On Measurably Indexed
Families of Hilbert Spaces

by

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to my parents

Contents

Abstract

Our aim is to study X-families of Hilbert spaces for X a measure space; the ultimate goal being the understanding of the classical (von Neumann) direct integral in the context of indexed category theory. Indeed, the diagram, $\int^{\oplus} : \underline{\mathbf{Hilb}}^X \leftrightarrows \underline{\mathbf{Hilb}} : \Delta$, provides a useful summary of our goal.

We first require a good base category of measure spaces and introduce, $\underline{\mathbf{Disint}}$, the category of disintegrations. $\underline{\mathbf{Disint}}$ does not have products (nor does any "useful" category of measure spaces) so we do not have the usual Paré-Schumacher style indexing. The diagram above cannot be interpreted as an adjunction.

We must approximate the situation as best possible and we put forth three approximations. Specifically, we propose three notions of X-family of Hilbert spaces: 1. measurable fields of Hilbert spaces on X, 2. Hilbert sheaves on X, and 3. Hilbert families over X. We will describe each of these approaches in detail including substitution with respect to base category morphisms and $\int^{\oplus}$. Finally, we will discuss connections between the three ideas and list some possible future directions for this work.

Notation

Most of the notation we use is standard. In some cases, however, our notation is slightly different for typographical reasons. The most notable examples are: **N**, **Z**, **Q**, **R**, and **C** denote, respectively, the set of natural, integral, rational, real, and complex numbers.

Categories are generally in bold face type and underlined (for example, **<u>Set</u>** denotes the category of (small) sets and functions; exceptions to this are $MEAS(X)$ and $Sh(\mathcal{A})$, categories with some sort of "argument"). Bicategories are in bold face and doubly underlined. Indexed categories are in bold face and "under-tilded." Composition of morphisms is in the functional (as opposed to algebraic) way. Arguments are generally written on the right and a "–" is sometimes used for an unspecified argument.

Definitions, remarks, and examples end in a $\square$. Proofs and some theorem statements (those for which no proof is supplied) end in a $\blacksquare$. All such structures, except examples, remarks, and corollaries, are numbered according to the section in which they appear. Bibliographic references generally follow some mnemonic of the author's name. For example, [PTJ] represents P.T. Johnstone. Other notation is standard or defined in the text.

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Introduction

There is no doubt that decompositions are useful. For example, a semi-simple module is, by definition, a sum of simple modules. In essence, then, to understand semi-simple modules, it is enough to understand simple modules (a smaller class) provided, of course the thing one is studying "commutes" with sum. The corresponding entity in vector spaces is, in fact, a theorem; the spectral theorem.

Let us recall some basic linear algebra. Suppose V is a finite dimensional **C**-Hilbert space (i.e. just $\mathbf{C}^n$). An operator, $T : V \longrightarrow V$, is just multiplication by a matrix A. We can compute the eigenvalues of an $n \times n$ matrix (λ such that $det(A - \lambda I) = 0$ or, such that $T - \lambda I$ is not invertible). If T is normal ($T^*T = TT^*$), then we can decompose it as $T = \sum_{i=1}^{k} \lambda_i E_i$ where the λ_i's are the eigenvalues and E_i is the projection onto the null space $\mathcal{N}(T - \lambda_i I)$.

In particular, if there are n distinct eigenvalues, we get a basis consisting of eigenvectors. We can decompose the space V into one-dimensional subspaces

$$V \simeq \mathcal{N}(T - \lambda_1 I) \oplus \mathcal{N}(T - \lambda_2 I) \oplus \cdots \oplus \mathcal{N}(T - \lambda_n I)$$

which, in itself, is not surprising since $\mathbf{C}^n = \bigoplus_{i=1}^{n} \mathbf{C}$. More importantly, however, we can decompose a normal operator into a sum of orthogonal projections. In essence, the action of T can be "simplified;" working with a diagonal matrix is much easier than working with a general matrix.

For the infinite dimensional case, we have the notion of spectrum:
$\sigma(T) := \{\lambda \in \mathbf{C} \mid T - \lambda I$ not invertible $\}$. We can also talk about eigenvalues; λ's for which there exists a nonzero vector, called an eigenvector, x, with $\lambda x = T(x)$ (the set of eigenvalues, called the *point spectrum*, is, in general, different from the spectrum for consider the example of the unilateral shift operator on $l^2(\mathbf{N})$; 0 is in the spectrum since this operator is not invertible but 0 is not an eigenvalue).

Furthermore, in any Hilbert space, it makes sense to talk about orthonormal bases.

For example, suppose $\mathcal{H}$ is a Hilbert space of countable dimension with an orthonormal basis given by the "eigenvectors" corresponding to some operator $T \in \mathcal{L}(\mathcal{H})$ (which has a countable spectrum). Then

$$\mathcal{H} = \sum_{i=1}^{\infty}{}^{\oplus} \mathcal{N}(T - \lambda_i I).$$

Now, if the spectrum is "continuous" as a subset of the complex plane, then we need some sort of "continuous direct sum:" the direct integral: $\int^{\oplus} \mathcal{H}(x) d\mu(x)$. More accurately, the direct integral of Hilbert spaces is a measurable analogue of the direct sum. That is to say, direct sums are a special case inasmuch as finite (or countable) counting measure is, in particular, an example of a measure.

And so, one way of looking at the direct integral is as a tool for the above mentioned decomposition of a normal operator for the infinite dimensional case. This generalized reduction theory for operators was developed by von Neumann in the late 1930's, although he didn't publish the results until 1949. The paper, in which the direct integral was first introduced, may be found in the "operator algebras" volume of his collected works, [vNeu], together with the collection of papers with Murray in which what were to become known as von Neumann algebras were described. At the beginning of chapter 2, we will provide a motivation for the direct integral of Hilbert spaces from the theory of unitary group representations.

Our project is to study the direct integral as an indexed notion. It appears that its initial construction was ad hoc (although immediately useful for the applications at hand (and others later)). We wish to provide a firm footing on which to found a systematic, categorical treatment of the direct integral of Hilbert spaces and related constructions; in short, indexing by measure spaces.

Let us expand on this indexing idea somewhat. We introduced the direct integral by first considering the (special) finite dimensional case. Let us introduce the notion of indexing by specializing, as well. Let k be a field and $I \in \underline{\mathbf{Set}}$.

We have an adjunction:

$$\begin{tikzcd}
(\underline{k\text{-}\mathbf{vect}})^I \arrow[r, shift left=3, "\bigoplus_I"] \arrow[r, phantom, shift left=1.5, "\perp"] \arrow[r, phantom, shift right=1.5, "\perp"] \arrow[r, shift right=3, "\prod_I"'] & \underline{k\text{-}\mathbf{vect}} \arrow[l, "\Delta" description]
\end{tikzcd}$$

Our task is to understand the analogous picture:

$$\begin{tikzcd}
(\underline{\mathbf{Hilb}})^X \arrow[r, shift left=2, "\int_X^{\oplus}"] \arrow[r, phantom, "\perp"] & \underline{\mathbf{Hilb}} \arrow[l, shift left=2, "\Delta"]
\end{tikzcd}$$

This amounts to understanding what X-indexed families of Hilbert spaces are for X a measure space (a related question: how to define the "co-direct integral", $\int_{\oplus}$, as a right adjoint to Δ, will not be treated in this paper). A true indexing procedure ("true" in the sense of [P&S]) cannot be found since interesting (from an analysis/operator theoretic point of view) categories of measure spaces don't have products. Indeed, we won't have a Δ per se. So, we must approximate the situation as closely as possible. We shall put forth three approximations for the unknown elements of the above diagram, each with its own merits, and will weigh them and describe connections between the three.

The understanding of the construction of the direct integral of Hilbert spaces in the context of category theory involves three aspects: measure theory, analysis (operator theory), and indexed category theory.

Measure theory provides the basic framework. It is, initially, in this context that the above constructions must be understood. That is to say, the project is, first and foremost, about measure theory in a categorical context. The next few paragraphs provide a partially chronological history of categorical measure theory and some related topics.

Essentially, the first application of categories to measure theory occurred in Linton's thesis ([FEJ]). His objective was to study Fubini's theorem in the context of Boolean rings and σ-rings (roughly speaking, their "concrete" realization being fields and σ-fields). This study required a "solid foundation in functor theory." Indeed, he studied measures by looking at them in the context of linear functional theory and used the power of Boolean algebras and the vector space of measures (Boolean algebras and vector spaces are well treated by category theory) and was able to give a purely Boolean proof of the Fubini theorem.

In related work, Börger ([Bör]) considered sequential Boolean algebras. His categorical treatment of integration theory grew out of a generalization of integration in three directions: integrate over abstract Boolean algebras as opposed to algebras of sets (akin to Linton above), admit vector-valued measures, and relax σ-additivity. His work went in a different direction than we will follow. We note that, in this paper, we do not wish to generalize integration in any of these directions. That is to say, "external forces" (for example, operator algebras) force the use of standard integration.

An important example in sheaf theory (developed in the 1960's) was given by Deligne ([SGAIV]). He constructed a topos from a measure space (we outline this example in detail in chapter 3, where we follow the work of Howlett [How]). It is the quintessential example of a topos without points. More precisely, this topos was constructed to provide an example of a topos with no points (assuming the measure space has no atoms). Analysis in sheaves and, more generally, topoi, is a rich subject and related to the work we present here. Many people have contributed (a short list is given below). Among those who contributed at the applications of sheaves conference ([FMS]) was Breitsprecher. He contributed two papers one of which ([Bre2]) has become quite important for our work here. In some sense, this was the beginning of "measurable sheaf theory" an, as yet, vague collection of results not coalesced into a discipline.

Probability theory is, in particular, measure theory ($\mu(X) = 1$). Indeed, it has been said that probability theory is measure theory plus "a point of view." Two applications of category theory to probability theory come to mind: Schiopu's master's thesis ([Sch]) on random variables and Bogdan's ([Bog]) application of algebraic categories to probability theory. Schiopu provided a categorical foundation of probability theory using a category whose morphisms are a continuous generalization of stocnastic matrices. Bogdan provided another categorical axiomatization of probability theory using algebraic categories and set up several isomorphisms of categories relevant tc probability spaces. These works are scmewhat distant from our discussions here (inasmuch as our "point of view" is not that of probability theory).

In 1973, Lawvere published his paper on metric spaces and closed categories ([Law2]). This work grew out of the formal comparison of the triangle inequality for metric spaces:

$$dist(a,b) + dist(b,c) \geq dist(a,c)$$

and the composition in the definition of a category:

$$hom(A,B) \otimes hom(B,C) \longrightarrow hom(A,C).$$

There is, of course, more than just formal similarity here and Lawvere developed a whole theory of such entities; the premise being that things with a formal hom-like appearance can be interpreted as such. Indeed, he described metric spaces as being enriched categories over **R**. One of his examples was that of the metric space constructed from a measure space ($dist(A, B) := \mu(A \triangle B)$ provides the σ-algebra with a pseudo-metric structure; see chapter 1). This was explored, with applications to convex sets (and, eventually, to stochastic programming) by Meng ([Meng]).

A new and growing field is quantum logic (see, for example, [Rum] or [R&R]). This is not immediately related to the material presented here. It does, however, make use of Hilbert space theory and, more importantly, quantales (the original paradigm being the lattice of closed right ideals of an arbitrary C^*-algebra; a good example of a locale being the lattice of closed ideals of a commutative C^*-algebra; these examples suggest that we may think of a quantale as a "non-commutative" locale, see, for example, [Bor]). So, if locales (and measurable sheaves) offer a way of understanding the direct integral from a classical logical point of view (or possibly, distantly, an intuitionistic point of view), one may hope that quantales offer a way of understanding Hilbert space methods in an alternate logic. Furthermore, C^*-algebras are sometimes thought of as providing a context for non-commutative integration ([Ped]). As we hinted above, however, the exploration of this interesting notion will await other work. And so, the future seems an appropriate place to end our historical remarks; we return to the description of the problem.

We search for an appropriate category of measure spaces. As hinted at above, certain requirements of analysis cannot be reconciled with certain requirements of indexed category theory. Specifically, reflecting sets of measure zero is incompatible with finite limits. These, as yet, vague remarks will be made more precise in chapter 1. In short, the search for a "good" category of measure spaces is difficult. We present three in chapter 1, but consider only one in detail. It is the "best" at hand for our purposes.

The second aspect of the understanding of direct integrals is the analytic one. We will be interested in operator theory (Hilbert spaces and operator algebraic notions; for example, von Neumann algebras and C^*-algebras). A small amount of necessary background about the direct integral of Hilbert spaces will be given at the beginning of chapter 2. However, we will be chiefly interested in categorical analysis. We will begin to describe operator theory inside a category (topos) $\underline{\mathbf{E}}$. A more complete discussion will await another paper as that would take us too far afield for our basic applications here.

By categorical analysis, we mean an analogy to the study of functional analysis and, in particular, Banach spaces inside a (usually Grothendieck) topos. This is well known and has been around for some time. For a useful collection of such results, see [FMS]; in particular, the reader is referred to the following papers relevant to our discussion here: [Ban], [Bu&Mu], [Fo&Hy], [Ho&Ke], [PTJ3], [Ke&Le], [Rou1], and [Tak]. Furthermore, the reader is also referred to [M&P], [WP&Ro], and [Rou2]. These papers provide a broad background to the subject. Operator theory of Hilbert spaces inside a topos is not so well known, though [Rou2] does address spectral decomposition of matrices in a topos.

A topos is to be thought of as a generalized logic. Indeed, the development of functional analysis inside a topos went hand in hand with intuitionistic analysis since, in general, the logic of a topos is not Boolean. We will look at these ideas in chapter three and explore analysis in a specific "measure theoretic" topos. It turns out, however, that this topos has the axiom of choice and is, in fact, Boolean. In essence, then, our logic is classical.

The final aspect is indexed category theory. In the late 1960's, it became increasingly clear that a generalized theory of indexing (generalized in the sense of indexing by objects other than sets) was necessary. Lawvere suggested the need to understand indexing via category theory. Among others, Paré and Schumacher, [P&S] developed such a theory. Subsequent to their work, many results were found and the general theory was applied, in a mutually self beneficial way, to enriched category theory, [Wd1], algebras, [RRb], topology, [Lev], and coalgebras, [G&P]. Related work was done by Tavakoli, [Tav]. Indeed, he studied vector spaces in topoi, which tied in with topos-based indexed category theory alluded to above.

Paré-Schumacher indexed category theory uses pseudo-functors. Another style was developed by Bénabou ([Bén]) using fibrations. We employ Paré-Schumacher and Bénabou style indexed category theory (the two being, more or less, equivalent). A brief outline of Paré-Schumacher style of indexing will be given in chapter 2. We have said that this paper is about indexing by measure spaces. More precisely, the project grows out of three "directives:" 1. Paré: understand the direct integral as an indexed functor (index by measure spaces), 2. Breitsprecher: understand disintegrations (of one measure space with respect to another) as a categorical notion, and 3. Lawvere: look at the Gros and Petit aspects of categorical measure theory (construct a "sheaf-based" operator theory and compare it to the well-known functional analysis in sheaves). We attempt to address each of these directives.

The first directive is, of course, the main impetus of this present research. The second directive is addressed in chapter 1 by the category **Disint**, a category which seems to exhibit the self indexing of measure spaces. Finally, the third directive is addressed in chapter 3, in which we introduce a sheaf category relevant to measure theory.

The third trinity to be discussed in this introduction is our three approaches to the problem: understand $(\underline{\mathbf{Hilb}})^X$ for X a measure space. We will approach the problem using measurable fields of Hilbert spaces, Hilbert space objects in a sheaf category, and Hilbert families. The following diagram provides a useful summary.

$$\begin{tikzcd}
\boxed{\text{1. Mble fields}} \arrow[dd, shift right, "4."'] \arrow[dr, shift left, "6."] & \\
 & \boxed{\text{2. Hilb Sheaves}} \arrow[ul, shift left, "7."] \arrow[dl, shift left, "8."] \\
\boxed{\text{3. Hilb Families}} \arrow[uu, shift right, "5."'] \arrow[ur, shift left, "9."] &
\end{tikzcd}$$

The essence of our present work is to describe this diagram; the three approaches and the six connections (boxes 2 and 3 are new approaches to box 1, although, we have inserted a great deal of category theory into box 1). That is, the important thing is to compare the three approaches. In chapter 1, we give the necessary measure theoretic background and attempt to address the Breitsprecher directive. In chapter 2, we give the necessary operator algebraic background and describe box 1 above. In chapter 3, we describe box 2 and address the Lawvere directive. The third approach to finding a suitable notion of measure indexed categories is discussed in chapter 4. In chapter 5, we note that each of the three approaches has merits and discuss connections (for example, some logical implications) between the three boxes above (arrows 4-9) and allude to possible future directions of this comparison.

Finally, we spend a few paragraphs listing what is new in this paper.

The category, $\underline{\mathbf{MP}}$, of measure preserving functions is not a new invention. The "base" categories, $\underline{\mathbf{M0R}}$ and $\underline{\mathbf{Disint}}$, are, however, new. $\underline{\mathbf{Disint}}$ is the best (in the sense that it has self indexing built into it) base category. It provides a powerful context to do measure indexed operator theory.

Each of the three approximations, of chapters 2, 3, and 4, contains new material. Measurable fields of Hilbert spaces are old, of course (von Neumann). But, the contexts of category theory and, indeed, indexed category theory are new. Box 1 represents a mostly operator theoretic approach to understanding X-families of Hilbert spaces.

Sheaves have been around for some time (Grothendieck) and sheaves on a measure space have also (Deligne). Applications to analysis, complex analysis, functional analysis, operator algebras, etc. have been done by Mulvey, Rousseau, Wick-Pelletier, et al (see the historical remarks above). Chapter 3 contains our description of the theory of Hilbert spaces in a (very specific) topos. As such, this topos on a measure space is a special case. Our point of view, that of indexed category theory, though, sheds a different light on these entities. The speciality of the topos allows us to prove (which, as far as we know, has not been published elsewhere) the existence of the completion of a preHilbert space to a Hilbert space (a result which we will find especially useful when discussing substitution). This is not a particularly difficult result since it is simply a (careful) translation of the classical proof to the sheaf world (again, we must emphasize, not the general (= possibly without the axiom of choice) sheaf world).

The material of chapter 4 (and the connections in chapter 5) is new. Box 3 is the "fibrations" approach to X-indexing. We set up an elaborate substitution machinery (partially for future considerations, as well) involving disintegrations. Globally, this is familiar to fibration enthusiasts. The details, however, are interesting and exhibit the utility of disintegrations as alluded to above.

Chapter 1

Categories of Measure Spaces

1.1 Introduction

This paper is, first and foremost. about measure theory and, in this chapter, we describe this measure theoretic background.

Specifically, we begin with **Mble**, the category of measurable spaces and measurable functions; a category not unlike **Top**, the category of topological spaces and continuous functions. **Mble** is, in some sense, the basic category. That is to say, all theory is based upon it as a foundation. Indeed, to talk about measure spaces, one must first understand their measurable structure.

In the introduction, we noted that the main thrust of this paper is to look at indexing by measure spaces: to develop a theory capable of describing the direct integral coherently. More importantly, there are three directives and three approaches to solving this problem. After describing the category, **Mble**, we introduce three "candidate" categories of measure spaces. The problem, of course, from an indexed categorical point of view, is that none of these candidates has products (specifically, the diagonal $X \xrightarrow{\Delta} X \times X$, which is measurable, is not necessarily in the category in question). In sections 1.3 and 1.4, we introduce two important "background" categories of measure spaces: **MP**, measure preserving functions (too restrictive to be useful), and **M0R**, measure zero reflecting functions (a useful base category).

In the last section of this chapter, we describe the category **Disint**. It is a useful starting point in that it seems to witness the self-indexing of measure spaces; a disintegration has indexing by a "measure-parameter-space" built into it. **Disint** is also our attempt at answering the Breitsprecher directive: understand measure theoretic disintegrations from a categorical perspective.

1.2 Measurable Spaces

We begin with the category of abstract measurable spaces and measurable functions. An excellent description of this category is given in [Sch]. (etymology: she used **Bsp**, for "Borel Space, " a term often used in probability theory. We use **M'ble**, for "Measurable," wishing to reserve Borel Space for its special meaning as the σ–algebra generated by the opens of a topological space.)

There are two ways of describing basic measure theory. The first, historically, and the approach most often followed in first courses in measure theory, is to take as basic notion measurable sets and build measurable functions. That is to say, to calculate the area under a curve, cut up the y-axis. Another approach is to take simple functions as basic and build measurable sets (see, for example [Bou]). While this approach lends itself well to such generalizations as vector measures, we shall use the (more algebraic) measurable sets approach; measure theory as described in [Roy], for example. We will, however, when needed, and not quite randomly, refer to the second style.

Definition 1.2.1 *A* σ-algebra *on a set, X, is a collection of subsets closed under countable (including finite) unions and complementation and containing $\emptyset$. A* Measurable Space *is a pair $(X, \mathcal{A})$ where $\mathcal{A}$ is a σ-algebra on the set X. The elements of $\mathcal{A}$ are called* Measurable Sets $\square$

Definition 1.2.2 *A* Measurable Function, $(X, \mathcal{A}) \xrightarrow{f} (Y, \mathcal{B})$, *is a function, $X \xrightarrow{f} Y$, for which $f^{-1}(B) \in \mathcal{A}\ \forall B \in \mathcal{B}$.* $\square$

Certainly, the identity function is measurable and the composition of two measurable functions is a measurable function so we have a category which is denoted by <u>**Mble**</u>.

The direct image of a measurable set under a measurable function need not be measurable. (Example: the cartesian product of $[0,1]$ and a (line) nonmeasurable set in the Lebesgue plane projected onto the second factor.) One could talk about the category <u>**Dmble**</u> of measurable spaces and directly measurable functions: $f(A) \in \mathcal{B},\ \forall A \in \mathcal{A}$. Except in this section, for completeness of discussion, we will not explore <u>**Dmble**</u> in this paper. Functions that are both measurable and direct measurable are quite rare.

Indeed, the axioms governing the class of subsets for a σ-algebra and those for a topology are similar (other examples: convexity, [Daw1] and [Daw2], or more generally, paving [K&T, p. 136]). A σ-algebra and topology on a set are both collections of subsets closed under various operations and in the latter part of this section, we will explore the similarities between <u>**Mble**</u> and <u>**Top**</u>, the category of abstract topological spaces and continuous functions.

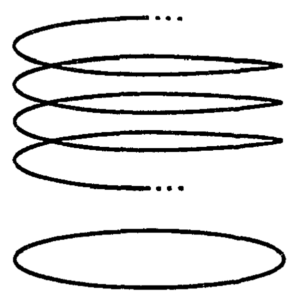
For the next few paragraphs, however, let us note some of the significant differences. The differences arise out of the arities of the operations that σ-algebras and topologies are to be closed under. Specifically, a topology is closed under arbitrary unions and finite intersections, whereas a σ-algebra is closed under countable unions and intersections. Essentially, these correspond, respectively, to arbitrary colimits and finite limits versus countable colimits and limits.

Topology lends itself well to categorical analysis and has been studied extensively from this point of view. Its *Gros* aspects are described by sheaf theory and *Petit* aspects by locales. The theory of locales is a particularly rich one (that is not to say that sheaf theory is not). We will study a measure theoretic locale in chapter 3. It is, however, not the obvious one.

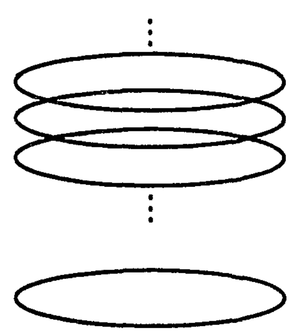
By obvious, we mean in analogy to the topological case. One may form the interior of a subset A of a topological space, X: $A^\circ := \bigcup\{O | O \text{ open and } O \subseteq A\}$. This yields a functor $\mathcal{P}(X) \xrightarrow{(\ )^\circ} \Omega(X)$ which is right adjoint to the inclusion, where $\mathcal{P}(X)$ is the power set of X and $\Omega(X)$ is the locale of open subsets of X. Furthermore, taking the points of a locale is left adjoint to $\Omega(-)$. This basic framework leads to the theory of Stone spaces and Stone's Representation Theorem and a version of Tychonoff's theorem free of the axiom of choice (for a categorical treatment of this subject, see [PTJ2]). An essential element of this theory, indeed the motivational paradigm for the definition of a locale, is the existence of finite limits = intersections and arbitrary colimits = unions and their distributivity. And, as such, the collection of measurable subsets of a measurable space does not form a locale (though, as we shall see, if we mod out by the ideal of the sets of measure zero, we get a locale; perhaps, the reader would consider this as the "obvious" locale to be constructed from a measure space after all). Furthermore, there is not a "measurable interior operator" that can be interpreted as a right adjoint, for if it were, the inclusion would have to preserve arbitrary unions. The difference between arbitrary and countable unions should not be underestimated and, in some sense, this paper is devoted to studying this difference in the context of indexed category theory. For a categorical treatment of indexing by topological spaces, see [Lev].

The difference between **<u>Top</u>** and **<u>Mble</u>** also becomes apparent when one naively translates topological notions into measure theory, encountering "mistakes" of triviality. For example, suppose we translate the notion of homotopy to measure theory by defining a "loop" as a measurable function $l : ([0,1], \mathcal{L}) \longrightarrow (X, \mathcal{A})$ such that $l(0) = l(1)$ (here, and always, when in an obvious context, $\mathcal{L}$ denotes Lebesgue measure) and homotopy in an obvious way. Unfortunately, this definition makes the "fundamental groups" of the disc and the annulus the same ($=\mathbf{1}$). In essence, the difference between "continuous" and "measurable" is that we are allowed to measurably cut a loop but not continuously cut it.

This brings another similar example to mind. In the study of covering spaces, one has the nontrivial spiral over the circle example:



If this is translated into measure theory, this example is trivial; it is simply a product of $\mathbf{Z}$ copies of the circle since we are allowed to measurably cut countably many times:



Now we will explore the similarities between **<u>Top</u>** and **<u>Mble</u>**. First, note that the construction of limits and colimits in **Mble** is analogous to that for **Top**. There is an underlying functor U : **Mble** $\longrightarrow$ **Set** = forget the measurable structure, and we have:

Proposition 1.2.1 *Discrete* $= D \dashv U \dashv I =$ *Indiscreet, where* $D(X) = (X, \mathcal{P}(X))$ *and* $I(X) = (X, \{\emptyset, X\})$ *for* $X \in$ **Set**

Proof: Every function out of a discrete space and every function into an indiscreet space is measurable. ∎

Predictably, since U preserves limits and colimits, this gives us their construction in **Mble**. The limit of a diagram in **Mble** is the limit formed in **Set** together with the coarsest (fewest measurable sets) σ-algebra to make the projections measurable. The colimit of a diagram in **Mble** is the colimit taken in **Set** together with the finest (most measurable sets) σ-algebra to make the injections measurable. For an explicit description of limits and colimits in **Mble**, see [Sch]. We will tacitly assume these descriptions. Note that, for **Dmble**, we have $I_0 \dashv U \dashv D_0$ and a similar construction of limits and colimits applies (here, the subscript notes that the functors have "different" codomain even though they are "the" indiscreet and discrete functors).

Mble (and **Dmble**) is both complete and cocomplete. In fact, using the total opfibrations of [Wd2], **Mble** is seen to be totally cocomplete. That is, $\exists L$ such that $L \dashv Y$, the covariant Yoneda functor: $Y(X) :=$ **Mble**$(-, X)$. **Mble** is also cototally cocomplete: $\exists R$ such that $Z \dashv R$, where $Z(X) :=$ **Mble**$(X, -)$ is the contravariant Yoneda functor. Furthermore, U is both continuous and cocontinuous (for the analogous case of **Top**, see [Wd2] or [Wen, pp.43-47]).

Finally, we note that **Mble**, like **Top** is not a topos, for $(X, \mathcal{A}) \xrightarrow{1} (X, \mathcal{B})$ is an epimorphism and a monomorphism but need not be an isomorphism (topoi are balanced).

It is, of course, important to note at this stage that the above mentioned similarities are essentially a consequence of the fact that the categories **Mble** and **Top** are both topological over **Set** (i.e., the forgetful functors are faithful bifibrations with large-complete fibres; see, for example [AHS, pp. 333- 354]).

In each of the next three sections, we will introduce a category of measure spaces. Recall,

Definition 1.2.3 *A* measure, *μ, on a measurable space $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \longrightarrow \mathbf{R}^{\geq 0}$ such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{i \in N} A_i) = \sum_{i \in N} \mu(A_i)$ for each disjoint collection of measurable sets $\{A_i\}_{i=1}^{\infty}$. A* measure space *is a triple $(X, \mathcal{A}, \mu)$ where $(X, \mathcal{A})$ is a measurable space and μ is a measure on it.* □

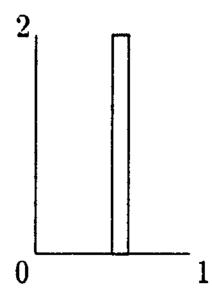
Of course, one may consider extended-real valued measures (or even measures with values in more exotic spaces; vector measures, for example). We will chiefly be concerned with finite measures (as defined above) and, to a lesser extent σ-finite measures (where we allow $\mu(X) = \infty$ but with a countable collection of measurable sets each of finite measure and whose union is X).

Bringing measures into measure theory results in a whole new level of difficulty from a categorical point of view. All the "nice" properties of **Mble** seem to disappear. The problem of major concern as far as indexed category theory is concerned is the disappearance of products. It turns out that, in some sense, the best we can hope for is a monoidal category. As we shall see, we are forced to take a more complex approach to indexing by measure spaces and this complexity is the essence of the difference between topology and measure theory or, more precisely, the difference between continuous families and measurable families.

1.3 Measure Preserving Functions

The first category of measure spaces we shall study is <u>**MP**</u>, the category whose objects are measure spaces and whose morphisms, $(X, \mathcal{A}, \mu) \xrightarrow{f} (Y, \mathcal{B}, \nu)$, are such that f is measurable and measure preserving: $\mu(f^{-1}B) = \nu(B)$, $\forall B \in \mathcal{B}$. Finiteness of measure is not an issue here, so we may have $\mu(X) = \infty$. Notice that <u>**MP**</u> is the conjunction of <u>**IMD**</u> (etymology: inverse measure decreasing), measurable functions such that $\mu(f^{-1}B) \leq \nu(B)$, $\forall B \in \mathcal{B}$ and <u>**IMI**</u> (etymology: inverse measure increasing), measurable functions such that $\mu(f^{-1}B) \geq \nu(B)$, $\forall B \in \mathcal{B}$.

Inasmuch as there are not many examples of measure preserving functions (the identity and $f : [0,1] \longrightarrow [0,8]$; $f(x) = x + 7$ being obvious examples), <u>**MP**</u> is not a very interesting category. Neither this category, nor <u>**IMD**</u> or <u>**IMI**</u>, have products, for consider the "skinny rectangle" suggested by the picture:



with $([0,1] \times [0,2], \lambda \times \lambda) \xrightarrow{proj} ([0,1], \lambda)$ where λ is Lebesgue measure. The measure of the rectangle is not equal to the length of one side.

We will, however, require some terminology, to be introduced here, and some results about isomorphisms; an isomorphism in <u>**Mble**</u> is a function, f, which is one-to-one and onto and such that f and f^{-1} are measurable.

Proposition 1.3.1 *Let* $(X, \mathcal{A}, \mu) \xrightarrow{f} (Y, \mathcal{B}, \nu)$ *be an isomorphism in* <u>**Mble**</u> *(and hence in* <u>**Dmble**</u>*).* $f \in$ <u>**MP**</u> $\Rightarrow f^{-1} \in$ <u>**MP**</u>.

Proof: $\nu(f(B)) = \mu(f^{-1}f(B)) = \mu(B)$. ∎

Proposition 1.3.2 *Let* $(X, \mathcal{A}, \mu) \xrightarrow{f} (Y, \mathcal{B}, \nu)$ *be an isomorphism in* $\underline{\textbf{Mble}}$. $f \in \underline{\textbf{IMD}}$ *and* $f^{-1} \in \underline{\textbf{IMI}} \Rightarrow f \in \underline{\textbf{MP}}$.

Proof: $f \in \underline{\textbf{IMD}} \Rightarrow \mu(f^{-1}(B)) \leq \nu(B)$ and $f^{-1} \in \underline{\textbf{IMI}} \Rightarrow \mu(f^{-1}(B)) \geq \nu(ff^{-1}(B)) = \nu(B)$. ∎

Definition 1.3.1 *A* measure isomorphism *is a measure preserving function which is an isomorphism in* $\underline{\textbf{Mble}}$. □

1.4 Measure Zero Reflecting Functions

1.4.1 Definitions

The next category of measure spaces we introduce involves measure zero reflecting functions:

Definition 1.4.1 *A function* $(X, \mathcal{A}, \mu) \xrightarrow{f} (Y, \mathcal{B}, \nu)$ *is said to be* measure zero reflecting *if it is measurable and if* $\nu(B) = 0 \Rightarrow \mu(f^{-1}B) = 0$. □

Remark: A comment to the term "reflecting". In analogy to "functor reflecting isomorphisms," one might consider measure zero reflecting as being $\nu(f(A)) = 0 \Rightarrow \mu(A) = 0$. For complete measure spaces, these two definitions are equivalent: assume the former and suppose $\nu(f(A)) = 0$. Then $\mu(A) \leq \mu(f^{-1}f(A)) = 0$. Conversely, assume the latter definition and suppose $\nu(B) = 0$. Then, since $ff^{-1}(B) \subseteq B$, $\nu(f^{-1}f(B)) = 0 \Rightarrow \mu(f^{-1}(B))$ as required. □

There is a certain amount of haziness (etymological note: "fuzzy" has already been used) in the mathematician's practical research world. In algebra and category theory, one often hears the phrase "up to isomorphism." In analysis, one hears the phrase "to within ϵ." Now, we do not wish to give the impression that mathematics is a hazy subject, nor do we make any deep philosophical statements about the nature of mathematical research. It is evident, however, that many theorems have a popular statement and a precise statement. As an example of the "to within ϵ" statement for measure theory, consider Littlewood's three principles (see [Roy p.71]) for Lebesgue measure: every measurable set is nearly a union of intervals, every measurable function is nearly continuous, and every convergent sequence of measurable functions is nearly uniformly convergent. These may be "translated" as the following propositions:

Proposition 1.4.1 *[Roy p.62]: Let E be a Lebesgue measurable set. Then given $\epsilon > 0$, there is an open set $O \supset E$ with $m^*(O \setminus E) < \epsilon$ where m^* is (Lebesgue) outer measure.* ∎

Proposition 1.4.2 *[Roy p.72]: (Lusin's Theorem): Let $f : [a,b] \longrightarrow \mathbf{R}$ be measurable. Given $\epsilon > 0$, there is a continuous $\phi : [a,b] \longrightarrow \mathbf{R}$ such that $m\{x|f(x) \neq \phi(x)\} < \epsilon$.* ∎

Proposition 1.4.3 *[Roy p.72]: (Egoroff's Theorem): Let $\langle f_n \rangle$ be a sequence of measurable functions which converge almost everywhere to a real-valued function f on a measurable set E. Then, given $\epsilon > 0$, $\exists A \subset E$ with $m(A) < \epsilon$ such that f_n converges uniformly on $E \setminus A$.* ∎

Notice that, in the previous proposition, almost everywhere convergence arises. In measure theory, the caveat is often "up to a set of measure zero" or "almost everywhere." For example, to say $\int_E f d\mu = 0$ for a positive measurable function is to say $f = 0$ almost everywhere (i.e. $\mu\{x|f(x) \neq 0\} = 0$). Measure zero sets must be considered. In the previous section, we noted that measure preservation is too stringent a requirement for morphisms. It is our contention that measure zero reflecting (abbreviated M0R) is the least requirement for a reasonable category of measure spaces. In chapter three, we will describe a topos and a locale to be constructed from a measure space. For this construction to be functorial, we will require M0R's. In fact, such functions are required whenever one considers the (Boolean) algebraic properties of $\mathcal{A}$ and $\mathcal{N}$, its ideal of measure zero sets(note: in general, we do not require our measure spaces to be complete so when we say $\mathcal{N}$ is downclosed, for example, this means $N \in \mathcal{N},\ A \in \mathcal{A},\ A \subseteq N \Rightarrow A \in \mathcal{N}$ from the monotonicity of the measure). For example, we may define a metric on $\mathcal{A}/\mathcal{N}$ by $d([A],[B]) := \mu(A \triangle B)$ where $\triangle$ denotes the usual symmetric difference of sets (see [Law2] or [A&G p.31] for more on this metric). If $(X,\mathcal{A},\mu) \xrightarrow{f} (Y,\mathcal{B},\nu)$ is M0R, then we have a map $\mathcal{B}/\mathcal{M} \xrightarrow{f^{-1}} \mathcal{A}/\mathcal{N}$. We see that measure zero reflecting is the least requirement for this map to be defined (after which, one may explore various properties of interest to metric space enthusiasts).

In practice, we will be interested in finite measure spaces. The identity is measure zero reflecting and measure zero reflecting functions compose so:

Definition 1.4.2 <u>**M0R**</u> *is the category whose objects are finite measure spaces and whose morphisms are measure zero reflecting functions.* $\square$

We will call two M0R's, $f, g : (X,\mathcal{A},\mu) \longrightarrow (Y,\mathcal{B},\nu)$ *equivalent* if $\mu\{x|f(x) \neq g(x)\} = 0$ and define <u>**M0RE**</u> as <u>**M0R**</u> with morphisms equivalence classes of M0R's (there being, actually, fewer morphisms in <u>**M0RE**</u>). We will most often work with <u>**M0R**</u> but mod out by sets of measure zero when necessary.

1.4.2 Examples

It is time for some examples. In some of the examples below, we will temporarily ignore the finiteness requirement.

Example 1: Let $(X,\mathcal{A},\mu)$, $(Y,\mathcal{B},\nu)$ be two finite measure spaces. The projection $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu) \xrightarrow{p} (X,\mathcal{A},\mu)$ is a M0R for $\mu(A) = 0 \Rightarrow (\mu \times \nu)(p^{-1}A) =$

$\mu(A) \cdot \nu(Y) = 0$ since $\nu(Y) < \infty$. If we use the common convention $\infty \cdot 0 = 0$, we can allow the spaces to have infinite measure. □

At this point, it is necessary to insert some comments about products and completeness of measures. Suppose $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are two σ-finite, complete measure spaces. We can use the Carathéodory procedure to construct the product measure.

Start with the semi-algebra $\mathcal{R}$ of all measurable rectangles, on which there is a measure $(\mu \times \nu)(A \times B) = \mu(A) \cdot \nu(B)$ and form the algebra $\mathcal{R}'$ consisting of finite disjoint unions of these with $\mu \times \nu$ extended in an obvious way. There is an outer measure induced by $\mu \times \nu$ defined by $(\mu \times \nu)^*(E) := \inf \sum_{i=1}^{\infty} (\mu \times \nu)(A_i \times B_i)$, where the infimum is taken over all covers of E by countable families of members of $\mathcal{R}'$ (members of $\mathcal{R}$ is enough). Define a measurable set as an E for which $(\mu \times \nu)^*(E) = (\mu \times \nu)^*(A \cap E) + (\mu \times \nu)^*(A \cap E^c)$ for all A.

This procedure yields a σ-algebra, $\mathcal{A} \otimes \mathcal{B}$, which contains the measurable rectangles and a measure, $\mu \otimes \nu$, which is complete. An important property is:

Proposition 1.4.4 *([Roy, p. 256]): Let μ be a measure on an algebra, $\mathcal{A}$, μ^* the outer measure induced by μ, and E any set. There is a set $B \in \mathcal{A}_{\sigma\delta}$ (countable intersections of countable unions of members of $\mathcal{A}$) with $E \subseteq B$ and $\mu^*(E) = \mu^*(B)$.* ∎

In the case of the product, this means that every $D \in \mathcal{A} \otimes \mathcal{B}$ is of the form $D = E \setminus F$ with $E \in \mathcal{R}_{\sigma\delta}$ and F a subset of a set of measure zero.

If μ and ν are not complete, the Carathéodory procedure still works (provided X and Y are σ-finite) and we get a measure $\mu \times \nu$ on $\mathcal{A} \times \mathcal{B}$, the smallest σ-algebra containing the measurable rectangles ($\mathcal{A} \times \mathcal{B}$ is the product in <u>Mble</u> since the rectangles are generated (as a semi-algebra) by $p_1^{-1}(A)$, $p_2^{-1}(B)$). $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$ is the completion of $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$ (and $\mu \times \nu$ is the restriction of $\mu \otimes \nu$ to $\mathcal{A} \times \mathcal{B}$). We will work with $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$ since $\mathcal{A} \times \mathcal{B}$ is the product in <u>**Mble**</u> and since we do not require the measure spaces to be complete (although they may be). It should be noted however, that neither $\otimes$ nor $\times$ gives the product in <u>**M0R**</u>.

(Counter)example 2: $([0,1], \mathcal{L}, \lambda) \xrightarrow{\delta} ([0,1] \times [0,1], \mathcal{L} \otimes \mathcal{L}, \lambda \otimes \lambda)$ is not M0R. Any subset of the diagonal has (Lebesgue) plane measure zero but may have (Lebesgue) length nonzero. $\square$

Remark: $\delta[0,1]$ is, in fact, an $\mathcal{R}_{\sigma\delta}$ (take intersections of unions of "little squares" that cover the diagonal) so this is also a counterexample for $\mathcal{L} \times \mathcal{L}$ and λ restricted to $\mathcal{R}_{\sigma\delta}$ subsets of the diagonal. $\square$

Example 3: Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(A) = 0$, $\forall A \in \mathcal{A}$ (i.e. $\mu(X) = 0$). Then any measurable function out of X is M0R. $\square$

Example 4: Let $(Y, \mathcal{B}, \nu)$ be a discrete space with counting measure. Then any measurable function into it is M0R since the only set of measure zero is the empty set. $\square$

Example 5: A terminal object of <u>**M0R**</u> is $(1, \mathcal{I}, \iota)$ where $1 = \{\star\}$ is a one point set, $\mathcal{I} = \{\emptyset, \{\star\}\}$, and ι is the counting measure. This follows from example 3 and the fact that $(1, \mathcal{I})$ is a terminal object in <u>**Mble**</u>. $\square$

Example 6: As another "special case" of example 3, consider the measure space, $(\mathbf{N}, \mathcal{P}(\mathbf{N})$,counting), where $\mathbf{N}$ is the set of natural numbers. Now, this space is not finite (it is σ-finite) but any measurable function into it is M0R. In fact a M0R, $(X, \mathcal{A}, \mu) \xrightarrow{f} (\mathbf{N}, \mathcal{P}(\mathbf{N})$, counting), is the same as a measurable partition of X; $\langle f^{-1}(i) \rangle_{i \in N}$. $\square$

Remark: From example 4, we see that, but for finiteness, we would have an adjunction:

$$\underline{\mathbf{Set}} \underset{D}{\overset{U}{\leftrightarrows}} \underline{\mathbf{M0R}}$$

with $U \dashv D$. Notice that, in example 3, we allowed an arbitrary measurable space structure so a left adjoint to the underlying functor does not exist. $\square$

Colimits in **MOR** seem to be more well-behaved than limits:

Proposition 1.4.5 **MOR** *has (a) an initial object given by* $(\emptyset, \{\emptyset\}, 0)$*, (b) binary coproducts, and (c) these coproducts are disjoint.*

Proof: a): There is only one measurable function out of $(\emptyset, \{\emptyset\})$ and it is MOR.
b) The coproduct of $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ is $(X+Y, \mathcal{A}+\mathcal{B}, \mu+\nu)$; $X+Y$ is the disjoint union of X and Y, $\mathcal{A}+\mathcal{B}$ consists of sets of the form $A+B$ where $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $(\mu+\nu)(A+B) := \mu A + \nu B$. It is a simple matter to check that this does indeed define the coproduct. Notice, for example, that if $(X, \mathcal{A}, \mu) \xrightarrow{i} (X+Y, \mathcal{A}+\mathcal{B}, \mu+\nu)$ denotes the injection and $(\mu+\nu)(A+B) = 0$, then $\mu(A) = \nu(B) = 0$ so $\mu(i^{-1}(A+B)) = 0$.
c) Consider the diagram:

$$\begin{tikzcd}
(T, \mathcal{D}, \tau) \arrow[drr, "g"] \arrow[ddr, "f"'] & & \\
& (\emptyset, \{\emptyset\}, 0) \arrow[r, "!_Y"] \arrow[d, "!_X"] & (Y, \mathcal{B}, \nu) \arrow[d, "j"] \\
& (X, \mathcal{A}, \mu) \arrow[r, "i"'] & (X+Y, \mathcal{A}+\mathcal{B}, \mu+\nu)
\end{tikzcd}$$

Now, $j!_Y = i!_X = !_{X+Y}$. Since coproducts are disjoint in **Set**, there are no maps f, g satisfying $jg = if$ (and no map $(T, \mathcal{D}, \tau) \longrightarrow (\emptyset, \{\emptyset\}, 0)$), if $T \neq \emptyset$, and exactly one map, the identity, which is M0R, if $T = \emptyset$. Thus the diagram is a pullback square as required. ∎

(Counter)example 7: Constant functions are not, in general, M0R. Even something as benign as a continuous, one-to-one function is not necessarily M0R (as example 6 shows). □

(Counter)example 8: The "element of" function, $1 \xrightarrow{\lceil x \rceil} X$, is not, in general M0R (unless $x \in X$ is an atom; $\mu(\{x\}) > 0$). □

We think of the category <u>**M0R**</u> as the basic category upon which to build our theory. In the sequel, we will describe the "fibrations" in <u>**M0R**</u>.

<u>**Mble**</u> has products which make it into a monoidal category. The unit is a (fixed) terminal object $(1, 2)$. Now, <u>**M0R**</u> is also a monoidal category. The unit in this case is $(1, 2, \text{counting})$ and the $\otimes$ is $(X, \mathcal{A}, \mu) \otimes (Y, \mathcal{B}, \nu) = (X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$.

Proposition 1.4.6 *Suppose* $(X, \mathcal{A}, \mu) \xrightarrow{f} (Y, \mathcal{B}, \nu)$ *and* $(Z, \mathcal{C}, \rho) \xrightarrow{g} (T, \mathcal{D}, \delta)$ *are in* <u>**M0R**</u>. *Then* $(X \times Y, \mathcal{A} \times \mathcal{C}, \mu \times \rho) \xrightarrow{f \times g} (Y \times T, \mathcal{B} \times \mathcal{D}, \nu \times \delta)$ *is in* <u>**M0R**</u>.

Proof: If $K = B \times D$ is a measurable rectangle, then so is $(f \times g)^{-1}(K) = f^{-1}(B) \times g^{-1}(D)$. Since $(f \times g)^{-1}$ preserves $\cap$, $\cup$, and $\setminus$, $f \times g$ is a measurable function and we need only check that if K is a measurable rectangle of measure zero, so is $(f \times g)^{-1}(K)$. We may assume $\nu(B) = 0$ (the other case is similar). But, $(\mu \times \rho)(f^{-1}(B) \times g^{-1}(D)) = \mu(f^{-1}(B)) \cdot \rho(g^{-1}(D)) = 0$ since $f \in$ <u>**M0R**</u>. ∎

1.5 Disintegrations

1.5.1 Introduction

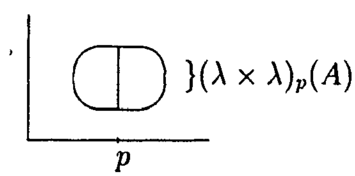
Breitspecher [Bre2] suggests that disintegrations should be studied from a categorical point of view. We now construct a category whose morphisms are "disintegration-like" (we employ the concept of disintegration in a new way). This turns out to be a useful category in the sense that a disintegration is like a family of measure spaces indexed by a measure space and, needless to say, (see [P&S]) this is a good thing as far as indexed category theory is concerned. As we shall see, disintegrations have a forgetful functor to **MOR**.

1.5.2 Naive Theory of Disintegrations

Let $(X, \mathcal{A}, \mu)$ be a measure space and $(P, \mathcal{P}, \rho)$ be another measure space, the *parameter space.* A *disintegration* of μ with respect to ρ is a collection of measures, μ_p, on X,indexed by $p \in P$, such that $\forall A \in \mathcal{A}$, $\mu_p(A)$ is a measurable function of p and $\int_P \mu_p(A) d\rho = \mu(A)$.

Example 1: Constant: Let $\rho(P) = 1$ and let $\mu_p(A) = \mu(A)\ \forall A \in \mathcal{A}$. Then $\mu_p(A)$ is measurable (as a constant function) and $\int_P \mu_p(A) d\rho = \mu(A) \cdot 1 = \mu(A)$ (note: if $\rho(P) \neq 0$, then we can take $\mu_p(A) = \dfrac{\mu(A)}{\rho(P)}$). □

Example 2: Product: Let $X = (\mathbf{R} \times \mathbf{R}, \mathcal{L} \times \mathcal{L}, \lambda \times \lambda)$. Let $P = (\mathbf{R}, \mathcal{L}, \lambda)$. For a measurable $A \subseteq \mathbf{R} \times \mathbf{R}$, put $(\lambda \times \lambda)_p(A) := \lambda(\{y | (p, y) \in A\})$.



Now, by Fubini's theorem (applied to χ_A), $A_p := \{y | (p, y) \in A\}$ is a measurable subset of the real line. and $\int_P \lambda_p(A) d\lambda = (\lambda \times \lambda)(A)$. □

Remarks: 1. In the space $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$, if $D \in \mathcal{A} \times \mathcal{B}$ then $D_x \in \mathcal{B}$, $\forall x \in X$ (fix $x \in X$, let $\mathcal{K}_x$ be the set of all $E \subseteq X \times Y$ such that $E_x \in \mathcal{B}$, then $\mathcal{K}_x$ contains the measurable rectangles and is a σ-algebra, hence contains $\mathcal{A} \times \mathcal{B}$, the smallest σ-algebra that contains the measurable rectangles).

2. Again, we note that $(\mathbf{R} \times \mathbf{R}, \mathcal{L} \times \mathcal{L}, \lambda \times \lambda)$ is not the Lebesgue plane. Fubini's theorem says, for an $A \in \mathcal{L} \otimes \mathcal{L}$, A_p is measurable for almost all $p \in \mathbf{R}$. "Slicing"

by p, however, works for members of $\mathcal{R}_{\sigma\delta}$ (remark 1). The "almost all p" arises out of subsets of sets of measure zero (i.e. during the completion part of the process and not before) so $(\lambda \otimes \lambda)_p(A) = \lambda(A_p)$ would provide an "almost everywhere" example of a disintegration. $\square$

This is an important example for our purposes, as will be seen below. We will describe many more examples in a later section. Given two measure spaces, one doesn't necessarily possess a disintegration with respect to the other. The main thrust of research in this field is to determine conditions for the existence of such. A definitive answer has not yet been given although there are some important existence theorems (see [T&T]).

1.5.3 The Category <u>Disint</u>

An object of **<u>Disint</u>** is a finite measure space. We will use the projection from the product as suggested by example 2 above as the motivation for our notion of morphism.

Definition 1.5.1 *A* morphism $(X, \mathcal{A}, \mu) \longrightarrow (Y, \mathcal{B}, \nu)$ *in* **<u>Disint</u>** *consists of*

- $f : (X, \mathcal{A}) \longrightarrow (Y, \mathcal{B}) \in$ **<u>Mble</u>**
- *a family* $(X_y, \mathcal{A}_y, \mu_y)_{y \in Y}$ *of finite measure spaces, where* $X_y := f^{-1}(y)$ *and* $\mathcal{A}_y = \{A \cap f^{-1}(y) \mid A \in \mathcal{A}\}$

subject to the axioms:

ax1: $\forall A \in \mathcal{A}$, *the map* $y \mapsto \mu_y(A \cap f^{-1}(y))$ *is measurable and bounded*

ax2: $\forall A \in \mathcal{A},\ \mu(A) = \int_Y \mu_y(A \cap f^{-1}(y))d\nu(y)$. $\square$

Remarks: 1. $\mathcal{A}_y = \{A \cap f^{-1}(y) \mid A \in \mathcal{A}\}$ is a σ-algebra (this follows immediately from the fact that $\mathcal{A}$ is a σ-algebra).

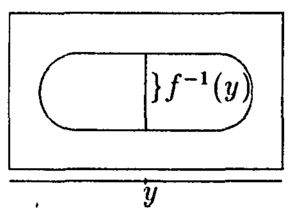
2. We call these morphisms *disintegrations* as well and will refer to axiom 1 as "measure boundedness."

3. For boundedness, it is enough to say $\mu_y(X \cap f^{-1}(y)) \in L^\infty(Y)$ because of monotonicity of measures (of course, the measurability condition for all $A \in \mathcal{A}$ is still necessary).

4. Every disintegration has a "norm" via $\|\mu_y(A \cap f^{-1}(y))\|_\infty$. We will not explore this in this paper.

5. Each $\mu_y(X_y) < \infty$. Measure boundedness is a condition on the $\mu_y(X_y)$'s over $y \in Y$. $\square$

Since the paradigm for a morphism of **<u>Disint</u>** is the product example above, we think of the fibres over the y's as slicing up A:



Notation: The fibre measurable spaces depend solely upon f. We write $(f, (\mu_y)_{y \in Y})$ or (f, μ_y) for a morphism in **<u>Disint</u>**. $\square$

1.5.4 Category Axioms

Identity: Define the identity in **<u>Disint</u>** as $(X, \mathcal{A}, \mu) \xrightarrow{(1_X, \iota_x)} (X, \mathcal{A}, \mu)$ where 1_X is the identity function and ι_x is counting measure on $\mathcal{I}_x = \{A \cap 1^{-1}(x) \mid A \in \mathcal{A}\}$, the discrete σ-algebra on $\{x\}$.

Axiom 1: $x \mapsto \iota_x(A \cap \{x\})$ is measurable and bounded since it is just χ_A and A is a measurable set. $\square$

Axiom 2: $\int_X \iota_x(A \cap \{x\})d\mu(x) = \int_X \chi_A d\mu(x) = \mu(A)$ as required. $\square$

Composition: Consider the diagram:

$$\begin{tikzcd} & (Y,\mathcal{B},\nu) \arrow[dr, "(g,\nu_z)"] & \\ (X,\mathcal{A},\mu) \arrow[ur, "(f,\mu_y)"] \arrow[rr, "(gf,\theta_z)"'] & & (Z,\mathcal{C},\rho) \end{tikzcd}$$

where θ_z is defined as:

$$\theta_z(E) := \int_{g^{-1}(z)} \mu_y(E \cap f^{-1}(y))d\nu_z(y) \text{ for } E \in \mathcal{E}_z = \{A \cap f^{-1}g^{-1}(z) \mid A \in \mathcal{A}\}.$$

Note that ν_z is defined on $g^{-1}(z)$ and $f^{-1}g^{-1}(z) = \bigcup_{y \in g^{-1}(z)} f^{-1}(y)$, the union being disjoint and $A \cap f^{-1}g^{-1}(z) \cap f^{-1}(y) = \begin{cases} A \cap f^{-1}(y) & y \in g^{-1}(z) \\ \emptyset & y \notin g^{-1}(z). \end{cases}$

Axiom 1: We wish to show that $\theta_z(E) = \int_{g^{-1}(z)} \mu_y(E \cap f^{-1}(y))d\nu_z(y)$ is a measurable function of z. Before we do that, however, we must determine that the integral makes sense.

Proposition 1.5.1 *For each $z \in Z$ and for each $E \in \mathcal{A}$, $\mu_y(E \cap f^{-1}(y))$ is a ν_z-measurable function.*

Proof: $\mu_y(E \cap f^{-1}(y))$ is a ν-measurable function (by axiom 1 for the μ_y's). Let $\alpha \in \mathbf{R}$, then $B = \{y \in Y \mid \mu_y(E \cap f^{-1}(y)) < \alpha\} \in \mathcal{B}$ and $B \cap g^{-1}(z) = \{y \in g^{-1}(z) \mid \mu_y(E \cap f^{-1}(y)) < \alpha\} \in \mathcal{B}_z$ for all $z \in Z$. $\blacksquare$

Proposition 1.5.2 *$z \mapsto \int_{g^{-1}(z)} k(y)d\nu_z(y)$ is a measurable function of z for $k(y)$ a non-negative ν-measurable function (in particular for $k(y) = \mu_y(E \cap f^{-1}(y))$)*

Proof: Case $k = \chi_E$, $E \in \mathcal{B}$: $z \mapsto \int_{g^{-1}(z)} \chi_E d\nu_z(y) = \nu_z(E \cap g^{-1}(z))$ which is z-measurable by axiom 1 for ν_z.

Case $k =$ a simple function: Apply the above case and linearity of the integral.

Case $k =$ a non-negative measurable function. Let $\langle \phi_n(y) \rangle$ be a sequence of simple functions increasing to k. Then $z \mapsto \int_{g^{-1}(z)} k(y) d\nu_z(y) = \int_{g^{-1}(z)} \lim \phi_n(y) d\nu_z(y)$ $= \lim \int_{g^{-1}(z)} \phi_n(y) d\nu_z(y)$ by the monotone convergence theorem. Each $z \mapsto \int_{g^{-1}(z)} \phi_n(y) d\nu_z$ is z-measurable by the above case and the limit of a sequence of measurable functions is measurable. ∎

Remark: The technique used in the above proposition is a very useful one. We will use the "build it up from simple functions" idea in many of our results. □

Proposition 1.5.3 *θ_z is a measure for each z.*

Proof: $\theta_z(\emptyset) = \int_{g^{-1}(z)} \mu_y(\emptyset \cap f^{-1}(y)) d\nu_z(y) = \int_{g^{-1}(z)} 0 d\nu_z(y) = 0$.

$$\begin{aligned}
\theta_z(\dot{\bigcup} A_i \cap f^{-1}g^{-1}(z)) &= \int_{g^{-1}(z)} \mu_y(\dot{\bigcup} A_i \cap f^{-1}(y)) d\nu_z(y) \\
&= \int_{g^{-1}(z)} \sum_i \mu_y(A_i \cap f^{-1}(y)) d\nu_z(y) \\
&= \sum_i \int_{g^{-1}(z)} \mu_y(A_i \cap f^{-1}(y)) d\nu_z(y) \\
&= \sum_i \theta_z(A_i \cap f^{-1}g^{-1}(z)). \quad ∎
\end{aligned}$$

Proposition 1.5.4 *θ_z is a bounded function (over $z \in Z$).*

Proof: Certainly, $\theta_z(A \cap f^{-1}g^{-1}(z)) = \int_{g^{-1}(z)} \mu_y(A \cap f^{-1}(y)) d\nu_z(y)$ is finite for all $z \in Z$ (since $\mu_y(A \cap f^{-1}(y))$ is bounded and ν_z is a finite measure). Furthermore, suppose ν_z and μ_y are bounded by K and M respectively, say. Then $\int_{g^{-1}(z)} \mu_y(A \cap f^{-1}(y)) d\nu_z(y) \leq \int_{g^{-1}(z)} K d\nu_z(y) \leq M \cdot K < \infty$. ∎

Axiom 2:

Proposition 1.5.5 $\mu(A) = \int_Z \theta_z(A \cap f^{-1}g^{-1}(z))d\rho(z)$ *(axiom 2).*

Proof: By axiom 2 for the μ_y's, we have $\mu(A) = \int_Y \mu_y(A \cap f^{-1}(y))d\nu(y) = (*)$. Now, $\int_Z \theta_z(A \cap f^{-1}g^{-1}(z))d\rho(z) = \int_Z \int_{g^{-1}(z)} \mu_y(A \cap f^{-1}(y))d\nu_z(y)d\rho(z) = (**)$ Thus, we must show $(*) = (**)$. We will show that $\int_Y k(y)d\nu(y)$ $= \int_Z \int_{g^{-1}(z)} k(y)d\nu_z(y)d\rho(z)$ for all (positive) measurable functions $k(y)$.
Case $k(y) = \chi_E$, $E \in \mathcal{B}$: $\int_Y \chi_E d\nu(y) = \nu(E)$ and $\int_Y \int_{g^{-1}(z)} \chi_E d\nu_z(y)d\rho(z)$ $= \int_Y \nu_z(E \cap g^{-1}(z))d\rho = \nu(E)$ by axiom 2 for ν_z.
Case $k(y) =$ a simple function: $\int = \int\int$ by linearity of the integral and the above case.

