DOMINATION POLYNOMIALS:
A BRIEF SURVEY AND ANALYSIS

by

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Table of Contents

List of Tables

List of Figures

Abstract

A dominating set S of a graph G of order n is a subset of the vertices of G such that every vertex is either in S or adjacent to a vertex of S. The domination number G, denoted $\gamma(G)$, is the cardinality of the smallest dominating set of G. The domination polynomial is defined by $D(G,x) = \sum_{\gamma(G)}^{n} = d(G,i)x^i$ where $d(G,i)$ is the number of dominating sets in G with cardinality i. Two graphs G and H are considered $\mathcal{D}$-equivalent if $D(G,x) = D(H,x)$. The equivalence class of G, denoted $[G]$, is the set of all graphs $\mathcal{D}$-equivalent to G. We provide some results on constructing $\mathcal{D}$-equivalent graphs as well as determine $[P_n]$. We also explore some bounds on the coefficients of $D(G,x)$ for a given graph, and for some families of graphs. We conclude with a few open problems and possible research directions.

List of Abbreviations and Symbols Used

$E(G)$ .... Edge set of a graph G

$V(G)$ .... Vertex set of a graph G

$N_G[v]$ .... Closed neighbourhood of vertex v on a graph G

$N_G(v)$ .... Open neighbourhood of vertex v on a graph G

$\overline{G}$ .... Complement of a graph G

P_n .... Path graph on n vertices

K_n .... Complete graph on n vertices

C_n .... Cycle graph on n vertices

$K_{n_1,n_2,\ldots,n_k}$ .... Complete multipartite graph

$G \cup H$ .... Disjoint union of graphs G and H

$G \vee H$ .... Join of graphs G and H

$G \circ H$ .... Corona of graphs G and H

$G \square H$ .... Cartesian product of graphs G and H

$\gamma(G)$ .... Domination number of a graph G

$D(G,x)$ .... Domination polynomial of a graph G

$p_v(G)$ ... The graph polynomial generated by the dominating sets of $G - N[v]$ which also dominate $N(v)$

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Chapter 1

Introduction

1.1 Definitions

A graph $G = (V, E)$ is a set of vertices $V(G)$ together with an edge set $E(G)$ of unordered pairs of vertices. The cardinality of the vertex set $V(G)$ is referred to as the **order** of G. Two vertices $u, v \in V(G)$ are said to be **adjacent** if there exists an edge $e \in E(G)$ with $e = \{u, v\}$. Furthermore, in such a case u and v are **incident** with e (e is incident with u and v). It is common for edge $\{u, v\}$ to be denoted uv. The **degree** of vertex $v \in V(G)$ is the number of edges incident with v, which is the same as the number of vertices adjacent to v. We denote the degree of v as $deg(v)$. The maximum and minimum degree of any vertex in G are denoted $\Delta(G)$ and $\delta(G)$ respectively. If $\Delta(G) = \delta(G) = k$ we say the graph is $k-$**regular**.

The set of vertices $N_G(v) = \{u | uv \in E(G)\}$ is called the **open neighbourhood** of v. Similarly $N_G[v] = N(v) \bigcup \{v\}$ is called the **closed neighbourhood** of v. It is common for the subscript G to be omitted from the notation when only referring to one graph. For $S \subseteq V(G)$, the closed neighbourhood $N[S]$ of S is simply the union of the closed neighbourhoods for each vertex in S. For vertices $u, v \in V(G)$, if v has degree 1 and $N(v) = \{u\}$ then we refer to v as a **leaf** vertex and u as a **stem** vertex.

If $deg(v) = 0$ then v is called **isolated**. The **complement** of a graph G, denoted $\bar{G}$, has $V(\bar{G}) = V(G)$ and $E(\bar{G}) = \{uv | u \neq v \text{ and } uv \notin E(G)\}$.

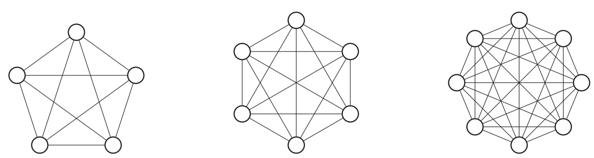
A **spanning** subgraph H of G is a graph with $V(H) = V(G)$ and $E(H) \subseteq E(G)$. An **induced** subgraph H of G is a copy of G with some vertices removed. That is, $V(H) \subseteq V(G)$ and $E(H) = \{uv | uv \in G \text{ and } u, v \in V(H)\}$. When a vertex is removed from a graph it is assumed each edge incident with said vertex is also removed. If an induced subgraph is complete, then we call it a **clique**. A **subgraph** H of G is an induced subgraph of a spanning subgraph of G. For a subset of vertices $S \subseteq V(G)$ we refer to $V(H) \cap S$ as the vertices of S **restricted** to H.

The **join** of two disjoint graphs G_1 and G_2 is denoted $G_1 \vee G_2$, with vertex set

$V(G_1) \cup V(G_2)$, and edge set $E(G_1 \vee G_2) = E(G_1) \bigcup E(G_2) \cup \{uv | u \in V(G_1), v \in V(G_2)\}$. The **corona** of two disjoint graphs G_1 and G_2, as defined by Frucht and Harary in [16] and denoted $G_1 \circ G_2$, is one copy of G_1 and $|V(G_1)|$ copies of G_2 where each vertex of G_1 is joined to every vertex in a unique copy of G_2. The **Cartesian** product of two disjoint graphs G_1 and G_2 is denoted $G_1 \square G_2$, with vertex set $V(G_1 \square G_2) = \{(u, v) | u \in V_1, u \in V_2\}$ and (u, v) is adjacent to (u', v') if either $u = u'$ and $vv' \in E(V_2)$ or $v = v'$ and $uu' \in E(V_2)$. The **disjoint union** of G and H, denoted $G \cup H$, has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

There are many families of graphs we will discuss in this thesis. The following are common families of graphs on a given labelled vertex set of n vertices, $V = \{v_i | 1 \leq i \leq n\}$.

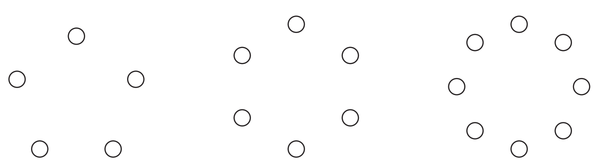
- A **complete** graph denoted K_n has edge set $E(K_n) = \{v_i v_j | v_i, v_j \in V, v_i \neq v_j\}$. Examples of K_5, K_6, and K_8 are in Figure 1.1.



(a) K_5 (b) K_6 (c) K_8

Figure 1.1: Examples of complete graphs

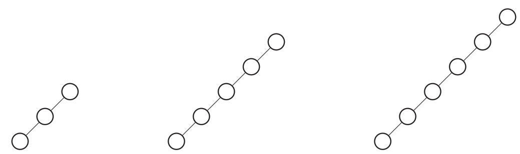
- The **empty** graph on n vertices is the complement of K_n, denoted $\overline{K_n}$. It has edge set $E(\overline{K_n}) = \emptyset$. Examples of $\overline{K_5}$, $\overline{K_6}$, and $\overline{K_8}$ are in Figure 1.2.



(a) $\overline{K_5}$ (b) $\overline{K_6}$ (c) $\overline{K_8}$

Figure 1.2: Examples of empty graphs

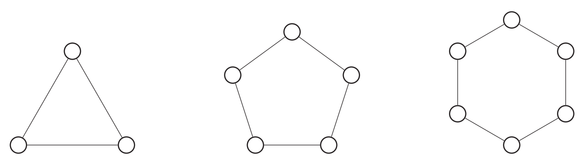
- A **path** graph, denoted P_n, has edges set $E(P_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\}$. The **length** of a path is the cardinality of its edge set. Examples of P_3, P_5, and P_6 are in Figure 1.3.



(a) P_3 (b) P_5 (c) P_6

Figure 1.3: Examples of path graphs

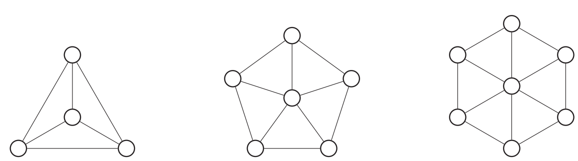
- A **cycle** graph (or n-cycle), denoted C_n, has edges set $E(C_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_1 v_n\}$. Examples of C_3, C_5, and C_6 are in Figure 1.4. Note C_3 is also a complete graph on three vertices.



(a) C_3 (b) C_5 (c) C_6

Figure 1.4: Examples of cycle graphs

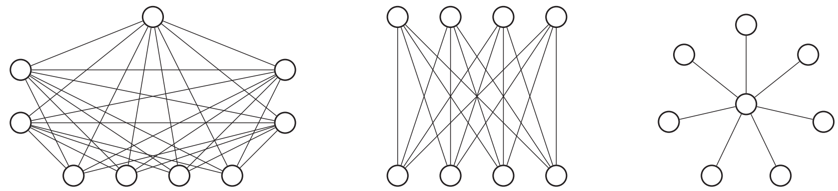
- A **wheel** graph, denoted W_n, is the join of K_1 and C_{n-1}. Examples of W_4, W_6, and W_7 are in Figure 1.5. Note W_4 is also a complete graph on four vertices.



(a) W_4 (b) W_6 (c) W_7

Figure 1.5: Examples of Wheel graphs

A **complete multipartite** graph, denoted $K_{n_1,n_2,\ldots,n_k}$, has vertex set $\{v_{i,j} | 1 \leq i \leq k, 1 \leq j \leq n_k\}$ where $v_{i,j}$ and $v_{k,l}$ are adjacent if $i \neq k$. Equivalently the vertices of the k-partite graph $K_{n_1,n_2,\ldots,n_k}$ are partitioned into to k sets of size $n_1, n_2, \ldots, n_k$ respectively and edges are added between each pair of vertices except pairs of vertices in the same set. A complete multipartite graph with only two partitions is called **complete bipartite**. A **star** graph, denoted $K_{1,n}$, is a special case of a complete bipartite graphs where one of the two partitions has exactly one vertex. An example of each is shown in Figure 1.6



(a) $K_{1,2,2,4}$ (b) $K_{4,4}$ (c) $K_{1,7}$

Figure 1.6: Examples of complete multipartite graphs

1.2 Overview

Domination and graph polynomials are each areas of graph theory with extensive research. A 1991 bibliography on domination in graphs [17] by Hedetniemi and Laskar traced domination back to the graph theory texts of König (1950), Berge (1958) and Ore (1962). Graph polynomials have also been of interest since 1912 when Birkhoff first defined the chromatic polynomial [15] in an attempt to prove the Four Colour Conjecture.

Although domination and graph polynomials have been areas of interest for quite some time, the domination polynomial was only introduced by Arocha and Llano in their 2000 paper [14]. In fact no other papers were published until 2008 when a seemingly independent work [8] was published by Alikhani and Peng. Results that have been of interest for the domination polynomial include computing the domination polynomial for families and products of graphs, finding recurrence relations, locating the roots, and finding the domination equivalence classes of families of graphs. The reader is directed to Alikhani's 2009 Ph.D. thesis [4] which is the culmination of six fundamental papers [2, 5, 8–10, 12] covering each area of domination polynomials studied today.

Chapter 2

Domination and the Domination Polynomial

In this chapter we introduce the domination polynomial and give an overview of previous results we will be using in later sections. In Section 2.1 we define dominating sets and the domination polynomial. In Section 2.2 we examine some general properties about the domination polynomial. In Section 2.3 we state some known recurrence relations and compute domination polynomials for particular families of graphs. In Section 2.4 we introduce domination equivalence and domination uniqueness. We also state some families of graphs which are known to have these properties.

2.1 Domination and the Domination Polynomial

For a graph G, $S \subseteq V(G)$ is a **dominating set** of G if $N[S] = V(G)$. That is to say, if S is a dominating set, then for each $v \in V(G)$, either $v \in S$ or there exists $u \in S$ which is adjacent to v. The **domination number** of G, denoted $\gamma(G)$ is the cardinality of the smallest dominating set of G. A dominating with cardinality $\gamma(G)$ is called a **minimum dominating** set. For a subgraph H we say a set A of G dominates H if $V(H) \subseteq N[A]$.

For example, consider the graph G in Figure 2.1, and a subset of its vertices, $S = \{v_1, v_2, v_5, v_7\}$. As $v_1, v_2, v_5, v_7 \in S$ and $v_3, v_4, v_6 \in N[v_7]$, S is a dominating set. Alternatively, $N[v_1] = \{v_1, v_2, v_3\}$, $N[v_2] = \{v_1, v_2, v_4, v_7\}$, $N[v_5] = \{v_4, v_5, v_6\}$, $N[v_7] = \{v_2, v_3, v_4, v_6\}$. So $N[S] = N[v_1] \cup N[v_2] \cup N[v_5] \cup N[v_7] = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\} = V(G)$. S is not a minimum dominating set as $\{v_2, v_6\}$ is also a dominating set. The domination number of G is 2 as we have a dominating set of cardinality 2 and there is no vertex in G which is adjacent to all other vertices, and hence no dominating set of cardinality one.

As demonstrated in our previous example, a graph can have multiple dominating sets. Another example of a graph with multiple dominating sets is K_n ($n \geq 2$), as

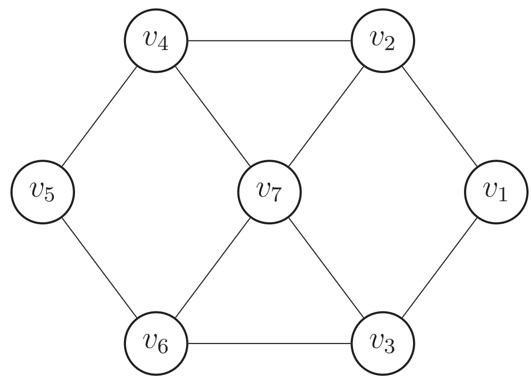


Figure 2.1: An graph on seven vertices

every non-empty subset of vertices is a dominating set. This leads us to our definition of the domination polynomial.

Definition 2.1.1 *Let $\mathcal{D}(G,i)$ be the collection of dominating sets of a graph G, each with cardinality i, and let $d(G,i) = |\mathcal{D}(G,i)|$. Then the **domination polynomial** $D(G,x)$ of G is defined as*

$$D(G,x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$$

where $\gamma(G)$ is the domination number of G.

Consider every subset of vertices for the path of length three shown in Figure 2.2. The empty set is not dominating so $d(P_3, 0) = 0$. For subsets of size one: $\{v_2\}$ is dominating but $\{v_1\}$ and $\{v_3\}$ are not, so $d(P_3, 1) = 1$. For subsets of size two: $\{v_1, v_2\}$, $\{v_2, v_3\}$ and $\{v_1, v_3\}$ are all dominating hence $d(P_3, 2) = 3$. The only subset of size three is the set of all vertices and hence dominating thus $d(P_3, 3) = 1$. Now we can conclude $D(P_3, x) = x^3 + 3x^2 + x$.

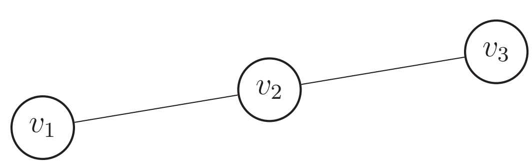


Figure 2.2: A path on three vertices

The exhaustive approach of checking if each subset of vertices is dominating is clearly not efficient. Unfortunately, in general, there seems to be no alternative that is significantly better. However the domination polynomial is known for some families of graphs. Furthermore we can deduce some coefficients based on particular properties of the graphs.

Consider once more the graph G in Figure 2.1. The order of G is 7, so manually checking each of its $2^7 = 128$ subsets of vertices would be rather time consuming. However $\delta(G) = 2$, so for each vertex in G the size of its closed neighbourhood is at least three. If a subset omits fewer than three vertices of $V(G)$, it must intersect the closed neighbourhood of each vertex in $V(G)$ and hence dominate G. Thus $d(G,7) = \binom{7}{0} = 1$, $d(G,6) = \binom{7}{1} = 7$, and $d(G,5) = \binom{7}{2} = 21$. For a subset of size four there are only two vertices 1 and 5 with closed neighbourhoods of size three. As those two neighbourhoods do not contain the same vertices then the only subsets of such size which do not dominate G are missing exactly the closed neighbourhoods of those two vertices. Thus $d(G,4) = \binom{7}{3} - 2 = 33$. For a subset of size three, consider each vertex with neighbourhoods of size three and four. As no two of these neighbourhoods are equal then the only subsets which do not dominate G are exclude either an entire closed neighbourhood of size four or a closed neighbourhood of size 3 and one of the remaining four vertices. Hence $d(G,3) = \binom{7}{4} - 4 - 2 \cdot 4 = 23$. As stated earlier $\gamma(G) = 2$, so $d(G,0) = d(G,1) = 0$. It is easy enough to see that the only dominating sets of size 2 are $\{2,6\}$ and $\{3,4\}$ so $d(G,2) = 2$ and $D(G,x) = x^7 + 7x^6 + 21x^5 + 33x^4 + 23x^3 + 2x^2$.

2.2 Basic Results

The following theorem states some basic consequences of the definition of the domination polynomial.

Theorem 2.2.1 [12] *Let G be a graph of order n. Then:*

(i) $d(G,n) = 1$.

(ii) If G is a connected graph of order at least 2, then $d(G,n-1) = n$.

(iii) $d(G,i) = 0$ *if and only if* $i < \gamma(G)$ *or* $i > n$.

(iv) $D(G,x)$ *has no constant term.*

(v) $D(G,x)$ *is a strictly increasing function on* $[0,\infty)$.

(vi) 0 *is a root of* $D(G,x)$*, with multiplicity* $\gamma(G)$.

□

Naturally when computing graph polynomials of product graphs we seek relationships with the graph polynomials of the smaller factor graphs. The domination polynomial is no different. Relationships for the disjoint union, join, and corona of graphs are detailed in the next three theorems.

Let G be disjoint union of connected graphs $G_1 \cup G_2 \cdots \cup G_k$. We call each G_i subgraph a **component** of G. Note that a connected graph only has one component. If S is a dominating set of G, then for each G_i, the vertices of S restricted to G_i must also dominate G_i. Moreover, to dominate G, simply choose a dominating set for each of its components. The dominating set for one component does not affect the dominating set for another component. From this it is clear that domination polynomials have a very useful property under disjoint unions.

Theorem 2.2.2 [12] *If a graph* G *consists of* m *components* $G_1, G_2, \ldots, G_m$ *the* $D(G,x) = D(G_1,x)D(G_2,x)\cdots D(G_m,x)$.

□

Consider a subset of vertices S for the join of two graphs $G_1 \vee G_2$. Any non-empty subset of $V(G_1)$ dominates G_2. Similarly any non-empty subset of $V(G_1)$ dominates G_2. Therefore if S dominates $G_1 \vee G_2$ and $S \subseteq G_1$ (or $S \subseteq G_2$) then S must also dominate G_1 (or G_2). As the binomial expansion of $(1+x)^n$ is a generating polynomial for the number of subsets of each size, the domination polynomial of $G_1 \vee G_2$ can be written in terms of $D(G_1,x)$ and $D(G_2,x)$.

Theorem 2.2.3 [12] *Let G_1 and G_2 be graphs of order n_1 and n_2 respectively. Then*

$$D(G_1 \vee G_2, x) = [(x+1)^{n_1} - 1][(x+1)^{n_2} - 1] + D(G_1, x) + D(G_2, x).$$

$\square$

For the corona of two graphs $G_1 \circ G_2$ the domination polynomial is not as obvious. However, Kotek, Preen, and Simon classified what we call irrelevant edges. In Section 2.4, Theorem 2.4.14 will state that irrelevant edges can be deleted from a graph without changing the domination polynomial. Each edge in G_1 becomes irrelevant in $G_1 \circ G_2$ and can be deleted. The resultant graph is a disjoint union of $|V(G_1)|$ copies of $G_2 \vee K_1$ which can easily be calculated with Theorems 2.2.3 and Theorems 2.2.2.

Theorem 2.2.4 [19] *Let G_1 and G_2 be non-empty graphs of order n_1 and n_2 respectively. Then*

$$D(G_1 \circ G_2, x) = [D(K_1 \vee G_2, x)]^{n_1} = [x(x+1)^{n_2} + D(G_2, x)]^{n_1}.$$

$\square$

Another area of interest with graph polynomials is finding which graph properties are preserved in our polynomial. Can a graph be determined simply by its graph polynomial? For domination polynomials, as we will find out in Section 2.4, the answer is no. However some graph properties can be taken from its domination polynomial.

Theorem 2.2.5 [12] *Let G be a graph of order n with t vertices of degree one and r isolated vertices. If $D(G, x) = \sum_{i=1}^{n} d(G, i)x^i$ is its domination polynomial then the following hold:*

(i) $r = n - d(G, n-1)$.

(ii) If G has s K_2-components, then $d(G, n-2) = \binom{n}{2} - t + s - r(n-1) + \binom{r}{2}$.

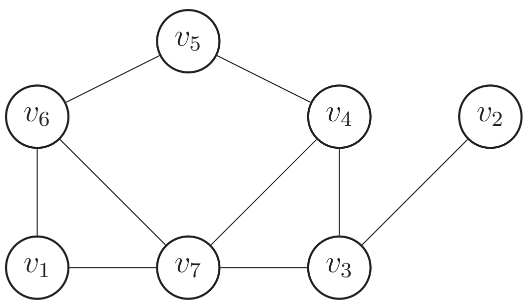
(iii) If G has no isolated vertices (i.e. $r = 0$) and $D(G, -2) \neq 0$ (i.e. $s = 0$), then $t = \binom{n}{2} - d(G, n-2)$.

(iv) $d(G,1) = |\{v \in V(G) | deg(v) = n-1\}|$.

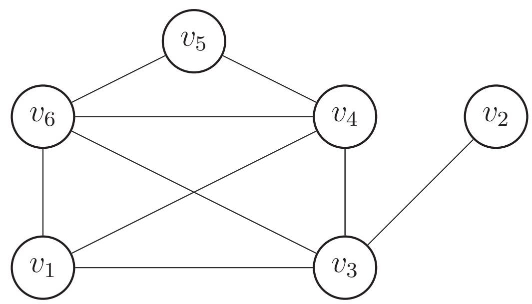
$\square$

In Section 2.4 we will show $\delta(G)$ can also be determined by the domination polynomial as it equals $n-i$ for the largest i where $d(G,i)$ is less than $\binom{n}{i}$.

Another area of interest with graph polynomials is finding recurrence relations. These are handy for inductions or even just computing domination polynomials. For the next theorem we introduce the **vertex contraction** G/v of a graph G, where all vertices in $N(v)$ are joined to each other and then v is deleted. An example of this is shown in Figure 2.3. In G/v_7, v_7 is deleted and $\{v_1, v_3, v_4, v_6\}$ is now a clique.



(a) G



(b) G/v_7

Figure 2.3: An example of a vertex contraction of G

We also introduce the polynomial $p_v(G)$ which enumerates the dominating sets of $G - N[v]$ which additionally dominate $N(v)$ in G. For example, let G be the graph in Figure 2.3 (a) and consider $p_{v_4}(G)$. The only subsets of $G - N[v_4]$ which also dominate $N(v_4)$ are $\{v_2, v_6\}$ and $\{v_2, v_6, v_1\}$. Therefore $p_{v_4}(G) = x^3 + x^2$. Furthermore $p_{v_3}(G) = 0$ as no subset of $G - N[v_3]$ also dominates $v_2 \in N(v_3)$.

Theorem 2.2.6 [4] *For any vertex v in a graph G we have*

$$D(G,x) = xD(G/v, x) + D(G-v, x) + xD(G-N[v], x) - (x+1)p_v(G).$$

$\square$

The following corollaries are very useful when determining the domination polynomial for trees.

Corollary 2.2.7 [4] *If* $u, v \in V(G)$*,* $uv \in E(G)$ *and* $N[v] \subseteq N[u]$ *then*

$$D(G, x) = xD(G/u, x) + D(G - u, x) + xD(G - N[u], x).$$

$\square$

Corollary 2.2.8 [4] *If* $u, v \in V(G)$*,* $uv \notin E(G)$ *and* $N(v) = N(u)$ *then*

$$D(G, x) = xD(G/u, x) + D(G - u, x) - D(G - N[u], x).$$

$\square$

In [19] Kotek, Preen, and Simon derived a recurrence relation for removing triangles based around a new operation on edges incident to a vertex u: Let $G \odot u$ be the graph obtained from G by the removal of all edges between any pair of neighbours of u. Note u is not removed from the graph.

Theorem 2.2.9 [19] *Let* $G = (V, E)$ *be a graph. For any* $u \in V$ *we have*

$$D(G, x) = D(G - u, x) + D(G \odot u, x) - D(G \odot u - u, x).$$

$\square$

2.3 Domination Polynomials for Families of Graphs

Although it is difficult to find the domination polynomial for a general graph, it is known for some families of graphs. The equations in the next theorem are either straightforward or follow directly from results in Section 2.2.

Theorem 2.3.1 [12] *Let* G *be a graph of order* n*. Then:*

(i) $D(K_n, x) = (x + 1)^n - 1$.

(ii) $D(K_{m,n}, x) = [(x + 1)^m - 1][(x + 1)^n - 1] + x^m + x^n$.

(iii) $D(K_{1,n}, x) = x^n + x(x + 1)^n$.

(iv) *If* $n \geq 4$, *then* $D(W_n, x) = x(x+1)^{n-1} + D(C_{n-1}, x)$.

□

We will now examine the coefficients of domination polynomials for paths and cycles. The results in Theorem 2.3.3 and Theorem 2.3.5 will be utilized in Section 4.2.

Consider the domination polynomial for paths of order three or greater. These paths all contain a leaf l adjacent to a stem s. Moreover, $N[l] = \{s, l\} \subseteq N[s]$. That is, the closed neighbourhood of l is contained in the closed neighbourhood of s. Therefore the recurrence relation in Corollary 2.2.7 applies. Each G/s, $G - s$, and $G - N[s]$ are either a path, or a path and an isolated vertex, giving us a useful recurrence.

Theorem 2.3.2 [9] *For every* $n \geq 4$

$$D(P_n, x) = x(D(P_{n-1}, x) + D(P_{n-2}, x) + D(P_{n-3}, x))$$

□

The recurrence in Theorem 2.3.2 allows us to generate the domination polynomial for paths quickly. Table 2.1 shows the number of dominating sets of P_n with cardinality j, for $n = 3, 4, \ldots 14$.

With the recurrence relation for paths and the large sample of domination polynomials of paths we are capable of generating, it makes it very easily to determine properties of $d(P_n, j)$. In most cases only an observation is needed as induction and the recurrence relation will handle the proof. The next theorem shows what is known about some coefficients of $d(P_n, j)$.

Theorem 2.3.3 [9]

(i) *For every* $n \geq 2$, $d(P_n, n-1) = n$.

(ii) *For every* $n \geq 3$, $d(P_n, n-2) = \binom{n}{2} - 2$.

(iii) *For every* $n \geq 4$, $d(P_n, n-3) = \binom{n}{3} - (3n-8)$.

$n\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
3	1	3	1											
4		4	4	1										
5		3	8	5	1									
6		1	10	13	6	1								
7			8	22	19	7	1							
8			4	26	40	26	8	1						
9			1	22	61	65	34	9	1					
10				13	70	120	98	43	10	1				
11				5	61	171	211	140	53	11	1			
12				1	40	192	356	343	192	64	12	1		
13					19	171	483	665	526	255	76	13	1	
14					6	120	534	1050	1148	771	330	89	14	1

Table 2.1: $d(P_n, j)$, the number of dominating sets of P_n with cardinality j

(iv) For every $n \geq 5$, $d(P_n, n-4) = \binom{n}{4} - (2n^2 - 13n + 20)$.

(v) For every $n \in \mathbb{N}$, $d(P_{3n}, n) = 1$.

(vi) For every $n \in \mathbb{N}$, $d(P_{3n+1}, n+1) = \dfrac{(n+2)(n+3)}{2} - 2$.

(vii) For every $n \in \mathbb{N}$, $d(P_{3n+2}, n+1) = n + 2$.

$\square$

We now turn our attention to cycles. Unlike paths, the recurrences of the previous section are not useful for cycles. However with about 15 pages of tedious work, Alikhani showed paths and cycles have the same recurrence relation with differences only in initial polynomials.

Theorem 2.3.4 [8] *For every* $n \geq 4$

$$D(C_n, x) = x(D(C_{n-1}, x) + D(C_{n-2}, x) + D(C_{n-3}, x))$$

$\square$

Similar to paths, the recurrence in Theorem 2.3.4 allows us to generate the domination polynomial for cycles quickly. Table 2.1 shows the number of dominating sets

$n \backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
3	3	3	1											
4		6	4	1										
5		5	10	5	1									
6		3	14	15	6	1								
7			14	28	21	7	1							
8			8	38	48	28	8	1						
9			3	36	81	75	36	9	1					
10				25	102	150	110	45	10	1				
11				11	99	231	253	154	55	11	1			
12				3	72	282	456	399	208	66	12	1		
13				0	39	273	663	819	598	273	78	13	1	
14				0	14	210	786	1372	1372	861	350	91	14	1

Table 2.2: $d(C_n, j)$, the number of dominating sets of C_n with cardinality j

of C_n with cardinality j, for $n = 3, 4, \ldots 14$. The next theorem shows what is known about some coefficients of $d(C_n, j)$.

Theorem 2.3.5 [8]

(i) For every $n \geq 3$, $d(C_n, n-1) = n$.

(ii) For every $n \geq 3$, $d(C_n, n-2) = \binom{n}{2}$.

(iii) For every $n \geq 4$, $d(C_n, n-3) = \binom{n}{3} - n$.

(iv) For every $n \in \mathbb{N}$, $d(C_{3n}, n) = 3$.

(v) For every $n \in \mathbb{N}$, $d(C_{3n+1}, n+1) = \dfrac{n(3n+7)+2}{2}$.

(vi) For every $n \in \mathbb{N}$, $d(C_{3n+2}, n+1) = 3n + 2$.

□

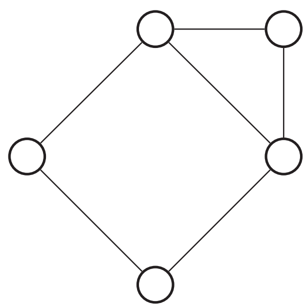
2.4 Domination Equivalence

It is possible for two graphs to have the same domination polynomial. If the two graphs are isomorphic then it is trivial. However, in some cases non-isomorphic graphs have the same domination polynomial. This leads us to the following definition first defined by Akbari, Alikhani and Peng in [2].

Definition 2.4.1 *Two graphs G and H are* ***dominating equivalent*** *or simply* ***$\mathcal{D}$-equivalent*** *(written $G \sim H$) if $D(G,x) = D(H,x)$.*

Recall $\mathcal{D}(G,i)$ is the set of dominating sets in a given graph G with cardinality i. Two graphs G and H are $\mathcal{D}$-equivalent if and only if there exists a bijection $\phi_i : \mathcal{D}(G,i) \mapsto \mathcal{D}(G,i)$ for every i. The union of every bijection, denoted ϕ, is simply a bijection from the dominating sets of G to the dominating sets H which preserves cardinality.

An example of two non-isomorphic graphs G and H which are $\mathcal{D}$-equivalent is shown in Figure 2.4. Each G and H have domination polynomial $x^5+5x^4+10x^3+7x^2$.


(a) G

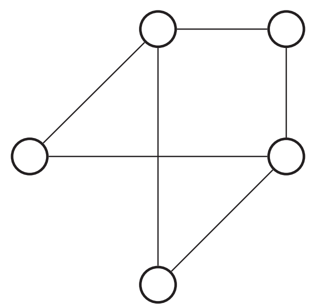

(b) H

Figure 2.4: Two $\mathcal{D}$-equivalent graphs

As in [2], we let $[G]$ denote the equivalence class determined by G, that is $[G] = \{H|H \sim G\}$. A graph G is said to be **dominating unique** or simply **$\mathcal{D}$-unique** if $[G] = \{H|H \text{ is ismorphic to } G\}$.

Two problems arise from equivalence classes: Which graphs are $\mathcal{D}$-unique? Can we determine the $\mathcal{D}$-equivalence classes for some families of graphs?

In [3] Akbari and Oboudi showed all cycles are $\mathcal{D}$-unique. Anthony and Picollelli classified all complete r-partite graphs which are $\mathcal{D}$-unique in [13]. In [11] Alikhani and Peng showed most cubic graphs of order 10 (including the Peterson graph) are $\mathcal{D}$-unique.

