are given by $(f \times_C g)_0 = F_0 \times_{C_0} G_0$ and $(f \times_C g)_1 = F_1 \times_{C_1} G_1$ respectively. In the sequel, by either '$f \times g$', or '$\mathcal{F} \times_{\mathcal{C}} \mathcal{G}$' the product in this sense will always be meant, depending upon whether the morphisms or total categories need to be emphasized.

Now, if $e\colon \mathcal{E} \to \mathcal{C}$ is 2-filtered as in Definition 4.4.8, there are three internal groupoids, namely, $\mathbb{S}_{e,f\times g}$ and $\mathbb{S}_{e,f}$ and $\mathbb{S}_{e,g}$ built from the respective objects of spans and objects of related spans according to the right calculus of fractions construction. These admit projection morphisms to $\mathbb{S}_e$. Denote these by subscripting with the name of the fibration, as in

$$\pi_{f\times g}\colon \mathbb{S}_{e,f\times g} \to \mathbb{S}_e \qquad \pi_f\colon \mathbb{S}_{e,f} \to \mathbb{S}_e \qquad \pi_g\colon \mathbb{S}_{e,g} \to \mathbb{S}_e.$$

Lemma 6.2.2. *Suppose that the discrete 2-opfibration $e\colon \mathcal{E} \to \mathcal{C}$ is 2-filtered as in Definition 4.4.8. There is then an isomorphism of internal discrete fibrations*

$$\pi_{f\times g} \cong \pi_f \times \pi_g$$

in $\mathbf{DFib}(\mathbb{S}_e)$*. In particular, there is an isomorphism*

$$\mathbb{S}_{e,f\times g} \cong \mathbb{S}_{e,f} \times_{\mathbb{S}_e} \mathbb{S}_{e,g}$$

of internal groupoids.

Proof. The argument is essentially that all constructions involved in formation of the π's are pullbacks and universally induced arrows. The key to the argument, however, is the observation that the squares

$$\begin{tikzcd} R_{f\times g} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R_g \arrow[d] \\ R_f \arrow[r] & R_e \end{tikzcd} \qquad \begin{tikzcd} S_{f\times g} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S_g \arrow[d] \\ S_f \arrow[r] & S_e. \end{tikzcd}$$

formed by the induced projection morphisms are both pullbacks by construction. The argument for the square on the right is straightforward. For the square on the left, it should first be observed that there are isomorphisms $P_{f\times g} \cong P_f \times_{P_e} P_g$ and $Q_{f\times g} \cong Q_f \times_{Q_e} Q_g$ by the construction of the P's and Q's as in §5.2.1. Since the respective R's are pullbacks of these, the conclusion follows. $\square$

Corollary 6.2.3. *If the discrete 2-opfibration $e\colon \mathcal{E} \to \mathcal{C}$ is 2-filtered as in Definition 4.4.8, then the tensor $\mathcal{E} \otimes_{\mathcal{C}} -$ preserves binary products.*

Proof. Since ρ filters each of the tensor products, they each arise by the right calculus of fractions construction. Thus, at the level of objects, there is the computation

$$\begin{aligned}(\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_0 \times_{(\mathcal{E}\otimes_{\mathcal{C}} 1)_0} (\mathcal{E} \otimes_{\mathcal{C}} \mathcal{G})_0 &= (E_0 \times_{C_0} F_0) \times_{E_0} (E_0 \times_{C_0} G_0) \\ &\cong E_0 \times_{C_0} (F_0 \times_{C_0} G_0) \\ &= (\mathcal{E} \otimes_{\mathcal{C}} (\mathcal{F} \times_{\mathcal{C}} \mathcal{G}))_0.\end{aligned}$$

Now, at the level of morphisms, compute that

$$\begin{aligned}(\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_1 \times_{(\mathcal{E}\otimes_{\mathcal{C}} 1)_1} (\mathcal{E} \otimes_{\mathcal{C}} \mathcal{G})_1 &\cong \varinjlim_{\mathbb{S}_e^{op}} \pi_f \times_{\varinjlim_{\mathbb{S}_e^{op}} 1} \varinjlim_{\mathbb{S}_e^{op}} \pi_g && \text{(by Cor. 6.1.7)} \\ &\simeq \varinjlim_{\mathbb{S}_e^{op}} \pi_f \times_{E_0\times E_0} \varinjlim_{\mathbb{S}_e^{op}} \pi_g && \text{(by Lemma 6.2.1)} \\ &\cong \varinjlim_{\mathbb{S}_e^{op}} \pi_f \times \pi_g && \text{(by Theorem 6.1.9)} \\ &\cong \varinjlim_{\mathbb{S}_e^{op}} \pi_{f\times g} && \text{(by Lemma 6.2.2)} \\ &\cong (\mathcal{E} \otimes_{\mathcal{C}} (\mathcal{F} \times_{\mathcal{C}} \mathcal{G}))_1 && \text{(by Cor. 6.1.7)}\end{aligned}$$

using the fact that $\varinjlim$ is exact and the previous result. $\square$

Lemma 6.2.4. *Fix $f\colon \mathcal{F} \to \mathcal{C}$ and $g\colon \mathcal{G} \to \mathcal{C}$, internal discrete 2-fibrations over an internal 2-category $\mathcal{C}$ as in Definition 3.4.12. For an equalizer diagram*

$$\mathcal{Q} \xrightarrow{r} \mathcal{F} \underset{k}{\overset{h}{\rightrightarrows}} \mathcal{G}$$

in $\mathfrak{DFib}(\mathcal{C})$, the canonically induced sequence of internal functors

$$\mathbb{S}_{e,q} \xrightarrow{r} \mathbb{S}_{e,f} \underset{k}{\overset{h}{\rightrightarrows}} \mathbb{S}_{e,g}$$

is an equalizer diagram in $\mathbf{DFib}(\mathbb{S}_e)$.

Proof. This is a tedious argument by finite limit construction. Note that the strictness condition, Equation 3.4.1 is used to induce required map $\Sigma_{e,f} \rightrightarrows \Sigma_{e,g}$. □

Corollary 6.2.5. *If the discrete 2-opfibration $e\colon \mathcal{E} \to \mathcal{C}$ is 2-filtered as in Definition 4.4.8, then the induced tensor 2-functor $\mathcal{E} \otimes_{\mathcal{C}} -$ preserves equalizers.*