Case $k(y) =$ a positive measurable function: Let $\phi_n \uparrow k(y)$ be a sequence of simple functions increasing to k. Then $\int_Z \int_{g^{-1}(z)} k(y)d\nu_z(y)d\rho(z) =$
$\int_Z \int_{g^{-1}(z)} k(y)d\nu_z(y)d\rho(z) = \int_Z \int_{g^{-1}(z)} \lim \phi_n(y)d\nu_z(y)d\rho(z)$
$= \int_Z \lim \int_{g^{-1}(z)} \phi_n(y)d\nu_z(y)d\rho(z) = \lim \int_Z \int_{g^{-1}(z)} \phi_n(y)d\nu_z(y)d\rho(y) = \clubsuit$, by repeated application of the monotone convergence theorem. Now, by the above case, $\clubsuit = \lim \int_Y \phi_n(y)d\rho(z)$ and applying the monotone convergence theorem again, we have $\clubsuit = \int_Y k(y)d\nu$. ∎

Unit laws: Consider:

$$\begin{tikzcd}
 & (Y, \mathcal{B}, \nu) \arrow[dr, "{(1_Y, \iota_y)}"] & \\
(X, \mathcal{A}, \mu) \arrow[ur, "{(f, \mu_y)}"] \arrow[rr, "{(1_Y \circ f, \theta_y)}"'] & & (Y, \mathcal{B}, \nu)
\end{tikzcd}$$

Now, $\theta_y(E \cap f^{-1}1^{-1}(y)) = \int_{1^{-1}(y)} \mu_y(E \cap f^{-1}(y))d\iota_y(x) = \mu_y(E \cap f^{-1}(y))$.

In a similar way, write $(f \circ 1_X, \gamma_y) = (f, \mu_y) \circ (1_X, \iota_x)$. Then $\gamma_y(E \cap 1^{-1}f^{-1}(y))$ $= \int_{f^{-1}(y)} \iota_x(E \cap 1^{-1}(x); d\mu_y(x) = \int_{f^{-1}(y)} \chi_E d\mu_y(x) = \mu_y(E \cap f^{-1}(y))$ as required. □

Associativity: Consider the diagram:

$$
\begin{tikzcd}
 & (Y,\mathcal{B},\nu) \arrow[rr, "{(hg,\rho_t)}"] \arrow[dr, "{(g,\nu_z)}"] & & (T,\mathcal{D},\delta) \\
(X,\mathcal{A},\mu) \arrow[ur, "{(f,\mu_y)}"] \arrow[rr, "{(gf,\theta_z)}"'] \arrow[urrr, bend left=20, "{(hgf,\eta_t)}"] \arrow[urrr, bend left=5, "{(hgf,\gamma_t)}"'] & & (Z,\mathcal{C},\rho) \arrow[ur, "{(h,\rho_t)}"'] &
\end{tikzcd}
$$

To prove associativity, we must show $\eta_t = \gamma_t$ for all $t \in T$. But,

$$
\begin{aligned}
\eta_t(F) &= \int_{h^{-1}(t)} \theta_z(F \cap f^{-1}g^{-1}(z))d\rho_t(z) \\
&= \int_{h^{-1}(t)} \int_{g^{-1}(z)} \mu_y(F \cap f^{-1}(y))d\nu_z(y)d\rho_t(z) \\
\text{and } \gamma_t(F) &= \int_{g^{-1}h^{-1}(t)} \mu_y(F \cap f^{-1}(y))d\beta_t(y).
\end{aligned}
$$

and we have:

Proposition 1.5.6 $\int_{g^{-1}h^{-1}(t)} k(y)d\beta_t(y) = \int_{h^{-1}(t)} \int_{g^{-1}(z)} k(y)d\nu_z(y)d\rho_t(z)$ *for all positive, measurable functions* $k(y)$.

Proof: Apply the proof of proposition 1.5.5 with $Y := g^{-1}h^{-1}(t)$, $Z := h^{-1}(t)$, $\nu := \beta_t$, and $\rho := \rho_t$. ∎

1.5.5 Examples and Basic Properties

Proposition 1.5.7 $(X,\mathcal{A},\mu) \xrightarrow{(f,\mu_y)} (Y,\mathcal{B},\nu) \in$ <u>Disint</u> $\Rightarrow f \in$ <u>MOR</u>

Proof: Let $\nu(B) = 0$. Then we have:

$$\begin{aligned}\mu(f^{-1}B) &= \int_Y \mu_y(f^{-1}B \cap f^{-1}y)d\nu(y)\\ &= \int_Y \mu_y(f^{-1}(B \cap \{(y)\}))d\nu(y)\\ &= \int_B \mu_y(f^{-1}(y))d\nu(y) = 0. \quad \blacksquare\end{aligned}$$

Remark: In view of this proposition and counterexample 6 above, we see that the diagonal is not in **Disint**. □

Example 1: Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two finite measure spaces. Define $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu) \xrightarrow{(p,(\mu\times\nu)_x)} (X, \mathcal{A}, \mu)$ as follows: p is the projection onto the first factor, $p^{-1}(x) = \{x\} \times Y$ and $(\mathcal{A} \times \mathcal{B})_x = \{D \cap p^{-1}(x) \mid D \in \mathcal{A} \times \mathcal{B}\} = \{x\} \times \mathcal{B}$ (certainly, we have $\supseteq$ (take $D = \{x\} \times B$); conversely, for $D = A \times B \in \mathcal{A} \times \mathcal{B}$, $D \cap p^{-1}(x) = \begin{cases} \{x\} \times B & x \in A \\ \emptyset & x \notin A \end{cases}$ both of which are in $\{x\} \times \mathcal{B}$ and since $\{x\} \times \mathcal{B}$ is a σ-algebra, we have $\subseteq$). So, define $(\mu \times \nu)_x(D \cap p^{-1}(x)) := \nu(D_x)$ where D_x is considered as an element of $\mathcal{B}$. We have already noted that the slices D_x are all measurable. Axioms 1 and 2 follow from:

Lemma 1.5.1 *([Roy, p. 266]): Let E be an $\mathcal{R}_{\sigma\delta}$ with $(\mu \times \nu)(E) < \infty$. Then the function $g(x)$ defined by $g(x) = \nu(E_x)$ is a measurable function of x and* $\int g(x)d\mu(x) = (\mu \times \nu)(E)$. ∎

Note that $g(x)$ is bounded by $\nu(Y) < \infty$. As mentioned above, we consider this example as the motivating one. □

Example 2: Let A_0 be a measurable subset of $X = (X, \mathcal{A}, \mu)$. We may interpret the inclusion $(A_0, \mathcal{A}_0, \mu_0) \xrightarrow{\imath} (X, \mathcal{A}, \mu)$, where $\mathcal{A}_0 = \{A \subseteq A_0 | A \in \mathcal{A}\}$ and $\mu_0(A) = \mu(A)$, as a disintegration. If $x \in A_0$, $\mathcal{I}_x = \{A \cap \imath^{-1}(x) \mid A \in \mathcal{A}\} = \{\emptyset, \{x\}\}$; put μ_{0x} = counting measure. If $x \notin A_0$, $\mathcal{I}_x = \{\emptyset\}$; put $\mu_{0x} = 0$. So,

$\mu_{0x}(A \cap i^{-1}(x)) = \chi_{A \cap A_0}$. The proof that axioms 1 and 2 hold is exactly the same as that for the identity disintegration. □

Remark: This example does not "contradict" the fact that the diagonal is not a disintegration. Interpreting the diagonal as a *subspace* of the plane would give $(X, \mathcal{A}, 0) \longrightarrow (X \times X, \mathcal{A} \times \mathcal{A}, \mu \times \mu)$. □

Example 3: Let $(X, \mathcal{A}, \mu)$ be such that $\mu(A) = 0$, $\forall A \in \mathcal{A}$. Then any measurable function $(X, \mathcal{A}, \mu) \xrightarrow{f} (Y, \mathcal{B}, \nu)$ may be interpreted as a disintegration by defining $\mu_y(A \cap f^{-1}(y)) = 0$, for all $A \in \mathcal{A}$ and $y \in Y$. □

Example 4: A terminal object of <u>**Disint**</u> is (1, 2, counting). The unique map, $(X, \mathcal{A}, \mu) \xrightarrow{!_X} (1, 2, \text{counting})$ has $(X_\star, \mathcal{A}_\star) = (X, \mathcal{A})$ and $\mu_\star = \mu$. Suppose $(X, \mathcal{A}, \mu) \xrightarrow{(!, \beta_\star = \beta)} (1, 2, \text{counting})$ is another disintegration. By axiom 2 for $\beta_\star$, we have $\mu(A) = \int_\star \beta_\star(A \cap !^{-1}(\star)) d(\text{counting}) = \beta(A)$. □

Example 5: The initial object of <u>**Disint**</u> is $(\emptyset, \{\emptyset\}, 0)$, which is a special case of example 2. □

Proposition 1.5.8 <u>Disint</u> *has (a) binary coproducts and (b) these coproducts are disjoint.*

Proof: a) Referring to proposition 1.4.5, the injection is a disintegration $(X, \mathcal{A}, \mu) \xrightarrow{(\iota, \mu_t)} (X + Y, \mathcal{A} + \mathcal{B}, \mu + \nu)$, with $\mathcal{A}_t = \{\emptyset\}$ and $\mu_t = 0$ if $t \in Y$ and $\mathcal{A}_t = \{\emptyset, \{t\}\}$ and $\mu_t(A \cap i^{-1}(t)) = \chi_A(t)$ if $t \in X$.

b) Again, referring to the diagram of proposition 1.4.5. If $T = \emptyset$, then we may insert the identity disintegration, $T \longrightarrow \emptyset$. If $T \neq \emptyset$, then there is no map, $T \longrightarrow \emptyset$ and no maps making the "outside" square commute. ∎

Example 6: Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two finite, discrete spaces. That is, X and Y are finite sets, $\mathcal{A} = 2^X$, $\mathcal{B} = 2^Y$, and μ and ν are counting measures. Every function, $X \xrightarrow{f} Y$, is measurable. Let $(X, \mathcal{A}, \mu) \xrightarrow{(f, \mu_y)} (Y, \mathcal{B}, \nu)$ be a disintegration. $\mathcal{A}_y = \{A \cap f^{-1}(y) \mid A \in \mathcal{A}\} = 2^{f^{-1}(y)}$ for all $y \in Y$. To satisfy axiom 2, μ_y

must be counting measure. And, such will automatically satisfy axiom 1. Thus, every measurable function yields a unique disintegration. That is, there is a full functor

$$\underline{\mathbf{Set}}_f \xrightarrow{D} \underline{\mathbf{Disint}}. \ \square$$

At the end of section 1.4, we have described the $\otimes$ for $\underline{\mathbf{MOR}}$; more precisely, we have interpreted the product of measure spaces as $\otimes$ in $\underline{\mathbf{MOR}}$. We now consider the case of $\underline{\mathbf{Disint}}$. Let $(X, \mathcal{A}, \mu) \xrightarrow{(f,\mu_y)} (Y, \mathcal{B}, \nu)$ and $(Z, \mathcal{C}, \rho) \xrightarrow{(g,\rho_t)} (T, \mathcal{D}, \delta)$ be in $\underline{\mathbf{Disint}}$ and form $f \otimes g$ in $\underline{\mathbf{MOR}}$:
$(X \times Z, \mathcal{A} \times \mathcal{C}, \mu \times \rho) \xrightarrow{f \times g} (Y \times T, \mathcal{B} \times \mathcal{D}, \nu \times \delta)$. We may make $f \times g$ into a disintegration as follows: $(\mathcal{A} \times \mathcal{C})_{(y,t)} = \{D \cap f^{-1}(y) \times g^{-1}(t) \mid D \in \mathcal{A} \times \mathcal{C}\} = \mathcal{A}_y \times \mathcal{C}_t$ (since $A \times C \cap f^{-1}(y) \times g^{-1}(t) = A \cap f^{-1}(y) \times C \cap g^{-1}(t)$ and since these are the generators for the σ-algebras, they are equal). So, define $(\mu \times \rho)_{(y,t)}(D) = (\mu_y \times \rho_t)(D)$ with D considered as an element of $\mathcal{A}_y \times \mathcal{C}_t$.

Proposition 1.5.9 *$\mu_y \times \rho_t$ satisfies axioms 1 and 2.*

Proof: Axiom 1: $k(y,t) = (\mu_y \times \rho_t)(D \cap f^{-1}(y) \times g^{-1}(t))$ is measurable and bounded:

If $D = A \times C$ is a measurable rectangle, then $k(y,t) = \mu_y(A \cap f^{-1}(y)) \cdot \rho_t(C \cap g^{-1}(t))$ is measurable and bounded since it is a product of two such (axiom 1 for μ_y and ρ_t). Furthermore, $k(y,t) \leq (\mu_y \times \rho_t)(X \times Z \cap f^{-1}(y) \times g^{-1}(t)) < \infty$. That is, k is bounded for any D. We need only check that it is measurable.

If $D = \bigcup_{i=1}^{\infty} A_i \times C_i$ is a disjoint union of rectangles, then $k(y,t)$ $= \sum_{i=1}^{\infty} \mu_y(A_i \cap f^{-1}(y)) \cdot \rho_t(C_i \cap g^{-1}(t))$ is a sum of measurable functions so is measurable. Now an arbitrary (countable) union can be written as a disjoint (countable) union (for example, for $\bigcup_{i=1}^{\infty} A_i$, put $B_i = A_i \setminus \bigcup_{j<i} A_j$ then $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$ and

the B_i's are disjoint). For finite intersections, use, for γ a finite measure, $\gamma(D_1 \cap D_2) = \gamma(D_1) + \gamma(D_2) - \gamma(D_1 \cup D_2)$. Finally, for countable intersections, use $\gamma(\bigcap_{i=1}^{\infty} D_i) = \lim_{n\to\infty} \gamma(\bigcap_{i=1}^{n} D_i)$ (with $\gamma(D_1) < \infty$).

Axiom 2: Again, the process is exactly as for axiom 1 (disjoint unions use additivity; increasing limits and $\sum$'s pull out of integrals by the monotone convergence theorem). We only check the basic case, $D = A \times C$:

$$\begin{aligned}
&\int (\mu_y \times \rho_t)(A \times C \cap f^{-1}(y) \times g^{-1}(t))d(\nu \times \delta)(y,t) \\
&= \int \mu_y(A \cap f^{-1}(y))d\nu(y) \cdot \int \rho_t(C \cap g^{-1}(t))d\delta(t) \\
&= \mu(A) \cdot \rho(C) \\
&= (\mu \times \rho)(A \times C)
\end{aligned}$$

(the first equality is Fubini's theorem and the second equality is axiom 2 for μ_y and ρ_t). ∎

Certainly, in $(X \times X, \mathcal{A} \times \mathcal{A}, \mu \times \mu) \xrightarrow{(1\times 1, (\iota \times \iota)_{(x,y)})} (X \times X, \mathcal{A} \times \mathcal{A}, \mu \times \mu)$, we have $(\iota \times \iota)_{(x,y)} = \iota_x \times \iota_y = \iota_{(x,y)}$. Now, suppose

$$\begin{tikzcd}
 & (Y, \mathcal{B}, \nu) \arrow[dl, "{(f, \mu_y)}"'] \arrow[dr, "{(g, \nu_z)}"] & \\
(X, \mathcal{A}, \mu) \arrow[rr, "{(gf, \gamma_z)}"'] & & (Z, \mathcal{C}, \rho)
\end{tikzcd}$$

and

$$\begin{tikzcd}
& (M,\mathcal{E},\eta) \arrow[dl, "{(h,\delta_m)}"'] \arrow[dr, "{(k,\eta_n)}"] & \\
(L,\mathcal{D},\delta) \arrow[rr, "{(kh,\beta_n)}"'] & & (N,\mathcal{F},\tau)
\end{tikzcd}$$

denote two compositions in **Disint** and consider:

$$\begin{tikzcd}
& (Y \times M, \mathcal{B} \times \mathcal{E}, \mu \times \eta) \arrow[dl, "{(f \times h, (\mu \times \delta)_{(y,m)})}"'] \arrow[dr, "{(g \times k, (\nu \times \eta)_{(z,n)})}"] & \\
(X \times L, \mathcal{A} \times \mathcal{D}, \mu \times \delta) \arrow[rr, shift left, "{(g \times f \circ f \times h, \alpha_{(z,n)})}"] \arrow[rr, shift right, "{(gf \times kh, \gamma_z \times \beta_n)}"'] & & (Z \times N, \mathcal{C} \times \mathcal{F}, \rho \times \tau)
\end{tikzcd}$$

where $\alpha_{(z,n)} = (\nu \times \eta)_{(z,n)} \circ (\mu \times \delta)_{(y,m)}$. We show that $\alpha_{(z,n)} = \gamma_z \times \beta_n$ on the generators of $(\mathcal{A} \times \mathcal{D})_{(z,n)} = \mathcal{A}_z \times \mathcal{D}_n$.

$$\begin{aligned}
&\alpha_{(z,n)}(A \times D \cap (g \times k \circ f \times h)^{-1}(z,n) \\
&= \int_{(g \times k)^{-1}(z,n)} (\mu \times \delta)_{(y,m)}(A \times D \cap (f \times h)^{-1}(y,m)) d(\nu \times \eta)_{(z,n)}(y,m) \\
&= \int_{g^{-1}(z)} \int_{k^{-1}(n)} (\mu \times \delta)_{(y,m)}(A \times D \cap (f \times h)^{-1}(y,m)) d\nu_z(y) d\eta_n(m) \\
&= \int_{g^{-1}(z)} \mu_y(A \cap f^{-1}(y)) d\nu_z(y) \cdot \int_{k^{-1}(n)} \delta_m(D \cap h^{-1}(m)) d\eta_n(m) \\
&= \gamma_z(A \cap f^{-1}g^{-1}(z)) \cdot \beta_n(D \cap h^{-1}k^{-1}(n)) \\
&= (\gamma_z \times \beta_n)(A \cap f^{-1}g^{-1}(z) \times D \cap h^{-1}k^{-1}(n)).
\end{aligned}$$

And so, we have shown that $\otimes$ is a bifunctor. It is a straightforward matter to check the $(1, 2, \text{counting})$ is the unit for this tensor. Thus,

Proposition 1.5.10 *($\underline{\mathbf{Disint}}, \otimes, (1, 2, counting)$) is a monoidal category.* ∎

1.5.6 Slice Categorical Examples

Let $(X, \mathcal{A}, \mu) \in \underline{\mathbf{Disint}}$. Then $\underline{\mathbf{Disint}}/X$ denotes the slice category of disintegrations over X. We give a list of examples of objects of $\underline{\mathbf{Disint}}/X$ and of disintegrations over specific objects (these examples will be useful in chapter 4). Proofs are omitted since they follow from general slice categorical nonsense.

Example 1: The terminal object of $\underline{\mathbf{Disint}}/X$ is $(X, \mathcal{A}, \mu) \xrightarrow{(1, \iota_x)} (X, \mathcal{A}, \mu)$, the identity. □

Example 2: The initial object of $\underline{\mathbf{Disint}}/X$ is the inclusion of the empty set: $(\emptyset, \{\emptyset\}, 0) \xrightarrow{(!_X, 0_x)} (X, \mathcal{A}, \mu)$ with $\{\emptyset\}_x = \{\emptyset\}$ and $0_x = 0$ for all $x \in X$. □

Example 3: More generally, any measurable subset $A \subseteq X$, with the inclusion, gives an object of $\underline{\mathbf{Disint}}/X$. □

Example 4: Examples 1 and 2 are special cases of $X \times I \xrightarrow{proj} X$ where I is a discrete space. □

Proposition 1.5.11 .

a) $\underline{\mathbf{Disint}}/\emptyset \simeq \mathbf{1}$

b) $\underline{\mathbf{Disint}}/1 \simeq \underline{\mathbf{Disint}}$

c) $\underline{\mathbf{Disint}}/2 \simeq \underline{\mathbf{Disint}} \times \underline{\mathbf{Disint}}$

d) $\underline{\mathbf{Disint}}/(X+Y) \simeq \underline{\mathbf{Disint}}/X \times \underline{\mathbf{Disint}}/Y$

e) $\underline{\mathbf{Disint}}/\mathbf{N} \simeq \prod_{N} \underline{\mathbf{Disint}}$. ∎

Remarks: 1. c) is a special case of d).

2. Strictly speaking, $\mathbf{N} \notin \underline{\mathbf{Disint}}$ but e) works nonetheless. □

Chapter 2

Measurable Fields of Hilbert Spaces

2.1 Introduction

We have given the necessary measure theoretic background. Specifically, we have some useful categories of measure spaces upon which to base our indexing ideas. In this chapter, we will begin to glue (in a non-technical sense) the three elements measure theory, operator theory, and indexed category theory together. Indeed, in this chapter, we concern ourselves with box 1 of the diagram in the introduction.

We must, however, fill in more background and, in the next two sections, we first provide a brief outline of direct integral theory (to fix notations and set definitions), and then a brief outline of indexed category theory. One final remark: in our categorical analysis, it is best to construct the direct integral in stages. That is to say, we will gradually introduce more and more elements of category theory into the construction.

2.2 Measurable Fields

2.2.1 Motivation

In this section, we motivate the direct integral of Hilbert spaces. We will draw on some folklore about unitary group representations and the (related) decomposition of a Hilbert space with respect to an algebra. "Folklore" is perhaps not a completely accurate term, for the results we give here are well documented in the literature (the principle texts to which we refer the reader for details are [Dix1], [Dix2], [Nai], and [Nie]). However, "folklore" may be assumed by the reader to accurately describe the style of the (short) exposition we give here.

A. Unitary Group Representations

Definition 2.2.1 *Let G be a topological group and H a Hilbert space. $U(H)$ denotes the group of unitary operators in H (u is unitary if $u^*u = 1_H = uu^*$). A* continuous unitary representation *of G on H is a continuous (in the strong topology) function $G \xrightarrow{u} U(H)$ such that $u(gh) = u(g)u(h)$ and $u(e) = 1_H$.* □

Remarks: 1. The condition $u(gh) = u(g)u(h)$ implies $u(e) = 1_H$.
2. Continuous in the strong topology means: for each $\xi \in H$, the function $g \mapsto u(g)\xi$ is continuous with respect to the norm topology for H and the given topology for G. □

Let u be a continuous representation. The set $u(G)$ is not linear since the sum of two unitaries is not necessarily a unitary. But, it is a group with respect to multiplication of operators. Furthermore, $u(g)^* = u(g)^{-1} = u(g^{-1})$. That is, $u(G)$ is self adjoint (= symmetric in the terminology of [Nai]).

Proposition 2.2.1 *Let $A \subseteq L(H)$. A subspace $M \leq H$ is A-invariant iff $M^\perp$ is A^*-invariant.*

Proof: $\Rightarrow$: Let $a \in A$ and consider $a^* \in A^*$ and $\xi \in M^\perp$. Now, $\langle a^*\xi|\eta\rangle = \langle\xi|a\eta\rangle = 0$, for all $\eta \in M$ since a is M-invariant. And so, $M^\perp$ is A^*-invariant. ∎

Since $u(G)$ is self adjoint, if $M \leq H$ is $u(G)$-invariant, so is $M^\perp$ and we write $H = M \oplus M^\perp$. So we must have $u(g) = \begin{pmatrix} u_1(g) & 0 \\ 0 & u_2(g) \end{pmatrix}$ with $u_1(g)$ and $u_2(g)$ unitary projections onto the subspaces M and $M^\perp$ respectively. And so, to study u, it is better to study u_1 and u_2 instead. More accurately, one should look at representations which do not have (nontrivial) invariant subspaces.

Definition 2.2.2 *u is said to be* irreducible *if $M \leq H$ $u(G)$-invariant implies $M = 0$ or $M = H$.* □

Proposition 2.2.2 *([Dix2, p.35]): u is irreducible iff $u(G)' = \mathbf{C} \cdot 1_H$ ($u(G)'$ denotes the commutant of $u(G)$, the set of all operators in $L(H)$ which commute with everything in $u(G)$; and $\mathbf{C} \cdot 1_H$ denotes the scalar operators on H).* ∎

Now, suppose G is Abelian so that $u(G) \subseteq u(G)'$ $(u(g)u(h) = u(gh) = u(hg) = u(h)u(g))$. By the above proposition, we have $\mathbf{C} \cdot 1_H = u(G)' \supseteq (\mathbf{C} \cdot 1_H)' = L(H)$. That is $L(H) \subseteq \mathbf{C} \cdot 1_H$ which implies $H = \mathbf{C}$ or "irreducible representations of an Abelian group are all one dimensional."

Notation: $\hat{G}$ denotes the set of irreducible representations of the Abelian group G; $\hat{G}$ is called the *dual group* of G. □

Example 1: Suppose $G = \mathbf{Z}$. Let $u(1) = z_0 \in U(\mathbf{C})$ (which means $|z_0| = 1$). Then u is completely determined, for $u(n) = z_0^n$. Thus, $\hat{\mathbf{Z}} \xrightarrow{\sim} \mathbf{T}$, $u \mapsto u(1) = z_0$, where $\mathbf{T}$ denotes the circle group in the complex plane. □

Example 2: Suppose $G = \mathbf{R}$. A continuous unitary representation must be of the form $u_s(t) = e^{2\pi i t s}$, one for each $s \in \mathbf{R}$. So, $\hat{\mathbf{R}} = \mathbf{R}$. □

Now, suppose $G = \mathbf{Z}$ and $H = L^2(\mathbf{T}, \lambda)$, where as usual λ denotes Lebesgue measure. Let $u(n)$ be multiplication by z^n for $z \in \mathbf{T}$. Specifically, $u(n)(f)(z) =$

$z^n f(z)$ for $f \in L^2(\mathbf{T}, \lambda)$. Let $\mathbf{S}$ be an open subset of $\mathbf{T}$ and let $M = L^2(\mathbf{S}, \lambda)$. It is invariant under u, so u is reducible (which we also know from the fact that irreducible representations are one dimensional and $L^2(\mathbf{T}, \lambda)$ is not one dimensional). We would like to break up u (which means breaking up H) into a direct sum of irreducible (= one dimensional) representations; ideally $H = \bigoplus_{i=1}^{\infty} \mathbf{C}$. But, we cannot chop up H using $L^2(\mathbf{S}, \lambda)$ for open subsets $\mathbf{S}$ (since these are not $\mathbf{C}$) and we cannot use L^2(point) (since this is 0, not $\mathbf{C}$). So, there are no points on which the irreducible representations are acting. To repair this, we need some sort of measurable direct sum (specifically, $u = \int_{z_0 \in T = \hat{Z}}^{\oplus} u_{z_0} d\lambda$). That is to say, we need to construct an entity $\int^{\oplus}$, so that $\int_T^{\oplus} \mathbf{C} d\lambda = L^2(\mathbf{T}, \lambda)$. To first describe this so-called direct integral, we must understand what a family of Hilbert spaces (not necessarily the constantly $\mathbf{C}$ family) indexed by a measure space is.

B. Decomposition With Respect to an Algebra

Let us look at the above example again in a slightly more general context. Let U be a self adjoint subalgebra of $L(H)$ and suppose that H is finite dimensional. If U is reducible, there is a non-trivial subspace $M \leq H$ that is U-invariant. So, by proposition 2.2.1, $M^{\perp}$ is also U-invariant. Furthermore, we have $H = M \oplus M^{\perp}$. Let U_M denote the algebra of operators of U restricted to M. If U_M is reducible, we can repeat this procedure. Since H is finite dimensional, this procedure terminates and we can write $H = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ with each corresponding U_{M_i} irreducible. If H is infinite dimensional, the above procedure may not terminate. We might expect to be able to write $H = \bigoplus_{i=1}^{\infty} M_i$ (formally, $\bigoplus_{i=1}^{\infty} M_i$ consists of (norm) square summable sequences, the ith member of which is an element of M_i). We may, however, have to write $H = \int^{\oplus} M_x d\mu$ (which consists of, in analogy to the above, (norm) square integrable families). The details of its construction follow.

2.2.2 Measurable Fields of Hilbert Spaces

In the next two subsections, we outline direct integral theory as given in [Dixl]. Most of the proofs are omitted although a few are inserted to give the reader a bit of the flavour of the techniques used.

Let $(X, \mathcal{A}, \mu)$ be a measure space. The operator theory literature gives the construction of the direct integral over a standard measure on a Polish space (for example, a separable, compact or locally compact topological space; the paradigm being the spectrum of a symmetric operator as a compact subset of **C**). Since, we wish to study indexing by measure spaces in a general (measure theoretic) setting, we will not assume anything about the measure space at this point. However, we will feel free to add assumptions throughout this discussion. For details on standard measures and the "simple functions to measurable sets" approach to measure theory, the reader is referred to [Bou] or [Nai].

Definition 2.2.3 *A* field *of complex Hilbert spaces on X is a family, $(H(x))_{x\in X}$, such that each $H(x)$ is a* **C**-*Hilbert space.* $\square$

Write $\mathcal{F} := \prod_{x\in X} H(x)$. It is a **C**-vector space (with pointwise operations). An element of $\mathcal{F}$, an x-tuple, $(f(x))_{x\in X}$, is called a *field of vectors.*

Definition 2.2.4 *A* measurable field of Hilbert spaces *is a family,$(H(x))_{x\in X}$, together with a $\mathcal{G} \subseteq \mathcal{F} = \prod_{x\in X} H(x)$ such that:*

1. $\forall g \in \mathcal{G}$, $x \mapsto \|g(x)\|$ is measurable.

2. If $f \in \mathcal{F}$ is such that $x \mapsto \langle f(x)|g(x)\rangle$ is measurable for all $g \in \mathcal{G}$, then $f \in \mathcal{G}$.

3. There exists a sequence, g_1, $g_2, \ldots$ in $\mathcal{G}$ such that $\forall x \in X$, $(g_n(x))_{n=1}^{\infty}$ forms a total sequence in $H(x)$. $\square$

Remarks: 1. "Total," in this context, means "dense span."

2. Axiom 2 ensures that $\mathcal{G}$ is the "largest" measurable subcollection of $\mathcal{F}$ in some sense. In practice, we use this axiom to prove that a thing is an element of $\mathcal{G}$.

3. We call the elements of $\mathcal{G}$ *measurable fields of vectors* or MFV's. A sequence satisfying axiom 3 is called a *fundamental sequence* of MFV's. The whole structure is called an MFHS.

4. Let g and h be MFV's. Since linear combinations of $\|(g+h)(x)\|^2$, $\|(g-h)(x)\|^2$, $\|(g+ih)(x)\|^2$, and $\|(g-ih)(x)\|^2$ are measurable, $x \mapsto \langle g(x)|h(x)\rangle$ is measurable by the polarization identity: $\langle g|h\rangle = \frac{1}{4}\|g+h\|^2 - \frac{1}{4}\|g-h\|^2 + \frac{i}{4}\|g+ih\|^2$ $-\frac{i}{4}\|g-ih\|^2$. Indeed, the (pointwise) product of an MFV with a $\mathbf{C}$-valued measurable function is an MFV. The limit of a sequence of MFV's, converging weakly for each $x \in X$, is an MFV.

5. Our first assumption on X is that it is a finite measure space. Furthermore, we assume the sequence in Axiom 3 has each $\|g_n(x)\|$ bounded. We get, through linear combinations, as in remark 4 above, a sequence of MFV's, $h_1, h_2, \ldots$, such that for each $x \in X$, $(h_n(x))_{n=1}^{\infty}$ is dense in $H(x)$. In particular, each $H(x)$ is separable.

6. If μ' is equivalent to μ (i.e. $\mu \ll \mu'$ and $\mu' \ll \mu$) then $\mathcal{G}$ is also a μ'-measurable field. In essence, then, we are looking at equivalence classes of MFHS's. $\square$

Example 1: Suppose X is discrete so that every function out of it is measurable. The only $\mathcal{G}$ that can satisfy axiom 2 is $\mathcal{G} = \mathcal{F}$. To satisfy axiom 3, we need each $H(x)$ separable. $\square$

Example 2: Let H_0 be a fixed separable Hilbert space and let $H(x) = H_0$ for all $x \in X$. Then MFV's are simply measurable functions $X \longrightarrow H_0$ (i.e. $\mathcal{G} = \underline{\mathbf{Mble}}(X, H_0)$. H_0 with its Borel structure). $\square$

Example 3: If $X' \subseteq X$, then an X-MFHS restricts to an X'-MFHS in an obvious way. We may also include an X'-MFHS in an X-MFHS by defining $H(x) = \mathbf{0}$ for

$x \notin X'$. □

More examples will be given in the sequel.

Definition 2.2.5 *Let $((H(x))_{x\in X}, \mathcal{G})$ and $((H'(x))_{x\in X}, \mathcal{G}')$ be two MFHS's. A* morphism *is a family $(T(x))_{x\in X}$ of linear maps, $H(x) \xrightarrow{T(x)} H'(x)$, such that for each $g \in \mathcal{G}$, $x \mapsto T(x)g(x) \in \mathcal{G}'$.* □

We reserve the term *field of operators* for the case when the T's are bounded linear operators. In particular, an *isomorphism* of $\mathcal{G}$ onto $\mathcal{G}'$ is a morphism with each $T(x)$ an isomorphism. An MFHS which is isomorphic to a constant field (example 2) is called *trivial.*

At first glance, axiom 3 seems somewhat mysterious. As we remarked above, it makes the $H(x)$'s separable. It will also ensure that the Hilbert space we construct in the sequel, the direct integral, is separable. Furthermore, an important consequence of axiom 3 is the following:

Proposition 2.2.3 *([Dix1, p. 144]): (i) Let $X_p = \{x \in X | dim(H(x)) = p\}$. Then each X_p, $p = 0,\ 1,\ \ldots, \aleph_0$ is a measurable subset of X.*

(ii) There is a sequence $(g_1,\ g_2, \ldots)$ of MFV's such that

a. if $d(x) = dim(H(x)) = \aleph_0$, $(g_1(x),\ g_2(x), \ldots)$ is an ONB in $H(x)$

b. if $d(x) < \aleph_0$, $(g_1(x),\ g_2(x), \ldots,\ g_{d(x)}(x))$ is an ONB of $H(X)$ and $g_i(x) = 0$ when $i > d(x)$. ∎

We call the sequence above a measurable field of ONB's.

A fundamental sequence is sufficient for axiom 2.

Proposition 2.2.4 *([Dix1, p. 144]): Let $(g_n(x))_{n=1}^{\infty}$ be a fundamental sequence of MFV's. A field of vectors $(g(x))_{x\in X}$ is measurable iff all the functions $x \mapsto \langle g(x) | g_n(x)\rangle$ are measurable.* ∎

Remark: Proposition 2.2.4 follows immediately from proposition 2.2.3, part (ii) and Parseval's identity: $\langle g(x)|h(x)\rangle = \sum_{i=1}^{\infty}\langle g(x)|g_i(x)\rangle\overline{\langle h(x)|g_i(x)\rangle}$ for g_i an ONB. We will use this idea, also, for change of base, below. □

Finally, we note that we have a version of "local triviality." Recall, an MFHS is trivial if it is isomorphic to a constant field.

2.2.3 The Direct Integral

Definition 2.2.6 *An MFV, $g(x)$, on X is* square integrable *if*
$\int_X \|g(x)\|^2 d\mu < \infty$. □

The collection of square integrable MFV's forms a $\mathbf{C}$-vector space K. For g and h in K, $x \mapsto \langle g(x)|h(x)\rangle$ is measurable and square integrable by the Hölder inequality, and setting $\langle g|h\rangle := \int_X \langle g(x)|h(x)\rangle d\mu$ gives a pseudo inner product on K. Let $H = K/\sim$ (as before, $f \sim g$ iff $f = g$ a.e.). Then H is a pre-Hilbert space.

Theorem 2.2.1 *H is a Hilbert space (i.e. is complete).*

Proof: Let $(g_n)_{n=1}^{\infty}$ be a Cauchy sequence in H. It suffices to show that a subsequence converges almost everywhere to an element $g \in H$. Since (g_n) is Cauchy, we can pick a subsequence (which, for brevity, we also denote by (g_n)) such that $\sum_{n=1}^{\infty} \|g_{n+1} - g_n\| < \infty$ which means $\sum_{n=1}^{\infty} \|g_{n+1}(x) - g_n(x)\| < \infty$ $\forall x \notin N$ for some N with $\mu(N) = 0$.

For $x \notin N$, $g_1(x) + \sum_{n=1}^{\infty}(g_{n+1}(x) - g_n(x))$ converges in $H(x)$ (by completeness of $H(x)$) to an element, say $g(x) \in H(x)$. So $\|g(x)\| = \|g_1(x) + \sum_{n=1}^{\infty}(g_{n+1}(x) - g_n(x))\|$ $\leq \|g_1(x)\| + \sum_{n=1}^{\infty} \|g_{n+1}(x) - g_n(x)\|$. Put $g(x) = 0$ for $x \in N$. We must show that g is a square integrable MFV.

The field of vectors, $g(x)$, is measurable since it is a limit (a.e.) of $g_1(x)+\sum_{n=1}^{p}(g_{n+1}(x) - g_n(x))$ each of which is measurable. Now, $\int \|g(x)\|^2 d\mu$ $\leq \int \|g_1(x)\|^2 d\mu + \sum_{n=1}^{\infty} \int \|g_{n+1}(x) - g_n(x)\|^2 d\mu < \infty$ (note that we have assumed that $\mu(X) < \infty$). This completes the proof. ∎

Definition 2.2.7 *H is called "the"* direct integral *of the $H(x)$'s and is denoted by $\int^{\oplus} H(x)d\mu$. If g is a square integrable MFV, we write $\int^{\oplus} g(x)d\mu$ for its equivalence class in H.* □

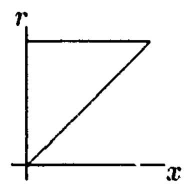
Remark: Suppose μ' is an equivalent measure to μ. Put $\mu' = \rho\mu$ where ρ is a μ-measurable function of x such that $0 < \rho(x) < \infty$. Then $x \mapsto \rho(x)^{-\frac{1}{2}}g(x)$ induces an isomorphism of $\int^{\oplus} H(x)d\mu$ onto $\int^{\oplus} H(x)d\mu'$ since $\int \|\rho(x)^{-\frac{1}{2}}g(x)\|^2 d\mu'$ $= \int \|g(x)\|^2 \rho(x)^{-1}\rho(x)d\mu = \int \|g(x)\|^2 d\mu$. For fixed μ and μ', this isomorphism does not depend on ρ and we call it the *canonical isomorphism* of $\int^{\oplus} H(x)d\mu$ onto $\int^{\oplus} H(x)d\mu'$. □

Example 1: Let X be discrete. As noted above, any field of vectors is measurable. If X is finite, with counting measure, then the direct integral is precisely $H_1 \oplus H_2 \oplus \cdots \oplus H_n$, the usual (finite) direct sum (= cartesian product with pointwise operations) of Hilbert spaces. If X is countable, again with counting measure, then the direct integral is simply $\bigoplus_{n=1}^{\infty} H_i$ where this is taken to mean square summable sequences. □

Example 2: For each $x \in X$, let $H(x) = H_0$ be a fixed Hilbert space. Then $\int^{\oplus} H(x)d\mu = L^2(X, H_0)$. We see that theorem 2.2.1 above generalizes the Reisz-Fischer theorem. □

Example 3: For each $r \in [0,1]$, let $H(r) = L^2([0,r],\lambda)$. Then $\int^{\oplus} H(r)dr$ $= \{(f(r,-))_{r\in[0,1]} | \int_0^1 \int_0^r |f(r,x)|^2 dxdr < \infty\}$ which is just $L^2(A,\lambda^2)$ where $A = \{(r,x) | 0 \leq x \leq r,\ 0 \leq r \leq 1\}$.

Picture:



□

Example 3 is a special case of the following:

Example 4: Let A be a plane measurable subset of $[0,1]\times[0,1]$. By use of Tonelli's theorem and in a similar manner to example 4 above, we have $L^2(A, \lambda \times \lambda)$ $\simeq \int^{\oplus} L^2(A_r)d\lambda$ where, as before A_r denotes the "rth slice," $A_r := \{x|(x,r) \in A\}$. □

In view of our discussions in the motivational section (section 2.2.1), a good way to slice up $L^2(A, \lambda \times \lambda)$ is as the direct integral of "points." Specifically,

Example 5: Let H_0 be $\mathbf{C}$ in example 2. Then $L^2(A, \lambda \times \lambda) \simeq \int_A^{\oplus} \mathbf{C} d(\lambda \times \lambda)$. □

In the rest of this section, we list some important properties (given in [Dix1]) which will be used in section 2.4.

Proposition 2.2.5 *Let $(H(x))_{x \in X}$ be a field of Hilbert spaces on X and let (g_i) be a sequence of fields of vectors such that: 1. $x \mapsto \langle g_i(x)|g_j(x)\rangle$ is measurable for all i and j and 2. $\{g_i(x)\}_{i=1}^{\infty}$ is total in $H(x)$ for each x. Then, there is a unique MFHS structure on the $H(x)$'s to make the g_i's MFV's.* ∎

Proposition 2.2.6 *Let s_i be an MF of ONB's for $(H(x), \mathcal{G})$. Then*

i) $s \in H = \int^{\oplus} H(x)d\mu(x)$ iff $x \mapsto \langle s(x)|s_i(x)\rangle$ is square integrable for all i and $\sum_{i=1}^{\infty} \int |\langle s(x)|s_i(x)\rangle|^2 d\mu(x) < \infty$.

ii) For $s, t \in H$, $\langle s|t\rangle = \sum_{i=1}^{\infty} \int \langle s(x)|s_i(x)\rangle \overline{\langle t(x)|s_i(x)\rangle} d\mu(x)$.

iii) $t \in H$ *is the (strong) limit of* $t_n(x) = \sum_{i=1}^{n} \langle t(x)|s_i(x)\rangle s_i(x)$. ∎

Definition 2.2.8 *The measure space,* $(X, \mathcal{A}, \mu)$, *is said to be* standard *if* $L^2(X, \mathcal{A}, \mu)$ *is separable.* □

Proposition 2.2.7 *If* $(X, \mathcal{A}, \mu)$ *is standard,* $\int^{\oplus} H(x)d\mu(x)$ *is separable.* ∎

Remarks: 1. Proposition 2.2.7 is really a corollary of proposition 2.2.6.

2. Proposition 2.2.6, ii) is Parceval's identity and iii) is Fourier series expansion in this context.

3. We have omitted almost all proofs in this section (they are in [Dix1]). However, we will refer to elements of the proofs of propositions 2.2.5 and 2.2.7. Specifically, the MFHS structure $\mathcal{G}$ of the former consists of all those g's such that $x \mapsto \langle g(x)|s_i(x)\rangle$ is μ-measurable for all $s_i(x)$ in an MF of ONB's. For the latter, we suppose $\alpha_i(x)$ is a dense sequence in $L^2(X, \mathcal{A}, \mu)$, then MFV's of the form $\sum_{i=1}^{n} \alpha_{n_i}(x)s_i(x)$ are dense in $\int^{\oplus} H(x)d\mu(x)$. □

2.2.4 Other Measurable Fields

There are other fields. One direction to move is to continuous fields (roughly speaking: replace "measurable" by "continuous" in the above definitions) which leads to the theory of Hilbert bundles and, more generally, vector bundles. We will not explore continuous fields in this paper (for exposition on continuous fields of C^*-algebras and Hilbert bundles, see [Dix2, pp. 211-249]). Indeed, rather than specializing the base category (i.e. from measurability to continuity), we will generalize the indexed categories. More precisely, in this section, we describe other entities indexed measurably culminating in the statement of a decomposition theorem alluded to in section 2.2.1. Again, we present an overview of [Dix1].

We have already defined morphism of MFHS's. Since operators (= bounded linear operators = continuous linear operators) are the entities to be studied in operator theory, a more important notion of morphism is:

Definition 2.2.9 *Let* $((H(x))_{x\in X}, \mathcal{G})$ *and* $((H'(x))_{x\in X}, \mathcal{G}')$ *be two MFHS's. A* measurable field of bounded linear maps *(or MFBLM) is a family,* $(H(x) \xrightarrow{T(x)} H'(x))_{x\in X}$*, of bounded linear maps such that for each MFV,* $g \in \mathcal{G}$*,* $(T(x)g(x))_{x\in X} \in \mathcal{G}'$*. We use MFO,* measurable field of operators*, if the* $T(x)$*'s are endomorphims.* □

$x \mapsto \|T(x)\|$ is measurable. In adition, $(T(x))_{x\in X}$ is an MFBLM iff $\langle T(x)g_i(x)|g_j'(x)\rangle$ is measurable for each g_i, g_j' of two fundamental sequences of MFV's $g_i \in \mathcal{G}$, $g_j' \in \mathcal{G}'$ (see [Dix1, p. 156] for details).

Definition 2.2.10 *An MFO* $(T(x))_{x\in X}$ *is* essentially bounded *if* $\|T(x)\|$ *is essentially bounded (i.e. there is an* M *such that* $\|T(x)\| \leq M$ *a.e.).* □

Remarks: 1. The product of an L^2-function by an L^∞-function is an L^2-function so if $(T(x))_{x\in X}$ is essentially bounded then we have a bounded linear operator $T \in B(H)$, where $H = \int^{\oplus} H(x)d\mu$. Furthermore, $\|T\| = \|T(x)\|_\infty$. Conversely, if $T \in B(H)$ is induced by an essentially bounded MFO, we say T is *decomposable* and write $T = \int^{\oplus} T(x)d\mu$.
2. Two essentially bounded MFO's which induce the same element of $B(H)$ are equal almost everywhere.
3. Operators of the form $\int^{\oplus} T(x)d\mu$, where $T(x)$ is scalar for each x are called *diagonalizable.* The set D of diagonalizable operators forms a commutative von

Neumann algebra and D', the commutant of D, is the set of decomposable operators. $\square$

Definition 2.2.11 *Let* $((H(x))_{x\in X}, \mathcal{G})$ *be an MFHS. A* measurable field of von Neumann algebras *(MFvNA) consists of a family,* $(A(x))_{x\in X}$*, such that each* $A(x)$ *is a von Neumann algebra on* $H(x)$*, together with a countable family,* $(T_i)_{i\in N}$*, of MFO's such that for almost all* $x \in X$*,* $A(x)$ *is the von Neumann algebra generated by the* $T_i(x)$*'s. In analogy to remark 1 above, a von Neumann algebra,* $A \subseteq B(H)$*, is called* decomposable *if it is induced by an MFvNA and we write*
$A = \int^{\oplus} A(x)d\mu.$ $\square$

Definition 2.2.12 *Let* $((H(x))_{x\in X}, \mathcal{G})$ *and* $((H'(x))_{x\in X}, \mathcal{G}')$ *be two MFHS's and* $((A(x))_{x\in X}, (T_i)_{i\in N})$ *and* $((A'(x))_{x\in X}, (T_i')_{i\in N})$ *be two MFvNA's (on* $\mathcal{G}$ *and* $\mathcal{G}'$ *respectively). Write* $A = \int^{\oplus} A(x)d\mu$ *and* $A' = \int^{\oplus} A'(x)d\mu$*. For each* $x \in X$ *let* $A(x)\xrightarrow{\varphi(x)}A'(x)$ *be a homomorphism of von Neumann algebras. The family,* $(\varphi(x))_{x\in X}$*, is called a* measurable field of homomorphisms *(MFH) if for each* $x \mapsto T(x) \in A(x)$*,* $x \mapsto \varphi(x)(T(x)) \in A'(x)$ *is measurable. For* $T = \int^{\oplus} T(x)d\mu \in A$*, write* $\varphi(T) = \int^{\oplus} \varphi(x)(T(x))d\mu \in A'$. $\square$

The reader may note the similarities between the definitions of MFHS and MFvNA. The basic format is: measurable field of things = family of things indexed by X + measurability requirement + countability requirement. The last condition is not so universally required (it is possible to talk about measurable fields of non-operator theoretic entities) but it is important in the operator theoretic world (e.g. separabilty of H).

Of course, we are interested in the question of when a Hilbert space is decomposable into a direct integral, more importantly, of when an operator decomposes into an MFO (in analogy to the finite dimensional spectral theorem) or when a

Hilbert space decomposes with respect to a von Neumann algebra (as alluded to in section 2.2.1).

Theorem 2.2.2 *Let H be a separable Hilbert space and A a von Neumann algebra on H. There exists a compact metrizable space X, a measure μ on X, an MFHS $(H(x))_{x\in X}$, such that each $H(x) \neq \mathbf{0}$, and an isomorphism of H onto $\int^{\oplus} H(x)d\mu$ which takes A to the algebra of diagonalizable operators.* ∎

Remark: We would like to decompose the algebra A. In order to do this, however, we must first know how to decompose the Hilbert space H it is acting on. And so, the theory is built up as: MFHS $\rightsquigarrow$ MFvNA $\rightsquigarrow$ Decomposition. □

2.3 Categorical Indexing Concepts

2.3.1 Introduction

Given a category $\underline{\mathbf{S}}$, one may form the category $\underline{\mathbf{Grp}}(\underline{\mathbf{S}})$ of group objects in $\underline{\mathbf{S}}$ (provided $\underline{\mathbf{S}}$ has finite (including empty) products). This is done, as is well known, by simply translating group theory data and axioms into categorical statements about objects of $\underline{\mathbf{S}}$ (for example, "multiplication" (data) is a morphism $G \times G \xrightarrow{\mu} G$). If $\underline{\mathbf{S}} = \underline{\mathbf{Set}}$, one gets the usual category of all (small) groups.

One might ask about the existence of limits in $\underline{\mathbf{Grp}}(\underline{\mathbf{S}})$. In particular, one can ask whether the product $G_1 \times G_2$ can be formed. Of course, we would like to from such products for any pair of group objects. This amounts to asking the question: given a family of size 2, can we form Π of this family? And similarly for "larger" sized families. And so, the notion of family is central to any discussion about completeness.

Now, if $\underline{\mathbf{S}} = \underline{\mathbf{Set}}$ and $I \in \underline{\mathbf{Set}}$, we know what an I-family of sets is and we know how to formulate questions about I-indexed products (="I-sized" products) in any category on $\underline{\mathbf{S}}$ (for example, in $\underline{\mathbf{Grp}} = \underline{\mathbf{Grp}}(\underline{\mathbf{Set}})$). In categorical language, an I-family may be interpreted as a functor $\underline{\mathbf{I}} \longrightarrow \underline{\mathbf{Set}}$, where $\underline{\mathbf{I}}$ is the discrete category whose objects are the elements of I.

For a general $\underline{\mathbf{S}}$, there is not necessarily a notion of I-family for I an object of $\underline{\mathbf{S}}$. For some significant examples, there is, however. Indeed, if $\underline{\mathbf{S}}$ is "$\underline{\mathbf{Set}}$-like," for example, then we have a good notion of family. The paradigm for "$\underline{\mathbf{Set}}$-like," in this context, is: $\underline{\mathbf{S}}$ a topos. If $\underline{\mathbf{S}}$ is a topos, we can form $\underline{\mathbf{S}}/I$. This is, of course, itself a topos (the fundamental theorem of elementary toposes). We may think of it as the category of I-families of objects of $\underline{\mathbf{S}}$ (in analogy to and using the equivalence $\underline{\mathbf{Set}}^I \simeq \underline{\mathbf{Set}}/I$ as motivation).

We can now talk about $\underline{\mathbf{Grp}}(\underline{\mathbf{S}})$ and $\underline{\mathbf{Grp}}(\underline{\mathbf{S}}/I)$, the latter thought of as I-families of group objects, and can ask about completeness relative to $\underline{\mathbf{S}}$ (in fact, $\underline{\mathbf{S}}/I \underset{\Pi_I}{\overset{\Delta_I}{\leftrightarrows}} \underline{\mathbf{S}}$ yields $\underline{\mathbf{Grp}}(\underline{\mathbf{S}}/I) \underset{\Pi}{\overset{\Delta}{\leftrightarrows}} \underline{\mathbf{Grp}}(\underline{\mathbf{S}})$). Δ_I is a special type of substitution. More generally, we have, for each $J \xrightarrow{\alpha} I$, a functor $\underline{\mathbf{S}}/I \xrightarrow{\alpha^*} \underline{\mathbf{S}}/J$ given by pulling back along α. The categories $\underline{\mathbf{S}}/I$ and the functors α^* will be the central data for our notion of $\underline{\mathbf{S}}$-indexed category. We give these abstractly.

Incidentally, $\underline{\mathbf{Grp}}(\underline{\mathbf{S}})$ is algebraic over $\underline{\mathbf{S}}$ and so has the same sized limits as $\underline{\mathbf{S}}$. Colimits are not necessarily so well behaved, however. For example, the category of finite sets $\underset{\sim}{\mathbf{Set}}_f$, as a category indexed by itself, is both complete and cocomplete (for example, a finite product of finite sets is a finite set). The category of finite groups, however, is not closed under finite coproducts.

As another example, let $I \in \underline{\mathbf{Top}}$. We may index $\underline{\mathbf{Top}}$ by itself as above (this works for any category with finite limits). We can also index $\underline{\mathbf{Set}}$ by $\underline{\mathbf{Top}}$ by considering the category of I-families of sets as the category of sheaves on I; $\underline{\mathbf{Sh}}(I)$. Again, we get substitution functors α^*, which, in this case, are the inverse image functors.

One important aspect about $\underline{\mathbf{Set}}$ is that it indexes many categories (one can talk of I-families of groups, of topological spaces, of ...). **Top** indexes fewer categories in some sense (for details of **Top** indexing, see [Lev]). In all examples of $\underline{\mathbf{S}}$-indexing, there is a (sometimes delicate) balance between the richness of the base category $\underline{\mathbf{S}}$ and the quantity of categories indexed by it (although, **Set**-indexing has both).

At this point, we should note that there are five approaches to a categorical treatment of indexing by objects other than sets: 1. Lawvere style (using $\underline{\mathbf{S}}$-atlases), 2. Penon style (using locally internal categories), 3. Bénabou style (using fibrations), 4. Paré-Schumacher style (using pseudo-functors), and 5. Betti-Walters style (using categories enriched over a bicategory). We will follow 4. in our outline immediately below and, indeed, throughout the thesis.

2.3.2 An Outline of Indexed Category Theory

Definition 2.3.1 *([Ma&Pa], p.63): Let* $\underline{\mathbf{S}}$ *be a category with finite limits. An* $\underline{\mathbf{S}}$-indexed category *consists of the following data:*

- *for every object* I *of* $\underline{\mathbf{S}}$*, a category* $\underline{\mathbf{A}}^I$
- *for every morphism* $J \xrightarrow{\alpha} I$ *of* $\underline{\mathbf{S}}$*, a functor* $\underline{\mathbf{A}}^I \xrightarrow{\alpha^*} \underline{\mathbf{A}}^J$
- *for each composable pair* $K \xrightarrow{\beta} J \xrightarrow{\alpha} I \in \underline{\mathbf{S}}$*, a natural isomorphism* $\phi_{\alpha,\beta} : \beta^*\alpha^* \longrightarrow (\alpha\beta)^*$
- *for each* $I \in \underline{\mathbf{S}}$*, a natural isomorphism* $\psi_I : (1_I)^* \longrightarrow 1_{A_I}$

subject to the (coherence) axioms:

1. *for each composable triple* $L \xrightarrow{\gamma} K \xrightarrow{\beta} J \xrightarrow{\alpha} I$ *in* $\underline{\mathbf{S}}$*, the following commutes:*

$$\begin{tikzcd}
\gamma^*\beta^*\alpha^* \arrow[r, "\gamma^*\phi_{\alpha,\beta}"] \arrow[d, "\phi_{\beta,\gamma}\alpha^*"'] & \gamma^*(\alpha\beta)^* \arrow[d, "\phi_{\alpha\beta,\gamma}"] \\
(\beta\gamma)^*\alpha^* \arrow[r, "\phi_{\alpha,\beta\gamma}"'] & (\alpha\beta\gamma)^*
\end{tikzcd}$$

2. *for each* $J \xrightarrow{\alpha} I \in \underline{\mathbf{S}}$, $\phi_{1_I,\alpha} = \alpha^*\psi_I : \alpha^* 1_I^* \longrightarrow \alpha^*$. □

Definition 2.3.2 *An* $\underline{\mathbf{S}}$-indexed functor, $\underset{\sim}{\mathbf{A}} \xrightarrow{\underset{\sim}{F}} \underset{\sim}{\mathbf{B}}$, *between two indexed categories consists of the following data:*

- *for every object* I *of* $\underline{\mathbf{S}}$, *a functor* $\underline{\mathbf{A}}^I \xrightarrow{F^I} \underline{\mathbf{B}}^I$
- *for each* $J \xrightarrow{\alpha} I \in \underline{\mathbf{S}}$, *a natural isomorphism* $\theta_\alpha : \alpha^* F^I \longrightarrow F^J \alpha^*$

subject to the axiom:

1. *for each composable pair* $K \xrightarrow{\beta} J \xrightarrow{\alpha} I$ *in* $\underline{\mathbf{S}}$, *the following commutes:*

$$\begin{tikzcd}
\beta^*\alpha^* F^I \arrow[r, "\phi_{\alpha,\beta}F^I"] \arrow[d, "\beta^*\theta_\alpha"'] & (\alpha\beta)^* F^I \arrow[dd, "\theta_{\alpha\beta}"] \\
\beta^* F^J \alpha^* \arrow[d, "\theta_\beta\alpha^*"'] & \\
F^K\beta^*\alpha^* \arrow[r, "F^K\phi_{\alpha,\beta}"'] & F^K(\alpha\beta)^*
\end{tikzcd}$$

□

Definition 2.3.3 *An* $\underline{\mathbf{S}}$-indexed natural transformation $\underset{\sim}{t} : \underset{\sim}{F} \longrightarrow \underset{\sim}{G}$ *between two indexed functors consists of a natural transformation* $t^I : F^I \longrightarrow G^I$ *for each* $I \in \underline{\mathbf{S}}$, *such that*

1. *for every* $J \xrightarrow{\alpha} I$, *the following commutes:*

$$\begin{tikzcd} \alpha^* F^I \arrow[r, "\alpha^* t^I"] \arrow[d, "\theta_\alpha"'] & \alpha^* G^I \arrow[d, "\theta_\alpha"] \\ F^J \alpha^* \arrow[r, "t^J \alpha^*"'] & G^J \alpha^* \end{tikzcd}$$ □

Remark: 1. The category $\underline{\mathbf{A}}^I$ is called the category of *I-indexed families* of objects of $\underline{\mathbf{A}}$. The functor α^* is called the *substitution* functor determined by α (in analogy to the substitution of example 1 below). □

We have already alluded to three important examples of indexing:

Example 1: Every category $\underline{\mathbf{A}}$ can be $\underline{\mathbf{Set}}$-indexed by taking, for $I \in \underline{\mathbf{Set}}$, $\underline{\mathbf{A}}^I := \{\langle A_i \rangle_{i \in I} \mid A_i \in \underline{\mathbf{A}}\}$, the I-fold product. For $J \xrightarrow{\alpha} I$, we define $\underline{\mathbf{A}}^I \xrightarrow{\alpha^*} \underline{\mathbf{A}}^J$ by $\langle A_i \rangle_{i \in I} \longmapsto \langle A_{\alpha(j)} \rangle_{j \in J}$. □

Example 2: $\underline{\mathbf{S}}$ indexes itself via $\underline{\mathbf{S}}^I := \underline{\mathbf{S}}/I$ and for $J \xrightarrow{\alpha} I$, $\underline{\mathbf{S}}/I \xrightarrow{\alpha^*} \underline{\mathbf{S}}/J$ sends $X \longrightarrow I$ to $Y \longrightarrow J$ determined by the pullback:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & J \arrow[d] \\ X \arrow[r] & I \end{tikzcd}$$

α^* has a left adjoint $\sum_\alpha$ given by composition. If $\underline{\mathbf{S}}$ is a topos, then α^* is logical and also has a right adjoint Π_α. In the case $\underline{\mathbf{S}} = \underline{\mathbf{Set}}$, we may think of α^* (as in example 1) as "relabeling along α." $\sum_\alpha$ and Π_α correspond, respectively, to forming coproducts and products over the fibres of α (see [PTJ1, pp. 35-37]). □

Example 3: $\underline{\mathbf{Top}}$ indexes $\underline{\mathbf{Set}}$ via, for $I \in \underline{\mathbf{Top}}$, $\underline{\mathbf{Set}}^I := Sh(I)$. For $J \xrightarrow{\alpha} I$ a continuous map, $Sh(I) \xrightarrow{\alpha^*} Sh(J)$ is pullback. A sheaf on I may be regarded as a local homeomorphism $X \longrightarrow I$. Pulling back along α yields another local homeomorphism which gives an element of $Sh(J)$. α^* is left exact and has a right adjoint $\alpha_*(F)(U) = F(f^{-1}(U))$. This gives a geometric morphism (indeed, the paradigm) $Sh(J) \xrightarrow{\alpha} Sh(I)$ (see [PTJ1, pp. 11-12] or chapter 3). □

Indexed category theory is useful for (at the very least and especially) two things: internal notions of smallness and limits. We will now describe these notions briefly.

