ON THE ROOTS OF INDEPENDENCE POLYNOMIALS

by

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Abstract

The *independence polynomial* of a graph is a polynomial whose coefficients give the number of independent sets of each size. Its roots are called *independence roots*. This thesis explores the analytic properties of independence polynomials and the interactions between these properties and the structure of the corresponding graphs. We begin by applying results that relate the independence roots to the coefficients of independence polynomials of very well-covered graphs. We will explore families of graphs whose independence roots all lie to the left of the imaginary axis (which appears to be most graphs at a first glance) and other families of graphs that have independence roots to the right of this line. We then prove exponential bounds on the maximum modulus of an independence root that a graph of order n can attain. Finally, we find graphs that are *independence equivalent*, that is have equivalent independence polynomial, to a path or a cycle of certain orders.

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Chapter 1

Introduction

Graph polynomials and their roots have arisen in a variety of applied and theoretical settings. *Chromatic polynomials* were introduced in 1912 by Birkhoff [8]; these are functions that count, for each positive integer λ, the number of ways to assign one of λ colours to each vertex such that adjacent vertices receive different colours (the interest arose out of what was known then as the *Four Colour Conjecture*, which claimed that any planar graph could be coloured with four colours).

Another graph polynomial, *all-terminal reliability*, was introduced to model robustness of a network. The salient model had vertices that were always operational, but edges that failed independently with probability $q \in [0, 1]$, and asks the probability that the spanning subgraph of operational edges forms a connected graph, that is, that all the vertices can communicate. The literature on all-terminal reliability is vast (see [32] for an early book on the topic).

Other graph polynomials have been introduced as generating polynomials to deeply explore counting sequences related to various graph parameters. The *domination polynomial* [3] has been proposed to study dominating sets in graphs. The *clique polynomial* [50] has been introduced to explore the number of complete subgraphs of different orders in a graph. Similarly, the neighbourhood polynomial [25] has been advanced to investigate subsets of vertices that have a common neighbour. Moreover, there has been much recent interest in independence polynomials of graphs, the generating polynomials for the sequence of numbers of independent sets of each size. Gutman and Harary [46] were the first to explore this generalization when they defined the independence polynomial in 1983. The independence polynomial is also known as the *partition function for the hard-core self-repulsion and pair interaction model* in statistical physics, an application explored by Scott and Sokal [78] using a surprising connection between the Lovász Local Lemma and the roots of independence polynomials of graphs.

In all cases, the roots of graph polynomials have been a centre of study, both for what they imply directly about the polynomial and their importance as to what they say about the sequence of coefficients. Birkhoff's motivation was to prove, with purely analytical methods, that 4 was never a root of the chromatic polynomial of a planar graph. This method was ultimately unsuccessful in proving the Four Colour Theorem, but the chromatic polynomial has proved to be one of the best-studied objects in graph theory [37]. Indeed the roots encode the *chromatic number* of the graph as the least positive integer that is not a root. In 1992, the roots of reliability polynomials were explored [15], leading to the well-known Brown-Colbourn conjecture that the roots of reliability polynomials all lie in the unit disk centred at 0. In spite of considerable evidence [15, 87, 30], the conjecture was shown to fail, but only by the slimmest of margins [76].

Work on the roots of domination, clique, and neighbourhood polynomials can be found in [26, 50, 25]. The research literature on the roots of independence polynomials (that is, on *independence roots*) is extensive. What follows in this thesis is our contributions to the study of independence polynomials and their roots.

1.1 Graph Theory Background

Throughout this thesis a *graph* G is the pair $(V(G), E(G))$, where $V(G)$ is the *vertex set* and $E(G)$ is the *edge set*. Each *edge* consists of unordered pairs of vertices. All graphs considered in this thesis are finite, undirected, and simple (that is, with no loops or parallel edges). The *order* of G is $|V(G)|$ and the *size* of G is $|E(G)|$. If two vertices u and v are joined by an edge, i.e. if $uv \in E(G)$, then we say u and v are *adjacent* and we write $u \sim v$ (equivalently $v \sim u$ since our graphs are undirected). A vertex that is the endpoint of an edge is said to be *incident* with the edge. For $v \in V(G)$, the set $N(v) = \{u \in V(G) : v \sim u\}$ is the *open neighbourhood* of v and $N[v] = N(v) \cup \{v\}$ is the *closed neighbourhood* of v. The *degree* of a vertex v, denoted $\deg(v)$, is defined by $\deg(v) = |N(v)|$. A vertex of degree 0 is called an *isolated vertex* and a vertex of degree 1 is called a *leaf*. The *maximum degree* of a graph is denoted $\Delta(G)$, and the *minimum degree* is denoted by $\delta(G)$.

A graph $H = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E'$. The graph H is an *induced subgraph* of G if the further condition that for all $u, v \in V'$,

u and v are adjacent in H if and only if they are adjacent in G. Subgraphs may be thought of as choosing vertices *and* edges, whereas induced subgraphs may be thought of as choosing vertices with no choice for edges. For $S \subseteq V(G)$, let $G - S$ be the graph obtained from G by deleting all vertices of S as well as their incident edges, i.e. the induced subgraph of G with vertex set $V(G) \setminus S$. If $S = \{v\}$, we will use the shorthand $G - v$ to denote $G - \{v\}$. A subset S of the vertex set of a graph G with $m = |S|$ that induces a graph with all $\binom{m}{2}$ possible edges is called a *clique*. On the other hand, if S induces a subgraph with no edges, then S is called an *independent set* (much more on these soon). A graph is *bipartite* if its vertex set can be partitioned into one or two independent sets.

A graph is *connected* if there is a path between every pair of vertices in its vertex set and *disconnected* otherwise. A disconnected graph has at least two maximal connected induced subgraphs called *components*.

Two graphs G and H are *isomorphic* if there is a bijection $f : V(G) \to V(H)$ such that $u \sim v$ if and only if $f(u) \sim f(v)$. Such a function is called an *isomorphism* and we write $G \cong H$ for G being isomorphic to H. We distinguish between isomorphic graphs only if the labelling of the vertices has some importance. The labels are not usually of concern for us as the independence polynomial does not take vertex labellings into account, so we consider isomorphic graphs essentially equal and hence sometimes write $G = H$. Isomorphism will play a role in Chapter 5.

The *complement* of G, denoted $\overline{G}$, is the graph on the same vertex set with u and v adjacent in G if and only if they are nonadjacent in $\overline{G}$. The *disjoint union* of two graphs G and H on disjoint vertex sets, denoted $G \cup H$, is the graph $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$ (if G and H are not disjoint, we use disjoint isomorphic copies of each). The *lexicographic product* (or graph substitution) is defined as follows. Given graphs G and H such that $V(G) = \{v_1, v_2, ..., v_n\}$ and $V(H) = \{u_1, u_2, ..., u_k\}$, the lexicographic product of G and H, which we will denote $G[H]$, is the graph such that $V(G[H]) = V(G) \times V(H)$ and $(v_i, u_l) \sim (v_j, u_m)$ if $v_i \sim_G v_j$ or $i = j$ and $u_l \sim_H u_m$. The graph $G[H]$, can be thought of as substituting a copy of H for each vertex of G. This intuitive way of understanding the product is why it is sometimes referred to as graph substitution. Other graph products and operations, like the join and corona will be used throughout but will be defined as needed.

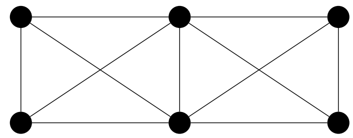
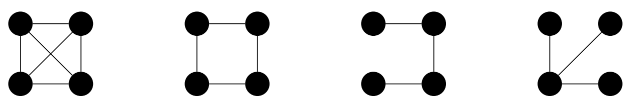


Figure 1.1: The lexicographic product $P_3[K_2]$.

Common graphs that we will use in this thesis include the *complete graph* on n vertices, denoted K_n (the graph whose vertex set is a clique), the *cycle* on n vertices, denoted C_n, the *path* on n vertices, denoted P_n, and the *complete k-partite graph* $K_{n_1,n_2,\ldots,n_k}$. One special case is the graph $K_{1,n-1}$, the *star* on n vertices. The *empty graph* on n vertices is the graph with no edges and is equal to $\overline{K_n}$. A *forest* is an acyclic graph (one without C_k subgraphs) and a connected forest is called a *tree*.



(a) K_4 (b) C_4 (c) P_4 (d) $K_{1,3}$

Figure 1.2: All connected graphs of order 4.

The reader is referred to [89] for any graph theory definitions or background omitted in this brief section.

1.2 Sequences, Polynomials, and Roots

As our objects of study are polynomials, we shall need to rely on various definitions and notions from algebra and analysis. The coefficients, degree, and leading and constant terms should be well-known to the reader. A *real polynomial* is one whose coefficients are all real, and such a polynomial is *standard* if it is either identically 0 or has positive leading coefficient. The *discriminant* of a polynomial with roots $r_1, r_2, \ldots, r_n$ is defined by $\prod_{i<j}(r_i - r_j)$. The discriminant is useful when considering small degree polynomials as there are expressions for it as a function of the coefficients of the polynomial [53].

A *root* (or *zero*) r of a polynomial $p(z)$ is a complex number such that $p(r) = 0$. The *left half-plane* (LHP) is the set of complex numbers whose real part is at most 0 (the *right half-plane* is defined analogously). A polynomial is *stable* if and only if all of its roots are in the left half-plane. Such polynomials are important in many applied settings [30]. The name "stable" comes from the fact that if a polynomial associated with a system of ordinary differential equations has all roots in the LHP, then the system is stable, that is the solution converges to an equilibrium [41].

A sequence $\langle a_1, \ldots, a_l \rangle$ of real numbers is *unimodal* if there is a positive integer k such that $a_0 \leq a_1 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_l$ and it is *log-concave* if $a_k^2 \geq a_{k-1}a_{k+1}$ for all $k = 1, 2, \ldots, l-1$ (the sequence is *strictly log-concave* if strict inequality always holds). Any log-concave sequence of positive numbers is unimodal. We say that a real polynomial is unimodal, log-concave or strictly log-concave if and only if its sequence of coefficients has the salient property.

1.3 A Retrospective of Independence Polynomials and Their Roots

Given a (finite, undirected) graph G, we formally define the *independence polynomial* of G, denoted $i(G, x)$, by

$$i(G, x) = \sum_{k=0}^{\alpha(G)} i_k x^k,$$

where i_k is the number of independent sets of size k in G, that is, the number of subsets of size k that do not contain any edge of G (by convention, every graph has $i_0 = 1$, as the empty set is always trivially independent). For every graph G of order n, it is clear that $0 \leq i_k \leq \binom{n}{k}$. As any subset of an independent set is also independent by definition so there cannot be any "internal zero coefficients" in an independence polynomial. More precisely, if $i_k \geq 1$, then $i_{k-j} \geq 1$ for all $j = 0, 1, \ldots k$.

For all $n \geq 1$, the complete graph, K_n, has $i(K_n, x) = 1 + nx$ since any subset of $V(K_n)$ on at least two vertices will induce a subgraph with an edge. At the opposite end of the spectrum we have $i(\overline{K_n}, x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1 + x)^n$. In both cases, the independence polynomials were easy to compute and we can easily see that $-\frac{1}{n}$ and -1 are the independence roots of K_n and $\overline{K_n}$ respectively.

Independence polynomials are not always straightforward to compute; in general it is an NP-hard problem to determine the independence number of a graph [43] (i.e. the degree of the independence polynomial) and therefore it is an NP-hard problem to compute the independence polynomial of a graph (even at any nonzero complex number c [48]).

One very important tool for studying independence polynomials is the following result that allows for independence polynomials to be computed recursively (but not efficiently).

Proposition 1.3.1 ([46])**.** *If G and H are graphs and $v \in V(G)$, then*

i) $i(G, x) = i(G - v, x) + x \cdot i(G - N[v], x)$, *and*

ii) $i(G \cup H, x) = i(G, x)i(H, x)$. □

Hoede and Li [50] generalized this formula to the deletion of a clique of any size and also gave a version for the deletion of an edge. Gutman [44] gave a recursive identity for the independence polynomial of a tree in terms of paths between two vertices in the tree.

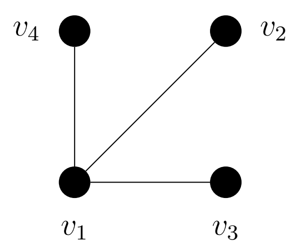


Figure 1.3: The graph $K_{1,3}$.

As an application of Proposition 1.3.1, let's calculate the independence polynomial of the star $K_{1,3}$. Let K_0 denote the graph with empty vertex set and note that $i(K_0, x) = 1$. From Proposition 1.3.1,

$$\begin{aligned} i(K_{1,3}, x) &= i(K_{1,3} - v_1, x) + x \cdot i(K_{1,3} - N[v_1], x) \\ &= i(\overline{K_3}, x) + x \cdot i(K_0, x) \\ &= (1 + x)^3 + x \cdot 1 \\ &= 1 + 4x + 3x^3 + x^3. \end{aligned}$$

Proposition 1.3.1 will be very important for our results in Chapters 4 and 5. The proof of this result is intuitive and instructive, so we outline the idea here. An independent set in G either contains or does not contain v. All independent sets not containing v are be counted by $i(G - v, x)$. All independent sets containing v consist of the union of $\{v\}$ and an independent set in G that does not contain any neighbour of v, so these are enumerated by $x \cdot i(G - N[v], x)$.

Even in cases when we can compute the polynomial, their roots are often difficult to find. From above, $i(K_{1,3}, x) = 1+4x+3x^2+x^3$. The roots of $i(K_{1,3}, x)$ can be found using the cubic formula, but it is messy. Instead we will remark on the nature of these roots as they differ from the independence roots that we found for the complete and empty graphs. The Rational Roots Theorem says that any rational independence root of $K_{1,3}$ must have numerator and denominator that divide 1. Therefore, the only possible rational independence roots of $K_{1,3}$ are 1 and -1, but neither are roots. Therefore, $K_{1,3}$ has no rational roots. Since $i(K_{1,3}, x)$ is a polynomial with real coefficients, every root comes with its complex conjugate pair, so $K_{1,3}$ either has three irrational roots, or one irrational root and two nonreal roots. Moreover, in chapter 10 of [53], the discriminant of a cubic polynomial is given as a function of its coefficients and it is shown that if this value is negative, then the polynomial has one real root and two nonreal roots. The discriminant of $i(K_{1,3}, x)$ is -31 so $K_{1,3}$ has two nonreal roots and one real (but irrational) root. Plots of independence roots suggest much intriguing structure (see Figures 1.4 and 1.5).

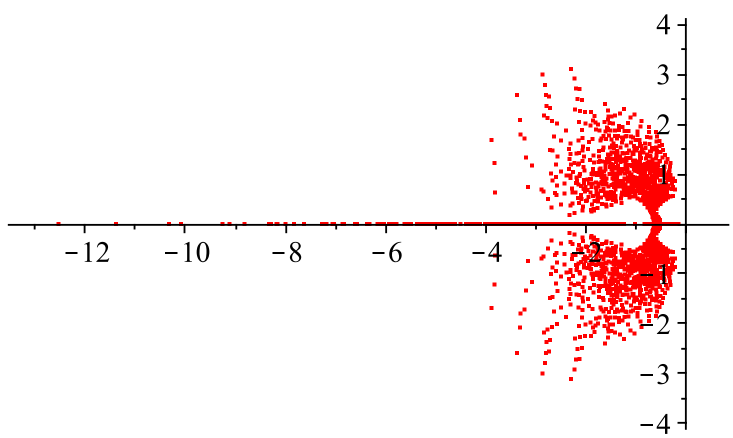


Figure 1.4: Independence roots of all graphs of order at most 8.

Interest in the independence polynomial since its introduction 36 years ago has focused on problems related to

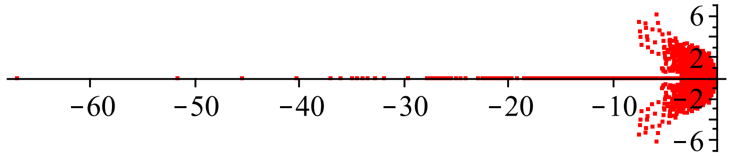


Figure 1.5: Independence roots of all trees of order at most 14.

- computing the independence polynomial [4, 50, 46, 48, 63, 19, 44],
- properties of the coefficient sequence [1, 69, 16, 47, 57, 58, 60, 62, 67],
- properties of the independence roots [16, 18, 19, 23, 24, 31, 34, 63, 71], and
- determining nonisomorphic graphs with equivalent independence polynomials [29, 50, 83, 66, 20, 72, 63, 64, 90].

Questions arise about the nature and location of independence roots in the complex plane. Brown and Nowakowski [24] use random graphs to show that almost all graphs have a nonreal independence root. Although almost all graphs have a nonreal independence root, no graph has all nonreal independence roots since the independence root of smallest modulus in every graph is real [16]. It was shown in [63] that the independence roots of a graph on n vertices lie in $|z| > \frac{1}{2n-1}$.

The independence roots of largest modulus have been considered as well. For a graph of order n with independence number α, a tight bound on the largest modulus of an independence root is $\left(\frac{n}{\alpha-1}\right)^{\alpha-1} + O(n^{\alpha-2})$ [23]. When restricted to well-covered graphs, however, independence roots lie in the disk $|z| \leq \alpha(G)$ [16].

In a beautiful paper, Chudnovsky and Seymour [31] extended the work done in both [49] and [47], showing that G has all real independence roots for every claw-free graph G. In general though, Brown et al. [19] showed that the set of independence roots of all graphs is dense in $\mathbb{C}$ and the set of all real independence roots is dense in $(-\infty, 0]$, even when the set is restricted to the independence roots of comparability graphs or well-covered graphs. If restrictions are put on different graph parameters their collection of independence roots may no longer be dense in $\mathbb{C}$. Sokal [81] conjectured that there exists a neighbourhood around the interval

$$\left[0, \frac{(\Delta(G)-1)^{\Delta(G)-1}}{(\Delta(G)-2)^{\Delta(G)}}\right)$$

such that no graph G has an independence root in this neighbourhood. This was recently shown to be true by Peters and Regts [73].

Inspired by all of the work that has come before on the independence polynomial, we will focus entirely on the independence polynomial throughout this thesis. The structure is as follows: In Chapter 2, we consider the log-concavity conjecture for very well-covered graphs using the independence roots and a generalization of Newton's result on polynomials with all real roots. Computations on the roots led us to become interested in graphs with independence roots with positive real part, so in Chapter 3, we undertake the first exploration of stable and nonstable independence polynomials. In Chapter 4, we bound the maximum modulus of an independence root of a graph on n vertices. The last chapter with original results is Chapter 5, where we look at independence equivalence classes of paths and cycles, extending the known results listed in this introduction. We end with Chapter 6, where we include open problems and ideas for future research. We also include four appendices with plots of independence roots of small graphs and trees of small order and graphs of small order whose independence roots have (imaginary part with) maximum modulus.

Chapter 2

On the Log-Concavity Conjecture for Independence Polynomials of Very Well-Covered Graphs

For many graph polynomials such as matching [49], chromatic [75, 51] and reliability [32, 52] polynomials, the absolute value of the coefficient sequence, have long been conjectured to be (or proven to be) *unimodal*, that is, nondecreasing then nonincreasing. In some cases, unimodality was proven by showing the stronger property of log-concavity. Problems on unimodality and log-concavity are not only of interest for graph polynomials; in fact, they permeate the literature in combinatorics, algebra, and geometry (see Stanley's survey [82] and Brenti's update [9]).

What can we say about the coefficients of independence polynomials – are they always unimodal or log-concave (a stronger property)? Unfortunately, the answer is no. For example, if $G = K_{70} + (K_3 \cup K_3 \cup K_3 \cup K_3)$, then $i(G, x) = 1 + 82x + 54x^2 + 108x^3 + 81x^4$ which is not unimodal. In fact, Alavi et al. [1] showed that independence polynomials can be as far from unimodal as possible, in that for every permutation π on $\{1, 2, \ldots, \alpha\}$, there exists a graph with independence number α such that $i_{\pi(1)} < i_{\pi(2)} < \cdots < i_{\pi(\alpha)}$. They refer to this property as being *unconstrained*

Despite this strong result against unimodality holding in general, there are many families of graphs that are conjectured to be or proved to be unimodal or log-concave. In the same paper where they showed independence polynomials are unconstrained in general, Alavi et al. [1] conjecture that $i(T, x)$ is unimodal for all trees T. This conjecture remains open with little progress, but the conjecture is reasonable, as their construction for unconstrained graphs forces many cycles. The conjecture was extended to all bipartite graphs in [61], but Bhattacharyya and Kahn [7] showed that there are bipartite graphs with nonunimodal independence polynomials.

Another highly structured family of graphs with respect to independence is the family of *well-covered* graphs, those whose maximal independent sets all have the same size, i.e. all maximal independent sets are maximum independent sets. Examples include complete graphs and the 5-cycle. Well-covered graphs were first introduced in

[74] and their structure has attracted considerable attention in the literature, including characterizations for those of high girth [40]. In [16], Brown et al. conjectured that the independence polynomials of well-covered graphs were unimodal, and showed that every graph G can be embedded as an induced subgraph of such a well-covered graph. Three years later, Michael and Traves proved that $i_0 \leq i_1 \leq \cdots \leq i_{\lfloor \alpha/2 \rfloor}$ for every well-covered graph with independence number α, which seems to provide more evidence for Brown et al.'s [16] conjecture. However, despite this partial result, Michael and Traves [69] also provided counterexamples for independence numbers $4, 5, 6$, and 7 and counter-conjectured the so called *Roller-Coaster Conjecture*. The Roller-Coaster Conjecture states that there exist well-covered graphs such that their independence polynomials are unconstrained from $i_{\lfloor \alpha/2 \rfloor}$ to i_α. The Roller-Coaster Conjecture was verified by Michael and Traves [69] for $\alpha \leq 7$, which was extended by Matchett [67] to $\alpha \leq 11$. Levit and Mandrescu [60] eventually extended the set of independence numbers for which there are well-covered graphs with nonunimodal independence polynomials to include all positive integers greater than or equal to 4. After Levit and Mandrescu's result, there was a period where little progress was made on the Roller-Coaster Conjecture until very recently when Cutler and Pebody [36] finally proved that the conjecture is indeed true.

At this point, the reader may be starting to question if there are any substantial families of graphs with unimodal independence polynomials. While the Roller-Coaster Conjecture is now the Roller-Coaster Theorem and much of the work on unimodality in other families of graphs consists of partial or negative results, there is a very important positive result. Hamidoune [47] showed that $i(G, x)$ is log-concave and therefore unimodal for every claw-free graph G, those are, graphs that do not contain a claw (the star $K_{1,3}$) as an induced subgraph.

The original unimodality conjecture on well-covered graphs was then amended as follows. A *very well-covered* graph G of order n is a well-covered graph for which every maximal independent set has size $n/2$; for example, the complete bipartite graphs $K_{\frac{n}{2},\frac{n}{2}}$ are very well-covered. Other examples are afforded by the graph star construction that we present in the next section. Very well-covered graphs were first considered in [39]. Levit and Mandrescu [59, 60] noted that the unimodality conjecture is still open for the independence polynomials of very well-covered graphs and eventually extended the conjecture to log-concavity in [62].

To date, the conjecture remains open. Some partial results have been proven on the tail of independence sequences of very well-covered graphs [62], and the first $\lceil \frac{\alpha}{2} \rceil$ terms have been shown to be nondecreasing for well-covered graphs [69]. The conjecture is known to hold when $\alpha(G) \leq 9$ [62] and for the star extension of any graph G where $\alpha(G) \leq 8$ [28], or where G is a path or star [58] (we will talk more about such extensions shortly).

There are many techniques for proving that a sequence is log-concave and therefore unimodal (see for example, [9] and [82]). One that has been frequently applied is due to Newton (c.f. [33, pp. 270-271]), who proved that if a polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ with positive coefficients has all real roots, then the sequence $\langle a_0, a_1, \ldots, a_n \rangle$ satisfies

$$a_i^2 \geq \frac{i+1}{i} \frac{n-i+1}{n-i} a_{i-1} a_{i+1},$$

and hence is log concave. Newton's elegant theorem has been used to prove that a variety of sequences (and polynomials) are unimodal, such as matching polynomials [49] and an alternate proof for the independence polynomials of claw-free graphs [31].

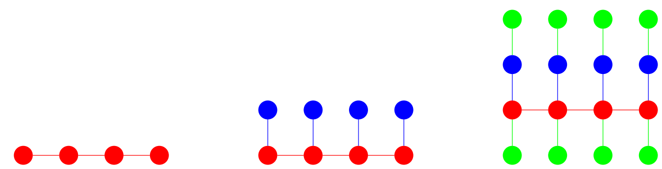
In this chapter, we shall show that for *any* graph G there exists a very well-covered extension G^{k*} such that the independence polynomial of G^{k*} is log-concave. We do this by using the independence roots and a generalization of Newton's Theorem. This provides some evidence for Levit and Mandrescu's [62] log-concavity conjecture for very well-covered graphs. It should be noted that this chapter contains and extends our work in [14].

2.1 Log-Concavity of Independence Polynomials of Graph k-stars and Sectors in the Complex Plane

Let G be any graph. Form G^*, the *graph star* of G [85, 59] from G by attaching, for each vertex v of G a new vertex v^* to v with an edge (such an edge is called a *pendant* edge). The construction is the special case of a more general graph product called the *corona* (more specifically, the corona of G with K_1) which we will define and use in Section 3.2. The graph star operation can also be iterated to form the *graph k-star* of G, denoted G^{k*}. For a graph G and positive integer k, let G^{k*} denote the *graph k-star* of G, that is, the graph formed by iteratively attaching pendant vertices, k times:

$$G^{k*} = \begin{cases} G^* & \text{if } k = 1, \\ (G^{(k-1)*})^* & \text{if } k \geq 2. \end{cases}$$

Figure 2.1 shows the graphs P_4, P_4^*, and P_4^{2*}.



(a) P_4 (b) P_4^* (c) P_4^{2*}

Figure 2.1: The graphs P_4, P_4^* and P_4^{2*}.

We start by justifying the claim that the graph star G^* of any graph $G = (V, E)$ of order n is always very well-covered. Clearly, $\alpha(G^*) \leq n$, as the graph has a perfect matching and no independent set can contain two vertices that are matched. Moreover, $\alpha(G^*) = n$ as any independent set I of G can be extended to one in G^* by adding in any subset of $(V - I)^* = \{v^* : v \in V - I\}$. Therefore, every maximal independent set has size $n = |V(G^*)|/2$. It follows (see also [63]) that if $i(G, x) = \sum i_k x^k$, then

$$\begin{aligned} i(G^*, x) &= \sum i_k x^k (1+x)^{n-k} \\ &= (1+x)^n \cdot i\left(G, \frac{x}{1+x}\right). \end{aligned} \tag{2.1}$$

We can extend formula (2.1) to higher iterations of the $*$ operation as follows.

Proposition 2.1.1. *For any graph G of order n and any positive integer k,*

$$i(G^{k*}, x) = i(G, \tfrac{x}{kx+1})(kx+1)^n \prod_{\ell=1}^{k-1} (\ell x + 1)^{n2^{k-\ell-1}}.$$

Proof. We proceed by induction on k, the number of iterations of the $*$ operation. The base case follows directly from (2.1), so we can assume that the result holds for some $k \geq 1$, i.e.,

$$i(G^{k*}, x) = i(G, \tfrac{x}{kx+1})(kx+1)^n \prod_{\ell=1}^{k-1} (\ell x + 1)^{n2^{k-\ell-1}}.$$

A trivial induction shows that G^{k*} has order $n2^k$. From this, formula (2.1), and the fact that $G^{(k+1)*} = (G^{k*})^*$ we obtain the desired result as follows:

$$\begin{aligned}
i(G^{(k+1)*}, x) &= (1+x)^{n2^k} i(G^{k*}, \tfrac{x}{x+1}) \\
&= (1+x)^{n2^k}\, i\left(G, \frac{\frac{x}{x+1}}{\frac{kx}{x+1}+1}\right) \left(k\left(\tfrac{x}{x+1}\right)+1\right)^n \prod_{\ell=1}^{k-1} \left(\ell\left(\tfrac{x}{x+1}\right)+1\right)^{n2^{k-\ell-1}} \\
&= (1+x)^{n2^k}\, i\left(G, \tfrac{x}{(k+1)x+1}\right) \left(\tfrac{(k+1)x+1}{x+1}\right)^n \prod_{\ell=1}^{k-1} \left(\tfrac{(\ell+1)x+1}{x+1}\right)^{n2^{k-\ell-1}} \\
&= \frac{(1+x)^{n2^k}}{(1+x)^{n2^{k-1}}} i\left(G, \tfrac{x}{(k+1)x+1}\right) ((k+1)x+1)^n \prod_{\ell=1}^{k-1} ((\ell+1)x+1)^{n2^{k-\ell-1}} \\
&= (1+x)^{n2^{k-1}} i\left(G, \tfrac{x}{(k+1)x+1}\right) ((k+1)x+1)^n \prod_{\ell=1}^{k-1} ((\ell+1)x+1)^{n2^{k-\ell-1}} \\
&= i\left(G, \tfrac{x}{(k+1)x+1}\right) ((k+1)x+1)^n \prod_{\ell=0}^{k-1} ((\ell+1)x+1)^{n2^{k-\ell-1}} \\
&= i\left(G, \tfrac{x}{(k+1)x+1}\right) ((k+1)x+1)^n \prod_{\ell=1}^{k} (\ell x+1)^{n2^{(k+1)-\ell-1}}.
\end{aligned}$$

□

From Proposition 2.1.1 we see at once that for any $k \geq 1$, all independence roots of G^{k*} are real if and only if the same is true of G. Since most independence polynomials have a non-real root [24], we won't be able to use Newton's Theorem to show that for *any* graph G the independence polynomial of G^{k*}, for some graph k, is log-concave. However, Newton's theorem is only a sufficient condition for the coefficient sequence to be log concave. Brenti et al. [10] weakened the conditions as follows:

Proposition 2.1.2 ([10]). *If all the roots of the polynomial $f(x) \in \mathbb{R}[x]$ are in the region*

$$\{z \in \mathbb{C} : |\arg(z)| < \tfrac{\pi}{3}\},$$

then the sequence of coefficients of $f(x)$ is strictly log concave and alternates in sign.