Proof. The preservation statement at the level of objects is similar to that in the product proof above. Now, for the arrows, use the lemma immediately above and the characterization of the object of arrows of the tensor product as the internal colimit, namely, Corollary 6.1.7. In the commutative diagram

$$\begin{tikzcd}
R_{e,q} \arrow[r] \arrow[d, shift right, "\partial_0"'] \arrow[d, shift left, "\partial_1"] & R_{e,f} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right, "\partial_0"'] \arrow[d, shift left, "\partial_1"] & R_{e,g} \arrow[d, shift right, "\partial_0"'] \arrow[d, shift left, "\partial_1"] \\
S_{e,q} \arrow[r] \arrow[d] & S_{e,f} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & S_{e,g} \arrow[d] \\
\varinjlim \pi_{e,q} \arrow[r] & \varinjlim \pi_{e,f} \arrow[r, shift left] \arrow[r, shift right] & \varinjlim \pi_{e,g}
\end{tikzcd}$$

the bottom row is a equalizer diagram by the exactness of the internal colimit result, Theorem 6.1.9. But this sequence is precisely the required sequence of arrow objects of tensor products

$$(\mathcal{E} \otimes_{\mathcal{C}} \mathcal{Q})_1 \xrightarrow{r} (\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_1 \underset{k}{\overset{h}{\rightrightarrows}} (\mathcal{E} \otimes_{\mathcal{C}} \mathcal{G})_1$$

by the cited corollary, as required. □

Theorem 6.2.6. *If the discrete 2-opfibration $e\colon \mathcal{E} \to \mathcal{C}$ is 2-filtered as in Definition 4.4.8, then the induced tensor $\mathcal{E} \otimes_{\mathcal{C}} -\colon \mathfrak{DFib}(\mathcal{C}) \to \mathfrak{K}$ preserves binary products and equalizers up to isomorphism and the terminal object up to equivalence.*

Proof. The statement follows now from Lemma 6.2.1 and Corollaries 6.2.3 and 6.2.5. □

6.3 Preservation of Ordinary Cotensors

Let $F\colon \mathfrak{F} \to \mathfrak{C}$ denote a discrete 2-fibration between ordinary 2-categories. Let $\mathbf{2} = \{0 \leq 1\}$ denote the usual ordinal category. Recall from Example 4.1.3 that the cotensor $\mathbf{2} \pitchfork F$ in the 2-category $\mathfrak{DFib}(\mathfrak{C})$ is given in the following way. Objects are vertical arrows $u\colon X \to Y$ of the total category $\mathfrak{F}$. Arrows are commutative squares between such vertical arrows. The 2-cells are those pairs yielding equalities

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\Rightarrow"] \arrow[dr, phantom, "="] & Y \arrow[d] \\ Z \arrow[r, "v"'] & W \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "="] & Y \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\Rightarrow"] \\ Z \arrow[r, "v"'] & W \end{tikzcd}$$

of composite 2-cells. Thus, $\mathbf{2} \pitchfork F$ is the full sub-2-category of the ordinary 2-arrow category $\mathfrak{F}^{\mathbf{2}}$ consisting of the vertical arrows relative to $F\colon \mathfrak{F} \to \mathfrak{C}$.

At the object-level, the canonical map $\mathfrak{E} \times_{\mathfrak{C}} (\mathbf{2} \pitchfork F) \to \mathbf{2} \pitchfork (E \otimes_{\mathfrak{C}} F)$ sends a pair (X, u) with u vertical in $\mathfrak{F}$ over EX to the the span

$$(A, B) \xleftarrow{(1,1)} (A, B) \xrightarrow{(1,u)} (A, W).$$

viewed modulo connected-components. For the following lemma, suppose that $E\colon \mathfrak{E} \to \mathfrak{C}$ is filtered by opcartesian arrows as in Definition 4.4.8. Since the resulting tensor $E \otimes_{\mathfrak{C}} F$ is thus formed through a right calculus of fractions as in Theorem 5.1.2, an arbitrary morphism of the tensor is represented as a span

$$(X, Y) \xleftarrow{(h,k)} (A, B) \xrightarrow{(f,g)} (Z, W).$$

with h opcartesian and k cartesian with each leg viewed modulo connected components. The following lemma and its proof show that if E is 2-filtered as in Definition 4.4.8, then every such map of the tensor, up to isomorphism, is of the form of those in the image of the canonical map above; and additionally that the vertical morphism u arises in a canonical way.

Lemma 6.3.1 (Factorization Lemma I)**.** *If the discrete 2-opfibration $E\colon \mathfrak{E} \to \mathfrak{C}$ is 2-filtered by opcartesian arrows as in Definition 4.4.8, then the arrow of the tensor above is isomorphic to one in the image of the canonical map $\mathfrak{E} \times_{\mathfrak{C}} (2 \pitchfork F) \to \mathbf{2} \pitchfork (E \otimes_{\mathfrak{C}} F)$.*

Proof. From Lemma 4.4.12, the morphism f fits into a 2-cell $\theta\colon fw \Rightarrow r$ with w and r opcartesian; let C denote the domain of w and r. Now, $\sigma(Er, W)$ and $\sigma(Ew, B)$ denote chosen cartesian arrows of $\mathfrak{F}$ over Er and Ew, respectively. Since F is locally a discrete fibration there is a lift in $\mathfrak{F}$ of the 2-cell $E\theta$ as appearing in

$$\begin{tikzcd}
E(w)^*B \arrow[r, "\sigma(Ew,B)"] \arrow[d, dashed, "\exists!\, u"'] \arrow[dr, dashed, bend right=15] \arrow[dr, phantom, shift left=3, "\exists!\ \Downarrow"] & B \arrow[d, "g"] \\
E(r)^*W \arrow[r, "\sigma(Er,W)"'] & W.
\end{tikzcd}$$

Since the target of the lifted 2-cell is over the morphism Er, there is a unique lift of the identity, $E(w)^*B \to E(r)^*W$, making a commutative triangle, as indicated by the other dashed arrow. This shows that there is a 2-cell $(f, g)(w, \sigma) \Rightarrow (r, \sigma)(1, u)$ in $\mathfrak{E} \times_{\mathfrak{C}} \mathfrak{F}$, which reduces to an equality modulo connected components.

Now, the claim is that the morphism above is then isomorphic to the morphism

$$(C, E(w)^*B) \xleftarrow{(1,1)} (C, E(w)^*B) \xrightarrow{(1,u)} (C, E(r)^*W).$$

The following diagram of spans in $\pi_0(\mathfrak{E} \times_{\mathfrak{C}} \mathfrak{F})$ produces the required isomorphism. The given span from the first display in the proof is the top row; and the span immediately above runs along the bottom. The vertical spans are evidently isomorphisms as each has both legs cartesian.

$$\begin{tikzcd}
(X,Y) & & (A,B) \arrow[ll, "(h,k)"'] \arrow[rr, "(f,g)"] & & (Z,W) \\
& (II) & & (C,E(w)^*B) \arrow[ul, dashed, "(w,\sigma)"'] \arrow[dr, dashed, "(1,u)"] & \\
(C,E(w)^*B) \arrow[uu, "(hw,k\sigma)"] \arrow[dd, "1"'] & & (C,E(w)^*B) \arrow[ur, dashed, "1"'] \arrow[dl, dashed, "1"'] & & (C,E(r)^*W) \arrow[uu, "(r,\sigma)"'] \arrow[dd, "1"] \\
& (C,E(w)^*B) \arrow[ul, dashed, "1"] \arrow[dr, dashed, "1"] & & (I) & \\
(C,E(w)^*B) & & (C,E(w)^*B) \arrow[ll, "1"] \arrow[rr, "(1,u)"'] & & (C,E(r)^*W).
\end{tikzcd}$$