A category object $\mathbf{C}$ in $\underline{\mathbf{S}}$ consists of a triple of objects, (C_2, C_1, C_0), and a sextuple of morphisms $(\pi_1, \pi_2, \circ, d_0, d_1, id)$, subject to axioms that make C_0 = objects, C_1 = morphisms, C_2 = composable pairs, π_1 and π_2 = projections, $\circ$ = composition, d_0 = domain, d_1 = codomain, and id = pick the identity (see, for example, [PTJ1, pp. 47-48] or [P&S, p.22]). Similarly, an internal functor is a triple (F_2, F_1, F_0) of morphisms exhibiting F_0 as the "object function" and F_1 as the "arrow function." With a suitable notion of internal natural transformation (a morphism from the objects of one to the morphisms of the other), we get a 2-category, $\underline{\mathbf{cat}}(\underline{\mathbf{S}})$.

For any $I \in \underline{\mathbf{S}}$, the hom functor, $\underline{\mathbf{S}}(I, -) : \underline{\mathbf{S}} \longrightarrow \underline{\mathbf{Set}}$, preserves category objects and internal functors. Indeed, $\underline{\mathbf{S}}(-, \mathbf{C})$, which takes I to the category (object in $\underline{\mathbf{Set}}$) $(\underline{\mathbf{S}}(I, C_2), \underline{\mathbf{S}}(I, C_1), \underline{\mathbf{S}}(I, C_0))$ is a contravariant functor from $\underline{\mathbf{S}}$ to $\underline{\mathbf{cat}}$. Now, an $F \in \underline{\mathbf{cat}}^{\underline{\mathbf{S}}^{op}}$ yields an $\underline{\mathbf{S}}$-indexed category: $\underset{\sim}{\mathbf{A}}^I = F(I)$ and $\alpha^* = F(\alpha)$. We call the $\underline{\mathbf{S}}$-indexed category determined by $\underline{\mathbf{S}}(-, \mathbf{C})$ the *externalization* of $\mathbf{C}$ and denote it by $[\mathbf{C}]$. We can formulate smallness for indexed categories:

Definition 2.3.4 *([P&S, p.26]): Two indexed categories $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}$ are said to be* equivalent *if there are indexed functors $\underset{\sim}{\mathbf{A}} \xrightarrow{\underset{\sim}{F}} \underset{\sim}{\mathbf{B}}$ and $\underset{\sim}{\mathbf{A}} \xleftarrow{\underset{\sim}{G}} \underset{\sim}{\mathbf{B}}$ and natural* *isomorphisms such that* $\underset{\sim}{F}\underset{\sim}{G} \cong 1_{\underset{\sim}{B}}$ *and* $\underset{\sim}{G}\underset{\sim}{F} \cong 1_{\underset{\sim}{A}}$.

An indexed category $\underset{\sim}{\mathbf{A}}$ *is called* small *if it is equivalent to* $[\mathbf{C}]$ *for some category object* $\mathbf{C} \in \underline{\mathbf{S}}$. □

This notion of smallness follows the idea that if $I \in \underline{\mathbf{S}}$ (think: I a set which is necessarily small; $\underline{\mathbf{Set}}$ being the large category consisting of all small sets), then a category "of size I" is to be considered small. For $\mathbf{C}$ a category object in $\underline{\mathbf{S}}$, $[\mathbf{C}]$ externalizes this object to the $\underline{\mathbf{cat}}$ (and $\underline{\mathbf{Set}}$) world in which we have utile notions of smallness and largeness. Paré and Schumacher go on to formulate other notions of smallness such as local smallness and well-poweredness.

Let $\mathbf{D}$ be a diagram ($=$ a small category) in a category $\underline{\mathbf{A}}$. we may formulate the existence of $\varprojlim_{\mathbf{D}}$ via the existence of a right adjoint to Δ:

$$\underline{\mathbf{A}}^{\mathbf{D}} \underset{\lim}{\overset{\Delta}{\rightleftarrows}} \underline{\mathbf{A}}$$

We already know what a small category is in the $\underline{\mathbf{S}}$-indexed world. We must translate each of the other notions.

Given two $\underline{\mathbf{S}}$-indexed categories $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}$, we can $\underline{\mathbf{S}}$-index the indexed functors from $\underset{\sim}{\mathbf{B}}$ to $\underset{\sim}{\mathbf{A}}$. $\underset{\sim}{\mathbf{A}}^{\underset{\sim}{B}}$ is indexed by $(\underset{\sim}{\mathbf{A}}^{\underset{\sim}{B}})^I := (\underset{\sim}{A}^I)^{\underset{\sim}{B}}$ and, for $J \xrightarrow{\alpha} I$, we get $\underline{\mathbf{A}}^I \xrightarrow{\alpha^*} \underline{\mathbf{A}}^J$ which yields $(\underset{\sim}{\mathbf{A}}^{\underset{\sim}{B}})^I \xrightarrow{\alpha^*} (\underset{\sim}{\mathbf{A}}^{\underset{\sim}{B}})^J$ via composition: $\underset{\sim}{\mathbf{B}} \longrightarrow \underset{\sim}{\mathbf{A}}^I \xrightarrow{\alpha^*} \underset{\sim}{\mathbf{A}}^J$ (note: $\underline{\mathbf{A}}^I$ is $\underline{\mathbf{S}}$-indexed via $(\underline{\mathbf{A}}^I)^J = \underline{\mathbf{A}}^{I \times J}$; we denote this by $\underset{\sim}{\mathbf{A}}^I$). We have a diagonal, $\Delta_{\underset{\sim}{B}} : \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{A}}^{\underset{\sim}{B}}$, given at I by $\Delta^I_{\underset{\sim}{B}} : \underset{\sim}{\mathbf{A}}^I \longrightarrow (\underset{\sim}{\mathbf{B}} \longrightarrow \underset{\sim}{\mathbf{A}}^I)$, $A \longmapsto (\underset{\sim}{\mathbf{B}} \xrightarrow{\underset{\sim}{F}} \underset{\sim}{\mathbf{A}}^I)$ where, for $B \in \underline{\mathbf{B}}^J$, $F^J(B)$ is constantly $\Delta_J(A)$; $\Delta_J : \underline{\mathbf{A}}^I \longrightarrow (\underline{\mathbf{A}}^I)^J$. We must now describe $\underline{\mathbf{S}}$-indexed adjunction.

Definition 2.3.5 *([P&S, pp. 68-69]): Let* $\underset{\sim}{U} : \underset{\sim}{\mathbf{B}} \longrightarrow \underset{\sim}{\mathbf{A}}$ *be an indexed functor. We say* $\underset{\sim}{U}$ *has a* indexed left adjoint *if there are an indexed functor,* $\underset{\sim}{F} : \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}$*, and indexed natural transformations,* $\underset{\sim}{\epsilon} : \underset{\sim}{F}\underset{\sim}{U} \longrightarrow 1_{\underset{\sim}{B}}$ *and* $\underset{\sim}{\eta} : 1_{\underset{\sim}{A}} \longrightarrow \underset{\sim}{U}\underset{\sim}{F}$*, such that* $\underset{\sim}{\epsilon}\underset{\sim}{F} \cdot \underset{\sim}{F}\underset{\sim}{\eta} = 1_{\underset{\sim}{F}}$ *and* $\underset{\sim}{U}\underset{\sim}{\epsilon} \cdot \underset{\sim}{\eta}\underset{\sim}{U} = 1_{\underset{\sim}{U}}$. □

And so, we can now formulate limits in the $\underline{\mathbf{S}}$-indexed world.

Definition 2.3.6 *([P&S, p.75]): Let $\underset{\sim}{\mathbf{A}}$ be an indexed category and* $\mathbf{C}$ *a category object in $\underline{\mathbf{S}}$. $\underset{\sim}{\mathbf{A}}$ has* $\mathbf{C}$*-limits if $\Delta_{\mathbf{C}} : \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{A}}^{\mathbf{C}}$ has a right adjoint* $\underleftarrow{\lim}$. □

Remark: 1. The two concepts of smallness and completeness seem to be opposed. If we "enlarge" the base category, $\underline{\mathbf{S}}$, things have more of a chance of being small but less things are complete. A good example of this point is the difference between $\underline{\mathbf{Set}}_f$ and $\underline{\mathbf{Set}}$ indexing. □

2.4 MFHS's as an Indexing Notion

2.4.1 Preamble

We have given some operator theoretic background and some indexed category theoretic background. It is time to blend the two (box 1 of the diagram in the introduction). Specifically, we now describe some categories of measurable fields of Hilbert spaces. As mentioned in the introduction, we will provide, in this paper, three approximations for "$\underline{\mathbf{Hilb}}^X$." These three approximations are to mix (the apparently opposing) elements of operator theory and indexed category theory. The first technique to describe $\underline{\mathbf{Hilb}}^X$ may be considered as mostly operator theoretic in nature.

In this section, we introduce two categories whose objects are measurable fields of Hilbert spaces; $MFHS(X)$ and $BMFHS(X)$. $MFHS(-)$ (the argument, here, is to be filled in with objects or morphisms of $\underline{\mathbf{MOR}}$) works well with substitution and is the translation of classical direct integral theory into categorical language. If we try, however, to interpret the direct integral as a functor, $MFHS(-)$ is not adequate so we introduce a new category, $BMFHS(X)$. This departs from the classical theory (we simply demand the morphisms of $BMFHS(X)$ be bounded over $x \in X$ and the classical $MFHS(X)$ and, indeed, [Dix1], allow essential boundedness over $x \in X$) but the gap is not too large. We can propose $\underline{\mathbf{Hilb}}^X := MFHS(X)$ but it seems that $\underline{\mathbf{Hilb}}^X := BMFHS(X)$ is better.

2.4.2 MFHS(X)

Definition 2.4.1 *Let $X = (X, \mathcal{A}, \mu)$ be a fixed measure space. The category $MFHS(X)$ (etymology: measurable fields of Hilbert spaces on X) has as objects, MFHS's, and as morphisms, essentially bounded MFBLM's.* □

Remarks: 1. An MFBLM, $((H(x))_{x\in X}, \mathcal{G}) \xrightarrow{(T(x))_{x\in X}} ((H'(x))_{x\in X}, \mathcal{G}')$, is essentially bounded if $\|T(x)\| \in L^\infty(X, \mathcal{A}, \mu)$. Such compose ($\|S \circ T(x)\|$ $\leq \|S(x)\|\|T(x)\|$) and the family of identities is essentially bounded so we do indeed have a category.

2. Following the remarks of definition 2.2.10, we require essential boundedness for the direct integral to be compatible with maps (that is, multiplying an L^2-function by an L^∞-function yields an L^2-function). □

And so, we have a functor, for each fixed X,

$$\int_X^\oplus : BMFHS(X) \longrightarrow \underline{\mathbf{Hilb}},$$

defined, in an obvious way, as $\int_X^\oplus ((H(x))_{x\in X}, \mathcal{G}) := \int^\oplus H(x)d\mu(x)$ and $\int_X^\oplus (T(x))_{x\in X} := \int^\oplus T(x)d\mu(x)$. Remark 2 above ensures functoriality. In particular, we have $\int^\oplus 1_{H(x)}d\mu(x) = 1_{\int^\oplus H(x)d\mu(x)}$.

Notice that if $H(x) = H'(x)$ $a.e$ on X, then $\int^\oplus H(x) = \int^\oplus H'(x)$ (indeed, the two families are effectively indistinguishable at the MFHS level). We can consider equivalence classes (of both objects and morphisms) but will, for now, keep such matters in the background.

The image of the functor $\int^{\oplus}$ consists of all (X-)decomposable Hilbert spaces and bounded linear maps (by definition). In fact, this image has a nice topological property as well. We recall a result about operators (endofunctions) from Dixmier:

Theorem 2.4.1 *([Dix2, p.388]):*
(a) Let $S = \int^{\oplus} S(x)d\mu(x)$ *and* $T = \int^{\oplus} T(x)d\mu(x)$ *be decomposable operators on* $H = \int^{\oplus} H(x)d\mu(x)$, *then*

$$S + T = \int^{\oplus}(S(x) + T(x))d\mu(x) \qquad ST = \int^{\oplus} S(x)T(x)d\mu(x)$$

$$\lambda S = \int^{\oplus} \lambda S(x)d\mu(x) \qquad S^* = \int^{\oplus} S(x)^* d\mu(x)$$

(b) Let $T_i = \int^{\oplus} T_i(x)d\mu(x)$ $(i = 1, 2, \ldots)$ *and* $T = \int^{\oplus} T(x)d\mu(x)$ *be decomposable operators. If* T_i *converges strongly to* T *(i.e. in the norm topology of* H*), then there is a subsequence* (T_{n_k}) *such that* $(T_{n_k}(x))$ *converges strongly to* $T(x)$ *almost everywhere. If* $T_i(x)$ *converges strongly to* $T(x)$ *almost everywhere and if* $\sup \|T_i\| < +\infty$, *then* T_i *converges strongly to* T. ∎

2.4.3 Substitution

The thesis is about X-indexed families of Hilbert spaces for X a measure space. Our first proposal (box 1) for $\underline{\mathbf{Hilb}}^X$ is $MFHS(X)$ (which we have just described). An essential aspect of indexed category theory is substitution. That is to say, one needs not only a notion of X-family but a useful way of getting a Y-family out of an X-family (so that one may talk of things like indexed functors, natural transformations, etc., as outlined in section 2.3). In this section, we describe this substitution for the "box 1 world." More precisely, suppose $(Y, \mathcal{B}, \nu) \xrightarrow{\phi} (X, \mathcal{A}, \mu)$ is measurable, we will describe $MHFS(X) \xrightarrow{\phi^*} MFHS(Y)$. ϕ^* sends an $((H(x))_{x \in X}, \mathcal{G} \subseteq \mathcal{E} = \prod_{x \in X} H(x))$ to $((H(\phi(y)))_{y \in Y}, \mathcal{H} \subseteq \mathcal{F} = \prod_{y \in Y} H(\phi(y)))$. Define $\mathcal{H} = \{h \in \mathcal{F} \mid y \mapsto \langle h(y) | g(\phi(y)) \rangle$ is measurable $\forall g \in \mathcal{G}\}$.

Remarks: 1. There is a map $\prod_{x \in X} H(x) \xrightarrow{\alpha} \prod_{y \in Y} H(\phi(y))$; $\alpha(e)(y) = e(\phi(y))$. If h is of the form $\alpha(g)$ for some $g \in \mathcal{G}$, then $y \mapsto \langle g(\phi(y)) | g'(\phi(y)) \rangle$ is measurable $\forall g' \in \mathcal{G}$ (it is the composite of $x \mapsto \langle g(x) | g'(x) \rangle$ and ϕ). Thus, $\mathcal{H}$ contains $\alpha(\mathcal{G})$.

2. If we take objects and morphisms as equivalence classes, under *a.e.* equality, of X-MFHS's (two such equivalent entities will produce the same direct integral), then it is appropriate to have $\phi \in$ <u>**MOR**</u>. □

Proposition 2.4.1 *$\mathcal{H} = \phi^*(\mathcal{G})$ is an MFHS on Y.*

Proof: We must exhibit the three axioms and the following order seems most appropriate.

3. Let $(g_i)_{i \in N}$ be a fundamental sequence in $\mathcal{G}$. Then $(g_i(\phi(y)))_{i \in N}$ forms a total set in $H(\phi(y))$ for each y (i.e. $\alpha(g_i)$ is a fundamental sequence in $\mathcal{H}$).

1. Let (g_i) be a measurable field of ONB's (proposition 2.2.3). By Parceval's identity, $\langle h(y) | h(y) \rangle = \sum_{i=1}^{\infty} \langle h(y) | g_i(\phi(y)) \rangle \overline{\langle h(y) | g_i(\phi(y)) \rangle}$. $\langle h(y) | g_i(\phi(y)) \rangle$ is measurable by definition and $(\bar{\ })$ is measurable so $y \mapsto \|h(y)\|$ is measurable $\forall h \in \mathcal{H}$.

2. Suppose $y \mapsto \langle f(y) | h(y) \rangle$ is measurable $\forall h \in \mathcal{H}$. Then, in particular, $\langle f(y) | g(\phi(y)) \rangle$ is measurable $\forall g \in \mathcal{G}$ ($\alpha(\mathcal{G}) \subseteq \mathcal{H}$). But this is precisely the criterion for being in $\mathcal{H}$, so $f \in \mathcal{H}$ as required. ∎

Given $H(x) \xrightarrow{T(x)} H'(x)$, a morphism in $MFHS(X)$, define $\phi^* T$ by $H(\phi(y)) \xrightarrow{T(\phi(y))} H'(\phi(y))$. Let $h \in \mathcal{H}$, we must show $T(\phi(y))h(y) \in \mathcal{H}'$. That is, we must show $y \mapsto \langle T(\phi(y))h(y) | g'(\phi(y)) \rangle$ is measurable $\forall g' \in \mathcal{G}'$. We know that $\langle T(\phi(y))g(\phi(y)) | g'(\phi(y)) \rangle$ is measurable $\forall g \in \mathcal{G}$ $\forall g' \in \mathcal{G}'$ (T takes g's to g''s and compose with ϕ) and $\langle h(y) | g(\phi(y)) \rangle$ is measurable $\forall g \in \mathcal{G}$ (by definition of

$\mathcal{H}$). Now, let (g_i) be a measurable field of ONB's. Use the Fourier expansion, $h(y) = \sum_{i=1}^{\infty}\langle h(y)|g_i(\phi(y))\rangle g_i(\phi(y))$, to get $\langle T(\phi(y))h(y)|g'(\phi(y))\rangle$
$= \sum_{i=1}^{\infty}\langle h(y)|g_i(\phi(y))\rangle\langle T(\phi(y))g_i(\phi(y))|g'(\phi(y))\rangle$ which is measurable as required.

If $T(x)$ is essentially bounded over $x \in X$, $T(\phi(y))$ is not necessarily essentially bounded over $y \in Y$. But, if $\phi \in \underline{\mathbf{M0R}}$, then it is. There is an M such that $\|T(x)\| \leq M$ except on A with $\mu(A) = 0$. $\|T(\phi(y))\| \leq M$ except on $\phi^{-1}(A)$ which has ν-measure zero. Thus, we have:

Theorem 2.4.2 *For $\phi \in \underline{\mathbf{M0R}}$, $MFHS(X) \xrightarrow{\phi^*} MFHS(Y)$ is a functor.* ∎

A special case is $MFHS(1) \xrightarrow{\Delta} MFHS(X)$ where $\Delta = !^*$, $X \xrightarrow{!} 1$. An MFHS on 1 is just a family of 1 Hilbert space. To satisfy axiom 3, the Hilbert space must be separable. Thus, there is a functor $\underline{\mathbf{SepHilb}} \xrightarrow{\Delta_X} MFHS(X)$ for each measure space X. However, Δ is not right adjoint to $\int^{\oplus}$ in general (see chapter 4).

Now, $1^* = 1$ (to say $\langle h(x)|g(1(x))\rangle = \langle h(x)|g(x)\rangle$ is measurable $\forall g \in \mathcal{G}$ is to say $h \in \mathcal{G}$ by axiom 2). Suppose we have $(Z,\mathcal{C},\rho) \xrightarrow{\psi} (Y,\mathcal{B},\nu) \xrightarrow{\phi} (X,\mathcal{A},\mu)$ two morphisms in $\underline{\mathbf{M0R}}$ and let $((H(x))_{x\in X},\mathcal{G}) \in MFHS(X)$. Put $\psi^*\phi^*(\mathcal{G}) = \mathcal{K}$ and $(\phi\psi)^*(\mathcal{G} = \mathcal{L}$ (we wish to show $\mathcal{K} = \mathcal{L}$ in order to show that $(\ )^*$ preserves composition in $\underline{\mathbf{M0R}}$). Then $k \in \mathcal{K}$ means $z \mapsto \langle k(z)|h(\psi(z))\rangle$ is measurable $\forall h \in \mathcal{H}$ and $l \in \mathcal{L}$ means $z \mapsto \langle l(z)|g(\phi\psi(z))\rangle$ is measurable $\forall g \in \mathcal{G}$.

$\mathcal{K} \subseteq \mathcal{L}$, since $\alpha(\mathcal{G}) \subseteq \mathcal{H}$. To get $\mathcal{L} \subseteq \mathcal{K}$, let (g_i) be a measurable field of ONB's, then $\langle k(z)|h(\psi(z))\rangle = \sum_{i=1}^{\infty}\langle k(z)|g_i(\phi\psi(z))\rangle\overline{\langle h(\psi(z))|g_i(\phi\psi(z))\rangle}$.

And so, we have a functor

$$\underline{\mathbf{M0R}}^{op} \xrightarrow{(\ )^*} \underline{\mathbf{Cat}},$$

where $X^* = MFHS(X)$. We denote the "$\underline{\mathbf{M0R}}$-indexed category" (quotation marks because $\underline{\mathbf{M0R}}$ does not have products, so is not a "real" $\underline{\mathbf{S}}$) by $\underset{\sim}{MFHS}$.

We call the fibration version of this $\underline{\mathbf{MFHS}} \xrightarrow{P} \underline{\mathbf{MOR}}$. The objects of the category $\underline{\mathbf{MFHS}}$ are triples $(X, (H(x))_{x\in X}, \mathcal{G})$, where $X = (X, \mathcal{A}, \mu)$ is in $\underline{\mathbf{MOR}}$ (i.e. is a finite measure space) and $((H(x))_{x\in X}, \mathcal{G})$ is an $MFHS(X)$.

A morphism is a pair, $(X, ((H(x))_{x\in X}, \mathcal{G}) \xrightarrow{(\phi,T)} (Y, (K(y))_{y\in Y}, \mathcal{K})$, where $(X, \mathcal{A}, \mu) \xrightarrow{\phi} (Y, \mathcal{B}, \nu)$ is in $\underline{\mathbf{MOR}}$ and T is an $MFHS(X)$ map, $((H(x))_{x\in X}, \mathcal{G}) \xrightarrow{T} \phi^*((K(y))_{y\in Y}, \mathcal{K})$. That is, T is a family of maps $\langle H(x) \xrightarrow{T_x} K(\phi(x))\rangle_{x\in X}$, (norm) essentially bounded over $x \in X$ and such that $x \longmapsto \langle T_x g(x) | k(\phi(x))\rangle$ is measurable $\forall k \in K$ and $g \in \mathcal{G}$. Finally, we note that P is projection onto the first factor.

2.4.4 Indexed Direct Integral

We have given necessary background information. In this section, we explore functoriality of the direct integral. More precisely, we will generalize the direct integral above in the box 1 world. Let $(X, \mathcal{A}, \mu) \xrightarrow{(\phi,\mu_y)} (Y, \mathcal{B}, \nu)$ be a disintegration. We seek a functor:

$$MFHS(X) \xrightarrow{\int_\phi^\oplus} MFHS(Y).$$

It is instructive to consider two examples first. If ϕ is the identity, $X \xrightarrow{1} X$, then $\int_\phi^\oplus (H(x), \mathcal{G})$ should be $(H(x), \mathcal{G})$. If ϕ is the unique map, $X \xrightarrow{!} 1$, then $\int_\phi^\oplus (H(x), \mathcal{G})$ should be $(\int^\oplus H(x)d\mu(x), \mathcal{D} = \int^\oplus H(x)d\mu(x))$. We first note that since $\mathcal{D}$ is to satisfy axiom 3, we require $\int^\oplus H(x)d\mu(x)$ to be separable. According to proposition 2.2.7, we could require $(X, \mathcal{A}, \mu)$ to be standard. In fact, we will insist upon a similar restriction on the measure spaces below.

Even more basic than a separabilty requirement is a boundedness requirement. We seek a functor $\int_\phi^\oplus$, taking X-MFHS's to Y-MFHS's, which generalizes $\int_!^\oplus$ as the classical direct integral (of section 2.4.2). Let $\int_\phi^\oplus (H(x), \mathcal{G}) = (D(y), \mathcal{D})$. With the above examples in mind, we should have $D(y) = \int_{\phi^{-1}(y)}^\oplus H(x) d\mu_y(x) =$ $\{g \in \mathcal{G} \mid \int_{\phi^{-1}(y)} \|g(x)\|^2 d\mu_y(x) < \infty\}/\sim$, with $g \sim g'$ iff $\mu_y\{x \in \phi^{-1}(y) \mid g(x) \neq g'(x)\} = 0$.

Now, suppose, for $(H(x), \mathcal{G}) \xrightarrow{(T(x))} (H'(x), \mathcal{G}')$, $\|T(x)\|$ is essentially bounded over $x \in X$ (i.e. $T \in MFHS(X)$). Then, the (only possible definition of the) map $D(y) \xrightarrow{S(y)} D'(y)$; $d(y) \mapsto [T(x)d(y)(x)]$ (an element of $D(y)$ is an equivalence class $d(y)$, which we also sometimes denote by $[d(y)(x)]$ to emphasize that a representative of $d(y)$ is a function $\phi^{-1}(y) \rightarrow \bigcup_{x \in \phi^{-1}(y)} H(x)$) does not necessarily land in $D'(y)$. We would require $\int_{\phi^{-1}(y)} \|T(x)d(y)(x)\|^2 d\mu_y(x) < \infty$ for all y. But, if $\|T(x)\|$ is (μ-)essentially bounded, then this is finite for only almost all y (making $S(y)$ defined for only almost all y). In short, $\|T(x)\|$ μ-essentially bounded implies $\|T(x)\||_{\phi^{-1}(y)}$ μ_y-essentially bounded for almost all y (and multiplying an essentially bounded function by a square integrable function yields a square integrable function) but not <u>all</u> y necessarily. We can surmount this problem in two ways: by considering as morphisms, almost everywhere defined T's (and, subsequently, S's) or by making $\|T(x)\|$ a bounded function ($\|T(x)\|$ bounded $\Rightarrow$ $\|T(x)\||_{\phi^{-1}(y)}$ bounded for all y and square integrability is preserved). The former is cumbersome and, indeed, some of the constructions below would not be well defined (for the first solution, we would have to require equivalence classes under almost everywhere equality). We choose the latter (departing from classical direct integral theory only slightly) and consider the subcategory:

Definition 2.4.2 *$BMFHS(X)$ (etymology: "B" for "bounded") is the subcategory of $MFHS(X)$ whose morphisms, $(H(x), \mathcal{G}) \xrightarrow{(T(x))} (H'(x), \mathcal{G}')$, have $\|T(x)\|$ bounded over $x \in X$.* $\square$

Remarks: 1. We say T is norm essentially bounded or norm bounded according to whether T is in $MFHS(X)$ or $BMFHS(X)$.

2. $BMFHS(1) = MFHS(1) =$ **SepHilb**.

3. Substitution, as described in section 2.4.3, restricts to this subcategory (if $\|T(y)\|$ is bounded over $y \in Y$, $\|T(\phi(x)\|$ is bounded over $x \in X$). $\square$

And so, we will describe $BMFHS(X) \xrightarrow{\int_\phi^\oplus} BMFHS(Y)$, $(H(x), \mathcal{G}) \longmapsto (D(y), \mathcal{D})$, with $D(y) = \int_{\phi^{-1}(y)}^\oplus H(x) d\mu_y(x)$. Now, $\mathcal{D}$ will have to have a fundamental sequence. Using proposition 2.2.7 as a clue, we make the:

Assumption: There is a sequence $a_i \in L^2(X, \mathcal{A}, \mu)$ such that $a_i(x)|_{\phi^{-1}(y)}$ is total in each $L^2(X_y, \mathcal{A}_y, \mu_y)$. $\square$

Example 1: $(X, \mathcal{A}, \mu) \xrightarrow{(1, \iota_x)} (X, \mathcal{A}, \mu)$. Here, $L^2(X_x, \mathcal{A}_x, \iota_x) \simeq \mathbf{C}$ for each x. We may take the family with one member $a_1 = \lceil 1 \rceil : X \longrightarrow \mathbf{C}$. $\square$

Example 2: $(X, \mathcal{A}, \mu) \xrightarrow{(!, \mu)} (1, 2, \text{counting})$ with $(X, \mathcal{A}, \mu)$ standard. Here, $X_* = X$ so the total sequence for "each" $L^2(X_*)$ is that for $L^2(X)$. $\square$

Example 3: For $([0,1] \times [0,1], \mathcal{L} \times \mathcal{L}, \lambda \times \lambda) \xrightarrow{(p, (\lambda \times \lambda)_x)} [0,1]$, with p=projection onto the first factor, take the sequence to be $\chi_{[a,b] \times [c,d]}$, $a, b, c, d \in \mathbf{Q} \cap [0,1]$. Every (square integrable) measurable function can be approximated by simple functions. These, in turn, can be approximated by simple functions over rational intervals. $\square$

Remarks: 1. As consequences of this assumption, we have $(X_y, \mathcal{A}_y, \mu_y)$ standard for all y and, indeed, $L^2(X) \simeq \int^\oplus L^2(X_y) d\nu(y)$.

2. The assumption does not necessarily imply that $(X, \mathcal{A}, \mu)$ is standard as the second example shows.

3. Example 3 requires a special property (density of the rationals) and, as such, is not a "good" example. There is a better sequence for this as the next example shows. $\square$

Example 4: $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu) \xrightarrow{(p_1,(\mu\times\nu)_x)} (X, \mathcal{A}, \mu)$ with $(Y, \mathcal{B}, \nu)$ standard. Let b_i be a total sequence in $L^2(Y)$ and put $a_i(x,y) := X \times Y \xrightarrow{p_2} Y \xrightarrow{b_i} \mathbf{C}$. Then, $a_i|_{p_1^{-1}(x)} = b_i$ is a total sequence for each x and $\int_{X\times Y} \|a_i(x,y)\|^2 d(\mu \times \nu)(x,y) = \int_Y \int_X \|a_i(x,y)\|^2 d\mu(x) d\nu(y) = \int_Y \|b_i(y)\|^2 \mu(X) d\nu(y) < \infty$. □

And so, before defining the MFHS structure $\mathcal{D}$, let us look at some specific members. Let $a_i(x)$ be an MF of ONB's for the $L^2(X_y)$'s and let $s_j(x)$ be an MF of ONB's for the $H(x)$'s. Each $a_i(x)s_j(x)|_{\phi^{-1}(y)} \in D(y)$ (since a_i and s_j are norm bounded over x (of norm ≤ 1, in fact), they are μ_y-square integrable for each y). By proposition 2.2.7, $(a_i(x)s_j(x)|_{\phi^{-1}(y)})_{(i,j)=(1,1)}^{(\infty,\infty)}$ forms a total set in each $D(y)$. Furthermore, $y \mapsto \langle a_i(x)s_j(x)|_{\phi^{-1}(y)} | a_{i'}(x)s_{j'}(x)|_{\phi^{-1}(y)}\rangle = \int_{\phi^{-1}(y)} \langle a_i(x)s_j(x) | a_{i'}(x)s_{j'}(x)\rangle d\mu_y(x)$ is ν-measurbale (see proposition 1.5.2). And so, by proposition 2.2.5, there is a unique MFHS structure $\mathcal{D}$, making the a_is_j's MFV's. $\mathcal{D}$ consists of all $\langle d(y)\rangle \in \prod D(y)$ such that $y \mapsto \langle d(y) | a_i(x)s_j(x)|_{\phi^{-1}(y)}\rangle$ is ν-measurable for all i and j.

Next, we put $((H(x), \mathcal{G}) \xrightarrow{T} (H'(x), \mathcal{G}')) \longmapsto ((D(y), \mathcal{D}) \xrightarrow{S} (D'(y), \mathcal{D}')$ with, as we have already noted, $(S(y)d(y))(x) := T(x)(d(y)(x))$. Then S is linear and well defined for suppose $d_0 = d_1$ in $D(y)$, then $\mu_y\{x \in \phi^{-1}(y) \mid T(x)d_0(y)(x) \neq T(x)d_1(y)(x)\} \leq \mu_y\{x \in \phi^{-1}(y) \mid d_0(y)(x) \neq d_1(y)(x)\} = 0$ ($T(x)$ is a function for each x).

Suppose $\|T(x)\| \leq M$, then $\|S(y)d(y)\|^2 = \int_{\phi^{-1}(y)} \|T(x)d(y)(x)\|^2 d\mu_y(x) \leq M^2 \int_{\phi^{-1}(y)} \|d(y)(x)\|^2 d\mu_y(x) = M^2 \|d(y)\|^2$. So, $S(y)d(y) \in D'(y)$ for all y and $\|S(y)\|$ is bounded over $y \in Y$.

Finally, let $\langle d(y)\rangle_{y\in Y} \in \mathcal{D}$. We wish to show $\langle S(y)d(y)\rangle_{y\in Y} \in \mathcal{D}'$. It is enough to show it for the "generators" of $\mathcal{D}$: $a_is_j|_{\phi^{-1}(y)}$. We must show $y \mapsto \langle S(y)a_is_j|_{\phi^{-1}(y)} | a_k s'_l|_{\phi^{-1}(y)}\rangle$ is ν-measurable (a_i, a_k from the same sequence, s_j an MF of ONB's for $\mathcal{G}$, s'_l an MF of ONB's for $\mathcal{G}'$). But, this is

$y \mapsto \int_{\phi^{-1}(y)} \langle T(x)a_i(x)s_j(x) | a_k(x)s_l'(x)\rangle d\mu_y(x)$. $T(x)a_i(x)s_j(x) \in \mathcal{G}'$ (T takes g's to g''s) and $a_k(x)s_l'(x) \in \mathcal{G}'$ so that the function under the integral is μ-measurable. Thus, the function is ν-measurable (again, as in proposition 1.5.2).

It is straightforward that $\int_\phi^\oplus$ preserves composition and identity, and so, we have a functor:

$$BMFHS(X) \xrightarrow{\int_\phi^\oplus} BMFHS(Y).$$

Let us revisit the two examples at the beginning of this section. For $\int_1^\oplus$, $a_1 = [1]$ and s_j is an MF of ONB's for $\mathcal{G}$. So, $a_1 s_j$ is an MF of ONB's for $\mathcal{G}$ (i.e. $\mathcal{D}$, being the unique MFHS structure, is $\mathcal{G}$). For $\int_!^\oplus$, the totalness of $a_i s_j$ is exactly proposition 2.2.7. These examples represent two extremes in some sense.

There is an alternate description of $\mathcal{D}$. Let $\overline{\mathcal{D}}$ consist of all those $\langle d(y)\rangle \in \prod D(y)$ for which there is a $g \in \mathcal{G}$ such that $g|_{\phi^{-1}(y)} = d(y)$ for all y. We will show that $\overline{\mathcal{D}} = \mathcal{D}$ by showing that $\overline{\mathcal{D}}$ is an MFHS structure (on the $D(y)$'s) containing the $a_i s_j$'s and applying the uniqueness property of $\mathcal{D}$. First, we have a notion of "well definedness" for $\overline{\mathcal{D}}$:

Proposition 2.4.2 *If $g'|_{\phi^{-1}(y)} = d(y) = g(y)|_{\phi^{-1}(y)}$ for all y, then $\mu\{x \mid g(x) \neq g'(x)\} = 0$.*

Proof: $\mu\{x \mid g(x) \neq g'(x)\} = \int_Y \mu_y\{x \in \phi^{-1}(y) \mid g'|_{\phi^{-1}(y)}(x) \neq g|_{\phi^{-1}(y)}(x)\} d\nu(y)$ $= \int_Y 0 d\nu(y) = 0$. ∎

Proposition 2.4.3 *$\overline{\mathcal{D}}$ is an MFHS structure (containing the $a_i s_j$'s).*

Proof: Axiom 1: $y \mapsto \|d(y)\|^2 = \int_{\phi^{-1}(y)} \|g(x)\|^2 d\mu_y(x)$ is measurable as in proposition 1.5.2, so the square root is measurable.

Axiom 3: the $a_i s_j|_{\phi^{-1}(y)}$'s form the fundamental sequence. This is immediate since, for any $g \in \mathcal{G}$, $\langle g|_{\phi^{-1}(y)}\rangle_{y \in Y} \in \overline{\mathcal{D}}$ by definition.

Axiom 2: Linear combinations (with coefficients measurable functions of x) of $a_i(x)s_j(x)$'s yield elements of $\mathcal{G}$. Thus, from the $a_i s_j|_{\phi^{-1}(y)}$'s, we can form a sequence $t_i(y)$, such that $y \mapsto \langle t_i(y)|t_j(y)\rangle$ is ν-measurable, $(t_i(y))_{i=1}^{dim(D(y))}$ is an ONB of $D(y)$ (and $t_i(y) = 0$ if $i > dim(D(y))$), <u>and</u> there are functions $g_i(x) \in \mathcal{G}$ such that $g_i|_{\phi^{-1}(y)} = t_i(y)$.

Now, let $\langle k(y)\rangle \in \prod D(y)$ and suppose $y \mapsto \langle k(y)|t_i(y)\rangle$ is ν-measurable for each i. For each y, we have $k(y) = \sum_{i=1}^{\infty}\langle k(y)|t_i(y)\rangle t_i(y) =: \sum_{i=1}^{\infty} b_i(y)t_i(y)$. Put $g(x) := \sum_{i=1}^{\infty} b_i(\phi(x))g_i(x)$, then this $\sum$ converges for each x (i.e. $g(x) \in H(x)$), $g \in \mathcal{G}$ ($\langle \sum b_i(\phi(x))g_i(x)|g'(x)\rangle = \sum b_i(\phi(x))\langle g_i(x)|g'(x)\rangle$ is measurable for all $g' \in \mathcal{G}$ since it is a sum of measurable functions), and $g(x)|_{\phi^{-1}(y)} = k(y)$ as required. ∎

We will use this alternate description to discuss pseudo-functoriality. Let $(X,\mathcal{A},\mu)\xrightarrow{(\phi,\mu_y)}(Y,\mathcal{B},\nu)\xrightarrow{(\psi,\nu_z)}(Z,\mathcal{C},\rho)$ be two disintegrations that satisfy the assumption and such that their composition $(\psi\phi,\theta_z)$ does as well (at this point, we do not know whether disintegrations that satisfy the assumption compose; it seems to be a difficult problem). Let $\int_\psi^\oplus \int_\phi^\oplus (H(x),\mathcal{G}) = (E(z),\mathcal{E})$ and $\int_{\psi\phi}^\oplus (H(x),\mathcal{G}) = (F(z),\mathcal{F})$. $E(z) = \{d \in \mathcal{D} \mid \int_{\psi^{-1}(z)} \|d(y)\|^2 d\nu_z(x) < \infty\}/\sim$, and $F(z) = \{g \in \mathcal{G} \mid \int_{\phi^{-1}\psi^{-1}(z)} \|g(x)\|^2 d\theta_z(x) < \infty\}/\sim$. We now define $E(z)\xrightarrow{S(z)}F(z)$ and $F(z)\xrightarrow{T(z)}E(z)$. Given $d \in \mathcal{D}$ (in $E(z)$), by the alternate description of $\mathcal{D}$, there is a $g \in \mathcal{G}$ such that $g|_{\phi^{-1}(y)} = d(y)$. Put $S(z)(d) = g$. Conversely, given $g \in F(z)$, put $T(z)(g) = \langle g|_{\phi^{-1}(y)}\rangle \in \mathcal{D}$.

Lemma 2.4.1 *$d \in E(z)$ iff $g \in F(z)$.*

Proof: $\int_{\psi^{-1}(z)} \|d(y)\|^2 d\nu_z(y) = \int_{\psi^{-1}(z)} \int_{\phi^{-1}(y)} \|g(x)\|^2 d\mu_y(x) d\nu_z(y) = \int_{\phi^{-1}\psi^{-1}(z)} \|g(x)\|^2 d\theta_z(x)$. ∎

Lemma 2.4.2 *$d_0 \sim d_1$ in $E(z)$ iff $g_0 \sim g_1$ in $F(z)$.*

Proof:

$$\begin{aligned}
&d_0 \sim d_1 \text{ in } E(z)\\
&\quad\text{iff}\quad \nu_z\{y \in \psi^{-1}(z) \mid d_0(y) \neq d_1(y)\} = 0\\
&\quad\text{iff}\quad \mu_y\{x \in \phi^{-1}(y) \mid g_0(x) \neq g_1(x)\} = 0 \ a.\ a.\ y \in \psi^{-1}(z)\\
&\quad\text{iff}\quad \int_{\psi^{-1}(z)} \mu_y\{x \in \phi^{-1}(y) \mid g_0(x) \neq g_1(x)\} d\nu_z(y) = 0\\
&\quad\text{iff}\quad \theta_z\{x \in \phi^{-1}\psi^{-1}(z) \mid g_0(x) \neq g_1(x)\} = 0\\
&\quad\text{iff}\quad g_0 \sim g_1 \text{ in } F(z). \quad \blacksquare
\end{aligned}$$

Lemma 2.4.3 *$S(z)$ and $T(z)$ are linear isometries.*

Proof: Linearity of $S(z)$ follows from proposition 2.4.2. Linearity of $T(z)$ is just "linearity of restriction." That both are isometries follows from the chain of integrals of lemma 2.4.1. ∎

Now, $S(z)T(z) : g \mapsto g|_{\phi^{-1}(y)} \mapsto g$ and $T(z)S(z) : d \mapsto g \mapsto g|_{\phi^{-1}(y)} = d$. And so, we have proved:

Theorem 2.4.3 *$E(z)$ and $F(z)$ are isometrically isomorphic for each z.* ∎

We need to show S and T respect $\mathcal{E}$ and $\mathcal{F}$. Let $e_i(z)$ be a MF of ONB's for $\mathcal{E}$, then there are $d_i \in \mathcal{D}$ such that $d_i|_{\psi^{-1}(z)} = e_i(z)$ and, furthermore, there are $g_i \in \mathcal{G}$ such that $g_i|_{\phi^{-1}(y)} = d_i(y)$. Let $f_i(z)$ be an MF of ONB's for $\mathcal{F}$, then there are $g_i' \in \mathcal{G}$ such that $g_i'|_{\phi^{-1}\psi^{-1}(z)} = f_i(z)$. We must show that $z \mapsto \langle S(z)e_i(z)|f_j(z)\rangle$ and $z \mapsto \langle T(z)f_i(z)|e_j(z)\rangle$ are ρ-measurable for each i and j.

The first function is $z \mapsto \int_{\phi^{-1}\psi^{-1}(z)} \langle g_i(x)|g_i'(x)\rangle d\theta_z(x)$. The function under the integral is μ-measurable so, as usual (proposition 1.5.2), the first function is

ρ-measurable. The second function is $z \mapsto \int_{\psi^{-1}(z)} \int_{\phi^{-1}(y)} \langle g_i'(x)|g_j(x)\rangle d\mu_y(x) d\nu_z(y)$ and is measurable for the same reason.

And so, we have exhibited a pseudo-functor:

$$\text{“}\underline{\textbf{Disint}}\text{”} \xrightarrow{\int_{-}^{\oplus}} \underline{\textbf{Cat}}$$

(**Disint** is in quotes because of the problem of composability of morphisms that satisfy the assumption).

Chapter 3

Measurable Sheaves

3.1 Introduction

In this chapter, we will describe two equivalent Grothendieck topoi to be constructed from a measure space, first invented by Deligne. For a detailed description of these sheaf categories, see [How] (for that matter, for a detailed description of topos theory, see [PTJ1], or for a detailed description of Grothendieck topoi, see [SGAIV]). We will follow a slightly different path (looking mainly at the locale of subobjects of 1) than [How], but will first recall a few results from his summary.

This chapter essentially follows the "Lawvere directive," as described in the introduction: understand the gros and petit aspects of categorical measure theory. We proceed to attempt that understanding here. More accurately, we will describe two topoi and the locale of subobjects of 1. Strictly speaking, we will explore only petit aspects (sheaves on a *single* measure space and subobjects of 1 in *that* topos). The Gros topos of a topological space uses as "site" open maps into the space; the philosophy being that this constructs a topos out of topological spaces (in the *plural*). Exploration of such Gros aspects in the case of measure theory will await future work.

We begin with the basic, background material; the construction of the two sheaf categories in the next section. In section 3.3, we describe the locale of subobjects of 1 in detail. It turns out that this object is, in fact, a complete Boolean algebra. In section 3.4, we describe some of the logic of the sheaf category. This category satisfies SS, supports split. In fact, a stronger property holds, the category has AC, the axiom of choice. This implies Booleanness and we see that our logic is essentially classical.

Of course, the important aspect of this chapter is its description of one of the approaches to categorical measure indexing. The idea, then, is to understand Hilbert spaces in the sheaf category. In the last section, we discuss such entities.

3.2 Sheaves on a Measure Space

3.2.1 Definitions

Let $(X, \mathcal{A}, \mu)$ be a measure space. Though not immediately necessary, we will assume that $\mu(X) < \infty$. We can make $(\mathcal{A}, \subseteq)$, considered as a poset category, into a site. A countable family $\{A_n \in \mathcal{A}\}_{n=1}^{\infty}$ will be a cover of $A \in \mathcal{A}$ if $A_n \subseteq A$, $\forall n \in \mathbf{N}$ and $\mu(A \setminus \bigcup_{n=1}^{\infty} A_n) = 0$.

Proposition 3.2.1 *These coverings define a pretopology on* $(\mathcal{A}, \subseteq)$.

We require a lemma from basic set theory:

Lemma 3.2.1 *1.* $C \subseteq B \subseteq A \Rightarrow A \setminus C = (A \setminus B) \cup (B \setminus C)$ *and*
2. $B_n \subseteq A_n \Rightarrow (\bigcup_n A_n) \setminus (\bigcup_n B_n) \subseteq \bigcup_n (A_n \setminus B_n)$. ∎

Proof: (of proposition 3.2.1): We must show the constant family is a cover and covers are stable under subcovers and pullback.

$\{A\} \in Cov(A)$: $\mu(A \setminus A) = 0$.

subcovers: Let $\{A_n\} \in Cov(A)$ and $\{A_{nm}\} \in Cov(A_n)$ for each n. Then $\mu(A \setminus \bigcup_n A_n) = 0$ and $\mu(A_n \setminus \bigcup_m A_{nm}) = 0$ for each n. Put $K_n := A_n \setminus \bigcup_m A_{nm}$, then $\mu(A \setminus \bigcup_{n,m} A_{nm}) \leq \mu((A \setminus \bigcup_n A_n) \cup \bigcup_n K_n) \leq \sum_n \mu(A \setminus \bigcup_n A_n) + \sum_n \mu(K_n) = 0$, the first inequality by the lemma.

pullback: Let $\{A_n\} \in Cov(A)$ and $A' \longrightarrow A \in (\mathcal{A}, \subseteq)$ (which means $A' \subseteq A$). Consider the pullback:

$$\begin{tikzcd}
A' \times_A A_n \arrow[r] \arrow[d] & A_n \arrow[d] \\
A' \arrow[r] & A
\end{tikzcd}$$

Now, in a poset, pullback is intersection and $\mu(A' \setminus \bigcup_n (A' \cap A_n))$ $= \mu(A' \setminus A' \cap (\bigcup_n A_n)) \leq \mu(A \setminus \bigcup_n A_n) = 0$. ∎

Recall,

Definition 3.2.1 *A* presheaf *is a functor* $(\mathcal{A}, \subseteq)^{op} \xrightarrow{F} \underline{\mathbf{Set}}$. *A* sheaf *is a presheaf, F, such that for all covers $\{A_n\}$ of A,*

$$F(A) \longrightarrow \prod_n F(A_n) \rightrightarrows \prod_{nm} F(A_n \cap A_m)$$

is an equalizer. □

Notation: "$|_{A'}$" denotes, $F(A' \subseteq A)$, the restriction to $A' \subseteq A$. We also write, when required, $\rho^A_{A'} : F(A) \longrightarrow F(A')$. □

The sheaf condition says that if we have elements $x_n \in F(A_n)$ which are compatible (i.e. $x_n|_{A_n \cap A_m} = x_m|_{A_n \cap A_m}$ $\forall n,\ m$), then we can "extend" to a unique $x \in F(A)$.

Notation: $PSh(\mathcal{A})$ denotes the (functor) category of presheaves on this site. $Sh(\mathcal{A})$ denotes the full subcategory whose objects are sheaves. ∎

$Sh(\mathcal{A})$ is an example of a topos with no points (if $\mu\{x\} = 0\ \forall x \in X$); see [How, p.47]. In general, representables are not sheaves, for consider the example:

(Counter)example 1: ([How, p.27]):Let $A,\ A' \in \mathcal{A},\ A \subseteq A',\ A \neq A'$, and $\mu(A' \setminus A) = 0$. Then $\mathcal{A}(A', A) = \emptyset$ and $\mathcal{A}(A, A) = 1$. Now $\{A'\}$ is a cover of A, so if $\mathcal{A}(-, A)$ were a sheaf, we would have $\mathcal{A}(A', A) = \mathcal{A}(A, A)$. □

The associated sheaf of $\mathcal{A}(-, A)$ is ([How, p.27]):

$$a(\mathcal{A}(-, A))(A') = \begin{cases} 1 & \mu(A' \setminus A) = 0 \\ \emptyset & \text{else} \end{cases}$$

"$\mu(A' \setminus A) = 0$," in the above, suggests an alternate site and an alternate sheaf category. We begin with the σ-algebra $\mathcal{A}$, and mod out by the ideal $\mathcal{N}$ of measure zero sets. Modding out means, in this case, with respect to the equivalence relation $A \sim A'$ iff $\mu(A \triangle A') = 0$ where $A \triangle A' = (A \setminus A') \cup (A' \setminus A)$ denotes the symmetric difference. $\mathcal{A}/\mathcal{N}$ is made into a site by giving $\overline{A} \longrightarrow \overline{B}$ iff there are two representatives, A_0 of $\overline{A}$ and B_0 of $\overline{B}$, such that $A_0 \subseteq B_0$. Given $\overline{A} \longrightarrow \overline{B} \longrightarrow \overline{C}$, with $A_0 \subseteq B_0$ and $B_1 \subseteq C_1$, we get $\overline{A} \longrightarrow \overline{C}$ by $A_0 \cap B_1 \subseteq B_0 \cap B_1 \subseteq C_1$ (note: $B_0 \sim B_1$ and $A_0 \subseteq B_0 \Rightarrow A_0 = A_0 \cap B_0 \sim A_0 \cap B_1$). We say $\{\overline{A_n}\}_{n=1}^{\infty}$ is a cover of $\overline{A}$ if $\bigcup_n \overline{A_n} = \overline{A}$ (note: we may define $\bigcup_n \overline{A_n} = \overline{\bigcup_n A_{0n}}$ where A_{0n} is any choice of representatives; since the countable union of measure zero sets has measure zero, this is well defined). We get two new categories $PSh(\mathcal{A}/\mathcal{N})$ and $Sh(\mathcal{A}/\mathcal{N})$.

Proposition 3.2.2 $Sh(\mathcal{A}) \simeq Sh(\mathcal{A}/\mathcal{N})$

Proof: Use the axiom of choice to pick a particular representative $r(\overline{A})$ of each equivalence class $\overline{A} \in \mathcal{A}/\mathcal{N}$. The equivalence is given by:

$$Sh(\mathcal{A}) \underset{(\ )_*}{\overset{(\ )^*}{\rightleftarrows}} Sh(\mathcal{A}/\mathcal{N})$$

For $F \in Sh(\mathcal{A})$, put $F_*(\overline{A}) = F(r(\overline{A}))$ and for $G \in Sh(\mathcal{A}/\mathcal{N})$, put $G^*(A) = G(\overline{A})$. ∎

Notation: Because we will work with $Sh(\mathcal{A})$ extensively, and think of it as depending on X and as "the" category of sheaves on a measure space, we write $MEAS(X) := Sh(\mathcal{A})$ and $L(X)$ for its locale of subobjects of 1. □

3.2.2 Examples

We now give an extensive list of objects of $MEAS(X)$. More (operator theoretic) examples will be described in the last section. We have already noted that:

Example 1: $a(\mathcal{A}(-,A))(A') := \begin{cases} 1 & \mu(A' \setminus A) = 0 \\ \emptyset & \text{else} \end{cases}$ is a sheaf. Representables are not sheaves, in general (but they are in $Sh(\mathcal{A}/\mathcal{N})$). We think of $a(\mathcal{A}(-,A))$ as the "representable," however. □

The empty family is a cover of $\emptyset \in \mathcal{A}$. So, if F is a sheaf, we have:

$$F(\emptyset) \longrightarrow \prod_{\emptyset} = 1 \rightrightarrows \prod_{\emptyset} = 1$$

which implies that $F(\emptyset) = 1$.

In general, "constant" presheaves (which means $F(A) = K$ if $A \neq \emptyset$ and $F(\emptyset) = 1$) are not sheaves. Suppose F is a sheaf with $F(A) = K$, $\forall A \in \mathcal{A}$. Suppose, further, that $A_1 \bigcup A_2 = A$ with A_i nonempty (A is "disconnected"). Then $\{A_1 \hookrightarrow A,\ A_2 \hookrightarrow A\}$ is a cover of A and the sheaf condition

$$F(A) \longrightarrow F(A_1) \times F(A_2) \rightrightarrows F(A_1) \times F(\emptyset) \times F(\emptyset) \times F(A_2)$$

which is

$$F(A) \longrightarrow F(A_1) \times F(A_2) \rightrightarrows F(A_1) \times F(A_2)$$

implies $F(A) \cong F(A_1) \times F(A_2)$ (or the diagonal from K to $K \times K$ is an isomorphism, so $K = \emptyset$ or $K = 1$). If $K = 1$, we have

Example 2: The constantly 1 sheaf: define $1(A) := 1$, $\forall A \in \mathcal{A}$. Then $1 \in MEAS(X)$. This is a terminal object of $MEAS(X)$. We shall return to this sheaf in section 3.3. □

And, if $K = \emptyset$, the other "constant" sheaf is:

Example 3: $0(A) := \begin{cases} 1 & \mu(A) = 0 \\ \emptyset & \text{else} \end{cases}$ is a sheaf. It is an initial object of $MEAS(X)$. Notice that this is $a(\mathcal{A}(-, \emptyset))$. □

Example 4: Let $(Y, \mathcal{B})$ be a fixed measurable space. Define $Mble_Y(A)$ $:= \{(A_0, f) \mid A_0 \in \mathcal{A},\ A_0 \subseteq A,\ \mu(A \setminus A_0) = 0,\ (A_0, \mathcal{A}|_{A_0}) \xrightarrow{f} (Y, \mathcal{B})$ $\in \underline{\textbf{Mble}}\}/\sim$, with $(A_0, f) \sim (A_0', f')$ iff $\mu\{x \in A_0 \cap A_0' \mid f(x) \neq f'(x)\} = 0$.

Remarks: 1. If f and f' are measurable, then $\{x \mid f(x) \neq f'(x)\}$ is measurable.
2. The (A_0, f)'s seem somewhat cumbersome but they are necessary for if we simply try $M(A) = \underline{\textbf{Mble}}(A, Y)/\sim$, say, then for $\mu(A) = 0$, $A \neq \emptyset$, and $Y = \emptyset$, there are no maps in $M(A)$ and we want there to be one. However, this is the only problem and, if $Y \neq \emptyset$, we may use $Mble_Y(-) = M(-)$. □

Proposition 3.2.3 *$Mble_Y(-)$ is a presheaf.*

Proof: Suppose $A' \subseteq A$ and let $(A_0, f) \in Mble_Y(A)$. Then, we claim, $(A_0 \cap A', f|_{A_0 \cap A'}) \in Mble_Y(A')$.

$\mu(A' \setminus (A' \cap A_0)) \leq \mu(A \setminus A_0) = 0$ and the restriction of a measurable function is measurable. Now, suppose $(A_0, f) \sim (A_0', f')$ in $Mble_Y(A)$. Then $\mu\{x \in (A_0 \cap A') \cap (A_0' \cap A') \mid f(x) \neq f'(x)\} = \mu\{x \in A_0 \cap A_0' \cap A' \mid f(x) \neq f'(x)\} \leq \mu\{x \in A_0 \cap A_0' \mid f(x) \neq f'(x)\} = 0$. ∎

Proposition 3.2.4 *$Mble_Y(A)$ is a sheaf.*

Proof: Let $\{A_n\}_{n=1}^{\infty}$ be a cover of A and let $A_{0n}, f_n)$ be the representatives of a compatible family in the $Mble_Y(A)$'s. Now, by lemma 3.2.1, $A \setminus \bigcup_n A_{0n}$ $= (A \setminus \bigcup_n A_N) \cup (\bigcup_n A_n \setminus \bigcup_n A_{0n}) \subseteq (A \setminus \bigcup_n A_n) \cup \bigcup_n (A_n \setminus A_{0n})$. So, $\mu(A \setminus \bigcup_n A_{0n})$ $\leq \mu(A \setminus \bigcup_n A_n) + \sum_n \mu(A_n \setminus A_{0n}) = 0$. Next. let $C_n = A_{0n} \setminus \bigcup_{I<n} A_{0i}$. Then the C_n's are pairwise disjoint Nan $\bigcup_n C_n = \bigcup_n A_{0n}$. Define $F : \bigcup_n C_n \longrightarrow Y$ as follows: $x \in \bigcup_n C_n \Rightarrow x$ is in a unique C_n; put $f(x) = f_n(x)$. Then $f|_{C_n} = f_n|_{C_n}$ by construction and f is measurable for if $B \in \mathcal{B}$, then $f^{-1}(B) = \bigcup_n f_n^{-1}(B) \cap C_n) \in \mathcal{A}$.