Replacing $f(x)$ by $f(-x)$, we derive that:

Corollary 2.1.3. *If all the roots of the polynomial $f(x) \in \mathbb{R}[x]$ are in the region*

$$\mathcal{R} = \{z \in \mathbb{C} : \tfrac{2\pi}{3} < |\arg(z)| < \tfrac{4\pi}{3}\},$$

then the sequence of coefficients of $f(x)$ is strictly log concave (and the sequence of coefficients of $f(x)$ is either all positive or all negative). □

We shall make use of this corollary to prove our main result on log-concavity of independence polynomials of graph k-stars.

Theorem 2.1.4. *For all graphs G let*

$$M = \max\left\{ \frac{\frac{1}{\sqrt{3}}|\operatorname{Im}(z)| + |\operatorname{Re}(z)|}{|z|^2} : z \textit{ is a root of } i(G,x) \right\}.$$

If $k > M$, then $i(G^{k*}, x)$ is strictly log concave.

Proof. From Proposition 2.1.1, we can partially factor $i(G^{k*}, x)$ as

$$\begin{aligned} i(G^{k*}, x) &= i(G, \tfrac{x}{kx+1})(kx+1)^n \prod_{\ell=1}^{k-1} (\ell x + 1)^{n2^{k-\ell-1}} \\ &= \left(i(G, \tfrac{x}{kx+1})(kx+1)^{\alpha(G)}\right)(kx+1)^{n-\alpha(G)} \prod_{\ell=1}^{k-1} (\ell x + 1)^{n2^{k-\ell-1}}. \end{aligned}$$

From this, it follows that if $r_1, \ldots, r_m$ are the roots of $i(G,x)$, then the roots of $i(G^{k*}, x)$ are $\frac{r_i}{1-kr_i}$ for $i = 1, 2, \ldots, m$ along with the rational numbers $\frac{-1}{\ell}$ for $\ell = 1, 2, \ldots, k-1$ (and $\ell = k$ if $\alpha(G) \neq n$, i.e. $G \neq \overline{K_n}$).

Let r be any root of $i(G,x)$, and set $a = \operatorname{Re}(r)$ and $b = \operatorname{Im}(r)$. Note that either a or b is nonzero (since 0 is not the root of any independence polynomial) and likewise, $r \neq 1/k$ for all $k \geq 0$ (as no independence root is positive). We expand a root of

$i(G^{k*}, x)$ to obtain,

$$\begin{aligned}
\frac{r}{1-kr} &= \frac{a+ib}{1-k(a+ib)} \\
&= \frac{a+ib}{(1-ka)-ikb} \cdot \frac{(1-ka)+ikb}{(1-ka)+ikb} \\
&= \frac{a(1-ka)+iakb+ib(1-ka)-kb^2}{(1-ka)^2+k^2b^2} \\
&= \frac{a(1-ka)-kb^2+i(akb+b(1-ka))}{(1-ka)^2+k^2b^2} \\
&= \frac{(a-ka^2-kb^2)+ib}{(1-ka)^2+k^2b^2}. \qquad (2.2)
\end{aligned}$$

We now wish to show that for sufficiently large k, the root $z = \frac{r}{1-kr}$ of $i(G^{k*}, x)$ lies in the sector $\{z \in \mathbb{C} : \frac{2\pi}{3} < |\arg(z)| < \frac{4\pi}{3}\}$; the result will then follow immediately from Corollary 2.1.3 (as the negative rational roots obviously lie in the sector). It is clear to see that z lies in the sector if and only if $\mathrm{Re}(z) < 0$ and $\left|\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right| < \sqrt{3}$. Now, $\mathrm{Re}(z) = \frac{a-ka^2-kb^2}{(1-ka)^2+k^2b^2}$ and $(1-ka)^2 + k^2b^2 > 0$ since if $b = 0$ then $a \neq 1/k$. We also have $a - ka^2 - kb^2 = -k(a^2+b^2) + a$ and so for $k > \frac{(1/\sqrt{3})|b|+|a|}{a^2+b^2} \geq \frac{|a|}{a^2+b^2}$, it follows that $\mathrm{Re}(z) < 0$. We note as well that for $k > \frac{|a|}{a^2+b^2}$, $k(a^2+b^2) - a$ is positive and increasing, as a function of k, and that $\frac{(1/\sqrt{3})|b|+|a|}{a^2+b^2} \geq \frac{|a|}{a^2+b^2}$. We now compute the ratio of the imaginary and real part of z for $k > \frac{(1/\sqrt{3})|b|+|a|}{a^2+b^2}$:

$$\begin{aligned}
\left|\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right| &= \left|\frac{b}{k(a^2+b^2)-a}\right| \\
&< \frac{|b|}{\left(\frac{(1/\sqrt{3})|b|+|a|}{a^2+b^2}\right)(a^2+b^2)-a} \\
&= \frac{|b|}{(1/\sqrt{3})|b|+|a|-a} \\
&\leq \sqrt{3}.
\end{aligned}$$

The result now follows from Corollary 2.1.3.

□

Corollary 2.1.5. *Every graph G on n vertices is an induced subgraph of a very well-covered graph H such that the sequence of coefficients of $i(H, x)$ is log-concave.* □

2.2 Discussion

While Theorem 2.1.4 shows that for any graph G, the independence polynomial of some graph k-star of G is log-concave, the question remains as to whether this is true for the graph k-star for *every* $k \geq 1$, and, in particular, for G^*.

The next result uses the properties of *Möbius transformations* (also called linear fractional transformations) which are rational functions of the form

$$T(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d, z \in \mathbb{C}$ and $ad - bc \neq 0$. Möbius transformations are one-to-one mappings of $\mathbb{C} \cup \infty$ onto itself and they map every circle onto another circle or a line and every line onto another line or a circle [41]. Therefore, the image of a line or a circle under a Möbius transformation is completely determined by the image of three points. Moreover, the interiors/exteriors of circles and half-planes are mapped onto the same set under a Möbius transformation. More background on Möbius transformations can be found in any complex analysis book, for example, in section 3.3 of Fisher's book [41]. From the properties of these functions, we can explicitly state where the independence roots of G need to lie to ensure its graph star has a log concave (and hence unimodal) independence polynomial.

Theorem 2.2.1. *If the roots of $i(G, x)$ lie outside of the region bounded by the union of circles with radii $\frac{\sqrt{3}}{3}$ centred at $\frac{1}{2} + \frac{\sqrt{3}i}{6}$ and $\frac{1}{2} - \frac{\sqrt{3}i}{6}$, respectively, then $i(G^*, x)$ is strictly log-concave.*

Proof. We will find the image of the region $\mathcal{R} = \{z \in \mathbb{C} : \frac{2\pi}{3} < |\arg(z)| < \frac{4\pi}{3}\}$ under the Möbius transformation $f(z) = \frac{z}{1+z}$ (the inverse of $\frac{r}{1-r}$). As noted, such a transformation sends lines and circles to lines and circles, and interiors/exteriors of circles and half-planes are sent to the same set of regions. We need only find the image of three points on the two line segments bounding the sector. The images of $-1 + \sqrt{3}i$, 0, and ∞ are $1 + \frac{\sqrt{3}i}{3}$, 0, and 1 respectively, yielding the circle C_1, centred at $\frac{1}{2} + \frac{\sqrt{3}i}{6}$ with radius $\frac{\sqrt{3}}{3}$. As $-\frac{1}{2}$ is below the line $\arg(z) = \frac{2\pi}{3}$ and gets mapped to -1, which is outside C_1, the half-plane below the line $\arg(z) = \frac{2\pi}{3}$ gets mapped to the exterior of C_1.

Similarly, the images of $-1-\sqrt{3}i$, 0, and ∞ are $1-\frac{\sqrt{3}i}{3}$, 0, and 1 respectively, yielding the circle C_2, centred at $\frac{1}{2}-\frac{\sqrt{3}i}{6}$ with radius $\frac{\sqrt{3}}{3}$. As $-\frac{1}{2}$ is above the line $\arg(z)=\frac{4\pi}{3}$ and gets mapped to -1, which is outside C_2, the half-plane above the line $\arg(z)=\frac{4\pi}{3}$ gets mapped to the exterior of C_2. Therefore, if we take a point above the line $\arg(z)=\frac{4\pi}{3}$ and below the line $\arg(z)=\frac{2\pi}{3}$ (i.e. in the region $\mathcal{R}$), then its image under f will be an exterior point to the union of C_1 and C_2. Therefore, by Corollary 2.1.3, $i(G^*,x)$ is strictly log-concave. □

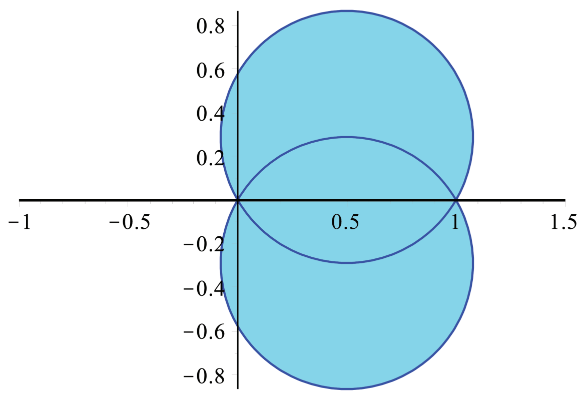


Figure 2.2: Region that ends up outside the sector $\mathcal{R}=\{z\in\mathbb{C}:\frac{2\pi}{3}<|\arg(z)|<\frac{4\pi}{3}\}$ under the Möbius transformation $f(z)=\frac{z}{1+z}$.

Theorem 2.2.1 assures us that as long as the roots of $i(G,x)$ are outside of a region in $\mathbb{C}$ with an area $\frac{4\pi}{9}+\frac{\sqrt{3}}{6}\approx 1.6849$, then $i(G^*,x)$ will be strictly log-concave. Although this result works for many graphs, Brown et al. [19] showed that the set of independence roots of all graphs is dense in $\mathbb{C}$, even when restricted to well-covered graphs. Therefore, there exist graphs with independence roots in the union of the interior of the two circles specified in the statement of Theorem 2.2.1 and hence graphs that have graph stars with independence roots outside of $\mathcal{R}$. Using the methods outlined in [19] to find independence roots throughout $\mathbb{C}$, we have found that $i(G^*,x)$ has a root outside of $\mathcal{R}$ for $G=P_5[\overline{K_6}]$ since G has independence roots in the two circle as shown in Figure 2.4 (recall $G[H]$ is the lexicographic product of G with H). Although the independence roots of G^* do not lie in $\mathcal{R}$ in this case, $i(G^*,x)$ is still log-concave so the conjecture stands.

Another point to note is that there exist graphs that are very well-covered but are not the graph k-star of another graph: some examples which were already pointed out

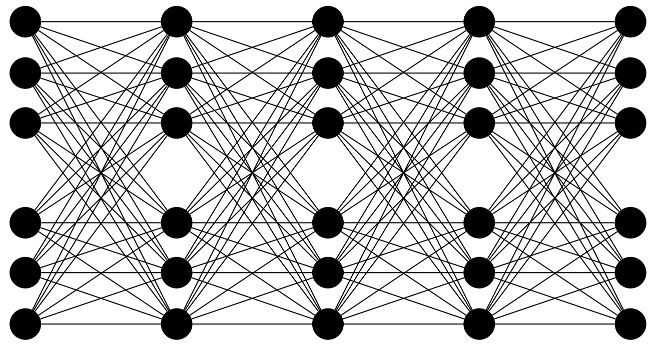


Figure 2.3: The graph $P_5[\overline{K_6}]$.

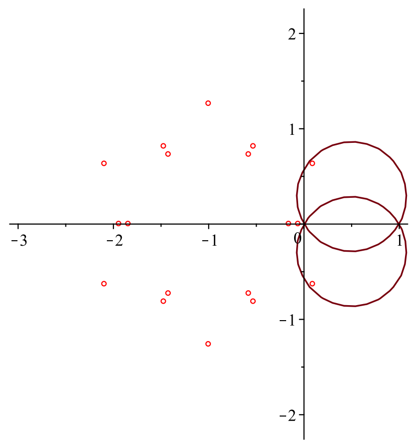


Figure 2.4: The independence roots of $P_5[\overline{K_6}]$ with two in the region in Figure 2.2.

are the bipartite graphs $K_{n,n}$ among others [39]. Our techniques do not encompass these graphs; however, Finbow et al. [40] showed that, with the exceptions of K_1 and C_7, a graph G with girth$(G) \geq 6$ is well-covered if and only if its pendant edges form a perfect matching. It is easy to see that the pendant edges of G forming a perfect matching is equivalent to $G = H^*$ for some graph H and therefore, in graphs with high girth the only (very) well-covered graphs are graph stars (or K_1 or C_7).

At the start of our work on log-concavity and graph stars, we plotted the independence roots of many graphs to see if they land in the sector that guarantees log-concavity. While many graphs have their independence roots contained in the sector $\mathcal{R}$, all independence roots lie in the open left half-plane. This observation leads to our next problem.

Chapter 3

Stability of Independence Polynomials

While Brown et al. [19] showed that the collection of the independence roots of all graphs is dense in the complex plane, plots of the independence roots of small graphs show a very different story (see Figures 1.4 and 1.5). All lie to the left of the imaginary axis, so we are left to wonder: how ubiquitous are graphs with stable independence polynomials, that is, with all their independence roots in the left half-plane (LHP) $\{z \in \mathbb{C} : \text{Re}(z) \leq 0\}$? Can there be any independence roots on the imaginary axis? Graphs with stable independence polynomials are a natural extension of graphs with all real independence roots as the LHP contains precisely the real numbers with negative real part, and all real independence roots must be negative as the coefficients are all positive. One important result on independence polynomials, that is considered a crown jewel of the field, is the Chudnovsky-Seymour result [31] that claw-free graphs have all real independence roots.

We shall call a graph itself *stable* if its independence polynomial is stable. It is known that the independence root of smallest modulus is always real and therefore negative (see [16]), so no independence polynomial has all its roots in the RHP, but it is certainly possible for it to have all roots in the LHP.

In this chapter, we shall consider the stability of independence polynomials, providing some families of graphs whose independence polynomials are indeed stable, while showing that graphs formed under various constructions have independence polynomials that are not only nonstable but have independence roots with arbitrarily large (positive) real part. We shall first consider stability for graphs with small independence number, and show that while all graphs with independence number at most 3 are stable, it is not the case for larger independence number. Then we shall turn to producing stable graphs as well as nonstable graphs. Graph operations will play key roles in both. We conclude with a discussion on purely imaginary independence roots, that is, independence roots that lie on the boundary of the left half-plane. It should be noted that this chapter contains and extends our work in [13].

3.1 Stability for Small Independence Number

We begin by proving that all graphs with independence number at most three are indeed stable. To do so, we shall utilize a necessary and sufficient condition, due to Hermite and Biehler, for a real polynomial to be stable. Prior to introducing the theorem, we shall need some notation.

Given a polynomial $P(x) = \sum_{i=0}^{d} a_i x^i$, let

$$P^{even}(x) = \sum_{i=0}^{\lfloor d/2 \rfloor} a_{2i} x^i,$$

and

$$P^{odd}(x) = \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} a_{2i+1} x^i;$$

$P^{even}(x)$ and $P^{odd}(x)$ are the *even* and *odd parts* of the polynomial, with

$$P(x) = P^{even}(x^2) + xP^{odd}(x^2).$$

For example, if $P(x) = i(K_{3,3}, x) = 1 + 6x + 6x^2 + 2x^3$, then $P^{even}(x) = 1 + 6x$ and $P^{odd}(x) = 6 + 2x$.

Finally, let $f(x)$ and $g(x)$ be two real polynomials with all real roots, say $s_1 \leq s_2 \leq \ldots \leq s_n$ and $t_1 \leq t_2 \leq \ldots \leq t_m$ being their respective roots. We say that

- f *interlaces* g if $m = n + 1$ and $t_1 \leq s_1 \leq t_2 \leq s_2 \leq \cdots \leq s_n \leq t_{n+1}$, and
- f *alternates left* of g if $m = n$ and $s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq s_n \leq t_n$.

We write $f \prec g$ for either f interlaces g or f alternates left of g. A key result that we shall rely on is the Hermite-Biehler Theorem which characterizes when a real polynomial is stable (see, for example, [87]).

Theorem 3.1.1 (Hermite-Biehler). *Let $P(x) = P^{even}(x^2) + xP^{odd}(x^2)$ be standard. Then $P(x)$ is stable if and only if both P^{even} and P^{odd} are standard, have only nonpositive real roots, and $P^{odd} \prec P^{even}$.* □

For example, $i(P_5, x) = x^3 + 6\,x^2 + 5\,x + 1$, so $i^{even}(P_5, x) = 1 + 6x$ and $i^{odd}(P_5, x) = 5 + x$. Both of which clearly have all real and nonpositive roots and $-5 < -\frac{1}{6}$, so $i^{odd}(P_5, x) \prec i^{even}(P_5, x)$. Therefore, P_5 is stable, by the Hermite-Biehler Theorem. (Note that we also know P_5 is stable since it is claw-free, so by the Chudnovsky-Seymour [31] result, all of its independence roots are real and therefore in the LHP.) On the other hand,

$$i(\overline{K_6} + K_9, x) = x^6 + 6\,x^5 + 15\,x^4 + 20\,x^3 + 15\,x^2 + 15\,x + 1$$

so

$$\begin{aligned} i^{even}(\overline{K_6} + K_9, x) &= x^3 + 15\,x^2 + 15\,x + 1 \\ i^{odd}(\overline{K_6} + K_9, x) &= 6\,x^2 + 20\,x + 15. \end{aligned}$$

The roots of $i^{even}(\overline{K_6}+K_9, x)$ are -13.928, -1, and -0.0718, and the roots of $i^{odd}(\overline{K_6}+K_9, x)$ are -2.194 and -1.14. Both $i^{odd}(\overline{K_6} + K_9, x)$ and $i^{even}(\overline{K_6} + K_9, x)$ have all nonpositive real roots, but $i^{odd}(\overline{K_6} + K_9, x)$ does not interlace $i^{even}(\overline{K_6} + K_9, x)$, so $\overline{K_6} + K_9$ is nonstable by the Hermite-Biehler Theorem.

We are now in a position to prove stability for small independence number.

Proposition 3.1.2. *If G is a graph with $\alpha(G) \leq 3$, then $i(G, x)$ is stable.*

Proof. For graphs with independence number 1 (that is, a complete graph), the independence polynomial is of the form $1 + nx$. These polynomials are obviously stable for all n. For graphs with independence number 2, the independence polynomial has the form $1 + nx + i_2x^2$. The complement of a graph with independence number 2 is triangle-free, and hence by Turán's famous theorem (see [89, pp. 30] for example), has at most $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil \leq \frac{n^2}{4}$ many edges. However, the number of edges in the complement is precisely i_2, so that $i_2 \leq \frac{n^2}{4}$, which implies that the independence polynomial's discriminant is $n^2 - 4i_2$, which is nonnegative, so the roots are real (and hence negative). Therefore, the independence polynomial of a graph with independence number 2 is necessarily stable.

For graphs with independence number 3, it is again the case that all independence polynomials are stable. To show this, we utilize the Hermite-Biehler Theorem. If

$\alpha(G) = 3$, then

$$i(G, x) = 1 + nx + i_2 x^2 + i_3 x^3 = P^{even}(x^2) + xP^{odd}(x^2)$$

where $P^{even} = 1 + i_2 x$ and $P^{odd} = n + i_3 x$. It is clear that P^{even} and P^{odd} each have only one real root, but we must show that $P^{odd} \prec P^{even}$, i.e. that $\frac{-n}{i_3} \leq \frac{-1}{i_2}$. Equivalently, we need to show that $ni_2 \geq i_3$, but this follows as every independent set of size 3 contains an independent set of size 2, so adjoining an outside vertex to each independent set of size 2 will certainly cover all independent sets of size 3 at least once (in fact, our argument shows that $(n-2)i_2 \geq i_3$). Thus by Theorem 3.1.1, $i(G, x)$ is stable for all $\alpha(G) = 3$. □

We now turn to independence number at least 4, and show, in contrast, that there are many graphs whose independence roots lie in the RHP – in fact, we can find roots in the RHP with arbitrarily large real part. We begin with a lemma. This lemma will be pivotal for many of the results in the remainder of this section as well as in Section 3.3.

Lemma 3.1.3. *Let $R > 0$ and $f(x) \in \mathbb{R}[x]$ be a polynomial of degree d with positive coefficients. Then:*

1. *If $d \geq 4$, then for m sufficiently large $f(x) + mx$ has a root with real part greater than R.*

2. *If $d \geq 3$, then for ℓ sufficiently large $f(x) + \ell$ has a root with real part greater than R.*

Proof. We consider the polynomial $g(x) = f(x + R)$. As per the Hermite-Biehler Theorem, let $g^{even}(x)$ and $g^{odd}(x)$ denote the even and odd part of $g(x)$, respectively, so that

$$g(x) = g^{even}(x^2) + xg^{odd}(x^2).$$

For the proof of part 1, consider the polynomial $P_m(x) = m(x + R) + g(x)$. Hence,

$$P_m^{even}(x) = mR + g^{even}(x)$$

and

$$P_m^{odd}(x) = m + g^{odd}(x).$$

Clearly $\deg(g^{even}(x)) \geq 2$, since $d \geq 4$. The leading coefficient of $P_m^{even}(x)$ is positive (as f has all positive coefficients and $R > 0$), so it follows that $\lim_{x\to\infty} g^{even}(x) = \infty$. Let

$$M = \max\{|g^{even}(z)| : (g^{even})'(z) = 0\},$$

that is, M is the maximum absolute value of the function $g_m^{even}(x)$ at the latter's critical points (which are the same as the critical points of $P_m^{even}(x)$, as the two functions differ by a constant). Thus, for any $m \geq \lfloor \frac{M}{R} \rfloor + 1$, $P_m^{even}(r) > 0$ for all of its real critical points r. It follows that the roots of $P_m^{even}(x) = mR + g^{even}(x)$ are simple (that is, have multiplicity 1), as if a root r of $P_m^{even}(x)$ had multiplicity larger than 1, then it would also be a critical point of $P_m^{even}(x)$, but for the chosen value of m, $P_m^{even}(r) > 0$. Moreover, $P_m^{even}(x)$ has at most one real root, as if it had two roots $a < b$, then by the simpleness of the roots, either the function $P_m^{even}(x)$ is negative at some point between a and b, or to the right of b, but in either case $P_m^{even}(x)$ would have a critical point c at which $P_m^{even}(c) < 0$, a contradiction. In any event, as $P_m^{even}(x)$ has at most one real root (counting multiplicities) and $\deg(P_m^{even}(x)) \geq 2$, $P_m^{even}(x)$ must have a nonreal root. By the Hermite-Biehler theorem, it follows that $P_m(x) = m(x+R) + f(x+R)$ has a root in the RHP. Note that $x = a + ib$ is a root of $P_m(x)$ if and only if $x + R = (a+R) + ib$ is a root of $f(x) + mx$. Since there exists a root x with $\text{Re}(x) > 0$ of $P_m(x)$, $x + R$ is a root of $f(x) + mx$ with $\text{Re}(x+R) > R$. Therefore, for sufficiently large m, $f(x) + mx$ has roots with real part greater than R.

A similar (but slightly simpler) argument holds for part 2, provided $d \geq 4$ and considering the even part of $P_\ell(x) = \ell + g(x)$. Therefore, all that remains is the case $d = 3$. In this case, let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Set

$$\begin{aligned} g(x) &= f(x+R) \\ &= a_0 + a_1R + a_2R^2 + a_3R^3 + (a_1 + 2Ra_2 + 3R^2a_3)x + (3Ra_3 + a_2)x^2 + a_3x^3. \end{aligned}$$

Now let $P_\ell = \ell + g(x)$. By Theorem 3.1.1, P_ℓ is stable if and only if

$$-\frac{a_1 + 2Ra_2 + 3R^2a_3}{a_3} \leq -\frac{a_0 + a_1R + a_2R^2 + a_3R^3 + \ell}{3Ra_3 + a_2},$$

that is, if and only if

$$\frac{a_1 + 2Ra_2 + 3R^2a_3}{a_3} \geq \frac{a_0 + a_1R + a_2R^2 + a_3R^3 + \ell}{3Ra_3 + a_2},$$

but clearly this fails if ℓ is large enough. Therefore, for ℓ sufficiently large, $P_\ell^{odd} \not\prec P_\ell^{even}$ and therefore, $f(x)$ has a root with real part greater than R. □

We shall shortly show that there are nonstable graphs of every independence number greater than 3 by combining the previous lemma with another tool from complex analysis, the well known and extremely useful *Gauss–Lucas* Theorem, see for example [41, pp. 381]. In particular, we will apply its contrapositive.

Theorem 3.1.4 (Gauss–Lucas). *If every root of the polynomial f lies in the half-plane $Re(Az + B) > 0$, then so does every root of its derivative f'.* □

We are now able to provide, for each $\alpha \geq 4$, infinitely many examples of graphs with independence number α that are nonstable. Moreover, we can embed *any* graph with independence number $\alpha \geq 4$ into another nonstable one with the same independence number, and we can even do so with a (nonreal) independence root as far to the right as we like. To do this, we use the join operation. The *join* of two graphs G and H, denoted $G + H$, is the graph obtained by joining all vertices of G with all vertices of H. For example, $K_{n,m}$ is just the join of $\overline{K_m}$ and $\overline{K_n}$. Note that $i(G + H, x) = i(G, x) + i(H, x) - 1$.

Proposition 3.1.5. *Let $G = H + \underbrace{F + F + \cdots + F}_{k}$, the join of a graph H and k copies of F. If $\alpha(H) \geq \alpha(F) + 3$, then for k sufficiently large, $i(G, x)$ has roots with arbitrarily large real part.*

Proof. Let $R > 0$ be given. Assume $\alpha(H) \geq \alpha(F) + 3$. Also let c be the coefficient of $x^{\alpha(F)}$ in $i(F, x)$ (it is the number of independent sets of F of maximum cardinality).

We take the $\alpha(F)$-th derivative of $i(G,x)$, denoted $i^{(\alpha(F))}(G,x)$. Since

$$i(G,x) = i(H,x) + k \cdot i(F,x) - (k-1),$$

we have

$$i^{(\alpha(F))}(G,x) = i^{(\alpha(F))}(H,x) + c \cdot k \cdot \alpha(F)!$$

Since $\alpha(H) \geq \alpha(F) + 3$, the polynomial $i^{(\alpha(F))}(H,x)$ has degree at least 3. We also know that c and $\alpha(F)!$ are both at least 1 so we may choose a sufficiently large k and apply Lemma 3.1.3 to show that $i^{(\alpha(F))}(G,x)$ has (nonreal) roots with arbitrarily large real parts. By the Gauss-Lucas Theorem, the same is true of $i(G,x)$. □

Since the independence number of a complete graph is 1, the following corollary follows immediately by taking $F = K_1$.

Corollary 3.1.6. *Let G be a graph with independence number at least 4, and let $R > 0$. Then for all m sufficiently large, $i(G + K_m, x)$ has a root with real part greater than R.* □

Corollary 3.1.7. *If G is a graph with $\alpha(G) \geq 4$, then G is an induced subgraph of a graph with independence number $\alpha(G)$ that is not stable.*

Proof. From Corollary 3.1.6, $H = G + K_m$ is not stable for m sufficiently large. Joining a clique does not change the independence number of the graph, so $\alpha(H) = \alpha(G)$ and G is a subgraph of H. □

We have shown that every graph with independence number at most 3 is stable, and there are nonstable graphs of all higher independence numbers. A complete characterization (in terms of the coefficients of $i(G,x)$) follows as a corollary of the Hermite-Biehler Theorem when $\alpha(G) = 4$, although it is difficult to extract meaningful information about the graph from it.

Proposition 3.1.8. *If G is a graph of order n with $\alpha(G) = 4$, then G is stable if and only if $i_2^2 - 4i_4 \geq 0$ and*

$$\frac{-i_2 - \sqrt{i_2^2 - 4i_4}}{2i_4} \leq -\frac{n}{i_3} \leq \frac{-i_2 + \sqrt{i_2^2 - 4i_4}}{2i_4}.$$

Proof. Let G be a graph of order n with $\alpha(G) = 4$, so $i(G,x) = 1 + nx + i_2x^2 + i_3x^3 + i_4x^4$. Therefore, $i^{even}(G,x) = 1 + i_2x + i_4x^2$ and $i^{odd}(G,x) = n + i_3x$. By the Hermite-Biehler Theorem, $i(G,x)$ is stable if and only if $i^{even}(G,x)$ and $i^{odd}(G,x)$ have only nonpositive roots and $i^{odd}(G,x) \prec i^{even}(G,x)$. The only nonpositive roots condition is always satisfied for $i^{odd}(G,x)$ and is satisfied for $i^{even}(G,x)$ if and only if its roots are real, which occurs if and only if $i_2^2 - 4i_4 \geq 0$. Provided that the even and odd parts have all nonpositive roots, the interlacing condition is satisfied if and only if

$$\frac{-i_2 - \sqrt{i_2^2 - 4i_4}}{2i_4} \leq -\frac{n}{i_3} \leq \frac{-i_2 + \sqrt{i_2^2 - 4i_4}}{2i_4}.$$

□

For example, $K_{19} + \overline{K_4}$, with independence polynomial $x^4 + 4\,x^3 + 6\,x^2 + 23\,x + 1$, is stable since $i_2^2 - 4i_4 = 36 - 4 \cdot 1 = 32 > 0$ and the inequality

$$\frac{-i_2 - \sqrt{i_2^2 - 4i_4}}{2i_4} \leq -\frac{n}{i_3} \leq \frac{-i_2 + \sqrt{i_2^2 - 4i_4}}{2i_4}$$

is satisfied as $-3 - 2\sqrt{2} \leq -\frac{23}{4} \leq -3 + \sqrt{2}$. However, if we increase the size of the clique by one vertex, we obtain the nonstable graph $K_{20} + \overline{K_4}$ with independence polynomial $x^4 + 4\,x^3 + 6\,x^2 + 24\,x + 1$. In this case, $i_2^2 - 4i_4$ is still equal to 32, but, we have the inequality

$$-\frac{n}{i_3} < \frac{-i_2 - \sqrt{i_2^2 - 4i_4}}{2i_4} < \frac{-i_2 + \sqrt{i_2^2 - 4i_4}}{2i_4}$$

as $-6 < -3 - 2\sqrt{2} < -3 + \sqrt{2}$.

