Factorization in C*-Algebras: Products of Positive Operators

by

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Signature of Author

For Fiona

and

my Parents

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Abstract

The main question of the thesis is the following: given a C*-algebra $\mathfrak{A}$ which elements of $\mathfrak{A}$ can be factored as, or approximated by, finite products of positive operators, with each factor also from $\mathfrak{A}$? We begin by extending Ballantine's theorem for matrices to the class of n-normal operators. This introduces measure theory, while in another direction we obtain approximation theorems for AF-algebras. Combining AF-algebras with n-normal operators we obtain Approximately Poly-Normal Algebras (APN) and give a characterization of those APN-algebras for which the set of products of four positive operators is dense. We conclude with partial results on the "direct integral" and the "compact direct integral", two algebras which arise in a natural way from a "measurable field of C*-algebras".

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Introduction and Basic Definitions

Introduction

One approach in trying to understand the structure of operators is to focus on particular families of operators; and distinguished here is the family of positive operators. Through classical theory and the Spectral Theorem their structure is reasonably well understood. Considering pairs of positive operators A and B, one easily finds that their product is again positive exactly when A and B commute. It is this very fact which helps make the question of factorization and approximation, by products of positive operators, an interesting one. For, on the one hand, there is the possibility that since positive operators themselves are understood, so too might finite products be. On the other hand, because forming products does not in general preserve positivity, we would obtain a significantly larger yet, we hope, tractable class of operators.

To determine the extent of this new class of operators, the question is naturally twofold, regarding exact factorization as well as approximation by these products. So far in the theory, pertinent factorization and approximation theorems have relied heavily on the fact that the factors can be chosen from the relatively large and well-structured C*-algebra of all bounded operators on the underlying Hilbert space. (Of particular importance is Herrero's work [especially vol. I, 1988] and the theorems of Fong and Sourour [1984, 1986].) Therefore, with positivity essentially an operator theoretic notion, the question properly belongs to the category of C*-algebras, and should be rephrased as follows: Letting $\mathfrak{A}$ be any C*-algebra, which elements of $\mathfrak{A}$ can be factored as, or approximated by, finite products of positive operators, with each factor also from $\mathfrak{A}$? In this framework, it is seen that Ballantine (1970) and Wu (1988) were considering factorization in the special C*-algebra $\mathcal{B}(\mathcal{H})$, for $\mathcal{H}$ respectively finite and infinite dimensional.

As it turns out, the question sheds light on both properties of the algebra and the elements therein. Thus, the question is fundamental, concretely intertwining single operator theory with the theory of operator algebras, and ultimately concerns deep structure theorems for both. Moreover, although not considered in this thesis, work so far suggests the possibility of obtaining a new invariant for C*-algebras.

For references on the primary question of factorization, two good expository papers are [Wu, 1] and [Halmos, 2]. For approximation theorems, our first results were inspired by [KLMR].

To give an outline of our paper we need some notation. So let $\mathcal{A}$ be a C*-algebra and $k \in \mathbb{N}$. We set

$$\mathcal{P}_k(\mathcal{A}) = \{A \in \mathcal{A} : A = P_1 \dots P_k,\ P_i \in \mathcal{A} \text{ positive and invertible}\} \qquad 1 \le k < \infty$$

with

$$\mathcal{P}_\infty(\mathcal{A}) = \bigcup_{k=1}^{\infty} \mathcal{P}_k(\mathcal{A})$$

and

$$\overline{\mathcal{P}}_k(\mathcal{A}) = \text{the norm closure of } \mathcal{P}_k(\mathcal{A}) .$$

Similarly

$$\mathcal{Q}_k(\mathcal{A}) = \{A \in \mathcal{A} : A = Q_1 \dots Q_k,\ Q_i \ge 0 \text{ (not necessarily invertible)}\} \qquad 1 \le k < \infty$$

$$\mathcal{Q}_\infty(\mathcal{A}) = \bigcup_{k=1}^{\infty} \mathcal{Q}_k(\mathcal{A})$$

and

$$\overline{\mathcal{Q}}_k(\mathcal{A}) = \text{the norm closure of } \mathcal{Q}_k(\mathcal{A}) .$$

We open the thesis by extending the work of Ballantine on $n \times n$ matrices [B] to the case of n-normal operators. We prove, among other things, that in the algebra of matrices of (equivalence classes of) bounded measurable functions, an invertible operator

T is a product of some finite number of positive operators if and only if $x \mapsto \det T(x)$ is greater than zero a.e. (almost everywhere), and that five factors will suffice. So

$$\mathcal{P}_\infty(\mathcal{M}_n(\mathcal{L}^\infty(X,\mu))) = \mathcal{P}_5(\mathcal{M}_n(\mathcal{L}^\infty(X,\mu)))$$
$$= \{T : \exists\delta > 0 \text{ s.t. } \det T(x) \geq \delta > 0 \text{ a.e.}\} .$$

In Chapter 2 we characterize those AF-algebras $\mathcal{A}$ for which $\overline{\mathcal{Q}_4}(\mathcal{A}) = \mathcal{A}$. We give two equivalent conditions which concern in turn (1) the ideal structure and so also (2) Bratteli diagrams. A consequence is a theorem on tensor products of AF-algebras.

In Chapter 3 we define APN-algebras (Approximately Poly-Normal), direct limits of direct integrals of finite dimensional C*-algebras. For an example, form the C*-algebra tensor product $c_0(\mathbb{N}) \otimes \mathcal{K}$, $\mathcal{K}$ being the compact operators.

In Chapter 4 we characterize those APN-algebras $\mathcal{A}$ for which $\overline{\mathcal{Q}_4}(\mathcal{A}) = \mathcal{A}$.

In Chapter 5 we introduce the notion of a measurable field of C*-algebras (as opposed to the usual fields of von Neumann algebras). We use this to define a direct integral of AF-algebras which is itself interesting and acts as an ambient space for the APN-algebras as well as for our so-called "compact direct integrals". Examples of these are $\ell^\infty(\mathbb{N},\mathcal{K})$ and the C*-tensor product $\ell^\infty(\mathbb{N}) \otimes \mathcal{K}$ respectively. We have partial results on the compact direct integrals, which seem to be algebras which behave very nicely with respect to the sets $\mathcal{Q}_k$. Our conjecture is basically that $\mathcal{Q}_4$ of such an algebra is dense if and only if $\mathcal{Q}_4$ of almost every integrand is dense. Note that this is not the case for APN-algebras (see Examples at the beginning of Chapter 4).

Basic Definitions

One preliminary result which is implicitly used several times through the thesis is the following:

Proposition 0.1: Let $\mathcal{A}$ be a unital C*-algebra. (a) Then $\mathcal{P}_{2k}(\mathcal{A})$ is similarity invariant; (b) $\overline{\mathcal{P}_k}(\mathcal{A}) = \overline{\mathcal{Q}_k}(\mathcal{A})$, $1 \leq k \leq \infty$.

Proof: (a) First recall that for an operator T in a C*-algebra $\mathfrak{A}$

$$T \in \mathcal{P}_2(\mathfrak{A}) \Leftrightarrow T \text{ is similar to some } P \in \mathcal{P}_1(\mathfrak{A}).$$

To see why this is true, just consider the following factorizations involving an invertible X and positive invertibles P_1, P_2:

$$P_1P_2 = P_1^{\frac{1}{2}}(P_1^{\frac{1}{2}}P_2P_1^{\frac{1}{2}})P_1^{-\frac{1}{2}}$$

and

$$X^{-1}P_1X = [X^{-1}(X^{-1})^*][X^*P_1X].$$

Now, if $T \in \mathcal{P}_{2k}(\mathfrak{A})$

$$T = P_1 \ldots P_{2k}$$

and X is invertible, then

$$X^{-1}TX = \prod_{i=1}^{k} (X^{-1}P_{2i-1}P_{2i}X)$$

$$= \prod_{i=1}^{k} (X^{-1}Y_i^{-1}S_iY_iX) \quad \text{for some invertible } Y_i \text{ and } S_i \in \mathcal{P}_1(\mathfrak{A})$$

$$= \prod_{i=1}^{k} [(Y_iX)^{-1}S_i(Y_iX)]$$

which is again in $\mathcal{P}_{2k}$.

(b) This follows from the spectral theorem for commutative C*-algebras.

Note: It might eventually be of use to know that similar calculations give

$$X^{-1}\mathcal{P}_{2k+1}X \subseteq \mathcal{P}_{2k+2}, \text{ for all } k \in \mathbb{N}.$$

(See Concluding Remarks.)

With that done we now proceed to establish the rest of our basic terminology and in doing so give a quick review of direct integral theory. The main source is [Tak], but we occasionally refer to [Nielsen].

<u>Direct Integrals of Hilbert Spaces</u>

We start with an example.

Let $X = [0,1]$, μ = Lebesgue measure on X and $\mathcal{H}$ be any fixed separable (finite or infinite dimensional) Hilbert space. Then we can construct the familiar Hilbert space tensor product $\mathcal{L}^2(X,\mu) \otimes \mathcal{H}$. This space can be identified with a Hilbert space of equivalence classes of $\mathcal{H}$-valued L^2-functions (See [Nielsen], Chapter 2). For consider functions $\xi : (X,\mu) \to \mathcal{H}$ which are measurable with respect to the Borel structure on $\mathcal{H}$ generated by the strong (or weak) topology, and for which

$$\int_X \|\xi(x)\|^2 d\mu < \infty .$$

Then, modulo those ξ which satisfy

$$\int_X \|\xi(x)\|^2 d\mu = 0$$

we obtain a Hilbert space with inner product

$$\langle \xi,\eta \rangle = \int_X \langle \xi(x),\eta(x) \rangle d\mu .$$

This Hilbert space is canonically isomorphic to $\mathcal{L}^2(x,\mu) \otimes \mathcal{H}$, associating to an elementary tensor $\xi \otimes f$ the class determined by the $\mathcal{H}$-valued function $x \mapsto \xi(x)\cdot f$, where $x \mapsto \xi(x)$ is a representative function for $\xi \in \mathcal{L}^2(X,\mu)$. We can therefore think of $\mathcal{L}^2(x,\mu) \otimes \mathcal{H}$ as a kind of direct sum of Hilbert spaces, where the index set $[0,1]$ is no longer discrete. For this interpretation we use the notation

$$\int_X^{\oplus} \mathcal{H} \, d\mu$$

and say that $\mathcal{L}^2(x,\mu) \otimes \mathcal{H}$ is the <u>direct integral of</u> $\mathcal{H}$ (over (X,μ)).

For the general direct integral, we allow the summands of the "continuous direct sum" to vary. Here the notation will be

$$\int_X^{\oplus} \mathcal{H}(x) d\mu .$$

In the case (X,μ) is $\mathbb{N}$ with discrete measure, this will reduce to the usual direct sum $\bigoplus_n \mathcal{H}(n)$. The elements of the new Hilbert space will be equivalence classes of certain L^2-functions. These functions ξ will have the property that $\xi(x) \in \mathcal{H}(x)$ almost everywhere. Since these functions must be measurable in some appropriate sense and the spaces $\mathcal{H}(x)$ may vary with the index $x \in X$, we have to specify how spaces sitting on distinct points of the measure space (X,μ) are bound together. So we now require some definitions.

Definition 0.2: (a) A Borel structure on a set X is a σ-field of subsets of X.

(b) A Borel space $(X,\mathcal{B})$ is a pair consisting of a set X and a Borel structure $\mathcal{B}$ on X; for convenience we often write X in place of $(X,\mathcal{B})$.

(c) A topological space is called Polish if it is separable and if its topology is generated by a complete metric.

(d) A Borel space X is called standard if there is a Polish space Z and a Borel subset Y of Z such that X is Borel isomorphic to the "Borel subspace of Z based on Y" (i.e. Y endowed with the relative Borel structure as a subset of Z). Another way of saying this is that X is Borel isomorphic to the Borel space of a Polish space generated by the topology.

(e) A Borel measure on a Borel space X is a countably additive measure defined on the Borel sets of X and taking values in $[0,\infty]$. A subset A of a Borel space X is called μ-measurable, where μ is some Borel measure on X, if there are Borel sets B and C in X such that $B \subsetneq A \subsetneq C$ and $\mu(C \setminus B) = 0$.

(f) A Borel measure μ on a Borel space X is called standard if there is a standard Borel set A in X with $\mu(X \setminus A) = 0$.

For several of the definitions and concepts to follow, it is enough to have only a Borel space. However, several of the structure theorems on direct integrals (of von

Neumann algebras) require that the Borel space $(X,\mu) = (X,\mathcal{B},\mu)$ be standard and σ-finite. Therefore, we assume from here on that unless otherwise stated (X,μ) is such a space (standard and σ-finite).

Definition 0.3: A measurable field of Hilbert spaces over (X,μ) is a family $\{\mathcal{H}(x) : x \in X\}$ of Hilbert spaces indexed by X together with a subspace $\mathcal{B}$ of the product space $\prod_{x\in X} \mathcal{H}(x)$ with the following properties:

(i) For any $\xi \in \mathcal{B}$ the function $x \mapsto \|\xi(x)\|$ is μ-measurable.

(ii) For any $\eta \in \prod_{x\in X} \mathcal{H}(x)$, if the function $x \mapsto \langle\xi(x),\eta(x)\rangle \in \mathbb{C}$ is μ-measurable for every $\xi \in \mathcal{B}$, then $\eta \in \mathcal{B}$.

(iii) There exists a "fundamental sequence", i.e. a countable subset $\{\xi_1,\xi_2,...\}$ of $\mathcal{B}$ such that for almost every $x \in X$, the set $\{\xi_n(x) : n \in \mathbb{N}\}$ is total in $\mathcal{H}(x)$.

Note: By (iii) $\mathcal{H}(x)$ is separable a.e.

Terminology: Members of $\mathcal{B}$ are called measurable vector fields.

Lemma 0.4: Suppose that $\{\xi_n : n \in \mathbb{N}\} \subseteq \prod_{x\in X} \mathcal{H}(x)$ is a countable subset of the product space satisfying:

(i) $x \mapsto \langle\xi_n(x),\xi_m(x)\rangle$ is μ-measurable for all $m,n \in \mathbb{N}$

and (ii) the set $\{\xi_n(x) : n \in \mathbb{N}\}$ is total in $\mathcal{H}(x)$ a.e.

Then the set

$$\mathcal{B} = \{\xi \in \prod_{x\in X} \mathcal{H}(x) : x \mapsto \langle\xi(x),\xi_n(x)\rangle \text{ is } \mu\text{-measurable for all } n \in \mathbb{N}\}$$

satisfies the conditions of the previous defiition.

Proof: [Tak], page 270.

Now let $\mathcal{H}'$ be the collection $\mathcal{B}$ of measurable vector fields ξ such that

$$\|\xi\|^2 = \int_X \|\xi(x)\|^2 d\mu < \infty .$$

With respect to the natural point-wise linear operations, $\mathcal{H}'$ is a vector space and the sesqui-linear form

$$\langle \xi,\eta \rangle^2 = \int_X \langle \xi(x),\eta(x) \rangle \, d\mu$$

gives a Hilbert space in the usual way, that is, by identifying two fields $\xi,\eta \in \mathcal{H}'$ if $\xi(x) = \eta(x)$ a.e.

Definition 0.5: We call this Hilbert space $\mathcal{H}$ the direct integral of the measurable field of Hilbert spaces $\{\mathcal{H}(x) : x \in X\}$ and denote it by

$$\mathcal{H} = \int_X^{\oplus} \mathcal{H}(x) d\mu .$$

Each vector $\xi \in \mathcal{H}$ is written as

$$\xi = \int_X^{\oplus} \xi(x) d\mu$$

or sometimes just

$$x \mapsto \xi(x)$$

where it is understood that this is a representative for the equivalence class ξ. Note that when $\mathcal{H}(x) = \mathcal{H}_0$ for some fixed Hilbert space $\mathcal{H}_0$ then the field $\{\mathcal{H}(x) : x \in \mathcal{H}\}$ is called the constant field and we have an isomorphism

$$\int_X^{\oplus} \mathcal{H}_0 d\mu \equiv \mathcal{L}^2(X,\mu) \otimes \mathcal{H}_0 .$$

In working with direct integral constructions it is often conceptually useful to stress the geometric character of these objects. Notice that the construction giving $\int_X^{\oplus} \mathcal{H}_0 d\mu$ is really a fibre-bundle construction. In geometry one usually constructs a vector bundle so that the fibre spaces fit together topologically, and then one considers cross-sections which are continuous. For direct integrals, we want the fibre spaces to fit together measurably so that we can talk about measurable cross-sections. As indicated by

Lemma 0.4, a good way to get at this question of measurability is to select a sequence of cross-sections which behave measurably with respect to each other and span point-wise; then consider all cross sections which are measurable with respect to this sequence.

Direct Integrals of Operator Fields

We can now define operators which are compatible with the direct integral (fibre-bundle) structure.

Definition 0.6: Given two measurable fields of Hilbert spaces $(\{\mathcal{H}_1(x) : x \in X\}, \mathcal{S}_1)$ and $(\{\mathcal{H}_2(x) : x \in X\}, \mathcal{S}_2)$ an operator field

$$T : x \mapsto T(x) \in \mathcal{B}(\mathcal{H}_1(x), \mathcal{H}_2(x))$$

is called measurable if for any measurable vector field $\xi \in \mathcal{S}_1$, the vector field

$$x \mapsto T(x)\xi(x) \in \mathcal{H}_2(x) \text{ is measurable, i.e. belongs to } \mathcal{S}_2 .$$

If a measurable operator field is essentially bounded (in the sense that the function $x \mapsto \|T(x)\|$ is essentially bounded) then for each

$$\xi = \int_X^{\oplus} \xi(x) d\mu \in \int_X^{\oplus} \mathcal{H}_1(x) d\mu \quad ,$$

$$T\xi = \int_X^{\oplus} T(x)\xi(x) d\mu \in \int_X^{\oplus} \mathcal{H}_2(x) d\mu \quad .$$

We write this operator as

$$T = \int_X^{\oplus} T(x) d\mu$$

and call it the direct integral of the (essentially bounded) measurable operator field $x \mapsto T(x)$.

Definition 0.7: (a) We call operators of this form decomposable.

(b) If $T(x)$ is scalar a.e. then T is called a diagonal operator.

(c) The algebra of all diagonal operators is called the <u>diagonal algebra</u> and is denoted $\mathcal{L}$.

<u>Example</u>: Let (X,μ) be a standard σ-finite measure space, and fix $n \in \mathbb{N}$. Let $\mathcal{L}^\infty(X,\mu)$ be the von Neumann algebra of equivalence classes of essentially bounded measurable functions, and $\mathcal{M}_n(\mathcal{L}^\infty(X,\mu))$ be the algebra of $n\times n$ matrices over $\mathcal{L}^\infty(X,\mu)$. This algebra acts naturally on the Hilbert space $\mathcal{H} = \int_X^\oplus \mathbb{C}^n d\mu$ and is seen to constitute the set of all decomposable operators on $\mathcal{H}$, while its commutant is exactly $\mathcal{L}$, the algebra of diagonal operators. So, in matrices,

$$\mathcal{M}_n(\mathcal{L}^\infty(X,\mu)) = \left\{ \begin{pmatrix} \phi_{11} & \cdots & \phi_{1n} \\ \vdots & & \\ \phi_{n1} & & \phi_{nn} \end{pmatrix} : \phi_{ij} \in \mathcal{L}^\infty(X,\mu) \right\}$$

$$\mathcal{L} = \mathcal{M}_n(\mathcal{L}^\infty(X,\mu))' = \left\{ \begin{pmatrix} \phi & & 0 \\ & \ddots & \\ 0 & & \phi \end{pmatrix} : \phi \in \mathcal{L}^\infty(X,\mu) \right\} .$$

The algebra $\mathcal{M}_n(\mathcal{L}^\infty(X,\mu))$ is often called the algebra of <u>n-normal operators</u>. (See [R&R], section 7.5).

Since we have now defined the algebra $\mathcal{L}^\infty(X,\mu)$, we take this opportunity to introduce some notation which will be used regularly through the thesis. For $\phi \in \mathcal{L}^\infty(X,\mu)$, with representative $x \mapsto \phi(x)$, we say that ϕ is <u>essentially bounded away from zero</u> if there exists $\delta > 0$ such that $|\phi(x)| \geq \delta > 0$ a.e., and denote this by $|\phi| >> 0$.

A basic fact about decomposable operators concerns their norm, that is

$$\left\| \int_X^\oplus T(x)d\mu \right\| = \text{ess sup}\|T(x)\| .$$

By considering the algebra of n-normal operators, one might anticipate the next proposition.

Proposition 0.8: Let $\mathcal{H} = \int_X^{\oplus} \mathcal{H}(x)d\mu$. A bounded operator T on $\mathcal{H}$ is decomposable if and only if it commutes with the algebra $\mathcal{L}$ of diagonal operators.

Proofs can be found in [Tak] (IV, 7.10) or [Nielsen] (6.2). They both rest on the same idea that is used to prove $\mathcal{L}^\infty(X,\mu)$ is a maximal abelian subalgebra of $\mathcal{B}(\mathcal{L}^2(X,\mu))$. When $\mu(X) < \infty$ and T commutes with $\mathcal{L}$, then let M_1 be the diagonal operator determined by the function $x \mapsto 1$ and use the fact $TM_1 = M_1T$ to define $x \mapsto T(x)$. Then use that $TM_\phi = M_\phi T$ for all $\phi \in \mathcal{L}^\infty(X,\mu)$ to show $x \mapsto T(x)$ is essentially bounded. If $\mu(X)$ is not finite, let h be some fixed L^2-function whose essential range is in $(0,1)$.

Other facts we will be needing are that if $x \mapsto T(x)$ is a measurable field of operators, then

$x \mapsto \ker T(x)$

$x \mapsto [\mathrm{rg}\, T(x)]$ (= closure)

are both measurable fields of Hilbert spaces; and that if

$x \mapsto \mathcal{W}(x) \subseteq \mathcal{H}(x)$

defines a measurable field of subspaces, then

$x \mapsto P_{\mathcal{W}(x)}$ = the orthogonal projection onto $\mathcal{W}(x)$

is a measurable field of operators such that, with

$$\mathcal{W} = \int_X^{\oplus} \mathcal{W}(x)d\mu$$

we have

$$P_{\mathcal{W}} = \int_X^{\oplus} P_{\mathcal{W}(x)}d\mu \ .$$

Direct Integrals of von Neumann Algebras

Definition 0.9: Let $x \mapsto \mathcal{H}(x)$ be a measurable field of Hilbert spaces and $x \mapsto (\mathcal{M}(x)\mathcal{H}(x))$ be a family of von Neumann algebras $\mathcal{M}(x)$ acting on $\mathcal{H}(x)$. The field of von Neumann algebras

$$x \mapsto (\mathcal{M}(x), \mathcal{H}(x))$$

is then said to be measurable if there exists a countable family

$$x \mapsto T_n(x) \ , n \in \mathbb{N}$$

of measurable fields of operators such that $\mathcal{M}(x)$ is generated by

$$\{T_n(x) : n \in \mathbb{N}\}$$

for almost all $x \in X$.

Note: It is the Effros-Borel structure which imposes the countability condition. This structure is a certain "standard Borel" structure on the set of all von Neumann algebras acting on a Hilbert space $\mathcal{H}$. A good reference is [Nielsen], Chapter 17.

A result of fundamental importance is that if $x \mapsto (\mathcal{M}(x), \mathcal{H}(x))$ is a measurable field of von Neumann algebras and $\mathcal{M}(x)'$ is the commutant of $\mathcal{M}(x)$, then so is $x \mapsto (\mathcal{M}(x)', \mathcal{H}(x))$ a measurable field of von Neumann algebras. The next theorem concerns those decomposable operators determined by the field $(\mathcal{M}(x), \mathcal{H}(x))$ and relates this measurable field to the field of commutants.