The dashed arrows indicate how the spans can be composed and that they are indeed related in $\mathfrak{E} \otimes_{\mathfrak{C}} \mathfrak{F}$. The square in the upper-right corner commutes by passing to connected-components;

the hexagons (I) and (II) evidently commute by construction. This shows that the original morphism is indeed isomorphic to the image of the constructed one. □

Remark 6.3.2. Lemma 6.3.1 shows, in other words, that each morphism of the tensor product factors as a morphism in the image of $E \times_{\mathfrak{C}} (\mathbf{2} \pitchfork F) \to \mathbf{2} \pitchfork (E \otimes_{\mathfrak{C}} F)$ pre- and post-composed with isomorphisms. Each of these three is determined by the data of the original morphism of the tensor. In this sense, Lemma 6.3.1 is a "Factorization Lemma."

Corollary 6.3.3. *Under the same hypotheses, the canonical map* $E \otimes_{\mathfrak{C}} (\mathbf{2} \pitchfork F) \to \mathbf{2} \pitchfork (E \otimes_{\mathfrak{C}} F)$ *is essentially surjective.*

Proof. The map $E \times_{\mathfrak{C}} (\mathbf{2} \pitchfork F) \to E \otimes_{\mathfrak{C}} (\mathbf{2} \pitchfork F)$ inverts the cartesian morphisms of the domain. Thus, there is an induced map from the tensor product as in the statement. The previous lemma is precisely the statement that it is essentially surjective. □

Lemma 6.3.1 and its corollaries show that the canonical map of cotensors

$$\Upsilon\colon E \otimes_{\mathfrak{C}} (\mathbf{2} \pitchfork F) \longrightarrow \mathbf{2} \pitchfork (E \otimes_{\mathfrak{C}} F)$$

is essentially surjective if E is filtered in the sense of Definition 4.4.8. But under these hypotheses, this canonical morphism is also a weak equivalence.

Lemma 6.3.4. *If* $E\colon \mathfrak{E} \to \mathfrak{C}$ *as above is 2-filtered by opcartesian arrows, the canonical map*

$$\Upsilon\colon E \otimes_{\mathfrak{C}} (\mathbf{2} \pitchfork F) \longrightarrow \mathbf{2} \pitchfork (E \otimes_{\mathfrak{C}} F)$$

is full.

Proof. An arrow between the images of (A, u) and (B, v) under Υ will be a commutative square in the target taking the following form. The image of (A, u) and (B, v) are the horizontal outside spans; the components of the morphism in the target are the vertical outside spans; the other interior arrows are any that compose and then relate the resulting compositions.

$$\begin{tikzcd}
(A,X) \arrow[r, "1"] \arrow[d, "{(s,t)}"'] & (A,X) \arrow[r, "{(1,u)}"] & (A,Y) \arrow[d, "{(p,q)}"] \\
(C,P) \arrow[d, "{(e,f)}"'] & (I,J) \arrow[l, "{(a,b)}"'] \arrow[r, "{(i,j)}"] \arrow[u, "{(pi,k)}"] \arrow[d, "{(gi,fb)}"'] & (D,Q) \arrow[d, "{(g,h)}"] \\
(B,Z) \arrow[r, "1"'] & (B,Z) \arrow[r, "{(1,v)}"'] & (B,W)
\end{tikzcd}$$

Without loss of generality, the legs (p, q) and (s, t) of the components of the morphism are cartesian. And note that by definition of the relation, the morphisms sa and pi are opcartesian; and that the morphisms tb and k are cartesian. Now, $j\colon J \to Q$ factors as $j = \sigma(Fj, Q)r$ for a vertical lift $r\colon J \to F(j)^*Q$ in the fiber over FJ. Thus, the horizontal arrows of the commutative square

$$\begin{tikzcd}
X \arrow[d, "u"'] \arrow[dr, phantom, "="] & J \arrow[l, "k"'] \arrow[r, "fb"] \arrow[d, "r"'] \arrow[dr, phantom, "="] & Z \arrow[d, "v"] \\
Y & F(j)^*Q \arrow[l, "q\sigma(Fj, Q)"] \arrow[r, "h\sigma(Fj, Q)"'] & W
\end{tikzcd}$$

define a morphism of the domain of Υ in the form of a span

$$(A, u) \xleftarrow{(pi, (k, q\sigma))} (I, r) \xrightarrow{(gi, (fb, h\sigma))} (B, v).$$

That the components of the image of this span under Υ are equivalent to the components of the given morphism in the target, displayed above, is straightforward to establish using the given morphisms. $\square$

Lemma 6.3.5. *If $E\colon \mathfrak{E} \to \mathfrak{C}$ as above is 2-filtered by opcartesian arrows, the canonical map*

$$\Upsilon\colon E \otimes_{\mathfrak{C}} (\mathbf{2} \pitchfork F) \longrightarrow \mathbf{2} \pitchfork (E \otimes_{\mathfrak{C}} F)$$

is faithful.

Proof. Take two objects of the domain (A, u) and (B, v) and two morphisms between them, represented by the spans

$$(A, u) \xleftarrow{(h, i, j)} (C, z) \xrightarrow{(k, p, q)} (B, v) \qquad (A, u) \xleftarrow{(f, a, b)} (D, w) \xrightarrow{(g, m, n)} (B, v)$$

Suppose that the images of these morphisms under Υ are equal. That is, the respective components of the two resulting morphisms under Υ are related in the manner specified by the calculus of fractions. This means that there are spans from vertices (M, N) and (R, S) in the following figure, making four commutative squares

$$
\begin{tikzcd}
 & (D,V) \arrow[dl, "{(f,b)}"'] \arrow[dr, "{(g,n)}"] & \\
(A,Y) & (M,N) \arrow[u] \arrow[d] & (B,W) \\
 & (C,Q) \arrow[ul, "{(h,j)}"] \arrow[ur, "{(k,q)}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (D,U) \arrow[dl, "{(f,a)}"'] \arrow[dr, "{(g,m)}"] & \\
(A,X) & (R,S) \arrow[u, "{(l,r)}"] \arrow[d, "{(s,t)}"'] & (B,Z) \\
 & (C,P) \arrow[ul, "{(h,i)}"] \arrow[ur, "{(k,p)}"'] &
\end{tikzcd}
$$

where the composites making the left-hand square in each diagram are cartesian. Now, in fact, only the figure at the right matters for the purpose of constructing a span relating the original arrows. The arrows l and s suffice for the E-component. The $\mathbf{2} \pitchfork F$-component requires more care. For this, take r and t and factor the composites with w and z respectively as a vertical followed by a chosen cartesian morphism as in the diagram

$$
\begin{tikzcd}
U \arrow[d, "w"'] & & S \arrow[ll, "r"'] \arrow[dl, "c"'] \arrow[dr, "d"] \arrow[rr, "t"] & & P \arrow[d, "z"] \\
V & \cdot \arrow[l, "{\sigma(F(wr),V)}"] & & \cdot \arrow[r, "{\sigma(F(zt),Q)}"'] & Q
\end{tikzcd}
$$