In [19] Kotek, Preen, and Simon defined and characterized irrelevant edges. These are edges which can be removed without changing the domination polynomial of a graph. From this they could show various trees (in particular paths [2]) barbell graphs [18], and other graphs are not $\mathcal{D}$-unique.

In this section we will examine previous known results to give operations and properties which can be used to determine if two graphs have the same domination polynomial. For two $\mathcal{D}$-equivalent first note some properties which easily follow from previous theorems.

Theorem 2.4.2 *If $G \sim H$ then the following must be true:*

(i) $|V(G)| = |V(H)| = n$.

(ii) If G and H are both connected and of order at least 3 then they must have the same number of vertices with degree 1*.*

(iii) G and H must have the same number of vertices with degree $n-1$.

Proof. Item (i) follows as the degree of a domination polynomial is the order of the underlying graph then G and H must have the same order. Statements (ii) and (iii) both follow directly from Theorem 2.2.5. □

For any graph G of order n, consider $S \subset V(G)$. If $S = V(G)$, clearly S forms a dominating set, but what happens when we start to remove vertices? Do the remaining vertices still dominate G? Generally that is a hard question to answer. If we remove more vertices from S, it becomes less likely S is a dominating set. But what if we only remove a few? Moreover, what is the fewest number of vertices needed to be removed from S so that S no longer dominates G? S is no longer a dominating set if there exists a $v \in V(G)$ where $N[v] \cap S = \emptyset$. The cardinality of the smallest closed neighbourhood is $\delta(G)+1$. Therefore if we remove fewer than $\delta(G)+1$ vertices from S, then S still dominates G. This idea is quite trivial but it is fundamental in proving that all cycles are $\mathcal{D}$-unique.

Lemma 2.4.3 [2] *Let G be a graph of order n. If $d(G,j) = \binom{n}{j}$ for some j, then $\delta(G) \geq n-j$. More precisely $\delta(G) = n-l$ where $l = min\{j|d(G,j) = \binom{n}{j}\}$, and there are at least $\binom{n}{n-\delta(G)-1} - d(G, n-\delta(G)-1)$ vertices of degree $\delta(G)$. Furthermore, if for every two vertices u and v of degree $\delta(G)$ we have $N[u] \neq N[v]$, then there are exactly $\binom{n}{n-\delta(G)-1} - d(G, n-\delta(G)-1)$ vertices of degree $\delta(G)$.* □

Corollary 2.4.4 [1] *If G and H are two graphs and $D(G,x) = D(H,x)$, then $\delta(G) = \delta(H)$.* □

Theorem 2.4.5 [2] *Let H be a k-regular graph with $N[u] \neq N[v]$, for all $u, v \in V(H)$. If $D(G,x) = D(H,x)$, then G is k-regular.* □

As cycles are 2-regular then for a different graph G to have the same domination polynomial it must be the disjoint union of smaller cycles. This fact combined with a few other properties known about the domination of cycles lead Akbari, Alikani and Peng in [2] and later Akbari and Oboudi in [3] to the following theorem.

Theorem 2.4.6 [3] *For every positive integer n, cycle C_n is $\mathcal{D}$-unique.* □

The proof for the next lemma will follow directly from Theorem 2.4.15.

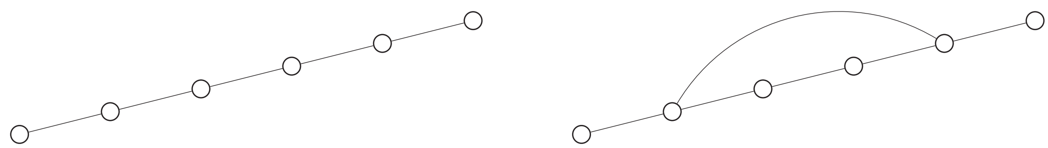
Lemma 2.4.7 [3] *If a graph G is $\mathcal{D}$-unique, then for every $m \geq 1$, $G \vee K_m$ is $\mathcal{D}$-unique.* □

Corollary 2.4.8 *For every two positive integers m and n, $K_m \vee C_n$ is $\mathcal{D}$-unique. In particular, W_n is $\mathcal{D}$-unique.* □

In [2] Akbari, Alikhani and Peng also determined $[P_{3n}]$ by restricting any graph $G \sim P_{3n}$ to some properties we know from the first few coefficients shown in Theorem 2.3.3. As $d(P_{3n}, n) = 1$, they completed their proof by showing $d(G, n) \neq 1$. However as P_{3n+1} and P_{3n+2} do not have unique minimum dominating sets their respective equivalence classes were left as an open problem which we will solve in section 4.

Theorem 2.4.9 [2] *Let n be a natural number. Then $[P_{3n}]$ contains just two graphs: P_{3n} and the graph generated by taking P_{3n} and adding an edge between its two stems.* □

In [13] Anthony and Picollelli proved most r-partite graphs are $\mathcal{D}$-unique; however, the equivalence class of $K_{n,n+1}$ is unknown.



(a) P_6 (b) P'_6

Figure 2.5: The domination equivalence class of P_6

Theorem 2.4.10 [1] *For every natural number n, $[K_{n,n}] = \{K_{n,n}, K_n \Box K_2\}$.* $\square$

Theorem 2.4.11 [13] *Let $r \in \mathbb{N}$ and let $n_1, n_2, \ldots n_r \in \mathbb{N}$. Then the complete r-partite graph $K_{n_1,n_2,\ldots,n_r}$ is $\mathcal{D}$-unique if and only if for all $1 \leq i < j \leq r$ either $max\{n_i, n_j\} \leq 2$ or $|n_i - n_j| \geq 2$.* $\square$

In Corollary 2.2.7, the domination polynomial is simplified when the closed neighbourhood of one vertex contains the closed neighbourhood of another vertex. In [19] Kotek, Preen and Simon gave a name to such vertices and stated some special properties they have that are useful when determining if two graphs are $\mathcal{D}$-equivalent.

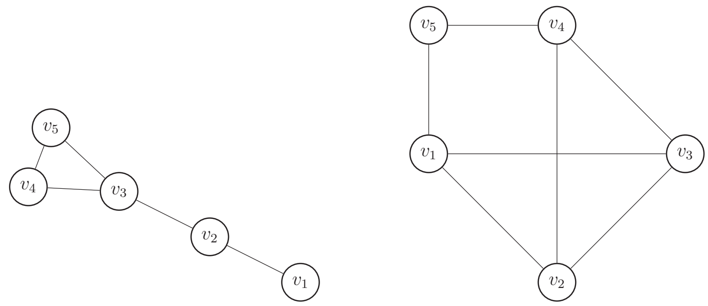
Definition 2.4.12 [19] *For a graph G, a vertex $v \in V(G)$ is* ***domination-covered*** *if every dominating set of $G - v$ includes at least one vertex adjacent to v in G.*

Theorem 2.4.13 [19] *For a graph G, a vertex $v \in V(G)$ is domination-covered if and only if there exists a $u \in N(v)$ such that $N[u] \subseteq N[v]$.* $\square$

Thus we say a vertex v is domination-covered by another vertex u if $N[u] \subseteq N[v]$. Some examples of domination-covered vertices are shown in Figure 2.6. In (a) vertices v_2, v_3, v_4 and v_5 are domination-covered and in (b) vertices v_2, v_3, v_4 and v_5 are domination-covered.

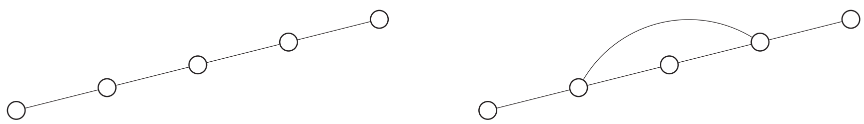
In [19] Kotek, Preen, and Simon defined an edge $e \in E(G)$ to be **irrelevant** if the domination polynomial was unchanged by its removal, that is, $D(G, x) = D(G-e, x)$. The next theorem classifies all such edges.

Theorem 2.4.14 [19] *Let G be a graph. An edge $e = \{u, v\} \in E$ is an irrelevant edge in G if and only if u and v are domination-covered in $G - e$* $\square$



(a) (b)

Figure 2.6: Examples of domination-covered vertices



(a) (b)

Figure 2.7: Each of these graphs have domination polynomial $x^5 + 5x^4 + 8x^3 + 3x^2$

Consider the two graphs in Figure 2.6. In Graph (a), vertices v_2 and v_3 are domination-covered. Now if we remove the edge v_2v_3, they still remain domination-covered therefore v_2v_3 is an irrelevant edge. In Graph (b) vertices v_1 and v_2 are domination-covered. However when we remove the edge v_1v_2, both v_1 and v_2 are no longer domination-covered, therefore the edge v_1v_2 is not irrelevant.

Theorem 2.4.14 can tell us much about domination equivalence. Consider adding an edge e between two domination-covered vertices in a graph G. Any dominating set of G is still a dominating set in $G+e$. Consider a dominating set S in $G+e$. Let u and v be the vertices incident with e. If $u, v \in S$ or $u, v \notin S$ then the clearly S is still a dominating set in G. Without loss of generality, suppose $u \in S$ and $v \notin S$. As v is domination-covered in G then there is a neighbour of v, other than u, which is in S. This means S is still a dominating set of G. Furthermore, e is irrelevant in $G+e$. In particular, adding an edge between two stems in a graph yields a domination-equivalent graph (the new edge is irrelevant). That is why the two graphs in Figure 2.7 are domination equivalent.

The following theorems are useful when determining the $\mathcal{D}$-equivalence of two graphs, depending on whether they contain vertices of degree n or $n-1$. We can also use these theorems to construct families of graphs which are $\mathcal{D}$-equivalent.

Theorem 2.4.15 [2] *Let G be a graph of order n with vertex $v \in V(G)$. If $deg(v) = n-1$, then G is $\mathcal{D}$-unique if and only if $G - v$ is $\mathcal{D}$-unique.* □

The proof of Theorem 2.4.15 follows directly from the result in Theorem 2.2.3. Furthermore, when trying to determine if two graphs are $\mathcal{D}$-equivalent, by Theorem 2.4.2 they must have the same number of vertices of degree $n-1$. So these vertices can be removed from each graph as they will not change the $\mathcal{D}$-equivalence. The next theorem is equivalent to example 3.5 in [19].

Theorem 2.4.16 [19] *Let G and H be $\mathcal{D}$-equivalent graphs. Let G' and H' be copies of G and H with an additional vertex connected to all but one vertex $v_G \in V(G)$ and $v_H \in V(H)$ respectively. If $deg_H(v_H) = deg_G(v_G)$ then $G' \sim H'$.*

□

Chapter 3

Coefficients of Domination Polynomials

Unfortunately a closed form of the domination polynomial is not known for many graphs. However, some coefficients are know for general graphs. In Theorem 2.2.5 we stated formulas for $d(G, n-1)$ and $d(G, n-2)$ in terms of certain properties within the graph. In Section 3.2 we will introduce a collection of graphs and give generalized formulas for $d(G, n-3)$ and $d(G, n-4)$. This will help us determine the equivalence class for paths in Section 4.2. First, though, in Section 3.1 we will give an efficient method to determine a lower bound on the $d(G, j)$ for connected G of given order.

3.1 Bounds on the Coefficients for Connected Graphs

We now introduce a novel area of interest for domination polynomials, namely determining bounds for its coefficients. For a graph G of order n, the coefficient a_i of x^i in $D(G, x)$ is bounded. By definition, each coefficient counts the number of dominating sets of size i. Therefore a_i is bounded below by zero. This bound is tight for all but a_n as $D(\overline{K_n}, x) = x^n$. As there are only $\binom{n}{i}$ subsets of $V(G)$ of cardinality i which can be dominating, therefore a_i is bounded above by $\binom{n}{i}$. This bound is tight because any non-empty subset of a complete graph is dominating. In this section we will give a method for tightening those bounds for a given graph. As well, we show some work done to find a lower bound on all connected G.

In [19] Kotek et al. showed that some edges of G can be added or removed without changing the domination polynomial. Generally though, adding edges increases the coefficients of $D(G, x)$ and removing edges decreases the coefficients of $D(G, x)$. For any edge $e \in E(G)$, a dominating set of $G - e$ is still a dominating set in G, therefore the coefficients of $D(G, x)$ are bounded below by the corresponding coefficients of $D(G - e, x)$. Let G^+ be any copy of G with some edges added and G^- be any copy of G which has edges removed. In general, $d(G^-, i) \leq d(G, i) \leq d(G^+, i)$ for all $i \in \mathbb{N}$.

The following algorithm removes edges from a given graph G until there is a disjoint union of stars. The resultant graph is denoted G^-. This will allow us to get a lower bound on the coefficients of $D(G, x)$ as there is a closed form formula for the domination polynomial of stars.

Set $r =$ the number of isolated vertices in G ;
Initialize G^- to be r isolated vertices;
Set $H = G - V(G^-)$;
Initialize F to be a spanning forest of H;
while *F is non-empty* **do**
  Choose stem s in F;
  Remove edges from s to neighbouring stems which are not also leaves;
  Set $G^- = G^- \cup N_F[s]$;
  Set $F = F - N_F[s]$;
end

Algorithm 1: Removes edges from G until G is a disjoint union of stars

In Algorithm 1 we remove the closed neighbourhood of the stems in F while F is not empty. If F is not empty and has no stems, Algorithm 1 will not terminate. In Lemma 3.1.1 we will show Algorithm 1 terminates if G is a finite graph. This will help us show that the resultant graph of Algorithm 1 has the same number of isolated vertices as G.

Lemma 3.1.1 *If G is a finite graph, Algorithm 1 will always terminate.*

Proof. To show a contradiction, suppose Algorithm 1 does not terminate and G is a finite graph. As Algorithm 1 does not terminate then the loop while F is not empty is endless. Now, choose any spanning forest F^∞ of H (the graph obtained by removing all isolated vertices from G) such that the loop is endless. Each iteration of the loop assigns F to a subgraph of F by removing vertices from F only if F contains a stem. As the loop is endless then F^∞ is non-empty and does not contain a stem.

We claim F^∞ is a collection of isolated vertices. We will now prove this claim through contradiction by showing if F^∞ is not a collection of isolated vertices, it has a stem. Suppose F^∞ is not a collection of isolated vertices. Then F^∞ contains a connected component, which we denote as C, of order two or greater. As F^∞ is a subgraph of the forest F, C is a connected subgraph of a forest and hence a tree. As C is a tree of order at least two, it has a leaf. Furthermore, C has order two or more, therefore the leaf has a neighbour which by definition is a stem. As this is a contradiction, F^{∞} is a collection of isolated vertices.

Finally we claim Algorithm 1 cannot reduce F to F^{∞} (a collection of isolated vertices) and hence will always terminate. We will now prove this claim through contradiction by showing each vertex of F will be removed before becoming isolated. Suppose Algorithm 1 reduces F to F^{∞}. Let $v \in F^{\infty}$. We now consider three cases: v is a stem in F, v is a leaf in F, and v is not a stem nor leaf in F.

In F, if v is a stem, then v will remain a stem until it is chosen to be removed from F. Similarly if v is a leaf in F, then v will remain a leaf until it and its corresponding stem s are chosen to removed from F.

Finally if v is not a leaf or stem in F, then at most one edge incident to v is removed per iteration. If two or more edges incident to v were removed then v would share at least two neighbours with another vertex which would induce a cycle. This is a contradiction as F is a forest. As v is isolated in F^{∞} then, at some point in the iteration, v is only incident to one edge and is therefore a leaf. However v will remain a leaf until it and its corresponding stem s are removed.

$\square$

In Lemma 3.1.2 we will show that the resultant graph of Algorithm 1 does not have more isolated vertices than the initial graph G. This is important because an isolated vertex will shrink the coefficients of our lower bound more than might be necessary. To achieve the best lower bound we would like to remove as few edges as possible. Thus removing all edges incident to one vertex is not ideal. Note also that we wish to form stars as the domination polynomial is known for stars. If G contains subgraphs with more edges (a clique component, perhaps) than a star, the algorithm could get a better lower bound by removing these subgraphs from G before beginning. We also choose stars because we wish to find a lower a bound for all connected graphs and Algorithm 1 shows every connected graph can be broken down into a disjoint union of stars.

Lemma 3.1.2 *Let G^- denote the resultant graph of Algorithm 1. Then G^- and G have the same number of isolated vertices.*

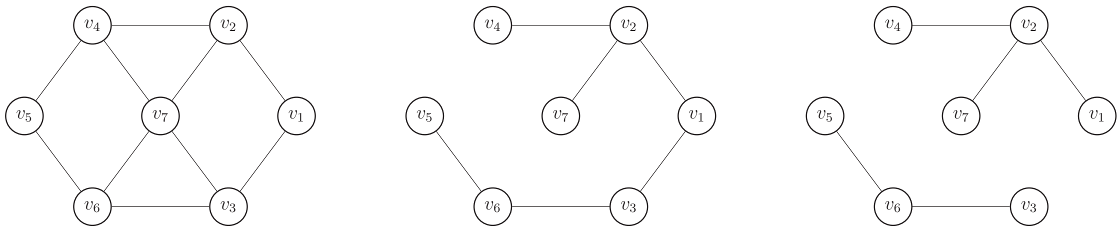
Proof. If G has isolated vertices then as the algorithm only removes edges then, the vertices will remain isolated. Therefore it is sufficient to show G^- does not have more isolated vertices than G.

Now suppose a vertex v is isolated in G^- but not in G. Note that Algorithm 1 only adds a vertex to G^- if it is in the closed neighbourhood of a stem. Therefore when v is added to G^- it is either a stem or adjacent to a stem (which may remove more than leaves). If v is added to G^- as a stem then by definition it is the neighbour of a leaf and hence not isolated. If v is added to G^- as a vertex adjacent to a stem then it has a stem as a neighbour of a leaf and hence not isolated.

□

An example for Algorithm 1 on a graph G is shown in Figure 3.1 (a). G has no isolated vertices so $r = 0$ and G^- is initialized to be empty. Furthermore we set $H = G$. We choose an arbitrary spanning forest F of G by removing the edges $\{v_3, v_7\}$, $\{v_4, v_5\}$, $\{v_4, v_7\}$, and $\{v_6, v_7\}$. We choose an arbitrary stem v_6. Remove any edges from v_6 to another stem, in this case there are none. Now we add $N[v_6]$ to G^- and remove $N[v_6]$ from F, removing the edge $\{v_1, v_3\}$. The reduced F has one stem, v_2. Remove any edges from v_2 to another stem, in this case there are none. Now we add $N[v_2]$ to G^- and remove $N[v_2]$ from F. F is now empty therefore we exit our while loop. The resultant graph, $K_{1,2} \cup K_{1,3}$, is shown in Figure 3.1 (c). A lower bound on the coefficients of G in Figure 3.1 are the coefficients of $D(K_{1,2} \cup K_{1,3}, x) = (x^2 + x(x+1)^2)(x^3 + x(x+1)^3) = x^7 + 7x^6 + 16x^5 + 14x^4 + 6x^3 + x^2$. Note that starting with a different spanning forest (or the same spanning forest, but different sequence of stem selections) may result in a different set of disjoint stars. For instance removing the edges $\{v_3, v_6\}$, $\{v_3, v_7\}$, $\{v_4, v_7\}$, and $\{v_5, v_6\}$ would yield a resultant graph $K_{1,2} \cup K_{1,1} \cup K_{1,1}$ with domination polynomial $x^7 + 7x^6 + 17x^5 + 16x^4 + 4x^3$. In some instances, choosing different stems for the same spanning forest may also result in a different set of disjoint stars.

In Lemma 3.1.3, we will give a condition where edges are irrelevant between two cliques in a graph. We will then use this in Algorithm 2 to make the calculation of our resultant graph easier.



(a) G (b) F (c) $K_{1,2} \cup K_{1,3}$

Figure 3.1: Example for Algorithm 1

Lemma 3.1.3 *Let G be a graph with $C_1, C_2 \subseteq V$ such that each induce cliques in G and $C_1 \cap C_2 = \emptyset$. If there exists $v_1 \in C_1$ and $v_2 \in C_2$ such that $N[v_1] = C_1$ and $N[v_2] = C_2$ then any edge incident to one vertex of $C_1 - v_1$ and one vertex of $C_2 - v_2$ is irrelevant.*

Proof. Let $e = \{u_1, u_2\}$ where $u_1 \in C_1 - v_1$ and $u_2 \in C_2 - v_2$. By Theorem 2.4.14 it is sufficient to show each u_1 and u_2 are each domination-covered in $G - e$. Without loss of generality, we will only consider u_1. As C_1 is a clique then $C_1 \subseteq N_{G-e}[u_1]$. Therefore $N_{G-e}[v_1] = C_1 \subseteq N_{G-e}[u_1]$ and u_1 is domination-covered in $G - e$. □

The next algorithm adds edges to a given graph G until every vertex is in a clique. Furthermore every clique C will contain a vertex v with $N[v] = C$. By Lemma 3.1.3, all the edges between the cliques are irrelevant and can be removed. This leaves us with a resultant graph which is a disjoint union of complete graphs. This will allow us to get an upper bound on the coefficients of $D(G, x)$ as there is a closed form formula for the domination polynomial of complete graphs.

The goal of Algorithm 2 is to partition the vertices of G into subgraphs so we can add edges to make each subgraph a clique and delete the edges between cliques to obtain a disjoint union of complete graphs. To delete the edges between the cliques we must ensure that, for each clique, there is a vertex which domination-covers every other vertex in its respective clique. That is, for every subgraph $N \in S$ there is a vertex with $N_G[v] \subseteq N$. We achieve this by adding $N_G[v]$ to S for some vertex v in B and removing $N_B[N_B[v]]$ from B. By removing $N_B[N_B[v]]$ from B, the collection of neighbourhoods S will not intersect and hence v will not be adjacent to vertices in

Initialize B to G;

Initialize S to the empty list;

while *B is non-empty* **do**

  Choose vertex v in B with the smallest degree in B;

  Set $B = B - N_B[N_B[v]]$;

  Append $N_G[v]$ to S.

end

for *each v remaining in $G - \cup S$* **do**

  Append v to the smallest set in S;

end

for *each N in S* **do**

  add an edge between each pair of vertices in N;

end

for *each N and M in S* **do**

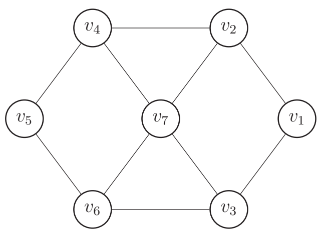
  remove any edge between vertices in N and vertices in M;

end

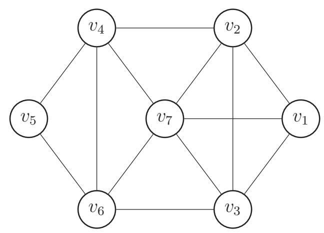
Algorithm 2: Adds edges to G until G is a disjoint union of complete graphs

other chosen neighbourhoods. We choose the vertex of smallest degree to get a better lower bound. As stated earlier, every graph of order n has its coefficients bounded above by the coefficients of $D(K_n, x)$. It is not clear which vertices will give the least upper bound on our coefficients. That being said, choosing a vertex in B with the highest degree puts us at risk of obtaining the trivial upper bound, $D(K_n, x)$. It is for that reason we choose the vertex with the smallest degree, though there may be room for improvement if a different set of vertices are chosen.

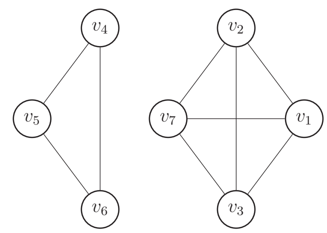
An example for Algorithm 2 on the same graph G used in the previous example is shown in Figure 3.2 (a). First we partition the vertices into cliques by choosing a vertex with the smallest degree. As $\delta(G) = 2$ and there are two degree two vertices,v_1 and v_5, then we arbitrarily choose v_1. Now append $N_G[v_1] = \{v_1, v_2, v_3\}$ to S and remove $N_B[N_B[v_1]] = \{v_1, v_2, v_3, v_4, v_6, v_7\}$ from B. As only the isolated vertex v_5 remains, append $N_G[v_5] = \{v_4, v_5, v_6\}$ to S. Removing $N_B[N_B[v_5]]$ from B leaves B empty so we exit our first loop. For each vertex in $G - \cup S = \{v_7\}$ we add it to the smallest neighbourhood in S. As both neighbourhoods have cardinality three and there is only one vertex in $G - \cup S$, we arbitrarily add v_7 to $N_G[v_1]$. Now that we have partitioned the vertices of G, add every edge between each vertex in $N_G[v_5]$, also add every edge between each vertex in $N_G[v_1] \cup \{v_7\}$. This is shown in Figure 3.2 (b). Now each vertex in $N_G(v_5)$ is domination-covered by v_5. Similarly each vertex in $N_G(v_1) \cup \{v_7\}$ is domination-covered by v_1. Therefore any edge between a vertex in $N_G[v_5]$ and a vertex in $N_G[v_1] \cup \{v_7\}$ is irrelevant and can be removed without changing the domination polynomial. The resultant graph $K_3 \cup K_4$ is shown in Figure 3.2 (c). Therefore an upper bound on the coefficients of G in Figure 3.2 is giving the coefficients of $D(K_3 \cup K_4, x) = ((x+1)^3 - 1)((x+1)^4 - 1) = x^7 + 7x^6 + 21x^5 + 34x^4 + 30x^3 + 12x^2$.



(a) G



(b) G'



(c) $K_3 \cup K_4$

Figure 3.2: Example for Algorithm 2

The following lemma gives a summation formula for the coefficients of star graphs and product of star graphs. We introduce some notation to simplify the proof of the following two lemmas. Given a sequence $n_1, n_2, \ldots, n_k$ of positive integers, for $S \subseteq \{1, 2, \ldots, k\}$ we set $\sum S = \sum_{i \in S} n_i$.

Lemma 3.1.4 *Let graph $G = K_{1,n_1} \cup K_{1,n_2} \cup \ldots \cup K_{1,n_k}$ have domination polynomial $D(G, x) = a_n x^n + a_{n-1} x^{n-1} + \ldots a_1 x$ and $[k] = \{1, 2, \ldots k\}$. Then*

$$a_{n-j} = \sum_{S \subseteq [k]} \binom{\sum S}{|S| + j - k}$$

Proof. By Theorem 2.2.2 the domination polynomial of G is the product of the domination polynomial of each of its components. Furthermore, from Theorem 2.3.1 $D(K_{1,n_i}, x) = x^{n_i} + x(1+x)^{n_i}$. Thus we have

$$\begin{aligned}
D(G,x) &= \prod_{i=1}^{k}(x^{n_i}+x(1+x)^{n_i}) \\
&= \prod_{i=1}^{k}(x^{n_i}+x^{n_i-(n_i-1)}(1+x)^{n_i}) \\
&= \sum_{S\subseteq[k]} x^{n-k-\sum S+|S|}(1+x)^{\sum S} \\
&= \sum_{S\subseteq[k]} x^{n-k-\sum S+|S|}\sum_{l=0}^{\sum S}\binom{\sum S}{l}x^{l} \\
&= \sum_{S\subseteq[k]}\sum_{l=0}^{\sum S}\binom{\sum S}{l}x^{n-k-\sum S+|S|+l}.
\end{aligned}$$

Now $n-k-\sum S+|S|+l=n-j$ iff $l=\sum S-|S|-j+k$, so we find that the coefficient a_{n-j} of $D(G,x)$ is equal to

$$\sum_{S\subseteq[k]}\binom{\sum S}{\sum S-|S|-j+k}=\sum_{S\subseteq[k]}\binom{\sum S}{|S|+j-k}.$$

□

The following lemma gives a summation formula for the coefficients of complete graphs and product of complete graphs.

Lemma 3.1.5 *Let graph $G=K_{n_1}\cup K_{n_2}\cup\ldots\cup K_{n_k}$ have domination polynomial $D(G,x)=a_nx^n+a_{n-1}x^{n-1}+\ldots a_1x$ and $[k]=\{1,2,\ldots k\}$. Then*

$$a_{n-j}=\sum_{S\subseteq[k]}(-1)^{k-|S|}\binom{\sum S}{n-j}$$

Proof. By Theorem 2.2.2 the domination polynomial of G is the product of the domination polynomial of each of its components. Furthermore from Theorem 2.3.1 $D(K_{n_i},x)=(1+x)^{n_i}-1$. Thus we have

$$\begin{aligned}
D(G,x) &= \prod_{i=1}^{k}((1+x)^{n_i}-1)\\
&= \sum_{S\subseteq[k]}(-1)^{k-|S|}(1+x)^{\sum S}\\
&= \sum_{S\subseteq[k]}(-1)^{k-|S|}\sum_{l=0}^{\sum S}\binom{\sum S}{l}x^l.
\end{aligned}$$

By considering the coefficient a_{n-j} in the last equality, we get our result.

□

For a given graph G, we can bound the coefficients of $D(G,x)$. Naturally we ask: can we bound the coefficients for all graphs? As mentioned previously, the tight trivial bounds for all graphs of order n are the coefficients of $D(\overline{K_n},x)$ and $D(K_n,x)$. But what about different families of graphs? Coefficients of connected graphs would still be bounded above by the coefficients of $D(K_n,x)$; however, their lower bounds are not as obvious. Coefficients of tree graphs would have the same lower bounds as connected graphs. However, their upper bounds are no longer trivial either. For connected graphs, coefficients a_n, and a_{n-1} are always 1 and n respectively. The next lemma will help us find a lower bound for the coefficient a_{n-j} in $D(G,x)$ for connected G of order n and $j \geq 2$.

Lemma 3.1.6 *For a fixed sum $n_1+n_2+\ldots+n_k$ and $j \geq 2$ the function*

$$f(L,n)=\sum_{S\subseteq L}\binom{\sum S}{|S|+j-k}$$

where $L=\{n_1,n_2,\ldots,n_k\}$ is minimized if $|n_p-n_q|\leq 1$ for all $n_p,n_q\in L$.

Proof. Suppose not, that is, suppose for some $n_p,n_q\in L$, we have $|n_p-n_q|>1$ and $f(L,n)$ is minimized. Without loss of generality suppose $n_q>n_p$. Let L' be a copy of L where n_q is replaced by n_q-1 and n_p is replaced by n_p+1. Note that L remains unchanged. Consider $f_r(L,n)$, which is the summation $\binom{\sum S}{|S|+j-k}$, where we restrict the cardinality of S to r. That is,

$$f_r(L,n) = \sum_{S \subseteq L, |S|=r} \binom{\sum S}{r+j-k}$$

.

Note that $f(L,n)$ can be written as the summation of $f_r(L,n)$ for r up to $|L| = k$. We will now show for each r, $f_r(L,n) > f_r(L',n)$ and thus arrive at a contradiction.

We partition the subsets $S \subseteq L$ where $|S| = r$ into four cases:

(i) $n_q, n_p \in S$

(ii) $n_q \in S$ and $n_p \notin S$

(iii) $n_q \notin S$ and $n_p \in S$

(iv) $n_q, n_p \notin S$

Pair each subset $S \subseteq L$ with $S' \subseteq L$ which is a copy of S where n_q is replaced by $n_q - 1$ (if $n_q \in S$) and n_p is replaced by $n_p + 1$ (if $n_p \in S$). Now consider $f_r(L,n) - f_r(L',n)$ as the sum of $\binom{\sum S}{r+j-k} - \binom{\sum S'}{r+j-k}$ for each of these pairings with $|S| = |S'| = r$. The pair will cancel if S is in case (i) or (iv) as $\sum S = \sum S'$.

We will now pair the remaining paired cases to create groups of four. Let S_q and S_p denote subsets from cases (ii) and (iii) respectively. Group each S_q with an S_p by replacing n_q with n_p. Let S'_q and S'_p denote their corresponding copies where n_q is replaced by $n_q - 1$ (if $n_q \in S$) and n_p is replaced by $n_p + 1$ (if $n_p \in S$). Let $t = r + j - k$ and $\sum S_q = C + n_q$. Then $\sum S_p = C + n_p$, $\sum S'_q = C + n_q - 1$ and $\sum S'_p = C + n_p + 1$. In $f_r(L,n) - f_r(L',n)$ we obtain

$$\binom{C+n_p}{t} + \binom{C+n_q}{t} - \binom{C+n_p+1}{t} - \binom{C+n_q-1}{t}$$

Now we factor out $t!$ in the denominator. We also factor $T_p = (C+n_p)\cdots(C+n_p+2-t)$ out of $\binom{C+n_p}{t} - \binom{C+n_p+1}{t}$ and $T_q = (C+n_q-1)\cdots(C+n_q+1-t)$ out of $\binom{C+n_q}{t} - \binom{C+n_q-1}{t}$. Note as $n_q - n_p > 1$ then $T_q > T_p$. Furthermore:

$$\begin{aligned}
& ((C+n_p+1-t) - (C+n_p+1))T_p + (C+n_q - (C+n_q-t))T_q \\
= \quad & -(t)T_p + (t)T_q \\
= \quad & t(T_q - T_p) > 0 \text{ for } t > 1
\end{aligned}$$

If $t \leq 1$ then our grouping is 0. Thus each grouping in $f_r(L,n) - f_r(L',n)$ is greater than or equal to zero and $f_r(L,n) - f_r(L',n) \geq 0$. As $0 \leq r \leq k$ then $r - k \leq 0$ and $max(t) = j \geq 2$. What this means is for every j and k, there exists an $r > 0$ such that $t > 1$ and thus $f_r(L,n) - f_r(L',n) > 0$. Furthermore $f(L,n) - f(L',n) > 0$ and we get our contradiction.