Theorem 0.10: Let $\mathcal{M}$ be the set of decomposable operators

$$T = \int_X^{\oplus} T(x)d\mu \quad \text{on} \quad \mathcal{H} = \int_X^{\oplus} \mathcal{H}(x)d\mu$$

such that

$$T(x) \in \mathcal{M}(x) \text{ a.e.}$$

and write

$$\mathcal{M} = \int_X^{\oplus} \mathcal{M}(x)d\mu \ .$$

Then $\mathcal{M}$ is a von Neumann algebra $\mathcal{H}$, and $\mathcal{M}'$ is given by

$$\mathcal{M}' = \int_X^{\oplus} \mathcal{M}'(x)d\mu \ .$$

Moreover, the center $\mathcal{Z}(\mathcal{M})$ contains the diagonal algebra $\mathcal{L}$.

Definition 0.11: The von Neumann algebra $\mathcal{M}$ in the preceding theorem is called the direct integral of the measurable field $x \mapsto (\mathcal{M}(x), \mathcal{H}(x))$.

Corollary 0.12: Let $(\mathcal{M}, \mathcal{H}) = (\int_X^{\oplus} \mathcal{M}(x)d\mu \, , \int_X^{\oplus} \mathcal{H}(x)d\mu)$ be a direct integral of von Naumann algebras. Then the center of $\mathcal{M}$ is also a direct integral,

$$\mathcal{Z} = \int_X^{\oplus} \mathcal{Z}(x)d\mu$$

with

$$\mathcal{Z}(x) = \mathcal{M}(x) \cap \mathcal{M}(x)' \text{ a.e.}$$

In particular, $\mathcal{Z}$ coincides with the diagonal algebra $\mathcal{L}$ if and only if $\mathcal{M}(x)$ is a factor a.e.

On Morphisms and Direct Integrals

Suppose that $\mathcal{Q}$ is a C*-algebra and $\{\mathcal{H}(x) : x \in X\}$ is a measurable field of Hilbert spaces over the standard σ-finite Borel space (X,μ). Suppose that for almost every $x \in X$ there is a representation $\pi(x)$ of $\mathcal{Q}$ on $\mathcal{H}(x)$ such that for each $A \in \mathcal{Q}$, the operator field $x \mapsto \pi(x)(A)$ is measurable. The field $\{\pi(x) : x \in X\}$ of representations is said to be measurable. We set

$$\pi(A) = \int_X^{\oplus} \pi(x)(A)d\mu \quad \text{on} \quad \mathcal{H} = \int_X^{\oplus} \mathcal{H}(x)d\mu \ .$$

Clearly π is a representation of $\mathcal{Q}$.

Definition 0.13: The representation π is called th direct integral of the field $\{\pi(x) : x \in X\}$ and is written as

$$\pi = \int_X^{\oplus} \pi(x) d\mu \ .$$

Theorem 0.14: Let $\mathcal{A}$ be a separable C*-algebra and let

$$\mathcal{H} = \int_X^{\oplus} \mathcal{H}(x) d\mu \ .$$

If π is a representation of $\mathcal{A}$ on $\mathcal{H}$ such that $\pi(\mathcal{A})$ commutes with the diagonal algebra, then there exists a measurable field $\{\pi(x) : x \in X\}$ of representations, essentially unique, such that

$$\pi = \int_X^{\oplus} \pi(x) d\mu \ .$$

Theorem 0.15: Let $\mathcal{A}$ be a separable C*-algebra and (X,μ) be a standard σ-finite measure space. For $j = 1,2$ let $\{(\pi_j(x), \mathcal{H}_j(x)) : x \in X\}$ be measurable fields of representations of $\mathcal{A}$ (respectively let $(\mathcal{M}_j(x), \mathcal{H}_j(x))$ be measurable fields of von Neumann algebras) such that

$$(\pi_j, \mathcal{H}) = \left(\int_X^{\oplus} \pi_j(x) d\mu \ , \int_X^{\oplus} \mathcal{H}_j(x) d\mu\right)$$

(resp. $$(\mathcal{M}_j, \mathcal{H}_j) = \left(\int_X^{\oplus} \mathcal{M}_j(x) d\mu \ , \int_X^{\oplus} \mathcal{H}_j(x) d\mu\right)) \ .$$

If $\pi_1(x)$ is unitarily equivalent to $\pi_2(x)$ (resp. $(\mathcal{M}_1(x), \mathcal{H}_1(x))$ is unitarily equivalent to $(\mathcal{M}_2(x), \mathcal{H}_2(x))$ a.e., then there exists a measurable field $x \mapsto U(x)$ of unitary operators such that

$$U(x)\pi_1(x)(A)U(x)^* = \pi_2(x)(A) \ , \ A \in \mathcal{A}$$

(resp. $$U(x)\mathcal{M}_1(x)U(x)^* = \mathcal{M}_2(x)) \text{ a.e.}$$

Hence the unitary operator

$$U = \int_X^{\oplus} U(x) d\mu$$

implements the unitary equivalence of π_1 and π_2 (resp. $\mathcal{M}_1$ and $\mathcal{M}_2$).

Definition 0.16: In this case we say that π_1 and π_2 (resp. $\mathcal{M}_1$ and $\mathcal{M}_2$) are direct integral unitarily equivalent. Sometimes we write $\pi_1 \underset{\oplus}{\equiv} \pi_2$ (resp. $\mathcal{M}_1 \underset{\oplus}{\equiv} \mathcal{M}_2$).

Proposition 0.17: Suppose that we have two direct integrals

$$\mathcal{M}_1 = \int_X^\oplus \mathcal{M}_1(x)d\mu$$

$$\mathcal{M}_2 = \int_X^\oplus \mathcal{M}_2(x)d\mu$$

of von Neumann algebras. If $\mathcal{M}_1(x)$ is isomorphic to $\mathcal{M}_2(x)$ a.e. then there exists a measurable field of isomorphisms $\pi(x)$ such that $\pi(x)\mathcal{M}_1(x) = \mathcal{M}_2(x)$ a.e.; hence the direct integral

$$\pi = \int_X^\oplus \pi(x)d\mu$$

is an isomorphism of $\mathcal{M}_1$ and $\mathcal{M}_2$.

Corollary 0.18: Suppose that

$$\mathcal{M} = \int_X^\oplus \mathcal{M}(x)d\mu$$

is a direct integral of von Neumann algebras. If each $\mathcal{M}(x)$ is isomorphic to a fixed $\mathcal{M}_0$, then

$$\mathcal{M} \text{ is isomorphic to } \mathcal{L}^\infty(X,\mu) \,\overline{\otimes}\, \mathcal{M}_0,$$

the von Neumann algebra tensor product.

Chapter 1

The Class of n-normal Operators

Section 1.1

We begin by considering operators which are defined by matrices of bounded measurable functions. More precisely, let (X,μ) be a standard Borel space with positive measure μ, and $\mathcal{M}_n(\mathbb{C})$ be the $n \times n$ matrices over the complex numbers, $\mathbb{C}$. Then $\mathcal{M}_n(\mathcal{L}^\infty(X,\mu))$ denotes the algebra $\mathcal{L}^\infty(X,\mu) \otimes \mathcal{M}_n(\mathbb{C})$ and may be identified with the algebra of $n \times n$ matrices over the von Neumann algebra $\mathcal{L}^\infty(X,\mu)$. Operators of this class are often called n-normal operators, and alternatively the algebra can be realized as the commutant of $\mathcal{L}^\infty(X,\mu)^{(n)}$ (the n-fold inflation) acting on the Hilbert space $\mathcal{L}^2(X,\mu)^{(n)}$ (see [R&R], Ch. 7). In terms of direct integrals, $\mathcal{M}_n(\mathcal{L}^\infty(X,\mu)) = \int_X^\oplus \mathcal{M}_n(\mathbb{C})d\mu$ (see def. 0.11).

Our first result generalizes a theorem of Ballantine ([B]) and characterizes products of five positive operators in $\mathcal{M}_n(\mathcal{L}^\infty(X,\mu))$. The proof is partly based on some ideas of A.R. Sourour found in his 1986 paper ([S]). There he gave an elementary and short proof of a certain factorization theorem for matrices. Correspondingly short proofs of various known results followed as corollaries. These included Ballantine's theorem on products of five positive definite matrices, as well as the commutator theorem of Shoda-Thompson for fields with sufficiently many elements. Our main technical theorem is an extension of Sourour's factorization theorem to the class of n-normal operators. One way of saying this is that Sourour's theorem is measurable (in an appropriate sense). For completeness, we include in section 1.2 the "n-normal analogues" to the corollaries mentioned.

To move into the more technical details, let $\mathcal{L} = \mathcal{L}(\mathcal{M}_n(\mathbb{C}))$ be the algebra of scalar multiples of the identity. For any matrix M in $\mathcal{M}_n(\mathbb{C})$, define the distance from M to the closed set $\mathcal{L}$ in the usual way.

$$d(M,\mathcal{L}) = \inf\{\|M-D\| : D \in \mathcal{L}\},$$

where $\| \ \|$ is the operator norm in $\mathcal{M}_n(\mathbb{C})$.

Now, the algebra $\mathcal{M}_n(\mathcal{L}^\infty(X,\mu))$ is naturally identified with $\int_X^\oplus \mathcal{M}_n(\mathbb{C})d\mu$, so given A in $\mathcal{M}_n(\mathcal{L}^\infty(X,\mu))$, we choose a measurable map from X to $\mathcal{M}_n(\mathbb{C})$ which is a representative for A (see def.). This allows us to define the "essential range" of A,

$$\text{ess rg } A = \{M \in \mathcal{M}_n(\mathbb{C}) : \mu\{x : \|A(x)-M\| < \varepsilon\} > 0 \text{ for all } \varepsilon > 0\}.$$

For the first lemma we give an alternate characterization of ess rg A.

<u>Lemma 1.1</u>. Let A be an n-normal operator. Then

$$\text{ess rg } A = \cap\{\text{clos } A(Y) : Y \text{ is measurable and } \mu(X\backslash Y) = 0\}.$$

<u>Proof</u>: Suppose $M \in$ ess rg A and let Y be a measurable set with $\mu(X\backslash Y) = 0$. Then $\mu(Y \cap \{x : \|T(x)-M\| < \varepsilon\}) > 0$ for every $\varepsilon > 0$, so that $M \in$ clos T(Y).

If $M \notin$ ess rg A, then there exists $\varepsilon_0 > 0$ such that $\mu\{x : \|A(x)-M\| < \varepsilon_0\} = 0$. Therefore $M \notin \text{clos}(X\backslash\{x : \|T(x)-M\| < \varepsilon_0)$, which implies that $M \notin \cap\{\text{clos } A(Y) : Y$ is measurable and $\mu(X\backslash Y) = 0\}$.

Some notation of which we will occasionally make use is the following: for a set of matrices $\mathcal{M}$

$$\sigma(\mathcal{M}) = \cup\{\sigma(M) : M \in \mathcal{M}\}.$$

In particular, we have

<u>Lemma 1.2</u>. Let A be an n-normal operator.

(a) If Y is a measurable subset of X, then

$$\sigma(\text{clos } A(Y)) = \text{clos } \sigma(A(Y)).$$

(b) Suppose that $\lambda \in \mathbb{C}$ and $\varepsilon > 0$. If $d(\sigma(A(x)),\lambda) < \varepsilon$ a.e. and $M \in \text{ess rg } A$, then $d(\sigma(M),\lambda) < \varepsilon$.

Proof: (a) Suppose that $\lambda \in \sigma(M)$, $M = \lim_n A(y_n)$, $y_n \in Y$. Then by finite dimensionality, $\sigma(M) = \lim_n \sigma(A(y_n))$. Therefore $\lambda \in \text{clos } \sigma(A(Y))$.

If $\lambda \in \text{clos } \sigma(A(Y))$, then there exists a sequence $A(y_n)$, $y_n \in Y$ such that $d(\lambda,\sigma(A(y_n))) \to 0$. Choose a convergent subsequence $A(y_{n_j})$ with limit M say. Then again, by continuity of σ in finite dimensions, $\lambda \in \sigma(M)$ and $M \in \text{clos } A(Y)$.

(b) Let $Y = \{x: d(\sigma(A(x),\lambda) < \varepsilon\}$. Then $\mu(X \backslash Y) = 0$. By Lemma 1.1, $M \in \text{clos } A(Y)$, so there exists $y_n \in Y$, $n = 1,2,3,\ldots$ such that $M = \lim_n A(y_n)$. Because σ is continuous, $d(\sigma(M),\lambda) < \varepsilon$, as claimed.

Lemma 1.3. An operator A in $\mathfrak{M}_n(\mathcal{L}^\infty(X,\mu))$ is invertible if and only if $|\det A(x)| \geq \varepsilon > 0$ a.e., for some $\varepsilon > 0$.

Proof: Lemma 1.1 follows easily from Theorem 7.20 of [R&R], which says that n-normal operators may be "triangularized". For a triangular matrix of $\mathcal{L}^\infty$-functions is invertible if and only if the diagonal entries are invertible.

We can now state our technical result. As above, $\mathcal{L}$ is the diagonal algebra of $\mathfrak{M}_n(\mathbb{C})$.

Theorem 1.4. Let A be invertible in $\mathfrak{M}_n(\mathcal{L}^\infty(X,\mu))$ and $\beta_1,\ldots,\beta_n$, $\gamma_1,\ldots,\gamma_n$ be bounded measurable functions such that

$$\prod_{j=1}^{n} (\beta_j\gamma_j)(x) = \det A(x) \quad \text{a.e.}$$

Suppose also that

$$\text{ess rg } A \cap \mathcal{L} = \phi . \tag{1}$$

Then there are operators B , C and Q in $\mathfrak{M}_n(\mathcal{L}^\infty(X,\mu))$, with Q invertible and B , C of the form

$$B = \begin{pmatrix} \beta_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ * & & & \cdot \\ & & & & \beta_n \end{pmatrix} , \quad C = \begin{pmatrix} \gamma_1 & & & * \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \cdot \\ & & & & \gamma_n \end{pmatrix}$$

so that

$$A = Q^{-1}(BC)Q = (Q^{-1}BQ)(Q^{-1}CQ) .$$

Proof: (by induction on n). The case $n = 1$ is obvious.

The case $n = 2$: By Theorem 7.20 of [K&R], n-normal operators may be triangularized (unitarily), so there is no loss of generality in assuming that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} , \quad a_{ij} \in \mathcal{L}^\infty(X,\mu) .$$

It will be shown that A is similar to an operator of the form

$$\begin{pmatrix} \beta_1\gamma_1 & y \\ z & r \end{pmatrix} .$$

From here the determinant condition uniquely determines the coefficient r , and the required factorization

$$\begin{pmatrix} \beta_1\gamma_1 & y \\ z & r \end{pmatrix} = \begin{pmatrix} \beta_1 & 0 \\ \gamma_1^{-1}z & \beta_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & \beta_1^{-1}y \\ 0 & \gamma_2 \end{pmatrix}$$

is established. So for the case $n = 2$, it remains to prove that

$$A \text{ is similar to } \begin{pmatrix} \beta_1\gamma_1 & y \\ z & r \end{pmatrix} .$$

To this end, write

$$A - \beta_1\gamma_1 = \begin{pmatrix} \phi_1 & \psi \\ 0 & \phi_2 \end{pmatrix}$$

where $\phi_i = a_{ii} - \beta_1\gamma_1$, $i = 1,2$, and $\psi = a_{12}$. Observe that $\phi_1 - \phi_2 = a_{11} - a_{22}$. Also, by standard arguments we assume (X,μ) is finite, and $\mu(X) < \infty$ implies that $\mathcal{L}^\infty(X,\mu) \subsetneq \mathcal{L}^2(X,\mu)$. So all that we need is

(i) $e_1 \in \mathcal{L}^\infty(X,\mu) \oplus \mathcal{L}^\infty(X,\mu) \subsetneq \mathcal{L}^2(X,\mu) \oplus \mathcal{L}^2(X,\mu)$

such that for

$$e_2 = (A-\beta_1\gamma_1)\, e_1$$

(ii) the operator G given by the matrix-valued function

$$x \mapsto (e_1(x), e_2(x))$$

is invertible in $\mathcal{M}_2(\mathcal{L}^\infty(X,\mu))$. For then the required similarity would follow from the computation

$$G^{-1}AG = \begin{pmatrix} \beta_1\gamma_1 & y \\ 1 & r \end{pmatrix} .$$

How do we produce the vector e_1 ? It is enough to choose e_1 so that $e_1(x)$ is almost everywhere not an iegenvector (see [S]). To accomplish this we partition the measure space into two measurable subsets, namely,

$$\mathcal{D} = \{x \in X : |a_{11}(x) - a_{22}(x)| \geq \tfrac{\delta}{2}\}$$

and

$$\mathcal{D}^c = \{x \in X : |a_{11}(x) - a_{22}(x)| < \tfrac{\delta}{2}\} ,$$

where $\delta : = d(\text{essrg } A, \mathcal{L}) > 0$.

Clearly $\mathcal{D}$ and $\mathcal{D}^c$ are both measurable, and we have $\mathcal{D}^c \cap \mathcal{D} = \phi$, $\mathcal{D}^c \cup \mathcal{D} = X$.

The idea behind the partition is that on $\mathcal{D}$, the operator $A - \beta_1\gamma_1$ is similar to an "essentially non-scalar" diagonal matrix, while over $\mathcal{D}^c$, condition (1) forces ψ , the north-east corner of $A - \beta_1\gamma_1$, to be "essentially away from zero". Each of these situations can be dealt with and then, since the sets $\mathcal{D}$ and $\mathcal{D}^c$ are measurable we can put the pieces together to obtain our solution. We make this rigorous.

Over the set $\mathcal{D}$ define H in $\mathcal{M}_2(\mathcal{L}^\infty(\mathcal{D},\mu))$ by

$$H = \begin{pmatrix} 1 & \frac{-\psi}{\phi_1-\phi_2} \\ 0 & 1 \end{pmatrix}.$$

Notice that, restricted to $\mathcal{D} \subseteq X$, H is invertible. This is because, for almost all x in $\mathcal{D}$, $|\phi_1(x) - \phi_2| \geq \frac{\delta}{2} > 0$. Hence H is in $\mathcal{M}_2(\mathcal{L}^\infty(\mathcal{D},\mu))$ and Lemma 1.3 applies. Also, it is easy to see that

$$H^{-1}AH = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}.$$

This suggests that a good candidate for the vector e_1 is

$$e_1 = H\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

And a calculation shows that, for $x \in \mathcal{D}$,

$$\begin{aligned} \det(e_1(x),e_2(x)) &= \det(e_1(x),(A-\beta_1\gamma_1)e_1(x)) \\ &= \det\left(H(x)\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ H(x)\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}\right) \\ &= \det H(x) \cdot (\phi_2(x) - \phi_1(x)) \\ &= (a_{22}(x) - a_{11}(x)). \end{aligned}$$

Therefore, $|\det(e_1(x), e_2(x))| \geq \frac{\delta}{2}$, and by Lemma 1.3 again, (e_1,e_2) defines an invertible operator in $\mathcal{M}_2(\mathcal{L}^\infty(\mathcal{D},\mu))$.

So we are half-way there. We must still obtain e_1 defined over the set $\mathcal{D}^c$. For this situation, a good example to keep in mind is the matrix of complex numbers

$$M = \begin{pmatrix} a & \varepsilon \\ 0 & a \end{pmatrix}, \quad a \neq 0, \ \varepsilon \neq 0.$$

Here

$$\begin{pmatrix} a & \varepsilon \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \varepsilon \\ a \end{pmatrix},$$

making the matrix

$$\begin{pmatrix} 0 & \varepsilon \\ 1 & a \end{pmatrix}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, M\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

invertible.

Over $\mathcal{D}^c$,

$$A - \beta_1\gamma_1 = \begin{pmatrix} \phi_1 & \psi \\ 0 & \phi_2 \end{pmatrix}$$

has $|\phi_1(x) - \phi_2| < \frac{\delta}{2}$.

Therefore, modulo $\frac{\delta}{2}$ and the entry ψ, $A - \beta_1\gamma_1$ is diagonal. But as mentioned earlier, condition (1) in the statement of the theorem forces $|\psi|$ to be essentially bounded above zero. So, taking our matrix M as a cue, we define

$$e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{so that} \quad e_2 = (A - \beta_1\gamma_1)e_1 = \begin{pmatrix} \psi \\ \phi_2 \end{pmatrix}.$$

Clearly (e_1,e_2) defines an element of $\mathcal{M}_2(\mathcal{L}^\infty(\mathcal{D},\mu))$. Also, $\det(e_1,e_2) = -\psi$.

So to complete the proof it just remains to show that $|\psi|$ is "essentially bounded above zero", that is, $\exists\varepsilon > 0$ such that $|\psi(x)| \geq \varepsilon > 0$ a.e. But if this were not the case, by the measurability of the functions concerned, there would exist a set $P \subsetneq X$ of positive measure with the property that for x in P

$$\|(A - \phi_1)(x)\| = \left\|\begin{pmatrix} 0 & \psi(x) \\ 0 & \phi_2(x) - \phi_1(x) \end{pmatrix}\right\| < \frac{3}{4}\delta, \quad \delta = d(\operatorname{essrg} A, \mathcal{L}),$$

which would yield a contradiction to (1). To finish the case $n = 2$, simply define G to be the invertible operator given by

$$G(x) = (e_1(x), e_2(x)).$$

<u>The induction</u>: $(n \geq 3)$

Because of its length, we first give an outline of the proof.

Assume that $n \geq 3$ and that we have the theorem for $n - 1$. Invoking theorem 7.20 of [R&R] again, we suppose, without loss of generality, that A is "upper-triangular",

$$A = \begin{pmatrix} a_{11} & \cdot & \cdot & \cdot & \cdot & a_{1n} \\ & \cdot & & & & \cdot \\ & & \cdot & & & \cdot \\ & 0 & & \cdot & & \cdot \\ & & & & \cdot & \cdot \\ & & & & & a_{nn} \end{pmatrix}, \quad a_{ij} \in \mathcal{L}^\infty(X,\mu) .$$

The main thing is to show that A is similar to an operator of the form

$$\begin{pmatrix} \beta_1\gamma_1 & u \\ z & S \end{pmatrix} \tag{2}$$

where $u = (u_1,\ldots,u_{n-1})$

$z = (z_1,\ldots,z_{n-1})^t \;,\; u_i,z_i \in \mathcal{L}^\infty(X,\mu)$,

and the operator

$$S - \frac{1}{\beta_1\gamma_1} zu'$$

satisfies condition (1) of the Theorem for $\mathfrak{M}_{n-1}(\mathcal{L}^\infty(X,\mu))$.

For then we use the induction hypothesis on S together with some algebra to get the result for A.

<u>Remark</u>: The motivation for considering the operator $S - \frac{1}{\beta_1\gamma_1} zu'$ is the following factorization, a special case of which is to be found in [GPR]: In general, given γ in $\mathbb{C}\backslash\{0\}$, we have the matrix equation

$$\begin{pmatrix} \gamma & C \\ B & D \end{pmatrix} = \begin{pmatrix} \gamma & 0 \\ B & D-\gamma^{-1}BC \end{pmatrix} \begin{pmatrix} 1 & \gamma^{-1}C \\ 0 & I \end{pmatrix} .$$

Applying this to the matrix at (2), we get

$$\begin{pmatrix} \beta_1\gamma_1 & u' \\ z & S \end{pmatrix} = \begin{pmatrix} \beta_1 & 0 \\ z & S-(\beta_1\gamma_1)^{-1}zu' \end{pmatrix} \begin{pmatrix} 1 & (\beta_1\gamma_1)^{-1}u' \\ 0 & I \end{pmatrix} ,$$

which yields to algebraic manipulations.

The similarity at (2) is obtained in two steps. We first show that there is some R in $\mathfrak{M}_n(\mathcal{L}^\infty(X,\mu))$ such that A is similar to

$$A_1 = \begin{pmatrix} \beta_1\gamma_1 & y' \\ 1 & \\ 0 & R \\ \vdots & \\ 0 & \end{pmatrix} . \qquad (3)$$

The second step is then showing that A_1 is similar to the special operator

$$\begin{pmatrix} \beta_1\gamma_1 & u' \\ z & S \end{pmatrix}$$

given in (2).

This is all accomplished in the same spirit as the case $n = 2$, but the division of the measure space X must be done with somewhat more care.

