REDUCIBILITY RESULTS ON OPERATOR SEMIGROUPS

By

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With gratitude,

to my parents, and to Heydar, my mathematical father.

"tcheh saaz bood keh dar pardeh mizad aan motreb

keh raft 'omr-o hanoozam demaagh por ze havaast

nedaa-y-e 'mehr'-e to doosham dar andaroon daadand

fazaa-y-e sineh-y-e 'jaanam' hanooz por ze sadaast"

aari, ...

"..., keh aatashi keh namirad, hamisheh dar

del-e maast..."

..., for the fire that never dies, always remains in the

heart...

—Hafez, the Persian Poet.

Contents

Acknowledgments

It/ Happened/ Again last/ Night:
Love/ Popped the cork on itself / Splattered my brains/ Across the / Sky.
I imagine now for ages/ Something of Hafiz/ Will appear
To fall like/ Stars.

"The Gift" – versions of Hafiz, the Persian poet. by Daniel Ladinsky.

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...Yet I believe another wonderful day,
And perhaps even a sweeter height of rare, inspired insanity,
O Hafiz, has just begun.

"The Subject Tonight Is Love" – versions of Hafiz, the Persian poet, by Daniel Ladinsky.

Abstract

Everything that can be said, can be said clearly.

–Ludwig Wittgenstein

This thesis focuses on reducibility and triangularizability of collections of linear transformations on a vector space over a general field as well as compact operators on a real or complex Banach space. It consists of three parts.

In part one, we extend triangularization results due to Levitzki, Kolchin, and others. For a given $n > 1$, we characterize all fields F such that Burnside's Theorem holds in $M_n(F)$. We consider irreducible semigroups and F-algebras of matrices in $M_n(K)$ with traces in a subfield F. We prove Wedderburn-Artin type theorems for such F-algebras of matrices. We use our main results to generalize some other classical triangularization results, e.g., those due to Guralnick, Kaplansky, McCoy, and others, and present applications in finite dimensions over a general field. We also consider semigroups and F-algebras of compact operators on an arbitrary Banach space and $\mathcal{C}_p$ class operators on an arbitrary Hilbert space. We present new proofs of certain classical theorems as well as some new triangularization results in this infinite-dimensional setting.

In part two, we show that triangularizability is stable under certain limit operations. This is then used to prove an invariant subspace theorem for certain bounded operators. We also prove that in finite dimensions reducibility remains intact under these limit operations provided the underlying space is complex or it is real with odd dimension.

In part three, we are interested in extending the triangularization theory to collections of matrices on division rings. We give a new proof of a well-known Theorem of Levitzki and prove an analogue of one of the main results of part one on division rings. We define the concept of permutability of trace on a collection of matrices over a division ring and prove that under a slight condition on the characteristic of the division ring, every irreducible family on which trace is permutable is commutative.

Chapter 1

Introduction

But leave the Wise to wrangle, and with me
The Quarrel of the Universe let be:
And, in some corner of the Hubbub coucht,
Make game of that which makes as much of thee.

–Khayyam, the Persian Mathematician, Astronomer, Philosopher, and Poet.
Rendered into English verse by Edward Fitzgerald.

This thesis deals with triangularization and reducibility results on operator semigroups. We are particularly interested in reducibility and triangularizability results on collections of linear transformations on finite-dimensional vector spaces over general fields and compact operators on real or complex Banach spaces, and also their applications. Reducibility and triangularizability results in some sense shed light on the structure of linear transformations. They also have applications in other areas of mathematics such as representation theory of groups, semigroups and algebras (e.g., Theorem 2.4.1 and Theorem 2.4.2). We refer the reader to [RR] page 25 and 189 respectively for historical comments on both finite- and infinite-dimensional triangularization theory.

1.1 Some basic concepts and lemmas

We commence by recalling some definitions and standard notations. Throughout this thesis, unless otherwise stated, $\mathcal{X}$ stands for a separable real or complex Banach space. As is usual, by $\mathbb{F}$ we mean $\mathbb{R}$ or $\mathbb{C}$. The terms *subspace* and *operator* or *linear operator* will, respectively, be used to describe a closed subspace of a Banach space $\mathcal{X}$ and a bounded linear operator on $\mathcal{X}$. The subspaces $\{0\}$ and $\mathcal{X}$ are called the trivial subspaces of $\mathcal{X}$.

If F is a field and $\mathcal{V}$ is a finite-dimensional vector space over F, then we use the term *linear transformation* to describe a vector space homomorphism on $\mathcal{V}$; and we use $\mathcal{L}(\mathcal{V})$ to denote the set (in fact the algebra) of linear transformations on $\mathcal{V}$. We use $\mathcal{B}(\mathcal{X})$ to denote the set (in fact the algebra) of bounded operators on $\mathcal{X}$; $\mathcal{B}_0(\mathcal{X})$ is used to denote the set (in fact the ideal) of compact operators on $\mathcal{X}$, $\mathcal{B}_{00}(\mathcal{X})$ is used to denote the set (in fact the ideal) of finite-rank operators on $\mathcal{X}$. We note that if $\mathcal{X}$ is a finite-dimensional real or complex Banach space, then $\mathcal{L}(\mathcal{X}) = \mathcal{B}(\mathcal{X}) = \mathcal{B}_0(\mathcal{X})$, and that every linear subspace of $\mathcal{X}$ is necessarily closed.

By a *subalgebra* $\mathcal{A}$ in $\mathcal{B}(\mathcal{X})$ (resp. $\mathcal{L}(\mathcal{V})$, $M_n(F)$), we mean a subring of $\mathcal{B}(\mathcal{X})$ (resp. $\mathcal{L}(\mathcal{V})$, $M_n(F)$) that is closed under scalar multiplication by the elements of $\mathbb{F}$ (resp. of the field F). Note that a subalgebra of $\mathcal{B}(\mathcal{X})$ (resp. $\mathcal{L}(\mathcal{V})$, $M_n(F)$) is not necessarily unital.

For a collection $\mathcal{F}$ of operators on $\mathcal{X}$ (resp. transformations on $\mathcal{V}$), the symbol $\mathcal{F}'$ is used to denote the *commutant* of $\mathcal{F}$, i.e., the set of all operators (resp. transformations) that commute with all elements of $\mathcal{F}$ (more precisely, $\mathcal{F}' := \{T \in \mathcal{B}(\mathcal{X})$ (resp. $T \in \mathcal{L}(\mathcal{V})) : ST = TS$ for all $S \in \mathcal{F}\}$). It is plain that $\mathcal{F}'$ is a unital subalgebra of $\mathcal{B}(\mathcal{X})$ (resp. $\mathcal{L}(\mathcal{V})$). A subspace $\mathcal{M}$ is *invariant* for a collection $\mathcal{F}$ of bounded operators (resp. linear transformations) if $T\mathcal{M} \subseteq \mathcal{M}$ for all $T \in \mathcal{F}$; $\mathcal{M}$ is *hyperinvariant* for a collection $\mathcal{F}$ of bounded operators if $T\mathcal{M} \subseteq \mathcal{M}$ for all $T \in \mathcal{F} \cup \mathcal{F}'$. A collection $\mathcal{F}$ of bounded operators (resp. linear transformations) on a space of dimension greater than one is called *reducible* if it has a nontrivial invariant subspace. In case the dimension of the underlying space is one, then the collection $\mathcal{F}$ is called *reducible* if $\mathcal{F} = \{0\}$. This definition is slightly unconventional, but it simplifies most of the statements in what follows. A collection $\mathcal{F}$ of bounded operators on $\mathcal{X}$ (resp. linear transformations on finite-dimensional $\mathcal{V}$) is called *transitive* if the set $\{Tx : T \in \mathcal{F}\}$ is dense in $\mathcal{X}$ (resp. is equal to $\mathcal{V}$) whenever $x \in \mathcal{X}$ (resp. $x \in \mathcal{V}$) is a nonzero vector. It is easily seen that for an algebra of operators (resp. linear transformations) the two concepts of transitivity and irreducibility coincide. For a collection $\mathcal{C}$ of vectors, the symbol $\langle\mathcal{C}\rangle$ denotes *the (not necessarily closed) linear manifold spanned by* $\mathcal{C}$.

A collection $\mathcal{F}$ of operators (resp. linear transformations) is called *simultaneously triangularizable* or simply *triangularizable* if there exists a maximal chain of subspaces of $\mathcal{X}$ each of which is invariant for $\mathcal{F}$. In case the underlying space is finite-dimensional, it is easily seen that triangularizability of a family of linear transformation is equivalent to the existence of a basis for the vector space such that all transformations in the family have upper triangular matrix representation with respect to that basis.

It is plain that a family $\mathcal{F}$ of linear operators (resp. linear transformations) is triangularizable iff $\mathrm{Sem}(\mathcal{F})$, *the semigroup generated by* $\mathcal{F}$, is triangularizable; or iff $\mathrm{Alg}(\mathcal{F})$, *the algebra generated by* $\mathcal{F}$, is triangularizable.

Also we note that for every family $\mathcal{F}$ of bounded operators (resp. linear transformations)

$$\mathcal{F}' = (\mathrm{Alg}(\mathcal{F}))' = (\mathrm{Sem}(\mathcal{F}))'.$$

Thus $\mathcal{F}$ has a non-trivial hyperinvariant subspace iff $\mathrm{Sem}(\mathcal{F})$ does, or iff $\mathrm{Alg}(\mathcal{F})$ does.

It is known that if $\mathcal{F} \subseteq \mathcal{B}_0(\mathcal{X})$ is a triangularizable family of compact operators, then $AB - BA$ is quasinilpotent for all $A, B \in \mathrm{Alg}(\mathcal{F})$. Recall that an operator T is called *quasinilpotent* if $\sigma(T) = \{0\}$ where $\sigma(T)$ denotes the spectrum of T.

A collection $\mathcal{F}$ of linear transformations in $\mathcal{L}(\mathcal{V})$ is called *absolutely irreducible* if $\mathrm{Alg}(\mathcal{F}) = \mathcal{L}(\mathcal{V})$. It is plain that an absolutely irreducible family of transformations in $\mathcal{L}(\mathcal{V})$ is irreducible and its commutant consists of scalars. In view of Theorem 2.2.21 below, the two concepts of irreducibility and absolute irreducibility are the same for collections of linear transformations in $\mathcal{L}(\mathcal{V})$ if and only if for each k dividing $\dim\mathcal{V}$ with $k > 1$ there is no irreducible polynomial of degree k over the ground field F. In particular, a collection $\mathcal{F}$ in $M_n(F)$ $(n > 1)$ is absolutely irreducible if and only if it is irreducible as a collection in $M_n(\overline{F})$ where $\overline{F}$ denotes the algebraic closure of F. Note that if $\dim\mathcal{V} = 1$, then a collection $\mathcal{F}$ of linear transformations in $\mathcal{L}(\mathcal{V})$ irreducible if and only if it is absolutely irreducible.

We start off with an elementary lemma.

Lemma 1.1.1. *Let $\mathcal{V}$ be an n-dimensional vector space over a field F, and let $T \in \mathcal{L}(\mathcal{V})$. The following are equivalent.*

(i) The transformation T is irreducible.

(ii) Every nonzero $\alpha \in \mathcal{V}$ is a cyclic vector for T.

(iii) The characteristic polynomial of T is irreducible over F.

(iv) The algebra

$$F[T] := \{\sum_{i=0}^{n-1} c_i T^i : c_i \in F, T^0 := I\}$$

is an n-dimensional extension field of F. In particular, $F[T] = F(T)$.

Furthermore, if any of the above conditions holds, then $S \in \{T\}'$ iff S is a polynomial in T.

Proof. "(i)$\Longrightarrow$ (ii)" It follows from the Cayley-Hamilton Theorem that

$$F[T] = \{\sum_{i=0}^{n-1} c_i T^i : c_i \in F, T^0 := I\}$$

is a subalgebra of $\mathcal{L}(\mathcal{V})$. Suppose that $\alpha \in \mathcal{V}$ is nonzero. Obviously, $\mathcal{W} := F[T]\alpha$ is a nonzero invariant subspace for the irreducible transformation T. So $\mathcal{W} = \mathcal{V}$. That is $\{\alpha, T\alpha, ..., T^{n-1}\alpha\}$ spans the n-dimensional vector space $\mathcal{V}$. Thus $\{\alpha, T\alpha, ..., T^{n-1}\alpha\}$ is a basis for $\mathcal{V}$. i.e., α is a cyclic vector for T.

"(ii)$\Longrightarrow$ (iii)" Suppose not. Thus the characteristic polynomial of T, ch(T), is reducible over F. So there are polynomials f and g, of degree at least one, such that ch$(T) = fg$. It follows from the Cayley-Hamilton Theorem that $f(T)g(T) = 0$. We note that $f(T) = 0$ or $g(T) = 0$ contradicts the hypothesis that $\{\alpha, T\alpha, ..., T^{n-1}\alpha\}$ is a basis, hence an independent set, for every nonzero α. So $f(T) \neq 0$ and $g(T) \neq 0$. Now $f(T)g(T) = 0$ implies that either $f(T)$ or $g(T)$ is not invertible. So without loss of generality we may assume that $f(T)$ is not invertible and that $f(T) \neq 0$. This plainly implies that ker$(f(T))$ is a nontrivial invariant subspace for T. So every nonzero element of ker$(f(T))$ cannot be a cyclic vector for T, contradicting the hypothesis.

"(iii)$\Longrightarrow$ (iv)" Again it follows from the Cayley-Hamilton Theorem that $F[T]$ is indeed an algebra. First we note that the hypothesis that ch(T) is irreducible over F

along with the Cayley-Hamilton Theorem implies that $\mathrm{ch}(T) = \mathrm{m}_T$ where m_T denotes that minimal polynomial of T. This obviously in turn implies that $\{I, T, ..., T^{n-1}\}$ is independent. We only need to show that every nonzero element of $F[T]$ is invertible and that the inverse is indeed in $F[T]$. To see this, let $f(T) = f_0 + f_1 T + ... + f_{n-1} T^{n-1}$ be an arbitrary nonzero element of $F[T]$. Since $\mathrm{ch}(T) = \mathrm{m}_T$, we conclude that $f(T) \neq 0$, for $\deg(f) < \deg(\mathrm{m}_T) = n$. Thus $\ker(f(T)) \neq \mathcal{V}$. On the other hand, $\ker(f(T))$ is an invariant subspace for T. So if $\ker(f(T)) \neq 0$, then $\mathrm{ch}(T|_{\ker(f(T))})$ divides $\mathrm{ch}(T)$ which contradicts the hypothesis that $\mathrm{ch}(T)$ is irreducible over F. So $\ker(f(T)) = 0$, and hence $f(T)$ is invertible. That the inverse of $f(T)$ is in $F[T]$ follows from the fact that the inverse of any linear transformation is a polynomial in that transformation. We have shown that $F[T] = F(T)$ is an n-dimensional extension field of F.

"(iv)$\Longrightarrow$ (i)" Suppose not. Let $\mathcal{M}$ be a nontrivial invariant subspace for T. It is plain that $\mathrm{ch}(T|_{\mathcal{M}}) = f$ divides $\mathrm{ch}(T)$. So there is a polynomial g with $\deg(g) \geq 1$ such that $\mathrm{ch}(T) = fg$. Now it follows from the Cayley-Hamilton Theorem that $f(T)g(T) = 0$. This is impossible, for $f(T)$ and $g(T)$ are nonzero elements of $F[T] = F(T)$. Therefore, T is irreducible.

To see the rest, it suffices to show that if $S \in \{T\}'$, then S is a polynomial in T. To this end, suppose that $S \in \{T\}'$ and that $0 \neq \alpha \in \mathcal{V}$ is a cyclic vector for T. Hence $\{\alpha, T\alpha, ..., T^{n-1}\alpha\}$ is a basis for $\mathcal{V}$. Therefore, we can write

$$S\alpha = c_0\alpha + ... + c_{n-1}T^{n-1}\alpha,$$

where $c_i \in F$ for each $i = 0, ..., n-1$. Hence

$$(S - (c_0 I + ... + c_{n-1}T^{n-1}))\alpha = 0.$$

Now since $S \in \{T\}'$ we can write

$$(S - (c_0 I + ... + c_{n-1}T^{n-1}))T^k\alpha = T^k(S - (c_0 I + ... + c_{n-1}T^{n-1}))\alpha = 0,$$

for each $k = 0, ..., n-1$. As $\{\alpha, T\alpha, ..., T^{n-1}\alpha\}$ is a basis for $\mathcal{V}$, we see that $S - (c_0 I + ... + c_{n-1}T^{n-1}) = 0$. i.e., $S = c_0 I + ... + c_{n-1}T^{n-1}$, finishing the proof. □

If $\mathcal{S}$ is a multiplicative semigroup, a subset $\mathcal{J}$ of $\mathcal{S}$ is called a *semigroup ideal* of $\mathcal{S}$ if $JS, SJ \in \mathcal{J}$ whenever $J \in \mathcal{J}$ and $S \in \mathcal{S}$. In what follows, we make frequent use of the elementary well-known lemma below.

Lemma 1.1.2. *(i) Let $\mathcal{V}$ be a finite-dimensional vector space over a field F, and $\mathcal{S}$ a semigroup in $\mathcal{L}(\mathcal{V})$. If $\mathcal{S}$ is (absolutely) irreducible, then so is every nonzero semigroup ideal of $\mathcal{S}$.*

(ii) Let $\mathcal{X}$ be a real or complex Banach space, and $\mathcal{S}$ a semigroup in $\mathcal{B}(\mathcal{X})$. If $\mathcal{S}$ is irreducible, then so is every nonzero semigroup ideal of $\mathcal{S}$.

Proof. (i) If $\dim \mathcal{V} = 1$, then the assertion trivially holds. So we may assume, with no loss of generality, that $\dim \mathcal{V} > 1$. Let $\mathcal{J}$ be a nonzero semigroup ideal of the semigroup $\mathcal{S}$.

First suppose that $\mathcal{S}$ is irreducible. To show that $\mathcal{J}$ is irreducible, use contradiction. So assume that $\mathcal{M}$ is a nontrivial invariant subspace for the nonzero ideal $\mathcal{J}$. Define

$$\mathcal{M}_1 := \langle \mathcal{J}\mathcal{M} \rangle, \qquad \mathcal{M}_2 := \bigcap_{J \in \mathcal{J}} \ker J.$$

As $\mathcal{J}$ is a semigroup ideal of $\mathcal{S}$ we easily see that both $\mathcal{M}_1$ and $\mathcal{M}_2$ are invariant under the whole semigroup. Two cases to consider: If $\mathcal{M}_1 \neq 0$, then $\mathcal{M}_1$ would be a nontrivial subspace since $\mathcal{M}_1 \subseteq \mathcal{M}$, a contradiction. If $\mathcal{M}_1 = 0$, then $\mathcal{M}_2$ would be nonzero and, on the other hand, $\mathcal{M}_2$ is proper for $\mathcal{J}$ is nonzero. So $\mathcal{M}_2$ would be a nontrivial invariant subspace for $\mathcal{S}$, a contradiction again (See [RR], Lemma 2.1.10, page 29).

Next suppose that $\mathcal{S}$ is absolutely irreducible. We need to show that $\mathrm{Alg}(\mathcal{J}) = \mathcal{L}(\mathcal{V}) = \mathrm{Alg}(\mathcal{S})$. This is obvious for $\mathrm{Alg}(\mathcal{J})$ is a nonzero ideal of $\mathrm{Alg}(\mathcal{S}) = \mathcal{L}(\mathcal{V})$ (here we have used a theorem from elementary algebra that $\mathcal{L}(\mathcal{V})$ is a simple ring).

(ii) The proof is identical to that of (i) except that

$$\mathcal{M}_1 := \overline{\langle \mathcal{J}\mathcal{M} \rangle}, \qquad \mathcal{M}_2 := \bigcap_{J \in \mathcal{J}} \ker J,$$

where $\overline{\langle \mathcal{J}\mathcal{M} \rangle}$ denotes the closure of $\langle \mathcal{J}\mathcal{M} \rangle$ (See [RR], Lemma 8.2.1, page 200). □

Another elementary lemma which we use very frequently is "The Triangularization Lemma" (See [RR], Lemma 1.1.4 and Lemma 7.1.11).

Lemma 1.1.3 (The Triangularization Lemma). *Let $\mathcal{P}$ be a set of properties of families of linear operators (resp. linear transformations) each of which is inherited by quotients. If every family of operators (resp. transformations) satisfying $\mathcal{P}$ on a space of dimension greater than one is reducible, then every family satisfying $\mathcal{P}$ is triangularizable.*

Proof. See [RR], Lemma 1.1.4 and Lemma 7.1.11. □

Chapter 2

On semigroups with traces and spectra in a subfield

With them the seed of Wisdom I did sow,
And with my own hand labour'd it to grow;
And this was all the Harvest that I reap'd–
I came like Water, and like wind I go.

–Khayyam, the Persian Mathematician, Astronomer, Philosopher, and Poet.
Rendered into English verse by Edward Fitzgerald.

2.1 Introduction

In this chapter we consider semigroups of linear transformations acting on finite-dimensional vector spaces over a general field K with traces in a subfield F. We extend a celebrated theorem of Burnside and prove a block matrix representation theorem for irreducible F-algebras of matrices in $M_n(K)$ with traces in F. We generalize some other classical triangularization results and present applications in finite dimensions over a general field. We extend our main result to semigroups of $\mathcal{C}_p$ class operators on a real or complex Hilbert space as well as semigroups of finite-rank operators on a real or complex Banach space. We present new proofs of certain classical theorems as well as some new triangularization results in this infinite-dimensional setting.

2.2 Some Preliminary and Basic Results

Motivated by Lemma 2.1.12 of [RR] we start off with the following lemma.

Lemma 2.2.1. *Let $\mathcal{V}$ be a finite-dimensional vector space over a field F, $\mathcal{S}$ a semigroup in $\mathcal{L}(\mathcal{V})$, and T a nonzero linear transformation in $\mathcal{L}(\mathcal{V})$. If $\mathcal{S}$ is irreducible, then so is $T\mathcal{S}|_{\mathcal{R}}$ where $\mathcal{R} = T\mathcal{V}$ is the range of T.*

Proof. If $\dim \mathcal{V} = 1$, then the assertion trivially holds. So we may assume, with no loss of generality, that $\dim \mathcal{V} > 1$. There are two cases to consider.

(a) $\text{rank}(T) = 1$.

To prove the assertion by contradiction suppose $T\mathcal{S}|_{\mathcal{R}}$ is reducible. Since $\dim \mathcal{R} = 1$ in this case, it follows from definition that $T\mathcal{S}|_{\mathcal{R}} = \{0\}$. Therefore, $T\mathcal{S}T = \{0\}$. Pick a nonzero $x \in \mathcal{V}$ such that $Tx \neq 0$. Now either $\mathcal{S}Tx = \{0\}$ in which case $\langle Tx \rangle$ is a nontrivial invariant subspace for $\mathcal{S}$, or else $\langle \mathcal{S}Tx \rangle$ is a nontrivial invariant subspace for $\mathcal{S}$, because $T\mathcal{S}T = \{0\}$ and $\mathcal{S}$ is a semigroup. This contradicts the hypothesis that $\mathcal{S}$ is irreducible.

(b) $\text{rank}(T) > 1$.

First note that a semigroup $\mathcal{S}$ is irreducible iff the algebra $\mathcal{A}$ generated by the semigroup is irreducible. That being noted, it suffices to prove that $T\mathcal{A}|_{\mathcal{R}}$ is irreducible because every invariant subspace of $T\mathcal{S}|_{\mathcal{R}}$ is invariant for $T\mathcal{A}|_{\mathcal{R}}$ as well. To prove that $T\mathcal{A}|_{\mathcal{R}}$ is irreducible, we use contradiction. Suppose that $T\mathcal{A}|_{\mathcal{R}}$ is reducible. So there exists a nontrivial subspace $\mathcal{M}$ of $\mathcal{R} = T\mathcal{V}$ such that $T\mathcal{A}\mathcal{M} \subseteq \mathcal{M}$. Choose a nonzero $x \in \mathcal{M}$ and note that $T\mathcal{A}x \subseteq \mathcal{M}$. The subspace $\mathcal{A}x$ is an invariant subspace of $\mathcal{A}$. Furthermore it is proper, for $T\mathcal{A}x \subseteq \mathcal{M} \subset \mathcal{R}$. If $\mathcal{A}x = 0$ then $\langle x \rangle$ is a nontrivial invariant subspace for $\mathcal{A}$, otherwise $\mathcal{A}x$ will be a nontrivial invariant subspace for $\mathcal{A}$. So in any event we conclude that $\mathcal{A}$ is reducible, a contradiction. □

Lemma 2.2.2. *Let $\mathcal{V}$ be a finite-dimensional vector space over a field F, $\mathcal{S}$ a semigroup in $\mathcal{L}(\mathcal{V})$, and T a nonzero linear transformation in $\mathcal{L}(\mathcal{V})$. If $\mathcal{S}$ is absolutely irreducible, then so is $T\mathcal{S}|_{\mathcal{R}}$ where $\mathcal{R} = T\mathcal{V}$ is the range of T.*

Proof. Let $\mathcal{A}$ denote the algebra generated by the semigroup $\mathcal{S}$. From definition we have $\mathcal{A} = \mathcal{L}(\mathcal{V})$. In particular, $T \in \mathcal{A} = \langle \mathcal{S} \rangle$. It is now easily seen that

$$\mathcal{L}(\mathcal{R}) \supseteq \mathrm{Alg}(T\mathcal{S}|_{\mathcal{R}}) \supseteq T\mathcal{A}|_{\mathcal{R}} = T\mathcal{L}(\mathcal{V})|_{\mathcal{R}} = \mathcal{L}(\mathcal{R}),$$

completing the proof. □

Recall that a linear transformation T in $\mathcal{L}(\mathcal{V})$ is called idempotent if $T^2 = T$. The corollary below is a quick consequence of the preceding two lemmas.

Corollary 2.2.3. *Let $\mathcal{V}$ be a finite-dimensional vector space over a field F, $\mathcal{S}$ a semigroup in $\mathcal{L}(\mathcal{V})$, and T a nonzero idempotent in $\mathcal{L}(\mathcal{V})$. If $\mathcal{S}$ is (absolutely) irreducible, then so is $T\mathcal{S}T|_{T\mathcal{V}}$.*

Proof. Lemma 2.2.1 and 2.2.2.

Different versions of the following lemma are well-known.

Lemma 2.2.4. *Let K be a field with $\mathrm{ch}(K) = 0$ or $> n$ where $n \in \mathbb{N}$, F a subfield of K, and $A \in M_n(K)$. Then the characteristic polynomial of A, denoted by c_A, is in $F[X]$ iff $\mathrm{tr}(A^k) \in F$ for all $1 \leq k \leq n$.*

Proof. "$\Longrightarrow$" Suppose that λ_k $(1 \leq k \leq n)$ are the eigenvalues of A in the algebraic closure of K. For each $k = 1, ..., n$, let S_k denote the elementary symmetric polynomial in $\lambda_1, ..., \lambda_n$ of degree k, i.e., $S_1 = \lambda_1 + ... + \lambda_n$, ..., $S_n = \lambda_1 ... \lambda_n$; and let T_k denote the sum of the k-th power of λ_k's, i.e., $T_k = \lambda_1^k + ... + \lambda_n^k$. It is well known that

$$T_k - T_{k-1}S_1 + ... + (-1)^{k-1}T_1S_{k-1} + (-1)^k kS_k = 0 \qquad (*)$$

for all for $k = 1, ..., n$. This identity together with the hypothesis that $\mathrm{ch}(K) = 0$ or $>$ n enables us to determine each T_k in terms of the S_j's and vice versa. It is also well known that $\mathrm{tr}(A^k) = T_k$ for all $k \in \mathbb{N}$ and that

$$\mathrm{c}_A = x^n - S_1x^{n-1} + ... + (-1)^n S_n \qquad (*')$$

Now suppose $c_A \in F[X]$. It follows from $(*')$ that $S_k \in F$ for all $k = 1, ..., n$. This fact in turn accompanied by $(*)$ implies that $\mathrm{tr}(A^k) = T_k \in F$ for all $k = 1, ..., n$. (As a matter of fact, using the Cayley-Hamilton Theorem, one can conclude that $\mathrm{tr}(A^k) = T_k \in F$ for all $k \in \mathbb{N}$).

"$\Longleftarrow$" Suppose that $\mathrm{tr}(A^k) = T_k \in F$ for all $k = 1, ..., n$. It follows from $(*)$ that $S_k \in F$ for all $k = 1, ..., n$, and thus by $(*')$, $c_A \in F[X]$. □

It is worth mentioning that Lemma 2.2.5 and Lemma 2.2.6 below are slight generalizations of Lemma 2.1.15 and Theorem 2.1.16 of [RR].

Lemma 2.2.5. *(i) Let F be a field, and $\{\lambda_{i1}, ..., \lambda_{in_i}\}, \{c_{i1}, ..., c_{in_i}\}$ $(n_i \in \mathbb{N}, i = 1, 2)$ subsets of F consisting of distinct nonzero elements and nonzero elements of F respectively and such that*

$$\sum_{j=1}^{n_1} c_{1j}\lambda_{1j}^k = \sum_{j=1}^{n_2} c_{2j}\lambda_{2j}^k,$$

for each $k = m, m+1, ..., m + (n_1 + n_2 - 1)$ where m is a given integer. Then $n_1 = n_2 = n$ and there is a permutation σ on n letters such that $c_{2j} = c_{1\sigma(j)}$ and $\lambda_{2j} = \lambda_{1\sigma(j)}$ for all $j = 1, .., n$.

(ii) Let F be a field with $\mathrm{ch}(F) = 0$ or $> n$, $\{\lambda_{i1}, ..., \lambda_{iN_i}\}$ $(N_i \in \mathbb{N}, N_i \leq n, i = 1, 2)$ subsets of $F \setminus \{0\}$ such that

$$\sum_{j=1}^{N_1} \lambda_{1j}^k = \sum_{j=1}^{N_2} \lambda_{2j}^k,$$

for each $k = m, m+1, ..., m + (n_1 + n_2 - 1)$ where m is a given integer and n_i is the number of distinct elements of $\{\lambda_{i1}, ..., \lambda_{iN_i}\}$ $(i = 1, 2)$. Then $n_1 = n_2$, $N_1 = N_2 = N$ and there is a permutation σ on N letters such that $\lambda_{2j} = \lambda_{1\sigma(j)}$ for all $j = 1, .., N$.

(iii) Let F be a field with $\mathrm{ch}(F) = 0$ or $> n$, $\{\lambda_{i1}, ..., \lambda_{in}\}$ $(n \in \mathbb{N}, i = 1, 2)$ subsets of F such that

$$\sum_{j=1}^{n} \lambda_{1j}^k = \sum_{j=1}^{n} \lambda_{2j}^k,$$

for each $k = m, m+1, ..., m+(n_1+n_2-1)$ where m is a given member of $\mathbb{N} \cup \{0\}$ and n_i is the number of distinct nonzero elements of $\{\lambda_{i1}, ..., \lambda_{in}\}$ $(i = 1, 2)$. Then $n_1 = n_2$

and there is a permutation σ on n letters such that $\lambda_{2j} = \lambda_{1\sigma(j)}$ for all $j = 1, .., n$.

Proof. (i) We prove the assertion by induction on $n_1 + n_2$. If $n_1 + n_2 = 2$, the assertion is easily verified. Suppose that the assertion holds for $n_1 + n_2 < n$, we prove the assertion for $n_1 + n_2 = n > 2$. We have

$$\sum_{j=1}^{n_1} c_{1j}\lambda_{1j}^k - \sum_{j=1}^{n_2} c_{2j}\lambda_{2j}^k = 0,$$

for all $k = m, ..., m + (n_1 + n_2 - 1)$ where $m \in \mathbb{Z}$. That is the nonzero column vector $(c_{11}, ..., c_{1n_1}, -c_{21}, ..., -c_{2n_2})$ is in the kernel of the following $(n_1 + n_2) \times (n_1 + n_2)$ matrix.

$$\begin{pmatrix} \lambda_{11}^m & \cdots & \lambda_{1n_1}^m & \lambda_{21}^m & \cdots & \lambda_{2n_2}^m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_{11}^{m+(n_1+n_2-1)} & \cdots & \lambda_{1n_1}^{m+(n_1+n_2-1)} & \lambda_{21}^{m+(n_1+n_2-1)} & \cdots & \lambda_{2n_2}^{m+(n_1+n_2-1)} \end{pmatrix}.$$

Using the fact that each $\{\lambda_{i1}, ..., \lambda_{in_i}\}$ (i=1, 2) consists of distinct nonzero elements, if necessary after renaming λ_{ij}'s, it follows from Vandermonde's Determinant Formula that $\lambda_{11} = \lambda_{21}$. We claim that $c_{11} = c_{21}$. Suppose not, then $c_{11} - c_{21} \neq 0$, and we can write

$$\sum_{j=2}^{n_1} c_{1j}\lambda_{1j}^k = (c_{11} - c_{21})\lambda_{21}^k + \sum_{j=2}^{n_2} c_{2j}\lambda_{2j}^k,$$

for each $k = m, m+1, ..., m + (n_1 + n_2 - 1)$. Since $(n_1 - 1) + n_2 < n_1 + n_2 = n$, it follows from the induction hypothesis that $n_1 - 1 = n_2$ and that in particular there exists $2 \leq j_0 \leq n_1$ such that $\lambda_{11} = \lambda_{21} = \lambda_{1j_0}$ contradicting the assumption that $\{\lambda_{11}, ..., \lambda_{1n_1}\}$ consists of distinct elements of F. Therefore $c_{11} = c_{21}$. Now we can write

$$\sum_{j=2}^{n_1} c_{1j}\lambda_{1j}^k = \sum_{j=2}^{n_2} c_{2j}\lambda_{2j}^k,$$

for each $k = m, m+1, ..., m+(n_1+n_2-1)$. Since $(n_1 - 1) + (n_2 - 1) < n_1 + n_2 = n$, the induction hypothesis establishes the proof.

(ii) Plainly, due to the characteristic condition on F, the proof is a quick consequence of (i).

(iii) It easily follows from (ii) that $n_1 = n_2$. Then again the assertion easily follows from (ii). □

Lemma 2.2.6. *Let F be a field with* $\mathrm{ch}(F) = 0$ or $> n$, *and* $\{\lambda_1, ..., \lambda_n\} \subset K$ *where K is a field extension of F.*

(i) If

$$\lambda_1^k + ... + \lambda_n^k = c^{k-m}(\lambda_1^m + ... + \lambda_n^m),$$

for all $k = m+1, ..., m+n$ where m is a given member of $\mathbb{N} \cup \{0\}$ and $c \in F$, then $\lambda_i = 0$ or c for all $i = 1, ..., n$.

(ii) If

$$\lambda_1^k + ... + \lambda_n^k = C,$$

for all $k = m, ..., m+n$ where m is a given member of $\mathbb{N} \cup \{0\}$ and $C \in F$, then C is an integer and $\lambda_i = 0$ or 1 for all $i = 1, ..., n$.

(iii) Let $A \in M_n(F)$ and $m \in \mathbb{N} \cup \{0\}$. Then A is nilpotent iff

$$\mathrm{tr}(A^k) = 0,$$

for each $k = m+1, ..., m+n$.

Proof. (i) We prove the assertion by induction on n. If $n = 1$, the proof is obvious. Suppose that the assertion holds for all $k < n$. Let $\{\lambda_1, ..., \lambda_n\} \subset K$ be a given subset as described in the statement of the theorem. If $c = 0$ or $\sum_{i=1}^n \lambda_i^m = 0$, then $\lambda_i = 0$ for all $i = 1, ..., n$ by Lemma 2.2.5(iii), and that settles the assertion in this case. So we may assume that $c \neq 0$ and $\sum_{i=1}^n \lambda_i^m \neq 0$. Let S_k's be as in the proof of Lemma 2.2.4. We obviously have

$$x^n - S_1 x^{n-1} + ... + (-1)^n S_n = (x - \lambda_1)...(x - \lambda_n). \qquad (*)$$

Thus

$$\lambda_i^{n+m} - S_1 \lambda_i^{n+m-1} + ... + (-1)^n S_n \lambda_i^m = 0,$$

for each $i = 1, ..., n$. Adding up the preceding equations we get

$$\sum_{i=1}^{n} \lambda_i^{n+m} - S_1 \sum_{i=1}^{n} \lambda_i^{n+m-1} + ... + (-1)^n S_n \sum_{i=1}^{n} \lambda_i^m = 0.$$

So using the hypothesis we can write

$$c^n(\sum_{i=1}^{n} \lambda_i^m) - S_1 c^{n-1} \sum_{i=1}^{n} \lambda_i^m + ... + (-1)^n S_n (\sum_{i=1}^{n} \lambda_i^m) =$$

$$\sum_{i=1}^{n} \lambda_i^{n+m} - S_1 \sum_{i=1}^{n} \lambda_i^{n+m-1} + ... + (-1)^n S_n \sum_{i=1}^{n} \lambda_i^m = 0,$$

Thus

$$(\sum_{i=1}^{n} \lambda_i^m)(c^n - S_1 c^{n-1} + ... + (-1)^n S_n) = 0,$$

implying

$$c^n - S_1 c^{n-1} + ... + (-1)^n S_n = 0.$$

So it follows from (*) that $c = \lambda_j$ for some $j = 1, ..., n$. Now obviously the induction hypothesis can be applied to the set $\{\lambda_1, ..., \lambda_n\} \setminus \{c\} \subset F$ and the proof is complete.

(ii) Take $c = 1$ in (i).

(iii) Necessity, in fact on any field, easily follows from the proof of Lemma 2.2.4 and the Cayley-Hamilton Theorem. To see sufficiency, let $\lambda_1, ..., \lambda_n$ be eigenvalues of A in the algebraic closure of F. It follows from the hypothesis that

$$\lambda_1^k + ... + \lambda_n^k = 0,$$

for each $k = m+1, ..., m+n$. Applying (i) with $c = 0$ (or Lemma 2.2.5(iii)) we conclude that $\lambda_i = 0$ for each $i = 1, ..., n$. Therefore A is nilpotent again by the proof of Lemma 2.2.4 and the Cayley-Hamilton Theorem. □

Recall that if $\mathcal{V}$ is a vector space over a field, then the *dual space* of $\mathcal{V}$, denoted by $\mathcal{V}^*$, is the vector space of all linear functionals on $\mathcal{V}$. Inspired by the Halperin-Rosenthal proof of Burnside's Theorem (See [HR] or Theorem 1.2.2 of [RR]), we give new proofs of Levitzki's results.

Theorem 2.2.7. *Let $\mathcal{V}$ be a finite-dimensional vector space over a field F. Then every algebra of nilpotent transformations in $\mathcal{L}(\mathcal{V})$ is triangularizable.*

Proof. The assertion trivially holds if $\dim \mathcal{V} = 1$. So we may assume, with no loss of generality, that $\dim \mathcal{V} > 1$. Let $\mathcal{A}$ be an algebra of nilpotent transformations. Since nilpotency is inherited by quotients, so, in light of the Triangularization Lemma (Lemma 1.1.3), it suffices to show that $\mathcal{A}$ is reducible. We prove this by contradiction. Suppose that $\mathcal{A}$ is irreducible. First we show that $\mathcal{A}$ contains a rank-one transformation. To this end, suppose that T is a nonzero element of $\mathcal{A}$ with minimal rank. We must show that rank$(T) = 1$. Suppose not. Hence there exist $x_1, x_2 \in \mathcal{V}$ such that $\{Tx_1, Tx_2\}$ is linearly independent. Since $Tx_1 \neq 0$ and $\mathcal{A}$ is irreducible, it follows that $\mathcal{A}Tx_1 = \mathcal{V}$. This in turn implies that there exists $A \in \mathcal{A}$ such that $ATx_1 = x_2$. Note that by hypothesis $TA \in \mathcal{A}$ is nilpotent. and that obviously $T\mathcal{V}$ is invariant under TA. Thus the restriction of TA to $T\mathcal{V}$ is also nilpotent, and hence not invertible. Therefore, the range of TAT is properly contained in that of T. On the other hand, we have $TAT \neq 0$ for $TATx_1 = Tx_2 \neq 0$. In other words, we have $TAT \neq 0$ and rank$(TAT) <$ rank(T) contradicting the minimality of the rank of T. Hence rank$(T) = 1$. Now choose a nonzero vector y_0 in the range of T. So there exists a nonzero functional $\phi_0 \in \mathcal{V}^*$ such that $T = \phi_0 \otimes y_0$, i.e., $Tx = \phi_0(x)y_0$ for all $x \in \mathcal{V}$.

Now by proving that $\mathcal{A} = \mathcal{L}(\mathcal{V})$ we obviously get a contradiction, finishing the proof (e.g., note that the identity transformation on $\mathcal{V}$ is not nilpotent). To see this, since every rank-one linear transformation on $\mathcal{V}$ is of the form $\phi \otimes y$ for some linear functional $\phi \in \mathcal{V}^*$ and $y \in \mathcal{V}$, and since every linear transformation on $\mathcal{V}$ is a sum of rank-one linear transformations, it suffices to show that $\phi \otimes y \in \mathcal{A}$ for all linear functionals $\phi \in \mathcal{V}^*$ and $y \in \mathcal{V}$.