□

For every connected graph G of order $n \geq 3$, with domination polynomial $D(G,x) = a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x$, the coefficients a_n and a_{n-1} are 1 and n respectively. The remaining coefficients are not so easily determined. Using Lemma 2.4.3 we can deduce $a_{n-j} = \binom{n}{j}$ for j up to $\delta(G)$. If the vertices with degree $\delta(G)$ have distinct closed neighbourhoods then $a_{n-\delta(G)} = \binom{n}{\delta(G)} - t_{\delta(G)}$ where $t_{\delta(G)}$ is the number of vertices with degree $\delta(G)$.

We will now explore a method to find the global lower bound for coefficients of domination polynomials for connected graphs of order n.

By Algorithm 1 we know that the coefficients of $D(G,x)$ are bounded below by the domination polynomial for a disjoint union of k star graphs. By Lemma 3.1.6, the coefficients of the domination polynomial for a disjoint union of k star graphs can be bounded below by making the number of leafs in each star graph roughly equal. That is, for any two stars the number of leafs in each star can differ by at most one. For k star graphs there is only one way to divide the number of leafs in such a way. Let $S_{n,k} = K_{1,n_1} \cup K_{1,n_2} \cup \ldots \cup K_{1,n_k}$ where $|n_i - n_j| \leq 1$ for each n_i and n_j. As G is connected, it is has no isolated vertices, so any resultant graph from Algorithm 1 will have no isolated vertices. Moreover, each star has at least two vertices and $k \leq \lfloor \frac{n}{2} \rfloor$. Finally by computing $S_{n,k}$ for k from one up to $\lfloor \frac{n}{2} \rfloor$ then taking the smallest coefficient a_{n-j} for each j we find a lower bound for a connected graphs of order n.

As an example, we will determine the lower bound for coefficients of the Domination Polynomial of connected graphs of order 10. We begin with the five possibilities for $S_{10,k}$:

$$\begin{aligned}
S_{10,1} &= K_{1,9} \\
S_{10,2} &= K_{1,4} \cup K_{1,4} \\
S_{10,3} &= K_{1,2} \cup K_{1,2} \cup K_{1,3} \\
S_{10,4} &= K_{1,1} \cup K_{1,1} \cup K_{1,2} \cup K_{1,2} \\
S_{10,5} &= K_{1,1} \cup K_{1,1} \cup K_{1,1} \cup K_{1,1} \cup K_{1,1}
\end{aligned}$$

As we know the domination polynomial for the product of stars we can obtain each domination polynomial for $S_{10,k}$:

$$\begin{aligned}
D(S_{10,1}, x) &= x^{10} + 10x^9 + 36x^8 + 84x^7 + 126x^6 + 126x^5 + 84x^4 + 36x^3 + 9x^2 + x \\
D(S_{10,2}, x) &= x^{10} + 10x^9 + 37x^8 + 68x^7 + 78x^6 + 58x^5 + 28x^4 + 8x^3 + x^2 \\
D(S_{10,3}, x) &= x^{10} + 10x^9 + 38x^8 + 69x^7 + 64x^6 + 33x^5 + 9x^4 + x^3 \\
D(S_{10,4}, x) &= x^{10} + 10x^9 + 39x^8 + 74x^7 + 69x^6 + 28x^5 + 4x^4 \\
D(S_{10,5}, x) &= x^{10} + 10x^9 + 40x^8 + 80x^7 + 80x^6 + 32x^5
\end{aligned}$$

Then by taking the minimum coefficient for each power we obtain the lower bound for the coefficients of all domination polynomials of degree 10. Although a disjoint union of stars is not connected, the center of each star is domination-covered. Irrelevant edges can be added to connect each star without changing the domination polynomial. Therefore the following bounds are tight for connected graphs:

$$\begin{array}{lllll}
a_{10} \geq 1 & a_9 \geq 10 & a_8 \geq 36 & a_7 \geq 68 & a_6 \geq 64 \\
a_5 \geq 28 & a_4 \geq 0 & a_3 \geq 0 & a_2 \geq 0 & a_1 \geq 0
\end{array}$$

For graphs of order ten the value of k which minimizes a_8, a_7, a_6, and a_5 seems linear. We find that a_8 is minimized when $k = 1$, a_7 is minimized when $k = 2$, a_6 is minimized when $k = 3$, and a_5 is minimized when $k = 4$. This is true for other small values of n but not all n. It remains open which values of k will minimize the coefficients of the domination polynomial of connected graphs. Below we give a table of the lower bound on coefficients of domination polynomials for connected graphs up to order 11.

$n\backslash j$	1	2	3	4	5	6	7	8	9	10	11
1	1										
2	2	1									
3	1	3	1								
4	0	3	4	1							
5	0	2	6	5	1						
6	0	0	6	10	6	1					
7	0	0	4	14	15	7	1				
8	0	0	0	13	26	21	8	1			
9	0	0	0	8	30	44	28	9	1		
10	0	0	0	0	28	64	68	36	10	1	
11	0	0	0	0	16	69	117	100	45	11	1

Table 3.1: The lower bound on $d(G, j)$ for connected graphs up to order 11

For comparison to Table 3.2 we also give a table of the uppers bound on coefficients of domination polynomials for connected graphs up to order 11. Note these are just the coefficients of the complete graphs as every non-empty subset is a dominating set.

$n\backslash j$	1	2	3	4	5	6	7	8	9	10	11
1	1										
2	2	1									
3	3	3	1								
4	4	6	4	1							
5	5	10	10	5	1						
6	6	15	20	15	6	1					
7	7	21	35	35	21	7	1				
8	8	28	56	70	56	28	8	1			
9	9	36	84	126	126	84	36	9	1		
10	10	45	120	210	252	210	120	45	10	1	
11	11	55	165	330	462	462	330	165	55	11	1

Table 3.2: The upper bound on $d(G, j)$ for connected graphs up to order 11

3.2 Coefficients of domination polynomials using graph properties

In [2] Akbari, Alikhani and Peng used $d(P_n, n-1)$, $d(P_n, n-2)$, and $d(P_n, n-3)$ to determine some properties of the graphs which are $\mathcal{D}$-equivalent to P_n. For a general graph G with order n, $d(G, n-1)$ and $d(G, n-2)$ are known. In this section we will determine $d(G, n-3)$ for a general graph and $d(G, n-4)$ for graphs in a new collection of graphs denoted, $G^k(m)$.

When counting the number of dominating sets with cardinality close to n, it may simplify things to count the number of subsets which are not dominating. A subset $S \subseteq V(G)$ is not dominating if there exists a vertex v in G such that none of its neighbours, nor itself, is in S. That is, $N[v] \cap S = \emptyset$. The next lemma will help us identify which subsets are not dominating by looking at the subset's complement (with respect to V). If there exists a vertex v and subset $S \subseteq V$ such that $N[v] \subseteq S$ then we say S **encompasses** v or v is encompassed by S.

Lemma 3.2.1 *For a graph G and $S \subseteq V(G)$, S is not dominating if and only if there exists a vertex $v \in \overline{S}$ which is encompassed by $\overline{S}$.*

Proof. If there exists a vertex $v \in \overline{S}$ which is encompassed by $\overline{S}$ then $N[v] \subseteq \overline{S}$. Thus $N[v] \cap S = \emptyset$ and S is not a dominating set. On the other hand, if S is not dominating then there exists a vertex v in G such that $N[v] \cap S = \emptyset$. However $N[v] \subseteq V(G)$ and $V(G) = S \cup \overline{S}$ therefore $N[v] \cap (S \cup \overline{S}) = N[v]$. Moreover

$$N[v] \cap (S \cup \overline{S}) = (N[v] \cap S) \cup (N[v] \cap \overline{S}) = N[v] \cap \overline{S}.$$

Hence $N[v] \cap \overline{S} = N[v]$ and there exists a vertex $v \in \overline{S}$ such that $N[v] \subseteq \overline{S}$.

□

Using Lemma 3.2.1 we can now determine the number of dominating sets by counting the number of subsets of vertices with a given cardinality which contain the closed neighbourhood of one of its vertices. In the next lemma we will use the same methods to determine $d(G, n-k)$ for all k, for a graph G of order n with no isolated vertices and no K_2 components. We first define some graph variables

- T_r: The set of degree r vertices in G which are not stems.
- W: The set of all stems in G.
- ω: The number of stems in G.
- $S_1, S_2, \ldots S_\omega$: The sets of leaves corresponding to each stem in G.
- H_k: A subset of vertices in G with cardinality k (*also referred to as a k-subset*).
- $f_G(H_k, U)$: Returns the number of vertices of U, which are encompassed by H_k.

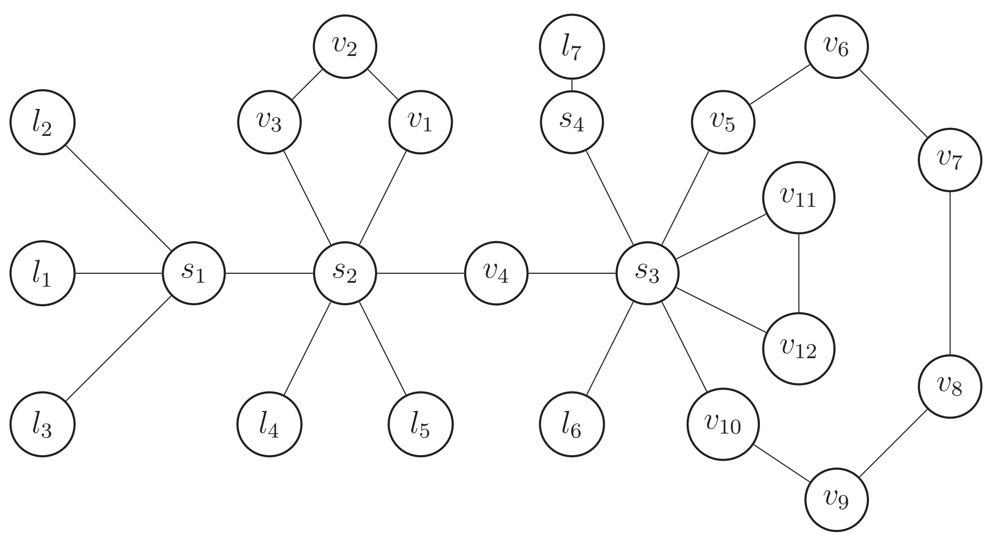


Figure 3.3: An example of a graph

For an example of the aforementioned variables see the graph in Figure 3.3. The set of stems W is $\{s_1, s_2, s_3, s_4\}$ and $\omega = 4$. There are no degree zero vertices therefore $T_0 = \emptyset$. There are six degree one vertices (leaves), none of which are stems, therefore $T_1 = \{l_1, l_2, l_3, l_4, l_5, l_6, l_7\}$. There are 13 degree two vertices, one of which (s_4) is a stem, so $T_2 = \{v_i | 1 \leq i \leq 12\}$. The sets of leaves are $S_1 = \{l_1, l_2, l_3\}$, $S_2 = \{l_4, l_5\}$, $S_3 = \{l_6\}$, and $S_4 = \{l_7\}$. An example of an 8-subset is $H_8 = \{s_3, s_4, l_1, l_6, l_7, v_2, v_{11}, v_{12}\}$. The vertices which are encompassed by H_8 are s_4, l_6, l_7, v_{11}, and v_{12}. Therefore $f_G(H_8, V) = 5$ and $f_G(H_8, V - W) = 4$ as $s_4 \in W$.

Lemma 3.2.2 *For a graph G of order n with no K_2 components and $k \in \mathbb{N}$, where $2 \leq k \leq n - \gamma(G)$, then*

$$d(G, n-k) = \binom{n}{k} - \left(\sum_{i=0}^{k-1} |T_i| \binom{n-i-1}{k-i-1} - \sum_{\substack{H_k \subseteq V \\ |H_k|=k}} max(f_G(H_k, V-W) - 1, 0) \right).$$

Proof. As there are $\binom{n}{k}$ k-subsets of vertices in G, $\binom{n}{k} - d(G, n-k)$ is the number of $(n-k)$-subsets of G which are not dominating. Thus by Lemma 3.2.1, the number of $(n-k)$-subsets which are not dominating is equivalent to the number of k-subsets which encompass at least one vertex. Therefore it is sufficient to show the number of k-subsets which encompass at least one vertex is

$$\sum_{i=1}^{k-1} |T_i| \binom{n-i-1}{k-i-1} - \sum_{\substack{H_k \subseteq V \\ |H_k|=k}} \max(f_G(H_k, V-W) - 1, 0).$$

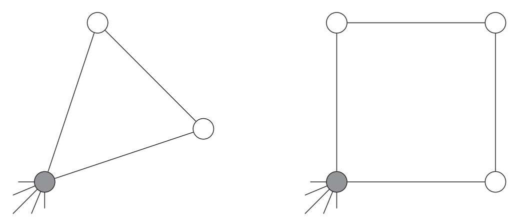
For each vertex $v \in T_i$, its closed neighbourhood has order $i+1$. If $i \leq k-1$ then there are $\binom{n-i-1}{k-i-1}$ k-subsets which encompass v. If $i > k-1$ then $|N[v]| > k$ and no k-subset can encompass v. Therefore the first term's count includes every k-subset which encompasses a non-stem vertex. We omit the stems of G as any k-subset which encompasses a stem s must also encompass one of its leaves l as $N[l] \subseteq N[s]$. Hence each of these k-subsets are counted when we count every k-subset which contains l.

If a k-subset H_k encompasses at least one vertex, we wish only count it once. However our first term counts each k-subset for each non-stem vertex it encompasses. That is, each H_k is counted $f_G(H_k, V-W)$ times and hence over counted $f_G(H_k, V-W)-1$ times. In the case where $f_G(H_k, V-W) \leq 1$ then we have not over counted. As this implies $f_G(H_k, V-W)-1 \leq 0$, it is sufficient subtract $max(f_G(H_k, V-W)-1, 0)$ for each H_k of G. This gives us the second term.

□

Note in Lemma 3.2.2 we require G to have no K_2 components as, in a K_2, each vertex is both a stem and a leaf and not in T_0. Therefore we would not count every k-subset which encompasses the vertices of the K_2 component.

We can now use Lemma 3.2.2 to determine if graphs which are $\mathcal{D}$-equivalent to G have particular subgraphs of a given order k. In this regard we now define a particular subgraph in G. An **r-loop** is an induced r-cycle in G such that all but one vertex has degree two in G. Of course, the one vertex which is not degree two must have degree greater than two. Examples of L_3 and L_4 are shown in Figure 3.4. The vertex which is not of degree two is shaded gray. Other examples of r-loops can be found in Figure 3.3; the vertices s_3, v_{11}, and v_{12} form a 3-loop, and the vertices $s_3, v_5, v_6, \ldots, v_{10}$ form a 7-loop. The vertices s_4, v_1, v_2, and v_3 also form a 4-loop.



(a) L_3 (b) L_4

Figure 3.4: Examples of L_n sub-graphs of G

In Lemma 3.2.2, the value of k limits $f_G(H_k, V - W)$ and H_k. As H_k has order k, it can encompass at most k vertices hence $f_G(H_k, V - W) \leq k$. Furthermore if $f_G(H_k, V - W) > 1$ then H_k encompasses a vertex v. Therefore $N[v] \subseteq H_k$ and hence $|N[v]| \leq k$. Furthermore any vertex H_k encompasses must have degree less than k. In the next lemma we will use Lemma 3.2.2 to determine $d(G, n-3)$ for a graph G of order n with no isolated vertices and no K_2 components. Before we begin, we define some graph variables which represent certain geometrical features for a given graph G which will be used in Theorem 3.2.3 and Theorem 3.2.6.

- $\mathcal{L}_r$: The set of r-loop subgraphs in G.
- $\mathcal{L}_r^i$: The set of r-loop subgraphs in G which contain stem i.
- $\mathcal{C}_r$: The set of components which are cycles of order r in G.

Theorem 3.2.3 *For a graph G of order n where G has no isolated vertices and no K_2 components,*

$$d(G, n-3) = \binom{n}{3} - \left(|T_1| \cdot (n-2) + |T_2| - \sum_{i=1}^{\omega} \binom{|S_i|}{2} - |\mathcal{L}_3| - 2|\mathcal{C}_3| \right).$$

Proof. By Lemma 3.2.2 we know

$$d(G, n-3) = \binom{n}{3} - \left(\sum_{i=0}^{2} |T_i| \binom{n-i-1}{3-i-1} - \sum_{H_3 \subseteq V} \max(f_G(H_3, V-W) - 1, 0) \right).$$

As G has no isolated vertices, $|T_0| = 0$ and $\sum_{i=0}^{2} |T_i| \binom{n-i-1}{3-i-1} = |T_1| \cdot (n-2) + |T_2|$. Now it is sufficient to show

$$\sum_{H_3 \subseteq V} \max(f_G(H_3, V-W) - 1, 0) = \sum_{i=1}^{\omega} \binom{|S_i|}{2} + |\mathcal{L}_3| + 2|\mathcal{C}_3|. \tag{3.1}$$

$max(f_G(H_3, V-W) - 1, 0)$ is only non-zero when $f_G(H_3, V-W) \geq 2$. Therefore we wish to find 3-subsets of G which encompass two or more non-stem vertices. Let H be an arbitrary 3-subset of G which encompass two or more non-stem vertices. As H has order three then the non-stem vertices which it encompasses have degree at most two. As G has no isolated vertices then the vertices which H encompass are either all in T_1 or all in T_2 or both. Let S be the set of vertices which H encompasses. We now count each H in the three aforementioned cases.

Case 1: $S \cap T_1 \neq \emptyset$ *and* $S \cap T_2 = \emptyset$

Let the three vertices of H be u, v, and w. As H has order three then it either encompasses two or three vertices in T_1. Without loss of generality let u and v be two of the vertices encompassed by H. Then $u, v \in T_1$ and as G has no K_2 components then u and v are not adjacent. Furthermore as $\deg(u) = \deg(v) = 1$ and H encompasses both u and v then $N(u) = \{w\}$ and $N(v) = \{w\}$. Therefore u and v are each leaves on the same stem w.

Case 2: $S \cap T_1 = \emptyset$ *and* $S \cap T_2 \neq \emptyset$

Let the three vertices of H be u, v, and w. As H has order three then it either encompasses two or three vertices in T_2. Without loss of generality let u and v be two of the vertices encompassed by H. Then $u, v \in T_2$ and $N(u) = \{v, w\}$ and $N(v) = \{u, w\}$. Therefore H induces a 3-cycle in G and $\{u, v\} \subseteq N(w)$. This leaves us with two possibilities for H, either $deg(w) = 2$ or $deg(w) > 2$. If $deg(w) = 2$ then H is a 3-cycle component of G. If $deg(w) > 2$ then H is a 3-loop in G.

Case 3: $S \cap T_1 \neq \emptyset$ *and* $S \cap T_2 \neq \emptyset$

We claim this case is impossible. If S contains at least one vertex from both T_1 and T_2 then let $v \in T_2$ with $N(v) = \{u, w\}$. As H encompasses v then u and w are both vertices in H. Moreover as H is order three and S contains at least one vertex in T_1 then either $u \in T_1$ or $w \in T_1$. Without loss of generality let $u \in T_1$. Then u is a leaf and has only one neighbour, v. But then v is a stem and by definition not in T_2 which is a contradiction.

Our three cases have produced three possible 3-subsets which encompass two or more non-stem vertices: two leafs on the same stem, 3-cycle components, and 3-loops. The three cases for H are shown in Figure 3.5. Note the vertices which are encompassed by H are shaded.

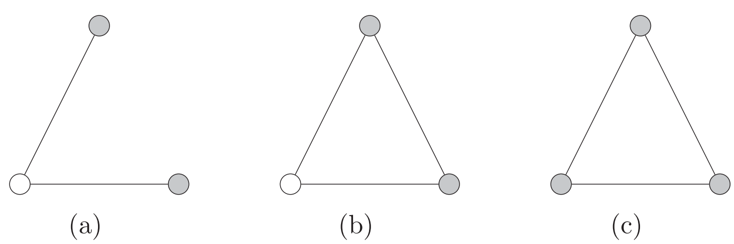


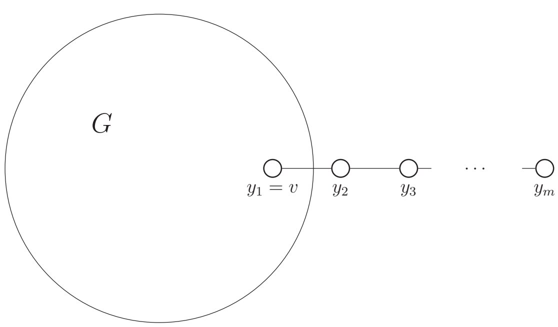
Figure 3.5: Every 3-subset which encompasses two or more non-stems

Now we need only to sum $f_G(H, V - W) - 1$ for each 3-subset. We will sum each $f_G(H, V - W) - 1$ by evaluating $f_G(H, V - W) - 1$ each case then multiplying it by the number of times it occurs in G.

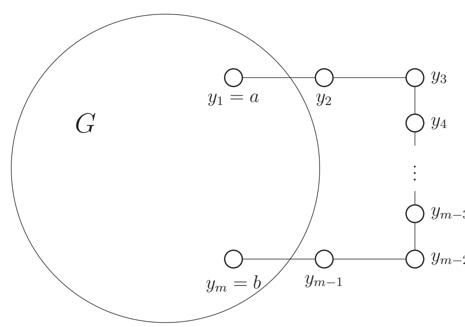
If H is two leafs on the same stem then $f_G(H, V - W) - 1 = 1$. This 3-subset will occur $\binom{|S_i|}{2}$ times for each stem. If H is a 3-loop then $f_G(H, V - W) - 1 = 1$. This 3-subset will occur $|\mathcal{L}_3|$ times. If H is a 3-cycle component then $f_G(H, V - W) - 1 = 2$. This 3-subset will occur $|\mathcal{C}_3|$ times. Taking the sum of each of the cases gives use the right hand side of equation (3.1).

□

We now introduce the collection of graphs $G^k(m)$. First we define the two generalized graphs $G(m)$ and $G'(m)$. Let P_m be a path with vertices labelled $y_1, \dots, y_m$. Let v be a specific vertex in a graph G. Denote by $G_v(m)$ (or simply $G(m)$) a graph obtained from G identifying the vertex v with y_1. This is illustrated in Figure 3.6 (a). Although $G_v(m)$ can change depending on v, we will refer to a graph which takes the form of a $G_v(m)$ graph as a $G(m)$ graph. It is common to refer to $G(m)$ multiple times with varying length paths (i.e. $G(m-1)$, $G(m-2)$, $G(m-3)$). In this case we assume the graph G and vertex are fixed. Similarly let a, b be two specific vertices in G (it is possible for $a = b$). Denote $G'_{a,b}(m)$ (or simply $G'(m)$) a graph obtained from G identifying the vertices a and b with end vertices y_1 and y_m. This is illustrated in Figure 3.6 (b). Although $G'_{a,b}(m)$ can change depending on a and b, we will refer to a graph which takes the form of a $G'_{a,b}(m)$ graph as a $G'(m)$ graph. It is common to refer to $G'(m)$ multiple times with varying length paths (i.e. $G'(m-1)$, $G'(m-2)$, $G'(m-3)$). In this case we assume the graph G and vertices a and b are fixed



(a) $G(m)$



(b) $G'(m)$

Figure 3.6: $G(m)$ and $G'(m)$

Note a path is a $G(m)$ graph where G is a single vertex. Moreover any graph with a leaf is a $G(m)$ graph as the leaf and its respective stem are a path of order two. A cycle is a $G'(m)$ graph where $G = K_2$. In [4] Alikhani proved the recurrences for graphs $G(m)$ and $G'(m)$ before showing the recursion also applied to paths and cycles with the only difference in the initial conditions.

Theorem 3.2.4 [4] *For every* $m \geq 5$

$$D(G_v(m), x) = x(D(G_v(m-1), x) + D(G_v(m-2), x) + D(G_v(m-3), x))$$

and

$$D(G'_{a,b}(m), x) = x(D(G'_{a,b}(m-1), x) + D(G'_{a,b}(m-2), x) + D(G'_{a,b}(m-3), x))$$

□

We will use Theorem 3.2.4 to show $D(G, -2) \neq 0$ if certain conditions hold. The domination polynomial is multiplicative across components and $D(K_2, -2) = 0$. Therefore, if G has a K_2 component then $D(G, -2) = 0$. That is, $D(G, -2) \neq 0$ implies G has no K_2 components. This is vital to proving the domination equivalence classes of paths.

We now introduce a new collection of graphs.

Definition 3.2.5 $G^k(m)$ *denotes the set of all* $G(m)$ *and* $G'(m)$ *graphs restricted to those with maximum non-stem degree* k*. In other words, a vertex is either a stem or has degree at most* k*.*

Our focus will be when $k = 2$. Two familiar families of graphs in $G^2(m)$ are paths and cycles. Another example of a graph in $G^2(m)$ was shown in Figure 3.3. For a graph H of order n, if $\omega + |T_1| + |T_2| = n$, clearly the highest non-stem degree is no more than two. However, we claim this also implies H takes the form of either a $G(m)$ or $G'(m)$ and hence $H \in G^2(m)$. If $|T_1| > 0$, H has a leaf and hence takes the form of a $G(m)$ graph. If $|T_1| = 0$, $\omega = 0$ and H is 2-regular. As the only 2-regular graphs are cycles, H takes the form of a $G'(m)$ graph. Note if $G \in G^2(m)$ and G has a r-loop then the one vertex of the r-loop which is not degree two is a stem.

In the next lemma we will determine $d(G, n-4)$ for a graph $G \in G^2(m)$ of order n with no isolated vertices and no K_2 components. This will allow us to characterize any graphs $\mathcal{D}$-equivalent to graphs in $G^2(m)$. Before we begin, we will partition T_2 into subsets based on the number of neighbouring stems.

- V_0: The subset of T_2 with no adjacent stems.
- V_1^i: The subset of T_2 adjacent to exactly one stem, stem i.
- V_2^{ij}: The subset of T_2 adjacent to exactly two stems, stems i and j (*denoted* V_2 *when* G *only has two stems*).

Theorem 3.2.6 *Let $G \in G^2(m)$ be a graph of order n with no isolated vertices and no K_2 components. Then*

$$d(G, n-4) = \binom{n}{4} - \left(|T_1|\binom{n-2}{2} + |T_2|(n-3) - \alpha_1 - \alpha_2 - \alpha_3\right)$$

where

$$\begin{aligned}
\alpha_1 &= \sum_{i=1}^{\omega} \tbinom{|S_i|}{2}(n - |S_i| - 1) + \sum_{i=1}^{\omega} \tfrac{|S_i|}{2}(|T_1| - |S_i|) + 2\sum_{i=1}^{\omega} \tbinom{|S_i|}{3}, \\
\alpha_2 &= \sum_{i=1}^{\omega} |V_1^i||S_i| + \sum_{i \neq j} |V_2^{ij}|(|S_i| + |S_j|), \text{ and} \\
\alpha_3 &= |V_0| + \sum_{i=1}^{\omega} \tfrac{|V_1^i|}{2} + \sum_{i \neq j} \tbinom{|V_2^{ij}|}{2} - |\mathcal{C}_4| + |\mathcal{C}_3|(2n-9) + \sum_{i=1}^{\omega} |\mathcal{L}_3^i|(n - 4 - |S_i|).
\end{aligned}$$

Proof. By Lemma 3.2.2 we know

$$d(G, n-4) = \binom{n}{4} - \left(\sum_{i=0}^{3} |T_i|\binom{n-i-1}{4-i-1} - \sum_{H_4 \subseteq V} max(f_G(H_4, V-W) - 1, 0)\right).$$

As G has no isolated vertices, $|T_0| = 0$ and $\sum_{i=0}^{3} |T_i|\binom{n-i-1}{4-i-1} = |T_1|\binom{n-2}{2} + |T_2|(n-3)$ (since $G \in G^2(m)$ implies $|T_3| = 0$). Now it is sufficient to show

$$\sum_{H_4 \subseteq V} max(f_G(H_4, V-W) - 1, 0) = \alpha_1 + \alpha_2 + \alpha_3. \tag{3.2}$$

$max(f_G(H_4, V-W) - 1, 0)$ from the left hand side of equation 3.2 is only non-zero when $f_G(H_4, V-W) \geq 2$. Therefore we wish to find 4-subsets of G which encompass two or more non-stem vertices. Let H be an arbitrary 4-subset of G which encompass two or more non-stem vertices. As $G \in G^2(m)$ then each non-stem vertex has degree at most two. Furthermore as G has no isolated vertices then the vertices which H

encompass are either all in T_1 or all in T_2 or both. Let S be the set of non-stem vertices which H encompasses. We now count each H in the three aforementioned cases.

Case 1: $S \cap T_1 \neq \emptyset$ *and* $S \cap T_2 = \emptyset$

Let the four vertices of H be w, x, y, and z. As H encompasses at least two non-stem vertices then, without loss of generality, let w and x be encompassed by H. Then $w, x \in T_1$ and and as G has no K_2 components then w and x are not adjacent. Therefore they are either leafs on the same stem, or leaves on different stems. If w and x are leaves on different stems then y and z are each stems and the only non-stems H encompasses are w and x. If w and x are leaves on the same stem then without loss of generality let y be the stem adjacent to w and x. Then z is either a third leaf on y or not. If z is a leaf on the stem y then H encompasses x, w and z. If z is not a leaf on y then H encompasses only w and x. The subgraphs which H induce are shown in Figure 3.7. Dark gray vertices are stems, light gray vertices are encompassed by H and dashed edges are possible edges.

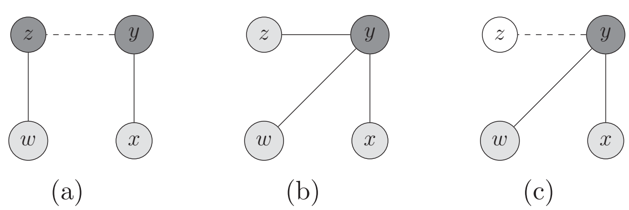


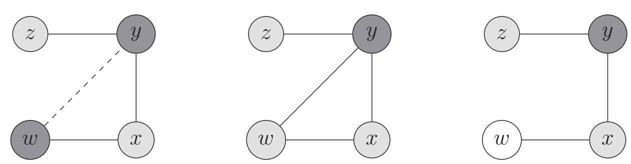
Figure 3.7: Every 4-subset which encompasses two or more vertices, all of which are in T_1

Now we need only to sum $f_G(H, V - W) - 1$ for each 4-subset. We will sum each $f_G(H, V - W) - 1$ by evaluating $f_G(H, V - W) - 1$ for each case then multiplying it by the number of times it occurs in G.

If H encompasses two leaves on different stems (Figure 3.7 (a)) then $f_G(H, V - W) - 1 = 1$. This 4-subset will occur $\sum_{i=1}^{\omega} \frac{|S_i|}{2}(|T_1| - |S_i|)$ times. If H encompasses three leaves on the same stem (Figure 3.7 (b)) then $f_G(H, V - W) - 1 = 2$. This 4-subset will occur $\sum_{i=1}^{\omega} \binom{|S_i|}{3}$ times. If H encompasses two leaves on the same stem and another vertex which is not on that stem (Figure 3.7 (c)) then $f_G(H, V - W) - 1 = 1$. This 4-subset will occur $\sum_{i=1}^{\omega} \binom{|S_i|}{2}(n - |S_i| - 1)$ times. Taking the sum of each of the cases gives us α_1.