We can provide complete characterizations for stability of graphs with independence number 5 and 6, but these start to lose meaning as the roots of the even and odd parts of the independence polynomials become more complicated, so instead we give a result with conditions for a nonstable graph with independence number 5 or 6 in terms of only a few coefficients.

Proposition 3.1.9. *Let G be a graph of order n and α denote its independence number. If $\alpha = 5$, then G is nonstable if conditions i) or ii) are satisfied and if $\alpha = 6$, then G is nonstable if condition ii) is satisfied. The conditions are:*

i) $i_2^2 - 4i_4 < 0$

ii) $i_3^2 - 4ni_5 < 0$. $\square$

For example, for the graph $\overline{K_6} + K_{11}$ with independence polynomial $x^6 + 6\,x^5 + 15\,x^4 + 20\,x^3 + 15\,x^2 + 17\,x + 1$, we have $20^2 - 4 \cdot 17 \cdot 6 = -8 < 0$. Hence the graph is nonstable.

3.2 Graphs with Stable Independence Polynomials

While we have seen that graphs with small independence number are stable, what other families of graphs are stable? By direct calculations, graphs on up to at least 10 vertices and trees on up to at least 20 vertices have all their independence roots in the LHP. As noted earlier, a graph with all real independence roots is necessarily stable since the real independence roots must be negative (as independence polynomials have all positive coefficients). The Chudnovsky-Seymour result therefore implies that all claw-free graphs are stable. What about families of stable graphs with whose independence polynomials do not have all real roots? We begin by showing that stars (which include the claw $K_{1,3}$) are examples of such graphs. We make use of another well-known result from complex analysis, Rouché's Theorem (see, for example, [41]).

Theorem 3.2.1 (Rouché's Theorem). *Let f and g be analytic functions on an open set containing γ, a simple piecewise smooth closed curve, and its interior. If $|f(z) + g(z)| < |f(z)|$ for all $z \in \gamma$, then f and g have the same number of zeros inside γ, counting multiplicities.* $\square$

Proposition 3.2.2. *The roots of $i(K_{1,n}, x)$ are in the left half-plane, and more precisely, in the rectangle shown in Figure 3.2.*

Proof. Let $G = K_{1,n}$; then $i(G, x) = x + (1 + x)^n$. Let $f(z) = -(1 + z)^n$ and $g(z) = (1 + z)^n + z$ and set

- $\gamma_1 = \{z : \mathrm{Re}(z) = 0 \text{ and } -2 \leq \mathrm{Im}(z) \leq 2\}$,
- $\gamma_2 = \{z : -3 \leq \mathrm{Re}(z) \leq 0 \text{ and } \mathrm{Im}(z) = 2\}$,
- $\gamma_3 = \{z : \mathrm{Re}(z) = -3 \text{ and } -2 \leq \mathrm{Im}(z) \leq 2\}$, and

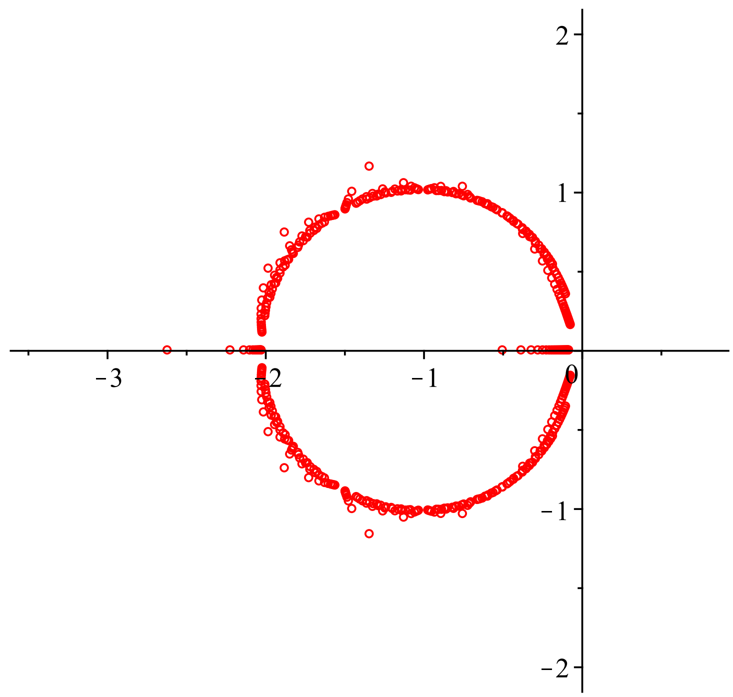


Figure 3.1: The independence roots of $K_{1,n}$ for $1 \leq n \leq 30$.

- $\gamma_4 = \{z : -3 \leq \mathrm{Re}(z) \leq 0 \text{ and } \mathrm{Im}(z) = -2\}$.

Let γ be the curve consisting of four line segments $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, i.e. $\gamma = \gamma_1+\gamma_2+\gamma_3+\gamma_4$, see Figure 3.2. The functions f and g are clearly analytic on $\mathbb{C}$ which contains γ and its interior. The curve γ is a simple piecewise smooth closed curve so the hypotheses of Rouché's Theorem are satisfied.

We now show that $|f(z) + g(z)| < |f(z)|$ for all $z \in \gamma$ (we actually consider their squares to simplify computations). Note that $|f(z)+g(z)| = |z|$. As $i(K_{1,1}, x) = 1+2x$ has only one root at $-1/2$, we will assume $n \geq 2$.

Case 1: If $z \in \gamma_1$, then $z = ki$ where $-2 \leq k \leq 2$. Now, $|z|^2 = k^2$ and $|f(z)|^2 = |1+x|^{2n} = (1+k^2)^n$. Clearly $k^2 < (1+k^2)^n$, so it follows that $|f(z)+g(z)|^2 < |f(z)|^2$, and hence $|f(z) + g(z)| < |f(z)|$ for all $z \in \gamma_1$.

Case 2: If $z \in \gamma_2$, then $z = k + 2i$ where $-3 \leq k \leq 0$. In this case $|z|^2 = k^2 + 4$ and $|(1 + x)^n|^2 = ((1 + k)^2 + 4)^n$. As in case 1, it suffices to show $((1 + k)^2 + 4)^2 > k^2 + 4$ since $n \geq 2$. Now $h(k) = ((1 + k)^2 + 4)^2 - k^2 - 4$ takes on the value 59 at $k = 1$ and

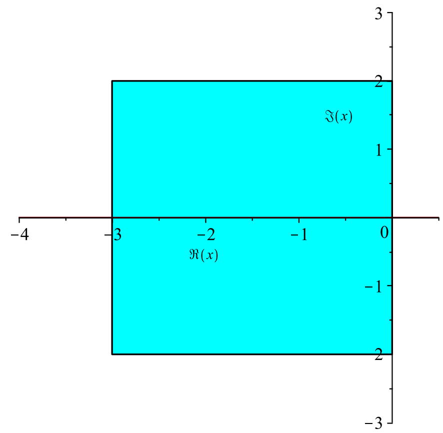


Figure 3.2: The region γ in Proposition 3.2.2.

it is easy to show that $h(k)$ has no real roots. Therefore, $((1+k)^2+4)^2 > k^2+4$ for all k and hence $|f(z)+g(z)| < |f(z)|$ for all $z \in \gamma_2$.

Case 3: If $z \in \gamma_3$, then $z = -3+ki$ where $-2 \leq k \leq 2$. Now, $|z|^2 = 9+k^2$ and $|(1+z)^n|^2 = (4+k^2)^n$. It suffices to show that $9+k^2 < (4+k^2)^2$ since $n \geq 2$. Evaluating at $k=0$, $(4+k^2)^2 - k^2 - 9$ takes on the value 7 and it has no real roots. Hence the inequality holds for all k, and so $|f(z)+g(z)| < |f(z)|$ for all $z \in \gamma_3$.

Case 4: If $z \in \gamma_4$, then $z = k-2i$ where $-3 \leq k \leq 0$. If we set $w = \bar{z} = k+2i$, then $|z|^2 = |w|^2 = k^2+4$ and $|(1+z)^n| = |(1+w)^n| = ((1+k)^2+4)^2$ so the rest of the argument is the same as that for case 2.

All cases together show that for all $z \in \gamma$, $|f(z)+g(z)| < |f(z)|$. Hence, by Rouché's Theorem, we know that f and g have the same number of zeros inside γ counting multiplicities. We know that f has one root of multiplicity n at $z=-1$ which is inside γ. Therefore $g(z) = i(G,z)$ has all n of its roots in γ which is contained in the (open) left half-plane. □

We now extend the star family to a much larger family of graphs that are also stable. The *corona* of a graph G with a graph H, denoted $G \circ H$, is defined by starting with the graph G, and for each vertex v of G, joining a new copy H_v of H to v. The graph $G \circ H$ has $|V(G)| + |V(G)||V(H)|$ vertices and $|E(G)| + |V(G)||E(H)| + |V(G)||V(H)|$ edges. For example, the star $K_{1,n}$ can be thought of as $K_1 \circ \overline{K_n}$.

See Figure 3.3 for an example of the corona of two other graphs. There is a nice relationship between the independence polynomials of G, H, and $G \circ H$ that was first described by Gutman [45].

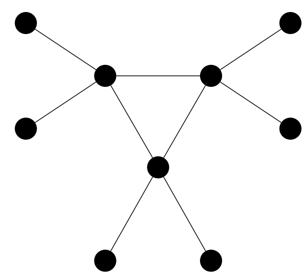


Figure 3.3: The graph $K_3 \circ \overline{K_2}$

Theorem 3.2.3 ([45])**.** *If G and H are graphs with G on n vertices, then*

$$i(G \circ H, x) = i\left(G, \tfrac{x}{i(H,x)}\right) i(H,x)^n. \quad \square$$

One special case of the corona product that is particularly useful is the corona with K_1, or the graph star that we have already seen in Chapter 2. Recall that

$$i(G^*, x) = i\left(G, \tfrac{x}{1+x}\right)(1+x)^n.$$

We now show that the graph star operation preserves the stability of independence polynomials. Similarly to the proof of Theorem 2.2.1, the argument uses properties of Möbius transformations.

Proposition 3.2.4. *If the roots of $i(G,x)$ lie outside of the region bounded by the circle with radius $\frac{1}{2}$ centred at $\frac{1}{2}$, then $i(G^*,x)$ is stable.*

Proof. Let C be the circle with center $z = 1/2$ and radius $1/2$. Note that the image of the imaginary axis, $\{z : \mathrm{Re}(z) = 0\}$, under the Möbius transformation $f(z) = \frac{z}{1+z}$ is C (one need only observe that the image of the points 0, i, and $-i$ are 0, $\frac{1}{2} + \frac{1}{2}i$, and $\frac{1}{2} - \frac{1}{2}i$, respectively). Moreover, as Möbius transformations send lines and circles to lines and circles, and the interiors/exteriors of circles and half-planes of lines to the same set, we find that the open right half-plane gets mapped to the interior of the circle C (as $\frac{1}{2}$, which is in the open RHP, gets mapped to $\frac{1}{3}$, which is in the interior of C). It follows that the open LHP gets mapped to the exterior of C.

The roots of $i(G^*, x)$ are -1 and the roots found by solving $f(z) = r$ for every root r of $i(G, x)$ since $i(G^*, x) = (1+x)^n i(G, \frac{x}{1+x})$ by Proposition 3.2.3. Therefore, if $i(G, x)$ has roots outside of C, then $i(G^*, x)$ is stable.

□

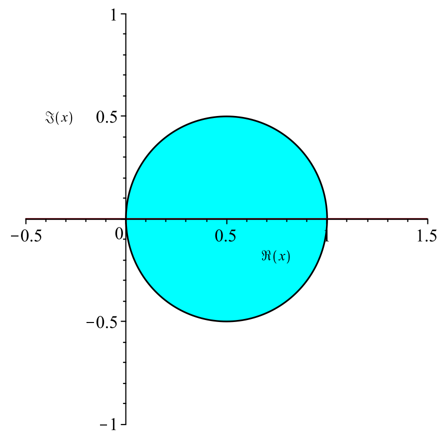


Figure 3.4: Region in Proposition 3.2.4.

In Section 3.1 we showed that for $\alpha(G) \leq 3$, $i(G, x)$ is always stable. This together with Proposition 3.2.4 and Theorem 3.2.2 proves the following corollary.

Corollary 3.2.5. *If G is a claw-free graph, $G = K_{1,n}$, or $\alpha(G) \leq 3$, then the graph k–star of G is stable for all $k \geq 1$.* □

Corollary 3.2.5 provides more families of stable graphs, but can the k–star be used to construct more families? It turns out it can be used to show that every graph is eventually stable after iterating the star operation enough times. Since the sector in Theorem 2.1.4 is contained in the LHP, we already know that this is true. We can however, get away with a smaller k if we only care about getting into the LHP and not necessarily in the sector. Our next result is proved in an analogous manner to Theorem 2.1.4, and the k in question is clearly smaller in some cases. We first recall that for any G or order n and any positive integer k,

$$i(G^{k*}, x) = i(G, \tfrac{x}{kx+1})(kx+1)^n \prod_{\ell=1}^{k-1} (\ell x + 1)^{n2^{k-\ell-1}}).$$

Proposition 3.2.6. *Let G be a graph and S be the set of its independence roots. If*

$$k > \max_{r \in S} \left\{ \frac{\mathrm{Re}(r)}{|r|^2} \right\},$$

then G^{k*} is stable.

Proof. Let $|V(G)| = n$ and

$$k > \max_{r \in S} \left\{ \frac{\mathrm{Re}(r)}{|r|^2} \right\}.$$

Then by Proposition 2.1.1,

$$i(G^{k*}, x) = i(G, \tfrac{x}{kx+1})(kx+1)^n \prod_{\ell=1}^{k-1} (\ell x + 1)^{n2^{k-\ell-1}}.$$

We know that the rational roots of the form $-\frac{1}{\ell}$ will surely all lie in the LHP so we must only consider the roots of $i(G, \frac{x}{kx+1})(kx+1)^{\alpha(G)}$ which can be found by solving for z in $r = \frac{z}{kz+1}$ where $r \in S$, that is, r is an independence root of G. Let $r \in S$, with $r = a + ib$ and consider the independence root of G^{k*} of the form $z = \frac{r}{1-kr}$. Now a calculation shows that

$$\mathrm{Re}(z) = \frac{(a^2+b^2)(-k)+a}{(1-ka)^2+k^2b^2} \tag{3.1}$$

Thus the sign of $\mathrm{Re}(z)$ is the sign of $(a^2+b^2)(-k)+a = |r|^2(-k) + \mathrm{Re}(r)$ and

$$\begin{aligned} |r|^2(-k) + a &< |r|^2 \left(-\frac{a}{|r|^2} \right) + a \\ &= -a + a \\ &= 0. \end{aligned}$$

Therefore, $\mathrm{Re}(z) < 0$ for all independence roots z of G^{k*}. Hence G^{k*} is stable.

□

The next corollary provides an interesting contrast with different graph operations when compared with Corollary 3.1.7.

Corollary 3.2.7. *Every graph is an induced subgraph of a stable graph.* □

3.3 Nonstable Families of Graphs

We have seen that joining a large clique to a graph with independence number at least 4 produces a nonstable graph. In this section we provide more constructions that will produce families of nonstable graphs, using the lexicographic product and the corona product. The last construction preserves acyclicity (as $G \circ \overline{K_m}$ preserves acyclicity) and therefore provides families of nonstable trees, which the construction of the previous section does not (and is surprising, given that we have noted that there are no roots in the RHP for trees of order at most 20).

We note that there are families of complete multipartite graphs that are stable. For example, we have shown that stars are stable and they are complete bipartite graphs. As well, it is not hard to see that

$$i(K_{n,n,\ldots,n}, x) = k(1+x)^n - (k-1).$$

The roots of this are $z_k = \left(\frac{k-1}{k}\right)^{1/n} e^{2k\pi/n} - 1$ for $k = 0, 1, \ldots, n-1$. Since $\left(\frac{k-1}{k}\right)^{1/n} < 1$ for all $k \geq 1$, it follows that $\mathrm{Re}(z_k) < 0$ for all k. Therefore, $K_{n,n,\ldots,n}$ is stable for all n and k.

It may seem that all complete multipartite graphs are stable, but such is not the case. We will consider the graphs $K_{1,2,3,\ldots,n}$, the complete multipartite graph with one part of each of the sizes $1, 2, \ldots, n$, and use Corollary 3.3.2 to prove that these graphs are not stable if $n \geq 15$. The graph $K_{1,2,\ldots,n}$ can be thought of as taking the graph K_n and replacing vertex v_i with a copy of $\overline{K_i}$ and joining all vertices in this copy of $\overline{K_i}$ to all vertices in the copy of $\overline{K_j}$ for all $i \neq j$. See Figure 3.5 for the graph $K_{1,2,3}$.

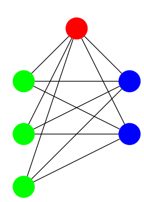


Figure 3.5: The graph $K_{1,2,3}$ with vertices coloured according to their part in the partition.

To prove that for large n, $K_{1,2,3,\ldots,n}$ is not stable, we use Sturm's sequences. For a real polynomial f, the *Sturm sequence* of f is the sequence $f_0, f_1, \ldots, f_k$ where $f_0 = f$,

$f_1 = f'$, $f_i = -\text{rem}(f_{i-1}, f_{i-2})$ for $i \geq 2$, where $\text{rem}(f_{i-1}, f_{i-2})$ is the remainder when f_{i-1} is divided by f_{i-2}, and f_k is the last nonzero term in the sequence of polynomials of strictly decreasing degrees. Sturm sequences are a very useful tool for determining the nature of polynomial roots due to the following result (see [54]).

Theorem 3.3.1 (Sturm's Theorem). *Let f be a polynomial with real coefficients and $(f_0, f_1, \ldots, f_k)$ be its Sturm sequence. Let $a < b$ be two real numbers that are not roots of f. Then the number of distinct roots of f in (a, b) is $V(a) - V(b)$, where $V(c)$ is the number of changes in sign in $(f_0(c), f_1(c), \ldots, f_k(c))$.* □

The Sturm sequence $(f_0, f_1, \ldots, f_k)$ of f is said to have *gaps in degree* if there is a $j \leq k$ such that $\deg(f_j) < \deg(f_{j-1}) - 1$. If there is a $j \leq k$ such that f_j has a negative leading coefficient, the Sturm sequence is said to have a *negative leading coefficient*. We now have the terminology to state the corollary of Sturm's Theorem (see [17]) that will be useful for our purposes.

Corollary 3.3.2. *Let f be a real polynomial whose degree and leading coefficient are positive. Then f has all real roots if and only if its Sturm sequence has no gaps in degree and no negative leading coefficients.* □

We are now ready to prove that infinitely many complete multipartite graphs are nonstable.

Theorem 3.3.3. *$i(K_{1,2,\ldots,n}, x)$ is not stable for $n \geq 15$.*

Proof. The cases where $n = 15$ and $n = 16$ require different arguments than the general ones we will apply for $n \geq 17$, so we will handle these two cases directly first.

If $n = 15$, then

$$i^{odd}(K_{1,2,\ldots,15}, x) = x^7 + 120\,x^6 + 1820\,x^5 + 8008\,x^4 + 12870\,x^3 + 8008\,x^2 + 1820\,x + 120$$

and

$$i^{even}(K_{1,2,\ldots,15}, x) = 16\,x^7 + 560\,x^6 + 4368\,x^5 + 11440\,x^4 + 11440\,x^3 + 4368\,x^2 + 560\,x + 1,$$

which have roots (found with at least 15 decimal points of accuracy using Maple)

$$-103.086868919817448, -10.8672960124875662, -3.50015487115602886,$$
$$-1.48449017876442602, -0.676272827195300907, -0.273430546648278205,$$
$$-0.111486643930936116$$

and

$$-25.2741423636056801, -5.82844291180558383, -2.23870542516616533,$$
$$-1.01405263705034243, -0.332554659001808528, -0.310290823327695198,$$
$$-0.00181118004271995899$$

respectively. Writing the roots of $i^{odd}(K_{1,2,\ldots,15}, x)$ in bold and $i^{even}(K_{1,2,\ldots,15}, x)$ in normal font, the roots are ordered as follows:

-103.08686891981744 $< -25.2741423636056801 <$ **-10.8672960124875662** $<$
$-5.8284429118055838 <$ **- 3.50015487115602886** < -2.23870542516616533
-1.48449017876442602 $< -1.01405263705034243 <$ **- 0.676272827195300907** $<$
$-0.332554659001808528 < -0.310290823327695198$**- 0.273430546648278205** $<$
-0.111486643930936116 $< -0.00181118004271995899.$

These roots do not interlace and therefore $K_{1,2,\ldots,15}$ is not stable by the Hermite-Biehler Theorem.

If $n = 16$, let $f_0 = i^{even}(K_{1,2,\ldots,16}, x) = x^8 + 136\,x^7 + 2380\,x^6 + 12376\,x^5 + 24310\,x^4 +$

$19448\,x^3 + 6188\,x^2 + 680\,x + 1$ and we compute the Sturm sequence:

$$f_1 = 8\,x^7 + 952\,x^6 + 14280\,x^5 + 61880\,x^4 + 97240\,x^3 + 58344\,x^2 + 12376\,x + 680$$

$$f_2 = 1428\,x^6 + 25704\,x^5 + 119340\,x^4 + 194480\,x^3 + 119340\,x^2 + 25704\,x + 1444$$

$$f_3 = \frac{6528\,x^5}{7} + \frac{141440\,x^4}{21} + \frac{282880\,x^3}{21} + \frac{65280\,x^2}{7} + \frac{776864\,x}{357} + \frac{48928}{357}$$

$$f_4 = \frac{335920\,x^4}{27} + \frac{1136960\,x^3}{27} + \frac{1932839\,x^2}{51} + \frac{4782274\,x}{459} + \frac{375395}{459}$$

$$f_5 = \frac{404843256\,x^3}{272935} + \frac{643306536\,x^2}{272935} + \frac{240797448\,x}{272935} + \frac{5360760}{54587}$$

$$f_6 = \frac{80617898366255329 69\,x^2}{1612423062955779} + \frac{178608898351090977 38\,x}{4837269188867337} + \frac{3198476546318302015}{4837269188867337}$$

$$f_7 = \frac{160559105190893635413988853041849816 2848\,x}{6537702611175257624330967220569238073} + $$

$$\frac{44872442470971720473043508889847480 7072}{6537702611175257624330967220569238073}$$

$$f_8 = -\frac{15774489377967441282026196375240878520276582902203759257352 60560}{7962713616255106549978377942920923565242492929835603174267 0249}$$

Since f_8 has negative leading coefficient, Theorem 3.3.1 gives that $i^{even}(K_{1,2,\ldots,16}, x)$ has nonreal roots and therefore $K_{1,2,\ldots,16}$ is not stable by Theorem 3.1.1.

Now let $n \geq 17$ and $G = K_{1,2,\ldots,n}$. The independence polynomial of G can be written as

$$\begin{aligned} i(G,x) &= \sum_{k=1}^{n}(1+x)^k - (n-1) \\ &= \frac{(1+x)^{n+1}-(1+x)}{x} - (n-1) \\ &= \frac{(1+x)^{n+1}-nx-1}{x}. \end{aligned}$$

Since $i(G,0) = 1$, it follows that the nonzero roots of $g = (1+x)^{n+1} - nx - 1$ are precisely the roots of $i(G,x)$. Let g^{odd} be the odd part of g as in the Hermite-Biehler Theorem, so $g^{odd} = 1 + \binom{n+1}{3}x + \binom{n+1}{5}x^2 + \cdots + \binom{n+1}{\ell}x^{(\ell-1)/2}$ where

$$\ell = \begin{cases} n & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even} \end{cases}$$

is the largest odd number for which G has an independent with that size. Note that $g^{odd}(0) = 1$ as well so 0 is not a root of g^{odd}, therefore, the roots of g^{odd} are all real if and only if the roots of $f = x^n g^{odd}\left(\frac{1}{x}\right) = x^n + \binom{n+1}{3}x^{n-1} + \binom{n+1}{5}x^{n-2} + \cdots + \binom{n+1}{\ell}x^{n-(\ell-1)/2}$ are all real. We will find the Sturm sequence of f and show that it has a negative leading coefficient to prove that f has nonreal roots.

For our calculation of the Sturm sequence of f, we first note that for general polynomials $h(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots$ and $j(x) = a'x^{n-1} + b'x^{n-1} + c'x^{n-3} + \cdots$, we only require the two leading coefficients to fill in the equation $h(x) = q(x)j(x) + r(x)$ as,

$$\begin{array}{r|llll} & & & \frac{a}{a'}x+ & \frac{\left(b-\frac{b'}{a'}\right)}{a'} \\ \hline a'x^{n-1}+ \quad b'x^{n-2}+\cdots & ax^n+ & bx^{n-1}+\cdots & & \\ & ax^n+ & bx^{n-1}+\cdots & & \\ \hline & & \frac{\left(b-\frac{b'}{a'}\right)}{a'}x^{n-1}+\cdots & & \\ & & \frac{\left(b-\frac{b'}{a'}\right)}{a'}x^{n-1}+\cdots & & \\ \hline & & & r(x) & \end{array}$$

Therefore,

$$r(x) \quad = \quad h(x) - q(x)j(x) = h(x) - \left(\frac{a}{a'}x + \frac{(b - \frac{b'}{a'})}{a'} \right) j(x). \qquad (3.2)$$

Let the Sturm sequence of f be $(f_0, f_1, \ldots, f_k)$ (where $f_0 = f$ and $f_1 = f'$). Both f_0 and f_1 are nonzero and have positive leading coefficient. For the calculation of f_2, f_3, and f_4, we use (3.2) and Maple as the calculations become quite involved. This allows us to obtain the leading coefficients of the first 5 terms in the Sturm sequence of f.

The leading coefficient of f_2 is calculated as

$$c_2 = \frac{n^5}{36} - \frac{2\,n^4}{45} + \frac{n^3}{36} - \frac{n^2}{36} - \frac{n}{18} + \frac{13}{180},$$

a polynomial in n. This polynomial has its largest real root at approximately 1.454, so $c_2 > 0$ for $n \geq 2$. The third term in the sequence, f_3, has leading coefficient

$$c_3 = \frac{(n-2)(n-3)\left(105\,n^8+5719\,n^7-34103\,n^6+63299\,n^5-79478\,n^4+34046\,n^3+5068\,n^2-15584\,n+55488\right)n}{35280\,(5\,n^3-8\,n^2+10\,n-13)^2}.$$

The denominator of c_3 is defined and positive for all n as it has no integer roots (easily verified by the Rational Roots Theorem). The numerator's largest real root is approximately 3.587037796, and thus $c_3 > 0$ for $n \geq 4$.

We now consider the term f_4. The leading coefficient of this term is

$$c_4 = -\frac{\gamma(n+1)(n-1)(n-4)(n-5)\left(5\,n^3-8\,n^2+10\,n-13\right)^2(n+2)}{40772160\,(105\,n^8+5719\,n^7-34103\,n^6+63299\,n^5-79478\,n^4+34046\,n^3+5068\,n^2-15584\,n+55488)^2}$$

where

$$\begin{aligned}\gamma = {} & 1036035\,n^{14} - 18710307\,n^{13} + 60715080\,n^{12} - 1252685357\,n^{11} + 16301479454\,n^{10} \\ & - 71027287359,\,n^9 + 150542755560\,n^8 - 194411482671\,n^7 + 73908295527\,n^6 \\ & + 81621340094\,n^5 - 183113161400\,n^4 + 127579216128\,n^3 - 28712745216\,n^2 \\ & + 24221417472\,n + 78617640960.\end{aligned}$$

The denominator of c_4 has its largest root at approximately 3.587037796, so for

$n \geq 4$, the denominator is defined and positive. The largest root of the numerator is approximately 16.22715983, therefore $c_4 < 0$ for $n \geq 17$.

There are no gaps in degree in the Sturm sequence of f as we have ensured c_2, c_3, and c_4 are nonzero, but there is a negative leading coefficient, c_4. Hence, it follows by Corollary 3.3.2 that f, and therefore g^{odd}, has a nonreal root. Thus, by the Hermite-Biehler Theorem g, and therefore $i(G, x)$, is not stable.

□

The join has given us much to discuss in terms of nonstable graphs, but we turn now to another graph operation, the lexicographic product, for constructing other nonstable graphs. The reason the lexicographic product has been so important to the study of the independence polynomial is due to the way the independence polynomials interact.

Theorem 3.3.4 ([19]). *If G and H are graphs, then $i(G[H], x) = i(G, i(H, x) - 1)$.*

□

In [19] it was shown that the independence roots of the family $\{P_n\}_{n \geq 1}$ are dense in $(-\infty, -\frac{1}{4}]$. This leads to another application of Lemma 3.1.3.