Now suppose that $y \in \mathcal{V}$ and $\phi \in \mathcal{V}^*$ are given. Since $\mathcal{A}$ is an algebra and $T = \phi_0 \otimes y_0 \in \mathcal{A}$, it follows that

$$BTC = \phi_0 C \otimes By_0 \in \mathcal{A},$$

for all $B, C \in \mathcal{A}$. Plainly the sets $\{\phi_0 C : C \in \mathcal{A}\}$ and $\{By_0 : B \in \mathcal{A}\}$ are subspaces of $\mathcal{V}^*$ and $\mathcal{V}$ respectively. Since $y_0 \neq 0$ and $\mathcal{A}$ is irreducible, it follows that there exists $B_0 \in \mathcal{A}$ such that $B_0 y_0 = y$. We claim that $\{\phi_0 C : C \in \mathcal{A}\} = \mathcal{V}^*$. Suppose not. Since, by Theorem IV.4.12 of [H], finite-dimensional vector spaces over division rings are reflexive, it follows that there would exist a nonzero $x_0 \in \mathcal{V}$ such that $\phi_0 C x_0 = \phi_0(C x_0) = 0$ for all $C \in \mathcal{A}$. But irreducibility of $\mathcal{A}$ along with the fact that $0 \neq x_0 \in \mathcal{V}$ implies that $\{C x_0 : C \in \mathcal{A}\} = \mathcal{V}$. Hence we must have $\phi_0 = 0$, a contradiction. Therefore $\{\phi_0 C : C \in \mathcal{A}\} = \mathcal{V}^*$. Now pick $C_0 \in \mathcal{A}$ such that $\phi_0 C_0 = \phi$. We can write $\phi \otimes y = \phi_0 C_0 \otimes B_0 y_0 = B_0 T C_0 \in \mathcal{A}$ which is what we wanted. □

Remark. By the second and the third paragraph of the proof above, we conclude that *an irreducible subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{V})$ is equal to $\mathcal{L}(\mathcal{V})$ iff $\mathcal{A}$ contains a rank-one linear transformation.*

Theorem 2.2.7 implies the following well-known result which shows that triangularizability of a collection of triangularizable matrices does not depend on the ground field.

Corollary 2.2.8. *Let F be a field, K a field extension of F, and $\mathfrak{F}$ a family of triangularizable matrices in $M_n(F)$ where $n \in \mathbb{N}$. Then $\mathfrak{F}$ is triangularizable over F iff $\mathfrak{F}$ is triangularizable over K.*

Proof. We may assume, with no loss of generality, that $n > 1$. The "only if" part is obvious. To see the "if" part, suppose that $\mathfrak{F} \subset M_n(F)$ is a family of triangularizable matrices, and that $\mathfrak{F}$ is triangularizable over K, we need to show that $\mathfrak{F}$ is triangularizable over F. Suppose that $\mathcal{M} \subset \mathcal{N}$ are two invariant subspaces for $\mathfrak{F}$ with $\dim \frac{\mathcal{N}}{\mathcal{M}} > 1$, it suffices to show that $\hat{\mathfrak{F}}$, the set of all quotient transformations $\hat{A}$ on $\frac{\mathcal{N}}{\mathcal{M}}$ $(A \in \mathfrak{F})$, is reducible. Since, by hypothesis, every $A \in \mathfrak{F}$ is triangularizable over F, so is every quotient transformation $\hat{A}$ on $\frac{\mathcal{N}}{\mathcal{M}}$ where $A \in \mathfrak{F}$. Therefore if $\hat{\mathfrak{F}}$ is commutative, then reducibility easily follows. So we may assume that there are $A, B \in \mathfrak{F}$ such that $\hat{A}\hat{B} \neq \hat{B}\hat{A}$. By proving that $\hat{\mathcal{A}}$, the algebra generated by $\hat{\mathfrak{F}}$, is reducible we settle the proof. Let $\mathcal{A}$ denote the algebra generated by $\mathfrak{F}$ over F. Note that since $\mathfrak{F}$ is triangularizable over K, it is easily seen that the ideal $\mathcal{J}$ generated by $AB - BA$ in $\mathcal{A}$ is an ideal of nilpotent matrices in $\mathcal{A}$. Hence the corresponding set of quotient transformations $\hat{\mathcal{J}}$ is indeed an ideal of nilpotent quotient transformations in $\hat{\mathcal{A}}$. The ideal $\hat{\mathfrak{J}}$ is nonzero for $0 \neq \hat{A}\hat{B} - \hat{B}\hat{A} \in \hat{\mathfrak{J}}$. On the other hand, the ideal $\hat{\mathfrak{J}}$ is in particular an algebra of nilpotent transformations on $\frac{\mathcal{N}}{\mathcal{M}}$ with $\dim \frac{\mathcal{N}}{\mathcal{M}} > 1$. Hence, it follows from Theorem 2.2.7 that $\hat{\mathfrak{J}}$ is triangularizable, and hence reducible, over F. Now reducibility of $\hat{\mathcal{A}}$ follows from that of the nonzero ideal $\hat{\mathfrak{J}}$ in light of Lemma 1.1.2(i), completing the proof. □

Theorem 2.2.9 (Levitzki's Theorem). *Let F be a field, and $n \in \mathbb{N}$. Then every semigroup $\mathcal{S}$ of nilpotent matrices in $M_n(F)$ is triangularizable.*

Proof. Since every nilpotent matrix in $M_n(F)$ is triangularizable, in view of Corollary 2.2.8, without loss of generality we may assume that the ground field F is algebraically closed. Now since nilpotency is inherited by quotients, it suffices to prove that every semigroup $\mathcal{S}$ of nilpotent matrices in $M_n(F)$ is reducible. To this end, let $\mathcal{A}$ denote the algebra generated by $\mathcal{S}$. Plainly trace is zero on $\mathcal{A}$, therefore reducibility of $\mathcal{A}$, and hence $\mathcal{S}$, follows from Burnside's Theorem (see Theorem 2.2.21 below), for the ground field F is algebraically closed. □

Remark. In Chapter 4, we use the noncommutative version of Lemma 2.2.1 to give a new proof of Levitzki's Theorem on division rings.

The following theorem is a finite-dimensional version of Theorem 5 of [Y1] over general fields.

Theorem 2.2.10. *Let $\mathcal{V}$ be a finite-dimensional vector space over a field F, $\mathfrak{F}$ a nonscalar triangularizable family of linear transformations on $\mathcal{V}$. Then $\mathfrak{F}$ has a nontrivial hyperinvariant subspace.*

Proof. We note that for every family $\mathfrak{F}$ of linear transformations

$$\mathfrak{F}' = (\mathrm{Alg}(\mathfrak{F}))' = (\mathrm{Sem}(\mathfrak{F}))'.$$

Thus $\mathfrak{F}$ has a nontrivial hyperinvariant subspace iff $\mathrm{Alg}(\mathfrak{F})$ does, or iff $\mathrm{Sem}(\mathfrak{F})$ does. Thus it suffices to prove the assertion for any nonscalar triangularizable algebra, say $\mathcal{A}$, of linear transformations.

Now either the algebra $\mathcal{A}$ is commutative or not. If it is a commutative algebra, note that by hypothesis there exists $A \in \mathcal{A}$ that is not scalar. That is, the transformation A is not a multiple of identity. So let λ be any eigenvalue of A, and $\mathcal{M}$ the corresponding eigenspace of A. Since $\mathcal{A}$ is commutative, for all $B \in \mathcal{A} \cup \mathcal{A}'$ and $x \in \mathcal{M}$ we have

$$ABx = BAx = \lambda Bx,$$

i.e., $Bx \in \mathcal{M}$, so $\mathcal{M}$ is invariant under $\mathcal{A} \cup \mathcal{A}'$. Now if the algebra $\mathcal{A}$ is not commutative, then there exist $A, B \in \mathcal{A}$ such that $AB - BA \neq 0$. Set $K_0 = AB - BA$. Then K_0 is a nonzero nilpotent transformation in $\mathcal{A}$ for $\mathcal{A}$ is assumed to be triangularizable.

Define $\mathcal{A}_1 := \mathcal{A}' + \mathcal{A} * \mathcal{A}'$ where

$$\mathcal{A} * \mathcal{A}' := \{\sum_{i=1}^{k} A_i A_i' : k \in \mathbb{N}, \quad A_i \in \mathcal{A}, \quad A_i' \in \mathcal{A}', \ (1 \leq i \leq k)\}.$$

Clearly, in view of the fact that $\mathcal{A}'$ is a unital subalgebra of $\mathcal{L}(\mathcal{V})$, we see that $\mathcal{A}_1$ is a subalgebra of $\mathcal{L}(\mathcal{V})$ which contains both $\mathcal{A}$ and $\mathcal{A}'$. It suffices to prove that $\mathcal{A}_1$ has a nontrivial invariant subspace.

For the nonzero nilpotent transformation $K_0 \in \mathcal{A}$, first we claim that $\mathcal{A}_1 K_0$, and hence $\mathcal{A}_1 K_0 \mathcal{A}_1$, the semigroup ideal generated by K_0 in $\mathcal{A}_1$, consists of nilpotents. To this end, let $A_0 = A' + \sum_{i=1}^{k} A_i A_i' \in \mathcal{A}_1$ with $A_i \in \mathcal{A}$, where $A', A_i' \in \mathcal{A}'$, $(1 \leq i \leq k, k \in \mathbb{N})$ are arbitrary. We prove that $A_0 K_0$ is nilpotent: first of all we notice that $A_0 K_0 = A' K_0 + \sum_{i=1}^{k} A_{i_0} A_i'$ where $A_{i_0} = A_i K_0 \in \mathcal{A}$. Let $n = \dim \mathcal{V}$. Set

$$\mathcal{S} := \{A \in \mathcal{A} : A^n = 0\}.$$

Since $\mathcal{A}$ is triangularizable it follows that $\mathcal{S}$ is a nonzero semigroup ideal of $\mathcal{A}$ consisting of nilpotent transformations (note that $0 \neq K_0 \in \mathcal{S}$).

The set $\mathcal{S}\mathcal{A}'$ is indeed a semigroup consisting of nilpotents because for all $A \in \mathcal{A}, A' \in \mathcal{A}'$ we have $AA' = A'A$ and that $\mathcal{S}$ is a semigroup of nilpotents. Thus Levitzki's Theorem (Theorem 2.2.9) shows that $\mathcal{S}\mathcal{A}'$ is triangularizable. Therefore

$\mathrm{Alg}(\mathcal{S}\mathcal{A}')$, the algebra generated by $\mathcal{S}\mathcal{A}'$, consists of nilpotents. We have

$$A_0K_0 = K_0A' + \sum_{i=1}^{k} A_{i_0}A_i'$$

where $A_{i_0} = A_iK_0 \in \mathcal{A}$. In fact $A_{i_0} = A_iK_0 \in \mathcal{S}$, for $K_0 \in \mathcal{S}$ and $\mathcal{A}$ is triangularizable. Now clearly $A'K_0 = K_0A' \in \mathcal{S}\mathcal{A}'$ and $A_{i_0}A_i' \in \mathcal{S}\mathcal{A}'$. Therefore $A_0K_0 \in \mathrm{Alg}(\mathcal{S}\mathcal{A}')$ and hence A_0K_0 is a nilpotent transformation. Thus $\mathcal{A}_1K_0\mathcal{A}_1$ is a nonzero semigroup ideal of $\mathcal{A}_1$ consisting of nilpotents which must be triangularizable, and hence reducible, by Levitzki's Theorem. Now reducibility of the nonzero ideal $\mathcal{A}_1K_0\mathcal{A}_1$ implies that of $\mathcal{A}_1$ in light of Lemma 1.1.2(i). □

Here is a finite-dimensional extension of a result due to V.S. Shulman to general fields (see [S] or [Y1]).

Corollary 2.2.11. *Let $\mathcal{V}$ be a finite-dimensional vector space over a field F. Then every nonzero semigroup $\mathcal{S}$ of nilpotent linear transformations in $\mathcal{L}(\mathcal{V})$ has a nontrivial hyperinvariant subspace.*

Proof. Since every such semigroup $\mathcal{S}$ is triangularizable by Levitzki's Theorem, Theorem 2.2.10 applies. □

Theorem 2.2.10 immediately implies the following corollary which is a finite-dimensional version of a result due to Yu.V. Turovskii on general fields (see [ST] or [Y1]).

Corollary 2.2.12. *Let $\mathcal{V}$ be a finite-dimensional vector space over a field F, $\mathcal{M}$ and $\mathcal{N}$ triangularizable sets of linear transformations in $\mathcal{L}(\mathcal{V})$ such that $\mathcal{N} \subset \mathcal{M}'$. Then $\mathcal{M} \cup \mathcal{N}$ is triangularizable.*

Proof. In view of The Triangularization Lemma (Lemma 1.1.3) we just need to show that $\mathcal{M} \cup \mathcal{N}$ is reducible. Without loss of generality we may assume that $\mathcal{M}$ is nonscalar; otherwise we have nothing to prove. Since $\mathcal{M} \cup \mathcal{N} \subset \mathcal{M} \cup \mathcal{M}'$, the assertion is a quick consequence of Theorem 2.2.10. □

For $m \in \mathbb{N}$, and a collection $\mathcal{F}$ of matrices by $\mathcal{F}^m$ we mean the set of products of the members of $\mathcal{F}$ of "length" m, i.e.,

$$\mathcal{F}^m = \{A_1 ... A_m : A_i \in \mathcal{F}, i = 1, ..., m\}.$$

The following theorem extends Levitzki's Theorem.

Theorem 2.2.13. *(i) Let $n \in \mathbb{N}$, F a field, and $\mathcal{S}$ a semigroup in $M_n(F)$ such that every $S \in \mathcal{S}$ can be written as a linear combination of nilpotent elements from the algebra generated by $\mathcal{S}$. Then the semigroup $\mathcal{S}$ is triangularizable and $\mathcal{S}^n = \{0\}$. In particular, $\mathcal{S}$ is a semigroup of nilpotents.*

(ii) Let $n \in \mathbb{N}$, F a field, $\mathcal{A}$ an algebra in $M_n(F)$. If the algebra $\mathcal{A}$ is generated by nilpotents as a vector subspace of $M_n(F)$, then the algebra $\mathcal{A}$ is triangularizable and $\mathcal{A}^n = \{0\}$. In particular, $\mathcal{A}$ is an algebra of nilpotents.

Proof. It suffices to prove (i).

(i) The assertion trivially holds if $n = 1$. So suppose that $n > 1$. Since nilpotency does not depend on the ground field, we may assume, without loss of generality, that the ground field F is algebraically closed. First we show that the semigroup $\mathcal{S}$ is triangularizable. Suppose that $\mathcal{M} \subset \mathcal{N}$ are two invariant subspaces for $\mathcal{S}$ with $\dim \frac{\mathcal{N}}{\mathcal{M}} > 1$, it suffices to show that $\hat{\mathcal{S}}$, the set of all quotient transformations $\hat{S}$ on $\frac{\mathcal{N}}{\mathcal{M}}$ ($S \in \mathcal{S}$), is reducible. It easily follows from the hypothesis that every $\hat{S} \in \hat{\mathcal{S}}$ can be written as a linear combination of nilpotent elements from the algebra generated by $\hat{\mathcal{S}}$. Thus, $\operatorname{tr}(\hat{\mathcal{S}}) = \{0\}$. Since the ground field F is algebraically closed, from Burnside's Theorem (see Theorem 2.2.21 below), we see that $\hat{\mathcal{S}}$ is reducible, as desired. Therefore, the semigroup $\mathcal{S}$ is triangularizable. That is, there is a maximal chain of subspaces each of which is invariant for $\mathcal{S}$ as follows.

$$\{0\} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset ... \subset \mathcal{M}_n = F^n,$$

where $\dim \frac{\mathcal{M}_i}{\mathcal{M}_{i-1}} = 1$ for each $i = 1, ..., n$. Now since $\dim \frac{\mathcal{M}_i}{\mathcal{M}_{i-1}} = 1$ for each $i = 1, ..., n$ and that every quotient transformations $\hat{S}_i$ on $\frac{\mathcal{M}_i}{\mathcal{M}_{i-1}}$ ($S \in \mathcal{S}$) is a linear combination of nilpotent elements from the algebra generated by $\hat{\mathcal{S}}_i$, we see that $\hat{S}_i = 0$ for all

$S \in \mathcal{S}$. In other words, there exists a basis for the vector space F^n such that every $S \in \mathcal{S}$ has strictly upper triangular matrix representation with respect to that basis. This obviously yields $\mathcal{S}^n = 0$, completing the proof. □

Remarks.

1. Let F be a field, and $\mathcal{A}$ an irreducible algebra in $M_n(F)$ where $n \in \mathbb{N}$. Then $\mathcal{A}$ cannot be generated by nilpotents as a subspace of $M_n(F)$, for otherwise by the preceding theorem $\mathcal{A}$ would be triangularizable, and hence reducible, a contradiction.

2. In view of Theorem 2.2.7, the proof of the preceding theorem provides a second proof for Levitzki's Theorem.

3. Let D be a division ring whose center is a field F. By an *F-algebra* $\mathcal{A}$ in $M_n(D)$, we mean a subring of $M_n(D)$ that is closed under scalar multiplication by the elements of the field F. Let $\mathcal{A}$ be an F-algebra in $M_n(D)$ where $n \in \mathbb{N}$. We conjecture that the F-algebra $\mathcal{A}$ is nilpotent, more precisely $\mathcal{A}^n = 0$, if and only if $\mathcal{A}$ is spanned by its nilpotents as a vector space over F. This conjecture, if true, immediately extends Levitzki's Theorem, Kolchin's Theorem (Corollary 2.2.15 below), and Theorem 2.3.2 below to division rings.

The following extends a well-known theorem of Kolchin.

Corollary 2.2.14. *(i) Let $n \in \mathbb{N}$, F a field, and $\mathfrak{F}$ a family of matrices in $M_n(F)$ with the following properties: (a) every A in $\mathfrak{F}$ can be written as a linear combination of nilpotent elements from the algebra generated by $\mathfrak{F}$; (b) if A and B are in $\mathfrak{F}$, then $AB + A + B$ is in $\mathfrak{F}$. Then $\mathfrak{F}$ is triangularizable, and $\mathfrak{F}^n = \{0\}$.*

(ii) Let $n \in \mathbb{N}$, F a field, and $\mathfrak{F}$ a family of matrices in $M_n(F)$ such that every A in $\mathfrak{F}$ can be written as a linear combination of nilpotent elements from the algebra generated by $\mathfrak{F}$. Then every semigroup of matrices of the form $I + N$ with N in $\mathfrak{F}$ is triangularizable. Therefore, such a semigroup of matrices is indeed a semigroup of unipotents (i.e., of the form $I + N$ with N nilpotent).

Proof. (i) Let $\mathcal{S}$ denote the semigroup generated by $\mathfrak{F}$. Note that every $S \in \mathcal{S}$ is a product of a length m in $\mathfrak{F}$ (i.e., $S = A_1 ... A_m$ with A_i in $\mathfrak{F}$ for each $i = 1, ..., m$). Using induction on the length m, in view of (a) and (b), it is easily seen that every

$S \in \mathcal{S}$ can be written as a linear combination of nilpotent elements from the algebra generated by $\mathcal{S}$. So the assertion follows from the preceding theorem.

(ii) Let $\mathcal{S}$ be a semigroup of matrices of the form $I + N$ with $\mathcal{F}$ as described in the hypothesis. Then the family $\mathcal{F}$ obviously satisfies (a) and (b) of part (i). So (i) applies, completing the proof. □

Corollary 2.2.15 (Kolchin's Theorem). *(i) Let $\mathcal{V}$ be a finite-dimensional vector space over a field F. Let $\mathcal{N}$ be a set of nilpotent linear transformations with the following property: if A and B are in $\mathcal{N}$, then $AB + A + B$ is in $\mathcal{N}$. Then $\mathcal{N}$ is triangularizable.*

(ii) Let $\mathcal{V}$ be a finite-dimensional vector space over a field F. Then every semigroup $\mathcal{S}$ of unipotent linear transformations in $\mathcal{L}(\mathcal{V})$ is triangularizable.

Proof. Corollary 2.2.14. □

Corollary 2.2.16 (Kaplansky). *Let $n \in \mathbb{N}$, F a field. Then every semigroup $\mathcal{S}$ of matrices in $M_n(F)$ of the form $\alpha I + N$ with $\alpha \in F$ and N nilpotent is triangularizable. In other words, every semigroup of $n \times n$ matrices over a field whose spectra are singletons is triangularizable.*

Proof. Without loss of generality, we may also assume that $F\mathcal{S} \subseteq \mathcal{S}$. In light of the Triangularization Lemma (Lemma 1.1.3) it suffices to prove that $\mathcal{S}$ is reducible. We recognize two cases.

(i) The semigroup $\mathcal{S}$ contains a nonzero nilpotent.

We show that the set of nilpotents in $\mathcal{S}$ forms a semigroup ideal of $\mathcal{S}$. To this end, suppose that $N \in \mathcal{S}$ is nilpotent and $\alpha I + N' \in \mathcal{S}$ where $\alpha \in F$ and N' is nilpotent. It follows from hypothesis that

$$N(\alpha I + N') = \alpha' I + N'',$$

where $\alpha' \in F$ and N'' is nilpotent. It suffices to show that $\alpha' = 0$. Suppose not. Therefore $\alpha' I + N''$ is invertible, hence so is $N(\alpha I + N')$ and hence N, a contradiction.

Thus $N(\alpha I + N') = N''$ is nilpotent. Similarly $(\alpha I + N')N$ is nilpotent. So we have shown that set of nilpotents in $\mathcal{S}$ forms a nonzero semigroup ideal of $\mathcal{S}$. Now Levitzki's Theorem along with Lemma 1.1.2(i) proves the reducibility of $\mathcal{S}$ in this case.

(ii) The semigroup $\mathcal{S}$ contains no nonzero nilpotent.

In this case, it is plain that every element of $\mathcal{S}$ is of the form $\alpha I + N$ with $0 \neq \alpha \in F$ and N nilpotent. Thus the semigroup $\mathcal{S}$ is reducible iff the semigroup $\mathcal{S}' \subset \mathcal{S}$ is reducible where

$$\mathcal{S}' = \{I + N : \alpha I + \alpha N \in \mathcal{S}, \text{ for some } 0 \neq \alpha \in F \;\&\; N^n = 0\}.$$

But $\mathcal{S}'$ is indeed triangularizable, hence reducible, by Corollary 2.2.15, finishing the proof. □

In what follows, we will make free use of the Jacobson radical theory as well as some classical theorems of the structure of rings such as the Wedderburn-Artin Theorem and its consequences (see Chapter IX of [H] for a nice exposition of the classical theorems on the structure of rings). Recall that Burnside's Theorem asserts that *the only irreducible algebra in $M_n(F)$ $(n > 1)$ is $M_n(F)$ provided that F is algebraically closed.* We wish to characterize all fields F such that Burnside's Theorem holds in $M_n(F)$ $(n > 1)$. To do so, we need the following lemma.

Lemma 2.2.17. *Let F be a field, and $\mathcal{A}$ an irreducible algebra in $M_n(F)$. Then $\mathcal{A}$ is semisimple both as a subring and as a subalgebra of $M_n(F)$. Furthermore, the algebra $\mathcal{A}$ is unital, its identity element equals the identity matrix, and $\mathcal{A}$ is simple.*

Proof. In light of Theorem IX.5.2(ii) of [H], it suffices to show that $\mathcal{A}$ is semisimple as a subalgebra of $M_n(F)$. To this end, note that $\mathcal{A}$ is an algebraic algebra. Therefore, it follows from Exercise IX.5.6 of [H] (or Theorem 14 on page 89 of [K1]) that the Jacobson radical of the algebra $\mathcal{A}$, denoted by $\text{Rad}(\mathcal{A})$, is nil (i.e., every element of $\text{Rad}(\mathcal{A})$ is nilpotent). Now Levitzki's Theorem together with Lemma 1.1.2(i) yields $\text{Rad}(\mathcal{A}) = \{0\}$. That is, $\mathcal{A}$ is semisimple as a subalgebra, and hence as a subring, of $M_n(F)$. That $\mathcal{A}$ is unital follows from Theorem IX.5.4 of [H]. Let I and I_n denote the identity element of $\mathcal{A}$ and $M_n(F)$ respectively. We need to show that $I = I_n$. To this end, by contradiction, suppose $I \neq I_n$. Obviously, I is not zero and $I^2 = I$, hence $I(I - I_n) = 0$. This in turn, in view of the contradiction hypothesis, implies that $\ker(I - I_n)$ is a nontrivial subspace which is invariant under $\{I\}' \supset \mathcal{A}$, contradicting the hypothesis that $\mathcal{A}$ is irreducible. Thus $I = I_n$, as desired. To see that $\mathcal{A}$ is simple, suppose $\mathcal{J}$ is a nonzero ideal of $\mathcal{A}$. We show that $\mathcal{J} = \mathcal{A}$. Since the algebra $\mathcal{A}$ is irreducible, it follows from Lemma 1.1.2(i) that the nonzero ideal $\mathcal{J}$ is an irreducible algebra in $M_n(F)$. Hence, the ideal $\mathcal{J}$ is unital and its unit equals I_n. Therefore, $\mathcal{J} = \mathcal{A}$, completing the proof. □

Remark. Let F be a field, $n > 1$, $\mathcal{S}$ an irreducible semigroup in $M_n(F)$, and $\mathcal{J}$ a nonzero ideal of $\mathcal{S}$. It is easily verified that the trace functional is permutable (resp. zero) on $\mathcal{S}$ if and only if it is permutable (resp. zero) on $\mathrm{Alg}(\mathcal{S})$. That being noted, in view of the preceding lemma, we see that the trace functional is permutable (resp. zero) on $\mathcal{S}$ if and only if it is permutable (resp. zero) on $\mathcal{J}$, for $\mathrm{Alg}(\mathcal{S}) = \mathrm{Alg}(\mathcal{J})$.

Theorem 2.2.18. *Let F be a field, and $\mathcal{S}$ an irreducible semigroup in $M_n(F)$ where $n \in \mathbb{N}$. Let $r \in \mathbb{N}$ be the smallest nonzero rank present in* $\mathrm{Alg}(\mathcal{S})$. *Then*

(i) After a similarity, $\mathrm{Alg}(\mathcal{S})$ *contains an idempotent $E = I_r \oplus 0_{n-r}$ where I_r is the identity matrix of size r and 0_{n-r} is the zero matrix of size $n - r$.*

(ii) The integer r divides n and after a similarity $E\mathrm{Alg}(\mathcal{S})E = \mathcal{D}_r \oplus 0_{n-r}$ where $\mathcal{D}_r$ is an irreducible division algebra in $M_r(F)$. Furthermore, the minimal polynomial of every $D \in \mathcal{D}_r$, denoted by m_D, is irreducible over F and $\deg(\mathrm{m}_D)$ divides r.

(iii) $\mathrm{Alg}(\mathcal{S}) = M_n(F)$ *iff $r = 1$.*

Proof. (i) The assertion trivially holds if $n = 1$. So suppose that $n > 1$. Set $\mathcal{A} := \mathrm{Alg}(\mathcal{S})$. It is plain that $\mathcal{A}$ is an irreducible subalgebra of $M_n(F)$. So it follows from the preceding lemma that $\mathcal{A}$ is semisimple both as a subring and as a subalgebra of $M_n(F)$ and that the algebra $\mathcal{A}$ contains I_n, the identity matrix.

Let r be the smallest nonzero rank present in $\mathcal{A}$. If $r = n$, we have nothing to prove since the algebra $\mathcal{A}$ contains I_n. So suppose that $r < n$. We can assume that there exists a nonzero element T of $\mathcal{A}$ with minimal rank, i.e., $\mathrm{rank}(T) = r$, such that $\mathrm{m}_T = x^{n_0} f$ where $f \in F[X]$, $\deg(f) \geq 1$, and $f(0) \neq 0$ (in fact it will turn out that $n_0 = 1$). Such a T exists because otherwise the set of T's whose ranks are r or 0 forms a nonzero semigroup ideal $\mathcal{J}$ of nilpotent matrices (whose index of nilpotency is in fact 2 by minimality of r) of the algebra, hence semigroup, $\mathcal{A}$. Thus it follows from Levitzki's Theorem that the nil ideal $\mathcal{J}$ is triangularizable, contradicting irreducibility of $\mathcal{A}$ by Lemma 1.1.2(i). Now it follows from the Primary Decomposition Theorem (Theorem 6.8.12 of [HK]) that there exist complementary T-invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ such that $F^n = \mathcal{M} \oplus \mathcal{N}$, $T = T_1 \oplus T_2$ where $T_1 = T|_{\mathcal{M}}$, $T_2 = T|_{\mathcal{N}}$, $\mathrm{m}_{T_1} = f$, $\mathrm{m}_{T_2} = x^{n_0}$, (thus T_1 is invertible for $f(0) \neq 0$). Now since T has minimal rank it follows that $T_2 = 0_{\mathcal{N}}$, for otherwise $\mathrm{rank}(T_2) > 0$ and

$$\mathrm{rank}(T^{n_0}) = \mathrm{rank}(T_1^{n_0} \oplus T_2^{n_0}) = \mathrm{rank}(T_1^{n_0} \oplus 0_{\mathcal{N}}) = \mathrm{rank}(T_1^{n_0})$$

$$= \mathrm{rank}(T_1) < \mathrm{rank}(T_1) + \mathrm{rank}(T_2) = \mathrm{rank}(T) = r,$$

contradicting the minimality of r. Thus $T_2 = 0_{\mathcal{N}}$, yielding $n_0 = 1$ for $x = \mathrm{m}_{0_{\mathcal{N}}} = \mathrm{m}_{T_2} = x^{n_0}$. Since $\mathrm{m}_{T_1} = f$ and $f(0) \neq 0$, it follows that there exists a polynomial $p = -(f - f(0))/f(0)$ such that $p(T_1) = I_{\mathcal{M}}$. So we have $p(T) = I_{\mathcal{M}} \oplus 0_{\mathcal{N}} \in \mathcal{A}$. Therefore, after a similarity $\mathcal{A}$ contains the desired idempotent $E = I_r \oplus 0_{n-r}$.

(ii) Find $E = I_r \oplus 0_{n-r}$ as described in (i). It is easily seen that one can write $E\mathcal{A}E = \mathcal{D}_r \oplus 0_{n-r}$ where $\mathcal{D}_r \subseteq M_r(F)$. That $\mathcal{D}_r$ is an algebra in $M_r(F)$ is trivial. If $r \geq 2$, then $\mathcal{D}_r$ is an irreducible division algebra by Corollary 2.2.3 and minimality of r; if $r = 1$, this is trivial (in fact by part (iii) of the theorem we have $r = 1$ iff $\mathcal{A} = M_n(F)$). So it remains to show that r divides n. To this end, use induction on n. If $n = 1$, we have nothing to prove. Suppose that the assertion holds for all irreducible semigroups of matrices of size less than n. For a given irreducible semigroup $\mathcal{S}$ of matrices in $M_n(F)$, set $\mathcal{A} := \mathrm{Alg}(\mathcal{S})$ and find $E = I_r \oplus 0_{n-r}$ as described in (i). If $r = 1$, we have nothing to prove. So without loss of generality assume that $r \geq 2$ and $E \in \mathcal{A}$ (note that rank is invariant under similarity). Thus $n - r \geq 2$ since $I - E \in \mathcal{A}$. From $I - E = (I_r \oplus I_{n-r}) - (I_r \oplus 0_{n-r}) = 0_r \oplus I_{n-r}$, it is easily seen that

$$\mathcal{A}' := (I - E)\mathcal{A}(I - E) = 0_r \oplus \mathcal{A}_r,$$

where $\mathcal{A}_r \subseteq M_{n-r}(F)$. Since $\mathcal{A}$ is an irreducible algebra in $M_n(F)$, it follows from Corollary 2.2.3 that so is $\mathcal{A}_r$ in $M_{n-r}(F)$. Let r' be the smallest nonzero rank present in $\mathcal{A}_r \subseteq M_{n-r}(F)$. We conclude from the induction hypothesis that r' divides $n - r$.

So to prove that r divides n, it suffices to show that $r' = r$. The fact that $0_r \oplus \mathcal{A}_r = (I-E)\mathcal{A}(I-E) \subseteq \mathcal{A}$, implies that $r \leq r'$. To see $r' \leq r$, first we claim that $(I-E)\mathcal{A}E \neq 0$. Suppose $(I-E)\mathcal{A}E = 0$. It is evident that $\mathcal{M} := EF^n$ is a nontrivial subspace of F^n. We have

$$\mathcal{AM} = \mathcal{A}EF^n = (E+(I-E))\mathcal{A}EF^n$$

$$= E\mathcal{A}EF^n + (I-E)\mathcal{A}EF^n = E\mathcal{A}EF^n \subseteq EF^n = \mathcal{M}.$$

Therefore, $\mathcal{AM} \subseteq \mathcal{M}$. That is, $\mathcal{A}$ is reducible, a contradiction. So there exists $A \in \mathcal{A}$ such that $(I-E)AE \neq 0$. Note that $\mathcal{A}$ is an irreducible subalgebra in $M_n(F)$ and $0 \neq (I-E)AE \in \mathcal{A}$. Hence, Lemma 2.2.17 together with Exercise IX.2.5(i) of [H] implies that there exists $B \in \mathcal{A}$ such that $(I-E)^2AEB = (I-E)AEB$ is not nilpotent. Therefore, $(I-E)AEB(I-E)$ is not nilpotent either which in turn implies that $(I-E)AEB(I-E) \neq 0$. It is now plain that

$$0 < \operatorname{rank}((I-E)AEB(I-E)) \leq \operatorname{rank}(E) = r.$$

Since $0 \neq AEB \in \mathcal{A}$, we conclude that

$$r' \leq \operatorname{rank}((I-E)AEB(I-E)) \leq r.$$

So $r' \leq r$, hence $r = r'$, finishing the proof. Finally, since $\mathcal{D}_r$ is a division algebra in $M_r(F)$, the minimal polynomial m_D of every $D \in \mathcal{D}_r$ is irreducible over F, implying that $\deg(\mathrm{m}_D)$ divides r by the Rational Canonical Form Theorem (see Theorem VII.4.2(i) and Theorem VII.4.6(i) of [H]).

(iii) The assertion follows from the remark following Theorem 2.2.7, for Alg($\mathcal{S}$) is an irreducible subalgebra of $M_n(F)$ which contains a rank-one matrix. □

Recall that a field F is called *perfect* if every algebraic extension field of F is separable over F, or equivalently, either $\mathrm{ch}(F) = 0$ or $\mathrm{ch}(F) = p$ and $F = \{a^p : a \in F\}$ (see Ex. 5.6.13 on page 289 of [H]), e.g., finite fields and algebraically closed fields are perfect.

Theorem 2.2.19. *Let F be a field, $n > 1$, $\mathcal{S}$ an irreducible semigroup in $M_n(F)$, $\mathcal{J}$ a nonzero semigroup ideal of $\mathcal{S}$ such that* $\mathrm{tr}(\mathcal{J}) = \{c\}$ *for some $c \in F$. Then trace is zero on $\mathcal{S}$, hence $c = 0$. Also the field F is not perfect, and* $\mathrm{ch}(F)$ *is nonzero and divides r, hence n, where $r \in \mathbb{N}$ is the smallest nonzero rank in* $\mathrm{Alg}(\mathcal{S})$.

Proof. Set $\mathcal{A} := \mathrm{Alg}(\mathcal{S})$. Plainly, the algebra $\mathcal{A}$ is irreducible, and hence simple in view of Lemma 2.2.18. Thus $\mathcal{A} = \mathrm{Alg}(\mathcal{J})$, for $\mathrm{Alg}(\mathcal{J})$ is a nonzero ideal of $\mathcal{A}$. We can write

$$\mathcal{A} = \{\sum_{i=1}^{k} c_i S_i : k \in \mathbb{N},\ c_i \in F, S_i \in \mathcal{J},\ 1 \le i \le k\}.$$

To show that $c = 0$, we use contradiction. Suppose that $c \neq 0$. There are two cases to consider.

(a) $\{S, S'\}$ *is linearly dependent for all $S, S' \in \mathcal{J}$.*

Fix $S \in \mathcal{J}$; it follows that for every $S' \in \mathcal{J}$ we have $S' = fS$ for some $f \in F$. Taking trace of both sides yields $\mathrm{tr}(S') = f\mathrm{tr}(S)$. Hence $c = fc$ implying $f = 1$. Therefore, we have $\mathcal{J} = \{S\}$ and this in turn implies $S^2 = S$, for $\mathcal{J}$ is a semigroup. From this we obviously see that $\mathcal{J}$ is reducible, a contradiction in light of Lemma 1.1.2(i).

(b) $\{S, S'\}$ *is linearly independent for some $S, S' \in \mathcal{J}$.*

Since the set $\{S, S'\}$ ($S, S' \in \mathcal{J}$) is linearly independent, we see that the set $\mathfrak{J} := \{A \in \mathcal{A} : \mathrm{tr}(A) = 0\}$ is nonzero. It follows from constancy of trace on $\mathcal{J}$ that $\mathfrak{J}$ is an ideal of the irreducible algebra $\mathcal{A}$. Therefore, $\mathfrak{J} = \mathcal{A}$ since $\mathcal{A}$ is simple. This yields $c = 0$, a contradiction. Therefore, $c = 0$. Now let r and $\mathcal{D}_r$ be as in Theorem 2.2.18. Since $\mathrm{tr}(\mathcal{A}) = \{0\}$, from Theorem 2.2.18 we see that r divides n and that $\mathrm{tr}(\mathcal{D}_r) = \{0\}$. In particular, we have $\mathrm{tr}(I_r) = 0$. Therefore, $\mathrm{ch}(F)$ is nonzero and divides r. Finally, the fact that the field F is not perfect follows from Theorem 2.5 of [FGG]. □

Remarks.

1. It will turn out that every semigroup in $M_n(F)$ with constant trace is indeed triangularizable provided that $\mathrm{ch}(F) > n$ (see Corollary 2.4.10 below).

2. Let $n > 1$, F a field that is perfect or its characteristic does not divide n. It follows from the preceding theorem that every semigroup of matrices in $M_n(F)$ with constant trace is reducible.

Recall that a matrix A in $M_n(F)$ is called *reducible* if A as a linear transformation on F^n is reducible, i.e., it has a nontrivial invariant subspace. As we saw in Lemma 1.1.1, it follows from the Cayley-Hamilton Theorem that: a matrix A in $M_n(F)$ is irreducible if and only if the characteristic polynomial for A is irreducible over F; if and only if every nonzero x in F^n is a cyclic vector for A, i.e., $\{x, Ax, ..., A^{n-1}x\}$ spans F^n. Furthermore,

$$\{A\}' = F[A] = \{f(A) : f \in F[X] \text{ with } \deg(f) \leq n-1\},$$

where $\{A\}'$ denotes the commutant of A. In contrast to Lemma 1.1.1, we prove the following.

Lemma 2.2.20. *Let F be a field and $n > 1$. A matrix A in $M_n(F)$ has no nontrivial hyperinvariant subspace if and only if the minimal polynomial for A is irreducible over F. Furthermore, after a similarity $\{A\}' = M_{\frac{n}{r}}(F[C])$ where $r = \deg(\mathrm{m}_A)$ (divides n) and $C = \mathrm{C}(\mathrm{m}_A)$ in $M_r(F)$ denotes the companion matrix of the minimal polynomial of A.*

Proof. The "only if" part of the assertion is easy. To prove it by contradiction suppose m_A, the minimal polynomial of A, is reducible over F. So there exists a polynomial $f \in F[X]$ different from m_A that divides m_A. It is now easily seen that $\ker(f(A))$ is a nontrivial hyperinvariant subspace for A, a contradiction. To see the "if" part, suppose that m_A is irreducible over F. From the Rational Canonical Form Theorem (see Theorem VII.4.2(i) and Theorem VII.4.6(i) of [H]), we see that $r = \deg(\mathrm{m}_A)$ divides n and that A is similar to a direct sum of copies of the companion matrix of m_A. More precisely, A is similar to $C \oplus ... \oplus C \in M_{\frac{n}{r}}(F[C])$ where C denotes the the companion matrix of m_A. With no loss of generality we may assume that $A = C \oplus ... \oplus C \in M_{\frac{n}{r}}(F[C])$. In view of the aforementioned elementary exercise, a straightforward calculation shows that $\{A\}' = M_{\frac{n}{r}}(F[C])$. Since m_A is irreducible over F, we conclude that $F[C]$ is an irreducible algebra in $M_r(F)$ which in turn implies the irreducibility of the algebra $\{A\}' = M_{\frac{n}{r}}(F[C])$ in $M_n(F)$. That is, the matrix A has no nontrivial hyperinvariant subspace. $\square$

For a given field F and $k \in \mathbb{N}$ with $k > 1$, we say that F is *k-closed* if every polynomial of degree k over F is reducible over F. It is plain that a field F is algebraically closed iff F is k-closed for all $k \in \mathbb{N}$ with $k > 1$. It can be shown that finite fields are not k-closed for all $k \in \mathbb{N}$ with $k > 1$. Recall that a collection $\mathfrak{F}$ in $M_n(F)$ $(n > 1)$ is absolutely irreducible if $\mathrm{Alg}(\mathfrak{F}) = M_n(F)$. It follows from the following theorem that a collection $\mathfrak{F}$ in $M_n(F)$ is absolutely irreducible iff it is irreducible as a collection in $M_n(\overline{F})$ where $\overline{F}$ denotes the algebraic closure of F.