Case 2: *$S \cap T_1 \neq \emptyset$ and $S \cap T_2 \neq \emptyset$*

Let the four vertices of H be w, x, y, and z. As H encompasses at least one vertex in T_2 and at least one vertex in T_1 then without loss of generality let x be encompassed by H and $x \in T_2$ where $N(x) = \{w, y\}$. As $x \in T_2$ then x is not a stem. Therefore w and y are not leaves and hence not in T_1. As H must encompass at least one vertex in T_1 then $z \in T_1$. Without loss of generality let $N(z) = \{y\}$. Note that the vertices of H are uniquely determined by the neighbourhoods of x and z. Furthermore y must be a stem and w can either be a stem, in T_2 and encompassed by H, or in T_2 and not encompassed by H. Each case induces a subgraph shown in Figure 3.8. Dark gray vertices are stems, light gray vertices are encompassed by H and dashed edges are possible edges.



(a) (b) (c)

Figure 3.8: Every 4-subset which encompasses at least one vertex from T_1 and at least one vertex from T_2

If w is a stem (Figure 3.8 (a)) then $f_G(H, V - W) - 1 = 1$ and $x \in V_2^{ij}$ for some stems i and j. This 4-subset will occur $|S_i| + |S_j|$ times for every T_2 vertex adjacent to stems i and j. As the number of T_2 vertices adjacent to stems i and j is V_2^{ij} then this 4-subset will occur $\sum\limits_{i \neq j} |V_2^{ij}|(|S_i| + |S_j|)$ times.

If w is not a stem (Figure 3.8 (b) and (c)) then $x \in V_1^i$ and $z \in S_i$ for some stem i. As H is uniquely determined by the closed neighbourhoods of x and z then we can count these by choosing one vertex from V_1^i and one vertex from S_i for each stem i. This gives us the term $\sum\limits_{i=1}^{\omega} |V_1^i||S_i|$. The subgraph in Figure 3.8 (b) will be counted twice for each instance in G and subgraph in Figure 3.8 (c) will be counted once for each instance in G. But that is exactly equal to $f_G(H, V - W) - 1$ for each of these cases. Hence $\sum\limits_{i=1}^{\omega} |V_1^i||S_i|$ is equal to $f_G(H, V - W) - 1$ multiplied by the number of times it occurs in G.

Taking the sum of terms for when w is a stem and when w is not a stem gives us α_2.

Case 3: $S \cap T_1 = \emptyset$ *and* $S \cap T_2 \neq \emptyset$

We will generate every possible such subgraph by first constructing the induced subgraphs of 4-subsets which encompass at least one degree two vertex. Clearly the smallest (fewest edges) such subgraph is $P_3 \cup K_1$ as shown in Figure 3.9 (a). We can then construct the other such subgraphs by adding every combination of the four omitted edges and removing any isomorphisms. This generates the seven other subgraphs shown in Figure 3.9.

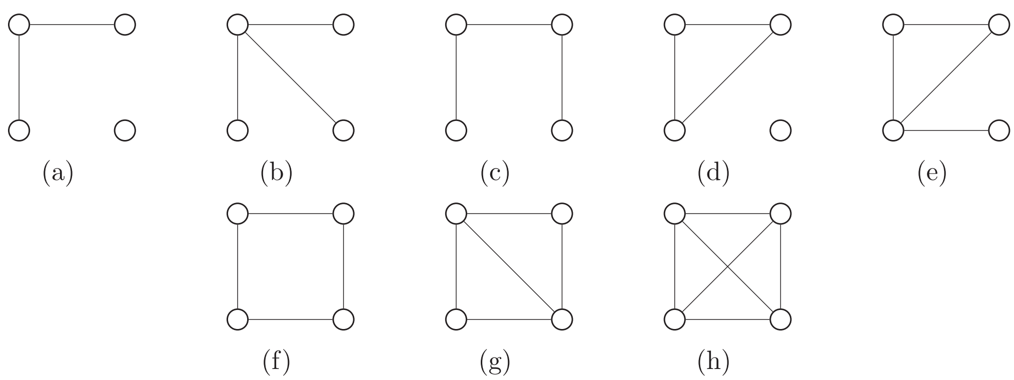


Figure 3.9: Every subgraph with four vertices containing at least one degree two vertex

We now narrow the subgraphs in Figure 3.9 to subgraphs which encompass two or more vertices in T_2. Simply put, each subgraph must contain at least two degree two vertices which are not stems. As subgraph (a) and (b) only contain one vertex for degree greater than one, they do not fit our criteria. As these subgraphs are from a graph in $G^2(m)$, any vertex with degree greater than two must be a stem and hence not in T_2. As each vertex in subgraph (h) is degree three then they are all stems and not in T_2. Therefore subgraph (h) does not fit our criteria and we need only to consider subgraphs (c), (d), (e), (f), and (g).

Of the remaining subgraphs we must consider the possibility that some degree two vertices are not in T_2 or are not encompassed by H. As each subgraph must contain at least two T_2 vertices then the degree two vertices in subgraphs (c), (e), and (g) cannot be stems. Each case is shown in Figure 3.10 where stems are the dark gray vertices and the vertices in T_2 are in light gray.

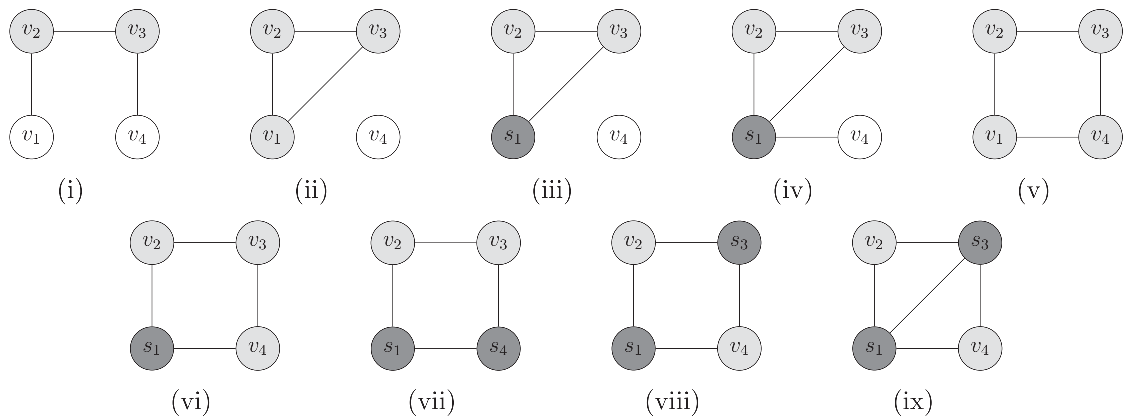


Figure 3.10: Every subgraph with four vertices containing two or more vertices in T_2

Note that in Figure 3.10 the white vertices are not encompassed and can either be stems, T_1, or T_2 vertices. However as we are examining the case where H only encompasses T_2 vertices then v_1, v_4 from (i) and v_4 from (iv) are not T_1 vertices.

Now we need only to sum $f_G(H, V - W) - 1$ for each 4-subset. We will sum each $f_G(H, V - W) - 1$ by evaluating $f_G(H, V - W) - 1$ for each case, then multiplying the result by the number of times the subgraph occurs in G. We may also group some cases for simplicity.

Cases (i) – (vii) from Figure 3.10 all encompass the adjacent T_2 vertices v_2 and v_3. Furthermore $N[v_2] \neq N[v_3]$ in the cases (i), (v), (vi), and (vii). Therefore $|N[v_2] \cup N[v_3]| = 4$ and any subset would require four vertices to encompass both v_2 and v_3. Therefore there is exactly one 4-subset of G which encompasses v_2 and v_3. As there is exactly one edge between v_2 and v_3 in G then we can relate the number of edges between T_2 vertices in G and the sum of $f_G(H, V - W) - 1$ for cases (i), (v), (vi), and (vii). We will count the total number edges between T_2 vertices in G by sum half the number of T_2 vertices each T_2 vertex is adjacent to. We will then subtract the number of edges between T_2 vertices which have the same closed neighbourhood as there are multiple 4-subsets which encompass them. We will also adjust for cases where the number of edges between T_2 vertices does not equal $f_G(H, V - W) - 1$.

In a $G^2(m)$ graph, the neighbours of a T_2 vertex are either stems or other T_2 vertices. Each vertex in V_0 is adjacent to two other T_2 vertices. Each vertex in V_1^i, for any stem i, is adjacent to one other T_2 vertex. Each vertex in V_2^{ij}, for any stems

i and j, is adjacent to no other T_2 vertices. Therefore the number of edges between T_2 vertices in G is

$$\frac{1}{2}\left(2|V_0| + \sum_{i=1}^{\omega} |V_1^i|\right).$$

If two adjacent T_2 vertices have the same closed neighbourhood then they induce a 3-cycle. Furthermore, as at least two of the vertices of the 3-cycle are in T_2, then the induced 3-cycle is either a 3-loop or 3-cycle component in G. As each 3-loop contains one edge between T_2 vertices and each 3-cycle component contains three edges between T_2 vertices we subtract $|\mathcal{L}_3| + 3|\mathcal{C}_3|$ from the total number of edges between T_2 vertices.

In cases (i), (vi), and (vii) the number of edges between T_2 vertices equals $f_G(H, V - W) - 1$. However in case (v), which is a C_4 component of G, $f_G(H, V - W) - 1 = 3$ and there are 4 edges between T_2 vertices. Hence we must also subtract one for each C_4 component of G. Therefore the sum of $f_G(H, V - W) - 1$ for cases (i), (vi), (v), and (vii) is

$$\frac{1}{2}\left(2|V_0| + \sum_{i=1}^{\omega} |V_1^i|\right) - |\mathcal{L}_3| - 3|\mathcal{C}_3| - |\mathcal{C}_4|.$$

For case (ii), $f_G(H, V - W) - 1 = 2$. Case (ii) is a C_3 component with any other vertex. Thus for each C_3 component there is $n - 3$ such 4-subsets. Therefore the number of instances of case (ii) is $|\mathcal{C}_3|(n - 3)$.

For cases (iii) and (iv), $f_G(H, V - W) - 1 = 1$. Cases (iii) and (iv) are 3-loops in G with any other vertex which is not a T_1 adjacent to the stem. This is true because H does not encompass any T_1 vertices and hence cannot contain both a stem and one of its leaves. The number of instances of the cases (iii) and (iv) in G is $\sum_{i=1}^{\omega} |\mathcal{L}_3^i|(n - 3 - |S^i|)$.

For cases $(viii)$ and (ix), $f_G(H, V - W) - 1 = 1$. Cases $(viii)$ and (ix) are two T_2 vertices adjacent to the same two stems. Therefore for each pair of stems i and j, there are $\binom{|V_2^{ij}|}{2}$ such 4-subsets. Therefore the number of instances of cases $(viii)$ and (ix) is $\sum_{i \neq j} \binom{|V_2^{ij}|}{2}$.

The sum of $f_G(H, V - W) - 1 = 1$ for cases $(i) - (ix)$ yields

$$\frac{1}{2}\left(2|V_0|+\sum_{i=1}^{\omega}|V_1^i|\right)+\sum_{i\neq j}\binom{|V_2^{ij}|}{2}-|\mathcal{L}_3|-3|\mathcal{C}_3|-|\mathcal{C}_4|+2|\mathcal{C}_3|(n-3)+\sum_{i=1}^{\omega}|\mathcal{L}_3^i|(n-3-|S_i|).$$

Combining like terms and the fact $|\mathcal{L}_3| = \sum_{i=1}^{\omega}|\mathcal{L}_3^i|$ gives us α_3,

$$|V_0| + \sum_{i=1}^{\omega}\frac{|V_1^i|}{2} + \sum_{i\neq j}\binom{|V_2^{ij}|}{2} - |\mathcal{C}_4| + |\mathcal{C}_3|(2n-9) + \sum_{i=1}^{\omega}|\mathcal{L}_3^i|(n-4-|S_i|).$$

□

Let G_m be a graph in $G^k(m)$. In the next lemma we will show that, if the magnitude of $D(G_i, -2)$ is non-zero, increasing and of alternating sign for the four consecutive integers $i = N, N+1, N+2, N+3$, then $D(G_m, -2) \neq 0$ for $m \geq N$. This allows us to show that any $G \sim G_m$ does not have any K_2 components, since $D(K_2, -2) = 0$, if G had a K_2 component then $D(G, -2) = 0$ as well.

Lemma 3.2.7 *Fix $k \geq 1$. Suppose we have a sequence of graphs $(G_m)_{m\geq 1}$ so that $G_m \in G_v^k(m)$ for all m. If for some $N \in \mathbb{N}$*

$$0 < |D(G_N, -2)| < |D(G_{N+1}, -2)| < |D(G_{N+2}, -2)| < |D(G_{N+3}, -2)|$$

and $D(G_N, -2)$, $D(G_{N+1}, -2)$, $D(G_{N+2}, -2)$, $D(G_{N+3}, -2)$ have alternating sign, then $D(G_m, -2) \neq 0$ for $m \geq N$.

Proof. Before we begin we will recall the recurrence from Theorem 3.2.4 valued at $x = -2$:

$$D(G_m, -2) = -2(D(G_{m-1}, -2) + D(G_{m-2}, -2) + D(G_{m-3}, -2)).$$

Through induction we will show the magnitude of $D(G_m, -2)$ is increasing in absolute value and alternating in sign for all $m \geq N+3$. As $|D(G_N, -2)| > 0$ then this will imply $D(G_m, -2) \neq 0$ for $m \geq N$.

Suppose for some $k \geq N+3$, $D(G_{N+3}, -2), \ldots, D(G_k, -2)$ alternate in signs and increase in absolute value. Then we will first show $D(G_{k+1}, -2)$ has opposite sign to $D(G_k, -2)$. First assume $D(G_k, -2) > 0$ (a similar argument holds when $D(G_k, -2) < 0$). Then $D(G_{k-1}, -2) < 0$ and $D(G_{k-2}, -2) > 0$. By our induction assumption, the magnitude $D(G_m, -2)$ is strictly increasing for $N + 3 \leq m \leq k$. Therefore

$$D(G_k, -2) + D(G_{k-1}, -2) + D(G_{k-2}, -2) > 0$$

.

When we multiply the left side of the above inequality by -2, from the recurrence relation for $D(G_k, -2)$ we will obtain $D(G_{k+1}, -2)$. The signs continue to alternate.

We now show $|D(G_{k+1}, -2)| > |D(G_k, -2)|$. We consider the two cases: $D(G_k, -2) > 0$ and $D(G_k, -2) < 0$.

If $D(G_k, -2) > 0$ then $D(G_{k-1}, -2) < 0$, $D(G_{k-2}, -2) > 0$, and $D(G_{k-3}, -2) < 0$. By our induction assumption, the magnitude $D(G_m, -2)$ is strictly increasing. Therefore

$$D(G_{k-1}, -2) + D(G_{k-2}, -2) + D(G_{k-3}, -2) < D(G_{k-1}, -2) + D(G_{k-2}, -2) < 0$$

.

By the recurrence relation for $D(G_k, -2)$ we deduce

$$\begin{aligned} D(G_k, -2) &= -2(D(G_{k-1}, -2) + D(G_{k-2}, -2) + D(G_{k-3}, -2)) \\ &> -2(D(G_{k-1}, -2) + D(G_{k-2}, -2)) \end{aligned}$$

As $D(G_{k+1}, -2) < 0$ then

$$\begin{aligned} |D(G_{k+1}, -2)| &= -D(G_{k+1}, -2) \\ &= -(-2(D(G_k, -2) + D(G_{k-1}, -2) + D(G_{k-2}, -2))) \\ &= 2D(G_k, -2) + 2D(G_{k-1}, -2) + 2D(G_{k-2}, -2) \\ &> D(G_k, -2) - 2(D(G_{k-1}, -2) + D(G_{k-2}, -2)) \\ &\quad +2D(G_{k-1}, -2) + 2D(G_{k-2}, -2) \\ &= D(G_k, -2) \\ &= |D(G_k, -2)| \end{aligned}$$

Therefore $|D(G_{k+1}, -2)| > |D(G_k, -2)|$ and our claim is true. A similar argument holds when $D(G_k, -2) < 0$.

$\square$

Chapter 4

Equivalence Classes of Domination Polynomials

As stated in section 2.4, two non-isomorphic graphs can have the same domination polynomial. In this chapter we will determine the equivalence class of paths, and give some families of graphs which are $\mathcal{D}$-equivalent.

4.1 Families of $\mathcal{D}$-equivalent graphs

We now define what it is for a subset of vertices S to be domination-covered.

Definition 4.1.1 *For a graph G, a subset of vertices S is* ***domination-covered*** *if for each $v \in S$ there exists an $u \in V(G-S)$ such that $N_G[u] \subseteq N_G[v]$.*

Lemma 4.1.2 *For a graph G and a domination-covered subset of vertices S every dominating set on $G-S$ is also a dominating set in G.*

Proof. Consider a dominating set D of $G-S$. As S is domination-covered then each vertex of $v \in S$ has a neighbour $u \in V(G-S)$ such that $N_G[u] \subseteq N_G[v]$. As D dominates $G-S$ and $u \in V(G-S)$ there exists $u' \in D$ such that $u' \in N_{G-S}[u] \subseteq N_G[u] \subseteq N_G[v]$. Therefore each dominating set of $G-S$ also dominates each $v \in S$ and thus is a dominating set in G. $\square$

Domination-covered subsets form a simplicial complex. That is, if S is a domination-covered subset then any $S' \subseteq S$ is also a domination-covered subset. The next theorem simplifies the recurrence relation from Theorem 2.2.6 if G has a vertex whose neighbourhood is a domination-covered subset. We will then obtain a corollary which allows us to construct $\mathcal{D}$-equivalent graphs from smaller $\mathcal{D}$-equivalent graphs.

Theorem 4.1.3 *For a graph G, Let $S \subseteq V(G)$ be a domination-covered subset. The new graph $G' = G+v$ which is a copy of G with an added vertex v such that $N(v) = S$ has domination polynomial*

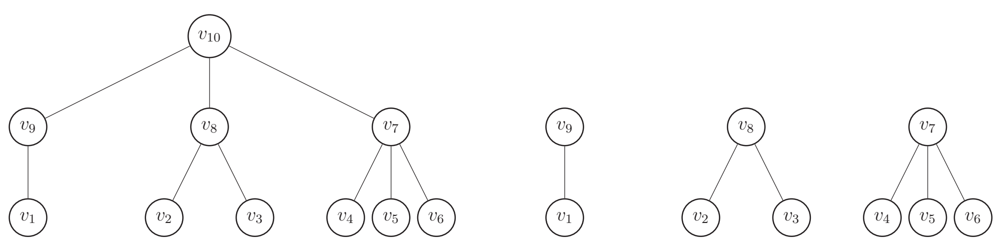
$$D(G', x) = (x+1)D(G, x) - D(G-S, x)$$

Proof. Consider the domination polynomial of $G+v$. By Theorem 2.2.6, we have

$$D(G', x) = xD(G'/v, x) + D(G, x) + xD(G-S, x) - (x+1)p_v(G').$$

As $S = N_{G'}(v)$ is a dominating-covered subset, each vertex of $w \in N_{G'}(v)$ has a neighbour $u \in V(G-S)$ such that $N_G[u] \subseteq N_G[w]$. Thus each $w \in N_{G'}(v)$ is domination-covered. The graph G'/v is just G with edges added which we claim are irrelevant edges. As each $w \in N_{G'}(v)$ is domination-covered then by Theorem 2.4.14, any edge added between two vertices of $N_{G'}(v)$ is irrelevant. Therefore $D(G'/v, x) = D(G, x)$. Furthermore $p_v(G')$ enumerate dominating sets of $G-S$ which also dominate S. As S is a domination-covered subset, by Lemma 4.1.2 each dominating set of $G-S$ also dominates S and hence $p_v(G') = D(G-S, x)$. Thus we have $D(G', x) = (x+1)D(G, x) - D(G-S)$. □

An example of Theorem 4.1.3 is shown in Figure 4.1. Every neighbour of v_{10} is a stem and therefore domination-covered. G is just the disjoint union $K_{1,1} \cup K_{1,2} \cup K_{1,3}$ and $G + v_{10} - N_G[v_{10}]$ is six isolated vertices. Therefore $D(G, x) = (x+1)(x^2 + 2x)(x(x+1)^2 + x^2)(x(x+1)^3 + x^3) - x^6$.



(a) $G + v_{10}$ (b) G

Figure 4.1: An example of Theorem 4.1.3

Corollary 4.1.4 *For two $\mathcal{D}$-equivalent graphs G and H, if $S_G \subset V(G)$ and $S_H \subset V(H)$ are each domination-covered subsets and $G - S_G \sim H - S_G$, then the two new graphs $G+v$ and $H+u$, where $N_G(v) = S_G$ and $N_H(u) = S_H$, are also $\mathcal{D}$-equivalent.*

Examples of Corollary 4.1.4 are shown in Figure 4.2. The open neighbourhood of v_7 in both (a) and (b) form a domination-covered subset. Furthermore, as the graph obtained by deleting v_7 and $N[v_7]$ are isomorphic in both cases, (a) and (b) have the same domination polynomial.

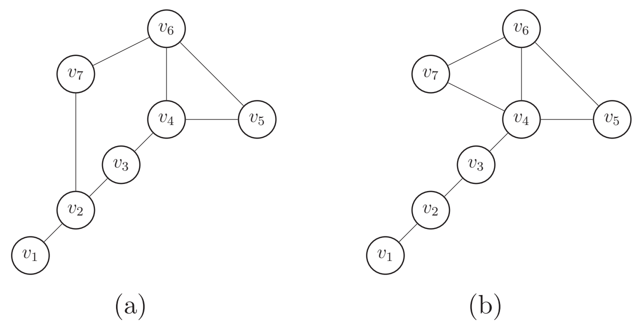
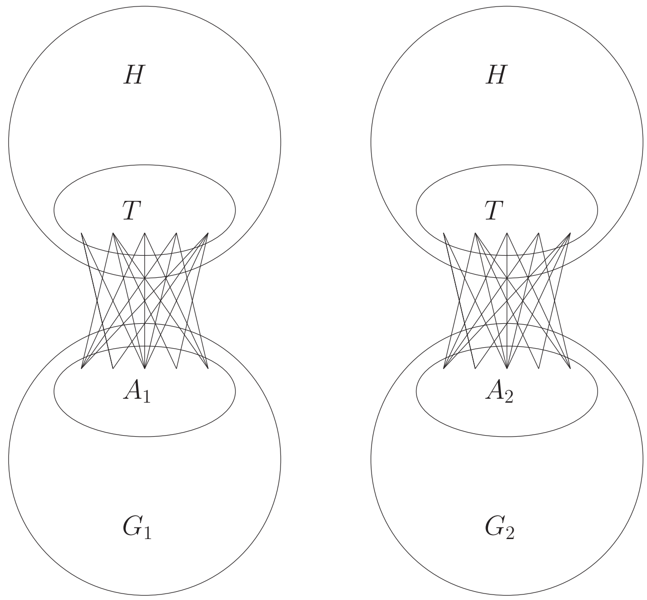


Figure 4.2: Examples of corollary 4.1.4

Recall for a graph G and vertex $v \in V(G)$, $p_v(G)$ is the polynomial generated by the dominating sets of $G - N[v]$ which also dominate $N(v)$. For a subset of vertices $S \subseteq V(G)$ let $p_S(G)$ be the polynomial generated by the dominating sets of $G - N[S]$ which also dominate $N(S)$. The next theorem will state some general equivalence conditions which make $H_1 \sim H_2$ for the graphs in Figure 4.3.

Theorem 4.1.5 *Consider a graph H with domination-covered subset T, and two $\mathcal{D}$-equivalent graphs G_1 and G_2 with $A_1 \subseteq V(G_1)$ and $A_2 \subseteq V(G_2)$. Suppose there exists a bijection $\phi : A_1 \mapsto A_2$ such that for every $B \subseteq A_1$, $p_B(G_1) = p_{\phi(B)}(G_2)$. Let H_1 be a copy of H and G_1 with any set of edges added from A_1 to T and H_2 be a copy of H and G_2 with the corresponding edges added from $\phi(A_1)$ to T. Then $H_1 \sim H_2$.*

Proof. Let S be any dominating set of H_1. Partition S in to $S_{G_1} = S \cap V(G_1)$ and $S_H = S \cap V(H)$. For simplicity we will say S_G dominates the induced subgraph G $(G = G_1, H)$ to mean the closed neighbourhood of S_G in H_1 contains every vertex in G. Unless otherwise stated the neighbourhood of a vertex, or set of vertices, will



(a) H_1 (b) H_2

Figure 4.3: A representation of Theorem 4.1.5

be its neighbourhood in H_1. As $G_1 \sim G_2$ then there is another bijection ψ from the dominating sets of G_1 to the dominating sets of G_2 with equal cardinality.

We claim there is a map from every S to a unique dominating set S' in H_2. We consider S in the following four cases:

Case 1: *S_{G_1} dominates G_1 and S_H dominates H.*

As $G_1 \sim G_2$ then there is another bijection ψ from the dominating sets of G_1 to the dominating sets of G_2 that preserves cardinality. Therefore $\psi(S_{G_1})$ is a corresponding dominating set of G_2 and $S' = \psi(S_{G_1}) \cup S_H$ is a dominating set which dominates both G_2 and H and thus H_2.

Case 2: *S_{G_1} dominates G_1 but S_H does not dominate H.*

Let $C \subseteq V(H)$ be the vertices of H not dominated by S_H (i.e. $C = H - N[S_H]$). Note S_H dominates $H - C$ but not H. We will show this case results in a contradiction by showing S_H dominates H. As S is a dominating set, then each vertex of C has a neighbour in S_{G_1}. Therefore each vertex in C is adjacent to a vertex in G_1. The only vertices of H adjacent to G_1 are in T, therefore $C \subseteq T$. As T is a domination-covered subset in H then so to is C. By Lemma 4.1.2 any dominating set of $H - C$ also dominates H. Therefore S_H dominates H, which is a contradiction, and this case is not possible.

Case 3: *S_H dominates H but S_{G_1} does not dominate G_1.*

Let $C \subseteq V(G_1)$ be the vertices of G_1 not dominated by S_{G_1} (i.e. $C = G_1 - N[S_{G_1}]$). Then S_{G_1} dominates $G_1 - C$ but does not intersect the neighbourhood of C. We observe that such sets S_{G_1} contribute to $p_C(G_1)$. As S is a dominating set, then each vertex of C has a neighbour in S_H. Therefore each vertex in C is adjacent to a vertex in H. The only vertices of G_1 adjacent to H are in A_1. Therefore $C \subseteq A_1$ and there is a corresponding set $\phi(C) \subseteq A_2$ such that $p_C(G_1) = p_{\phi(C)}(G_2)$. Therefore there exists another bijection ρ from the sets in G_1 counted by the coefficients of $p_C(G_1)$ to the sets in G_2 counted by the coefficients of $p_{\phi(C)}(G_2)$ preserving cardinality. Therefore in H_2, $\rho(S_{G_1})$ is a set of G_2 which dominates $G_2 - \phi(C)$. As C is not dominated by S_{G_1} then each vertex in C is adjacent to a vertex of S_H. Therefore in H_2, each vertex in $\phi(C)$ is adjacent to the corresponding vertex of S_H. It follows that S_H still dominates H and also dominates $\phi(C)$. As $\rho(S_{G_1})$ is unique for each S_{G_1} then $S' = \rho(S_{G_1}) \cup S_H$ is the corresponding dominating set of H_2.

Case 4: *S_{G_1} does not dominate G_1 and S_H does not dominates H.*

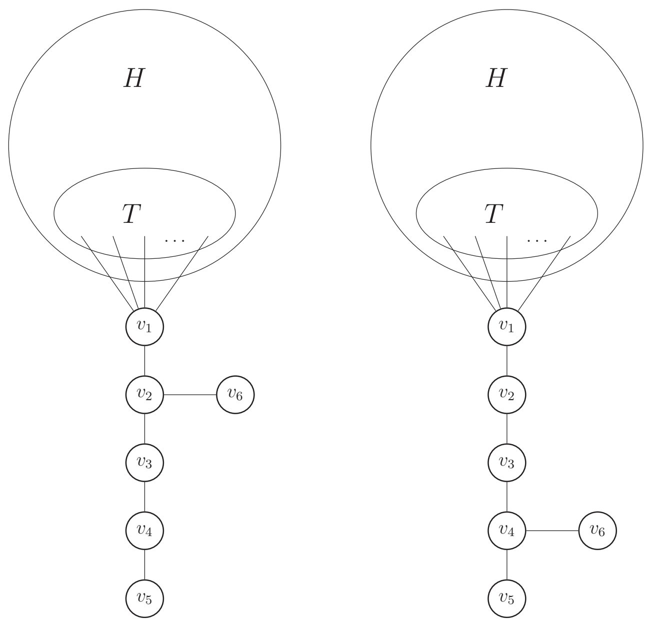
By the same arguments used in case 2, S cannot dominate H_1 without S_H dominating H. Hence we get a contradiction and this case is not possible.

For each dominating set S of G_1 there is a dominating set of equal cardinality S' of G_2. By symmetry the same is true for each dominating set S' of G_2, and so $H_1 \sim H_2$. □

The polynomial $p_v(G)$ is not always intuitive to find. However once two $\mathcal{D}$-equivalent graphs are found with the properties stated in Theorem 4.1.5, you can build infinitely many $\mathcal{D}$-equivalent graphs. Consider a case where our subset $A_1 \subseteq G_1$ is only of size one. Then $A_2 \subseteq G_2$ is only of size one and our bijection is trivial.

For example, in Figure 4.4 let G_1 and G_2 be the respective subgraphs of H_1

and H_2 induced by vertices $v_1, v_2, \ldots, v_6$. Furthermore, $A_1 = \{v_1\}$ and $A_2 = \{v_1\}$ with bijection $\phi : A_1 \mapsto A_2$ defined by $\phi(v_1) = v_1$. G_1 and G_2 are the isomorphic thus $G_1 \sim G_2$. Moreover any dominating set of $G_1 - N[v_1]$ which also dominates $N(v_1) = v_2$ must include v_6 and dominate a path of three so $p_{v_1}(G_1) = xD(P_3, x)$. Any dominating set of $G_2 - N[v_1]$ which also dominates $N(v_1) = v_2$ must include v_3 and also must dominate a path of three. Even though v_4 is already dominated by v_3 it is easy to see $p_{v_1}(G_2) = xD(P_3, x)$. Thus for every subset for every $B \subseteq A_1$, $p_B(G_1) = p_{\phi(B)}(G_2)$. So if T is a domination-covered subset of H then $H_1 \sim H_2$.



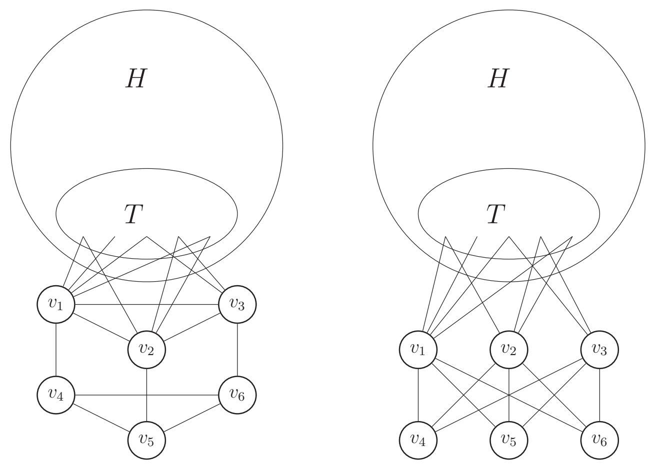
(a) H_1 (b) H_2

Figure 4.4: An example of graphs for Theorem 4.1.5

For a more general case consider $G_1 = K_{n,n}$ and $G_2 = K_n \Box K_2$. In [1] it was shown $K_{n,n} \sim K_n \Box K_2$. Both graphs have two copies of n vertices. Let $A_1 \subseteq V(K_{n,n})$ be one of the two partite sets of n vertices in $K_{n,n}$ which are disjoint. Let $A_2 \in K_n \Box K_2$ be one of the two sets of n vertices in $K_n \Box K_2$ which are complete. Now fix a bijection $\phi : A_1 \mapsto A_2$. Let B be any non-empty subset of A_1. If $B = A_1$ then $K_{n,n} - N[B] = \emptyset$ and $p_B(K_{n,n}) = 0$. Furthermore, $\phi(B) = A_2$, so $K_n \Box K_2 - N[\phi(B)] = \emptyset$ and $p_{\phi(B)}(K_{n,n}) = 0$. If $B \neq A_1$, then the closed neighbourhood of B in $K_{n,n}$ will be every vertex not in $A_1 \cup B$. Then the remaining $n - |B|$ vertices are isolated and $p_B(K_{n,n}) = x^{n-|B|}$. Furthermore, the closed neighbourhood of $\phi(B)$ in $K_n \Box K_2$ includes each vertex in

A_2. As $K_n \square K_2 - N_{K_n \square K_2}[\phi(B)]$ is complete on $n - |\phi(B)|$ vertices any non-empty subset will dominate it. However only the subset of all $n - |\phi(B)|$ vertices dominates $N(\phi(B)) = A_2 - \phi(B)$ thus $p_{\phi(B)}(K_n \square K_2) = x^{n-|B|}$. So if the neighbourhood of A_1 restricted to some other graph H is a domination-covered subset of H then $H_1 \sim H_2$.