Proposition 3.3.5. *If H is a graph with $\alpha(H) \geq 3$, then for some n sufficiently large, $P_n[H]$ has independence roots with arbitrarily large real part.*

Proof. Suppose H is a graph with $\alpha(H) \geq 3$. By Theorem 3.3.4, we know that $i(P_n[H], x) = i(P_n, i(H, x) - 1)$ and therefore the independence roots of $P_n[H]$ are found by solving $i(H, x) - 1 = r$, that is, $i(H, x) - r - 1 = 0$ for all independence roots r of P_n. Since we know that the independence roots of $\{P_n\}_{n \geq 1}$ are dense in $(-\infty, \frac{1}{4}]$, it follows that we can make $-r - 1$ as large as we like. Finally, since $\alpha(H) \geq 3$, Lemma 3.1.3(2) applies and $i(P_n[H], x)$ has roots with arbitrarily large real parts for n sufficiently large. □

Finally, we consider the stability of trees (we recall that all trees of order 20 and less have been found to be stable by computations in Maple). Could this be true in general? As we have learned from the Chudnovsky-Seymour result on independence roots of claw-free graphs, a small restriction in the graph structure can have a large impact on the independence roots. Our previous constructions for producing graphs with roots arbitrarily far in the RHP did not turn up any trees, and it would be reasonable to speculate that perhaps all trees are stable, but this, in fact, turns out to be false. Before we can provide a family of trees with nonstable independence polynomials, we first must show that there exist trees with real independence roots arbitrarily close to 0.

Lemma 3.3.6. *Fix $\varepsilon > 0$. Then for n sufficiently large, there exists a real root r of $i(K_{1,n}, x)$ with $|r| < \varepsilon$.*

Proof. From Theorem 8 in [26], it was shown that the polynomial $f(x) = x^n + x(1+x)^n$ has a real root in the interval $(-2n, -\ln(n))$ for n sufficiently large. Note that

$$\begin{aligned} x^{n+1} f\left(\frac{1}{x}\right) &= x + x^n \left(\frac{x+1}{x}\right)^n \\ &= x + (1+x)^n \\ &= i(K_{1,n}, x). \end{aligned}$$

Therefore, $i(K_{1,n}, x)$ has a real root in the interval $\left(-\frac{1}{\ln(n)}, -\frac{1}{2n}\right)$ for sufficiently large n. The result follows. □

It is interesting to note that the polynomial $f(x)$ in the previous proof is the domination polynomial of $K_{1,n}$, which we will discuss more in Chapter 6.

We can now prove that trees do not necessarily have stable independence polynomials (and in fact can have independence roots with arbitrarily large real part).

Proposition 3.3.7. *If G is a graph with $\alpha(G) \geq 4$ and $R > 0$, then for sufficiently large n, $K_{1,n} \circ G$ has independence roots in the RHP with real part at least R.*

Proof. Set $H = K_{1,n} \circ G$. By Theorem 3.2.3,

$$i(H, x) = (i(G, x))^{n+1}\, i\left(K_{1,n}, \frac{x}{i(G, x)}\right)$$

so the independence roots of H are the roots of $i(G, x)$ together with the roots of the polynomials $f(x) = -\frac{x}{r} + i(G, x)$ for all independence roots r of $i(K_{1,n}, x)$. By Lemma 3.3.6, there exist real roots r that are negative and arbitrarily close to 0 for sufficiently large n. In this case, $-\frac{x}{r} = px$ for some $p > 0$, and by choosing r sufficiently close to 0, we can make p as large as we like. Since $\alpha(G) \geq 4$, we can apply Lemma 3.1.3 to show that for any $R > 0$ and sufficiently large n, $f(x)$ has a root with real part greater than R, and so the same holds for $i(H, x)$. □

This proposition implies that trees are not necessarily stable as $K_{1,n} \circ \overline{K_m}$ is a tree for all $m \geq 1$, and will have independence roots in the RHP for all $m \geq 4$ and sufficiently large n. Thus independence roots of trees can be found with arbitrarily large real parts. For example, the tree $T = K_{1,25} \circ \overline{K_6}$ (order 182) is nonstable as can be seen from the plot of its roots in Figure 3.6.

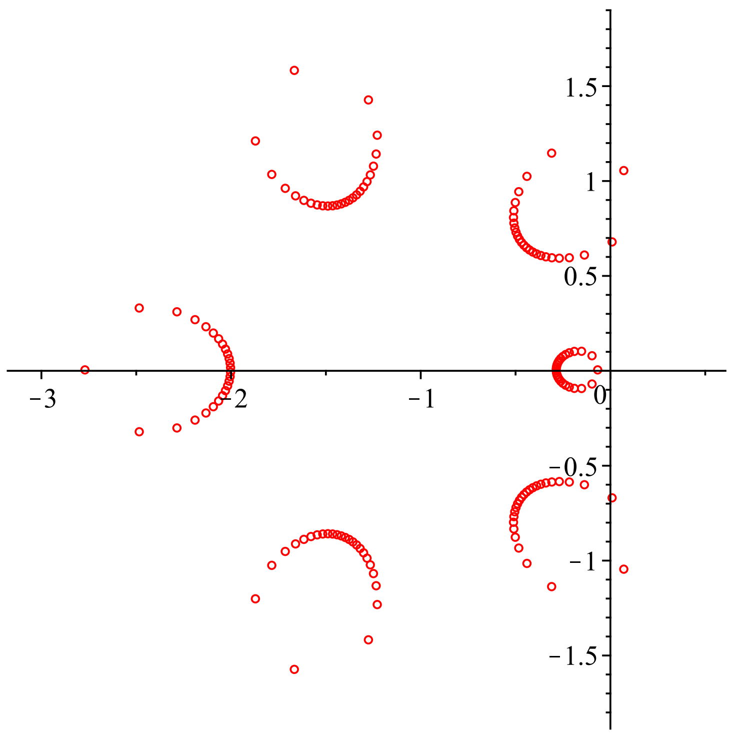


Figure 3.6: The independence roots of $K_{1,25} \circ \overline{K_6}$

3.4 Graphs with Purely Imaginary Roots

In generating nonstable graphs of small order, we came across $\overline{K_6} + K_8$, which has independence polynomial

$$x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 14x + 1 = \left(x^2 + 1\right)\left(x^4 + 6x^3 + 14x^2 + 14x + 1\right)$$

and therefore has independence roots at i and $-i$. This is interesting because Brown, Mol, and Oellermann [22] note that it is not known if independence roots could be purely imaginary. In fact, there are very few graph polynomials known to have nonzero purely imaginary roots (chromatic and reliability polynomials fall into this class). We first show that there are infinitely many connected graphs that have independence roots at i and $-i$ (of course, we can construct infinitely many disconnected graphs with a purely imaginary independence roots by taking disjoint unions of any graph with one that is known to have roots at i and $-i$).

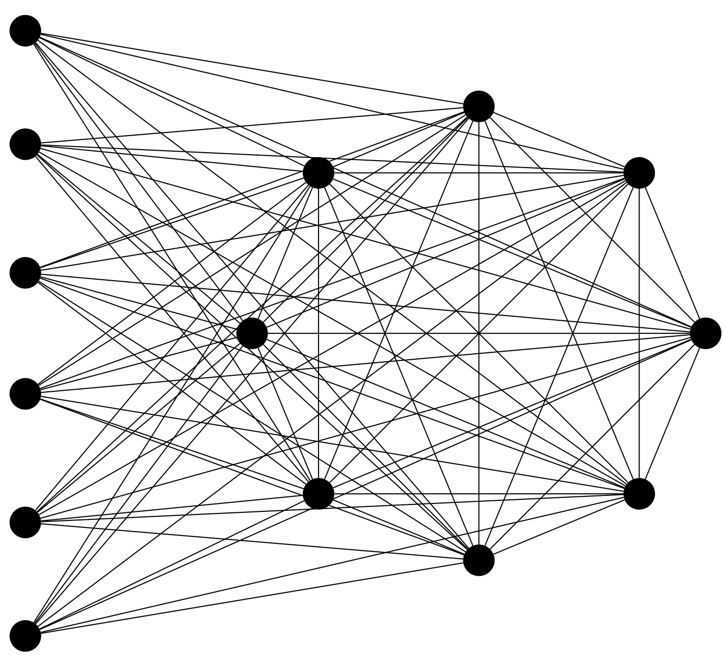


Figure 3.7: The graph $\overline{K_6} + K_8$.

Proposition 3.4.1. *If $n = 8k - 2$ for some integer $k \geq 1$, then the graph $G = \overline{K_n} + K_{2^{\frac{n}{2}}}$ has independence roots at i and $-i$.*

Proof. Let $n = 8k-2$ for some integer $k \geq 1$. Now,

$$\begin{aligned} i(K_n + K_{2^{\frac{n}{2}}}, i) &= (1+i)^n + 2^{\frac{n}{2}} i \\ &= \left(\sqrt{2} e^{i\frac{\pi}{4}}\right)^{8k-2} + 2^{4k-1} e^{i\frac{\pi}{2}} \\ &= 2^{4k-1} e^{i\frac{(4k-1)\pi}{2}} + 2^{4k-1} e^{i\frac{\pi}{2}} \\ &= 2^{4k-1} \left(e^{i(\frac{-\pi}{2} + 2k\pi)} + e^{i\frac{\pi}{2}}\right) \\ &= 2^{4k-1}(-i+i) \\ &= 0. \end{aligned}$$

Since complex roots of polynomials with real coefficients come in conjugate pairs, $K_n + K_{2^{\frac{n}{2}}}$ has independence roots at i and $-i$. □

Corollary 3.4.2. *There are infinitely many connected graphs with* i *and* $-i$ *as independence roots.* □

The following is a result that may be useful to search for other purely imaginary independence roots. Note the similarities between it and the Hermite-Biehler Theorem.

Proposition 3.4.3. *If* $p(x)$ *is a polynomial with real coefficients and* $b \in \mathbb{R}$ $(b \neq 0)$*, then* $p(x)$ *has a root at* bi *if and only if* $p^{even}(x)$ *and* $p^{odd}(x)$ *both have roots at* $-b^2$.

Proof. Let $p(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}[x]$ and $b \in \mathbb{R}$ be nonzero. Suppose that $p(bi) = 0$. Let n_e be the largest even power of x in $p(x)$ and n_o be the largest odd power of x in $p(x)$. Note that n is one of n_e or n_o, depending on its parity. Therefore,

$$\begin{aligned} 0 = p(bi) &= \sum_{k=0}^{n} a_k (bi)^k \\ &= a_0 + a_1(bi) + a_2(-b^2) + a_3(-ib^3) + a_4(b^4) + \cdots + a_n(ib)^n \\ &= a_0 - a_2(b^2) + a_4(b^4) - a_6(b^6) + \cdots + (-1)^{n_e/2} a_{n_e}(b^{n_e}) + \\ & a_1(bi) - a_3(b^3 i) + a_5(b^5 i) - a_7(b^7 i) + \cdots + (-1)^{(n_o-1)/2} a_{n_o}(b^{n_0} i) \\ &= p^{even}(-b^2) + bi \cdot p^{odd}(-b^2). \end{aligned}$$

Therefore, both $\text{Re}(p(bi)) = p^{even}(-b^2)$ and $\text{Im}(p(bi)) = b \cdot p^{odd}(-b^2)$ must equal zero. □

A corollary of Proposition 3.4.3 and our arguments in the proof of Proposition 3.1.2 is that is that no graph G with $\alpha(G) \leq 3$ can have any purely imaginary independence roots. This is because the proof of Proposition 3.1.2 actually showed that if $\alpha(G) \leq 2$, then G has all real independence roots, and if $\alpha(G) = 3$, then $r < s$ where r is the root of $i^{odd}(G, x)$ and s is the root of $i^{even}(G, x)$. Another corollary rules out certain kinds of purely imaginary independence roots for graphs in general.

Corollary 3.4.4. *If $b \in \mathbb{Z}$, $b \neq 1$, and $b \neq -1$, then bi is not an independence root of any graph.*

Proof. From Proposition 3.4.3, if bi is an independence root of G, then $-b^2$ is a root of $i^{even}(G, x)$. By the Rational Roots Theorem, any rational root must have numerator which divides the constant term of $i^{even}(G, x)$, which is 1 for every graph G. Therefore, bi is not an independence root of any graph if b is an integer and $b \neq 1$, $b \neq -1$. □

Proposition 3.4.3 does give us an idea of how to find many graphs with independence roots at i and $-i$ even when the calculations are not quite as nice as in Proposition 3.4.1. Since joining a large clique to a graph only changes the first coefficient of its independence polynomial and the change is an increase by a positive integer, graphs G for which $G + K_m$ has independence roots at i and $-i$ for some m must have -1 as a root of $i^{even}(G, x)$. Moreover, $i^{odd}(G, -1)$ must equal a negative integer.

Corollary 3.4.5. *If G is a graph such that $i^{even}(G, -1) = 0$ and $i^{odd}(G, -1) = -m$ for some positive integer m, then $G + K_m$ has independence roots at i and $-i$.* □

To generate further families of graphs with independence roots at i and $-i$, places to look are graphs with palindromic independence polynomial and independence number congruent to 2 mod 4. By palindromic independence polynomial, we mean that $i_k = i_{\alpha - k}$ for all k. This problem was first considered by Stevanović in [84], where constructions for graphs with palindromic independence polynomials were provided.

Observation 3.4.6. *If G is a graph with palindromic independence polynomial with $\alpha(G) \equiv 2$ mod 4, then $i^{even}(G, -1) = 0$.*

Therefore, looking at such graphs is a good idea if more families of graphs with i and $-i$ as independence roots are desired. It should be noted however, that the conditions in the above observation do not guarantee $i^{odd}(G, -1)$ will be negative. Consider $G = \overline{K_{10}}$. This is a graph with palindromic independence polynomial and $\alpha(G) \equiv 2 \mod 4$, but i is not an independence root of $\overline{K_{10}} + K_m$ for any m since $i^{odd}(\overline{K_{10}}, -1) = 32 > 0$. One construction that always results in a palindromic independence polynomial is to take the corona product of any graph with $\overline{K_2}$ [84]. We checked for i and $-i$ as independence roots in $G \circ \overline{K_2}$ for all graphs of order at most 9 and found that $(G \circ \overline{K_2}) + K_m$ has independence roots at i and $-i$ for *all* G of order 7. However, this construction has only worked for graphs of order 7 so far.

We were unable to find bi as an independence root for any value of b other than 1 or -1, but, other than integers, it is not known if other values of b will result in independence roots. We have also been unable to find a graph without a universal vertex that has a purely imaginary independence root.

Chapter 4

Maximum Modulus of Independence Roots of Graphs and Trees

In the preceding chapters, we have looked very closely at independence roots. The roots of other graph polynomials have also been of interest and the location in $\mathbb{C}$ of these roots can vary considerably depending on the polynomial (see [65]). Determining bounds on the moduli of these roots is an important question. In 1992, Brown and Colbourn [15] conjectured that the roots of reliability polynomials lie in the unit disk. The Brown-Colbourn conjecture stood for 12 years until it was shown to be false (although just barely) in [76]. It was later shown that if G is a connected graph on n vertices and q is a reliability root, then $|q| \leq n-1$, yet the largest known modulus of a counterexample to the Brown-Colbourn Conjecture is just approximately 1.113486 [21]. The reliability roots may still be bounded by some constant. A polynomial that is more closely related to the independence polynomial is the edge cover polynomial and it was recently shown that its roots are bounded, in fact contained in the disk $|z| < \frac{(2+\sqrt{3})^2}{1+\sqrt{3}}$ [35].

In contrast, the collection of all roots of independence polynomials [19], domination polynomials [26], and chromatic polynomials [80] are each known to be dense in $\mathbb{C}$. Although these polynomials have roots with arbitrarily large moduli, an interesting question to ask is: for fixed n, how large can the modulus of a root of one of these polynomials be for a graph of order n? Sokal [79] showed that all simple graphs on n vertices have their chromatic roots contained in the disk $|z| \leq 7.963907(n-1)$, so that the maximum modulus of chromatic roots grows at most linearly in n. The growth rate of domination roots is unknown. There has been work done on bounding the independence roots; for example, it was shown in [23] that for fixed α, the largest modulus of an independence root of a graph with independence number α on n vertices is $\left(\frac{n}{\alpha-1}\right)^{\alpha-1} + O(n^{\alpha-2})$. Appendix C shows the graphs up to order 9 with independence roots of maximum modulus .

In this paper, we consider the problem determining the maximum modulus of an independence root over all graphs on n vertices. We will show that the growth rate is indeed exponential. To that end, let $\text{maxmod}(n)$ denote the maximum modulus of an independence root over all graphs on n vertices and $\text{maxmod}_T(n)$ denote the maximum modulus of an independence root over all trees on n vertices. We show that, in contrast to Sokal's linear bound for chromatic roots, $\text{maxmod}(n)$ and $\text{maxmod}_T(n)$ are both exponential in n: in Section 4.1, we prove that

$$3^{\frac{n-r}{3}} \leq \text{maxmod}(n) \leq 3^{\frac{n}{3}} + n - 1,$$

where $1 \leq r \leq 5$, while in Section 4.2, we prove that

$$2^{\frac{n-1}{2}} \leq \text{maxmod}_T(n) \leq 2^{\frac{n-1}{2}} + \frac{n-1}{2}$$

if n is odd, and

$$2^{\frac{n-6}{2}} \leq \text{maxmod}_T(n) \leq 2^{\frac{n-2}{2}} + \frac{n}{2}$$

if n is even. The results of this chapter have appeared in [12].

We shall need some notation. The number of maximum independent sets in G is denoted by $\xi(G)$, while the number of maximal independent sets in G is denoted $\mu(G)$. Note that $\xi(G) \leq \mu(G)$ and that $\xi(G) = i^G_{\alpha(G)}$, the leading coefficient of the independence polynomial of G. We will deal with multiple independence polynomials in various calculations throughout this chapter. Thus when necessary, we will distinguish the coefficients with a superscript to avoid confusion, so that i^G_k is the number of independent sets of size k in G.

4.1 Bounds on the Maximum Modulus of Independence Roots

To bound the roots of independence polynomials, we will make extensive use of the classical *Eneström-Kakeya Theorem* which uses the ratios of consecutive coefficients of a given polynomial to describe an annulus in $\mathbb{C}$ that contains all its roots.

Theorem 4.1.1 (Eneström-Kakeya [38, 56])**.** *If $f(x) = a_0 + a_1x + \cdots + a_nx^n$ has positive real coefficients, then all complex roots of f lie in the annulus $r \leq |z| \leq R$ where*

$$r = \min\left\{\frac{a_i}{a_{i+1}} : 0 \leq i \leq n-1\right\} \quad \textit{and} \quad R = \max\left\{\frac{a_i}{a_{i+1}} : 0 \leq i \leq n-1\right\}. \quad \square$$

We will also need to make extensive use of Proposition 1.3.1, which we recall now: If G and H are graphs and $v \in V(G)$, then

i) $i(G, x) = i(G - v, x) + x \cdot i(G - N[v], x)$, and

ii) $i(G \cup H, x) = i(G, x)i(H, x)$.

Note that from Proposition 1.3.1, for the disjoint union $G \cup H$ of G and H, $\xi(G \cup H) = \xi(G) \cdot \xi(H)$. Our proofs are inductive and often require upper bounds $\xi(G)$ for all graphs on n vertices (a larger collection of these can also can be found in [55]).

Theorem 4.1.2 ([70])**.** *If G is a graph of order $n \geq 2$, then*

$$\xi(G) \leq \mu(G) \leq g(n) = \begin{cases} 3^{\frac{n}{3}} & \text{if } n \equiv 0 \mod 3 \\ 4 \cdot 3^{\frac{n-4}{3}} & \text{if } n \equiv 1 \mod 3 \\ 2 \cdot 3^{\frac{n-2}{3}} & \text{if } n \equiv 2 \mod 3 \end{cases}. \qquad \square$$

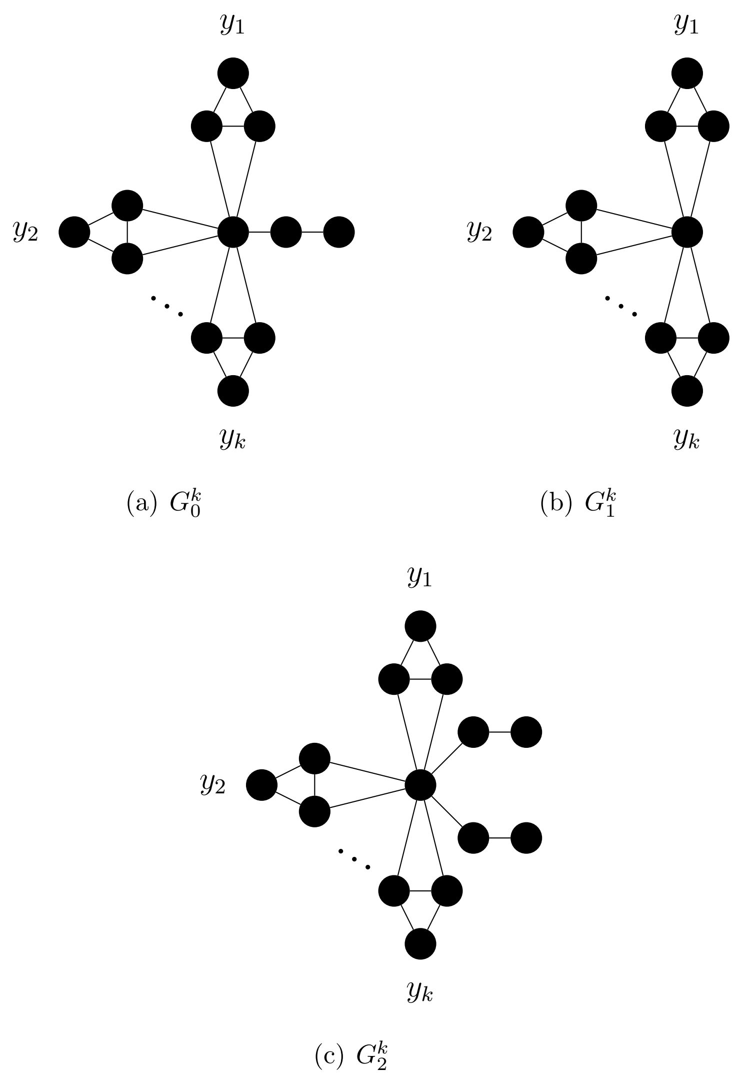
An easy corollary of this is that for a graph on n vertices, $\xi(G) \leq \mu(G) \leq 3^{\frac{n}{3}}$.

We are now ready to prove a lower bound on the maximum modulus of an independence root of a graph of order n.

Theorem 4.1.3. *For all $n \geq 1$,*

$$\text{maxmod}(n) \geq \begin{cases} 3^{\frac{n-3}{3}} & \text{if } n \equiv 0 \mod 3 \\ 3^{\frac{n-1}{3}} & \text{if } n \equiv 1 \mod 3 \\ 3^{\frac{n-5}{3}} & \text{if } n \equiv 2 \mod 3 \end{cases}.$$

Proof. The proof is in three cases depending on n mod 3. Each relies on independence polynomials of graphs G_0^k, G_1^k, and G_2^k, respectively, shown in Figure 4.1, where G_1^k is obtained by joining a central vertex to all but one vertex in each of k copies of K_3, G_0^k is obtained by joining one vertex in K_2 to the central vertex in G_1^k, and G_2^k is obtained by joining one vertex in another copy of K_2 to the central vertex in G_0^k.



(a) G_0^k (b) G_1^k

(c) G_2^k

Figure 4.1: Graphs with independence roots of large moduli.

Note that the orders of G_0^k, G_1^k, and G_2^k are congruent to 0, 1, and 2, respectively, mod 3. We then use the Intermediate Value Theorem (IVT) to prove that each has a real root of large modulus. (We shall say that a polynomial *alternates in sign* on an interval if it takes on both positive and negative values in the interval, and hence by the IVT has a root in the interval.) From Proposition 1.3.1, it easily follows that

$$
\begin{aligned}
i(G_0^k, x) &= (1 + 3x)^k (1 + 2x) + x(1 + x)^{k+1}, \\
i(G_1^k, x) &= (1 + 3x)^k + x(1 + x)^k, \text{ and} \\
i(G_2^k, x) &= (1 + 3x)^k (1 + 2x)^2 + x(1 + x)^{k+2}.
\end{aligned}
$$

It is now straightforward to determine that

$$\operatorname{sign}\left(\lim_{x\to-\infty} i(G_0^k, x)\right) = (-1)^k$$
$$\operatorname{sign}\left(\lim_{x\to-\infty} i(G_1^k, x)\right) = (-1)^{k+1}$$
$$\operatorname{sign}\left(\lim_{x\to-\infty} i(G_2^k, x)\right) = (-1)^{k+1}.$$

We now prove the lower bounds for maxmod(n) by exhibiting, in each one of the cases, a graph with a real independence root with modulus larger than the bound.

Case 0: $n \equiv 0 \mod 3$

If $n = 3$, then we can use the quadratic formula to find that P_3 has independence roots $\frac{-3\pm\sqrt{5}}{2}$ and therefore an independence root with modulus approximately $2.618 > 1$. So we may assume that $n \geq 6$ and thus $k = \frac{n-3}{3} \geq 1$ for our analysis of G_0^k. For all $k \geq 1$, we have that

$$\begin{aligned} i(G_0^k, -3^k) &= \left(1 - 3^{k+1}\right)^k \left(1 - 2\cdot 3^k\right) - 3^k \left(1 - 3^k\right)^{k+1} \\ &= (-1)^k \left[(1 - 3^k)\left((3^{k+1} - 1)^k - (3^{k+1} - 3)^k\right) - 3^k(3^k - 1)^k\right] \end{aligned}$$

which has the same sign as $(-1)^{k+1}$ since $0 < (3^{k+1}-1)^k - (3^{k+1}-3)^k$. Thus, $i(G_0^k, x)$ alternates sign on $(-\infty, -3^k) = (-\infty, -3^{\frac{n-3}{3}})$, so by the IVT, $i(G_0^k, x)$ has a root in this interval.

Case 1: $n \equiv 1 \mod 3$

If $n = 1$, then K_1 is the only graph to consider and the result clearly holds. So we may assume that $n \geq 4$ and therefore $k = \frac{n-1}{3} \geq 1$ for our analysis of G_1^k. Since $i(G_1^k, -3^k) = (-1)^k((3^{k+1}-1)^k - (3^{k+1}-3)^k)$, it follows that $i(G_1^k, -3^k)$ has the same sign as $(-1)^k$. Thus $i(G_1^k, x)$ alternates sign on $(-\infty, -3^k) = (-\infty, -3^{\frac{n-1}{3}})$ and by IVT it must have a root in the interval.

Case 2: $n \equiv 2 \mod 3$

If $n = 2$, then the graph $\overline{K_2}$ has -1 as an independence root and $|-1| = 1 = 3^0$. If $n = 5$, then P_5 has independence polynomial $1 + 5x + 6x^2 + x^3$ and $i(P_5, -5) = 1$ while $i(P_5, -6) = -29$, so P_5 has a real independence root in the interval $(-6, -5)$ by IVT. This independence root has modulus greater than 3. We now consider $n \geq 8$ and therefore $k = \frac{n-5}{3} \geq 1$ for our analysis of the graph G_2^k. We now have,

$$
\begin{aligned}
i(G_2^k, -3^k) &= (-1)^k \left[(1 - 2 \cdot 3^k)^2(3^{k+1} - 1)^k - 3^k(1 - 3^k)^2(3^k - 1)^k\right] \\
&= (-1)^k \left[(1 - 4 \cdot 3^k + 4 \cdot 3^{2k})(3^{k+1} - 1)^k - (1 - 3^k)^2(3^{k+1} - 3)^k\right] \\
&= (-1)^k \left[(1 - 4 \cdot 3^k + 3^{2k} + 3^{2k+1})(3^{k+1} - 1)^k - (1 - 3^k)^2(3^{k+1} - 3)^k\right] \\
&= (-1)^k \left[(1 - 3^k)^2 \left((3^{k+1} - 1)^k - (3^{k+1} - 3)^k\right) + \right. \\
&\qquad \left. (3^{2k+1} - 2 \cdot 3^k)(3^{k+1} - 1)^k\right]
\end{aligned}
$$

which has sign $(-1)^k$ since

$$
\begin{aligned}
&(3^{k+1} - 1)^k - (3^{k+1} - 3)^k > 0, \\
&(1 - 3^k)^2 > 0, \text{ and} \\
&(3^{2k+1} - 2 \cdot 3^k)(3^{k+1} - 1)^k > 0.
\end{aligned}
$$

Therefore, IVT yields that $i(G_2^k, x)$ must have a root in the interval $(-\infty, -3^k) = (-\infty, -3^{\frac{n-5}{3}})$.

This completes the proof. □

Therefore, $\text{maxmod}(n)$ is at least exponential in n and perhaps $3^{\frac{n}{3}}$ is a close approximation. We will indeed put an upper bound on $\text{maxmod}(n)$ but we first require the next two lemmas.