We now begin the rigorous proof of the induction. Consider

$$A - \beta_1\gamma_1 = \begin{pmatrix} \phi_1 & \psi_{12} & \cdot & \cdot & \cdot & \psi_{1n} \\ & \cdot & & & & \cdot \\ & & \cdot & & & \cdot \\ & 0 & & \cdot & & \cdot \\ & & & & \cdot & \psi_{n-1,n} \\ & & & & & \phi_n \end{pmatrix} ,$$

where $\phi_i = a_{ii} - \beta_1\gamma_1$, $i = 1,2,\ldots,n$

and $\psi_{ik} = a_{ik}$, $i < k \leq n$.

Step 1:

(A is similar to A_1 — see equation (3).) Let

$$\mathcal{D} = \{x \in X : |\phi_i(x) - \phi_j(x)| \geq \frac{\delta}{2} \text{ for some } i \neq j\}$$

and

$$\mathcal{D}^c = \{x \in X : |\phi_i(x) - \phi_j(x)| < \frac{\delta}{2} \text{ for all } i, j\} ,$$

where $\delta = d(\text{ess rg } A, \mathcal{L})$.

As in the case $n = 2$, we deal with A defined over $\mathcal{D}$ and $\mathcal{D}^c$ separately.

Over $\mathcal{D}^c$: Let

$$\Psi_i = \begin{pmatrix} \psi_{1i} \\ \vdots \\ \psi_{i-1,i} \\ 0 \\ \vdots \\ 0 \end{pmatrix} , \quad i = 2,\ldots,n$$

be $\mathcal{L}^\infty$-column vectors, so that the n-normal operator matrix

$$(0, \Psi_2, \Psi_3, \ldots, \Psi_n)$$

is the strictly "upper-triangular" part of $A - \beta_1\gamma_1$.

Let $\mathcal{S}_i = \{x \in \mathcal{D}^c : \|\Psi_i(x)\| \geq \|\Psi_j(x)\| \ , \ j = 2,\ldots,n\}$, where $\|\cdot\|$ is the standard Euclidean norm. It now follows that restricted to $\mathcal{S}_i$, $\|\Psi_i(x)\|$ is essentially bounded away from zero. For otherwise, since $\|\Psi_i(x)\|$ dominates the other columns, we would obtain a set of positive measure over which $A - \beta_1\gamma_1$ is close to $\mathcal{L}$ relative to δ , contradicting our hypothesis, condition (1). Since

$$\bigcup_{i=2}^{n} \mathcal{S}_i = X$$

we may define new measurable sets $\mathcal{T}_i$ by

$$\mathcal{T}_2 = \mathcal{S}_2$$
$$\mathcal{T}_3 = \mathcal{S}_3 \setminus \mathcal{S}_2$$
$$\vdots \quad \vdots \quad \vdots$$
$$\mathcal{T}_n = \mathcal{S}_n \setminus (\mathcal{S}_2 \cup \ldots \cup \mathcal{S}_{n-1})$$

so that

$$\cup\, \mathcal{T}_i = \cup\, \mathcal{S}_i \quad \text{but} \quad \mathcal{T}_i \cap \mathcal{T}_j = \phi \quad \text{for} \quad i \neq j .$$

We now require even a finer partition, and to do this we need to consider the "essentially best lower bounds"

$$\eta_i = \text{ess inf}\{\|\Psi_i(x)\| : x \in \mathcal{T}_i\} .$$

The new partition is obtained by setting

$$\mathcal{T}'_{ik} = \{x \in \mathcal{T}_i : |\psi_{ki}(x)|^2 \geq \eta_i^2/(n-1)\} \quad , \quad k = 1,2,\ldots,i-1 , \qquad (*)$$

and defining measurable sets $\mathcal{T}_{ik}$ so that

$$\mathcal{T}_{i1} \subseteq \mathcal{T}'_{ik}$$
$$\mathcal{T}_{ik} \cap \mathcal{T}'_{i\ell} = \phi \quad \text{for} \quad k \neq \ell$$

and

$$\bigcup_{k=1}^{i-1} \mathcal{T}_{ik} = \bigcup_{k=1}^{i-1} \mathcal{T}'_{ik} = \mathcal{T}_i$$

On $\mathcal{T}_{ik}$ define

$$e_1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - i^{th} \quad \text{place} .$$

Hence

$$e_2 := (A - \beta_1\gamma_1)e_1 = \begin{pmatrix} \psi_{1i} \\ \vdots \\ \psi_{i-1i} \\ \phi_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \Psi_i + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \phi_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} - i^{\text{th}} \text{ place.}$$

We will now define column vectors of $\mathcal{C}^\infty$-functions $e_3,\dots,e_n$ so that

$$G_{ik} = (e_1,\dots,e_n)$$

will be invertible in $\mathcal{M}_n(\mathcal{C}^\infty(\mathcal{T}_{ik},\mu))$. For e_3 choose any $r \neq k,i$ and let

$$e_3 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \overline{\psi}_{ki} \\ 0 \\ \vdots \\ 0 \end{pmatrix} - r^{\text{th}} \text{ place .}$$

For e_4 ,

$$e_4 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with a "1" <u>not</u> in the r^{th}, k^{th} or i^{th} place. Define $e_5,\dots,e_n$ similarly. It follows by (*) that

$$|\det(e_1,\dots,e_n)(x)| = |\psi_{ki}(x)|^2 \geq \eta_i^2/(n-1) \ , \ \ k = 1,\dots,i-1 \ .$$

Finally, let $\eta = \min\{\eta_i : i = 2,\ldots,n\}$ and define

$$G(x) = G_{ik}(x) \quad \text{for} \quad x \in \mathcal{T}_{ik} \ .$$

Since $\mathcal{D}^c$ is equal to the disjoint union $\bigcup_{i=2}^{n} \bigcup_{k=1}^{i-1} \mathcal{T}_{ik}$, G gives us an element of $\mathcal{M}_n(\mathcal{L}^\infty(\mathcal{D}^c,\mu))$. Moreover,

$$|\det G(x)| \geq \eta^2/(n-1) \ .$$

Hence G is invertible.

But by definition, $e_2 = (A-\beta_1\gamma_1)\, e_1$, therefore, restricted to $\mathcal{D}^c$

$$G^{-1}AG = \begin{pmatrix} \beta_1\gamma_1 & * & \cdots & * \\ 1 & * & \cdots & * \\ 0 & & & \\ \vdots & & & \vdots \\ 0 & * & \cdots & * \end{pmatrix} .$$

So we have the required similarity over $\mathcal{D}^c$, and to complete Step 1 of the case $n \geq 3$ we must turn our attention to the set $\mathcal{D}$.

Over $\mathcal{D}$: By definition

$$\mathcal{D} = \{x \in X : |\phi_i(x)-\phi_j(x)| \geq \tfrac{\delta}{2} \ \text{for some} \ i \neq j\} ,$$

$$\delta = \text{ess } d(A,\mathcal{L}) \ .$$

Let $\mathcal{D}_{ij} = \{x \in \mathcal{D} : |\phi_i(x)-\phi_j(x)| \geq \frac{\delta}{2}\}$, $i,j = 1,2,\ldots,n$, $i \neq j$,

so that

$$\mathcal{D} = \bigcup_{i<j} \mathcal{D}_{ij}$$

Now choose pairwise disjoint measurable sets $\mathcal{E}_{ij} \subsetneq \mathcal{D}_{ij}$ such that

$\bigcup_{i,j} \mathcal{E}_{ij} = \mathcal{D}$. We will define e_1 on $\mathcal{E}_{ij}$. For each i,j, $i < j$, let P_{ij} be the orthogonal projection onto the space $\mathcal{H}_{ij}$ generated by the vectors

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - i^{th} \text{ place} \;, \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - j^{th} \text{ place} \;.$$

Thus

$$P_{ij}(A - \beta_1\gamma_1)P_{ij} = \begin{pmatrix} 0 & & 0 & & 0 \\ & \phi_i & & \psi_{ij} & \\ 0 & & 0 & & 0 \\ & & & \phi_j & \\ 0 & & 0 & & 0 \end{pmatrix} .$$

Invoking the case $n = 2$, we find e_1 such that e_1 and $e_2 := P_{ij}(A-\beta_1\gamma_1)P_{ij}e_1$ define an element of $\mathcal{M}_2(\mathcal{L}^\infty(\mathcal{F}_k,\mu))$ satisfying

$$|\det_{\mathcal{H}_{ij}}(e_1(x), e_2(x)| \geq |\phi_i(x) - \phi_j(x)) \geq \frac{\delta}{2} .$$

Define

$$e_3 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - r^{th} \text{ place} \;, \quad r \neq i,j \;, \quad \text{etc.}$$

Then a calculation shows that

$$|\det(e_1(x),\ldots,e_n(x))| \geq |\phi_i(x) - \phi_j(x)) \geq \frac{\delta}{2} ,$$

and it follows that for $x \in \mathcal{E}_{ij}$,

$$G_{ij}(x) = (e_1(x),\ldots,e_n(x))$$

defines an invertible operator in $\mathcal{M}_n(\mathcal{L}^\infty(\mathcal{F}_k,\mu))$.

Forming the disjoint union $\dot{\bigcup} \mathcal{E}_{ij}$ we retrieve $\mathcal{D}$. Therefore, the operator

$$G(x) = G_{ij}(x) \quad \text{for} \quad x \in \mathcal{E}_{ij}$$

$$G = (e_1,e_2,\dots,e_n)$$

yields an invertible element of $\mathcal{M}_n(\mathcal{L}^\infty(\mathcal{D},\mu))$ such that

$$e_2 = (A - \beta_1\gamma_1)\, e_1 \ .$$

Combining this with the operator already defined over $\mathcal{D}^c$ we obtain an invertible G in $\mathcal{M}_n(\mathcal{L}^\infty(X,\mu))$ for which

$$G^{-1}AG = \begin{pmatrix} \beta_1\gamma_1 & y' \\ 1 & \\ 0 & R \\ \vdots & \\ 0 & \end{pmatrix} := A_1$$

as required.

This completes Step 1 of the proof.

Step 2. (A_1 is similar to $\begin{pmatrix} \beta_1\gamma_1 & u' \\ z & S \end{pmatrix}$ - see equation (3).)

This part of the proof consists in showing that

$$A_1 := \begin{pmatrix} \beta_1\gamma_1 & y' \\ z & R \end{pmatrix} , \quad z = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

is similar to an operator of the form

$$\begin{pmatrix} \beta_1\gamma_1 & u' \\ z & S \end{pmatrix}$$

with $S - \frac{1}{\beta_1\gamma_1} zu'$ satisfying condition (1) in $\mathcal{M}_{n-1}(\mathcal{L}^\infty(X,\mu))$. If $R - \frac{1}{\beta_1\gamma_1} zy'$ already satisfies (1) then there is nothing to prove. Otherwise write $\gamma = \beta_1\gamma_1$ and

$$A_1 = \begin{pmatrix} \gamma & y' \\ z & R \end{pmatrix} .$$

In this case we claim that we can find $w' = (w_1,\dots,w_{n-1})$ so that the similarity is implemented by an operator of the form

$$\begin{pmatrix} 1 & w' \\ 0 & I \end{pmatrix} = P .$$

To focus our search for w', suppose such a vector exists and consider

$$P^{-1}A_1P = \begin{pmatrix} \gamma-\langle w',z\rangle & (\gamma-\langle w',z\rangle)w'+y'-w'R \\ \\ z & zw'+R \end{pmatrix}$$

where $\langle f,g\rangle(x) = \Sigma\ f_i(x)\overline{g_i}(x)$, f,g in $\mathbb{C}^{n-1}(\mathcal{L}^2(X,\mu))$. So it would be enough to find w' for which

(a) $\langle w',z\rangle = 0$

and (b) $(zw'+R) - \frac{1}{\gamma}z(\gamma w'+y'-w'R)$ satisfies (1),

that is, for which

(a) $\langle w',z\rangle = 0$

and (b) $R - \frac{1}{\gamma}zy' + \frac{1}{\gamma}zw'R$ satisfies (1) .

Let us digress a moment.

Roughly speaking we need w' which is perpendicular to z and such that the almost everywhere rank-one operator $\frac{1}{\gamma}zw'R$ keeps $(R - \frac{1}{\gamma}zy') + \frac{1}{\gamma}zw'R$ away from the scalars. It is helpful at this point to have the matrices in mind:

$$z = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} , \quad zy' = \begin{pmatrix} y_1 & \cdots & y_{n-1} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

$$zw' = \begin{pmatrix} z_1w_1 & \cdots & z_1w_{n-1} \\ 0 & & 0 \\ \vdots & & \vdots \\ & & \\ 0 & \cdots & 0 \end{pmatrix} = (w_1z,\dots,w_{n-1}z)$$

$$zw'R = (\langle w',R^1\rangle\, z,\dots,\langle w',R^{n-1}\rangle\, z)\ ,$$

$$R = (R^1,\dots,R^{n-1})\ ,\ R^i = i^{th}\text{ column of } R\ ,$$

$$R = (r_{ij})_{i,j=1,\dots,n}\ .$$

In the case where X is a singleton, all operators are just matrices of complex numbers. The matrix A being invertible implies that not all columns of R are multiples of z. Hence there exists a column of R (R^1 say) such that z and R^1 are linearly independent. So with w' a sufficiently large positive multiple of the component of R^1 which is perpendicular to z, it is ensured both that $(R - \frac{1}{\gamma}zy' + zw'R)$ is non-scalar and that $\langle w',z\rangle = 0$. The object in what follows is to do this measurably, while keeping the norm of $w'(x)$ bounded.

We require two lemmas and some notation.

Lemma 1.5. Let z be in $\mathbb{C}^{n-1}(\mathcal{L}^2(X,\mu))$. Then there exists an orthogonal projection E in $\mathcal{M}_{n-1}(\mathcal{L}^\infty(X,\mu))$ such that

$$\operatorname{rg} E(x) = \operatorname{span}\{z(x)\} \quad \text{a.e.}$$

Proof. Let $z_0(x) = z(x)/\|z(x)\|$. Then just define E by

$$(Ef)(x) = \langle f(x),z_0(x)\rangle\, z_0(x)$$

which proves the lemma.

Now recall that

$$A_1 = \begin{pmatrix} \gamma & y' \\ z & R \end{pmatrix} \ ; \quad z = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \ , \quad R = (R^1, \ldots, R^{n-1})$$

with $R^1,\ldots,R^{n-1}$ and z in $\mathbb{C}^{m-1}(\mathcal{L}^\infty(X,\mu))$.

By Lemma 1.5 there is a projection E in $\mathfrak{M}_{n-1}(\mathcal{L}^\infty(X,\mu))$ such that $\operatorname{rg} E(x) = \operatorname{span}\{z(x)\}$ a.e. So for any $v \in \mathbb{C}^{n-1}(\mathcal{L}^\infty(X,\mu))$ it makes sense to write

$$v_{/\!/} = Ev \ , \quad v_{/\!/}(x) = E(x)v(x)$$

$$v_\perp = E^\perp \ , \quad v_\perp(x) = E^\perp(x)v(x) \ , \quad E^\perp = I - E \ .$$

In particular, for $i = 1,\ldots,n-1$

$$R^i{}_\perp = E^\perp R^i \ .$$

<u>Lemma 1.6</u>. The essential range of the function

$$x \mapsto \sum_{i=1}^{n-1} \|R^i_\perp(x)\|$$

does not contain zero.

<u>Proof</u>: (By contradiction) Suppose

$$0 \in \operatorname{ess\ rg} \sum_{i=1}^{n-1} \|R^i_\perp\| \ .$$

Write $A_1(x)$ as

$$A_1(x) = \left(\begin{pmatrix} \gamma(x) \\ z(x) \end{pmatrix}, \begin{pmatrix} y_1(x) \\ R^1_{/\!/}(x) \end{pmatrix} + \begin{pmatrix} 0 \\ R^1_\perp(x) \end{pmatrix}, \ldots, \begin{pmatrix} y_{n-1}(x) \\ R^{n-1}_{/\!/}(x) \end{pmatrix} + \begin{pmatrix} 0 \\ R^{n-1}_\perp(x) \end{pmatrix} \right) .$$

Since $n-1 \geq 2$, it follows by multi-linear expansion that

$$0 \in \operatorname{ess\ rg}(\det A_1)$$

which is a contradiction since A_1 is an invertible n-normal operator. This proves the lemma.

Now, let η be the essential infimum of the function in Lemma 1.6. Thus

$$\text{ess inf} \sum_{i=1}^{n-1} \|R_\perp^i\| = \eta > 0 \quad .$$

It follows that the measure space X is the union of the sets

$$X_i = \{x \in X : \|R_\perp^i(x)\| \geq \rho/n\text{-}1\} \quad , \quad i = 1,2,\ldots,n\text{-}1 \quad .$$

By the now-familiar "partitioning", we obtain

$$Y_i \subseteq X_i \quad \text{such that} \quad \bigcup_{i=1}^{n-1} Y_i = X$$

$$\text{and} \quad Y_i \cap Y_j = \phi \quad \text{for} \quad i \neq j \quad .$$

At last, we are in a position to produce the vector w' which will satisfy conditions (a) and (b) from the beginning of Step 2. In fact, w' is defined piece-wise, according to the partition given by $\{Y_i\}_{i=1}^{n-1}$. More precisely, we claim that there exists a measurable function $\alpha : X \to \mathbb{C}$ such that

$$w'(x) = \alpha(x)(R_\perp^i(x))^t \quad , \quad x \in Y_i \quad , \quad i = 1,2,\ldots,n\text{-}1$$

satisfies our requirements. To prove this, suppose $i \geq 2$. For x in Y_i define

$$\alpha(x) = \text{ess sup}\left(\frac{|\gamma(x)|}{\|R_\perp^i(x)\|^2}\left(|R_{1i}(x)| + \left|\frac{y_i(x)}{\gamma(x)}\right| + \rho/n-1\right)\right) \quad .$$

Thus, if $\beta \in \mathbb{C}$ and $w' = \alpha \cdot (R_\perp^i)^t$, then

$$\|(R - \tfrac{1}{\gamma}zy + \tfrac{1}{\gamma}zw'R - \beta)(x)\|$$

$$\geq \|R^i(x) - \frac{1}{\gamma(x)}\begin{pmatrix} y_1(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \frac{\alpha(x)}{\gamma(x)}\begin{pmatrix} \|R^i_\perp(x)\|^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta \\ 0 \\ \vdots \\ 0 \end{pmatrix}\| \quad (i^{th} \text{ column})$$

which, by the way $\alpha(x)$ is defined, turns out to be greater than or equal to $\rho/n-1$. If $i = 1$, define $\alpha(x) = 0$ for x in Y_1. Then, for $\beta \in \mathbb{C}$ and $w' = 0$,

$$\|(R - \tfrac{1}{\gamma}zy + \tfrac{1}{\gamma}zw'R - \beta)(x)\|$$

$$\geq \|R^1(x) - \begin{pmatrix} \frac{y_1(x)}{\gamma(x)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} \beta \\ 0 \\ \vdots \\ 0 \end{pmatrix}\| \quad (1^{st} \text{ column})$$

$$\geq \|R^1(x)\| \quad (\text{Recall } R^1_\perp \text{ is orthogonal to } z = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix})$$

$$\geq \rho/n-1 \quad .$$

The claim follows and we may now finish the proof of the theorem.

By construction, with

$$P = \begin{pmatrix} 1 & w' \\ 0 & I \end{pmatrix},$$

$$P^{-1}A_1P = A_2 = \begin{pmatrix} \gamma & u' \\ z & S \end{pmatrix},$$

where $S - \frac{1}{\gamma}zu'$ satisfies condition (1) in $\mathfrak{M}_{n-1}(\mathfrak{L}^\infty(X,\mu))$ and $\gamma = \beta_1\gamma_1$. But now, as pointed out in the Remark preceding <u>Step 1</u>

$$A_2 = \begin{pmatrix} \beta_1\gamma_1 & 0 \\ z & S - \frac{1}{\beta_1\gamma_1} zu' \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{\beta_1\gamma_1} u' \\ 0 & I \end{pmatrix}$$

and we can apply our induction hypothesis to

$$S - \frac{1}{\beta_1\gamma_1} zu' \ .$$

Since

$$\det A_2(x) = \det A(x)$$

we have that

$$\det\left(S - \frac{1}{\beta_1\gamma_1} zu' \right)(x) = \prod_{j=2}^{n-1} \beta_j(x)\gamma_j(x) \ .$$

By induction, there exist B_0, C_0 and Q_0 in $\mathfrak{M}_{n-1}(\mathcal{L}^\infty(X,\mu))$, with Q_0 invertible, such that

$$B_1 := Q_0^{-1} B_0 Q_0 = \begin{pmatrix} \beta_2 & & 0 \\ & \ddots & \\ * & & \beta_n \end{pmatrix} ,$$

$$C_1 := Q_0^{-1} B1 Q_0 = \begin{pmatrix} \gamma_2 & & * \\ & \ddots & \\ 0 & & \gamma_n \end{pmatrix} ,$$

It follows that with

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & Q_0 \end{pmatrix}$$

$$Q^{-1} A_2 Q = \begin{pmatrix} \beta_1 & & & 0 \\ & \beta_2 & & \\ & & \ddots & \\ * & & & \beta_n \end{pmatrix} \begin{pmatrix} \gamma_1 & & & * \\ & \gamma_2 & & \\ & & \ddots & \\ 0 & & & \gamma_n \end{pmatrix} .$$

Since A is similar to A_2, the theorem is now established.

Taking Sourour's example, we offer a corollary on "unipotent" operators.

Definition: An n-normal operator A is called unipotent if it is of the form $I+N$, for some n-normal nilpotent operator N.

Corollary 1.7:

(i) If A is n-normal, $d(\text{ess rg } A, \mathbb{C}) > 0$ and $\det A(x) = +1$ a.e., then A is a product of two unipotent operators.

(ii) If $A = \alpha \cdot I$, $\alpha \in \mathcal{L}^\infty(X,\mu)$, is a product of two unipotent operators $I+N_1$, $I+N_2$ then

$$\alpha = 1 \quad \text{and} \quad N_1+N_2 = -N_1N_2 = -N_2N_1 \ .$$

Proof:

(i) This is a special case of Theorem 1.4 with $\beta_j = \gamma_j = 1$, $1 \le j \le n$.

(ii) If we have

$$A = \alpha \cdot I = (I+N_1)(I+N_2)$$

and

$$N_1^r = 0 \ , \quad N_1^{r-1} \neq 0 \ , \quad r \in \mathbb{N}$$

then

$$\alpha(I+N_1)^{-1} = (I+N_2)$$

so that

$$(\alpha - 1)I - N_1 + N_1^2 - \ldots + (-1)^{r-1} N_1^{r-1} = N_2 \ .$$

It follows that

$$N_2N_1 = N_1N_2$$

making $(\alpha-1)$ nilpotent. We conclude that

$$\alpha = 1 \quad \text{and} \quad N_1+N_2 = -N_1N_2 = -N_2N_1 \ .$$

Remark: An easy extension of (i) is the following: If $d(A(x),\mathcal{L}) > 0$ and $\det A(x) = +1$ almost everywhere then for each $\varepsilon > 0$ there exists $X_\varepsilon \subsetneq X$ such that $\mu(X \setminus X_\varepsilon) < \varepsilon$ and restricted to X_ε, A is a product of two unipotent operators. Amongst the theorems to follow, similar extensions are possible but will not be stated.

Section 1.2

The next theorem includes a characterization of $\mathcal{P}_5(\mathcal{M}_n(\mathcal{L}^\infty(X,\mu)))$, the set of products of five positive invertible n-normal operators, as well as results on $\mathcal{P}_4$.

Theorem 1.8: Let A be an n-normal operator, and $\mathcal{L}_{np}$ be the subset of $\mathcal{L} = \mathcal{L}(\mathcal{M}_n(\mathbb{C}))$ consisting of non-positive scalar multiples of the identity.

(a) If there exists $\varepsilon > 0$ such that $\det A(x) \geq \varepsilon > 0$ a.e. and $d(\text{ess rg } A, \mathcal{L}_{np}) > 0$, then $A \in \mathcal{P}_4$. (Note that $d(\text{ess rg } A, \mathcal{L}) = d(\text{ess rg } A, \mathcal{L}_{np})$.)