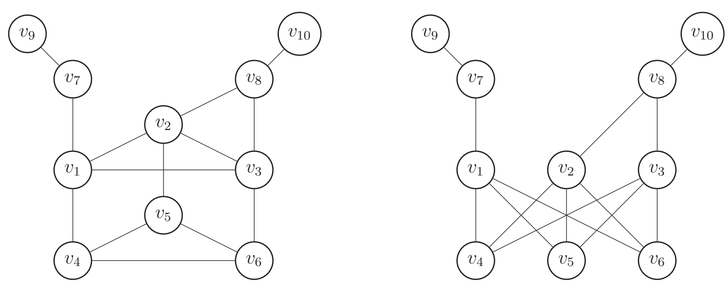
For example, in Figure 4.5 the vertices v_1, v_2, v_3, v_4, v_5 and v_6 form $K_3 \square K_2$ in (a) and $K_{3,3}$ in (b). If T is domination-covered in H and the same edges are added from $\{v_1, v_2, v_3\}$ to H in both H_1 and H_2 then $H_1 \sim H_2$.



(a) H_1 (b) H_2

Figure 4.5: Examples of graphs for Theorem 4.1.5 and Theorem 4.1.6

A more specific example of Theorem 4.1.5 is in Figure 4.6. Vertices v_1, v_2, v_3, v_4, v_5 and v_6 form $K_3 \square K_2$ in (a) and $K_{3,3}$ in (b). Furthermore, the neighbourhood of v_1, v_2 and v_3 in both graphs is $\{v_7, v_8\}$ which is a domination-covered subset therefore $H_1 \sim H_2$.



(a) H_1 (b) H_2

Figure 4.6: Specific examples of Theorem 4.1.5 on $K_{3,3}$ and $K_3 \square K_2$

The previous theorem gave conditions to make H_1 and H_2 in Figure 4.3 $\mathcal{D}$-equivalent. In the next theorem we will fix $G_1 = K_n \Box K_2$ and $G_2 = K_{n,n}$ and give more conditions such that $H_1 \sim H_2$. Also fix A_1 to be one of the two sets of n disjoint vertices in $K_n \Box K_2$. Fix A_2 to be one of the two partite sets of n vertices in $K_{n,n}$. We will refer to the vertices of A_1 and A_2 as the inside vertices, denoted I_1 and I_2 respectively. We refer to the other vertices of G_1 and G_2 as the outside vertices, denoted O_1 and O_2 respectively.

Theorem 4.1.6 *For a graph H, let H_1 be a copy of H and $G_1 = K_n \Box K_2$ with any set of edges added from I_1 to $T \subset V(H)$ and H_2 is a copy of H and $G_2 = K_{n,n}$ with the corresponding edges added from I_2 to $T \subset V(H)$. If T is a clique then $H_1 \sim H_2$.*

Proof. It is sufficient to show that every dominating set S of H_1 can be mapped to a unique dominating set S' of H_2 of equal cardinality, and conversely each S of H_2 can be mapped to a unique dominating set S' of H_1 of equal cardinality. We will now define a bijection $\phi : V(G_1) \mapsto V(G_2)$. Label the vertices of I_1 and O_1 to be $i_1, i_2, \ldots i_n$ and $o_1, o_2, \ldots o_n$, respectively. Label the vertices of I_2 and O_2 to match the labels of I_1 and O_1. Then for each vertex $v \in V(G_1)$, $\phi(v)$ is the vertex in G_2 with the same label as v. For simplicity we say a subset of vertices A dominates some other subset of vertices B, when $B \subseteq N[A]$. Note if A dominates $T' \subseteq T$ in H_1 then $\phi(A)$ dominates $T' \subseteq T$ in H_2.

For a dominating set S of H_1, partition the vertices of S into $S_H = S \cap V(H)$ and $S_{G_1} = S \cap G_1$. We consider the following three cases:

Case 1: *$I_1 \cap S_{G_1} = I_1$.*

Then S_{G_1} dominates G_1 and T, so S_H dominates $H - T$. Also $\phi(S_{G_1})$ contains every vertex in I_2. Therefore in H_2, $\phi(S_{G_1}) = I_2$ and dominates G_2 and T. As S_H still dominates $H - T$, then $S' = \phi(S_{G_1}) \cup S_H$ dominates G_2 and $|S'| = |S|$. Note that $S' \cap I_2 = I_2$.

Case 2: *$I_1 \cap S_{G_1} \subset I_1$ and $I_1 \cap S_{G_1} \neq \emptyset$.*

Let $A = I_1 \cap S_{G_1}$. As $A \neq I_1$, A does not dominate all of O_1. As O_1 induces a clique, and only has neighbours in G_1, then some non-empty $B \subseteq O_1$ is in S_{G_1}. More precisely $S_{G_1} = A \cup B$. Now let T' be the vertices in H dominated by S_{G_1}. Then S_H dominates $H - T'$. Therefore in H_2, $\phi(S_{G_1})$ dominates T' and S_H dominates $H - T'$. Furthermore, $\phi(S_{G_1})$ intersects O_2 and also intersects I_2. As G_2 is complete bipartite, $\phi(S_{G_1})$ dominates G_2 and $S' = \phi(S_{G_1}) \cup S_H$ dominates G_2 with $|S'| = |S|$. Note $S' \cap I_2 = \phi(A) \neq I_2$; therefore, it is distinct from each S' in case 1.

The following is to help show each S' in this case is distinct from each S' in case 3: for subcase $3a$ note that $S' \cap I_2 = \phi(A) \neq \emptyset$ and for subcase $3b$ note that $S' \cap O_2 = \phi(B) \neq \emptyset$.

Case 3: $I_1 \cap S_{G_1} = \emptyset$.

As the only neighbours of O_1 are in G_1, and O_1 induces a clique, some non-empty $B \subseteq O_1$ is in S_{G_1}. More precisely $S_{G_1} = B$. We now consider two subcases for B.

Subcase a: $B = O_1$.

As $B = O_1$, B dominates every vertex in G_1. As S_{G_1} does not intersect I_1 then S_H dominates H. Therefore in H_2, S_H still dominates H. Furthermore, $\phi(B)$ contains all of O_2. As G_2 is complete bipartite in H_2, $\phi(B)$ dominates G_2 and $S' = \phi(B) \cup S_H$ dominates G_2. Since $|B| = |S_{G_1}|$, $|S'| = |S|$. Note that $S' \cap I_2 = \emptyset$, so it is distinct from each S' in case 1 and case 2.

Subcase b: $B \neq O_1$.

Here B does not dominate every vertex in I_1. Let A be the vertices of I_1 dominated by B and A' be the vertices of I_1 not dominated by B. Note $A \cup A' = I_1$, $|A| = |B|$, and $A \neq I_1$. Then S_H dominates A' along with all of H (as S does not intersect I_1). Therefore in H_2, S_H dominates $\phi(A')$ along with all of H. Furthermore, $\phi(A)$ dominates $\phi(A)$ and O_2 are G_2 is complete bipartite. As $A \cup A' = I_1$, then $\phi(A) \cup \phi(A') = I_2$. Therefore $S' = \phi(A) \cup S_H$ dominates G_2. Since $|A| = |B| = |S_{G_1}|$, $|S'| = |S|$. As $A \neq I_1$, $\phi(A) \neq I_2$ and $S' \cap I_2 = \phi(A) \neq I_2$, therefore S' in this case is distinct from each S' in case 1. As $S' \cap O_2 = \emptyset$ it is distinct from each S' in case 1 and case 2.

We now turn to consider a dominating set S of H_2. Partition the vertices of S into $S_H = S \cap V(H)$ and $S_{G_2} = S \cap G_2$. We consider the following three cases:

Case 1: $O_2 \cap S_{G_2} = O_2$.

Then S_{G_2} dominates G_2. As S_{G_2} may also intersect I_2, let T' be the vertices in H which are dominated by S_{G_2}. Then S_H dominates $H - T'$ in H_2. Therefore in H_1, $\phi^{-1}(S_{G_2})$ dominates T' and S_H still dominates $H - T'$. Furthermore $\phi^{-1}(S_{G_2})$ contains every vertex in O_1, so $\phi^{-1}(S_{G_2})$ also dominates G_1. Thus $S' = \phi^{-1}(S_{G_2}) \cup S_H$ dominates G_1 and $|S'| = |S|$. Note that $S' \cap O_1 = O_1$.

Case 2: $O_2 \cap S_{G_2} \subset O_2$ *and* $O_2 \cap S_{G_1} \neq \emptyset$.

Let $A = O_2 \cap S_{G_2}$. As $A \neq O_2$, A does not dominate all of O_2. As G_2 is complete bipartite and O_2 only has neighbours in G_2, some non-empty $B \subseteq I_2$ is in S_{G_2}. More precisely, $S_{G_2} = A \cup B$. Now let T' be the vertices in H dominated by S_{G_2}. Then S_H dominates $H - T'$. Therefore in H_1, $\phi^{-1}(S_{G_2})$ dominates T' and S_H dominates $H - T'$. Furthermore, $\phi^{-1}(S_{G_2})$ intersects O_1 and O_1 induces a clique in H_1, so $\phi(S_{G_2})$ dominates I_1. Similarly $\phi^{-1}(S_{G_2})$ intersects I_1 and I_1 induces a clique in H_1, so $\phi^{-1}(S_{G_2})$ dominates I_1. As $V(G_2) = I_1 \cup O_1$, then $\phi^{-1}(S_{G_1})$ dominates G_2 and $S' = \phi^{-1}(S_{G_2}) \cup S_H$ dominates G_1 with $|S'| = |S|$. Note $S' \cap O_1 = \phi^{-1}(B) \neq O_1$ therefore it is distinct from each S' in case 1.

The following is to help show each S' in this case is distinct from each S' in case 3: for subcase $3a$ note that $S' \cap O_1 = \phi^{-1}(B) \neq \emptyset$ and for subcase $3b$ note that $S' \cap I_1 = \phi^{-1}(A) \neq \emptyset$

Case 3: $O_2 \cap S_{G_2} = \emptyset$.

As the only neighbours of O_2 are in G_2, and G_2 is complete bipartite, then some non-empty $B \subseteq I_2$ is in S_{G_2}. More precisely $S_{G_2} = B$. We now consider two subcases for B.

Subcase a: $B = I_2$.

Then B dominates every vertex in G_2 and T, so S_H dominates $H - T$. Therefore in H_1, $\phi^{-1}(S_{G_2}) = I_1$ and dominates G_1 and T. As S_H still dominates $H - T$, $S' = \phi^{-1}(S_{G_2}) \cup S_H$ dominates H_1 and $|S'| = |S|$. Note that $S' \cap O_1 = \emptyset$. Therefore it is distinct from each S' in case 1 and case 2.

Subcase b: $B \neq I_2$.

Then B does not dominate every vertex in I_2. Let A be the non-empty set of vertices in I_2 not dominated by B. Note that $A \cup B = I_2$. Each vertex in A has a neighbour in T which is also in S_H. As T induces a clique and T intersects S_H, S_H dominates all of T and hence all of H and A. Therefore in H_1, S_H dominates H and $\phi^{-1}(A)$. Because $G_1 = K_n \Box K_2$, the edges between I_1 and O_1 form a bijection from I_1 to O_1. Let $B' \subset O_1$ be the image of $\phi^{-1}(B)$ in said bijection. Note $|B'| = |\phi^{-1}(B)| = |B|$ and B' dominates $\phi^{-1}(B)$. As $A \cup B = I_2$, $\phi^{-1}(A) \cup \phi^{-1}(B) = I_1$ and $B' \cup S_H$ dominates I_1, As O_1 induces a clique, B' dominates O_1. Therefore $S' = B' \cup S_H$ dominates H_1. Note that $S' \cap O_1 = \phi^{-1}(B) \neq O_1$. Therefore it is distinct from each S' in case 1. As $S' \cap I_1 = \emptyset$ then it is distinct from each S' in case 2.

□

The condition that T is a clique was only required to map dominating sets of H_2 to H_1. This means for any H, each coefficient of $D(H_1, x)$ is bounded above by the corresponding of $D(H_2, x)$.

An example of Theorem 4.1.6 is in Figure 4.7. Vertices v_1, v_2, v_3 and v_4 form $K_2 \Box K_2$ in (a) and $K_{2,2}$ in (b). H is the graph formed by v_5, v_6, v_7, v_8, v_9 and v_{10}. In (a), $I_1 = \{v_1, v_4\}$. Similarly in (b), $I_2 = \{v_1, v_4\}$. The edges from I_1 to H correspond to the edges from I_2 to H. The neighbourhood of I_1 and I_2 in each graph is one single vertex v_5 so $T = \{v_5\}$. As T is a clique, $H_1 \sim H_2$.

We will now consider the graph in Figure 4.8. It is a copy of C_4 with the graph substitution K_{n_i} for each $v_i \in V(C_4)$. That is, each vertex $v_i \in V(C_4)$ is replaced by K_{n_i} and if v_i is adjacent to v_j in C_4, then every vertex of K_{n_i} is adjacent to every vertex of K_{n_j}. We denote this graph $C(S_4)$, where S_4 is the ordered 4-tuple (n_1, n_2, n_3, n_4). In the next theorem we will show that changing the order of the n_i

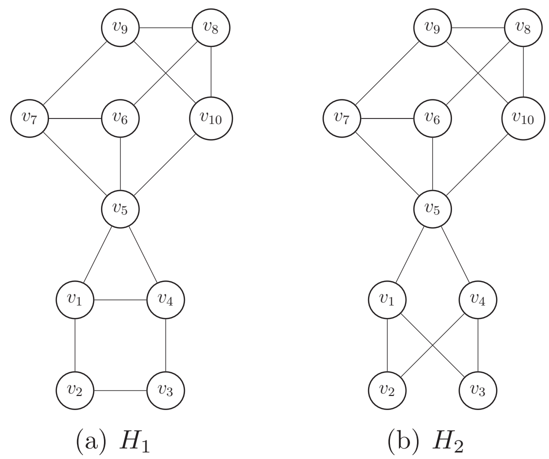


Figure 4.7: Specific examples of Theorem 4.1.6 on $K_{2,2}$ and $K_2 \square K_2$

in S_4 will not change $D(C(S_4), x)$.

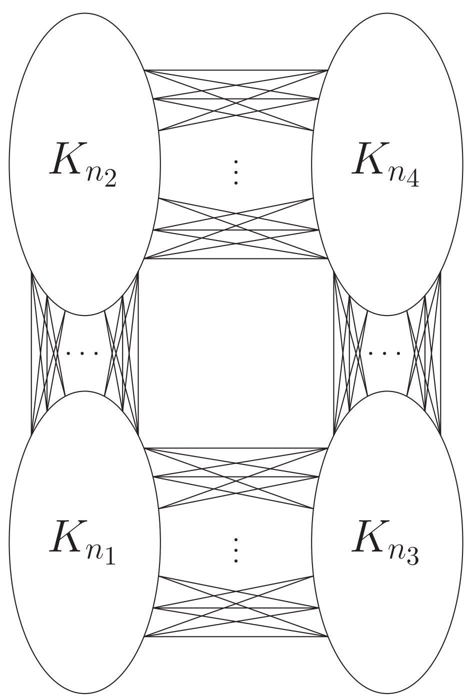


Figure 4.8: An example of $C(n_1, n_2, n_3, n_4)$

Theorem 4.1.7 *For any $n_1, n_2, n_3, n_4 \in \mathbb{N}$ let $S_4 = (n_1, n_2, n_3, n_4)$ and $\sigma(S_4)$ be any permutation of S_4. Then $C(S_4) \sim C(\sigma(S_4))$.*

Proof. As C_4 is isomorphic to $K_{2,2}$ then we write can write

$$C(S_4) = (K_{n_1} \cup K_{n_3}) \vee (K_{n_2} \cup K_{n_4})$$

.

By Theorem 2.2.3

$$\begin{aligned} D(C(S_4), x) &= ((x+1)^{n_1+n_3} - 1)((x+1)^{n_2+n_4} - 1) + D(K_{n_1} \cup K_{n_3}, x) \\ &\quad + D(K_{n_2} \cup K_{n_4}, x) \\ &= ((x+1)^{n_1+n_3} - 1)((x+1)^{n_2+n_4} - 1) \\ &\quad + ((x+1)^{n_1} - 1)((x+1)^{n_3} - 1) \\ &\quad + ((x+1)^{n_2} - 1)((x+1)^{n_4} - 1) \\ &= ((x+1)^{n_1+n_2+n_3+n_4} - 1 - \textstyle\sum_{i=1}^{4}((x+1)^{n_i} - 1) \end{aligned}$$

As addition is commutative, for any permutation $\sigma(S_4)$, $D(C(\sigma(S_4)), x)$ can be simplified to the same formula, so $C(S_4) \sim C(\sigma(S_4))$. $\square$

For many permutations $\sigma(S_4)$, $C(\sigma(S_4))$ is isomorphic to $C(S_4)$. This would leave us to believe Theorem 4.1.7 is not needed. However, for distinct values of n_1, n_2, n_3, and n_4, $C(n_1, n_2, n_3, n_4), C(n_1, n_2, n_4, n_3)$ and $C(n_1, n_3, n_4, n_2)$ are not in general isomorphic as each have a distinct number of edges. Furthermore, for every other $\sigma(S_4)$, $C(\sigma(S_4))$ is isomorphic to one of those three cases.

4.2 Equivalence Classes for Paths

For a path P_n let P'_n be a copy of P_n with an added edge between its two stems. In [2] Akbari, Alikhani and Peng showed $[P_{3n}] = \{P_{3n}, P'_{3n}\}$. In this section we will use the results of Section 3.2 to determine $[P_n]$ for $n \geq 9$.

We call on some previous results which will be used to show any graph $G \sim P_n$ does not have any 4-cycle components.

Lemma 4.2.1 [3] *If n is a positive integer, then*

$$D(C_n, -1) = \begin{cases} 3 & n \equiv 0 \ mod\ 4 \\ -1 & otherwise \end{cases}$$

$\square$

Lemma 4.2.2 [6] *Let F be a forest. Then $D(F,-1) \in \{1,-1\}$ and therefore $D(P_n,-1) \in \{1,-1\}$.* □

Corollary 4.2.3 *If a G is $\mathcal{D}$-equivalent to P_n with a component H, then $|D(H,-1)| = 1$. Moreover G does not have any 4-cycle components.* □

Proof. Suppose not, that is suppose G has component H with $|D(H,-1)| \neq 1$. As $D(H,x)$ has all integer coefficients then $D(H,-1)$ is an integer and $|D(H,-1)| > 1$. Let $|D(H,-1)| = k$. As G is $\mathcal{D}$-equivalent to P_n then by Lemma 4.2.2 $D(G,-1) \in \{-1,1\}$. Moreover $|D(G,-1)| = 1$. Let $G = G' \cup H$ then

$$|D(G,-1)| = |D(H,-1)D(G',-1)| = k|D(G',-1)|$$

.

Therefore $|D(G',-1)| = \frac{1}{k}$ and hence not an integer, which is a contradiction. Therefore $|D(H,-1)| = 1$. By Lemma 4.2.1 $D(C_4,-1) = 3$, therefore G has no 4-cycle components. □

In the next Lemma we use the results from Lemma 3.2.7, Theorem 3.2.3, and Theorem 3.2.6 to show, for large enough n, any graph $G \sim P_n$ must be the disjoint union of one path and an arbitrary number of cycles.

Lemma 4.2.4 *For $n \geq 9$, if $G \sim P_n$ then $G = H \cup C$ where $H \in \{P_k, P'_k\}$ and C is a disjoint union of cycles.*

Proof. We will first show $D(P_n,-2) \neq 0$ for $n \geq 9$. This fact will be sufficient to show any graph $G \sim P_n$ has no K_2 components. This is because $D(K_2,-2) = 0$, so if G had a K_2 component then $D(G,-2) = 0$ and hence $D(G,-2) \neq D(P_n,-2)$. So we begin by computing $D(P_n,-2)$.

$$\begin{array}{lll}
D(P_1,-2) = -2 & D(P_6,-2) = 4 & D(P_{11},-2) = -16 \\
D(P_2,-2) = 0 & D(P_7,-2) = 0 & D(P_{12},-2) = 16 \\
D(P_3,-2) = 2 & D(P_8,-2) = 0 & D(P_{13},-2) = -32 \\
D(P_4,-2) = 0 & D(P_9,-2) = -8 & D(P_{14},-2) = 64 \\
D(P_5,-2) = -4 & D(P_{10},-2) = 16 & D(P_{15},-2) = -96
\end{array}$$

Note $D(P_n, -2) \neq 0$ and is alternating for $9 \leq n \leq 15$. Furthermore, $0 < |D(P_{12}, -2)| < |D(P_{13}, -2)| < |D(P_{14}, -2)| < |D(P_{15}, -2)|$. As P_n is a $G(m)$ graph and has maximum non-stem degree two, $P_n \in G^2(m)$. Then by Lemma 3.2.7, $D(P_n, -2) \neq 0$ for $n \geq 9$. Thus G has no K_2 components for $n \geq 9$.

Let G be a graph with $D(G, x) = D(P_n, x)$ where $n \geq 9$. Then $d(G, i) = d(P_n, i)$ for all i. Furthermore by Theorem 2.3.3 we have

(i) $d(G, n-1) = n$.

(ii) $d(G, n-2) = \binom{n}{2} - 2$.

(iii) $d(G, n-3) = \binom{n}{3} - (3n - 8)$.

(iv) $d(G, n-4) = \binom{n}{4} - (2n^2 - 13n + 20)$.

By Theorem 2.2.5 the number of isolated vertices in G is $n - d(G, n-1) = 0$. By Lemma 3.2.7 $D(G, -2) \neq 0$ and again by Theorem 2.2.5 the number of leaves is $|T_1| = \binom{n}{2} - d(G, n-2) = 2$. By Theorem 3.2.3, as G has no K_2 components nor isolated vertices then

$$d(G, n-3) = \binom{n}{3} - \left(|T_1| \cdot (n-2) + |T_2| - \sum_{i=1}^{\omega} \binom{|S_i|}{2} - |\mathcal{L}_3| - 2|\mathcal{C}_3| \right).$$

Furthermore, from $|T_1| = 2$ and item (iii) we know

$$n - 4 = |T_2| - \sum_{i=1}^{\omega} \binom{|S_i|}{2} - |\mathcal{L}_3| - 2|\mathcal{C}_3|.$$

By rearranging for $|T_2|$ we get

$$|T_2| = n - 4 + \sum_{i=1}^{\omega} \binom{|S_i|}{2} + |\mathcal{L}_3| + 2|\mathcal{C}_3|.$$

We claim for G, $|\mathcal{L}_3| = 0$, $|\mathcal{C}_3| = 0$ and $G \in G^2(m)$. Recall that $G^2(m)$ is the set of all $G(m)$ and $G'(m)$ graphs restricted to those with maximum non-stem degree two. Recall as well that ω is the number of stems in G. We will show our claim is true using the fact that $n = \omega + \sum_{i \in \mathbb{N}} |T_i|$ so $n \geq \omega + |T_1| + |T_2|$. As $|T_1| = 2$ then $T_2 \leq n - (2 + \omega)$. Also, if $n = \omega + |T_1| + |T_2|$ then $G \in G^2(m)$. This is because

$|T_1| > 0$, hence G has a leaf and thus takes the form of a $G(m)$ graph. Furthermore if $n = \omega + |T_1| + |T_2|$ then $|T_i| = 0$ for $i \geq 3$ and thus the maximum non-stem degree is two. As G has leaves then it has stems, more specifically as G has two leaves then it either has one or two stems. We now consider the two cases for G.

Case 1: *G has one stem.*

Then $\omega = 1$, $|S_1| = 2$, and $|T_2| \leq n - 3$. Thus

$$|T_2| = n - 4 + \binom{2}{2} + |\mathcal{L}_3| + 2|\mathcal{C}_3|.$$

As $|\mathcal{L}_3| + 2|\mathcal{C}_3| \geq 0$ then $|T_2| \geq n - 3$ and therefore $|T_2| = n - 3$. Furthermore $|\mathcal{L}_3| + 2|\mathcal{C}_3| = 0$ so $|\mathcal{L}_3| = 0$ and $|\mathcal{C}_3| = 0$. As $\omega + |T_1| + |T_2| = n$, $G \in G^2(m)$.

Case 2: *G has two stems.*

Then $\omega = 2$,$|S_1| = 1$,$|S_2| = 1$, and $|T_2| \leq n - 4$. Thus

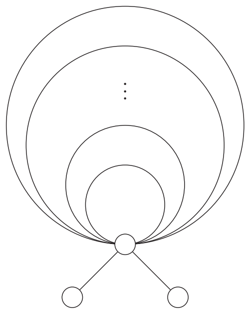
$$|T_2| = n - 4 + 0 + |\mathcal{L}_3| + 2|\mathcal{C}_3|.$$

As $|\mathcal{L}_3| + 2|\mathcal{C}_3| \geq 0$ then $|T_2| \geq n - 4$ and therefore $|T_2| = n - 4$. Furthermore $|\mathcal{L}_3| + 2|\mathcal{C}_3| = 0$ so $|\mathcal{L}_3| = 0$ and $|\mathcal{C}_3| = 0$. As $\omega + |T_1| + |T_2| = n$, $G \in G^2(m)$.

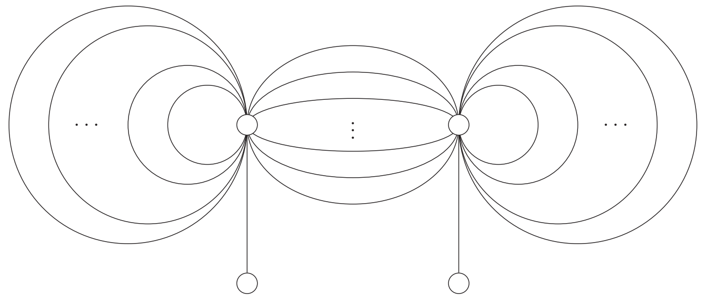
For a graph in $G^2(m)$, a T_2 vertex can only be adjacent to stems or other T_2 vertices. Therefore the T_2 vertices form paths between stems, r-loops, and disjoint cycles in $G^2(m)$ graphs. As $G \in G^2(m)$, G will be the disjoint union of some number of cycles and a subgraph H which has one of the two forms shown in Figure 4.9.

Recall from Section 3.2, we partitioned T_2 into subsets based on the number of neighbouring stems.

- V_0: The subset of T_2 with no adjacent stems.
- V_1^i: The subset of T_2 adjacent to exactly one stem, stem i.
- V_2^{ij}: The subset of T_2 adjacent to exactly two stems, stems i and j (*denoted* V_2 *when G only has two stems*).



(a) One stem



(b) Two stems

Figure 4.9: The two possible structures of H

We wish to show that the subgraph H of G is either a path or a path with an edge between its stems. This is equivalent of showing H has two stems with one path between them and no r-loops. Note as we do not specify the degree of the stems, this allows for the possibility of an edge between them. If G has exactly two stems, and no r-loops, then the number of paths between the stems is exactly $\frac{1}{2}(|V_1^1|+|V_1^2|)+|V_2|$. Furthermore, if $|V_1^1| \leq 1$ and $|V_1^2| \leq 1$ then H has no r-loops. Therefore it is sufficient to show H has two stems and either $|V_1^1| = |V_1^2| = 0$ and $|V_2| = 1$, or $|V_1^1| = |V_1^2| = 1$ and $|V_2| = 0$. We will show this by examining $d(G, n-4)$.

By Theorem 3.2.6, as G has no K_2 components, no isolated vertices, and $G \in G^2(m)$, we have that

$$d(G, n-4) = \binom{n}{4} - \left(|T_1|\binom{n-2}{2} + |T_2|(n-3) - \alpha_1 - \alpha_2 - \alpha_3\right)$$

where

$$\begin{aligned}
\alpha_1 &= \sum_{i=1}^{\omega}\tbinom{|S_i|}{2}(n-|S_i|-1) + \sum_{i=1}^{\omega}\tfrac{|S_i|}{2}(|T_1|-|S_i|) + 2\sum_{i=1}^{\omega}\tbinom{|S_i|}{3} \\
\alpha_2 &= \sum_{i=1}^{\omega}|V_1^i||S_i| + \sum_{i\neq j}|V_2^{ij}|(|S_i|+|S_j|) \\
\alpha_3 &= |V_0| + \sum_{i=1}^{\omega}\tfrac{|V_1^i|}{2} + \sum_{i\neq j}\tbinom{|V_2^{ij}|}{2} - |\mathcal{C}_4| + |\mathcal{C}_3|(2n-9) + \sum_{i=1}^{\omega}|\mathcal{L}_3^i|(n-4-|S_i|)
\end{aligned}$$

As $|\mathcal{L}_3| = 0$ then $|\mathcal{L}_3^i| = 0$ for every i. Furthermore $|\mathcal{C}_3| = 0$ and by corollary 4.2.3 $|\mathcal{C}_4| = 0$. We again consider the two cases where G has one stem and G has two stems. Note that $|T_2| = |V_0| + \sum_{i=1}^{\omega}|V_1^i| + \sum_{i\neq j}|V_2^{ij}|$.

Case 1: *G has one stem.*

We claim this case results in a contradiction. As G has one stem then $\omega = 1$, $|S_1| = 2$, $|T_2| = n-3$. As G only has one stem, there are no degree two vertices adjacent to two stems and $|V_2^{ij}| = 0$ for all i and j. Furthermore $|V_0| + |V_1^1| = |T_2| = n-3$. Using this we can simplify α_1, α_2, and α_3 to be

$$\begin{aligned}
\alpha_1 &= \tbinom{2}{2}(n-2-1) + \tfrac{2}{2}(2-2) + 2\cdot 0 = n-3. \\
\alpha_2 &= 2|V_1^1|. \\
\alpha_3 &= |V_0| + \tfrac{|V_1^1|}{2}.
\end{aligned}$$

As $|V_0| + |V_1^1| = n-3$ then $\alpha_1 + \alpha_2 + \alpha_3 = 2n - 6 + \frac{3|V_1^1|}{2}$ and

$$d(G, n-4) = \binom{n}{4} - \left(2n^2 - 13n + 21 - \frac{3|V_1^1|}{2}\right)$$

However by item (iv), $d(G, n-4) = \binom{n}{4} - (2n^2 - 13n + 20)$ and therefore $|V_1^1| = \frac{2}{3}$. But $|V_1^1|$ is a positive integer, which gives us a contradiction.

Case 2: *G has two stems.*

As G has two stems, we find that $\omega = 2$,$|S_1| = 1$, $|S_2| = 1$, and $|T_2| = n-4$. As there are only two stems, let the set of T_2 vertices which are adjacent to both be denoted V_2. Furthermore $|V_0| + |V_1^1| + |V_1^2| + |V_2| = |T_2| = n-4$. Using this we can simplify α_1, α_2, and α_3 to be

$$\begin{aligned}
\alpha_1 &= 0 + \sum_{i=1}^{2} \tfrac{1}{2}(2-1) = 1 \\
\alpha_2 &= \sum_{i=1}^{2} |V_1^i| + 2|V_2| \\
\alpha_3 &= |V_0| + \sum_{i=1}^{2} \tfrac{|V_1^i|}{2} + \tbinom{|V_2|}{2}
\end{aligned}$$

As $|V_0| + |V_1^1| + |V_1^2| + |V_2| = n-4$ then $\alpha_1 + \alpha_2 + \alpha_3 = n - 3 + \sum_{i=1}^{2} \frac{|V_1^i|}{2} + |V_2| + \binom{|V_2|}{2}$ and

$$d(G, n-4) = \binom{n}{4} - \left(2n^2 - 13n + 21 - \sum_{i=1}^{2} \frac{|V_1^i|}{2} - |V_2| - \binom{|V_2|}{2}\right)$$

However by item (iv), $d(G, n-4) = \binom{n}{4} - (2n^2 - 13n + 20)$ and therefore

$$\sum_{i=1}^{2} \frac{|V_1^i|}{2} + |V_2| + \binom{|V_2|}{2} = 1.$$

As each summand is non-negative and $|V_2| + \binom{|V_2|}{2}$ is a non-negative integer, the only solutions to this are $\sum_{i=1}^{2} \frac{|V_1^i|}{2} = 1, |V_2| = 0$ or $\sum_{i=1}^{2} \frac{|V_1^i|}{2} = 0, |V_2| = 1$.