We are now in a position to extend Burnside's Theorem as follows.

Theorem 2.2.21. *Let F be a field and $n > 1$. The following are equivalent.*

(i) The only irreducible algebra in $M_n(F)$ is $M_n(F)$.

(ii) Every irreducible family of matrices in $M_n(F)$ is absolutely irreducible.

(iii) The commutant of every irreducible family of matrices in $M_n(F)$ consists of scalars.

(iv) Every nonscalar matrix in $M_n(F)$ has a nontrivial hyperinvariant subspace.

(v) The field F is k-closed for all k dividing n with $k > 1$.

Proof. "$(i) \Longrightarrow (ii)$" Obvious.

"$(ii) \Longrightarrow (iii)$" Let $\mathfrak{F}$ be an irreducible family of matrices in $M_n(F)$ and $\mathcal{A}$ denote the algebra generated by $\mathfrak{F}$. It is plain that $\mathfrak{F}' = \mathcal{A}'$. On the other hand, by the hypothesis we must have $\mathcal{A} = M_n(F)$. Therefore, $\mathfrak{F}' = \mathcal{A}' = M_n(F)' = \{cI_n : c \in F\}$, proving the assertion.

"$(iii) \Longrightarrow (iv)$" Use contradiction. Suppose that the nonscalar matrix A in $M_n(F)$ has no nontrivial hyperinvariant subspace. Therefore, $\{A\}'$ must be an irreducible family, in fact algebra, of matrices in $M_n(F)$. Since $A \in (\{A\}')'$, it follows from the hypothesis that A is scalar, a contradiction.

"$(iv) \Longrightarrow (v)$" Use contradiction again. Suppose that there exists an irreducible polynomial f over F such that $\deg f = r > 1$ divides n. Let $C \in M_r(F)$ denote the companion matrix for f. Set $A = C \oplus ... \oplus C \in M_{\frac{n}{r}}(F[C])$. It is plain that A is nonscalar and that $\mathrm{m}_A = f$. Now irreducibility of m_A contradicts the hypothesis in view of Lemma 2.2.20.

"$(v) \Longrightarrow (i)$" Let $\mathcal{A}$ be an irreducible algebra in $M_n(F)$. Let r and $\mathcal{D}_r$ be as in Theorem 2.2.18. In light of Theorem 2.2.18(ii), it is evident that $r = 1$ and $\mathcal{D}_r = F$. It now follows from Theorem 2.2.18(iii) that $\mathcal{A} = M_n(F)$, as desired. □

Remarks.

1. Since the field of real numbers is k-closed whenever k is an odd number, a quick consequence of the preceding theorem is what we can call **Burnside's Theorem for real vector spaces**: *Let $n \in \mathbb{N}$ be an odd number. Then the only irreducible algebra of linear transformations on an n-dimensional real vector space $\mathcal{V}$ is the algebra of all linear transformations on $\mathcal{V}$.*

2. Let $n > 1$, and F a field that is k-closed for all k dividing n with $k > 1$. In view of Theorem 2.2.21 we see that a collection $\mathcal{F}$ in $M_n(F)$ is irreducible if and only if it is absolutely irreducible; if and only if it is irreducible as a collection in $M_n(K)$ where K is any extension field of F.

Let F be a field, and $\mathcal{F}$ an irreducible family of matrices in $M_n(F)$ $(n > 1)$. We say that $\mathcal{F}$ is *trivially irreducible* if $\mathcal{F} \subseteq F[A]$ where $A \in \mathcal{F}$ is an irreducible matrix in $M_n(F)$, i.e., the matrix A has no nontrivial invariant subspace or, equivalently, the characteristic polynomial of A is irreducible over F.

Theorem 2.2.22. *(i) Let $n > 1$, F a field with $\mathrm{ch}(F) = 0$ or $> n/2$, and $\mathcal{A}$ an algebra in $M_n(F)$ on which the trace is zero. Then either the algebra $\mathcal{A}$ is reducible, or else $\mathrm{ch}(F) = n$, every nonscalar $A \in \mathcal{A}$ is irreducible in $M_n(F)$, and $\mathcal{A} = F[A]$.*

(ii) Let F be a field with $\mathrm{ch}(F) = 0$ or $> n/2$ where $n > 1$, and $\mathcal{S}$ a semigroup in $M_n(F)$ on which trace is constant. Then either the semigroup $\mathcal{S}$ is reducible, or else $\mathrm{ch}(F) = n$, trace is zero on $\mathcal{S}$, every nonscalar $A \in \mathrm{Alg}(\mathcal{S})$ is irreducible in $M_n(F)$, and $\mathcal{S} \subseteq F[A]$.

Proof. (i) If the algebra $\mathcal{A}$ is reducible we have nothing to prove. So suppose that the algebra $\mathcal{A}$ is irreducible. Let r and $\mathcal{D}_r$ be as in Theorem 2.2.18. From Theorem

2.2.19 we see that $\mathrm{ch}(F)$ is nonzero and that $\mathrm{ch}(F)$ divides r. Hence $r \geq \mathrm{ch}(F) > n/2$. Since $\mathrm{ch}(F)$ divides r and r divides n, we conclude that $\mathrm{ch}(F) = r = n$. From $r = n$ and the hypothesis that the algebra $\mathcal{A}$ is irreducible we see that $\mathcal{A}$ is an irreducible division algebra on which trace is zero. Now since $\mathrm{ch}(F) = n = r$, from the proof of Lemma 2.2.5(i) it is easily seen that every A in $\mathcal{A}$ is of the form $\alpha I_n + N \in M_n(\overline{F})$ where $\overline{F}$ denotes the algebraic closure of F and N nilpotent. Thus, we conclude form Corollary 2.2.16 that the algebra $\mathcal{A}$ is triangularizable on $\overline{F}$. This along with the fact that $\mathcal{A}$ is a division algebra implies that the algebra $\mathcal{A}$ is commutative, for $AB - BA$ must be nilpotent, hence 0, for all $A, B \in \mathcal{A}$. The minimal polynomial for every $A \in \mathcal{A}$ is irreducible over F because $\mathcal{A}$ is a division algebra in $M_n(F)$ (equivalently $r = n$). That being noted, let A be any nonscalar matrix in $\mathcal{A}$. Using Exercise V.6.6 of [H] and the fact that $\mathrm{ch}(F) = n = r$, the characteristic polynomial of A, say f, equals the minimal polynomial of A and we have $f = x^n - a$ for some $a \in F$. Now since f is irreducible over F, we see that the matrix A is irreducible as an element of $M_n(F)$. Finally, irreducibility of the matrix A together with the fact that the algebra $\mathcal{A}$ is commutative easily yields $\mathcal{A} = F[A]$, as desired.

(ii) Suppose that the semigroup $\mathcal{S}$ is irreducible. Set $\mathcal{A} := \mathrm{Alg}(\mathcal{S})$. The algebra $\mathcal{A}$ is irreducible and hence it is simple in view of Lemma 2.2.17. We can write

$$\mathcal{A} = \{\sum_{i=1}^{k} c_i S_i : k \in \mathbb{N},\ c_i \in F, S_i \in \mathcal{S},\ 1 \leq i \leq k\}.$$

Since $\mathcal{S}$ is irreducible, it follows from the proof of Theorem 2.2.13 that there exist $S, S' \in \mathcal{S}$ such that $\{S, S'\}$ is linearly independent. This in turn implies that the set $\mathcal{J} := \{A \in \mathcal{A} : \mathrm{tr}(A) = 0\}$ is a nonzero ideal of the simple algebra $\mathcal{A}$. Hence, $\mathcal{J} = \mathcal{A}$ for $\mathcal{A}$ is simple. Now (i) completes the proof. □

Remarks.

1. Under the hypotheses of the preceding theorem, the semigroup $\mathcal{S}$ would be reducible provided that $\mathcal{S}$ is not trivially irreducible, or noncommutative, or contains a reducible matrix that is not scalar, or n is not a prime number.

2. Let $\mathbb{Z}_2$ be the field of integers modulo 2, $\mathbb{Z}_2[X]$ denote the ring of polynomials over $\mathbb{Z}_2$, and $F := \mathbb{Z}_2(X)$ denote the quotient field of $\mathbb{Z}_2[X]$ (note that $\mathbb{Z}_2[X]$ is an

integral domain and that $\mathrm{ch}(F) = 2$). It is easily seen that the matrix $A \in M_2(F)$ defined by

$$A := \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix}$$

is irreducible in $M_2(F)$, and that trace is zero on the irreducible algebra $\mathcal{A} := \mathrm{Alg}(A) = F[A]$.

2.3 Main Results

Let K be a field and F a subfield of K. By an *F-algebra* $\mathcal{A}$ in $M_n(K)$, we mean a subring of $M_n(K)$ that is closed under scalar multiplication by the elements of the subfield F. For a semigroup $\mathcal{S}$ in $M_n(K)$, let $\mathrm{Alg}_F(\mathcal{S})$ denote the F-algebra generated by $\mathcal{S}$, i.e.,

$$\mathrm{Alg}_F(\mathcal{S}) := \{\sum_{i=1}^{k} \alpha_i S_i : k \in \mathbb{N}, \alpha_i \in F, S_i \in \mathcal{S}\}.$$

By $\mathrm{Alg}(\mathcal{S})$ we simply mean $\mathrm{Alg}_K(\mathcal{S})$. If $m \in \mathbb{N}$, we use $\mathcal{S}^m$ to denote the semigroup ideal of $\mathcal{S}$ consisting of products of the members of $\mathcal{S}$ of "apparent length" m, i.e.,

$$\mathcal{S}^m = \{S_1 ... S_m : S_i \in \mathcal{S}, i = 1, ..., m\}.$$

A semigroup ideal $\mathcal{J}$ of $\mathcal{S}$ is called an *absorbing semigroup ideal* of $\mathcal{S}$ if there exists $m \in \mathbb{N}$ such that $\mathcal{S}^m \subseteq \mathcal{J}$. Note that $\mathcal{S}^m$ is an absorbing semigroup ideal of $\mathcal{S}$ for each $m \in \mathbb{N}$.

For a given $T \in M_n(F)$, we use $\bar{\sigma}(T)$ to denote the spectrum of T in the algebraic closure of F. Note that, by the proof of Lemma 2.2.4 and the Cayley-Hamilton Theorem, $\bar{\sigma}(T) = \{0\}$ iff T is nilpotent.

In what follows the following three theorems are crucial.

Theorem 2.3.1. *Let $n \in \mathbb{N}$, F a field, $\mathcal{S}$ an irreducible semigroup in $M_n(F)$ on which trace is not identically zero, and $\mathcal{J}$ a nonzero semigroup ideal of $\mathcal{S}$. Then*

$$\{A \in \mathrm{Alg}(\mathcal{S} \cup \{I\}) : \mathrm{tr}(A\mathcal{J}) = \{0\}\} = \{0\}.$$

Proof. Denote the left hand side of the asserted identity by $\mathfrak{J}$. Set $\mathcal{A} := \mathrm{Alg}(\mathcal{S})$. The algebras $\mathcal{A}$ and $\mathrm{Alg}(\mathcal{J})$ are irreducible algebras in $M_n(F)$ since $\mathcal{S}$ is an irreducible semigroup in $M_n(F)$ and $\mathcal{J}$ is a nonzero semigroup ideal of $\mathcal{S}$. Note that $\mathrm{Alg}(\mathcal{J})$ is a nonzero ideal of $\mathcal{A}$ because $\mathcal{J}$ a nonzero semigroup ideal of $\mathcal{S}$ and that $\mathcal{A}$ is the linear span of $\mathcal{S}$. Therefore, $\mathcal{A} = \mathrm{Alg}(\mathcal{J})$, for the algebra $\mathcal{A}$ is simple by Lemma 2.2.17. Again from Lemma 2.2.17, we see that $\mathcal{A} = \mathrm{Alg}(\mathcal{S}) = \mathrm{Alg}(\mathcal{S} \cup \{I\})$. Due to linearity of the trace functional and the fact that $\mathcal{A} = \mathrm{Alg}(\mathcal{J})$ is unital, it is not difficult to see that

$$\mathfrak{J} = \{A \in \mathrm{Alg}(\mathcal{S}) : \mathrm{tr}(A\mathcal{J}) = \{0\}\} = \{A \in \mathcal{A} : \mathrm{tr}(A\mathcal{A}) = \{0\}\}$$

is an ideal of the simple algebra $\mathcal{A}$ consisting of matrices with traces zero. Thus $\mathfrak{J} = \{0\}$, for otherwise $\mathfrak{J} = \mathcal{A}$, and hence $\mathrm{tr}(\mathcal{A}) = \{0\}$, contradicting the hypothesis that trace is not identically zero on $\mathcal{S}$. □

Remarks.

1. Let $n \in \mathbb{N}$, F a field, $\mathcal{S}$ an irreducible semigroup in $M_n(F)$, and $\mathcal{J}$ a nonzero semigroup ideal of $\mathcal{S}$. From irreducibility of $\mathcal{S}$, in view of Lemma 1.1.2 and Lemma 2.2.17, we obviously have $\mathrm{Alg}(\mathcal{J}) = \mathrm{Alg}(\mathcal{J}^2)$. This in turn implies that

$$\{A \in \mathrm{Alg}(\mathcal{S} \cup \{I\}) : \mathrm{tr}(A\mathcal{J}) = \{0\}\} = \{A \in \mathrm{Alg}(\mathcal{S} \cup \{I\}) : \mathrm{tr}(\mathcal{J}A\mathcal{J}) = \{0\}\}.$$

Therefore, either trace is identically zero on the irreducible semigroup $\mathcal{S}$, or else

$$\{A \in \mathrm{Alg}(\mathcal{S} \cup \{I\}) : \mathrm{tr}(\mathcal{J}A\mathcal{J}) = \{0\}\} = \{0\}.$$

2. In the preceding theorem if $n > 1$ and the ground field F happens to be perfect, or the field F is such that $\mathrm{ch}(F) = 0$ or $\mathrm{ch}(F)$ does not divide n, then, in view of Theorem 2.2.19, it follows that the hypothesis that trace is not identically zero on $\mathcal{S}$ is redundant.

3. In light of Theorem 2.2.22, it is easily seen that if F is a field with $\mathrm{ch}(F) = 0$ or $> n/2$ where $n > 1$, $\mathcal{S}$ an irreducible semigroup in $M_n(F)$, and $\mathcal{J}$ a nonzero

semigroup ideal of $\mathcal{S}$, then either the conclusion of the theorem above holds or $\text{ch}(F) = n$, trace is zero on $\mathcal{S}$, every nonscalar $A \in \text{Alg}(\mathcal{S})$ is irreducible in $M_n(F)$, and $\text{Alg}(\mathcal{S}) = F[A]$. Therefore, the preceding theorem holds provided $\text{ch}(F) = 0$ or $> n/2$ and that $\mathcal{S}$ is a nontrivial irreducible semigroup in $M_n(F)$. This also shows that there are irreducible semigroups on which trace is identically zero.

4. Let F and $A \in M_2(F)$ be as in the remark following Theorem 2.2.22. Note that $\text{ch}(F) = 2$, the field F is not perfect nor is it 2-closed. Set $\mathcal{A} = F[A]$ and let $\mathcal{S} = \mathcal{J} = \mathcal{A}$. Trace is zero on the irreducible algebra $\mathcal{A}$. This gives an explicit example of an irreducible semigroup with zero trace.

5. It is worth mentioning that the equality in the preceding theorem does not imply irreducibility of the semigroup $\mathcal{S}$. For instance the equality holds for the diagonal semigroup $\mathcal{S} := \{\text{diag}(d_1, ..., d_n) : d_j \in F,\ i = 1, ..., n\}$ where $n > 1$.

6. In Chapter 4, we prove an analogue of the above theorem for division rings.

In the preceding theorem the asserted identity holds with "$\bar{\sigma}$" replacing "tr" and with no condition imposed on the ground field F nor on the irreducible semigroup in terms of trace. More precisely, we have the following theorem.

Theorem 2.3.2. *Let $n \in \mathbb{N}$, F a field, $\mathcal{S}$ an irreducible semigroup in $M_n(F)$, and $\mathcal{J}$ a nonzero semigroup ideal of $\mathcal{S}$. Then*

(i)
$$\{A \in \text{Alg}(\mathcal{S} \cup \{I\}) : \bar{\sigma}(\mathcal{J}A\mathcal{J}) = \{0\}\} = \{0\}.$$

(ii)
$$\{A \in \text{Alg}(\mathcal{S} \cup \{I\}) : \bar{\sigma}(A\mathcal{J}) = \{0\}\} = \{0\}.$$

Proof. It suffices to prove (i).

(i) Denote the left hand side of the asserted identity by $\mathfrak{J}$. We prove that $\mathfrak{J} = \{0\}$. To this end, let $A \in \mathfrak{J}$ be arbitrary, we show that $A = 0$. Plainly the set $\mathcal{J}A\mathcal{J} = \{J_1AJ_2 : J_1, J_2 \in \mathcal{J}\}$ is a subset of $\text{Alg}(\mathcal{J})$ consisting of nilpotents. The algebra $\text{Alg}(\mathcal{J})$ is irreducible in $M_n(F)$, for $\mathcal{S}$ is an irreducible semigroup in $M_n(F)$ and $\mathcal{J}$ is a nonzero semigroup ideal of $\mathcal{S}$. It is easily seen that $\text{Alg}(\mathcal{J}A\mathcal{J})$ is an ideal of the irreducible, hence simple, algebra $\text{Alg}(\mathcal{J})$. We note that $\text{Alg}(\mathcal{J}A\mathcal{J}) \neq \text{Alg}(\mathcal{J})$, for otherwise the irreducible algebra $\mathrm{Alg}(\mathcal{J})$ would be generated by nilpotents as a vector subspace of $M_n(F)$ which is a contradiction in view of Remark 1 following Theorem 2.2.13. Hence $\mathrm{Alg}(\mathcal{J}A\mathcal{J}) = \{0\}$. Therefore, $\mathcal{J}A\mathcal{J} = \{0\}$, and hence $A = 0$, for $\mathrm{Alg}(\mathcal{J}) = \langle\mathcal{J}\rangle$ is unital by Lemma 2.2.17. □

Remark. If $F = \mathbb{R}$ or $\mathbb{C}$, we may write ρ instead of $\bar{\sigma}$ in (i) and (ii) where ρ stands for the spectral radius.

If the ground field happens to be k-closed for each k dividing n with $k > 1$, then in Theorem 2.3.1 not only the condition that trace is not identically zero on $\mathcal{S}$ can be dropped, but also a lot more can be said. More precisely, we have:

Theorem 2.3.3. *Let $n > 1$, F a field that is k-closed for each k dividing n with $k > 1$ (resp. F be a field) , $\mathcal{S}$ an irreducible (resp. absolutely irreducible) semigroup in $M_n(F)$, and $\mathcal{J}$ a nonzero semigroup ideal of $\mathcal{S}$. Then* $M_n(F) = \mathrm{Alg}(\mathcal{S}) = \mathrm{Alg}(\mathcal{S} \cup \{I\}) = \mathrm{Alg}(\mathcal{J})$, *moreover*

(i)

$$\{A \in M_n(F) : \mathrm{tr}(A\mathcal{J}) = \{0\}\} = \{0\}.$$

(ii)

$$\{A \in M_n(F) : \bar{\sigma}(A\mathcal{J}) = \{0\}\} = \{0\}.$$

Proof. First we note that in view of Lemma 1.1.2(i), irreducibility (resp. absolutely irreducibility) of $\mathcal{S}$ implies that of the nonzero ideal $\mathcal{J}$. Thus it follows from Theorem 2.2.21 (resp. the definition) that $M_n(F) = \mathrm{Alg}(\mathcal{S}) = \mathrm{Alg}(\mathcal{S} \cup \{I\}) = \mathrm{Alg}(\mathcal{J})$. It is evident that it suffices to prove (i).

(i) Denote the left hand side of the asserted identity by $\mathfrak{J}$. Let $J \in \mathfrak{J}$ be given. Since $M_n(F) = \mathrm{Alg}(\mathcal{J})$, linearity of trace yields $\mathrm{tr}(JM_n(F)) = \{0\}$. It is now easily seen that $\mathfrak{J}$ is indeed a two-sided ideal of the ring $M_n(F)$. It follows from elementary algebra that $\mathfrak{J} = \{0\}$ or $\mathfrak{J} = M_n(F)$. On the other hand, the fact that $\mathrm{tr}(\mathfrak{J}M_n(F)) = \{0\}$ implies $\mathrm{tr}(\mathfrak{J}) = \{0\}$ and so $\mathfrak{J} \neq M_n(F)$. Hence $\mathfrak{J} = \{0\}$. □

Remark. Under the hypotheses of the preceding theorem we indeed have

$$\{A \in M_n(F) : \mathrm{tr}(\mathcal{J}A\mathcal{J}) = \{0\}\} = \{0\}.$$

Part (i) of the following corollary characterizes all irreducible families of matrices over a field on which trace is permutable. Part (iii) is a special case of a result in [R2].

Corollary 2.3.4. *(i) Let $n > 1$, F a field, and $\mathfrak{F}$ an irreducible subset of $M_n(F)$ on which trace is permutable. Then either* $\mathrm{tr}(\mathrm{Alg}(\mathfrak{F})) = \{0\}$ *or else* $\mathrm{Alg}(\mathfrak{F})$ *is an extension field of F.*

(ii) Let $n > 1$, F be a field that is perfect or such that $\mathrm{ch}(F)$ *does not divide n, and $\mathfrak{F}$ an irreducible collection of matrices in $M_n(F)$ on which trace is permutable. Then there exists an irreducible matrix A in* $\mathrm{Alg}(\mathfrak{F})$ *such that $\mathfrak{F} \subseteq F[A]$. Therefore, such a collection $\mathfrak{F} \subset M_n(F)$ exists iff there exists an irreducible polynomial of degree n over F.*

(iii) Let $n > 1$, F a field with $\mathrm{ch}(F) = 0$ or $> n$. *Then a collection $\mathfrak{F}$ of triangularizable matrices in $M_n(F)$ is triangularizable if and only if trace is permutable on $\mathfrak{F}$. In particular, the assertion holds for any collection $\mathfrak{F}$ of matrices in $M_n(F)$ if the underlying field F is algebraically closed or, more generally, is k-closed for each $k = 2, ..., n$.*

Proof. (i) Suppose that trace is not identically zero on $\mathcal{A} := \mathrm{Alg}(\mathfrak{F})$, we first show that $\mathfrak{F}$, hence $\mathrm{Alg}(\mathfrak{F})$, is commutative. To this end, let $A, B \in \mathfrak{F}$ be arbitrary. Since trace is permutable on $\mathfrak{F}$, it follows that $\mathrm{tr}((AB - BA)\mathrm{Alg}(\mathfrak{F})) = \{0\}$, and hence $AB = BA$ by Theorem 2.3.1. So we have proved that $\mathrm{Alg}(\mathfrak{F})$ is commutative. From this together with irreducibility of $\mathrm{Alg}(\mathfrak{F})$ we see that every nonzero element of $\mathrm{Alg}(\mathfrak{F})$ is invertible because the kernel of every element of $\mathrm{Alg}(\mathfrak{F})$ is invariant under $\mathfrak{F}$. Therefore, $\mathrm{Alg}(\mathfrak{F})$ is indeed an extension field of F, completing the proof.

(ii) The hypothesis together with the second remark following Theorem 2.2.19 implies that trace cannot be identically zero on $\mathrm{Alg}(\mathfrak{F})$. So it follows from part (i) that $\mathrm{Alg}(\mathfrak{F})$ is an extension field of F. If F is perfect, the assertion is obvious in view of the Primitive Element Theorem (Proposition V.6.15 of [H]). If F is a field such that $\mathrm{ch}(F)$ does not divide n, then $\mathrm{ch}(F)$ does not divide $\dim_F \mathrm{Alg}(\mathfrak{F})$ either. To see this, use contradiction. If $\mathrm{ch}(F)$ divides $\dim_F \mathrm{Alg}(\mathfrak{F})$, then $\mathrm{ch}(F)$ divides n^2 for $\dim_F \mathrm{Alg}(\mathfrak{F})$ divides $\dim_F M_n(F) = n^2$ by Lemma 7.4 of [D]. Hence $\mathrm{ch}(F)$ divides

n because ch(F) is a prime number, a contradiction. Thus, ch(F) does not divide $\dim_F \mathrm{Alg}(\mathcal{F})$. Now the assertion follows from Exercise V.6.8 of [H] and the Primitive Element Theorem, finishing the proof.

(iii) If the collection $\mathcal{F}$ is triangularizable, it is not hard to see that trace is permutable on $\mathcal{F}$. So it remains to prove the "if part" of the assertion. To this end, suppose that trace is permutable. Using permutability of trace and induction on k, it is not hard to show that $\mathrm{tr}(A_1 ... A_n - A_{\sigma 1} ... A_{\sigma n})^k = 0$ for all $k \in \mathbb{N}$, $A_1, ..., A_n \in \mathcal{F}$, and all permutations σ on n letters. In other words $A_1 ... A_n - A_{\sigma 1} ... A_{\sigma n}$ is nilpotent (here we have used the characteristic condition on F). This in turn implies that the property of permutability of trace is inherited by quotients. Thus to show that $\mathcal{F}$ is triangularizable we only need to show that $\mathcal{F}$ is reducible. If $\mathcal{F}$ is commutative, then it is easily seen that $\mathcal{F}$ is reducible. So suppose that $\mathcal{F}$ is not commutative. Hence there are $A, B \in \mathcal{F}$ with $AB - BA \neq 0$. Now let $\mathcal{S}$ be the semigroup generated by $\mathcal{F}$. Note that $\mathcal{F}$ is reducible iff $\mathcal{S}$ is. Note that $0 \neq AB - BA \in \mathrm{Alg}_F(\mathcal{S})$ and, by permutability of trace on $\mathcal{F}$, we have $\mathrm{tr}((AB - BA)C) = 0$ for all $C \in \mathcal{S}$. That is $0 \neq AB - BA \in \mathrm{Alg}_F(\mathcal{S})$ and $\mathrm{tr}((AB - BA)\mathcal{S}) = 0$. Therefore, Theorem 2.3.1 (with $\mathcal{J} = \mathcal{S}$) implies that $\mathcal{S}$, and hence $\mathcal{F}$, is reducible, finishing the proof. □

Remarks.

1. The conclusion of Corollary 2.3.4(iii) holds for any collection $\mathcal{F}$ of matrices that contains a matrix whose eigenspaces are all one-dimensional. Note that no condition on other members of $\mathcal{F}$ is imposed.

2. If the underlying field F is k-closed for each k dividing n with $k > 1$ (no condition on ch(F) is imposed), then permutability of trace on a collection $\mathcal{F}$ of matrices in $M_n(F)$ implies reducibility of the collection.

3. As shown in [R2], the conclusion of Corollary 2.3.4(iii) holds true under the weaker hypothesis that $\mathrm{ch}(F) = 0$ or $> n/2$.

4. Let p be a prime number, and F a field. In view of Corollary II.8.17 of [M], and the proof of Theorem 2.2.22, it is not difficult to see that if trace is permutable on an irreducible family $\mathcal{F}$ in $M_p(F)$, then there exists a matrix $A \in \mathrm{Alg}(\mathcal{F})$ such that $\mathrm{Alg}(\mathcal{F}) = F[A]$. Therefore, an irreducible family of matrices in $M_p(F)$ on which trace is permutable is a subset of $F[A]$ where A is an irreducible matrix in the algebra generated by the family.

Corollary 2.3.5. *(i) Let $n > 1$, F a field, $\mathcal{S}$ a semigroup in $M_n(F)$ on which trace is not identically zero, and $\mathcal{J}$ a nonzero semigroup ideal of $\mathcal{S}$. If there exists a nonzero functional f on $M_n(F)$ defined by $f(X) := \mathrm{tr}(AX)$ where $A \in \mathrm{Alg}(\mathcal{S} \cup \{I\})$ such that f is zero on $\mathcal{J}$, then $\mathcal{S}$ is reducible.*

(ii) Let $n > 1$, F a field with $\mathrm{ch}(F) = 0$ or $> n$, $m \in \mathbb{N}$, and $\mathcal{F}$ a family of triangularizable matrices in $M_n(F)$. Then $\mathcal{F}$ is triangularizable iff $\mathrm{tr}((AB-BA)\mathcal{S}^m) = 0$ for all $A, B \in \mathcal{F}$ where $\mathcal{S}$ denotes the semigroup generated by $\mathcal{F}$.

(iii) Let $n > 1$, F a field with $\mathrm{ch}(F) = 0$ or $> n$, $m \in \mathbb{N}$, and A, B triangularizable matrices in $M_n(F)$. Then $\{A, B\}$ is triangularizable iff $\mathrm{tr}((AB - BA)S) = 0$ for all words S in A and B of length at least m.

(iv) Let $n > 1$, F a field with $\mathrm{ch}(F) = 0$ or $> n$, $m \in \mathbb{N}$, $\mathcal{S}$ a semigroup of triangularizable matrices in $M_n(F)$. Then $\mathcal{S}$ is triangularizable iff $\mathrm{tr}((AB-BA)C) = 0$ for all $A, B \in \mathcal{S}$, $C \in \mathcal{S}^m$.

(v) Let $n > 1$, F a field, and $\mathcal{A}$ a ring of matrices in $M_n(F)$. Then $\mathcal{A}$ is triangularizable iff $\{A, B\}$ is triangularizable for all $A, B \in \mathcal{A}$.

Proof. (i) Theorem 2.3.1.

(ii) Necessity follows from the Spectral Mapping Theorem (Theorem 1.1.8 of [RR]). To prove sufficiency, first we note that the property that $\mathrm{tr}((AB - BA)\mathcal{S}^m) = 0$ is inherited by quotients. To see this we simply note that if $A, B \in \mathcal{F}$ and $S \in \mathcal{S}^m$, then

$$((AB - BA)S)^k = (AB - BA)C,$$

for some $C \in \mathrm{Alg}(\mathcal{S}^m)$. This in turn, along with the hypothesis, implies that $\mathrm{tr}(((AB-BA)S)^k) = 0$ for all $k \in \mathbb{N}$, $A, B \in \mathcal{F}$, and $S \in \mathcal{S}^m$. Hence $(AB - BA)\mathcal{S}^m$ is nilpotent for every $A, B \in \mathcal{F}$. Therefore, the property that $\mathrm{tr}((AB - BA)\mathcal{S}^m) = 0$ for all $A, B \in \mathcal{F}$ is inherited by quotients. So in view of the Triangularization Lemma (Lemma 1.1.3) it suffices to prove reducibility of $\mathcal{F}$. If $AB = BA$ for all $A, B \in \mathcal{F}$, then reducibility easily follows. If $AB \neq BA$ for some $A, B \in \mathcal{F}$, then $AB - BA \neq 0$ and we would have $\mathrm{tr}((AB - BA)\mathcal{S}^m) = 0$. Now reducibility follows from part (i) or from Theorem 2.3.1, completing the proof.

(iii) This is a special case of (ii) where $\mathcal{F}$ is a pair $\{A, B\}$ of triangularizable matrices over F.

(iv) This is a special case of (ii) where $\mathcal{F} = \mathcal{S}$ is in fact a semigroup of triangularizable matrices.

(v) In light of Corollary 2.2.8 without loss of generality we may assume that the ground field F is algebraically closed. Plainly it suffices to prove sufficiency. To this end, again in view of the Triangularization Lemma (Lemma 1.1.3) it suffices to prove reducibility of $\mathcal{A}$. If $AB = BA$ for all $A, B \in \mathcal{A}$, then reducibility easily follows. If $AB \neq BA$ for some $A, B \in \mathcal{A}$, then $0 \neq AB - BA \in \mathcal{A}$ and $\{A, B\}$ would be a triangularizable pair of matrices. Now let $C \in \mathcal{A}$ be arbitrary. It follows from the hypothesis that the pair $\{AB - BA, C\}$ is triangularizable. Since $AB - BA$ is nilpotent, it follows from the Spectral Mapping Theorem that so is $(AB - BA)C$. In particular, we would have $\operatorname{tr}((AB - BA)C) = 0$. That is $\operatorname{tr}((AB - BA)\mathcal{A}) = 0$. Hence reducibility follows from Theorem 2.3.3 with $\mathcal{S} = \mathcal{A}$. □

Remarks.

1. Corollary 2.3.5(ii), Corollary 2.3.5(iii), and Corollary 2.3.5(iv) are generalizations of results due to Guralnick, McCoy, and Radjavi respectively.

2. The conclusion of Corollary 2.3.5(ii) (resp. Corollary 2.3.5(iv), Corollary 2.3.5(v)) holds for any collection $\mathcal{F}$ (resp. semigroup $\mathcal{S}$, subring $\mathcal{A}$) of matrices that contains a matrix whose eigenspaces are all one-dimensional.

Corollary 2.3.6. *(i) Let F be a field that is k-closed for each k dividing n with $k > 1$ where $n \in \mathbb{N}$ is > 1, and $\mathcal{S}$ a semigroup in $M_n(F)$, and $\mathcal{J}$ a nonzero semigroup ideal of $\mathcal{S}$. If there exists a nonzero functional f on $M_n(F)$ defined by $f(X) := \operatorname{tr}(AX)$ where $A \in M_n(F)$ such that f is zero on $\mathcal{J}$, then $\mathcal{S}$ is reducible.*

(ii) Let F be a field, $n, m \in \mathbb{N}$, and $\mathcal{F}$ a family of triangularizable matrices in $M_n(F)$. Then $\mathcal{F}$ is triangularizable iff $(AB - BA)\mathcal{S}^m$ is nilpotent for all $A, B \in \mathcal{F}$ where $\mathcal{S}$ denotes the semigroup generated by $\mathcal{F}$.

(iii) Let F be a field, $n, m \in \mathbb{N}$, and A, B triangularizable matrices in $M_n(F)$. Then $\{A, B\}$ is triangularizable iff $(AB - BA)S$ is nilpotent where S is any word in A and B of length at least m.

(iv) Let F be a field, $n, m \in \mathbb{N}$, $\mathcal{S}$ a semigroup of triangularizable matrices in $M_n(F)$. Then $\mathcal{S}$ is triangularizable iff $(AB - BA)C$ is nilpotent for all $A, B \in \mathcal{S}$, $C \in \mathcal{S}^m$.

Proof. Note that, in view of Corollary 2.2.8, in (ii)-(iv) without loss of generality we may assume that F is algebraically closed for all matrices are assumed to be individually triangularizable over F. That being noted, the proof is similar to that of Corollary 2.3.5 except that we need to use Theorem 2.3.3. We omit the proof for the sake of brevity. □

For a subset $C \subseteq K$, the symbol $\langle C\rangle_F$ is used to denote the linear manifold spanned by C over F. Suppose that K is a field, F a subfield of K, and $\mathcal{S}$ a semigroup $\mathcal{S}$ in $M_n(K)$. It is easily verified that $\text{Alg}_F(\mathcal{S}) = \langle \mathcal{S}\rangle_F$.

Lemma 2.3.7. *Let $n > 1$, K a field, F a subfield of K, $\mathcal{S}$ an irreducible semigroup in $M_n(K)$ such that $\{0\} \neq \text{tr}(\mathcal{S}) \subseteq F$. Suppose that $\{S_1, ..., S_m\}$ is a subset of $\mathcal{S}$ that is linearly independent over F. If $A \in \text{Alg}_F(\mathcal{S} \cup \{I\})$ and $A = c_1S_1 + ... + c_mS_m$ where $c_i \in K$ for all $i = 1, ..., m$, then $c_i \in F$ for all $i = 1, ..., m$. Therefore, a subset $\{S_1, ..., S_m\}$ of $\mathcal{S}$ is linearly independent over F if and only if it is linearly independent over K.*

Remark. Although we shall soon see that $\text{Alg}_F(\mathcal{S} \cup \{I\}) = \text{Alg}_F(\mathcal{S})$, at this point we need the above lemma in its current form.

Proof. By the hypothesis we have

$$A = c_mS_m + ... + c_1S_1, \tag{$*$}$$

where $c_i \in K$ for all $1 \leq i \leq m$. Set $c_{m+1} := 1 \in F$. By proving that $c_j \in \langle c_{j+1}, ..., c_m, 1\rangle_F$ for each $j = 1, ..., m$, we show that $c_j \in F$ for all $1 \leq j \leq m$. First note that $c_1 \in \langle c_2, ..., c_m, 1\rangle_F$. To see this, since $0 \neq S_1 \in \mathcal{S} \subset \text{Alg}_F(\mathcal{S}\cup\{I\})$, it follows from Theorem 2.3.1 that there exists $S \in \mathcal{S}$ such that $\text{tr}(S_1S) \neq 0$. Multiplying ($*$) by S from the right, then taking trace of both sides, and dividing by $\text{tr}(S_1S)$, we conclude that $c_1 \in \langle c_2, ..., c_m, 1\rangle_F$. Let j_0 be the largest j for which $c_i \in \langle c_{i+1}, ..., c_m, 1\rangle_F$ for $i = 1, ..., j$. If $j_0 = m$, we are done. Suppose, otherwise, that $j_0 < m$, we show that $c_{j_0+1} \in \langle c_{j_0+2}, ..., c_m, 1\rangle_F$, contradicting the fact that j_0 is the largest index having the aforementioned property. It is plain that

$$c_i \in \langle c_{j_0+1}, ..., c_m, 1\rangle_F,$$

for all $1 \leq i \leq j_0$. So for each $i = 1, ..., j_0$ we can write

$$c_i = r_i c_{j_0+1} + n_i$$

where $r_i \in F$, $n_i \in \langle c_{j_0+2}, ..., c_m, 1\rangle_F \subseteq K$. Thus we can write

$$A = c_m S_m + ... + c_1 S_1$$

$$= c_m S_m + c_{m-1} S_{m-1} + ... + c_{j_0+1} S_{j_0+1} + ... + c_1 S_1$$

$$= c_m S_m + c_{m-1} S_{m-1} + ... + c_{j_0+1}(S_{j_0+1} + r_{j_0} S_{j_0} + ... + r_1 S_1)$$

$$+ n_{j_0} S_{j_0} + ... + n_1 S_1.$$

We have $0 \neq S_{j_0+1} + r_{j_0} S_{j_0} + ... + r_1 S_1 := B \in \mathrm{Alg}_F(\mathcal{S})$ for $\{S_1, ..., S_m\} \subset M_n(K)$ is independent over F. So from Theorem 2.3.1 we see that there exists $S' \in \mathcal{S}$ such that $\mathrm{tr}(BS') \neq 0$. By multiplying both sides of $(*)$ by S' from the right we can write

$$AS' =$$

$$c_m S_m S' + c_{m-1} S_{m-1} S' + ... + c_{j_0+1}(BS') + n_{j_0} S_{j_0} S' + ... + n_1 S_1 S'.$$

Taking trace of both sides and dividing by $\mathrm{tr}(BS') \in F$, we conclude that $c_{j_0+1} \in \langle c_{j_0+2}, ..., c_m, 1\rangle_F$, a contradiction. Therefore, $j_0 = m$ and so $c_i \in F$ for all $1 \leq i \leq m$, finishing the proof. □

Corollary 2.3.8. *Let $n > 1$, K a field, F a subfield of K, $\mathcal{S}$ an irreducible semigroup in $M_n(K)$ such that $\{0\} \neq \mathrm{tr}(\mathcal{S}) \subseteq F$. Then $\mathrm{Alg}_F(\mathcal{S} \cup \{I\}) = \mathrm{Alg}_F(\mathcal{S})$, the minimal polynomial of every element of $\mathrm{Alg}_F(\mathcal{S})$ is in $F[X]$, and $\mathrm{Alg}_F(\mathcal{S})$ is semisimple both as as a ring and an F-algebra.*

Proof. To prove $\mathrm{Alg}_F(\mathcal{S} \cup \{I\}) = \mathrm{Alg}_F(\mathcal{S})$, it suffices to show that $I \in \mathrm{Alg}_F(\mathcal{S})$.