(b) A is in $\mathcal{P}_5$ if and only if there exists $\varepsilon > 0$ such that $\det A(x) \geq \varepsilon > 0$ a.e.

As a partial converse to (a), we have

(c) If $A \in \mathcal{P}_4$, $\lambda \in \mathbb{C}$ and $\lambda \cdot I \in \text{ess rg } A$, then $\lambda > 0$.

Definition 1.9. Bounded measurable functions ϕ_λ, $\lambda \in \Lambda$ are said to be essentially distinct if there exists $\varepsilon > 0$ such that $|\phi_{\lambda_1}(x) - \phi_{\lambda_2}(x)| \geq \varepsilon > 0$ for all $\lambda_1 \neq \lambda_2$, a.e. .

Proof of Theorem: a) By hypothesis, A is invertible. Choose positive bounded measurable functions $\beta_1,\ldots,\beta_n,\gamma_1,\ldots,\gamma_n$ such that the β_i's are essentially distinct and the γ_i's are essentially distinct and satisfy

$$\prod_{i=1}^{n} (\beta_i\gamma_i)(x) = \det A(x) \quad \text{a.e.} .$$

Now, in $\mathcal{M}_n(\mathbb{C})$, $\mathcal{L}_{np}$ is dense in $\mathcal{L}$. So $d(\text{ess rg } A, \mathcal{L}) > 0$, and by Theorem 1.4 and the fact that the β_i's, γ_i's are essentially distinct there exist n-normal operators B and C such that $A = BC$ with B and C similar (separately) within the algebra of n-normal operators to diagonal operators B_1 and C_1, with entries $\beta_1,\ldots,\beta_n$ and $\gamma_1,\ldots,\gamma_n$ respectively. Considering B first, we have that for some invertible n-normal operator R,

$$B = R^{-1}B_1R = [(R^{-1})(R^{-1})^*][R^*BR]$$

so that B is a product of two positive invertible operators. A similar calculation works for C, and (a) is proven.

(b) One way is clear. We prove the converse. That is, if $\varepsilon > 0$ and $\det A(x) \geq \varepsilon > 0$ a.e. then $A \in \mathcal{P}_5$. To see this, assume, without loss of generality, that A is of the form

$$A = \begin{pmatrix} a_{11} & \cdot & \cdot & \cdot & \cdot & a_{1n} \\ & \cdot & & & & \cdot \\ & & \cdot & & & \cdot \\ & & & \cdot & & \cdot \\ & 0 & & & \cdot & \cdot \\ & & & & & a_{nn} \end{pmatrix}.$$

Since A is invertible,

$$\eta = \text{ess inf } |a_{11}| > 0 \ .$$

Let

$$\mathcal{D} = \{x \in X : d(A(x), \mathcal{L}) \geq \eta/4\} \ .$$

From part (a), we conclude that, restricted to $\mathcal{D}$, we may factor A into four positive invertible operators.

For $\mathcal{D}^c$,

$$\mathcal{D}^c = \{x \in X : d(A(x), \mathcal{L}) < \eta/4\}$$

consider the n-normal operator

$$P = \begin{pmatrix} 1 & 0 & . & . & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & . \\ \frac{1}{2} & . & . & . & . & 1 \end{pmatrix} .$$

The operator P is positive and invertible. Moreover

$$AP = \begin{pmatrix} * & \cdots & * & \frac{1}{2}a_{11} + a_{1n} \\ * & \cdots & * & * \\ \vdots & & \ddots & \vdots \\ \frac{a_{nn}}{2} & & 0 \cdots 0 & 1 \end{pmatrix} .$$

Hence, for $x \in \mathcal{D}^c$ and $\lambda \in \mathbb{C}$

$$\begin{aligned} \|(AP)(x) - \lambda\| &\geq |\tfrac{1}{2}a_{11}(x) + a_{1n}(x)| \\ &\geq \tfrac{1}{2}|a_{11}(x)| - |a_{1n}(x)| \\ &\geq \tfrac{1}{2}\eta - \tfrac{\eta}{4} \\ &= \tfrac{\eta}{4} . \end{aligned}$$

Therefore, restricted to $\mathcal{D}^c$

$$d(AP, \mathcal{L}) \geq \eta/4 > 0 .$$

So $AP \in \mathcal{P}_4$ and $A = (AP)P^{-1} \in \mathcal{P}_5$ as required.

(c) Suppose P_1, P_2, P_3, P_4 are positive invertible n-normal operators such that $A = P_1P_2P_3P_4$, and that $\lambda \in \mathbb{C}$, $\lambda \cdot I \in \text{ess rg } A$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that in $\mathcal{M}_n(\mathbb{C})$

(1) $A(x_n)$ converges to $\lambda \cdot I$

(2) $P_i(x_n)$ converges to Q_i, $i = 1,2,3,4$

(3) $P_i(x_n)$ and Q_i are positive invertible matrices, $i = 1,2,3,4$, $n = 1,2,3,\ldots$.

It follows that

$$Q_1Q_2 = \lambda Q_4^{-1}Q_3^{-1} .$$

Since Q_1Q_2 and $Q_4^{-1}Q_3^{-1}$ are both similar to positive matrices, it must be that $\lambda > 0$.

Remark: Part (c) (ii) of the theorem means that $\mathcal{P}_5$ is the optimal set. For let $n = 4$ and $A = iI$. Then $\det A(x) \equiv 1 > 0$. But i is not positive, so A is not in $\mathcal{P}_4$. Of course, by part (b), A is in $\mathcal{P}_5$.

We now consider products of a specified number of positive n-normal operators and so determine the sets $\overline{\mathcal{P}}_2$, $\overline{\mathcal{P}}_4$, $\overline{\mathcal{P}}_k$, $\overline{\mathcal{Q}}_k$, $4 \leq k \leq \infty$.

Lemma 1.10: Let T be n-normal. Then

$$\sigma(T) = \cup\{\sigma(S) : S \in \text{ess rg } T\} = \sigma(\text{ess rg } T).$$

Proof: To begin, we assume (by Theorem 7.20 [RR]) that the operator is of the form

$$T' = \begin{pmatrix} \phi_1 & & & * & \\ & \cdot & & & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & \phi_n \end{pmatrix}.$$

Suppose now that $\lambda \in \sigma(\text{ess rg } T)$. So there exists $M \in \text{ess rg } T$ such that $\lambda \in \sigma(M)$. Therefore

$$\det(M-\lambda) = 0$$

which, because "det" is continuous, implies that

$$0 \in \text{ess rg} \det(T-\lambda).$$

Therefore, by Lemma 1.3, $\lambda \in \sigma(T)$.

On the other hand, if $\lambda \notin \sigma(\text{ess rg } T)$, then we claim that there exists $\delta > 0$ such that $|\phi_i(x) - \lambda| \geq \delta$ a.e. To see why this is so, suppose not. Then for each $\delta > 0$, there exist both i and a set Y of positive measure over which $|\phi_i(x) - \lambda| < \delta$. Letting $\delta_m = \frac{1}{m}$, $m = 1,2,3,\ldots$, and using the fact there there are only fiitely many indices

$i = 1,2,\ldots,n$, we obtain i_0, $1 \le i_0 \le n$, together with a sequence Y_{m_j} of sets of positive measure such that for $x \in Y_{m_j}$, $|\phi_{i_0}(x) - \lambda| < \frac{1}{m_j}$, for all j. Because $\{M_j\}_{j=1}^{\infty}$ is a bounded sequence of $n \times n$ matrices, we may choose a convergent subsequence M_{j_k} with limit $M \in$ ess rg T. But now continuity of σ in finite dimensions gives that $d(\sigma(M),\lambda) = 0$, that is, $\lambda \in \sigma(M)$. This is the required contradiction.

Thus, our claim is true. It follows that

$$0 \notin \text{ess rg} \det(T-\lambda)$$

and this implies that

$$\lambda \notin \sigma(T) .$$

This finishes the proof.

Remarks: 1) One consequence of the lemma is that σ(ess rg T) is a closed subset of $\mathbb{C}$. This, however, can be proven directly, using finite dimensionality and continuity of σ.

2) For results on the spectrum of more general types of direct integrals, see [Chow].

Lemma 1.11: Let $\mathcal{Q}$ be a C*-algebra. Then $T \in \mathcal{P}_2(\mathcal{Q}) \Leftrightarrow T$ is similar, within $\mathcal{Q}$, to a positive n-normal operator.

Proof: If $T = AB$, A, B positive invertible, then $T = A^{1/2}(A^{1/2}BA^{1/2})A^{-1/2}$, so T is similar to $A^{1/2}BA^{1/2}$ which is positive invertible; and all operators involved in the factorization are from $\mathcal{Q}$.

If $T = X^{-1}PX$, X, P invertible, P positive, then $T = [X^{-1}(X^{-1})^*][X^*PX]$.

This proves the lemma.

Lemma 1.12: Suppose that $\phi_1,\ldots,\phi_n \in \mathcal{L}^\infty(X,\mu)$ are invertible and $\varepsilon > 0$. Then there exist δ, $0 < \delta < \varepsilon$, and $\psi_1,\ldots,\psi_n \in \mathcal{L}^\infty(X,\mu)$ such that

(i) $|\phi_j(x) - \psi_j(x)| < \varepsilon$ a.e. for all j

(ii) $|\psi_j(x)| \geq \delta$ a.e. for all j

(iii) $|\psi_i(x) - \psi_j(x)| \geq \delta$ a.e. for all $i \neq j$.

Moreover, if $\phi_j(x) \geq 0$ then $\psi_j(x) \geq 0$ a.e. for all j .

Proof. (By induction): Suppose $n = 2$. Since ϕ_1, ϕ_2 are invertible, there exists $\delta_0 > 0$ such that $|\phi_1(x)|, |\phi_2(x)| \geq \delta_0 > 0$ a.e. Let $\delta = \min(\frac{\varepsilon}{2}, \frac{\delta_0}{4})$ and $\mathcal{D} = \{x: |\phi_1(x)|, |\phi_2(x)| \geq \delta\}$. On $\mathcal{D}$ we set $\psi_1(x) = \phi_1(x)$ and $\psi_2(x) = \phi_2(x)$. On $\mathcal{D}^c$ we set $\psi_1(x) = \phi_1(x)$ and $\psi_2(x) = \phi_1(x) + \delta$. Then

$$|\psi_2(x)| = |\phi_1(x) + \delta| \geq \delta_0 - \delta = \frac{3\delta_0}{4} > \delta ;$$

$$|\psi_2(x) - \psi_1(x)| = \delta ; \ |\psi_1(x) - \phi_2(x)| = 0 < \varepsilon ;$$

and $$|\psi_2(x) - \phi_2(x)| = |\phi_1(x) + \delta - \phi_2(x)| \leq \frac{\varepsilon}{2} + \delta < \varepsilon .$$

Now suppose that the result is true for $n-1$. Therefore we may assume that $\phi_1,\dots,\phi_{n-1}$ already satisfy (a.e.) $|\phi_i(x) - \phi_j(x)| \geq \delta_0 > 0$, $i < j \leq n-1$ and $|\phi_i(x)| \geq \delta_0 > 0$ for all $i = 1,2,\dots,n$

Let

$$\delta = \min\{\tfrac{1}{4} \text{ ess inf}|\phi_i - \phi_j| , \tfrac{\varepsilon}{2} : i < j\}$$

and set

$$\mathcal{D} = \{x : |\phi_n(x) - \phi_i(x)| \geq \delta, \ i=1,\dots,n-1\} .$$

On $\mathcal{D}$ define $\psi_n(x) = \phi_n(x)$. The set $\mathcal{D}^c$ partitions in the usual way into $\bigcup_{i=1}^{n-1} \mathcal{S}_i$

$$\mathcal{S}_i \subseteq \{x : |\phi_n(x) - \phi_i(x)| < \delta\} \ , i=1,\dots,n-1 .$$

On $\mathcal{S}_i$ define $\psi_n(x) = \phi_i(x) + \delta$. The result follows.

Corollary 1.13: The same result holds even if the $\phi_1,\dots,\phi_n$ are not necessarily invertible.

Proof: Let $\eta > 0$. For each i, let

$$\mathcal{D}_i = \{x : \phi_i(x) \neq 0\} .$$

On $\mathcal{D}_i$ define $\phi'_i(x) = \phi_i(x) + \eta \dfrac{\phi_i(x)}{|\phi_i(x)|}$. On $\mathcal{D}^c{}_i$ define $\phi'_i(x) = \eta$. In this way we replace the elements ϕ_i by elements ϕ'_i which are invertible and for which

$$|\phi_i(x) - \phi'_i(x)| \leq \eta \quad \text{a.e. for all } i .$$

Since $\eta > 0$ was arbitrary, the result follows.

Proposition 1.14: Let $T \in \mathcal{A} = \mathcal{M}_n(\mathcal{L}^\infty(X,\mu))$. Then

$$T \in \overline{\mathcal{P}}_2(\mathcal{A}) \Leftrightarrow \sigma(T) \geq 0 .$$

Proof: Suppose $T = \lim_n P_nQ_n$, P_n, Q_n positive invertible,

then $T = \lim_n P_nQ_n$ in the strong operator topology,

so $T(x) = \lim_n P_n(x)Q_n(x)$ SOT, a.e.

([Nielsen], Theorem 7.1). But $T(x) \in \mathcal{M}_n(\mathbb{C})$ a.e. therefore

$$T(x) = \lim_n P_n(x)Q_n(x) \quad \text{in the operator norm a.e.}$$

and since the spectrum is continuous on finite dimensional algebras, it follows that $\sigma(T(x)) \geq 0$, and $\sigma(S) \geq 0$ for all $S \in \operatorname{essrg} T$. By Lemma 1.10 we conclude that $\sigma(T) \geq 0$.

Conversely, suppose $\sigma(T) \geq 0$. By Theorem 7.20 [R&R], T is unitarily equivalent to an operator of the form

$$T' = \begin{pmatrix} \phi_1 & & & * \\ & \cdot & & \\ & & \cdot & \\ & 0 & & \cdot \\ & & & & \phi_n \end{pmatrix} .$$

And, viat he determinant condition, $\sigma(T) = \bigcup_{i=1}^{n} \text{essrg}\ \phi_i$, which implies that essrg $\phi_i \geq 0$ for $i = 1,\ldots,n$. Let $\varepsilon > 0$ be small enough so that we may replace each φ_i by ψ_i such that

(i) $\psi_i(x) \geq \frac{\varepsilon}{4}$ a.e.

(ii) $|\psi_i(x) - \phi_i(x)| < \frac{\varepsilon}{4}$ a.e.

(iii) $|\psi_i(x) - \psi_j(x)| \geq \frac{\varepsilon}{4}$, $i \neq j$ a.e.

By applying Corollary 0.15 [R&R] (of Rosenblum's Corollary) to the operator

$$S_\varepsilon = \begin{pmatrix} \psi_1 & & & * & \\ & \cdot & & & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & \psi_n \end{pmatrix}$$

(where the off-diagonals are the same as for T') we find that S_ε is similar to a "diagonal" positive invertible operator, namely,

$$\begin{pmatrix} \psi_1 & & & & \\ & \cdot & 0 & & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & \psi_n \end{pmatrix}.$$

So by Lemma 1.11, $S_\varepsilon \in \mathcal{P}_2(\mathcal{A})$. But $\|S_\varepsilon - T'\| < \varepsilon$ and $\varepsilon > 0$ was arbitrary, so the result follows.

<u>Proposition 1.15</u>: Let $\mathcal{A} = \mathcal{M}_n(\mathcal{L}^\infty(X,\mu))$. Then

$$\overline{\mathcal{P}_4(\mathcal{A})} = \{T : \det T(x) \geq 0 \text{ a.e.}\}.$$

Proof: If $T = \lim_n T_n$, $T_n \in \mathcal{P}_4$

then $T = \lim_n T_n$ (SOT), which ([N], Theorem 7.1) implies

$$T(x) = \lim_n T_n(x) \quad \text{(SOT) a.e.}$$

Hence $T(x) = \lim_n T_n(x)$ (operator norm) a.e.

and so $\det T(x) = \lim_n T_n(x) \geq 0$ a.e.

For the reverse inclusion, we may assume T is of the form

$$T = \begin{pmatrix} \phi_1 & & & \\ & \cdot & * & \\ & & \cdot & \\ & 0 & & \cdot \\ & & & \phi_n \end{pmatrix}$$

so that

$$\det T(x) = \phi_1(x) \ldots \phi_n(x) \geq 0 \quad \text{a.e.}$$

Let $\varepsilon > 0$.

We use the usual trick. Replace the ϕ_i's by ψ_i's in such a way that for some $\delta > 0$ the following conditions are satisfied:

(i) $\psi_1(x) \ldots \psi_n(x) \geq \delta > 0$ a.e.

(ii) $|\psi_i(x) - \psi_j(x)| \geq \delta > 0$ a.e. for $i \neq j$

(iii) $|\psi_i(x) - \phi_i(x)| < \varepsilon$ a.e.

Then let

$$T_\varepsilon = \begin{pmatrix} \psi_1 & & & \\ & \cdot & * & \\ & & \cdot & \\ & 0 & & \cdot \\ & & & \psi_n \end{pmatrix}.$$

It follows that

$$\|T - T_\varepsilon\| < \varepsilon$$

and by Theorem 1.8

$$T_\varepsilon \in \mathcal{P}_4 .$$

Corollary 1.16: As in the case of matrices, we have

$$\overline{\mathcal{P}_4} = \overline{\mathcal{P}_5} = \overline{\mathcal{Q}_4} = \overline{\mathcal{Q}_5} = \overline{\mathcal{P}_\infty} = \overline{\mathcal{Q}_\infty} .$$

We now return to the question of exact factorization (as opposed to approximation). When the measure space (X,μ) is a singleton, Theorem 1.8 determines exactly which operators are in $\mathcal{P}_4(\mathcal{M}_n(\mathbb{C}))$. This is Ballantine's theorem for matrices and can be stated as follows:

Let A be a real or complex $n \times n$ matrix.

(1) A is a product of four positive-definite matrices if and only if $\det A > 0$ and A is not a scalar $\alpha \cdot I$, α not positive.

(2) A is a product of five positive definite matrices if and only if $\det A > 0$.

Moreover, if $A = \alpha \cdot I$, $\det A > 0$ and α is not positive, then five factors is the smallest number possible.

Ideally, we would like to fully extend Ballantine's result to our setting of n-normal operators and give a complete characterization of $\mathcal{P}_4(\mathcal{M}_n(\mathcal{L}^\infty(X,\mu)))$, as distinguished from $\mathcal{P}_5$. What we have so far are sufficient conditions from part (a) of Theorem 1.8 and necessary conditions from part (c). What is missing? If $A = P_1P_2P_3P_4$ then $A(x) = P_1(x)P_2(x)P_3(x)P_4(x)$ a.e., and so by Ballantine's theorem, $A(x)$ is (a.e.) not a non-positive scalar. So what remains are precisely those operators A for which

(i) $d(\text{essrg } A, \mathcal{L}_{np}) = 0 = d(\text{essrg } A, \mathcal{L})$ $(\mathcal{L} = \overline{\mathcal{L}_{np}})$

(ii) $\text{essrg } A \cap \mathcal{L}_{np} = \phi$

(iii) $\det A(x) \geq \varepsilon > 0$ a.e., for some $\varepsilon > 0$.

In other words, it remains to characterize those operators of essentially positive determinant which provide the link between essentially non-scalar elements of $\mathcal{P}_k$ and those of the form $\alpha \cdot I$, $\alpha(x) > 0$ a.e. And it is easy to show by example that this set is non-empty.

<u>Examples</u>: 1. Let $X = \mathbb{N}$, $\mu\{n\} = \frac{1}{2^n}$, and suppose $\varepsilon : \mathbb{N} \to \mathbb{C}$ is such that $\varepsilon(n) > 0$ and $\varepsilon(n) \underset{n}{\to} 0$. Define

$$A = \begin{pmatrix} 1+i\varepsilon & 0 \\ 0 & 1-i\varepsilon \end{pmatrix}.$$

2. Let X and ε be as in 1. Define

$$A = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}.$$

3. Again X, ε are as in 1. Let $\psi \in \ell^\infty(X)$. Define

$$A = \begin{pmatrix} 1+i\varepsilon & \psi \\ 0 & 1-i\varepsilon \end{pmatrix}.$$

As we will show, every one of the operators above belongs to $\mathcal{P}_4$. For the general case we have the following conjecture.

<u>Conjecture</u>: Let (X,μ) be a standard σ-finite measure space. Then

$A \in \mathcal{P}_4(\mathcal{M}_4(\mathcal{L}^\infty(X,\mu))) \Leftrightarrow \det A(x)$ is essentially bounded away from aero and essrg $A \cap \mathcal{L}_{np} = \phi$.

We have partial results, including a discussion of the case $n = 2$.

<u>Proposition 1.17</u>: Let $X = \mathbb{N}$, μ be counting measure and suppose that $A \in \mathcal{M}_2(\ell^\infty(\mathbb{N}))$ is of the form

$$A = \begin{pmatrix} 1+i\varepsilon & \psi \\ 0 & 1-i\varepsilon \end{pmatrix}$$

where $\varepsilon, \psi \in \ell^\infty(\mathbb{N})$, $\varepsilon(n) \neq 0$, $\psi(n) \neq 0$ for all n, $\varepsilon(n)$, $\psi(n)$ both converge to zero and $\varepsilon(n)$ is real-valued. Then $A \in \mathcal{P}_4(\mathcal{M}_2(\ell^\infty(\mathbb{N})))$.

Proof: Let $X_1 = \{n : |\psi(n)| \geq 2|\varepsilon(n)|\}$

$X_2 = \{n : |\psi(n)| < 2|\varepsilon(n)|\}$.

Then X_1 and X_2 are measurable and partition $\mathbb{N}$. We consider each of these sets in turn. If X_1 is finite, then by Ballantine's theorem, $A|_{X_1} \in \mathcal{P}_4(\mathcal{M}_2(\ell^\infty(X_1)))$. If X_1 is not finite, then

$$X_1 = \{k_1, k_2, k_3, \ldots\}, \text{ where } k_\ell < k_{\ell+1} \text{ for all } \ell .$$

In this case we claim that there is a sequence of unitary matrices $U(\ell)$ so that, restricted to X_1, $\int_{X_1}^{\oplus} U(\ell)d\mu$ implements a unitary equivalence between A and an operator B of the form

$$B = \begin{pmatrix} 1+\varepsilon & z \\ y & 1-\varepsilon \end{pmatrix}$$

where y, z are real-valued and converge to zero. And we will show that B is then in $\mathcal{P}_4$ over X .

To see why the claim is true we use the fact that for 2×2 matrices, unitary equivalence is determind completely by the trace, the determinant and the Hilbert-Schmidt norm. (To see this, assume that the operator is upper-triangular. Then choose an orthonormal basis so that the off-diagonal becomes non-negative.) We are therefore reduced to solving the equations:

1) $yz = -2\varepsilon^2$

2) $|y|^2 + |z|^2 = |\psi|^2$.

Substituting equation 2) into equation 1) we obtain

$$|z|^2 = \frac{|\psi|^2 \pm \sqrt{|\psi|^4 - 16\varepsilon^4}}{2} .$$

Since $|\psi(n)| \geq 2|\varepsilon(n)|$ for all $n \in X_1$, we obtain a solution so that the claim is established. But now

$$\begin{pmatrix} 1+\varepsilon & z \\ y & 1-\varepsilon \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{y}{1+\varepsilon} & \frac{w}{p} \end{pmatrix} \begin{pmatrix} 1+\varepsilon & z \\ 0 & pv \end{pmatrix}$$

which, by Rosenblum's corollary and Lemma 1.11, is in $\mathcal{P}_2 \mathcal{P}_2 = \mathcal{P}_4$ for $p > 0$ a sufficiently large constant and w, v invertible positive ℓ^∞-functions appropriately chosen.