In the case $\sum_{i=1}^{2} \frac{|V_1^i|}{2} = 1, |V_2| = 0$, then as G has no K_2 components and there are no vertices adjacent to both stems ($|V_2| = 0$) then $|V_1^i| \geq 1$ for each i. Furthermore as $\sum_{i=1}^{2} \frac{|V_1^i|}{2} = 1$, then $|V_1^1| = |V_1^2| = 1$, and $|V_2| = 0$. In the case $\sum_{i=1}^{2} \frac{|V_1^i|}{2} = 0, |V_2| = 1$ as $|V_1^i| \geq 0$ for each i then $|V_1^1| = |V_1^2| = 0$, and $|V_2| = 1$. Both cases result in H having one path between its two stems and no r-loops. As we do not specify the degree of the stems, this allows for the possibility of an edge to be between them, and proves our result.

□

From Lemma 4.2.4 all that is needed to determine the equivalence class of paths is to show is a path is not $\mathcal{D}$-equivalent to the disjoint union of a smaller path and cycles. Until this point we have used the highest coefficients of $D(P_n, x)$ to reduce the number of graphs which could be $\mathcal{D}$-equivalent to a path. However, let $G_{p,c_1,c_2,\ldots c_k} = P_p \cup C_{c_1} \cup C_{c_2} \cup \ldots \cup C_{c_k}$ and $n = p + \sum_{i=1}^{k} c_i$. With the help of Maple, it seems the highest m coefficients of $D(G_{p,c_1,c_2,\ldots c_k}, x)$ are the same as the highest m coefficients of $D(P_n, x)$, where $m = \min\{p, c_1, c_2, \ldots c_k\}$. We believe for any m, we can find large a large enough n such that the highest m coefficients of $D(G_{p,c_1,c_2,\ldots c_k}, x)$ are the same as $D(P_n, x)$. Clearly looking at the highest coefficients will not yield much more.

Showing a path is not $\mathcal{D}$-equivalent to the disjoint union of a smaller path and cycles has proven to be a difficult problem. We will show this by creating a system of equations which has no solutions. As we have two polynomials $D(G_{p,c_1,c_2,\ldots c_k}, x) = D(P_n, x)$, they must agree at every value of x. Furthermore, their first, second, and third derivatives must also agree at every value of x. Domination polynomials of paths and cycles have shown to have nice patterns when evaluated at -1. We will first limit the number of cycles in $G_{p,c_1,c_2,\ldots c_k}$ and then show a system of equations created by the domination polynomial and its first three derivatives evaluated at -1

has no solutions. This method is very cumbersome, and alternative approaches could be pursued.

Let $n \in \mathbb{Z}$ and p be a prime number. If n is not zero, there is a nonnegative integer a such that $p^a \mid n$ but $p^{a+1} \nmid n$; we let $ord_p(n) = a$. In other words, a is the exponent of p in the prime decomposition of n. Furthermore let $ord_p(0) = 0$. In a similar method used by Akbari and Oboudi [3] we will determine $ord_3(D(P_n, -3))$ and show if a graph G is $\mathcal{D}$-equivalent to a path, then G is the disjoint union of a path and at most two cycles.

Lemma 4.2.5 *[3] For $n \in \mathbb{N}$*

$$ord_3(D(C_n, -3)) = \begin{cases} \lceil \frac{n}{3} \rceil + 1 & n \equiv 0 \ mod\ 3 \\ \lceil \frac{n}{3} \rceil \ or\ \lceil \frac{n}{3} \rceil + 1 & n \equiv 1 \ mod\ 3 \\ \lceil \frac{n}{3} \rceil & n \equiv 2 \ mod\ 3 \end{cases}$$

□

We will now extend there argument to paths.

Lemma 4.2.6 *For $n \in \mathbb{N}$*

$$ord_3(D(P_n, -3)) = \begin{cases} \lceil \frac{n}{3} \rceil & n \equiv 0 \ mod\ 3 \\ \lceil \frac{n}{3} \rceil & n \equiv 1 \ mod\ 3 \\ \lceil \frac{n}{3} \rceil \ or\ \lceil \frac{n}{3} \rceil + 1 & n \equiv 2 \ mod\ 3 \end{cases}$$

Proof. Let $a_n = D(P_n, -3)$. We set $a_n = (-1)^n 3^{\lceil \frac{n}{3} \rceil} b_n$. We claim that b_n is an integer satisfying the recurrence relation

$$b_n = \begin{cases} 3b_{n-1} - 3b_{n-2} + b_{n-3} & \text{when } n \equiv 0 \text{ mod } 3 \\ b_{n-1} - b_{n-2} + b_{n-3} & \text{when } n \equiv 1 \text{ mod } 3 \\ 3b_{n-1} - b_{n-2} + b_{n-3} & \text{when } n \equiv 2 \text{ mod } 3. \end{cases} \quad (1)$$

We will first prove our claim that b_n satisfies recurrence relation (1). We first check the base cases by computing a_n for $1 \leq n \leq 6$. Since $D(P_1, x) = x$, $D(P_2, x) =$

$x^2 + 2x$, and $D(P_3, x) = x^3 + 3x^2 + x$ then $a_1 = -3$, $a_2 = 3$, and $a_3 = -3$. Then as $a_{k+1} = D(P_{k+1}, -3)$, by Theorem 2.3.2 we get the following recursion:

$$a_{k+1} = -3(a_k + a_{k-1} + a_{k-2}). \tag{2}$$

We now compute a_4, a_5, and a_6 using recurrence relation (2):

$$\begin{aligned} a_4 &= -3((-3) + (3) + (-3)) = 9 \\ a_5 &= -3((9) + (-3) + (3)) = -27 \\ a_6 &= -3((-27) + (9) + (-3)) = 63 \end{aligned}$$

Therefore $b_1 = 1$, $b_2 = 1$, $b_3 = 1$, $b_4 = 1$, $b_5 = 3$, and $b_6 = 7$. By recurrence relation (1):

$$\begin{aligned} b_4 &= (1) - (1) + (1) = 1 \\ b_5 &= 3(1) - (1) + (1) = 3 \\ b_6 &= 3(3) - 3(1) + (1) = 7 \end{aligned}$$

Therefore recurrence relation (1) holds for our base cases. Now suppose our claim is true for all $n \leq k$. We now consider the three cases $k+1 \equiv 0, 1, 2 \bmod 3$.

Case 1: $k + 1 \equiv 0 \ mod\ 3$.

Then $\frac{k+1}{3}$ and $\frac{k-2}{3}$ are integers. Thus $\lceil \frac{k}{3} \rceil = \frac{k+1}{3}$, $\lceil \frac{k-1}{3} \rceil = \frac{k+1}{3}$, and $\lceil \frac{k-2}{3} \rceil = \frac{k-2}{3}$. Furthermore $\lceil \frac{k+1}{3} \rceil = \frac{k+1}{3}$. By our recursion we obtain

$$\begin{aligned} a_{k+1} &= -3(a_k + a_{k-1} + a_{k-2}) \\ (-1)^{k+1}3^{\lceil \frac{k+1}{3} \rceil} b_{k+1} &= -3((-1)^k 3^{\lceil \frac{k}{3} \rceil} b_k + (-1)^{k-1} 3^{\lceil \frac{k-1}{3} \rceil} b_{k-1} + (-1)^{k-2} 3^{\lceil \frac{k-2}{3} \rceil} b_{k-2}) \\ (-1)^{k+1}3^{\lceil \frac{k+1}{3} \rceil} b_{k+1} &= 3(-1)^{k+1}(3^{\frac{k+1}{3}} b_k - 3^{\frac{k+1}{3}} b_{k-1} + 3^{\frac{k-2}{3}} b_{k-2}) \\ (-1)^{k+1}3^{\lceil \frac{k+1}{3} \rceil} b_{k+1} &= (-1)^{k+1} 3^{\frac{k+1}{3}} (3b_k - 3b_{k-1} + b_{k-2}) \\ (-1)^{k+1}3^{\lceil \frac{k+1}{3} \rceil} b_{k+1} &= (-1)^{k+1} 3^{\lceil \frac{k+1}{3} \rceil} (3b_k - 3b_{k-1} + b_{k-2}) \\ b_{k+1} &= 3b_k - 3b_{k-1} + b_{k-2} \end{aligned}$$

Case 2: $k + 1 \equiv 1 \ mod\ 3$.

Then $\frac{k}{3}$ is an integer and $\lceil \frac{k}{3} \rceil = \frac{k}{3}$, $\lceil \frac{k-1}{3} \rceil = \frac{k}{3}$, and $\lceil \frac{k-2}{3} \rceil = \frac{k}{3}$. Furthermore $\lceil \frac{k+1}{3} \rceil = \frac{k+3}{3}$.

$$
\begin{aligned}
a_{k+1} &= -3(a_k + a_{k-1} + a_{k-2}) \\
(-1)^{k+1}3^{\lceil\frac{k+1}{3}\rceil}b_{k+1} &= -3((-1)^k 3^{\lceil\frac{k}{3}\rceil}b_k + (-1)^{k-1}3^{\lceil\frac{k-1}{3}\rceil}b_{k-1} + (-1)^{k-2}3^{\lceil\frac{k-2}{3}\rceil}b_{k-2}) \\
(-1)^{k+1}3^{\lceil\frac{k+1}{3}\rceil}b_{k+1} &= 3(-1)^{k+1}(3^{\frac{k}{3}}b_k - 3^{\frac{k}{3}}b_{k-1} + 3^{\frac{k}{3}}b_{k-2}) \\
(-1)^{k+1}3^{\lceil\frac{k+1}{3}\rceil}b_{k+1} &= (-1)^{k+1}3^{\frac{k+3}{3}}(b_k - b_{k-1} + b_{k-2}) \\
(-1)^{k+1}3^{\lceil\frac{k+1}{3}\rceil}b_{k+1} &= (-1)^{k+1}3^{\lceil\frac{k+1}{3}\rceil}(b_k - b_{k-1} + b_{k-2}) \\
b_{k+1} &= b_k - b_{k-1} + b_{k-2}
\end{aligned}
$$

Case 3: $k + 1 \equiv 2 \ mod\ 3$.

Then $\frac{k+2}{3}$ and $\frac{k-1}{3}$ are integers. Thus $\lceil\frac{k}{3}\rceil = \frac{k+2}{3}$, $\lceil\frac{k-1}{3}\rceil = \frac{k-1}{3}$, and $\lceil\frac{k-2}{3}\rceil = \frac{k-1}{3}$. Furthermore, $\lceil\frac{k+1}{3}\rceil = \frac{k+2}{3}$.

$$
\begin{aligned}
a_{k+1} &= -3(a_k + a_{k-1} + a_{k-2}) \\
(-1)^{k+1}3^{\lceil\frac{k+1}{3}\rceil}b_{k+1} &= -3((-1)^k 3^{\lceil\frac{k}{3}\rceil}b_k + (-1)^{k-1}3^{\lceil\frac{k-1}{3}\rceil}b_{k-1} + (-1)^{k-2}3^{\lceil\frac{k-2}{3}\rceil}b_{k-2}) \\
(-1)^{k+1}3^{\lceil\frac{k+1}{3}\rceil}b_{k+1} &= 3(-1)^{k+1}(3^{\frac{k+2}{3}}b_k - 3^{\frac{k-1}{3}}b_{k-1} + 3^{\frac{k-1}{3}}b_{k-2}) \\
(-1)^{k+1}3^{\lceil\frac{k+1}{3}\rceil}b_{k+1} &= (-1)^{k+1}3^{\frac{k+2}{3}}(3b_k - b_{k-1} + b_{k-2}) \\
(-1)^{k+1}3^{\lceil\frac{k+1}{3}\rceil}b_{k+1} &= (-1)^{k+1}3^{\lceil\frac{k+1}{3}\rceil}(3b_k - b_{k-1} + b_{k-2}) \\
b_{k+1} &= 3b_k - b_{k-1} + b_{k-2}
\end{aligned}
$$

As recurrence relation (1) holds for $k + 1$ then by induction it holds for $n \in \mathbb{N}$. A trivial induction shows as b_1, b_2, and b_3 are integers that b_n is always an integer. We can simplify the computation of $ord_3(a_n)$ to

$$ord_3(a_n) = \lceil\frac{n}{3}\rceil + ord_3(b_n) \tag{3}$$

Therefore we can determine $ord_3(a_n)$ by computing $ord_3(b_n)$. We will now compute the first 32 values of b_n mod 9 to show $b_n \equiv b_{n+27}$ mod 9 for $n \geq 3$.

k	b_{3k} mod 9	b_{3k+1} mod 9	b_{3k+2} mod 9
0	-	1	1
1	1	1	3
2	7	5	2
3	7	1	7
4	7	1	3
5	4	2	5
6	4	1	4
7	4	1	3
8	1	8	8
9	1	1	1
10	1	1	3

As the three entries in row $k = 1$ correspond to the three entries in row $k = 10$ then a trivial induction shows that $b_n \equiv b_{n+27} \bmod 9$ for all $n \geq 3$. Note that $b_n \not\equiv 0 \bmod 9$ and therefore $ord_3(b_n) < 2$. We will now compute the first 14 values of $b_n \bmod 3$ to show $b_n \equiv b_{n+9} \bmod 3$.

k	b_{3k} mod 3	b_{3k+1} mod 3	b_{3k+2} mod 3
0	-	1	1
1	1	1	0
2	1	2	2
3	1	1	1
4	1	1	0

As the three entries in row $k = 1$ correspond to the three entries in row $k = 4$ then a trivial induction shows that $b_n \equiv b_{n+9} \bmod 3$ for all n. Note that the only zeroes are in the b_{3k+2} column. Therefore

$$ord_3(b_n) = \begin{cases} 0 & n \equiv 0 \bmod 3 \\ 0 & n \equiv 1 \bmod 3 \\ 0 \text{ or } 1 & n \equiv 2 \bmod 3 \end{cases}$$

Together with equation (3) this concludes our proof.

□

The next two lemmas will allow us to restrict the number of disjoint cycles in G if $G \sim P_n$.

Lemma 4.2.7 *For every $n \in \mathbb{N}$, $\gamma(P_n) = \lceil \frac{n}{3} \rceil$.* □

Lemma 4.2.8 *For every $n \in \mathbb{N}$, $\gamma(C_n) = \lceil \frac{n}{3} \rceil$.* □

From Lemma 4.2.4 we know if G is $\mathcal{D}$-equivalent to P_n then G is the disjoint union of H and some number of cycles where $H \in \{P_k, P'_k\}$ and $k \leq n$. In the next lemma we will show the number of cycles is at most two.

Lemma 4.2.9 *For $n \in \mathbb{N}$*

For $n \geq 9$, If $G \sim P_n$ then $G = H \cup C$ where $H \in \{P_k, P'_k\}$, $k \leq n$, and C is a disjoint union of at most two cycles.

Proof. Let $n = 3m + r$ and G be a graph with $D(G, x) = D(P_{3m+r}, x)$ where $r \in \{0, 1, 2\}$. By Lemma 4.2.4,

$$G = P_{3m_1+r_1} \cup C_{3m_2+r_2} \cup \ldots \cup C_{3m_k+r_k},$$

where $3m + r = \sum_{i=1}^{k}(3m_i + r_i)$ and for each i, $r_i \in \{0, 1, 2\}$. In this proof we will begin by restricting the number of non-zero r_i, and then restrict the number of r_i which are zero. By Lemma 4.2.7 and Lemma 4.2.8 we know

$$\gamma(G) = \sum_{i=1}^{k} \lceil \frac{3m_i + r_i}{3} \rceil = \sum_{i=1}^{k} m_i + \sum_{i=1}^{k} \lceil \frac{r_i}{3} \rceil.$$

As $3m + r = \sum\limits_{i=1}^{k}(3m_i + r_i)$ then $\sum\limits_{i=1}^{k} m_i = m + \frac{r}{3} - \sum\limits_{i=1}^{k} \frac{r_i}{3}$ and

$$\gamma(G) = m + \frac{r}{3} + \sum_{i=1}^{k} \left(\lceil \frac{r_i}{3} \rceil - \frac{r_i}{3} \right).$$

As $\gamma(G) = \gamma(P_{3m+r})$ and $\gamma(P_{3m+r}) = \lceil \frac{3m+r}{3} \rceil = m + \lceil \frac{r}{3} \rceil$ then

$$\sum_{i=1}^{k}\left(\lceil\frac{r_i}{3}\rceil-\frac{r_i}{3}\right)=\lceil\frac{r}{3}\rceil-\frac{r}{3}.$$

Let $f(r_i)=\left\lceil\frac{r_i}{3}\right\rceil-\frac{r_i}{3}$. As $r_i\in\{0,1,2\}$, then $f(0)=0, f(1)=\frac{2}{3}$, and $f(2)=\frac{1}{3}$. Now consider the number of $r_i\neq 0$ for the cases $r=0,1,$ and 2

- If $r=0$ then $\sum f(r_i)=0$ and no $r_i\neq 0$.
- If $r=1$ then $\sum f(r_i)=\frac{2}{3}$ and at most two $r_i\neq 0$.
- If $r=2$ then $\sum f(r_i)=\frac{1}{3}$ and at most one $r_i\neq 0$.

We now count the $r_i=0$. For a graph H, let $g(H)=ord_3(D(H,-3))-\gamma(H)$. Using Lemma 4.2.5, Lemma 4.2.6 and the fact that $\gamma(C_{3m+r})=\gamma(P_{3m+r})=\lceil\frac{3m+r}{3}\rceil$ we can obtain $g(P_{3m+r})$ and $g(C_{3m+r})$:

$$g(P_{3m+r})=\begin{cases}0 & r=0\\ 0 & r=1\\ 0\text{ or }1 & r=2\end{cases}\quad,\quad g(C_{3m+r})=\begin{cases}1 & r=0\\ 0\text{ or }1 & r=1\\ 0 & r=2\end{cases}$$

For simplicity we will denote $g(P_{3m+r})$ and $g(C_{3m+r})$ with $g_P(r)$ and $g_C(r)$. Because G is the disjoint union of a path and cycles then $\gamma(G)$ is just the sum of domination numbers of each of the paths and cycles. Similarly $ord_3(D(G,-3))$ is just the sum of the orders of each of its components. From this we get the following equality:

$$g_P(r)=g_P(r_1)+\sum_{i=2}^{k}g_C(r_i)$$

Now consider the number of $r_i=0$ for the cases $r=0,1,$ and 2

- If $r=0$ then $g_P(r_1)+\sum_{i=2}^{k}g_C(r_i)=0$ and no $r_i=0$ for $i\geq 2$.
- If $r=1$ then $g_P(r_1)+\sum_{i=2}^{k}g_C(r_i)=0$ and no $r_i=0$ for $i\geq 2$.
- If $r=2$ then $g_P(r_1)+\sum_{i=2}^{k}g_C(r_i)=0$ or 1 and at most one $r_i=0$ for $i\geq 2$

Together with the three cases counting the number $r_i \neq 0$, we can easily see there are at most two r_i for $i \geq 2$. Therefore there are at most two cycle components.

$\square$

We have narrowed the number of cycle components to two in graphs which are $\mathcal{D}$-equivalent to paths. We will now evaluate the domination polynomial and each of the first three derivatives of paths and cycles at -1.

Lemma 4.2.10 *[3] For $n \in \mathbb{N}$*

$$D(C_n, -1) = \begin{cases} 3, & n \equiv 0 \ mod\ 4 \\ -1, & n \equiv 1 \ mod\ 4 \\ -1, & n \equiv 2 \ mod\ 4 \\ -1, & n \equiv 3 \ mod\ 4 \end{cases}$$

$\square$

Lemma 4.2.11 *[3] For $n \in \mathbb{N}$*

$$D'(C_n, -1) = \begin{cases} -n, & n \equiv 0 \ mod\ 4 \\ n, & n \equiv 1 \ mod\ 4 \\ 0, & n \equiv 2 \ mod\ 4 \\ 0, & n \equiv 3 \ mod\ 4 \end{cases}$$

$\square$

Lemma 4.2.12 *[3] For $n \in \mathbb{N}$*

$$D''(C_n, -1) = \begin{cases} \frac{1}{4} n\left(n-4\right), & n \equiv 0 \ mod\ 4 \\ -\frac{1}{2} n\left(n-1\right), & n \equiv 1 \ mod\ 4 \\ \frac{1}{4} n\left(n+2\right), & n \equiv 2 \ mod\ 4 \\ 0, & n \equiv 3 \ mod\ 4 \end{cases}$$

$\square$

Lemma 4.2.13 *For $n \in \mathbb{N}$*

$$D'''(C_n, -1) = \begin{cases} -\frac{1}{16}n^3 + \frac{3}{4}n^2 - 2n, & n \equiv 0 \ mod\ 4 \\ \frac{3}{16}n^3 - \frac{9}{8}n^2 + \frac{15}{16}n, & n \equiv 1 \ mod\ 4 \\ -\frac{3}{16}n^3 + \frac{3}{4}n, & n \equiv 2 \ mod\ 4 \\ \frac{1}{16}n^3 + \frac{3}{8}n^2 + \frac{5}{16}n, & n \equiv 3 \ mod\ 4 \end{cases} \tag{4}$$

Proof. We will prove equation (4) by induction. We first compute $D'''(C_n,-1)$ for $n=3,4,5,6$. In these cases equation (4) reduces to

$$\begin{array}{rclcr} D'''(C_3,-1) & = & \frac{1}{16}3^3+\frac{3}{8}3^2+\frac{5}{16}3 & = & 6 \\ D'''(C_4,-1) & = & -\frac{1}{16}4^3+\frac{3}{4}4^2-2\cdot 4 & = & 0 \\ D'''(C_5,-1) & = & \frac{3}{16}5^3-\frac{9}{8}5^2+\frac{15}{16}5 & = & 0 \\ D'''(C_6,-1) & = & -\frac{3}{16}6^3+\frac{3}{4}6 & = & -36. \end{array}$$

From Table 2.2 we have

$$\begin{array}{rcl} D(C_3,x) & = & x^3+3x^2+3x \\ D(C_4,x) & = & x^4+4x^3+6x^2 \\ D(C_5,x) & = & x^5+5x^4+10x^3+5x^2 \\ D(C_6,x) & = & x^6+6x^5+15x^4+14x^3+3x^2. \end{array}$$

Therefore

$$\begin{array}{rcl} D'''(C_3,x) & = & 6 \\ D'''(C_4,x) & = & 24x+24 \\ D'''(C_5,x) & = & 60x^2+120x+60 \\ D'''(C_6,x) & = & 120x^3+360x^2+360x+84. \end{array}$$

Then by evaluating each equation at $x=-1$ we obtain

$$D'''(C_3,-1)=6 \quad D'''(C_4,-1)=0 \quad D'''(C_5,-1)=0 \quad D'''(C_6,-1)=-36.$$

Therefore our claim is true for $n=3,4,5,6$.

Now suppose that our claim is true for all n up until $4k-1$. It is sufficient to show our claim is true for $4k, 4k+1, 4k+2$ and $4k+3$. Recall the following recurrence relation from Theorem 2.3.4:

$$D(C_n,x)=x\sum_{i=1}^{3}D(C_{n-i},x).$$

Therefore by taking the derivative three times and evaluating it at $x=-1$, we obtain

$$D'''(C_n,-1)=3\sum_{i=1}^{3}D''(C_{n-i},-1)-\sum_{i=1}^{3}D'''(C_{n-i},-1).$$

Recall from Theorem 4.2.12

$$D''(C_n,-1) = \begin{cases} \frac{1}{4}\,n\,(n-4)\,, & n \equiv 0 \bmod 4 \\ -\frac{1}{2}\,n\,(n-1)\,, & n \equiv 1 \bmod 4 \\ \frac{1}{4}\,n\,(n+2)\,, & n \equiv 2 \bmod 4 \\ 0, & n \equiv 3 \bmod 4 \end{cases}$$

Now consider the four cases $4k, 4k+1, 4k+2$ and $4k+3$.

$$\begin{aligned} D'''(C_{4k},-1) &= 3\sum_{i=1}^{3} D''(C_{4k-i},-1) - \sum_{i=1}^{3} D'''(C_{4k-i},-1) \\ &= 3(0+\tfrac{1}{4}\,(4k-2)\,((4k-2)+2) - \tfrac{1}{2}\,(4k-3)\,((4k-3)-1))- \\ &\quad (\tfrac{1}{16}(4k-1)^3 + \tfrac{3}{8}(4k-1)^2 + \tfrac{5}{16}(4k-1)- \\ &\quad \tfrac{3}{16}(4k-2)^3 + \tfrac{3}{4}(4k-2)+ \\ &\quad \tfrac{3}{16}(4k-3)^3 - \tfrac{9}{8}(4k-3)^2 + \tfrac{15}{16}(4k-3)) \\ &= -\tfrac{1}{16}(4k)^3 + \tfrac{3}{4}(4k)^2 - 2(4k) \end{aligned}$$

$$\begin{aligned} D'''(C_{4k+1},-1) &= 3\sum_{i=1}^{3} D''(C_{4k+1-i},-1) - \sum_{i=1}^{3} D'''(C_{4k+1-i},-1) \\ &= 3(\tfrac{1}{4}\,(4k)\,(4k-4) + 0 + \tfrac{1}{4}\,(4k-2)\,((4k-2)+2))- \\ &\quad (-\tfrac{1}{16}(4k)^3 + \tfrac{3}{4}(4k)^2 - 2(4k)+ \\ &\quad \tfrac{1}{16}(4k-1)^3 + \tfrac{3}{8}(4k-1)^2 + \tfrac{5}{16}(4k-1)- \\ &\quad \tfrac{3}{16}(4k-2)^3 + \tfrac{3}{4}(4k-2)) \\ &= \tfrac{3}{16}(4k+1)^3 - \tfrac{9}{8}(4k+1)^2 + \tfrac{15}{16}(4k+1) \end{aligned}$$

$$\begin{aligned} D'''(C_{4k+2},-1) &= 3\sum_{i=1}^{3} D''(C_{4k+2-i},-1) - \sum_{i=1}^{3} D'''(C_{4k+2-i},-1) \\ &= 3(-\tfrac{1}{2}\,(4k+1)\,((4k+1)-1) + \tfrac{1}{4}\,(4k)\,(4k-4) + 0)- \\ &\quad (\tfrac{3}{16}(4k+1)^3 - \tfrac{9}{8}(4k+1)^2 + \tfrac{15}{16}(4k+1)- \\ &\quad \tfrac{1}{16}(4k)^3 + \tfrac{3}{4}(4k)^2 - 2(4k)+ \\ &\quad \tfrac{1}{16}(4k-1)^3 + \tfrac{3}{8}(4k-1)^2 + \tfrac{5}{16}(4k-1)) \\ &= -\tfrac{3}{16}(4k+2)^3 + \tfrac{3}{4}(4k+2) \end{aligned}$$

$$\begin{aligned}
D'''(C_{4k+3},-1) &= 3\textstyle\sum_{i=1}^{3} D''(C_{4k+3-i},-1) - \sum_{i=1}^{3} D'''(C_{4k+3-i},-1) \\
&= 3(\tfrac{1}{4}(4k+2)((4k+2)+2) - \\
&\quad \tfrac{1}{2}(4k+1)((4k+1)-1) + \tfrac{1}{4}(4k)(4k-4)) - \\
&\quad (-\tfrac{3}{16}(4k+2)^3 + \tfrac{3}{4}(4k+2) - \\
&\quad (\tfrac{3}{16}(4k+1)^3 - \tfrac{9}{8}(4k+1)^2 + \tfrac{15}{16}(4k+1) - \\
&\quad \tfrac{1}{16}(4k)^3 + \tfrac{3}{4}(4k)^2 - 2(4k)) \\
&= \tfrac{1}{16}(4k+3)^3 + \tfrac{3}{8}(4k+3)^2 + \tfrac{5}{16}(4k+3)
\end{aligned}$$

□

Lemma 4.2.14 *For $n \in \mathbb{N}$*

$$D(P_n,-1) = \begin{cases} 1, & n \equiv 0 \ mod\ 4 \\ -1, & n \equiv 1 \ mod\ 4 \\ -1, & n \equiv 2 \ mod\ 4 \\ 1, & n \equiv 3 \ mod\ 4 \end{cases} \tag{5}$$

Proof. We will prove equation (5) by induction. We first compute $D(P_n,-1)$ for $n = 3,4,5,6$. From Table 2.1 we have

$$\begin{aligned}
D(P_3,x) &= x^3 + 3x^2 + x \\
D(P_4,x) &= x^4 + 4x^3 + 4x^2 \\
D(P_5,x) &= x^5 + 5x^4 + 8x^3 + 3x^2 \\
D(P_6,x) &= x^6 + 6x^5 + 13x^4 + 10x^3 + x^2.
\end{aligned}$$

Then by evaluating each equation at $x = -1$ we obtain

$$D(P_3,-1) = 1 \quad D(P_4,-1) = 1 \quad D(P_5,-1) = -1 \quad D(P_6,-1) = -1.$$

Therefore our claim is true for $n = 3,4,5,6$. Now suppose that our claim is true for all n up until $4k-1$. It is sufficient to show our claim is true for $4k, 4k+1, 4k+2$ and $4k+3$. Recall the following recurrence relation from Theorem 2.3.2:

$$D(P_n,x) = x\sum_{i=1}^{3} D(P_{n-i},x).$$

Now consider the four cases $4k, 4k+1, 4k+2$ and $4k+3$.

$$\begin{aligned} D(P_{4k},-1) &= -(D(P_{4k-1},-1)+D(P_{4k-2},-1)+D(P_{4k-3},-1)) \\ &= -(1-1-1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} D(P_{4k+1},-1) &= -(D(P_{4k},-1)+D(P_{4k-1},-1)+D(P_{4k-2},-1)) \\ &= -(1+1-1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} D(P_{4k+2},-1) &= -(D(P_{4k+1},-1)+D(P_{4k},-1)+D(P_{4k-1},-1)) \\ &= -(-1+1+1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} D(P_{4k+3},-1) &= -(D(P_{4k+2},-1)+D(P_{4k+1},-1)+D(P_{4k},-1)) \\ &= -(-1-1+1) \\ &= 1 \end{aligned}$$

$\square$

Lemma 4.2.15 *For $n \in \mathbb{N}$*

$$D'(P_n,-1) = \begin{cases} 0, & n \equiv 0 \ mod\ 4 \\ \frac{n+1}{2}, & n \equiv 1 \ mod\ 4 \\ 0, & n \equiv 2 \ mod\ 4 \\ -\frac{n+1}{2}, & n \equiv 3 \ mod\ 4 \end{cases} \tag{6}$$

Proof. We will prove equation (6) by induction. We first compute $D'(P_n,-1)$ for $n = 3,4,5,6$. For these values of n equation (6) reduces to

$$\begin{aligned} D'(P_3,-1) &= -\tfrac{3+1}{2} = -2 \\ D'(P_4,-1) &= 0 \\ D'(P_5,-1) &= \tfrac{5+1}{2} = 3 \\ D'(P_6,-1) &= 0. \end{aligned}$$

From Table 2.1 we have

$$\begin{aligned}
D(P_3,x) &= x^3+3x^2+x \\
D(P_4,x) &= x^4+4x^3+4x^2 \\
D(P_5,x) &= x^5+5x^4+8x^3+3x^2 \\
D(P_6,x) &= x^6+6x^5+13x^4+10x^3+x^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
D'(P_3,x) &= 3x^2+6x+1 \\
D'(P_4,x) &= 4x^3+12x^2+8x \\
D'(P_5,x) &= 5x^4+20x^3+24x^2+6x \\
D'(P_6,x) &= 6x^5+30x^4+52x^3+30x^2+2x.
\end{aligned}$$

Then by evaluating each equation at $x=-1$ we obtain

$$D'(P_3,-1)=-2 \quad D'(P_4,-1)=0 \quad D'(P_5,-1)=3 \quad D'(P_6,-1)=0.$$

Therefore our claim is true for $n=3,4,5,6$.