To this end, first note that $\text{Alg}(\mathcal{S}) = \langle \mathcal{S} \rangle$ is an irreducible algebra in $M_n(K)$. By Lemma 2.2.17, $I \in \langle \mathcal{S} \rangle$. Now this together with the fact that $I \in \text{Alg}_F(\mathcal{S} \cup \{I\})$ yields $I \in \langle \mathcal{S} \rangle_F = \text{Alg}_F(\mathcal{S})$ in view of the preceding lemma. To prove the rest of the assertion, set $\mathcal{A}_F := \text{Alg}_F(\mathcal{S})$ and let $A \in \mathcal{A}_F$ be given. From the hypothesis we easily see that $\mathcal{A}_F$ is an irreducible F-algebra and $\text{tr}(\mathcal{A}_F) \subseteq F$. Suppose that

$$m = x^k - m_{k-1}x^{k-1} - ... - m_1 x - m_0$$

with $k \leq n$ is the minimal polynomial of the given A. We need to show that $m_i \in F$ for each $i = 0, ..., k-1$. We have

$$A^k = m_{k-1}A^{k-1} + ... + m_1 A + m_0 I. \qquad (*)$$

By minimality of m, the set $\{A^{k-1}, ..., A, I\} \subset \mathcal{A}_F$ is independent over K, hence over the subfield F. On the other hand, $A^k \in \mathcal{A}_F$. This together with $(*)$ shows that $m_i \in F$ for each $i = 0, ..., k-1$ in light of the preceding lemma. Finally, since the minimal polynomial of every element of the F-algebra $\text{Alg}_F(\mathcal{S})$ is in $F[X]$, it follows that $\text{Alg}_F(\mathcal{S})$ is an algebraic F-algebra. From the proof of Lemma 2.2.17 we conclude that $\text{Alg}_F(\mathcal{S})$ is semisimple as an F-algebra, hence as a ring for $\text{Alg}_F(\mathcal{S})$ is unital, completing the proof. □

Corollary 2.3.9. *Let $n > 1$, K a field, F a subfield of K, and $\mathcal{A}$ an irreducible F-algebra in $M_n(K)$ such that $\{0\} \neq \text{tr}(\mathcal{A}) \subseteq F$. Then the F-algebra $\mathcal{A}$ is unital, the minimal polynomial of every element of $\mathcal{A}$ is in $F[X]$, and $\mathcal{A}$ is semisimple both as as a ring and an F-algebra.*

Proof. Corollary 2.3.8. □

Remark. *Let $n > 1$, K a field that is k-closed for each k dividing n with $k > 1$, and F a subfield of K. Then every irreducible F-algebra $\mathcal{A}$ in $M_n(K)$ with traces in F is central, i.e., the center of $\mathcal{A}$ consists of cI_n's where $c \in F$.* To see this, by Theorem 2.2.21, $\mathcal{A}' = \{cI_n : c \in K\}$. As well, trace is not identically zero on the F-algebra $\mathcal{A}$ by Theorem 2.3.3. So, due to the fact that the F-algebra $\mathcal{A}$ is unital by the preceding corollary, it suffices to show that if $cI_n \in \mathcal{A}$ for some nonzero $c \in K$, then $c \in F$. To prove this, from Theorem 2.3.3 we see that there exists $A_0 \in \mathcal{A}$ such that $\operatorname{tr}(cI_nA_0) = c\operatorname{tr}(A_0) \neq 0$. Therefore, $\operatorname{tr}(A_0)$ and $c\operatorname{tr}(A_0)$ are nonzero elements of F. Hence $c \in F$ as desired.

Corollary 2.3.10. *Let $n > 1$, K a field, F a subfield of K, $\mathcal{S}$ an irreducible semigroup in $M_n(K)$ such that $\{0\} \neq \operatorname{tr}(\mathcal{S}) \subseteq F$. Then, $\operatorname{Alg}_F(\mathcal{S})$ is a finite-dimensional F-algebra and*

$$\dim_F \operatorname{Alg}_F(\mathcal{S}) = \dim_K \operatorname{Alg}_K(\mathcal{S}).$$

Proof. It suffices to prove the equality. We note that $\operatorname{Alg}_K(\mathcal{S}) = \langle \mathcal{S} \rangle_K$ and $\operatorname{Alg}_F(\mathcal{S}) = \langle \mathcal{S} \rangle_F$ for $\mathcal{S}$ is a semigroup. Let $\{S_1, ..., S_m\} \subset \mathcal{S}$ be a basis for $\langle \mathcal{S} \rangle_K$. It suffices to show that $\{S_1, ..., S_m\}$ is a basis for $\langle \mathcal{S} \rangle_F$ as well. The subset $\{S_1, ..., S_m\}$ is linearly independent over the subfield F for it is independent over K. To show that $\{S_1, ..., S_m\}$ spans $\langle \mathcal{S} \rangle_F$, suppose that $A \in \langle \mathcal{S} \rangle_F$ is given. Since $\langle \mathcal{S} \rangle_F \subset \langle \mathcal{S} \rangle_K$, we can write

$$A = c_1S_1 + ... + c_mS_m,$$

for some $c_i \in K$ $(1 \leq i \leq m)$. By Corollary 2.3.7 we obtain $c_i \in F$ for $1 \leq i \leq m$, completing the proof. □

Motivated by Theorem 4 of [ORR] and Theorem 3.4 of [RRa] we were able to prove the following theorem which is a generalization of Theorem 4 of [ORR]. Theorem 2.3.12 below, and its consequences as explained in the remarks following the theorem, can be regarded as Wedderburn-Artin type theorems as follows: (a) for irreducible F-algebras of matrices in $M_n(K)$ with traces in the subfield F but not identically zero, (b) for irreducible algebras of matrices in $M_n(K)$ with zero trace, and (c) for irreducible algebras of matrices in $M_n(K)$. Recall that by the Wedderburn-Artin Theorem every simple algebra $\mathcal{A}$ of matrices is isomorphic to $M_m(D)$ where m is a unique integer and D is a division algebra that is unique up to isomorphism. However, the theorem does not say how m and D are related to the simple algebra $\mathcal{A}$. In comparison to the Wedderburn-Artin Theorem, Theorem 2.3.12 below and its consequences give a more precise description of irreducible (F-)algebras of types (a), (b), and (c).To be more precise, by Theorem 2.3.12 below and its consequences every irreducible (F-)algebra $\mathcal{A}$ in $M_n(K)$ of types (a), (b), and (c) is, up to a similarity, equal to $M_{n/r}(\mathfrak{D}_r)$. Here r is the smallest nonzero rank in $\mathcal{A}$ and divides n, $\mathfrak{D}_r \oplus I_{n-r} \subset \mathcal{A}$, and $\mathfrak{D}_r$ is an irreducible division (F-)algebra in $M_r(K)$ of types (a), (b), and (c) respectively.

Theorem 2.3.11. *Let $n \in \mathbb{N}$, K a field, F a subfield of K, and $\mathcal{A}$ an irreducible F-algebra in $M_n(K)$ such that $\{0\} \neq \mathrm{tr}(\mathcal{A}) \subseteq F$. Let $r \in \mathbb{N}$ be the smallest nonzero rank present in $\mathcal{A}$. Then*

(i) After a similarity, $\mathcal{A}$ contains an idempotent $E = I_r \oplus 0_{n-r}$ where I_r is the identity matrix of size r and 0_{n-r} is the zero matrix of size $n-r$.

(ii) The integer r divides n and after a similarity $E\mathcal{A}E = \mathfrak{D}_r \oplus 0_{n-r}$ where $\mathfrak{D}_r$ is an irreducible division F-algebra in $M_r(K)$ with $\{0\} \neq \mathrm{tr}(\mathfrak{D}_r) \subseteq F$. Furthermore, the minimal polynomial of every $D \in \mathfrak{D}_r$, which is an element of $F[X]$, is irreducible over F.

(iii) After a similarity, $\mathcal{A} = M_{n/r}(\mathfrak{D}_r)$ where $\mathfrak{D}_r$ is the irreducible division F-algebra of (ii). Therefore, $\mathcal{A}$ is simple as a ring. Conversely, let K be an arbitrary field, and F a subfield of K. If $\mathcal{A} \subseteq M_n(K)$ is similar to $M_{n/r}(\mathfrak{D}_r)$ where $\mathfrak{D}_r$ is an irreducible division F-algebra in $M_r(K)$ with $\mathrm{tr}(\mathfrak{D}_r) \subseteq F$, then $\mathcal{A}$ is an irreducible unital F-algebra in $M_n(K)$ with $\mathrm{tr}(\mathcal{A}) \subseteq F$ and r is the smallest nonzero rank present in $\mathcal{A}$.

(iv) After a similarity, $\mathcal{A} = M_n(F)$ iff $r = 1$.

Proof. **(i)** First note that by Corollary 2.3.9, the minimal polynomial of every $T \in \mathcal{A}$, denoted by m_T, is in $F[X]$. If $r = n$ we have nothing to prove for $\mathcal{A}$ is unital by Corollary 2.3.9. So suppose that $r < n$. From this point on the proof is identical to that of Theorem 2.2.18(i).

(ii) The proof is very much like that of Theorem 2.2.18(ii), we however include the proof for the sake of completeness. Find $E = I_r \oplus 0_{n-r}$ as described in (i). It is easily seen that one can write $E\mathcal{A}E = \mathfrak{D}_r \oplus 0_{n-r}$ where $\mathfrak{D}_r \subseteq M_r(K)$. That $\mathfrak{D}_r$ is an F-algebra in $M_r(K)$ follows from the fact that $\mathcal{A}$ is an F-algebra in $M_n(K)$, and that $\mathrm{tr}(\mathfrak{D}_r) \subseteq F$ follows from the fact that $\mathfrak{D}_r \oplus 0_{n-r} = E\mathcal{A}E \subseteq \mathcal{A}$. Since $E \in \mathcal{A}$ and

$E \neq 0$, from Theorem 2.3.1 we see that there exists $B \in \mathcal{A}$ such that $\text{tr}(EB) \neq 0$. We can write

$$\text{tr}(EBE) = \text{tr}(E^2B) = \text{tr}(EB) \neq 0.$$

Therefore, $\{0\} \neq \text{tr}(\mathcal{D}_r) \subseteq F$. If $r \geq 2$, then $\mathcal{D}_r$ is an irreducible division F-algebra by Corollary 2.2.3 and minimality of r; if $r = 1$, this is trivial (in fact it follows from part (iii) of the theorem that $r = 1$ iff after a similarity $\mathcal{A} = M_n(F)$). So it remains to show that r divides n. To this end, use induction on n. If $n = 1$, we have nothing to prove. Suppose that the assertion holds for all irreducible F-algebras of matrices of size less than n with traces in F but not identically zero. For a given irreducible F-algebra $\mathcal{A}$ of matrices in $M_n(K)$ with $\{0\} \neq \text{tr}(\mathcal{A}) \subseteq F$, find $E = I_r \oplus 0_{n-r}$ as described in (i). If $r = 1$ we have nothing to prove. So without loss of generality assume that $r \geq 2$ and $E \in \mathcal{A}$ (note that rank is invariant under similarity). Thus $n - r \geq 2$ since $I - E \in \mathcal{A}$. From $I - E = (I_r \oplus I_{n-r}) - (I_r \oplus 0_{n-r}) = 0_r \oplus I_{n-r}$, it is easily seen that

$$\mathcal{A}' := (I - E)\mathcal{A}(I - E) = 0_r \oplus \mathcal{A}_r,$$

where $\mathcal{A}_r \subseteq M_{n-r}(K)$. Since $\mathcal{A}$ is an irreducible F-algebra in $M_n(K)$, it follows from from Corollary 2.2.3 that indeed $\mathcal{A}_r$ is an irreducible F-algebra in $M_{n-r}(K)$. Since trace is not identically zero on $\mathcal{A}$, $I - E \in \mathcal{A}$ and $I - E \neq 0$, in light of Theorem 2.3.1 we conclude that $\{0\} \neq \text{tr}(\mathcal{A}_r) \subseteq F$. Now let r' be the smallest nonzero rank present in $\mathcal{A}_r \subseteq M_{n-r}(K)$. It follows from the induction hypothesis that r' divides $n - r$. So to prove that r divides n, it suffices to show that $r' = r$. Since $0_r \oplus \mathcal{A}_r = (I - E)\mathcal{A}(I - E) \subseteq \mathcal{A}$, it follows that $r \leq r'$. To see $r' \leq r$, first we claim that $(I - E)\mathcal{A}E \neq 0$. Suppose $(I - E)\mathcal{A}E = 0$. It is evident that $\mathcal{M} := EK^n$ is a nontrivial subspace of K^n. We have

$$\mathcal{A}\mathcal{M} = \mathcal{A}EK^n = (E + (I - E))\mathcal{A}EK^n$$

$$= E\mathcal{A}EK^n + (I - E)\mathcal{A}EK^n = E\mathcal{A}EK^n \subseteq EK^n = \mathcal{M}.$$

Therefore $\mathcal{A}\mathcal{M} \subseteq \mathcal{M}$. That is, $\mathcal{A}$ is reducible, a contradiction. So there exists $A \in \mathcal{A}$ such that $(I - E)AE \neq 0$. Note that $0 \neq (I - E)AE \in \mathcal{A}$, and $\mathcal{A}$ is irreducible, so it follows from Theorem 2.3.1 that there exists $B \in \mathcal{A}$ such that $\text{tr}((I - E)AEB) \neq 0$.

We can write

$$\operatorname{tr}((I-E)AEB(I-E)) = \operatorname{tr}((I-E)^2AEB) = \operatorname{tr}((I-E)AEB) \neq 0.$$

Hence $(I-E)AEB(I-E) \neq 0$. It is now plain that

$$0 < \operatorname{rank}((I-E)AEB(I-E)) \leq \operatorname{rank}(E) = r.$$

Since $0 \neq AEB \in \mathcal{A}$, we conclude that

$$r' \leq \operatorname{rank}((I-E)AEB(I-E)) \leq r.$$

So $r' \leq r$, hence $r = r'$, finishing the proof. Finally, since $\mathcal{D}_r$ is a division F-algebra in $M_r(K)$ with $\{0\} \neq \operatorname{tr}(\mathcal{D}_r) \subseteq F$, the minimal polynomial of every $D \in \mathcal{D}_r$, which is in $F[X]$, is irreducible over F.

(iii) We prove the assertion by induction on n. If $n = 1$, we have nothing to prove. Suppose that the assertion holds for all F-algebras of matrices of size less than n with traces in F. We prove the assertion for all F-algebras of matrices of size n with traces in F. Let an irreducible F-algebra $\mathcal{A}$ in $M_n(K)$ with $\{0\} \neq \operatorname{tr}(\mathcal{A}) \subseteq F$ be given. Applying (i) and (ii) after a similarity we can write

$$E\mathcal{A}E = \mathcal{D}_r \oplus 0_{n-r}, \; (I-E)\mathcal{A}(I-E) = 0_r \oplus \mathcal{A}_r,$$

where $E = I_r \oplus 0_{n-r} \in \mathcal{A}$, and $\mathcal{D}_r \subseteq M_r(K)$, $\mathcal{A}_r \subseteq M_{n-r}(K)$ are, respectively, an irreducible division F-algebra and an irreducible F-algebra with traces in F but not identically zero. By the proof of (ii) the smallest nonzero rank present in $\mathcal{A}_r$ is r. Since $n-r < n$, it follows from the induction hypothesis that after a similarity of the form $T_r^{-1}(.)T_r$ with $T_r \in M_{n-r}(K)$, we have $\mathcal{A}_r = M_{\frac{n-r}{r}}(\mathcal{D}'_r)$ where $\mathcal{D}'_r \subseteq M_r(K)$ is an irreducible division F-algebra with traces in F but not identically zero. Applying the similarity $I_r \oplus T_r$ to $\mathcal{A}$ we may assume that

$$E\mathcal{A}E = \mathcal{D}_r \oplus 0_{n-r} \subset \mathcal{A}, \; (I-E)\mathcal{A}(I-E) = 0_r \oplus \mathcal{A}_r \subset \mathcal{A}, \qquad (*)$$

where $E = I_r \oplus 0_{n-r} \in \mathcal{A}$, $\mathcal{A}_r = M_{\frac{n-r}{r}}(\mathcal{D}'_r)$ and $\mathcal{D}_r$, $\mathcal{D}'_r$, $\mathcal{A}_r$ are as described above. Note that every element of $A \in \mathcal{A}$ can be represented in the block form, i.e., $A = (a_{ij})_{\frac{n}{r}\times\frac{n}{r}}$ where the blocks, i.e., a_{ij}'s, are matrices of size r over K. For $A = (a_{ij}) \in \mathcal{A}$, $A_{ij} \in M_{\frac{n}{r}}(M_r(K))$ is used to denote the block matrix with $a_{ij} \in M_r(K)$ in the ij place and $0_r \in M_r(K)$ elsewhere. Let $E_{ij} \in M_{\frac{n}{r}}(M_r(K))$ denote the block matrix with the identity $I_r \in M_r(K)$ in the ij place and $0_r \in M_r(K)$ elsewhere. It follows from $(*)$ that $E_{ij} \in \mathcal{A}$ for $i = j = 1$ and for all $2 \le i, j \le \frac{n}{r}$. Thus if $A \in \mathcal{A}$, then $A_{1j} = E_{11}AE_{jj}, A_{i1} = E_{ii}AE_{11} \in \mathcal{A}$ for all $1 \le i, j \le \frac{n}{r}$.

As we saw in the proof of (ii), it follows from irreducibility of $\mathcal{A}$ that $(I_n - E_{11})\mathcal{A}E_{11} \neq 0$ (note that in fact $E_{11} = E = I_r \oplus 0_{n-r} \in \mathcal{A}$). Since $I_n - E_{11} = E_{22} + \dots + E_{\frac{n}{r}\frac{n}{r}}$, it follows that there exists $2 \le i_0 \le \frac{n}{r}$ such that $E_{i_0 i_0}\mathcal{A}E_{11} \neq 0$. That is, there exists $A \in \mathcal{A}$ such that $0 \neq A_{i_0 1} = E_{i_0 i_0}AE_{11} \in \mathcal{A}$. This along with minimality of r implies that $a_{i_0 1} \in M_r(K)$ is invertible (note that $A = (a_{ij}) \in \mathcal{A}$). Similarly, it follows from irreducibility of $\mathcal{A}$ that $E_{11}\mathcal{A}(I_n - E_{11}) \neq 0$, and therefore there exists $2 \le j_0 \le \frac{n}{r}$ such that $E_{11}\mathcal{A}E_{j_0 j_0} \neq 0$. That is, there exists $A' \in \mathcal{A}$ such that $0 \neq A'_{1j_0} = E_{11}A'E_{j_0 j_0} \in \mathcal{A}$. This together with minimality of r implies that $a'_{1j_0} \in M_r(K)$ is invertible (note that $A' = (a'_{ij}) \in \mathcal{A}$).

We claim that after applying the similarity $T = \operatorname{diag}(I_r, a, \dots, a) \in M_n(K)$ to $\mathcal{A}$ where $a := a_{i_0 1} \in M_r(K)$ we have $\mathcal{A}' := T^{-1}\mathcal{A}T = M_{n/r}(\mathcal{D}_r)$ where $\mathcal{D}_r$ is the irreducible division F-algebra of (ii) and this would finish the proof. To this end, first it is easily verified that

$$T^{-1}BT =$$

$$\begin{pmatrix} I_r & 0_r & \dots & 0_r \\ 0_r & a^{-1} & \dots & 0_r \\ \vdots & \vdots & \ddots & \vdots \\ 0_r & \dots & 0_r & a^{-1} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1\frac{n}{r}} \\ b_{21} & b_{22} & \dots & b_{2\frac{n}{r}} \\ \vdots & \vdots & \ddots & \vdots \\ b_{\frac{n}{r}1} & \dots & \dots & b_{\frac{n}{r}\frac{n}{r}} \end{pmatrix} \begin{pmatrix} I_r & 0_r & \dots & 0_r \\ 0_r & a & \dots & 0_r \\ \vdots & \vdots & \ddots & \vdots \\ 0_r & \dots & 0_r & a \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} & b_{12}a & \dots & b_{1\frac{n}{r}}a \\ a^{-1}b_{21} & a^{-1}b_{22}a & \dots & a^{-1}b_{2\frac{n}{r}}a \\ \vdots & \vdots & \ddots & \vdots \\ a^{-1}b_{\frac{n}{r}1} & a^{-1}b_{\frac{n}{r}2}a & \dots & a^{-1}b_{\frac{n}{r}\frac{n}{r}}a \end{pmatrix}.$$

From the preceding block matrix identity we see that $E_{ij} \in \mathcal{A}'$ whenever $i = j = 1$ or

$2 \leq i, j \leq \frac{n}{r}$. This in particular implies that if $C = (c_{ij}) \in \mathcal{A}'$ then $C_{ij} = E_{ii}CE_{jj} \in \mathcal{A}'$ for all $1 \leq i, j \leq \frac{n}{r}$.

Since $0 \neq A_{i_0 1} = E_{i_0 i_0} A E_{11} \in \mathcal{A}$ (note that $A_{i_0 1}$, in its block matrix representation, has $a_{i_0 1} \in M_r(K)$ in the $i_0 1$ place and $0_r \in M_r(K)$ elsewhere), again it follows from the block matrix identity above that $E_{i_0 1} = T^{-1} A_{i_0 1} T \in \mathcal{A}'$. Having

$$E_{i1} = E_{ii_0} E_{i_0 1}, \;\; E_{ii_0}, E_{i_0 1} \in \mathcal{A}',$$

for all $2 \leq i \leq \frac{n}{r}$, we conclude that $E_{i1} \in \mathcal{A}'$ for all $2 \leq i \leq \frac{n}{r}$. In particular, $E_{j_0 1} \in \mathcal{A}'$ where j_0 is as in the above.

Recall that $0 \neq A'_{1j_0} = E_{11} A' E_{j_0 j_0} \in \mathcal{A}$. So by the block matrix identity above, we see that $B_{1j_0} := T^{-1} A'_{1j_0} T \in \mathcal{A}'$, that B_{1j_0}, in its block matrix representation, has $b_{1j_0} = a'_{1j_0} a = a'_{1j_0} a_{i_0 1} \in M_r(K)$ in the $1j_0$ place and $0_r \in M_r(K)$ elsewhere. Let $B'_{j_0 1} = (b'_{ij}) \in M_{\frac{n}{r}}(M_r(K))$ denote the block matrix with $b'_{j_0 1} = b^{-1}_{1j_0} \in M_r(K)$ and $0_r \in M_r(K)$ otherwise. As $B_{1j_0}, E_{j_0 1}, E_{ii} \in \mathcal{A}'$ for all $1 \leq i \leq \frac{n}{r}$ and $\mathcal{A}'$ is an F-algebra, we can write

$$C := B_{1j_0} + E_{j_0 1} + \sum_{j_0 \neq i = 2}^{\frac{n}{r}} E_{ii} \in \mathcal{A}'.$$

Now it is a matter of a straightforward calculation to see that

$$C^{-1} = B'_{j_0 1} + E_{1j_0} + \sum_{j_0 \neq i = 2}^{\frac{n}{r}} E_{ii}.$$

Since $\mathcal{A}'$ is an irreducible F-algebra with $\{0\} \neq \operatorname{tr}(\mathcal{A}') \subseteq F$ and $C \in \mathcal{A}'$, it follows from Corollary 2.3.9 that the minimal polynomial for C is in $F[X]$ and hence $C^{-1} \in \mathcal{A}'$. This in turn implies that

$$E_{1j_0} = E_{11} C^{-1} E_{j_0 j_0} \in \mathcal{A}',$$

for $E_{ii} \in \mathcal{A}'$ for all $1 \leq i \leq \frac{n}{r}$. Now since $E_{1j_0}, E_{j_0 1} \in \mathcal{A}'$, we conclude that

$$E_{ij} = E_{ij_0} E_{j_0 1} E_{1j_0} E_{j_0 j} \in \mathcal{A}',$$

for all $1 \leq i, j \leq \frac{n}{r}$. It is now easy to see that we in fact have $\mathcal{A}' = M_{n/r}(\mathcal{D}_r)$ where $\mathcal{D}_r$ is the irreducible division F-algebra of (ii). It is evident that if $S = (s_{ij}) \in \mathcal{A}'$, then $s_{11} \in \mathcal{D}_r$. Suppose that $S = (s_{ij}) \in \mathcal{A}'$ is given. Since $E_{1i}E_{ii}SE_{jj}E_{j1} \in \mathcal{A}'$, it follows from the preceding remark that $s_{ij} \in \mathcal{D}_r$. Thus $\mathcal{A}' \subseteq M_{n/r}(\mathcal{D}_r)$. On the other hand, let $S = (s_{ij}) \in M_{n/r}(\mathcal{D}_r)$ be an arbitrary element. Since $s_{ij} \in \mathcal{D}_r$ and $\mathcal{D}_r \oplus 0_{n-r} \subset \mathcal{A}'$, it follows that $S_{ij} = E_{i1}(s_{ij} \oplus 0_{n-r})E_{1j} \in \mathcal{A}'$ and hence

$$S = \sum_{i,j=1}^{\frac{n}{r}} S_{ij} \in \mathcal{A}'.$$

This yields $M_{n/r}(\mathcal{D}_r) \subseteq \mathcal{A}'$, finishing the proof.

For the converse, since $\mathcal{D}_r$ is a division F-algebra with $\operatorname{tr}(\mathcal{D}_r) \subseteq F$, it easily follows that $M_{n/r}(\mathcal{D}_r)$ is a unital F-algebra with $\operatorname{tr}(M_{n/r}(\mathcal{D}_r)) \subseteq F$ whose smallest nonzero rank is r. Therefore, if after a similarity $\mathcal{A} = M_{n/r}(\mathcal{D}_r)$, then $\mathcal{A}$ is a unital F-algebra with traces in F whose smallest nonzero rank is r. So it suffices to show that $M_{n/r}(\mathcal{D}_r)$ is irreducible in $M_n(K)$. To see this, it suffices to show that the K-algebra generated by $M_{n/r}(\mathcal{D}_r)$ is irreducible or equivalently transitive. That is, we need to show that every nonzero vector of K^n is a cyclic vector for the K-algebra generated by $M_{n/r}(\mathcal{D}_r)$. Suppose that $X, Y \in K^n$ are given where X is nonzero. We need to show that there exists A in the K-algebra generated by $M_{n/r}(\mathcal{D}_r)$ such that $AX = Y$. It is plain that one can write $X = (x_1, ..., x_{\frac{n}{r}})$, $Y = (y_1, ..., y_{\frac{n}{r}})$ where $x_i, y_i \in K^r$ for each $i = 1, ..., \frac{n}{r}$ and $x_j \neq 0$ for some $1 \leq j \leq \frac{n}{r}$. Since $\mathcal{D}_r$ is irreducible in $M_r(K)$ by hypothesis, it follows that the K-algebra generated by $\mathcal{D}_r$ is transitive. Therefore, for $x_j, y_i \in K^r$ where $1 \leq i \leq \frac{n}{r}$, there exists

$$a_{ij} = \sum_{k=1}^{n_{ij}} \alpha_{ijk} d_{ijk} \in \operatorname{Alg}_K(\mathcal{D}_r),$$

such that $a_{ij}x_j = y_i$ where $\alpha_{ijk} \in K$, $d_{ijk} \in \mathcal{D}_r$, and $n_{ij} \in \mathbb{N}$. Let $A_{ij}, D_{ijk} \in M_{\frac{n}{r}}(M_r(K))$ denote the block matrix with a_{ij}, d_{ijk} in ij's place and 0_r elsewhere respectively. Note that $D_{ijk} \in M_{\frac{n}{r}}(\mathcal{D}_r)$ and hence

$$A := \sum_{i=1}^{\frac{n}{r}} A_{ij} = \sum_{i=1}^{\frac{n}{r}} \sum_{k=1}^{n_{ij}} \alpha_{ijk} D_{ijk} \in \mathrm{Alg}_K(M_{\frac{n}{r}}(\mathcal{D}_r)).$$

It is now straightforward to see that we have

$$AX = \sum_{i=1}^{\frac{n}{r}} A_{ij} X = \sum_{i=1}^{\frac{n}{r}} \sum_{k=1}^{n_{ij}} \alpha_{ijk} D_{ijk} X = Y.$$

Therefore the K-algebra generated by $M_{\frac{n}{r}}(\mathcal{D}_r)$ is transitive, hence irreducible. Thus $M_{\frac{n}{r}}(\mathcal{D}_r)$ is also irreducible, finishing the proof.

(iv) The "only if" part trivially holds. To see the "if" part, it is plain that if $r = 1$, then $\mathcal{D}_1 = F$, and hence after a similarity $\mathcal{A} = M_n(F)$. □

Theorem 2.3.12. *Let $n \in \mathbb{N}$, K a field, F a subfield of K, and $\mathcal{S}$ an irreducible semigroup in $M_n(K)$ with $\{0\} \neq \mathrm{tr}(\mathcal{S}) \subseteq F$. Let $r \in \mathbb{N}$ be the smallest nonzero rank present in $\mathrm{Alg}_F(\mathcal{S})$. Then*

(i) After a similarity, $\mathrm{Alg}_F(\mathcal{S})$ contains an idempotent $E = I_r \oplus 0_{n-r}$ where I_r is the identity matrix of size r and 0_{n-r} is the zero matrix of size $n - r$.

(ii) The integer r divides n and after a similarity $E\mathrm{Alg}_F(\mathcal{S})E = \mathcal{D}_r \oplus 0_{n-r}$ where $\mathcal{D}_r$ is an irreducible division F-algebra in $M_r(K)$ with $\{0\} \neq \mathrm{tr}(\mathcal{D}_r) \subseteq F$. Furthermore, the minimal polynomial of every $D \in \mathcal{D}_r$, which is an element of $F[X]$, is irreducible over F.

(iii) After a similarity $\mathrm{Alg}_F(\mathcal{S}) = M_{n/r}(\mathcal{D}_r)$ where $\mathcal{D}_r$ is the irreducible division F-algebra of (ii). Therefore, $\mathrm{Alg}_F(\mathcal{S})$ is simple as a ring. Conversely, let K be an arbitrary field, and F a subfield of K. If $\mathcal{S}$ is a semigroup in $M_n(K)$ and after a similarity $\mathrm{Alg}_F(\mathcal{S}) = M_{n/r}(\mathcal{D}_r)$ where $\mathcal{D}_r$ is an irreducible division F-algebra in $M_r(K)$ with $\mathrm{tr}(\mathcal{D}_r) \subseteq F$, then $\mathcal{S}$ is an irreducible semigroup in $M_n(K)$ with $\mathrm{tr}(\mathcal{S}) \subseteq F$ and r is the smallest nonzero rank present in $\mathrm{Alg}_F(\mathcal{S})$.

(iv) After a similarity $\mathrm{Alg}_F(\mathcal{S}) = M_n(F)$ iff $r = 1$.

Proof. Since $\mathcal{A} := \mathrm{Alg}_F(\mathcal{S})$ is an irreducible F-algebra in $M_n(K)$ with traces in the subfield F, Theorem 2.3.11 applies. □

Remarks.

1. In light of Theorem 2.2.18, it is easily seen that if $K = F$, then the conclusions of Theorem 2.3.11 (resp. Theorem 2.3.12) hold and no condition on the irreducible algebra $\mathcal{A}$ (resp. semigroup $\mathcal{S}$) in terms of trace is needed. That being noted, it is also clear from the proof of Theorem 2.3.11 (resp. Theorem 2.3.12) that for every irreducible algebra $\mathcal{A}$ (resp. irreducible semigroup $\mathcal{S}$) on which trace is zero, up to a similarity, we have $\mathcal{A} = M_{n/r}(\mathcal{D}_r)$ (resp. $\mathrm{Alg}(\mathcal{S}) = M_{n/r}(\mathcal{D}_r)$) where r is as in Theorem 2.3.11 (resp. Theorem 2.3.12) and $\mathcal{D}_r$ is an irreducible division algebra in $M_r(F)$ on which trace is zero.

2. Let $n > 1$, K be a field, and F a subfield of K. If the ground field K is perfect; or $\mathrm{ch}(K)$ is not a divisor of n; or is k-closed for each k dividing n with $k > 1$, then in light of Theorem 2.2.19 and Theorem 2.3.3, we see that the trace functional cannot be identically zero on an irreducible semigroup in $M_n(K)$. Therefore the conclusions of Lemma 2.3.7-Corollary 2.3.14 below (together with those results that are stated as remarks) hold for every irreducible semigroup or F-algebra with traces in the subfield F provided that the ground field K is perfect; or $\mathrm{ch}(K)$ is not a divisor of n; or the field K is k-closed for each k dividing n with $k > 1$.

3. Let F be a field and $\mathcal{F} \subset M_n(F)$ an irreducible family of matrices in $M_n(F)$. In light of the preceding remark and Theorem 2.3.11, we conclude that if the smallest nonzero rank in $\mathrm{Alg}(\mathcal{F})$ is 1, then $\mathrm{Alg}(\mathcal{F}) = M_n(F)$ and hence $\mathcal{F}$ is absolutely irreducible and

$$\mathcal{F}' = \{cI : c \in F\}.$$

4. In light of Lemma 1.1.2(i), Corollary 2.2.3, and Theorem 2.3.3, if the semigroup or the F-algebra in Lemma 2.3.7-Theorem 2.3.12 happens to be absolutely irreducible, then the same arguments shows that, with no condition imposed on the field K, similar conclusions hold except that everywhere "irreducible" would be replaced by "absolutely irreducible". As well, in some of the results to follow one may assume absolutely irreducibility to get similar results. For the sake of brevity we avoid mentioning these similar results in detail.

5. Let $\mathfrak{D}_r$ be as in Theorem 2.3.11 (resp. Theorem 2.3.12). By Corollary 2.3.10, we have

$$\dim_F \mathfrak{D}_r = \dim_K \mathrm{Alg}_K(\mathfrak{D}_r) = \dim_K \langle \mathfrak{D}_r \rangle_K.$$

Since $\mathfrak{D}_r$ is irreducible in $M_r(K)$, it follows that $\mathrm{Alg}_K(\mathfrak{D}_r) = \langle \mathfrak{D}_r \rangle_K$ is a simple subalgebra of $M_r(K)$ by Lemma 2.2.17. So from Lemma 7.4 of [D] we see that $\dim_F \mathfrak{D}_r = \dim_K \langle \mathfrak{D}_r \rangle_K$ is a divisor of r^2, hence a divisor of n^2.

In case the subfield F happens to be finite, we can give a more precise description of irreducible (F-)algebras of types (a), (b), and (c) as described on page 43 following Corollary 2.3.10. The following two results serve this purpose.

Lemma 2.3.13. *Let $n > 1$, K a field, F a finite subfield of K, and $\mathfrak{D}$ an irreducible division F-algebra in $M_n(K)$ with $\{0\} \neq \mathrm{tr}(\mathfrak{D}) \subseteq F$. Then $\mathfrak{D}$ is a field and there exists a K-irreducible matrix $A \in M_n(F)$ such that after a similarity $\mathfrak{D} = F[A]$. Therefore, $\mathfrak{D}$ is indeed a simple extension field of F.*

Proof. Let $\{I_n, D_1, ..., D_p\}$ be a basis in $\mathfrak{D}$ for $\langle \mathfrak{D} \rangle_K$, the linear manifold spanned by $\mathfrak{D}$ over K. By Corollary 2.3.10 we have $\mathfrak{D} = \langle I_n, D_1, ..., D_p \rangle_F$. As F is a finite field, we see that $\mathfrak{D}$ is a finite division ring. Now by Wedderburn's Theorem (see Corollary IX.6.9 of [H]), $\mathfrak{D}$ must be a field. The fact that $\mathfrak{D} = \langle I_n, D_1, ..., D_p \rangle_F$ implies that $\mathfrak{D}$ is a finite-dimensional extension field of F. Since F is finite, we conclude that F is a perfect field. Thus, there exists a matrix $B \in M_n(K)$ such that after a similarity $\mathfrak{D} = F[B]$. The irreducibility of $\mathfrak{D}$ yields that of the matrix B in $M_n(K)$. It is now plain that the characteristic polynomial of the matrix $B \in M_n(K)$ equals its minimal polynomial which is an element of $F[X]$ by Corollary 2.3.9. Therefore, there exists a K-irreducible $A \in M_n(F)$ such that after a similarity $\mathfrak{D} = F[A]$ (in fact A can be taken as the companion matrix of B). □

Remark. In the preceding lemma if K happens to be k-closed for each k dividing n with $k > 1$ ($\mathrm{ch}(K) \neq 0$), then it follows from the proof of the lemma that there is no irreducible division F-algebra $\mathfrak{D}$ in $M_n(K)$ with $\mathrm{tr}(\mathfrak{D}) \subseteq F$. Therefore, under the above hypothesis, there is no irreducible division ring on which trace is zero. To see this, note that such an irreducible division ring is in particular an irreducible division F-algebra in $M_n(K)$ with zero trace, hence in the prime subfield F that is finite.

The following result is a quick consequence of Theorem 2.3.12 and Lemma 2.3.13.

Corollary 2.3.14. *Let $n > 1$, K a field, F a finite subfield of K, and $\mathcal{S}$ an irreducible semigroup in $M_n(K)$ with $\{0\} \neq \mathrm{tr}(\mathcal{S}) \subseteq F$. Let $r \in \mathbb{N}$ be the smallest nonzero rank present in $\mathrm{Alg}_F(\mathcal{S})$. Then r divides n and*

(i) if $r = 1$, then after a similarity $\mathrm{Alg}_F(\mathcal{S}) = M_n(F)$.

(ii) if $r > 1$, then there exists a K-irreducible matrix $A \in M_r(F)$ such that after a similarity $\mathrm{Alg}_F(\mathcal{S}) = M_{n/r}(F[A])$.

Therefore, in any case $\mathrm{Alg}_F(\mathcal{S})$ is indeed a finite irreducible F-algebra in $M_n(K)$. Furthermore, after a similarity, the commutant of $\mathrm{Alg}_F(\mathcal{S})$ in $M_n(K)$, which is the same as that of $\mathcal{S}$ in $M_n(K)$, is equal to $F[A] \oplus \dots \oplus F[A]$.

Proof. Theorem 2.3.12 together with Lemma 2.3.13 easily settles the proof. □

Remark. In Corollary 2.3.14, if K happens to be k-closed for each k dividing n with $k > 1$ ($\mathrm{ch}(K) \neq 0$), then $r = 1$ by the proof of Lemma 2.3.13. Hence, after a similarity, we would have $\mathrm{Alg}_F(\mathcal{S}) = M_n(F)$. In particular, it follows that up to a similarity the only irreducible F-algebra in $M_n(K)$ with traces in F is $M_n(F)$ and that every irreducible F-algebra in $M_n(K)$ with traces in F is central, i.e., its center consists of cI where $c \in F$.

2.4 Some Applications in Finite Dimensions

For a collection $\mathcal{C} \subset M_n(K)$, where K is a field, we say that $\mathcal{C}$ *is defined over a subfield F of K* if there exists an invertible $S \in M_n(K)$ such that $S^{-1}\mathcal{C}S \subset M_n(F)$. For instance suppose that $\mathcal{A}$ and $\mathcal{S}$ are as in Theorem 2.3.11 and Theorem 2.3.12 respectively, then $\mathcal{A}$ and $\mathcal{S}$ are defined over F if $r = 1$. The following question was asked in [RRa]. As pointed out there, an affirmative answer to the following question would be a sweeping generalization of a celebrated theorem of Brauer.

Question. *Letting $\mathcal{S}$ be any irreducible semigroup in $M_n(K)$ with $\bar{\sigma}(\mathcal{S}) \subseteq F$ where F is a subfield of K, must $\mathcal{S}$ be defined on F? In other words, does there exist an invertible matrix $T \in M_n(K)$ such that $T^{-1}\mathcal{S}T \subseteq M_n(F)$?*

By proving that every such semigroup $\mathcal{S}$ in the above question is absolutely irreducible, one may assume, with no loss of generality, that the field K is algebraically closed. We have not yet been able to prove or disprove this for irreducible semigroups with spectra in a subfield F. However, as shown in the following theorem, this is indeed the case for irreducible F-algebras of matrices in $M_n(K)$ with spectra in the subfield F. The theorem below not only gives an affirmative answer to the above question for irreducible F-algebras of matrices in $M_n(K)$ with spectra in the subfield F, but it also characterizes all subfields F as well as irreducible F-algebras of matrices in $M_n(K)$ with spectra in F. It is also worth noting that the following theorem, in some sense, is a generalization of Burnside's Theorem.

Theorem 2.4.1. *Let K be a field, F a subfield of K, $n > 1$, and $\mathcal{A}$ an irreducible F-algebra in $M_n(K)$ such that $\bar{\sigma}(\mathcal{A}) \subseteq F$. Then after a similarity $\mathcal{A} = M_n(F)$. Therefore, $\mathcal{A}$ is defined over F, $\mathcal{A}$ is absolutely irreducible, and the subfield F is k-closed for each $k = 2, ..., n$.*

Proof. Plainly the minimal polynomial of every $A \in \mathcal{A}$ is in $F[X]$. Therefore, $\mathcal{A}$ is an algebraic algebra over F. This together with irreducibility of $\mathcal{A}$, in view of Theorem 14 on page 89 of [K1], implies that the Jacobson radical of the F-algebra $\mathcal{A}$ consists of nilpotents. Therefore, the Jacobson radical of $\mathcal{A}$ is zero, for otherwise $\mathcal{A}$ would be reducible in light of Levitzki's Theorem and Lemma 1.1.2(i). That being noted, in view of the hypothesis that $\bar{\sigma}(\mathcal{A}) \subseteq F$, a proof identical to those of Theorem 2.2.18(i) and Theorem 2.2.18(ii) shows that the smallest nonzero rank in $\mathcal{A}$ is 1. Now adjusting the proof of Theorem 2.3.11(iii), we conclude that after a similarity $\mathcal{A} = M_n(F)$. The rest easily follows from $\mathcal{A} = M_n(F)$ and the hypothesis that $\bar{\sigma}(\mathcal{A}) \subseteq F$. □

We now give an affirmative answer to the Brauer type question above under the weaker hypothesis that the semigroup in the question has traces in the subfield F provided the subfield F is k-closed for each k dividing n with $k > 1$, or it is finite.