For the set X_2 we have $0 \leq |\psi(n)| < 2|\varepsilon(n)|$. If X_2 is finite, then as before, $A|_{X_2} \in \mathcal{P}_4$ over X_2. Suppose then that X_2 is infinite. Observe that the operator $\begin{pmatrix} 1+i\varepsilon & 0 \\ 0 & 1-i\varepsilon \end{pmatrix}$ is similar to $\begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & 1 \end{pmatrix}$ via the constant $Q = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$; and $\begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & 1 \end{pmatrix}$ is in $\mathcal{P}_4$ via Ballantine's theorem.

This suggests that we try to solve the similarity

$$\begin{pmatrix} 1+i & \psi \\ 0 & 1-i\varepsilon \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & 1 \end{pmatrix}.$$

We obtain the equations

$$ai\varepsilon + c\psi = -b\varepsilon \ , \quad Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ |\det Q| >> 0 .$$
$$ic = d$$

If there exists a solution, $|c| = |d| >> 0$. So we try $c = 1$, $d = i$. The equations become

$$(ai+b)\varepsilon = -\psi$$

subject to $\quad |ai-b| >> 0$.

Therefore it is enough to solve

$$ai + b = -\tfrac{\psi}{\varepsilon}$$

$$ai - b = 2e^{i\theta}$$

for θ a measurable real-valued function, e.g. $\theta = 0$. We obtain solutions

$$a = \frac{1}{2i}\left(-\frac{\psi}{\varepsilon} + 2\right)$$

$$b = \frac{1}{2}\left(-\frac{\psi}{\varepsilon} - 2\right)$$

which are indeed ℓ^∞-functions, since $\left|\frac{\psi(n)}{\varepsilon(n)}\right| < 2$ for $n \in X_2$.

We now can put the solutions from X_1 and X_2 together to conclude that $A \in \mathcal{P}_4(\mathcal{M}_2(\ell^\infty))$. This finishes the proof of the proposition.

Remark: It might eventually be of use (see Concluding Remarks) to consider the qualitative behaviour of $A = \begin{pmatrix} 1+i\varepsilon & \psi \\ 0 & 1-i\varepsilon \end{pmatrix}$. From the proof of the proposition, there are two cases.

Case 1. $|\psi| \geq 2|\varepsilon| > 0$.

Here A is "asymptotic" to $I+R_n$, where R_n is an upper triangular nilpotent converging to zero.

Case 2. $|\psi| < 2|\varepsilon|$.

Here A is "asymptotic" to I.

Similar calculations will prove the conjecture for the general 2-normal operator over (X,μ). The additional measure-theoretic details shed little new light on how to obtain the factorization in its full generality regarding n-normal operators. We therefore do not include a proof. However, we do present the reduction to a canonical 2-normal operator, like the one considered in the proposition above.

Reduction for the case $n = 2$: We will show that the problem can be reduced to considering an operator

$$A = \begin{pmatrix} 1+i\varepsilon & \psi \\ 0 & \alpha - i\alpha\varepsilon \end{pmatrix}$$

which satisfies the following:

(i) the set $Z_n = \{x : 0 < d(A(x),\mathcal{L}) \le \frac{1}{n}\}$ has positive measure for each $n \in \mathbb{N}$.

(ii) $\|\varepsilon|_{Z_n}\| \to 0$, $\|\psi|_{Z_n}\| \to 0$, $\varepsilon(x) \in \mathbb{R}$ a.e.

(iii) $\alpha >> 0$ and $\|(\alpha-1)|_{Z_n}\| \to 0$.

From the start we assume, without loss of generality, that

$$A = \begin{pmatrix} \phi_1 & \psi \\ 0 & \phi_2 \end{pmatrix}$$

and suppose that A satisfies the conditions stated in the conjecture, that is

$$\det A >> 0 \quad \text{and} \quad \text{essrg}\, A \cap \mathcal{L}_{np} = \phi .$$

If $A = \alpha\cdot I$ or $\text{ess}\, d(A,\mathcal{L}) > 0$, then by Theorem 1.8 we are done, with $A \in \mathcal{P}_4$ as required. Otherwise, A must satisfy condition (i) above.

Now, for each n

$$|\phi_1(x) - \phi_2(x)| \le \frac{1}{n} \quad \text{a.e. on } Z_n .$$

For if not, there is some n_0 and a set $P_{n_0} \subsetneq Z_{n_0}$ of positive measure over which $|\phi_1(x) - \phi_2(x)| > \frac{1}{n}$ a.e. Let $\delta = \phi_1 - \phi_2$. Then there is some $\delta_0 > \frac{1}{n}$, $\delta_0 \in \text{essrg}\, \delta|_{P_{n_0}}$. Let $Y_k = \{x \in P_{n_0} \mid \|\delta(x) - \delta_0\| < \frac{1}{k}\}$. Then for each k , Y_k has positive measure. Let $\lambda_k \in \text{essrg}\, \phi_2|_{Y_k}$. The sequence $(\lambda_k)_{k=1}^{\infty}$ has a convergent subsequence λ_{k_ℓ} with limit λ . But, by considering $A(x)-\lambda$, we obtain a contradiction to condition (i).

Now write $\phi_j = \alpha_j + i\beta_j$. For n sufficiently large $\alpha_j|_{Z_n} >> 0$. For otherwise there exists a non-positive $\lambda \in \text{essrg} \det A$, where

$$\det A(x) = (\alpha_1(x)\alpha_2(x) - \beta_1(x)\beta_2(x)) .$$

Similarly $||\beta_j|_{Z_n}|| \underset{n}{\rightarrow} 0$, $j = 1,2$, for if not, then there is a $\lambda \in \mathbb{C}_{np}$ such that $\lambda \in \text{essrg}\, A$. We therfore have

$$A = \begin{pmatrix} \alpha_2 + \mu + i\beta_1 & \psi \\ 0 & \alpha_2 + i\beta_2 \end{pmatrix}$$

where $\mu|_{Z_n}(x) \geq 0$ a.e. for n large enough. Dividing A by the invertible positive $\mathbb{C}^\infty$-function $\alpha_2 + \mu$, we obtain the reduction.

<u>Further Special Cases</u>

1) If

$$A = \begin{pmatrix} \phi_1 & & & * \\ & \cdot & & \\ & & \cdot & \\ & 0 & & \cdot \\ & & & & \phi_n \end{pmatrix}$$

with $\phi_i >> 0$, $i = 1,\ldots,n$, then there exists a large positive constant $p > 0$ so that the factorization

$$A = \begin{pmatrix} \sqrt{\phi_1} & & & * \\ & p\sqrt{\phi_2} & & \\ & & \ddots & \\ 0 & & & p^{n-1}\sqrt{\phi_n} \end{pmatrix} \begin{pmatrix} \sqrt{\phi_1} & & & * \\ & \frac{1}{p}\sqrt{\phi_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{p^{n-1}}\sqrt{\phi_n} \end{pmatrix}$$

shows A is in $\mathcal{P}_4$.

2) If

$$A = \begin{pmatrix} 1 + i\varepsilon_1 & & & & \\ & \cdot & & (\psi_{ij})_{i<j} & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & 1 + i\varepsilon_n \end{pmatrix},$$

and

$$|\psi_{ij}(x)| < c_{ij}\, f_{ij}(\{\varepsilon_k - \varepsilon_\ell)(x)$$

for suitable functions f_{ij} and constants $c_{ij} > 0$, then by induction we obtain that A is direct integral unitarily equivalent to the operator

$$\begin{pmatrix} 1+i\varepsilon_1 & & & \\ & \cdot & 0 & \\ & & \cdot & \\ & 0 & \cdot & \\ & & & 1+i\varepsilon_n \end{pmatrix}$$

thereby reducing the problem in this situation to the case of a diagonal operator.

3) A last observation is that when

$$y_{ij}(m) \underset{m}{\rightarrow} 0 \qquad i > j$$

$$z_{ij}(m) \underset{m}{\rightarrow} 0 \qquad i < j$$

define real-valued functions $\mathbb{N} \to \mathbb{R}$, then

$$\begin{pmatrix} 1 & & & \\ & \cdot & (z_{k\ell}) & \\ & & \cdot & \\ (y_{ij}) & & & \cdot \\ & & & & 1 \end{pmatrix}$$

is equal to

$$\begin{pmatrix} 1 & & & & \\ & p\cdot\alpha_1 & 0 & & \\ & & \cdot & & \\ & (v_{ij}) & & \cdot & \\ & & & \cdot & \\ & & & & p^{n-1}\alpha_{n-1} \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & \frac{1}{p}\beta_1 & (w_{k\ell}) & & \\ & & \cdot & & \\ & & & \cdot & \\ & 0 & & \cdot & \\ & & & & \frac{1}{p^{n-1}}\beta_{n-1} \end{pmatrix}$$

which is in $\mathcal{P}_4(\mathcal{M}_n(\ell^\infty))$ for suitably chosen positive invertible ℓ^∞-functions α_j, β_j and large positive constant $p > 0$.

<u>Some Applications:</u>

We now give the "n-normal analogues" to the corollaries from Sourour's 1986 paper. The first of these concerns commutators and extends the Shoda-Thompson Theorem (5, 6 of [Sourour]). For the classical theorem, F is a field not of characteristic 2 and n is a positive integer not equal to 2. It was shown that the set of commutators of invertible matrices over F coincides with the matrices of determinant equal to 1, that is

$$\{BCB^{-1}C^{-1} : B,C \in GL(n,F)\} = SL(n,F) .$$

For our theorem we replace the field F by the commutative von Neumann algebra $\mathcal{L}^\infty(X,\mu)$ and so find ourselves in the setting of n-normal operators. Our notation here will be

$$\mathcal{G} = \{A \in \mathcal{M}_n(\mathcal{L}^\infty(X,\mu)) : A \text{ is invertible}\}$$

$$\mathcal{S} = \{A \in \mathcal{M}_n(\mathcal{L}^\infty(X,\mu)) : \det A(x) = +1 \text{ a.e.}\} .$$

We can now state the result.

Theorem 1.18: Let $A \in \mathcal{S}$.

(a) If $A = \alpha \cdot I$, then A is a commutator of operators in $\mathcal{G}$.

(b) If $d(\text{essrg } A, \mathcal{L}) > 0$ then A is a commutator of operators in $\mathcal{S}$.

(c) If $d(\text{essrg } A, \mathcal{L}) > 0$ then A is a commutator of operators in $\mathcal{G}$ with arbitrarily prescribed determinant invertible in $\mathcal{L}^\infty(X,\mu)$.

Proof: (a) If $A = \alpha \cdot I$, then $\alpha^n(x) = 1$ a.e. Let $B = \text{diag}(\alpha, \alpha^2, \ldots, \alpha^n)$ and $D = \text{diag}(1, \alpha^{-1}, \ldots, \alpha^{-n+1})$. Then there exists $C \in \mathcal{G}$ such that $D = CB^{-1}C^{-1}$. Therefore $A = BD = BCB^{-1}C^{-1}$.

(b) Since $\text{ess } d(A, \mathcal{L}) > 0$, by Theorem 1.8 we can write A as a product $B \cdot D$ where B has essentially distinct "eigenvalues" $\beta_1, \ldots, \beta_n$ and D has "eigenvalues" $\beta_1^{-1}, \ldots, \beta_n^{-1}$. Therefore D is similar to B^{-1}, that is, there exists $C \in \mathcal{G}$ such that $D = CB^{-1}C^{-1}$, which makes $A = BCB^{-1}C^{-1}$.

But let us be more careful in our choice of $\{\beta_j\}$. If n is odd, let $\beta_1 = 1$ and take $\frac{n-1}{2}$ essentially distinct pairs of the form $\{\beta, \beta^{-1}\}$. If n is even take $\frac{n}{2}$ pairs $\{\beta, \beta^{-1}\}$. In this way $\det B(x) = 1$ a.e.

Now observe that the operator C in the first paragraph may be replaced by CD for any invertible D commuting with B. Since B can be "diagonalized" (use Theorem 7.20 of [R&R] together with Rosenblum's Corollary) there exists a "diagonalizable" D which commutes with B and has arbitrary invertible determinant in $\mathcal{L}^\infty(X,\mu)$. So in particular, we can replace C by an operator in $\mathcal{S}$.

(c) The proof is similar to (b).

The next theorem of this section deals with products of involutions. An <u>involution</u> is an operator whose square is the identity. (In forthcoming research we consider involutions in the algebra of decomposable operators, where the integrand Hilbert spaces are a.e. separable and infinite dimensional. This is in contrast to our present setting where the underlying space is a direct integral of n-dimensional spaces $\mathbb{C}^n$, $n < \infty$.) The relevant matrix result, proven by Gustafson, Halmos and Radjavi [GHR], is that over an arbitrary field, every $n \times n$ matrix with determinant ± 1 is the product of at most four involutions. Again, it is possible to replace the field F by the algebra $\mathcal{L}^\infty(X,\mu)$ to obtain analogous results.

<u>Theorem 1.19</u>: Let $A \in \mathcal{M}_n(\mathcal{L}^\infty(X,\mu))$ with $\det A(x) = \pm 1$ a.e. If either $d(\text{essrg } A, \mathcal{L}) > 0$ or $A = \alpha \cdot I$, then A is the product of at most four involutions.

<u>Proof</u>: Since $x \mapsto \det A(x)$ is a measurable function, the measure space X partitions measurably into

$$X = X_+ \cup X_-$$

where

$$X_+ = \{x : \det A(x) = +1\}$$

and

$$X_- = \{x : \det A(x) = -1\}.$$

So first consider the case where $d(\text{essrg } A, \mathcal{L}) > 0$ and $\det A(x) = +1$ a.e. As in the proof of the last theorem, we may write A as a product BD where each of B and D has distinct "eigenvalues" of the form $\{\beta_1, \beta_1^{-1}, \ldots, \beta_m, \beta_m^{-1}\}$ or $\{1, \beta_1, \beta_1^{-1}, \ldots, \beta_m, \beta_m^{-1}\}$, according as n is even or odd. Since each of B and D is diagonalizable and being an involution is similarity invariant, it suffices to show that the operator $\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$ is a product of two involutions, but as in [Sourour] this follows easily since

$$\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \beta^{-1} \\ \beta & 0 \end{pmatrix} .$$

Assume now that $d(\text{essrg } A, \mathbb{C}) > 0$ and $\det A(x) = -1$ a.e. If n is odd use $-A$. If n is even write $A = BD$ with B and D as above except that β_1 and β_1^{-1}, in the list of "eigenvalues" for B are replaced by 1 and -1 (leave D as it was). This contributes a direct summand $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which is itself an involution.

For the scalar case we require a separate proof. As above, suppose $\det A(x) = +1$ a.e., $A = \alpha \cdot I$ and $\alpha(x)^n = 1$ a.e. If $n = 2k+1$ is odd then

$$A = \text{diag}(\alpha,\ldots,\alpha) = \text{diag}(\alpha,\alpha^2,\ldots,\alpha^{2k+1}) \cdot (1,a^{-1},\ldots,\alpha^{-2k}) .$$

Each factor is similar to $\text{diag}(1,\alpha,\alpha^{-1},\ldots,\alpha^k,\alpha^{-k})$ which, as explained in the non-scalar case, is a product of two involutions, from which it follows that A is a product of four. If $n = 2k$ is even then a similar calculation gives

$$A = \text{diag}(\alpha,\ldots,\alpha) = \text{diag}(\alpha,\ldots,\alpha^{2k}) \cdot \text{diag}(1,\alpha^{-1},\ldots,\alpha^{-2k+1})$$

and each factor is similar to

$$\text{diag}(1,\alpha^k,\alpha^2,\alpha^{-2},\ldots,\alpha^{k-1},\alpha^{-(k-1)})$$

which equals

$$\text{diag}(1,\alpha^k) \oplus \text{diag}(\alpha^2,\alpha^{-2},\ldots,\alpha^{k-1},\alpha^{-(-1)}) .$$

The first summand is already an involution while the second is a product of two.

The next case to consider is $\alpha(x)^n = -1$ a.e. If n is odd, use $-A$. For the case when n is even we draw on the original proof of the matrix theorem in [GHR]. We have

$$A = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \alpha & & & \\ \alpha & 0 & & & \\ & & \ddots & & \\ & & & 0 & \alpha \\ & & & \alpha & 0 \end{pmatrix} = J \cdot R \ .$$

Consider the permutation $\rho = (1\ 2)(3\ 4)\ \ldots\ (n\text{-}1\ n)$. Let $\beta = (2\ 3)\ \ldots\ (n\text{-}2\ n\text{-}1)$ so that $\beta\rho = (13\ \ldots\ n\text{-}3\ n\text{-}1\ n\ n\text{-}2\ n\text{-}4\ \ldots\ 42) = \sigma$ and $\rho = \beta\sigma$, where β is an involution and σ is cyclic with no fixed points. The operator J is an involution, so we focus on R . We have $R = BS$, where B is the permutation matrix corresponding to β and S is the "weighted permutation matrix" corresponding to σ .

Now the weights in $S(x)$ are $\alpha(x)$, but the permutation σ is independent of x . Therefore there exists a fundamental sequence $e_0, e_1, \ldots, e_{n-1}$ for $\mathcal{L}^2(X,\mu) \otimes \mathbb{C}^n = \int_X^{\oplus} \mathbb{C}^n d\mu$, such that

$$S(x)e_i(x) = \alpha(x)e_{i+1}(x) \quad \text{a.e.}$$

so that S is similar (within the algebra of n-normal operators) to

$$S' = \begin{pmatrix} 0 & 0 & & 0 & \alpha \\ \alpha & 0 & & . & 0 \\ 0 & \alpha & & . & . \\ . & 0 & & . & . \\ . & \vdots & & 0 & . \\ 0 & 0 & & \alpha & 0 \end{pmatrix} .$$

But now S' is similar to the operator

$$S'' = \begin{pmatrix} 0 & 0 & & 0 & 1 \\ -1 & 0 & & . & 0 \\ 0 & 1 & & . & . \\ . & 0 & & . & . \\ . & \vdots & & 0 & . \\ 0 & 0 & & 1 & 0 \end{pmatrix} .$$

For we can replace e_1 by αe_1 , e_2 by $\alpha^2 e_2, \ldots, e_{n-1}$ by $\alpha^{n-1}e_{n-1}$ and e_0 by

$\alpha^n e_0 = -e_0$ and the associated change of basis operator is n-normal and invertible. And S" is the product of two involutions C , D as is seen by defining $C(x)e_i(x) = e_{1-i}(x)$ for all i ,

$$D(x)e_i(x) = e_{-i}(x) \qquad \text{for } i \neq 0$$

and

$$D(x)e_0(x) = -e_0(x) .$$

Since being the product of two involutions is similarity invariant, and S" is similar to S , it follows that A = JBS is a product of four involutions.

This completes the proof.

Chapter 2

AF-Algebras

An AF-algebra, or approximately finite dimensional C*-algebra, is an inductive limit of a sequence of finite dimensional C*-algebras. A first comprehensive treatment of such algebras is to be found in [Br]. Other good references are [Eff] and [Bl]. We give a quick review of the basic facts and definitions.

To start with we need a sequence of C*-algebras $\mathfrak{A}_n$ together with a "coherent" family of *-homomorphisms $\phi_{mn} : \mathfrak{A}_m \to \mathfrak{A}_n$ for $m \leq n$ (so $\phi_{mp} = \phi_{np} \circ \phi_{mn}$ when $m \leq n \leq p$). Now consider the algebraic direct limit which is obtained as a quotient of the algebraic direct sum $\bigoplus_n \mathfrak{A}_n$. The defining equivalence relation is generated by $\phi_{mm+1}(A) \sim A$ for A in $\mathfrak{A}_m$. We endow this with a semi-norm $|||\cdot|||$ defined by

$$|||A||| = \limsup_n \|\phi_{mn}(A)\| .$$

Each ϕ_{mn} is bounded in norm by 1, so this quantity is finite. It can be shown that we lose no generality by assuming that each ϕ_{mn} is injective, and for *-homomorphisms of C*-algebras, this means isometric. Therefore our semi-norm is actually a norm. Completing in this norm we obtain the C*-algebra direct limit, denoted $\varinjlim(\mathfrak{A}_m,\phi_{mn})$. Since the maps ϕ_{mn} are isometric, we may think of the algebraic direct limit as the union $\bigcup_n \mathfrak{A}_n$, while the C*-algebra direct limit is the completion of this.

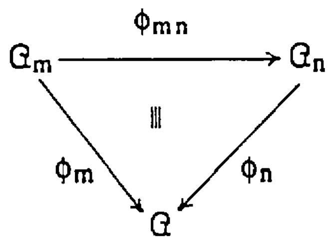
For an example, let $\mathfrak{A}_n = C\{1,\ldots,n\}$, the continuous complex-valued functions on the set $\{1,\ldots,n\}$. Define $\phi_{nn+1}(f)(j) = f(j)$ for $1 \leq j \leq n$ and 0 for $j = n+1$. Then $\bigcup_n \mathfrak{A}_n$ is the set of functions mapping $\mathbb{N}$ to $\mathbb{C}$ with finite support, while $\varinjlim(\mathfrak{A}_m,\phi_{mn}) \cong c_0(\mathbb{N})$.

Another example, as important as it is fundamental, is the non-commutative version of the algebra just given. Let $\mathcal{K}$ be the C*-algebra of compact operators and $\mathcal{A}_n$ be the $n\times n$ matrices of complex numbers. Define $\phi_{nn+1} : \mathcal{A}_n \to \mathcal{A}_{n+1}$ by

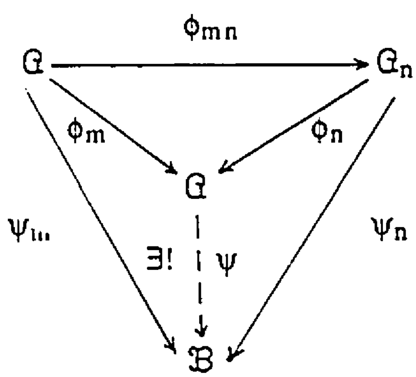
$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$. Then $\varinjlim(\mathcal{A}_m, \phi_{mn}) \cong \mathcal{K}$.

Note that in general, if $\mathcal{A} = \varinjlim \mathcal{A}_n$ and the symbol "~" denotes unitization, then $\mathcal{A}^\sim = \varinjlim \mathcal{A}_n^\sim$ under unital maps.

In our present setting where the morphisms ϕ_{mn} are all injective, we obtain naturally defined injective *-homomorphisms $\phi_n : \mathcal{A}_n \to \mathcal{A}$, $\mathcal{A} = \varinjlim(\mathcal{A}_n, \phi_{mn})$ such that the following diagram commutes:



In fact, the direct limit $\mathcal{A} = \varinjlim(\mathcal{A}_n, \phi_{mn})$ enjoys a universal property: if $\mathcal{B}$ is any C*-algebra and ψ_n is a sequence of *-homomorphisms, $\psi_n : \mathcal{A}_n \to \mathcal{B}$, such that for all $m \leq n$, $\psi_n \circ \phi_{mn} = \psi_m$, then there exists a unique *-homomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that $\psi \circ \phi_n = \psi_n$ for all n . There is a concise diagram for this.



This universal property characterizes the direct limit up to isomorphism.

Lemma 2.1: Let $\mathcal{A} = \lim\limits_{\rightarrow}(\mathcal{A}_n, \phi_{mn})$ with unital maps, and let A be invertible in $\mathcal{A}$. Then for each $\varepsilon > 0$, and for all sufficiently large n, there is an invertible B in $\mathcal{A}_n$ with $\|\phi_n(B) - A\| < \varepsilon$.

Proof. See [Bl], Prop. 3.3.3, p. 22.

Definition: An AF-algebra is a direct limit of a sequence of finite dimensional C*-algebras.