Now suppose that our claim is true for all n up until $4k-1$. It is sufficient to show our claim is true for $4k, 4k+1, 4k+2$ and $4k+3$. Recall the following recurrence relation from Theorem 2.3.2:

$$D(P_n,x)=x\sum_{i=1}^{3} D(P_{n-i},x).$$

Therefore by taking the derivative and evaluating it at $x=-1$ we obtain

$$D'(P_n,-1)=\sum_{i=1}^{3} D(P_{n-i},-1)-\sum_{i=1}^{3} D'(P_{n-i},-1).$$

Recall from Theorem 4.2.14

$$D(P_n,-1)=\begin{cases} 1, & n\equiv 0 \bmod 4 \\ -1, & n\equiv 1 \bmod 4 \\ -1, & n\equiv 2 \bmod 4 \\ 1, & n\equiv 3 \bmod 4 \end{cases}$$

Now consider the four cases $4k, 4k+1, 4k+2$ and $4k+3$.

$$\begin{aligned}
D'(P_{4k},-1) &= \textstyle\sum_{i=1}^{3} D'(P_{4k-i},-1) - \sum_{i=1}^{3} D'(P_{4k-i},-1)\\
&= (1-1-1) - (-\tfrac{(4k-1)+1}{2} + 0 + \tfrac{(4k-3)+1}{2})\\
&= -1 - \tfrac{-2}{2}\\
&= 0
\end{aligned}$$

$$\begin{aligned}
D'(P_{4k+1},-1) &= \textstyle\sum_{i=1}^{3} D'(P_{4k+1-i},-1) - \sum_{i=1}^{3} D'(P_{4k+1-i},-1)\\
&= (1+1-1) - (0 - \tfrac{(4k-1)+1}{2} + 0)\\
&= 1 + \tfrac{4k}{2}\\
&= \tfrac{(4k+1)+1}{2}
\end{aligned}$$

$$\begin{aligned}
D'(P_{4k+2},-1) &= \textstyle\sum_{i=1}^{3} D'(P_{4k+2-i},-1) - \sum_{i=1}^{3} D'(P_{4k+2-i},-1)\\
&= (-1+1+1) - (\tfrac{(4k+1)+1}{2} + 0 - \tfrac{(4k-1)+1}{2})\\
&= 1 - \tfrac{2}{2}\\
&= 0
\end{aligned}$$

$$\begin{aligned}
D'(P_{4k+3},-1) &= \textstyle\sum_{i=1}^{3} D'(P_{4k+3-i},-1) - \sum_{i=1}^{3} D'(P_{4k+3-i},-1)\\
&= (-1-1+1) - (\tfrac{(4k+1)+1}{2} + 0)\\
&= -1 - \tfrac{4k+2}{2}\\
&= -\tfrac{(4k+3)+1}{2}
\end{aligned}$$

□

Lemma 4.2.16 *For $n \in \mathbb{N}$*

$$D''(P_n,-1) = \begin{cases} -\frac{1}{8}n(n+4), & n \equiv 0 \ mod\ 4 \\ -\frac{1}{8}(n-1)^2, & n \equiv 1 \ mod\ 4 \\ \frac{1}{8}(n+2)^2, & n \equiv 2 \ mod\ 4 \\ \frac{1}{8}(n-3)(n+1), & n \equiv 3 \ mod\ 4 \end{cases} \tag{7}$$

Proof. We will prove our equation (7) by induction. We first compute $D''(P_n,-1)$ for $n = 3,4,5,6$. For these values of n, equation (7) reduces to

$$\begin{aligned}
D''(P_3,-1) &= \tfrac{1}{8}(3-3)(3+1) &= 0\\
D''(P_4,-1) &= -\tfrac{1}{8}(4)(4+4) &= -4\\
D''(P_5,-1) &= -\tfrac{1}{8}(5-1)^2 &= -2\\
D''(P_6,-1) &= \tfrac{1}{8}(6+2)^2 &= 8.
\end{aligned}$$

From Table 2.1 we have

$$\begin{aligned}
D(P_3,x) &= x^3+3x^2+x\\
D(P_4,x) &= x^4+4x^3+4x^2\\
D(P_5,x) &= x^5+5x^4+8x^3+3x^2\\
D(P_6,x) &= x^6+6x^5+13x^4+10x^3+x^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
D''(P_3,x) &= 6x+6\\
D''(P_4,x) &= 12x^2+24x+8\\
D''(P_5,x) &= 20x^3+60x^2+48x+6\\
D''(P_6,x) &= 30x^4+120x^3+156x^2+60x+2.
\end{aligned}$$

Then by evaluating each equation at $x=-1$ we obtain

$$D''(P_3,-1)=0 \quad D''(P_4,-1)=-4 \quad D''(P_5,-1)=-2 \quad D''(P_6,-1)=8.$$

Therefore our claim is true for $n=3,4,5,6$.

Now suppose that our claim is true for all n up until $4k-1$. It is sufficient to show our claim is true for $4k, 4k+1, 4k+2$ and $4k+3$. Recall the following recurrence relation from Theorem 2.3.2:

$$D(P_n,x)=x\sum_{i=1}^{3}D(P_{n-i},x).$$

Therefore by taking the derivative two times and evaluating it at $x=-1$ we obtain

$$D''(P_n,-1)=2\sum_{i=1}^{3}D'(P_{n-i},-1)-\sum_{i=1}^{3}D''(P_{n-i},-1).$$

Recall from Theorem 4.2.14

$$D'(P_n,-1)=\begin{cases} 0, & n\equiv 0 \bmod 4\\ \frac{n+1}{2}, & n\equiv 1 \bmod 4\\ 0, & n\equiv 2 \bmod 4\\ -\frac{n+1}{2}, & n\equiv 3 \bmod 4\end{cases}$$

Now consider the four cases $4k, 4k+1, 4k+2$ and $4k+3$.

$$\begin{aligned} D''(P_{4k},-1) &= 2\sum_{i=1}^{3} D'(P_{4k-i},-1)-\sum_{i=1}^{3} D''(P_{4k-i},-1)\\ &= 2(-\tfrac{(4k-1)+1}{2}+0+\tfrac{(4k-3)+1}{2})-\\ &\quad (\tfrac{1}{8}((4k-1)-3)((4k-1)+1)+\\ &\quad \tfrac{1}{8}((4k-2)+2)^2-\tfrac{1}{8}((4k-3)-1)^2)\\ &= -\tfrac{1}{8}(4k)((4k)+4)\end{aligned}$$

$$\begin{aligned} D''(P_{4k+1},-1) &= 2\sum_{i=1}^{3} D'(P_{4k+1-i},-1)-\sum_{i=1}^{3} D''(P_{4k+1-i},-1)\\ &= 2(0-\tfrac{(4k-1)+1}{2}+0)-\\ &\quad (-\tfrac{1}{8}(4k)((4k)+4)+\tfrac{1}{8}((4k-1)-3)((4k-1)+1)+\\ &\quad \tfrac{1}{8}((4k-2)+2)^2)\\ &= -\tfrac{1}{8}((4k+1)-1)^2\end{aligned}$$

$$\begin{aligned} D''(P_{4k+2},-1) &= 2\sum_{i=1}^{3} D'(P_{4k+2-i},-1)-\sum_{i=1}^{3} D''(P_{4k+2-i},-1)\\ &= 2(\tfrac{(4k+1)+1}{2}+0-\tfrac{(4k-1)+1}{2})-\\ &\quad (-\tfrac{1}{8}((4k+1)-1)^2-\tfrac{1}{8}(4k)((4k)+4)+\\ &\quad \tfrac{1}{8}((4k-1)-3)((4k-1)+1))\\ &= \tfrac{1}{8}((4k+2)+2)^2\end{aligned}$$

$$\begin{aligned} D''(P_{4k+3},-1) &= 2\sum_{i=1}^{3} D'(P_{4k+3-i},-1)-\sum_{i=1}^{3} D''(P_{4k+3-i},-1)\\ &= 2(0+\tfrac{(4k+1)+1}{2}+0)-\\ &\quad (\tfrac{1}{8}((4k+2)+2)^2-\tfrac{1}{8}((4k+1)-1)^2-\\ &\quad \tfrac{1}{8}(4k)((4k)+4))\\ &= \tfrac{1}{8}((4k+3)-3)((4k+3)+1)\end{aligned}$$

$\square$

Lemma 4.2.17 *For $n \in \mathbb{N}$*

$$D'''(P_n, -1) = \begin{cases} \frac{1}{16}n^3 - n, & n \equiv 0 \ mod\ 4 \\ -\frac{9}{16}n^2 + \frac{3}{8}n + \frac{3}{16}, & n \equiv 1 \ mod\ 4 \\ -\frac{1}{16}n^3 + \frac{1}{4}n, & n \equiv 2 \ mod\ 4 \\ \frac{9}{16}n^2 + \frac{3}{8}n - \frac{3}{16}, & n \equiv 3 \ mod\ 4 \end{cases} \tag{8}$$

Proof. We will prove equation (8) by induction. We first compute $D'''(P_n, -1)$ for $n = 3, 4, 5, 6$. For these values of n, equation (8) reduces to

$$\begin{aligned} D'''(P_3, -1) &= \frac{9}{16}(3)^2 + \frac{3}{8}(3) - \frac{3}{16} &&= 6 \\ D'''(P_4, -1) &= \frac{1}{16}(4)^3 - 4 &&= 0 \\ D'''(P_5, -1) &= -\frac{9}{16}(5)^2 + \frac{3}{8}(5) + \frac{3}{16} &&= -12 \\ D'''(P_6, -1) &= -\frac{1}{16}(6)^3 + \frac{1}{4}(6) &&= -12. \end{aligned}$$

From Table 2.1 we have

$$\begin{aligned} D(P_3, x) &= x^3 + 3x^2 + x \\ D(P_4, x) &= x^4 + 4x^3 + 4x^2 \\ D(P_5, x) &= x^5 + 5x^4 + 8x^3 + 3x^2 \\ D(P_6, x) &= x^6 + 6x^5 + 13x^4 + 10x^3 + x^2. \end{aligned}$$

Therefore

$$\begin{aligned} D'''(P_3, x) &= 6 \\ D'''(P_4, x) &= 24x + 24 \\ D'''(P_5, x) &= 60x^2 + 120x + 48 \\ D'''(P_6, x) &= 120x^3 + 360x^2 + 312x + 60. \end{aligned}$$

Then by evaluating each equation at $x = -1$ we obtain

$$D'''(P_3, -1) = 6 \quad D'''(P_4, -1) = 0 \quad D'''(P_5, -1) = -12 \quad D'''(P_6, -1) = -12.$$

Therefore our claim is true for $n = 3, 4, 5, 6$. Now suppose that our claim is true for all n up until $4k - 1$. It is sufficient to show our claim is true for $4k, 4k + 1, 4k + 2$ and $4k + 3$. Recall the following recurrence relation from Theorem 2.3.2:

$$D(P_n, x) = x \sum_{i=1}^{3} D(P_{n-i}, x).$$

Therefore by taking the derivative three times and evaluating it at $x = -1$ we obtain

$$D'''(P_n, -1) = 3 \sum_{i=1}^{3} D''(P_{n-i}, -1) - \sum_{i=1}^{3} D'''(P_{n-i}, -1).$$

Recall from Theorem 4.2.14

$$D''(P_n, -1) = \begin{cases} -\frac{1}{8}n(n+4), & n \equiv 0 \bmod 4 \\ -\frac{1}{8}(n-1)^2, & n \equiv 1 \bmod 4 \\ \frac{1}{8}(n+2)^2, & n \equiv 2 \bmod 4 \\ \frac{1}{8}(n-3)(n+1), & n \equiv 3 \bmod 4 \end{cases}$$

Now consider the four cases $4k, 4k+1, 4k+2$ and $4k+3$.

$$\begin{aligned} D'''(P_{4k}, -1) &= 3\textstyle\sum_{i=1}^{3} D''(P_{4k-i}, -1) - \sum_{i=1}^{3} D'''(P_{4k-i}, -1) \\ &= 3(\tfrac{1}{8}((4k-1)-3)((4k-1)+1)+ \\ &\quad \tfrac{1}{8}((4k-2)+2)^2 - \tfrac{1}{8}((4k-3)-1)^2)- \\ &\quad (\tfrac{9}{16}(4k-1)^2 + \tfrac{3}{8}(4k-1) - \tfrac{3}{16}- \\ &\quad \tfrac{1}{16}(4k-2)^3 + \tfrac{1}{4}(4k-2)- \\ &\quad \tfrac{9}{16}(4k-3)^2 + \tfrac{3}{8}(4k-3) + \tfrac{3}{16}) \\ &= \tfrac{1}{16}(4k)^3 - (4k) \end{aligned}$$

$$\begin{aligned} D'''(P_{4k+1}, -1) &= 3\textstyle\sum_{i=1}^{3} D''(P_{4k+1-i}, -1) - \sum_{i=1}^{3} D'''(P_{4k+1-i}, -1) \\ &= 3(-\tfrac{1}{8}(4k)((4k)+4) + \tfrac{1}{8}((4k-1)-3)((4k-1)+1)+ \\ &\quad \tfrac{1}{8}((4k-2)+2)^2)- \\ &\quad (\tfrac{1}{16}(4k)^3 - (4k)+ \\ &\quad \tfrac{9}{16}(4k-1)^2 + \tfrac{3}{8}(4k-1) - \tfrac{3}{16}- \\ &\quad \tfrac{1}{16}(4k-2)^3 + \tfrac{1}{4}(4k-2)) \\ &= -\tfrac{9}{16}(4k+1)^2 + \tfrac{3}{8}(4k+1) + \tfrac{3}{16} \end{aligned}$$

$$\begin{aligned}
D'''(P_{4k+2},-1) &= 3\sum_{i=1}^{3} D''(P_{4k+2-i},-1) - \sum_{i=1}^{3} D'''(P_{4k+2-i},-1) \\
&= 3(-\tfrac{1}{8}((4k+1)-1)^2 - \tfrac{1}{8}(4k)((4k)+4)+ \\
&\quad \tfrac{1}{8}((4k-1)-3)((4k-1)+1))- \\
&\quad (-\tfrac{9}{16}(4k+1)^2 + \tfrac{3}{8}(4k+1) + \tfrac{3}{16}+ \\
&\quad \tfrac{1}{16}(4k)^3 - (4k)+ \\
&\quad \tfrac{9}{16}(4k-1)^2 + \tfrac{3}{8}(4k-1) - \tfrac{3}{16}) \\
&= -\tfrac{1}{16}(4k+2)^3 + \tfrac{1}{4}(4k+2)
\end{aligned}$$

$$\begin{aligned}
D'''(P_{4k+3},-1) &= 3\sum_{i=1}^{3} D''(P_{4k+3-i},-1) - \sum_{i=1}^{3} D'''(P_{4k+3-i},-1) \\
&= 3(\tfrac{1}{8}((4k+2)+2)^2 - \tfrac{1}{8}((4k+1)-1)^2- \\
&\quad \tfrac{1}{8}(4k)((4k)+4))- \\
&\quad (-\tfrac{1}{16}(4k+2)^3 + \tfrac{1}{4}(4k+2)- \\
&\quad \tfrac{9}{16}(4k+1)^2 + \tfrac{3}{8}(4k+1) + \tfrac{3}{16}+ \\
&\quad \tfrac{1}{16}(4k)^3 - (4k)) \\
&= \tfrac{9}{16}(4k+3)^2 + \tfrac{3}{8}(4k+3) - \tfrac{3}{16}
\end{aligned}$$

□

We now present our main result, the equivalence class of paths. The next theorem will show $[P_n] = \{P_n, P_n'\}$ for $n \geq 9$. However, first we will discuss the $[P_n]$ for $n \leq 8$ as shown in Table 4.1. For $n \neq 4, 7, 8$, $[P_n] = \{P_n, P_n'\}$ (P_3 only has one stem, so P_3 and P_3' are isomorphic). Recall from the proof of Lemma 4.2.4, $D(P_n, -2) = 0$ when $n = 4, 7, 8$. This is evident in Table 4.1 as P_4, P_7, and P_8 are each $\mathcal{D}$-equivalent to graphs with K_2 components. Note that $[P_7]$ and $[P_8]$ each have four graphs, however they effectively only have two graphs as the other two graphs are just copies with an irrelevant edge added (the edge between two stems).

Theorem 4.2.18 *For $n \geq 9$, $[P_n] = \{P_n, P_n'\}$.*

Proof. For $n \geq 9$, let G be a graph which is $\mathcal{D}$-equivalent to P_n. By Lemma 4.2.9 $G = H \cup C$ where $H \in \{P_{n_1}, P_{n_1}'\}$ with $n_1 \leq n$ and C is the disjoint union of at most two cycles. Therefore either $G = H$, $G = H \cup C_{n_2}$, or $G = H \cup C_{n_2} \cup C_{n_3}$. It is sufficient to show the latter two cases result in a contradiction. We will do so by evaluating $D(P_n, -1), \ldots, D'''(P_n, -1)$ and $D(G, -1), \ldots, D'''(G, -1)$ for all cases $n_1, n_2, n_3 \equiv 0, 1, 2, 3 \bmod 4$ and showing each case results in a contradiction.

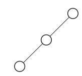
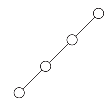
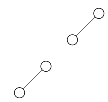
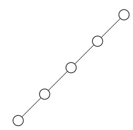
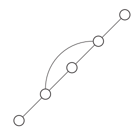
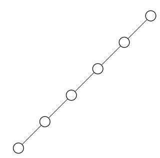
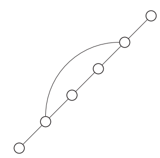
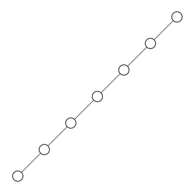
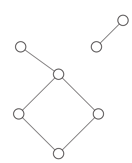
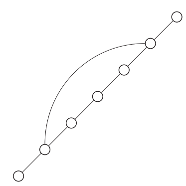
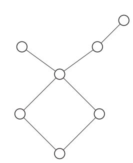
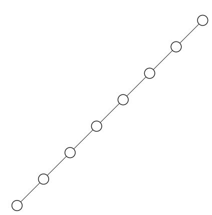
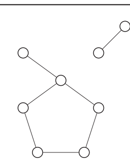
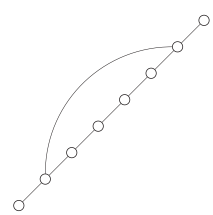
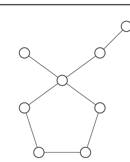
n	$D(P_n, x)$	$[P_n]$	
3	$x^3 + 3x^2 + x$		
4	$x^4 + 4x^3 + 4x^2$		
5	$x^5 + 5x^4 + 8x^3 + 3x^2$		
6	$x^6 + 6x^5 + 13x^4 + 10x^3 + x^2$		
7	$x^7 + 7x^6 + 19x^5 + 22x^4 + 8x^3$	 	 
8	$x^8 + 8x^7 + 26x^6 + 40x^5 + 26x^4 + 4x^3$	 	 

Table 4.1: The domination equivalence classes for paths up to length eight

By Lemma 4.2.10 – 4.2.13 we know that $D(C_n, -1)$, $D'(C_n, -1)$, $D''(C_n, -1)$ and $D'''(C_n, -1)$. Similarly by Lemma 4.2.14 - 4.2.17 we know $D(P_n, -1)$, $D'(P_n, -1)$, $D''(P_n, -1)$ and $D'''(P_n, -1)$. As G is a disjoint union of paths and cycles then its domination polynomial is the product of the domination polynomials of paths and cycles. Furthermore we can then use the product rule to equate $D(P_n, -1)$ and $D(G, -1)$, $D'(P_n, -1)$ and $D'(G, -1)$, and so on. This gives us a system of equations. If the system has no solutions then the case results in a contradiction.

We claim that each case results in a contradiction. We omit the cases were $n_2, n_3 \equiv 0 \bmod 4$ because then $D(C_{n_2}, -1) = D(C_{n_3}, -1) = 3$ and by Corollary 4.2.3 any component H of G must have $D(H, -1) \in \{1, -1\}$. Note the following facts which will be used to show a contradiction:

- $n_2, n_3 \geq 3$ as they are the order of cycles.
- $n_1 \geq 0$ as it is the order of a path.
- $n_1 \neq n$ as $n_2, n_3 \geq 3$.
- $n \geq 9$.

Case A: One Cycle

Then let $G = P_{n_1} \cup C_{n_2}$. We will examine the twelve cases for $n_1 \equiv 0, 1, 2, 3 \bmod 4$ and $n_2 \equiv 1, 2, 3 \bmod 4$. By taking the first three derivatives of $D(G, x)$ we obtain the following system of equations:

$$D(P_n, -1) = D(P_{n_1}, -1)D(C_{n_2}, -1) \qquad (PC0)$$

$$D'(P_n, -1) = D'(P_{n_1}, -1)D(C_{n_2}, -1) + D(P_{n_1}, -1)D'(C_{n_2}, -1) \qquad (PC1)$$

$$\begin{aligned} D''(P_n, -1) = {} & D''(P_{n_1}, -1)D(C_{n_2}, -1) + 2D'(P_{n_1}, -1)D'(C_{n_2}, -1) \\ & + D(P_{n_1}, -1)D''(C_{n_2}, -1) \end{aligned} \qquad (PC2)$$

$$\begin{aligned} D'''(P_n, -1) = {} & D'''(P_{n_1}, -1)D(C_{n_2}, -1) + 3D''(P_{n_1}, -1)D'(C_{n_2}, -1) \\ & + 3D'(P_{n_1}, -1)D''(C_{n_2}, -1) + D(P_{n_1}, -1)D'''(C_{n_2}, -1) \end{aligned} \qquad (PC3)$$

In each case we will substitute the results from Lemma 4.2.10 - 4.2.17 into the appropriate equation to show our contradiction.

Case 1: $n_1 \equiv 0$, $n_2 \equiv 1$ *(mod 4)*

As $n \equiv n_1 + n_2 \pmod 4$ then $n \equiv 1 \pmod 4$ and therefore equation $(PC1)$ reduces to

$$\frac{n+1}{2} = n_2$$

and equation $(PC2)$ reduces to

$$-\frac{(n-1)^2}{8} = \frac{n_1(n_1+4)}{8} - \frac{n_2(n_2-1)}{2}.$$

Therefore $n = 2n_2 - 1$. As $n = n_1 + n_2$ then $n_1 = n_2 - 1$. So by substituting this into the reduced equation $(PC2)$ and multiplying both sides by 8 we obtain

$$-(2n_2-2)^2 = (n_2-1)(n_2+3) - 4n_2(n_2-1).$$

By bringing everything to one side and simplifying we are left with

$$(n_2-1)^2 = 0.$$

Therefore $n_2 = 1$. However as $n_2 \geq 3$, this is a contradiction.

Case 2: $n_1 \equiv 0$, $n_2 \equiv 2$ *(mod 4)*

As $n \equiv n_1 + n_2 \pmod 4$ then $n \equiv 2 \pmod 4$ and therefore equation $(PC2)$ reduces to

$$\frac{(n+2)^2}{8} = \frac{n_1(n_1+4)}{8} + \frac{n_2(n_2+2)}{4},$$

and equation $(PC3)$ reduces to

$$-\frac{n^3}{16} + \frac{n}{4} = -\frac{n_1^3}{16} + n_1 - \frac{3n_2^3}{16} + \frac{3n_2}{4}.$$

We will now substitute $n = n_1 + n_2$ into the reduced equation $(PC2)$:

$$\begin{aligned}
0 &= \tfrac{1}{8}(n_1+n_2+2)^2 - (\tfrac{1}{8}n_1(n_1+4) + \tfrac{1}{4}n_2(n_2+2)) \\
0 &= (n_1+n_2+2)^2 - n_1(n_1+4) - 2n_2(n_2+2) \\
0 &= n_1^2 + n_2^2 + 4 + 2n_1n_2 + 4n_1 + 4n_2 - n_1^2 - 4n_1 - 2n_2^2 - 4n_2 \\
0 &= -n_2^2 + 2n_1n_2 + 4
\end{aligned}$$

Therefore $n_1 = \frac{n_2^2-4}{2n_2}$. By substituting this and $n = n_1 + n_2$ into the reduced equation $(PC3)$ and multiplying by n_2 we obtain

$$0 = -\frac{1}{4}n_2^4 - 2n_2^2 + 12.$$

Using Maple, we obtain the solutions $n_2 = -2, 2, -2\sqrt{3}i$ or $2\sqrt{3}i$. As n_2 is order of the cycle then $n_2 \geq 3$ and real. This is a contradiction for all four solutions.

Case 3: $n_1 \equiv 0,\ n_2 \equiv 3$ *(mod 4)*

As $n \equiv n_1 + n_2 \pmod 4$ then $n \equiv 3 \pmod 4$ and $D(P_n, -1) = 1$. However $D(G, -1) = D(P_{n_1}, -1)D(C_{n_2}, -1) = (1)(-1) = -1$ which is a contraction.

Case 4: $n_1 \equiv 1,\ n_2 \equiv 1$ *(mod 4)*

Then $n \equiv 2 \pmod 4$ and $D(P_n, -1) = -1$. However $D(G, -1) = D(P_{n_1}, -1)D(C_{n_2}, -1) = (-1)(-1) = 1$ which is a contraction.

Case 5: $n_1 \equiv 1,\ n_2 \equiv 2$ *(mod 4)*

As $n \equiv n_1 + n_2 \pmod 4$ then $n \equiv 3 \pmod 4$ and therefore equation $(PC1)$ reduces to

$$-\frac{1}{2}(n+1) = -\frac{1}{2}(n_1+1).$$

This implies $n = n_1$. However $n = n_1 + n_2$, so $n_2 = 0$, which is a contradiction.

Case 6: $n_1 \equiv 1,\ n_2 \equiv 3$ *(mod 4)*

As $n \equiv n_1 + n_2 \pmod 4$ then $n \equiv 0 \pmod 4$ and therefore equation $(PC1)$ reduces to

$$0 = -\frac{1}{2}(n_1 + 1).$$

This implies $n_1 = -1$ which is a contradiction.

Case 7: $n_1 \equiv 2$, $n_2 \equiv 1$ *(mod 4)*

As $n \equiv n_1 + n_2 \pmod 4$ then $n \equiv 3 \pmod 4$ and therefore equation $(PC1)$ reduces to

$$-\frac{n+1}{2} = -n_2,$$

and equation $(PC2)$ reduces to

$$\frac{1}{8}(n-3)(n+1) = -\frac{1}{8}(n_1+2)^2 + \frac{1}{2}n_2(n_2-1).$$

Therefore $n = 2n_2 - 1$. As $n = n_1 + n_2$ then $n_1 = n_2 - 1$. So by substituting this into the reduced equation $(PC2)$ and multiplying both sides by 8 we obtain

$$(2n_2 - 4)(2n_2) = -(n_2+1)^2 + 4n_2(n_2-1).$$

By bringing everything to one side and simplifying we are left with

$$(n_2 - 1)^2 = 0.$$

Therefore $n_2 = 1$. However as $n_2 \geq 3$, this is a contradiction.

Case 8: $n_1 \equiv 2$, $n_2 \equiv 2$ *(mod 4)*

As $n \equiv n_1 + n_2 \pmod 4$ then $n \equiv 0 \pmod 4$ and therefore equation $(PC2)$ reduces to

$$-\frac{1}{8}n(n+4) = -\frac{1}{8}(n_1+2)^2 - \frac{1}{4}n_2(n_2+2),$$

and equation $(PC3)$ reduces to

$$\frac{1}{16}n^3 - n = \frac{1}{16}n_1^3 - \frac{1}{4}n_1 + \frac{3}{16}n_2^3 - \frac{3}{4}n_2.$$

We will now substitute $n = n_1 + n_2$ into the reduced equation $(PC2)$.

$$\begin{aligned}
0 &= -\tfrac{1}{8}(n_1+n_2)(n_1+n_2+4) - (-\tfrac{1}{8}(n_1+2)^2 - \tfrac{1}{4}n_2(n_2+2)) \\
0 &= -(n_1+n_2)(n_1+n_2+4) + (n_1+2)^2 + 2n_2(n_2+2) \\
0 &= -n_1^2 - 2n_1n_2 - n_2^2 - 4n_1 - 4n_2 + n_1^2 + 4n_1 + 4 + 2n_2^2 + 4n_2 \\
0 &= -2n_1n_2 + 4 + n_2^2
\end{aligned}$$

Therefore $n_1 = \frac{n_2^2+4}{2n_2}$. By substituting this and $n = n_1 + n_2$ into the reduced equation $(PC3)$ and multiplying by n_2 we obtain

$$0 = \frac{1}{4}n_2^4 + 2n_2^2 - 12.$$

Using Maple, we obtain the solutions $n_2 = -2, 2, -2\sqrt{3}i$ or $2\sqrt{3}i$. As n_2 is order of the cycle then $n_2 \geq 3$ and real. This is a contradiction for all four solutions.

Case 9: $n_1 \equiv 2,\ n_2 \equiv 3$ *(mod 4)*

As $n \equiv n_1 + n_2 \pmod 4$ then $n \equiv 1 \pmod 4$ and $D(P_n, -1) = -1$. However $D(G,-1) = D(P_{n_1}, -1)D(C_{n_2}, -1) = (-1)(-1) = 1$ which is a contraction.

Case 10: $n_1 \equiv 3,\ n_2 \equiv 1$ *(mod 4)*

As $n \equiv n_1 + n_2 \pmod 4$ then $n \equiv 0 \pmod 4$ and $D(P_n, -1) = 1$. However $D(G,-1) = D(P_{n_1}, -1)D(C_{n_2}, -1) = (1)(-1) = -1$ which is a contraction.

Case 11: $n_1 \equiv 3,\ n_2 \equiv 2$ *(mod 4)*

As $n \equiv n_1 + n_2 \pmod 4$ then $n \equiv 1 \pmod 4$ and equation $(PC1)$ reduces to

$$\frac{1}{2}(n+1) = \frac{1}{2}(n_1+1).$$

This implies $n = n_1$. However $n = n_1 + n_2$, so $n_2 = 0$, which is a contradiction.

Case 12: $n_1 \equiv 3,\ n_2 \equiv 3$ *(mod 4)*

As $n \equiv n_1 + n_2 \pmod 4$ then $n \equiv 2 \pmod 4$ and equation $(PC1)$ reduces to

$$0 = \frac{1}{2}(n_1 + 1).$$

This implies $n_1 = -1$ which is a contradiction.

As each of the twelve cases result in a contradiction then G is not a disjoint union of H and one cycle, where $H \in \{P_{n_1}, P'_{n_1}\}$. We will now consider whether G can be a disjoint union of H and two cycles.

Case B: Two Cycles

Then let $G = P_{n_1} \cup C_{n_2} \cup C_{n_3}$. We will examine all cases with $n_1 \equiv 0, 1, 2, 3 \bmod 4$, $n_2 \equiv 1, 2, 3 \bmod 4$, and $n_3 \equiv 1, 2, 3 \bmod 4$. Without loss of generality we will also only consider the cases where n_2 is less than or equal to n_3 modulus 4. For example, if $n_3 \equiv 2 \bmod 4$ then we will only consider the cases where $n_2 \equiv 1, 2 \bmod 4$.

By taking the first three derivatives of $D(G, x)$ we obtain the following system of equations:

$$D(P_n, -1) = D(P_{n_1}, -1)D(C_{n_2}, -1)D(C_{n_3}, -1) \quad (PCC0)$$

$$\begin{aligned} D'(P_n, -1) = {} & D'(P_{n_1}, -1)D(C_{n_2}, -1)D(C_{n_3}, -1) + D(P_{n_1}, -1)D'(C_{n_2}, -1)D(C_{n_3}, -1) \\ & + D(P_{n_1}, -1)D(C_{n_2}, -1)D'(C_{n_3}, -1) \end{aligned} \quad (PCC1)$$

$$\begin{aligned} D''(P_n, -1) = {} & D''(P_{n_1}, -1)D(C_{n_2}, -1)D(C_{n_3}, -1) + D(P_{n_1}, -1)D''(C_{n_2}, -1)D(C_{n_3}, -1) \\ & + D(P_{n_1}, -1)D(C_{n_2}, -1)D''(C_{n_3}, -1) + 2D'(P_{n_1}, -1)D'(C_{n_2}, -1)D(C_{n_3}, -1) \\ & + 2D'(P_{n_1}, -1)D(C_{n_2}, -1)D'(C_{n_3}, -1) + 2D(P_{n_1}, -1)D'(C_{n_2}, -1)D'(C_{n_3}, -1) \end{aligned} \quad (PCC2)$$

$$\begin{aligned} D'''(P_n, -1) = {} & D'''(P_{n_1}, -1)D(C_{n_2}, -1)D(C_{n_3}, -1) + D(P_{n_1}, -1)D'''(C_{n_2}, -1)D(C_{n_3}, -1) \\ & + D(P_{n_1}, -1)D(C_{n_2}, -1)D'''(C_{n_3}, -1) + 3D''(P_{n_1}, -1)D'(C_{n_2}, -1)D(C_{n_3}, -1) \\ & + 3D'(P_{n_1}, -1)D''(C_{n_2}, -1)D(C_{n_3}, -1) + 3D''(P_{n_1}, -1)D(C_{n_2}, -1)D'(C_{n_3}, -1) \\ & + 3D'(P_{n_1}, -1)D(C_{n_2}, -1)D''(C_{n_3}, -1) + 3D(P_{n_1}, -1)D''(C_{n_2}, -1)D'(C_{n_3}, -1) \\ & + 3D(P_{n_1}, -1)D'(C_{n_2}, -1)D''(C_{n_3}, -1) + 6D'(P_{n_1}, -1)D'(C_{n_2}, -1)D'(C_{n_3}, -1) \end{aligned} \quad (PCC3)$$

In each case we will substitute the results from Lemma 4.2.10 - 4.2.17 into the appropriate equation to show our contradiction.