We need only show that this definition of f respects $\sim$ (which will also show uniqueness of the extension). Suppose $(A_{0n}, f_n) \sim (A_{1n}, g_n)$, $n = 1, 2, 3, \ldots$ Then $(C_n, f_n|_{C_n}) \sim (D_n, g_n|_{D_n})$ where $D_n = A_{1n} \setminus \bigcup_{i<n} A_{i1}$, for $\mu\{x \in C_n \cap D_n \mid f_n \neq g_n\} \leq \mu\{x \in A_{0n} \cap A_{1n} \mid f_n \neq g_n\} = 0$.

We claim $\bigcup_n C_n, f) \sim (\bigcup_n D_n, g)$. Let $x \in \bigcup_n C_n \cap \bigcup_n D_n$ and $f(x) = f_{n_0}(x)$, $g(x) = g_{n_1}(x)$. Then $f(x) \neq g(x) \Rightarrow f_{n_1}(x) \neq g_{n_1}(x)$ or $f_{n_0}(x) \neq f_{n_1}(x)$. Each of the latter two occurs on a set of measure zero and taking the union over $n_0, n_1 = 1, 2, 3, \ldots$, we get $f \sim g$ as claimed. ∎

As special cases of this, we have

Example 5: $\mathbf{R}(-) := Mble_{\mathbf{R}}(-)$ where $(\mathbf{R}, \mathcal{L}, \lambda)$ is the (Lebesgue) real line. □

Example 6: $\mathbf{C}(-) := Mble_{\mathbf{C}}(-)$ where $(\mathbf{C}, \mathcal{L} \otimes \mathcal{L}, \lambda \otimes \lambda)$ is the (Lebesgue) complex plane. □

In the last section of this chapter, we will see that $\mathbf{R}(-)$ is the object of (Dedekind) reals in $MEAS(X)$ and $\mathbf{C}(-)$ will be a complex numbers object (and a Hilbert space object, the "one dimensional" space over itself). Obvious measure theoretic constructions may not always be interpreted as sheaves, however:

(Counter)example 7: $L^2(-)$ defined by $L^2(A) := \{A_0 \xrightarrow{f} \mathbf{C} | A_0 \in \mathcal{A},\ A_0 \subseteq A,$ $\mu(A \setminus A_0) = 0,\ \int_A |f|^2 d\mu < \infty\}/\sim$, is not a sheaf. Let $X := [0,1]$, $A := (0,1)$

with cover $A_n := (\frac{1}{n+1}, \frac{1}{n})$ (all with Lebesgue measure). Let $A_n \xrightarrow{f_n} \mathbf{C},\ x \mapsto \frac{1}{x}$. Then, on each piece, $\int_A |f_n|^2 d\mu < \infty$, but extending to (the only possible function) $f(x) = \frac{1}{x}$ on $(0,1)$, we see that $\int_0^1 |f|^2 d\mu \not< \infty$.

Of course, this example works for any L^p space ($p \geq 1$). $L^2(X, \mathbf{C})$ is a Hilbert space in real life, so we see the difficulty in studying Hilbert space objects in $MEAS(X)$.

$L^2(-)$ is a presheaf, however (restricting a square integrable function to a smaller set yields a square integrable function), and it is easy to compute its associated sheaf. $L^2(-) \subseteq \mathbf{C}(-)$ as presheaves $\Rightarrow (aL^2)(-) \subseteq (a\mathbf{C})(-) = \mathbf{C}(-)$ (a preserves monomorphisms and $\mathbf{C}(-)$ is already a sheaf). In fact,

Proposition 3.2.5 $\mathbf{C}(-)$ *is the associated sheaf of* $L^2(-)$.

Proof: Let $A \xrightarrow{f} \mathbf{C}$ be measurable. We must exhibit a cover of A such that $f \in L^2$ on each piece. Let $A_n := \{x \mid |f(x)| < n\}$. Then each A_n is measurable and $A = \bigcup_{n=1}^{\infty} A_n$ and $\int_{A_n} |f(x)|^2 d\mu \leq \int_{A_n} n^2 d\mu = n^2 \mu(A_n) < \infty$ ($\mu(X) < \infty$ is our standing assumption). ∎

In a similar manner, $\mathbf{C}(-)$ is the associated sheaf of all the $L^p(-)$ presheaves. We may think of $\mathbf{C}(-)$ as acting the role of all L^p spaces simultaneously in $MEAS(X)$. □

Example 4 suggests another example.

Example 8: $M0R(A, Y) := \{A_0 \xrightarrow{f} Y \mid A_0 \in \mathcal{A},\ A_0 \subseteq A,\ \mu(A \setminus A_0) = 0,$ $f \in \underline{\mathbf{M0R}}\}/\sim$ is a sheaf. □

There is an interesting function from $M0R(A, Y)$ to $Mble_{\mathbf{R}}(A)$ which is constructed using the Radon-Nikodym theorem. Recall,

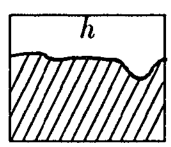
Theorem 3.2.1 *[Roy, p.238]: (Radon-Nikodym): Let* $(X, \mathcal{A}, \mu)$ *be a* σ*−finite measure space and* ν *a measure defined on* $\mathcal{A}$ *such that* $\nu \ll \mu$ *(i.e.* $\mu(A) = 0$

$\Rightarrow \nu(A) = 0$). Then there exists a nonnegative function f, such that $\forall E \in \mathcal{A}$, $\nu(E) = \int_E f d\mu$. f is unique in the sense that if g is any other function with this property, then $g = f$ a.e. μ. ∎

Let $A \xrightarrow{f} Y \in M0R(A, Y)$. We get another measure on $(Y, \mathcal{B})$ by $\alpha(B) := \mu(f^{-1}(B))$. Evidently, $\alpha \ll \mu$ since $f \in$ <u>**M0R**</u> so, by the Radon-Nikodym theorem, there is a $Y \xrightarrow{g} \mathbf{R}$ such that $\alpha(E) = \int_E g(y) d\nu$. Composing, we get $A \xrightarrow{f} Y \xrightarrow{g} \mathbf{R}$. That is, we have a function $M0R(A, Y) \xrightarrow{\tau_A} Mble_{\mathbf{R}}(A)$, $f \mapsto g \circ f$. Unfortunately, this map is not natural in A as the following example shows (we thank Ian Putnam for suggesting this example). Consider:

$$\begin{tikzcd} M0R(A,Y) \arrow[r, "\tau_A"] \arrow[d, "\rho^A_{A'}"'] & Mble(A,\mathbf{R}) \arrow[d, "\sigma^A_{A'}"] \\ M0R(A',Y) \arrow[r, "\tau_{A'}"'] & Mble(A',\mathbf{R}) \end{tikzcd}$$

Let $X = A = [0,1]^2$ and $Y = [0,1]$ with Lebesgue measure. Let $[0,1]^2 \xrightarrow{f} [0,1]$, $(x,y) \mapsto x$ be the first projection, so that $\tau_A \circ \sigma^A_{A'}$ is $\lceil 1 \rceil$ for any A'. Suppose $[0,1] \xrightarrow{h} [0,1]$ is a continuous function and let A' be the set under h:



Then $\tau_{A'} \circ \rho^A_{A'}(x,y) = h(x) \neq \sigma^A_{A'} \circ \tau_A(x,y)$.

If we try to interpret the collection of disintegrations from A to Y as a sheaf, the same problem as with $L^2(-)$ arises.

(Counter)example 9: *$Disint(A,Y) := \{(f, (\mathcal{A}|_{A_0})_y, (\mu|_{A_0})_y) : A_0 \longrightarrow Y \mid$ $f \in$ <u>**Disint**</u>$\}/\!\sim$* is not a sheaf.

For two disintegrations, $(X, \mathcal{A}, \mu) \xrightarrow{(f,\mu_y)} (Y, \mathcal{B}, \nu)$ and $(X, \mathcal{A}, \mu) \xrightarrow{(g,\eta_y)} (Y, \mathcal{B}, \nu)$, we say $f \sim g$ if two conditions hold. The first is $\mu\{x \mid f(x) \neq g(x)\} = 0$. Let $G = \{x \mid f(x) = g(x)\}$ be the "good" set for f and g. We can restrict f and g to G to get disintegrations, $(G, \mathcal{A}|_G \mu|_G) \xrightarrow{(f|_G,\beta_y)} (Y, \mathcal{B}, \nu)$ and $(G, \mathcal{A}|_G, \mu|_G) \xrightarrow{(g|_G,\alpha_y)} (Y, \mathcal{B}, \nu)$. On G, $f = g$, so $f^{-1}(y) \cap G = g^{-1}(y) \cap G$ for all $y \in Y$. The second condition we demand for $\sim$ is that the measure structures are equal: $\beta_y = \alpha_y$ for all $y \in Y$. Furthermore, we say $(A_0, f) \sim (A_1, f')$ in $Disint(A, Y)$ if $\mu(A_0 \triangle A_1) = 0$ and $f|_{A_0 \cap A_1} \sim f'|_{A_0 \cap A_1}$ as disintegrations.

We have shown (in chapter 1) that restriction of a disintegration to a subspace yields a disintegration. Thus, $Disint(-, Y)$ is a presheaf (on $\mathcal{A}$). Furthermore, for the sheaf condition, this allows us to choose as representatives for a compatible family, $(C_n, (f_n, (\mu|_{C_n})_y)$ with the C_n's disjoint ($C_n = A_{0n} \setminus \bigcup_{i<n} A_{0i}$ as in example 4 above). Define $f : \bigcup_n C_n \longrightarrow Y, (\mu|_{\cup C_n})_y)$ as follows: Let $x \in C_n$ (unique n) and put $f(x) = f_n(x)$. Then f is measurable as in example 4.

Lemma 3.2.2 *$(\mu|_{\cup C_n})_y$ is a measure for each y and $y \longmapsto (\mu|_{\cup C_n})_y$ is ν-measurable.*

Proof: $(\mu|_{\cup C_n})_y(\emptyset \cap \bigcup_n C_n \cap f_n^{-1}(y)) = \sum (\mu|_{C_n})_y(\emptyset \cap C_n \cap f_n^{-1}(y)) = \sum 0 = 0.$

$(\mu|\cup C_n)_y(\bigcup_i \cap \bigcup_n C_n \cap f^{-1}(y)) = \sum_n \sum_i (\mu|_{\cup C_n})_y(K_i \cap C_n \cap f_n^{-1}(y)) = \sum_i \sum_n$

$= \sum_i (\mu|_{\cup C_n})_y(K_i \cap \bigcup_n C_n \cap f^{-1}(y)).$

Since $(\mu|_{\cup C_n})_y$ is a sum of nonnegative y-measurable functions (the latter is axiom 1 for the $(\mu|_{C_n})_y$'s), it is nonnegative and y-measurable. ∎

Remark: If the $(\mu|_{C_n})_y$'s are bounded, there is no guarantee that these are bounded *over* n. Thus, $Disint(-, Y)$ is not a sheaf (this is essentially the same problem as with $L^2(-)$). But, it almost is; everything works except boundedness (the extension respects $\sim$ and even axiom 2 holds). □

Lemma 3.2.3 *Axiom 2 holds for $(\mu|_{\cup C_n})_y$.*

Proof:

$$\begin{aligned}
&\int_Y (\mu|_{\cup C_n})_y(A \cap \dot{\bigcup_n} C_n \cap f_n^{-1}(y))d\nu(y) \\
&= \sum_n \int_Y (\mu|_{C_n})_y(A \cap C_n \cap f_n^{-1}(y))d\nu(y) \quad \text{(MCT)} \\
&= \sum_n (\mu|_{C_n}(A \cap C_n)) \quad \text{(Axiom 2)} \\
&= \mu|_{\cup C_n}(A \cap \dot{\bigcup_n} C_n) \quad (\mu \text{ a measure}). \ \blacksquare
\end{aligned}$$

Finally, suppose $(C_n, f_n) \sim (D_n, g_n)$ and let G_n be the good set for f_n and g_n. Then $G = \bigcup G_n$ is the good set for f and g, so $(G, f|_G) = (G, g|_G)$.

3.3 The Locale L(X)

3.3.1 Subobjects of 1

As we noted above, the constantly 1 sheaf is terminal in $MEAS(X)$. Write $L(X) = Sub(1)$. A subpresheaf, $F \hookrightarrow G$, means $F(A) \subseteq G(A)$, $\forall A \in \mathcal{A}$ (limits in $MEAS(X) = Sh(\mathcal{A})$ are as in $PSh(\mathcal{A})$; in particular, monomorphisms are the same). So, U a subpresheaf (= subfunctor) of 1 means that $U(A) \subseteq 1$, $\forall A \in \mathcal{A}$ which implies $U(A) = 1$ or $U(A) = \emptyset$. We consider U as a "characteristic function," put $\mathcal{S} := \{A | U(A) = 1\}$, and translate $U \in Sub(1)$ in terms of $\mathcal{S}$.

Subpresheaf:

$$\begin{array}{ccccc}
A & & U(A') & \longrightarrow & 1(A') = 1 \\
\big\downarrow \cap & \rightsquigarrow & \big\downarrow & & \big\downarrow \\
A' & & U(A) & \longrightarrow & 1(A) = 1
\end{array}$$

If $A' \in \mathcal{S}$, (i.e. $U(A') = 1$), then we must have $U(A) = 1$ which means $A \in \mathcal{S}$; i.e. $\mathcal{S}$ is downclosed.

Sheaf condition: Let $\{A_n\}_{n=1}^{\infty} \in Cov(A)$, then

$$U(A) \longrightarrow \prod_{n=1}^{\infty} U(A_n) \rightrightarrows \prod_{n,m=1}^{\infty} U(A_n \cap A_m)$$

is an equalizer. If any $U(A_{n_0}) = \emptyset$ then $\prod U(A_n) = \emptyset$ and $\prod U(A_n \cap A_m) = \emptyset$ (in particular, $U(A_{n_0} \cap A_{n_0}) = \emptyset$) so we have

$$U(A) \longrightarrow \emptyset \rightrightarrows \emptyset$$

which implies $U(A) = \emptyset$. If all the $U(A_n) = 1$ (i.e. $A_n \in \mathcal{S}\ \forall n$), then $\prod U(A_n) = 1$ and $\prod U(A_n \cap A_m) = 1$ (since $U(A_n) = 1 \longrightarrow U(A_n \cap A_m) \Rightarrow U(A_n \cap A_m) \neq \emptyset$). So, the sheaf condition says

$$U(A) \longrightarrow 1 \rightrightarrows 1$$

is an equalizer which implies $U(A) = 1$ (i.e. $A \in \mathcal{S}$). And so, $\mathcal{S}$ is closed under countable unions and sets of measure zero.

Theorem 3.3.1 *The elements of $L(X)$ form a set of generators for $MEAS(X)$ (i.e. $MEAS(X)$ satisfies (SG) of [PTJ1,p.145]).*

We first require an obvious but important lemma.

Lemma 3.3.1 *Let $A' \subseteq A$, $\mu(A \setminus A') = 0$. Then $\rho^A_{A'} : F(A) \longrightarrow F(A')$ is an isomorphism.*

Proof: A' covers A so

$$F(A) \longrightarrow F(A') \rightrightarrows F(A' \cap A')$$

being an equalizer implies $F(A) \longrightarrow F(A')$ is an isomorphism. ∎

Remark: Not only are $F(A)$ and $F(A')$ isomorphic, they are canonically isomorphic. □

Proof: (of theorem 3.3.1): Let $\alpha, \beta : F \longrightarrow G \in MEAS(X)$ with $\alpha \neq \beta$. Then there is an A_0 and an $x \in F(A_0)$ such that $\alpha_{A_0}(x) \neq \beta_{A_0}(x)$. Let $U_{A_0}(A)$ $:= \begin{cases} 1 & \text{if } \mu(A \setminus A_0) = 0 \\ \emptyset & \text{else} \end{cases}$ be in $L(X)$. We define $U_{A_0}(A) \xrightarrow{\eta_A} F(A)$ as follows: If $U_{A_0}(A) = \emptyset$, define η_A as the unique map. If $U_{A_0}(A) = 1$, consider the composite:

$$F(A_0) \xrightarrow{\rho^{A_0}_{A \cap A_0}} F(A \cap A_0) \xrightarrow{(\rho^{A}_{A \cap A_0})^{-1}} F(A)$$

(note: $\mu(A \setminus (A \cap A_0)) = 0$ and $A \cap A_0 \subseteq A$; apply lemma 3.3.1). Define $\eta_A(\star) := (\rho^A_{A \cap A_0})^{-1} \rho^{A_0}_{A \cap A_0}(x)$.

η_A is natural: Let $A' \subseteq A$ and consider:

$$\begin{tikzcd} U_{A_0}(A) \arrow[r] \arrow[d] & F(A) \arrow[d] \\ U_{A_0}(A') \arrow[r] & F(A') \end{tikzcd}$$

If $U_{A_0}(A) = \emptyset$, then both composites are the unique map to $F(A')$ so the square commutes. If $U_{A_0}(A) = 1$, then $U_{A_0}(A') = 1$ $(\mu(A \setminus A_0) = 0 \Rightarrow \mu(A' \setminus A_0) = 0)$ and we have $\star \mapsto (\rho^A_{A \cap A_0})^{-1} \rho^{A_0}_{A \cap A_0}(x) \mapsto \rho^A_{A'}(\rho^A_{A \cap A_0})^{-1} \rho^{A_0}_{A \cap A_0}(x)$ as the top right composite and $\star \mapsto \star \mapsto (\rho^{A'}_{A' \cap A_0})^{-1} \rho^{A_0}_{A' \cap A_0}(x)$ as the left bottom composite. Now,

$$\begin{tikzcd}
F(A_0) \arrow[r, "\rho^{A_0}_{A\cap A_0}"] \arrow[d, "\rho^{A_0}_{A'\cap A_0}"'] & F(A\cap A_0) \arrow[r, "(\rho^{A}_{A\cap A_0})^{-1}"] \arrow[dl, "\rho^{A\cap A_0}_{A'\cap A_0}"] & F(A) \arrow[d, "\rho^{A}_{A'}"] \\
F(A'\cap A_0) \arrow[rr, "(\rho^{A'}_{A'\cap A_0})^{-1}"'] & & F(A')
\end{tikzcd}$$

commutes by functoriality of F (the triangle commutes and the trapezoid without inverses commute by functoriality of F, so the trapezoid with the inverses commutes).

η separates α and β: $U_{A_0}(A)\xrightarrow{\eta_{A_0}}F(A_0)\xrightarrow{\alpha_{A_0}}G(A_0)$ is $\star\mapsto(\rho^{A_0}_{A_0\cap A_0})^{-1}\rho^{A_0}_{A_0\cap A_0}(x)=x\mapsto\alpha_{A_0}(x)$ and $U_{A_0}(A_0)\xrightarrow{\eta_{A_0}}F(A_0)\xrightarrow{\beta_{A_0}}G(A_0)$ is $\star\mapsto x\mapsto\beta_{A_0}(x)\neq\alpha_{A_0}(x)$ as required. ∎

3.3.2 L(X) as a Locale

We saw that $L(X)=\{\mathcal{S}\subseteq\mathcal{A}\mid\mathcal{S}$ downclosed, $\mathcal{S}$ contains measure zeroes, $\mathcal{S}$ closed under countable unions $\}$. Explicitly, these mean:

1. downclosed: $A\in\mathcal{A}\ \exists S\in\mathcal{S}(A\subseteq S)\Rightarrow A\in\mathcal{S}$
2. measure zeroes: $A\in\mathcal{A},\ \mu(A)=0\Rightarrow A\in\mathcal{S}$
3. unions: $S_i\in\mathcal{S}\Rightarrow\bigcup_{i=1}^{\infty}S_i\ \in\mathcal{S}$

$L(X)$ is the locale of subobjects of 1 in the (localic) topos $MEAS(X)$. It is a poset under $\subseteq$. In the next section, we shall show that $L(X)$ is a quotient and a complete Boolean algebra. It is instructive, however, to study it as a locale first. We will continue, in the next few paragraphs, to describe various operations on $L(X)$ and then discuss functoriality.

Join: $\mathcal{S}\vee\mathcal{T}=\{S\cup T\mid S\in\mathcal{S},\ T\in\mathcal{T}\}$

$\mathcal{S} \cup \mathcal{T}$ is not closed under binary unions for pick $S \in \mathcal{S} \setminus \mathcal{T}$, $T \in \mathcal{T} \setminus \mathcal{S}$ then $S \cup T$ is not in $\mathcal{S} \cup \mathcal{T}$. So, we must "close up" $\mathcal{S} \cup \mathcal{T}$ under binary unions to produce the larger $\mathcal{S} \vee \mathcal{T}$. Since $\mathcal{S} \subseteq \mathcal{S} \cup \mathcal{T} \subseteq \mathcal{S} \vee \mathcal{T}$, we see that the latter contains the measure zero sets. Furthermore, suppose $K \subseteq K_1 \cup K_2 \in \mathcal{S} \cup \mathcal{T}$, then $K = (K_1 \cap K) \cup (K_2 \cap K)$ and $K \cap K_j \in \mathcal{S} \cup \mathcal{T}$ since each is downclosed.

More generally, $\bigvee_{i\in I} \mathcal{S}_i = \{\bigcup_{j=1}^{\infty} K_j \mid K_j \in \bigcup_{i\in I} \mathcal{S}_i\}$. We have $\mathcal{S}_i \subseteq \mathcal{T}\ \forall i \Leftrightarrow \bigvee_{i\in I} \mathcal{S}_i \subseteq \mathcal{T}$. Suppose $\mathcal{S}_i \subseteq \mathcal{T}\ \forall i$ and let $\bigcup_{j=1}^{\infty} K_j \in \bigvee_{i\in I} \mathcal{S}_i$. Then $K_j \in \mathcal{S}_{i_j} \subseteq \mathcal{T}\ \forall j$ so $\bigcup K_j \in \mathcal{T}$ since $\mathcal{T}$ is closed under countable unions. The converse is trivial since $\mathcal{S}_i \subseteq \bigvee_{i\in I} \mathcal{S}_i\ \forall i$.
□

Meet: $\mathcal{S} \wedge \mathcal{T} = \mathcal{S} \cap \mathcal{T}$, more generally, $\bigwedge_{i\in I} \mathcal{S}_i = \bigcap_{i\in I} \mathcal{S}_i$.

Both $\mathcal{S}$ and $\mathcal{T}$ contain the measure zeroes, so $\mathcal{S} \wedge \mathcal{T}$ does. Next, suppose $A \subseteq K \in \mathcal{S} \wedge \mathcal{T}$. Then $A \subseteq K \in \mathcal{S} \Rightarrow A \in \mathcal{S}$ and similarly with $\mathcal{T}$ whence $A \in \mathcal{S} \wedge \mathcal{T}$. Finally, let $A_i \in \mathcal{S} \wedge \mathcal{T}$ then $A_i \in \mathcal{S} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{S}$ and $A_i \in \mathcal{T} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{T} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{S} \wedge \mathcal{T}$.

Now, suppose $\mathcal{R} \subseteq \mathcal{S} \cap \mathcal{T}$ then $\mathcal{R} \subseteq \mathcal{S}$ and $\mathcal{R} \subseteq \mathcal{T}$. Conversely, suppose $\mathcal{R} \subseteq \mathcal{S}$ and $\mathcal{R} \subseteq \mathcal{T}$ then $\mathcal{R} \subseteq \mathcal{S} \cap \mathcal{T}$ (and similarly for arbitrary infima). □

The above shows that $L(X)$ is a complete lattice with the following definitions of top and bottom.

Top: $1_X := \mathcal{A}$ □

Bottom: $0_X := \mathcal{N} := \{A \in \mathcal{A} \mid \mu(A) = 0\}$. □

Locale: We must show $\mathcal{S} \wedge \bigvee_{i\in I} \mathcal{T}_i = \bigvee_{i\in I} (\mathcal{S} \wedge \mathcal{T}_i)$.

Let $A \in \bigvee_{i\in I} \mathcal{S} \wedge \mathcal{T}_i$ then $A = \bigcup_{j=1}^{\infty} K_j$, $K_j \in \bigcup_{i\in I} \mathcal{S} \cap \mathcal{T}_i = \mathcal{S} \cap \bigcup_{i\in I} \mathcal{T}_i$, so $K_j \in \mathcal{S}$ and $K_j \in \bigcup \mathcal{T}_i$ whence $K_j \in \mathcal{S} \wedge \bigvee \mathcal{T}_i$ and so $A \in \mathcal{S} \wedge \bigvee \mathcal{T}_i$. Conversely, let $A \in \mathcal{S} \wedge \bigvee \mathcal{T}_i$ then $A \in \mathcal{S}$ and $A \in \bigvee \mathcal{T}_i$ or $A = \bigcup_{j=1}^{\infty} K_j$, $K_j \in \bigcup \mathcal{T}_i$. Now, each $K_j \in \bigcup_{i\in I} \mathcal{S} \cap \mathcal{T}_i$

so $A \in \bigvee_{i \in I} \mathcal{S} \wedge \mathcal{T}_i$ $\square$

Implies: The above shows that $\mathcal{S} \wedge -$ is cocontin ous, which, of course, is true in any locale, so the adjoint functor theorem gives its right adjoint as:

$\mathcal{S} \to \mathcal{T} = \bigvee\{\mathcal{R} \in L(X) \mid \mathcal{S} \wedge \mathcal{R} \subseteq \mathcal{T}\}$. $\square$

Not: Recall that in a Heyting Algebra (in particular, a locale) H, $\neg a := a \to 0$. So, in $L(X)$, $\neg\mathcal{S} = \mathcal{S} \to 0_X = \bigvee\{\mathcal{R} \in L(X) \mid \mathcal{S} \cap \mathcal{R} = 0_X\}$. $\square$

Remark: There are better descriptions of implies and not. Let $A \in \mathcal{A}$ and put $\mathcal{U}_A = \{A' \in \mathcal{A} \mid \mu(A' \setminus A) = 0\}$ = the smallest element of $L(X)$ which contains A (in section 3.3.4, we will study this in detail and will denote it by $\overline{A}$). Then

$$
\begin{aligned}
A \in (\mathcal{S} \longrightarrow \mathcal{T}) &\Leftrightarrow \mathcal{U}_A \subseteq \mathcal{S} \longrightarrow \mathcal{T} \\
&\Leftrightarrow \mathcal{U}_A \wedge \mathcal{S} \subseteq \mathcal{T} \\
&\Leftrightarrow \forall S \in \mathcal{S},\ \mu(S \setminus A) = 0 \Rightarrow S \in \mathcal{T}
\end{aligned}
$$

and

$$
\begin{aligned}
A \in \neg\mathcal{S} &\Leftrightarrow A \in \mathcal{S} \longrightarrow \mathbf{0} \\
&\Leftrightarrow \forall S \in \mathcal{S},\ \mu(S \setminus A) = 0 \Rightarrow \mu(S) = 0. \quad \square
\end{aligned}
$$

Next, we look at the action of L on **<u>MOR</u>** morphisms. Suppose $(X, \mathcal{A}, \mu) \xrightarrow{f} (Y, \mathcal{B}, \nu) \in$**<u>MOR</u>**. We have the direct image of a geometric morphism $MEAS(X) \xrightarrow{f_*} MEAS(Y)$ given by $(f_*F)(B) := F(f^{-1}(B))$ for $F \in MEAS(X)$ and $B \in \mathcal{B}$.

$L(X)$ is the locale of subobjects of $1 \in MEAS(X)$. To each $\mathcal{S} \in L(X)$, there corresponds a sheaf $U_S \in MEAS(X)$ defined by $U_S = \begin{cases} 1 & \text{if } f^{-1}(B) \in \mathcal{S} \\ \emptyset & \text{else.} \end{cases}$

Now, $f.U_S(B) = U_S f^{-1}(B) = \begin{cases} 1 & \text{if } f^{-1}(B) \in \mathcal{S} \\ \emptyset & \text{else.} \end{cases}$ So, this suggests a function $L(X) \xrightarrow{\hat{f}} L(Y);\ \mathcal{S} \mapsto \{B \in \mathcal{B} \mid f^{-1}(B) \in \mathcal{S}\} =: \mathcal{K}$.

Proposition 3.3.1 $\mathcal{K} \in L(Y)$

Proof: The three axioms:

1. downclosed: Suppose $B' \subseteq B \in \mathcal{K}$. Then $f^{-1}(B') \subseteq f^{-1}(B) \in \mathcal{S}$ but $\mathcal{S}$ is downclosed so $f^{-1}(B') \in \mathcal{S}$, whence $B' \in \mathcal{K}$.

2. measure zeroes: Suppose $\nu(B) = 0$. Then $\mu(f^{-1}(B)) = 0$ since $f \in$ <u>**MOR**</u>. So, $f^{-1}(B) \in \mathcal{S}$ since $\mathcal{S}$ contains measure zeroes, whence $B \in \mathcal{K}$.

3. countable unions: Suppose $B_i \in \mathcal{K}$. Then $f^{-1}(\bigcup_{i=1}^{\infty} B_i) = \bigcup_{i=1}^{\infty} f^{-1}(B_i) \in \mathcal{S}$ since $\mathcal{S}$ is closed under countable unions. So $\bigcup_{i=1}^{\infty} B_i \in \mathcal{K}$. ∎

Proposition 3.3.2 $\hat{1}_X = 1_{L(X)}$ *and* $\widehat{fg} = \hat{f}\hat{g}$.

Proof: $\hat{1}_X(\mathcal{S}) = \{B \in \mathcal{B} \mid 1_X^{-1}(B) \in \mathcal{S}\} = \mathcal{S}$.
Consider $L(X, \mathcal{A}, \mu) \xrightarrow{\hat{f}} L(Y, \mathcal{B}, \nu) \xrightarrow{\hat{g}} L(Z, \mathcal{C}, \rho)$

$\widehat{gf}(\mathcal{S}) = \{C \in \mathcal{C} \mid f^{-1}g^{-1}(C) \in \mathcal{S}\}$ and $\hat{g}(\hat{f}(\mathcal{S})) = \hat{g}\{B \in \mathcal{B} \mid f^{-1}(B) \in \mathcal{S}\} = \{C \in \mathcal{C} \mid g^{-1}(C) \in \mathcal{K}\} = \{C \in \mathcal{C} \mid f^{-1}g^{-1}(C) \in \mathcal{S}\}$ as required. ∎

Proposition 3.3.3 *If $f, g : X \longrightarrow Y$ are measurable and agree except on a set of measure zero, then $\hat{f} = \hat{g}$.*

Proof: Let $D = \{x \in X \mid f(x) \neq g(x)\}$ and let $\hat{f}(\mathcal{S}) = \{B \in \mathcal{B} \mid f^{-1}(B) \in \mathcal{S}\}$ $=: \mathcal{K}$ and $\hat{g}(\mathcal{S}) = \{B \in \mathcal{B} \mid g^{-1}(B) \in \mathcal{S}\} =: \mathcal{L}$.
We will prove $\mathcal{K} \subseteq \mathcal{L}$, the other direction being similar.
Let $B \in \mathcal{K}$. Now, $B \subseteq (B \setminus D) \cup D$ and $g^{-1}(B) \subseteq (g^{-1}(B \setminus D)) \cup g^{-1}(D)$. But $g^{-1}(B \setminus D) \in \mathcal{S}$ since $g^{-1}(B \setminus D) = f^{-1}(B \setminus D) \in \mathcal{S}$ and $g^{-1}(D) \in \mathcal{S}$ since $g \in$<u>**MOR**</u> and $\mathcal{S}$ contains the measure zeroes. And so $g^{-1}(B) \in \mathcal{S}$, whence $\mathcal{K} \subseteq \mathcal{L}$. ∎

Thus, $\hat{f}$ corresponds to an equivalence class, although we won't explicitly consider it as such. In the examples below, we shall see that, in general, $\hat{f} = \hat{g} \not\Rightarrow f \sim g$, but it often does.

Proposition 3.3.4 *$\hat{f}$ has a left adjoint, $\check{f}$, i.e. $\check{f}(\mathcal{T}) \subseteq \mathcal{S}$ iff $\mathcal{T} \subseteq \hat{f}(\mathcal{S})$ for $\mathcal{S} \in L(X)$, $\mathcal{T} \in L(Y)$.*

Proof: By the adjoint functor theorem for posets, we need only show $\hat{f}$ preserves order and infima; in which case $\check{f}(\mathcal{T}) = \bigwedge\{\mathcal{S} \in L(X) \mid \mathcal{T} \subseteq \hat{f}(\mathcal{S})\}$
$= \cap\{\mathcal{S} \in L(X) \mid \mathcal{T} \subseteq \hat{f}(\mathcal{S})\}$.
order: Let $\mathcal{S} \subseteq \mathcal{S}'$ and let $B \in \hat{f}(\mathcal{S})$. Then $f^{-1}(B) \in \mathcal{S} \Rightarrow f^{-1}(B) \in \mathcal{S}'$
$\Rightarrow B \in \hat{f}(\mathcal{S}')$.
infima: $\hat{f}(\bigcap \mathcal{S}_i) = \{B \in \mathcal{B} \mid f^{-1}(B) \in \bigcap \mathcal{S}_i\} = \{B \in \mathcal{B} \mid f^{-1}(B) \in \mathcal{S}_i\ \forall i\}$. On the other hand, $\bigcap \hat{f}\mathcal{S}_i = \bigcap\{B \in \mathcal{B} \mid f^{-1}(B) \in \mathcal{S}_i\} = \{B \in \mathcal{B} \mid f^{-1}(B) \in \mathcal{S}_i\ \forall i\}$. ∎
Remark: 1. If $(X, \mathcal{A}, \mu) \xrightarrow{f} (Y, \mathcal{B}, \nu)$ is in **<u>M0R</u>**, then f^{-1} is a morphism of sites (i.e. preserves covers; see [B&W, p.233]) and so yields a geometric morphism $MEAS(X) \underset{f_*}{\overset{f^*}{\leftrightarrows}} MEAS(Y)$. And, thus, we could have gotten $\check{f}$ by considering f^*, $f^*(G)(A) := a(\mathrm{colim}_{A \subseteq f^{-1}(B)} G(B))$. □

Before checking the left exactness of $\tilde{f}$, we give another description, indeed a working definition.

Definition 3.3.1 $\tilde{f}(\mathcal{T}) := \{A \in \mathcal{A} \mid \exists B \in \mathcal{T}$ *such that* $\mu(A \setminus f^{-1}(B)) = 0\}$. □

Lemma 3.3.2 $\tilde{f}(\mathcal{T}) \in L(X)$.

Proof: measure zeroes: Put $B = \emptyset$. Then $f^{-1}(B) = \emptyset$ and $\mu(A \setminus f^{-1}(B)) = \mu(A)$. If $\mu(A) = 0$ then $A \in \tilde{f}(\mathcal{T})$.
downclosed: If $A' \subseteq A$ and $\mu(A \setminus f^{-1}(B)) = 0$, then $\mu(A' \setminus f^{-1}(B)) = 0$.

countable unions: Let $A_i \in \tilde{f}(\mathcal{T})$ then $\exists B_i$ such that $\mu(A_i \setminus f^{-1}(B_i)) = 0$. We claim that $\bigcup_{i=1}^{\infty} B_i$ is the "B" for $A = \bigcup_{i=1}^{\infty} A_i$. $\mu(\bigcup A_i \setminus f^{-1}(\bigcup B_i)) \leq \sum \mu(A_i \setminus f^{-1}(\bigcup B_i))$ $= \sum \mu(A_i \setminus \bigcup f^{-1}(B_i)) \leq \sum \mu(A_i \setminus f^{-1}(B_i)) = \sum 0 = 0$ as required (the last inequality follows from lemma 3.2.1). ∎

Lemma 3.3.3 $\tilde{f} \dashv \hat{f}$

Proof: We wish to show $\tilde{f}(\mathcal{T}) \subseteq \mathcal{S}$ iff $\mathcal{T} \subseteq \hat{f}(\mathcal{S})$ or $\{A \in \mathcal{A} \mid \exists B \in \mathcal{T}$ $\mu(A \setminus f^{-1}(B)) = 0\} \subseteq \mathcal{S}$ iff $\mathcal{T} \subseteq \{B \in \mathcal{B} \mid f^{-1}(B) \in \mathcal{S}\}$.

$\Rightarrow$: Let $T \in \mathcal{T}$. $\mu(f^{-1}(B) \setminus f^{-1}(B)) = 0$ so $f^{-1}(T) \in \tilde{f}(\mathcal{T})$ and so $f^{-1}(T) \in \mathcal{S}$, whence $T \in \hat{f}(\mathcal{S})$.

$\Leftarrow$: Let $A \in \tilde{f}(\mathcal{T})$. Then, $\exists B \in \mathcal{T}$ with $\mu(A \setminus f^{-1}(B)) = 0$. Now, $B \in \mathcal{T} \Rightarrow f^{-1}(B) \in \mathcal{S}$ and $A \subseteq (A \setminus f^{-1}(B)) \cup f^{-1}(B)$. But, $A \setminus f^{-1}(B) \in \mathcal{S}$ since it has measure 0 and $f^{-1}(B) \in \mathcal{S}$ so $A \in \mathcal{S}$ since $\mathcal{S}$ is closed under binary unions and is downclosed. ∎

Lemma 3.3.4 $\hat{f}$ *is order preserving.* ∎

Lemma 3.3.5 $\hat{f}$ *is left exact.*

Proof: We must show that $\tilde{f}$ sends 1_Y to 1_X and preserves finite, nonempty infima.

Top: $\tilde{f}(\mathcal{B}) = \{A \in \mathcal{A} \mid \exists B \in \mathcal{B},\ \mu(A \setminus f^{-1}(B)) = 0\}$. Let $A \in \mathcal{A}$. Then $A \subseteq X$ so $\mu(A \setminus f^{-1}(Y)) = 0$ (and $Y \in \mathcal{B}$) so $A \in \tilde{f}(\mathcal{B})$ whence $\tilde{f}(\mathcal{B}) = \mathcal{A}$.

$\cap$: $\tilde{f}(\mathcal{T} \cap \mathcal{U}) = \{A \in \mathcal{A} \mid \exists B \in \mathcal{T} \cap \mathcal{U},\ \mu(A \setminus f^{-1}(B)) = 0\} = \{A \in \mathcal{A} \mid \exists B \in \mathcal{T}$ and $B \in \mathcal{U},\ \mu(A \setminus f^{-1}(B)) = 0\}$

On the other hand, $\tilde{f}(\mathcal{T}) \cap \tilde{f}(\mathcal{U}) = \{A \in \mathcal{A} \mid \exists B' \in \mathcal{T}\ \mu(A \setminus f^{-1}(B')) = 0\} \cap \{A \in \mathcal{A} \mid \exists B'' \in \mathcal{T},\ \mu(A \setminus f^{-1}(B'')) = 0\}$

Now, $\tilde{f}(\mathcal{T} \cap \mathcal{U}) \subseteq \tilde{f}(\mathcal{T}) \cap \tilde{f}(\mathcal{U})$, for the B works as both a B' and a B''. Conversely, let $A \in \tilde{f}(\mathcal{T}) \cap \tilde{f}(\mathcal{U})$ and let $B = B' \cap B''$. Note that $B \in \mathcal{T} \cap \mathcal{U}$ and $\mu(A \setminus B' \cap B'') = \mu(A \setminus B' \cup A \setminus B'') \leq \mu(A \setminus B') + \mu(A \setminus B'') = 0 + 0 = 0$ as required. ∎

Remarks: 1. The above proof is easily extended to show that $\hat{f}$ preserves countable (but not arbitrary) infima.

2. Since $\hat{f}$ is a right adjoint, it preserves 0 but we can prove this directly:

$\hat{f}(0_Y) = 0_X$: $\hat{f}(0_Y) = \{A \in \mathcal{A} \mid \exists B \in 0_Y,\ \mu(A \setminus f^{-1}(B)) = 0\}$

$\subseteq$: Let $A \in \hat{f}(0_Y)$. Since $B \in 0_Y$ and f is **MOR**, $\mu(f^{-1}(B)) = 0$ so $A \subseteq A \setminus f^{-1}(B) \cup f^{-1}(B) \Rightarrow \mu(A) \leq \mu(A \setminus f^{-1}(B)) + \mu(f^{-1}(B)) = 0 + 0 = 0$.

$\supseteq$: Take $B = \emptyset \in 0_Y$ $\square$

In view of the above lemata, we have proved the following:

Theorem 3.3.2 *$\hat{f}$ is the inverse image of a continuous morphism (etymology: [PTJ2, p. 39]) $L(X) \to L(Y)$ which preserves countable limits.* ∎

And so, we have a functor: $\underline{\mathbf{MOR}} \xrightarrow{L(-)} \underline{\mathbf{Loc}}$. Recall that if $f \sim g$, then $\hat{f} = \hat{g}$ and adjoints are unique (up to isomorphism, which is $=$ in case the 2-cells are $\subseteq$). So, we have a functor $\underline{\mathbf{MORE}} \xrightarrow{L(-)} \underline{\mathbf{Loc}}$.

3.3.3 Examples

Above, we implicitly referred to an example of an $L(X)$. It is time for some more detailed examples.

Example 1: $L(\emptyset, \{\emptyset\}, 0) = 1$ is initial $\underline{\mathbf{Loc}}$. Let $L \in \underline{\mathbf{Loc}}$. Define $1 \overset{i^*}{\underset{i_*}{\leftrightarrows}} L$ by $i_*(0) = 1 \in L$, $i^*(l) = 0$ for all $l \in L$. $\square$

Example 2: $L(1, 2, \text{counting}) = 2 = \{0, 1\}$ is terminal in $\underline{\mathbf{Loc}}$. For $L \in \underline{\mathbf{Loc}}$, define $L \overset{t^*}{\underset{t_*}{\leftrightarrows}} 2$ as $t^*(0) = 0$, $t^*(1) = 1$, and $t_*(l) = \begin{cases} 0 & l \neq 1 \\ 1 & l = 1. \end{cases}$ $\square$

Example 3: Examples 1 and 2 are special cases of $(X, \mathcal{P}(X), \text{counting})$. In particular, $L(\mathbf{N}, \mathcal{P}(\mathbf{N}), \text{counting}) = \{\mathcal{P}(A) \mid A \subseteq \mathbf{N}\}$ (since $\mathbf{N}$ is countable). Although this is not a finite measure space, we will study it in some detail. $\mathcal{P}(\mathbf{N}) \xrightarrow{\varphi} L(\mathbf{N})$;

$A \mapsto \mathcal{P}(A)$, is an isomorphism of locales (φ preserves $\bigvee$, $\cap$, $0_N = \{\emptyset\}$, and $1_N = \mathcal{P}(\mathbf{N})$).

Let $\mathbf{N} \xrightarrow{f} \mathbf{N}$ be a function (every function is measurable and **MOR**). There is a nice description of $\check{f}(\mathcal{T})$.

Lemma 3.3.6 $\check{f}(\mathcal{T}) = \{A \in \mathcal{A} \mid \exists B \in \mathcal{T},\ A \subseteq f^{-1}(B)\}$ *in this case.*

Proof: For the counting measure, $\mu(C) = 0 \Rightarrow C = \emptyset$. ∎

Furthermore, we have a "fullness" of $L(-)$:

Theorem 3.3.3 *Let* $L(\mathbf{N}) \xleftarrow{\alpha} L(\mathbf{N})$ *preserve binary* $\wedge = \cap$, 0_N, 1_N, *and countable* $\bigvee$. *Then* $\alpha = \check{f}$ *for some* $\mathbf{N} \xrightarrow{f} \mathbf{N}$.

Proof: Elements of $L(\mathbf{N})$ are $\mathcal{P}(A)$'s so let $\alpha(\mathcal{P}(x)) =: \mathcal{P}(A_x)$ for $x \in \mathbf{N}$. The A_x's are disjoint for

$$\begin{aligned}
\mathcal{P}(A_x) \cap \mathcal{P}(A_y) &= \alpha(\mathcal{P}(x)) \cap \alpha(\mathcal{P}(y)) \\
&= \alpha(\mathcal{P}(x) \cap \mathcal{P}(y)) \\
&= \alpha(\mathcal{P}(\{x\} \cap \{y\})) \\
&= \begin{cases} \alpha(\mathcal{P}(x)), & x = y \\ \alpha(\emptyset), & x \neq y \end{cases} \\
&= \begin{cases} \mathcal{P}(A_x), & x = y \\ \{\emptyset\}, & x \neq y. \end{cases}
\end{aligned}$$

Furthermore, the A_x's partition $\mathbf{N}$: $\mathcal{P}(\mathbf{N}) = \alpha(\mathcal{P}(\mathbf{N})) = \alpha(\mathcal{P}(\bigvee_{x \in N} x))$ $= \alpha(\bigvee_{x \in N} \mathcal{P}(x)) = \bigvee_{x \in N} \alpha(\mathcal{P}(x)) = \bigvee_{x \in N} \mathcal{P}(A_x)$. So, if $y \in \mathbf{N}$, $\exists x$ such that $y \in A_x$ for, if not, then $\bigvee \mathcal{P}(A_x) \neq \mathcal{P}(\mathbf{N})$. Define $\mathbf{N} \xrightarrow{f} \mathbf{N}$ by $f(y) = x$ if $y \in A_x$. Then $\check{f} = \alpha$. We need only check that $\check{f}(\mathcal{P}(x)) = \alpha(\mathcal{P}(x)) = \mathcal{P}(A_x)$ $\forall x \in \mathbf{N}$ $\check{f}(\mathcal{P}(x)) = \{A \in \mathcal{A} \mid \exists B \in \mathcal{P}(x),\ A \subseteq f^{-1}(B)\}$

$\subseteq$: Let $A \in \check{f}(\mathcal{P}(x))$. The only B's are $B = \emptyset$ and $B = \{x\}$. If $A \subseteq f^{-1}(\emptyset)$ then $A = \emptyset \Rightarrow A \subseteq A_x \Rightarrow A \in \mathcal{P}(A_x)$. If $A \subseteq f^{-1}(x)$ then $A \subseteq A_x \Rightarrow A \in \mathcal{P}(A_x)$.

$\supseteq$: Suppose $A \subseteq A_x$. Then $f(A) \subseteq f(A_x) = \{x\}$ and $A \subseteq f^{-1}f(A)$ $= f^{-1}(x) \Rightarrow A \in \check{f}(\mathcal{P}(x))$. ∎

Remarks:

1. This proof may be modified to prove $\check{f} = \check{g} \Rightarrow f \sim g$ (which means $f = g$) for $f, g : \mathbf{N} \to \mathbf{N}$.

2. It turns out that $L(\mathbf{N}) \xleftarrow{\check{f}} L(\mathbf{N})$ is $\check{f}(\mathcal{P}(A)) = \mathcal{P}(\forall_f A))$ since $f^{-1}(B) \subseteq A$ iff $B \subseteq \forall_f A$ by definition. □

It is not true, in general, that $\hat{f} = \hat{g} \Rightarrow f \sim g$ as the following counterexample shows.

(Counter)example 4: Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ with Y indiscreet. If $\nu(Y) \neq 0$ then f and g are both in **<u>MOR</u>**. Now, $\hat{f} = \hat{g}$ says $\hat{f}(\mathcal{S}) = \hat{g}(\mathcal{S})$, $\forall \mathcal{S}$ which says $\{B \in \mathcal{B} | f^{-1}(B) \in \mathcal{S}\} = \{B \in \mathcal{B} | g^{-1}(B) \in \mathcal{S}\}$. For any functions f and g we have $f^{-1}(\emptyset) = \emptyset = g^{-1}(\emptyset)$ and $f^{-1}(Y) = X = g^{-1}(Y)$, so $\hat{f} = \hat{g}$. But, there are many pairs for which $f \not\sim g$ (if $|Y| \geq 2$). □

Example 5: Countable cocountable Space: Again, this is a "pathological" example and it isn't of finite measure.

Let X be an uncountable set and let $\mathcal{A}$ consist of all subsets which are either countable or cocountable, the complement of a countable set. This is a σ-algebra.

Define $\mu(A) = \begin{cases} 0 & A \text{ countable} \\ \infty & A \text{ cocountable.} \end{cases}$

Let $\mathcal{S} \in L(X)$. One possible $\mathcal{S}$ is the collection of all countable subsets (it is 0_X). Now, suppose $A \in \mathcal{S}$ with $X \setminus A$ countable. Then $X = A \cup (X \setminus A) \in \mathcal{S}$ so $\mathcal{S} = \mathcal{A}$. Thus, $L(X)$ has only two members, 0_X and 1_X. In particular, $L(-)$ is not 1-1 in the sense that we may have $L(X) = L(Y)$ for $X \neq Y$; viz. examples 2 and 5. □

In section 3.3.5. we will look at another example, the Lebesgue unit interval, in some detail.

3.3.4 L(X) is Boolean

In this section, we will show that the locale $L(X)$ is a complete Boolean Algebra. That is, $L(X)$ satisfies $\neg\neg\mathcal{S} = \mathcal{S}\ \forall\mathcal{S} \in L(X)$ (recall: a locale is, in particular, a Heyting algebra, and a Heyting algebra H is a Boolean algebra iff $\neg\neg a = a,\ \forall a \in H$ [PTJ2, p.9]). In fact, we will show that $L(X)$ is simply the quotient, $\mathcal{A}/\mathcal{N}$, where $\mathcal{N}$ is the collection of sets of measure zero in X. To do this, we consider the function $(\overline{\ \ }) : \mathcal{A} \longrightarrow L(X)$; $\overline{A} := \{A_0 \in \mathcal{A} \mid \mu(A_0 \setminus A) = 0\}$. $\overline{A}$ is to be thought of as a *downclosure* of $A \in \mathcal{A}$, a "best approximation" of A in $L(X)$. We first check that $(\overline{\ \ })$ is well defined.

Lemma 3.3.7 $\overline{A} \in I(X),\ \forall A \in \mathcal{A}$.

Proof: 1. measure zero: if B is of measure zero, then $\mu(B \setminus A) = 0$, so $B \in \overline{A}$.
2. downclosed: Let $A_0 \in \overline{A}$ and $B \subseteq A_0$ Then $\mu(B \setminus A) \le \mu(A_0 \setminus A) = 0$, whence $B \in \overline{A}$.
3. countable unions: Let $\{A_i\}_{i=1}^{\infty}$ be a countable family of elements of $\overline{A}$. Then $\mu((\bigcup A_i) \setminus A) = \mu(\bigcup(A_i \setminus A)) \le \sum \mu(A_i \setminus A) = 0$, whence $\bigcup A_i \in \overline{A}$. ∎

Two important examples are:

Proposition 3.3.5 $\overline{X} = 1_X = \mathcal{A}$ *and* $\overline{\emptyset} = 0_X = \mathcal{N}$.

Proof: $\overline{X} = \{A_0 \mid \mu(A_0 \setminus X) = 0\}$. But, $\mu(A_0 \setminus X) = 0,\ \forall A_0 \in \mathcal{A}$.
$\overline{\emptyset} = \{A_0 \mid \mu(A_0 \setminus \emptyset) = 0\} = \{A_0 \mid \mu(A_0) = 0\} = 0_X$. ∎

To prove $L(X)$ is Boolean, we will proceed in two steps: 1. the collection of $\overline{A}$'s is a Boolean Algebra in the operations inherited from $L(X)$ and 2. every $\mathcal{S} \in L(X)$ is an $\overline{A}$ for some $A \in \mathcal{A}$.

Proposition 3.3.6 *1.* $\overline{A \cup B} = \overline{A} \vee \overline{B}$ *2.* $\overline{A \cap B} = \overline{A} \wedge \overline{B}$

Proof:

1. $\overline{A \cup B} = \{C_0 \mid \mu(C_0 \setminus (A \cup B)) = 0\}$ and $\overline{A} \vee \overline{B} = \{A_0 \mid \mu(A_0 \setminus A) = 0\}$ $\vee \{B_0 \mid \mu(B_0 \setminus B) = 0\}$.

$\subseteq$: let $\mu(C_0 \setminus (A \cup B)) = 0$. Now, $A \cup B \in \overline{A} \vee \overline{B}$ so $C_0 \in \overline{A} \vee \overline{B}$ since $\overline{A} \vee \overline{B}$ is closed under extension by a measure zero set.

$\supseteq$: let $C \in \overline{A} \vee \overline{B}$. Then $C = C_1 \cup C_2$, $C_1 \in \overline{A}$ and $C_2 \in \overline{B}$. So $\mu(C \setminus (A \cup B)) \leq \mu(C_1 \setminus (A \cup B)) + \mu(C_2 \setminus (A \cup B))$. Now, $C_k \in \overline{A} \cup \overline{B}$ so $\mu(C_k \setminus (A \cup B)) = 0$ (i.e. C_k is an A_0 or a B_0 and $A \subseteq A \cup B$, $B \subseteq A \cup B$; subtracting off a la· ;er set is smaller) And so, $\mu(C \setminus (A \cup B)) = 0$.

2. $\overline{A \cap B} = \{C_0 \mid \mu(C_0 \setminus (A \cap B)) = 0\}$ and $\overline{A} \wedge \overline{B} = \{D_0 \mid \mu(D_0 \setminus A) = 0$ $= \mu(D_0 \setminus B)\}$.

$\subseteq$: $\mu(C_0 \setminus A \cap B) = \mu((C_0 \setminus A) \cup (C_0 \setminus B)) \leq \mu(C_0 \setminus A) + \mu(C_0 \setminus B) = 0 + 0 = 0$.

$\supseteq$: $A \cap B \subseteq B \Rightarrow D_0 \setminus A \subseteq D_0 \setminus (A \cap B)$ so $\mu(D_0 \setminus A \cap B) = 0 \Rightarrow \mu(D_0 \setminus A) = 0$. Similarly, with $\mu(D_0 \setminus B)$. ∎

Corollary (to 2.): $\overline{A} \wedge \overline{B} = 0_X$ iff $\mu(A \cap B) = 0$. ∎

Lemma 3.3.8 *1. For* $\mathcal{S} \in L(X)$, $\mathcal{S} = \bigvee_{C \in \mathcal{S}} \overline{C}$ *and*
2. $\overline{A} \subseteq \overline{B}$ *iff* $\mu(A \setminus B) = 0$

Proof: 1. $\subseteq$: Let $C \in \mathcal{S}$. Now, $C \in \overline{C}$ so $C \in \bigvee\{\overline{C} \mid C \in \mathcal{S}\}$.

$\supseteq$: Let $C_0 \in \bigvee\{\overline{C} \mid C \in \mathcal{S}\}$. Then $C_0 = \bigcup_{k=1}^{\infty} C_k$, $C_k \in \bigcup\{\overline{C} \mid C \in \mathcal{S}\}$. $\mathcal{S}$ is closed under countable unions so we need only show $C_k \in \mathcal{S}$ $\forall k$. But, $C_k \in \overline{C}$ for some $C \in \mathcal{S}$ (in particular, for example, $C_k \in \overline{C_k}$) so $\mu(C_k \setminus C) = 0$ and $C \in \mathcal{S} \Rightarrow C_k \in \mathcal{S}$.

2. $\Rightarrow$: ($A \in \overline{A} \Rightarrow A \in \overline{B}$) $\Rightarrow \mu(A \setminus B) = 0$.

$\Leftarrow$: Suppose $\mu(A \setminus B) = 0$ and $A_0 \in \overline{A}$. Now, $A_0 \setminus B \subseteq (A_0 \setminus A) \cup (A \setminus B)$ so $\mu(A_0 \setminus B) \leq \mu(A_0 \setminus A) + \mu(A \setminus B) = 0 + 0 = 0$. ∎

Proposition 3.3.7 *1.* $\overline{A} \to \overline{B} = \overline{A^c \cup B}$, *2.* $\overline{A} \leftrightarrow \overline{B} = \overline{X \setminus (A \triangle B)}$, *and* *3.* $\neg \overline{A} = \overline{A^c}$

Proof:

$$\begin{aligned}
1.\ \overline{A} \to \overline{B} &= \bigvee\{\mathcal{S} \in L(X) \mid \overline{A} \wedge \mathcal{S} \subseteq \overline{B}\} \\
&= \bigvee\{\overline{C} \mid \overline{A} \wedge \overline{C} \subseteq \overline{B}\} \text{ since } \mathcal{S} = \bigvee_{C \in \mathcal{S}} \overline{C} \text{ and } L(X) \text{ is a locale.} \\
&= \bigvee\{\overline{C} \mid \overline{A \cap C} \subseteq \overline{B}\} \text{ by proposition 3.3.6 above.} \\
&= \bigvee\{\overline{C} \mid \mu(A \cap C \setminus B) = 0\} \text{ by lemma 3.3.8 above.}
\end{aligned}$$

Now, $\overline{A^c \cup B} = \{D_0 \mid \mu(D_0 \setminus A^c \cup B) = 0\}$. But, $A \cap C \setminus B = C \setminus A^c \cup B$ so $\mu(A \cap C \setminus B) = 0$ iff $\mu(C \setminus A^c \cup B) = 0$ and this completes the proof.

$$\begin{aligned}
2. \qquad \overline{A} \leftrightarrow \overline{B} &= (\overline{A} \to \overline{B}) \wedge (\overline{B} \to \overline{A}) \\
&= \overline{(A^c \cup B)} \cap \overline{(B^c \cup A)} \\
&= \overline{(A^c \cup B) \cap (B^c \cup A)} \\
&= \overline{((A^c \cup B) \cap B^c) \cup ((A^c \cup B) \cap A)} \\
&= \overline{(A^c \cap B^c) \cup (B \cap B^c) \cup (A^c \cap A) \cup (B \cap A)} \\
&= \overline{(A^c \cap B^c) \cup (B \cap A)} \\
&= \overline{X \setminus A \triangle B}
\end{aligned}$$

3. $\neg\overline{A} = \overline{A} \to 0_X = \bigvee\{\overline{C} | \overline{A} \wedge \overline{C} = 0_X\} = \bigvee\{\overline{C} | \mu(C \cap A) = 0_X\}$. On the other hand, $\overline{A^c} = \{C | \mu(C \setminus A^c) = 0\}$. But, $C \setminus A^c = C \cap A$ so $C \in A^c$ iff $C \in \neg\overline{A}$. ∎

Corollary (to 1.): $\neg\mathcal{S} = \bigvee\{\overline{B} | \mu(B \cap A) = 0\ \forall A \in \mathcal{S}\}$. ∎

Corollary (to 3.): $\neg\neg\overline{A} = \overline{A}$.

Proof: $\neg\neg\overline{A} = \neg(\overline{A^c}) = \overline{A^{cc}} = \overline{A}$. ∎

In view of this corollary, we need only show that each $\mathcal{S} \in L(X)$ is an $\overline{A}$ for some $A \in \mathcal{A}$. Let $\alpha = \sup_{A \in \mathcal{S}} \mu(A)$. This supremum exists since X is a finite measure space.