To understand AF-algebras it is necessary to know the structure of the morphisms ϕ_{mn}. Recall that a finite dimensional C*-algebra is a direct sum of matrix algebras over the complex numbers. So if $m_1,\ldots,m_r$, $n_1,\ldots,n_s$ are positive integers, $\mathcal{A} = \mathcal{M}_{m_1} \oplus \ldots \oplus \mathcal{M}_{m_r}$, $\mathcal{B} = \mathcal{M}_{n_1} \oplus \ldots \oplus \mathcal{M}_{n_s}$ and $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is a *-homomorphism, then ϕ is unitarily equivalent to a unique "canonical homomorphism" $\psi : \mathcal{A} \rightarrow \mathcal{B}$ (implemented by a unitary in $\mathcal{B}$). The canonical map ψ may be described as follows: write ψ as $(\psi_1,\ldots,\psi_s)$, where for each $i = 1,\ldots,s$, $\psi_i : \mathcal{A} \rightarrow \mathcal{M}_{m_i}$. For each i let $m_{i1},\ldots,m_{ir}$ be non-negative integers. Given $A = (A_1,\ldots,A_r)$ in $\mathcal{A}$, define ψ_i by

$$\psi_r : (A_1,\ldots,A_r) \mapsto \begin{pmatrix} A_1 & (m_{i1}) & & & & & & & & & & \\ & \ddots & & & & & & & & & & \\ & & A_1 & & & & & & 0 & & & \\ & & & A_2 & (m_{i2}) & & & & & & & \\ & & & & \ddots & & & & & & & \\ & & & & & A_2 & & & & & & \\ & & & & & & \ddots & & & & & \\ & & & & & & & A_r & (m_{ir}) & & & \\ & & 0 & & & & & & \ddots & & & \\ & & & & & & & & & A_r & & \\ & & & & & & & & & & 0 & \\ & & & & & & & & & & & \ddots \\ & & & & & & & & & & & & 0 \end{pmatrix}.$$

We call the matrix $(m_{ij})_{\substack{i=1,\ldots,s \\ j=1,\ldots,r}}$ the multiplicity matrix. This is because m_{ij} is the multiplicity of the embedding of $\mathcal{M}_{m_j}$ into $\mathcal{M}_{n_i}$. (See also [Eff], p. 8) The relevant theorem can be found in [Tak], Chapter I:

<u>Proposition 2.2</u>: Let $\phi : \mathfrak{A} \to \mathfrak{B}$ be a *-homomorphism of finite dimensional C*-algebras. Then ϕ is characterized up to unitary equivalence by the matrix (m_{ij}) of multiplicities. That is, if $\phi,\psi : \mathfrak{A} \to \mathfrak{B}$ with $(m_{ij}) = (m'_{ij})$ then there exists a unitary U in $\mathfrak{B}$ such that $\psi(A) = U\phi(A)U^*$ for all A in $\mathfrak{A}$.

To study AF-algebras, we may in fact restrict our attention to "canonical systems" $(\mathfrak{A}_n,\phi_{mn})$, that is, those for which each ϕ_{nn+1} is canonical in the sense just described. To sketch the proof of this we require some definitions and lemmas. (See [Eff].)

<u>Definition</u>: Let $\mathfrak{A}$ be a unital C*-algebra and $\alpha : \mathfrak{A} \to \mathfrak{A}$ be an automorphism. Then α is called an <u>inner automorphism</u> if there exists a unitary U in $\mathfrak{A}$ such that $\alpha(A) = UAU^*$ for all A in $\mathfrak{A}$.

<u>Definition</u>: Let $\phi,\psi : \mathcal{A} \to \mathcal{B}$ be *-homomorphisms of unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$. We say that ϕ and ψ are <u>inner equivalent</u> if there exist inner automorphisms $\gamma : \mathcal{A} \to \mathcal{A}$ and $\delta : \mathcal{B} \to \mathcal{B}$ for which the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ \phi\ } & \mathcal{B} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\delta} \\ \mathcal{A} & \xrightarrow[\ \psi\ ]{} & \mathcal{B} \end{array}$$

commutes.

<u>Lemma</u>: Given coherent systems $(\mathcal{A}_n,\phi_{mn})$ and $(\mathcal{A}_n,\psi_{mn})$ such that, for each n, ϕ_{nn+1} and ψ_{nn+1} are inner equivalent, it follows that $\varinjlim(\mathcal{A}_n,\phi_{mn})$ is *-isomorphic to $\varinjlim(\mathcal{A}_n,\psi_{mn})$.

<u>Corollary</u>: Let $\mathcal{A}$ be an AF-algebra, $\mathcal{A} = \varinjlim(\mathcal{A}_n,\phi_{mn})$. Then there exists a canonical system $(\mathcal{A}_n,\psi_{mn})$ such that for each n, ϕ_{nn+1} and ψ_{nn+1} are inner equivalent, and therefore such that

$$\varinjlim(\mathcal{A}_n,\phi_{mn}) \overset{*}{\cong} \varinjlim(\mathcal{A}_n,\psi_{mn}) \ .$$

Here are some examples.

(1) $$\mathcal{M}_1 \xrightarrow{\ \phi_{12}\ } \mathcal{M}_2 \xrightarrow{\ \phi_{23}\ } \mathcal{M}_3 \longrightarrow \cdots$$

where

$$\phi_{nn+1} : A \mapsto \begin{pmatrix} & A & & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ 0 & \cdots & 0 & 0 \end{pmatrix} .$$

As we mentioned earlier, the limit is isomorphic to the C*-algebra of compact operators.

(2) $$\mathcal{M}_1 \xrightarrow{\phi_{01}} \mathcal{M}_1 \oplus \mathcal{M}_1 \xrightarrow{\phi_{12}} \mathcal{M}_1 \oplus \mathcal{M}_1 \oplus \mathcal{M}_1 \oplus \mathcal{M}_1 \longrightarrow \ldots$$

where

$$\phi_{nn+1} : (A_1, \ldots, A_{2^n}) \mapsto (A_1, A_1, A_2, A_2, \ldots, A_{2^n}, A_{2^n}) .$$

The limit here is isomorphic to the continuous functions on the Cantor set K. The next example combines (1) and (2).

(3) $$\mathcal{M}_1 \xrightarrow{\phi_{01}} \mathcal{M}_2 \oplus \mathcal{M}_2 \xrightarrow{\phi_{12}} \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \mathcal{M}_3 \oplus \mathcal{M}_3 \longrightarrow \ldots$$

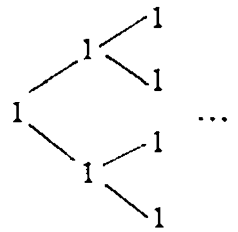
where

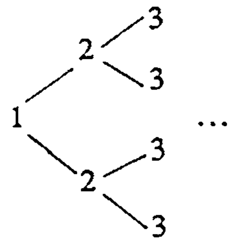
$$\phi_{nn+1} : (A_1, \ldots, A_{2^n}) \mapsto \left(\begin{pmatrix} A_1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} A_1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} A_2 & \\ & 0 \end{pmatrix}, \begin{pmatrix} A_2 & \\ & 0 \end{pmatrix}, \ldots, \begin{pmatrix} A_{2^n} & \\ & 0 \end{pmatrix}, \begin{pmatrix} A_{2^n} & \\ & 0 \end{pmatrix} \right)$$

The limit is isomorphic to $C(K) \otimes \mathcal{K}$, norm continuous functions from the Cantor set K to the algebra $\mathcal{K}$ of compact operators.

From now on, all morphisms and systems are canonical. Bratteli introduced a convenient notation for describing such systems [Br]. To each system $(\mathcal{A}_n, \phi_{mn})$ with limit $\mathcal{A}$, we associate a diagram (a graph) denoted $\mathcal{D}(\mathcal{A})$, and called the Bratteli diagram for the system. It contains the key features of the system, namely the sizes of the matrix summands together with the multiplicity numbers. Avoiding general notation we give instead the diagrams for the three examples above.

(1) $1 — 2 — 3 — \ldots$ $\qquad\qquad$ $\mathcal{K}$ = compact operators.

(2)  C(K) = continuous functions on the Cantor set.

(3)  $C(K)\otimes\mathcal{K}$ = compact operator-valued continuous functions on K.

A fourth example yields the famous CAR algebra, or Fermion algebra, of mathematical physics.

(4) $2 \overset{2}{\text{———}} 4 \overset{2}{\text{———}} 8 \text{———} \ \ldots$.

Each $\mathfrak{A}_n$ is $\mathcal{M}_{2^n}$, while $\phi_{nn+1} : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ is given by

$$\phi_{nn+1} : A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} .$$

Now that we have our terms defined and some notation established, we may return to our question of approximation by products of positive operators. In the last chapter we quoted Ballantine's theorem for matrix algebras which gives necessary and sufficient conditions for an operator A in $\mathcal{M}_n$ to be a product of four or five positive invertibles. A consequence is that $\overline{\mathcal{Q}_4}(\mathcal{M}_n) = \overline{\mathcal{Q}_\infty}(\mathcal{M}_n) = \{A \in \mathcal{M}_n : \det A \geq 0\}$. Our question is: to what extent can Ballantine's theorem be used for the analysis of products of non-negative operators in an AF-algebra? The first example to consider is the algebra $\mathcal{K}$ of compact operators.

Claim: $\overline{\mathcal{Q}}_4(\mathcal{K}) = \mathcal{K}$.

Proof: Corollary 2 of [KLMR] gives that every A in $\mathcal{K}$ may be approximated by products of four positive invertible operators. Thus the corollary states that $\mathcal{K} \subseteq \overline{\mathcal{P}}_4(\mathcal{B}(\mathcal{H}))$. But their proof of the corollary actually gives more. For by Theorem 6 of [FS], every compact operator is a product of two quasi-nilpotent operators, both of which may also be chosen to be compact. So, we can avoid Herrero's deep approximation theorem [H1, Theorem 5.1] by using the special structure of $\mathcal{K}$. That is, given a quasi-nilpotent compact operator B , we can show that B is a limit of products of two non-negative compact operators. For since B is quasi-nilpotent and restricted to $\mathcal{K}$ the spectrum is norm continuous, each of its canonical approximants may be assumed to be nilpotent and of finite rank (hence algebraic). But arguing as in Proposition 1 of [KLMR] we find that every such nilpotent is a limit of products of two non-negative finite rank operators. The claim follows.

We therefore have that for compact operators, $\mathcal{Q}_4(\mathcal{K})$ is dense in $\mathcal{K}$, whereas for any of the finite dimensional subalgebras $\mathcal{M}_n$, $\mathcal{Q}_4(\mathcal{M}_n)$ is a rather thin subset of $\mathcal{M}_n$. What is it that gives rise to this phenomenon? Corollary 2 of [KLMR] relies on a non-trivial result of Fong and Sourour [FS], whose own proof makes use of the structure of $\mathcal{B}(\mathcal{H})$ as well as a theorem of Anderson and Stampfli [AS]. We hope for an elementary approach which uses only that $\mathcal{K}$ is an AF-algebra. And since our proof given above (that $\overline{\mathcal{Q}}_4(\mathcal{K}) = \mathcal{K}$) also depends on Fong and Sourour's result, it is not yet clear what rôle the finite dimensional subalgebras might play in the analysis; nor that this line of thinking should extend to other AF-algebras.

Actually it is Theorem 2 of [KLMR] which points us in the right direction and we find that the determinant function is the right tool. To get a first clue into how this couples with the nest of finite dimensional subalgebras, consider the following compact operator:

Fix λ in $\mathbb{C}$, $\lambda \neq 0$, and let

$$T = \begin{pmatrix} \lambda & & & 0 \\ & \frac{\lambda}{2} & & \\ & & \frac{\lambda}{3} & \\ 0 & & & \ddots \end{pmatrix} .$$

Let $\varepsilon > 0$ and choose n such that

$$\|T-T_n\| < \varepsilon/2$$

with

$$T_n = \begin{pmatrix} \lambda & & & & & & \\ & \frac{\lambda}{2} & & & 0 & & \\ & & \ddots & & & & \\ & & & \frac{\lambda}{n} & & & \\ & & & & 0 & & \\ & 0 & & & & 0 & \\ & & & & & & 0 \\ & & & & & & & \ddots \end{pmatrix} .$$

Choose m in $\mathbb{N}$ and z in $\mathbb{C}$ so that $\lambda^n z^m > 0$, $|\frac{\lambda}{n+1} - z| < \frac{\varepsilon}{2}, \ldots, |\frac{\lambda}{n+m} - z| < \frac{\varepsilon}{2}$

and $|z| < \frac{\varepsilon}{2}$. Let

$$S = \begin{pmatrix} \lambda & & & & & & & & & & \\ & \frac{\lambda}{2} & & & & & & & & & \\ & & \ddots & & & & & & & & \\ & & & \frac{\lambda}{n} & & & & & & & \\ & & & & z & & & & & & \\ & & & & & \ddots & (m) & & & & \\ & & & & & & z & & & & \\ & & & & & & & 0 & & & \\ & & & & & & & & 0 & & \\ & & & & & & & & & 0 & \\ & & & & & & & & & & \ddots \end{pmatrix}$$

and observe that

$$\|T-S\| < \frac{\varepsilon}{2} .$$

Now we use the determinant function on the $(n+m)^{th}$ canonical approximant to S, so that by Ballantine's theorem there exist positive invertible $(n+m)\times(n+m)$ matrices $P_1,\ldots,P_5$ such that

$$P_1 \cdots P_5 = \begin{pmatrix} \lambda & & & & & & \\ & \frac{\lambda}{2} & & & & & \\ & & \ddots & & & & \\ & & & \frac{\lambda}{n} & & & \\ & & & & z & & \\ & & & & & \ddots & \\ & & & & & & z \end{pmatrix} .$$

But $\overline{\mathcal{Q}_4}(\mathcal{M}_{n+m}) = \overline{\mathcal{P}_5}(\mathcal{M}_{n+m})$. Thus, we may approximate $P_1 \ldots P_5$ by $Q_1 \ldots Q_4$ in $\mathcal{Q}_4(\mathcal{M}_{n+m})$ ensuring that

$$\|P_1 \ldots P_5 - Q_1 \ldots Q_4\| < \frac{\varepsilon}{2} .$$

Identifying $Q_1 \ldots Q_4$ with its canonical inclusion into $\mathcal{K}$, we obtain

$$\|Q_1 \ldots Q_4 - T\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Note that since the inclusion is a *-homomorphism, it follows that $Q_1 \ldots Q_4$ belongs to $\mathcal{Q}_4(\mathcal{K})$. It follows that T is in $\overline{\mathcal{Q}_4}(\mathcal{K})$.

What makes this work is that not only can we approximate T in $\mathcal{K}$ by operators S chosen from canonically embedded subalgebras, but there is "enough room" to insist that $\det S > 0$. A loose way of saying this is that the algebra of compact operators is "large enough" but not "too large". (Compare this to the algebra $\mathcal{B}(\mathcal{H})$, where despite the extra flexibility engendered by its intrinsic "largeness", it seems that it is this very trait which forces us to increase the number of positive factors required in our approximation theorems (see [Wu]$_2$). In fact, working with these ideas and using various approximation arguments we are able to completely characterize which AF-algebras have the property that $\overline{\mathcal{Q}_4}(\mathcal{A}) = \mathcal{A}$. Note that for any AF-algebra $\mathcal{A}$, $\overline{\mathcal{Q}_4}(\mathcal{A}) = \overline{\mathcal{Q}_\infty}(\mathcal{A})$,
so it is enough to consider $\mathcal{Q}_4$ (see Lemma 2.4).

Our main result here is in terms of the ideal structure of $\mathcal{A}$. We have that $\overline{\mathcal{Q}_4}(\mathcal{A}) = \overline{\mathcal{A}}$ if and only if $\mathcal{A}$ has neither finite dimensional ideals nor finite dimensional quotients. Because of Bratteli's work [Br] this translates into a statement about diagrams. The condition on the Bratteli diagrams requires some definitions and notation, as does the proof of the theorem. So let us tend to that next.

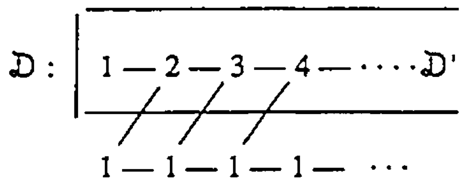
Suppose $\mathcal{D}$ is a Bratteli diagram for $\mathcal{A}$ and that $\mathcal{D}'$ is a "subdiagram" of $\mathcal{D}$. (We only use some of the matrix summands and all lines joining them: if we have part of $\mathcal{D}$ which looks like

$$k \xrightarrow{m>0} \ell$$

and k is a part of $\mathcal{D}'$ then so is ℓ .) We get an ideal by taking $\overline{(\bigcup_n \mathcal{A}_n')} = \mathcal{J}$, where

$\mathcal{A}'_n$ is the subalgebra of $\mathcal{A}_n$ with zero in all of the components not in $\mathcal{D}'$. Note that $\mathcal{D}'$ alone is a diagram for $\mathcal{J}$. In fact every ideal arises in this way. Furthermore, the quotient $\mathcal{A}/\mathcal{J}$ is also an AF-algebra with diagram $\mathcal{D}\backslash\mathcal{D}'$.

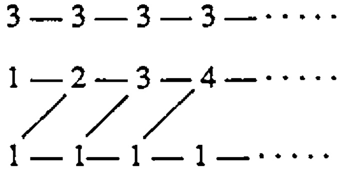
Example:



($\mathcal{D}$ is the diagram for the unitization $\mathcal{K}^\sim$ of the compact operators $\mathcal{K}$. $\mathcal{D}'$ is the diagram for the ideal $\mathcal{K} \leq \mathcal{K}^\sim$.)

The claim now is that the existence of either finite dimensional ideals or finite dimensional quotients corresponds to the existence of what we call "constant edges" in the Bratteli diagram. A constant edge is any infinite path through the diagram such that each vertex along this path is simple and of the same dimension, and so the connecting edges are non-zero and of multiplicity 1.

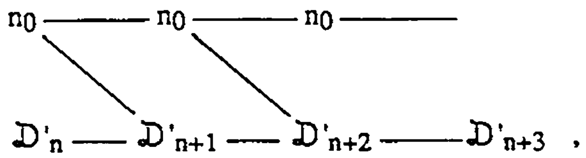
Example:



This is a diagram for $\mathcal{K}^\sim \oplus \mathcal{M}_3$. The constant edges correspond to the ideal $\mathcal{M}_3$ and the quotient $\mathcal{M}_1$.

Proposition 2.3. Let $\mathcal{A}$ be an AF-algebra and $\mathcal{D}$ be a Bratteli diagram for $\mathcal{A}$. Then $\mathcal{A}$ has a finite dimensional quotient if and only if there exists a constant edge in $\mathcal{D}$.

Proof: Suppose there is a constant edge E in $\mathcal{D}$, and that the dimension of each vertex in E is n_0^2. By considering multiplicity we find that $\mathcal{A}_n = \mathcal{A}'_n \oplus \mathcal{M}_{n_0}$ and that $\mathcal{D}$ must eventually look like



where $\mathcal{D}'$ corresponds to $\mathcal{D} \backslash E$. In this way we see that $\mathcal{D}'$ determines an ideal $\mathcal{J}$ in $\mathcal{A}$ such that $\mathcal{A}/\mathcal{J} \equiv \mathcal{M}'_{n_0}$. Observe that $\mathcal{A}/\mathcal{J}$ is a direct summand (i.e. a finite dimensional unital ideal) if and only if the diagonal lines in the above scheme are eventually zero.

For the converse, suppose $\mathcal{D}_0$ is a Bratteli diagram for some AF-algebra $\mathcal{B}$ which has no constant edges. Then along every edge of $\mathcal{D}_0$, the dimensions of the vertices are eventually increasing, making the dimension of $\mathcal{B}$ infinite. Assume now that $\mathcal{D}$ is a Bratteli diagram for our given AF-algebra $\mathcal{A}$, with no constant edge, and that $\mathcal{J}$ is an ideal with diagram $\mathcal{D}'$. Then $\mathcal{D} \backslash \mathcal{D}'$ and $\mathcal{D}'$ have no constant edges, so that by the above remarks, $\mathcal{J}$ and $\mathcal{A}/\mathcal{J}$ are both infinite dimensional. This completes the proof of the proposition.

Lemma 2.4: Let $\mathcal{A}$ be an AF-algebra, $\mathcal{A} = \overline{(\bigcup_n \mathcal{A}_n)}$. Then

(i) $\mathcal{Q}_1(\mathcal{A}) = \overline{\bigcup_n \mathcal{Q}_1(\mathcal{A}_n)}$, and

(ii) $\overline{\mathcal{Q}_4}(\mathcal{A}) = \overline{\mathcal{Q}_\infty}(\mathcal{A})$.

Proof: (i) Let A be in $\mathcal{Q}_1(\mathcal{A})$, so $A \geq 0$, and suppose A_{n_j} converges to A, A_{n_j} in $\mathcal{A}_{n_j}$. Then $(A^*_{n_j} A_{n_j})^{1/2}$ converges to A as well, so that $\mathcal{Q}_1(\mathcal{A}) \subseteq \overline{\bigcup_n \mathcal{Q}_1(\mathcal{A}_n)}$.

The other inclusion is obvious.

(ii) Since for each n $\mathcal{Q}_\infty(\mathcal{A}_n) = \overline{\mathcal{P}}_\infty(\mathcal{A}_n) = \overline{\mathcal{P}}_4(\mathcal{A}_n) = \overline{\mathcal{Q}}_4(\mathcal{A}_n) = \mathcal{Q}_4(\mathcal{A}_n)$, we therefore have the following chain of inclusions:

$$\begin{aligned}
\mathcal{Q}_\infty(\mathcal{A}) &\subseteq \overline{\bigcup_n \mathcal{Q}_\infty(\mathcal{A}_n)} && \text{(by (i))} \\
&= \overline{\bigcup_n \mathcal{Q}_4(\mathcal{A}_n)} && \text{(by Ballantine's Theorem)} \\
&\subseteq \overline{\mathcal{Q}}_4(\mathcal{A}) \\
&\subseteq \overline{\mathcal{Q}}_\infty(\mathcal{A}) .
\end{aligned}$$

The result follows.

We now state our main theorem on AF-algebras.

Theorem 2.5: Let $\mathcal{A}$ be an AF-algebra with diagram $\mathcal{D}$. Then the following are equivalent:

(i) $\overline{\mathcal{Q}}_4(\mathcal{A}) = \mathcal{A}$

(ii) $\mathcal{D}$ has no constant edges

(iii) $\mathcal{A}$ has no finite dimensional quotients.

At this point we require a series of lemmas.

Lemma 2.6: Suppose $\mathcal{F}$ is a finite dimensional C*-algebra. Then $\overline{\mathcal{Q}}_4(\mathcal{F})$ is properly contained in $\mathcal{F}$.

Proof: Because $\mathcal{F}$ is finite dimensional,

$$\mathcal{J} \equiv \bigoplus_{j=1}^{m} \mathcal{M}_{n_j} .$$

From Ballantine's theorem

$$\overline{\mathcal{Q}_4}(\mathcal{J}) = \{ \bigoplus_{j=1}^{n} A_j : \det A_j \geq 0 \} .$$

Since the the determinant function is continuous, and we can construct $X = \bigoplus_{j=1}^{m} X_j$ in $\mathcal{J}$ such that $\det X_j = -1$ for $j = 1,\ldots,m$, it follows that X is not in $\overline{\mathcal{Q}_4}(\mathcal{J})$.

Lemma 2.7. Suppose $\mathcal{A}$ is an AF-algebra and $\mathcal{J}$ is an ideal such that $\mathcal{A}/\mathcal{J}$ is finite dimensional. Then $\overline{\mathcal{Q}_4}(\mathcal{A})$ is properly contained in $\mathcal{A}$.

Proof: Consider the canonical surjection π.

$$\pi : \mathcal{A} \to \mathcal{A}/\mathcal{J} .$$

If $\overline{\mathcal{Q}_4}(\mathcal{A}) = \mathcal{A}$, then because π is a *-epimorphism of C*-algebras,

$$\mathcal{A}/\mathcal{J} = \pi(\overline{\mathcal{Q}_4}(\mathcal{A})) \subseteq \overline{\mathcal{Q}_4}(\mathcal{A}/\mathcal{J}) \subseteq \mathcal{A}/\mathcal{J} ,$$

that is

$$\overline{\mathcal{Q}_4}(\mathcal{A}/\mathcal{J}) = \mathcal{A}/\mathcal{J} .$$

This is a contradiction since $\mathcal{A}/\mathcal{J}$ is finite dimensional.