Lemma 4.1.4. *For all graphs G with at least one edge, there exists a non-isolated vertex, v, such that $\alpha(G) = \alpha(G - v) \geq \alpha(G - N[v]) + 1$.*

Proof. Let G be a graph with at least one edge. It is clear that for any vertex v of G, $\alpha(G) \geq \alpha(G - N[v]) + 1$, since any maximum independent set in $G - N[v]$ will still be independent in G with the addition of v. Suppose that for all vertices $v \in V(G)$, that $\alpha(G) > \alpha(G - v)$. Then every vertex belongs to every maximum independent set. However, G has at least one edge, so the vertices incident with this edge cannot belong to the same independent set, which contradicts both of these vertices being in every maximum independent set. Therefore, there exists some $v \in V(G)$ incident with some edge such that

$$\alpha(G) = \alpha(G - v) \geq \alpha(G - N[v]) + 1.$$

□

Lemma 4.1.5. *If G is a graph on n vertices such that $\xi(G) = 1$, then*

$$i_{\alpha(G)-1} \leq 3^{\frac{n}{3}} + n - 1.$$

Proof. Let G be a graph on n vertices such that $\xi(G) = 1$. Every independent set of size $\alpha(G) - 1$ is either maximal or is a subset of the one independent set of size $\alpha(G)$. Therefore, $i_{\alpha(G)-1} \leq \mu(G) - 1 + \alpha(G) \leq 3^{\frac{n}{3}} + n - 1$ (subtracting 1 from $\mu(G)$ to account for the one maximum independent set) by the note following Theorem 4.1.2 □

Theorem 4.1.6. *For all $n \geq 1$, $\mathrm{maxmod}(n) \leq 3^{\frac{n}{3}} + n - 1$.*

Proof. We actually prove the stronger result that, for a graph on n vertices, the ratios of consecutive coefficients of its independence polynomial are bounded above by $3^{\frac{n}{3}} + n - 1$. It then follows directly from the Eneström-Kakeya Theorem that the roots are bounded by this value. We proceed by induction on n.

The results hold for graphs on $n \leq 5$ vertices by straightforward checking the ratios of consecutive coefficients of the independence polynomials of all 52 graphs in Maple, shown in Table 4.1 along with the values for $\mathrm{maxmod}(n)$. In this table let M_n denote the maximum value $\frac{i_k^G}{i_{k+1}^G}$ can take over all k and all graphs G of order n.

Now suppose the result holds for all $5 \leq k < n$, and let G be a graph on n vertices. If G has no edges, then we are done, since G has only -1 as an independence root in this case. Therefore, G has at least one edge. Let v be a nonisolated vertex in G such that $\alpha(G) = \alpha(G - v) \geq \alpha(G - N[v]) + 1$, noting that v exists by Lemma 4.1.4. Now by Proposition 1.3.1,

n	$3^{\frac{n-r}{3}}$	maxmod(n)	M_n	$3^{\frac{n}{3}}+n-1$
1	1	1	1	1.442249570
2	1	1	2	3.080083823
3	1	2.618033989	3	5
4	3	3.732050808	4	7.326748710
5	3	5.04891733952231	6	10.24025147

Table 4.1: maxmod(n) for $n \leq 5$

$$\begin{aligned} i(G,x) &= i(G-v,x) + x \cdot i(G-N[v],x) \\ &= \sum_{k=0}^{\alpha(G-v)} i_k^{G-v} x^k + x \sum_{k=0}^{\alpha(G-N[v])} i_k^{G-N[v]} x^k \\ &= 1 + \sum_{k=1}^{\alpha(G-v)} i_k^{G-v} x^k + \sum_{k=1}^{\alpha(G-N[v])+1} i_{k-1}^{G-N[v]} x^k. \end{aligned} \tag{1}$$

We now have two cases.

Case 1: $\alpha(G) = \alpha(G-v) = \alpha(G-N[v]) + 1$.

In this case, (1) gives

$$i(G,x) = 1 + \sum_{k=1}^{\alpha(G-v)} \left(i_k^{G-v} + i_{k-1}^{G-N[v]} \right) x^k.$$

This gives $1/n$ and

$$\frac{i_k^{G-v} + i_{k-1}^{G-N[v]}}{i_{k+1}^{G-v} + i_k^{G-N[v]}} \text{ for } k = 1, 2, \ldots, \alpha(G-N[v])$$

for the ratios between coefficients. For all $n \geq 1$, $\frac{1}{n} < 3^{\frac{n}{3}} + n - 1$, and by the inductive hypothesis,

$$\frac{i_k^{G-v} + i_{k-1}^{G-N[v]}}{i_{k+1}^{G-v} + i_k^{G-N[v]}} < \frac{\left(3^{\frac{n-1}{3}} + n - 2\right) i_{k+1}^{G-v} + \left(3^{\frac{n-|N[v]|}{3}} + n - |N[v]| - 1\right) i_k^{G-N[v]}}{i_{k+1}^{G-v} + i_k^{G-N[v]}}$$
$$\leq \frac{\left(3^{\frac{n-1}{3}} + n - 2\right)\left(i_{k+1}^{G-v} + i_k^{G-N[v]}\right)}{i_{k+1}^{G-v} + i_k^{G-N[v]}}$$
$$= 3^{\frac{n-1}{3}} + n - 2$$
$$< 3^{\frac{n}{3}} + n - 1.$$

Case 2: $\alpha(G) = \alpha(G-v) > \alpha(G-N[v]) + 1$.

In this case, the independence polynomial is obtained from (1) as,

$$i(G,x) = 1 + \sum_{k=1}^{\alpha(G-N[v])+1} \left(i_k^{G-v} + i_{k-1}^{G-N[v]}\right) x^k + \sum_{\alpha(G-N[v])+2}^{\alpha(G-v)} i_k^{G-v} x^k.$$

This gives four different forms for $\frac{i_k^G}{i_{k+1}^G}$. The first two, namely $\frac{1}{n}$ and $\frac{i_k^{G-v}+i_{k-1}^{G-N[v]}}{i_{k+1}^{G-v}+i_k^{G-N[v]}}$, are less than or equal to $3^{\frac{n}{3}} + n - 1$ for each $k = 1, 2, \ldots, \alpha(G-N[v])$ by the same argument as Case 1. This leaves,

$$\frac{i_{\alpha(G-N[v])+1}^{G-v} + i_{\alpha(G-N[v])}^{G-N[v]}}{i_{\alpha(G-N[v])+2}^{G-v}} \text{ and } \frac{i_k^{G-v}}{i_{k+1}^{G-v}}, \text{ for } k \geq \alpha(G-N[v]) + 2$$

By the inductive hypothesis, $\frac{i_k^{G-v}}{i_{k+1}^{G-v}} \leq 3^{\frac{n-1}{3}} + n - 2 < 3^{\frac{n}{3}} + n - 1$, so we are left only with $\frac{i_{\alpha(G-N[v])+1}^{G-v} + i_{\alpha(G-N[v])}^{G-N[v]}}{i_{\alpha(G-N[v])+2}^{G-v}}$.

In this case, we first show that we may assume that $|N[v]| \geq 3$. As v is not isolated, $|N[v]| \geq 2$. If $|N[v]| = 2$, then v is a leaf, and since v was chosen such that $\alpha(G) = \alpha(G-v) \geq \alpha(G-N[v]) + 1$, v is not in every maximum independent set in G. But every maximum independent set in G must contain either v or its neighbour, so $\alpha(G-v) = \alpha(G-N[v]) + 1$, which was covered in Case 1. Therefore, we may assume $|N[v]| \geq 3$. We also note that

$$\frac{i^{G-v}_{\alpha(G-N[v])+1} + i^{G-N[v]}_{\alpha(G-N[v])}}{i^{G-v}_{\alpha(G-N[v])+2}} = \frac{i^{G-v}_{\alpha(G-N[v])+1}}{i^{G-v}_{\alpha(G-N[v])+2}} + \frac{\xi(G-N[v])}{i^{G-v}_{\alpha(G-N[v])+2}}.$$

There are three subcases to consider.

Case 2a: $\alpha(G-N[v]) + 2 < \alpha(G-v) = \alpha(G)$.

If $\alpha(G-N[v]) + 2 < \alpha(G-v)$, then $G-v$ has an independent set of size $\alpha(G-N[v]) + 3$. Therefore, $i^{G-v}_{\alpha(G-N[v])+2} \geq \alpha(G-N[v]) + 3 \geq 3$, since any independent set of size k, contains at least $\binom{k}{k-1} = k$ independent sets of size $k-1$. Now by the inductive hypothesis and the note following Theorem 4.1.2,

$$\begin{aligned}\frac{i^{G-v}_{\alpha(G-N[v])+1}}{i^{G-v}_{\alpha(G-N[v])+2}} + \frac{\xi(G-N[v])}{i^{G-v}_{\alpha(G-N[v])+2}} &\leq 3^{\frac{n-1}{3}} + n - 2 + \frac{\xi(G-N[v])}{3} \\ &\leq 3^{\frac{n-1}{3}} + n - 2 + 3^{\frac{n-|N[v]|-3}{3}} \\ &\leq 3^{\frac{n-1}{3}} + n - 2 + 3^{\frac{n-6}{3}} \\ &= 3^{\frac{n}{3}}\left(3^{\frac{-1}{3}} + \frac{1}{9}\right) + n - 2 \\ &\leq 3^{\frac{n}{3}} + n - 1.\end{aligned}$$

Case 2b: $\alpha(G-N[v]) + 2 = \alpha(G-v) = \alpha(G)$ and $|N[v]| \geq 4$.

In this case, by the inductive hypothesis and the note following Theorem 4.1.2,

$$\frac{i^{G-v}_{\alpha(G-N[v])+1}}{i^{G-v}_{\alpha(G-N[v])+2}} + \frac{\xi(G-N[v])}{i^{G-v}_{\alpha(G-N[v])+2}} \leq 3^{\frac{n-1}{3}} + n - 2 + \xi(G-N[v])$$
$$\leq 3^{\frac{n-1}{3}} + n - 2 + 3^{\frac{n-|N[v]|}{3}}$$
$$\leq 3^{\frac{n-1}{3}} + n - 2 + 3^{\frac{n-4}{3}}$$
$$= 3^{\frac{n}{3}}\left(3^{\frac{-1}{3}} + 3^{\frac{-4}{3}}\right) + n - 2$$
$$< 3^{\frac{n}{3}} + n - 1.$$

Case 2c: $\alpha(G-N[v]) + 2 = \alpha(G-v)$ and $|N[v]| = 3$.

We break this final case into two subcases based on the size of $i^{G-v}_{\alpha(G-N[v])+2}$. First, if $i^{G-v}_{\alpha(G-N[v])+2} \geq 2$, then by the inductive hypothesis and the note following Theorem 4.1.2,

$$\frac{i^{G-v}_{\alpha(G-N[v])+1}}{i^{G-v}_{\alpha(G-N[v])+2}} + \frac{\xi(G-N[v])}{i^{G-v}_{\alpha(G-N[v])+2}} \leq 3^{\frac{n-1}{3}} + n - 2 + \frac{\xi(G-N[v])}{2}$$
$$\leq 3^{\frac{n-1}{3}} + n - 2 + \frac{3^{\frac{n-3}{3}}}{2}$$
$$= 3^{\frac{n}{3}}\left(3^{\frac{-1}{3}} + \frac{1}{3}\right) + n - 2$$
$$\leq 3^{\frac{n}{3}} + n - 1.$$

Note that if some maximum independent set in G contained v, then this set with v removed would be an independent set of size $\alpha(G)-1 = \alpha(G-N[v])+1$ in $G-N[v]$, which is a contradiction. Therefore, the maximum independent sets in G and $G-v$ are exactly the same sets and, in particular, $\xi(G) = \xi(G-v)$. Now, if

$$1 = i^{G-v}_{\alpha(G-N[v])+2} = \xi(G-v) = \xi(G),$$

then Lemma 4.1.5 applied to G gives a bound on $i^{G}_{\alpha(G)-1}$ that we can use to show that

$$\frac{i^{G-v}_{\alpha(G-N[v])+1}+i^{G-N[v]}_{\alpha(G-N[v])}}{i^{G-v}_{\alpha(G-N[v])+2}} = \frac{i^{G}_{\alpha(G)-1}}{i^{G}_{\alpha(G)}}$$
$$= i^{G}_{\alpha(G)-1}$$
$$\leq 3^{\frac{n}{3}}+n-1.$$

All cases together show that if z is an independence root of G, then, by the Eneström-Kakeya Theorem, $|z| \leq 3^{\frac{n}{3}}+n-1$.

□

Theorems 4.1.3 and 4.1.6 yield the following corollary.

Corollary 4.1.7.
$$\frac{\log_3(\text{maxmod}(n))}{n} = \frac{1}{3}+o(1).$$

Proof. From Theorems 4.1.3 and 4.1.6, we have the following chain of equalities:

$$3^{\frac{n-r}{3}} \leq \text{maxmod}(n) \leq 3^{\frac{n}{3}}+n-1$$
$$\frac{n}{3}-\frac{r}{3} \leq \log_3(\text{maxmod}(n)) \leq \frac{n}{3}+\log_3\left(1+\frac{n-1}{3^{\frac{n}{3}}}\right)$$
$$\frac{1}{3}-\frac{r}{3n} \leq \frac{\log_3(\text{maxmod}(n))}{n} \leq \frac{1}{3}+\frac{\log_3\left(1+\frac{n-1}{3^{\frac{n}{3}}}\right)}{n}$$

Thus
$$\frac{\log_3(\text{maxmod}(n))}{n} = \frac{1}{3}+o(1).$$

□

4.2 Bounds for Trees

Now that we have determined bounds on maxmod(n), a natural extension of this is to determine the largest modulus an independence root can obtain among all graphs of order n in a specific family of graphs. In particular, the bound we obtained for maxmod(n) seems to be much too large when we restrict our attention to trees. In this section, we consider $\mathrm{maxmod}_T(n)$, the maximum modulus of an independence root over all trees on n vertices.

Let T_k be the tree obtained by identifying k copies of P_3 together at a leaf (see Figure 4.2). This tree is known [77] to have the largest number of maximal independent sets among trees on $2k+1$ vertices and we will show that it also has the largest ratio of consecutive coefficients among all such trees, and therefore provides an upper bound on $\mathrm{maxmod}_T(n)$.

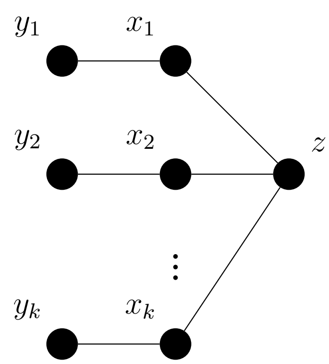


Figure 4.2: The tree T_k on $2k+1$ vertices that has independence root in $[-2^k-k, -2^k)$.

We need to extend our notation to $\mathrm{maxmod}_F(n)$, the maximum modulus of an independence root of a forest of order n; clearly $\mathrm{maxmod}_T(n) \leq \mathrm{maxmod}_F(n)$. We will also require a technical lemma for proving an upper bound on the ratio of consecutive coefficients of the independence polynomials of forests. The proof of this lemma relies on the following theorem.

Theorem 4.2.1 ([88]). *If G is a tree of order $n \geq 2$, then*

$$\xi(G) \leq t(n) = \begin{cases} 2^{\frac{n-3}{2}} & \text{if } n \text{ is odd} \\ \\ 2^{\frac{n-2}{2}}+1 & \text{if } n \text{ is even} \end{cases} . \quad \square$$

Lemma 4.2.2. *If F is a forest on n vertices, $n \geq 2$, and $v \in V(F)$, then*

$$\frac{\xi(F)}{\xi(F-v)} \leq \begin{cases} 2^{\frac{n-3}{2}}+1 & \text{if } n \text{ is odd} \\ \\ 2^{\frac{n-2}{2}}+1 & \text{if } n \text{ is even} \end{cases} .$$

Proof. Let $F = H_1 \cup H_2 \cup \cdots \cup H_k$, where $k \geq 1$, and each H_i is a connected component of F. Suppose, without loss of generality, that $v \in H_k$. Then

$$F - v = H_1 \cup H_2 \cup \cdots H_{k-1} \cup F',$$

where F' is the forest obtained from deleting v from H_k (note that if v was an isolated vertex in F, then F' may have no vertices and $\xi(H_k) = 1$). Now we have,

$$\begin{aligned}
\frac{\xi(F)}{\xi(F-v)} &= \frac{\xi(H_1)\cdot\xi(H_2)\cdots\xi(H_k)}{\xi(H_1)\cdot\xi(H_2)\cdots\xi(H_{k-1})\cdot\xi(F')} \\
&= \frac{\xi(H_k)}{\xi(F')} \\
&\leq \xi(H_k) \\
&\leq \max\{t(i) : 1 \leq i \leq n\} \qquad \text{(from Theorem 4.2.1)} \\
&= \begin{cases} 2^{\frac{n-3}{2}} + 1 & \text{if } n \text{ is odd} \\ \\ 2^{\frac{n-2}{2}} + 1 & \text{if } n \text{ is even} \end{cases}.
\end{aligned}$$

□

We are now ready for the main result of this section.

Theorem 4.2.3. *For $n \geq 1$,*

$$\operatorname{maxmod}_F(n) \leq \begin{cases} 2^{\frac{n-1}{2}} + \frac{n-1}{2} & \text{if } n \text{ is odd} \\ \\ 2^{\frac{n-2}{2}} + \frac{n}{2} & \text{if } n \text{ is even} \end{cases}.$$

Proof. As in the proof of Theorem 4.1.6, we actually prove a stronger result, bounding the ratios of consecutive coefficients. The Eneström-Kakeya Theorem then applies to obtain the bound the roots. We proceed by induction on n.

For $n = 1, 2, 3, 4$ the results hold by checking all forests of order at most 4 (see Table 4.2 and Table 4.4). Suppose the result holds for all $4 \leq k \leq n-1$ and let F be a forest on n vertices. Note that if $F = \overline{K_n}$, then the largest ratio of consecutive coefficients of $i(F, x)$ can easily be verified to be n which is less than the result in either case, so suppose F has at least one edge and therefore at least one leaf. Let v be a leaf of F and let u be adjacent to v. Note that $\alpha(F-\{u,v\}) \leq \alpha(F-v) \leq \alpha(F-\{u,v\})+1$ and for our argument, we assume that $\alpha(F-v) = \alpha(F-\{u,v\})$ and will address the easier case when $\alpha(F-v) = \alpha(F-\{u,v\})+1$ shortly. To simplify notation, let $\alpha = \alpha(F-v) = \alpha(F-\{u,v\})$. By Proposition 1.3.1, we have

$$\begin{aligned} i(F,x) &= i(F-v,x) + x \cdot i(F-\{u,v\},x) \\ &= \sum_{k=0}^{\alpha} i_k^{F-v} x^k + x \sum_{k=0}^{\alpha} i_k^{F-\{u,v\}} x^k \\ &= 1 + \sum_{k=1}^{\alpha} \left(i_k^{F-v} + i_{k-1}^{F-\{u,v\}} \right) x^k + i_{\alpha}^{F-\{u,v\}} x^{\alpha+1}. \end{aligned} \tag{2}$$

(Note that if $\alpha(F-v) = \alpha(F-\{u,v\})+1$, then

$$i(F,x) = 1 + \sum_{k=1}^{\alpha(F-v)} \left(i_k^{F-v} + i_{k-1}^{F-\{u,v\}} \right) x^k.$$

This yields $\frac{1}{n}$ and $\frac{i_k^{F-v}+i_{k-1}^{F-\{u,v\}}}{i_{k+1}^{F-v}+i_k^{F-\{u,v\}}}$ for $k = 1, 2, \ldots \alpha(F-v)-1$ as the ratios of coefficients. We will cover both of these ratios under our assumption that $\alpha(F-v) = \alpha(F-\{u,v\})$, so we do not need to consider $\alpha(F-v) = \alpha(F-\{u,v\})+1$ separately.)

We need to show that $\frac{i_k^F}{i_{k+1}^F}$ is bounded above by the desired value and from (2), we see that $\frac{i_k^F}{i_{k+1}^F}$ can take on the following forms:

$$\frac{1}{n}, \frac{i_k^{F-v} + i_{k-1}^{F-\{u,v\}}}{i_{k+1}^{F-v} + i_k^{F-\{u,v\}}} \text{ for } k = 1, 2, \ldots \alpha(F-\{u,v\})-1, \quad \text{and} \quad \frac{i_{\alpha}^{F-v} + i_{\alpha-1}^{F-\{u,v\}}}{i_{\alpha}^{F-\{u,v\}}}.$$

The first ratio, $\frac{1}{n}$, clearly satisfies the desired bound regardless of the parity of n. We now only need to verify the remaining two forms of $\frac{i_k^F}{i_{k+1}^F}$. We will do this in two cases depending on the parity of n.

Case 1: n is odd.

We apply the inductive hypothesis to get,

$$\begin{aligned}
\frac{i_k^{F-v} + i_{k-1}^{F-\{u,v\}}}{i_{k+1}^{F-v} + i_k^{F-\{u,v\}}} &\leq \frac{\left(2^{\frac{n-3}{2}} + \frac{n-1}{2}\right) i_{k+1}^{F-v} + \left(2^{\frac{n-3}{2}} + \frac{n-3}{2}\right) i_k^{F-\{u,v\}}}{i_{k+1}^{F-v} + i_k^{F-\{u,v\}}} \\
&\leq \frac{\left(2^{\frac{n-3}{2}} + \frac{n-1}{2}\right)\left(i_{k+1}^{F-v} + i_k^{F-\{u,v\}}\right)}{i_{k+1}^{F-v} + i_k^{F-\{u,v\}}} \\
&= 2^{\frac{n-3}{2}} + \frac{n-1}{2} \\
&< 2^{\frac{n-1}{2}} + \frac{n-1}{2}.
\end{aligned}$$

For the last ratio, we have,

$$\begin{aligned}
\frac{i_\alpha^{F-v} + i_{\alpha-1}^{F-\{u,v\}}}{i_\alpha^{F-\{u,v\}}} &= \frac{i_\alpha^{F-v}}{i_\alpha^{F-\{u,v\}}} + \frac{i_{\alpha-1}^{F-\{u,v\}}}{i_\alpha^{F-\{u,v\}}} \\
&\leq \frac{\xi(F-v)}{\xi(F-\{u,v\})} + 2^{\frac{n-3}{2}} + \frac{n-3}{2} \qquad \text{(by the inductive hypothesis)} \\
&\leq 2^{\frac{n-3}{2}} + 1 + 2^{\frac{n-3}{2}} + \frac{n-3}{2} \qquad \text{(by Lemma 4.2.2)} \\
&= 2^{\frac{n-1}{2}} + \frac{n-1}{2}.
\end{aligned}$$

Therefore, the result holds when n is odd by the Eneström-Kakeya Theorem.

Case 2: Suppose that n is even.

Then we apply the inductive hypothesis to get,

$$\begin{aligned}
\frac{i_k^{F-v} + i_{k-1}^{F-\{u,v\}}}{i_{k+1}^{F-v} + i_k^{F-\{u,v\}}} &\leq \frac{\left(2^{\frac{n-2}{2}} + \frac{n-2}{2}\right) i_{k+1}^{F-v} + \left(2^{\frac{n-4}{2}} + \frac{n-2}{2}\right) i_k^{F-\{u,v\}}}{i_{k+1}^{F-v} + i_k^{F-\{u,v\}}} \\
&\leq \frac{\left(2^{\frac{n-2}{2}} + \frac{n-2}{2}\right)\left(i_{k+1}^{F-v} + i_k^{F-\{u,v\}}\right)}{i_{k+1}^{F-v} + i_k^{F-\{u,v\}}} \\
&= 2^{\frac{n-2}{2}} + \frac{n-2}{2} \\
&< 2^{\frac{n-2}{2}} + \frac{n}{2}.
\end{aligned}$$

For the last ratio, we have,

$$\begin{aligned}\frac{i_\alpha^{F-v}+i_{\alpha-1}^{F-\{u,v\}}}{i_\alpha^{F-\{u,v\}}} &= \frac{i_\alpha^{F-v}}{i_\alpha^{F-\{u,v\}}}+\frac{i_{\alpha-1}^{F-\{u,v\}}}{i_\alpha^{F-\{u,v\}}} \\ &\leq \frac{\xi(F-v)}{\xi(F-\{u,v\})}+2^{\frac{n-4}{2}}+\tfrac{n-2}{2} \qquad \text{(by the inductive hypothesis)} \\ &\leq 2^{\frac{n-4}{2}}+1+2^{\frac{n-4}{2}}+\tfrac{n-2}{2} \qquad \text{(by the Lemma 4.2.2)} \\ &= 2^{\frac{n-2}{2}}+\tfrac{n}{2}.\end{aligned}$$

Therefore, the result holds when n is even by the Eneström-Kakeya Theorem. □

Corollary 4.2.4. *For $n \geq 1$,*

$$\operatorname{maxmod}_T(n) \leq \begin{cases} 2^{\frac{n-1}{2}}+\frac{n-1}{2} & \text{if } n \text{ is odd} \\ \\ 2^{\frac{n-2}{2}}+\frac{n}{2} & \text{if } n \text{ is even}\end{cases}.$$

□

We remark that, at least in terms of the bounds on the ratio of consecutive coefficients, this is best possible as there are trees when n is odd and forests when n is even that achieve these bounds. Let n be odd, and consider the graph $T_{\frac{n-1}{2}}$ as previously defined and pictured in Figure 4.2. The independence polynomial of this tree is $(1+2x)^{\frac{n-1}{2}}+x(1+x)^{\frac{n-1}{2}}$, which has $2^{\frac{n-1}{2}}+\frac{n-1}{2}$ as its last ratio of consecutive coefficients. If n is even, then consider the forest $T_{\frac{n-2}{2}} \cup K_1$, whose independence polynomial has $2^{\frac{n-2}{2}}+\frac{n}{2}$ as its last ratio of consecutive coefficients.

We have shown that the bounds on the ratio of consecutive coefficients are tight, but are these bounds tight on the roots? It is not always the case that the upper bound on the modulus of the roots of a polynomial from the Eneström-Kakeya Theorem is tight, even for trees and forests. Take for example, the tree $K_{1,30}$ which has 30 as an upper bound on the roots from the Eneström-Kakeya Theorem but its actual root of largest modulus is approximately 2.023777128. It gets even worse when we consider taking the disjoint union of k copies of $K_{1,30}$. This forest will have the same root of maximum modulus but the Eneström-Kakeya bound is $30k$, which is unbounded. Fortunately, it turns out that the bound we found in Theorem 4.2.3 is quite good for trees when n is odd.

For the case where n is even in the next proof we require the definition of the tree T'_k as shown in Figure 4.3. Let T'_k be the graph obtained by adding two leaves to each vertex in K_2 (i.e, $K_2 \circ \overline{K_2}$) and then identifying a leaf of the resulting graph to the central vertex of T_k (see Figure 4.3).

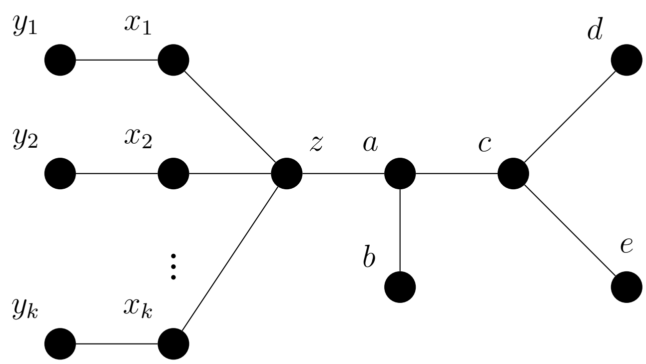


Figure 4.3: The tree T'_k on $2(k+3)$ vertices that has an independence root in $[-2^{k+1} - k - 4, -2^k)$.

Proposition 4.2.5. *For all* $n \geq 1$*,*

$$\text{maxmod}_T(n) \geq \begin{cases} 2^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ & \\ 2^{\frac{n-6}{2}} & \text{if } n \text{ is even} \end{cases}.$$

Proof. The proof is similar to the proof of Theorem 4.1.3, in that it relies on finding trees that have real independence roots of large modulus.

Case 1: n is odd.

If $n = 1$, then the result clearly holds so we may assume $n \geq 3$. Let $n = 2k + 1$, so that $k = \frac{n-1}{2} \geq 1$, and set $T = T_k$ as in Figure 4.2. A simple calculation via Proposition 1.3.1 shows that $i(T, x) = (1 + 2x)^k + x(1 + x)^k$. We will use the Intermediate Value Theorem to show that $i(T, x)$ has a real root to the left of -2^k. Now

$$
\begin{aligned}
i(T, -2^k) &= (1 - 2^{k+1})^k - 2^k(1 - 2^k)^k \\
&= (-1)^k \left((2^{k+1} - 1)^k - (2^{k+1} - 2)^k \right),
\end{aligned}
$$

so $i(T, -2^k)$ has sign $(-1)^k$. On the other hand, $i(T, x)$ has sign $(-1)^{k+1}$ as x tends to ∞. Thus, $i(T_k, x)$ alternates sign on $(-\infty, -2^k]$, so by IVT it must have a real root in the interval $(-\infty, -2^k)$. We remark that from Theorem 4.2.3, that $i(T, x)$ actually has a real root in the interval $[-2^k - k, -2^k)$.

Case 2: n is even.

For $n = 2$ and 4, the result holds by straightforward checking (see Table 4.3 where M_n^T is the maximum of consecutive coefficients over all independence polynomials of order n). For $n \geq 6$, we will show that T'_k, the graph in Figure 4.3, has a real root to the left of -2^k. Let $n = 2(k+3)$ for $k \geq 0$, so that $k = \frac{n-6}{2}$. If $k = 0$, then $i(T'_k, x) = (1+x)^2(1+4x+x^2)$ which has roots -1, $-2+\sqrt{3}$, and $-2-\sqrt{3}$, with $-2-\sqrt{3}$ being to the left of $-2^{\frac{6-6}{2}} = -1$. If $k = 1$, then $i(T'_k, x) = x^5+9\,x^4+22\,x^3+21\,x^2+8\,x+1$, which has its largest root at approximately -5.7833861, which is to the left of $-2^{\frac{8-6}{2}} = -4$. Since the result holds for $k = 0, 1$, we may now assume that $k \geq 2$.