Theorem 2.4.2. *(i) Let $n > 1$, K a field, F a subfield of K that is k-closed for each k dividing n with $k > 1$, and $\mathcal{S}$ an irreducible semigroup in $M_n(K)$ such that $\{0\} \neq \operatorname{tr}(\mathcal{S}) \subseteq F$. Then after a similarity $\operatorname{Alg}_F(\mathcal{S}) = M_n(F)$, and hence $\mathcal{S}$ is defined over F. In particular, if F is algebraically closed, then after a similarity $\operatorname{Alg}_F(\mathcal{S}) = M_n(F)$ and so $\mathcal{S}$ is defined over F.*

(ii) Let $n > 1$, K a field, F a finite subfield of K, and $\mathcal{S}$ an irreducible semigroup in $M_n(K)$ such that $\{0\} \neq \operatorname{tr}(\mathcal{S}) \subseteq F$. Then $\mathcal{S}$ is finite and is defined over F.

Proof. (i) Let $\mathcal{A} := \operatorname{Alg}_F(\mathcal{S})$ denote the F-algebra generated by $\mathcal{S}$, and let r and $\mathfrak{D}_r$ be as in Theorem 2.3.12. It suffices to show that $r = 1$. To prove it by contradiction, suppose $r > 1$. First we show that $\mathrm{Z}(\mathfrak{D}_r) = \{cI : c \in F\}$ where $\mathrm{Z}(\mathfrak{D}_r)$ denotes the center of $\mathfrak{D}_r$. To this end, suppose that $D \in \mathrm{Z}(\mathfrak{D}_r) \subset \mathfrak{D}_r$. By Theorem 2.3.12(ii), the minimal polynomial for D, denoted by m_D, is an irreducible polynomial in $F[X]$. On the other hand, since $D \in \mathrm{Z}(\mathfrak{D}_r)$ and $\mathfrak{D}_r$ is irreducible in $M_r(K)$, it follows that m_D is irreducible over K, for otherwise by Lemma 2.2.20 the matrix $D \in \mathrm{Z}(\mathfrak{D}_r)$ would have a nontrivial hyperinvariant subspace, contradicting irreducibility of $\mathfrak{D}_r$ in $M_r(K)$. Now since m_D is irreducible over K, it follows from the Rational Canonical Form Theorem (see Theorem VII.4.2(i) and Theorem VII.4.6(i) of [H]) that $\deg(\mathrm{m}_D)$ divides r. This yields $\deg(\mathrm{m}_D) = 1$, for otherwise $\mathrm{m}_D \in F[X]$ would be an irreducible polynomial whose degree divides $r > 1$, hence n, contradicting the hypothesis that F is k-closed for all k dividing n with $k > 1$. Therefore, $\deg(\mathrm{m}_D) = 1$ and so $D = cI$ for some $c \in F$ because m_D is the minimal polynomial of D, $\mathrm{m}_D \in F[X]$, and $\deg(\mathrm{m}_D) = 1$. Thus, $\mathrm{Z}(\mathfrak{D}_r) = \{cI : c \in F\}$. Now let F_m denote a maximal subfield of $\mathfrak{D}_r$ (such a maximal subfield exists by Exercise IX.6.4 of [H]). The maximal subfield F_m contains the center of $\mathfrak{D}_r$, namely F, for otherwise F_m and F generate a subfield of $\mathfrak{D}_r$ properly containing F_m, a contradiction. Since F is indeed the center of $\mathfrak{D}_r$, it follows from Theorem IX.6.6 of [H] that $\dim_F \mathfrak{D}_r = (\dim_F F_m)^2$. By showing that $F_m = F$ we complete the proof. Plainly $\operatorname{Alg}_K(\mathfrak{D}_r) = \langle \mathfrak{D}_r \rangle_K$ and $\operatorname{Alg}_F(\mathfrak{D}_r) = \langle \mathfrak{D}_r \rangle_F = \mathfrak{D}_r$. That being noted, by Corollary 2.3.10, we have $\dim_K\langle\mathfrak{D}_r\rangle_K = \dim_F\langle\mathfrak{D}_r\rangle_F = \dim_F \mathfrak{D}_r$. Set

$$\mathcal{A} := \{A \oplus ... \oplus A \in M_n(K) : A \in \langle\mathfrak{D}_r\rangle_K \subseteq M_r(K)\}.$$

Since the algebra $\langle\mathfrak{D}_r\rangle_K$ is irreducible in $M_r(K)$, it follows from the remark following Theorem 2.3.12 that $\langle\mathfrak{D}_r\rangle_K$ is simple as a ring. Therefore, so is $\mathcal{A}$ in $M_n(K)$. It is also evident that $\dim_K \mathcal{A} = \dim_K\langle\mathfrak{D}_r\rangle_K$. That being noted, since F and $\mathcal{A}$ are simple subrings of $M_n(K)$, in view of Lemma 7.4 of [D], we can write

$$n^2 = \dim_K M_n(K)$$

$$= [M_n(K) : \mathcal{A}].[\mathcal{A} : K] = [M_n(K) : \mathcal{A}] \dim_K \mathcal{A}$$

$$= [M_n(K) : \mathcal{A}] \dim_K\langle\mathfrak{D}_r\rangle_K = [M_n(K) : \mathcal{A}] \dim_F\langle\mathfrak{D}_r\rangle_F$$

$$= [M_n(K) : \mathcal{A}] \dim_F \mathfrak{D}_r = [M_n(K) : \mathcal{A}](\dim_F F_m)^2.$$

Therefore, $(\dim_F F_m)^2$ divides n^2 and so $\dim_F F_m$ divides n. Now since F_m is an extension field of F, we conclude that the minimal polynomial of every $D \in F_m$ is irreducible over F. On the other hand, we can write

$$\dim_F F_m = [F_m : F] = [F_m : F[D]].[F[D] : F].$$

But $[F[D] : F] = \deg(\mathrm{m}_D)$. Therefore, $\deg(\mathrm{m}_D)$ divides $\dim_F F_m$. Thus $\deg(\mathrm{m}_D)$ divides n, for, as we just saw, $\dim_F F_m$ divides n. From this we obtain $\deg(\mathrm{m}_D) = 1$ for otherwise, due to the fact that $\mathrm{m}_D \in F[X]$ is irreducible over F and the hypothesis that F is k-closed for all k dividing n with $k > 1$, we obtain a contradiction. So we have shown that the minimal polynomial for every $D \in F_m$ has degree one. That is, $F_m = F$ so $\dim_F \mathfrak{D}_r = (\dim_F F_m)^2 = 1$, a contradiction.

(ii) The proof is a quick consequence of Corollary 2.3.14 and the remark following the corollary. □

Remarks.

1. Let $\mathcal{A}$ in $M_2(\mathbb{C})$ be the following representation of quaternions as an $\mathbb{R}$-algebra

$$\mathcal{A} := \{\begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \in M_2(\mathbb{C}) : z_1, z_2 \in \mathbb{C}\}.$$

The $\mathbb{R}$-algebra $\mathcal{A}$ is irreducible in $M_2(\mathbb{C})$ for $\dim_{\mathbb{C}}\langle\mathcal{A}\rangle_{\mathbb{C}} = 4$. It is easily seen that $\mathcal{A}$ is a division algebra and has traces in $\mathbb{R}$. Therefore, the $\mathbb{R}$-algebra $\mathcal{A}$ is not defined on $\mathbb{R}$ for $\dim_{\mathbb{R}}\mathcal{A} = \dim_{\mathbb{C}}\langle\mathcal{A}\rangle_{\mathbb{C}} = 4$. This shows that the hypothesis that "F is k-closed for each k dividing n with $k > 1$" cannot be dropped.

2. Let $\mathbb{Z}_2$ be the field of integers modulo 2, let $\overline{\mathbb{Z}}_2$ denote the algebraic closure of $\mathbb{Z}_2$, $\overline{\mathbb{Z}}_2[X]$ the ring of polynomials over $\overline{\mathbb{Z}}_2$, and $F := \overline{\mathbb{Z}}_2(X)$ the quotient field of $\overline{\mathbb{Z}}_2[X]$ (note that $\overline{\mathbb{Z}}_2[X]$ is an integral domain). The field F is not perfect nor it is 2-closed, and we have $\mathrm{ch}(F) = 2$. It is easily seen that the matrix $A \in M_2(F)$ defined by

$$A := \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix}$$

is irreducible in $M_2(F)$ and that the irreducible algebra $\mathcal{A} := \mathrm{Alg}(A) = F[A]$ has traces zero, hence in the subfield $\overline{\mathbb{Z}}_2$ which is algebraically closed. The algebra $\mathcal{A}$, however, is not defined on $\overline{\mathbb{Z}}_2$, for otherwise the characteristic polynomial for the matrix A would be in $\overline{\mathbb{Z}}_2[X]$, a contradiction. Therefore, the irreducible algebra $\mathcal{A}$ is not defined on $\mathbb{Z}_2$ either. Therefore, in both parts of the preceding theorem the hypothesis that trace in not identically zero on the semigroup cannot be dropped.

3. In the preceding theorem if the ground field K satisfies one of the following conditions: (a) $\mathrm{ch}(K) = 0$ or $\mathrm{ch}(K)$ does not divide n, (b) K is a perfect field, (c) K is k-closed for each k dividing n with $k > 1$, then the hypothesis that trace is not identically zero on the irreducible semigroup $\mathcal{S}$ is easily seen to be redundant.

Theorem 2.4.3. *Let $n > 1$, K a field, F a subfield of K that is not k-closed for some $2 \leq k \leq n$, and $\mathcal{A}$ an F-algebra in $M_n(K)$ such that $\bar{\sigma}(\mathcal{A}) \subseteq F$. Then $\mathcal{A}$ is reducible. Conversely, let a subfield F of K be given. If every F-algebra $\mathcal{A}$ in $M_n(K)$ with $\bar{\sigma}(\mathcal{A}) \subseteq F$ is reducible, then F is not k-closed for some $2 \leq k \leq n$.*

Proof. The first assertion is a quick consequence of Theorem 2.4.1.

For the converse, we use contradiction. Suppose that F is k-closed for all $2 \leq k \leq n$. This hypothesis easily implies that the irreducible F-algebra $\mathcal{A} := M_n(F)$ has the property that $\bar{\sigma}(\mathcal{A}) \subseteq F$. On the other hand, $\mathcal{A}$ must be reducible by hypothesis, a contradiction. □

Remark. In Chapter 3, we give a second, as well as a simpler, proof for Theorem 2.4.3, and Corollary 2.4.4(ii) below in the special case $F = K$.

Let K be a field, F a subfield of K, and $\mathcal{V}$ a vector space over K. We use the symbol $\mathcal{V}_F$ to denote the vector space $\mathcal{V}$ having the subfield F as its field of scalars.

Corollary 2.4.4. (Generalized Burnside Theorem) *Let $n > 1$, and let K be a field, F a subfield of K, and $\mathcal{V}$ an n-dimensional vector space over K.*

(i) If F is k-closed for each k dividing n with $k > 1$, then up to a similarity the only irreducible F-algebra of linear transformations on $\mathcal{V}$ with traces in F but not identically zero is the F-algebra of all linear transformations on $\mathcal{V}_F$.

(ii) If there exists an irreducible F-algebra $\mathcal{A}$ of linear transformations on $\mathcal{V}$ with $\bar{\sigma}(\mathcal{A}) \subseteq F$, then F is k-closed for each $k = 2, ..., n$. Therefore, after a similarity $\mathcal{A} = \mathcal{L}(\mathcal{V}_F)$.

(iii) If up to a similarity the only irreducible F-algebra of linear transformations on $\mathcal{V}$ with traces in F is the F-algebra of all linear transformations on $\mathcal{V}_F$, then for each k dividing n with $k > 1$ every polynomial $f \in F[X]$ of degree k is reducible over K. In particular, if $F = K$ and the conclusion of Burnside's Theorem holds, then F is k-closed for each k dividing n where $k > 1$.

Proof. (i) Fix a basis for $\mathcal{V}$. Relative to that basis we end up dealing with F-algebras of matrices with traces in the subfield F. Thus it suffices to prove the matrix version of the assertion. Namely, we need to prove that up to a similarity the only irreducible F-algebra in $M_n(K)$ on which trace is not identically zero and with traces in F is the F-algebra $M_n(F)$. This is an immediate consequence of Theorem 2.4.2(i).

(ii) This easily follows from Theorem 2.4.1.

(iii) Fixing a basis for $\mathcal{V}$, it suffices to prove the matrix version of the assertion. Use contradiction: Suppose there exists a polynomial $f \in F[X]$ of degree r dividing n with $r > 1$ that is irreducible over K. Let $A \in M_r(F)$ denote the companion matrix of f. It follows from elementary algebra that $\mathcal{D}_r := F[A]$ is indeed an irreducible commutative division F-algebra in $M_r(K)$ with traces in F. By the proof of the converse part of Theorem 2.3.11(iii), we see that $\mathcal{A} := M_{n/r}(\mathcal{D}_r)$ is an irreducible F-algebra in $M_n(K)$ with traces in F which is not similar to $M_n(F)$ for the smallest nonzero rank in $\mathcal{A} := M_{n/r}(\mathcal{D}_r)$ is $r > 1$ (note that the smallest nonzero rank in $M_n(F)$ is 1), contradicting the hypothesis. □

Remarks.

1. The irreducible $\overline{\mathbb{Z}}_2$-algebra $\mathcal{A}$ in the second remark following Theorem 2.4.2 shows that the converse of Corollary 2.4.4(iii) is not true. It is however worth mentioning that if $K = F$, then the converse holds by Theorem 2.2.21. In other words, if $K = F$ then the only irreducible algebra of linear transformations on $\mathcal{V}$ is the algebra of all linear transformations on $\mathcal{V}$ if and only if F is k-closed for each k dividing n with $k > 1$ where $n > 1$ is the dimension of the underlying space $\mathcal{V}$.

2. In view of the remark following Corollary 2.3.14, we see that the Generalized Burnside Theorem above holds whenever the field K is k-closed for all k dividing n with $k > 1$ and the subfield F finite.

The first part of the following corollary gives a new proof of a special case of a result of Guralnick (see [G]). Part (ii) gives a new proof of a special case of the finite-dimensional version of Theorem 9.2.10 of [RR] on general fields.

Corollary 2.4.5. *Let K be a field, F a subfield of K, and $\mathcal{V}$ an n-dimensional vector space over K. Let $\mathcal{A}$ be an F-algebra of linear transformations with $\bar{\sigma}(\mathcal{A}) \subseteq F$.*

(i) The F-algebra $\mathcal{A}$ is triangularizable iff $AB-BA$ is nilpotent for each $A, B \in \mathcal{A}$.

(ii) If $\operatorname{rank}(AB - BA) \leq 1$ for each $A, B \in \mathcal{A}$, then the F-algebra $\mathcal{A}$ is triangularizable.

Proof. (i) Necessity is obvious. So we need to prove sufficiency. Since every $A \in \mathcal{A}$ is triangularizable, in view of Corollary 2.2.8, we may assume, without loss of generality that K is algebraically closed. In light of the Triangularization Lemma (Lemma 1.1.3) we need to prove reducibility. Suppose otherwise. So $\mathcal{A}$ is an irreducible F-algebra of linear transformations with $\bar{\sigma}(\mathcal{A}) \subseteq F$. It follows from Corollary 2.4.4(ii) that after a similarity $\mathcal{A} = \mathfrak{L}(\mathcal{V}_F)$ which obviously does not satisfy the condition that $AB - BA$ is nilpotent for each $A, B \in \mathcal{A} = \mathfrak{L}(\mathcal{V}_F)$.

(ii) It follows from the Cayley-Hamilton Theorem that every rank-one transformation with trace zero is nilpotent. That being noted, since $\text{rank}(AB - BA) \leq 1$ and $\text{tr}(AB - BA) = 0$, we conclude that $AB - BA$ is nilpotent for each $A, B \in \mathcal{A}$. Thus (i) applies. □

Let R be a subring of a field K. We say that a matrix $A \in M_n(K)$ is *triangularizable over R* if $\bar{\sigma}(A) \subset R$. The following result gives a criterion for triangularizability of F-algebras of matrices with spectra in the subfield F provided that F is not 2-closed.

Theorem 2.4.6. *Let K be a field, F a subfield of K that is not 2-closed, and $\mathcal{A}$ an F-algebra in $M_n(K)$ with $\sigma(\mathcal{A}) \subseteq F$. Then $\mathcal{A}$ is triangularizable iff every $A \in \mathcal{A}$ is triangularizable. Conversely, let a subfield F of K be given. If every F-algebra $\mathcal{A}$ in $M_n(K)$ with $\bar{\sigma}(\mathcal{A}) \subseteq F$ is triangularizable, then F is not 2-closed.*

Proof. It suffices to prove the "if part" of the assertion. To this end, suppose that every $A \in \mathcal{A}$ is triangularizable over F or equivalently $\bar{\sigma}(A) \subset F$. Now since every $A \in \mathcal{A}$ is triangularizable, without loss of generality we may assume that K is algebraically closed. In view of the Triangularization Lemma (Lemma 1.1.3), it suffices to show that $\mathcal{A}$ is reducible. But $\mathcal{A}$ is reducible by Theorem 2.4.3, finishing the proof.

For the converse, we use contradiction. Suppose that F is 2-closed. This hypothesis easily implies that the nontriangularizable F-algebra $\mathcal{A} := \text{diag}(M_2(F), 0_{n-2})$ has the property that $\bar{\sigma}(\mathcal{A}) \subseteq F$. On the other hand, $\mathcal{A}$ must be triangularizable by hypothesis, a contradiction. □

Corollary 2.4.7. *Let K be a field, the subfield F a finite-dimensional extension of the prime field of K, and $\mathcal{A}$ an F-algebra in $M_n(K)$ with $\sigma(\mathcal{A}) \subseteq F$. Then $\mathcal{A}$ is* *triangularizable iff every $A \in \mathcal{A}$ is triangularizable. In particular, the assertion holds when F is the prime field of K.*

Proof. In view of Theorem 2.4.6, it suffices to show that the subfield F is not 2-closed. There are three cases to consider; $\mathrm{ch}(K) = 0$, $\mathrm{ch}(K) = 2$, and $\mathrm{ch}(K) = p$ with $p > 2$. If $\mathrm{ch}(K) = 0$, Lemma V.2.8 together with Exercise V.3.16(c) of [H] implies that F is not 2-closed. If $\mathrm{ch}(K) = 2$, the F would be a finite subfield of K. On the other hand, since $a \neq 1-a$ but $a^2+a = (1-a)^2+(1-a)$ for all $a \in F$, it follows that there exists $c \in F$ such that the polynomial $x^2 + x - c$ is irreducible over F. That is, F is not 2-closed in this case. Finally, if $\mathrm{ch}(K) = p$ with $p > 2$, again F would be a finite subfield of K. Now, since $a \neq -a$ but $a^2 = (-a)^2$ for all nonzero elements a of F, it follows that there exists $c \in F$ such that the polynomial $x^2 - c$ is irreducible over F. That is, F is not 2-closed in this case as well. □

Remarks.

1. Note that the conclusion of the preceding corollary holds whenever F is a finite subfield of K.

2. In the case of characteristic zero, it is not difficult to see that the corollary above still holds for any Z-algebra in $M_n(K)$ where Z is the subring of integers in K, i.e., $Z = \mathbb{Z}1$. As a matter of fact we have the following theorem.

Theorem 2.4.8. *Let K be a field, R a subring of K whose field of quotients, denoted by F, is not 2-closed, and $\mathcal{A}$ an R-algebra in $M_n(K)$ with $\sigma(\mathcal{A}) \subseteq R$. Then $\mathcal{A}$ is triangularizable iff every $A \in \mathcal{A}$ is triangularizable over R. Conversely, let a subring R of K be given, and let R be a UFD as a ring. If every R-algebra $\mathcal{A}$ in $M_n(K)$ with $\bar{\sigma}(\mathcal{A}) \subseteq R$ is triangularizable, then the field of quotients of R, i.e., F, is not 2-closed.*

Proof. It suffices to prove the "if" part of the assertion. To this end, suppose that every $A \in \mathcal{A}$ is triangularizable over R or equivalently $\bar{\sigma}(\mathcal{A}) \subset R$. Let $\mathcal{A}_1$ denote the F-algebra generated by $\mathcal{A}$ where F is the field of quotients of R. It is easily seen that $\bar{\sigma}(\mathcal{A}_1) \subset F$. Thus $\mathcal{A}_1$ is triangularizable by Theorem 2.4.6, and hence so is $\mathcal{A}$.

As for the converse, again we use contradiction. Suppose that F is 2-closed. This hypothesis implies that the nontriangularizable R-algebra $\mathcal{A} := \mathrm{diag}(M_2(R), 0_{n-2})$

has the property that $\bar{\sigma}(\mathcal{A}) \subseteq R$ (here we have used the Gauss Lemma which implies that a monic polynomial $f \in R[X]$ of positive degree is reducible over R iff it is reducible over F, e.g., see Lemma 6.6.13 of [H]). On the other hand, $\mathcal{A}$ must be triangularizable by hypothesis, a contradiction. $\square$

Corollary 2.4.9. *Let $\mathcal{V}$ be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, and $\mathcal{A}$ an $\mathbb{R}$-algebra of linear transformations in $\mathcal{B}(\mathcal{V})$ whose spectra are subsets of $\mathbb{R}$. Then $\mathcal{A}$ is triangularizable iff every element of $\mathcal{A}$ is triangularizable.*

Proof. The field of reals has zero characteristic and is not 2-closed, so Theorem 2.4.6 applies. $\square$

Corollary 2.4.10 (Kaplansky's Theorem). *Let $n > 1$, K be a field with $\mathrm{ch}(K) =$ 0 or $> n$. Then every semigroup $\mathcal{S}$ of matrices in $M_n(K)$ with constant trace is triangularizable. Furthermore, every diagonal entry in a triangularization of such a semigroup is either constantly 0 or constantly 1.*

Proof. Let F be the prime field of K and $\mathcal{A}$ be the F-algebra generated by $\mathcal{S}$. In view of Corollary 2.4.7, it suffices to show that every $A \in \mathcal{A}$ is triangularizable over F. Suppose that $\mathrm{tr}(\mathcal{S}) = \{C\}$ for some $C \in K$. Suppose that $A = c_1S_1 + ... + c_kS_k \in \mathcal{A}$ where $k \in \mathbb{N}$, $c_j \in F$, $S_j \in S$ for each $j = 1, ..., k$ is given. Since $\mathrm{tr}(\mathcal{S}) = \{C\}$, it is easily seen that $\mathrm{tr}(A^j) = C(c_1 + ... + c_k)^j$ for all $j \in \mathbb{N}$. If $c_1 + ... + c_k = 0$ it follows from Lemma 2.2.5(iii) that A is nilpotent and hence is triangularizable over F. If $c := c_1 + ... + c_k \neq 0$, then $\mathrm{tr}(\frac{A}{c})^j = C$ for all $j \in \mathbb{N}$. So it follows from Lemma 2.2.6(ii), C is an integer, i.e., $C \in F$, and that $\bar{\sigma}(\frac{A}{c}) \subset \{0, 1\}$. Hence $\bar{\sigma}(A) \subset \{0, c\}$ where $A = c_1S_1 + ... + c_kS_k \in \mathcal{A}$ and $c := c_1 + ... + c_k \in F$ (In particular, $\bar{\sigma}(S) \subset \{0, 1\}$ for all $S \in \mathcal{S}$). Thus $\mathcal{A}$, and hence $\mathcal{S}$, is triangularizable over F. Since $\bar{\sigma}(S) \subset \{0, 1\}$ for all $S \in \mathcal{S}$, in any triangularization of $\mathcal{S}$, the diagonal zero entries of any $S, T \in \mathcal{S}$ do occur at the same position, for otherwise we would have $\mathrm{tr}(ST) \neq \mathrm{tr}(S)$, contradicting the constancy of the trace functional on $\mathcal{S}$ (see Corollary 2.2.3 of [RR]). $\square$

Remark. We would like to point out that Kaplansky's Theorem is a quick consequence of Corollary 2.3.4(iii) above because, in view of the proof above, each member of such a semigroup is triangularizable over F; indeed $\bar{\sigma}(S) \subset \{0, 1\}$ for all $S \in \mathcal{S}$. The detailed proof above is given here not only because of its independent interest, but also because the same proof will be used again in infinite dimensions.

2.5 Some Applications in Infinite Dimensions

We start this section with an infinite-dimensional analogue of Lemma 2.2.1 and its consequences. The proof of the following lemma is very much like Lemma 2.2.1. However, we include the proof for the sake of completeness.

Lemma 2.5.1. *Let $\mathcal{X}$ be a real or complex Banach space, $\mathcal{S}$ a semigroup in $\mathcal{B}(\mathcal{X})$, and T a nonzero linear operator in $\mathcal{B}(\mathcal{X})$. If $\mathcal{S}$ is irreducible, then so is $T\mathcal{S}|_{\mathcal{R}}$ where $\mathcal{R} = \overline{T\mathcal{X}}$ is the closure of the range of T.*

Proof. Just as in Lemma 2.2.1, if $\dim \mathcal{X} = 1$, then the assertion trivially holds. So we may assume, with no loss of generality, that $\dim \mathcal{X} > 1$. There are two cases to consider.

(a) $\text{rank}(T) = 1$.

To prove the assertion by contradiction suppose $T\mathcal{S}|_{\mathcal{R}}$ is reducible. Since $\dim \mathcal{R} = 1$ in this case, it follows from definition that $T\mathcal{S}|_{\mathcal{R}} = \{0\}$. Therefore, $T\mathcal{S}T = \{0\}$. Pick a nonzero $x \in \mathcal{X}$ such that $Tx \neq 0$. Now either $\mathcal{S}Tx = \{0\}$ in which case $\langle Tx \rangle$ is a nontrivial invariant subspace for $\mathcal{S}$, or else $\overline{\langle \mathcal{S}Tx \rangle}$ is a nontrivial invariant subspace for $\mathcal{S}$, because $T\mathcal{S}T = \{0\}$ and that $\mathcal{S}$ is a semigroup. This contradicts the hypothesis that $\mathcal{S}$ is irreducible.

(b) $\text{rank}(T) > 1$.

First note that a semigroup $\mathcal{S}$ is irreducible iff the algebra $\mathcal{A}$ generated by the semigroup is irreducible. That being noted, it suffices to show that that $T\mathcal{A}|_{\mathcal{R}}$ is irreducible because every invariant subspace of $T\mathcal{S}|_{\mathcal{R}}$ is invariant for $T\mathcal{A}|_{\mathcal{R}}$ as well. To prove that $T\mathcal{A}|_{\mathcal{R}}$ is irreducible, we use contradiction. Suppose that $T\mathcal{A}|_{\mathcal{R}}$ is reducible.

So there exists a nontrivial subspace $\mathcal{M}$ of $\mathcal{R} = \overline{T\mathcal{X}}$ such that $T\mathcal{A}\mathcal{M} \subseteq \mathcal{M}$. Choose a nonzero $x \in \mathcal{M}$ and note that $T\overline{\mathcal{A}x} \subseteq \mathcal{M}$. The subspace $\overline{\mathcal{A}x}$ is an invariant subspace of $\mathcal{A}$. Furthermore, it is proper, for $\overline{T\overline{\mathcal{A}x}} \subseteq \mathcal{M} \subset \mathcal{R}$. If $\mathcal{A}x = 0$ then $\langle x \rangle$ is a nontrivial invariant subspace for $\mathcal{A}$, otherwise $\overline{\mathcal{A}x}$ will be a nontrivial invariant subspace for $\mathcal{A}$. So in any event we conclude that $\mathcal{A}$ is reducible, a contradiction. □

Recall that an operator T in $\mathcal{B}(\mathcal{X})$ is called idempotent if $T^2 = T$. The corollary below is a quick consequence of the preceding corollary.

Corollary 2.5.2. *Let $\mathcal{X}$ be a real or complex Banach space, $\mathcal{S}$ a semigroup in $\mathcal{B}(\mathcal{X})$, and T a nonzero idempotent in $\mathcal{B}(\mathcal{X})$. If $\mathcal{S}$ is irreducible, then so is $T\mathcal{S}T|_{\overline{T\mathcal{X}}}$.*

Proof. Lemma 2.5.1. □

Let $\mathcal{X}$ be a complex (resp. real) Banach space, and S a subset of $\mathbb{C}$ (resp. $\mathbb{R}$). By an *S-semigroup* $\mathcal{S}$ of $\mathcal{B}(\mathcal{X})$, we mean a multiplicative semigroup $\mathcal{S}$ of bounded operators that is closed under scalar multiplication by the elements of S.

Lemma 2.5.3. *Let $\mathcal{X}$ be a complex (resp. real) Banach space and $\mathcal{S}$ a uniformly closed $\mathbb{R}^+$-semigroup of compact (resp. compact triangularizable) operators on $\mathcal{X}$ where $\mathbb{R}^+$ denotes the set of positive real numbers. If $\mathcal{S}$ contains an operator that is not quasinilpotent, then $\mathcal{S}$ contains a nonzero finite-rank operator that is either idempotent or nilpotent.*

Proof. If $\mathcal{X}$ is a complex Banach space, then the proof is the same as that of Lemma 7.4.5 of [RR]; in case $\mathcal{X}$ is a real Banach space, then the idea of proof is identical. First note that, multiplying by an appropriate sequence of positive reals, we can assume that there is a $K \in \mathcal{S}$ of spectral radius 1. Since $\sigma(K) \subset \mathbb{R}$ (note that K is triangularizable over the real Banach space $\mathcal{X}$), it follows that K has either one or two eigenvalues of absolute value 1, namely 1 or -1. If necessary, by repeated application of Corollary 6.4.13 of [RR], one can conclude that there are complementary invariant subspaces $\mathcal{N}$ and $\mathcal{R}$ of K such that $\mathcal{N}$ is finite dimensional, $\emptyset \neq \sigma(K|_{\mathcal{N}}) \subseteq \{-1, 1\}$, and $\rho(K|_{\mathcal{R}}) < 1$. From this point on, the proof is identical to that of Lemma 7.4.5 of [RR]. □

Theorem 2.5.4. *Let $\mathcal{X}$ be a real or complex Banach space of dimension greater than 1, F a subfield of $\mathbb{R}$, and $\mathcal{A} \leq \mathcal{B}_0(\mathcal{X})$ an F-algebra of triangularizable compact operators whose spectrum is in F. Then $\mathcal{A}$ is reducible.*

Proof. It suffices to show that $\bar{\mathcal{A}}$, the uniform closure of $\mathcal{A}$, is reducible. First note that $\bar{\mathcal{A}}$ is an $\mathbb{R}$-algebra in $\mathcal{B}_0(\mathcal{X})$ with $\bar{\sigma}(\bar{\mathcal{A}}) \subseteq \mathbb{R}$. To see this, note that $\bar{F} = \mathbb{R}$ because $\mathbb{Q} \subseteq F \subseteq \mathbb{R}$. Now it follows from the hypothesis and Lemma 5 on page 1091 of [DS] that $\bar{\sigma}(\bar{\mathcal{A}}) \subseteq \mathbb{R}$. To prove reducibility of $\bar{\mathcal{A}}$, we use contradiction. If $\bar{\mathcal{A}}$ is a Volterra $\mathbb{R}$-algebra, i.e., an $\mathbb{R}$-algebra of quasinilpotent operators, then $\mathcal{A}$ is triangularizable, hence reducible, by Turovskii's Theorem (Theorem 8.1.11 of [RR] or see [T]) which is a contradiction (note that every Volterra $\mathbb{R}$-algebra is in particular a Volterra semigroup). So suppose that $\bar{\mathcal{A}}$ contains an operator that is not quasinilpotent. It follows from the preceding lemma that $\bar{\mathcal{A}}$ then contains a nonzero finite-rank operator T that is either idempotent or nilpotent. Since $\bar{\mathcal{A}}$ is assumed to be irreducible, without loss of generality we may assume that rank(T) > 1. Let $\mathcal{R}$ denote the range of T, by Lemma 2.5.1 the $\mathbb{R}$-algebra $T\bar{\mathcal{A}}|_{\mathcal{R}}$, on the finite-dimensional space $\mathcal{R}$ over $\mathbb{R}$ or $\mathbb{C}$ of dimension greater than 1, is irreducible. On the other hand, by Corollary 2.4.9 the $\mathbb{R}$-algebra $T\bar{\mathcal{A}}|_{\mathcal{R}}$ is triangularizable, hence reducible, for $T\bar{\mathcal{A}}|_{\mathcal{R}}$ is an $\mathbb{R}$-algebra of triangularizable linear transformations on the finite-dimensional vector space $\mathcal{R}$ over $\mathbb{F}$ with $\sigma(T\bar{\mathcal{A}}|_{\mathcal{R}}) \subset \mathbb{R}$. This contradiction proves the assertion. □

Corollary 2.5.5. *Let $\mathcal{X}$ be a real or complex Banach space of dimension greater than 1, R a subring of $\mathbb{R}$, and $\mathcal{A} \leq \mathcal{B}_0(\mathcal{X})$ an R-algebra of triangularizable compact operators whose spectra are subsets of R. Then $\mathcal{A}$ is reducible.*

Proof. Let F denote the field of quotients of R, and $\mathcal{A}_F$ be the F-algebra generated by $\mathcal{A}$. It is plain that F is a subfield of $\mathbb{R}$, and that $\mathcal{A}_F \leq \mathcal{B}_0(\mathcal{X})$ is an F-algebra of triangularizable compact operators whose spectrum is in F. Now it follows from Theorem 2.5.4 that $\mathcal{A}_F$ is reducible and so is $\mathcal{A} \subset \mathcal{A}_F$, finishing the proof. □

Corollary 2.5.6. *Let $\mathfrak{X}$ be a real or complex Banach space, R a subring of $\mathbb{R}$, and $\mathcal{A} \leq \mathcal{B}_0(\mathfrak{X})$ an R-algebra of compact operators with spectra in R. Then $\mathcal{A}$ is triangularizable iff every element of $\mathcal{A}$ is triangularizable.*

Proof. The "only if" part trivially holds. So it suffices to prove the "if" part. The proof of the "if" part is established by the Triangularization Lemma (Lemma 1.1.3) and Corollary 2.5.5. □

Now we plan to establish analogues of Theorem 2.3.1 for irreducible semigroups of $\mathcal{C}_1$ operators. First we start with an analogue of Lemma 2.2.5 and 2.2.6. It is worth mentioning that Lemma 2.5.7(i) below is taken from [R1] and that Lemma 2.5.7(ii) and Lemma 2.5.8(ii) are slight generalizations of an observation made in the proof of Theorem 5 of [R1].

Lemma 2.5.7. *(i) Let $\sum_{i=1}^{\infty} a_i$ be an absolutely convergent series in $\mathbb{C}$ with $|a_i| < 1$ for all $i \in \mathbb{N}$. Then*

$$\lim_n \sum_{i=1}^{\infty} a_i^n = 0.$$

(ii) Let $a_i \in \mathbb{C}$ with $|a_i| = 1$ $(1 \leq i \leq m)$ be such that $\lim_n \sum_{i=1}^{m} a_i^n = c$ where $c \in \mathbb{C}$. Then $c = m$ and $a_i = 1$ for all $1 \leq i \leq m$.

(iii) Let $a_i, b_j \in \mathbb{C}$ with $|a_i| = |b_j| = 1$ $(1 \leq i \leq m, 1 \leq j \leq n)$ be such that $\lim_k (\sum_{i=1}^{m} a_i^k - \sum_{j=1}^{n} b_j^k) = 0$. Then $m = n$ and there is a permutation σ on m letters such that $b_i = a_{\sigma(i)}$ for all $1 \leq i \leq m$.

Proof. (i) Since $\sum_{i=1}^{\infty} a_i$ is absolutely convergent, it follows that there exists $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} |a_i| < 1.$$

We can write

$$|\sum_{i=1}^{\infty} a_i^n| \leq \sum_{i=1}^{\infty} |a_i|^n = \sum_{i=1}^{N} |a_i|^n + \sum_{i=N+1}^{\infty} |a_i|^n$$

$$\leq \sum_{i=1}^{N} |a_i|^n + (\sum_{i=N+1}^{\infty} |a_i|)^n$$

Letting $n \longrightarrow \infty$ completes the proof.

(ii) First we claim there is a subsequence k_l such that $\lim_l a_i^{k_l} = 1$ for all $1 \leq i \leq m$. To see this, we use induction on m. If $m = 1$ the assertion is obvious. Suppose the assertion holds for $m-1$, we prove the assertion for m. By the induction hypothesis there exists a subsequence k_l such that $\lim_l a_i^{k_l} = 1$ for all $1 \leq i \leq m-1$. If necessary, by taking a subsequence of n_l with no loss of generality we may assume that $\lim_l a_i^{k_l} = 1$ for all $1 \leq i \leq m-1$ and $\lim_l a_m^{k_l} = b_m$ for some $b_m \in \mathbb{C}$ with $|b_m| = 1$. Define

$$A := \{(a_1^k, ..., a_{m-1}^k, a_m^k) : k \in \mathbb{N}\},$$

and let A' denote the set of limit points of A. It is now plain that

$$(1, ..., 1, b_m^k) \in A',$$

for all $k \in \mathbb{N}$. Since there is a subsequence k'_j such that $\lim_j b_m^{k'_j} = 1$, it follows that $(1, ..., 1, 1) \in (A')' \subset A'$. Thus there exists a subsequence k_n such that $\lim_n (a_1^{k_n}, ..., a_{m-1}^{k_n}, a_m^{k_n}) = (1, ..., 1, 1)$ and hence $\lim_n a_i^{k_n} = 1$ for all $1 \leq i \leq m$, completing the proof of the claim.

Now using the hypothesis we can write

$$\lim_n \sum_{i=1}^{m} a_i^{(k_n+1)q} = c,$$

where $q \in \mathbb{N}$ is arbitrary. Since $\lim_n a_i^{k_n} = 1$ for all $1 \leq i \leq m$, we conclude that

$$\sum_{i=1}^{m} a_i^q = c,$$

for each $q \in \mathbb{N}$. Applying Lemma 2.2.6(ii) we conclude that $a_i = 1$ for each $i = 1, ..., m$. Therefore $c = m$ and the proof is complete.

(iii) It follows from the proof of (ii) that there is a subsequence k_l such that $\lim_l a_i^{k_l} = 1, \lim_l b_j^{k_l} = 1$ for all $1 \leq i \leq m, 1 \leq j \leq n$. Now using the hypothesis we can write

$$\lim_k (\sum_{i=1}^{m} a_i^{(k_l+1)q} - \sum_{j=1}^{n} b_j^{(k_l+1)q}) = 0,$$

where $q \in \mathbb{N}$ is arbitrary. Hence

$$\sum_{i=1}^{m} a_i^q - \sum_{j=1}^{n} b_j^q = 0,$$

for each $q \in \mathbb{N}$. Applying Lemma 2.2.5(ii) we conclude that $m = n$ and that there is a permutation σ on m letters such that $b_i = a_{\sigma(i)}$ for all $1 \leq i \leq m$, finishing the proof. □

Lemma 2.5.8. *Let $\sum_{j=1}^{\infty} \lambda_j$ and $\sum_{j=1}^{\infty} \mu_j$ be two absolutely convergent series in $\mathbb{C}$, and let $m \in \mathbb{N}$ be given.*

(i) If $\lambda_j, \mu_j \in \mathbb{C} \setminus \{0\}$ for all $j \in \mathbb{N}$, and

$$\sum_{j=1}^{\infty} \lambda_j^k = \sum_{j=1}^{\infty} \mu_j^k,$$

for all $k \in \mathbb{N}$ with $k \geq m$, then there is a permutation σ on $\mathbb{N}$ such that $\mu_j = \lambda_{\sigma(j)}$.

(ii) If for some $C \in \mathbb{C}$

$$\sum_{j=1}^{\infty} \lambda_j^k = C,$$

for all $k \in \mathbb{N}$ with $k \geq m$, then C is a nonnegative integer and $\lambda_j = 0$ or 1 for all $j \in \mathbb{N}$.

(iii) If for some $c \in \mathbb{C}$

$$\sum_{j=1}^{\infty} \lambda_j^k = c^{k-m} \sum_{j=1}^{\infty} \lambda_j^m,$$

for all $k \in \mathbb{N}$ with $k \geq m$, then $\lambda_j = 0$ or c for all $j \in \mathbb{N}$.

(iv) Let $\mathcal{H}$ be a real or complex Hilbert space, and $A \in \mathcal{C}_p(\mathcal{H})$. Then A is quasinilpotent iff

$$\operatorname{tr}(A^k) = 0,$$

for each $k \in \mathbb{N}$ with $k \geq m$ where $m \in \mathbb{N}$ with $m > p$.