Lemma 2.8. Let $\lambda_1,\ldots,\lambda_n$ be non-zero complex numbers. Then for each $\varepsilon > 0$ there exists $N = N(\varepsilon)$ in $\mathbb{N}$ such that the following is true:

If $m_1,\ldots,m_n,m_{n+1}$ are non-negative integers such that $m_1 + \ldots + m_n + m_{n+1} \geq N$ then there exist complex numbers $\mu_1,\ldots,\mu_n,\rho$ with $|\mu_j - \lambda_j| < \varepsilon$, $|\rho| < \varepsilon$ and $\mu_1^{m_1}\ldots\mu_n^{m_n}\,\rho^{m_{n+1}} > 0$.

Remark: This generalizes the fact that for $\lambda \neq 0$ and $\varepsilon > 0$ there exists n_0 in $\mathbb{N}$ and μ in $\mathbb{C}$ such that $|\mu-\lambda| < \varepsilon$ and $\mu^{n_0} > 0$.

Proof: Let θ be a branch of the argument function with $\theta(\lambda_j) \in [0,2\pi)$. $j = 1,\ldots,n$, and suppose $\varepsilon > 0$. Choose $\delta > 0$ small enough so that $0 < \delta < \frac{\varepsilon}{2}$, and if both $|\lambda_j| = |\mu_j|$ and $|\theta(\lambda_i) - \theta(\mu_i) < \delta$, then $|\mu_j - \lambda_j| < \varepsilon$. Choose $n_0 \in \mathbb{N}$ so that $n_0 \cdot 2\delta > 2\pi$, and let $S_1,\ldots,S_{n+1}$ be intervals defined by $S_j = (\theta(\lambda_j) - \delta_1, \theta(\lambda_j) + \delta)$, $j = 1,\ldots,n$ and $S_{n+1} = (-\delta,\delta)$. Then the length of each interval $n_0 \cdot S_j$ is greater than 2π for all $j = 1,\ldots,n+1$. Now, if $m_1 + \ldots + m_{n+1} \geq (n+1)n_0$, then there is a j_0, $1 \leq j_0 \leq n+1$ for which $m_{j_0} \geq n_0$. It follows that the interval $m_1S_1 + \ldots + m_{n+1}S_{n+1}$ has length greater than 2π. Therefore, there exist $\theta_j \in S_j$ satisfying $m_1\theta_1 + \ldots + m_{n+1}\theta_{n+1} \equiv 0 \bmod 2\pi$. To finish the proof just let $N = (n+1)n_0$. Then with $\mu_j = |\lambda_j|e^{i\theta_j}$, $j = 1,\ldots,n$ and $\rho = \varepsilon e^{i\theta_{n+1}}$, $\mu_1^{m_1} \ldots \mu_n^{m_n} \rho^{m_{n+1}} > 0$, $|\rho| < \varepsilon$ and $|\mu_j - \lambda_j| < \varepsilon$ as required.

Remark: If $m_{n+1} > 0$ then there is always a solution to $\lambda \cdot \rho^{m_{n+1}} > 0$, $\lambda \in \mathbb{C} \backslash \{0\}$ arbitrary, $|\rho| < \varepsilon$. What the proof shows is that ρ may be chosen from within an arbitrarily small prescribed arc centered at any θ_0 -- in particular $\theta_0 = 0$.

From Lemma 2.8 we obtain a useful corollary.

Corollary 2.9. Suppose $\lambda_1,\ldots,\lambda_n$ are non-zero complex numbers and that we have a sequence of upper-triangular matrices $\{A_k\}$, A_k in $\mathfrak{M}_{m_k}$,

$$A_k = \begin{pmatrix} \lambda_1 & & & & & & & & & & & \\ & \ddots & & & & & & & & * & & \\ (m_{k_1}) & & \lambda_1 & & & & & & & & & \\ & & & \lambda_2 & & & & & & & & \\ & & & & \ddots & & & & & & & \\ & & & (m_{k_2}) & & \lambda_2 & & & & & & \\ & & & & & & \ddots & & & & & \\ & & & & & & & \lambda_n & & & & \\ & & & & & & & & \ddots & & & \\ & 0 & & & & & & (m_{k_n}) & & \lambda_n & & \\ & & & & & & & & & & 0 & \\ & & & & & & & & & & & \ddots \\ & & & & & & & & & & (z) & & 0 \end{pmatrix} .$$

If $m_k = m_{k_1} + \ldots + m_{k_n} + z$ is an unbounded sequence of positive integers, then for each $\varepsilon > 0$, there exists $k = k(\varepsilon)$ and B_k in $\mathcal{M}_{m_k}$ such that

$$B_k \in \mathcal{Q}_4(\mathcal{M}_{m_k}) \quad \text{and} \quad \|A_k - B_k\| < \varepsilon .$$

Proof: Let $\varepsilon > 0$. By Lemma 2.8 we may choose k large enough to allow us to perturb the diagonals of A_k so that the resulting matrix B'_k has positive determinant and $\|A_k - B'_k\| < \frac{\varepsilon}{2}$. Since $\det B'_k > 0$, B_k is in $\mathcal{P}_5(\mathcal{M}_{m_k})$. But $\overline{\mathcal{Q}_4}(\mathcal{M}_{m_k}) = \overline{\mathcal{P}_5}(\mathcal{M}_{m_k})$, hence there exists B_k in $\mathcal{Q}_4(\mathcal{M}_{m_k})$ satisfying $\|B_k - B'_k\| < \frac{\varepsilon}{2}$. It follows that $\|A_k - B_k\| < \varepsilon$, as required.

Proof of Theorem 2.5: By Lemmas 2.6 and 2.7, (i) implies (iii). By Proposition 2.3, (iii) is equivalent to (ii). Therefore it remains to prove that (ii) implies (i). (It's a good idea to keep our example of the diagonal compact operator in mind.)

Let A be in $\mathcal{Q}$. Then by definition,

$$A = \lim_n A_n , \quad A_n = \phi_n(A'_n) ,$$

where $\phi_n : \mathcal{G}_n \to \mathcal{G}$ is the canonical *-monomorphism. Moreover, for each n, the invertibles of $\mathcal{G}_n$ are dense in $\mathcal{G}_n$. So to prove that $\overline{Q_4}(\mathcal{G}) = \mathcal{G}$ we show that if A in $\mathcal{G}_n$ is invertible and $\varepsilon > 0$, then there exists $\ell \geq n$ and B in $Q_4(\mathcal{G}_j)$ such that

$$\|\phi_{n\ell}(A) - B\| < \varepsilon .$$

To do this first note that there is a unitary U in $\mathcal{G}_n$ such that $A' = U^*AU$ is of the form

$$A' = \bigoplus_{k=1}^{r} \begin{pmatrix} \lambda_1^{(k)} & & & * \\ & \cdot & & \\ & & \cdot & \\ & 0 & & \cdot \\ & & & & \lambda_{n_k}^{(k)} \end{pmatrix}$$

with $\lambda_1^{(k)} \ldots \lambda_{n_k}^{(k)} \neq 0$ for $k = 1,2,\ldots,r$.

What we do now is follow A' through the system. For each $\ell \geq n$, $\phi_{n\ell}$ is of the form

$$\phi_{n\ell} = (\psi_1,\ldots,\psi_s) , \quad s \in \mathbb{N} \quad .$$

For the moment consider only one component of $\phi_{n\ell}$, and denote this by ψ. Then A' is mapped via the canonical morphisms, to an element C, which is itself unitarily equivalent, via W say, to an element C' which is upper triangular and whose diagonal is of the form

$$\mathrm{diag}(\lambda_1^{(1)},\ldots,\lambda_1^{(1)},\ldots,\lambda_{n_r}^{(r)},\ldots,\lambda_{n_r}^{(r)},0,0,\ldots,0) .$$

Suppose that λ_j^k appears with multiplicity m_j^k and that 0 appears with multiplicity z_j. By the hypothesis on $\mathcal{D}$

$$\sum_{k,j} m_j^k + z_j$$

is unbounded as $\ell \to \infty$. By Corollary 2.9 we choose N in $\mathbb{N}$ such that $\sum m_j^k + z_j \geq N$ implies the existence of B' in Q_4 with $\|C' - B'\| < \varepsilon$. Therefore

$$
\begin{aligned}
\varepsilon &> \|C' - B'\| \\
&= \|W^*CW - B'\| \\
&= \|C - WB'W^*\| \\
&= \|\psi(A') - WB'W^*\| \\
&= \|\psi(U^*AU) - WB'W^*\| \\
&= \|\psi(U)^*\psi(A)\psi(U) - WB'W^*\| \ .
\end{aligned}
$$

But U is unitary so that $\psi(U)$ is a partial isometry that extends to a unitary V which satisfies $\psi(U)^*\psi(A)\psi(U) = V^*\psi(A)V$. Hence this last quantity equals

$$\|V^*\psi(A)V - WB'W^*\| = \|\psi(A) - (VW)B(VW)^*\| ,$$

and since VW is unitary and B' is in $\mathcal{Q}_4$, $(VW)B'(VW)^*$ is also in $\mathcal{Q}_4$. But there exists $\ell > n$ such that every component of $\phi_{n\ell}(A)$ satisfies

$$\sum m_j^k + z_j \geq N \ ,$$

for otherwise we could deduce the existence of a constant edge. Therefore the distance from $\phi_{n\ell}(A)$ to $\mathcal{Q}_4(\mathcal{A}_j)$ is less than ε. Since $\varepsilon > 0$ was arbitrary, we conclude that $\overline{\mathcal{Q}_4}(\mathcal{A}_j) = \mathcal{A}$, as claimed.

This finishes our proof of Theorem 2.5.

<u>Examples</u>:

1. Let $\mathcal{K}$ be the compact operators. A diagram for $\mathcal{K}$ is

1 — 2 — 3 — 4 — ...

which has no constant edges. So $\overline{\mathcal{Q}_4}(\mathcal{K}) = \mathcal{K}$.

2. Let $\mathcal{K}^\sim$ be the unitization of the compact operators. A diagram for $\mathcal{K}^\sim$ is

```
1 — 2 — 3 — 4 — ...
   /   /   /
1 — 1 — 1 — 1 — ...
```

which has a constant edge (of 1's). So $\overline{\mathcal{Q}}_4(\mathcal{K}^\sim) \subsetneq \mathcal{K}^\sim$.

Corollary 2.10: Let $\mathcal{A}$ be an AF-algebra and $\mathcal{J}$ be an ideal. If $\overline{\mathcal{Q}}_4(\mathcal{A}/\mathcal{J}) = \mathcal{A}/\mathcal{J}$ and $\overline{\mathcal{Q}}_4(\mathcal{J}) = \mathcal{J}$ then $\overline{\mathcal{Q}}_4(\mathcal{A}) = \mathcal{A}$.

Proof: Let $\mathcal{D}$ be a diagram for $\mathcal{A}$. Then use condition (ii) of the theorem together with the correspondence in AF-algebras between ideals and subdiagrams.

Remarks: Throughout this thesis our concern is not only factorization and approximation, but doing so within a prescribed algebra, or even class, of operators. Theorem 2.5 as well as Theorem 2 of [KLMR] are definite results along these lines. And, as mentioned earlier, the proof for Theorem 2.5 was motivated in part by the result of [KLMR] together with a consideration of the (AF) algebra of compact operators. So these theorems are not independent of each other. It is with this in mind that three related points seem worth mentioning.

1. The proof of Theorem 2 [KLMR] actually gives a sharper result than the statement of the theorem, which says that if A is algebraic then A is in $\overline{\mathcal{P}}_4(B(\mathcal{H}))$. (Note that by "algebraic" we mean an operator which satisfies a polynomial equation over $\mathbb{C}$ in the single variable z .) Letting $\mathcal{A}\ell g$ denote the set of all algebraic operators, we in fact have the inclusion

$$\mathcal{A}\ell g \subseteq \overline{\mathcal{P}}_4(\mathcal{A}\ell g)$$

that is, the factors in approximation may themselves be chosen to be algebraic.

To see why this is so requires only a small observation in their proof. but also brings us to the second point. We therefore review the main steps of the argument.

If A is algebraic, then using the primary decomposition theorem,

$$A = \begin{pmatrix} \alpha_1 & A_{12} & \cdots & A_{1k} \\ & \cdot & \ddots & \vdots \\ & & \cdot & A_{k-1k} \\ 0 & & & \alpha_k \end{pmatrix}, \quad \alpha_j \in \mathbb{C} \ ,$$

with respect to some decomposition of the Hilbert space $\mathcal{H}$,

$$\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_k .$$

Approximate A by an operator B

$$B = \begin{pmatrix} \beta_1 & A_{12} & \cdots & A_{1k} \\ & \cdot & \ddots & \vdots \\ & & \cdot & A_{k-1,k} \\ 0 & & & \beta_k \end{pmatrix}$$

obtained from A by perturbing the scalars α_j so that the β_j's are all distinct. By a corollary to Rosenblum's Corollary (see Chapter 0 of [R&R]), B is similar, via R say, to the diagonal operator

$$B' = R^{-1}BR = \begin{pmatrix} \beta_1 & & & & \\ & \cdot & & 0 & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & \beta_k \end{pmatrix}.$$

But now, using the fact that $\mathcal{H}$ is infinite dimensional, it is possible to use the determinant function to construct a finite number of diagonal (scalar) matrices D_j such that D_j is not a multiple of the identity, $\det D_j > 0$ and the operator D built out of inflations of the D_j's approximates B'. It is then an application of Ballantine's theorem to see that D, which is of the form

$$D = \sum_j^{\oplus} D_j^{(\ell_j)}$$

belongs to $\mathcal{P}_4(\mathcal{B}(\mathcal{H}))$. But $\mathcal{P}_4(\mathcal{B}(\mathcal{H}))$ is similarity invariant. Hence RDR^{-1} at once approximates A and is a product of four positive invertible operators.

The observation to be made is that D actually factors as a product of four (positive invertible) algebraic operators, for we are just applying Ballantine's theorem to a finite number of matrices (inflations) and therefore obtaining matrices (inflations) as factors, each of which is of course algebraic. And any operator similar to an algebraic operator is also algebraic (the set $\mathcal{Alg}$ is similarity invariant). So if

$$D = P_1P_2P_3P_4 \ , \ P_i \text{ algebraic,}$$

then

$$RDR^{-1} = (RP_1R^{-1})(RP_2R^{-1})(RP_3R^{-1})(RP_4R^{-1}) \ , RP_iR^{-1} \text{ algebraic}$$

is a member of $\mathcal{P}_4(\mathcal{Alg})$, as claimed.

2. The second point is really a question, namely, what is the relation between the fact that

$$\mathcal{Alg} \subseteq \overline{\mathcal{P}}_4(\mathcal{Alg}) = \overline{\mathcal{Q}}_4(\mathcal{Alg})$$

and that for certain AF-algebras $\mathcal{A}$ we have

$$\mathcal{A} \subsetneq \overline{\mathcal{Q}}_4(\mathcal{A}) \ ?$$

Some light is shed on this by defining "AF-operators": Let $T \in \mathcal{B}(\mathcal{H})$. Then T is an <u>AF-operator</u> if and only if there exists a sequence of finite dimensional C*-subalgebras $\mathcal{A}_n \subsetneq \mathcal{B}(\mathcal{H})$ together with $A_n \in \mathcal{A}_n$ such that

$$T = \lim_n A_n \quad \text{(operator norm)} \ .$$

Let $\mathcal{AF}$ denote the union of all AF-algebras in $\mathcal{B}(\mathcal{H})$ we have

$$T \text{ is an AF-operator} \Leftrightarrow T \in \overline{\mathcal{AF}} \ .$$

Now let $\{A_j\}_{j=1,\ldots,k}$ be any finite sequence of $n_j \times n_j$ matrices of complex numbers, c_j be cardinal numbers, $1 \le c_j \le \infty$ and $A_j^{(c_j)}$ be the c_j-fold inflation of A_j. Consider the operator T in $\mathcal{B}(\mathcal{H})$ $(\dim \mathcal{H} = \infty)$ given by

$$T = \sum_{j=1}^{k}{}^{\oplus} A_j^{(c_j)} \quad .$$

Then T is clearly an AF-operator. For just let

$$\mathcal{A} = \bigoplus_{j=1}^{k} \mathcal{M}_{nj}$$

and define $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by

$$\pi : (M_1,\ldots,M_k) \mapsto \sum_{j=1}^{k}{}^{\oplus} M_j^{(c_j)} \quad .$$

Then $\pi(\mathcal{A})$ is a finite dimensional subalgebra of $\mathcal{B}(\mathcal{H})$ and T is in $\pi(\mathcal{A})$.

It now follows from the first remark that if A is an algebraic operator, then

$$A = \lim_n A_n$$

is a limit of operators A_n, each of which is similar to an operator B_n from a finite dimensional algebra $\mathcal{A}_n$ such that

$$B_n \in \mathcal{P}_4(\mathcal{A}_n) \subseteq \mathcal{P}_4(\mathcal{A}\mathcal{F}) .$$

This now accounts for why the proof of our Theorem 2.5 and that of Theorem 2 of [KLMR] have some bearing on each other.

To think of this more set-theoretically, define

$$\mathcal{S}im_4(\mathcal{A}\mathcal{F}) = \{X^{-1}AX : X \text{ invertible in } \mathcal{B}(\mathcal{H}),\ A \text{ belongs to } \mathcal{Q}_4(\mathcal{A}\mathcal{F})\} .$$

and for completeness

$$\mathcal{S}im(\mathcal{AF}) = \{X^{-1}AX : X \text{ invertible in } \mathcal{B}(\mathcal{H}),\ A \text{ in } \mathcal{AF}\}\ .$$

Then we've shown that

$$\overline{\mathcal{A}lg} \subsetneq \overline{\mathcal{S}im_4(\mathcal{AF})} \subseteq \overline{\mathcal{S}im(\mathcal{AF})}\ .$$

On the other hand, if T is any AF-operator then T is a limit of algebraic operators, so

$$\mathcal{AF} \subsetneq \overline{\mathcal{AF}} \subsetneq \overline{\mathcal{A}lg}\ .$$

Thus, the question we started with raises the more precise question of whether or not the chain of inclusions

$$\overline{\mathcal{AF}} \subsetneq \overline{\mathcal{A}lg} \subsetneq \overline{\mathcal{S}im_4(\mathcal{AF})} \subsetneq \overline{\mathcal{S}im(\mathcal{AF})}\ .$$

is actually an equality. By returning to the proof of Theorem 2 [KLMR] we find that a first step towards resolving this would be deciding whether or not operators of the form

$$\begin{pmatrix} 1 & X & Y \\ 0 & 2 & Z \\ 0 & 0 & 3 \end{pmatrix}$$

are in $\overline{\mathcal{AF}}$. We emphasize that X, Y, Z are arbitrary; also that every AF-operator T is a limit of operators A_n such that for each n, A_n is a finite complex matrix relative to some orthogonal decomposition of $\mathcal{H}$.

3) We therefore come to another question. If it happens that $\overline{\mathcal{AF}}$ is properly contained in $\overline{\mathcal{A}lg}$, how can we measure the difference? What function or invariant would be sensitive to the finite dimensionality of AF-operators? This could be related to "points of spectral continuity" and the Remarks at the end of this chapter. See also Theorem 6.15 of [H1].

For completeness we make one last remark.

4) George Elliott proved that for AF-algebras $\mathcal{A}$, the scaled ordered group $K_0(\mathcal{A})$ is a complete isomorphism invariant. So what is $K_0(\mathcal{A})$ for an AF-algebra $\mathcal{A}$ such that $\overline{\mathcal{Q}_4}(\mathcal{A}) = \mathcal{A}$?

Our next proposition deals with C*-tensor products of AF-algebras. A C*-algebra $\mathcal{A}$ is said to be nuclear if for every C*-algebra $\mathcal{B}$, there is only one C*-norm on the algebraic tensor product. By Proposition 11.3.12 of [KR]$_2$, every AF-algebra is nuclear. Thus, given two AF-algebras $\mathcal{A}$ and $\mathcal{B}$, the C*-tensor product $\mathcal{A}\otimes\mathcal{B}$ is well-defined.

Proposition 2.11: Suppose $\mathcal{A}$ and $\mathcal{B}$ are AF-algebras. Then $\overline{\mathcal{Q}_4}(\mathcal{A}\otimes\mathcal{B}) = \mathcal{A}\otimes\mathcal{B}$ if and only if $\overline{\mathcal{Q}_4}(\mathcal{A}) = \mathcal{A}$ or $\overline{\mathcal{Q}_4}(\mathcal{B}) = \mathcal{B}$.

To prove this we require some notation and some preliminary results.

For any AF-algebra $\mathcal{A} = \varinjlim(\mathcal{A}_n, \phi_{mn})$, a Bratteli diagram for $\mathcal{A}$, $\mathcal{D}(\mathcal{A})$, naturally decomposes into

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}_1(\mathcal{A}) \overset{E_1}{\text{———}} \mathcal{D}_2(\mathcal{A}) \overset{E_2}{\text{———}} \mathcal{D}_3(\mathcal{A}) \text{———} \cdots ,$$

where $\mathcal{D}_n(\mathcal{A})$ is the set of weighted vertices corresponding to the n^{th} algebra and E_n denotes the connecting edges, with multiplicity numbers, between $\mathcal{D}_n(\mathcal{A})$ and $\mathcal{D}_{n+1}(\mathcal{A})$. If $\mathcal{A}$ and $\mathcal{B}$ are AF-algebras with diagrams $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{B})$, is there an easy way to obtain a diagram for $\mathcal{D}(\mathcal{A}\otimes\mathcal{B})$? Indeed, all we have to do is form a "vertex-wise" tensor product of the diagrams:

Lemma 2.12: Let $\mathcal{A} = \varinjlim(\mathcal{A}_n, \phi_{mn})$ and $\mathcal{B} = \varinjlim(\mathcal{B}_n, \psi_{mn})$ be two AF-algebras. Then

$$\mathcal{A}\otimes\mathcal{B} \equiv \lim_{\rightarrow}(\mathcal{A}_n\otimes\mathcal{B}_n\,,\,\phi_{mn}\otimes\psi_{mn})\,.$$

Proof: Firstl, note that $(\mathcal{A}_n\otimes\mathcal{B}_n\,,\,\tau_{mn} = \phi_{mn}\otimes\psi_{mn})$ is a coherent system. Therefore the direct limit $\mathcal{C}$ exists, with canonical monomorphisms $\tau_n : \mathcal{A}_n\otimes\mathcal{B}_n \rightarrow \mathcal{C}_n$.

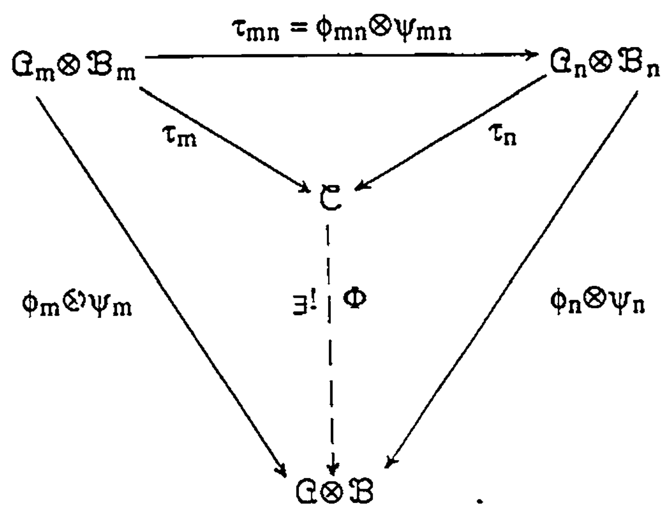
Secondly, if $\phi_n : \mathcal{A}_n \rightarrow \mathcal{A}$ with $\psi_n : \mathcal{B}_n \rightarrow \mathcal{B}$ are the canonical monomorphisms for $\mathcal{A}$ and $\mathcal{B}$, then we obtain *-homomorphisms $\phi_n\otimes\psi_n : \mathcal{A}_n\otimes\mathcal{B}_n \rightarrow \mathcal{A}\otimes\mathcal{B}$. Moreover, for $m \leq n$

$$(\phi_n\otimes\psi_n)\circ(\phi_{mn}\otimes\psi_{mn}) = \phi_n\otimes\psi_n\,.$$

Therefore, by the universal property of direct limits, we obtain a unique *-homomorphism

$$\Phi : \mathcal{C} \rightarrow \mathcal{A}\otimes\mathcal{B}\,.$$

By considering the algebraic tensor product $\mathcal{A}\otimes_{alg}\mathcal{B}$ which is dense in $\mathcal{A}\otimes\mathcal{B}$, we find that Φ is one-one and onto and therefore an isomorphism. The usual kind of diagram for this proof is below.