Lemma 3.3.9 $\mathcal{S}$ *attains its supremum* α.

Proof: We wish to find an $A \in \mathcal{S}$ with $\mu(A) = \alpha$. Now, $\alpha = \sup_{A \in \mathcal{S}} \mu(A)$ so for each n, there is an $A_n \in \mathcal{S}$ such that $\mu(A_n) > \alpha - \frac{1}{n}$. Let $A = \bigcup_{n=1}^{\infty} A_n$ then $\mu(A) \geq \mu(A_n)$ for all n. Taking limits, we have $\mu(A) \geq \alpha$. But, α is the supremum so $\alpha \geq \mu(A)$, whence $\mu(A) = \alpha$ as required. ∎

Proposition 3.3.8 $\mathcal{S} = \overline{A}$ *for some* $A \in \mathcal{S}$.

Proof: Let A and α be as above. Let $B \in \mathcal{S}$ and consider $B \setminus A$. If $\mu(B \setminus A) > 0$, then $\mu(A \cup (B \setminus A)) = \mu(A) + \mu(B \setminus A) > \alpha$. But, $A \cup (B \setminus A) \in \mathcal{S}$ (since $A \in \mathcal{S}$ and $B \in \mathcal{S}$ and $B \setminus A \subseteq B \Rightarrow B \setminus A \in \mathcal{S} \Rightarrow A \cup (B \setminus A) \in \mathcal{S}$) which contradicts the maximality of α. ∎

Theorem 3.3.4 $L(X)$ *is a complete Boolean algebra.*

Proof: $L(X) \simeq \mathcal{A}/\mathcal{N}$. Let $\mathcal{S} \in L(X)$ and let $\mathcal{S} = \overline{A}$ as in proposition 3.3.8. Put $\mathcal{S} \longmapsto [A] \in \mathcal{A}/\mathcal{N}$. This is well defined since any other such set is within measure zero of A. Conversely, send $[A]$ in $\mathcal{A}/\mathcal{N}$ to $\overline{A}$. Again, if $[A] = [A']$, then $\overline{A} = \overline{A'}$. ∎

Finally, we note that we may interpret the α of lemma 3.3.9 as left Kan extension:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\overline{(\ )}"] \arrow[d, "\mu"'] & L(X) \arrow[dl, "\overline{\mu}"] \\
(\mathbf{R}^{\geq 0}, \leq) &
\end{tikzcd}$$

where $\alpha = \overline{\mu}(\mathcal{S}) := \sup_{A \in \mathcal{S}} \mu(A)$. $\square$

3.3.5 Consequences of $(\overline{\ })$

In this section, we explore some of the consequences of the $(\overline{\ })$ operation. In short, calculations are made easier by the fact that every $\mathcal{S} \in L(X)$ is an $\overline{A}$ for some A and Boolean operations behave well with respect to $(\overline{\ })$ (for example, $\overline{A^c} = \neg\overline{A}$).

Our first application is in giving a counterexample to show L does not, in general, preserve products. Recall, [PTJ2, p.61], for X a topological space, $\Omega(X)$ denotes the locale of open subsets of X and if X is locally compact, $\Omega(X) \times_{Loc} \Omega(Y) \simeq \Omega(X \times Y)$. In general, however, this is not true (a counterexample, [PTJ2, p. 61]: $\Omega(\mathbf{Q}) \times_{Loc} \Omega(\mathbf{Q})$ is not spatial where $\mathbf{Q}$ is topologized as a subspace of $\mathbf{R}$).

As with Ω, $L(X \times Y)$ is not necessarily the product (in <u>**Loc**</u>) of $L(X)$ and $L(Y)$. We use the $(\overline{\ })$ operation to describe a counterexample (the "diagonal," in fact). Suppose $L(X \times Y) \simeq L(X) \times_{Loc} L(Y)$ and consider the diagram:

$$\begin{tikzcd}
 & T \arrow[dl, shift right, "a^*"'] \arrow[d, "\alpha"] \arrow[dr, shift left, "b^*"] & \\
L(X) \arrow[ur, shift right, "a_*"'] \arrow[r, shift left, "\tilde{p}_1"] & L(X \times Y) \arrow[l, shift left, "\hat{p}_1"] \arrow[r, shift right, "\hat{p}_2"'] & L(Y) \arrow[ul, shift left, "b_*"] \arrow[l, shift right, "\tilde{p}_2"']
\end{tikzcd}$$

$X \times Y \xrightarrow{p_1} X$ induces $\tilde{p}_1 \dashv \hat{p}_1$: $\hat{p}_1(\mathcal{U}) = \{A \in \mathcal{A} | p_1^{-1}(A) \in \mathcal{U}\} = \{A \in \mathcal{A} | A \times Y \in \mathcal{U}\}$ and $\tilde{p}_1(\mathcal{S}) = \{D \in \mathcal{A} \times \mathcal{B} | \exists A \in \mathcal{S},\ (\mu \times \nu)(D \setminus p_1^{-1} A) = 0\} = \{D \in \mathcal{A} \times \mathcal{B} | \exists A \in \mathcal{S},\ (\mu \times \nu)(D \setminus A \times Y) = 0\}$. We begin by giving an alternate description of $\tilde{p}_1$.

Lemma 3.3.10 *For* $X \xrightarrow{f} Y \in$ <u>**MOR**</u>, $\tilde{f}(\overline{B}) = \overline{f^{-1}(B)}$.

Proof: $\tilde{f}(\mathcal{T}) = \{A \in \mathcal{A} | \exists B \in \mathcal{T},\ \mu(A \setminus f^{-1}(B)) = 0\}$ so $\tilde{f}(\overline{B}) = \{A \in \mathcal{A} | \exists B_0 \in \{B_0 | \nu(B_0 \setminus B) = 0\},\ \mu(A \setminus f^{-1}(B_0)) = 0\}$ and $\overline{f^{-1}(B)} = \{A' \in \mathcal{A} | \mu(A' \setminus f^{-1}(B)) = 0\}$.

$\supseteq$: Let $A' \in \overline{f^{-1}(B)}$ and let $B_0 = B \in \overline{B}$. Then $A' \in \tilde{f}(\overline{B})$.

$\subseteq$: Let $A \in \tilde{f}(\overline{B})$. Then there is a B_0 with $\nu(B_0 \setminus B) = 0$ and $\mu(A \setminus f^{-1}(B_0)) = 0$. We must show $\mu(A \setminus f^{-1}(B)) = 0$. Now, $\nu(B_0 \setminus B) = 0 \Rightarrow \mu(f^{-1}(B_0 \setminus B)) = 0$, since $f \in$ **<u>MOR</u>**, so $\mu(f^{-1}(B_0) \setminus f^{-1}(B)) = 0$. Furthermore, $A \setminus f^{-1}(B) = ((A \setminus f^{-1}(B_0)) \setminus f^{-1}(B)) \cup (A \cap (f^{-1}(B_0) \setminus f^{-1}(B)))$ so $\mu(A \setminus f^{-1}(B)) \leq \mu((A \setminus f^{-1}(B_0)) \setminus f^{-1}(B))) + \mu(A \cap (f^{-1}(E_0) \setminus f^{-1}(B))) = 0 + 0 = 0$. ∎

Corollary: $\tilde{p_1}(\overline{A}) = \overline{A \times Y}$. ∎

In the diagram above, we require the triangles to commute for the universal property. In particular, we must have

$$a^*(\overline{A}) = \alpha^* \tilde{p_1}(\overline{A}) = \alpha^*(\overline{A \times Y})$$

and

$$b^*(\overline{B}) = \alpha^*(\overline{X \times B})$$

Lemma 3.3.11 $\overline{A \times Y} \wedge \overline{X \times B} = \overline{A \times B}$.

Proof: By proposition 3.3.6, # 2, $\overline{A \times Y} \cap \overline{X \times B} = \overline{A \times Y \cap X \times B}$ $= \overline{A \times B}$. ∎

α^*, being left exact, must preserve $- \wedge -$, so we must have:

$$\alpha^*(\overline{A \times B}) = a^*(\overline{A}) \wedge b^*(\overline{B})$$

Recall, lemma 3.3.8 says $\mathcal{S} = \bigvee\{\overline{C} \mid C \in \mathcal{S}\}$. For the product space, measurable rectangles are enough:

Lemma 3.3.12 *For* $\mathcal{U} \in L(X \times Y)$, $\mathcal{U} = \bigvee\{\overline{A \times B} \mid A \times B \in \mathcal{U}\}$.

Proof: $\subseteq$: Let $D \in \mathcal{U}$ be an $\mathcal{R}_{\sigma\delta}$ (recall from section 1.4 that every $D \in \mathcal{A} \times \mathcal{B}$ is within measure zero of an $\mathcal{R}_{\sigma\delta}$ and elements of the locales, here, are closed under extension by sets of measure zero). But, if $D = \bigcap_{i=1}^{\infty} D_i$, $D_i \in \mathcal{R}_\sigma$, then, in particular, $D \subseteq D_1$ and $\bigvee$ is downclosed so it is enough to show $D \in \mathcal{R}_\sigma$, $D \in \mathcal{U} \Rightarrow D \in \bigvee$. Let $D = \bigcup_{i=1}^{\infty} A_i \times B_i$. Since $A_i \times B_i \in \overline{A_i \times B_i}$ and $\bigvee$ is closed under countable unions, $D \in \bigvee$.

$\supseteq$: Let $D \in \bigvee$. Then $D = \bigcup_{k=1}^{\infty} D_k$, $D_k \in \bigcup\{\overline{A \times B} \mid A \times B \in \mathcal{U}\} \Rightarrow D_k \in \overline{A \times B}$ for some A and B. Now, $D_k \subseteq (D_k \setminus A \times B) \cup (A \times B) \in \mathcal{U}$, so $D_k \in \mathcal{U}$ for all k. Thus, $D \in \mathcal{U}$ as required. ∎

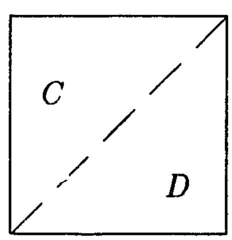
And so, α^*, being a left adjoint, must preserve suprema. Thus, we must have

$$\alpha^*(\mathcal{U}) = \alpha^*(\bigvee\{\overline{A \times B} \mid A \times B \in \mathcal{U}\}) = \bigvee\{a^*(\overline{A}) \wedge b^*(\overline{B}) \mid A \times B \in \mathcal{U}\}.$$

Now, consider the special case:

$$\begin{tikzcd}
& L(X) & \\
L(X) \arrow[ur, "1"] \arrow[r, "\tilde{p}_1"'] & L(X \times X) \arrow[u, "\alpha^*"] & L(X) \arrow[ul, "1"'] \arrow[l, "\tilde{p}_2"]
\end{tikzcd}$$

with $X = [0,1]$. Let C and D be the subsets of the plane:



$\alpha^*(\overline{C}) = \bigvee\{\overline{A} \wedge \overline{B} \mid A \times B \in \overline{C}\} = \bigvee\{\overline{A \cap B} \mid A \times B \in \overline{C}\}$. But, if $A \times B \in \overline{C}$, then $A \cap B \sim \emptyset$ (since $(a,b) \in C \Rightarrow a > b$), so $\alpha^*(\overline{C}) = \bigvee\{\overline{\emptyset}\} = 0$. By a similar argument, $\alpha^*(\overline{D}) = 0$. And so, $\alpha^*(\overline{C}) \vee \alpha^*(\overline{D}) = 0$. But, $\overline{C} \vee \overline{D} = 1$ so $\alpha^*(\overline{C} \vee \overline{D}) = 1$, which is a contradiction.

Remark: We do not know if there is some condition, in analogy to local compactness, for example, that ensures $L(X) \times_{Loc} L(Y) \simeq L(X \times Y)$. $\square$

We next look at some preservation properties of $\tilde{f}$ and $\hat{f}$ for $f \in \underline{\mathbf{M0R}}$.

Proposition 3.3.9 *$\tilde{f}$ preserves $\neg$.*

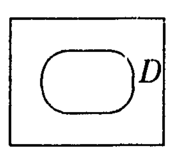
Proof: $\tilde{f}(\neg\overline{B}) = \tilde{f}(\overline{B^c}) = \overline{f^{-1}(B^c)} = \overline{(f^{-1}(B))^c} = \neg\overline{f^{-1}(B)} = \neg\tilde{f}(\overline{B})$. ∎

Combining this with the remarks about $\tilde{f}$ preserving finite infima, we see that we have a functor:

$$\begin{array}{ccc}
\underline{\mathbf{M0R}}^{op} & \longrightarrow & \underline{\mathbf{BAlg}} \\
X & & L(X) \\
\big\downarrow f & \mapsto & \big\uparrow \tilde{f} \\
Y & & L(Y)
\end{array}$$

In fact, as we noted above, $\tilde{f}$ preserves countable infima (of course, $\tilde{f}$ is a morphism of σ-frames; see, for example [B&G]).

In general, $\hat{f}$ does not preserve $\neg$. Consider $L(X \times Y) \xrightarrow{\hat{p}} L(X)$ as above. For $\mathcal{U} \in L(X \times Y)$, $\hat{p}(\mathcal{U}) = \{A \in \mathcal{A} | p^{-1}(A) \in \mathcal{U}\} = \{A \in \mathcal{A} | A \times Y \in \mathcal{U}\}$. Let $\mathcal{U} = \overline{D}$:



$\hat{p}(\overline{D}) = 0_X$ for many D's (for $A \times Y \in \overline{D}$ in the above picture, for example, then $\mu(A)$ should be zero) and $\hat{p}(\neg\overline{D}) = \hat{p}(\overline{D^c})$ will be mostly 0_X, as well, so $\hat{p}(\neg\overline{D}) \neq \neg\hat{p}(\overline{D})$.

As another argument, $\hat{f}(0_X) = \{B \in \mathcal{B} | \mu(f^{-1}(B)) = 0\}$, there may be many B's for which $\nu(B) \neq 0$ and $\mu(f^{-1}(B)) = 0$. However, we can characterize when $\hat{f}(0) = 0$.

Proposition 3.3.10 *$\hat{f}$ preserves 0 iff $\hat{f}\check{f}(\overline{B}) = \overline{B}$, $\forall B \in \mathcal{B}$.*

Proof: $\Leftarrow$: $\hat{f}\check{f}(0) = 0$ and $\check{f}(0) = 0$ since $\check{f}$ is a left adjoint.

$\Rightarrow$: First, we require a lemma (which is an exercise in [PTJ2, p.40]). Suppose A and B are Heyting algebras and $B \underset{g}{\overset{f}{\rightleftarrows}} A$ with $f \dashv g$. Then f preserves $- \wedge -$ iff $g(fa \to b) = a \to gb$.

$$\Rightarrow: \quad \begin{array}{c} \underline{fa \wedge fb \le c} \\ \underline{fa \le fb \to c} \\ \underline{a \le g(fb \to c)} \\ \underline{a \le b \to gc} \\ \underline{a \wedge b \le gc} \\ f(a \wedge b) \le c \end{array} \qquad \Leftarrow: \quad \begin{array}{c} \underline{c \le g(fa \to b)} \\ \underline{fc \le fa \to b} \\ \underline{fc \wedge fa \le b} \\ \underline{f(c \wedge a) \le b} \\ \underline{c \wedge a \le gb} \\ c \le a \to gb \end{array}$$

In our case, "f"$= \check{f}$ and "g"$= \hat{f}$ and $A = L(Y)$, $B = L(X)$ so $\hat{f}(\check{f}(\overline{B}) \longrightarrow \overline{A}) = \overline{B} \longrightarrow \hat{f}(\overline{A})$. If $\overline{A} = 0$, then $\hat{f}(\check{f}(\overline{B}) \longrightarrow 0) = \overline{B} \longrightarrow \hat{f}(0) = \overline{B} \longrightarrow 0 = \neg\overline{B}$. On the other hand, $\hat{f}(\check{f}(\overline{B}) \longrightarrow 0) = \hat{f}(\neg\check{f}(\overline{B})) = \hat{f}(\check{f}(\neg\overline{B}))$. We have actually shown $\hat{f}\check{f}(\neg\overline{B})) = \neg\overline{B}$ but every $\overline{C}$ is $\neg\overline{B}$ for some B (take $B = C^c$). ∎

Remark: $\hat{f}\check{f}(\overline{B}) = \overline{B}$ says $\check{f}$ is fully faithful. □

The next consequence of the $(\overline{\ \ })$ operation we look at is its application to the Lebesgue unit interval.

Example 6: Let $f, g : X \longrightarrow [0,1] \in$ **<u>M0R</u>** and suppose $\check{f} = \check{g}$. Since elements of $L([0,1])$ are $\overline{B}$, for measurable B, and $\check{f}(\overline{B}) = \overline{f^{-1}(B)}$, this means $\mu(f^{-1}(B) \triangle g^{-1}(B)) = 0$ for all measurable B.

Let $D = \{x \in X | f(x) \neq g(x)\}$ and $D_n = \{x \in X || f(x) - g(x)| > \frac{1}{n}\}$. then $D = \bigcup_{i=1}^{\infty} D_n$. Suppose $\mu(D) > 0$, then $\mu(D_n) > 0$ for some n. Write $[0, 1] = I$ $= \bigcup_{k=1}^{n} I_k$, where $I_k = [\frac{k-1}{n}, \frac{k}{n}]$ and put $A_k = f^{-1}(I_k)$, $A'_k = g^{-1}(I_k)$ so, in particular, $A_k \sim A'_k$.
Now, $\bigcup_{k=1}^{n} A_k = X = \bigcup_{k=1}^{n} A'_k$ so $\bigcup_{k=1}^{n}(A_k \cap D_n) = D_n$ and so there is a k with $\mu(A_k \cap D_n) > 0$. $A_k \cap D_n \cap A'_k = \emptyset$ since $x \in D_n$ implies that f and g are more than $\frac{1}{n}$ apart. Thus, $\mu(A_k \setminus A'_k) \geq \mu((A_k \cap D_n) \setminus A'_k) = \mu(A_k \cap D_n) > 0$ which contradicts the fact that $A_k \sim A'_k$ and so we must have $\mu(D) = 0$, whence $f \sim g$. $\square$

3.4 MEAS(X) Revisited

3.4.1 MEAS(X) has SS

Definition 3.4.1 *[PTJ1, p.141]: We say that a topos $\mathcal{E}$ satisfies (SS), or* supports split, *if every subobject of* 1 *is projective in $\mathcal{E}$.*

Definition 3.4.2 *[Mac, p.114]: An object, P, is* projective *if every $P \xrightarrow{h} C$ factors through every epimorphism $B \xrightarrow{g} C$:*

$$\begin{tikzcd}
 & P \arrow[dl, "h'"'] \arrow[d, "h"] \\
B \arrow[r, two heads, "g"'] & C
\end{tikzcd}$$

We first show that epimorphisms in $MEAS(X)$ (i.e. locally onto natural transformations) are coordinatewise onto.

Proposition 3.4.1 *$F \xrightarrow{\eta} G$ locally onto implies $F(A) \xrightarrow{\eta_A} G(A)$ is onto for each $A \in \mathcal{A}$.*

Proof: Locally onto means for all $y \in G(A)$ there is a cover, $\{A_i\}$ of A and $x_i \in F(A_i)$ such that $\eta_{A_i}(x_i) = \rho^A_{A_i}(y)$. Now, we may take the cover to be disjoint for define $B_i := A_i \setminus (A_1 \cup \cdots \cup A_{i-1})$ and define $x'_i \in F(B_i)$ as $x'_i = \sigma^{A_i}_{B_i}(x_i)$ (here σ is the restriction for F).

Since the A_i's are disjoint, the family $\{x_i\}$ is compatible, so $\exists! x \in F(A)$ with $\sigma^A_{A_i}(x) = x_i$ and η_A sends this x to $y \in G(A)$ and so η_A is onto. ∎

Proposition 3.4.2 *$MEAS(X)$ has (SS)*

Proof: Let $U \in Sub(1)$. We wish to exhibit an α to make

$$\begin{tikzcd} & U \arrow[dl, "\alpha"'] \arrow[d, "\pi"] \\ F \arrow[r, "\eta"'] & G \end{tikzcd}$$

commute, where π and η are given. Since $U \in Sub(1)$, there is an A_0 such that $U(A) = \begin{cases} 1 & \text{if } \mu(A \setminus A_0) = 0 \\ \emptyset & \text{else.} \end{cases}$ Consider first, the situation at A_0. $U(A_0) = 1 = \{\star\}$. Let $\pi_{A_0}(\star) = y_0 \in G(A_0)$. Since η_{A_0} is onto, there is an $x_0 \in F(A_0)$ such that $\eta_{A_0}(x_0) = y_0$. Now, for $A \in \mathcal{A}$ with $U(A) = 1$, $\mu(A \setminus A_0) = 0$, so $\{A \cap A_0 \hookrightarrow A\}$ is a cover, so define $\alpha_A(\star)$ to be the unique $x \in F(A)$ such that $\rho^A_{A\cap A_0}(x) = \rho^{A_0}_{A\cap A_0}(x_0)$ (here, ρ denotes the restriction for F and we use the sheaf property for F).

We only show that α is natural (the proof that the triangle commutes uses similar ideas (naturality and functoriality)). Let $A' \subseteq A$ and consider:

$$\begin{tikzcd} U(A) \arrow[r, "\alpha_A"] \arrow[d, "\sigma^A_{A'}"'] & F(A) \arrow[d, "\rho^A_{A'}"] \\ U(A') \arrow[r, "\alpha_{A'}"'] & F(A') \end{tikzcd}$$

If $U(A) = \emptyset$ then both composites are the unique map to $F(A')$. If $U(A) = 1$ then $U(A') = 1$ and the left-bottom composite is $\star \mapsto \star \mapsto$ (unique x' such that $\rho^{A'}_{A'\cap A_0}(x') = \rho^{A_0}_{A'\cap A_0}(x_0)$). The top-right composite is $\star \mapsto$ (unique x such that $\rho^{A}_{A\cap A_0}(x) = \rho^{A_0}_{A\cap A_0}(x_0)) \mapsto \rho^A_{A'}(x)$. We must show $\rho^A_{A'}(x) = x'$. That is, we must show $\rho^{A'}_{A'\cap A_0}\rho^A_{A'}(x) = \rho^{A_0}_{A'\cap A_0}(x_0)$. Now, we know, $\rho^{A}_{A\cap A_0}(x) = \rho^{A_0}_{A\cap A_0}(x_0)$, so $\rho^{A'}_{A'\cap A_0}\rho^A_{A'}(x) = \rho^{A}_{A'\cap A_0}(x) = \rho^{A\cap A_0}_{A'\cap A_0}\rho^{A}_{A\cap A_0}(x) = \rho^{A\cap A_0}_{A'\cap A_0}\rho^{A_0}_{A\cap A_0}(x_0) = \rho^{A_0}_{A'\cap A_0}(x_0)$ as required. ∎

3.4.2 MEAS(X) as a Topos over Set

$MEAS(X)$ is a Grothendieck topos over **Set**. In this section, we will give an explicit description of

$$MEAS(X) \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \underline{\mathbf{Set}}.$$

$\Gamma(F) := F(X)$ and for $K \in \underline{\mathbf{Set}}$, we define $\Delta(K)(A) := \{(B,f) | \mu(A \triangle B) = 0$, $B, \xrightarrow{f} K$, $f(B)$ countable , $f^{-1}(k) \in \mathcal{A} \forall k \in K\}/\sim$, where $(B,f) \sim (B',f')$ iff $\mu(B \triangle B') = 0$ and $\mu\{x \in B \cap B' | f(x) \neq f'(x)\} = 0$.

Γ is made into a functor by $(F \xrightarrow{\eta} G) \mapsto (F(X) \xrightarrow{\eta_X} G(X))$. It is easy to see that Δ is a presheaf, for suppose that $A' \subseteq A$, then we have $\Delta(A) \longrightarrow \Delta(A')$, $(B,f) \mapsto (B \cap A', f|_{B\cap A'})$. Furthermore, Δ is a sheaf in exactly the same way as $Mble_Y(-)$ is (see proposition 3.2.4; indeed, for $\{A_i\}$ a cover of A and $(B_i, f_i) \in$

$\Delta(K)(A_i)$, we can extend to $(\cup B_i, f)$ where $f(x) = f_i(x)$ with i the smallest index for which $x \in B_i$. Then $f(\cup B_i) \subseteq \cup f_i(B_i)$ so is countable and $f^{-1}(k) = \cup f_i^{-1}(k))$. Finally, we note that we make Δ into a functor as follows: let $K \xrightarrow{t} L$ be a function; define $\Delta(K)(A) \xrightarrow{\beta_A} \Delta(L)(A)$ by $\beta_A(B, f) := B \xrightarrow{f} K \xrightarrow{t} L$ (note: $f(B)$ countable implies $t(f(B))$ countable).

Next, we look at the adjunction. Consider:

$$\frac{\Delta K \xrightarrow{\alpha} F}{K \xrightarrow{t} F(X)} \quad \Downarrow \phi \Uparrow \psi$$

Given α, we know $\Delta(K)(X) \xrightarrow{\alpha_X} F(X)$. In particular, $X \xrightarrow{\lceil k \rceil} K$ is an element of $\Delta(K)(A)$. Define $t(k) = \phi(\alpha)(k) := \alpha_X \lceil k \rceil$.

Given t, $t(k) \in F(X)$. Let $B \xrightarrow{f} K \in \Delta(K)(A)$. Since $f(B)$ is countable, $\{f^{-1}(k)\}_{k \in f(B)}$ is a disjoint cover of A. Put $A_k := f^{-1}(k)$ (we assume $A_k \neq \emptyset$ for all k's) and consider $F(X) \xrightarrow{\rho^X_{A_k}} F(A_k)$, $t(k) \mapsto y_k$. Since the cover is disjoint, the family $\{y_k\}$ is compatible, so, by the sheaf condition, $\exists! y \in F(A)$ such that $\rho^A_{A_k}(y) = y_k \ \forall k$. Define $\alpha = \psi(t)(A)(B, f) = y$.

We have an extensive list of things to check: ψ is well defined (with respect to $(B, f) \sim (B', f')$), $\psi(t)$ is natural in A, ϕ and ψ are natural in K and F, $\phi\psi = 1$, and $\psi\phi = 1$.

Lemma 3.4.1 *If $(B, f) \sim (B', f')$ in $\Delta(K)(A)$, then $\psi(t)(A)(B, f) = \psi(t)(A)(B', f')$.*

Proof: Let $C \subseteq B \cap B'$ be the "good" set (i.e. $C = \{x \in B \cap B' | f(x) = f'(x)\}$. On C, $A_k = A'_k$, so, on C, $y_k = y'_k$. By uniqueness (sheaf property), y_k and y'_k

both extend to the same $y \in F(C)$. But $C \cap A$ is a cover of A ($\mu(A \Delta C) = 0$) so this extends to a unique $y \in F(A)$ as required. ∎

Lemma 3.4.2 *$\psi(t)$ is natural in A.*

Proof: Let $A' \subseteq A$ and consider:

$$\begin{tikzcd} \Delta(K)(A) \arrow[r, "\psi(t)(A)"] \arrow[d, "\sigma^A_{A'}"'] & F(A) \arrow[d, "\rho^A_{A'}"] \\ \Delta(K)(A') \arrow[r, "\psi(t)(A')"'] & F(A') \end{tikzcd}$$

The top right-composite is $(B, f) \mapsto y \mapsto \rho^A_{A'}(y)$ and the left-bottom composite is $(B, f) \mapsto (B \cap A', f|_{B \cap A'}) \mapsto y'$ where y' is unique such that $\rho^{A'}_{A'_k}(y') = \rho^X_{A'_k}(t(k))$. We must show $\rho^{A'}_{A'_k}\rho^A_{A'}(y) = \rho^X_{A'_k}$. But,

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d] & F(A_k) \arrow[d] \\ F(A') \arrow[r] & F(A'_k) \end{tikzcd}$$

commutes, by functoriality, so $\rho^{A'}_{A'_k}\rho^A_{A'}(y) = \rho^{A_k}_{A'_k}\rho^A_{A_k}(y) = \rho^{A_k}_{A'_k}\rho^X_{A_k}(t(k)) = \rho^X_{A'_k}(t(k))$ as required. ∎

Lemma 3.4.3 *ϕ is natural in F.*

Proof: Let $F \xrightarrow{\beta} F'$ be natural. We must show

$$\begin{tikzcd}
MEAS(\Delta K, F) \arrow[r, "\phi_{KF}"] \arrow[d, "MEAS(\Delta K,\beta)"'] & \underline{\mathbf{Set}}(K, \Gamma(F)) \arrow[d, "\underline{\mathbf{Set}}(K,\Gamma(\beta))"] \\
MEAS(\Delta K, F') \arrow[r, "\phi_{KF'}"'] & \underline{\mathbf{Set}}(K, \Gamma(F'))
\end{tikzcd}$$

commutes. The top-right composite is $(\Delta K \xrightarrow{\alpha} F) \mapsto (K \xrightarrow{\alpha_X\lceil k\rceil} F(X)) \mapsto (K \xrightarrow{\alpha_X\lceil k\rceil} F(X) \xrightarrow{\beta} F'(X))$ (i.e. it sends (B, f) to the map that sends $k \mapsto \beta_X(\alpha_X\lceil k\rceil))$. The left- bottom composite is $(\Delta K \xrightarrow{\alpha} F) \mapsto (\Delta K \xrightarrow{\alpha} F \xrightarrow{\beta} F') \mapsto (K \xrightarrow{(\beta\circ\alpha)_X\lceil k\rceil} F'(X) = (K \xrightarrow{\beta_x(\alpha_X\lceil k\rceil)} F'(X))$ as required. ∎

Lemma 3.4.4 $\phi\psi = 1$

Proof: $\phi(\psi(t))(k_0) = \psi(t)(X)\lceil k_0\rceil$. Now $\lceil k_o\rceil(k) = \begin{cases} X & \text{if } k = k_0 \\ \emptyset & \text{else} \end{cases}$, so k_0 is mapped to the unique $y \in F(X)$ such that $\rho^X_X(y) = \rho^X_X(t(k_0))$ and so $y = t(k_0)$ as required. ∎

Lemma 3.4.5 $\psi\phi = 1$.

Proof: $\psi(\phi(\alpha))(A)(B, f) = y$ where y is unique such that $\rho^A_{A_k}(y) = \rho^X_{A_k}(\phi(\alpha)(k)) = \rho^X_{A_k}(\alpha_X\lceil k\rceil)$ for all k. Now, on $A_k = f^{-1}(k)$, f is constantly k so α_A evaluated at f is equal to $\alpha_X\lceil k\rceil$ restricted to A_k by the naturality of α, so $\rho^A_{A_k}(\alpha_A(B, f)) = \rho^X_{A_k}(\alpha_X\lceil k\rceil)$ as required. ∎

Lemma 3.4.6 *ψ is natural in K and F.*

Proof: This follows immediately from the three previous lemmas. ∎

We may ask whether Δ is logical or has a left adjoint. These questions are related and the answer for $MEAS(X)$ is negative. Recall [B&D],

Theorem 3.4.1 *Let $\mathcal{E}$ be a Grothendieck topos (with $\Delta \dashv \Gamma$). Then the following are equivalent:*

(i) $\mathcal{E}$ is the category of sheaves for an atomic site,

(ii) Δ is logical,

(iii) the subobject lattice of every object of $\mathcal{E}$ is a complete atomic Boolean algebra. ∎

Theorem 3.4.2 *Let $(\Delta, \Gamma) : \mathcal{E} \longrightarrow S$ be a geometric morphism between toposes. Then Δ is logical iff there is an object function $\Lambda : \mathcal{E} \longrightarrow S$ such that for any $E \in \mathcal{E}$ the partially ordered set objects $\Gamma(\Omega^E)$ and $\Omega^{\Lambda E}$ of B are isomorphic. In that case, Λ can be extended to a functor left adjoint to Δ.* ∎

$MEAS(X)$ is not atomic (if, for example, X has no atoms). From condition (i) and (ii), we see immediately that Δ is not logical. Furthermore, by the second theorem, this means that, in general, Δ does not have a left adjoint.

3.4.3 MEAS(X) has AC

In section 3.4.1, we showed that $MEAS(X)$ satisfies "supports split." In fact, a stronger result is true. We have collected up all the material to prove that $MEAS(X)$ satisfies the axiom of choice (which, of course, implies (SS)). Recall,

Definition 3.4.3 *[PTJ1, p.141]: A topos, $\underline{\mathbf{E}}$, satisfies (AC), the* axiom of choice, *if supports split in $\mathcal{E}/X$ for every $X \in \mathcal{E}$ (equivalently, every object of $\mathcal{E}$ is projective or every epimorphism in $\mathcal{E}$ splits).* □

(AC) can be characterized as:

Theorem 3.4.3 *[PTJ1, p.151]: Let* $\underline{\mathbf{E}} \xrightarrow{\Gamma} \underline{\mathbf{Set}}$ *be a topos over* $\underline{\mathbf{Set}}$ *(for which we assume the axiom of choice). Then the following conditions are equivalent:*

(i) $\mathcal{E}$ *satisfies (AC)*

(ii) $\mathcal{E}$ *is Boolean and satisfies (SG)*

(iii) There exists a complete Boolean algebra B *such that* $\mathcal{A} \simeq Sh_C(B)$ *where* C *is the canonical topology on* B. ∎

In section 3.3.1, we showed that $MEAS(X)$ satisfies (SG) and, in section 3.3.4, we showed that $MEAS(X)$ is Boolean. Furthermore, in section 3.4.2, we showed that $MEAS(X)$ is a topos over $\underline{\mathbf{Set}}$. Finally, we note that our remarks about $Sh(\mathcal{A})$ and $Sh(\mathcal{A}/\mathcal{N})$ in section 3.2.1 amount to condition (iii) above.

3.5 Hilbert Sheaves

3.5.1 Analysis in MEAS(X); Preamble

The next bit of background that we must fill in is about analysis in $MEAS(X)$ (more generally, in a topos, $\mathcal{E}$). Bits and pieces of this will show up throughout the rest of the chapter. The purpose of this section is to set up notation and recall a few basic results from the literature. Details will be given later, if and when required.

We assume notation of the Mitchell-Bénabou language (see, for example, [PTJ1, pp. 152-161] for details). Since we are working in a topos with the axiom of choice (hence Boolean), our logic is much "easier" than intuitionistic logic. Most importantly, we have the logical principle of the excluded middle. In fact, in $MEAS(X)$, logic is essentially pointwise with the caveat: almost everywhere.

3.5.2 Number Systems in MEAS(X)

In this section, we describe number systems in $MEAS(X)$. Specifically, we define $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$, the objects, respectively, of natural, integral, rational, real, and complex numbers (see also section 3.2.2 for $\mathbf{R}$ and $\mathbf{C}$). If necessary, we will use the notation, $\mathbf{N}_X$ to distinguish this from the ordinary natural numbers $\mathbf{N}$. When the context is clear, however, we will omit the subscript.

Definition 3.5.1 *Let $\mathcal{E}$ be a topos. A* Natural numbers object *(NNO) in $\mathcal{E}$ consists of a triple $1 \xrightarrow{0} N \xleftarrow{s} N$ which is initial among all such triples (i.e. for all $1 \xrightarrow{z} X \xleftarrow{f} X$ in $\mathcal{E}$, $\exists! N \xrightarrow{t} X$ such that*

$$\begin{tikzcd}
 & N \arrow[d, "t"] & N \arrow[l, "s"'] \arrow[d, "t"] \\
1 \arrow[ur, "0"] \arrow[dr, "z"'] & & \\
 & X & X \arrow[l, "f"]
\end{tikzcd}$$

commutes). □

In $MEAS(X)$, the NNO is given by $\mathbf{N}_X(A) = Mble_{\mathbf{N}}(A)$ with $\mathbf{N}$ discrete. The 0 is the function that sends $\star \in 1(A)$ to the constantly 0 function. The successor s is the function that sends $k(x) \in \mathbf{N}(A)$ to $k(x) + 1 \in \mathbf{N}(A)$. The existence and uniqueness of t is as in the proof of $\Delta \dashv \Gamma$ (see section 3.4.2) for $\mathbf{N}(A) = \Delta(\mathbf{N})(A)$.

We construct $\mathbf{Z}_X$ as $(\mathbf{N} \times \mathbf{N})/\!\sim$, where we identify (a, b) and $(a + c, b + c)$, so $\mathbf{Z}_X$ is the coequalizer of

$$\mathbf{N}\times\mathbf{N}\times\mathbf{N} \underset{(p_1,p_2)}{\overset{(+\times+)\circ((p_1,p_3)\times(p_1,p_2))}{\rightrightarrows}} \mathbf{N}\times\mathbf{N},$$

where the top arrow is $(a,b,c)\mapsto(a+c,b+c)$ and the bottom arrow is $(a,b,c)\mapsto(a,b)$. In our case, $\mathbf{Z}_X(A)=Mble_{\mathbf{Z}}(A)$ with $\mathbf{Z}$ discrete.

For $\mathbf{Q}_X$, we want $\frac{m}{n}$'s where $m\in\mathbf{Z}_X$ and $n\in\mathbf{N}^+$; we want a morphism $\mathbf{Z}\times\mathbf{N}^+\longrightarrow\mathbf{Q}$, $(m,n)\mapsto\frac{m}{n}$ and we must identify $\frac{m}{n}\sim\frac{p}{q}$ iff $mq=np$. We take the equalizer:

$$E\longrightarrow\mathbf{Z}\times\mathbf{N}^+\times\mathbf{Z}\times\mathbf{N}^+ \underset{(p_2,p_3)}{\overset{(p_1,p_4)}{\rightrightarrows}} \mathbf{Z}\times\mathbf{N}^+ \xrightarrow{\cdot|_{\mathbf{Z}\times\mathbf{N}^+}} \mathbf{Z}$$

where the top arrow is $(m,n,p,q)\mapsto mq$ and the bottom arrow is $(m,n,p,q)\mapsto np$. This gives two maps $E\rightrightarrows\mathbf{Z}\times\mathbf{N}^+$, and we take their coequalizer to get $\mathbf{Q}$. As might be expected, in our case, $\mathbf{Q}_X(A)=Mble_{\mathbf{Q}}(A)$ with $\mathbf{Q}$ discrete. Since $\mathbf{Q}$ is countable, this is the same as equivalence classes of (almost everywhere defined) locally constant $\mathbf{Q}$-valued measurable functions ($\mathbf{Q}$ in $Sh(X)$, for X a topological space, consists of locally constant continuous functions into the discrete $\mathbf{Q}$); [PTJ1, p. 213].

Remark: Arithmetic is pointwise. For example, for $p,\ q\in\mathbf{Q},\ q\neq 0$ (which means $q(x)\neq 0$ for almost all x), $\frac{p}{q}(x)=\frac{p(x)}{q(x)}$. □

Next, we look at real numbers.

Definition 3.5.2 *([PTJ1, p.211]): Let $\mathcal{E}$ be a topos with NNO. A* Dedekind real number *in $\mathcal{E}$ is an ordered pair, $r=(L,U)$ of subobjects of $\mathbf{Q}$ satisfying:*

D1 $\forall q(q\in L\leftrightarrow\exists q'(q'\in L\wedge q'>q))$

D2 $\forall q(q\in U\leftrightarrow\exists q'(q'\in U\wedge q'<q))$

D3 $\forall q \forall q'(q \in L \wedge q' \in U \rightarrow q < q')$

D4 $\forall n \exists q \exists q'(q \in L \wedge q' \in U \wedge n(q' - q) < 1)$ □

Example: The object of Dedekind reals in $MEAS(X)$ is $\mathbf{R}_X(A)$ (see [PTJ1, p. 213]). □

Remarks: 1. We will often write, simply, $f \in Mble_{\mathbf{R}}(A)$ and tacitly assume the domain A_0.

2. There are two other good notions of real numbers object in a topos with NNO: the MacNeille reals $\mathbf{R}_m$ (see [M&P]), and the Cauchy reals $\mathbf{R}_c$ (see [PTJ1, p.218]). In general, these are not the same (though we do have $\mathbf{R}_c \hookrightarrow \mathbf{R}_d \hookrightarrow \mathbf{R}_m$). However, in $MEAS(X)$, $\mathbf{R}_c = \mathbf{R}_d$ ([PTJ1, p.220]) and, furthermore, in any Boolean topos, $\mathbf{R}_m \simeq \mathbf{R}_d$ ([PTJ3, p. 483]). We will use $\mathbf{R} = \mathbf{R}_d$. Various properties of $\mathbf{R}$ (order relations, for example) will be described when needed. The interested reader is referred to [M&P], [PTJ1], [PTJ3], and [Rou1] for further discussion on real numbers. □

We will devote the rest of this section to a study of the algebraic properties of $\mathbf{C}(-) := Mble_{\mathbf{C}}(-)$, the complex numbers object.

Remark: Taking the real or imaginary part of a complex-valued, measurable function yields a real-valued, measurable function and we have $\mathbf{C}(-) \subsetneq \mathbf{R}(-) \times \mathbf{R}(-)$. In [Rou1], Rousseau notes that a suitable object of complex numbers is one for which $\mathbf{C} \simeq \mathbf{R} \times \mathbf{R}$ (suitable in any topos for which $\mathbf{R}_c$ is complete). She goes on to give axioms for $\mathbf{C}$ which work in a more general setting. We will exhibit all her axioms (suffice it to say that our $\mathbf{C}$ will have "all" the properties appropriate for complex numbers). □

In section 3.2.2, we showed that $\mathbf{C}(-)$ was a sheaf. We have operations on $\mathbf{C}(A)$:

$$\lceil 0 \rceil : 1 \longrightarrow \mathbf{C};\ \lceil 0 \rceil_A(a) := 0 \text{ for all } a \in A$$

$$\lceil 1 \rceil : 1 \longrightarrow \mathbf{C};\ \lceil 1 \rceil_A(a) = 1 \text{ for all } a \in A$$

More generally, for c a complex number,

$$\lceil c \rceil : 1 \longrightarrow \mathbf{C};\ \lceil c \rceil_A(a) = c \text{ for all } a \in A$$

$$- : \mathbf{C} \longrightarrow \mathbf{C};\ -(A_0, f) = (A_0, -f)$$

$$(^-) : \mathbf{C} \longrightarrow \mathbf{C};\ \overline{(A_0, f)} = (A_0, \overline{f})$$

$$+ : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C};\ (A_0, f) + (A_1, g) = (A_0 \cap A_1, f + g)$$

$$\cdot : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C};\ (A_0, f) \cdot (A_1, g) = (A_0 \cap A_1, f \cdot g)$$

These operations are compatible with $\sim$ (for example, if $f = f'$ except on D and $g = g'$ except on E, with $\mu(D) = \mu(E) = 0$, then $f + g = f' + g'$ except possibly on $D \cup E$) and make $\mathbf{C}(A)$ an involutive commutative ring. Thus, $\mathbf{C}(-)$ is also since this involves only finite limits and equations (for sheaves, finite limits are computed pointwise). In fact, $\mathbf{C}(-)$ is a field.

Recall, [Tav], [Mul], [PTJ1] for $\underline{\mathbf{E}}$ a topos and K a commutative ring,

Definition 3.5.3 *K satisfies the* axiom of nontriviality *if*

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 1 \arrow[d, "\lceil 1 \rceil"] \\
1 \arrow[r, "\lceil 0 \rceil"'] & K
\end{tikzcd}$$

is a pullback diagram. □

Definition 3.5.4 *The* group of units, *U, of K, is defined by the following diagram:*

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & 1 \arrow[d, "\lceil 1 \rceil"] \\
K \times K \arrow[r, "\cdot"'] & K
\end{tikzcd}$$

which is to be a pullback diagram. □

Definition 3.5.5 *Suppose K satisfies the axiom of nontriviality and U is the group of units for K. We say K is*

- *a* geometric field *if $K \simeq U + \lceil 0 \rceil$*
- *a* field of fractions *if $\neg\lceil 0 \rceil \subseteq U$*
- *a* field of quotients *if $\neg U \subseteq \lceil 0 \rceil$.* □

Proposition 3.5.1 *([Mul1]): If $\underline{\mathbf{E}}$ is Boolean, then these three notions of field agree.* ∎

We proceed to show that $\mathbf{C}(-)$ is a geometric field (in which case, it will be a field in the other two senses also in view of the Booleanness of $MEAS(X)$).

Proposition 3.5.2 $\mathbf{C}(-)$ *satisfies the axiom of nontriviality.*

Proof: Since pullbacks in $MEAS(X)$ are computed pointwise, we need only show

$$\begin{tikzcd} 0(A) \arrow[r] \arrow[d] & 1(A) \arrow[d, "\ulcorner 1 \urcorner"] \\ 1(A) \arrow[r, "\ulcorner 0 \urcorner"'] & \mathbf{C}(A) \end{tikzcd}$$

is a pullback for each $A \in \mathcal{A}$. There are two cases to consider.

case $\mu(A) \neq 0$:

$$\begin{tikzcd} T(A) \arrow[dr] \arrow[drr, "b"] \arrow[ddr, "a"'] & & \\ & \emptyset \arrow[r] \arrow[d] & 1 \arrow[d, "\ulcorner 1 \urcorner"] \\ & 1 \arrow[r, "\ulcorner 0 \urcorner"'] & \mathbf{C}(A) \end{tikzcd}$$

If $T(A) = \emptyset$, there are two maps a and b and a unique map $T(A) \xrightarrow{1_\emptyset} \emptyset$. If $T(A) \neq \emptyset$, there are no maps a and b and no map $T(A) \longrightarrow \emptyset$.

case $\mu(A) = 0$:

$$\begin{tikzcd} T(A) \arrow[dr] \arrow[drr, "b"] \arrow[ddr, "a"'] & & \\ & 1 \arrow[r] \arrow[d] & 1 \arrow[d] \\ & 1 \arrow[r] & 1 \end{tikzcd}$$

In this case, everything collapses to the unique map into 1. ∎

In a similar manner, we may compute the group of units of $\mathbf{C}(-)$ as:

$$\begin{aligned} U(A) &= \{(A_0, f) \in \mathbf{C}(A) \mid \exists (A_1, g) \in \mathbf{C}(A),\ (A_0, f) \cdot (A_1, g) \sim [1]\} \\ &= \{(A_0, f) \in \mathbf{C}(A) \mid \mu\{x \in A_0 \mid f(x) = 0\} = 0\} \end{aligned}$$

Proposition 3.5.3 $\mathbf{C}(-)$ *is a geometric field.*

Proof: We must show that $1 \xrightarrow{[0]} \mathbf{C} \longleftarrow U$ is a coproduct diagram. Specifically, we must show $\forall f \in \mathbf{C}(A)$, there is a cover $\{A_i \hookrightarrow A\}$ such that $f|_{A_i} \in U(A_i)$ or $f|_{A_i} \sim 0$.

Consider the two sets $A_z = \{a \in A_0 \mid f(a) = 0\}$ and $A_n = \{a \in A_0 \mid f(a) \neq 0\}$ both sets being measurable. In fact, $\{A_z, A_n\}$ forms a cover of A. Furthermore, $f|_{A_z} = 0$. We next show that $f|A_n \in U(A_n)$. We claim that $\frac{1}{f}$ is measurable on A_n (this will be the required g).

Write $f = a + ib$, then $\frac{1}{f} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$. If a and b are measurable real functions, then $\frac{1}{f}$ will be also, provided that f is measurable and real implies $\frac{1}{f}$ measurable. But, for f real, we have

$$\{x \mid \frac{1}{f(x)} > \alpha\} = \begin{cases} \{x \mid \frac{1}{\alpha} > f(x) > 0\}, & \alpha > 0 \\ \{x \mid f(x) > 0\}, & \alpha = 0 \\ \{x \mid \frac{1}{\alpha} < f(x) < 0\}, & \alpha < 0. \end{cases}$$

And, in each of these cases, we get a measurable set. ∎

We will explore the topological properties of $\mathbf{C}(-)$ (as a normed algebra over itself) and like objects (Hilbert space objects) in section 3.5.4.

3.5.3 A Sheaf From a Measurable Field

In this section, we describe a sheaf, G, to be constructed from a measurable field of Hilbert spaces, $(H(x)_{x\in X}, \mathcal{G})$. Inasmuch as this is a connection (between boxes 1 and 2), the material here belongs in chapter 5. We insert it here, however, since we will use the sheaves G and **C** as motivating examples for Hilbert space objects in $MEAS(X)$ (= Hilbert sheaves).

Definition 3.5.6 $G(A) := \{g \in \mathcal{G} | g(x) = 0\ \forall x \notin A\}/\sim$.

Proposition 3.5.4 $G(-) : (\mathcal{A}, \subseteq)^{op} \longrightarrow \underline{\mathbf{Set}}$ *is a presheaf.*

Proof: The proof of this is similar to that for converges proposition 3.2.3. Suppose $A' \subseteq A$. We have a restriction given by

$$G(A) \longrightarrow G(A'),\ (g(x))_{x\in X} \mapsto (g'(x))_{x\in X},\ g'(x) = \begin{cases} g(x) & x \in A' \\ 0 & \text{else.} \end{cases}$$

Certainly $g'(x) = 0$ for $x \notin A'$. We must check that $g' \in \mathcal{G}$ and the restriction is well defined with respect to $\sim$.

$g' \in \mathcal{G}$: $(x \mapsto \langle h(x)|g'(x)\rangle) = (x \mapsto \langle h(x)|g(x)\rangle \cdot \chi_{A'})$ which is measurable for all $h \in \mathcal{G}$, whence, by axiom 2, $g' \in \mathcal{G}$.

Well defined: Suppose $h(x) = g(x)$ except on B with $\mu(B) = 0$. Then $h'(x) = g'(x)$ except on $B \cap A'$ and $\mu(B \cap A') = 0$. ∎

Proposition 3.5.5 $G(-)$ *is a sheaf.*

Proof: Again, this proof is similar to that for proposition 3.2.4. The only question is whether g, the unique extension of a compatible family $\{g_i\}$ on a cover $\{A_i\}$ of A is in $\mathcal{G}$. As in the above, we use axiom 2 for an MFHS and note that $x \mapsto \langle h(x)|g_i(x)\rangle$ is measurable for each g_i and for all $h \in \mathcal{G}$ (each g_i is in $\mathcal{G}$). ∎

We can make each $G(A)$ into a $\mathbf{C}(A)$-module by defining operations pointwise (for example, $(g+h)(x) := g(x)+h(x)$ and if $g \sim g'$ and $h \sim h'$, then $g+g' \sim h+h'$ as before). Now, since $\mathbf{C}$ is a field, we have:

Proposition 3.5.6 *G is a $\mathbf{C}$-vector space.* ∎

In fact, G can be made into a normed vector space. The sheaf of nonnegative reals is given by $\mathbf{R}^{\geq 0}_X = Mble_{R^{\geq 0}}$:

$$\mathbf{R}^{\geq 0}(A) = \{f : A \longrightarrow \mathbf{R}^{\geq 0} \text{ measurable}\}/\sim$$

For each A, there is a function $G(A) \xrightarrow{\| \|_A} \mathbf{R}^{\geq 0}(A)$ given by $g \longmapsto \|g\|$ where $\|g\|(x) = \|g(x)\|_{H(x)}$ $\forall x \in A$. $x \mapsto \|g(x)\|_{H(x)}$ is measurable by axiom 1 for an MFHS and $g \mapsto \|g\|$ is well defined since if $g = g'$ except on B with $\mu(B) = 0$, then $\|g\| = \|g'\|$ except on B as well. Furthermore, $\|\cdot\|$ is a natural transformation. Suppose $A' \subseteq A$ and consider:

$$\begin{tikzcd}
G(A) \arrow[r] \arrow[d] & \mathbf{R}^{\geq 0}(A) \arrow[d] \\
G(A') \arrow[r] & \mathbf{R}^{\geq 0}(A')
\end{tikzcd}$$

The top-right composite is $g \mapsto \|g\| \mapsto \|g\||_{A'}$, the left-bottom composite is $g \mapsto g|_{A'} \mapsto \|g|_{A'}\|$ and these two are equal. And so, we have a map $G \xrightarrow{\|\cdot\|} \mathbf{R}^{\geq 0}$ in $MEAS(X)$. The norm axioms follow from those for the $H(x)$'s (modding out by a.e. equivalence ensures $\|g\| = 0 \Rightarrow g = \mathbf{0}$). We will describe other topological properties of G in the sequel.

3.5.4 Hilbert Sheaves (Definitions and Topology)

We use the above discussion (about G, $\mathbf{C}$, norm, etc.) as motivation for our notion of Hilbert space object in $MEAS(X)$. In this section, we define such and discuss topological notions (for example, completeness).

Definition 3.5.7 *A* Hilbert Space Object (= Hilbert Sheaf) *in* $MEAS(X)$ *is a Cauchy complete inner product space over* $\mathbf{C}$. □

Such an $H \in MEAS(X)$ has as part of its data, natural maps $1 \xrightarrow{\lceil 0 \rceil} H$, $H \xrightarrow{-(\ )} H$, $H \times H \xrightarrow{+} H$, $\mathbf{C} \times H \xrightarrow{\cdot} H$. These operations make H into a $\mathbf{C}$-vector space.

In ordinary functional analysis, there are two equivalent ways to describe distance in a Hilbert space. One is to give an inner product, $\langle - | - \rangle$, which yields a norm (via $\| \cdot \| = \sqrt{\langle \cdot | \cdot \rangle}$) that satisfies the parallelogram law, $\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$. Another way is to give a norm that satisfies the parallelogram law and define an inner product using the polarization identity, $\langle f|g \rangle = \frac{1}{4}\|f+g\|^2 - \frac{1}{4}\|f-g\|^2 + \frac{i}{4}\|f+ig\|^2 - \frac{i}{4}\|f-ig\|^2$. In our case, we have:

1. a natural transformation $\langle - | - \rangle : H \times H \longrightarrow \mathbf{C}$

2. a natural transformation $\| \cdot \| : H \longrightarrow \mathbf{R}^{\geq 0}$ as in the previous section

These are to satisfy the obvious axioms (the classical ones translated as equations for morphisms). As a simple example, positive definiteness may be regarded as the existence of a factorization of $H \xrightarrow{\Delta} H \times H \xrightarrow{\langle \cdot | \cdot \rangle} \mathbf{C}$ through $\mathbf{R}^{\geq 0}$ considered as a subsheaf of $\mathbf{C}$.

An inner product yields a norm and a norm yields an inner product as in the classical case.

Given $\langle -|-\rangle : H \times H \longrightarrow \mathbf{C}$, we get $\|\cdot\| : H \longrightarrow \mathbf{R}^{\geq 0}$ as follows: positive definiteness of $\langle -|-\rangle$ ensures the image of $H \xrightarrow{\Delta} H \times H \xrightarrow{\langle -|-\rangle} \mathbf{C}$ is contained in $\mathbf{R}^{\geq 0}$; we take $\|\cdot\| : H \longrightarrow \mathbf{R}^{\geq 0}$ as $H \longrightarrow Im(\langle -|-\rangle \circ \Delta) \xrightarrow{\sqrt{-}} \mathbf{R}^{\geq 0}$. To check that this works, we need only check that $\mathbf{R}^{\geq 0} \xrightarrow{\sqrt{-}} \mathbf{R}^{\geq 0}$ is natural (everything else is as in classical functional analysis). But, certainly, for a nonnegative measurable function, $t(x)$, $\sqrt{t(x)}$ is also. Furthermore, the square root of the restriction of t is equal to the restriction of the square root of t. Finally, we note that if $t = t'$ except on a set A with $\mu(A) = 0$, then $\sqrt{t} = \sqrt{t'}$ except on A.

Conversely, given a norm satisfying the parallelogram law, we define $\langle -|-\rangle$ by the polarization identity. In a similar way, we see that $\langle -|-\rangle$ is natural and well defined. Indeed, addition, subtraction, and scalar multiplication in H are all natural (this gives $f+g$, $f - ig$, etc), taking the norm in $\mathbf{C}$, squaring in $\mathbf{C}$, and addition, subtraction, scalar multiplication of complex numbers are all natural and well defined as well (this gives $\frac{1}{4}\|f+g\|^2$, $\frac{i}{4}\|f+ig\|^2$, etc.). Thus, we have:

Proposition 3.5.7 *A natural inner product, $\langle -|-\rangle : H \times H \longrightarrow \mathbf{C}$ yields a natural norm $\|\cdot\| : H \longrightarrow \mathbf{R}^{\geq 0}$ satisfying polarization and the parallelogram law and conversely.* ∎

So far, we have described preHilbert space objects in $MEAS(X)$. A morphism of such is a natural transformation $H(A) \xrightarrow{\tau_A} K(A)$, which is linear ($\tau_A(f +_{H(A)} g) = \tau_A(f) +_{K(A)} \tau_A(g)$) and bounded (there is a $b \in \mathbf{R}^{\geq 0}_X$ such that $\forall h \in H$, $\|\tau(h)\|_K \leq b\|h\|_H$; it is enough to have τ_X bounded). Note that, if τ is bounded, we can find a $b \geq 1$ (b bounded away from zero) such that $\|\tau(h)\| \leq b\|h\|$. Furthermore, the restrictions $\rho^A_{A'}$ are linear and bounded (by 1). We get a category which we denote by <u>**PreHilb**</u>$(MEAS(X))$.

A sequence in H is, by definition, a map $\mathbf{N}_X \xrightarrow{s} H$. Now, in a Grothendieck topos, $\mathbf{N}_X = \sum_{n \in \mathbf{N}} 1_X$, so, in our case, a sequence is simply a(n ordinary) sequence of global elements: $\mathbf{N} \xrightarrow{s} H(X)$ (for convenience, we will often start sequences off at $n = 1$). We may formulate convergence and Cauchyness using this fact.

Definition 3.5.8 *The sequence* $\mathbf{N} \xrightarrow{(s_n)} H(X)$ *is said to be* convergent *if* $\exists s \in H(X)(\forall k \in \mathbf{N}_X^+ \ \exists$ *a cover* $\{A_i\}_{i=1}^{\infty}$ *of* X *and* $\exists N_i$, $i = 1, 2, 3, \ldots$, *such that* $\forall n \geq N_i$, $\|s_n - s\| < \frac{1}{k}$ *on* A_i). □

Definition 3.5.9 *The sequence* $\mathbf{N} \xrightarrow{(s_n)} H(X)$ *is said to be* Cauchy *if* $\forall k \in \mathbf{N}_X^+$, $\exists$ *a cover* $\{A_i\}_{i=1}^{\infty}$ *of* X *and* $\exists N_i$, $i = 1, 2, 3, \ldots$, *such that* $\forall n, m \geq N_i$ $\|s_n - s_m\| < \frac{1}{k}$ *on* A_i. □

Remarks: 1. $\|s_n - s\| < \frac{1}{k}$ on A_i means $\|s_n - s\|_{A_i}(x) < \frac{1}{k(x)}$ for almost all $x \in A_i$.