But of course the two chosen cartesian morphisms fit into a figure together with b and j over a commutative square of $\mathfrak{C}$. Since all these morphisms are cartesian, the domains of the chosen cartesian arrows are isomorphic; the isomorphism commutes with the vertical fills c and d by uniqueness. But vertical isomorphims are cartesian; so, effectively, this induced isomorphism can be ignored. In any event, the figure immediately above makes two commutative squares with the center arrow vertical with respect to the fibration F; each horizontal span consists of cartesian arrows. And, together with the lifted isomorphism, the chosen cartesians on the bottom make commutative squares with b and j on the one hand and with u and q on the other. Thus, the original morphisms of the domain are related by the span

$$
(D,w) \xleftarrow{(l,r,\sigma)} (R,c) \xrightarrow{(s,t,\sigma)} (C,z)
$$

as can be seen by a computation from the constructions given in the proof. $\square$

Theorem 6.3.6. *If $E\colon \mathfrak{E} \to \mathfrak{C}$ as above is 2-filtered by opcartesian arrows, the canonical map*

$$\Upsilon\colon E \otimes_{\mathfrak{C}} (\mathbf{2} \pitchfork F) \longrightarrow \mathbf{2} \pitchfork (E \otimes_{\mathfrak{C}} F)$$

is a weak equivalence.

Proof. Lemma 6.3.1 and Lemmas 6.3.4 and 6.3.4 show that cotensors are preserved up to equivalence. ☐

6.4 Preservation of Cotensors: Internalization

The results of the previous section can be translated into the internal category theory of an exact category $\mathscr{E}$ with pullback-stable coequalizers of reflexive pairs. For this elementary development, fix throughout $e\colon \mathcal{E} \to \mathcal{C}$, an internal discrete 2-opfibration; and let $f\colon \mathcal{F} \to \mathcal{C}$ denote an internal discrete 2-fibration, each as in Definition 2.2.15. The ideas for the internalization are already in the foregoing proofs and the internalization itself is purely technical. For this reason, here is proved the essential surjectivity, while the proof of fully faithful is mostly left to the reader.

First observe that the iso-construction of Construction 3.1.1 can be applied to any internal arrow category, yielding the object $\mathbf{Iso}(\mathbb{C}^{\mathbf{2}})$, for any internal category $\mathbb{C}$. In the case of $\mathscr{E} = \mathbf{Set}$, this object will consist of commutative squares

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\cong"'] \arrow[dr, phantom, "="] & \cdot \arrow[d, "\cong"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

with the two vertical sides isomorphisms. Think of the top horizontal arrow as the domain and the bottom as the codomain. In more detail, consider the following.

Construction 6.4.1. *In the internal case, $\mathbf{Iso}(\mathbb{C}^{\mathbf{2}})$ can be given in terms of $\mathbf{Iso}(\mathbb{C})$. That is, it occurs as the corner object of the pullback*

$$\begin{tikzcd} \mathbf{Iso}(\mathbb{C}^{\mathbf{2}}) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \times_{C_0} \mathbf{Iso}(\mathbb{C}) \arrow[d, "\circ"] \\ \mathbf{Iso}(\mathbb{C}) \times_{C_0} C_1 \arrow[r, "\circ"'] & C_1 \end{tikzcd}$$

by restricting the composition of $\mathbb{C}$ *to the subobject* $\mathbf{Iso}(\mathbb{C})$. *Consistent with Construction 3.1.1, declare the "domain" map to be the composite projection*

$$\pi_1\pi_2 \colon \mathbf{Iso}(\mathbb{C}^{\mathbf{2}}) \to C_1$$

and the codomain to be the composite projection $\pi_2\pi_1$.

Now, use the notation $S = S_{e,f}$ and $\Sigma = \Sigma_{e,f}$ for the constructions from §5.3.1. Additionally, let $q \colon S \to (\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_1$ denote the quotient map to the object of morphisms of the tensor product in $\mathscr{E}/E_0 \times E_0$ as in Definition 5.2.8. The following development constructs a generalized object of $\mathbf{Iso}((\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})^{\mathbf{2}})$. By the construction above, this can be given by specifying two morphisms to the object of isomorphisms and two to $(\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_1$, all mimicking in an elementary the construction of Lemma 6.3.1. These morphisms given over the course of the subsequent three constructions. As set-up, establish the following notation. Declare

1. $h := \rho\pi_1 \colon S \to E_1$
2. $k := \sigma\pi_1 \colon S \to F_1$
3. $j := \pi_1\pi_2 \colon S \to E_1$
4. $g := \pi_2\pi_2 \colon S \to F_1$.

And set

1. $x := d_1 h$ and $a := d_0 h = d_0 f$
2. $y := d_1 k$ and $b := d_0 k = d_0 g$
3. $z := d_1 f$ and $w := d_1 g$.

Thus, set-theoretically, S is interpreted as yielding a span of generalized internal objects and arrows of the form

$$(x, y) \xleftarrow{(h,k)} (a, b) \xrightarrow{(j,g)} (z, w).$$

as in the set-up preceding Lemma 6.3.1. Now, by the filteredness assumption, Lemma 5.3.4 implies that there is a regular epimorphism $p \colon Z \to S$ and opcartesian generalized arrows $r, l \colon Z \to F_1$, appropriately composable with j, and a generalized 2-cell $\theta \colon Z \to F_1$ with $\theta \colon l \circ jp \Rightarrow r$. This cell plays a crucial role in what follows as it did in the proof of the Factorization Lemma 6.3.1.