Proof. (i) Without loss of generality we may assume that

$$|\lambda_1| = ... = |\lambda_{n_1}| > |\lambda_{n_1+1}| = ... = |\lambda_{n_2}| > ...$$

$$> |\lambda_{n_j+1}| = ... = |\lambda_{n_{j+2}}| > ...$$

$$|\mu_1| = ... = |\mu_{m_1}| > |\mu_{m_1+1}| = ... = |\mu_{m_2}| > ....$$

$$> |\mu_{m_j+1}| = ... = |\mu_{m_{j+2}}| > ...$$

We prove that $n_1 = m_1$ and that there is a permutation σ_1 on n_1 letters such that $\mu_j = \lambda_{\sigma_1(j)}$ for all $1 \leq j \leq n_1$. Using the same argument the assertion follows by induction on j finishing the proof. First we claim that $|\lambda_1| = |\mu_1|$. To see this we use contradiction. So without loss of generality suppose that $|\lambda_1| < |\mu_1|$. We can write

$$|\frac{\lambda_1}{\mu_1}|^k(\sum_{j=1}^{n_1}(\frac{\lambda_j}{|\lambda_1|})^k + \sum_{j=n_1+1}^{\infty}(\frac{\lambda_j}{|\lambda_1|})^k) = \sum_{j=1}^{m_1}(\frac{\mu_j}{|\mu_1|})^k + \sum_{j=m_1+1}^{\infty}(\frac{\mu_j}{|\mu_1|})^k, \qquad (*)$$

for all $k \in \mathbb{N}$ with $k \geq m$. By Lemma 2.5.7(i)

$$\lim_k \sum_{j=n_1+1}^{\infty}(\frac{\lambda_j}{|\lambda_1|})^k = 0 = \lim_k \sum_{j=m_1+1}^{\infty}(\frac{\mu_j}{|\mu_1|})^k.$$

On the other hand it is plain that

$$\lim_k |\frac{\lambda_1}{\mu_1}|^k = 0, |\sum_{j=1}^{n_1}(\frac{\lambda_j}{|\lambda_1|})^k| \leq n_1.$$

Therefore $\lim_k \sum_{j=1}^{m_1}(\frac{\mu_j}{|\mu_1|})^k = 0$. So it follows from Lemma 2.5.7(ii) that $m_1 = 0$, a contradiction. Hence $|\lambda_1| = |\mu_1|$. This along with (*) and Lemma 2.5.7(i) implies that

$$\lim_k[\sum_{j=1}^{n_1}(\frac{\lambda_j}{|\lambda_1|})^k - \sum_{j=1}^{m_1}(\frac{\mu_j}{|\mu_1|})^k] = 0.$$

Now it follows from Lemma 2.5.7(iii) that $n_1 = m_1$ and that there is a permutation σ_1 on n_1 letters such that $\mu_j = \lambda_{\sigma_1(j)}$ for all $1 \leq j \leq n_1$, completing the proof.

(ii) there are two cases to consider.

(a) $C = 0$

We show that $\lambda_i = 0$ for all $i \in \mathbb{N}$. To see this we use contradiction. By rearranging λ_i's we may assume that for some $n \in \mathbb{N}$

$$0 \neq |\lambda_1| = \dots = |\lambda_n| > |\lambda_{n+1}| \geq |\lambda_{n+2}| \geq \cdots$$

Set $\mu_i := \lambda_i/|\lambda_1|$ for each $i \in \mathbb{N}$. Plainly $|\mu_i| = 1$ for all $i \in \mathbb{N}$ with $1 \leq i \leq n$ and $|\mu_i| < 1$ for all $i \in \mathbb{N}$ with $i > n$. Also

$$\sum_{j=1}^{\infty} \mu_j^k = 0,$$

for all $k \in \mathbb{N}$ with $k \geq m$. We have

$$\sum_{j=1}^{n} \mu_j^k + \sum_{j=n+1}^{\infty} \mu_j^k = 0,$$

for all $k \in \mathbb{N}$ with $k \geq m$. Letting $k \longrightarrow \infty$ and using the fact that $\lim_k \sum_{i=n+1}^{\infty} \mu_i^k = 0$ (by Lemma 2.5.7(i)) we conclude that

$$\lim_k \sum_{j=1}^{n} \mu_j^k = 0.$$

Now since $|\mu_i| = 1$ for all $1 \leq i \leq n$ it follows from Lemma 2.5.7(ii) that $n = 0$, a contradiction.

(b) $C \neq 0$

Without loss of generality we may assume that

$$|\lambda_1| = \dots = |\lambda_n| > |\lambda_{n+1}| \geq |\lambda_{n+2}| \geq \cdots$$

We show that $\lambda_i = 1$ for all $1 \leq i \leq n$ and that $\lambda_i = 0$ for $i \in \mathbb{N}$ with $i > n$, finishing the proof.

First we show that $|\lambda_1| = 1$. Use contradiction.

If $|\lambda_1| > 1$, then we would have

$$\sum_{j=1}^{n} \mu_j^k + \sum_{j=n+1}^{\infty} \mu_j^k = C/|\lambda_1|^k,$$

for all $k \in \mathbb{N}$ with $k \geq m$ where μ_i's are as in (a). Again letting $k \longrightarrow \infty$ and using the fact that $\lim_k \sum_{i=n+1}^{\infty} \mu_i^k = 0$ (by Lemma 2.5.7(i)) we conclude that

$$\lim_k \sum_{j=1}^{n} \mu_j^k = 0.$$

Now since $|\mu_i| = 1$ for all $1 \leq i \leq n$, it follows from Lemma 2.5.7(ii) that $n = 0$, a contradiction.

If $|\lambda_1| < 1$, then we would have

$$|\lambda_1|^k (\sum_{j=1}^{n} \mu_j^k + \sum_{j=n+1}^{\infty} \mu_j^k) = C,$$

for all $k \in \mathbb{N}$ with $k \geq m$. On the other hand $\lim_k |\lambda_1|^k = 0$, $|\sum_{j=1}^{n} \mu_j^k| \leq n$, and $\lim_k \sum_{j=n+1}^{\infty} \mu_j^k = 0$. Thus letting $k \longrightarrow \infty$, it follows that $C = 0$, a contradiction. Therefore $|\lambda_1| = 1$. So we can write

$$\sum_{j=1}^{n} \lambda_j^k + \sum_{j=n+1}^{\infty} \lambda_j^k = C,$$

for all $k \in \mathbb{N}$ with $k \geq m$. Once again letting $k \longrightarrow \infty$, we conclude that

$$\lim_k \sum_{j=1}^{n} \lambda_j^k = C.$$

Now since $|\lambda_j| = 1$ for all $1 \leq j \leq n$, it follows from Lemma 2.5.7(ii) that $\lambda_j = 1$ for all $1 \leq j \leq n$ and therefore $C = n$.

Since $\lambda_j = 1$ for all $1 \le j \le n$ and $C = n$, we can write

$$\sum_{j=n+1}^{\infty} \lambda_j^k = 0,$$

for all $k \in \mathbb{N}$ with $k \ge m$. Now it follows from (a) that $\lambda_i = 0$ for all $j > n$, finishing the proof.

(iii) If $c = 0$, then (ii) applies with $C = 0$. If $c \ne 0$, again (ii) applies to the series $\sum_{j=1}^{\infty}(\lambda_j/c)$ with $C = \sum_{j=1}^{\infty}(\lambda_j/c)^m$.

(iv) Necessity easily follows from Lidskii's Theorem (Theorem XI.9.19 of [DS], page 1104). To see sufficiency, let $(\lambda_i)_{i=1}^{\infty}$ be the eigenvalues of A in $\mathbb{C}$ (counting multiplicities). It is known that $A^k \in \mathcal{C}_1$ for all $k \in \mathbb{N}$ with $k > p$ (see Lemma XI.9.9(c) of [DS]). So it follows from the hypothesis that

$$\sum_{j=1}^{\infty} \lambda_j^k = 0,$$

for all $k \in \mathbb{N}$ with $k \ge m$. Now applying (iii) with $c = 0$ we conclude that $\lambda_i = 0$ for all $i \in \mathbb{N}$. Thus A is quasinilpotent, finishing the proof. It is worth mentioning that if A had only finitely many eigenvalues the assertion would follow from Lemma 2.2.6(i) with $c = 0$. □

Recall that a semigroup (resp. algebra) of compact quasinilpotent operators on a Banach space is called a *Volterra semigroup* (resp. *Volterra algebra*). As usual, by $\mathbb{F}$ we mean $\mathbb{R}$ or $\mathbb{C}$. The following well-known lemma which extends Kaplansky's Theorem (Corollary 2.4.10) to trace class operators is proved in [R1].

Lemma 2.5.9. *Let $\mathcal{H}$ be a real or complex Hilbert space, $\mathcal{S}$ a semigroup in $\mathcal{C}_1$ on which trace is constant. Then the semigroup $\mathcal{S}$ is triangularizable. In particular, if trace is zero on a semigroup $\mathcal{S}$ in $\mathcal{C}_1$, then the algebra generated by $\mathcal{S}$ is a Volterra algebra of $\mathcal{C}_1$ operators.*

Proof. In view of Corollary 2.5.6 and Lemma 2.5.8, the proof of the first assertion is identical to that of Corollary 2.4.10. For the rest, in view of the preceding lemma, it is easily seen that the algebra $\mathcal{A}$ generated by $\mathcal{S}$ is indeed a Volterra algebra of $\mathcal{C}_1$ operators. □

Remarks.

1. In view of Lemma 2.5.8, it follows from the proof Corollary 2.4.10 that if trace is constant on a semigroup $\mathcal{S}$ of trace class operators, then the constant is indeed an integer and that $\bar{\sigma}(c_1S_1 + ... + c_nS_n) \subseteq \{0, c_1 + ... + c_n\}$ where c_i's are scalars and S_i's are in $\mathcal{S}$.

2. A quick consequence of the second part of the preceding lemma is that *if an algebra $\mathcal{A}$ of $\mathcal{C}_1$ operators is generated by quasinilpotents as a vector subspace of $\mathcal{C}_1$, then the algebra $\mathcal{A}$ is a Volterra algebra of $\mathcal{C}_1$ operators, and therefore the algebra $\mathcal{A}$ is triangularizable.* This suggests the following conjecture.

Conjecture. *Let $\mathcal{X}$ be a real or complex Banach space of infinite dimension, and $\mathcal{S}$ a semigroup in $\mathcal{B}_0(\mathcal{X})$. If every S in $\mathcal{S}$ can be written as a linear combination of quasinilpotent operators from the algebra generated by $\mathcal{S}$, then the algebra generated by $\mathcal{S}$ is a Volterra algebra, and therefore the semigroup $\mathcal{S}$ is triangularizable.*

The above conjecture, if true, immediately extends Turovskii's Theorem (Theorem 8.1.11 of [RR]), Kolchin's Theorem, and Theorem 2.5.11 below for compact operators on a real or complex Banach space.

Theorem 2.5.10. *Let $\mathcal{H}$ be a real or complex Hilbert space, $\mathcal{S}$ an irreducible semigroup of $\mathcal{C}_1$ operators, and $\mathcal{J}$ a semigroup ideal of $\mathcal{S}$. Then*

(i)

$$\{A \in \mathrm{Alg}_{\mathbb{F}}(\mathcal{S} \cup \{I\}) : \mathrm{tr}(A\mathcal{J}) = \{0\}\} = \{0\}.$$

(ii)

$$\{A \in \mathrm{Alg}_{\mathbb{F}}(\mathcal{S} \cup \{I\}) : \rho(A\mathcal{J}) = \{0\}\} = \{0\}.$$

Proof. It suffices to prove (i).

(i) Without loss of generality we may assume that $\mathcal{H}$ is infinite-dimensional. Denote the left hand side of the asserted identity by $\mathfrak{J}$. We prove that $\mathfrak{J} = \{0\}$. To this end, let $A \in \mathfrak{J}$ be arbitrary, we show that $A = 0$. Plainly the set $\mathcal{J}A\mathcal{J} = \{J_1AJ_2 :$

$J_1, J_2 \in \mathcal{J}\}$ is a subset of $\mathrm{Alg}(\mathcal{J})$ consisting of quasinilpotents by Lemma 2.5.8(iv). The algebra $\mathrm{Alg}(\mathcal{J})$ is irreducible for $\mathcal{S}$ is an irreducible semigroup of $\mathcal{C}_1$ operators and $\mathcal{J}$ is a nonzero semigroup ideal of $\mathcal{S}$. It is easily seen that $\mathrm{Alg}(\mathcal{J}A\mathcal{J})$ is an ideal of the irreducible algebra $\mathrm{Alg}(\mathcal{J})$. We note that $\mathrm{Alg}(\mathcal{J}A\mathcal{J}) = \{0\}$, for otherwise the algebra $\mathrm{Alg}(\mathcal{J}A\mathcal{J})$ would be generated by quasinilpotents as a vector subspace of $\mathcal{C}_1$ which is a contradiction in view of the preceding theorem. Hence $\mathrm{Alg}(\mathcal{J}A\mathcal{J}) = \{0\}$. Therefore, $\mathcal{J}A\mathcal{J} = \{0\}$, and hence $A = 0$, for $\mathrm{Alg}(\mathcal{J})$ is transitive. □

Remark. Let $\mathcal{H}$, $\mathcal{S}$, and $\mathcal{J}$ be as in the preceding theorem. It is clear from the proof above that

$$\{A \in \mathrm{Alg}_{\mathbb{F}}(\mathcal{S} \cup \{I\}) : \mathrm{tr}(\mathcal{J}A\mathcal{J}) = \{0\}\} = \{0\}.$$

Theorem 2.5.11. *Let $\mathcal{H}$ be a real or complex Hilbert space, $\mathcal{S}$ an irreducible semigroup of $\mathcal{C}_p$ operators with $p > 1$, and $\mathcal{J}$ a semigroup ideal of $\mathcal{S}$. Then*

(i)

$$\{A \in \mathrm{Alg}_{\mathbb{F}}(\mathcal{S} \cup \{I\}) : \rho(\mathcal{J}A\mathcal{J}) = 0\} = \{0\}.$$

(ii)

$$\{A \in \mathrm{Alg}_{\mathbb{F}}(\mathcal{S} \cup \{I\}) : \rho(A\mathcal{J}) = 0\} = \{0\}.$$

Proof. It suffices to prove (i).

(i) Note that $\mathrm{Alg}_{\mathbb{F}}(\mathcal{S} \cup \{I\})\mathcal{J}^m \subseteq \mathcal{C}_1$ for each integer m with $m > p$. That being noted, the proof is similar to that of the preceding Theorem. Fix an integer m with $m > p$. Again denote the left hand side of the asserted identity by $\mathfrak{J}$. We prove that $\mathfrak{J} = \{0\}$. To this end, let $A \in \mathfrak{J}$ be arbitrary, we show that $A = 0$. Plainly the set $\mathcal{J}^m A \mathcal{J}^m = \{J_1 A J_2 : J_1, J_2 \in \mathcal{J}^m\}$ is a subset of $\mathrm{Alg}(\mathcal{J}^m)$ consisting of quasinilpotents. The algebra $\mathrm{Alg}(\mathcal{J}^m)$ is irreducible, for $\mathcal{S}$ is an irreducible semigroup of $\mathcal{C}_p$ operators and $\mathcal{J}$, hence $\mathcal{J}^m$, is a nonzero semigroup ideal of $\mathcal{S}$. It is easily seen that $\mathrm{Alg}(\mathcal{J}^m A \mathcal{J}^m)$ is an ideal of the irreducible algebra $\mathrm{Alg}(\mathcal{J}^m)$. We note that $\mathrm{Alg}(\mathcal{J}^m A \mathcal{J}^m) = \{0\}$, for otherwise the algebra $\mathrm{Alg}(\mathcal{J}^m A \mathcal{J}^m)$ would be generated by quasinilpotents as a vector subspace of $\mathcal{C}_1$ which is a contradiction in view of Lemma

2.5.9. Hence $\mathrm{Alg}(\mathcal{J}^m A \mathcal{J}^m) = \{0\}$. Therefore, $\mathcal{J}^m A \mathcal{J}^m = \{0\}$, and hence $A = 0$ for $\mathrm{Alg}(\mathcal{J}^m)$ is a transitive algebra. □

Having proved the $\mathcal{C}_p$ version of Theorem 2.3.1 and Theorem 2.3.2, namely Theorem 2.5.10 and Theorem 2.5.11, one can prove analogues of Corollary 2.3.4(iii), and Corollary 2.3.5 for $\mathcal{C}_p$ class operators acting on a real or complex Hilbert space. To establish this, one follows the line of argument deducing Corollary 2.3.4(iii) and Corollary 2.3.5 from Theorem 2.3.1. It is worth mentioning that $\mathcal{C}_p$ class operators can naturally be defined on any real or complex Banach space that is isomorphic to a real or complex Hilbert space respectively. As well, as pointed out in [KR], it is shown by König and others that on arbitrary Banach spaces there exist ideals of compact operators (denoted by $\mathrm{S}_a^1(\mathcal{X})$ and $\Pi_2^2(\mathcal{X})$, see [Kö]) on which trace is well-defined as the continuous linear extensions of the trace of finite-rank operators and that Lidskii's Theorem holds on these ideals. Similarly, one can prove analogues of Corollary 2.3.4(iii), and Corollary 2.3.5 for $\mathcal{C}_p$ class operators acting on such a real or complex Banach space as well as for semigroups in $\mathrm{S}_a^1(\mathcal{X})$ and $\Pi_2^2(\mathcal{X})$ on arbitrary Banach spaces. For the sake of brevity we omit the details of proofs.

In infinite dimensions Corollary 2.2.16 (Kaplansky) can be strengthened as follows. It is worth mentioning that this stronger result is due to Nordgren-Radjavi-Rosenthal (see Theorem 8.6.13 of [RR] or [NRR]) and that it does not hold in finite dimensions (e.g., if $n > 1$, and F is a field such that $\mathrm{ch}(F) = 0$ or $\mathrm{ch}(F)$ is not a divisor of n, then every matrix in $M_n(F)$ can be written as $\alpha I + N$ where N is a matrix with $\mathrm{tr}(N) = 0$).

Corollary 2.5.12. *Let $\mathcal{H}$ be an infinite-dimensional real or complex Hilbert space. Then every semigroup $\mathcal{S}$ of operators of the form $\alpha I + N$ where N is a trace class operator with $\mathrm{tr}(N) = 0$ and with $\alpha \in \mathbb{F}$ is triangularizable.*

Proof. Since the underlying space is infinite-dimensional, a straightforward induction shows that trace is zero on the semigroup generated by N's as described in the statement of the theorem. Now since trace is zero on the semigroup generated by the N's, it follows from Lemma 2.2.8(iv) that the algebra generated by the N's is a

Volterra algebra and hence triangularizable. So is the semigroup $\mathcal{S}$, completing the proof. □

Recall that we use $\mathcal{B}_{00}(\mathcal{X})$ to denote the ideal of finite-rank operators on a Banach space $\mathcal{X}$. Note that the trace on $\mathcal{B}_{00}(\mathcal{X})$ is defined by the finite sum of the spectrum over $\mathbb{C}$, counting multiplicities. It can be shown that trace, defined this way, is indeed a continuous linear functional on $\mathcal{B}_{00}(\mathcal{X})$ having all the basic properties of the finite-dimensional trace functional that one expects. That being noted, here is the infinite-dimensional version of Theorem 2.3.1 for finite-rank operators acting on a real or complex Banach space.

Theorem 2.5.13. *Let $\mathcal{X}$ be a real or complex Banach space, $\mathcal{S}$ an irreducible semigroup of finite-rank operators on $\mathcal{X}$, and $\mathcal{J}$ a semigroup ideal of $\mathcal{S}$. Then all the assertions of Theorem* 2.5.10 *hold.*

Proof. The proof is almost identical to that of Theorem 2.5.10. □

We can now present the infinite-dimensional version of Corollary 2.3.4(iii) above for collections of triangularizable finite-rank operators acting on a real or complex Banach space.

Theorem 2.5.14. *Let $\mathcal{X}$ be a real or complex Banach space. Then a collection $\mathcal{F}$ of triangularizable finite-rank operators is triangularizable iff trace is permutable on $\mathcal{F}$.*

Proof. In light of Theorem 2.5.13, the proof is identical to that of Corollary 2.3.4(iii). □

Again having proved the infinite-dimensional version of Theorem 2.3.1 and Corollary 2.3.4 for collections of triangularizable finite-rank operators, one can prove the infinite-dimensional version of Corollary 2.3.5, and Lemma 2.3.7 and its consequences for collections of triangularizable finite-rank operators on every real or complex Banach space. As well, again in view of Corollary 2.5.6 and Lemma 2.5.8 it is possible to prove Corollary 2.4.10 (Kaplansky's Theorem) for collections of finite-rank operators acting on a real or complex Banach space. It is worth noting that the Corollary 2.5.12 remains true for semigroups of the form $\alpha I + N$ where N is a finite-rank operator acting on an infinite-dimensional real or complex Banach space with $\mathrm{tr}(N) = 0$ and $\alpha \in \mathbb{F}$. Again for the sake of brevity we omit the details.

Chapter 3

Near triangularizability

AH, my beloved, fill the cup that clears
To-day of past Regrets and future fears–
To-Morrow? –Why, To-morrow I may be
Myself with Yesterday's Sev'n Thousand Years.

–Khayyam, the Persian Mathematician, Astronomer, Philosopher, and Poet.
Rendered into English verse by Edward Fitzgerald.

3.1 Introduction

This chapter is devoted to collections of compact operators on a real or complex Banach space including linear operators on finite-dimensional vector spaces over $\mathbb{R}$ or $\mathbb{C}$. We show that such a collection is simultaneously triangularizable if and only if it is arbitrarily close to a simultaneously triangularizable collection of compact operators. Nearness is measured by the Hausdorff metric induced by the operator norm. We also prove analogous results for $\mathcal{C}_p$ class operators. As an application of these results we obtain an invariant subspace theorem for certain bounded operators. We further prove that in finite dimensions near reducibility implies reducibility if the ground field is $\mathbb{C}$ or if the ground field is $\mathbb{R}$ and the dimension of the underlying space is odd.

3.2 Near triangularizability in finite dimensions

We start off with a well-known theorem due to O. Perron (See [RR], Theorem 1.6.2).

Theorem 3.2.1 (O. Perron). *If $\mathcal{A}$ is the algebra of $n \times n$ upper triangular matrices on $\mathbb{F}$ ($\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$), relative to a given basis, then for every $\epsilon > 0$ there is an invertible matrix $S_\epsilon = \mathrm{diag}(\eta, ..., \eta^n)$ where $\eta = \eta(\epsilon)$ depends on ϵ such that*

$$\|S_\epsilon^{-1} A S_\epsilon - D(A)\| < \epsilon \|A\|$$

for all $A \in \mathcal{A}$ and where $D(A)$ is the diagonal matrix with the same entries as A on its main diagonal.

Perron's Theorem immediately implies the following

Corollary 3.2.2. *Let $\mathfrak{F} = \{A_\alpha : \alpha \in \Lambda\}$ be a norm bounded triangularizable family of linear transformations on $\mathbb{F}$ (i.e., on $\mathbb{R}$ or $\mathbb{C}$). Then there is a diagonalizable, thus commutative, family $\{D_\alpha : \alpha \in \Lambda\}$ (relative to the triangularizing basis for $\mathfrak{F}$) such that for every $\epsilon > 0$ there is an invertible transformation T_ϵ satisfying*

$$\|T_\epsilon^{-1} A_\alpha T_\epsilon - D_\alpha\| < \epsilon,$$

for all $\alpha \in \Lambda$

Proof. Triangularize $\mathfrak{F}$ by a similarity T. Set

$$D_\alpha = D(T^{-1} A_\alpha T); \; T_\epsilon = T S_{\frac{\epsilon}{M+1}}$$

where $M = \sup\{\|A_\alpha\| : \alpha \in \Lambda\}$. □

Motivated by Theorem 1.6.4 of [RR] and its proof, due to A.A. Jafarian, H. Radjavi, P. Rosenthal, and A.R. Sourour, we were able to prove the following generalization.

Theorem 3.2.3 (Near Triangularizability Theorem). *Let $\mathcal{F}$ be a family of linear transformations on a finite-dimensional vector space $\mathcal{V}$ over $\mathbb{C}$ with the following property: for each finite subfamily $\{A_1, ..., A_m\}$ of $\mathcal{F}$, there is a constant $K > 0$ such that for every $\epsilon > 0$ there exist a triangularizable family $\{T_1, ..., T_m\}$, and an invertible transformation $S = S_\epsilon$ satisfying*

$$\|T_j\| \leq K, \ \|S^{-1}A_jS - T_j\| < \epsilon,$$

for every $1 \leq j \leq m$. Then $\mathcal{F}$ is triangularizable.

Proof. First, we note that if $\mathcal{F}$ is a singleton, then we have nothing to prove, so we may assume that $|\mathcal{F}| > 1$. Secondly, it is plain that without loss of generality we may assume that $\mathcal{F}$ contains the identity transformation. Let $\mathcal{A}$ be the algebra generated by $\mathcal{F}$. In view of Theorem 1.5.4(iv) of [RR], it suffices to show that the trace of $(BC-CB)^2$ is 0 for all B and C in $\mathcal{A}$. Given $B, C \in \mathcal{A}$, there are $A_i \in \mathcal{F}$, $(1 \leq i \leq m)$, and noncommutative polynomials p and q such that

$$B = p(A_1, ..., A_m), \ C = q(A_1, ..., A_m).$$

Since all norms on $\mathcal{B}(\mathcal{V})$ are equivalent (for $\mathcal{B}(\mathcal{V})$ is a finite-dimensional vector space, see Theorem 3.3.1 of [C] on page 69), without loss of generality we may assume that $\|.\| = \|.\|_1$ with respect to a fixed basis of the space. So in particular for every $T \in \mathcal{B}(\mathcal{V})$ we have:

$$|\text{tr}(T)| \leq \|T\|,$$

where "tr" means the trace linear functional. Let $K > 0$ be the appropriate constant for $\{A_1, ..., A_m\}$. Define: $h : \mathcal{B}(\mathcal{V})^m \longrightarrow \mathcal{B}(\mathcal{V})$ by

$$h(X_1, ..., X_m) =$$

$$(p(X_1, ..., X_m)q(X_1, ..., X_m) - q(X_1, ..., X_m)p(X_1, ..., X_m))^2.$$

We observe that h is a noncommutative polynomial in m linear transformations. It is easily seen that every such h is a uniformly continuous function of its arguments on any bounded set in $\langle \mathcal{B}(\mathcal{V})^m, \|.\|_\infty \rangle$ where $\|(X_1, ..., X_m)\|_\infty = \max\{\|X_1\|, ..., \|X_m\|\}$.

In particular, for every $\eta > 0$, there is a positive δ with $0 < \delta < 1$ such that

$$\|h(X_1, ..., X_m) - h(Y_1, ..., Y_m)\| < \eta,$$

whenever $\|X_j - Y_j\| < \delta$, $\|X_j\| \le K+1$, $\|Y_j\| \le K+1$ for all $1 \le j \le m$. Now for a given $\eta > 0$, find the corresponding δ with $0 < \delta < 1$. By hypothesis, for this δ, there exists a triangularizable family $\{T_1, ..., T_m\}$, and an invertible transformation $S = S_\delta$ satisfying

$$\|T_j\| \le K, \ \|S^{-1}A_jS - T_j\| < \delta,$$

for every $1 \le j \le m$. So we can write

$$\|T_j\| \le K+1, \ \|S^{-1}A_jS\| \le \delta + \|T_j\| < 1 + K,$$

for every $1 \le j \le m$. Hence we get

$$\|S^{-1}A_jS - T_j\| < \delta, \|S^{-1}A_jS\| \le K+1, \|T_j\| \le K+1,$$

for every $1 \le j \le m$. It follows from uniform continuity of h that

$$\|h(S^{-1}A_1S, ..., S^{-1}A_mS) - h(T_1, ..., T_m)\| < \eta.$$

We note that $\mathrm{tr}(h(T_1, ..., T_m)) = 0$ for

$$h(T_1, ..., T_m) = (p(T_1, ..., T_m)q(T_1, ..., T_m) - q(T_1, ..., T_m)p(T_1, ..., T_m))^2,$$

and $\{T_1, ..., T_m\}$ is triangularizable. So we can write

$$|\mathrm{tr}(h(A_1, ..., A_m))| = |\mathrm{tr}(S^{-1}h(A_1, ..., A_m)S)|$$

$$= |\mathrm{tr}(h(S^{-1}A_1S, ..., S^{-1}A_mS))|$$

$$= |\mathrm{tr}(h(S^{-1}A_1S, ..., S^{-1}A_mS)) - \mathrm{tr}(h(T_1, ..., T_m))|$$

$$= |\mathrm{tr}(h(S^{-1}A_1S, ..., S^{-1}A_mS) - h(T_1, ..., T_m))|.$$

On the other hand, as we mentioned before

$$|\mathrm{tr}(T)| \leq \|T\|,$$

for every $T \in \mathcal{B}(\mathcal{V})$. So we can write

$$|\mathrm{tr}(h(A_1, ..., A_m))| \leq \|h(S^{-1}A_1S, ..., S^{-1}A_mS) - h(T_1, ..., T_m)\| < \eta.$$

Thus $|\mathrm{tr}(h(A_1, ..., A_m))| < \eta$. Since $\eta > 0$ was arbitrary, it follows that

$$\mathrm{tr}(h(A_1, ..., A_m)) = 0.$$

That is, $\mathrm{tr}((BC - CB)^2) = 0$. Thus $\mathcal{A}$, and therefore $\mathcal{F}$, is triangularizable by Theorem 1.5.4(iv) of [RR]. □

Remark. Using the same argument one can prove that the Near Triangularizability Theorem above holds for families of linear transformations on a finite-dimensional vector space over an algebraically closed complete field F with a nontrivial absolute value. (See Chapter XII of [L] for a nice exposition of fields with absolute values).

We recall that the *Hausdorff metric* D of a metric space (X, d) is defined, on the collection $\mathcal{H}$ of all nonempty closed bounded subsets of (X, d), as follows:

$$D(A, B) = \sup_{a \in A, b \in B} \{\mathrm{dist}(a, B), \mathrm{dist}(A, b)\},$$

where $\mathrm{dist}(a, B) = \inf\{\mathrm{d}(a, b) : b \in B\}$. It can be shown that if (X, d) is a complete metric space, so is $(\mathcal{H}, D)$. It can also be shown that D is a pseudometric on the collection $\mathcal{B}$ of all nonempty bounded subsets of (X, d). We note, however, that $D(A, B)$ may be defined and be finite even for unbounded subsets of a metric space (X, d). Using the definition, it is easy to see that if $\mathcal{F}_i$ ($i \in \mathbb{N}$) are nonempty bounded subsets of a normed space $(X, \|.\|)$, and if $\mathcal{F}_i \longrightarrow \mathcal{F}$ in the Hausdorff metric D induced by $\|.\|$, then $\mathcal{F}_i$'s and $\mathcal{F}$ are uniformly bounded, i.e., there exists $K > 0$ such that $\|x\| < K$ for all $x \in (\cup_{i \in \mathbb{N}} \mathcal{F}_i) \cup \mathcal{F}$.

We are now ready to prove that for a collection $\mathcal{F}$ of linear transformations near triagularizability implies triangularizability where nearness is measured by the Hausdorff metric induced by any norm on $\mathcal{B}(\mathcal{V})$.

Corollary 3.2.4. *Let $\mathcal{F}_i, \mathcal{F}$ $(i \in \mathbb{N})$ be nonempty families of linear transformations on a finite-dimensional complex vector space $\mathcal{V}$ satisfying (i) or (ii) below.*

(i) Each family $\mathcal{F}_n$ $(n \in \mathbb{N})$ is triangularizable and $\lim_n \operatorname{dist}(\mathcal{F}_n, f) = 0$ for all $f \in \mathcal{F}$.

(ii) Each family $\mathcal{F}_n$ $(n \in \mathbb{N})$ is triangularizable and $\mathcal{F}_i \longrightarrow \mathcal{F}$ in the Hausdorff metric D induced by any norm on $\mathcal{B}(\mathcal{V})$.

Then $\mathcal{F}$ is triangularizable.

Proof. (i) We use Theorem 3.2.3. Suppose that $\{A_1, ..., A_m\} := \mathcal{G}$ is a finite subfamily of $\mathcal{F}$. Let $1/n > 0$ $(n \in \mathbb{N})$ be given. There exists $N_n \in \mathbb{N}$ such that $\operatorname{dist}(\mathcal{F}_i, A_j) < 1/n$ for all $i \geq N_n$ and $1 \leq j \leq m$. In particular, $\operatorname{dist}(\mathcal{F}_{N_n}, A_j) < 1/n$ for all $1 \leq j \leq m$. So it follows from the definition that

$$\inf_{T \in \mathcal{F}_{N_n}} \|T - A_j\| < 1/n,$$

for all $1 \leq j \leq m$. Thus for every $1 \leq j \leq m$, there exists $T_{jN_n} \in \mathcal{F}_{N_n}$ such that $\|T_{jN_n} - A_j\| < 1/n$. Set $\mathcal{G}_n := \{T_{1N_n}, ..., T_{mN_n}\}$. Plainly $\mathcal{G}_n \subseteq \mathcal{F}_{N_n}$ and so each $\mathcal{G}_n$ is simultaneously triangularizable by hypothesis. It is easily seen that $\|x\| < K = M+1$ for all $x \in (\cup_{i \in \mathbb{N}} \mathcal{G}_i) \cup \mathcal{G}$ where $M = \max\{\|A_1\|, ..., \|A_m\|\}$. Now for given $\epsilon > 0$ choose $n \in \mathbb{N}$ such that $1/n < \epsilon$. Find the triangularizable set $\mathcal{G}_n = \{T_{1N_n}, ..., T_{mN_n}\}$ and let $S = I$ where I denotes the identity transformation, and $K = 1 + \max\{\|A_1\|, ..., \|A_m\|\}$ we obviously have

$$\|T_{jN_n}\| \leq K, \ \|A_j - T_{jN_n}\| < 1/n < \epsilon,$$

for all $1 \leq j \leq m$. That is, we have shown that the hypotheses of Theorem 3.2.3 are met, so $\mathcal{F}$ is triangularizable by Theorem 3.2.3.

(ii) It is easily seen that if $\mathcal{F}_i \longrightarrow \mathcal{F}$ in the Hausdorff metric D induced by any norm on $\mathcal{B}(\mathcal{V})$, then $\lim_n \operatorname{dist}(\mathcal{F}_n, f) = 0$ for all $f \in \mathcal{F}$. Therefore (i) applies. □

We need the following lemma.

Lemma 3.2.5. *Let A be a linear transformation on a real or complex finite-dimensional vector space. Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of simultaneously diagonalizable linear transformations and $(S_n)_{n\in\mathbb{N}}$ a sequence of invertible linear transformations such that $\lim_n \|S_n^{-1}AS_n - D_n\| = 0$, then $(D_n)_{n\in\mathbb{N}}$ is bounded.*

Proof. Diagonalize $(D_n)_{n\in\mathbb{N}}$ by a similarity T. Set

$$N_n = T^{-1}D_nT; \; S'_n = S_nT.$$

We note that $(N_n)_{n\in\mathbb{N}}$ is a sequence of diagonal, hence normal, linear transformations. We can write

$$\lim_n \|S_n'^{-1}AS'_n - N_n\| = \|T^{-1}S_n^{-1}AS_nT - T^{-1}D_nT\|$$

$$\leq \|T^{-1}\|\|T\| \lim_n \|S_n^{-1}AS_n - D_n\| = 0$$

So $\lim_n \|S_n'^{-1}AS'_n - N_n\| = 0$. Now it follows from Lemma 1.6.5 of [RR] that $(N_n)_{n\in\mathbb{N}}$ is bounded. This implies that $(D_n)_{n\in\mathbb{N}}$ is bounded. □

Corollary 3.2.6. *Let $\mathcal{F}$ be a collection of linear transformations on a complex finite-dimensional vector space $\mathcal{V}$. Then the following assertions are equivalent:*

(i) The collection $\mathcal{F}$ is triangularizable.

(ii) There is a basis $\mathcal{B}$ for the space such that for each finite subset $\{A_1, ..., A_m\}$ of $\mathcal{F}$, there exists a diagonalizable, hence commutative, set $\{D_1, ..., D_m\}$, relative to $\mathcal{B}$, of linear transformations such that for every $\epsilon > 0$ there is an invertible transformation $S = S_\epsilon$ satisfying

$$\|S^{-1}A_jS - D_j\| < \epsilon,$$

for all $1 \leq j \leq m$.

(iii) There is a basis $\mathcal{B}$ for the space such that for each finite subset $\{A_1, ..., A_m\}$ of $\mathcal{F}$ and every $\epsilon > 0$, there exists a diagonalizable, hence commutative, set of linear transformations $\{D_1, ..., D_m\}$, relative to $\mathcal{B}$, and an invertible linear transformation $S = S_\epsilon$ such that

$$\|S^{-1}A_jS - D_j\| < \epsilon,$$

for all $1 \leq j \leq m$.

(iv) For each finite subfamily $\{A_1, ..., A_m\}$ *of* $\mathcal{F}$, *there is a constant* $K > 0$ *such that for every* $\epsilon > 0$ *there exist a triangularizable family* $\{T_1, ..., T_m\}$, *and an invertible transformation* $S = S_\epsilon$ *satisfying*

$$\|T_j\| \leq K, \ \|S^{-1}A_jS - T_j\| < \epsilon,$$

for every $1 \leq j \leq m$.

(v) There exist triangularizable families $\mathcal{F}_n$ $(n \in \mathbb{N})$ *of linear transformations on* $\mathcal{V}$ *such that* $\lim_n \operatorname{dist}(\mathcal{F}_n, f) = 0$ *for all* $f \in \mathcal{F}$.

(vi) There exist triangularizable families $\mathcal{F}_n$ $(n \in \mathbb{N})$ *of linear transformations on* $\mathcal{V}$ *such* $\mathcal{F}_i \longrightarrow \mathcal{F}$ *in the Hausdorff metric* D *induced by any norm on* $\mathcal{B}(\mathcal{V})$. *i.e., the collection* $\mathcal{F}$ *is, in the Hausdorff metric, arbitrarily close to a triangularizable collection of linear transformations on* $\mathcal{V}$.

Proof. Obviously Corollary 3.2.2 shows that (i) implies (ii). That (ii) implies (iii) is obvious. That (iii) implies (iv) follows from Lemma 3.2.5. Taking $\epsilon = 1/n$ in (iii), we get a diagonalizable set $\{D_{n1}, ..., D_{nm}\}$ of linear transformations, relative to $\mathcal{B}$, and an invertible linear transformation S_n such that $\|S_n^{-1}A_jS_n - D_{nj}\| < 1/n$ for all $1 \leq j \leq m$. Hence $\lim_n \|S_n^{-1}A_jS_n - D_{nj}\| = 0$ for all $1 \leq j \leq m$. So Lemma 3.2.5 implies that $\{\|D_{nj}\|\}_{n\in\mathbb{N}}$ is bounded for all $1 \leq j \leq m$. Thus there exists $0 < K \in \mathbb{R}$ such that $\|D_{nj}\| \leq K$ for all $i \in \mathbb{N}$, $1 \leq j \leq m$. Now it is obvious that (iii) implies (iv). That (iv) implies (i) is nothing but Theorem 3.2.3. Finally, (i) obviously implies (v) and (vi). That (v) or (vi) implies (i) is a quick consequence of Corollary 3.2.4. □

In order to prove the Near Triangularizability Theorem, i.e., Theorem 3.2.3, for real vector spaces, we basically need a criterion for triangularizability of an algebra of linear transformations on real vector spaces. It is known that an algebra $\mathcal{A}$ of linear transformations on a finite-dimensional complex vector space is triangularizable iff $AB - BA$ is nilpotent for all $A, B \in \mathcal{A}$, iff each pair $\{A, B\}$ is triangularizable (see Corollary 3.2.8 below). The criterion for triangularizability of an algebra of linear transformations over reals is rather surprising. In fact it follows from Theorem 2.4.6 that individual triangularizability of the members of an algebra of transformations implies triangularizability of the algebra provided the ground field F is not 2-closed (e.g., $F = \mathbb{R}$). Below we will give a simpler proof of this fact.

Recall that if $\mathcal{V}$ is a finite-dimensional vector space over a field F, then a linear transformation T on $\mathcal{V}$ is triangularizable iff the characteristic polynomial for T is a product of linear polynomials over F (use The Triangularization Lemma, Lemma 1.1.3, or see Theorem 6.4.5 of [HK]), or equivalently $\bar{\sigma}(T) \subset F$ where $\bar{\sigma}(T)$ denotes the spectrum of T in the algebraic closure of F. Therefore there exists a nontriangularizable linear transformation in $\mathcal{L}(\mathcal{V})$ iff the underlying field F is not k-closed for some $1 < k \leq \dim \mathcal{V}$, i.e., there is a monic polynomial $f \in F[X]$ of degree k with $1 < k \leq \dim \mathcal{V}$ that is irreducible over F.

As promised in the remark following Theorem 2.4.3, we present a simpler proof of a special case of that theorem as well as Theorem 2.4.1, Corollary 2.4.5, Theorem 2.4.6, and Corollary 2.4.9.