Using Proposition 1.3.1, we find that

$$
i(T'_k, x) = (1 + 2x)^k(1 + 5x + 6x^2 + 2x^3) + x(1 + x)^k(1 + 4x + 4x^2 + x^3).
$$

Let $g(x) = 1 + 5x + 6x^2 + 2x^3$ and $h(x) = 1 + 4x + 4x^2 + x^3$. We can easily verify that $g(x) < 0$ for all $x \leq -2$ and $h(x) < 0$ for all $x \leq -3$. Moreover, $h(x) = g(x) - x(x+1)^2$. We consider the function

$$
f(x) = (-2x - 1)^k(1 + 5x + 6x^2 + 2x^3) + x(-x - 1)^k(1 + 4x + 4x^2 + x^3),
$$

so that $i(T'_k, x) = (-1)^k f(x)$. Now,

$$\begin{aligned} f(-2^k) &= (2^{k+1}-1)^k g(-2^k) - 2^k(2^k-1)^k h(-2^k) \\ &= (2^{k+1}-1)^k g(-2^k) - 2^k(2^k-1)^k (g(-2^k) + 2^k(1-2^k)^2) \\ &= g(-2^k)((2^{k+1}-1)^k - (2^{k+1}-2)^k) - 2^{2k}(2^k-1)^{k+2} \end{aligned}$$

and since $g(-2^k)$ and $-2^{2k}(2^k-1)^{k+2}$ are both negative for $k \geq 2$, it follows that $f(-2^k) < 0$. Therefore, $i(T'_k, x)$ has sign $(-1)^k(-1) = (-1)^{k+1}$. On the other hand, $i(T'_k, x)$ has sign $(-1)^{k+4} = (-1)^k$ as x tends to ∞. Thus, by the IVT, $i(T'_k, x)$ has a real root to the left of -2^k. From Theorem 4.2.3 and the Eneström-Kakeya Theorem, $i(T, x)$ has no roots in $(-\infty, -2^{k+1/2} - k - 3)$, so $i(T'_k, x)$ has a root in the interval $[-2^{k+1/2} - k - 3, -2^k)$. □

Tables 4.2 and 4.3 show values of $\text{maxmod}_T(n)$ for small values of n in comparison to our bounds. We also include Table 4.4 as the values are different for trees and forests of even order, but they are indeed the same for odd order. In these tables, let M_n^T be the maximum value of $\frac{i_k^T}{i_{k+1}^T}$ over all values of k and all trees T of order n. Let M_n^F be defined analogously for forests.

n	$2^{\frac{n-1}{2}}$	$\text{maxmod}_T(n) = \text{maxmod}_F(n)$	$M_n^T = M_n^F$	$2^{\frac{n-1}{2}} + \frac{n-1}{2}$
3	2	2.61803398900000	3	3
5	4	5.04891733952231	6	6
7	8	9.49699733952714	11	11
9	16	17.9705962347393	20	20
11	32	34.4632033453548	37	37
13	64	66.9662907779610	70	70
15	128	131.473379027662	135	135
17	256	259.980782682655	264	264

Table 4.2: Comparing $\text{maxmod}_T(n)$ to our bounds for odd n.

Although the bounds on $\text{maxmod}_T(n)$ are not as tight for even n as for odd n, Corollary 4.2.4 and Proposition 4.2.5 give the following corollary for all n.

Corollary 4.2.6.

$$\frac{\log_2(\text{maxmod}_T(n))}{n} = \frac{1}{2} + o(1).$$

n	$2^{\frac{n-6}{2}}$	$\text{maxmod}_T(n)$	M_n^T	$2^{\frac{n-2}{2}} + \frac{n}{2}$
2	0.25	0.5	1	2
4	0.5	1.77423195656734	3	4
6	1	3.732050808	6	7
8	2	5.78338611675281	9	12
10	4	10.0833151322046	14	21
12	64	18.5001015662614	23	38
14	8	34.9710040067543	40	71
16	16	67.4665144832128	73	136

Table 4.3: Comparing $\text{maxmod}_T(n)$ to our bounds for even n

n	M_n^F	$2^{\frac{n-2}{2}} + \frac{n}{2}$
2	2	2
4	4	4
6	7	7
8	12	12
10	21	21
12	38	38

Table 4.4: Comparing our bounds on $\text{maxmod}_F(n)$ and M_n^F.

Proof. From Corollary 4.2.4 and Proposition 4.2.5 we have the following. If n is odd, then

$$2^{\frac{n-1}{2}} \leq \text{maxmod}_T(n) \leq 2^{\frac{n-1}{2}} + \frac{n-1}{2}$$

$$\frac{n}{2} - \frac{1}{2} \leq \log_2(\text{maxmod}_T(n)) \leq \frac{n}{2} - \frac{1}{2} + \log_2\left(1 + \frac{n-1}{2^{\frac{n+1}{2}}}\right)$$

$$\frac{1}{2} - \frac{1}{2n} \leq \frac{\log_2(\text{maxmod}_T(n))}{n} \leq \frac{1}{2} - \frac{1}{2n} + \frac{\log_2\left(1 + \frac{n-1}{2^{\frac{n+1}{2}}}\right)}{n}$$

Therefore

$$\frac{\log_2(\text{maxmod}_T(n))}{n} = \frac{1}{2} + o(1).$$

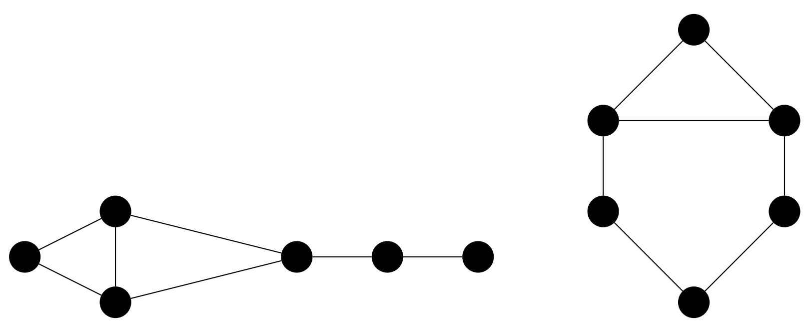
If n is even, then

$$
\begin{aligned}
2^{\frac{n-6}{2}} &\le \text{maxmod}_T(n) \le 2^{\frac{n-2}{2}} + \frac{n}{2} \\
\frac{n}{2} - 3 &\le \log_2(\text{maxmod}_T(n)) \le \frac{n}{2} - 1 + \log_2\left(1 + \frac{n}{2^{\frac{n}{2}}}\right) \\
\frac{1}{2} - \frac{3}{n} &\le \frac{\log_2(\text{maxmod}_T(n))}{n} \le \frac{1}{2} - \frac{1}{n} + \frac{\log_2\left(1 + \frac{n}{2^{\frac{n}{2}}}\right)}{n}
\end{aligned}
$$

Therefore

$$\frac{\log_2(\text{maxmod}_T(n))}{n} = \frac{1}{2} + o(1).$$

□



(a) (b)

Figure 4.4: Both graphs that achieve maxmod(6).

When the precise values of maxmod(n) and maxmod$_T(n)$ have been determined it is natural to wonder if these values are obtained by only one graph. Computational evidence suggests that this is the case for most n, but not all n. For $n = 6$, there are two graphs that achieve maxmod(6). Both graphs are shown in Figure 4.4, and it is interesting to note that both graphs have $1 + 6x + 8x^2 + x^3$ as their independence polynomial. The problem of determining graphs with equivalent independence polynomials is a fascinating one and it is our focus for the next chapter.

Chapter 5

Independence Equivalence Classes of Paths and Cycles

Properties of the coefficients and roots of independence polynomials tell us interesting things about the graph or family of graphs in question, but how good is the independence polynomial at distinguishing nonisomorphic graphs? Since graphs can be completely defined by the set of all independent sets (even just those of size 2!), it is natural to wonder how much information is lost when we only have the number of independent sets of each size, that is, the independence polynomial. As we saw at the end of the last chapter, the combinatorial information encoded in the independence polynomial is not enough to completely distinguish a graph. Studying graphs with equivalent independence polynomials is also of interest in analogy to the corresponding notion for the chromatic polynomial. The *chromaticity* of a graph, that is, the study of graphs have unique chromatic polynomials and families of graphs that share a chromatic polynomial, has been a very active area of research (see chapters 4, 5, and 6 of [37]).

We say that two unlabelled graphs G and H, are *independence equivalent*, denoted $G \sim H$, if they have the same independence polynomial. Independence equivalence is clearly an equivalence relation, so we define the *independence equivalence class* of a graph G, denoted $[G]$, to be the set of all graphs that are independence equivalent to G. We say G is *independence unique* if $[G] = \{G\}$.

As an example, $i(P_4, x) = i(K_3 \cup K_1, x) = 1 + 4x + 3x^2$, so P_4 and $K_3 \cup K_1$ are independence equivalent. This straightforward example shows that the independence polynomial does not even distinguish between connected and disconnected graphs! On the other hand, each complete graph K_n, is independence unique as it is the only graph with independence polynomial $1+nx$. In general, are most graphs independence unique? Makowsky and Zhang [66] answered this question by showing

$$\lim_{n\to\infty} \frac{\mathcal{UG}_n}{\mathcal{G}_n} = 0$$

where $\mathcal{G}_n$ is the number of nonisomorphic graphs of order n and $\mathcal{UG}_n$ is the number of nonisomorphic independence unique graphs of order n. In other words, independence unique graphs are a rarity.

Independence equivalence was first considered by Hoede and Li [50] from the perspective of the clique polynomial. Stevanović [83] showed that every threshold graph is independence unique, Brown and Hoshino [20] completely characterized independence unique circulant graphs, and Levit and Mandrescu [63] showed that well-covered spiders are independence unique among well-covered graphs. In Chism's thesis [29], the independence equivalence classes of paths were considered. In [64], it was shown that the only tree in the independence equivalence class of a given path is the path itself. A similar result for cycles was shown in [72].

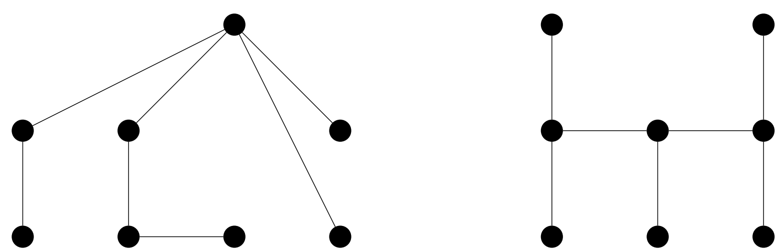


Figure 5.1: Independence equivalent trees on 8 vertices.

Even for the path and cycle of order n, determining their independence equivalence classes is tricky and subtle (much more so than for the chromatic polynomial, where the class of P_n consists of all trees of order n, and that of C_n is just itself). Chism [29] showed that $[P_{2n}]$ contains a few families of graphs (we will expand upon this in Section 5.1) and Zhang [90] proved the same results via different techniques. In [64], it was shown that the only tree in $[P_n]$ is P_n itself. Most recently, Oboudi [72] completely determined all connected graphs in the independence equivalence classes of cycles. In this work, we extend the results of Oboudi [72] and Li [64] by considering which <u>disconnected</u> graphs can be in $[P_n]$ and $[C_n]$ respectively.

This chapter is structured as follows: Section 5.1 is devoted to exploring $[P_n]$. For odd n we show that P_n is independence unique, whereas for even n there can be arbitrarily many nonisomorphic graphs in $[P_n]$. In Section 5.2, we consider $[C_n]$, using very different methods depending on the parity of n. We find that when n is even (and $n \neq 6$), or a prime power where the base is at least 5, then $[C_n] = \{C_n, D_n\}$.

Throughout this chapter, D_n is the graph obtained by identifying a leaf of P_{n-2} with one vertex of a triangle (see Figure 5.2). Our results for paths and even cycles involve combinatorial analysis that comes from analyzing the coefficients. Our results for prime cycles and prime power cycles, however, are proved using algebraic results by examining the reducibility of the polynomials. The results of this chapter have appeared in [6].

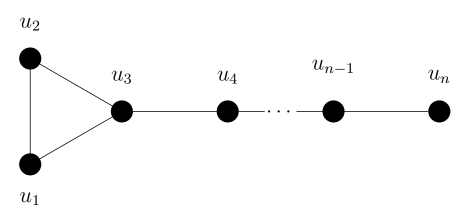


Figure 5.2: The graph D_n

5.1 Independence Equivalence Classes of Paths

The highly structured nature of paths allows for an explicit formula for $i(P_n, x)$ given in the following theorem due to Arocha.

Theorem 5.1.1 (Arocha, [4]). *The independence polynomial of a path of order n is given by*

$$i(P_n, x) = \sum_{j=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-j}{j} x^j. \quad \square$$

Despite this closed formula, $[P_n]$ has remained elusive. Recently, Li, Liu, and Wu [64] completely classified all *connected* graphs in $[P_n]$ for all n

Theorem 5.1.2 ([64]). *For any connected graph G and $n \in \mathbb{N}$, if $i(G, x) = i(P_n, x)$ then $G \cong P_n$.* $\square$

However, independence equivalence does not necessarily put a restriction on connectivity. In this section we will consider what disconnected graphs can belong to $[P_n]$, showing that P_{2k+1} is independence unique for all $k \geq 0$. We start by showing that even paths are very different in the disconnected case; there can be arbitrarily many graphs in the independence equivalence classes of even paths. To do this, we build on the basic results in [29, 90] that provide an example of a disconnected graph in $[P_n]$ for even n.

Proposition 5.1.3 ([29, 90])**.** *$P_{2n} \sim P_{n-1} \cup C_{n+1}$ for $n \geq 2$.* □

Proposition 5.1.4 ([29, 90])**.** *For $n \geq 3$, $C_n \sim D_n$ (where D_n is formed from a triangle by adding a pendant path – see Figure 5.2).* □

Proposition 5.1.5. *There exists arbitrarily large even integers n such that $||[P_n]|| \geq \frac{n}{2}$.*

Proof. Let N be a positive integer, and set $n = 2^{\lceil N/2 \rceil + 2} - 2$. We claim that P_n has at least $\frac{n}{2}$ non-isomorphic graphs in its independence equivalence class. From Proposition 5.1.3, $P_{2^{\lceil N/2 \rceil + 2 - k} - 2}$ is equivalent to $P_{2^{\lceil N/2 \rceil + 1 - k} - 2} \cup C_{2^{\lceil N/2 \rceil + 1 - k}}$ for all $k = 0, 1, \ldots, \lceil N/2 \rceil - 1$. Therefore, by iteratively applying Proposition 5.1.3, we obtain

$$P_n \sim P_{2^{\lceil N/2 \rceil + 1 - k} - 2} \cup \bigcup_{\ell=0}^{k} C_{2^{\lceil N/2 \rceil + 1 - \ell}}. \tag{5.1}$$

By Proposition 5.1.4, for $0 \leq \ell \leq k$, $C_{2^{\lceil N/2 \rceil + 1 - \ell}} \sim D_{2^{\lceil N/2 \rceil + 1 - \ell}}$. Therefore, for each value of k, the cycles in (5.1) can be replaced by equivalent graphs in 2^{k+1} ways. This, together with the graph P_n, gives $1 + 2 + 2^2 + \cdots 2^{\lceil N/2 \rceil} = 2^{\lceil N/2 \rceil + 1} - 1 = \frac{n}{2}$ distinct graphs in $[P_n]$. Therefore, $||[P_n]|| \geq \frac{n}{2}$. □

The surprising difference between the disconnected and connected graphs that are independence equivalent to even paths begs the question of what happens with odd paths. In the odd case, we completely characterize $[P_{2n+1}]$ for all n by showing, in stark contrast to Proposition 5.1.5, P_{2n+1} is independence unique for all $n \geq 0$.

Theorem 5.1.6. *P_{2n+1} is independence unique for all $n \geq 0$.*

Proof. Suppose that there exists a graph G such that $G \sim P_{2n+1}$. Note that $i(P_{2n+1}, x)$ is monic for every $n \geq 0$, since there is exactly one independent set of maximum size, $n+1$, by taking a leaf and then every second vertex along the path. So $i(G, x)$ must be monic. (We can also see that $i(P_{2n+1}, x)$ is monic from Theorem 5.1.1.) Therefore, G must have exactly one independent set of size $n+1$; call this set S. If there is a vertex in $V(G) - S$ that is adjacent to at most one vertex in S, then we can take this vertex and n vertices in S that are not adjacent with it to make a second independent set of size $n+1$, a contradiction. Therefore every vertex in $V(G)-S$ is adjacent to at least 2 vertices in S, requiring at least $2n$ edges between $V(G)-S$ and S. From the second coefficient of $i(P_{2n+1}, x)$, we know that G has exactly $2n$ edges and therefore G is a bipartite graph with bipartition $(V(G)-S, S)$. Hence G is triangle-free.

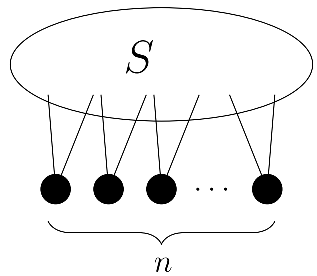


Figure 5.3: G

If $G \not\cong P_{2n+1}$, then from Theorem 5.1.2 we know that G must be disconnected. Let $G_1, G_2, \ldots, G_k$ be the connected components of G for some $k \geq 2$. Let $S_i = S \cap V(G_i)$ and $\overline{S_i} = V(G_i) - S_i$ for $i = 1, 2, \ldots, k$. Each G_i is bipartite with bipartition $(S_i, \overline{S_i})$. Suppose that for some i, $|S_i| \leq |\overline{S_i}|$. Now, $\bigcup_{j \neq i} S_j \cup \overline{S_i}$ is an independent set with at least $n+1$ vertices in it, which contradicts $i(G, x)$ being monic and of degree $n+1$. Therefore, $|S_i| \geq |\overline{S_i}| + 1$ for $i = 1, 2, \ldots, k$. Therefore,

$$2n+1 = |V(G)| = \sum_{i=1}^{k} |V(G_i)| = \sum_{i=1}^{k} \left(|S_i| + |\overline{S_i}|\right) \geq \sum_{i=1}^{k} \left(2|\overline{S_i}| + 1\right) = 2n+k \geq 2n+2,$$

a contradiction. Therefore, G must be connected, and by Corollary 5.1.2, $G \cong P_{2n+1}$. Therefore P_{2n+1} is independence unique. $\square$

It is interesting to note the dichotomy between the independence equivalence classes of even and odd paths respectively given by Proposition 5.1.5 and Theorem 5.1.6. It may be that the key distinction between the independence equivalence classes of odd and even paths is the number of independent sets of maximum size. An even path on n vertices has $\frac{n}{2}+1$ maximum independent sets, while an odd path has only one. As seen in the proof of Theorem 5.1.6, a graph having few maximum independent sets determines some structure. We will use a similar approach in the next section for even cycles.

5.2 Independence Equivalence Class of Cycles

An early result in chromaticity is that cycles are chromatically unique [27]. Clearly this is not the case for independence polynomials as Proposition 5.1.4 shows $C_n \sim D_n$ for $n \geq 4$. In this section, we will show that $[C_n] = \{C_n, D_n\}$ for n even, or n a prime at least 5 to any power. Along with these results, we have used the computational tools of nauty [68] and Maple to show that $[C_n] = \{C_n, D_n\}$ for $1 \leq n \leq 32$ with the exceptions of C_6, C_9, and C_{15}. We will present the independence equivalence classes of each of these three exceptional graphs as we proceed.

Like paths, all connected graphs which are independence equivalent to cycles have been determined.

Theorem 5.2.1 ([72])**.** *For $n \geq 3$, if G is a connected graph such that $i(G, x) = i(C_n, x)$, then $G \cong C_n$ or $G \cong D_n$.* □

Given Theorem 5.2.1, we need only consider disconnected graphs to determine $[C_n]$. We will use an argument on the degree sequence to show that there are no disconnected graphs in $[C_{2n}]$ for $n \geq 2$, and one disconnected graph in $[C_6]$. As is shown in the next theorem, using the principle of inclusion-exclusion, some information about the degree sequence of a graph is encoded in the coefficient of x^3 in its independence polynomial. We note that after personal correspondence with Hailiang Zhang, it appears that the next theorem was first stated in [91]. However, the result was presented without proof, so we prove the theorem here.

Theorem 5.2.2. *For any graph $G = (V, E)$ with n vertices and m edges*

$$i_3(G) = \binom{n}{3} - m(n-2) + \sum_{v \in V} \binom{\deg(v)}{2} - n(C_3),$$

where $i_3(G)$ is the number of independent sets in G with cardinality three and $n(C_3)$ is the number of 3-cycles in G.

Proof. It is sufficient to show the number of 3-subsets which are not independent is $m(n-2) - \sum_{v \in V} \binom{\deg(v)}{2} + n(C_3)$. Any 3-subset of V induces one of the graphs in Figure 5.4.

We can construct each non-independent 3-subset by taking an edge uv and a vertex w not incident to the edge. As G has m edges, we will construct $m(n-2)$

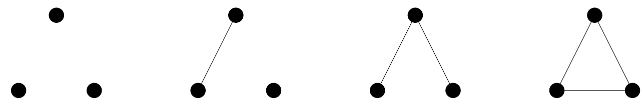


Figure 5.4: All graphs on 3 vertices.

subsets. If w is not adjacent to u nor v then we induce the subgraph (b) in Figure 5.4 and construct it once. If w is adjacent to u (or v) then we induce the subgraph (c) in Figure 5.4. However this 3-subset will have been constructed in two ways: the edge uv and vertex w, and the edge uw (or vw) and vertex v. Therefore we have counted each 3-subset which induces a subgraph of type (c) twice and (d) three times.

We can construct each 3-subset which induces a subgraph of type (c) by taking a vertex and choosing any two of its neighbours. Hence there are $\sum_{v\in V}\binom{\deg(v)}{2}$ such subsets. Note this counts the number of 3-subsets which induces subgraph (d) three times as well. Clearly the number of 3-subsets which induces subgraph (d) is $n(C_3)$. Thus the number of non-independent 3-subsets is $m(n-2)-\sum_{v\in V}\binom{\deg(v)}{2}+n(C_3)$. □

Lemma 5.2.3. *Let $n \geq 4$ and G be a graph with $n(C_3)$ many 3-cycles and g_i many vertices of degree i. If $G \sim C_n$ then*

(i) $\sum_{i=0}^{n-1} g_i = n$,

(ii) $\sum_{i=1}^{n-1} i \cdot g_i = 2n$,

(iii) $\sum_{i=2}^{n-1} \binom{i}{2} g_i = n + n(C_3)$, *and*

(iv) $n(C_3) \geq g_0 + \sum_{i=3}^{n-1} g_i$, *that is, there are at most $n(C_3)$ vertices not of degree one or two.*

Proof. Suppose G is a graph such that $G \sim C_n$. Then G has n vertices and n edges making (i) and (ii) trivial. To prove (iii), we note that by Theorem 5.2.2,

$$i_3(G) = \binom{n}{3} - n(n-2) + \sum_{i=2}^{n-1}\binom{i}{2} g_i - n(C_3)$$

.

Furthermore $i_3(C_n)$ can easily be computed to be $\binom{n}{3} - n(n-2) + n$. As $i_3(G) = i_3(C_n)$ it follows that (iii) holds. Finally by adding (i) and (iii) and subtracting (ii) we obtain:

$$n(C_3) = \sum_{i=0}^{n-1} g_i + \sum_{i=2}^{n-1} \binom{i}{2} g_i - \sum_{i=1}^{n-1} i \cdot g_i = g_0 + \sum_{i=3}^{n-1} \left(\binom{i}{2} - i + 1 \right) g_i \geq g_0 + \sum_{i=3}^{n-1} g_i.$$

Hence (iv) holds as well. □

5.2.1 Even Cycles

For even n, we can completely determine $[C_n]$.

Theorem 5.2.4. *Let $K_4 - e$ denote the graph which consists of a K_4 with one edge removed. Then*

- $[C_6] = \{C_{2n}, D_{2n}, (K_4 - e) \cup K_2\}$, *and*
- $[C_{2n}] = \{C_{2n}, D_{2n}\}$ *for* $n \geq 2,\ n \neq 3$.

Proof. Suppose $G \sim C_{2n}$ and $G \not\cong C_{2n}$. Then G has $2n$ vertices and $2n$ edges. For $n = 2$ there is only one graph, D_4, with 4 edges and 4 vertices which is not isomorphic to C_4. As $C_4 \sim D_4$ by Proposition 5.1.4 then $[C_4] = \{C_{2n}, D_{2n}\}$. We now consider when $n \geq 3$. By Theorem 5.1.1 and Proposition 1.3.1 it can be shown that $i(G, x)$ is degree n with leading coefficient equal to 2. That is, there are exactly two maximum independent sets in G of size n.

We begin by showing G contains a triangle. Suppose not, that is G is triangle-free and let g_i be the number of vertices of degree i in G. By Lemma 5.2.3 (iii) and (iv), the fact that G is triangle-free (i.e. $n(C_3) = 0$) and $G \sim C_{2n}$, we have,

$$\sum_{i=2}^{2n-1} \binom{i}{2} g_i = 2n \quad \text{and} \quad 0 \geq g_0 + \sum_{i=3}^{2n-1} g_i.$$

Hence $g_i = 0$ for $i \geq 3$ and thus $\sum_{i=2}^{2n-1} \binom{i}{2} g_i = 2n$ implies G is 2-regular. However as $G \not\cong C_{2n}$ then G is a disjoint union of cycles. It is easy to see each cycle has at least two maximum independent sets, meaning G must have at least 4 maximum independent sets which is a contradiction. Thus G contains a triangle.

As G contains a triangle, it is not bipartite, and hence the two maximum independent sets (of cardinality n) in G are not disjoint. Thus we can partition the vertices into non-empty sets A, A', B, B' such that $A \cup A'$ and $A \cup B$ are the two maximum independent sets of size n. Note $|A \cup A'| = |A \cup B| = |B \cup B'| = n$ and all sets are disjoint so $|A'| = |B|$. It follows that $|A| = |B'|$ and so $|A'| + |B'| = n$.

Each vertex in B' is adjacent to at least two vertices in $A \cup A'$. Otherwise we can form another independent set of size at least n which is not $A \cup A'$ nor $A \cup B$. Thus our partially constructed G looks like Figure 5.5.

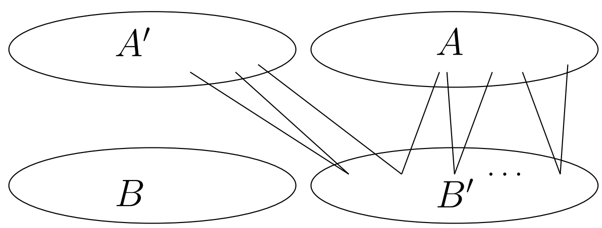


Figure 5.5: Partially constructed G.

We now consider two cases: $|B| \geq 2$ and $|B| = 1$. If $|B| \geq 2$, then by the same argument used for B' and $A \cup A'$, each vertex in A' is adjacent to at least two vertices in B. Thus G has is at least $2(|A'| + |B'|) = 2n$ edges. However, as G is not bipartite and has exactly $2n$ edges, there must be an edge between two vertices of $B \cup B'$, a contradiction.

Now suppose $|B| = 1$. In this case, there are $2(|B'|) = 2(n-1) = 2n-2$ edges between B' and $A \cup A'$ leaving only 2 edges to account for in G. As $|B| = |A'|$, we now have that $|A'| = 1$. We will label the vertex in A' and the vertex in B to be a' and b, respectively. Note a' and b are adjacent, as otherwise $A \cup A' \cup B$ forms a independent set of size $n+1$. Thus our partially constructed G (omitting one edge in $B \cup B'$) looks like Figure 5.6.

We will consider the placement of the final edge in G which must connect two vertices in $B \cup B'$ (otherwise G is triangle free). We break this into two cases.

Case 1: The edge is from b to some vertex $v \in B'$.

Then as G contains a triangle, v must be adjacent to a' (note this is only triangle

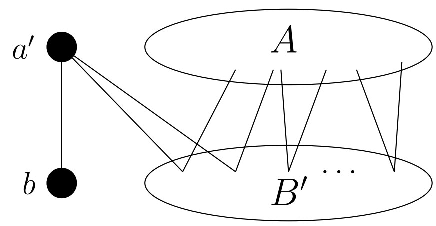


Figure 5.6: Partially constructed G with $|A'| = |B| = 1$.

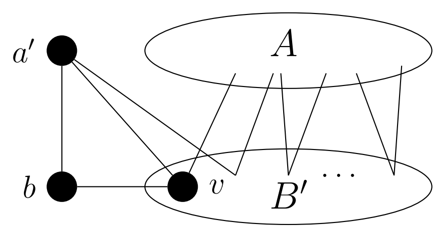


Figure 5.7: G with edge bv.

in G). All vertices in $B \cup B'$ are now degree two with the exception of v which has degree three. Thus G now looks like Figure 5.7.

We now know that G has exactly one triangle (i.e. $n(C_3) = 1$) and $G \sim C_{2n}$ so, by Lemma 5.2.3 (iv), G has at most one vertex which is not degree one or two. As v is degree three then every other vertex must either have degree one or two. Again let g_i be the number of vertices of degree i in G. Note $g_i = 0$ for $i \neq 1, 2, 3$, $g_3 = 1$, and $g_1 + g_2 + g_3 = 2n$. Furthermore by Lemma 5.2.3 (iii),

$$2n + 1 = \sum_{i=2}^{2n-1} \binom{i}{2} g_i = \binom{2}{2} g_2 + \binom{3}{2} g_3 = g_2 + 3.$$

Thus $g_2 = 2n - 2$, $g_3 = 1$ and $g_1 = 1$. Note a' must have degree two. We now construct G. Begin with the one triangle in G which is formed by the vertices a', b, and v. As v is a degree three vertex it must have a neighbour in A. Let ℓ denote the only vertex of degree 1. As all other vertices in $V(G) - \{v\}$ are all of degree two, ℓ must be in the same component as v, otherwise the component with ℓ has exactly one vertex of odd degree which contradicts the Handshaking Lemma. Therefore, there must be an induced path connecting v and ℓ. This induced path together with a' and b forms a D_r component in G for some $r \leq n$. If $r = n$, then $G \cong D_n$. If $r < n$, then G is the disjoint union of cycles and a D_r for $r < n$. However as D_n has two maximum independent sets, if G has any cycle components it would have at least four maximum independent sets which is a contradiction.