Construction 6.4.2. *For the first map to* $\mathbf{Iso}(\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})$*, let* ϕ *denote the morphism*

$$\phi := \langle l \circ hp, \sigma\langle e_1 l, bp\rangle \circ k\rangle \colon Z \to E_1 \times_{C_1} F_1.$$

This arrow corresponds to the non-identity side of the leftmost vertical span in the last diagram in the proof of Lemma 6.3.1. Now, let ψ *denote the arrow*

$$\psi := \langle d_0 l, i, d_0\sigma\langle e_1 l, bp\rangle\rangle \colon Z \to \Sigma.$$

This arrow corresponds to the identity leg of the same span. Thus, by construction and the normalization of the cleavage and opcleavage, the outside of the following diagram commutes:

$$\begin{tikzcd}
Z \arrow[drr, bend left, "\phi"] \arrow[dr, dashed] \arrow[ddr, bend right, "\psi"'] & & \\
& S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_1 \times_{C_1} F_1 \arrow[d, "d_0 \times d_0"] \\
& \Sigma \arrow[r, "d_0\rho \times d_0\sigma"'] & E_0 \times_{C_0} F_0.
\end{tikzcd}$$

The dashed arrow reconstructs the required span, viewed as a generalized element of S*. Since both legs of the span* $Z \to S$ *are cartesian, this morphism induces one* $Z \to \mathbf{Iso}(\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})$ *by Remark 5.2.24.*

Construction 6.4.3. *For the second map, use the morphism* $\psi \colon Z \to \Sigma$ *from above. As in the proof of Lemma 6.3.1, the cell* $e_1\alpha$ *of* $\mathcal{C}$ *lifts to one* $\widetilde{e_1\alpha}$ *of* $\mathcal{F}$*. And since* $f \colon \mathcal{F} \to \mathcal{C}$ *is a fibration, there is a unique lift* $u \colon Z \to F_1$ *of an identity morphism such that* $t\widetilde{e_1\alpha} = u \circ \sigma\langle e_1 r, wp\rangle$*. Let* χ *denote the morphism*

$$\chi := \langle id_o l, u\rangle \colon Z \to E_1 \times_{C_1} F_1.$$

These fit into the following diagram, whose outside commutes by construction

$$\begin{tikzcd}
Z \arrow[drr, bend left, "\chi"] \arrow[dr, dashed] \arrow[ddr, bend right, "\psi"'] & & \\
& S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_1 \times_{C_1} F_1 \arrow[d, "d_0 \times d_0"] \\
& \Sigma \arrow[r, "d_0\rho \times d_0\sigma"'] & E_0 \times_{C_0} F_0.
\end{tikzcd}$$

The dashed universal arrow is the required span. Followed by the canonical reduction to the tensor product, this gives the required morphism $Z \to (\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_1$*.*

Construction 6.4.4. *For the last required morphism, let ζ denote the morphism*

$$\zeta := \langle d_0 r, i, d_0\sigma\langle e_1 r, wp\rangle\rangle \colon Z \to \Sigma.$$

This is the identity side of the rightmost vertical span in the last diagram of the proof of Lemma 6.3.1. Let ξ denote the morphism

$$\xi := \langle r, \sigma\langle e_1 r, wp\rangle\rangle \colon Z \to E_1 \times_{C_1} F_1.$$

These fit into the following diagram, the outside of which commutes by construction:

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, bend left, "\xi"] \arrow[ddr, bend right, "\zeta"'] & & \\
& S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_1 \times_{C_1} F_1 \arrow[d, "d_0 \times d_0"] \\
& \Sigma \arrow[r, "d_0\rho \times d_0\sigma"'] & E_0 \times_{C_0} F_0
\end{tikzcd}$$

The dashed arrow thus exists. And since each side of the span represented by this arrow is cartesian, this arrow induces the last required morphism $Z \to \mathbf{Iso}(\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})$ by Remark 5.2.24.

Lemma 6.4.1. *The three induced maps given in Constructions 6.4.2, 6.4.3, and 6.4.4 compose and thus yield a morphism to the object of isomorphisms of the tensor as in the diagram*

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "\langle\langle\psi,\chi\rangle,\langle\zeta,\xi\rangle\rangle"] \arrow[ddr, bend right, "\langle\langle\psi,\phi\rangle,p\rangle"'] & & \\
& \mathbf{Iso}((\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})^{\mathbf{2}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_1 \times_{(\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_0} \mathbf{Iso}(\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F}) \arrow[d, "\circ"] \\
& \mathbf{Iso}(\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F}) \times_{(\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_0} (\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_1 \arrow[r, "\circ"'] & (\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_1
\end{tikzcd}$$

with the pullback square as appearing in Construction 6.4.1.

Proof. Let V denote the corner object of the following pullback as in Construction 5.2.2. The object Z then admits two morphisms to V induced by universality as in

$$\begin{tikzcd}
Z \arrow[dr, dashed, "x"] \arrow[drr, bend left, "\langle\langle l,\sigma\rangle, p\rangle"] \arrow[ddr, bend right, "\langle p,\langle\psi,\phi\rangle\rangle"'] & & \\
& V \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & (E_1 \times_{C_0} F_1) \times_{E_0 \times_{C_0} F_0} S \arrow[d, "- \circ s"] \\
& \Sigma \times_{E_0 \times_{C_0} F_0} S \arrow[r, "s \circ -"'] & E_1 \times_{C_1} F_1
\end{tikzcd}$$

and

$$\begin{tikzcd}
Z \arrow[dr, dashed, "y"] \arrow[drr, bend left, "\langle\chi,\langle\zeta,\xi\rangle\rangle"] \arrow[ddr, bend right, "\langle p,\langle\psi,\chi\rangle\rangle"'] & & \\
& V \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \times_{C_0} S \arrow[d, "- \circ s"] \\
& \Sigma \times_{C_0} S \arrow[r, "s \circ -"'] & C_1
\end{tikzcd}$$

The induced composition $c\colon V \to S$ of Construction 5.2.2 coequalizes x and y, as can be calculated directly by checking on components. Now, recall that, by construction of c, the composite qc factors through the composition morphism

$$\circ\colon (\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_1 \times_{(\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_0} (\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_1 \to (\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_1.$$

via the reduction map

$$v\colon V \to (\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_1 \times_{(\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_0} (\mathcal{E} \otimes_{\mathcal{C}} \mathcal{F})_1$$

of 5.2.13. Thus, the morphism $\circ v$ coequalizes x and y. By construction of x, y and v, this implies that the outside of the diagram in the statement commutes, as required. $\square$

Now, since $u\colon S \to F_1$ is a vertical morphism of $f\colon \mathcal{F} \to \mathcal{C}$, it factors through the object of objects of the arrow category of f, namely, $(\mathbf{2} \pitchfork f)_0$, as given in the pullback

$$\begin{tikzcd}
(\mathbf{2} \pitchfork f)_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_1 \arrow[d, "f_1"] \\
C_0 \arrow[r, "i"'] & C_1
\end{tikzcd}$$

as in Example 4.1.3. Now, this means that, by construction, the outside square in the following diagram commutes, yielding a canonical morphism indicated by the dashed arrow:

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\langle u,\theta\rangle"] \arrow[drr, bend left, "\theta"] \arrow[ddr, bend right, "u"'] & & \\
& B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Iso}((\mathcal{E}\otimes_{\mathcal{C}}\mathcal{F})^{\mathbf{2}}) \arrow[d, "d_0"] \\
& E_0 \times_{C_0} (\mathbf{2} \pitchfork f)_0 \arrow[r] & (\mathcal{E}\otimes_{\mathcal{C}}\mathcal{F})_1
\end{tikzcd}$$

Now the map indicated by 'd_0' above is the codomain morphism $\pi_1\pi_2$ of the iso object viewed as a subobject of the internal arrow category. Thus, $d_0\theta = \langle \psi, \chi\rangle$. On the other hand, the domain morphism

$$\pi_2\pi_1 \colon \mathbf{Iso}((\mathcal{E}\otimes_{\mathcal{C}}\mathcal{F})^{\mathbf{2}}) \to (\mathcal{E}\otimes_{\mathcal{C}}\mathcal{F})_1$$

has $d_1\theta = qp$. Thus, the constructions and foregoing discussion proves the following result.