Theorem 3.2.7. *(i) Let $\mathcal{V}$ be a finite-dimensional vector space over a field F of dimension greater than 1. Suppose that F is not k-closed for some $2 \leq k \leq \dim \mathcal{V}$, and $\mathcal{A}$ is an algebra in $\mathcal{L}(\mathcal{V})$ such that $\bar{\sigma}(\mathcal{A}) \subseteq F$. Then $\mathcal{A}$ is reducible. Conversely, let $\mathcal{V}$ be as before. If every F-algebra $\mathcal{A}$ in $\mathcal{L}(\mathcal{V})$ with $\bar{\sigma}(\mathcal{A}) \subseteq F$ is reducible, then F is not k-closed for some $2 \leq k \leq \dim \mathcal{V}$.*

(ii) If there exists an irreducible algebra $\mathcal{A}$ of linear transformations with $\bar{\sigma}(\mathcal{A}) \subseteq F$, then F is k-closed for each $k = 2, ..., n$, and therefore $\mathcal{A} = \mathcal{L}(\mathcal{V})$. In particular, Burnside's Theorem holds in $\mathcal{L}(\mathcal{V})$.

Proof. (i) The proof is almost identical to that of Burnside's Theorem due to I. Halperin and P. Rosenthal (see Theorem 1.2.2 of [RR], or [HR]): Use contradiction. Assume irreducibility and show that the minimal nonzero rank of the elements of $\mathcal{A}$ is 1; from there again use irreducibility to show that $\mathcal{A}$ contains all rank-one transformations and thus $\mathcal{A} = \mathcal{L}(\mathcal{V})$, for every linear transformation on a finite-dimensional vector space is a sum of rank-one linear transformations in $\mathcal{L}(\mathcal{V})$. Hence every linear transformation on $\mathcal{V}$ is triangularizable. On the other hand since F is not k-closed for some $2 \leq k \leq \dim \mathcal{V}$, it follows that there is a non-triangularizable transformation in $\mathcal{L}(\mathcal{V})$, by the comment preceding Theorem 3.2.7, a contradiction. Thus $\mathcal{A}$ is reducible.

For the converse, we again use contradiction. Suppose that F is k-closed for all $2 \leq k \leq \dim \mathcal{V}$. This hypothesis easily implies that the irreducible algebra $\mathcal{A} := \mathcal{L}(\mathcal{V})$ has the property that $\bar{\sigma}(\mathcal{A}) \subseteq F$. On the other hand $\mathcal{A}$ must be reducible by hypothesis, a contradiction.

(ii) This easily follows from (i). □

The preceding theorem implies the following which is a special case of Corollary 2.4.5.

Corollary 3.2.8. *Let F be a field, and let $\mathcal{V}$ be an n-dimensional vector space over F with $n > 1$. Let $\mathcal{A}$ be an algebra of linear transformations in $\mathcal{L}(\mathcal{V})$ with $\bar{\sigma}(\mathcal{A}) \subseteq F$.*

(i) The algebra $\mathcal{A}$ is triangularizable iff $AB - BA$ is nilpotent for each $A, B \in \mathcal{A}$.

(ii) If $\operatorname{rank}(AB - BA) \leq 1$ for each $A, B \in \mathcal{A}$, then the algebra $\mathcal{A}$ is triangularizable.

Proof. In view of the preceding theorem the proof is identical to that of Corollary 2.4.5. □

Theorem 3.2.7 implies the following. Later on, we will use part (ii) of the following theorem to prove the Near Triangularizability Theorem on real vector spaces.

Theorem 3.2.9. *(i) Let F be a field that is not 2-closed, and $\mathcal{V}$ an n-dimensional vector space over F with $n > 1$. Let $\mathcal{A}$ be an algebra in $\mathcal{L}(\mathcal{V})$. Then $\mathcal{A}$ is triangularizable iff every $A \in \mathcal{A}$ is triangularizable. Conversely, let a field F be given. If every algebra $\mathcal{A}$ in $\mathcal{L}(\mathcal{V})$ with $\bar{\sigma}(\mathcal{A}) \subseteq F$ is triangularizable, then F is not 2-closed.*

(ii) Let $\mathcal{V}$ be a finite-dimensional vector space over $\mathbb{R}$, and let $\mathcal{A} \leq \mathcal{B}(\mathcal{V})$ be a subalgebra of linear transformations. Then $\mathcal{A}$ is triangularizable iff every element of $\mathcal{A}$ is triangularizable.

Proof. (i) "$\Longrightarrow$" Obvious.

"$\Longleftarrow$" The proof is just a quick consequence of Theorem 3.2.7 together with the Triangularization Lemma (Lemma 1.1.3).

As for the converse, fixing a basis for $\mathcal{V}$ we need to prove the matrix version of the assertion. Again we use contradiction. Suppose that F is 2-closed. This hypothesis easily implies that the nontriangularizable algebra $\mathcal{A} := \text{diag}(M_2(F), 0_{n-2})$ has the property that $\bar{\sigma}(\mathcal{A}) \subseteq F$. On the other hand, $\mathcal{A}$ must be triangularizable by hypothesis, for $\bar{\sigma}(\mathcal{A}) \subseteq F$, a contradiction.

(ii) This is a quick consequence of (i), for $\mathbb{R}$ is not 2-closed. □

Corollary 3.2.10. *Let $\mathcal{V}$ be a finite-dimensional vector space over $\mathbb{R}$, and let $\mathcal{A} \leq \mathcal{B}(\mathcal{V})$ be a subalgebra of triangularizable linear transformations. Then $BC - CB$ is nilpotent for all B, C in $\mathcal{A}$.*

Proof. The proof is evident by Theorem 3.2.9(ii) and the Spectral Mapping Theorem (See Theorem 1.1.8 of [RR]). □

Now we are going to use Theorem 3.2.9(ii) to prove the Near Triangularizability Theorem for families of linear transformations on a finite-dimensional real vector space. We would however like to point out that the Near Triangularizability Theorem below holds for families of linear transformations on a finite-dimensional vector space over a complete field F with a nontrivial absolute value provided that F is not 2-closed, i.e., there is an irreducible polynomial f of degree 2 over F.

Theorem 3.2.11. *Let $\mathcal{F}$ be a family of linear transformations on a finite-dimensional vector space $\mathcal{V}$ over $\mathbb{R}$ with the following property: for each finite subfamily $\{A_1, ..., A_m\}$ of $\mathcal{F}$, there is a constant $K > 0$ such that for every $\epsilon > 0$ there exist a triangularizable family $\{T_1, ..., T_m\}$ of transformations, and an invertible linear transformation $S = S_\epsilon$ with $||S^{-1}|| ||S|| \leq K$ satisfying*

$$||T_j|| \leq K, \ ||S^{-1}A_jS - T_j|| < \epsilon,$$

for every $1 \leq j \leq m$. Then $\mathcal{F}$ is triangularizable.

Proof. Let $\mathcal{A}$ be the algebra generated by $\mathcal{F}$. In view of Theorem 3.2.9(ii), it suffices to show that every element of $\mathcal{A}$ is triangularizable. Given $A \in \mathcal{A}$, there are $A_i \in \mathcal{F}, (1 \leq i \leq m, m \in \mathbb{N})$, and noncommutative polynomial p such that

$$A = p(A_1, ..., A_m).$$

Let $K > 0$ be the appropriate constant for $\{A_1, ..., A_m\}$. As we mentioned before, it is easily seen that every such p is a uniformly continuous function of its arguments on any bounded set in $\langle(\mathcal{B}(\mathcal{V}))^m, \|.\|_\infty\rangle$ where $\|(X_1, ..., X_m)\|_\infty = \max\{\|X_1\|, ..., \|X_m\|\}$, and where $\|.\|$ is any norm on $\mathcal{B}(\mathcal{V})$ (note that all norms on $\mathcal{B}(\mathcal{V})$ are equivalent). In particular, for every $n > 0$, there is a δ_n with $0 < \delta_n < 1$ such that

$$\|p(X_1, ..., X_m) - p(Y_1, ..., Y_m)\| < \frac{1}{n(K+1)}, \qquad (*)$$

whenever

$$\|X_j - Y_j\| < \delta_n, \ \|X_j\| \leq K+1, \ \|Y_j\| \leq K+1$$

for all $1 \leq j \leq m$.

Now by the hypothesis for this δ_n there is a triangularizable family $\{T_{n1}, ..., T_{nm}\}$ of linear transformations, and an invertible linear transformation S_n with $\|S_n^{-1}\|\|S_n\| \leq K$ satisfying

$$\|T_{nj}\| \leq K, \ \|S_n^{-1}A_jS_n - T_{nj}\| < \delta_n,$$

for every $1 \leq j \leq m$. Clearly,

$$\|S_n^{-1}A_jS_n\| \leq K+1, \ \|T_{nj}\| \leq K+1, \ \|S_n^{-1}A_jS_n - T_{nj}\| < \delta_n,$$

for every $1 \leq j \leq m$. Thus it follows from $(*)$ that

$$\|p(S_n^{-1}A_1S_n, ..., S_n^{-1}A_mS_n) - p(T_{n1}, ..., T_{nm})\| < \frac{1}{n(K+1)}.$$

Plainly

$$p(S_n^{-1}A_1S_n, ..., S_n^{-1}A_mS_n) = S_n^{-1}p(A_1, ..., A_m)S_n.$$

So we can write

$$\begin{aligned}
&\|p(A_1, ..., A_m) - S_n p(T_{n1}, ..., T_{nm}) S_n^{-1}\| \\
&= \|S_n(S_n^{-1} p(A_1, ..., A_m) S_n - p(T_{n1}, ..., T_{nm})) S_n^{-1}\| \\
&\leq \|S\| \|h(S^{-1}A_1 S, ..., S^{-1}A_m S) - h(T_1, ..., T_m)\| \|S^{-1}\| \\
&< \|S_n\| \|S_n^{-1}\| \frac{1}{n(K+1)} \leq \frac{K}{n(K+1)} < \frac{1}{n}
\end{aligned}$$

Thus

$$\|p(A_1, ..., A_m) - T_n\| < \frac{1}{n},$$

where $T_n = S_n p(T_{n1}, ..., T_{nm}) S_n^{-1}$. Obviously T_n is a triangularizable transformation, for $\{T_{n1}, ..., T_{nm}\}$ is a triangularizable family of linear transformations. In particular $\sigma(T_n) \subseteq \mathbb{R}$, $n \in \mathbb{N}$. So we have $p(A_1, ..., A_m) = \lim_n T_n$ and $\sigma(T_n) \subseteq \mathbb{R}$, $n \in \mathbb{N}$; hence it follows from Lemma 3.1.2 of [RR] that $\sigma(p(A_1, ..., A_m)) \subseteq \mathbb{R}$. Therefore $A = p(A_1, ..., A_m)$ is triangularizable by the remark preceding Theorem 3.2.7. □

Remark. Having proved the Near Triangularizability Theorem for real vector spaces, one can prove an analogue of Corollary 3.2.4 for collections of linear transformations on finite-dimensional vector spaces over $\mathbb{R}$. The proof is similar to that of Corollary 3.2.4 using Theorem 3.2.3.

We now prove a near reducibility theorem.

Theorem 3.2.12 (Near Reducibility Theorem). *Let $\mathcal{F}$ be a family of linear transformations on a finite-dimensional vector space $\mathcal{V}$ over $\mathbb{C}$ with the following property: for each finite subfamily $\{A_1, ..., A_m\}$ of $\mathcal{F}$, there is a constant $K > 0$ such that for every $\epsilon > 0$ there exist a reducible family $\{T_1, ..., T_m\}$ satisfying*

$$\|T_j\| \leq K, \ \|A_j - T_j\| < \epsilon,$$

for every $1 \leq j \leq m$. Then $\mathcal{F}$ is reducible.

Proof. Let $\{A_1, ..., A_m\}$ be a basis for $\langle \mathcal{F} \rangle \leq \mathcal{L}(\mathcal{V})$, the subspace generated by $\mathcal{F}$. It suffices to show that the algebra generated by $\{A_1, ..., A_m\}$, denoted by $\mathcal{A}$, is reducible.

Suppose not. It follows from Burnside's Theorem that $\mathcal{A} = \mathcal{L}(\mathcal{V})$. Let $n = \dim \mathcal{V}$. Therefore, there are noncommutative polynomials $p_i(x_1, ..., x_m)$ $(i = 1, ..., n^2)$ such that $\{p_i(A_1, ..., A_m)\}_{1 \leq i \leq n^2}$ is a basis for $\mathcal{L}(\mathcal{V})$. Since $\mathcal{L}(\mathcal{V})$ is finite-dimensional and $\{p_i(A_1, ..., A_m)\}_{1 \leq i \leq n^2}$ is a basis for $\mathcal{L}(\mathcal{V})$, it follows that there exists $\epsilon > 0$ with the property that: If $B_i \in \mathcal{L}(\mathcal{V})$ $(i = 1, ..., n^2)$ and $||p_i(A_1, ..., A_m) - B_i|| < \epsilon$ for each $i = 1, ..., n^2$, then $\{B_i\}_{1 \leq i \leq n^2}$ is linearly independent, hence a basis for $\mathcal{L}(\mathcal{V})$. Let $K > 0$ be the appropriate constant for $\{A_1, ..., A_m\}$.

We observe that each $p_i : \mathcal{B}(\mathcal{V})^m \longrightarrow \mathcal{B}(\mathcal{V})$ $(i = 1, ..., n^2)$ is a noncommutative polynomial in m linear transformations. We note that every such p_i is a uniformly continuous function of its arguments on any bounded set in $\langle \mathcal{B}(\mathcal{V})^m, ||.||_\infty \rangle$ where

$$||(X_1, ..., X_m)||_\infty = \max\{||X_1||, ..., ||X_m||\}.$$

In particular, for every $\eta > 0$, there is a positive δ with $0 < \delta < 1$ such that

$$||p_i(X_1, ..., X_m) - p_i(Y_1, ..., Y_m)|| < \eta, \qquad (*)$$

whenever $||X_j - Y_j|| < \delta$, $||X_j|| \leq K + 1$, $||Y_j|| \leq K + 1$ for all $1 \leq j \leq m$ and $1 \leq i \leq n^2$.

Now for $\epsilon > 0$ with the aforementioned property, find the corresponding δ with $0 < \delta < 1$. By the hypothesis, for this δ, there exists a reducible family $\{T_1, ..., T_m\}$ satisfying

$$||T_j|| \leq K, \ ||A_j - T_j|| < \delta,$$

for every $1 \leq j \leq m$. So we can write

$$||T_j|| \leq K + 1, \ ||A_j|| \leq \delta + ||T_j|| < 1 + K,$$

for every $1 \leq j \leq m$. Hence we get

$$||A_j - T_j|| < \delta, ||A_j|| \leq K + 1, ||T_j|| \leq K + 1,$$

for every $1 \leq j \leq m$. It follows from $(*)$ that

$$\|p_i(A_1, ..., A_m) - p_i(T_1, ..., T_m)\| < \epsilon,$$

for all $1 \leq i \leq n^2$. Now it follows from the aforementioned property of $\epsilon > 0$ that $\{p_i(T_1, ..., T_m)\}_{1 \leq i \leq n^2}$ is a basis for $\mathcal{L}(\mathcal{V})$ which contradicts the fact that $\{T_1, ..., T_m\}$ is reducible. This contradiction completes the proof. □

Remarks.

1. Using the same argument one can prove that the near reducibility theorem above holds for families of linear transformations on a finite-dimensional vector space over a an algebraically closed complete field F with a nontrivial absolute value.

2. By the remark following Theorem 2.2.21, Burnside's Theorem holds on finite-dimensional vector spaces over $\mathbb{R}$ whose dimensions are odd. Therefore, the same proof shows that the Near Reducibility Theorem above holds on finite-dimensional vector spaces over $\mathbb{R}$ whose dimensions are odd.

3. Having proved the above Near Reducibility Theorem, one can prove an analogue of Corollary 3.2.4 for reducible collections of transformations on a complex vector space or a real vector space of odd dimension. To establish this, one follows the line of argument deducing Corollary 3.2.4 from Theorem 3.2.3. More precisely, one can prove: *Let $\mathcal{V}$ be a finite-dimensional complex vector space or a finite-dimensional real vector space whose dimension is odd, $\mathcal{F}_i, \mathcal{F}$ $(i \in \mathbb{N})$ nonempty families of linear transformations on $\mathcal{V}$ satisfying one of the following conditions.*

(i) Each family $\mathcal{F}_n$ $(n \in \mathbb{N})$ is reducible and $\lim_n \mathrm{dist}(\mathcal{F}_n, f) = 0$ for all $f \in \mathcal{F}$.

(ii) Each family $\mathcal{F}_n$ $(n \in \mathbb{N})$ is reducible and $\mathcal{F}_i \longrightarrow \mathcal{F}$ in the Hausdorff metric D induced by any norm on $\mathcal{B}(\mathcal{V})$.

Then $\mathcal{F}$ is reducible.

3.3 Near triangularizability in infinite dimensions

In this section we prove the infinite-dimensional version of near triangularizability. First we show that Theorem 3.2.3 holds for $\mathcal{C}_1$ operators on a Hilbert space.

Theorem 3.3.1. *Let $\mathcal{F}$ be family of $\mathcal{C}_1$ operators on a complex Hilbert space with the following property: for each finite subfamily $\{A_1, ..., A_m\}$ of $\mathcal{F}$, there is a constant $K > 0$ such that for every $\epsilon > 0$ there exist a triangularizable family $\{T_1, ..., T_m\}$ of $\mathcal{C}_1$ operators, and an invertible linear operator $S = S_\epsilon$ satisfying*

$$||T_j||_1 \leq K, \ ||S^{-1}A_jS - T_j||_1 < \epsilon,$$

for every $1 \leq j \leq m$. Then $\mathcal{F}$ is triangularizable.

Proof. First we note that if $\mathcal{F}$ is a singleton, then we have nothing to prove for every $\mathcal{C}_1$ operator is compact and hence triangularizable. So we may assume that $|\mathcal{F}| > 1$. By Lemma 2.5.8(iv) a trace class operator A is quasinilpotent iff $\text{tr}(A^n) = 0$ for all $n \in \mathbb{N}$, and $|\text{tr}(A)| \leq ||A||_1$ for all $A \in \mathcal{C}_1$ (see Corollary 6.5.13 of [RR]). Let $\mathcal{A}$ be the algebra generated by $\mathcal{F}$.

In view of Theorem 7.6.1 of [RR], it suffices to prove that each commutator $BC - CB$ is quasinilpotent for all B and C in $\mathcal{A}$. To do so, we need to show that the trace of $(BC - CB)^n$ is 0 for all B and C in $\mathcal{A}$ and all $n \in \mathbb{N}$. From this point on, the proof is almost identical to that of Theorem 3.2.3. Here is a sketch:

Given $B, C \in \mathcal{A}$, there are $A_i \in \mathcal{F}, (1 \leq i \leq m)$, and noncommutative polynomials p and q such that

$$B = p(A_1, ..., A_m), \ C = q(A_1, ..., A_m).$$

Let $K > 0$ be the appropriate constant for $\{A_1, ..., A_m\}$. Define: $h_n : (\mathcal{C}_1)^m \longrightarrow \mathcal{C}_1$ by

$$h_n(X_1, ..., X_m)$$
$$= (p(X_1, ..., X_m)q(X_1, ..., X_m) - q(X_1, ..., X_m)p(X_1, ..., X_m))^n.$$

We observe that h_n, $n \in \mathbb{N}$ is a noncommutative polynomial in m trace class (i.e., $\mathcal{C}_1$) operators. It is easily seen that every such h_n is a uniformly continuous function of its arguments on any bounded set in $\langle (\mathcal{C}_1)^m, ||.||_\infty \rangle$ where $||(X_1, ..., X_m)||_\infty = \max\{||X_1||_1, ..., ||X_m||_1\}$. In particular, for every $\eta > 0$, there is a δ with $0 < \delta < 1$ such that

$$||h_n(X_1, ..., X_m) - h_n(Y_1, ..., Y_m)||_1 < \eta,$$

whenever $\|X_j - Y_j\|_1 < \delta$, $\|X_j\|_1 \leq K+1$, $\|Y_j\|_1 \leq K+1$ for all $1 \leq j \leq m$. The rest of proof is identical to that of Theorem 3.2.3 except that instead of $\|.\|$ we have the $\mathcal{C}_1$ norm i.e., $\|.\|_1$ and instead of h we have h_n. So just as in the proof of Theorem 3.2.3, we can conclude that $\operatorname{tr}(h_n(A_1, ..., A_m)) = 0$ where $n \in \mathbb{N}$ is arbitrary. That is $\operatorname{tr}((BC - CB)^n) = 0$ for all $n \in \mathbb{N}$. Hence $BC - CB$ is quasinilpotent. Thus $\mathcal{A}$, and therefore $\mathcal{F}$, is triangularizable by Theorem 7.6.1 of [RR]. □

Remark. The preceding argument provides a second proof for Theorem 3.2.3.

It is not hard to extend Theorem 3.3.1 to $\mathcal{C}_p$ operators ($1 \leq p \in \mathbb{R}$) on a complex Hilbert space. To do so, we need the following lemma.

Lemma 3.3.2. *Let $p \in \mathbb{N}$ and let $h(x_1, ..., x_m)$ be a noncommutative polynomial in m variables with complex coefficients whose constant coefficient is 0, and such that every monomial in $h(x_1, ..., x_m)$ is of degree greater than or equal to p. Then*

$$h : \langle (\mathcal{C}_p)^m, \|.\|_\infty \rangle \longrightarrow \langle \mathcal{C}_1, \|.\|_1 \rangle$$

defines a function that is uniformly continuous on bounded subsets of $\langle (\mathcal{C}_p)^m, \|.\|_\infty \rangle$ where $\|(X_1, ..., X_m)\|_\infty = \max\{\|X_1\|_p, ..., \|X_m\|_p\}$ for $(X_1, ..., X_m) \in (\mathcal{C}_p)^m$.

Proof. We present a sketch of the proof. Since a finite sum of uniformly continuous functions is a uniformly continuous function, it suffices to prove the assertion for the case when h is a monomial. We recall that by Lemma 14(c) on page 1098 of [DS] we know: if $\frac{1}{p_1} + ... + \frac{1}{p_n} = 1$ and $A_i \in \mathcal{C}_{p_i}$, then $A_1 ... A_n \in \mathcal{C}_1$; moreover,

$$\|A_1 ... A_n\|_1 \leq \|A_1\|_{p_1} ... \|A_n\|_{p_n}.$$

Also we have

$$\|A\| = \inf\{\|A - F\| : \operatorname{rank} F \leq 0\} = s_1(A) \leq (\sum_{i=1}^{\infty} s_i^p(A))^{\frac{1}{p}}.$$

Hence $\|A\| \leq \|A\|_p$ for all $A \in \mathcal{C}_p$, $1 \leq p \in \mathbb{R}$ (see Lemma 6.5.15. of [RR]). So if $A_1, ..., A_m \in \mathcal{C}_p$ where $m > p$, then, in view of Lemma XI.9.9(d) on page 1093 of [DS] and the comments above, we can write

$$\|A_1 \dots A_m\|_1 = \|A_1 \dots A_{p-1}(A_p \dots A_m)\|_1$$

$$\leq \|A_1\|_p \dots \|A_{p-1}\|_p \|A_p \dots A_m\|_p \leq \|A_1\|_p \dots \|A_{p-1}\|_p (\|A_p\|_p \|A_{p+1} \dots A_m\|)$$

$$\leq \|A_1\|_p \dots \|A_{p-1}\|_p \|A_p\|_p (\|A_{p+1}\| \dots \|A_m\|)$$

$$\leq \|A_1\|_p \dots \|A_{p-1}\|_p \|A_p\|_p \|A_{p+1}\|_p \dots \|A_m\|_p$$

Thus

$$\|A_1 \dots A_m\|_1 \leq \|A_1\|_p \dots \|A_m\|_p.$$

Now, using induction, it is a matter of a straightforward calculation to see that every monomial $h : \langle (\mathcal{C}_p)^m, \|\cdot\|_\infty \rangle \longrightarrow \langle \mathcal{C}_1, \|\cdot\|_1 \rangle$ is a uniformly continuous function of its arguments on every bounded subset of $\langle (\mathcal{C}_p)^m, \|\cdot\|_\infty \rangle$. □

Theorem 3.3.3. *Let $1 \leq p \in \mathbb{R}$ be given, and let $\mathcal{F}$ be a family of $\mathcal{C}_p$ operators on a complex Hilbert space with the following property: for each finite subfamily $\{A_1, \dots, A_m\}$ of $\mathcal{F}$, there is a constant $K > 0$ such that for every $\epsilon > 0$ there exist a triangularizable family $\{T_1, \dots, T_m\}$ of $\mathcal{C}_p$ operators, and an invertible linear operator $S = S_\epsilon$ satisfying*

$$\|T_j\|_p \leq K, \ \|S^{-1} A_j S - T_j\|_p < \epsilon,$$

for every $1 \leq j \leq m$. Then $\mathcal{F}$ is triangularizable.

Proof. By Lemma 9(a) on page 1092 of [DS], $\mathcal{C}_p \subseteq \mathcal{C}_{p'}$ if $p \leq p'$ and $\|A\|_p$ decreases as p increases. Thus, if necessary, by changing p to $\lfloor p \rfloor + 1$, where $\lfloor p \rfloor$ denotes the integer part of p, we may assume that $p \in \mathbb{N}$. Again let $\mathcal{A}$ be the algebra generated by $\mathcal{F}$.

We note, as before, that if $\mathcal{F}$ is a singleton, then we have nothing to prove, for every $\mathcal{C}_p$ operator is compact and hence triangularizable. So we may assume that $|\mathcal{F}| > 1$. In view of Theorem 7.6.1 of [RR], it suffices to prove that each commutator $BC - CB$ is quasinilpotent for all B and C in $\mathcal{A}$.

To do so, we will show that $(BC - CB)^p$ is quasinilpotent. To see this, it suffices to show that the trace of $(BC-CB)^{pn}$ is 0 for all B and C in $\mathcal{A}$ and all $n \in \mathbb{N}$. The proof is similar to that of Theorem 3.2.3 or Theorem 3.3.1.

Given $B, C \in \mathcal{A}$, there are $A_i \in \mathcal{F}, (1 \le i \le m)$, and noncommutative polynomials p and q such that

$$B = p(A_1, ..., A_m), \; C = q(A_1, ..., A_m).$$

Let $K > 0$ be the appropriate constant for $\{A_1, ..., A_m\}$. Define

$$h_n(x_1, ..., x_m) = (p(x_1, ..., x_m)q(x_1, ..., x_m) - q(x_1, ..., x_m)p(x_1, ..., x_m))^{pn}.$$

It is plain that h_n, $n \in \mathbb{N}$ is a polynomial function from $\langle (\mathcal{C}_p)^m, \|.\|_\infty \rangle$ into $\langle \mathcal{C}_1, \|.\|_1 \rangle$. By Lemma 3.3.2

$$h_n : \langle (\mathcal{C}_p)^m, \|.\|_\infty \rangle \longrightarrow \langle \mathcal{C}_1, \|.\|_1 \rangle$$

is a uniformly continuous function of its arguments on every bounded subset of $\langle (\mathcal{C}_p)^m, \|.\|_\infty \rangle$ where $\|(X_1, ..., X_m)\|_\infty = \max\{\|X_1\|_p, ..., \|X_m\|_p\}$ for $(X_1, ..., X_m) \in (\mathcal{C}_p)^m$.

In particular, for every $\eta > 0$, there is a δ with $0 < \delta < 1$ such that

$$\|h_n(X_1, ..., X_m) - h_n(Y_1, ..., Y_m)\|_1 < \eta,$$

whenever $\|X_j - Y_j\|_p < \delta$, $\|X_j\|_p \le K+1$, $\|Y_j\|_p \le K+1$ for all $1 \le j \le m$. By an argument similar to that used in the proof of Theorem 3.2.3, we can conclude that $\mathrm{tr}(h_n(A_1, ..., A_m)) = 0$ where $n \in \mathbb{N}$ is arbitrary. That is, $\mathrm{tr}((BC-CB)^{pn}) = 0$ for all $n \in \mathbb{N}$. Therefore $(BC-CB)^p$, and hence $BC-CB$, is quasinilpotent. Thus $\mathcal{A}$, and hence $\mathcal{F}$, is triangularizable by Theorem 7.6.1 of [RR]. □

Here is the Near Triangularizability Theorem for arbitrary collections of compact operators on a real or complex Banach space.

Theorem 3.3.4. *Let $\mathcal{F}$ be a family of compact operators on a real or complex Banach space with the following property: for each finite subfamily $\{A_1, ..., A_m\}$ of $\mathcal{F}$, there is a constant $K > 0$ such that for every $\epsilon > 0$ there exist a triangularizable family $\{T_1, ..., T_m\}$ of compact operators, and an invertible linear operator $S = S_\epsilon$ with*

$||S^{-1}||||S|| \leq K$ satisfying

$$||T_j|| \leq K, \ ||S^{-1}A_jS - T_j|| < \epsilon,$$

for every $1 \leq j \leq m$. Then $\mathfrak{F}$ is triangularizable.

Proof. We present the proof for the case when the underlying space is a complex Banach space. The proof for the real case is similar to that of Theorem 3.2.11 using Corollary 2.5.6 and Lemma 5 on page 1091 of [DS]. As before, we note that if $\mathfrak{F}$ is a singleton, then we have nothing to prove, for every compact operator is triangularizable. So we may assume that $|\mathfrak{F}| > 1$. Let $\mathcal{A}$ be the algebra generated by $\mathfrak{F}$. In view of Theorem 7.6.1 of [RR], it suffices to prove that each commutator $BC - CB$ is quasinilpotent for all B and C in $\mathcal{A}$.

To do so, we will show that $\rho(BC-CB) = 0$ where "ρ" denotes the spectral radius. As before, given $B, C \in \mathcal{A}$, there are $A_i \in \mathfrak{F}, (1 \leq i \leq m)$, and noncommutative polynomials p and q such that

$$B = p(A_1, ..., A_m), \ C = q(A_1, ..., A_m).$$

Let $K > 0$ be the appropriate constant for $\{A_1, ..., A_m\}$. Define:

$$h(x_1, ..., x_m) = p(x_1, ..., x_m)q(x_1, ..., x_m) - q(x_1, ..., x_m)p(x_1, ..., x_m).$$

Since the spectral radius is continuous at $h(A_1, ..., A_m)$, for $h(A_1, ..., A_m)$ is a compact operator, it follows that for a given $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that $|\rho(h(A_1, ..., A_m)) - \rho(A)| < \epsilon$ whenever $||h(A_1, ..., A_m) - A|| < \delta$.

Now, for this $\delta = \delta(\epsilon) > 0$ there is an η with $0 < \eta < 1$ such that

$$||h(X_1, ..., X_m) - h(Y_1, ..., Y_m)|| < \frac{\delta}{K+1}, \tag{$*$}$$

whenever

$$||X_j - Y_j|| < \eta, \ ||X_j|| \leq K+1, \ ||Y_j|| \leq K+1$$

for all $1 \leq j \leq m$. By the hypothesis, for this $0 < \eta < 1$ there is a triangularizable family $\{T_1, ..., T_m\}$ of compact operators, and an invertible linear operator $S = S_\eta$

with $||S^{-1}||||S|| \leq K$ satisfying

$$||T_j|| \leq K, \; ||S^{-1}A_jS - T_j|| < \eta,$$

for every $1 \leq j \leq m$. Clearly,

$$||S^{-1}A_jS|| \leq K+1, \; ||T_j|| \leq K+1, \; ||S^{-1}A_jS - T_j|| < \eta,$$

for every $1 \leq j \leq m$. Thus it follows from $(*)$ that

$$||h(S^{-1}A_1S, ..., S^{-1}A_mS) - h(T_1, ..., T_m)|| < \frac{\delta}{K+1}.$$

Plainly

$$h(S^{-1}A_1S, ..., S^{-1}A_mS) = S^{-1}h(A_1, ..., A_m)S.$$

We can write

$$\begin{aligned} &||h(A_1, ..., A_m) - Sh(T_1, ..., T_m)S^{-1}|| \\ &= ||S(S^{-1}h(A_1, ..., A_m)S - h(T_1, ..., T_m))S^{-1}|| \\ &\leq ||S||||h(S^{-1}A_1S, ..., S^{-1}A_mS) - h(T_1, ..., T_m)||||S^{-1}|| \\ &< ||S||||S^{-1}||\frac{\delta}{K+1} \leq \frac{K}{K+1}\delta < \delta. \end{aligned}$$

Thus

$$||h(A_1, ..., A_m) - Sh(T_1, ..., T_m)S^{-1}|| < \delta.$$

Now it follows from the continuity of spectral radius at $h(A_1, ..., A_m)$ that

$$|\rho(h(A_1, ..., A_m)) - \rho(Sh(T_1, ..., T_m)S^{-1})| < \epsilon.$$

It is plain that

$$\rho(Sh(T_1, ..., T_m)S^{-1}) = \rho(h(T_1, ..., T_m)).$$

But $\rho(h(T_1, ..., T_m)) = 0$, for $\{T_1, ..., T_m\}$ is triangularizable. Thus we conclude that

$$|\rho(h(A_1, ..., A_m))| < \epsilon,$$

for all $\epsilon > 0$. Hence, $\rho(BC - CB) = \rho(h(A_1, ..., A_m)) = 0$ for each commutator $BC - CB$, $B, C \in \mathcal{A}$. Therefore, $BC - CB$ is quasinilpotent for all $B, C \in \mathcal{A}$. □

Corollary 3.3.5. *Let $p \in \mathbb{R}$ with $p \geq 1$ be given, and let $\mathcal{F}$ be a family of $\mathcal{C}_p$ operators on a real or complex Hilbert space with the following property: for each finite subfamily $\{A_1, ..., A_m\}$ of $\mathcal{F}$, there is a constant $K > 0$ such that for every $\epsilon > 0$ there exist a triangularizable family $\{T_1, ..., T_m\}$ of $\mathcal{C}_p$ operators, and an invertible linear operator $S = S_\epsilon$ with $||S^{-1}||||S|| \leq K$ satisfying*

$$||T_j||_p \leq K, \ ||S^{-1}A_jS - T_j||_p < \epsilon,$$

for every $1 \leq j \leq m$. Then $\mathcal{F}$ is triangularizable.

Proof. Theorem 3.3.4 applies because $||A|| \leq ||A||_p$ for each $\mathcal{C}_p$ operator A. □

Remarks.

1. Having proved the above Near Triangularizability Theorem, one can prove an analogue of Corollary 3.2.4 for collections of compact operators on a real or complex Banach space where nearness is measured by the operator norm. To accomplish this, one follows the line of argument deducing Corollary 3.2.4 from Theorem 3.2.3.

2. It is worth mentioning that $\mathcal{C}_p$ class operators can naturally be defined on any real or complex Banach space that is isomorphic to a real or complex Hilbert space respectively. Similarly, one can prove analogues of Theorem 3.3.1, and Theorem 3.3.3 and Corollary 3.3.5 for $\mathcal{C}_p$ class operators acting on such a real or complex Banach space.

3.4 A Reducibility Result

In this section we use the Near Triangularizability Theorem to prove a rather surprising reducibility result. Let $\mathcal{X}$ be a real or complex Banach space, and $A_n, A \in \mathcal{B}(\mathcal{X})$. By s-lim$_n A_n = A$ we mean A is the limit of A_n's in the strong operator topology on $\mathcal{B}(\mathcal{X})$, i.e., $\lim_n ||A_n x - Ax|| = 0$ for all $x \in \mathcal{X}$. To present our reducibility result, we need the following two results.

Lemma 3.4.1. *Let $\mathcal{X}$ be a real or complex Banach space, $A_n, A \in \mathcal{B}(\mathcal{X})$, and $K_n, K \in \overline{\mathcal{B}_{00}(\mathcal{X})}$ $(n \in \mathbb{N})$. If* s-$\lim_n A_n = A$ *and* $\lim_n K_n = K$, *then* $\lim_n A_n K_n = AK$.

Proof. We give the proof in three stages.

(i) *If* s-$\lim_n A_n = A$, *then* $\lim_n A_n F = AF$ *for all* $F \in \mathcal{B}_{00}(\mathcal{X})$.

Since F is a finite-rank operator, it follows that we can write

$$F = \sum_{i=1}^{m} \phi_i \otimes x_i,$$

where $m \in \mathbb{N}$, $\phi_i \in \mathcal{X}^*$, $x_i \in \mathcal{X}$ $(1 \leq i \leq m)$, and $\phi_i \otimes x_i$ is the rank-one operator defined on $\mathcal{X}$ by $\phi_i \otimes x_i(x) = \phi_i(x)x_i$. It is easily seen that $AF = \sum_{i=1}^{m} \phi_i \otimes Ax_i$. Therefore, we can write

$$||A_n F - AF|| = ||(A_n - A)F|| = \sup_{||y||=1} ||(A_n - A)F(y)||$$

$$= \sup_{||y||=1} ||\sum_{i=1}^{m} \phi_i(y)(A_n x_i - Ax_i)||.$$

On the other hand, since s-$\lim_n A_n = A$, it follows that $\lim_n A_n x_i = Ax_i$ for each $i = 1, ..., m$. Hence, for given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$||A_n x_i - Ax_i|| < \frac{\epsilon}{2mM},$$

for all $n \geq N$ and $1 \leq i \leq m$ where $M = \max_{1 \leq i \leq m} ||\phi_i||$. So for all $n \geq N$ we can write

$$||A_n F - AF|| \leq \sum_{i=1}^{m} ||\phi_i||.||(A_n x_i - Ax_i)|| \leq \sum_{i=1}^{m} M \frac{\epsilon}{2mM} = \frac{\epsilon}{2} < \epsilon.$$

That is, $\lim_n A_n F = AF$.

(ii) *If* s-$\lim_n A_n = A$ *and* $\lim_n F_n = K$ *where* $F_n \in \mathcal{B}_{00}(\mathfrak{X})$ *(*$n \in \mathbb{N}$*), then* $\lim_n A_n K = AK$.

Since s-$\lim_n A_n = A$, it follows from the Principle of Uniform Boundedness (Theorem III.14.1 of [C]) that there exists $M > 0$ such that $||A||, ||A_n|| \leq M$ for each $n \in \mathbb{N}$. We have $\lim_n F_n = K$. Therefore, for a given $\epsilon > 0$ there exists $N_1 > 0$ such that

$$||F_n - K|| < \frac{\epsilon}{3(M+1)},$$

for all $n \geq N_1$. We can write

$$||A_n K - AK|| \leq ||A_n K - A_n F_{N_1}|| + ||A_n F_{N_1} - AF_{N_1}|| + ||AF_{N_1} - AK||$$

$$\leq ||A_n||.||K - F_{N_1}|| + ||A_n F_{N_1} - AF_{N_1}|| + ||A||.||F_{N_1} - K||$$

On the other hand, (i) implies that $\lim_n ||A_n F_{N_1} - AF_{N_1}|| = 0$. Hence there exists $N_2 > 0$ such that

$$||A_n F_{N_1} - AF_{N_1}|| < \frac{\epsilon}{3},$$

for all $n \geq N_2$. Now for all $n \geq \max(N_1, N_2)$ we can write

$$||A_n K - AK|| < M\frac{\epsilon}{3(M+1)} + \frac{\epsilon}{3} + M\frac{\epsilon}{3(M+1)} < \epsilon.$$

That is $\lim_n ||A_n K - AK|| = 0$. In other words, $\lim_n (A_n K - AK) = 0$ which is what we wanted.

(iii) *We now prove the general statement.*

Again in view of the Principle of Uniform Boundedness, it is easily seen that there exists $M > 0$ such that $||A||, ||A_n||, ||K||, ||K_n|| \leq M$ for each $n \in \mathbb{N}$. Since $\lim_n K_n = K$, we conclude that for a given $\epsilon > 0$ there exists $N_1 > 0$ such that

$$||K_n - K|| < \frac{\epsilon}{2(M+1)},$$

for all $n \geq N_1$. The fact that $K \in \overline{\mathcal{B}_{00}(\mathfrak{X})}$ along with (ii) implies that $\lim_n ||A_n K -$

$AK|| = 0$. Thus, there exists $N_2 > 0$ such that

$$||A_nK - AK|| < \frac{\epsilon}{2},$$

for all $n \geq N_2$. Now for all $n \geq \max(N_1, N_2)$ we can write

$$||A_nK_n - AK|| \leq ||A_nK_n - A_nK|| + ||A_nK - AK||$$

$$\leq ||A_n||.||K_n - K|| + ||A_nK - AK|| < M\frac{\epsilon}{2(M+1)} + \frac{\epsilon}{2} < \epsilon.$$

That is, $\lim_n ||A_nK_n - AK|| = 0$. In other words, $\lim_n A_nK_n = AK$ which is what we wanted. □

Theorem 3.4.2. *Let $\mathcal{X}$ be a real or complex Banach space of dimension greater than one, $\mathcal{S}$ a semigroup of operators in $\mathcal{B}(\mathcal{X})$, and $B \in \mathcal{B}(\mathcal{X})$ a bounded operator with* rank$(B) \geq 2$. *If $\mathcal{S}B$ is triangularizable, then $\mathcal{S}$ has a nontrivial invariant subspace.*

Proof. Let $\mathcal{A}$ denote the algebra generated by the semigroup $\mathcal{S}$. We note that $\mathcal{A} = \langle \mathcal{S} \rangle$. That being noted, it suffices to prove the assertion for an algebra $\mathcal{A}$ of operators in $\mathcal{B}(\mathcal{X})$. Let $\mathcal{C}$ be a maximal chain of subspaces each of which is invariant for $\mathcal{A}B$. Let $\mathcal{X}_- := \overline{\cup_{\mathcal{X} \neq \mathcal{Y} \in \mathcal{C}} \mathcal{Y}}$. We now distinguish two cases.