Case 2: The edge in $B \cup B'$ is between two vertices $u, v \in B'$.

As G contains a triangle, u and v must have at least one common neighbour in $A \cup A'$. Note that we now know the number of vertices of each degree in $B \cup B'$; b is degree one, u and v are degree three and every other vertex in $B \cup B'$ is degree two. Thus we consider two subcases: u and v have one or two common neighbours.

Case 2a: u and v have exactly one common neighbour

Then G has exactly one triangle and now looks like Figure 5.8.

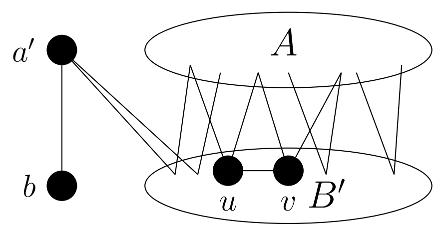


Figure 5.8: G with exactly one triangle.

As G has exactly one triangle (i.e. $n(C_3) = 1$) and $G \sim C_{2n}$, Lemma 5.2.3 (iv) gives that $g_3 \leq 1$. However u and v both have degree three which is a contradiction.

Case 2b: u and v have exactly two common neighbours.

Then G has exactly two triangles and looks like Figre 5.9.

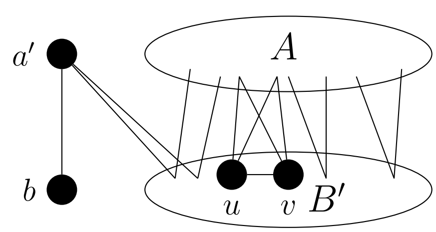


Figure 5.9: G with exactly two triangles.

Since G has exactly two triangles (i.e. $n(C_3) = 2$) and $G \sim C_{2n}$, Lemma 5.2.3 (iv) implies that $\sum_{i \neq 1,2} g_i \leq 2$. Both u and v have degree three so every other vertex must either have degree one or two. Note $g_i = 0$ for $i \neq 1, 2, 3$, $g_3 = 2$, and $g_1 + g_2 + g_3 = 2n$. Furthermore by Lemma 5.2.3 (iii),

$$2n + 2 = \sum_{i=2}^{2n-1} \binom{i}{2} g_i = \binom{2}{2} g_2 + \binom{3}{2} g_3 = g_2 + 6.$$

Thus $g_2 = 2n - 4$, $g_3 = 2$ and $g_1 = 2$. Note that every vertex in $A \cup A'$ has degree at most two, thus u, v and their two common neighbours form a K_4 less an edge component of G. Furthermore G has two vertices of degree one. As b is degree one and every vertex in B' is degree two or three, the second vertex of degree one is a' or some vertex in A.

First suppose some vertex $\ell \in A$ is degree one. At this point our graph looks like Figure 5.10.

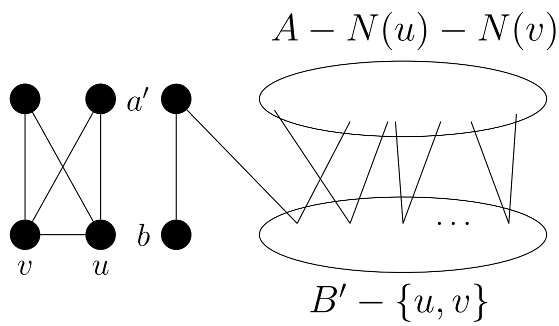


Figure 5.10: G with a $K_4 - e$ component.

Note that every vertex in $B' - \{u, v\}$ and $A - N(u) - N(v)$, other than ℓ, has degree 2. Therefore one component in G is a path of even order from b to ℓ. However, every even path with more than two vertices has at least three maximum independent sets, which is a contradiction as G only has two maximum independent sets.

Now suppose a' is degree one. Then a' and b form a K_2 component in G and the remaining vertices in $(B' - \{u, v\}) \cup (A - N(u))$ must induce a disjoint union of cycles. In the case where $n = 3$, that is $G \sim C_6$, G has no cycle components and $G \cong (K_4 - e) \cup K_2$. For $n \geq 4$, $(B' - \{u, v\}) \cup (A - N(u))$ contains at least one cycle. However, as K_2 and cycles each have at least two maximum independent sets, G has at least four maximum independent sets, which is again a contradiction.

The only two cases which didn't result in a contradiction yielded $G \cong D_{2n}$ and, when G was of order 6, $G \cong (K_4 - e) \cup K_2$. As $D_{2n} \sim C_{2n}$ for all $n \geq 3$, we have shown that $[C_6] = \{C_6, D_6, (K_4 - e) \cup K_2\}$ and $[C_{2n}] = \{D_{2n}, D_{2n}\}$ for $n \geq 4$. $\square$

5.2.2 Prime Power Cycles

In Theorem 5.2.4, we used an involved construction to show that there is only one disconnected graph that is independence equivalent to C_{2n}. This construction relies on the fact that the leading coefficient of $i(C_{2n}, x)$ is 2. This argument will not hold for odd cycles as the leading coefficient of $i(C_{2n+1}, x)$ is $2n + 1$. However, if we can show that for certain n any graph independence equivalent to C_n must be connected, then it will follow from Theorem 5.2.1 that $[C_n] = \{C_n, D_n\}$. There are other ways to show connectivity than constructing the graph from its independence polynomial, and we shall do so via irreducibility of polynomials over the rationals. We will require Eisenstein's famous criterion for irreducibility that we state here

Theorem 5.2.5 (c.f. [42] pp. 215)**.** *Let $p \in \mathbb{Z}$ be a prime and $f(x) = a_0 + a_1x + \ldots + a_nx^n$ be a polynomial of degree n with integer coefficients. If p divides each of $a_0, a_1, \ldots, a_{n-1}$ but p does not divide a_n, and p^2 does not divide a_0, then f is irreducible over the rationals.* □

Proposition 5.2.6. *If p is an odd prime, then $[C_p] = \{C_p, D_p\}$ (note $C_p \cong D_p$ when $p = 3$).*

Proof. We show that $i(C_p, x)$ is irreducible over the rationals and therefore C_p has no disconnected graphs in its equivalence class. The result will then follow by Theorem 5.2.1. Let p be an odd prime. By Theorem 5.1.1 and Proposition 1.3.1 we know that

$$\begin{aligned}
i(C_p, x) &= i(P_{p-1}, x) + x i(P_{p-3}, x) \\
&= \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p-j}{j} x^j + \sum_{j=0}^{\lfloor \frac{p-2}{2} \rfloor} \binom{p-2-j}{j} x^{j+1} \\
&= \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p-j}{j} x^j + \sum_{j=1}^{\lfloor \frac{p}{2} \rfloor} \binom{p-j-1}{j-1} x^j \\
&= 1 + \sum_{j=1}^{\lfloor \frac{p}{2} \rfloor} \left(\binom{p-j}{j} + \binom{p-j-1}{j-1} \right) x^j
\end{aligned}$$

$$= 1 + \sum_{j=1}^{\lfloor \frac{p}{2} \rfloor} \left(\frac{(p-j)!}{j!(p-2j)!} + \frac{(p-j-1)!}{(j-1)!(p-2j)!} \right) x^j$$
$$= 1 + \sum_{j=1}^{\lfloor \frac{p}{2} \rfloor} \frac{(p-j)!}{j!(p-2j)!} \left(1 + \frac{j}{p-j} \right) x^j.$$
$$= 1 + \sum_{j=1}^{\lfloor \frac{p}{2} \rfloor} \binom{p-j}{j} \left(\frac{p}{p-j} \right) x^j.$$

The coefficients above must be integers and since p is a prime, it follows that $p-j$ does not divide p for any $j = 1, 2, \ldots, \lfloor \frac{p}{2} \rfloor$, so $p - j$ must divide the integer $\binom{p-j}{j}$. Therefore, $\binom{p-j}{j}\left(\frac{p}{p-j}\right)$ is a multiple of p for $j = 1, 2, \ldots, \lfloor \frac{p}{2} \rfloor$. We now consider the coefficient of $x^{\lfloor \frac{p}{2} \rfloor}$,

$$\binom{p - \lfloor \frac{p}{2} \rfloor}{\lfloor \frac{p}{2} \rfloor} \left(\frac{p}{p - \lfloor \frac{p}{2} \rfloor} \right) = \left(\frac{(p - \lfloor \frac{p}{2} \rfloor - 1)!}{\lfloor \frac{p}{2} \rfloor!(p - 2\lfloor \frac{p}{2} \rfloor)!} \right) p$$
$$= \left(\frac{(p - \lceil \frac{p}{2} \rceil)!}{\lfloor \frac{p}{2} \rfloor!(\lceil \frac{p}{2} \rceil) - \lfloor \frac{p}{2} \rfloor)!} \right) p$$
$$= \left(\frac{\lfloor \frac{p}{2} \rfloor!}{\lfloor \frac{p}{2} \rfloor!} \right) p$$
$$= p.$$

Therefore, applying Eisenstein's famous criterion to the polynomial $x^{\alpha(C_p)} i(C_p, \frac{1}{x})$ with the prime p, it follows that $i(C_p, x)$ is irreducible over the rationals. Since $i(C_p, x)$ is irreducible, C_p cannot be independence equivalent to any disconnected graph. It follows that $[C_p] = \{C_p, D_p\}$ by Theorem 5.2.1.

□

The ideas used to show irreducibility of cycles of prime length given in Proposition 5.2.6 can be partially extended to cycles of order p^n for all n and all odd primes $p \geq 5$. These polynomials are reducible but considering each irreducible factor will lead us to the same conclusion as the case for $n = 1$.

We say that a polynomial $p(x) = \sum_{i=0}^{n} p_i x^i$ with integer coefficients is *unicyclic* if

$p_0 = 1$, $p_1 = k$ and $p_2 = \binom{k}{2} - k$ for some integer k. Note that a unicyclic polynomial is one that shares the same first three coefficients with the independence polynomial of some unicyclic graph. If a connected graph has a unicyclic independence polynomial, then that graph must be unicyclic. This is because the graph has k vertices, k edges, and is connected.

Lemma 5.2.7. *If $h(x) = g(x)f(x)$ and $h(x)$, $g(x)$ are unicyclic, then $f(x)$ is unicyclic.*

Proof. Assuming the hypothesis, let the first three terms of $g(x)$ be $1, nx, \left(\binom{n}{2} - n\right)x^2$, and the first three terms of $f(x)$ be $1, kx, \left(\binom{k}{2} - k + \ell\right)x^2$ where ℓ is some integer (note $f(x)$ must have integer coefficients from well-known factorization results, see [42, pp. 215] for example). Since $h(x)$ is unicyclic, the first three terms of $h(x)$ are $1, (n+k)x, \left(\binom{n+k}{2} - (n+k)\right)x^2$. Since $h(x) = f(x)g(x)$, they must be equal coefficient-wise so we must have,

$$\begin{aligned}\binom{n+k}{2} - (n+k) &= \binom{n}{2} - n + \binom{k}{2} - k + \ell + nk \\ &= \frac{n(n-1) + k(k-1) + 2nk}{2} - (n+k) + \ell \\ &= \frac{(n+k)((n+k)-1)}{2} - (n+k) + \ell \\ &= \frac{(n+k)^2 - (n+k)}{2} - (n+k) + \ell \\ &= \binom{n+k}{2} - (n+k) + \ell.\end{aligned}$$

Therefore, $\ell = 0$, and $f(x)$ is unicyclic. □

When looking at the factors of $i(C_n, x)$, it is very helpful to know the independence roots of C_n. Luckily, the roots of $i(C_n, x)$ have been completely determined by Alikahni and Peng [2] and we will make use of a corollary that can be derived from their results.

Theorem 5.2.8 ([2])**.** *For $n \geq 3$, the roots of $i(C_n, x)$ are given by*

$$r_i = -\frac{1}{2\left(1+\cos\left(\frac{(2i-1)\pi}{n}\right)\right)}$$

for $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$, *and these roots are all distinct.* □

Corollary 5.2.9. *For odd* n *and* $k \neq 1$, $k|n$ *if and only if* $i(C_k, x)|i(C_n, x)$.

Proof. Let n be odd. First suppose $k|n$. Then let $n = qk$ for some positive integer q. By Theorem 5.2.8, we only have to show that for all $j = 1, 2, \ldots, \lfloor \frac{k}{2} \rfloor$ there exists an i from $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ such that $\frac{(2i-1)\pi}{n} = \frac{(2j-1)\pi}{k}$. This happens if and only if

$$i = \frac{(2j-1)q+1}{2}.$$

Since n is odd, it follows that q is also odd and therefore i is indeed an integer and since $j \leq \lfloor \frac{k}{2} \rfloor$, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Thus every root of $i(C_k, x)$ is also a root of $i(C_n, x)$. Let $i(C_k, x) = (x - r_1)(x - r_2)\ldots(x - r_{\lfloor \frac{k}{2} \rfloor})$ where the r_i's are the roots of $i(C_k, x)$. Since all roots of $i(C_k, x)$ are also roots of $i(C_n, x)$, it follows that $i(C_n, x) = (x - r_1)(x - r_2)\ldots(x - r_{\lfloor \frac{k}{2} \rfloor})g(x)$ for some polynomial $g(x)$ and therefore $i(C_k, x)|i(C_n, x)$.

Conversely suppose $i(C_k, x)|i(C_n, x)$. Then the leading coefficient of $i(C_k, x)$ must divide the leading coefficient of $i(C_n, x)$. From Theorem 5.1.1 and Proposition 1.3.1, as n is odd then the leading coefficient of $i(C_n, x)$ is n. Furthermore the leading coefficient of $i(C_k, x)$ is either 2 if k is even or k if k is odd. As n is odd then $2 \nmid n$ and hence $k|n$. □

Lemma 5.2.10. *Let* p *be an odd prime and* $n \geq 1$. *Then every irreducible factor of* $i(C_{p^n}, x)$ *is unicyclic.*

Proof. The proof is by induction on n. For $n = 1$, that case was handled in Proposition 5.2.6. Suppose the result holds for $n \leq k$ for some $k \geq 1$. Now from Corollary 5.2.9, we know that $i(C_{p^k}, x)|i(C_{p^{k+1}}, x)$. Let $i(C_{p^{k+1}}, x) = i(C_{p^k}, x)r(x)$. We claim that $r(x)$ is irreducible and unicyclic. The fact that $r(x)$ is unicyclic follows from the inductive hypothesis and Lemma 5.2.7.

Similarly to the proof of Proposition 5.2.6, we derive an expression for the coefficients of $i(C_{p^k}, x)$

$$i(C_{p^k}, x) = 1 + \sum_{j=1}^{\lfloor \frac{p^k}{2} \rfloor} \binom{p^k - j}{j} \left(\frac{p^k}{p^k - j} \right) x^j.$$

Note that p divides each coefficient above except the constant term as $\binom{p^k-j}{j}\frac{p^k}{p^k-j}$ must be an integer and $p^k - j$ has at most $k-1$ factors of p for all $1 \leq j \leq p^k - 1$.

Let $r(x) = r_0 + r_1x + r_2x^2 + \cdots + r_mx^m$. Since $i(C_{p^{k+1}}, x) = i(C_{p^k}, x)r(x)$, we must have

$$\binom{p^{k+1} - j}{j} \left(\frac{p^{k+1}}{p^{k+1} - j} \right) = \sum_{i=0}^{j} \left(r_i \binom{p^k - (j-i)}{j-i} \left(\frac{p^k}{p^k - (j-i)} \right) \right) \tag{5.2}$$

for $j = 0, 1, \ldots, \lfloor \frac{p^{k+1}}{2} \rfloor$.

As noted earlier, since $p | \frac{p^{k+1}}{p^{k+1}-j}$ for $1 \leq j \leq p^{k+1} - 1$, p must divide the sum on the right hand side of (5.2). Since we know p divides each coefficient of $i(C_{p^k}, x)$ except the constant term, it follows that $p | r_j$ for all $j = 1, 2, \ldots, m$. Also, since $p^k r_m = p^{k+1}$, it follows that $r_m = p$. So by Eisenstein's Criterion applied to $x^m r(\frac{1}{x})$, it follows that $r(x)$ is irreducible. □

We are now ready to extend Proposition 5.2.6.

Theorem 5.2.11. *For $k, p \in \mathbb{N}$ where $p \geq 5$ is prime, $[C_{p^k}] = \{C_{p^k}, D_{p^k}\}$.*

Proof. Suppose $G \sim C_n$ and $G \ncong C_n$ where $n = p^k$. Then G has n vertices and n edges. Then by Lemma 5.2.3 we obtain the following three equations:

$$\sum_{i=0}^{n-1} g_i = n, \quad \sum_{i=1}^{n-1} i \cdot g_i = 2n, \quad \sum_{i=2}^{n-1} \binom{i}{2} g_i = n + n(C_3).$$

Thus,

$$n(C_3) = \sum_{i=0}^{n-1} g_i + \sum_{i=2}^{n-1} \binom{i}{2} g_i - \sum_{i=1}^{n-1} i \cdot g_i = g_0 + \sum_{i=3}^{n-1} \left(\binom{i}{2} - i + 1 \right) g_i. \tag{1}$$

Furthermore, G has no C_3 components, otherwise $i(C_3, x) | i(C_n, x)$ and hence by Corollary 5.2.9, $3|n$ which is a contradiction as $n = p^k$ for prime $p \geq 5$. Hence every induced C_3 has a vertex with degree 3 or greater. By Lemma 5.2.10, every irreducible factor of $i(C_n, x)$ is unicyclic and hence every connected component of G has the same number of vertices and edges and is therefore unicyclic. Therefore every vertex is part of at most one induced C_3. As every induced C_3 has a vertex with degree 3 or greater then

$$n(C_3) \leq \sum_{i=3}^{n-1} g_i.$$

Therefore by subtracting this inequality from equation (1) we obtain

$$0 \geq g_0 + \sum_{i=3}^{n-1} \left(\binom{i}{2} - i \right) g_i.$$

As $\binom{i}{2} - i \geq 2$ for $i \geq 4$ and $\binom{3}{2} - 3 = 0$, it follows that $g_i = 0$ for $i \neq 1, 2$ or 3. Therefore, by equation (1) we have $g_3 = n(C_3)$. We can also now simplify the sums given in Lemma 5.2.3 to get $g_1 + g_2 + g_3 = n$ and $g_1 + 2g_2 + 3g_3 = 2n$ and subtracting 2 times the former from the latter we obtain $g_1 = g_3$.

Consider the structure of G. Note that no two induced C_3 graphs intersect, as each vertex is in at most one. As G has no C_3 components, each of the induced C_3 must contain at least one degree three vertex. As $g_3 = n(C_3)$, each induced C_3 contains exactly one degree three vertex and there are no other degree three vertices in the graph. Now all that remains are degree one and two vertices. Hence the other neighbour of each degree three vertex is either a leaf or a degree two vertex. It is easy to see that if it is a degree two vertex, this must be the beginning of a path of degree two vertices ending in a leaf, otherwise we would contradict either the component being unicyclic or the number of degree three or greater vertices. This shows that each component is either a cycle or D_r for some $r \leq n$. As $D_{l_i} \sim C_{l_i}$, G must be independence equivalent to a disjoint union of cycles.

Now let $G \sim C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_r}$ for some $r \in \mathbb{N}$. Note $n_j \geq 3$ as each component must have an equal number of vertices and edges. As the independence polynomial is multiplicative across components we have

$$i(G, x) = i(C_{n_1}, x) \cdot i(C_{n_2}, x) \cdots i(C_{n_r}, x).$$

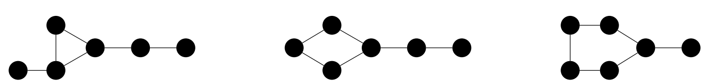
It is easy to see from Theorem 5.1.1 and Proposition 1.3.1 that the leading coefficient and the coefficient x of $i(C_{n_j}, x)$ are both n_j. Thus the leading coefficient of $i(G, x)$ is $n_1 \cdot n_2 \cdots n_r$ and the coefficient of x is $n_1 + n_2 + \cdots + n_r$. However as $i(G, x) = i(C_n, x)$ then the leading coefficient and the coefficient of x of $i(G, x)$ are both n. Thus $n_1 n_2 \cdots n_r = n_1 + n_2 + \cdots + n_r$. However a simple induction can show $n_1 \cdot n_2 \cdots n_r > n_1 + n_2 + \cdots + n_r$ for $r \geq 2$ and $n_j \geq 3$. As each $n_j \geq 3$, $r = 1$ and G is connected. By Theorem 5.2.1, we conclude that $[C_n] = \{C_n, D_n\}$.

□

One notable exception to these results is $[C_{3^n}]$ when $n > 1$. These cases are more difficult to deal with, as a graph in $[C_{3^n}]$ can have C_3 components which does not allow us the certainty of where the degree 3 vertices are located among the components. We suspect that if $[C_n]$ grows large for certain n, then n will be an odd multiple of 3. For example, the only cycles that we know of with graphs other than D_n and C_n in their independence equivalence classes are C_6, C_9 and C_{15}. Oboudi showed in [72] that

$$[C_9] = \{C_9, D_9, G_1 \cup C_3, G_2 \cup C_3, G_3 \cup C_3\}$$

where G_1, G_2, and G_3 are shown in Figure 5.11.



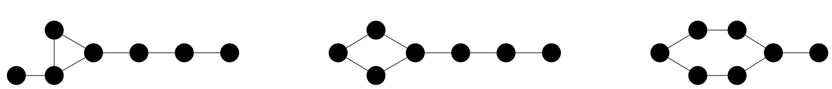
(a) G_1 (b) G_2 (c) G_3

Figure 5.11: Components of the disconnected graphs in $[C_9]$

Computationally, we were able to show that

$$[C_{15}] = \{C_{15}, D_{15}, G_1' \cup C_3 \cup C_5, G_2' \cup C_3 \cup C_5, G_3' \cup C_3 \cup C_5\}$$

where G_1', G_2', and G_3' are shown in Figure 5.12.



(a) G_1' (b) G_2' (c) G_3'

Figure 5.12: Components of the disconnected graphs in $[C_{15}]$

Despite the similarities between $[C_9]$ and $[C_{15}]$, we were able to computationally verify that $[C_{21}] = \{C_{21}, D_{21}\}$ and $[C_{27}] = \{C_{27}, D_{27}\}$.

Chapter 6

Conclusion

This thesis was primarily focused on the independence roots of graphs, although properties of the independent set sequence and independence equivalence classes were also explored. In this final chapter, we will focus on what salient open problems and conjectures arise from our work.

6.1 Unimodality, Log Concavity, and Independence Polynomials

In Chapter 2, we provided evidence for the log-concavity conjecture for very well-covered graphs by exploring a surprising connection between the roots and the log-concavity of a polynomial. We showed that every graph can be extended to a very well-covered graph that has a log-concave (and therefore unimodal) independence polynomial. The sector

$$\mathcal{R} = \{z \in \mathbb{C} : \tfrac{2\pi}{3} < |\arg(z)| < \tfrac{4\pi}{3}\}$$

played a large role in our results, and the distribution of independence roots with respect to this sector is fascinating. Computations suggest that most small graphs have their independence roots in the sector; in fact, out of 11,117 connected graphs of order 8, there are only 40 independence roots (counting multiplicities) outside the sector (see Figure 6.1). The observation seems to be true of trees of even higher order as well. Hence we propose the following conjecture.

Conjecture 6.1.1. The independence polynomial of almost every graph G has all of its independence roots lying in the sector $\{z \in \mathbb{C} : \frac{2\pi}{3} < |\arg(z)| < \frac{4\pi}{3}\}$, and hence is log concave.

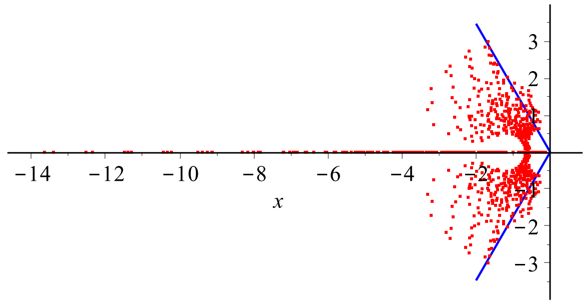


Figure 6.1: Independence roots of all connected graphs of order 8.

6.2 On the Stability of Independence Polynomials

In Chapter 3 we were able to provide constructions for families of graphs with all independence roots lying in the left half-plane as well as other constructions for families of graphs that have at least one independence root in the right half-plane. One of these constructions provided an infinite family of trees with nonstable independence polynomials.

While we have seen that stars are stable, other complete multipartite graphs are not. In the concluding remarks of [13], we asked if all complete bipartite graphs were stable and we have shown here that they are not. From computations in Maple we found a handful of nonstable complete bipartite graphs, all of which have relatively large order. The nonstable complete bipartite graph of smallest order is the graph $K_{9,22}$ which has independence roots with real part approximately 0.0006577811540.

While we have provided a number of families (and constructions) of nonstable graphs, we still feel that stability is a common property, as small graphs suggest.

Problem 6.2.1. Are almost all graphs stable?

We have also proved that trees are not necessarily stable, but the structure of a tree that ensures that its independence polynomial is stable seems elusive.

Problem 6.2.2. Characterize when a tree is stable.

We do not know the smallest nonstable graph with respect to order or number of edges. Calculations in Maple show that the order is greater than 10, but after this point it becomes infeasible to check the stability of all graphs of a fixed order. From

Corollary 3.1.6, we can search for nonstable graphs by joining a clique to all graphs of small order with independence number at least 4. We were able to iteratively join larger cliques to all graphs of order at most 8 and check the stability of the resulting graph. In [13], we said that the smallest order of a nonstable graph was between 11 and 24, as we were only looking at joining cliques to graphs with independence number 4 and the smallest graph we found was $\overline{K_4} + K_{20}$. We have since extended our search in Maple for nonstable graphs with higher independence number which has turned up several nonstable graphs of order 15 and with fewer edges than $\overline{K_4} + K_{20}$. The first was $\overline{K_6} + K_9$ (independence number 6); we found many other graphs of the same order that were nonstable. The graph $\overline{K_6} + K_9$ has $\binom{9}{2} + 6 \cdot 9 = 90$ edges though the size can be reduced by the graph $(K_{1,7} \cup K_1) + K_6$, which has $\binom{6}{2} + 7 + 6 \cdot 9 = 76$ edges, and is the lone nonstable graph with fewer than 77 edges that we found. Therefore, the smallest order of a nonstable graph is somewhere between 11 and 15 inclusive and the fewest number of edges in a nonstable graph is at most 76, but the open problem remains.

Problem 6.2.3. What is the smallest nonstable graph with respect to order and with respect to edges?

We ended Chapter 3 with a brief exploration of graphs with purely imaginary independence roots, where the only such roots we could find were i and $-i$. Our work in Section 3.4 leads us to suspect that these are the only possibilities.

Conjecture 6.2.4. If G is a graph with bi as an independence root, then $b = 1$ or $b = -1$.

6.3 The Maximum Modulus of Independence Roots

In Chapter 4, we proved bounds on the maximum modulus of an independence root of a graph and a tree on n vertices, respectively. Our work shows that the maximum modulus grows exponentially in n in both cases as

$$\frac{\log_3(\text{maxmod}(n))}{n} = \frac{1}{3} + o(1)$$

and

$$\frac{\log_2(\text{maxmod}_T(n))}{n} = \frac{1}{2} + o(1).$$

From computations with Maple and nauty [68], we have the following conjecture.

Conjecture 6.3.1. If G is a graph on n vertices, then for $n \geq 3$,

$$\text{maxmod}(n) \leq \begin{cases} 2 \cdot 3^{\frac{n-3}{3}} + \frac{n}{3} & \text{if } n \equiv 0 \mod 3 \\ 3^{\frac{n-1}{3}} + \frac{n-1}{3} & \text{if } n \equiv 1 \mod 3 \\ 4 \cdot 3^{\frac{n-5}{3}} + \frac{n+1}{3} & \text{if } n \equiv 2 \mod 3 \end{cases}$$

Conjecture 6.3.1 comes from the true upper bound on the ratios of consecutive coefficients for the independence polynomials of the graphs G_0^k, G_1^k, and G_2^k, as

$$\begin{aligned} i(G_0^k, x) &= x^{k+2} + \left(2 \cdot 3^{\frac{n-3}{3}} + \frac{n}{3}\right) x^{k+1} + \cdots + 1 \\ i(G_1^k, x) &= x^{k+1} + \left(3^{\frac{n-1}{3}} + \frac{n-1}{3}\right) x^{k} + \cdots + 1 \\ i(G_2^k, x) &= x^{k+3} + \left(4 \cdot 3^{\frac{n-5}{3}} + \frac{n+1}{3}\right) x^{k+2} + \cdots + 1. \end{aligned}$$

Furthermore, we pose that the only extremal graphs are the ones above.

Conjecture 6.3.2. The graphs G_0^k, G_1^k, and G_2^k are the only graphs to achieve $\text{maxmod}(n)$ for all $n \neq 6, 7$.

Turning now to trees, the ratio of coefficients of even order trees is smaller than the bound in Theorem 4.2.3, achieved uniquely by T'_k, which leads us to conjecture an improvement on an upper bound for $\text{maxmod}_T(n)$ for even n.