Theorem 6.4.2 (Factorization Lemma II)**.** *The diagram of maps from the above discussion*

$$\begin{tikzcd}
& Z \arrow[dl, "\langle u,\theta\rangle"'] \arrow[dr, "qp"] & \\
B \arrow[r] & \mathbf{Iso}((\mathcal{E}\otimes_{\mathcal{C}}\mathcal{F})^{\mathbf{2}}) \arrow[r, "d_1"'] & (\mathcal{E}\otimes_{\mathcal{C}}\mathcal{F})_1.
\end{tikzcd}$$

is commutative.

Canonical Map of Cotensors is Essentially Surjective

Let I denote the image object of the arrow $B \to (\mathcal{E}\otimes_{\mathcal{C}}\mathcal{F})_1$ at the base of the triangle above. Thus, the arrow $B \to (\mathcal{E}\otimes_{\mathcal{C}}\mathcal{F})_1$ factors as mp for a regular epimorphism $p\colon B \to I$ and a monic $m\colon I \to (\mathcal{E}\otimes_{\mathcal{C}}\mathcal{F})_1$.

Lemma 6.4.3. *The commutative square*

$$\begin{tikzcd}
S \arrow[r, "1"] \arrow[d, "p\langle u,\theta\rangle"'] \arrow[dr, phantom, "="] & S \arrow[d, "q"] \\
I \arrow[r, tail, "m"'] & (\mathcal{E}\otimes_{\mathcal{C}}\mathcal{F})_1
\end{tikzcd}$$

of arrows in the triangle immediately above is a pullback. In particular, $p\langle u,\theta\rangle$*, and hence the monic arrow* $m\colon I \to (\mathcal{E}\otimes_{\mathcal{C}}\mathcal{F})_1$*, are regular epimorphisms.*

Proof. Straightforward computation. The arrow m is regular epi since q and $p\langle u,\theta\rangle$ are. Notice that since m is thus monic and regular epi it is an isomorphism by Lemma 2.3.3. $\square$

Corollary 6.4.4. *The canonical map of cotensors*

$$\mathcal{E}\otimes_{\mathcal{C}}(\mathbf{2}\pitchfork f)\longrightarrow(\mathcal{E}\otimes_{\mathcal{C}}\mathcal{F})^{\mathbf{2}}$$

is essentially surjective.

Proof. Lemma 6.4.3 immediately above shows that the arrow running along the top row of

$$\begin{tikzcd}
B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Iso}((\mathcal{E}\otimes_{\mathcal{C}}\mathcal{F})^{\mathbf{2}}) \arrow[r, "d_1"] \arrow[d, "d_0"] & (\mathcal{E}\otimes_{\mathcal{C}}\mathcal{F})_1 \\
E_0\times_{C_0}(\mathbf{2}\pitchfork f)_0 \arrow[r] & (\mathcal{E}\otimes_{\mathcal{C}}\mathcal{F})_1 &
\end{tikzcd}$$

is a regular epimorphism, in the sense that the codomain $(\mathcal{E}\otimes_{\mathcal{C}}\mathcal{F})_1$ is isomorphic to its image I. This is precisely the condition required by Definition 3.1.16. $\square$

Theorem 6.4.5. *If* $e\colon\mathcal{E}\to\mathcal{C}$ *is 2-filtered as in Definition 4.4.8, then the induced tensor product 2-functor*

$$\mathcal{E}\otimes_{\mathcal{C}} -\colon \mathfrak{DFib}(\mathcal{C})\to\mathfrak{K}$$

preserves up to equivalence finite products, equalizers, and cotensors with **2***.*

Proof. Theorem 6.2.6 shows that $\mathcal{E}\otimes_{\mathcal{C}} -$ preserves finite conical limits. The previous result shows that the canonical internal functor of cotensors is internally essentially surjective. That this is also internally fully faithful in the sense of Definition 3.1.14 involves showing that a certain square is a pullback. That this is the case is another exercise in cone-building in the internal category theory of $\mathscr{E}$ inspired by the proofs of Lemmas 6.3.4 and 6.3.5. $\square$

Chapter 7

Conclusion: Future Work

7.1 Limit Preservation

It is clearly unsatisfactory not to have a complete statement as to whether the tensor product

$$E \otimes_{\mathfrak{C}} - \colon \mathfrak{DFib}(\mathfrak{C}) \to \mathfrak{Cat}$$

is finite-limit preserving if $E \colon \mathfrak{E} \to \mathfrak{C}$ is 2-filtered with respect to opcartesian morphisms. The results of Chapter 6 do prove that $E \otimes_{\mathfrak{C}} -$ preserves the terminal object, binary products, equalizers, and cotensors with $\mathbf{2}$, but only up to equivalence. In particular, it is the terminal object and cotensors with $\mathbf{2}$ that are only preserved up to equivalence. Where the present account falters is in the question of whether or not the construction of finite 2-limits from these primitive 2-limit shapes is also preserved by the tensor. Only in the case that it is can it be stated with confidence that finite 2-limits are preserved. It is not immediately clear that the tensor does preserve the construction. Thus, the obvious next step is to inquire into whether or not the construction of 2-limits and of pseudo-limits is preserved by the tensor product. Then an internal version of the same result should be sought.

7.2 Bicategories

Something about the fact that the limit preservation mentioned above is only an equivalence suggests that perhaps an "enriched" approach to 2-dimensional category theory is not the right one. Generally speaking "preservation" in enriched category theory means "up to isomorphism." Thus, the focus in the latter chapters of this thesis on 2-functors and 2-naturality, which are enriched notions, has a kind of artificiality to it. Rather it is suspected that the theory developed here is a fragment of a more genuinely *bicategorical* approach. What this looks like however is not yet clear.

Some musings, however, might be appropriate. For example, the domain of our representations might be boosted up to a bicategory $\mathcal{B}$ so that under consideration would be homomorphisms $E \colon \mathcal{B} \to \mathfrak{Cat}$ and $F \colon \mathcal{B}^{op} \to \mathfrak{Cat}$. The question would then be as to whether or not there is a tensor extension

$$E \otimes_{\mathcal{B}} - \colon \mathfrak{Hom}(\mathcal{B}^{op}, \mathfrak{Cat}) \to \mathfrak{Cat}$$

as some kind of bicolimit; and how or whether its exactness properties can be characterized by filteredness conditions on the bicategory of elements construction associated to E as in §3.3 of [Buc14]. This construction involves the tricategorical structure on the collection of bicategories.