2. We may also define, for $A \in \mathcal{A}$, A-convergent and A-Cauchy as the above with X replaced by A.

3. If $X = 1$, these are the usual notions in <u>**Hilb**</u>.

4. For $f, g \in \mathbf{R}_X(A)$, $f < g$ iff $f(x) < g(x)$ for almost all $x \in A$. $<$ is an internal order. That is, $\mathbf{R} \times \mathbf{R} \simeq \mathbf{R} + \boxed{<} + \boxed{>}$ (let $(f, g) \in (\mathbf{R} \times \mathbf{R})(A)$, then $A_1 = \{x \mid f(x) = g(x)\}$, $A_2 = \{x \mid f(x) < g(x)\}$, and $A_3 = \{x \mid f(x) > g(x)\}$ forms a cover of A). □

Definition 3.5.10 $H(X)$ *is said to be* complete *if every Cauchy sequence in* H *converges in* H. □

Proposition 3.5.8 $\mathbf{R}_X$ *is complete.*

Remarks: 1. Strictly speaking, $\mathbf{R}_X$ is not a Hilbert space (over $\mathbf{C}$). But, convergent, Cauchy, and complete can all be formulated in an obvious way.

2. The classical proof that $\mathbf{R}$ is Cauchy complete involves a sequence of steps: Cauchy $\Rightarrow$ bounded; sequence $\Rightarrow \exists$ monotone subsequence; monotone sequence + bounded $\Rightarrow$ convergent; Cauchy + convergent subsequence $\Rightarrow$ convergent. This does not translate to our case. For example, if $f_n \to f$ pointwise, then we do not necessarily have a subsequence that increases to f. □

Proof: (of proposition 3.5.8): Our proof requires only two steps: f_n Cauchy $\Rightarrow \exists f$, $f_n \to f$ pointwise and $f_n \to f$ pointwise $\Rightarrow f_n \to f$. Let $\{s_n\}$ be a Cauchy sequence. Then $\forall k \in \mathbf{N}_X^+$, there is a cover $\{A_i\}$ and N_i, such that $\forall n, m \geq N_i$, $\|s_n(x) - s_m(x)\| < \frac{1}{k}$ on A_i. In particular, $s_n(x)$ is a(n ordinary) Cauchy sequence for almost all $x \in X$ (can choose $k = \lceil k \rceil$). Since $\mathbf{R}$ is complete, there is an $s(x)$ such that $s_n(x) \to s(x)$. Since s is the pointwise limit of measurable functions, it is measurable and there is an N such that $\|s(x) - s_N(x)\| < \lceil 1 \rceil$ so, since $\|s_N(x)\| < \infty$ *a.e.* ($s_N \in \mathbf{R}_X$), $\|s(x)\| < 1 + \|s_N(x)\| < \infty$ *a.e.* so $s \in \mathbf{R}_X$.

We must show $s_n \to s$. W.L.O.G. we may assume $s_n(x) \to s(x)$ pointwise everywhere. Let $k \in \mathbf{N}_X^+$ and assume first that $k = \lceil k \rceil$ is constant. Let $G_n = \{x \mid \|s_n(x) - s(x)\| < \frac{1}{\lceil k \rceil}\}$ and $E_i = \bigcap_{n=i}^{\infty} G_n = \{x \mid \|s_n(x) - s(x)\| < \frac{1}{\lceil k \rceil}$ $\forall n \geq i\}$. Suppose $x \in A$, then since $s_n(x) \to s(x)$, there is an N such that $\|s_n(x) - s(x)\| < \frac{1}{\lceil k \rceil}$, $\forall n \geq N$. That is, $x \in E_N$ for some N. Thus, the E_i's cover A. Put $N_i = i$ and we have found our cover and the N_i for which $\|s_n - s\| < \frac{1}{\lceil k \rceil}$.

Finally, suppose $k \in \mathbf{N}_X^+$ is not necessarily constant. It is locally constant. By considering $A_j = \{x \mid k(x) = j\}$ and applying the above special case, we get $s_n \to s$. ∎

Remark: If we try to remove the "existence of cover" requirement from the definitions of convergence and Cauchy, we get for example, $\forall k \in \mathbf{N}_X^+ \exists N \in \mathbf{N} \forall n, m \geq N$ $\|s_n - s_m\|_X < \frac{1}{k(x)}$. This definition would yield an $s(x)$ ($s_n(x)$ would still be Cauchy for *a.a.* x). However, there would be no way to track the rates of convergence of $s_n(x)$ to $s(x)$ for various x's. $\square$

In a similar manner, we have,

Proposition 3.5.9 *Let G be the preHilbert sheaf constructed from an MFHS, $((H(x))_{x\in X}, \mathcal{G})$, then G is complete.*

Proof: The proof is exactly as above. The only thing at stake is whether $s(x) = \lim_{n\to\infty} s_n(x)$ (pointwise limit) is in $\mathcal{G}$. But $\forall g \in \mathcal{G},\ x \mapsto \langle s(x)|g(x)\rangle = \lim_{n\to\infty}(x \mapsto \langle s_n(x)|g(x)\rangle$, being the limit of measurable functions, is measurable. Whence, by axiom 2 for an MFHS, $s \in \mathcal{G}$ as required. ∎

Corollary: $\mathsf{C}(-)$ is complete. ∎

We end this section with a discussion about the completion of a preHilbert space object. We will prove a lengthy list of lemmas (many of the proofs mimic classical ones but require some translation to the sheaf case) culminating in a theorem about the existence and basic properties of the completion. We will exhibit a functor $\underline{\mathbf{PreHilb}}(MEAS(X)) \xrightarrow{c(\ )} \underline{\mathbf{Hilb}}(MEAS(X))$.

For $H \in \underline{\mathbf{PreHilb}}(MEAS(X))$, let $c(H)(A)$ as the set of equivalence classes of A-Cauchy sequences $\mathbf{N} \xrightarrow{s} H(A)$, with $\{s_n\} \equiv \{t_n\}$ iff $\lim_{n\to\infty} \|s_n - t_n\| = 0$ (this latter limit taken in $\mathbf{R}^{\geq 0}(A)$).

Lemma 3.5.1 *$\equiv$ is an equivalence relation.*

Proof: Certainly, $\equiv$ is reflexive and symmetric ($-(s_n - t_n) = (t_n - s_n)$ and $\|(-1)h\| = \|-1\|\|h\| = \|h\|$). Now suppose $\|s_n - t_n\| \to 0$ and $\|t_n - u_n\| \to 0$ in $\mathbf{R}^{\geq 0}(A)$. Let $k \in \mathbf{N}^+_X$. There is a cover $\{A_i\}$ of A and $\exists N_i \forall n \geq N_i \|s_n - t_n\| < \dfrac{1}{\lceil 2\rceil k}$ on A_i and there is a cover $\{A'_i\}$ of A and $\exists M_i \forall n \geq M_i \|t_n - u_u\| < \dfrac{1}{\lceil 2\rceil k}$ on A'_i. Let $P_i = \max\{M_i, N_i\}$ and $B_i = A_i \cap A'_i$. Then $\{B_i\}$ is a cover of A ($A \setminus \bigcup(A_i \cap A'_i) = (A \setminus \bigcup A_i) \cap (A \setminus \bigcup A'_i)$) and $\|s_n - u_n\| = \|s_n - t_n + t_n - u_n\| \leq \|s_n - t_n\| + \|t_n - u_n\| < \dfrac{1}{\lceil 2\rceil k} + \dfrac{1}{\lceil 2\rceil k} = \dfrac{1}{k}\ \forall n \geq P_i$ on B_i. ∎

Remark: This is one "translation" alluded to in our opening paragraph. The transitivity part of the proof above exhibits an "$\frac{\epsilon}{2}$-proof" in this context. □

Lemma 3.5.2 *$c(H)(-)$ is a presheaf.*

Proof: Suppose $A' \subseteq A$ and $\{s_n\} \in c(H)(A)$. We get a sequence in $H(A')$ by restriction (in the sheaf H): $\{s_n\} \longmapsto \{s_n|_{A'}\}$. Now suppose $\{s_n\} \equiv_A \{t_n\}$ and let $k \in \mathbf{N}_X^+(A')$. Put $\hat{k} \in \mathbf{N}_X^+(A)$ as $\hat{k}(x) \doteq \begin{cases} k(x) & x \in A' \\ 1 & x \notin A' \end{cases}$. Then there is a cover, $\{A_\iota\}$, of A and $\exists N_\iota \forall n \geq N_\iota \|s_n - t_n\|_{A_\iota} < \dfrac{1}{\hat{k}(x)}$ on A_ι. But, the same N_ι will work for the cover of A' given by $A_\iota \cap A'$ and so $\{s_n\} \equiv_{A'} \{t_n\}$. ∎

Lemma 3.5.3 *$c(H)(-)$ is a sheaf.*

Proof: Suppose $\{A_p\}_{p=1}^{\infty}$ covers A and let $\{s_{p\imath}\}_{\imath=1}^{\infty}$ be a compatible family of elements of $H(A_p)$ ($p = 1,\ 2,\ 3, \ldots$). Since $H(-)$ is a sheaf, we get a unique extension $s_r \in H(A)$ ($r = 1,\ 2,\ 3, \ldots$). We must show $\{s_r\} \in c(H)(A)$ and $\{s_{pr}\} \equiv_{A_p} \{t_{pr}\}$, $p = 1,\ 2,\ 3, \ldots$ implies $\{s_r\} \equiv_A \{t_r\}$. We will show the second (the first being similar).

Let $k \in \mathbf{N}_X^+$. For each p there is a cover $A_{p\imath}$ of A_p and $N_{p\imath}$ such that $\forall n \geq N_{pi} \|s_{pn} - t_{pn}\| < \dfrac{1}{k}$ on $A_{p\imath}$. But $\{A_{p\imath}\}_{(p,\imath)=(1,1)}^{(\infty,\infty)}$ forms a cover of A and $\forall n \geq N_{pi} \|s_n|_{A_{p\imath}} - t_n|_{A_{p\imath}}\|_{A_{p\imath}} = \|s_{pn} - t_{pn}\|_{A_{p\imath}} < \dfrac{1}{k}$ for *a.a.* $x \in A_{p\imath}$ and so $\|s_n - t_n\| \to 0$ in $\mathbf{R}_X^{\geq 0}(A)$ as required. ∎

There are operations on $c(H)(A)$ defined pointwise: $0 = \{0\}_{n=1}^{\infty}$, $-\{s_n\} = \{-s_n\}$, $\{s_n\} + \{t_n\} = \{s_n + t_n\}$, $\alpha \quad \{s_n\} = \{\alpha \cdot s_n\}$. these operations are well defined with respect to $\equiv$. For example, suppose $\{s_n\} \equiv \{s_n'\}$ and $\{t_n\} \equiv \{t_n'\}$, then $\|(s_n + t_n) - (s_n' + t_n')\| \leq \|s_n - s_n'\| + \|t_n - t_n'\| \longrightarrow 0$ as in the $\frac{\epsilon}{2}$-proof of lemma 3.5.1. We define a norm on $c(H)(A)$ by $\|\{s_n\}\| = \lim\limits_{n\to\infty} \|s_n\|$. Before, we show that this is well defined, we require some basic properties of limits.

Lemma 3.5.4 *In* $\mathbf{R}_X$*,*

1. $a \rightarrow a$.

2. *If* $a_n \rightarrow a$*,* a_n *positive, then* a *is nonnegative.*

3. *If* $a_n - b_n \rightarrow 0$ *and* $a_n \rightarrow a$*, then* $b_n \rightarrow a$.

4. *If* $a_n \rightarrow a$ *and* $b_n \rightarrow b$*, then* $a_n - b_n \rightarrow a - b$.

5. *If* $a_n \rightarrow a$*,* $b_n \rightarrow b$*, and if* $a_n < b_n \forall n$*, then* $a \leq b$.

6. *If* $\mathbf{R} \xrightarrow{\tau} \mathbf{R}$ *is bounded and* $a_n \rightarrow a$*, then* $\tau(a_n) \rightarrow \tau(a)$.

Proof: 1. The cover of A is $\{A\}$ and put $N_1 = 1$ for any choice of $k \in \mathbf{N}_X^+$.
2. Suppose a is not nonnegative. Then there is a set B of positive measure for which $a(x) < 0 \forall x \in B$. There is a $t \in \mathbf{N}$ with $A_t = \{x \mid a(x) < -\lceil \frac{1}{t} \rceil\}$ of positive measure (if not, then $\mu(B) = 0$). Since $a_n \rightarrow a$, there is a cover $\{A_i\}$ of A and N_i such that $\forall n \geq N_i$, $\|a_n - a\| < \lceil \frac{1}{t} \rceil$ on A_i. Since $\{A_i\}$ covers A, there is an i such that $\|a_n - a\| < \lceil \frac{1}{t} \rceil$ on $A_i \cap A_t$ and $\mu(A_i \cap A_t) > 0$ whence $a_n < 0$ on $A_i \cap A_t$ which is a contradiction.
3. and 4. are "$\frac{\epsilon}{2}$-proofs" as in lemma 3.5.1; $\|b_n - a\| = \|b_n - a_n + a_n - a\| \leq \|b_n - a_n\| + \|a_n - a\| \rightarrow 0$ and $\|(a_n - b_n) - (a - b)\| \leq \|a_n - a\| + \|b - b_n\| \rightarrow 0$.
5. follows immediately from 2. and 4.
6. Let $b \in \mathbf{N}_X^+$ be a bound for τ and let $k \in \mathbf{N}_X^+$. Choose a cover $\{A_i\}$ of A and natural numbers N_i such that $\|a_n - a\| < \frac{1}{bk}$ on A_i for $n \geq N_i$. Then, for this

cover and N_ι, $\|\tau(a_n) - \tau(a)\|$
$= \|\tau(a_n - a)\| \le b\|a_n - a\| < \frac{1}{k}$. ∎

Lemma 3.5.5 $\|\cdot\|$ *is well defined.*

Proof: Since $\{s_n\}$ is a Cauchy sequence and $|\|s_n\| - \|s_m\|| \le \|s_n - s_m\|$, $\{\|s_n\|\}$ is a Cauchy sequence in $\mathbf{R}^{\ge 0}_X$. But $\mathbf{R}^{\ge 0}_X$ is complete, so $\lim_{n\to\infty} \|s_n\|$ exists. Now, suppose $\{s_n\} \equiv \{s'_n\}$, then by lemma 3.5.4, $\lim_{n\to\infty} \|s_n\| = \lim_{n\to\infty} \|s'_n\|$. ∎

Lemma 3.5.6 $\|\cdot\|$ *is a norm on* $c(H)(A)$.

Proof: $\|\alpha\{s_n\}\| = \lim_{n\to\infty} \|\alpha s_n\| = \lim_{n\to\infty} \|\alpha\|\|s_n\| = \|\alpha\| \lim_{n\to\infty} \|s_n\|$ by lemma 3.5.4 # 6, since $\|\alpha\| \cdot -$ is a bounded linear transformation.
$\|\{s_n\} + \{t_n\}\| = \|\{s_n + t_n\}\| = \lim_{n\to\infty} \|s_n + t_n\| \le \lim_{n\to\infty} \|s_n\| + \|t_n\| = \lim_{n\to\infty} \|s_n\| + \lim_{n\to\infty} \|t_n\|$. The inequality follows from # 5 the proof of the last equality is essentially the same as that for # 4.
Finally, that $\|\{s_n\}\| \ge 0$ follows from # 2. Suppose that $\|\{s_n\}\| = 0$, then $\lim_{n\to\infty} \|s_n - 0\| = 0$ so $\{s_n\} \equiv \{0\}$. ∎

Lemma 3.5.7 *Let* $H^c(A)$ *consist of equivalence classes of constant sequences. Then* $H^c(A)$ *is isometric to* $H(A)$ *and* $cl(H^c(A)) = c(H)(A)$.

Proof: $H^c(A) = \{\{s\}_{n=1}^\infty \mid s \in H(A)\}/\equiv$. The isometry is $H(A) \longrightarrow H^c(A)$, $s \longmapsto \{s\}$ (note: by lemma 3.5.4 above, if $\{s\} \equiv \{t\}$ then $s = t$ and, furthermore, $\|\{s\}\| = \lim_{n\to\infty} \|s\| = \|s\|$).

Next we show, for any $\{s_n\} \in c(H)(A)$, there is a sequence of points of $H^c(A)$ c-converging to it. To fix notation, put $s^* = \{s_n\}_{n=1}^\infty \in c(H)(A)$ and let $s^*_p = \{s_p\}_{n=1}^\infty \in H^c(A)$ be the constantly s_p family. We claim $s_p* \to s^*$ in $c(H)(A)$. Now, $\|s^*_p - s^*\| = \lim_{n\to\infty} \|s_p - s_n\|$. $\{s_n\}$ is Cauchy, so $\forall k \in \mathbf{N}^+_X$ there is a cover $\{A_\iota\}$ of A

and N_i such that $\forall p, n \geq N_i \|s_p - s_n\| < \frac{1}{\lceil 2 \rceil k}$. By lemma 3.5.4, $\|s_p - s_n\| < \frac{1}{\lceil 2 \rceil k}$ implies $\lim_{n\to\infty} \|s_p - s_n\| \leq \frac{1}{\lceil 2 \rceil k} < \frac{1}{k}$. And so, we have found our cover and N_i for $\|s_p^* - s^*\| < \frac{1}{k}$. ∎

Lemma 3.5.8 *Cauchy sequences in $H^c(A)$ converge in $c(H)(A)$.*

Proof: Let $\{s_n^*\}$ be a Cauchy sequence in $H^c(A)$ with $s_n^* = [\{s_n, s_n, s_n, \ldots\}]$. Then $\forall k \in \mathbf{N}_X^+$, there is a cover $\{A_i\}$ and N_i, such that $\forall n, m \geq N_i$, $\|\{s_n^*\} - \{s_m^*\}\| < \frac{1}{k}$, so $\{s_n\}$ is Cauchy in $H(A)$, whence there is an $s \in H(A)$ to which it converges. And so $s_n^* \to s^* = [\{s, s, s, \ldots\}]$. ∎

Lemma 3.5.9 *Cauchy sequences in $c(H)(A)$ converge in $c(H)(A)$.*

Proof: Let $s_n^* = \{s_{nm}\}$ $(n = 1,\ 2,\ 3, \ldots)$ be a Cauchy sequence in $c(H)(A)$. By lemma 3.5.7, there is a sequence $s_{pn}^* = \{s_p, s_p, s_p, \ldots\} \to s_n^*$.

s_{pn}^* is Cauchy: Choose a cover (intersect if necessary) and N_i so that $\|s_{pn}^* - s_{pm}^*\|$ $= \|s_{pn}^* - s_n^* + s_n^* - s_m^* + s_m^* - s_{pm}^*\| \leq \|s_{pn}^* - s_n^*\| + \|s_n^* - s_m^*\| + \|s_m^* - s_{pm}^*\|$ $< \frac{1}{\lceil 3 \rceil k} + \frac{1}{\lceil 3 \rceil k} + \frac{1}{\lceil 3 \rceil k} = \frac{1}{k}$.

Since s_{pn}^* is Cauchy and by lemma 3.5.8, $s_{pn}^* \to s^*$. Now, choose a cover and N_i so that $\|s_n^* - s^*\| \leq \|s_n^* - s_{pn}^*\| + \|s_{pn}^* - s^*\| < \frac{1}{\lceil 2 \rceil k} + \frac{1}{\lceil 2 \rceil k}$. Thus, $s_n^* \to s^*$ as required. ∎

And so, we only need to prove the uniqueness part of the following theorem.

Theorem 3.5.1 *For H a preHilbert sheaf, there is a Hilbert sheaf, $c(H)$, which contains a dense, isometric copy of H. Furthermore, if K is another Hilbert sheaf with this property, then K is isometric to $c(H)$.*

Proof: Suppose $H \simeq H^c \subseteq c(H)$ and $H \simeq H^z \subseteq K$ with $H^c \xrightarrow{\phi} H^z$ an isometric isomorphism. The isometry between $c(H)$ and K is defined as follows:

Given $\{h_n\} \in c(H)$, put $k = \lim_{n\to\infty} \phi(h_n)$. Since $\{h_n\}$ is Cauchy and ϕ is an isometry, $\{\phi(h_n)\}$ is Cauchy and so converges in K. That k is a well defined element of K follows from lemma 3.5.4 #6.

Conversely, given $k \in K$, let $h_n \to k$ with $h_n \in H^z$. We get $\{\phi^{-1}(h_n)\} \in c(H)$. If h'_n is another sequence tending to k, then $\{\phi^{-1}(h_n)\} \equiv \{\phi^{-1}(h'_n)\}$ for $\|\phi^{-1}(h_n) - \phi^{-1}(h'_n)\| = \|h_n - h'_n\| \to k - k = 0$. That $\{\phi^{-1}(h_n)\}$ is Cauchy is the usual: $\|\phi^{-1}(h_n) - \phi^{-1}(h_m)\| = \|h_n - h_m\| \leq \|h_n - k\| + \|k - h_m\| < \frac{1}{\lceil 2 \rceil k} + \frac{1}{\lceil 2 \rceil k}$. ∎

Definition 3.5.11 $c(H)$ *is called the* Completion *of* H. □

Theorem 3.5.1 gives us an idea of how to make $c(\ )$ functorial. Suppose $H \xrightarrow{T} K \in \underline{\textbf{PreHilb}}(MEAS(X))$. Define $c(H) \xrightarrow{c(T)} c(K)$ by $c(T)\{s_n\} = \{T(s_n)\}$. As in the theorem, $\{s_n\}$ Cauchy implies $\{T(s_n)\}$ Cauchy and $\{s_n\} \equiv \{t_n\}$ implies $\{T(s_n)\} \equiv \{T(t_n)\}$. And so, there is a functor

$$\underline{\textbf{PreHilb}}(MEAS(X)) \xrightarrow{c(\ )} \underline{\textbf{Hilb}}(MEAS(X)).$$

This is left adjoint to the forgetful functor U. Consider

$$\frac{c(H) \xrightarrow{\quad T \quad} K}{H \xrightarrow{\quad S \quad} U(K)}$$

with $H \in \underline{\textbf{PreHilb}}(MEAS(X))$, $K \in \underline{\textbf{Hilb}}(MEAS(X))$. Given S, we define $T\{h_n\} = \lim_{n\to\infty} S(h_n)$ and conversely, given T, define $S(h) = T(\{h\})$.

3.5.5 Hilbert Sheaves as an Indexing Notion

We begin by describing the direct integral of a Hilbert sheaf. Let
$H \in \underline{\textbf{Hilb}}(MEAS(X))$, define

$$\int^{\oplus} H = \{s : 1 \to H(X) \mid \int \|s\|_X^2 d\mu < \infty \text{ for any choice of } \|s\|\}$$

Remarks: 1. For an $s \in H(X)$, $\|s\|$ is actually an equivalence class of a measurable functions from X to $\mathbf{R}^{\geq 0}$ also denoted by $\|s\|$. We require the integral to be finite for any choice of representative (see also the discussion for the definition of $\int_\phi^\oplus$ below).

2. We may replace "for any choice" by "for some choice." □

We have arithmetic operations on $\int^\oplus H$ inherited from those on H. For example, $s + t := s +_{H(X)} t$. For almost all x, we have $\|s+t\|^2 \leq 2^2(\|s\|^2 + \|t\|^2)$, so if s and t are square-integrable, so is $s + t$.

Put a norm on $\int^\oplus H$ by defining the inner product $\langle s|t\rangle_2 := \int \langle s|t\rangle_X d\mu$. The resulting norm is called the $\|\cdot\|_2$ norm.

Theorem 3.5.2 $\int^\oplus H$ *is complete.*

Proof: The proof is similar to that for theorem 2.2.1. Let s_n be a 2-Cauchy sequence in $\int^\oplus H$. We can choose a subsequence, also called s_n, such that $\sum_{n=1}^\infty \|s_{n+1} - s_n\|_2 < \infty$.

We claim that $t_N = s_1 + \sum_{n=1}^N (s_{n+1} - s_n)$ converges to a $t \in H$ and $t = \lim_{n\to\infty} s_n \in \int^\oplus H$. We must show that t_N is a Cauchy sequence. That is, for each $k \in \mathbf{N}_X^+$, we must find a cover and an N_i for the sequence t_N.

Suppose $k = \lceil k \rceil$ is constant. Since $\sum_{n=1}^\infty \|s_{n+1} - s_n\|_2 < \infty$, we have $\sum_{n=1}^\infty \|s_{n+1} - s_n\|(x) < \infty$ for almost all $x \in X$. Put $A_{M,k} = \{x \mid \sum_{n=M}^\infty \|s_{n+1} - s_n\|(x) < \frac{1}{k}\}$. Then $\{A_{M,k}\}_{M=1}^\infty$ forms a cover of X and $\|t_p - t_q\| < \frac{1}{k}$ for all $p, q \geq M$. For a general k, put $A_j = \{x \mid k(x) = j\}$. Then $A_{M,j} \cap A_j$ is the required cover and M is the required "N_i" of definition 3.5.9.

Since H is complete, $t_N \to t = s_1 + \sum_{n=1}^\infty (s_{n+1} - s_n)$. $\|t_N - t\|_2$

$\leq \sum_{n=N}^{\infty} \|s_{n+1} - s_n\|_2 \to 0$ as $N \to \infty$ so $t_N \to_2 t$ as well. Furthermore, $\|t\|_2 \leq \|s_1\|_2 + \sum_{n=1}^{\infty} \|s_{n+1} - s_n\|_2 < \infty$ so that $t \in \int^{\oplus} H$ as required. ∎

Remark: t_N and t are special in the above theorem. In general, $u_n \to u$ does not imply $u_n \to_2 u$ (if $\|u_{N_i} - u\|(x) < \frac{1}{k}$ on A_i, then we do not necessarily have $\|u_N - u\|_2 < \epsilon$, say, on all of X; the N_i's may increase (over i) without bound). In order to ensure 2-convergence, we would require some uniformity (a common bound) of the N_i's. However, if $u_n \to u$ and $u_n, u \in \int^{\oplus} H$, then $u_n \to_2 u$ iff $\|u_n\|_2 \to \|u\|_2$ (for the case of ordinary functions in L^p, see [Roy, p. 118]). □

Now, suppose $H \xrightarrow{\tau} K$ is a bounded (by $b \in \mathbf{R}^+_X$ say) linear transformation and let $s \in \int^{\oplus} H$. Then $\int \|\tau(s)\|^2_X d\mu \leq \int \|b\|^2_X \|s\|^2_X d\mu$. There is, however, no guarantee that this second integral is finite. So, bounded linear transformations are not adequate to make $\int^{\oplus}$ functorial. We need stronger conditions on the bound.

Definition 3.5.12 *Let the objects of the following two categories be Hilbert sheaves on X.*

<u>**CBHilb**</u>$(MEAS(X))$ *has as morphisms linear $H \xrightarrow{\tau} K$, for which there is a constant $b \in \mathbf{R}^{\geq 0}$ such that $\forall h \in H$, $\|\tau(h)\|_K \leq \lceil b \rceil \|h\|_H$.*

<u>**L2Hilb**</u>$(MEAS(X))$ *has $b \in L^2(X; \mathbf{R}^{\geq 0})$.* □

Remarks: 1. etymology: "CBHilb" = constantly bounded = bounded by a constant function; "L2Hilb"= bounded by an L2 function.

2. CBHilb $\subseteq$ L2Hilb.

3. The bound can be chosen to be "well away" from 0. □

Theorem 3.5.3 *On both the categories* <u>**CBHilb**</u>$(MEAS(X))$ *and* <u>**L2Hilb**</u>$(MEAS(X))$, $\int^{\oplus}$ *is functorial.* ∎

In fact, we get a Hilbert space, $\int_A^{\oplus} H$, for each $A \in \mathcal{A}$, by considering square-integrable sections of $H(A)$. And so, we get an element of $\Pi_{A \in \mathcal{A}} H(A)$. This $\mathcal{A}$-family is not arbitrary though, in view of the fact that the restrictions $\rho^A_{A'}$ are bounded (indeed, constantly bounded) linear transformations.

We next look at substitution and will consider the special case of Δ first. Recall from section 3.4.2, for $K \in$ <u>set</u>, $\Delta(K)(A) = \{(B, f) \mid \mu(A \triangle B) = 0,\ B \xrightarrow{f} K,$ $f(B)$ countable, $f^{-1}(k) \in \mathcal{A} \forall k \in K\}/\sim$ gives a sheaf in $MEAS(X)$.

Now, $\Delta(1)(A) = 1 = 1(A)$ since $f =!$ (indeed, $\Delta(K) = \Delta(1) + \cdots + \Delta(1)$). Furthermore, $\Delta(K \times L) = \Delta(K) \times \Delta(L)$. Given $B \xrightarrow{f} K \times L \in \Delta(K \times L)(A)$, we get a pair, $(B \xrightarrow{f_1} K, B \xrightarrow{f_2} L) \in \Delta(K)(A) \times \Delta(L)(A)$ where $f = (f_1, f_2)$. Conversely, given (B_1, f_1) and (B_2, f_2), we get $(B_1 \cap B_2, (f_1, f_2))$.

Suppose $H \in$ <u>**Hilb**</u>. We have operations on $\Delta(H)(A)$ defined in an obvious way: $0 \in \Delta(H)(A) = A \xrightarrow{[0]} H$; $-(B, f) = (B, -f)$; $(B, f) + (B', f') = (B \cap B', f + f')$. These make $\Delta(H)(A)$ into an additive group.

For $\alpha \in \mathbf{C}_X$ and $(B, f) \in \Delta(H)(A)$, $\alpha(x) \cdot f(x)$ does not necessarily have a countable image. Thus, $\Delta(H)$ is not a $\mathbf{C}_X$-vector space. However, if we consider the geometric field, $\mathbf{C}_{lc} = \Delta(\mathbf{C})$, of equivalence classes of locally constant $\mathbf{C}$-valued functions (the proof that $\mathbf{C}_{lc}$ is a geometric field is the same as that for $\mathbf{C}_X$; see proposition 3.5.3), then $\Delta(H)$ is a $\mathbf{C}_{lc}$-vector space.

We have a "norm" (satisfies positive definiteness and the triangle inequality) given by $\|(B, f)\| = (B, \|f\|) \subseteq \Delta(\mathbf{R}) \subseteq \mathbf{R}_X$. As one might expect, this is not complete but the last containment is dense in the following sense:

Proposition 3.5.10 *Every $f \in \mathbf{C}_X$ is the pointwise limit of some sequence in $\Delta(\mathbf{C})(X)$.*

Proof: This is exactly the statement that a measurable function is the limit of "step" functions (not quite step functions but functions with countable image).

$f \in \mathbf{C}_X$ can be written as $f = g+ih$, where $g,\ h \in \mathbf{R}_X$ and g and h are each the difference of two nonnegative functions. Thus, let $f(x) \geq 0\ a.a.\ x$. Consider $A_n = \{x \mid n-1 \leq f(x) < n\}$ $(n = 1,\ 2,\ 3, \ldots)$, $A_{nk} = \{x \in A_n \mid \frac{1}{k+1} \leq f(x) - (n-1) < \frac{1}{k}\}$ $(k = 1,\ 2,\ 3, \ldots)$ and put $f_{NK}(x) = \sum_{n=1}^{N} \sum_{k=1}^{K} ((n-1) + \frac{1}{k+1})\chi_{A_{nk}}$. Then $f_{NK} \rightarrow f$ pointwise. ∎

We note that Δ(Hilbert) $\neq$ Hilbert is not entirely surprising since, as we saw in section 3.4.2, Δ is not logical. It does preserve finite products but not necessarily (logically) more complicated entities like $\mathbf{C}$.

We saw in example 3 of section 2.3.2 that $Sh(I)$ realizes the **Top**-indexing of **Set**. For a continuous $\alpha : J \longrightarrow I$, we get α^* by regarding an element of $Sh(I)$ as a local homeomorphism over I and pulling back along α yields a local homeomorphism over J which we may regard as a J-sheaf. As we noted in section 1.2, simply translating topological notions (like local homeomorphism) into the measure theory world leads to problems of triviality. The pulling back idea does not work.

For $(X, \mathcal{A}, \mu) \xrightarrow{\phi} (Y, \mathcal{B}, \nu)$ in **MOR**, we get substitution $MEAS(X) \xleftarrow{\phi^*} MEAS(Y)$. Indeed, ϕ^{-1} is a morphism of sites, so we have a geometric morphism $\phi^* \dashv \phi_*$, where $\phi^*(G)(A) = a(\mathrm{colim}_{A \subseteq \phi^{-1}(B)} G(B))$ for $G \in MEAS(Y)$, $A \in \mathcal{A}$, and a is the associated sheaf functor, and $\phi_*(F)(B) = F(\phi^{-1}(B))$ for $F \in MEAS(X)$ and $B \in \mathcal{B}$ (see also the remark after proposition 3.3.4). As we have noted, ϕ^* does not preserve Hilbert space objects. However, the proof of proposition 3.5.10 actually gives:

Proposition 3.5.11 $c\Delta_X \mathbf{C} = \mathbf{C}_X$ *(the completion of the sheaf of locally constant functions is the sheaf of all measurable functions).* ∎

This gives a clue that we should complete. And so, we will describe our substitution,

$$\underline{\mathbf{Hilb}}(MEAS(X)) \xleftarrow{\phi^{\#}} \underline{\mathbf{Hilb}}(MEAS(Y)),$$

where $\phi^{\#}G = c\phi^{*}G$.

ϕ^{*} preserves finite limits so it preserves Abelian group objects. In addition, we have a commutative diagram,

$$\begin{tikzcd} MEAS(X) & & MEAS(Y) \arrow[ll, "\phi^{*}"'] \\ & \underline{\mathbf{Set}} \arrow[ul, "\Delta_X"] \arrow[ur, "\Delta_Y"'] & \end{tikzcd}$$

In particular, $\phi^{*}\Delta_Y \mathbf{C} = \phi^{*}\mathbf{C}_{Y,l.c.} = \Delta_X \mathbf{C} = \mathbf{C}_{X,l.c.}$. Thus, ϕ^{*} lifts to $\mathbf{C}_{l.c.}$-modules:

$$\underline{CLCMod}(MEAS(X)) \xleftarrow{\phi^{*}} \underline{CLCMod}(MEAS(Y)).$$

We next discuss the norm. More precisely, we shall show how a norm on $G \in \underline{CLCMod}(MEAS(Y))$ yields a pseudo-norm on $\phi^{*}G$. This will become a norm on $\phi^{\#}G$ after completion. A norm on $\phi^{*}G$ would be

$$\frac{\phi^{*}G \xrightarrow{\|\cdot\|^{*}} \mathbf{R}_X^{\geq 0}}{G \longrightarrow \phi_{*}\mathbf{R}_X^{\geq 0}}.$$

There is a map $G(B) \longrightarrow \phi_{*}\mathbf{R}_X^{\geq 0}(B)$ the composite $G(B) \xrightarrow{\|\cdot\|_B} \mathbf{R}_Y^{\geq 0}(B) \xrightarrow{-\circ\phi} \mathbf{R}_X^{\geq 0}(\phi^{-1}(B))$. We define $\|\cdot\|^{*}$ to be the transpose of this composite. Two comments are necessary:

Remarks: 1. $\|s\|_B$ is an equivalence class. But if, for two representatives, $\|s\|_B \sim_Y \|s\|'_B$, then $\|s\|_B \circ \phi \sim_X \|s\|'_B \circ \phi$, since $\phi \in \underline{\mathbf{MOR}}$.

2. The converse of M0R does not necessarily hold (that is, $\mu(f^{-1}(B)) = 0 \not\Rightarrow \nu(B) = 0$). As a consequence, $\|s\|_B \circ \phi = 0 \not\Rightarrow s = 0$ and we cannot expect $\|\cdot\|^*$ to be anything more than a pseudo-norm. □

We will find an equivalent definition of $\|\cdot\|^*$ more useful. Let t be the transpose:

$$\frac{\phi^*\mathbf{R}_Y^{\geq 0} \xrightarrow{\quad t \quad} \mathbf{R}_X^{\geq 0}}{\mathbf{R}_Y^{\geq 0} \xrightarrow{\quad -\circ\phi \quad} \phi_*\mathbf{R}_X^{\geq 0}}.$$

Then $\|\cdot\|^*$ is the composite $\phi^*G \xrightarrow{\phi^*\circ\|\cdot\|} \phi^*\mathbf{R}_Y^{\geq 0} \xrightarrow{t} \mathbf{R}_X^{\geq 0}$. We must exhibit the triangle inequality and homogeneity for $\|\cdot\|^*$.

Triangle Inequality: $G \xrightarrow{\|\,\|} \mathbf{R}_Y^{\geq 0}$ satisfies the triangle inequality iff there is a γ to make the following triangle commute:

$$\begin{tikzcd}
G \times G \arrow[r, "{(\alpha,\beta)}"] \arrow[dr, "\gamma"'] & \mathbf{R}_Y^{\geq 0} \times \mathbf{R}_Y^{\geq 0} \\
& \mathbf{R}_Y^{\geq 0} \times \mathbf{R}_Y^{\geq 0} \arrow[u, "{(p_1,+)}"']
\end{tikzcd}$$

where α is $G \times G \xrightarrow{+} G \xrightarrow{\|\cdot\|_\bullet} \mathbf{R}_Y^{\geq 0}$ and β is $G \times G \xrightarrow{\|\,\|_\bullet \times \|\cdot\|_\bullet} \mathbf{R}_Y^{\geq 0} \times \mathbf{R}_Y^{\geq 0} \xrightarrow{+} \mathbf{R}_Y^{\geq 0}$. This is simply a translation of the statement $\|s+s'\| \leq \|s\|+\|s'\|$ into diagrammatic form. For example, the monomorphism $(p_1, +)$ expresses what it means for a non-negative real to be less than or equal to another non-negative real. We must exhibit the above for $\|\cdot\|^*$. That is, we must show there is a γ' such that

$$\begin{tikzcd}
\phi^*G \times \phi^*G \arrow[r, "{(\alpha', \beta')}"] \arrow[dr, "\gamma'"'] & \mathbf{R}^{\geq 0}_{\overline{X}} \times \mathbf{R}^{\geq 0}_{\overline{X}} \\
& \mathbf{R}^{\geq 0}_{\overline{X}} \times \mathbf{R}^{\geq 0}_{\overline{X}} \arrow[u, "{(q_1, +)}"']
\end{tikzcd}$$

commutes. Take ϕ^* of the triangle for $\dot{G}$ and augment to get:

$$\begin{tikzcd}
\phi^*G \times \phi^*G \arrow[r, "{(\phi^*\alpha, \phi^*\beta)}"] \arrow[dr, "\phi^*\gamma"'] \arrow[rr, bend left, "{(t\phi^*\alpha, t\phi^*\beta)}"] & \phi^*\mathbf{R}^{\geq 0}_{\overline{Y}} \times \phi^*\mathbf{R}^{\geq 0}_{\overline{Y}} \arrow[r, "t \times t"] \arrow[dr, phantom, "\boxed{\mathrm{SQ1}}"] & \mathbf{R}^{\geq 0}_{\overline{X}} \times \mathbf{R}^{\geq 0}_{\overline{X}} \\
& \phi^*\mathbf{R}^{\geq 0}_{\overline{Y}} \times \phi^*\mathbf{R}^{\geq 0}_{\overline{Y}} \arrow[u, "{(\overline{p_1}, \phi^*+)}"] \arrow[r, "t \times t"'] & \mathbf{R}^{\geq 0}_{\overline{X}} \times \mathbf{R}^{\geq 0}_{\overline{X}} \arrow[u, "{(q_1, +)}"']
\end{tikzcd}$$

We need $t\phi^*\alpha$ to be "the α'" for ϕ^*G, $t\phi^*\beta$ to be "the β'," and SQ1 to commute; in which case $(t \times t) \circ \phi^*\gamma$ will be the required γ'. Now, for α', $\phi^*G \times \phi^*G \xrightarrow{\phi^*+ = +_{\phi^*G}} \phi^*G \xrightarrow{\phi^*\|\cdot\|} \phi^*\mathbf{R}^{\geq 0}_{\overline{Y}} \xrightarrow{t} \mathbf{R}^{\geq 0}_{\overline{X}}$ (which is $t\phi^*\alpha$) is the α' (note: $\phi^*\|\cdot\| \circ t = \|\cdot\|^*$). For β', we have

$$\begin{tikzcd}
\phi^*G \times \phi^*G \arrow[r, "{(\phi^*\|\cdot\| \times \phi^*\|\cdot\|}"] \arrow[dr, "\|\cdot\|^* \times \|\cdot\|^*"'] & \phi^*\mathbf{R}^{\geq 0}_{\overline{Y}} \times \phi^*\mathbf{R}^{\geq 0}_{\overline{Y}} \arrow[r, "\phi^*(+)"] \arrow[d, "t \times t"'] \arrow[dr, phantom, "\boxed{\mathrm{SQ2}}"] & \phi^*\mathbf{R}^{\geq 0}_{\overline{Y}} \arrow[d, "t"] \\
& \mathbf{R}^{\geq 0}_{\overline{X}} \times \mathbf{R}^{\geq 0}_{\overline{X}} \arrow[r, "+"'] & \mathbf{R}^{\geq 0}_{\overline{X}}
\end{tikzcd}$$

So $t\phi^*\beta$ will be the β' provided SQ2 commutes.

To show SQ1 commutes, it is enough to show that the composite with each projection, $r_1, r_2 : \mathbf{R}_X^{\geq 0} \times \mathbf{R}_X^{\geq 0} \longrightarrow \mathbf{R}_X^{\geq 0}$, commutes. Composing with the first projection commutes trivially. Composing with the second projection yields

$$\begin{tikzcd}
\phi^*\mathbf{R}_Y^{\geq 0} \arrow[r, "t"] & \mathbf{R}_X^{\geq 0} \\
\phi^*\mathbf{R}_Y^{\geq 0} \times \phi^*\mathbf{R}_Y^{\geq 0} \arrow[u, "\phi^*+"] \arrow[r, "t \times t"'] & \mathbf{R}_X^{\geq 0} \times \mathbf{R}_X^{\geq 0} \arrow[u, "+"']
\end{tikzcd}$$

This is precisely SQ2. To show that this commutes, we "detranspose" to get

$$\begin{tikzcd}
\mathbf{R}_Y^{\geq 0} \arrow[r, "- \circ \phi"] & \phi_*\mathbf{R}_X^{\geq 0} \\
\mathbf{R}_Y^{\geq 0} \times \mathbf{R}_Y^{\geq 0} \arrow[u, "+"] \arrow[r, "(- \circ \phi) \times (- \circ \phi)"'] & \phi_*\mathbf{R}_X^{\geq 0} \times \phi_*\mathbf{R}_X^{\geq 0} \arrow[u, "\phi^*+"']
\end{tikzcd}$$

which commutes because addition in $\mathbf{R}_X^{\geq 0}$ is defined pointwise. $\square$

Scalars: Homogeneity for G means that

$$\begin{tikzcd}
\mathbf{C}_{Y,l.c.} \times G \arrow[r, "\cdot"] \arrow[d, "|\cdot| \times \|\cdot\|"'] & G \arrow[d, "\|\cdot\|"] \\
\mathbf{R}_Y^{\geq 0} \times \mathbf{R}_Y^{\geq 0} \arrow[r, "\times"'] & \mathbf{R}_Y^{\geq 0}
\end{tikzcd}$$

commutes. Again, we apply ϕ^* and augment. We must show that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{C}_{X,l.c.} \times \phi^*\mathbf{C}_{Y,l.c.} \arrow[r, "\cong \times 1"] \arrow[ddr, "|\cdot| \times \|\cdot\|^*"'] & \phi^*\mathbf{C}_{Y,l.c.} \times \phi^*G \arrow[r, "\phi^*\cdot = \cdot^*"] \arrow[d, "\phi^*|\cdot\| \times \phi^*|\cdot|"'] & \phi^*G \arrow[d, "\phi^*\|\cdot\|"] \\
& \phi^*\mathbf{R}^{\geq 0}_Y \times \phi^*\mathbf{R}^{\geq 0}_Y \arrow[r, "\phi^*|\cdot|"'] \arrow[d, "t \times t"'] & \phi^*\mathbf{R}^{\geq 0}_Y \arrow[d, "t"] \\
& \mathbf{R}^{\geq 0}_X \times \mathbf{R}^{\geq 0}_X \arrow[r, "\cdot"'] & \mathbf{R}^{\geq 0}_X
\end{tikzcd}$$

The top square is ϕ^* of the square for G, so commutes. The bottom square is similar to SQ2 above. That is , to show that it commutes, "detranspose" and use the fact that multiplication is pointwise. The left triangle is the product of two triangles. The second factor commutes by definition of $\|\cdot\|^*$. The first factor commutes, since

$$\begin{tikzcd}
\mathbf{C}_{X,l.c.} \arrow[d, "|\cdot|"'] & \phi^*\mathbf{C}_{Y,l.c.} \arrow[l, "\cong"'] \arrow[d, "|\cdot|"] \\
\mathbf{R}^{\geq 0}_X & \phi^*\mathbf{R}^{\geq 0}_Y \arrow[l, "t"]
\end{tikzcd}$$

commutes because its transpose does. And so, we have shown that ϕ^* lifts to pseudo-normed $\mathbf{C}_{l.c.}$-modules:

$$\underline{PNCLCMod}(MEAS(X)) \xleftarrow{\phi^*} \underline{PNCLCMod}(MEAS(Y)).$$

Remark: The foregoing discussion is interesting in that we have lifted ϕ^* without ever explicitly describing it (being the associated sheaf of a colimit, it is difficult to calculate except in certain special cases). The lifting uses functoriality, lexness, and adjointness. □

Now, complete (as PNCLC-Modules) to get $\phi^\#$. But, we have:

Proposition 3.5.12 *Let G be normed with $\mathbf{C}_{l.c.}$-homogeneity and suppose it is complete in this norm. Then G can be made into a normed $\mathbf{C}$-module with $\mathbf{C}$-homogeneity and it is complete. In short,*

$$\frac{\mathbf{C}_{l.c.} \times G \xrightarrow{\quad\cdot\quad} G}{\mathbf{C} \times G \xrightarrow{\quad\cdot\quad} G}$$

Proof: $\Uparrow$ is free. For $\Downarrow$ and $\alpha(y) \in \mathbf{C}$, let $\alpha_n(y) \to \alpha(y)$ with α_n locally constant. Put $(\alpha(y)) \cdot g := \lim_{n\to\infty} (\alpha_n(y)) \cdot g$. This limit exists since the sequence is Cauchy in G. ∎

As we have already noted, completing a pseudo-norm yields a norm (this is precisely as in the classical sense). The completion is functorial (we have shown this for preHilbert spaces, but the same works here). It preserves products. A sequence in $H \times K$, say, is just a pair of sequences, convergence and Cauchy is just convergence and Cauchy in each coordinate (here, the norm on $H \times K$ is the Euclidean norm, $\sqrt{\|h\|^2 + \|k\|^2}$). Indeed, all this works for preHilbert spaces (satisfying the parallelogram law is just equational). And so, we have a functor:

$$\underline{\mathbf{Hilb}}(MEAS(X)) \xleftarrow{\quad\phi^\#\quad} \underline{\mathbf{Hilb}}(MEAS(Y)).$$

More precisely, $\phi^\#$ may be defined as the triple composite:

$$\begin{tikzcd}
H(X) & H(Y) \arrow[l, "\phi^{\#}"'] \arrow[d, "u"] \\
P(X) \arrow[u, "c"] & P(Y) \arrow[l, "\phi^{*}"]
\end{tikzcd}$$

with "H" for Hilbert and "P" for preHilbert. In this event, pseudo-functoriality follows from the pasting together:

$$\begin{tikzcd}
H(X) & H(Y) \arrow[l, "\phi^{\#}"'] \arrow[d, shift right, "u"'] & H(Z) \arrow[l, "\psi^{\#}"'] \arrow[d, "u"] \arrow[ll, bend right, "\phi^{\#}\psi^{\#} = (\psi\phi)^{\#}"'] \\
P(X) \arrow[u, "c"] & P(Y) \arrow[l, "\phi^{*}"] \arrow[u, shift right, "c"'] & P(Z) \arrow[l, "\psi^{*}"] \arrow[ll, bend left, "\phi^{*}\psi^{*} = (\psi\phi)^{*}"]
\end{tikzcd}$$

Indeed, the top equality (between $\#$'s) follows from the bottom equality and $uc = 1$. Furthermore, $1^{\#} = 1$ and so, we have a pseudo-functorial substitution:

$$\underline{\mathbf{M0R}}^{op} \longrightarrow \underline{\mathbf{BooleanGrTopos}}.$$

Let $(X, \mathcal{A}, \mu) \xrightarrow{(\phi,\mu_y)} (Y, \mathcal{B}, \nu)$ be a disintegration and $H \in \underline{\mathbf{Hilb}}(MEAS(X))$. Put $(\int_\phi^\oplus H)(B) = \{s \in H(\phi^{-1}(B)) \mid \int_{\phi^{-1}(y)} \|s\|^2(x) d\mu_y(x) < \infty$ *a.a.* y for any choice of $\|s\|\}$. We have already described the nature of this choice above. Let us expand on those remarks. For $A \in \mathcal{A}$, the norm is a map $H(A) \xrightarrow{\|\cdot\|_A} \mathbf{R}^{\geq 0}(A)$. For $s \in H(A)$, $\|s\|$ is an equivalence class in $\mathbf{R}^{\geq 0}(A)$. The definition requires $\int_{\phi^{-1}(y)} \|s\|^2(x) d\mu_y(x) < \infty$ for any choice of representative (also denoted by $\|s\|$). Since we will study $\int_{\phi^{-1}(y)} \|s\|^2(x) d\mu_y(x)$ as a function of y, a useful result is:

Proposition 3.5.13 *If* $f_1(x) \sim_X f_2(x)$ *then*
$\int_{\phi^{-1}(y)} f_1(x)d\mu_y(x) \sim_Y \int_{\phi^{-1}(y)} f_2(x)d\mu_y(x)$.

Proof: Let $f_3(x) = \min\{f_1(x), f_2(x)\}$ so that $f_3(x) \leq f_1(x)$ is a measurable function and $f_3 \sim_X f_1$.

$\int_Y \int_{\phi^{-1}(y)} f_3(x)d\mu_y(x)d\nu(y) = \int_X f_3(x)d\mu(x) = \int_X f_1(x)d\mu(x)$
$= \int_Y \int_{\phi^{-1}(y)} f_1(x)d\mu_y(x)d\nu(y)$ so $\int_Y \int_{\phi^{-1}(y)} (f_1 - f_3)(x)d\mu_y(x)d\nu(y) = 0$. Since $(f_1 - f_3)(x) \geq 0$, $\int_{\phi^{-1}(y)} (f_1 - f_3)(x)d\mu_y(x) = 0$ for almost all y so that $\int_{\phi^{-1}(y)} f_1(x)d\mu_y(x) \sim_Y \int_{\phi^{-1}(y)} f_3(x)d\mu_y(x)$. In a similar manner, we have $\int_{\phi^{-1}(y)} f_2(x)d\mu_y(x) \sim_Y \int_{\phi^{-1}(y)} f_3(x)d\mu_y(x)$. ∎

As a consequence of this proposition, we may replace "for any choice" in the definition by "for some choice." If the integral is finite almost everywhere for some choice, then it is finite almost everywhere for any choice. Another property that we will find useful is:

Proposition 3.5.14 *Given* $s \in (\int_\phi^\oplus H)(B)$*, there is a choice of* $\|s\|$ *such that* $\int_{\phi^{-1}(y)} \|s\|^2(x)d\mu_y(x)$ *is finite for all* $y \in B$.

Proof: Let $s \in (\int_\phi^\oplus H)(B)$ and let $\|s\|$ be some choice of the norm for which the integral is finite almost everywhere. Let $G \subseteq Y$ be the measurable set where the integral is finite. Define $\|s\|'(x) = \chi_{\phi^{-1}(G)}\|s\|(x)$. Then $\int_{\phi^{-1}(y)} \|s\|'^2(x)d\mu_y(x) < \infty$ for all y and $\|s\|' \sim_X \|s\|$ since $\phi \in \underline{\mathbf{MOR}}$. ∎

Proposition 3.5.15 $(\int_\phi^\oplus H)(-)$ *is a sheaf.*

Proof: $H(\phi^{-1}(-))$ is a sheaf (it is, in fact, $(\phi_*H)(-)$) and the finiteness condition is on the points y, independent of covers. So, if each s_i satisfies the condition, the unique extension s, will as well. ∎

We next look at the algebraic properties of $\int_\phi^\oplus$. $\lceil 0 \rceil$, $-$, and $+$ are as in $H(\phi^{-1}(B))$ (note: because $\|\cdot\|_A$ is a norm, we have $\|s+s'\|^2(x) \leq 2^2(\|s\|^2(x) + \|s'\|^2(x))$ as in the ordinary sense (see [B&N])).

For scalar multiplication, suppose $\beta \in \mathbf{C}(B)$. We can compose with ϕ to get $\beta \circ \phi \in \mathbf{C}(\phi^{-1}(B))$. For $s \in (\int_\phi^\oplus H)(B)$, define $\beta \cdot s = (\beta \circ \phi) \cdot_{\phi^{-1}(B)} s$. Now, $\int_{\phi^{-1}(y)} \|(\beta \circ \phi) \cdot s\|^2 d\mu_y(x) = \int_{\phi^{-1}(y)} \|\beta \circ \phi(x)\|^2 \|s\|^2(x) d\mu_y(x) =$
$\|\beta(y)\|^2 \int_{\phi^{-1}(y)} \|s\|^2(x) d\mu_y(x) < \infty$. Furthermore, if $\beta \sim_Y \beta'$, then $\beta \circ \phi \sim_X \beta' \circ \phi$ because $\mu\{x \mid \beta \circ \phi(x) \neq \beta' \circ \phi(x)\} = \mu(\phi^{-1}\{y \mid \beta(y) \neq \beta'(y)\}) = 0$, since $\phi \in \underline{\mathbf{M0R}}$.

We may put a norm on $(\int_\phi^\oplus H)(B)$ by $\|s\|_2^2(y) = [\int_{\phi^{-1}(y)} \|s\|^2(x) d\mu_y(x)]$. As usual, $[-]$ denotes equivalence class (in this case, in $\underline{\mathbf{Mble}}(B, \mathbf{R}^{\geq 0})/\sim$). This is indeed a norm. For example, suppose $\int_{\phi^{-1}(y)} \|s\|^2(x) d\mu_y(x) = 0$. Then
$\int_X \|s\|^2(x) d\mu(x) = \int_Y \int_{\phi^{-1}(y)} \|s\|^2(x) d\mu_y(x) d\nu(y) = 0$ which implies $s = 0$.

Completeness of this norm seems to be difficult in general (all of the examples below, however, are complete). Indeed, finding a subsequence such that $\sum_{n=1}^{\infty} \|s_{n+1} - s_n\|_2(y) < \infty$ for almost all y, a step crucial to theorem 3.5.2, is not easy. We avoid the issue as to whether $\int_\phi^\oplus$ is complete for general ϕ by defining *new* $\int_\phi^\oplus = c($*old* $\int_\phi^\oplus)$. This is good, since we have all the machinery for the completion (for example, functoriality).

Example 1: Identity: In this case, $(\int_1^\oplus H)(A) = \{s \in H(A) \mid \int_{\{x\}} \|s\|^2(t) d\iota_x(t)$ $< \infty$ for any choice of $\|s\|\} = H(A)$; the finiteness condition says $\|s\|^2(x) < \infty$ which is always true (norms are real-valued not extended-real-valued). □

Example 2: Terminal Object: Let ϕ be the unique disintegration, $(X, \mathcal{A}, \mu) \xrightarrow{(!,\mu)} (1, 2, \text{counting})$. Then $(\int_!^\oplus H)(B) = \{s \in H(!^{-1}(B)) \mid$ $\int_{!^{-1}(*)} \|s\|^2 d\mu(x) < \infty$ *a.a.* x for any choice of $\|s\|\}$. If $B = \{*\}$, this is the ordinary direct integral as described above. $\square$

Example 3: Finite Sets: If $X = (1, 2, \text{counting})$, $MEAS(n) = Sh(2^n) \simeq \underline{\mathbf{Set}}^n$. Here, $L(n) = 2^n$ and 2^n is equipped with the "topology of unions," a set is covered by a family if the union of the family equals the set. In this case, every set is covered by the collection of its points.

An $H \in \underline{\mathbf{Hilb}}(MEAS(n))$ corresponds to $(H_1, H_2, \ldots, H_n) \in \underline{\mathbf{Hilb}}(\underline{Set}^n)$. Such is $H(A) = \prod_{i \in A} H(i)$. In particular, $\mathbf{C}(A) = \prod_{i \in A} \mathbf{C}$ and the norm is $H(A) \longrightarrow \mathbf{R}(A)$, $\langle h_i \rangle_{i \in A} \longmapsto \langle \|h_i\| \rangle_{i \in A}$.

Now, suppose, $n \xrightarrow{\phi} m$ is a disintegration (as we noted in chapter 1, such is just a function from n to m). $(\int_\phi^\oplus H)(B) = H(\phi^{-1}(B)) = \prod_{i \in \phi^{-1}(B))} H(i)$. Operations are coordinatewise and the norm is the Euclidean norm:

$\prod_{i \in \phi^{-1}(j)} H(i) \longrightarrow \mathbf{R}$, $\langle h_i \rangle \longmapsto \sqrt{\sum_{i \in \phi^{-1}(i)} \|h_i\|^2}$. $\square$

Now, suppose we have $H \xrightarrow{\tau} H'$ in $\underline{\mathbf{CBHilb}}(MEAS(X))$. For $s \in (\int_\phi^\oplus H)(B)$, $\int_{\phi^{-1}(y)} \|\tau s\|^2(x) d\mu_y(x) \leq b^2 \int_{\phi^{-1}(y)} \|s\|^2(x) d\mu_y(x) < \infty$ for almost all y. And so, we have a functor:

$$\underline{\mathbf{CBHilb}}(MEAS(X)) \xrightarrow{\int_\phi^\oplus} \underline{\mathbf{CBHilb}}(MEAS(Y))$$

for each ϕ.

Example 1 above shows that $\int_1^\oplus = 1$. Let $(X, \mathcal{A}, \mu) \xrightarrow{(\phi, \mu_y)} (Y, \mathcal{B}, \nu) \xrightarrow{(\psi, \nu_z)} (Z, \mathcal{C}, \rho)$ be two disintegrations and let $(\psi\phi, \theta_z)$ denotes their composition. For $H \in \underline{\mathbf{Hilb}}(MEAS(X))$, $(\int_{\psi\phi}^\oplus H)(C) = \{s \in H(\phi^{-1}\psi^{-1}(C)) \mid \int_{\phi^{-1}\psi^{-1}(z)} \|s\|^2(x) d\theta_z(x) < \infty$ *a.a.* z for any choice of $\|s\|\}$ and $(\int_\psi^\oplus \int_\phi^\oplus H)(C) =$
$\{t \in (\int_\phi^\oplus H)(\psi^{-1}(C)) \mid \int_{\psi^{-1}(z)} \|t\|^2(y) d\nu_z(y) < \infty$ *a.a.* z for any choice of $\|t\|\} =$
$\{t \in H(\phi^{-1}\psi^{-1}(C)) \mid \int_{\phi^{-1}(y)} \|t\|^2(x) d\mu_y(x) < \infty$ *a.a.* y and

$\int_{\psi^{-1}(z)} \int_{\phi^{-1}(y)} \|t\|^2(x) d\mu_y(x) d\nu_z(y) < \infty$ $a.a.$ z for any choice of $\|t\|\}$. Note that the two choices are "absorbed" into one choice. We claim that these two sets are equal.