Conjecture 6.3.3. If T is a tree on n vertices with $n \geq 6$ even , then,

$$\text{maxmod}_T(n) \leq 2^{\frac{n-4}{2}} + \frac{n+2}{2}$$

Conjecture 6.3.4. The trees $T_{\frac{n-1}{2}}$ and $T'_{\frac{n-6}{2}}$ (recall Figures 4.2 and 4.3) are the only trees to achieve $\text{maxmod}_T(n)$ for n odd and even respectively.

Chapter 4 focused on two values, $\text{maxmod}(n)$ and $\text{maxmod}_T(n)$. However, questions remain open about $\text{maxmod}_{\mathcal{F}}(n)$, the maximum modulus of an independence root over all graphs of order n belonging to the family of graphs $\mathcal{F}$ for various families $\mathcal{F}$. One family of graphs of interest that we are very familiar with at this point in the thesis is the family of well-covered graphs. For each well-covered graph with independence number α, it is known that all of its independence roots lie in the disk $|z| \leq \alpha$ and there are well-covered graphs with independence roots arbitrarily close to the boundary [16]. This difference between the independence roots of graphs and well-covered graphs begs the question of what happens for well-covered trees? Finbow et al. [40] showed that every well-covered tree is necessarily equal to T^* (recall Figure 2.1) for some tree T. Recalling Proposition 2.1.1, we can obtain the independence roots of T^* by an easy Möbius transformation of the independence roots of T. This allows us to be more efficient when computing the independence roots of well-covered trees. Using Maple and nauty [68], we were able to verify that all well-covered trees on $n \leq 40$ vertices have their independence roots contained in the unit disk!

This makes it extremely tempting to conjecture that the independence roots of all well-covered trees are contained in the unit disk. However, we know that we can exploit the formula for $i(T^*, x)$ and the properties of Möbius transformations to determine the region where the independence roots of all trees would have to be contained for the independence roots of all well-covered trees to be contained in the unit disk. Any tree T with independence roots to the right of the line $\text{Re}(z) = \frac{1}{2}$, will yield a well-covered tree T^* with independence roots outside of the unit disk. From Proposition 3.3.7, we know that there are trees with independence roots arbitrarily far in the right half of $\mathbb{C}$; therefore, there are well-covered trees with independence roots outside of the unit disk. Note that all well-covered trees on n vertices have $n/2$ as an upper bound on the ratios of consecutive coefficients by Lemma 3.1 in [16], so it does not seem like the Eneström-Kakeya Theorem will be useful to solve the tantalizing question:

Question 6.3.5. What is the maximum modulus of an independence root of a well-covered tree on n vertices?

While Question 6.3.5 remains elusive, we can definitively answer which well-covered tree of order n has the independence root of *smallest* modulus. We will now briefly look at a lower bound on the modulus of an independence root of a well-covered tree to extend Oboudi's result [71] for trees.

Theorem 6.3.6 ([71]). *If T is a tree on n vertices, r is an independence root of T, and w is an independence root of $K_{1,n-1}$ of smallest modulus, then $|r| \geq |w|$.* □

The root of smallest modulus is unique and real and we will require this result as well.

Theorem 6.3.7 ([34]). *If G is a graph and w is an independence root of G of smallest modulus, then w is real and the unique root with modulus $|w|$.* □

Proposition 6.3.8. *Among all well-covered trees of order $2n$, S_n^* is the tree with the independence root of smallest modulus.*

Proof. Let $G = T^*$ be a well-covered tree on $2n$ vertices, so that T is on n vertices. We know from Theorem 6.3.7 that the independence root of smallest modulus of G is real. Therefore, we only need to consider real independence roots. Since no positive real number is an independence root of any graph, the root of smallest modulus will be the largest real root. Now let $r \neq 1$ be a real independence root of a well-covered tree on $2n$ vertices. From Proposition 2.1.1 and the Finbow et al. [40] result, we know that $r = \frac{s}{1-s}$ for some real independence root of a tree on n vertices. If we consider r as a function of s and differentiate, we obtain $r' = \frac{1}{(1-s)^2}$, which is always positive and therefore r is always increasing. Since the largest real root of a tree on n vertices is an independence root of $K_{1,n-1}$ [71], the largest real root of a well-covered tree on $2n$ vertices must be the largest real root of the $K_{1,n-1}^*$.

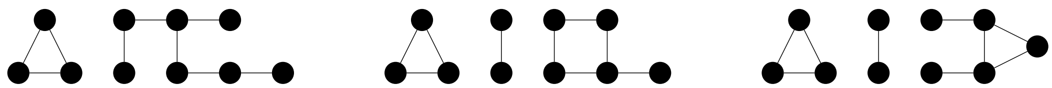
□

6.4 Independence Equivalence

Our goal in writing Chapter 5 was to completely determine $[P_n]$ and $[C_n]$. Although we did this for certain values of n, $[P_n]$ and $[C_n]$ are still unknown for the remaining values of n.

Problem 6.4.1. What graphs can be in $[P_{2n}]$?

We showed that $|[P_{2n}]|$ is unbounded for certain n, but this involved showing that $[P_{2n}]$ consisted of disjoint unions of cycles, graphs independence equivalent to cycles, and a path. However, using nauty [68], we were able to computationally determine that $|[P_{10}]| = 10$. In addition to the 7 graphs that we expected from Proposition 5.1.5, we found the 3 surprising graphs in Figure 6.2. What other graphs can belong to $[P_{2n}]$?



(a) H_1 (b) H_2 (c) H_3

Figure 6.2: Surprising graphs in $[P_{10}]$.

Problem 6.4.2. What graphs can be in $[C_{3n}]$?

Multiples of 3 make things more difficult when trying to characterize the equivalence classes of cycles as graphs in these classes can have triangle components. In fact, the only cycles we know of where $[C_n] \neq \{C_n, D_n\}$ are cycles with $n = 3k$ for k odd. Not every multiple of three has this property however, as C_{21} is only equivalent to itself and D_{21}. Does $[C_{3n}]$ eventually stabilize to the two graphs we expect, or can it grow like the independence equivalence classes of even paths?

Problem 6.4.3. Are there families of graphs such that the independence equivalence class is unbounded *and* each independence polynomial is irreducible?

We saw that $i(C_p, x)$ was irreducible and $|[C_p]| = 2$ for all primes $p \geq 3$. An irreducible independence polynomial implies that all graphs in the independence equivalence class are connected. The restriction to connected graphs via irreducibility seems that it would make it less likely to have large independence equivalence classes, but the question remains open. We also think that studying the irreducibility of independence polynomials can be useful when studying independence equivalence classes of other graphs.

Finally, we leave the reader with a conjecture that all of our results and computational work has lead us to believe is true.

Conjecture 6.4.4. If $3 \nmid n$ and $n \geq 4$, then $[C_n] = \{C_n, D_n\}$.

6.5 Future Work

While we have touched on a number of different problems related to our results, there are some other tantalizing questions related to independence polynomials that are worth pursuing.

Unimodality of the Domination Polynomial

We believe our methods for determining log-concavity of certain very well-covered graphs depending on their independence roots can be useful for log-concavity conjectures for other graph polynomials as well. One in particular, is the domination polynomial.

A set of vertices is a *dominating set* if every vertex outside of the set is joined to at least one vertex in the set. More precisely, $S \subseteq V(G)$ is dominating if $N[S] = V(G)$. In the same vein as the independence polynomial, the *domination polynomial* of a graph G, denoted $D(G,x)$, is defined by $D(G,x) = \sum_{k=0}^{n} d_k x^k$, where d_k is the number of dominating sets of size k in G. Alikhani and Peng [3] conjectured that the domination polynomial is unimodal for *all* graphs. Despite the intriguing unimodality conjecture for the domination polynomial, there has been very little progress made. There has been even less work on determining which families of graphs have log-concave domination polynomials. We computed the domination polynomials of all graphs on 9 and fewer vertices and checked their log-concavity using Maple. Surprisingly, there is exactly one graph whose domination polynomial is not log-concave, which is shown in Figure 6.3. This graph has domination polynomial $x^9 + 9x^8 + 35x^7 + 75x^6 + 89x^5 + 50x^4 + 7x^3 + x^2$, and $50 \cdot 1 > 7^2$, so it fails log-concavity at the first possible index.

Based on these findings, we hypothesize that most graphs actually have a log-concave domination polynomial. To provide evidence for this, we plan to use results relating the roots of a polynomial to its log-concavity. We expect that using the roots to show log-concavity (and therefore unimodality) will be a better approach than trying to show these properties combinatorially. We also want to study different random graph constructions in ways that the domination polynomial (or part of it) can still be computed to show that almost all graphs have a log-concave domination polynomial. This technique has been useful with other graph polynomials, including showing that almost all independence polynomials have a nonreal root [24]. We also expect that aiming for the stronger property of log-concavity will provide great insight on how to prove the unimodality conjecture for domination polynomials.

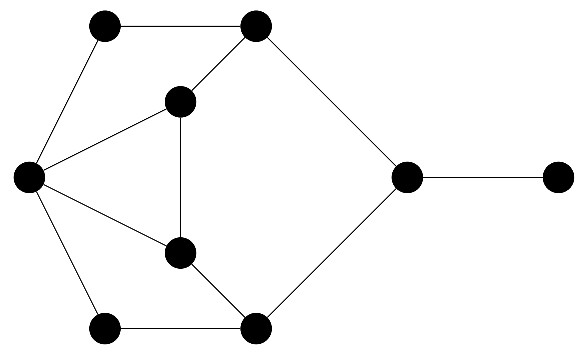


Figure 6.3: The lone graph of order 9 whose domination polynomial is not log-concave.

Multivariate Independence Polynomials

One of the disadvantages of studying the independence polynomial is that much of the information about the graph is not encoded in the polynomial. This can make for interesting results, for example all of the content in Chapter 5, but it can also make it difficult to exploit structural properties in a given family of graphs. One method to study graphs through polynomials without losing as much information is to extend the graph polynomial from a univariate polynomial to a multivariate polynomial. The trade off for the extra information about the graphs is that we now must work in the world of multivariate polynomials, although as we will see, this may not be such a disadvantage.

Definition 6.5.1. Let G be a graph on $V = \{v_1, v_2, \ldots, v_n\}$ with $\mathcal{I}(G)$ as the set of all independent sets of G and let $x_1, x_2, \ldots, x_n$ be variables. The *multivariate independence polynomial* of a graph G is defined as

$$i(G, x_1, x_2, x_3, \ldots, x_n) = \sum_{I \in \mathcal{I}(G)} \prod_{v_i \in I} x_i.$$

It is easy to see that by setting all $x_i = x$ that $i(G, x, x, x \ldots, x) = i(G, x)$, so the extension is a very natural one. It is also easy to see that the degree of x_i is at most 1 in each monomial term of $i(G, x_1, x_2, \ldots, x_n)$ as no vertex will appear in an independent set twice. Multivariate polynomials with this property are said to be *multiaffine* and these tend to be nicer multivariate polynomials to work with. The multivariate independence polynomial was used in [73] to prove the existence of a zero-free region (based on maximum degree) with applications for efficiently approximating $i(G, z)$ for certain $z \in \mathbb{C}$. It was also the main tool in [78] for studying a certain problem relating to statistical mechanics to the Lovász Local Lemma. While the many multivariate results in [30] cannot be directly applied to independence polynomials in general (the basic underlying structures in [30] are *matroids*), they do suggest that the multivariate extension of the independence polynomial may be a helpful tool in studying independence roots.

The Algebra of Independent Sets

Independent sets in a graph are crucial for understanding the graph and our methods for studying independent sets have been through the independence polynomial. Independent sets can of course be studied using graph theoretic techniques, but the collection of all independents has a deep algebraic structure as well, known as a *(simplicial) complex* (see [11, 32] for thorough discussions of complexes from a combinatorial viewpoint). As we noted in the introduction, every subset of an independent set is necessarily independent. Therefore, for a given graph $G = (V, E)$, $\text{Ind}(G) = (V, F)$ is a complex where F is the set of all independent sets of G. The complex $\text{Ind}(G)$ is called the *independence complex* of G.

We know from Proposition 5.2.6, a result whose proof used the reducibility of polynomials, that $[C_{p^k}] = \{D_{p^k}, C_{p^k}\}$ for all primes $p \geq 5$ and $k \geq 1$. It may be the case there are alternate proofs for classifying independence equivalence classes that rely on the properties of the independence complex. Perhaps invariants like homology of the independence complexes could be useful in providing necessary or sufficient conditions on when two graphs are independence equivalent.

Every complex has an associated f-polynomial, which is the generating polynomial for the number of faces of each size [11]. The f-polynomial of $\text{Ind}(G)$ is equal to $i(G, x)$, so, in this way, studying $\text{Ind}(G)$ is more general and has potential to lead to deeper results than by studying the independence polynomial alone. For example, it has eluded graph theorists for 32 years why general graphs can have unconstrained independence polynomials, but trees appear to always have log-concave, or at least

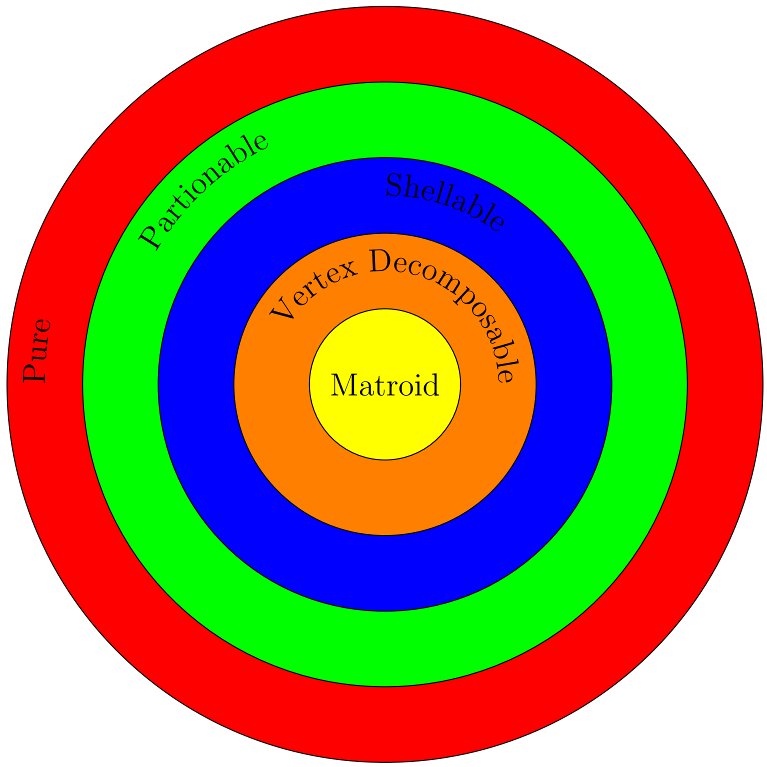


Figure 6.4: The containment of various simplicial complexes.

unimodal, independence polynomials. Perhaps the answer to this lies in the independence complexes of trees.

Complexes are objects of interest in algebra and topology, and as such there are different tools for studying independence complexes from an algebraic perspective (various recent results along these lines can be found in [86, 5], for example). There are also established notions that have analogous graph interpretations. For example, a graph G is well-covered if and only if $\text{Ind}(G)$ is pure. There are subclassifications of pure simplicial complexes, such as *vertex decomposable*, *shellable*, and *partionable* complexes as well as the previously mentioned matroids. It is known that $\text{Ind}(G)$ is a matroid if and only if every component of G is a clique [11]. Therefore, all independence roots of graphs whose independence complexes are matroids are not only real, but rational!

An interesting problem is to classify all graphs whose independence complexes have one of the specified properties, especially since there is the following implication: matroid $\Rightarrow$ vertex decomposable $\Rightarrow$ shellable $\Rightarrow$ partitionable $\Rightarrow$ pure (see Figure 6.4). One question that we are interested in is the behaviour of the independence roots of graphs in each category. It is known that the independence roots of graphs with pure independence complexes are still dense in $\mathbb{C}$, but their roots grow linearly in n as opposed to the exponential bound we showed in Chapter 4. As we move through the layers of Figure 6.4, are the independence roots still dense in $\mathbb{C}$? Are the graphs eventually stable? What is the maximum modulus of an independence root? For matroids we know every independence root is of the form $-1/n$ for some positive integer n, but all of these questions remain open for graphs whose independence complexes are partitionable, shellable, or vertex decomposable.

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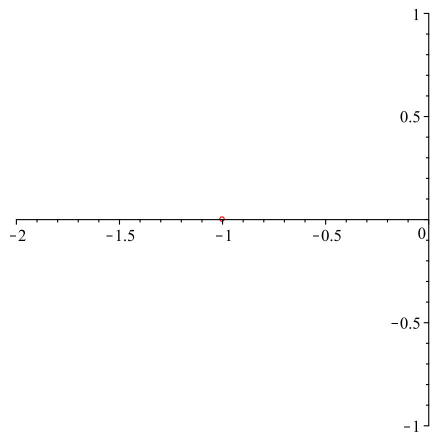
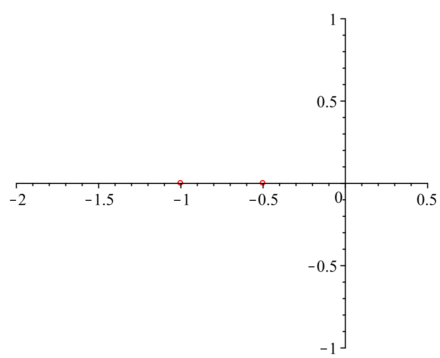
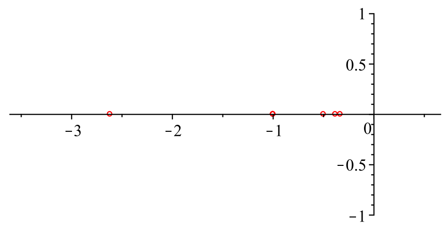
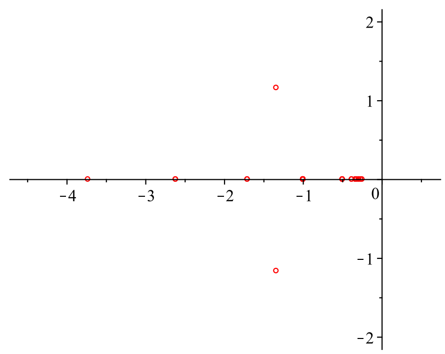
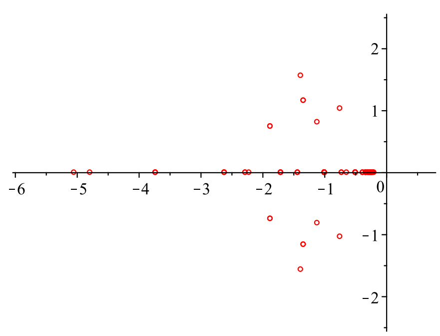
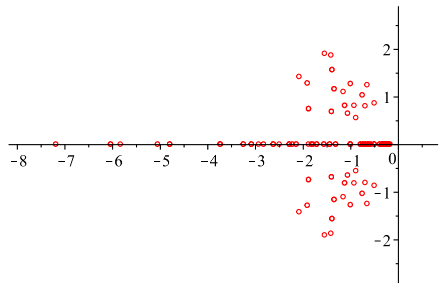
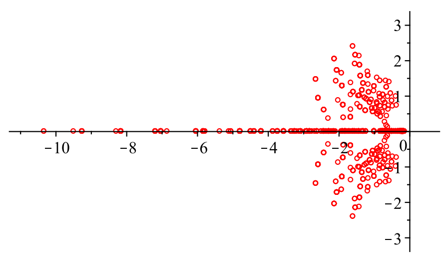
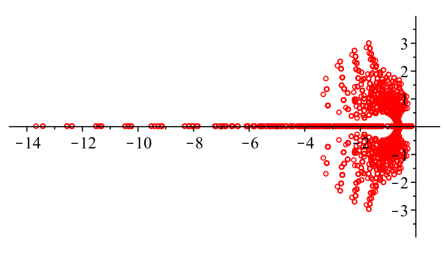
Appendix A

Plots of Independence Roots

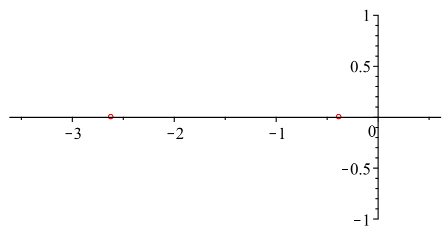
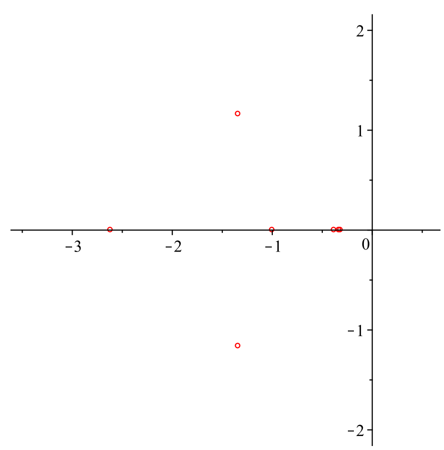
Order	Independence Roots
1	
2	

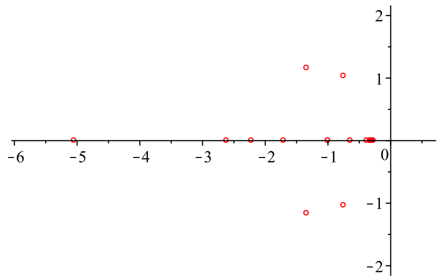
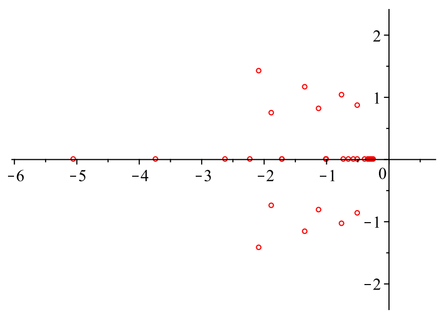
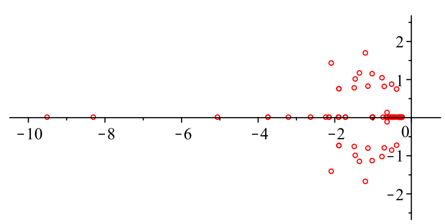
3	
4	
5	

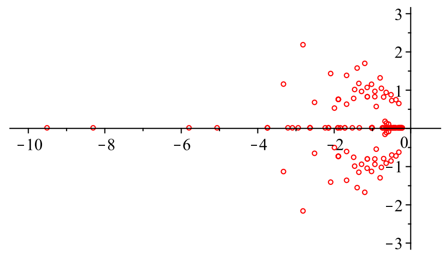
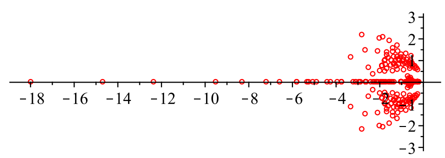
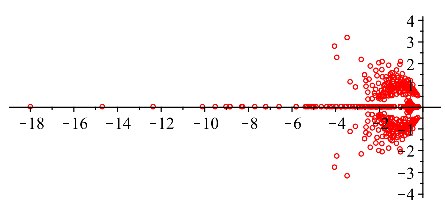
6	
7	
8	

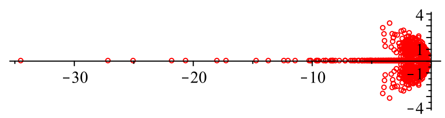
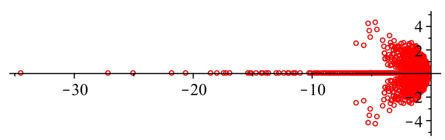
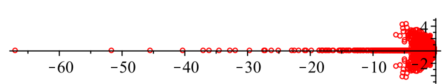
Appendix B

Plots of Independence Roots of Trees

Order	Independence Roots
3	
4	

5	
6	
7	

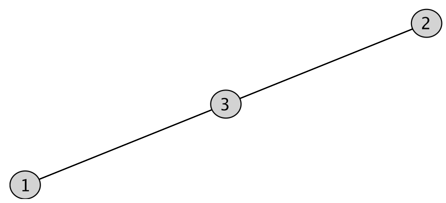
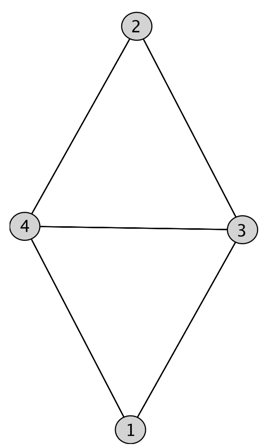
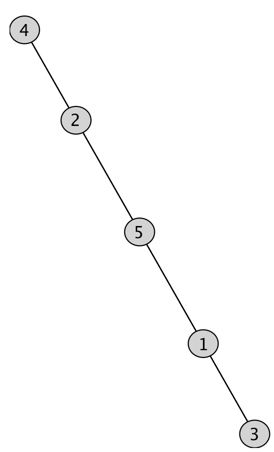
8	
9	
10	

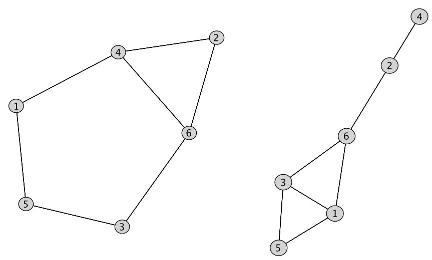
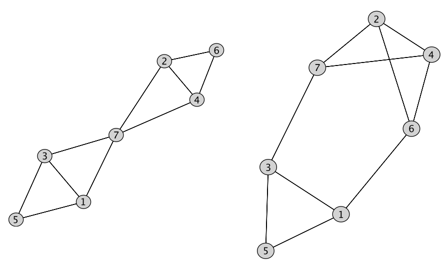
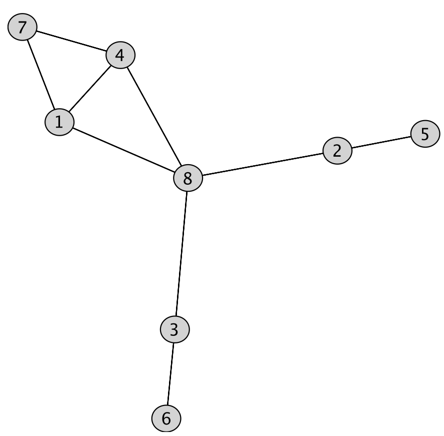
11	 -30 -20 -10 4 1 -1 -4
12	 -30 -20 -10 4 -4
13	 -60 -50 -40 -30 -20 -10 4

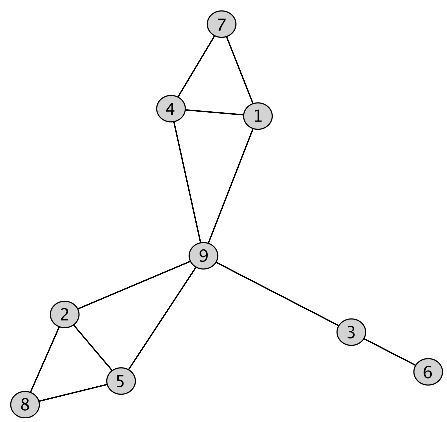
Appendix C

Independence Roots with Maximum Modulus

Order	Graph(s)	maxmod(n)
1		1
2		1

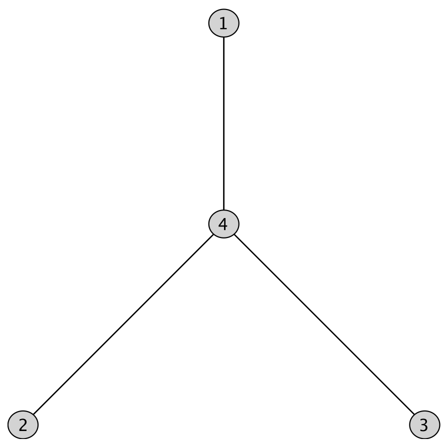
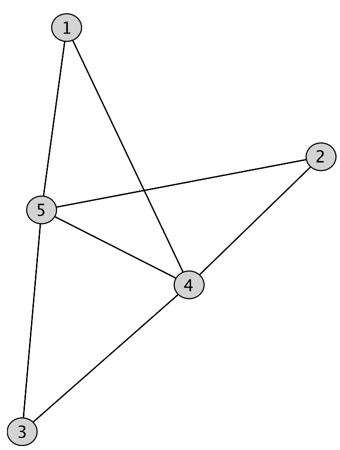
3		2.618033989
4		3.732050808
5		5.04891733952231

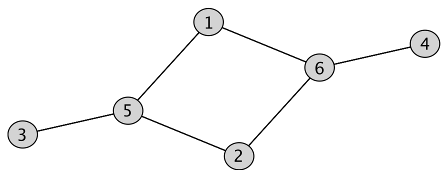
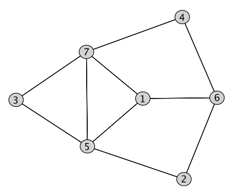
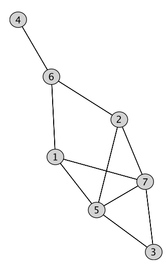
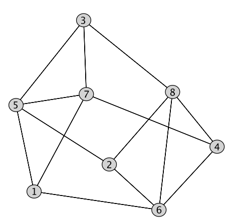
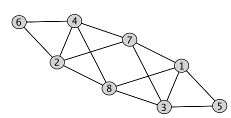
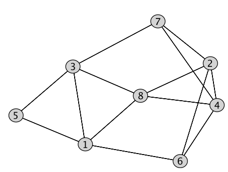
6		7.18421012919818
7		10.3318514126666
8		13.6506654518316

9		19.8113767295360

Appendix D

Independence Roots with Maximum Imaginary Part

Order	Graph(s)	maxImmod(G)
4		1.16154139999725
5		1.56388451052696

6		1.90626149836483
7	 	2.40107000622037
8	  	2.98709821496198