But it is not clear that the strict 2-category $\mathfrak{Cat}$ is the correct target of truly bicategorical representations. That is, would not a representation of a bicategory actually be a homomorphism into some "base" bicategory? The question is then what this would be. It might be the bicategory $\mathfrak{Prof}$ of categories, profunctors and their transformations. Then a representation of a bicategory would be a homomorphism $E\colon \mathcal{B} \to \mathfrak{Prof}$. One would then have to describe a tensor extension as a bicolimit in $\mathfrak{Prof}$. One would like a concrete computation. Of course $\mathfrak{Prof}$ is the bicategorical part of the double category $\mathbb{P}\mathbf{rof}$ of categories, functors, profunctors and their transformations. So, alternatively, if one views $\mathbb{S}\mathbf{et}$, the double category of sets, functions and spans, as a sort of "set-theoretic" base double category, then perhaps $\mathbb{C}\mathbf{at}$, the double category of categories, functors and spans, is the "category-theoretic" base double category. So, on this view, the bicategorical structure of categories and spans might provide the correct setting for representations of bicategories. Again the main task here would be giving a concrete computation of a tensor product as a bicolimit.

7.3 Further Internalization

One nagging question is about the existence of the tensor product in the internal account of Chapters 5 and 6. It was seen that the tensor exists under the conditions of 2-filteredness and that it was formed through a right calculus of fraction. However, it is not clear that the tensor exists whether or not the discrete 2-opfibration is filtered. That is, without the filteredness, there is no guarantee that the tensor is formed through a right calculus of fractions. It would be nice to be able to give conditions on $\mathscr{E}$ such that some kind of "internal category of fractions" construction can be carried out. The idea of course is that this should construct the "internal colimit." In the 1-dimensional case, a sufficient condition for internal cocompleteness was that the base category have coequalizers of reflexive pairs. Some sufficient condition on $\mathscr{E}$ for internal cocompletess of $\mathfrak{K} = \mathfrak{Cat}(\mathscr{E})$ is needed. This might have the form of further exactness properties or perhaps an axiomatization of some internal "free category modulo relations" construction.

The true goal of the present research was to get a purely elementary account of flatness and filteredness in a suitably exact and cocomplete 2-category on the model of Diaconescu's results generalizing the set-theoretic theory of flatness to elementary toposes. His main tools in constructing the basic objects of the theory (the internal colimit and the tensor product for example) were the *exactness* and *cocompleteness* properties of toposes, namely, that any elementary topos is an exact, hence a regular, category and that any topos admits all finite colimits. One of the problems with working in $\mathfrak{Cat}$ is that if it is to be the "base 2-topos" it is not yet clear how to understand which are the most essential of its exactness properties to be axiomatized in, or perhaps deduced from, general 2-topos axioms such as those of [Web07].

The paper [BG14] studies certain kernel-quotient systems defined on 2-categories as certain weighted diagram shapes and defines notions of regularity and exactness with respect to these kernel-quotient systems. Each kernel-quotient system comes with a natural notion of factorization system on the 2-category. The authors identify several choices in $\mathfrak{Cat}$ that fit this overall pattern. For example, essentially surjective functors on the one hand for the "epimorphism-like" class and on the other hand fully faithful functors form the "monomorphism-like" class; or one could take essentially surjective and full functors on the one hand and faithful functors on the other; or one could take surjective on objects functors on the one hand and fully faithful injective on objects functors on the other. The question, however, as to which choice is suitable for the 2-topos axioms seems to be unaddressed.

7.4 A Tricategory of Category-Valued Pseudo-Profunctors?

Whether or not the desired exactness results will hold in a purely elementary fashion, there are nonetheless interesting questions about the categorical structure of the collection of discrete 2-fibrations over some base and about category-valued pseudo-profunctors more generally.

Recall from §7.8 of [Bor94], for example, that a profunctor (or "distributor") between categories $M\colon \mathscr{C} \looparrowright \mathscr{D}$ is an ordinary functor $M\colon \mathscr{C}^{op} \times \mathscr{D} \to \mathbf{Set}$. Thus, ordinary functors $E\colon \mathscr{C} \to \mathbf{Set}$ are profunctors $E\colon \mathbf{1} \looparrowright \mathscr{C}$ and those $F\colon \mathscr{C}^{op} \to \mathbf{Set}$ are profunctors $F\colon \mathscr{C} \looparrowright \mathbf{1}$.

Profunctors $N\colon \mathscr{B} \looparrowright \mathscr{C}$ and $M\colon \mathscr{C} \looparrowright \mathscr{D}$ compose by a coend formula

$$N \otimes M(B, D) := \int^C N(C, D) \times M(B, C).$$

The tensor notation is justified by the considerations of IX.6 of [Mac98], where the composition of profunctors $E\colon \mathbf{1} \looparrowright \mathscr{C}$ and $F\colon \mathscr{C} \looparrowright \mathbf{1}$ is shown to be isomorphic to the tensor product of E and F as set-valued functors:

$$E \otimes_{\mathscr{C}} F \cong \int^C EC \times FC.$$

Composition of profunctors is associative up to isomorphism in the sense that there are natural isomorphisms

$$P \otimes (N \otimes M) \cong (P \otimes N) \otimes M.$$

This is part of the bicategory structure on $\mathfrak{Prof}$ whose objects are categories, whose morphisms are profunctors, and whose 2-cells natural transformations of profunctors.

A natural question about the work of the present thesis is as to whether the tensor product of a discrete 2-opfibration $E\colon \mathfrak{E} \to \mathfrak{C}$ and a discrete 2-fibration $F\colon \mathfrak{F} \to \mathfrak{C}$ given as

$$E \otimes_{\mathfrak{C}} F := \pi_0\Delta(E, F)[\Sigma^{-1}]$$

is a fragment of a more general composition law for certain category-valued profunctors on 2-categories. Our conjecture is that this is true. In fact, the work of the thesis has suggested that category-valued profunctors can be organized into a *tricategory* with objects small 2-categories whose composition law is given as a generalized bicoend having the tensor product above as a special case.

Tricategories seem first to have been studied in [GPS95]. These are essentially 3-dimensional categories obtained as somehow "weakly enriched over bicategories." The details of this formulation are formidable. Additionally, there is the issue that $\mathfrak{Cat}$ is not regular as a 1-category. In particular, regular epimorphisms are not stable under pullback. This makes trouble for even a definition of a canonical map between the two possible compositions of three category-valued profunctors. However, a relatively recent draft paper [Cor17] provides a calculus of certain bicoends that does give an associativity result that might be of use in this direction.