Certainly, we have $\supseteq$ since, as we noted for composition, $\int\int = \int$. For $\subseteq$, there is a choice of $\|s\|$ to make $\int_{\phi^{-1}\psi^{-1}(z)} \|s\|^2(x) d\theta_z(x) < \infty$ for all z. Thus, $\int_{\psi^{-1}(z)} \int_{\phi^{-1}(y)} \|s\|^2(x) d\mu_y(x) d\nu_z(y) < \infty$ for all z which implies the inside integral is finite for $a.a.$ y. We have already noted that if the integral is finite for some choice, then it is finite for any choice. And so, $\int_{-}^{\oplus}$ is also pseudo-functorial.

Chapter 4

Hilbert Families

4.1 Introduction

A fundamental fact in the indexing of sets by sets is the equivalence of categories:

$$\underline{\mathbf{Set}}/I \xrightarrow{\sim} \underline{\mathbf{Set}}^I$$

for $I \in \underline{\mathbf{Set}}$. This is the genesis of the indexing idea. Immediately, one may construct a utile and rich theory of indexing by the objects of a topos, which is to be thought of, in this context, as a generalized set theory, since $\underline{\mathbf{E}}/I$ makes sense for a topos $\underline{\mathbf{E}}$ and an object I (and is, in fact, a topos). Looking at Hilbert spaces in the special topos $MEAS(X)$, exhibited one way of attacking the problem at hand. This was the approach of the previous chapter.

On a much more basic level, however, is the notion that I-family of sets is equivalent to a function into I. In this chapter, we explore a similar idea appropriately translated into our measure theoretic context as our third approach to the problem of understanding indexing by measure spaces We simply take as a basic notion of "family," a measure space over X (etymology: we use $I \in \underline{\mathbf{Set}}$ and

$X \in \underline{\mathbf{Meas}}$, whatever the latter may be). More accurately, we will use the power of the built-in indexing in disintegrations as our measure spaces over X.

It is disintegrations that we consider as "fibrations." In this chapter, the basic premise is that an object of $\underline{\mathbf{Disint}}/X$ (some examples of which are given in section 1.5.6) represents the notion of X-family. An important aspect of any theory of indexing must be the notion of substitution. In sections 4.2-4.4, we describe substitution and its adjoint, composition.

Finally, we note that it is operator theory that we hope to study. To that end, we describe "Hilbert families" in section 4.5. Using the substitution machinery developed earlier, we introduce and describe two new categories (of Hilbert families over X).

4.2 Substitution Along a M0R

4.2.1 Definitions

In this section, we explore substitution (the "pullback") of a disintegration along a M0R. Consider the diagram:

$$\begin{tikzcd}
(Z, \mathcal{C}, \rho) \arrow[r, "r"] \arrow[d, "{(g, \rho_{x'})}"'] & (Y, \mathcal{B}, \nu) \arrow[d, "{(f, \nu_x)}"] \\
(X', \mathcal{A}', \mu') \arrow[r, "\phi"'] & (X, \mathcal{A}, \mu)
\end{tikzcd}$$

with $\phi \in \underline{\mathbf{M0R}}$, $(f, \nu_x) \in \underline{\mathbf{Disint}}$. We will slowly construct the elements of this diagram.

Notation: $Z := \sum_{x' \in X'} Y_{\phi(x')}$, where $Y_{\phi(x')} := f^{-1}(\phi(x'))$. In general, T_k denotes the fibre over k. A typical element of Z is (y, x'), where $x' \in X'$ and $y \in Y_{\phi(x')}$. $\square$

Z is the pullback of ϕ and f in **set**. g and r are the projections: $g(y, x') = x'$ and $r(y, x') = y$. Thus $g^{-1}(x') = Y_{\phi(x')} \times \{x'\} \cong Y_{\phi(x')}$ and, for $A' \in \mathcal{A}'$, $g^{-1}(A') = \sum_{x' \in X'} K_{x'}$, where $K_{x'} = \begin{cases} Y_{\phi(x')}, & x' \in A' \\ \emptyset, & x' \notin A' \end{cases}$. For $r^{-1}(y)$, suppose $y \in Y_x$ (i.e. $f(y) = x$) then $r^{-1}(y) = \{y\} \times \phi^{-1}(x)$. Furthermore, $r^{-1}(B) = \{(y, x') | y \in B \text{ and } f(y) = \phi(x')\} = \{(y, x') | y \in B \text{ and } y \in f^{-1}(\phi(x'))\} = \{(y, x') | y \in B \cap f^{-1}(\phi(x'))\}$. Thus, $(r^{-1}(B))_{x'} = B \cap f^{-1}(\phi(x'))$.

Let $\mathcal{C}$ be the σ-algebra generated by $g^{-1}(A')$, $r^{-1}(B)$ for $A' \in \mathcal{A}'$ and $B \in \mathcal{B}$.

Lemma 4.2.1 *Every* $C \in \mathcal{C}$ *is* $\sum_{x' \in X'} C_{x'} := \sum_{x' \in X'} C \cap \phi^{-1}(x')$ *with* $C_{x'} \in \mathcal{B}_{\phi(x')}$.

Proof: As noted above, $g^{-1}(A') = \sum_{x' \in X'} K_{x'}$ and $K_{x'}$ is either $Y_{\phi(x')}$ or $\emptyset$ so $K_{x'} \in \mathcal{B}_{\phi(x')}$ for all x'. $r^{-1}(B) = \sum_{x' \in X'} (B \cap f^{-1}(\phi(x'))$ and $B \cap f^{-1}(\phi(x')) \in \mathcal{B}_{\phi(x')}$.
We next show that sets of the form of the statement form a σ-algebra: $\emptyset = \sum_{x' \in X'} \emptyset$; $Z = \sum_{x' \in X'} Y_{\phi(x')}$; $(\sum_{x' \in X'} C_{x'})^c = \sum_{x' \in X'} C^c_{x'}$; $\bigcup_{i=1}^{\infty} \sum_{x' \in X'} C_{x'i} = \sum_{x' \in X'} \bigcup_{i=1}^{\infty} C_{x'i}$. This completes the proof since $\mathcal{C}$ is generated by $g^{-1}(A')$, $r^{-1}(B)$. ∎

And so, we have $\mathcal{C}_{x'} = \{C \cap g^{-1}(x') \mid C \in \mathcal{C}\} \subseteq \mathcal{B}_{\phi(x')} \times \{x'\}$ for each $x' \in X'$. The other containment holds as well:

Lemma 4.2.2 $\mathcal{B}_{\phi(x')} \times \{x'\} \subseteq \mathcal{C}_{x'}$.

Proof: Let $B \cap f^{-1}(\phi(x')) \in \mathcal{B}_{\phi(x')}$. Then $(B \cap f^{-1}(\phi(x'))) \times \{x'\} = r^{-1}(B) \cap g^{-1}(x') \in \mathcal{C}_{x'}$. ∎

Define $\rho(\sum_{x' \in X'} C_{x'}) := \int_{X'} \nu_{\phi(x')}(C_{x'}) d\mu'$ (here, we identify $B \cap f^{-1}(\phi(x')) \times \{x'\}$ with $B \cap f^{-1}(\phi(x'))$ to take $\nu_{\phi(x')}$ of it).

Lemma 4.2.3 *For* $C \in \mathcal{C}$, $x \mapsto \nu_{\phi(x')}(C_{x'})$ *is (measurable and) integrable (i.e.* ρ *"makes sense").*

Proof: We first show the statement holds in the case of a "measurable rectangle," $C = g^{-1}(A') \cap r^{-1}(B)$. $\nu_{\phi(x')}((g^{-1}(A' \cap r^{-1}(B))_{x'}) = \nu_{\phi(x')}(B \cap f^{-1}(\phi(x')) \cdot \chi_{A'}$. The second factor is integrable (since $\mu'(A') < \infty$). The first factor is integrable since it is the composite of $\nu_x(B \cap f^{-1}(x))$ and $\phi(x')$.

Now, for $C = Z$, $\nu_{\phi(x')}(Z_{x'}) = \nu_{\phi(x')}(Y \cap f^{-1}(\phi(x')) \cdot \chi_{X'}$ is integrable. Any $C \subseteq Z$ has $C_{x'} \subseteq Z_{x'}$ so $\nu_{\phi(x')}(C_{x'}) \leq \nu_{\phi(x')}(Z_{x'})$ for all x'. Thus, we need only show measurability. We do this in stages.

Disjoint unions: Let $C = \dot{\bigcup\limits_{i \in N}} C_i$. Then $\nu_{\phi(x')}((\dot{\bigcup\limits_{i \in N}})_{x'} C_i) = \nu_{\phi(x')}(\dot{\bigcup\limits_{i \in N}} C_{ix'})$ $= \sum\limits_{i \in N} \nu_{\phi(x')}(C_{ix'})$ is measurable, since it is a sum of measurable functions.

Arbitrary (countable) unions: We can write such as a disjoint union and apply the above case.

Finite intersections: Let $C = C_1 \cap C_2$. Then $\nu_{\phi(x')}(C_{x'}) = \nu_{\phi(x')}(C_{1x'} \cap C_{2x'}) = \nu_{\phi(x')}(C_{1x'}) + \nu_{\phi(x')}(C_{2x'}) - \nu_{\phi(x')}(C_{1x'} \cup C_{2x'})$ is measurable.

Countable intersections: Let $C = \bigcap\limits_{i=1}^{\infty} C_i$ and consider $C_N = \bigcap\limits_{i=1}^{N} C_i$. Then $\nu_{\phi(x')}(C_{x'})$ $= \lim\limits_{N \to \infty} \nu_{\phi(x')}(C_{Nx'})$. ∎

Lemma 4.2.4 *ρ is a finite measure on $\mathcal{C}$.*

Proof: Finiteness follows from lemma 4.2.3. $\rho \geq 0$, since the integral of a nonnegative function is nonnegative. $\rho(\emptyset) = \int_{X'} \nu_{\phi(x')}(\emptyset) d\mu'(x') = 0$.

$$
\begin{aligned}
\rho(\dot{\bigcup_{i \in N}} \sum_{x' \in X'} C_{x'i}) &= \rho(\sum_{x' \in X'} \dot{\bigcup_{i \in N}} C_{x'i}) \\
&= \int_{X'} \nu_{\phi(x')}(\dot{\bigcup_{i \in N}} C_{x'i}) d\mu'(x') \\
&= \int_{X'} \sum_{i \in N} \nu_{\phi(x')}(C_{x'i}) d\mu'(x'), \text{ since } \nu_{\phi(x')} \text{ is a measure} \\
&= \sum_{i \in N} \int_{X'} \nu_{\phi(x')}(C_{x'i}) d\mu'(x') \text{ by the MCT}
\end{aligned}
$$

$$= \sum_{i \in N} \rho(C_i). \quad \blacksquare$$

Notation: We will make use of two notations, $\dot{\bigcup}$ and $\sum$, for coproducts in **Set**. We use $\sum$ in a "categorical context" and $\dot{\bigcup}$ in a "measure theoretical context." $\square$

Now, certainly, r and g are measurable by construction. Recall, $g^{-1}(x') = Y_{\phi(x')} \times \{x'\}$ and $\mathcal{C}_{x'} = \mathcal{B}_{\phi(x')} \times \{x'\}$. Put $\rho_{x'}(C_{x'}) := \nu_{\phi(x')}(C_{x'})$ (again, identify $\mathcal{B}_{\phi(x'} \times \{x'\}$ with $\mathcal{B}_{\phi(x')}$).

Lemma 4.2.5 *$(g, \rho_{x'})$ is a disintegration.*

Proof: We have already shown above that $\rho_{x'}$ is measurable and bounded (each $C \subseteq Z$).

Axiom 2: $\rho(\sum_{x' \in X'} C_{x'}) = \int_{X'} \nu_{\phi(x')}(C_{x'}) d\mu'(x') = \int_{X'} \rho_{x'}(C_{x'}) d\mu'(x')$. $\blacksquare$

And so, we have proved:

Theorem 4.2.1 *Given $f \in$ **Disint** and $\phi \in$ **M0R** then*
$(Z, \mathcal{C}, \rho) \xrightarrow{(g, \rho_{x'})} (X', \mathcal{A}', \mu') \in$ **Disint**. $\blacksquare$

Remark: Z resembles the pullback. It is not universal, however. $\square$

4.2.2 Examples

Example 1: Product:

$$\begin{tikzcd}
(Z, \mathcal{C}, \rho) \arrow[r, "r"] \arrow[d, "{(g, \rho_{x'})}"'] & (Y, \mathcal{B}, \nu) \arrow[d, "{(!_Y, \nu)}"] \\
(X', \mathcal{A}', \mu') \arrow[r, "!_{X'}"'] & (1, \mathcal{I}, \iota)
\end{tikzcd}$$

Here, of course, $(1, \mathcal{I}, \iota)$ is "the" one point measure space. In this case,
$Z = \sum_{x' \in X'} Y_* = \sum_{x' \in X'} Y = Y \times X'$. Also, $g^{-1}(A') = \sum_{x' \in X'} K_{x'} \cong \sum_{x' \in A'} Y \cong Y \times A'$,
$r^{-1}(B) = \{(y, x') | y \in B \cap !_Y{}^{-1}(!_{X'}(x'))\} \cong B \times X'$, and $\mathcal{C} \cong \mathcal{B} \times \mathcal{A}'$.
Let $C \in \mathcal{B} \times \mathcal{A}'$, then by Tonelli's theorem, $C_{x'} = C \cap \{(y,t) | y \in Y,\ t = x'\} \in \mathcal{B}$ and $\int_{X'} \nu(C_{x'}) d\mu'(x') = (\nu \times \mu)(C)$.
Note that $\rho(\sum_{x' \in X'} C_{x'}) = \int_{X'} \nu_{!_{X'}(x')}(C_{x'}) d\mu'(x') = \int_{X'} \nu(C_{x'}) d\mu'(x')$. For example, there are two ways of viewing Z: as $g^{-1}(X')$ or as $r^{-1}(Y)$. Now,
$\rho(g^{-1}(X')) = \int_{X'} \nu_{!_{X'}(x')}(K_{x'}) d\mu'(x') = \int_{X'} \nu(Y) d\mu'(x') = \nu(Y) \cdot \mu'(X')$
$= (\nu \times \mu')(Z)$ and $\rho(r^{-1}(Y)) = \int_{X'} \nu_{!_{X'}(x')}(Y \cap !_Y^{-1}(!_{X'}(x'))) d\mu'(x')$
$= \int_{X'} \nu(Y) d\mu' x') = (\nu \times \mu')(Z)$. $\square$

Example 2: Terminal object:

$$\begin{tikzcd}
(Z, \mathcal{C}, \rho) \arrow[r, "!_Z"] \arrow[d, "{(g, \rho_{x'})}"'] & (1, \mathcal{I}, \iota) \arrow[d, "{(1, \iota)}"] \\
(X', \mathcal{A}', \mu') \arrow[r, "!_{X'}"'] & (1, \mathcal{I}, \iota)
\end{tikzcd}$$

Here, $r = !_Z$, $g^{-1}(A') = \sum_{x' \in X'} K_{x'} \cong \sum_{x' \in A'} 1 \cong A'$. Notice that $Z = \sum_{x' \in X'} 1 \cong X'$ so that $\mathcal{C} \cong \mathcal{A}'$ and, furthermore, $\rho(\sum_{x' \in A'} 1) = \int_{X'} \iota(K_{x'}) d\mu'(x') = \int_{A'} \iota(1) d\mu'(x')$
$= \iota(1) \cdot \mu'(A') = \mu'(A')$ and so $\rho = \mu'$. In this example, $\mathcal{C}_{x'} \cong \{\emptyset, \{x'\}\}$ and $\rho_{x'} =$ the counting measure. Thus, $(g, \rho_{x'})$ is the identity (up to isomorphism). $\square$

Example 3: Identity disintegration:

$$\begin{array}{ccc} (Z,\mathcal{C},\rho) & \xrightarrow{r} & (X,\mathcal{A},\mu) \\ {\scriptstyle (g,\rho_{x'})}\downarrow & & \downarrow{\scriptstyle (1,\mu_x)} \\ (X',\mathcal{A}',\mu') & \xrightarrow[\phi]{} & (X,\mathcal{A},\mu) \end{array}$$

Here, $\mathcal{A}_x = \{\emptyset, \{x\}\}$ and μ_x is the counting measure. $Z = \sum_{x' \in X'} X_{\phi(x')}$ $= \sum_{x' \in X'} \phi(x') \cong \sum_{x' \in X'} 1 \cong X'$ and $g^{-1}(A') \cong \sum_{x' \in A'} \phi(x') \cong A'$. We see that $\mathcal{C} \cong \mathcal{A}'$. Also, $r^{-1}(B) = \{(y, x')|y \in B \cap 1^{-1}\phi(x')\} = \{(y, x')|y \in B \cap \phi(x')\}$. Now, if $\phi(x') \in B$ then $B \cap \phi(x') = \{\phi(x')\}$ and if $\phi(x') \notin B$ then $B \cap \phi(x') = \emptyset$ so $r^{-1}(B) \cong \phi^{-1}(B)$. Furthermore, $\rho(A') = \rho(\sum_{x' \in A'} \phi(x')) = \int_{A'} \mu_{\phi(x')}(\phi(x'))d\mu'(x')$ $= \int_{A'} 1 d\mu'(x') = \mu' A'$. □

Example 4: Identity M0R:

$$\begin{array}{ccc} (Z,\mathcal{C},\rho) & \xrightarrow{r} & (Y,\mathcal{B},\nu) \\ {\scriptstyle (g,\rho_{x'})}\downarrow & & \downarrow{\scriptstyle (f,\nu_x)} \\ (X,\mathcal{A},\mu) & \xrightarrow[1]{} & (X,\mathcal{A},\mu) \end{array}$$

In this case, $Z = \sum_{x \in X} Y_x = Y$, $g^{-1}(A) = \sum_{x \in X} K_x \cong \sum_{x \in A} Y_x = f^{-1}(A)$, and $r^{-1}(B) = \{(y, x)|y \in B \cap f^{-1}(1(x))\} = \{(y, x)|y \in B \cap f^{-1}(x)\} \cong B$. Thus, $\mathcal{C} = \mathcal{B}$.

$\rho(B) = \rho(\sum_{x \in X} B_x) = \int_X \nu_x(B_x)d\mu(x) = \int_X \nu_x(B \cap f^{-1}(x))d\mu(x) = \nu(B)$ since (f, ν_x) is a disintegration. □

Example 5: Intersection: Let A_0 and A_1 be two measurable subsets of $(X, \mathcal{A}, \mu)$.

$$\begin{tikzcd}
(Z, \mathcal{C}, \rho) \arrow[r, "r"] \arrow[d, "(g, \rho_{x'})"'] & (A_0, \mathcal{A}_0, \mu_0) \arrow[d, "(i_0, \mu_{0x})"] \\
(A_1, \mathcal{A}_1, \mu_1) \arrow[r, "i_1"'] & (X, \mathcal{A}, \mu)
\end{tikzcd}$$

Now, $Z = \sum_{x \in A_1} A_{0 i_1(x)} = \sum_{x \in A_1} i_0^{-1}(x) \cong A_1 \cap A_0$, $g^{-1}(A \cap A_1) = \sum_{x \in A \cap A_1} i_0^{-1}(x)$ $\cong A \cap A_1$, and $r^{-1}(A \cap A_0) = \{(y, x) | y \in (A \cap A_0) \cap i_0^{-1}(i_1(x))\} \cong A \cap A_0$. Thus, $\mathcal{C} \cong \mathcal{A}|_{A_1 \cap A_0}$ (some "rectangles" will be $(A \cap A_1) \cap (A \cap A_0) = A \cap (A_1 \cap A_0)$).

We have $\rho(A \cap A_1) = \mu_1(A \cap A_1) = \mu(A \cap A_1)$ and $\rho(A \cap A_0) = \mu_0(A \cap A_0) = \mu(A \cap A_0)$. □

4.2.3 r is M0R

In this section, we will show that $r \in$ **M0R**. Let $B \in \mathcal{B}$ have $\nu(B) = 0$. We wish to show that $\rho(r^{-1}(B)) = 0$.

Recall, $\rho(r^{-1}(B)) = \int_{X'} \nu_{\phi(x')}(B \cap f^{-1}(\phi(x')) d\mu'(x') = \int_{X'} t_B(\phi(x')) d\mu'(x')$, where $t_B(x) = \nu_x(B \cap f^{-1}(x))$. Now, $0 = \nu(B) = \int_X \nu_x(B \cap f^{-1}(x)) d\mu(x)$, so we need only show the following:

Proposition 4.2.1 *If* $X' \xrightarrow{\phi} X \xrightarrow{t} \mathbf{R} \geq 0$, *with* $\phi \in$ **M0R**, *then* $\int_X t(x) d\mu(x) = 0$ *implies* $\int_{X'} (t \circ \phi)(x') d\mu'(x') = 0$.

Proof: As usual, we proceed in steps:

Case $t = \chi_A$, $\mu(A) = 0$: $t(\phi(x')) = \left\{ \begin{array}{ll} 1 & \phi(x') \in A \\ 0 & \phi(x') \notin A \end{array} \right\} = \left\{ \begin{array}{ll} 1 & x' \in \phi^{-1}(A) \\ 0 & x \notin \phi^{-1}(A) \end{array} \right.$ so $\int_{X'} t_\phi(x')) d\mu'(x') = \int_{\phi^{-1}(A)} 1 d\mu'(x') = \mu'(\phi^{-1} A) = 0$ (this last equality since $\phi \in$ **M0R**).

Case $t = a\chi_{A_1} + b\chi_{A_2}$ with $A_1 \cap A_2 = \emptyset$: $\int_{X'} t(\phi(x'))d\mu'(x') = \int_{\phi^{-1}(A_1)} ad\mu'(x')$
$+ \int_{\phi^{-1}(A_2)} bd\mu'(x') = 0 + 0 = 0.$
Case t is a nonnegative measurable function: Let $s_n \uparrow t$ be a sequence of simple functions increasing to t. Then $\int t\phi(x') = \int \lim s_n\phi = \lim \int s_n\phi = \lim 0$ $= 0$. ∎

4.2.4 Functoriality

Notation: We write $\phi^*(f)$ for $(g, \rho_{x'})$. □

In this section, we shall show that $(-)^*$ is a pseudo-functor; it preserves identity and composition up to isomorphism.

We have already shown that $1^* \cong 1$ in example 4 above. Now, consider the diagram:

$$\begin{tikzcd}
(W,\mathcal{E},\eta) \arrow[dr, shift left, "b"'] \arrow[drrr, "u"] \arrow[ddr, "{(k,\eta_{x''})}"'] & & & \\
& (T,\mathcal{D},\delta) \arrow[ul, shift left, "a"'] \arrow[r, "s"] \arrow[d, "{(h,\delta_{x''})}"] & (Z,\mathcal{C},\rho) \arrow[r, "r"] \arrow[d, "{(g,\rho_{x'})}"] & (Y,\mathcal{B},\nu) \arrow[d, "{(f,\nu_x)}"] \\
& (X'',\mathcal{A}'',\mu'') \arrow[r, "\psi"'] & (X',\mathcal{A}',\mu') \arrow[r, "\phi"'] & (X,\mathcal{A},\mu)
\end{tikzcd}$$

Lemma 4.2.6 $g^{-1}(\psi(x'')) \cong Y_{\phi\psi(x'')}$

Proof: $g^{-1}(\psi(x'')) = \sum_{x' \in X'} K_{x'} \cong \sum_{x' \in \{\psi(x'')\}} Y_{\phi(x')} = Y_{\phi\psi(x'')}$. ∎

In the above diagram, $T = \sum_{x'' \in X''} Z_{\psi(x''))} = \sum_{x'' \in X''} g^{-1}(\psi(x''))$. And so, by the lemma, $T = \sum_{x'' \in X''} Y_{\phi(\psi(x''))} = \sum_{x' \in X'} Y_{\phi\psi(x'')}$. On the other hand, $W = \sum_{x'' \in X''} Y_{\phi\psi(x'')}$. Thus, $W \cong T$ as sets (which, of course, makes sense, since in <u>**Set**</u> these are just pullbacks and pullbacks compose by the pullback lemma).

We will have use of the explicit form of the isomorphism a and its inverse b: $W = \{(y, x'') | \phi\psi(x'') = f(y)\}$, $T = \{(z, x'') | \psi(x'') = x' = g(y, x')\}$ $= \{((y, x'), x'') | \psi(x'') = x' \text{ and } \psi(x') = f(y)\}$, so define $W \xrightarrow{a} T$ as $(y, x'') \mapsto (y, \psi(x''), x'')$ and $T \xrightarrow{b} W$ as $(y, x', x'') \mapsto (y, x'')$.

We have $ab(y, x', x'') = a(y, x'') = (y, \psi(x''), x'') = (y, x', x'')$, since this is in T (i.e. for $(y, x', x'') \in T$, we must have $\psi(x'') = x'$) and $ba(y, x'') = b(y, \psi(x''), x'') = (y, x'')$. We must show that a is a measurable equivalence (recall from section 1.3 this means a and $a^{-1} = b$ are measurable and a is measure preserving). We will require the following equalities (which we shall prove by chasing elements, even though some are consequences of "pullback-ness"):

Lemm[illegible] *(a) $ha = k$, (b) $rsa = u$, and (c) $gs = \psi h$.*

Proof: a) $ha(y, x'') = h(y, \psi(x''), x'') = x'' = k(y. x'')$.
b) $rsa(y, x'') = rs(y, \psi(x''), x'') = r(y, \psi(x'')) = y = u(y, x'')$.
c) $gs(y, x', x'') = g(y, x') = x'$ and $\psi h(y, x', x'') = \psi(x'') = x'$. ∎

Lemma 4.2.8 *$(W, \mathcal{E}) \xrightarrow{a} (T, \mathcal{D})$ is measurable.*

Proof: Let $D \in \mathcal{D}$. We will check cases.

Case $D = h^{-1}(A'')$: $a^{-1}h^{-1}(A'') = k^{-1}(A'') \in \mathcal{E}$.

Case $D = s^{-1}C$: This breaks down into subcases:

subcase $C = r^{-1}(B)$: $a^{-1}s^{-1}r^{-1}(B) = u^{-1}(B) \in \mathcal{E}$.

subcase $C = g^{-1}(A')$: $a^{-1}s^{-1}g^{-1}(A') = a^{-1}h^{-1}\psi^{-1}(A') = k^{-1}\psi^{-1}(A')$. Now, $\psi^{-1}(A') \in \mathcal{A}''$ so $k^{-1}\psi^{-1}(A'') \in \mathcal{E}$ as required.

subcase $C = g^{-1}(A') \cap r^{-1}(B)$: $a^{-1}(g^{-1}(A') \cap r^{-1}(B))$
$= a^{-1}g^{-1}(A') \cap a^{-1}r^{-1}(B) \in \mathcal{E}$.

subcase $C = \bigcup_{i \in N} g^{-1}(A'_i) \cap r^{-1}(B_i)$: $a^{-1}s^{-1}(\bigcup) = \bigcup a^{-1}s^{-1} \in \mathcal{E}$.

subcase $C = \bigcap_{i \in N} C_i$, $C_i \in \mathcal{R}_\sigma$: $a^{-1}s^{-1}(\bigcap) = \bigcap a^{-1}s^{-1} \in \mathcal{E}$.

Case $D = h^{-1}(A'') \cap s^{-1}(C)$: $a^{-1}(h^{-1}(A'') \cap s^{-1}(C)) = a^{-1}h^{-1}(A'') \cap a^{-1}s^{-1}(C) \in \mathcal{E}$.

Case $D = \bigcup_{i \in N} h^{-1}(A'') \cap s^{-1}(C)$: $a^{-1}(\bigcup) = \bigcup a^{-1} \in \mathcal{E}$.

Case $D = \bigcap_{i \in N} D_i$, $D_i \in \mathcal{R}_\sigma$: $a^{-1}(\bigcap) = \bigcap a^{-1} \in \mathcal{E}$. ∎

Lemma 4.2.9 *a is direct measurable (i.e. b is measurable).*

Proof: We wish to show $b^{-1}(E) \in \mathcal{D}$, $\forall E \in \mathcal{E}$. Again, there are cases to check. We prove only the two "basic" cases.

Case $E = k^{-1}(A'')$: $b^{-1}k^{-1}(A'') = b^{-1}a^{-1}h^{-1}(A'') = h^{-1}(A'') \in \mathcal{D}$.

Case $E = u^{-1}(B)$: $b^{-1}u^{-1}(B) = b^{-1}a^{-1}s^{-1}r^{-1}(B) = s^{-1}r^{-1}(B) \in \mathcal{D}$. ∎

The next thing we must prove is that a and b preserve the measures η and δ, respectively.

Lemma 4.2.10 $\eta(a^{-1}(D)) = \delta(D)$, $\forall D \in \mathcal{D}$ *and* $\delta(b^{-1}(E)) = \eta(E)$, *for each* $E \in \mathcal{E}$.

Proof: By the remarks on measure equivalences in chapter 1, it is enough to show only one. We will prove the (easier) one, that for b.

Case $E = k^{-1}(A'')$: $\eta(k^{-1}(A'')) = \int_{X''} \mu_{\phi\psi(x'')}(Y_{\phi\psi(x'')}) \cdot \chi_{A''} d\mu''$ and
$\delta(b^{-1}k^{-1}(A'')) = \delta(b^{-1}a^{-1}h^{-1}(A'')) = \delta(h^{-1}(A'')) = \int_{X''} \rho_{\psi(x'')}(Z_{\psi(x'')}) \cdot \chi_{A''} d\mu''$
$= \int_{X''} \mu_{\phi\psi(x'')}(Y_{\phi\psi(x'')}) \cdot \chi_{A''} d\mu'' = \eta(k^{-1}(A''))$ (the second last by lemma 4.2.6).

Case $E = u^{-1}(B)$: $\eta(u^{-1}(B)) = \int_{X''} \mu_{\phi\psi(x'')}(B \cap f^{-1}(\phi\psi(x'')))d\mu''$ and
$\delta(b^{-1}u^{-1}(B)) = \delta(b^{-1}a^{-1}s^{-1}r^{-1}(B)) = \delta(s^{-1}r^{-1}(B))$
$= \int_{X''} \rho_{\psi(x'')}(r^{-1}(B) \cap g^{-1}(\psi(x'')))d\mu''$. Now, $r^{-1}(B) = \{(y, x') | y \in B \cap f^{-1}(\phi(x'))\}$

and $g^{-1}(\psi(x'')) = \{(y,x')|x' = \psi(x'') \text{ and } y \in Y_{\phi(x')}\} = \{(y,x')|x' = \psi(x'') \text{ and } y \in f^{-1}(\phi(x'))\}$. And so, $r^{-1}(B) \cap g^{-1}(\psi(x'')) = \{(y,x')|x' = \psi(x''),\ y \in B \cap f^{-1}(\phi\psi(x''))\}$. Thus, $\rho_{\psi(x'')}(r^{-1}(B) \cap g^{-1}(\psi(x''))) = \mu_{\phi\psi(x'')}(B \cap f^{-1}(\phi\psi(x'')))$ as required.

Case $E = k^{-1}(A'') \cap u^{-1}(B)$: $\eta(k^{-1}(A'') \cap u^{-1}(B))$
$= \int_{X''} \mu_{\phi\psi(x'')}(B \cap f^{-1}(\phi\psi(x''))) \cdot \chi_{A''} d\mu''$ and
$\delta(b(k^{-1}(A'') \cap u^{-1}(B))) = \delta(b^{-1}k^{-1}(A'') \cap b^{-1}u^{-1}(B))$
$= \delta(h^{-1}(A'') \cap s^{-1}r^{-1}(B)) = \int_{X''} \rho_{\psi(x'')}(r^{-1}(B) \cap g^{-1}(\psi(x''))) \cdot \chi_{A''} d\mu''$
$= \int_{X''} \mu_{\phi\psi(x'')}(B \cap f^{-1}(\phi\psi(x''))) \cdot \chi_{A''} d\mu''$.

Case $E = \dot{\bigcup\limits_{i \in N}} k^{-1}A''_i \cap u^{-1}B_i$: $\eta \dot{\bigcup} = \sum \eta$ and $\delta b^{-1} \dot{\bigcup} = \delta \dot{\bigcup} b^{-1} = \sum \delta b^{-1}$.
Arbitrary (countable) union and the complement can be written as a disjoint union.
∎

Thus, we have proved the following:

Theorem 4.2.2 ($)^*$ *is a pseudo functor* $\underline{\mathbf{M0R}} \longrightarrow \underline{\mathbf{Cat}}$ *with object function* $X \mapsto \underline{\text{Disint}}/X$. ∎

Moreover, for a fixed $\phi : X' \longrightarrow X$, ϕ^* is a functor $\underline{\text{Disint}}/X \longrightarrow \underline{\text{Disint}}/X'$ with

$$\begin{tikzcd}
(Y',\mathcal{B}',\nu') \arrow[rr, "{(k,\nu'_y)}"] \arrow[dr, "{(f',\nu'_x)}"'] & & (Y,\mathcal{B},\nu) \arrow[dl, "{(f,\nu_x)}"] & \mapsto & (Z',\mathcal{C}',\rho') \arrow[rr, "{(m,\rho'_x)}"] \arrow[dr, "{(h,\rho'_{x'})}"'] & & (Z,\mathcal{C},\rho) \arrow[dl, "{(g,\rho_{x'})}"] \\
& (X,\mathcal{A},\mu) & & & & (X',\mathcal{A}',\mu') &
\end{tikzcd}$$

Where $Z = \{(y,x')|f(y) = \phi(x')\}$, $Z' = \{(y',x')|f'(y') = \phi(x')\}$ and $m(y',x') = (k(y'),x')$ so $m^{-1}(y,x') = k^{-1}(y) \times \{x'\}$. $\mathcal{C}'_{(y,x')} := \mathcal{B}'_y \times \{x'\}$ and for $\mathcal{C}' = \sum\limits_{y' \in Y'} \mathcal{C}_{y'}$, put $\rho'_{(y,x')}(C' \cap m^{-1}(y,x')) := \nu'_y(\sum_{y' \in k^{-1}(y)} C_{y'})$. It is a simple matter to check that this makes ϕ^* into a functor.

We denote the "indexed category" determined by the pseudo-functor above by **Disint** (note: we actually have two indexed categories, **Disint** indexed by **MOR** and indexed by **Disint**; we will not have reason to (notationally) distinguish between the two at this point).

4.3 Substitution Along a Disintegration

4.3.1 A Characterization

We glibly described $g^{-1}(A') \cap r^{-1}(B)$, above, as a "measurable rectangle" in $(Z, \mathcal{C}, \rho)$. In this section, we will consider the case where $\phi \in$ **Disint** (the "pull-back" of a disintegration along a disintegration). Our ultimate goal is to prove that $r \in$ **Disint** and a symmetry result: if $\phi \in$ **Disint**, we can form $f^*(\phi)$ as well as $\phi^*(f)$; these are measurably equivalent. We begin by giving a characterization of ρ using measurable rectangles. More accurately, fibrewise ρ looks like the product measure. Consider:

$$
\begin{tikzcd}
(Z, \mathcal{C}, \rho) \arrow[r, "r"] \arrow[d, "{(g, \rho_{x'})}"'] & (Y, \mathcal{B}, \nu) \arrow[d, "{(f, \nu_x)}"] \\
(X', \mathcal{A}', \mu') \arrow[r, "{(\phi, \mu'_x)}"'] & (X, \mathcal{A}, \mu)
\end{tikzcd}
$$

$$\xrightarrow[(\phi g, \theta_x)]{}$$

with $(\phi, \mu'_x) \in$ **<u>Disint</u>**. Here, of course, θ_x is the composition of μ'_x and $\rho_{x'}$ with $\theta_x(C \cap g^{-1}\phi^{-1}(x)) = \int_{\phi^{-1}(x)} \rho_{x'}(C \cap g^{-1}(x'))d\mu'_x(x')$. Before we give our new description of ρ, we require a little "fibre-optics." As usual:

Notation: $Y_x := f^{-1}(x)$; $X'_x := \phi^{-1}(x)$. $\square$

Lemma 4.3.1 .

a) $g^{-1}\phi^{-1}(x) = Y_x \times X'_x$

b) $g^{-1}(A') \cap r^{-1}(B) \cap Y_x \times X'_x = (B \cap Y_x) \times (A' \cap X'_x)$

c) $g^{-1}(A' \cap \phi^{-1}(x)) = Y_x \times (A' \cap X'_x)$

Proof: a) $g^{-1}\phi^{-1}(x) = \{(y, x') | f(y) = \phi(x') \text{ and } x' \in \phi^{-1}(x)\} = f^{-1}(x) \times \phi^{-1}(x)$.
b) $g^{-1}(A') \cap r^{-1}(B) \cap Y_x \times X'_x = \{(y, x') | x' \in A',\ y \in B,\ f(y) = \phi(x') = x\} = (B \cap Y_x) \times (A' \cap X'_x)$.
c) $g^{-1}(A' \cap \phi^{-1}(x)) = \{(y, x') | f(y) = \phi(x'),\ x' \in A' \cap \phi^{-1}(x)\} = Y_x \times (A' \cap X'_x)$.
Of course, c) is a special case of b). ∎

Proposition 4.3.1 *For* $C \in \mathcal{C}$, $\rho(C) = \int_X (\nu_x \times \mu'_x)(C \cap Y_x \times X'_x)d\mu(x) = \spadesuit$

Proof: Since θ_x is a disintegration, $\rho(C) = \int_X \theta_x(C \cap g^{-1}\phi^{-1}(x))d\mu(x)$
$= \int_X \int_{\phi^{-1}(x)} \rho_{x'}(C \cap g^{-1}(x'))d\mu'_x(x')d\mu(x) = \clubsuit$. We must show $\spadesuit = \clubsuit$.
Case $C = g^{-1}(A')$: $\rho_{x'}(C \cap g^{-1}(x')) = \nu_{\phi(x')}(Y_{\phi(x')}) \cdot \chi_{A'}$. So

$$\begin{aligned}
\clubsuit &= \int_X \int_{\phi^{-1}(x)} \nu_{\phi(x')}(Y_{\phi(x')}) \cdot \chi_{A'} d\mu'_x(x')d\mu(x) \\
&= \int_X \int_{\phi^{-1}(x)} \nu_x(Y_x) \cdot \chi_{A'} d\mu'_x(x')d\mu(x) \\
&= \int_X \nu_x(Y_x) \int_{\phi^{-1}(x)} \chi_{A'} d\mu'_x(x')d\mu(x) \\
&= \int_X \nu_x(Y_x)\mu'_x(A' \cap \phi^{-1}(x))d\mu(x) = \heartsuit
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\spadesuit &= \int_X (\nu_x \times \mu'_x)(g^{-1}(A') \cap (Y_x \times X'_x))d\mu(x) \\
&= \int_X (\nu_x \times \mu'_x)(Y_x \times (A' \cap X'_x))d\mu(x) \\
&= \int_X \nu_x(Y_x) \cdot \mu'_x(A' \cap X'_x)d\mu(x) \\
&= \int_X \nu_x(Y_x) \cdot \mu'_x(A' \cap \phi^{-1}(x))d\mu(x) \\
&= \heartsuit \text{ as required.}
\end{aligned}$$

Case $C = r^{-1}(B)$: $\rho_{x'}(r^{-1}(B) \cap g^{-1}(x')) = \nu_{\phi(x')}(B \cap f^{-1}(\phi(x')))$.

$$\begin{aligned}
\clubsuit &= \int_X \int_{\phi^{-1}(x)} \nu_{\phi(x')}(B \cap f^{-1}(\phi(x')))d\mu'_x(x')d\mu(x) \\
&= \int_X \int_{\phi^{-1}(x)} \nu_x(B \cap f^{-1}(x))d\mu'_x(x')d\mu(x) \\
&= \int_X \nu_x(B \cap f^{-1}(x)) \int_{\phi^{-1}(x)} d\mu'_x(x')d\mu(x) \\
&= \int_X \nu_x(B \cap f^{-1}(x) \cdot \mu'_x(\phi^{-1}(x))d\mu(x) \\
&= \int_X \nu_x(B \cap f^{-1}(x)) \cdot \mu'_x(X'_x)d\mu(x) = \diamondsuit
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\spadesuit &= \int_X (\nu_x \times \mu'_x)(r^{-1}(B) \cap (Y_x \times X'_x))d\mu(x) \\
&= \int_X (\nu_x \times \mu'_x)((Y_x \cap B) \times X'_x)d\mu(x) \\
&= \int_X \nu_x(Y_x \cap B) \cdot \mu'_x(X'_x)d\mu(x) \\
&= \int_X \nu_x(B \cap f^{-1}(x)) \cdot \mu'_x(X'_x)d\mu(x) \\
&= \diamondsuit \text{ as required.}
\end{aligned}$$

Case $C = g^{-1}(A') \cap r^{-1}(B)$: $\rho_{x'}(g^{-1}(A') \cap r^{-1}(B) \cap g^{-1}(x')) = \nu_{\phi(x')}(B \cap f^{-1}(\phi(x'))) \cdot \chi_{A'}$, so

$$\begin{aligned}
\clubsuit &= \int_X \int_{\phi^{-1}(x)} \nu_{\phi(x')}(B \cap f^{-1}(\phi(x'))) \cdot \chi_{A'} d\mu'_x(x') d\mu(x) \\
&= \int_X \nu_x(B \cap f^{-1}(x)) \int_{\phi^{-1}(x)} \chi_{A'} d\mu'_x(x') d\mu(x) \\
&= \int_X \nu_x(B \cap f^{-1}(x)) \cdot \mu'_x(A' \cap \phi^{-1}(x)) d\mu(x) = \heartsuit\heartsuit
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\spadesuit &= \int_X (\nu_x \times \mu'_x)(g^{-1}(A') \cap r^{-1}(B) \cap (Y_x \times X'_x)) d\mu(x) \\
&= \int_X (\nu_x \times \mu'_x)((B \cap Y_x) \times (A' \cap X'_x)) d\mu(x) \\
&= \int_X \nu_x(B \cap Y_x) \cdot \mu'_x(A' \cap X'_x) d\mu(x) \\
&= \int_X \nu_x(B \cap f^{-1}(x)) \cdot \mu'_x(A' \cap f^{-1}(x)) d\mu(x) \\
&= \heartsuit\heartsuit \text{ as required.}
\end{aligned}$$

For disjoint unions of $g^{-1}(A') \cap r^{-1}(B)$'s use the fact that $\rho_{x'}$ is a measure to get $\rho(\bigcup_{i \in N}) = \sum_{i \in N} \rho$ and pull the sum out of the two integrals in $\clubsuit$ using the monotone convergence theorem. Finally, apply the above case. For arbitrary (countable) unions, we have already noted that such can be written as a disjoint union.

The rest of the proof is similar to lemma 4.2.3. For finite intersection, use $\rho(C_1 \cap C_2) = \rho(C_1) + \rho(C_2) - \rho(C_1 \cup C_2)$ and linearity of the integral. For countable intersections, use continuity of measures $(\rho(\bigcap_{i=1}^{\infty}) = \lim_{N \to \infty} \rho(\bigcap_{i=1}^{N})$; this requires measure finiteness of $\rho(C_1)$, which we have) and apply the monotone convergence theorem to pull limits out of the integrals. ∎

4.3.2 r is a Disintegration

Next, we will show that r is a disintegration and

$$\begin{tikzcd}
(Z,\mathcal{C},\rho) \arrow[r, "{(r,\mathcal{C}_y,\rho_y)}"] \arrow[d, "{(g,\rho_{x'})}"'] & (Y,\mathcal{B},\nu) \arrow[d, "{(f,\nu_x)}"'] \arrow[d, bend left, "{(fr,\gamma_x)}"] \\
(X',\mathcal{A}',\mu') \arrow[r, "{(\phi,\mu'_x)}"'] & (X,\mathcal{A},\mu)
\end{tikzcd}$$

$$\xrightarrow[(\phi g,\theta_x)]{}$$

commutes. $\mathcal{C}_y := \{y\} \times \mathcal{A}'_{f(y)}$ (see example 1 of section 1.5.5). We define ρ_y using $\mu'_{f(y)}$ in analogy to $\rho_{x'}$. $\rho_y(C \cap r^{-1}(y)) := \mu'_{f(y)}(C \cap \phi^{-1}(f(y))$ (or better: $\rho_y(g^{-1}(A') \cap r^{-1}(y)) := \mu'_{f(y)}(A' \cap \phi^{-1}(f(y))$ and $\rho_y(r^{-1}(B) \cap r^{-1}(y)) := \mu'_{f(y)}(\phi^{-1}(f(y)) \cdot \chi_B$). Again, that ρ_y is a bounded measurable function of y is exactly the same as for $\rho_{x'}$. It remains to show Axiom 2: $\rho(C) = \int_Y \rho_y(C \cap r^{-1}(y))d\nu(y)$.

Lemma 4.3.2 $\int_X \int_{f^{-1}(y)} \mu'_{f(y)}(C \cap \phi^{-1}(f(y))d\nu_x(y)d\mu(x)$
$= \int_Y \mu'_{f(y)}(C \cap \phi^{-1}(f(y))d\nu(y)$.

Proof: see proposition 1.5.5. ∎

Proposition 4.3.2 $\int_Y \rho_y(C \cap r^{-1}(y))d\nu(y) = \rho(C)$ *(axiom 2)*.

Proof: see lemma 4.2.5 and lemma 4.3.2 . ∎

4.3.3 Symmetry

Consider the diagram:

$$
\begin{tikzcd}
(T,\mathcal{D},\delta) \arrow[dr, "s"] \arrow[drr, "{(h,\delta_y)}"] \arrow[ddr, "{(p,\delta_{x'})}"'] & & \\
& (Z,\mathcal{C},\rho) \arrow[r, "{(r,\rho_y)}"] \arrow[d, "{(g,\rho_{x'})}"] & (Y,\mathcal{B},\nu) \arrow[d, "{(f,\nu_x)}"] \\
& (X',\mathcal{A}',\mu') \arrow[r, "{(\phi,\mu'_x)}"'] & (X,\mathcal{A},\mu)
\end{tikzcd}
$$

where $Z := \phi^*(f)$ and $T := f^*(\phi)$. That is, $Z = \sum_{x' \in X'} Y_{\phi(x')} = \{(y,x') | f(y) = \phi(x')\}$ and $T = \sum_{y \in Y} X'_{f(y)} = \{(x',y) | \phi(x') = f(y)\}$.

The map s (etymology: switch), defined as $s(x',y) := (y,x')$, is a measure equivalence for it is a measurable isomorphism and, furthermore, by the characterization above:

$$\rho(C) = \int_X (\nu_x \times \mu'_x)(C \cap Y_x \times X'_x) d\mu(x) \ \forall C \in \mathcal{C}$$

and

$$\delta(D) = \int_X (\mu'_x \times \nu_x)(D \cap X'_x \times Y_x) d\mu(x) \ \forall D \in \mathcal{D}.$$

Thus, $\delta(s^{-1}(C)) = \rho(C)$ and $\rho(s(D)) = \delta(D)$. $\square$

4.4 Composition

4.4.1 Definitions

If $(X',\mathcal{A}',\mu') \xrightarrow{(\phi,\mu'_x)} (X,\mathcal{A},\mu) \in$ <u>**Disint**</u>, we have a composition functor

$\underline{\mathbf{Disint}}/X' \xrightarrow{\Sigma_\phi} \underline{\mathbf{Disint}}/X$, given by

$$\begin{array}{ccc} (T,\mathcal{D},\delta) & & (T,\mathcal{D},\delta) \\ \Big\downarrow{\scriptstyle (h,\delta_{x'})} & & \Big\downarrow \\ (X',\mathcal{A}',\mu') & \mapsto & (X',\mathcal{A}',\mu') \quad \Big\downarrow{\scriptstyle (\phi h,\theta_x)} \\ & & \Big\downarrow \\ & & (X,\mathcal{A},\mu) \end{array}$$

$\sum_\phi$ is not, in general, left adjoint to ϕ^*. $\phi^* \sum_\phi$ sends $Y' \longrightarrow X'$ in $\underline{\mathbf{Disint}}/X$ to $Y' \longrightarrow X' \xrightarrow{\phi} X$ to $Z \longrightarrow X'$. Let $\phi = ! : X' \longrightarrow 1$ and $Y' \longrightarrow X' = X' \xrightarrow{1} X'$. Then $Z = X' \times X'$: (example 1 of section 4.2.2).

Remark: Since ϕ^* is the pullback in $\underline{\mathbf{Mble}}$, we have $\sum_\phi \dashv \phi^*$ in

$$\underline{\mathbf{Mble}}/X \underset{\phi^*}{\overset{\Sigma_\phi}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \underline{\mathbf{Mble}}/X'$$

4.4.2 Composition and colimits

Proposition 4.4.1 $\sum_\phi$*(initial) = initial*

Proof: The initial object of $\underline{\mathbf{Disint}}/X'$ is $(\emptyset, \{\emptyset,\}0) \xrightarrow{incl=!_{X'}} (X', \mathcal{A}', \mu')$. Composing with $(\phi, \mathcal{A}'_x, \mu'_x)$, we have $(\emptyset, \{\emptyset\}, 0) \xrightarrow{incl=!_X} (X, \mathcal{A}, \mu)$ which is the initial object of $\underline{\mathbf{Disint}}/X$. ∎

Proposition 4.4.2 $\sum_\phi$ *preserves countable coproducts*

Proof: We will only prove that $\sum_\phi$ preserves binary coproducts. The proof consists mainly of fixing notation, after which, the calculations are straightforward.

Let $(T,\mathcal{D},\delta)\xrightarrow{(h,\delta_{x'})}(X',\mathcal{A}',\mu')$ and $(S,\mathcal{C},\gamma)\xrightarrow{(g,\gamma_{x'})}(X',\mathcal{A}',\mu')$ be in **Disint**$/X'$. The coproduct of S and T in **Disint**$/X'$ is given by:

$$\begin{array}{c}(S+T,\mathcal{C}+\mathcal{D},\gamma+\delta)\\ \Big\downarrow{\scriptstyle (g+h,(\gamma+\delta)_{x'})}\\ (X',\mathcal{A}',\mu')\end{array}$$

with $S+T := S\dot{\bigcup}T$, $\mathcal{C}+\mathcal{D} := \{C\dot{\bigcup}D | C\in\mathcal{C},\ D\in\mathcal{D}\}$ and $(\gamma+\delta)(C\dot{\bigcup}D) := \gamma(C)+\delta(D)$. We define $(g+h)(t,i) := \begin{cases} g(t), & i=1\\ h(t), & i=2\end{cases}$ making $(g+h)^{-1}(x') = g^{-1}(x')\dot{\bigcup}h^{-1}(x')$. Finally, we note that $(\mathcal{C}+\mathcal{D})_{x'} = \mathcal{C}_{x'}+\mathcal{D}_{x'}$ and define $(\gamma+\delta)_{x'} := \gamma_{x'}+\delta_{x'}$.

Composing with (ϕ,μ'_x), we have:

$$\begin{array}{c}(S+T,\mathcal{C}+\mathcal{D},\gamma+\delta)\\ \Big\downarrow{\scriptstyle (\phi(g+h),\theta_x)}\\ (X,\mathcal{A},\mu)\end{array}$$

where $\theta_x(E\cap(g^{-1}\phi^{-1}(x)\dot{\bigcup}h^{-1}\phi^{-1}(x))$
$= \int_{\phi^{-1}(x)}(\gamma+\delta)_{x'}(E\cap(g^{-1}(x')\dot{\bigcup}h^{-1}(x'))d\mu'_x(x')$.

On the other hand, composing first then forming the coproduct, we get $(S,\mathcal{C},\gamma)\xrightarrow{(\phi g,\gamma_x)}(X,\mathcal{A},\mu)$ and $(T,\mathcal{D},\delta)\xrightarrow{(\phi h,\delta_x)}(X,\mathcal{A},\mu)$ which gives:

$$\begin{array}{c}(S+T,\mathcal{C}+\mathcal{D},\gamma+\delta)\\ \Big\downarrow{\scriptstyle (\phi g+\phi h,\gamma_x+\delta_x)}\\ (X,\mathcal{A},\mu)\end{array}$$

Certainly, $\phi(g+h) = \phi g+\phi h$. We must show the measures are the same.

Let $E = S\dot{\bigcup}T \in \mathcal{E}_x$, then $\theta_x(E \cap (g^{-1}\phi^{-1}(x)\dot{\bigcup}h^{-1}\phi^{-1}(x))$

$$
\begin{aligned}
&= \int_{\phi^{-1}(x)} (\gamma+\delta)_{x'}(E \cap (g^{-1}(x')\dot{\bigcup}h^{-1}(x')))d\mu'_x(x') \\
&= \int_{\phi^{-1}(x)} (\gamma+\delta)_{x'}((S\dot{\bigcup}T) \cap (g^{-1}(x')\dot{\bigcup}h^{-1}(x'))d\mu'_x(x') \\
&= \int_{\phi^{-1}(x)} \gamma_{x'}(S \cap g^{-1}(x'))d\mu'_x(x') + \int_{\phi^{-1}(x)} \delta_{x'}(T \cap h^{-1}(x'))d\mu'_x(x') \\
&= \gamma_x(S \cap g^{-1}\phi^{-1}(x)) + \delta_x(T \cap h^{-1}\phi^{-1}(x)) \\
&= (\gamma+\delta)_x(S\dot{\bigcup}T \cap (\phi(g+h))^{-1}(x)) \text{ as required. } \blacksquare
\end{aligned}
$$

As can be expected, $\sum_\phi$ does not behave well with limits. For example, $\sum_\phi(\text{terminal}) \neq$ terminal (i.e. $X' \xrightarrow{1} X'$ is terminal in **Disint**$/X'$ but $X' \xrightarrow{1} X' \xrightarrow{\phi} X$ is not terminal in **Disint**$/X$ unless ϕ is the identity).

4.4.3 Beck Condition

In this section, we will prove the Beck condition for $\sum$. Specifically, given a diagram of the form:

$$
\begin{tikzcd}
 & {} \arrow[d, "h"] \\
 & {} \\
 & {} \arrow[d, "f"] \\
{} \arrow[r, "\phi"'] & {}
\end{tikzcd}
$$

with $\phi \in$ **MOR** and $f, h \in$ **Disint**, we will show $\phi^*(\sum_f(h)) \cong \sum_{\phi^*(f)}(r^*(h))$ where $\cong$ is interpreted as measure equivalence. Let us clarify this statement by expanding and labeling the diagram:

$$\begin{tikzcd}
(V,\mathcal{G},\gamma) \arrow[dr, shift left, "q"] \arrow[drr, bend left, "u"] \arrow[dddr, bend right, "{(l,\gamma_{x'})}"'] & & \\
& (W,\mathcal{E},\eta) \arrow[ul, shift left, "p"] \arrow[r, "s"] \arrow[d, "{(k,\eta_z)}"] \arrow[dd, bend right, "{(gk,\beta_{x'})}"'] & (T,\mathcal{D},\delta) \arrow[d, "{(h,\delta_y)}"] \arrow[dd, bend left, "{(fh,\theta_x)}"] \\
& (Z,\mathcal{C},\rho) \arrow[r, "r"'] \arrow[d, "{(g,\rho_{x'})}"] & (Y,\mathcal{B},\nu) \arrow[d, "{(f,\nu_x)}"] \\
& (X',\mathcal{A}',\mu') \arrow[r, "\phi"'] & (X,\mathcal{A},\mu)
\end{tikzcd}$$

All squares are to be instances of $(-)^*$ and all long (down) arrows are compositions. That is, $g = \phi^*(f)$, $k = r^*(h)$, so $gk = \sum_{\phi^*(f)}(r^*(h))$ and $l = \phi^*(\sum_f(h))$. We would like to show that p and q form a measurable isomorphism which respects $\mathcal{G}_{x'}$, $\mathcal{H}_{x'}$, $\gamma_{x'}$, and $\beta_{x'}$ (which implies p and q respect γ and η since these are disintegrations). By "respects," we mean, for each $x' \in X'$, $\beta_{x'}(q^{-1}(G) \cap k^{-1}g^{-1}(x')) = \gamma_{x'}(G \cap l^{-1}(x'))$ and the corresponding equality for p.

We first note that p and q already respect $(V,\mathcal{G})$ and $(W,\mathcal{E})$ (as before, enumerate cases). Explicitly, $W = \{(t,z)|h(t) = r(z)\} = \{(t,y,x')|h(t) = r(y,x') = y,\ \phi(x') = f(y)\}$, $V = \{(t,x')|\phi(x') = fh(t)\}$, $p(t,x') = (t,h(t),x')$, and $q(t,y,x') = (t,x')$. Now, fix $x' \in X'$.

Lemma 4.4.1 $\beta_{x'}(q^{-1}(G) \cap k^{-1}g^{-1}(x')) = \gamma_{x'}(G \cap l^{-1}(x'))$

Proof: For brevity, we will only check the case when G is a "measurable rectangle," i.e. $G = l^{-1}(A') \cap u^{-1}(D)$. The other calculations are similar.

$$\begin{aligned}
&\beta_{x'}(q^{-1}(l^{-1}(A') \cap u^{-1}(D)) \cap k^{-1}g^{-1}(x')) \\
&= \beta_{x'}(q^{-1}l^{-1}(A') \cap q^{-1}u^{-1}(D) \cap k^{-1}g^{-1}(x')) \\
&= \int_{g^{-1}(x')} \eta_z(k^{-1}g^{-1}(A') \cap s^{-1}(D) \cap k^{-1}g^{-1}(x'))d\rho_{x'}(z) \\
&= \int_{g^{-1}(x')} \delta_{r(z)}(D \cap h^{-1}(r(z)) \cdot \chi_{g^{-1}(A')}d\rho_{x'}(z) \\
&= \clubsuit
\end{aligned}$$

$$\begin{aligned}
\gamma_{x'}(l^{-1}(A') \cap u^{-1}(D) \cap l^{-1}(x')) &= \theta_{\phi(x')}(D \cap h^{-1}f^{-1}(\phi(x'))) \cdot \chi_{A'} \\
&= \int_{f^{-1}(\phi(x'))} \delta_y(D \cap h^{-1}(y)d\nu_{\phi(x')}(y) \cdot \chi_{A'} \\
&= \spadesuit
\end{aligned}$$

If we put $a(y) := \delta_y(D \cap h^{-1}(y))$, we see that $\spadesuit = \int_{f^{-1}(\phi(x'))} a(y)d\nu_{\phi(x')}(y) \cdot \chi_{A'}$ and $\clubsuit = \int_{g^{-1}(x')} a(r(z)) \cdot \chi_{g^{-1}(A')}d\rho_{x'}(z)$. As usual, we build up the proof by looking at characteristic functions, simple functions, and increasing limits of simple functions.

Case $a(y) = \chi_B$, $B \in \mathcal{B}$: $\spadesuit = \int_{f^{-1}(\phi(x'))} \chi_B d\nu_{\phi(x')}(y) \cdot \chi_{A'}$
$= \nu_{\phi(x')}(B \cap f^{-1}(\phi(x')) \cdot \chi_{A'}$ and $a(r(z)) = \chi_{r^{-1}(B)}$
so $\clubsuit = \int_{g^{-1}(x')} \chi_{r^{-1}(B)} \cdot \chi_{g^{-1}(A')}d\rho_{x'}(z) = \rho_{x'}(g^{-1}(A') \cap r^{-1}(B) \cap g^{-1}(x'))$
$= \nu_{\phi(x')}(B \cap f^{-1}(\phi(x'))) \cdot \chi_{A'} = \spadesuit$

Case $a(y) =$ simple function: this follows from the linearity of the integrals in $\clubsuit$ and $\spadesuit$.