Case 1: $n_1 \equiv 0,\ n_2 \equiv 1,\ n_3 \equiv 1$ *(mod 4)*

As $n \equiv n_1+n_2+n_3 \pmod 4$ then $n \equiv 2 \pmod 4$ and by Lemma 4.2.14, $D(P_n,-1) = -1$. However by Lemma 4.2.10 and Lemma 4.2.14, $D(P_{n_1},-1) = 1$ and $D(C_{n_2},-1) = D(C_{n_3},-1) = -1$ so $D(G,-1) = 1$, which is a contradiction.

Case 2: $n_1 \equiv 0,\ n_2 \equiv 1,\ n_3 \equiv 2$ *(mod 4)*

As $n \equiv n_1 + n_2 + n_3 \pmod 4$ then $n \equiv 3 \pmod 4$ and therefore equation $(PCC1)$ reduces to

$$-\frac{n+1}{2} = -n_2$$

and equation $(PCC2)$ reduces to

$$\frac{1}{8}(n-3)(n+1) = -\frac{1}{8}n_1(n_1+4) - \frac{1}{4}n_3(n_3+2) + \frac{1}{2}n_2(n_2-1).$$

Therefore $n = 2n_2-1$. As $n = n_1+n_2+n_3$ then $n_2 = n_1+n_3+1$ and $n = 2n_1+2n_3+1$. So by substituting this into the reduced equation $(PCC2)$ and multiplying both sides by 8 we obtain

$$(2n_1+2n_3-2)(2n_1+2n_3+2) = -n_1(n_1+4) - 2n_3(n_3+2) + 4(n_1+n_3+1)(n_1+n_3),$$

which simplifies to

$$\begin{aligned}
4(n_1+n_3)^2 - 4 &= -n_1(n_1+4) - 2n_3(n_3+2) + 4(n_1+n_3)^2 + 4(n_1+n_3) \\
-4 &= -n_1^2 - 4n_1 - 2n_3^2 - 4n_3 + 4n_1 + 4n_3 \\
0 &= -n_1^2 - 2n_3^2 + 4.
\end{aligned}$$

As $n_3 \geq 3$, there are no solutions, which is a contradiction.

Case 3: $n_1 \equiv 0,\ n_2 \equiv 2,\ n_3 \equiv 2$ *(mod 4)*

As $n \equiv n_1 + n_2 + n_3 \pmod 4$ then $n \equiv 0 \pmod 4$ and therefore equation $(PCC2)$ reduces to

$$-\frac{1}{8}n(n+4) = -\frac{1}{8}n_1(n_1+4) - \frac{1}{4}n_3(n_3+2) - \frac{1}{4}n_2(n_2+2).$$

and equation $(PCC3)$ reduces to

$$\frac{1}{16}n^3 - n = \frac{1}{16}n_1^3 - n_1 + \frac{3}{16}n_3^3 - \frac{3}{4}n_3 + \frac{3}{16}n_2^3 - \frac{3}{4}n_2.$$

We will now substitute $n = n_1 + n_2 + n_3$ into the the reduced equation $(PCC2)$

$$n_2^2 + n_3^2 - 2n_1n_2 - 2n_1n_3 - 2n_2n_3 = 0.$$

Therefore if we isolate for n_1 we obtain

$$n_1 = \frac{(n_2 - n_3)^2}{2(n_2 + n_3)}.$$

By substituting this and $n = n_1 + n_2 + n_3$ into the the reduced equation $(PCC3)$, multiplying by $64n_2 + 64n_3$, and simplifying we obtain

$$n_2^4 - 8n_2^3n_3 + 30n_2^2n_3^2 - 8n_2n_3^3 + n_3^4 - 16n_2^2 - 32n_2n_3 - 16n_3^2 = 0 \tag{9}$$

We have plotted the non-negative solutions to equation (9) along with the line $n_3 = 8 - n_2$ in Figure 4.10. We will show that any line $n_3 = k - n_2$ which intersects the set of non-negative solutions to equation (9) must have $k \leq 8$. Therefore we will be able to bound all solutions to equation (9) with the bounds $n_3 \leq 8 - n_2$and $n_3, n_2 \geq 3$.

We will show the line $n_3 = k - n_2$ only intersects the set of non-negative solutions to equation (9) if $k \leq 8$. First substitute $n_3 = k - n_2$ into equation (9) to obtain

$$48n_2^4 - 96kn_2^3 + 60k^2n_2^2 - 12k^3n_2 + k^4 - 12k^2 = 0.$$

With the help of Maple we found the solution

$$n_2 = \frac{1}{2}k \pm \frac{1}{12}\sqrt{18k^2 \pm 6k\sqrt{-3k^2 + 192}}.$$

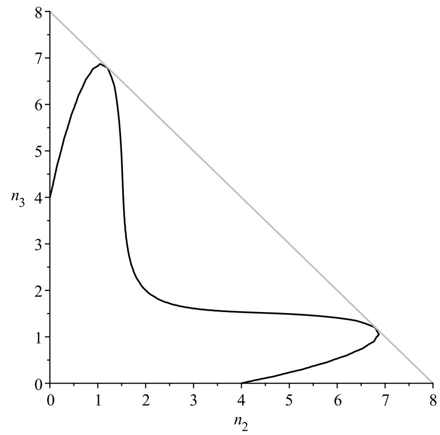


Figure 4.10: Solutions to $n_2^4 - 8n_2^3n_3 + 30n_2^2n_3^2 - 8n_2n_3^3 + n_3^4 - 12n_2^2 - 32n_2n_3 - 12n_3^2 = 0$

Therefore n_2 is real only if $-3k^2 + 192 \geq 0$ and hence $k \leq 8$. Therefore the only remaining viable solutions are the 6 integer pairs bounded by $n_2, n_3 \geq 3$ and $n_3 \leq 8 - n_2$. As none are solutions, this is a contradiction.

Case 4: $n_1 \equiv 0$, $n_2 \equiv 1$, $n_3 \equiv 3$ *(mod 4)*

As $n \equiv n_1 + n_2 + n_3 \pmod 4$ then $n \equiv 0 \pmod 4$ and by Lemma 4.2.15, $D'(P_n, -1) = 0$. However by Lemma 4.2.11 and Lemma 4.2.15 $D'(G, -1) = -n_2$, so $n_2 = 0$, which is a contradiction as $n_2 \geq 3$.

Case 5: $n_1 \equiv 0$, $n_2 \equiv 2$, $n_3 \equiv 3$ *(mod 4)*

As $n \equiv n_1 + n_2 + n_3 \pmod 4$ then $n \equiv 1 \pmod 4$ and by Lemma 4.2.14, $D(P_n, -1) = -1$. However by Lemma 4.2.10 and Lemma 4.2.14, $D(P_{n_1}, -1) = 1$ and $D(C_{n_2}, -1) = D(C_{n_3}, -1) = -1$ so $D(G, -1) = 1$, which is a contradiction.

Case 6: $n_1 \equiv 0$, $n_2 \equiv 3$, $n_3 \equiv 3$ *(mod 4)*

As $n \equiv n_1 + n_2 + n_3 \pmod 4$ then $n \equiv 2 \pmod 4$ and by Lemma 4.2.14, $D(P_n, -1) = -1$. However by Lemma 4.2.10 and Lemma 4.2.14, $D(P_{n_1}, -1) = 1$ and $D(C_{n_2}, -1) = D(C_{n_3}, -1) = -1$ so $D(G, -1) = 1$ which, is a contradiction.

Case 7: $n_1 \equiv 1,\ n_2 \equiv 1,\ n_3 \equiv 1$ *(mod 4)*

As $n \equiv n_1+n_2+n_3 \pmod 4$ then $n \equiv 3 \pmod 4$ and by Lemma 4.2.14, $D(P_n, -1) = 1$. However by Lemma 4.2.10 and Lemma 4.2.14, $D(P_{n_1}, -1) = -1$ and $D(C_{n_2}, -1) = D(C_{n_3}, -1) = -1$ so $D(G, -1) = -1$, which is a contradiction.

Case 8: $n_1 \equiv 1,\ n_2 \equiv 1,\ n_3 \equiv 2$ *(mod 4)*

As $n \equiv n_1+n_2+n_3 \pmod 4$ then $n \equiv 0 \pmod 4$ and by Lemma 4.2.14, $D(P_n, -1) = 1$. However by Lemma 4.2.10 and Lemma 4.2.14, $D(P_{n_1}, -1) = -1$ and $D(C_{n_2}, -1) = D(C_{n_3}, -1) = -1$ so $D(G, -1) = -1$, which is a contradiction.

Case 9: $n_1 \equiv 1,\ n_2 \equiv 2,\ n_3 \equiv 2$ *(mod 4)*

As $n \equiv n_1 + n_2 + n_3 \pmod 4$ then $n \equiv 1 \pmod 4$ and therefore equation $(PCC1)$ reduces to

$$\frac{n+1}{2} = \frac{n_1+1}{2}.$$

However this implies $n = n_1$, which is a contradiction.

Case 10: $n_1 \equiv 1,\ n_2 \equiv 1,\ n_3 \equiv 3$ *(mod 4)*

As $n \equiv n_1 + n_2 + n_3 \pmod 4$ then $n \equiv 1 \pmod 4$ and therefore equation $(PCC1)$ reduces to

$$\frac{n+1}{2} = \frac{n_1+1}{2} + n_2$$

Therefore $n = n_1 + 2n_2$. Furthermore equation $(PCC2)$ reduces to

$$-\frac{1}{8}(n-1)^2 = -\frac{1}{8}(n_1-1)^2 - n_2(n_1+1) - \frac{1}{2}n_2(n_2-1).$$

By substituting $n = n_1 + 2n_2$ into the reduced equation $(PCC2)$ and multiplying both sides by 8 we obtain

$$-(n_1 + 2n_2 - 1)^2 = -(n_1-1)^2 - 8n_2(n_1+1) - 4n_2(n_2-1).$$

This simplifies to

$$\begin{aligned}
0 &= -(n_1+2n_2-1)^2-(-(n_1-1)^2-8n_2(n_1+1)-4n_2(n_2-1)) \\
0 &= -(n_1+2n_2-1)^2+(n_1-1)^2+8n_2n_1+8n_2+4n_2^2-4n_2 \\
0 &= -(n_1-1)^2-2(2n_2)(n_1-1)-4n_2^2+(n_1-1)^2+8n_2n_1+4n_2+4n_2^2 \\
0 &= -2(2n_2)(n_1-1)+8n_2n_1+4n_2 \\
0 &= -4n_2n_1+4n_2+8n_2n_1+4n_2 \\
0 &= 4n_2n_1+8n_2
\end{aligned}$$

As $n_2 \geq 3$, this equation has no solutions, which is a contradiction.

Case 11: $n_1 \equiv 1$, $n_2 \equiv 2$, $n_3 \equiv 3$ *(mod 4)*

As $n \equiv n_1+n_2+n_3 \pmod 4$ then $n \equiv 2 \pmod 4$ and therefore equation $(PCC1)$ reduces to

$$0 = \frac{n_1+1}{2}.$$

Therefore $n_1 = -1$, which is a contradiction as $n_1 > 0$.

Case 12: $n_1 \equiv 1$, $n_2 \equiv 3$, $n_3 \equiv 3$ *(mod 4)*

As $n \equiv n_1+n_2+n_3 \pmod 4$ then $n \equiv 3 \pmod 4$ and by Lemma 4.2.14, $D(P_n,-1) = 1$. However by Lemma 4.2.10 and Lemma 4.2.14, $D(P_{n_1},-1) = -1$ and $D(C_{n_2},-1) = D(C_{n_3},-1) = -1$ so $D(G,-1) = -1$, which is a contradiction.

Case 13: $n_1 \equiv 2$, $n_2 \equiv 1$, $n_3 \equiv 1$ *(mod 4)*

As $n \equiv n_1+n_2+n_3 \pmod 4$ then $n \equiv 0 \pmod 4$ and by Lemma 4.2.14, $D(P_n,-1) = 1$. However by Lemma 4.2.10 and Lemma 4.2.14, $D(P_{n_1},-1) = -1$ and $D(C_{n_2},-1) = D(C_{n_3},-1) = -1$ so $D(G,-1) = -1$, which is a contradiction.

Case 14: $n_1 \equiv 2$, $n_2 \equiv 1$, $n_3 \equiv 2$ *(mod 4)*

As $n \equiv n_1+n_2+n_3 \pmod 4$ then $n \equiv 1 \pmod 4$ and therefore equation $(PCC1)$ reduces to

$$\frac{n+1}{2} = n_2,$$

and equation $(PCC2)$ reduces to

$$-\frac{1}{8}(n-1)^2 = \frac{1}{8}(n_1+2)^2 + \frac{1}{4}n_3(n_3+2) - \frac{1}{2}n_2(n_2-1).$$

Therefore $n = 2n_2 - 1$. As $n = n_1+n_2+n_3$ then $n_2 = n_1+n_3+1$ and $n = 2n_1+2n_3+1$. So by substituting this into the reduced equation $(PCC2)$ and multiplying both sides by 8 we obtain

$$-(2n_1+2n_3)^2 = (n_1+2)^2 + 2n_3(n_3+2) - 4(n_1+n_3+1)(n_1+n_3),$$

which simplifies to

$$\begin{aligned}
-4(n_1+n_3)^2 &= (n_1+2)^2 + 2n_3(n_3+2) - 4(n_1+n_3)^2 - 4(n_1+n_3) \\
0 &= (n_1+2)^2 + 2n_3(n_3+2) - 4(n_1+n_3) \\
0 &= n_1^2 + 4n_1 + 4 + 2n_3^2 + 4n_3 - 4n_1 - 4n_3 \\
0 &= n_1^2 + 4 + 2n_3^2.
\end{aligned}$$

As $n_3 \geq 3$, this equation has no solutions, which is a contradiction.

Case 15: *$n_1 \equiv 2$, $n_2 \equiv 2$, $n_3 \equiv 2$ (mod 4)*

As $n \equiv n_1 + n_2 + n_3 \pmod 4$ then $n \equiv 2 \pmod 4$ and therefore equation $(PCC2)$ reduces to

$$\frac{1}{8}(n+2)^2 = \frac{1}{8}(n_1+2)^2 + \frac{1}{4}n_3(n_3+2) + \frac{1}{4}n_2(n_2+2).$$

and equation $(PCC3)$ reduces to

$$-\frac{1}{16}n^3 + \frac{1}{4}n = -\frac{1}{16}n_1^3 + \frac{1}{4}n_1 - \frac{3}{16}n_3^3 + \frac{3}{4}n_3 - \frac{3}{16}n_2^3 + \frac{3}{4}n_2.$$

We will now substitute $n = n_1 + n_2 + n_3$ into the reduced equation $(PCC2)$

$$-n_2^2 - n_3^2 + 2n_1n_2 + 2n_1n_3 + 2n_2n_3 = 0.$$

Therefore if we isolate for n_1 we obtain

$$n_1 = \frac{(n_2 - n_3)^2}{2(n_2 + n_3)}.$$

By substituting this and $n = n_1 + n_2 + n_3$ into the reduced equation $(PCC3)$, multiplying by $-64n_2 - 64n_3$, and simplifying we obtain

$$n_2^4 - 8n_2^3 n_3 + 30n_2^2 n_3^2 - 8n_2 n_3^3 + n_3^4 + 32n_2^2 + 64n_2 n_3 + 32n_3^2 = 0$$

We now substitute $n_3 = k - n_2$ into the equation above to obtain

$$48n_2^4 - 96kn_2^3 + 60k^2 n_2^2 - 12k^3 n_2 + k^4 + 32k^2 = 0.$$

With the help of Maple we found the solution

$$n_2 = \frac{1}{2}k \pm \frac{1}{12}\sqrt{18k^2 \pm 6k\sqrt{-3k^2 - 384}}.$$

Therefore n_2 is real only if $-3k^2 - 384 \geq 0$. However $-3k^2 - 384 < 0$ and we have no real solutions for n_2, which is a contradiction.

Case 16: $n_1 \equiv 2,\ n_2 \equiv 1,\ n_3 \equiv 3$ *(mod 4)*

As $n \equiv n_1+n_2+n_3 \pmod 4$ then $n \equiv 2 \pmod 4$ and by Lemma 4.2.15, $D'(P_n, -1) = 0$. However by Lemma 4.2.11 and Lemma 4.2.15 $D'(G, -1) = n_2$, so $n_2 = 0$, which is a contradiction as $n_2 \geq 3$.

Case 17: $n_1 \equiv 2,\ n_2 \equiv 2,\ n_3 \equiv 3$ *(mod 4)*

As $n \equiv n_1+n_2+n_3 \pmod 4$ then $n \equiv 3 \pmod 4$ and by Lemma 4.2.14, $D(P_n, -1) = 1$. However by Lemma 4.2.10 and Lemma 4.2.14, $D(P_{n_1}, -1) = -1$ and $D(C_{n_2}, -1) = D(C_{n_3}, -1) = -1$ so $D(G, -1) = -1$, which is a contradiction.

Case 18: $n_1 \equiv 2,\ n_2 \equiv 3,\ n_3 \equiv 3$ *(mod 4)*

As $n \equiv n_1+n_2+n_3 \pmod 4$ then $n \equiv 0 \pmod 4$ and by Lemma 4.2.14, $D(P_n, -1) = 1$. However by Lemma 4.2.10 and Lemma 4.2.14, $D(P_{n_1}, -1) = -1$ and $D(C_{n_2}, -1) = D(C_{n_3}, -1) = -1$ so $D(G, -1) = -1$, which is a contradiction.

Case 19: $n_1 \equiv 3$, $n_2 \equiv 1$, $n_3 \equiv 1$ *(mod 4)*

As $n \equiv n_1 + n_2 + n_3 \pmod 4$ then $n \equiv 1 \pmod 4$ and by Lemma 4.2.14, $D(P_n, -1) = -1$. However by Lemma 4.2.10 and Lemma 4.2.14, $D(P_{n_1}, -1) = 1$ and $D(C_{n_2}, -1) = D(C_{n_3}, -1) = -1$ so $D(G, -1) = 1$, which is a contradiction.

Case 20: $n_1 \equiv 3$, $n_2 \equiv 1$, $n_3 \equiv 2$ *(mod 4)*

As $n \equiv n_1 + n_2 + n_3 \pmod 4$ then $n \equiv 2 \pmod 4$ and by Lemma 4.2.14, $D(P_n, -1) = -1$. However by Lemma 4.2.10 and Lemma 4.2.14, $D(P_{n_1}, -1) = 1$ and $D(C_{n_2}, -1) = D(C_{n_3}, -1) = -1$ so $D(G, -1) = 1$, which is a contradiction.

Case 21: $n_1 \equiv 3$, $n_2 \equiv 2$, $n_3 \equiv 2$ *(mod 4)*

As $n \equiv n_1 + n_2 + n_3 \pmod 4$ then $n \equiv 3 \pmod 4$ and therefore equation $(PCC1)$ reduces to

$$-\frac{n+1}{2} = -\frac{n_1+1}{2}$$

However this implies $n = n_1$, which is a contradiction.

Case 22: $n_1 \equiv 3$, $n_2 \equiv 1$, $n_3 \equiv 3$ *(mod 4)*

As $n \equiv n_1 + n_2 + n_3 \pmod 4$ then $n \equiv 3 \pmod 4$ and therefore equation $(PCC1)$ reduces to

$$-\frac{n+1}{2} = -\frac{n_1+1}{2} - n_2.$$

Therefore $n = n_1 + 2n_2$. Furthermore equation $(PCC2)$ reduces to

$$\frac{1}{8}(n-3)(n+1) = \frac{1}{8}(n_1-3)(n_1+1) + n_2(n_1+1) + \frac{1}{2}n_2(n_2-1).$$

By substituting $n = n_1 + 2n_2$ into the reduced equation $(PCC2)$ and multiplying both sides by 8 we obtain

$$(n_1 + 2n_2 - 3)(n_1 + 2n_2 + 1) = (n_1 - 3)(n_1 + 1) + 8n_2(n_1 + 1) + 4n_2(n_2 - 1),$$

which simplifies to

$$4n_2n_1 + 8n_2 = 0.$$

As $n_2 \geq 0$, this equation has no non-negative solutions, which is a contradiction.

Case 23: $n_1 \equiv 3$, $n_2 \equiv 2$, $n_3 \equiv 3$ *(mod 4)*

As $n \equiv n_1 + n_2 + n_3 \pmod 4$ then $n \equiv 0 \pmod 4$ and therefore equation $(PCC1)$ reduces to

$$0 = -\frac{n_1 + 1}{2}.$$

Therefore $n_1 = -1$, which is a contradiction as $n_1 > 0$.

Case 24: $n_1 \equiv 3$, $n_2 \equiv 3$, $n_3 \equiv 3$ *(mod 4)*

As $n \equiv n_1 + n_2 + n_3 \pmod 4$ then $n \equiv 1 \pmod 4$ and by Lemma 4.2.14, $D(P_n, -1) = -1$. However by Lemma 4.2.10 and Lemma 4.2.14, $D(P_{n_1}, -1) = 1$ and $D(C_{n_2}, -1) = D(C_{n_3}, -1) = -1$ so $D(G, -1) = 1$, which is a contradiction.

As each of the 24 cases result in a contradiction, G is not a disjoint union of H and two cycles, where $H \in \{P_{n_1}, P'_{n_1}\}$. Finally, we conclude G has no cycle components and $G \in \{P_n, P'_n\}$.

$\square$

Chapter 5

Conclusion

The focus of our work has been on coefficients and equivalence classes for domination polynomials. In Chapter 3 we gave methods to bound the coefficients of $D(G, x)$ for given G. This led to lower bounds on the coefficients of $D(G, x)$ for all connected G. We also found closed formulas for $d(G, n-3)$, where G is a general graph, and $d(G, n-4)$, where G is in the new collection of graphs $G^2(m)$. This helped us determine the domination equivalence class of P_n in Chapter 4. Also in Chapter 4 we constructed $\mathcal{D}$-equivalent graphs by connecting smaller $\mathcal{D}$-equivalent graphs to another graph. There are many open questions and directions for future research. We present a discussion of these here.

In Section 3.1 we showed for all connected G of order n and each i, there exists a k_i such that $d(G, i)$ is bounded below by $d(S_{n,k_i}, i)$. Recall S_{n,k_i} is the disjoint union of k_i star graphs, where the n vertices are roughly divided evenly amongst each star graph. That is, the order of two of these differs by at most one. We showed the lower bound of $d(G, i)$ was achieved by the lower bound of $d(S_{n,k_i}, i)$ for $\lceil \frac{n}{2} \rceil \leq k_i \leq n$. This is an polynomial time algorithm to find the lower bound of $d(G, i)$ for each i. Rather than going through each possible k_i, can we determine which k_i makes $d(S_{n,k_i}, i)$ the lower bound? See Table 5.1 for the k_{n-j} which gives a lower bound for $d(G, n-j)$. The boldfaced entries indicate there were multiple k_{n-j} which will give a lower bound for $d(G, n-j)$. If there were multiple k_{n-j} which will give a lower bound for $d(G, n-j)$, we only stated the lowest k_{n-j}.

For $j - 2$, we can show $d(G, n-2)$ is bounded below when $k_{n-2} = 1$. Recall from Theorem 2.2.5, for connected graphs $d(G, n-2) = \binom{n}{2} - t + s$ where t is the number of leaves and s is the number of K_2 components. Therefore the graph with no K_2 components and the most leaves (i.e. a star) gives a lower bound for $d(G, n-2)$. For n up to 11, there seems to be a simple pattern, $k_{n-j} = j - 1$. However for larger n it becomes less clear. This leads us to our first question.

$n\backslash j$	2	3	4	5	6	7	8	9
3	1							
4	1	2						
5	1	2						
6	1	2	3					
7	1	2	3					
8	1	2	3	4				
9	1	2	3	4				
10	1	2	3	4	5			
11	1	**2**	3	4	5			
12	1	3	**3**	4	5	6		
13	1	3	4	4	4	6		
14	1	3	4	**4**	5	**5**	7	
15	1	3	4	5	5	5	7	
16	1	3	4	5	5	6	8	8

Table 5.1: The k_{n-j} which minimizes $d(S_{n,k_{n-j}}, n-j)$ and therefore bounds below $d(G, n-j)$.

Question 1: Given n and i, can we determine k_i such that $d(S_{n,k_i}, i) \leq d(G, i)$ for all connected G of order n?

We are also interested in bounding other families of graphs. Trees would share the same lower bounds as connected graphs; however, their upper bounds would be much lower. It seems for each tree T of order n, $d(T, i)$ is bounded above by $d(P_n, i)$ for i closer to n and $d(K_{1,n-1}, i)$ otherwise. Table 5.2 shows the upper bound on $d(T, n-j)$ trees up to order 11.

For comparison to Table 5.2 we also give a table of the upper bounds on coefficients of domination polynomials for all graphs up to order 11. Note these are just the coefficients of the complete graphs as every non-empty subset is a dominating set.

Question 2: Given n and i, how can we bound $d(T, i)$ for all trees T of order n?

$n\backslash j$	1	2	3	4	5	6	7	8	9	10	11
3	1	3	1								
4	1	4	4	1							
5	1	5	8	4	1						
6	1	6	13	10	5	1					
7	1	7	19	22	15	6	1				
8	1	8	26	40	35	21	7	1			
9	1	9	34	65	70	56	28	8	1		
10	1	10	43	98	126	126	84	36	9	1	
11	1	11	53	140	211	252	210	120	45	10	1

Table 5.2: The upper bound of $d(T, n-j)$ for a tree T of order n

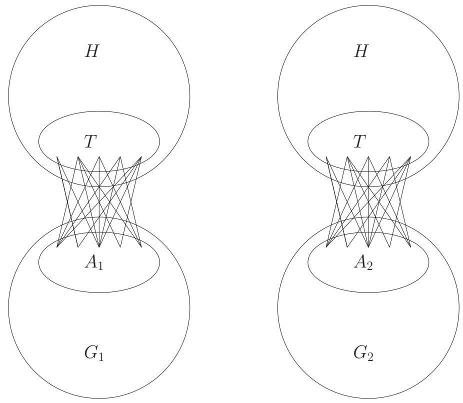
$n\backslash j$	1	2	3	4	5	6	7	8	9	10	11
3	3	3	1								
4	4	6	4	1							
5	5	10	10	5	1						
6	6	15	20	15	6	1					
7	7	21	35	35	21	7	1				
8	8	28	56	70	56	28	8	1			
9	9	36	84	126	126	84	36	9	1		
10	10	45	120	210	252	210	120	45	10	1	
11	11	55	165	330	462	462	330	165	55	11	1

Table 5.3: The upper bound of $d(G, n-j)$ for any graph G of order n

In Section 3.1 we also gave two algorithms to bound the coefficients of $D(G, x)$ of a given G. Algorithm 1 removed edges from the neighbourhood of a vertex to leave a disjoint union of stars. Algorithm 2 adds edges to the neighbourhood of some chosen to obtain a disjoint union of complete graphs. However in both cases we arbitrarily choose the vertices. Naturally this leads us to our next questions.

Question 3: Is there an optimal way to choose vertices in Algorithm 2 and Algorithm 1?

In Section 4.1 we gave conditions to make $H_1 \sim H_2$ in Figure 5.1. In each case $G_1 \sim G_2$ and for each edge from A_1 to T there was a corresponding edge from A_2 to T. However in Theorem 4.1.5, T was domination covered and there was a bijection ϕ from the subsets of A_1 to A_2 such that $p_B(G_1) = p_{\phi(B)}(G_2)$ for every $B \subseteq A_1$. In Theorem 4.1.6, T was a clique and G_1 and G_2 were specified graphs. This brings us to our next question.



(a) H_1 (b) H_2

Figure 5.1: A representation of Theorem 4.1.5

Question 4: Can we determine all conditions to make $H_1 \sim H_2$, as shown in Figure 5.1, given $G_1 \sim G_2$?

In Section 4.2 we showed $[P_n] = \{P_n, P_n'\}$ for $n \geq 9$ where P_n' is a copy of P_n with an edge added between its stems. We did so by using the following steps

- Show $D(P_n, -2) \neq 0$ for sufficiently large n.
- Show if $G \sim P_n$ then $G = H \cup C$, where $H \in [P_k]$ for $k \leq n$ and C is a disjoint union of cycles, by using the highest coefficients of $D(P_n, x)$
- Limiting the number of cycles in G by using $ord_3(D(P_n, -3))$ and the domination number of paths and cycles.
- Creating a system of equations by evaluating $D(P_n, x)$ and its first three derivatives at $x = -1$ and finding a contradiction.

We believe this process can be used for other graphs in $G^2(m)$ and can be a source for future research. However it has its limitations. Suppose we wish to determine $[G]$ for some graph $G \in G^2(m)$. If $D(G, -2) = 0$, we may not be able to determine $[G]$. If $D(G, -2) \neq 0$, we can easily determine the number of leaves in G; however, we have no way of determining the number of stems other than it is bounded above by the number of leaves. If G has many leafs, we must go through at least as many cases for the number of stems. Furthermore the domination polynomial does not encode the number of stems. In Figure 5.2 there are two graphs which are $\mathcal{D}$-equivalent but with a different number of stems. Although our process is still effective when $D(G,-2) \neq 0$ and G has few leaves, the final step of evaluating $D(G,x)$ at $x=-1$ is cumbersome. For paths, $D(P_n,-1)$ is determined modulus 4, meaning we needed 36 individual cases to finally determine $[P_n]$. However the purpose of those cases were to show if $H \sim P_n$ then H could not have any cycle components. This leads us to our next question.

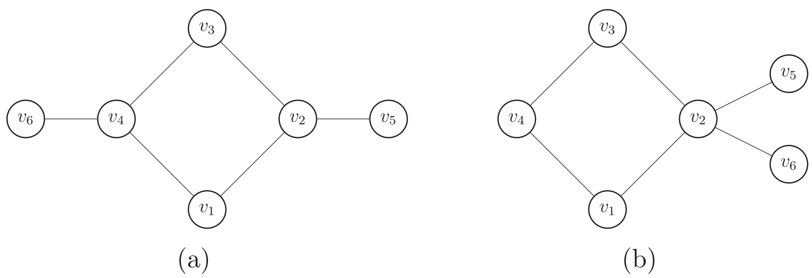


Figure 5.2: Two graphs with domination polynomial $x^6 + 6x^5 + 13x^4 + 10x^3 + 3x^2$

Question 5: Can we use the domination polynomial to determine if a graph has a component that is a cycle?

Despite its limitations, we still believe we can find more equivalence classes using the process outlined above. In particular for the graphs which take one of the three forms shown in Figure 5.3. This leads us to our final question.

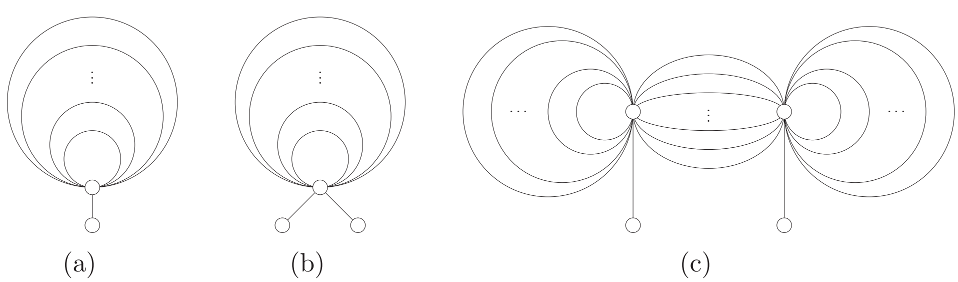


Figure 5.3: Forms of graphs for future research

Question 6: For which other graphs in $G^2(m)$ can we determine the domination equivalence class?

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