Bibliography

[AGV72] M. Artin, A. Grothendieck, and J. L. Verdier. *Théorie des Topos et Cohomologie Etale des Schémas (SGA 4) Tome 2*, volume 270 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1972.

[Bar71] M. Barr. Exact categories. In *Exact Categories and Categories of Sheaves*, Springer Lecture Notes in Mathematics 236, pages 1–120. 1971.

[Bar04] T. Bartels. *Higher Gauge Theory I: 2-bundles*. PhD thesis, U. C.-Riverside, 2004. arxiv.org/abs/math/0410328.

[BC04] J. Baez and A. Crans. Higher-dimensional algebra VI: Lie 2-algebras. *Theory Appl. Categ.*, 12:492–528, 2004.

[Bén67] J. Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar I*, Springer Lecture Notes in Mathematics 47, pages 1–77. 1967.

[BG14] J. Bourke and R. Garner. Two-dimensional regularity and exactness. *Jour. Pure Appl. Algebra*, 218:1346–1371, 2014.

[Bir84] G. J. Bird. *Limits in 2-Categories of Locally-Presentable Categories*. PhD thesis, University of Sydney, 1984.

[BL04] J. Baez and A. Lauda. Higher-dimensional algebra v: 2-groups. *Theory Appl. Categ.*, 12:423–491, 2004.

[Bor94] F. Borceux. *Handbook of Categorical Algebra 1: Basic Category Theory*, volume 50 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, Cambridge, 1994.

[BP79] M. Bunge and R. Paré. Stacks and equivalence of indexed categories. *Cahiers de Topologie et Géom. Diff. Catégoriques*, 20:373–399, 1979.

[BS07] J. Baez and U. Schreiber. Higher gauge theory. In A. Davydov, editor, *Categories in Algebra, Geometry and Mathematical Physics*, volume 431 of *Contemp. Math.*, pages 7–30. American Mathematical Society, 2007.

[Buc14] M. Buckley. Fibred 2-categories and bicategories. *Jour. Pure Appl. Algebra*, 218:1034–1074, 2014.

[Cor17] A. S. Corner. A Universal Characterisation of Codescent Objects. *arXiv e-prints*, page arXiv:1709.01332, Sep 2017.

[DDS18a] M.E. Descotte, E. J. Dubuc, and M. Szyld. A construction of certain weak colimits and an exactness property of the 2-category of categories. *Theory Appl. Categ.*, 33:193–215, 2018.

[DDS18b] M.E. Descotte, E. J. Dubuc, and M. Szyld. Sigma colimits in 2-categories and flat pseudofunctors. *Advances in Mathematics*, 333:266–313, 2018.

[Dia73] R. Diaconescu. *Change of Base for Some Toposes*. PhD thesis, Dalhousie University, 1973.

[Dia75] R. Diaconescu. Change of base for toposes with generators. *J. Pure Appl. Alg.*, (6):191–218, 1975.

[DS06] E. Dubuc and R. Street. A construction of 2-filtered bicolimits of categories. *Cahiers de Topologie et Géom. Diff. Catégoriques*, 47:83–106, 2006.

[GPS95] R. Gordon, J. Power, and R. Street. Coherence for tricategories. *Mem. Amer. Math Soc.*, 117(558), 1995.

[Gra66] J. Gray. Fibred and cofibred categories. In *Proc. Conf. on Cat. Alg., La Jolla 1965*. Springer-Verlag, 1966.

[Gra74] J. Gray. *Formal Category Theory: Adjointness for 2-Categories*, volume 391 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1974.

[GZ67] P. Gabriel and M. Zisman. *Calculus of Fractions and Homotopy Theory*, volume 35 of *Ergebnisse der Math.* Springer-Verlag, Berlin, 1967.

[Hun74] T. W. Hungerford. *Algebra*, volume 73 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, 1974.

[Jac99] B. Jacobs. *Categorical Logic and Type Theory*, volume 141 of *Studies in Logic and the Foundations of Math.* North-Holland, Amsterdam, 1999.

[Joh01] P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium, Volume 1*, volume 43 of *Oxford Logic Guides*. Clarendon Press, London, 2001.

[Joh14] P. T. Johnstone. *Topos Theory*. Dover, Minneapolis, 2014.

[Kel82a] G. M. Kelly. *Basic Concepts of Enriched Category Theory*, volume 64 of *London Math. Soc. Lecture Notes Series*. Cambridge University Press, Cambridge, 1982.

[Kel82b] G. M. Kelly. Structures defined by finite limits in the enriched context I. *Cahiers de Topologie et Géom. Diff. Catégoriques*, 23:3–42, 1982.

[Kel89] G. M. Kelly. Elementary observations on 2-categorical limits. *Bull. Austral. Math. Soc.*, 39:301–317, 1989.

[Ken92] J. F. Kennison. The fundamental localic groupoid of a topos. *J. Pure Appl. Algebra*, 77:67–86, 1992.

[KLW93] G. M. Kelly, S. Lack, and R. F. C. Walters. Coinverters and categories of fractions for categories with structure. *Appl. Cat. Structures*, 1:95–102, 1993.

[KS74] G. M. Kelly and R. Street. Review of the elements of 2-categories. In G. M. Kelly, editor, *Category Seminar: Proceedings Sydney Category Theory Seminar 1972 /1973*, Springer Lecture Notes in Mathematics 420, pages 75–103. 1974.

[Lac02] S. Lack. Codescent objects and coherence. *J. Pure Appl. Algebra*, 175:223–241, 2002.

[Law13] F. Lawler. *Fibrations of predicates and bicategories of relations*. PhD thesis, Trinity College, 2013. arxiv.org/abs/1502.08017.

[Laz64] D. Lazard. Sur les modules plats. *C. R. Acad. Sci. Paris*, 258:6313–6316, 1964.

[Mac98] S. Mac Lane. *Category Theory for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, 2nd edition, 1998.

[MLM92] S. Mac Lane and I. Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Springer-Verlag, Berlin, 1992.

[Moe95] I. Moerdijk. *Classifying Spaces and Classifying Topoi*, volume 1616 of *Springer Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1995.

[Sch05] Urs Schreiber. *From Loop Space Mechanics to Nonabelian Strings*. PhD thesis, U. Duisburg-Essen, 2005. arxiv.org/abs/hep-th/0509163.

[Str74] R. Street. Fibrations and Yoneda's lemma in a 2-category. In G. M. Kelly, editor, *Category Seminar: Proceedings Sydney Category Theory Seminar 1972 /1973*, Springer Lecture Notes in Mathematics 420, pages 104–133. 1974.

[Str76] R. Street. Limits indexed by category-valued 2-functors. *J. Pure Appl. Algebra*, 8:149–181, 1976.

[Str80] R. Street. Fibrations in bicategories. *Cahiers de Topologie et Géom. Diff.*, 21:111–160, 1980.

[Web07] M. Weber. Yoneda structures from 2-toposes. *Applied Categorical Structures*, 15:259–323, 2007.