(a) $\mathcal{X}_- = \mathcal{X}$.

Obviously, there exists a $\mathcal{Y} \in \mathcal{C}$ with $\mathcal{Y} \neq \mathcal{X}$, and a $y_0 \in \mathcal{Y}$ such that $By_0 \neq 0$. Define $\mathcal{M} := \overline{\mathcal{A}By_0}$. If $\mathcal{M} = \{0\}$, then $\langle By_0 \rangle$ is a nontrivial invariant subspace for $\mathcal{A}$. If $\mathcal{M} \neq \{0\}$, then we would have $0 \neq \mathcal{M} = \overline{\mathcal{A}By_0} \subseteq \mathcal{Y} \neq \mathcal{X}$. So $\mathcal{M}$ would then be a nontrivial invariant subspace for $\mathcal{A}$.

(b) $\mathcal{X}_- \neq \mathcal{X}$.

Since $\mathcal{C}$ is maximal, is follows that $\mathcal{X}_- \in \mathcal{C}$ is a closed subspace of $\mathcal{X}$ of codimension one, i.e., $\dim \frac{\mathcal{X}}{\mathcal{X}_-} = 1$. Since rank$(B) \geq 2$, it follows that there exists $x_0 \in \mathcal{X}_-$ such that $Bx_0 \neq 0$. Again define $\mathcal{M} := \overline{\mathcal{A}Bx_0}$. If $\mathcal{M} = \{0\}$, then $\langle Bx_0 \rangle$ is a nontrivial invariant subspace for $\mathcal{A}$. If $\mathcal{M} \neq \{0\}$, then we would have $0 \neq \mathcal{M} = \overline{\mathcal{A}Bx_0} \subseteq \mathcal{X}_- \neq \mathcal{X}$. So $\mathcal{M}$ would then be a nontrivial invariant subspace for $\mathcal{A}$. □

Remarks.

1. In the preceding theorem, if the triangularizing chain, say $\mathcal{C}$, for $\mathcal{S}B$ happens to have the property that $\mathcal{X}_- = \mathcal{X}$, e.g., any continuous chain, then, by case (a) of the proof above, the assertion holds under the weaker hypothesis that B is nonzero.

2. By adjusting case (b) of the proof above, it is easily seen that the preceding theorem holds on finite-dimensional vector spaces over general fields. It is worth noting that in the preceding theorem the hypothesis that $\mathrm{rank}(B) \geq 2$ cannot be weakened. To see this, let F be a field and $n > 1$. Note that $M_n(F)$ is irreducible whereas $M_n(F)E_{nn}$ is triangularizable where E_{nn} is the standard matrix with one in the (n,n)th place and zero elsewhere.

3. It is not difficult to see that in the preceding theorem if the operator B happens to be 1-1, then reducibility of $\mathcal{S}B$ implies that of $\mathcal{S}$. (See the remarks following Theorem 2.3 of [Y2].)

Here is the main theorem of this section.

Theorem 3.4.3. *Let $\mathcal{X}$ be a real or complex Banach space, $A_n, A \in \mathcal{B}(\mathcal{X})$, and $K_n, K \in \overline{\mathcal{B}_{00}(\mathcal{X})}$ ($n \in \mathbb{N}$) with* $\mathrm{rank}(K) \geq 2$. *If* $\text{s-lim}_n A_n = A$, $\lim_n K_n = K$, *and $\{A_n, K_n\}$ is triangularizable for each $n \in \mathbb{N}$, then A has a nontrivial invariant subspace.*

Proof. Let $\mathcal{S}$ denote the semigroup generated by A. In light of Theorem 3.4.2, it suffices to show that $\mathcal{S}K$ is triangularizable. That is, we need to show that the collection $\{A^iK\}_{i=1}^{\infty}$ is triangularizable. In view of Theorem 3.3.4, it suffices to show that $\{A^iK\}_{i=1}^{\infty}$ satisfies the hypotheses of Theorem 3.3.4. Suppose that a finite subfamily $\{A^{n_1}K, ..., A^{n_m}K\}$ is given. Since $\text{s-lim}_i A_i = A$, it easily follows that $\text{s-lim}_i A_i^{n_j} = A^{n_j}$ for each $j = 1, .., m$. Now since $\lim_i K_i = K$, it follows from Lemma 3.4.1 that $\lim_i A_i^{n_j} K_i = A^{n_j}K$ for each $j = 1, .., m$. By the Principle of Uniform Convergence there exists $M_1 > 0$ such that

$$||A_i||, ||A||, ||K_i||, ||K|| \leq M_1,$$

for all $i \in \mathbb{N}$. Set $n := \max_{1\leq j\leq m}(n_j) + 1$ and $M := \max_{1\leq i\leq n}\{M_1^i\}$. Now let $\epsilon > 0$ be given. Since $\{A_i, K_i\}$ is triangularizable, it follows that so is $\{A_i^{n_1}K_i, ..., A_i^{n_m}K_i\}$ for each $i \in \mathbb{N}$. We also have $||A_i^{n_j}K_i|| \leq M$ for each $j = 1,..,m$ and $i \in \mathbb{N}$. Since $\lim_i A_i^{n_j}K_i = A^{n_j}K$ for each $j = 1,..,m$, we conclude that for i large enough

$$||A_i^{n_j}K_i - A^{n_j}K|| < \epsilon,$$

for each $j = 1,..,m$. Therefore, the collection $\{A^iK\}_{i=1}^{\infty}$ is triangularizable, finishing the proof. □

Corollary 3.4.4. *(i) Let $\mathcal{X}$ be a real or complex Banach space, $A_n, A \in \mathcal{B}(\mathcal{X})$, and $K \in \overline{\mathcal{B}_{00}(\mathcal{X})}$ $(n \in \mathbb{N})$ with* rank$(K) \geq 2$. *If* s-lim$_n A_n = A$, *and $\{A_n, K\}$ is triangularizable for each $n \in \mathbb{N}$, then A has a nontrivial invariant subspace.*

(ii) Let $\mathcal{H}$ be a real or complex Hilbert space, $(\alpha_i)_{i\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$, and $A_n, A \in \mathcal{B}(\mathcal{H})$. If s-lim$_n A_n = A$, *and for each $n \in \mathbb{N}$ there exists a permutation π_n on $\mathbb{N}$ such that $(\alpha_{\pi_n(i)})_{i\in\mathbb{N}}$ is a triangularizing chain for A_n, then A has a nontrivial invariant subspace.*

Proof. (i) This is a special case of Theorem 3.4.3 when $K_n = K$ for all $n \in \mathbb{N}$.

(ii) Let K be the compact (and in fact normal) operator defined by diag$(1/j)_{j=1}^{\infty}$ relative to the orthonormal basis $(\alpha_i)_{i\in\mathbb{N}}$ for the space $\mathcal{H}$. Note that $K \in \overline{\mathcal{B}_{00}(\mathcal{H})}$, for $\overline{\mathcal{B}_{00}(\mathcal{H})} = \mathcal{B}_0(\mathcal{H})$. The hypothesis implies that $\{A_n, K\}$ is triangularizable for each $n \in \mathbb{N}$. So (i) applies. □

Chapter 4

Simultaneous triangularization over division rings

AH, make the most of what we yet may spend,
Before we too into the Dust descend;
Dust into dust, and under dust to lie,
Sans Wine, sans Song, sans Singer, and -sans End!

–Khayyam, the Persian Mathematician, Astronomer, Philosopher, and Poet.
Rendered into English verse by Edward Fitzgerald.

4.1 Introduction

In this chapter we study semigroups of linear transformations on (left) finite-dimensional vector spaces over division rings. We give a new proof of a well-known theorem of Levitzki about triangularizability of semigroups of nilpotent transformations on finite-dimensional vector spaces over division rings. It is emphasized that a result of Radjavi, which extends Engel's Theorem and as well as Jacobson's Theorem, also holds on finite-dimensional vector spaces over division rings. We define the concept of permutability of trace on a collection of matrices over a division ring and we use our main theorem to prove that with a slight condition on the characteristic of the underlying division ring an irreducible collection of matrices on which trace is permutable is commutative.

We start off with the noncommutative analogues of the definitions and notations we recalled in Chapter 1. Unless otherwise stated, D is used to denote a division ring, F the center of D, and $\mathcal{V}$ a (left) finite-dimensional vector space over D. We use the term *linear transformation* to describe a left vector space homomorphism on $\mathcal{V}$. We use $\mathcal{L}(\mathcal{V})$ to denote the set (in fact the ring) of linear transformations on $\mathcal{V}$. Recall that by Theorem VII.1.4 of [H], if $\mathcal{V}$ is an n-dimensional vector space over D, then there is a ring isomorphism $\mathcal{L}(\mathcal{V}) \cong M_n(D^{op})$ where D^{op} is the *opposite ring* of D (i.e., D^{op} has the same set of elements as D, same addition, and reversed multiplication). In order to avoid D^{op}, we may view $M_n(D)$ as a ring of linear transformations acting on the right of D^n where D^n is the vector space of n-tuple rows viewed as a left vector space over D. For instance, we can write

$$(\lambda X)A = \lambda\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda x & \lambda y \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} \lambda xa + \lambda yc & \lambda xb + \lambda yd \end{pmatrix} = \lambda\begin{pmatrix} xa + yc & xb + yd \end{pmatrix} = \lambda(XA),$$

where the entries are in D. This shows that λA viewed as a linear transformation does not have the property that $X(\lambda A) = \lambda(XA)$ unless λ is in the center of the division ring D. That being noted, we see that, unlike the commutative case, the matrix ring $M_n(D)$ viewed as a left vector space over D is not compatible with $M_n(D)$ as a ring of linear transformations as described above. However the matrix ring $M_n(D)$ may be viewed as an F-algebra of linear transformations acting on a left vector space as described above where F is the center of the division ring D. Recall that by an *F-algebra $\mathcal{A}$* in $M_n(D)$ we mean a subring of $M_n(D)$ that is closed under scalar multiplication by the elements of F. The concepts of reducibility and triangularizability can naturally be defined for collections of linear transformations on left finite-dimensional spaces (resp. matrices over division rings). Fixing a basis for $\mathcal{V}$, we can identify $\mathcal{L}(\mathcal{V})$ with the matrix ring $M_n(D)$ (see Theorem II.4.2 of [J]). Just as in the commutative case, the matrices of linear transformations with respect to two bases correspond to two similar matrices. We conclude that, to prove a reducibility (resp. triangularizability) result, it suffices to prove either the matrix version or linear transformation version of that result.

Just as in the commutative case, the following lemma is crucial in what follows.

Lemma 4.1.1. *Let $\mathcal{V}$ be a finite-dimensional vector space over a division ring D, and $\mathcal{S}$ a semigroup in $\mathcal{L}(\mathcal{V})$. If $\mathcal{S}$ is irreducible, then so is every nonzero semigroup ideal of $\mathcal{S}$.*

Proof. The proof is identical to that of Lemma 1.1.2. □

4.2 Some Preliminary Results

The following lemma is the counterpart of Lemma 2.2.1. Its proof is almost identical to the corresponding proof in the commutative case. However we include its proof for the sake of completeness.

Lemma 4.2.1. *Let $\mathcal{V}$ be a finite-dimensional vector space over a division ring D, $\mathcal{S}$ a semigroup in $\mathcal{L}(\mathcal{V})$, and T a nonzero linear transformation in $\mathcal{L}(\mathcal{V})$. If $\mathcal{S}$ is irreducible, then so is $T\mathcal{S}|_{\mathcal{R}}$ where $\mathcal{R} = T\mathcal{V}$ is the range of T.*

Proof. If $\dim \mathcal{V} = 1$, then the assertion trivially holds. So we may assume, with no loss of generality, that $\dim \mathcal{V} > 1$. There are two cases to consider.

(a) $\operatorname{rank}(T) = 1$.

To prove the assertion by contradiction suppose $T\mathcal{S}|_{\mathcal{R}}$ is reducible. Since $\dim \mathcal{R} = 1$ in this case, it follows from definition that $T\mathcal{S}|_{\mathcal{R}} = \{0\}$. Therefore, $T\mathcal{S}T = \{0\}$. Pick a nonzero $x \in \mathcal{V}$ such that $Tx \neq 0$. Now either $\mathcal{S}Tx = \{0\}$ in which case $\langle Tx \rangle = \{Tdx : d \in D\}$ is a nontrivial invariant subspace for $\mathcal{S}$, or else

$$\langle \mathcal{S}Tx \rangle = \{\sum_{i=1}^{k} S_i T x_i : k \in \mathbb{N}, S_i \in \mathcal{S}, x_i \in \langle x \rangle \ (1 \leq i \leq k)\}$$

is a nontrivial invariant subspace for $\mathcal{S}$, because $T\mathcal{S}T = \{0\}$ and $\mathcal{S}$ is a semigroup. This contradicts the hypothesis that $\mathcal{S}$ is irreducible.

(b) $\text{rank}(T) > 1$.

To prove that $T\mathcal{S}|_{\mathcal{R}}$ is irreducible we use contradiction. Suppose that $T\mathcal{S}|_{\mathcal{R}}$ is reducible. So there exists a nontrivial subspace $\mathcal{M}$ of $\mathcal{R} = T\mathcal{V}$ such that $T\mathcal{S}\mathcal{M} \subseteq \mathcal{M}$. Choose a nonzero $x \in \mathcal{M}$ and note that $T\mathcal{S}\langle x\rangle \subseteq \mathcal{M}$ where $\langle x\rangle = \{dx : d \in D\}$. The subspace

$$\langle \mathcal{S}x\rangle = \{\sum_{i=1}^{k} S_i x_i : k \in \mathbb{N}, S_i \in \mathcal{S}, x_i \in \langle x\rangle \ (1 \le i \le k)\}$$

is an invariant subspace of $\mathcal{S}$. Furthermore, it is proper, for $T\mathcal{S}\langle x\rangle \subseteq \mathcal{M} \subset \mathcal{R}$. If $\mathcal{S}x = 0$, then $\langle x\rangle$ is a nontrivial invariant subspace for $\mathcal{S}$, otherwise $\langle \mathcal{S}x\rangle$ would be a nontrivial invariant subspace for $\mathcal{S}$. So in any event we conclude that $\mathcal{S}$ is reducible, a contradiction. □

Recall that a linear transformation T in $\mathcal{L}(\mathcal{V})$ is called idempotent if $T^2 = T$. The corollary below is a quick consequence of the preceding lemma.

Corollary 4.2.2. *Let $\mathcal{V}$ be a finite-dimensional vector space over a division ring D, $\mathcal{S}$ a semigroup in $\mathcal{L}(\mathcal{V})$, and T a nonzero idempotent in $\mathcal{L}(\mathcal{V})$. If $\mathcal{S}$ is irreducible, then so is $T\mathcal{S}T|_{T\mathcal{V}}$.*

Proof. Lemma 4.2.1. □

As already mentioned in the remark following Theorem 2.2.9, we are going to use the preceding lemma to give a second proof of Levitzki's Theorem over division rings.

Corollary 4.2.3 (Levitzki's Theorem). *Let $\mathcal{V}$ be a finite-dimensional vector space over a division ring D. Then every semigroup of nilpotent transformations in $\mathcal{L}(\mathcal{V})$ is triangularizable.*

Proof. In light of the Triangularization Lemma (Lemma 1.1.3) for collections of transformations over a division ring, it suffices to show that $\mathcal{S}$ is reducible whenever $\dim \mathcal{V} > 1$. We prove reducibility by induction on $n = \dim \mathcal{V}$. If $n = 1$, then we have nothing to prove. Suppose that the assertion holds for every semigroup of nilpotent transformations on spaces of dimension less than n. Now suppose that $\dim \mathcal{V} = n$; and let $\mathcal{S}$ be a semigroup of nilpotent transformations in $\mathcal{L}(\mathcal{V})$. To show that $\mathcal{S}$ is reducible, we distinguish the following two cases and in each case we prove reducibility of $\mathcal{S}$.

(a) $\mathrm{rank}(\mathcal{S}) \leq 1$.

We claim that $T\mathcal{S}T = \{0\}$ for all $T \in \mathcal{S}$. To this end, suppose $T, S \in \mathcal{S}$ are given. We must show that $TST = 0$. Note that by the hypothesis $TS \in \mathcal{S}$ is nilpotent, and it is plain that $T\mathcal{V}$ is invariant under TS. Thus the restriction of TS to $T\mathcal{V}$ is also nilpotent, and hence not invertible. Since $\dim T\mathcal{V} \leq 1$, it follows that $TS|_{T\mathcal{V}} = 0$. That is, $TST = 0$ for all $S, T \in \mathcal{S}$. If $\mathcal{S} = \{0\}$, then the assertion trivially holds. If $\mathcal{S} \neq \{0\}$, then just as we saw in case (a) of the proof of Lemma 4.2.1, we conclude that $\mathcal{S}$ is reducible.

(b) $\mathrm{rank}(T) \geq 2$ *for some* $T \in \mathcal{S}$.

First note that by hypothesis and nilpotency of T, we would have $2 \leq \mathrm{rank}(T) < n = \dim \mathcal{V}$. Define $\mathcal{R} := T\mathcal{V}$. Therefore, in light of Lemma 4.2.1, reducibility of the semigroup $T\mathcal{S}|_{\mathcal{R}}$, implies that of the semigroup $\mathcal{S}$. Note that $T\mathcal{S}|_{\mathcal{R}}$ is a semigroup of nilpotent transformations on the vector space $\mathcal{R}$ whose dimension is less than n. Thus $T\mathcal{S}|_{\mathcal{R}}$ must be reducible by the induction hypothesis. $\square$

Remarks.

1. We would like to point out that the following theorem due to Radjavi (see Theorem 1.7.3 of [RR]) with the same proof also holds over division rings. It is worth mentioning that this result extends the celebrated result of Engel about triangularization of Lie algebras of nilpotent transformations (Corollary 1.7.6 of [RR]) as well as Jacobson's Theorem (Corollary 1.7.4 of [RR]) to finite-dimensional vector spaces over division rings. Here is the theorem: *Let* $\mathcal{V}$ *be a finite-dimensional vector space over*

a division ring D. A set $\mathcal{N}$ of nilpotent transformations in $\mathcal{L}(\mathcal{V})$ is triangularizable if it has the property that whenever A and B are in $\mathcal{N}$, there is a noncommutative polynomial p such that $AB + p(A, B)A$ is in $\mathcal{N}$. (the proof is identical to that of Theorem 1.7.3 of [RR]).

2. It can be shown that the following statement, if true, extends the well-known results of Kolchin, Kaplansky, McCoy, as well as Theorem 2.3.2 to finite-dimensional vector spaces over division rings (see Corollary 2.2.15, Corollary 2.2.16, and Corollary 2.3.6(ii)). *Let $n > 1$, D a division ring D, F the center of D, and $\mathcal{A}$ an F-algebra in $M_n(D)$. Then, $\mathcal{A}$ is nilpotent if it is spanned by nilpotents as a linear space over F.*

The following theorem is the counterpart of Theorem 2.2.10. Although its proof is almost identical to its counterpart, we include the proof for the sake of completeness.

Theorem 4.2.4. *Let D be a division ring, F its center, $\mathcal{V}$ a finite-dimensional vector space with dimension greater than one over D, and $\mathcal{F}$ a triangularizable family of linear transformations in $\mathcal{L}(\mathcal{V})$ such that the F-algebra generated by $\mathcal{F}$ contains a nonzero nilpotent transformation. Then $\mathcal{F}$ has a nontrivial hyperinvariant subspace.*

Proof. We note that for every family $\mathcal{F}$ of linear transformations

$$\mathcal{F}' = (\mathrm{Alg}_F(\mathcal{F}))' = (\mathrm{Sem}(\mathcal{F}))'.$$

Thus $\mathcal{F}$ has a nontrivial hyperinvariant subspace iff $\mathrm{Alg}_F(\mathcal{F})$ does, or iff $\mathrm{Sem}(\mathcal{F})$ does. So it suffices to prove the assertion for any triangularizable F-algebra, say $\mathcal{A}$, of linear transformations that contains a nonzero nilpotent transformation. Suppose $\mathcal{A}$ is such an F-algebra and that K_0 is a nonzero nilpotent transformation in $\mathcal{A}$. Define $\mathcal{A}_1 := \mathcal{A}' + \mathcal{A} * \mathcal{A}'$ where

$$\mathcal{A} * \mathcal{A}' := \{\sum_{i=1}^{k} A_i A_i' : k \in \mathbb{N},\ \ A_i \in \mathcal{A},\ \ A_i' \in \mathcal{A}',\ (1 \le i \le k)\}.$$

Clearly, $\mathcal{A}_1$ is an F-algebra in $\mathcal{L}(\mathcal{V})$ which contains both $\mathcal{A}$ and $\mathcal{A}'$. It suffices to prove that $\mathcal{A}_1$ has a nontrivial invariant subspace. For the nonzero nilpotent transformation

$K_0 \in \mathcal{A}$, first we claim that $\mathcal{A}_1 K_0$, and hence $\mathcal{A}_1 K_0 \mathcal{A}_1$, the semigroup ideal generated by K_0 in $\mathcal{A}_1$, consists of nilpotents. To this end, let $A_0 = A' + \sum_{i=1}^{k} A_i A_i' \in \mathcal{A}_1$ with $A_i \in \mathcal{A}$, $A', A_i' \in \mathcal{A}'$, $(1 \leq i \leq k, k \in \mathbb{N})$ be arbitrary. We prove that $A_0 K_0$ is nilpotent: first of all we notice that $A_0 K_0 = A' K_0 + \sum_{i=1}^{k} A_{i_0} A_i'$ where $A_{i_0} = A_i K_0 \in \mathcal{A}$. Set

$$\mathcal{S} := \{A \in \mathcal{A} : A^n = 0\}.$$

Since $\mathcal{A}$ is triangularizable, it follows that $\mathcal{S}$ is a nonzero semigroup ideal of $\mathcal{A}$ consisting of nilpotent transformations (note that $0 \neq K_0 \in \mathcal{S}$).

The set $\mathcal{S}\mathcal{A}'$ is indeed a semigroup consisting of nilpotents because for all $A \in \mathcal{A}, A' \in \mathcal{A}'$ we have $AA' = A'A$ and that $\mathcal{S}$ is a semigroup of nilpotents. Thus Levitzki's Theorem (Theorem 4.2.3) shows that $\mathcal{S}\mathcal{A}'$ is triangularizable. Therefore, $\mathrm{Alg}_F(\mathcal{S}\mathcal{A}')$, the F-algebra generated by $\mathcal{S}\mathcal{A}'$, consists of nilpotents. We have

$$A_0 K_0 = K_0 A' + \sum_{i=1}^{k} A_{i_0} A_i'$$

where $A_{i_0} = A_i K_0 \in \mathcal{A}$. In fact $A_{i_0} = A_i K_0 \in \mathcal{S}$, for $K_0 \in \mathcal{S}$ and $\mathcal{A}$ is triangularizable. Now clearly $A' K_0 = K_0 A' \in \mathcal{S}\mathcal{A}'$ and $A_{i_0} A_i' \in \mathcal{S}\mathcal{A}'$. Therefore, $A_0 K_0 \in \mathrm{Alg}_F(\mathcal{S}\mathcal{A}')$, and hence $A_0 K_0$ is a nilpotent transformation. Thus $\mathcal{A}_1 K_0 \mathcal{A}_1$ is a nonzero semigroup ideal of $\mathcal{A}_1$ consisting of nilpotents which must be triangularizable, and hence reducible, by Levitzki's Theorem. Now reducibility of the nonzero ideal $\mathcal{A}_1 K_0 \mathcal{A}_1$ implies that of $\mathcal{A}_1$ in light of Lemma 4.1.1, finishing the proof. □

Remark. *Let $\mathcal{V}$ be an n-dimensional vector space over a division ring D. Then $A \in \mathcal{L}(\mathcal{V})$ is nilpotent (i.e., some power of A is zero) iff $A^n = 0$.* Necessity is trivial. To see sufficiency, suppose that A is nilpotent. Since nilpotency is inherited by quotients, it follows from the Triangularization Lemma that A is triangularizable. So A can be put in triangular form. This along with the fact that A is nilpotent and that D is a division ring implies that A can be put in strict triangular form, yielding $A^n = 0$.

Corollary 4.2.5. *Let $\mathcal{V}$ be a finite-dimensional vector space over a division ring D. Then every nonzero semigroup $\mathcal{S}$ of nilpotent linear transformations on $\mathcal{V}$ has a* *nontrivial hyperinvariant subspace.*

Proof. Corollary 4.2.3 and Theorem 4.2.4. □

Lemma 4.2.6. *Let D be a division ring with* $\mathrm{ch}(D) = 0$ or $> n$ *where* $n \in \mathbb{N}$, $A \in M_n(D)$, *and* $m \in \mathbb{N}$. *Then A is nilpotent if*

$$\mathrm{tr}(S^{-1}A^iS) = 0,$$

for all invertible matrices $S \in M_n(D)$, *and all* $i \in \mathbb{N}$ *with* $i \geq m$.

Proof. We prove the assertion by induction on n. If $n = 1$, then the assertion trivially holds. Assuming that the assertion holds for matrices of size less than n, we prove the assertion for matrices of size n. Since $M_n(D)$ is a finite-dimensional vector space over D, it follows that there exists a monic polynomial $f = x^k + f_{k-1}x^{k-1} + \dots + f_1x + f_0 \in D[X]$ of minimal degree such that

$$A^k + f_{k-1}A^{k-1} + \dots + f_1A + f_0I = 0. \qquad (*)$$

Let m_0 be the smallest positive integer for which $\mathrm{tr}(A^i) = 0$ for all $i \in \mathbb{N}$ with $i \geq m_0$. We note that if $m_0 = 1$, then by the characteristic condition on D we have $\mathrm{tr}(A^{m_0-1}) = \mathrm{tr}(I) \neq 0$. Multiplying both sides of $(*)$ by A^{m_0-1} from the right, taking trace of both sides, and then dividing by $\mathrm{tr}(A^{m_0-1}) \neq 0$ from the right, we conclude that $f_0 = 0$. Hence $(A^{k-1} + f_{k-1}A^{k-2} + \dots + f_1)A = 0$. This together with the fact that f is the minimal polynomial of A implies that A is not invertible. Therefore, there exists $e_1 \in D^n$ such that $Ae_1 = 0$. It is obvious that after a similarity we can write

$$A = \begin{pmatrix} 0 & X \\ 0 & \hat{A} \end{pmatrix}$$

for some $\hat{A} \in M_{n-1}(D)$ and $X \in M_{1,n-1}(D)$. Plainly it follows from the hypothesis and the above matrix representation that

$$\mathrm{tr}(\hat{S}^{-1}\hat{A}^i\hat{S}) = 0,$$

for all $i \in \mathbb{N}$ with $i \geq m$ and for all invertible matrices $\hat{S} \in M_{n-1}(D)$. Now it follows from the induction hypothesis that $\hat{A}$ is nilpotent, hence so is A, finishing the proof. □

Remark. The trace of a nilpotent matrix on a division ring is not necessarily zero. To see this, let $\mathbb{H}$ denote the division ring of real quaternions. Let $A \in M_2(\mathbb{H})$ be the matrix defined by

$$\begin{pmatrix} 1 & i \\ j & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & i \\ j & 1 \end{pmatrix}.$$

Plainly $A^2 = 0$. It is just a matter of a straightforward calculation to see that

$$A = \frac{1}{2} \begin{pmatrix} -i-j & -1+k \\ -1-k & -i+j \end{pmatrix}.$$

Thus, $\operatorname{tr}(A) = -i$.

The following theorem is crucial in the proof of our main theorem below.

Theorem 4.2.7. *Let D be a division ring with* $\operatorname{ch}(D) = 0$ *or* $> n$ *with* $n > 1$, *$T \in M_n(D)$, and $\mathcal{S}$ a semigroup in $M_n(D)$. If*

$$\operatorname{tr}(S^{-1}\mathcal{S}S) = \{\operatorname{tr}(S^{-1}TS)\},$$

for all invertible matrices $S \in M_n(D)$, then $\mathcal{S}$ is reducible.

Proof. Let $\mathcal{S}$ be a semigroup in $M_n(D)$ satisfying the above condition. If $\operatorname{tr}(S^{-1}TS) = 0$ for all invertible matrices $S \in M_n(D)$, then it follows from Lemma 4.2.6 that $\mathcal{S}$ is a semigroup of nilpotent matrices which would be triangularizable, hence reducible, by Levitzki's Theorem. So we may assume that $c_0 := \operatorname{tr}(S_0^{-1}TS_0) \neq 0$ for some invertible matrix $S_0 \in M_n(D)$. Now we recognize two cases.

(a) $\{S_0^{-1}SS_0, S_0^{-1}S'S_0\}$ *is linearly dependent for all $S, S' \in \mathcal{S}$.*

Pick $S \in \mathcal{S}$, it follows that for every $S' \in \mathcal{S}$ we have $S_0^{-1}S'S_0 = dS_0^{-1}SS_0$ for some $d \in D$. Taking trace of both sides yields $\operatorname{tr}(S_0^{-1}S'S_0) = d\operatorname{tr}(S_0^{-1}SS_0)$. Hence $c_0 = dc_0$

implying $d = 1$. Therefore we have $\mathcal{S} = \{S\}$ and this in turn implies $S^2 = S$ for $\mathcal{S}$ is a semigroup. It is now plain that $\mathcal{S}$ is reducible.

(b) $\{S_0^{-1}SS_0, S_0^{-1}S'S_0\}$ *is linearly independent for some* $S, S' \in \mathcal{S}$.

Define $\mathcal{A} := \mathrm{Alg}_F(\mathcal{S})$ where F is the center of D. Suppose that $A = c_1S_1 + ... + c_kS_k \in \mathcal{A}$ where $k \in \mathbb{N}$, $c_j \in F$, $S_j \in S$ $(j = 1, ..., k)$ is given. Set $c_S := \mathrm{tr}(S^{-1}TS)$. Since $\mathrm{tr}(S^{-1}\mathcal{S}S) = \{c_S\}$, it is easily seen that $\mathrm{tr}(S^{-1}A^jS) = c_S(c_1 + ... + c_k)^j$ for all $j \in \mathbb{N}$ and for all invertible matrices $S \in M_n(D)$. So if $c_1 + ... + c_k = 0$, it follows from Lemma 4.2.6 that A is nilpotent. In particular $0 \neq S' - S \in \mathcal{A}$ would be nilpotent. Now the trace condition on $\mathcal{S}$ easily implies that the set of $A = c_1S_1 + ... + c_kS_k \in \mathcal{A}$ with $c_1 + ... + c_k = 0$ in $\mathcal{A}$ is indeed a nonzero semigroup ideal of $\mathcal{A}$ consisting of nilpotents. Therefore by Levitzki's Theorem and Lemma 4.1.1 reducibility of $\mathcal{A}$, and hence that of $\mathcal{S}$, follows. □

4.3 Main Results

Let D be a division ring, and F the center of D. For a semigroup $\mathcal{S}$ in $M_n(D)$, let $\mathrm{Alg}_F(\mathcal{S})$ denote the F-algebra generated by $\mathcal{S}$, i.e.,

$$\mathrm{Alg}_F(\mathcal{S}) := \{\sum_{i=1}^{k} \alpha_i S_i : k \in \mathbb{N}, \alpha_i \in F, S_i \in \mathcal{S}\}.$$

If $m \in \mathbb{N}$, we use $\mathcal{S}^m$ to denote the semigroup ideal of $\mathcal{S}$ consisting of words in $\mathcal{S}$ of "apparent length" m, i.e.,

$$\mathcal{S}^m = \{S_1 ... S_m : S_i \in \mathcal{S}, i = 1, ..., m\}.$$

Recall that a semigroup ideal $\mathcal{J}$ of $\mathcal{S}$ is called an absorbing semigroup ideal of $\mathcal{S}$ if there exists $m \in \mathbb{N}$ such that $\mathcal{S}^m \subseteq \mathcal{J}$. Plainly $\mathcal{S}^m$ is an absorbing semigroup ideal of $\mathcal{S}$ for each $m \in \mathbb{N}$. We use $GL_n(D)$ to denote the group of invertible matrices in $M_n(D)$. The following result can be considered as an analogue of Theorem 2.3.1 over division rings.

Theorem 4.3.1. *Let D be a division ring with* $\mathrm{ch}(D) = 0$ or $> n$, *F the center of D, $\mathcal{S}$ an irreducible semigroup in $M_n(D)$, and $\mathcal{J}$ an absorbing semigroup ideal of $\mathcal{S}$. Then*

$$\{A \in \mathrm{Alg}_F(\mathcal{S} \cup \{I\}) : \mathrm{tr}(T^{-1}\mathcal{J}A\mathcal{J}T) = \{0\} \text{ for all } T \in GL_n(D)\} = \{0\}.$$

Proof. First we note that since $\mathcal{J}$ is an absorbing semigroup ideal of $\mathcal{S}$, there exists $m \in \mathbb{N}$ such that $\mathcal{S}^m \subseteq \mathcal{J}$. Denote the left hand side of the asserted identity by $\mathcal{J}_1$. To show that $\mathcal{J}_1 = \{0\}$, we use contradiction. Suppose that $\mathcal{J}_1 \neq \{0\}$. Indeed $\mathcal{J}_1$ is an ideal of the F-algebra, hence semigroup, $\mathrm{Alg}_F(\mathcal{S} \cup \{I\})$. To see this, note that due to linearity of the trace functional and the fact that $\mathcal{J}$ is a semigroup ideal of $\mathcal{S}$ we can write

$$\mathrm{tr}(T^{-1}S_1BAS_2T) = 0; \ \mathrm{tr}(T^{-1}S_1ABS_2T) = 0,$$

for a given $B \in \mathrm{Alg}_F(\mathcal{S} \cup \{I\})$, $A \in \mathcal{J}_1$ and for all $S_1, S_2 \in \mathcal{J}$, and $T \in GL_n(D)$. That is $BA, AB \in \mathcal{J}_1$ for all $B \in \mathrm{Alg}_F(\mathcal{S} \cup \{I\})$, $A \in \mathcal{J}_1$, i.e., $\mathcal{J}_1$ is an ideal of $\mathrm{Alg}_F(\mathcal{S} \cup \{I\})$. So irreducibility of $\mathcal{J}_1$ follows from that of $\mathrm{Alg}_F(\mathcal{S} \cup \{I\})$ in light of Lemma 4.1.1 (note that irreducibility of $\mathrm{Alg}_F(\mathcal{S} \cup \{I\})$ follows from that of $\mathcal{S}$). Now we note that if $A \in \mathcal{J}_1 \cap \mathrm{Alg}_F(\mathcal{S})$, then A is nilpotent. To see this, since $A \in \mathcal{J}_1 \cap \mathrm{Alg}_F(\mathcal{S})$ and trace is linear, it follows that $\mathrm{tr}(T^{-1}A^mAA^{k+m-1}T) = 0$, for all $T \in GL_n(D)$, and $k \in \mathbb{N}$ (note that A^{k+m-1} is in the F-algebra generated by $\mathcal{S}^m \subseteq \mathcal{J}$ for all $k \in \mathbb{N}$). In other words $\mathrm{tr}(T^{-1}A^{k+2m}T) = 0$ for all $T \in GL_n(D)$, and $k \in \mathbb{N}$. It follows from Lemma 4.2.6 that A is nilpotent. Now it is easily seen that $\mathcal{J}_1 \cap \mathrm{Alg}_F(\mathcal{S})$ is indeed an ideal of $\mathrm{Alg}_F(\mathcal{S} \cup \{I\})$. On the other hand, $\mathcal{J}_1 \cap \mathrm{Alg}_F(\mathcal{S})$ consists of nilpotent matrices. Therefore it follows from Levitzki's Theorem (Corollary 4.2.3 or see Theorem 35 of [K1], page 135) and Lemma 4.1.1 that $\mathcal{J}_1 \cap \mathrm{Alg}_F(\mathcal{S}) = \{0\}$. Now define the subsemigroup $\mathfrak{J}$ of $\mathcal{J}_1$ as follows

$$\mathfrak{J} := \mathcal{J}_1 \cap (I - \mathrm{Alg}_F(\mathcal{S})).$$

We have $\mathfrak{J} \neq \emptyset$ for $\mathcal{J}_1 \neq 0$. Plainly irreducibility of $\mathfrak{J}$ follows from that of $\mathcal{J}_1$ and the fact that $\mathcal{J}_1 \cap \mathrm{Alg}_F(\mathcal{S}) = \{0\}$. Next we show that $\mathfrak{J}$ is a singleton. Suppose otherwise. Then there are $I - A, I - B \in \mathfrak{J}$ such that $B - A \neq 0$. Since $I - A, I - B \in \mathfrak{J} \subset \mathcal{J}_1$, it follows that $B - A = (I - A) - (I - B) \in \mathfrak{I}_1$. On the other hand, we have $B - A \in \mathrm{Alg}_F(\mathcal{S})$ hence, $0 \neq B - A \in \mathfrak{I}_1 \cap \mathrm{Alg}_F(\mathcal{S})$, contradicting the fact that $\mathfrak{I}_1 \cap \mathrm{Alg}_F(\mathcal{S}) = \{0\}$. Thus $\mathcal{J} = \{J\}$ where $J = I - A \in \mathfrak{I}_1$ and $A \in \mathrm{Alg}(\mathcal{S})$. By showing that J, equivalently $\mathcal{J}$, is reducible we obtain a contradiction, finishing the proof. Since $J = I - A \in \mathfrak{I}_1$ and $A \in \mathrm{Alg}_F(\mathcal{S})$, it follows from linearity of the trace functional that $\mathrm{tr}(T^{-1}A^m(I - A)A^{k+m-1}T) = 0$ for all $T \in GL_n(D)$, and $k \in \mathbb{N}$. Thus $\mathrm{tr}(T^{-1}A^{k+2m}T) = \mathrm{tr}(T^{-1}A^{k+2m-1}T)$ for all $T \in GL_n(D)$, and $k \in \mathbb{N}$. Note that $\{A^i\}_{i \geq 2m}$ is a semigroup ideal of the semigroup $\{A^i\}_{i \in \mathbb{N}}$. On the other hand, we have $\mathrm{tr}(T^{-1}A^{k+2m}T) = \mathrm{tr}(T^{-1}A^{2m}T)$ for all $T \in GL_n(D)$, and $k \in \mathbb{N}$. Hence the semigroup $\{A^i\}_{i \geq 2m}$ is reducible by Theorem 4.2.7. Now it follows from Lemma 4.1.1 that reducibility of $\{A^i\}_{i \geq 2m}$ implies that of $\{A^i\}_{i \in \mathbb{N}}$ and hence that of $\mathcal{J} = \{I - A\}$, a contradiction. □

Remark. It is worth mentioning that the conclusion of the preceding theorem does not imply irreducibility of the semigroup $\mathcal{S}$. For instance, the conclusion holds for the diagonal semigroup $\mathcal{S} := \{\mathrm{diag}(d_1, ..., d_n) : d_j \in F,\ i = 1, ..., n\}$ where F is the center of D.

Let D be a division ring. We say trace is *permutable* on a collection $\mathcal{F} \subset M_n(D)$ if for all $m \in \mathbb{N}$, all $T \in GL_n(D)$, all $A_1, ..., A_m \in \mathcal{F}$, and all permutations σ on m letters, we have

$$\mathrm{tr}(T^{-1}A_{\sigma(1)}...A_{\sigma(m)}T) = \mathrm{tr}(T^{-1}A_1...A_mT)$$

The following theorem is an extension of a result due to Radjavi to matrices over division rings (see [R2]).

Theorem 4.3.2. *Let D be a division ring with* $\mathrm{ch}(D) = 0$ or $> n$ *where $n \in \mathbb{N}$, $\mathcal{F} \subset M_n(D)$ an irreducible family on which trace is permutable. Then $\mathcal{F}$ is commutative.*

Proof. We use contradiction. Suppose that $\mathcal{F}$ is not commutative. Hence there are $A, B \in \mathcal{F}$ with $AB - BA \neq 0$. Now let $\mathcal{S}$ be the semigroup generated by $\mathcal{F}$. Note that $\mathcal{F}$ is reducible iff $\mathcal{S}$ is. Also note that $0 \neq AB - BA \in \mathrm{Alg}_F(\mathcal{S})$ where F is the center of D and, by permutability of trace on $\mathcal{F}$, we have $\mathrm{tr}(T^{-1}C_1(AB - BA)C_2T) =$

0 for all $C_1, C_2 \in \mathcal{S}$, and $T \in GL_n(D)$. That is $0 \neq AB - BA \in \mathrm{Alg}_F(\mathcal{S})$ and $\mathrm{tr}(T^{-1}\mathcal{S}(AB - BA)\mathcal{S}T) = 0$ for all $T \in GL_n(D)$. Therefore, Theorem 4.3.1 (with $\mathcal{I} = \mathcal{S}$) implies that $\mathcal{S}$, and hence $\mathfrak{F}$, is reducible, a contradiction. $\square$

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