Case $a(y) =$ limit of an increasing sequence of simple functions: Let $t_n(y) \uparrow a(y)$. We first recall some basic facts:

1. $t(y)$ simple $\Rightarrow t(r(z))$ simple (proof: $t(y) = \sum_{i=1}^{m} b_i \chi_{B_i} \Rightarrow t(r(z)) = \sum_{i=1}^{m} b_i \chi_{r^{-1}(B_i)}$).

2. $t_n(y) \uparrow a(y) \Rightarrow t_n(r(z)) \uparrow a(r(z))$ (proof: that the limit works is obvious; for increasing, suppose $a(y) \geq t_n(y)$ $a.a.$ y, then $a(r(z)) \geq t_n(r(z))$ $a.a.$ z, since $r \in$ **<u>MOR</u>**).

With these facts in mind and using the monotone convergence theorem,

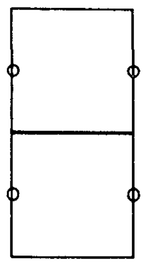
$$
\begin{aligned}
\spadesuit &= \int_{f^{-1}(\phi(x'))} \lim t_n(y) d\nu_{\phi(x')}(y) \cdot \chi_{A'} \\
&= \lim \int_{f^{-1}(\phi(x'))} t_n(y) d\nu_{\phi(x')}(y) \cdot \chi_{A'} \\
&= \lim \int_{g^{-1}(x')} t(n(r(z)) \cdot \chi_{g^{-1}(A')} d\rho_{x'}(z) \\
&= \int_{g^{-1}(x')} \lim t(n(r(z)) \cdot \chi_{g^{-1}(A')} d\rho_{x'}(z) \\
&= \int_{g^{-1}(x')} a(r(z)) \cdot \chi_{g^{-1}(A')} d\rho_{x'}(z) \\
&= \clubsuit. \quad \blacksquare
\end{aligned}
$$

And so, we have proved the following:

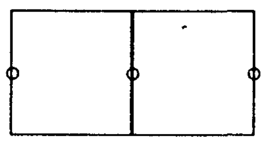
Theorem 4.4.1 $\sum$ *satisfies the Beck condition.* $\blacksquare$

4.4.4 Spans

The preceding results provide an immediate application to bicategories. More accurately, we have two bicategories. Essentially, the Beck condition tells us that we can vertically paste squares:

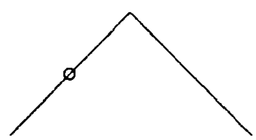


and (pseudo) functoriality tells us that we can horizontally paste squares:

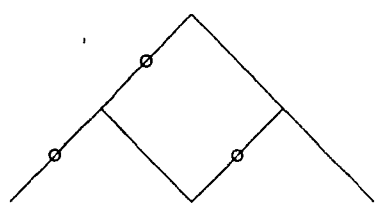


(here $\text{---}\!\circ\!\text{---}$ denotes disintegration). More importantly, we have two bicategories of spans:

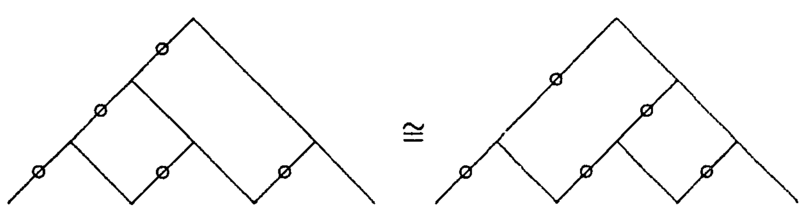
$\underline{\underline{Span}}_1$ has as objects, measure spaces, and as 1−cells, spans:



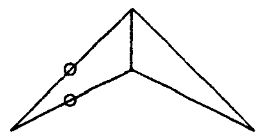
(all arrows point downward) with composition given by "pulling back":



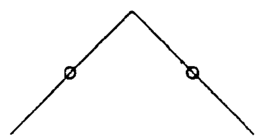
Pseudo functoriality and Beck (pasting in either direction) gives pseudo-associativity:



where, as usual, $\cong$ is interpreted as measure equivalence. 2−cells in $\underline{\underline{Span}}_1$ are disintegrations which make the small triangles commute:



$\underline{\underline{Span}}_2$ is the same as $\underline{\underline{Span}}_1$ except that 1−cells have two disintegrations:



Recall, from section 4.3, that "pulling back" a disintegration along a disintegration makes both legs disintegrations.

4.5 HF/X

4.5.1 Preamble

We have set up substitution machinery for **<u>Disint</u>**. It is time to apply this to operator theory. We now introduce a category HF/X of (measurable) Hilbert families over an $X \in$ **<u>Disint</u>**. In essence, we wish to interpret an $MFHS(X)$ as the collection of "global sections" of a measurable $Y \longrightarrow X$. That is, each fibre Y_x is to be a Hilbert space (we think of Y_x as $H(x)$) and global sections are measurable

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "1"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

But, we wish to be more "abstract" (avoiding reference to all the axioms for $\mathcal{G}$ in $(H(x), \mathcal{G})$ at this point). Before we give the axioms for an HF/X, let us look at the complex numbers. These will provide an important example.

Let $(X, \mathcal{A}, \mu) \in$ **<u>Disint</u>** be fixed. The category **<u>Mble</u>**$/X$ has as objects

$$\begin{tikzcd} (Y, \mathcal{B}) \arrow[d, "f"] \\ (X, \mathcal{A}, \mu) \end{tikzcd}$$

and, as morphisms, measurable $(Y, \mathcal{B}) \xrightarrow{t} (Y', \mathcal{B}')$ which make the evident triangle commute.

A particular object of **<u>Mble</u>**$/X$ is:

$$\begin{tikzcd} \mathbf{C} \times X \arrow[d, "p_2"] \\ X \end{tikzcd}$$

and we have a measurable operation

$$\begin{tikzcd} X \arrow[rr, "\lceil 0 \rceil"] \arrow[dr] & & \mathbf{C} \times X \arrow[dl] \\ & X & \end{tikzcd}$$

given by $x \mapsto (0, x)$ and other operations (defined over X):

- $\lceil 1 \rceil : X \longrightarrow \mathbf{C} \times X;\ x \longmapsto (1, x)$
- $+ : (\mathbf{C} \times X) \times_X (\mathbf{C} \times X) \longrightarrow \mathbf{C} \times X;\ ((c, x), (c', x) \longmapsto (c + c', x)$
- $\cdot : (\mathbf{C} \times X) \times_X (\mathbf{C} \times X) \longrightarrow \mathbf{C} \times X;\ ((c, x), (c', x)) \longmapsto (cc', x)$
- $- : \mathbf{C} \times X \longrightarrow \mathbf{C} \times X;\ (c, x) \longmapsto (-c, x)$
- $(^{-}) : \mathbf{C} \times X \longrightarrow \mathbf{C} \times X;\ (c, x) \longmapsto (\bar{c}, x)$.

With these operations, $\mathbf{C} \times X \xrightarrow{p_2} X$ is a $*$-algebra (scalar multiplication is the same as multiplication). It is commutative and satisfies the axiom of non-triviality. In fact, it is a geometric field (a statement which still makes sense in a "non-topos" like $\underline{\mathbf{Mble}}/X$). Here, $U = \mathbf{C} \setminus \{0\} \times X \longrightarrow X$ and $\lceil 0 \rceil = \{0\} \times X$ and $U + 0 = (\mathbf{C} \setminus \{0\} \times X) + (\{0\} \times X) \simeq \mathbf{C} \times X$ (over X) via $((c, x), 1) \longmapsto (c, x)$ and $((0, x), 2) \longmapsto (0, x)$. Thus, $\mathbf{C} \times X$ is a geometric field in $\underline{\mathbf{Mble}}/X$.

4.5.2 HF/X

An object of HF/X is $(Y, \mathcal{B}) \xrightarrow{f} (X, \mathcal{A}, \mu) \in \underline{\mathbf{Mble}}/X$ subject to three axioms.
Axiom a) $Y_x = f^{-1}(x)$ is a separable Hilbert space for each $x \in X$. □
Notation: x as a subscript will denote an entity of the xth fibre space, Y_x. Elements of Y are $y \in Y$. If it is necessary to emphasize that y is in a particular Y_x, we will write $y_x \in Y_x$ or $(y, x) \in Y$. □

Part of the data for axiom a) provides us with maps of algebra and topology like those for $\mathbf{C} \times X$. In more precise terms, we have maps, defined over X: $X \xrightarrow{\lceil 0 \rceil} Y$, $\lceil 0 \rceil(x) = 0_x \in Y_x$; $Y \xrightarrow{-(\,)} Y$, $-(y_x) = -_x y_x$; $Y \times_X Y \xrightarrow{+} Y$, $+(y, y', x) = y +_x y'$; $(\mathbf{C} \times X) \times_X Y \xrightarrow{\cdot} Y$, $\cdot((c, x), y_x) = c \cdot_x y_x$; and $Y \times_X Y \xrightarrow{\langle - | - \rangle} \mathbf{C} \times X$, $\langle - | - \rangle(y, y', x) = (\langle y | y' \rangle_x, x)$. These make Y into a $\mathbf{C} \times X$-vector space with an $\mathbf{R}^{\geq 0} \times X$-valued norm satisfying the parallelogram law.
Axiom b) These maps are all measurable. That is, $(Y, \mathcal{B})$ is a $(\mathbf{C} \times X, \mathrm{Borel} \times \mathcal{A})$-inner product space in $\underline{\mathbf{Mble}}/X$. □

We have completeness in each of the fibres but will require some form of "global" completeness (in fact, we will require a stability condition as well).

Definition 4.5.1 *A* sequence *in Y is a measurable map over X,* $\mathbf{N} \times X \xrightarrow{s} Y$.
□

Now, $\mathbf{N} \times X \xrightarrow{s} Y$ over X is a(n ordinary) sequence of measurable maps, $X \xrightarrow{s_n} Y$, over X, an $\epsilon \in \mathbf{R}^{>0} \times X$ is a measurable $X \xrightarrow{\epsilon} \mathbf{R}^{>0}$, and a natural number is a measurable $X \xrightarrow{N} \mathbf{N}$.

Definition 4.5.2 *A sequence, s_n, is said to* converge *if there is an $s \in Y$ (which means a measurable section $s : X \longrightarrow Y$) such that $\forall \epsilon(x) \in \mathbf{R}^{>0} \times X$, $\exists N(x) \in \mathbf{N} \times X$ such that $\forall n(x) \geq N(x)$, $\|s_{n(x)} - s\|(x) < \epsilon(x)$.* $\square$

Remarks: 1. $<$ and $\leq$ are interpreted as being everywhere as opposed to almost everywhere.

2. A *Cauchy sequence* is defined in a similar manner. Likewise, *completeness* of Y has an obvious definition. $\square$

Completeness of Y is not enough to make substitution work. We will need stability under pullbacks.

Axiom c) Y is stably complete. $\square$

This means, for all

$$\begin{tikzcd}
 & (Y, \mathcal{B}) \arrow[d, "f"] \\
(X', \mathcal{A}', \mu') \arrow[r, "\phi"'] & (X, \mathcal{A}, \mu)
\end{tikzcd}$$

and for all ϕ-sequences (i.e. measurable s's such that

$$\begin{tikzcd}
\mathbf{N} \times X' \arrow[r, "s"] \arrow[d, "p_2"'] & Y \arrow[d, "f"] \\
X' \arrow[r, "\phi"'] & X
\end{tikzcd}$$

commutes) ϕ-Cauchy (i.e. $\forall \epsilon(x') \in \mathbf{R}^{>0} \times X'$, $\exists N(x') \in \mathbf{N} \times X'$ such that $\forall n(x'), m(x') \geq N(x')$, $\|s_{n(x')} - s_{m(x')}\|(\phi(x')) < \epsilon(x')$) implies ϕ-convergent (which has a similar definition). Note: $\|\cdot\|$ is a measurable $Y \longrightarrow \mathbf{R} \times X$, over X; for each section, $X \xrightarrow{s} Y$, $\|s\|$ is a measurable function $X \longrightarrow \mathbf{R}$.

As we shall see below stable completeness implies that each "pullback," $(Z, \mathcal{C})$ is complete. In particular, along $\phi = 1$, so $(Y, \mathcal{B})$ is complete.

A morphism of HF/X is

$$\begin{tikzcd}
(Y, \mathcal{B}) \arrow[rr, "T"] \arrow[dr, "f"'] & & (Y', \mathcal{B}') \arrow[dl, "f'"] \\
& X &
\end{tikzcd}$$

measurable such that each $T_x : Y_x \longrightarrow Y'_x$ is a bounded linear map. Since we will eventually want to construct the direct integral, we will require:

Axiom: $\|T_x\|_{Y_x}$ is bounded over $x \in X$. $\square$

Remark: We actually have three categories: $PreHilb/X$, $Complete/X$ and $HF/X = StablyComplete/X$. $\square$

A result which we will find useful is:

Lemma 4.5.1 *Let H be a complete metric space with dense sequence $\{h_i\}$. Then the σ-algebra of Borel sets is generated by the open balls of rational radius about the h_i's.*

Proof: We must show that every open set is a countable union of such open balls. Let U be open and let $\mathcal{K}$ be the collection of such open balls contained in U. This is a countable collection and we claim that $U = \bigcup\{K \mid K \in \mathcal{K}\}$.

Certainly, we have $\supseteq$ since each $K \subseteq U$. For the other direction, suppose $x \in U$. Since U is open, there is an open ball, O, of radius ϵ, about x entirely contained in U. Let h_{i_0} be of distance less than $\frac{\epsilon}{3}$ to x and r be a rational such that $\frac{\epsilon}{3} < r < \frac{2\epsilon}{3}$. Then $x \in B(h_{i_0}, r) \subseteq O \subseteq U$ as required. ∎

Let us consider the special case $HF/1$ first. Specifically, we will describe an adjunction

$$\underline{\mathbf{SepHilb}} \underset{I}{\overset{F}{\rightleftarrows}} HF/1.$$

Define $I(H) = (H, \text{Borel}) \xrightarrow{!} (1, 2, \text{counting})$. Axioms a and b are satisfied (note: the relevant maps are all continuous so are all Borel measurable). We proceed to show axiom c holds. Let

$$\begin{tikzcd} & H \arrow[d, "!"] \\ X \arrow[ur, "s_n"] \arrow[r, "!"'] & 1 \end{tikzcd}$$

be a !-sequence (here, ϕ is always !) and suppose that it is !-Cauchy. We claim that $s_n(x)$ is pointwise Cauchy for each x. Fix x_0 and let $\epsilon > 0$ be given. Put $\epsilon(x) = \lceil \epsilon \rceil$, then there is an $N(x)$ such that $\forall n(x), m(m) \geq N(x)$, $\|s_{n(x)} - s_{m(x)}\| < \epsilon$. Now, let $N = N(x_0)$ and $p, q \geq N$. If we set $p(x) = \max\{\lceil p \rceil, N(x)\}$ and $q(x) = \max\{\lceil q \rceil, N(X)\}$, then $p(x)$ and $q(x)$ are measurable, $p(x), q(x) \geq N(x)$, $p(x_0) = p$, and $q(x_0) = q$, so $\|s_p - s_q\| < \epsilon$. And so, $s_n(x_0)$ is indeed Cauchy.

Since H is (ordinary) complete, there is an $s(x)$ such that $s_n(x) \to s(x)$ for each x. In addition, $\|s_n(x)\| \to \|s(x)\|$ since, in particular, $\mathbf{R}$ is (ordinary) complete and $\|\cdot\|$ is continuous. The pointwise limit (in $\mathbf{R}$) of measurable functions yields a measurable function That is, $\|s(x)\|$ is measurable. But, as a consequence of lemma 4.5.1, $s(x)$ is measurable as well (each $s^{-1}(B(0, r)) = s^{-1}\{h \in H \mid$

$\|h\| < r\} = \{x \in X \mid \|s(x)\| < r\} \in \mathcal{A}$ since $\|s(x)\|$ is measurable; then use the measurable translation to get other open balls).

To exhibit !-completeness of $H \longrightarrow 1$, we need only show $s_n(x) \rightarrow_{p.w.} s(x) \Rightarrow s_n \rightarrow_X s$ (the latter denotes convergence in the sense of this chapter).

Let $\epsilon(x)$ be given. Suppose, first, that it is constantly ϵ. For each x, there is an N such that $\|s_n(x) - s(x)\| < \epsilon$ for all $n \geq N$. Put $N(x) = \min\{N \mid \|s_n(x) - s_m(x)\| < \epsilon \forall n \geq N\}$. All we need to show is that $N(x)$ is measurable. But $N^{-1}(k) = A_k \setminus A_{k-1}$ where $A_k = \bigcup_{t=k}^{\infty} \{x \mid \|s_n(x)_s(x)\| < \epsilon\}$ is measurable.

A general $\epsilon(x)$ can be approximated below by simple functions. We may apply the above case repeatedly to arrive at the inequality for a simple function and hence the inequality for a general ϵ. And so, we have shown that axiom c holds.

Remark: In essence, for $I(H) \in HF/1$, we have complete iff stably complete (we actually have shown one direction for Cauchy and the other direction for convergence but the rest is similar). It is important to note that this does not generalize to HF/X, however. That is, fibrewise completeness $\not\Leftrightarrow$ stable completeness; neither direction holds (for FC to SC, we cannot assume s is measurable in general (lemma 4.5.1 is special); for SC to FC, we cannot take a sequence Cauchy in one, fixed fibre and produce a global Cauchy sequence since the fibres "are of global measure zero", for example $s_n(x) = \begin{cases} s_n(x_0), & x = x_0 \\ 0 & \text{else} \end{cases}$ is essentially the 0 function; of course, if x_0 is an atom, this works). For this reason, we impose both completeness conditions. Both together are strictly stronger than either one separately. $\square$

A morphism, $H \xrightarrow{T} K \in$ **<u>SepHilb</u>**, yields a morphism

$$\begin{tikzcd}
(H, \text{Borel}) \arrow[rr, "T"] \arrow[dr, "!"'] & & (K, \text{Borel}) \arrow[dl, "!"] \\
& (1, 2, \text{counting}) &
\end{tikzcd}$$

(note: T is continuous so it is Borel measurable). Furthermore, a

$$\begin{tikzcd}
(H, \mathcal{B}) \arrow[rr, "T"] \arrow[dr, "!"'] & & (K, \mathcal{C}) \arrow[dl, "!"] \\
& (1, 2, \text{counting}) &
\end{tikzcd}$$

is, in particular, a bounded linear transformation (forget measurability) from H to K. Thus, the functor, I is full.

Axiom b for $(H, \mathcal{B}) \xrightarrow{!} (1, 2, \text{counting})$ says, in particular, $\|\cdot\|$ and translation are measurable with respect to $\mathcal{B}$. And so, as in the lemma, $\mathcal{B}$ must contain the Borels. Thus, forgetting the measurable structure on $(H, \mathcal{B})$ provides a left adjoint F to I, i.e.

$$\frac{(H, \mathcal{B}) \longrightarrow (K, \text{Borel})}{(H, \text{Borel}) \longrightarrow (K, \text{Borel}).}$$

Suppose we are given $(X', \mathcal{A}', \mu') \xrightarrow{\phi} (X, \mathcal{A}, \mu)$ in <u>**MOR**</u>. we get:

$$\begin{tikzcd}
(Z, \mathcal{C}) \arrow[r, "r"] \arrow[d, "g"'] & (Y, \mathcal{B}) \arrow[d, "f"] \\
(X', \mathcal{A}', \mu') \arrow[r, "\phi"'] & (X, \mathcal{A}, \mu)
\end{tikzcd}$$

with $Z_{x'} = g^{-1}(x') = Y_{\phi(x')}$ a Hilbert space. The operations of arithmetic and the inner product are measurable when "pulled back" along ϕ. For example, $X' \xrightarrow{\lceil 0 \rceil} Z$ is $x' \longmapsto 0_{x'} = 0_{\phi(x')}$ which is just the composition of 0_Y and ϕ. For addition, the relevant picture is:

$$\begin{tikzcd}
Z \times Z \arrow[rr] \arrow[dr] \arrow[dd] & & Y \times Y \arrow[dr] \arrow[dd] & \\
& Z \arrow[rr] \arrow[dl] & & Y \arrow[dl] \\
X' \arrow[rr] & & X &
\end{tikzcd}$$

$Z = \sum_{x' \in X'} Y_{\phi(x')}$ and the measurable $+_Y$ yields a measurable $+_Z$ given by $(y, x') + (y', x') = (y +_{\phi(x')} y', x')$.

We must show $(Z, \mathcal{C}) \xrightarrow{g'} (X'\mathcal{A}', \mu')$ is ψ-complete for all $\psi : X'' \longrightarrow X'$. Let

$$\begin{tikzcd}
& Z \arrow[r, "r"] \arrow[d, "g"] & Y \arrow[d, "f"] \\
X'' \arrow[ur, "s_n"] \arrow[r, "\psi"'] & X' \arrow[r, "\phi"'] & X
\end{tikzcd}$$

be a ψ-sequnce in Z. Compose with r to get $\phi\psi$-sequnce in Y, $t_n = rs_n$. Let $\epsilon(x'') \in \mathbf{R}^{>0} \times X''$ be given. Then $\|rs_{n(x'')} - rs_{m(x'')}\|_Y(\phi\psi(x'')) < \epsilon(x'')$ iff $\|s_{n(x'')} - s_{m(x'')}\|_Z(\psi(x'')) < \epsilon(x'')$ since $Z_{x'} = Y_{\phi(x')}$ so, in particular, $Z_{\psi(x'')} = Y_{\phi\psi(x'')}$ and the two norms mean the same thing. Thus, s_n is ψ-Cauchy iff t_n is $\phi\psi$-Cauchy and similarly for convergence. Since Y is $\phi\psi$-complete for all ψ, Z is ψ-complete for all ψ.

Remark: It is this with which we rationalize the term *stable* completeness. It means "complete and complete stably under pullbacks." □

Pulling back a $Y \xrightarrow{T} Y' \in HF/X$ yields a $Z \xrightarrow{T^*} Z'$ in HF/X'. Pseudo-functorial substitution restricts to Hilbert families.

Example: This discussion provides us with an important example. For each $H \in \underline{\textbf{SepHilb}}$, $\Delta I(H) = (H \times X, \text{Borel} \times \mathcal{A}) \xrightarrow{p_2} (X, \mathcal{A}, \mu)$ is an object of HF/X. These are to be thought of as the constant X-families. □

4.5.3 Direct Integral and HF/X

We next construct the direct integral:

$$HF/X \xrightarrow{\int^{\oplus}} \underline{\textbf{Hilb}}.$$

Define $\int^{\oplus} Y = \int^{\oplus} (Y, \mathcal{B}) \xrightarrow{f} (X, \mathcal{A}, \mu) := \{s : X \longrightarrow Y \mid s \text{ measurable}, fs = 1_x, \text{ and} \int \|s(x)\|^2 d\mu < \infty\}/\sim$, with $s \sim s'$ iff $\mu\{x \mid s(x) \neq s'(x)\} = 0$. Furthermore, define: $\lceil 0 \rceil(x) = 0_x$, $(-s)(x) = -_x s(x)$, $(s + s')(x) = s(x) +_x s'(x)$, and $(\alpha \cdot s)(x) = \alpha \cdot_x s(x)$. With these definitions, $\int^{\oplus} Y$ is a $\mathbf{C}$-vector space.

Remarks: 1 If $\alpha(x) \in L^{\infty}(X, \mathbf{C})$, then modifying scalar multiplication to $(\alpha \cdot s)(x) = \alpha(x) \cdot_x s(x)$ makes $\int^{\oplus} Y$ into an $L^{\infty}(X, \mathbf{C})$-module.

2. That $s(x) + s'(x)$ is square integrable follows from the usual proof for L^2 (see, for example, [B&N]). □

We define an inner product on $\int^{\oplus} Y$ as $\langle s|s' \rangle = \int \langle s(x)|s'(x) \rangle_x d\mu$ which gives a norm $\|s\|_2 = \int \|s(x)\|_x^2 d\mu$ (note: since we have modded out by *a.e.* equality, $\|\cdot\|_2$ is indeed a norm).

Theorem 4.5.1 *$\int^{\oplus} Y$ is complete.*

Proof: see theorem 2.2.1. ∎

Remark: We actually get an object of $HF/1$, $(\int^{\oplus} Y, \text{Borel})$. $\square$

For

$$\begin{tikzcd} Y \arrow[rr, "T"] \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

in HF/X, define $\int^{\oplus} T : \int^{\oplus} Y \longrightarrow \int^{\oplus} Y'$; $s \longmapsto Ts$; $Ts(x) = T_x s(x)$. Now, $T(s+s')(x) = T_x s(x) +_x Ts'(x) = Ts(x) + Ts'(x)$ and $T(\alpha s)(x) = T_x \alpha \cdot_x s(x) = \alpha \cdot_x T_x s(x) = \alpha \cdot T(s)(x)$. Since $\|T_x\|_x$ is bounded (across x), we have $\int \|Ts(x)\|_x^2 d\mu = \int \|T_x s(x)\|_x^2 d\mu \le \int \|T_x\|_x^2 \|s(x)\|_x^2 d\mu \le k \int \|s(x)\|_x^2 d\mu < \infty$. And so, we have a functor: $\int^{\oplus} : HF/X \longrightarrow$ <u>**Hilb**</u>.

Remark: $\int^{\oplus} \Delta H = \int^{\oplus} H \times X \xrightarrow{p_2} X = \{s : X \longrightarrow H \times X \mid s \text{ measurable}, p_2 s = 1, \text{ and } \int \|s(x)\|_x^2 d\mu < \infty\} = L^2(X; H)$ (here we abuse notation and call $\Delta H = \Delta I H$).

Let us expand on this remark. $L^2(X;H)$ is functorial in H. Given a bounded linear map $F : H \longrightarrow H'$, we get a map, $L^2(X;H) \xrightarrow{L^2(X;F)} L^2(X;H')$, $f \longmapsto \hat{f}(x) = Ff(x)$. Since F is continuous, $Ff(x)$ is measurable and $\int \|Ff(x)\|^2 d\mu \le \int \|T\|^2 \|f(x)\|^2 d\mu < \infty$.

We have a map $H \xrightarrow{T} L^2(X;H)$ $h \longmapsto \lceil h \rceil$ (recall, $\mu(X) < \infty$) which is linear and bounded ($\|Th\| = (\int \|h\|^2 d\mu)^2 = \|h\| \mu(X)^{\frac{1}{2}}$ so $\|T\| = \mu(X)^{\frac{1}{2}}$). Furthermore, it is natural in H, for consider

$$\begin{tikzcd}
H \arrow[r, "T_H"] \arrow[d, "F"'] & L^2(X,H) \arrow[d, "L^2(X,F)"] \\
H' \arrow[r, "T_{H'}"'] & L^2(X,H')
\end{tikzcd}$$

The top-right composite is $h \mapsto \lceil h \rceil \mapsto F\lceil h \rceil$ and the left-bottom composite is $h \mapsto F(h) \mapsto \lceil F(h) \rceil$ and these are equal. This natural transformation is, in general, not an isomorphism (unless $X = 1$). □

We next look at the preservation properties of Δ and the relationship between Δ and $\int^{\oplus}$.

Proposition 4.5.1 $\Delta(H \oplus K) = \Delta(H \times K) = \Delta(H) \times \Delta(K)$.

Proof: We must show

$$\begin{tikzcd}
H \times K \arrow[dr] & H \times K \times X \arrow[l] \arrow[r] \arrow[d] & K \times X \arrow[dl] \\
& X &
\end{tikzcd}$$

is a product diagram. Consider

$$\begin{tikzcd}
& Y \arrow[dl, "a"'] \arrow[d, "t"] \arrow[dr, "b"] & \\
H \times X \arrow[r] & H \times K \times X \arrow[r] & K \times X
\end{tikzcd}$$

Let $y \in Y$ and set $a(y) = (h, x)$, $b(y) = (k, x)$ (same x). Put $t(y) = (h, k, x)$. ∎

Proposition 4.5.2 $\Delta(1) = 1$.

Proof: $\Delta(1) = 1 \times X \xrightarrow{p_2} X \simeq X \xrightarrow{1} X$. ∎

In view of the fact that $0 = 1$ in Hilb (both are the 1 point Hilbert space) and $0 \neq 1$ in HF/X (0 is $\emptyset \hookrightarrow X$ and 1 is $X \longrightarrow X$), we have:

Corollary 1: Δ does not preserve 0. ∎

Corollary 2: Δ does not have a right adjoint. ∎

Note that $\int^{\oplus}$ is not left adjoint to Δ. The unit would be

$$\begin{tikzcd} H \arrow[rr] \arrow[dr, "f"'] & & (\int^{\oplus} H) \times X \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

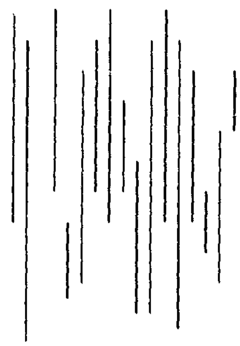
$h \in H_x$ gets sent to the function (in $\int^{\oplus} H$) that sends $x \mapsto h$ and everything else to 0. In the case X is a finite set with counting measure, everything works. But, if points have measure zero in X, then the function so described is the 0 map (after modding out by *a.e.* equality) and so there is no "injection."

Also, the counit would be a map $L^2(X; H) \longrightarrow H$ and given an L^2-function, there seems to be no *canonical* way of getting an element of H (we would need some sort of "indefinite" integral $h = \int f(x) d\mu$ and a square integrable function is not necessarily integrable).

Now, suppose $(X', \mathcal{A}', \mu') \xrightarrow{(\phi, \mu'_x)} (X, \mathcal{A}, \mu)$ is a disintegration. For $(T, \mathcal{D}) \xrightarrow{h} (X', \mathcal{A}', \mu')$, put $(\int_{\phi}^{\oplus} (T, \mathcal{D}))_x := \{s : \phi^{-1}(x) \longrightarrow T \mid s$ a measurable ϕ-section, $\int_{\phi^{-1}(y)} \|s(x')\|^2 d\mu'_x(x') < \infty\}/\sim$ where $s \sim s'$ iff $\mu'_x\{x' \in \phi^{-1}(x) \mid s(x') \neq s'(x')\} = 0$. Equivalently, we could take global measurable sections, $s : X \longrightarrow T$, with the same $\sim$. Next, take the coproduct to get $\sum_{x \in X} (\int_{\phi}^{\oplus} (T, \mathcal{D}))_x =: Y \xrightarrow{p} X$ with p the evident projection.

There is no obvious way to put a σ-algebra structure, $\mathcal{B}$, on Y (exceptions: $\int_1^{\oplus}(T,\mathcal{D}) = (T,\mathcal{D})$ and $\int_!^{\oplus}(T,\mathcal{D}) = (\int^{\oplus} T, \text{Borel}))$. Indeed, this is an interesting open problem. Let us briefly discuss this.

One idea is to take simply the Borels in each fibre (note: each $(\int_\phi^{\oplus}(T,\mathcal{D}))_x$ is a Hilbert space). This would be the σ-algebra of the infinite coproduct (= disjoint union). The problem is that this provides no compatibility across the fibres. Consider the example suggested by the picture:



In each fibre space, we have a Borel set. However, these may "slide back and forth" in a random (non-measurable) way to produce a globally non-measurable set (in the constant family special case, $H \times X$, this essentially means that a measurable set in the plane is not arrived at by arbitrarily gluing together slices; of course, the converse is problematic as well: slicing a Borel set does not necessarily produce a Borel set).

These are, in some sense, function spaces (a special case is $L^2(X)$ which works (take the Borels) except for the caveat about slicing a Borel just mentioned). A related question (and another idea), then, is how to put a useful σ-algebra structure on a function space. Obvious things such as the "infinite product" structure or the "measurable-measurable" σ-algebra (in analogy to the compact-open topology) do not seem to work. These lead to the problems of triviality alluded to in chapter 1 (we need, for example, a more appropriate translation of "compact set").

Our feeling is that disintegrations provide the answer. We need to make sense of "$\mathcal{A} = \int \mathcal{A}_y d\nu(y)$," in the same sense as we have made sense of "$\mu = \int \mu_y d\nu(y)$," in a way that does not conflict with square-integrability. We note that, given a measure, we may possibly disintegrate along slices. But, the converse, given slice spaces and gluing them together requires some sort of global compatibility conditions. It is possible that this is related to the unsolved "existence of (ordinary) disintegration" problem alluded to in chapter 1.

Finally, we make an important observation. All this works in $\underline{\mathbf{Set}}/X$ (substitution, direct integral, pseudo-functoriality (provided we use the special disintegrations of chapter 2)). For this reason, we believe this is the "correct" notion of direct integral in the box 3 world. The difficult part is putting a measurable structure on.

Chapter 5

Conclusions

5.1 Introduction

In this chapter, we provide concluding remarks. The main purpose is to summarize our approaches and to provide connections between them. This is done in sections 5.2 and 5.3 respectively. We also provide a brief discussion of the merits of each. It is too early in this research, however, to provide definitive claims as to which approach is better. This will await future work. This last point and a list of other future possible directions are given in the last section.

5.2 Goals and Approaches

Our aim is to study direct integral decompositions in the context of indexed category theory.; more precisely, measure indexed category theory. This is, perhaps, a bit too ambitious a project for one paper. Fortunately, "direct integration," as a theory, is built up in stages. That is to say, the spectral theorem of von Neumann and the direct integral of von Neumann algebras, of operators, of representations, etc. presuppose the direct integral of Hilbert spaces (to decompose $A \subseteq B(H)$, one must first decompose H). And so, in this paper, we study the latter as a measure-indexed notion.

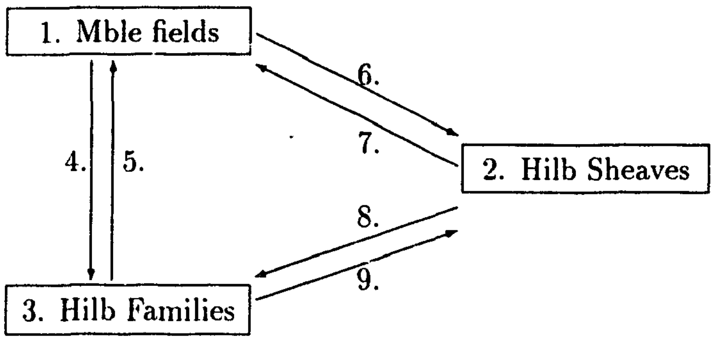
Recall, our task is to understand the picture:

$$\begin{tikzcd} (\underline{\mathbf{Hilb}})^X \arrow[r, shift left, "\int_X^{\oplus}"] & \underline{\mathbf{Hilb}} \arrow[l, shift left, "\Delta"] \end{tikzcd}$$

in analogy to $\oplus_I \dashv \Delta_I$ in the case of I-indexed families of k-vector spaces (I a set). The main point, then, is to arrive at a good notion of X-families of Hilbert spaces for X a measure space. We also need a good notion of substitution (along morphisms of a suitable base category). Δ is to be a special type of substitution and we would like to extend (interpret) $\int^{\oplus}$ to an indexed functor.

Thus, the first priority is to establish a good base category. Good, in the sense of Paré-Schumacher, means a category $\underline{\mathbf{S}}$ with finite limits (more specially, in the Penon style, a topos). We have supplied a suitable category. It does not have products. This is an artifact of the fact that the operator theoretic and the sheaf theoretic world requires us to consider measure zero reflecting as important. Indeed, in precisely that sense, no suitable category will be "good" in the sense of Paré-Schumacher (much less, in the sense of Penon; although it would be interesting to explore the topos-like properties of the category $\underline{\mathbf{Disint}}$; this will await future work). The indexed category theory we develop here is more general. It represents a balance between the classical indexing which works well and a desire to understand the (very real) examples in operator theory.

And so, with the feeling that we cannot "exactly" describe the diagram above (interpret the direct integral as an adjunction, that is, as an indexed limit or, simply as an entity with a tractable universal property), we must approximate the situation as best possible. **Hilb**, of course, is the category of Hilbert spaces and bounded linear maps. We have put forth three approximations for $\mathbf{Hilb}^X$. Recall the diagram of the introduction.



Box 1: We begin by simply translating classical direct integral theory into categorical (indeed, indexed categorical) language. This gives $MFHS(X)$ whose objects are measurable fields of Hilbert spaces (subsets of the product of an X-family of Hilbert spaces satisfying three axioms) and whose morphisms are norm essentially bounded (over the index $x \in X$). It is a good approach to presenting measurable indexing of Hilbert spaces. Substitution, along a M0R works well by simply putting, for $(Y, \mathcal{B}, \nu) \xrightarrow{\phi} (X, \mathcal{A}, \mu)$, $\phi^*((H(x), \mathcal{G}) \xrightarrow{(T(x))} (H'(x), \mathcal{G}'))$ $= (H(\phi(y)), \mathcal{H}) \xrightarrow{(T(\phi(y)))} (H'(\phi(y)), \mathcal{H}')$ where $\mathcal{H}$ is the inner product closure of the $g(\phi(y))$'s for $g \in \mathcal{G}$. Operator theory works as well, in this instance. As already noted, this is simply the classical material.

Indexing the direct integral, however, does not work for merely $\phi \in$ **M0R**. We require that ϕ be a disintegration to get a handle on the fibre spaces (disintegrations are *very* useful for this). We also need to make two assumptions: $(T(x))$ is norm bounded, instead of merely essentially bounded (we work with the category $BMFHS(X)$), and $L^2(Y) \simeq \int^{\oplus} L^2(Y_x)d\mu(x)$. The former assumption represents a slight deviation from the classical theory but is contained in it in the sense that bounded functions are amoung the essentially bounded functions. The second assumption is a structural (in this case, separability) requirement. This is necessary for axiom 3 of an MFHS (the "separability" condition) to be consistent with indexing of the direct integral.

And so, in the box 1 world, we propose $\underline{\mathbf{Hilb}}^X = BMFHS(X)$. Substitution may be interpreted as a pseudo-functor, $\underline{\mathbf{MOR}}^{op} \xrightarrow{(\ )^*} \underline{\mathbf{Cat}}$. The direct integral may almost be interpreted as a pseudo-functor, $\underline{\mathbf{Disint}} \xrightarrow{\int^{\oplus}} \underline{\mathbf{Cat}}$. The caveat is that the special disintegrations (that satisfy the second assumption above) must compose.

Finally, we note that this is not an indexed adjunction. Indeed, the special case of Δ and (ordinary) $\int^{\oplus}$ is not an adjunction. It is possible that the (indexed) direct integral may have some utile universal property but such is not forthcoming; it is not easily discernible. $\Box$

Box 2: Here, we consider Hilbert spaces in Sheaves. We decide that the appropriate sheaf category for measure indexing is $MEAS(X)$. This may alternately be described as sheaves on the site $\mathcal{A}$ with covers countable families of subsets whose union almost cover or as sheaves on the locale $\mathcal{A}/\mathcal{N}$ where $\mathcal{N}$ denotes the ideal of measure zero sets with cover = countable union. The locale is, in fact, a complete Boolean algebra. Moreover, we have the axiom of choice (which, of course, implies Booleanness) in this topos so our logic is essentially classical.

In the box 2 world, we, more or less, put $\underline{\mathbf{Hilb}}^X = \underline{\mathbf{Hilb}}(MEAS(X))$. We say "more or less" since some care must be taken as to what the appropriate morphisms are. As was noted in the box 1 world, boundedness, as opposed to essential boundedness, is appropriate. And so, we really put $\underline{\mathbf{Hilb}}^X = \underline{\mathbf{CBHilb}}(MEAS(X))$; "CB" means constantly bounded linear natural transformation.

To describe Hilbert sheaves, we must first understand basic arithmetic. $MEAS(X)$ has the usual objects of numbers. For example, $\mathbf{N}_X$, $\mathbf{R}_X$, and $\mathbf{C}_X$, at $A \in \mathcal{A}$ are equivalence classes of measurable functions into, respectively, $\mathbf{N}$, $\mathbf{R}$, and $\mathbf{C}$. $\mathbf{C}_X$ is a geometric field (and hence a field of quotients and a field of fractions since the topos is Boolean). We can talk of $\mathbf{C}_X$-vector spaces. An inner product may be defined in an obvious manner (a natural transformation, $H(A) \times H(A) \xrightarrow{\langle\,|\,\rangle} \mathbf{C}(A)$, satisfying equations). An object of $\underline{\mathbf{Hilb}}(MEAS(X))$ is to be, in particular, a $\mathbf{C}_X$-inner product space (i.e. a preHilbert sheaf). We require, of course, H to be complete in the language of $MEAS(X)$.

This classical logic allows us to construct a useful completion for preHilbert spaces. We use this to describe substitution and $\int^{\oplus}$. A $\phi \in \underline{\mathbf{MOR}}$, induces a geometric morphism, $\phi^* \dashv \phi_* : MEAS(X) \longrightarrow MEAS(Y)$. ϕ^* lifts to pseudo-normed $\mathbf{C}_{l.c}$-modules. Completing this produces a pseudo-functorial substitution, one for each ϕ, $\underline{\mathbf{Hilb}}(MEAS(X)) \xleftarrow{\phi^{\#}} \underline{\mathbf{Hilb}}(MEAS(Y))$. For $(\phi, \mu_y(x))$ a disintegration, put $(\int_{\phi}^{\oplus} H)(-)$ as the completion of $\{s \in H(\phi^{-1}(-)) \mid$ $\int_{\phi^{-1}(y)} \|s\|^2(x) d\mu_y(x) < \infty \; a.a. \; y$ for any choice of $\|s\|\}$. Again, this is pseudo-functorial and, again, an obvious universal property for it is not discernible. $\square$

Box 3: This is the slice categorical world. If box 1 represents the mostly operator theory world and box 2 represents the mostly category theory one, box 3 represents the mostly indexed category theory approach. Here we put an X-family of Hilbert spaces as a measurable Y over X subject to some axioms (paraphrased below).

$\underline{\mathbf{Mble}}/X$ is not a topos but it is almost as good. Importantly, there is a useful, pseudo-functorial substitution (indeed, this is developed for $\underline{\mathbf{Disint}}/X$ in hopes of understanding more general direct integral constructions in the future) which, together with composition, satisfies the Beck condition (this works in $\underline{\mathbf{Disint}}/X$ even though substitution is not universal (we still get commutativity of a "canonical" map); in $\underline{\mathbf{Mble}}/X$, this is just pullback so works as in the usual topos world).

In box 3, $\underline{\mathbf{Hilb}}^X$ is defined as HF/X (an object of HF/X is a $Y \longrightarrow X$ in

$\underline{\mathbf{Mble}}/X$ such that the fibres are Hilbert spaces, the induced global operations of arithmetic and inner product are measurable, and Y is stably complete (it and all its pullbacks are complete in the "language" of $\underline{\mathbf{Mble}}/X$)). Fibrewise completeness and stable (global) completeness are not equivalent so we impose both conditions. We have noted that "local homeomorphism" does not translate (naively) into the $\underline{\mathbf{Mble}}$ world. However, with this particular approach, we provide an account of how local homeomorphism-like indexing may be applied. In short, all is not lost. There is some useful information to be gained.

This approximation is open-ended. That is to say, substitution restricts to Hilbert families. The direct integral, defined as the collection of square-integrable sections, works in $\underline{\mathbf{Set}}/X$. It works in $\underline{\mathbf{Mble}}/X$ for some important special examples ($\int_!^\oplus = \int^\oplus$, the classical direct integral, and $\int_1^\oplus = 1$). However, lifting all this to $\underline{\mathbf{Mble}}/X$ for a general ϕ seems to be a difficult and interesting problem. It is perhaps related to the existence of disintegration of measures problem in (usual) disintegration theory.

We have provided three approaches to understanding $\underline{\mathbf{Hilb}}^X$ from an indexed categorical point of view. The philosophy is that $\int_\phi^\oplus$ is "fixed", or better, "has a forced and obvious definition," in each world (indeed, the basic direct integral, $\int_!^\oplus$, is the same in each and is the usual one studied by operator algebraists). The point, then, is to describe a universal property for this fixed direct integral. Failing that, we could also describe a pseudo-functorial substitution which is almost as good.

Each approximation has its merits and drawbacks. Which of the three is most useful depends, perhaps, on the context of the applications. As we have noted, the three blend, to varying proportions, operator theory, category theory, and indexed category theory. And, each of the three has elements of measure theory incorporated. In the next section, we provide a list of connections between the approaches.

It is, of course, highly possible, that there are other approaches combining these aspects more subtly. A true comparison will require much work. Furthermore, the interesting thing is to apply this material. We are attempting, after all, to set up a categorical framework to do direct integration.

5.3 Connections

5.3.1 MFHS and Sheaf

In this section, we provide a list of some functors between the three worlds. We begin with MFHS and Sheaves. Recall, from section 3.5.3, we get an X-Hilbert sheaf from an $MFHS(X)$ by

$$G(A) = \{g \in \mathcal{G} \mid g(x) = 0 \forall x \notin A\}/\sim .$$

This construction is functorial. Suppose we have, $((H(x))_{x\in X}, \mathcal{G}) \xrightarrow{(T(x))_{x\in X}} ((H'(x))_{x\in X}, \mathcal{G}')$, a morphism in $MFHS(X)$. We define $G(A) \xrightarrow{\tau} G'(A)$ by $(\tau g)(x) = T(x)g(x)$. Now, $\tau g \in \mathcal{G}'$ (by definition of morphism in $MFHS(X)$) and since $T(x)$ is linear, $T(x)(0) = 0$ so $\tau g \in G'(A)$. Also, since each $T(x)$ is linear, τ is linear (in the operations of G and G'). Finally, we suppose that $A' \subseteq A$ and consider:

$$\begin{tikzcd} G(A) \arrow[r, "\tau_A"] \arrow[d, "\rho^A_{A'}"'] & G'(A) \arrow[d, "\rho'^A_{A'}"] \\ G(A') \arrow[r, "\tau_{A'}"'] & G'(A') \end{tikzcd}$$

The top-right composite is $g \mapsto (x \mapsto T(x)g(x),\ x \in A) \mapsto (x \mapsto T(x)g(x),\ x \in A')$. The left-bottom composite is $g \mapsto g|_{A'} \mapsto (x \mapsto T(x)g(x),\ x \in A')$ so these are equal and τ is natural. Thus, we have a functor

$$MFHS(X) \xrightarrow{\widehat{(\ )}} \underline{\mathbf{Hilb}}(MEAS(X)).$$

Remark: We do not require boundedness of $\|T(x)\|$, over $x \in X$, but this restricts to a functor $BMFHS(X) \longrightarrow \underline{\mathbf{CBHilb}}(MEAS(X))$. □

The diagram:

$$\begin{tikzcd}
BMFHS(X) \arrow[r, "\widehat{(\ )}"] \arrow[d, "\int_\phi^\oplus"'] & \underline{\mathbf{CBHilb}}(MEAS(X)) \arrow[d, "\int_\phi^\oplus"] \\
BMFHS(Y) \arrow[r, "\widehat{(\ )}"'] & \underline{\mathbf{CBHilb}}(MEAS(Y))
\end{tikzcd}$$

commutes. Suppose $(H(x), \mathcal{G})$ is an object of $BMFHS(X)$. Before completion, the top-right composite at B is $\{s \in G(\phi^{-1}(B)) \mid \int_{\phi^{-1}(y)} \|s\|^2(x) d\mu_y(x) < \infty$ $a.a.$ y for any choice of $\|s\|\} = \{s \in \mathcal{G} \mid \int_{\phi^{-1}(y)} \|s(x)\|^2 d\mu_y(x) < \infty$ $a.a.$ y and $s(x) = 0$ for all $x \notin \phi^{-1}(B)\}/\sim$. Likewise, the left-bottom composite at B is $\{d \in \mathcal{D} \mid d(y) = 0$ for all $y \notin B\}$ with $\int_\phi^\oplus (H(x), \mathcal{G}) = (D(y), \mathcal{D})$ (notation as in chapter 2).

Suppose $s \in TR$. Then there is a choice of $\|s\|$ to make the integral finite for all y, in which case, $[s|_{\phi^{-1}(y)}] \in D(y)$ for all y. Let $y \in B^c$ (we wish to show $d(y) := [s|_{\phi^{-1}(y)}] = 0$), then $\phi^{-1}(y) \subseteq \phi^{-1}(B^c) = (\phi^{-1}(B))^c$. So, for all $x \in \phi^{-1}(y)$, $s(x) = 0$ which implies $[s|_{\phi^{-1}(y)}] = 0$.

Conversely, suppose $d \in LB$. By the "alternate description" of $\mathcal{D}$, there is a $g \in \mathcal{G}$ such that $[g|_{\phi^{-1}(y)}] = d(y)$ for all y. Now, certainly $g(x)$ satisfies the finiteness condition since $[g|_{\phi^{-1}(y)}] = d(y) \in D(y)$ (recall, if the integral is finite for some choice, then it is finite for any choice). In addition, $\mu\{x \in \phi^{-1}(B^c) \mid g(x) \neq 0\} = \int_{B^c} \mu_y\{x \in \phi^{-1}(y) \mid g(x) \neq o\} d\nu(y) = \int_{B^c} o d\nu(y) = 0$. Thus, $g(x) = 0$ for almost all $x \in \phi^{-1}(B)$. Replace $g(x)$ by $g'(x) = \chi_G g(x)$ where G is the set of x's for which $g(x) \neq 0$. Then $g \sim_X g'$ and $\int_{\phi^{-1}(y)} \|g'(x)\|^2 d\mu_y(x) < \infty$ $a.a.\ y$ (since $\int_Y \int_{\phi^{-1}(y)} = \int_X = 0 \Rightarrow$ the inside integral is finite $a.e.$). The composites are the same after completion (constant sequences are dense in Cauchy sequences).

The other square, with $\phi^{\#}$, is probably difficult inasmuch as there is not yet a nice, explicit description of $\phi^{\#}$ (in the general Grothendieck topos world, ϕ^* is given as a double colimit or with one of the colimits the associated sheaf functor (as was our description in chapter 3); these are quite complicated). That is to say, one direction of commutativity is free (follows from the adjunction). Indeed, consider the diagram:

$$\begin{array}{ccc} MFHS(X) & \xrightarrow{(\ )^+} & \underline{\mathbf{PreHilb}}(MEAS(X)) \\ \uparrow{\scriptstyle \phi^*} & & \uparrow{\scriptstyle \phi^*} \\ MFHS(Y) & \xrightarrow[(\ )^+]{} & \underline{\mathbf{PreHilb}}(MEAS(Y)) \end{array}$$

Let $(G(y), \mathcal{G})$ be an object of $MFHS(Y)$. The left-top composite sends this to L with $L(A) = \{h \in \mathcal{H} \mid h(x) = 0 \text{ for all } x \notin A\}/\sim$ with $\mathcal{H}$ the "inner product closure" of the $g(\phi(y))$'s (again, notation as in chapter 2).

On the other hand, $(G(y), \mathcal{G})^+(B) = \{g \in \mathcal{G} \mid g(y) = 0 \text{ for all } y \notin B\}/\sim$. By the adjunction,

$$\frac{\phi^* K \longrightarrow L}{K \longrightarrow \phi_* L = L(\phi^{-1}((-)).}$$

There is an obvious map, $K(B) \longrightarrow L(\phi^{-1}(B))$. It is $g(y) \longmapsto g(\phi(x))$. This is precisely the map, α, of remark 1 of section 2.4.3.

Of course, there is no functor in the other direction, from sheaves to fields. Such would require, in particular, the fibre spaces which could only be the stalks, $H_x = \mathrm{colim}_{x \in A} H(A)$, with $\in$ interpreted as $a.e$ containment of $\{x\}$. If x is not an atom, then x is in any measure zero set, in particular $\emptyset$, the initial object of $\mathcal{A}$. Thus, the colimit is $H(\emptyset) = 1$.

This brings an observation to mind. We have noted many times that the three worlds blend, to varying proportions, the aspects expected in the construction of the indexed direct integral. There is also a "fibre and global" mixture. That is to say, we have two other ingredients to vary, fibrewise structure and global structure. MFHS's and HF's retain the fibrewise Hilbert space structure, HF's have more of a global structure (represented by what we call stable completeness), and Hilbert sheaves have an entirely global structure.

5.3.2 Hilbert Family and Sheaf

Let $(Y, \mathcal{B}) \xrightarrow{f} (X, \mathcal{A}, \mu)$ be an HF/X. We wish to construct a sheaf. Put $S(Y)(A) = \{s : (A, \mathcal{A}|_A) \longrightarrow (Y, \mathcal{B}) \mid s \text{ measurable}, fs = incl_A\}/\sim$ where, as usual, $s \sim s'$ iff $\mu|_A\{x \in A \mid s(x) \neq s'(x)\} = 0$. Equivalently, $S(Y)(A) =$ $\{s : X \longrightarrow Y \mid s \text{ measurable}, fs = 1\}/\sim$.

This is a subsheaf of $Mble(-, Y)/\sim$. Arithmetic operations are defined from those in $Y \longrightarrow X$. For example, $+ : S(Y)(A) \times S(Y)(A) \longrightarrow S(Y)(A)$, $(s, s') \longmapsto [x \mapsto s(x) +_x s'(x)]$. Scalar multiplication, by elements of $\mathbf{C}(A)$, is pointwise as well: $\alpha \cdot s = [x \mapsto \alpha(x) \cdot_x s(x)]$ (that these are well-defined is the usual proof).

And so, $S(Y)(-) \in \underline{\mathbf{PreHilb}}(MEAS(X))$. But, completeness is the same in both as well. That is, suppose s_n is Cauchy in $S(Y)(A)$ so that $\forall k \in \mathbf{N}_X^+$, $\exists$ a cover, A_i, and N_i such that $\forall n, m \geq N_i$, $\|s_n - s_m\| < \dfrac{1}{k(x)}$ on A_i. This implies s_n is id-Cauchy in Y (for $\epsilon(x)$, pick $\dfrac{1}{k(x)} < \epsilon(x)$; the cover simply codifies the locally constant $n(x)$, $m(x)$, and $N(x)$; that $\|\cdot\| < \dfrac{1}{k(x)}$ $a.e.$ on A_i, and we require "everywhere" for id-Cauchy is taken care of by $\sim$; more precisely, for the x's for which $\|s_n(x) - s_m(x)\| \geq \dfrac{1}{k(x)}$, put $s_n(x) = s_m(x) = 0$ to arrive at a new, equivalent sequence which is id-Cauchy). In a similar way, id-convergent implies convergent in sheaves (here, $\epsilon(x) = \dfrac{1}{k(x)}$ is a special case). Thus, $S(Y)(-) \in \underline{\mathbf{Hilb}}(MEAS(X))$. Now, given

$$\begin{tikzcd} Y \arrow[rr, "T"] \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

in HF/X, put $S(Y)(A) \xrightarrow{\tau_A} S(Y')(A)$, $s \longmapsto [x \mapsto T_x s(x)]$. It is straightforward to check that τ is well defined, natural, linear, and bounded. For example,

$$\begin{tikzcd} S(Y)(A) \arrow[r, "\tau_A"] \arrow[d, "\rho^A_{A'}"'] & S(Y')(A) \arrow[d, "\delta^A_{A'}"] \\ S(Y)(A') \arrow[r, "\tau_{A'}"'] & S(Y')(A') \end{tikzcd}$$

The top-right composite is $s \longmapsto [x \mapsto t_x s(x)] \longmapsto [x \mapsto T_x s(x); x \in A']$. The left-bottom composite is $s \longmapsto s|_{A'} \longmapsto [x \mapsto T_x s|_{A'}(x); x \in A']$ and these are the same.

And so, we have a functor:

$$HF/X \xrightarrow{S} \underline{\mathbf{Hilb}}(MEAS(X)).$$

Whether this commutes with either substitution or direct integral will await future work (we do not yet have explicit descriptions of substitution for $\underline{\mathbf{Hilb}}(MEAS(X))$ nor $\int_\phi^\oplus$ for HF/X).

5.3.3 MFHS and Hilbert Family

Let $(Y,\mathcal{B}) \xrightarrow{f} (X,\mathcal{A},\mu)$ be a Hilbert family over X. We wish to make the collection of measurable sections into an MFHS structure on the $(Y_x)_{x\in X}$. Since $\|\cdot\|$ is measurable, $x \mapsto \|s(x)\|$ is measurable for any measurable section, s, of Y, so axiom 1 for an MFHS holds. However, axioms 2 and 3 do not hold in general. We must restrict our Hilbert families:

Definition 5.3.1 *The full subcategory, SHF/X (etymology: "S" for "separable"), of HF/X has as objects, Hilbert families, $Y \xrightarrow{f} X$, with the additional axioms:*
Axiom d: *If, for the section*

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ X \arrow[ur, "t"] \arrow[r, "1"'] & X, \end{tikzcd}$$

$x \mapsto \langle t(x)|s(x)\rangle$ is measurable for all measurable sections, s, then t is a measurable section. $\square$

Axiom e: *There is a sequence, s_i, of measurable sections, such that $\{s_i(x)\}_{i=1}^{\infty}$ forms a total set in Y_x for each x.* $\square$

This, of course, is just axioms 2 and 3 for an MFHS and provides us with a functor:

$$SHF/X \xrightarrow{\Psi} MFHS(X)$$

with $\Psi(Y,\mathcal{B}) = ((Y_x)_{x\in X}, \mathcal{S} = \{s : X \longrightarrow Y \mid s \text{ measurable and } fs = 1\})$ and, for $T : Y \longrightarrow Y'$, in SHF/X, we get a morphism, $(Y_x) \xrightarrow{(T_x)} (Y'_x$, of $MFHS(X)$ (note: if g is a $\mathcal{B}$-section, then, since T and g are measurable, $g' = Tg$ is a $\mathcal{B}'$-section).

Again, we must note that the other direction is difficult. It is similar to $\int_\phi^\oplus$ as discussed in chapter 4. That is, given an MHFS, $(H(x), \mathcal{G})$, we can put $Y = \dot{\bigcup\limits_{x\in X}} H(x)$. The problem is to put an appropriate σ-algebra on Y (appropriate means, in particular, make the g's measurable sections and make the arithmetic measurable). Again, the immediately obvious "just make all the g's measurable" does not work (indeed, in the 1-family case, this is not the Borels which it should be).

5.4 Future Considerations

Here is list a few interesting open problems (which have been described at various locations). We cannot speculate as to their relative level of difficulty; some may be quite easy, some may be quite hard.

-Do the special disintegrations introduced in chapter 2 compose?

-For H a Hilbert sheaf, is $\int_\phi^\oplus H$ complete, in the language of $MEAS(X)$, for a general ϕ?

-What is an appropriate measurable structure on $\int_\phi^\oplus (Y,\mathcal{B})$ for Hilbert families?

-Is there a functor, $MFHS(X) \xrightarrow{\Phi} HF/X$?

-If so, what is the relationship between Φ and Ψ?

-Is there a Beck condition for substitution and the direct integral even though they are not adjoint?

-Are there other approaches that blend the aspects in a better, more subtle way?

Finally, we note that, in this work, we have provided a categorical footing upon which to describe the direct integral of Hilbert spaces. We would like to generalize this to operators, to von Neumann algebras (to arrive at an indexed version of Von Neumann's decomposition theorem perhaps), to C^*-algebras, and a whole host of other interesting direct integral-like construction. There seems to be a very large possibility for application.

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