It now follows that a diagram for $\mathcal{A}\otimes\mathcal{B}$ can be obtained from the sequence

$$\mathcal{A}_1 \otimes \mathcal{B}_1 \xrightarrow{\phi_{12} \otimes \psi_{12}} \mathcal{A}_2 \otimes \mathcal{B}_2 \xrightarrow{\phi_{23} \otimes \psi_{23}} \mathcal{A}_3 \otimes \mathcal{B}_3 \longrightarrow \cdots$$

Therefore, our next lemma is:

Lemma 2.13: Let $\phi : \mathfrak{M}_r \to \mathfrak{M}_s$ and $\psi : \mathfrak{M}_k \to \mathfrak{M}_\ell$ be *-homomorphisms with multiplicities $m(\phi)$ and $m(\psi)$ respectively. Then $\phi \otimes \psi : \mathfrak{M}_r \otimes \mathfrak{M}_k \to \mathfrak{M}_s \otimes \mathfrak{M}_\ell$ has multiplicity

$$m(\phi \otimes \psi) = m(\phi) \cdot m(\psi) \ .$$

Proof: Use canonical homomorphisms.

Corollary 2.14: Suppose $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$ are finite dimensional C*-algebras and $\phi : \mathcal{A}_1 \to \mathcal{A}_2$, $\psi : \mathcal{B}_1 \to \mathcal{B}_2$ are *-homomorphisms. Let $M = (m_{ij})$ and $N = (n_{k\ell})$ be the multiplicity matrices for ϕ and ψ. Then the multiplicity matrix for $\phi \otimes \psi$ is given by $M \otimes N$.

Now let $\mathcal{A}$ and $\mathcal{B}$ be two AF-algebras, with diagrams

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}_1(\mathcal{A}) \overset{E_1}{\text{———}} \mathcal{D}_2(\mathcal{A}) \overset{E_2}{\text{———}} \mathcal{D}_3(\mathcal{A}) \text{———} \cdots ,$$

$$\mathcal{D}(\mathcal{B}) = \mathcal{D}_1(\mathcal{B}) \overset{E_1}{\text{———}} \mathcal{D}_2(\mathcal{B}) \overset{E_2}{\text{———}} \mathcal{D}_3(\mathcal{B}) \text{———} \cdots .$$

Then for each n, $\mathcal{D}_n(\mathcal{A}) \otimes \mathcal{D}_n(\mathcal{B})$ is the diagram for the tensor product of the two finite dimensional algebras $\mathcal{A}_n$, $\mathcal{B}_n$, while $E_n \otimes F_n$ is the set of weighted edges corresponding to $\phi_n \otimes \psi_n : \mathcal{A}_n \otimes \mathcal{B}_n \to \mathcal{A}_{n+1} \otimes \mathcal{B}_{n+1}$. It now follows from Lemma 2.12 and Corollary 2.14 that we obtain a diagram for $\mathcal{A} \otimes \mathcal{B}$, namely,

$$\mathcal{D}(\mathcal{A}\otimes\mathcal{B}) = \mathcal{D}_1(\mathcal{A})\otimes\mathcal{D}_1(\mathcal{B}) \overset{E_1\otimes F_1}{\text{———}} \mathcal{D}_2(\mathcal{A})\otimes\mathcal{D}_2(\mathcal{B}) \overset{E_2\otimes F_2}{\text{———}} \mathcal{D}_3(\mathcal{A})\otimes\mathcal{D}_3(\mathcal{B}) \text{———} \cdots .$$

Moreover, the multiplicity matrices are related by

$$M(\phi_n\otimes\psi_n) = M(\phi_n) \otimes M(\psi_n) \quad .$$

This fact, together with Theorem 2.5, allows us to analyze $\overline{\mathcal{Q}_4}(\mathcal{A}\otimes\mathcal{B})$ via the Bratteli diagrams for $\mathcal{A}$ and $\mathcal{B}$. In other words, we can now establish Proposition 2.11.

<u>Proof of Proposition 2.11</u>: Let $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{B})$ be diagrams for $\mathcal{A}$ and $\mathcal{B}$ respectively. If $\overline{\mathcal{Q}_4}(\mathcal{A})$ is properly contained in $\mathcal{A}$ and $\overline{\mathcal{Q}_4}(\mathcal{B})$ is properly contained in $\mathcal{B}$, then by Theorem 2.5 $\mathcal{D}(\mathcal{A})$ has a constant edge of weight j say, and $\mathcal{D}(\mathcal{B})$ has a constant edge of weight k. Now form the "vertex-wise" tensor product of the diagrams, and use the fact that the tensor product is distributive over direct sums, to conclude that $\mathcal{D}(\mathcal{A}) \otimes \mathcal{D}(\mathcal{B}) = \mathcal{D}(\mathcal{A}\otimes\mathcal{B})$ has a constant edge of weight $j\cdot k$. Hence, by Theorem 2.5 again, $\overline{\mathcal{Q}_4}(\mathcal{A}\otimes\mathcal{B})$ is properly contained in $\mathcal{A}\otimes\mathcal{B}$.

Conversely, it is a finite counting argument to show that any constant edge in $\mathcal{D}(\mathcal{A}) \otimes \mathcal{D}(\mathcal{B})$ must occur in this way. So if $\overline{\mathcal{Q}_4}(\mathcal{A}\otimes\mathcal{B})$ is properly contained in $\mathcal{A}\otimes\mathcal{B}$, then the same must be true of $\mathcal{A}$ and $\mathcal{B}$.

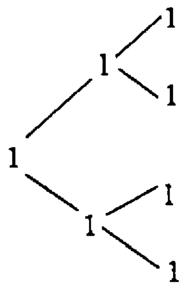
<u>Examples</u>:

1) $\mathcal{D}(\mathcal{A}) = 1 — 1 — 1 — \cdots$ $\qquad \overline{\mathcal{Q}_4}(\mathcal{A}) \subsetneq \mathcal{A}$

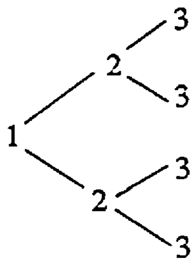
$\mathcal{D}(\mathcal{B}) = 1 — 2 — 3 — \cdots$ $\qquad \overline{\mathcal{Q}_4}(\mathcal{B}) = \mathcal{B}$

$\mathcal{D}(\mathcal{A}\otimes\mathcal{B}) = 1 — 2 — 3 — \cdots$ $\qquad \overline{\mathcal{Q}_4}(\mathcal{A}\otimes\mathcal{B}) = \mathcal{A}\otimes\mathcal{B}$

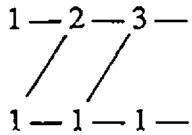
2)

$\mathcal{D}(\mathcal{A}) =$  $\cdots$ $\overline{\mathcal{Q}_4}(\mathcal{A}) \subsetneqq \mathcal{A}$

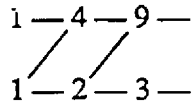
$\mathcal{D}(\mathcal{B}) = 1 — 2 — 3 — \cdots$ $\overline{\mathcal{Q}_4}(\mathcal{B}) = \mathcal{B}$

$\mathcal{D}(\mathcal{A}\otimes\mathcal{B}) =$  $\cdots$ $\overline{\mathcal{Q}_4}(\mathcal{A}\otimes\mathcal{B}) = \mathcal{A}\otimes\mathcal{B}$

3)

$\mathcal{D}(\mathcal{A}) =$  $\cdots$ $\overline{\mathcal{Q}_4}(\mathcal{A}) \subsetneqq \mathcal{A}$

$\mathcal{D}(\mathcal{B}) = 1 — 2 — 3 — \cdots$ $\overline{\mathcal{Q}_4}(\mathcal{B}) = \mathcal{B}$

$\mathcal{D}(\mathcal{A}\otimes\mathcal{B}) =$  $\cdots$ $\overline{\mathcal{Q}_4}(\mathcal{A}\otimes\mathcal{B}) = \mathcal{A}\otimes\mathcal{B}$

For our final result of this chapter we consider $\overline{\mathcal{Q}_2}(\mathcal{A})$ for an AF-algebra $\mathcal{A}$. By Proposition 3 of [KLMR], if A is a Riesz operator, then A is in $\overline{\mathcal{P}_2}(\mathcal{B}(\mathcal{H}))$ if and only if $\sigma(A)$ is contained in the non-negative real axis. And if A is compact (algebraic) then it is not hard to see that their proof gives information about the factors involved, that is,

$$A \text{ is in } \overline{\mathcal{Q}_2}(\mathcal{K})(\overline{\mathcal{P}_2}(\mathcal{Alg})) \Leftrightarrow \sigma(A) \geq 0 .$$

Is there an analogous result for AF-algebras? If $\mathcal{A}$ is an AF-algebra, is it true that

$$\overline{\mathcal{Q}}_2(\mathcal{A}) = \{A \in \mathcal{A} : \sigma(A) \geq 0\} \text{ ?}$$

As a partial answer we have the following proposition.

Proposition 2.15: Let $\mathcal{A}$ be an AF-algebra. If A is in $\mathcal{A}$ and $\sigma(A) \geq 0$, then A is in $\overline{\mathcal{Q}}_2(\mathcal{A})$, that is, we have the inclusion

$$\{A \in \mathcal{A} : \sigma(A) \geq 0\} \subseteq \overline{\mathcal{Q}}_2(\mathcal{A}) .$$

Note: Regarding the reverse inclusion, see Remarks below.

Proof: We suppose that $\mathcal{A} = \lim\limits_{\rightarrow} \mathcal{A}_n$, for each n, $\mathcal{A}_n$ is a direct sum of matrix algebras and that A belongs to $\mathcal{A}$ with non-negative spectrum, i.e. $\sigma(A) \geq 0$.

Let $\varepsilon > 0$ and $N_{\varepsilon/4}(\sigma(A))$ be an $\frac{\varepsilon}{4}$ neighbourhood of $\sigma(A)$ in $\mathbb{C}$, that is,

$$N_{\varepsilon/4}(\sigma(A)) = \{z \in \mathbb{C} : |z-\lambda| < \frac{\varepsilon}{4} \text{ for some } \lambda \text{ in } \sigma(A)\} .$$

Suppose also that A_n converges to A, A_n from $\mathcal{A}_n$. Since the set-valued function spectrum is upper semicontinuous (see [Halmos], Problem 103), there exists n_0 such that for $n \geq n_0$

$$\|A - A_n\| < \frac{\varepsilon}{4}$$

and

$$\sigma(A_n) \subseteq N_{\varepsilon/4}(\sigma(A)) .$$

For each $n \geq n_0$, A_n is unitarily equivalent, via U_n in $\mathcal{Q}_n$, to a direct sum of upper-triangular matrices, so $U^*_n A_n U_n$ is of the form

$$A'_n = U^*_n A_n U_n = \bigoplus_{j=1}^{r} \begin{pmatrix} \lambda_{j1} & & & * \\ & \cdot & & \\ & & \cdot & \\ & 0 & & \cdot \\ & & & & \lambda_{jn_j} \end{pmatrix}$$

with $\sigma(A_n) = \{\lambda_{jk} : j=1,\dots,r,\ k=1,\dots,r_j\}$. Now form a new operator B'_n obtained from A'_n by replacing the diagonals λ_{jk} by positive numbers μ_{jk} for which

$$\mu_{j1},\dots,\mu_{jn_j} \text{ are distinct, } j=1,\dots,r$$

and

$$|\mu_{jk} - \lambda_{jk}| < \frac{\varepsilon}{4} .$$

Then B'_n is in $\mathcal{P}_2(\mathcal{Q}_n)$. So letting $B_n = U_n B'_n U^*_n$ we have that B_n is in $\mathcal{P}_2(\mathcal{Q}_n)$ and that

$$\begin{aligned} \|A_n - B_n\| &= \|U^*_n(A_n - B_n)U_n\| \\ &= \|A'_n - B'_n\| \\ &< \frac{\varepsilon}{4} . \end{aligned}$$

Therefore

$$\begin{aligned}\|A - B_n\| &\le \|(A - A_n)\| + \|(A_n - B_n)\| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} \\ &< \varepsilon ,\end{aligned}$$

and it follows that A is in $\overline{\mathfrak{Q}_2(\mathfrak{A})}$. This finishes the proof.

Remarks: An operator T is called a point of spectral continuity if when $S_n \to T$ in norm it follows that $\sigma(S_n) \to \sigma(T)$ in the Hausdorff topology (see [Halmos], Problems 102-105). It is known that compact operators are such points (see [Aup], Corollary 3.4.5). Suppose that $\mathfrak{A}$ is an AF-algebra, $\mathfrak{A} = \overline{\cup \mathfrak{A}_n}$, $A \in \overline{\mathfrak{Q}_2(\mathfrak{A})}$ and that A is a point of spectral continuity. Then, without loss of generality, $A = \lim_j P_{n_j} Q_{n_j}$, where P_{n_j}, Q_{n_j} are positive invertible in $\mathfrak{A}_{n_j}$. Thus $A = \lim_j P_{n_j}^{1/2}(P_{n_j}^{1/2} Q_{n_j} P_{n_j}^{1/2}) P_{n_j}^{-1/2}$ so that $\sigma(A) = \lim_j \sigma(P_{n_j}^{1/2} Q_{n_j} P_{n_j}^{1/2}) \ge 0$.

So is it true that if T belongs to an AF-algebra then T is a "relative point of spectral continuity", in the sense that if $S_n \to T$ and $S_n \in \mathfrak{A}$ then $\sigma(S_n) \to \sigma(T)$. Is a stronger condition true, that is, are operators in $\overline{\mathfrak{A}\mathfrak{F}}$ points of spectral continuity? And would this be a discriminant, distinguishing $\overline{\mathcal{A}\ell g}$ from $\overline{\mathfrak{A}\mathfrak{F}}$?

Suppose $\mathfrak{A}$ is a C*-algebra, $T \in \mathfrak{A}$ and π is a faithful unital representation of $\mathfrak{A}$. If $\pi(T)$ is a point of spectral continuity then so is T. So one place to begin might be the analysis of the representation theory of a particular AF-algebra, e.g. the CAR-algebra (see $[KR]_2$, Example 10.4.19).

Chapter 3

Approximately Poly-Normal Algebras

The next family of algebras we wish to consider are direct limits of direct integrals of finite dimensional C*-algebras. These generalize both AF-algebras and algebras of n-normal operators, and for an example which is neither "AF" nor "n-normal" consider the tensor product of C*-algebras $\mathcal{L}^\infty(X,\mu) \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators and $X = [0,1]$ is endowed with Lebesgue measure. In general, for (X,μ) standard, what makes the algebra $\mathcal{L}^\infty(X,\mu) \otimes \mathcal{K}$ manageable in regard to questions of factorization, is that it can be realized as a particularly nice direct limit of almost everywhere finite dimensional subalgebras. Holding on precise definitions for now, we have that $\mathcal{K}$ is an AF-algebra,

$$\mathcal{K} = \lim_{\rightarrow}(\mathcal{M}_n, \phi_{mn})$$

where

$$\Phi_{nn+1}: \ A \mapsto \begin{pmatrix} & & 0 \\ & A & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

Therefore, by the universal property for direct limits we obtain a C*-algebra monomorphism

$$\lim_{\rightarrow}(\mathcal{L}^\infty(X,\mu) \otimes \mathcal{M}_n, J\otimes\phi_{mn}) \rightarrow \mathcal{L}^\infty(X,\mu) \otimes \mathcal{K}$$

where J is the identity map on $\mathcal{L}^\infty(X,\mu)$; and in fact this map is onto.

Using our results on n-normal operators in conjunction with "measurable versions" of our techniques developed for AF-algebras, we find that $\overline{\mathcal{Q}_4}(\mathcal{L}^\infty(X,\mu) \otimes \mathcal{K}) = \mathcal{L}^\infty(X,\mu) \otimes \mathcal{K}$.

So our next concern is to define these direct limit algebras in general. As a special case we will obtain the tensor product $\mathcal{L}^\infty(X,\mu) \otimes \mathcal{Q}$, where $\mathcal{Q}$ is any AF-algebra.

The system maps for the direct limits will be "canonical" in an appropriate sense, analogous to the canonical maps for AF-algebras. In the next chapter necessary and sufficient conditions will be obtained for $\overline{\mathcal{Q}_4}$ to be the whole algebra. This will include a statement in terms of "measurable fields" of Bratteli diagrams.

For a first definition of our objects, let (X_n,μ_n) be a sequence of Borel spaces, each equipped with a positive σ-finite measure μ_n, and let

$$\mathcal{A}_n = \int_{X_n}^{\oplus} \mathcal{A}_n(x)d\mu_n \,, \quad n = 1,2,3,...$$

be a direct integral of finite dimensional von Neumann algebras $(\mathcal{A}_n(x), \mathcal{H}_n(x))$, $\mathcal{H}_n(x)$ a finite dimensional Hilbert space for almost all x in X_n. So $\mathcal{A}_n$ acts on

$$\mathcal{H}_n = \int_{X_n}^{\oplus} \mathcal{H}_n(x)d\mu_n \;.$$

Let

$$\phi_{nn+1} : \mathcal{A}_n \to \mathcal{A}_{n+1}$$

be a *-monomorphism, and set

$$\phi_{mn} = \phi_{n-1n} \circ \cdots \circ \phi_{mm+1} \;.$$

Now define

$$\mathcal{A} = \lim_{\rightarrow}(\mathcal{A}_n \,, \phi_{mn})$$

Our first step is to show that, modulo null-spaces (in the sense of representation theory), each $\mathcal{A}_n$ is just a direct sum of n-normal operator algebras (the matrix sizes may vary). For notation recall that $\mathcal{Z}(\mathcal{A})$ is the center of a C*-algebra $\mathcal{A}$, while for positive integers r, $\mathcal{A}^{(r)}$ is the r-fold inflation of $\mathcal{A}$.

<u>Lemma 3.1</u>: Let $\mathcal{A}$ be a direct integral of finite dimensional von Neumann algebras

$$\mathcal{A} = \int_X^{\oplus} \mathcal{A}(x)d\mu \quad \text{acting on} \quad \mathcal{H} = \int_X^{\oplus} \mathcal{H}(x)d\mu \,, \ \dim \mathcal{H}(x) < \infty \quad \text{a.e.}$$

Then there is a measurable function $x \mapsto m(x) \in \mathbb{N}$ and a sequence of measurable fields of minimal central projections $P_k(x) \in \mathcal{Z}(\mathcal{A}_x)$ a.e. such that

$$\mathcal{A} \cong \int_X^{\oplus} \left(\sum_{k=1}^{m(x)} P_k(x)\mathcal{A}_x \right) d\mu \ .$$

Proof: By Corollary 0.12, the center of $\mathcal{A}$ is given by

$$\mathcal{Z}(\mathcal{A}) = \int_X^{\oplus} \mathcal{Z}(\mathcal{A}_x)d\mu \ .$$

Since $\mathcal{Z}(\mathcal{A})$ is a direct integral, there exists a countable family $\{Q_k\}$ in $\mathcal{Z}(\mathcal{A})$ which generate $\mathcal{Z}(\mathcal{A})$ and such that $\{Q_k(x)\}$ generates $\mathcal{Z}(\mathcal{A}_x)$ almost everywhere. And since $\mathcal{Z}(\mathcal{A})$ is a von Neumann algebra we may assume Q_k is a projection for each k. Using this sequence, we inductively define a new sequence of projections P_k which satisfy the following:

(i) $P_k(x) \perp P_\ell(x) \quad k \neq \ell$ a.e.

(ii) $1_x = \sum_{k=1}^{m(x)} P_k(x)$ a.e., and $x \mapsto m(x) \in \mathbb{N}$ is a measurable function

(iii) $1 = \int_X^{\oplus} 1_x d\mu = \int_X^{\oplus} \sum_{k=1}^{m(x)} P_k(x)d\mu$

(iv) $\{P_k\}$ generates $\mathcal{Z}(\mathcal{A})$ and

$\{P_{k(x)}\}$ generates $\mathcal{Z}(\mathcal{A}_x)$ a.e.

It follows that $P_k(x)\mathcal{A}(x)$ has trivial center a.e. and

$$\mathcal{A} = \int_X^{\oplus} \left(\sum_{k=1}^{m(x)} P_k(x)\mathcal{A}_x \right) d\mu \ .$$

Lemma 3.2: If n is fixed and $x \mapsto \mathcal{A}_x$ is a measurable field of simple finite dimensional von Neumann algebras, each acting on $\mathbb{C}^n$, then the following maps are measurable:

(i) $x \mapsto \dim[\mathcal{A}_x\mathbb{C}^n] = r(x)$

(ii) $x \mapsto \dim \mathcal{A}_x$

(iii) $x \mapsto m(x)$, where $m(x)$ = the multiplicity of the identity representation of $\mathcal{A}_x$ on $\mathbb{C}^n$.

Proof: (i) The quantity $\dim[\mathcal{A}_x\mathbb{C}^n] = r(x)$ is equal to the rank of 1_x. But $x \mapsto \mathcal{A}_x$ is a measurable field of von Neumann algebras, and as in the proof of Lemma 3.1 $x \mapsto 1_x$ defines a measurable field of operators. This implies that $x \mapsto [\mathrm{rg}\ 1_x]$ is a measurable field of Hilbert spaces and hence $x \mapsto \dim[\mathrm{rg}\ 1_x]$ is a measurable function, that is, $x \mapsto r(x)$ is measurable.

(iii) From the classical theory (e.g. See [Tak], sec I.11)

$$m(x) = \dim \mathcal{Z}(\mathcal{A}_x),$$

and we saw in the proof of the last lemma that this was a measurable function.

(ii) Again from the classical theory

$$m(x)\cdot(\dim \mathcal{A}_x)^{1/2} = r(x) \quad \text{a.e.}$$

from which (ii) follows.

Theorem 3.3: Let $\mathcal{A}$ be a direct integral of finite dimensional von Neumann algebras,

$$\mathcal{A} = \int_X^{\oplus} \mathcal{A}_x d\mu \quad \text{acting on} \quad \int_X^{\oplus} \mathcal{H}_x d\mu,\ n(x) = \dim \mathcal{H}_x < \infty \quad \text{a.e.}$$

Assume also that the identity representation of $\mathcal{A}_x$ on $\mathcal{H}_x$ is faithful a.e. Then there exist measurable functions

$$t, j_1, \ldots, j_t, r_1, \ldots, r_t : X \to \mathbb{N}$$

so that we have a direct integral unitary equivalence:

$$\mathcal{A} \equiv \int_X^{\oplus} \left(\sum_{k=1}^{t(x)\,\oplus} \mathcal{M}_{j_k(x)}^{(r_k(x))} \right) d\mu \subsetneq \int_X^{\oplus} \mathcal{M}_{n(x)} d\mu \ \text{ acting on } \int_X^{\oplus} \mathbb{C}^{n(x)} d\mu \ .$$

Remark: To help understand the notation, consider the example where $t(x) = t =$ constant, $r_k(x) = 1 =$ constant, $j_k(x) = \ell =$ constant and $n(x) = n =$ constant. The algebra $\mathcal{A}$ is then

$$\sum_{k=1}^{t\,\oplus} \mathcal{M}_{\ell}(\mathcal{L}^{\infty}(X,\mu)) = \left\{ \begin{pmatrix} A_1 & & & & & \\ & \ddots & & & 0 & \\ & & A_t & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix} \in \mathcal{M}_n(\mathcal{L}^{\infty}(X,\mu)) : \right.$$

$$\left. A_k \in \mathcal{M}_{\ell}(\mathcal{L}^{\infty}(X,\mu)) \right\} .$$

Proof of Theorem 3.3: By Lemma 3.1 we may assume that the center of $\mathcal{A}_x$ is trivial. Since $\mathcal{A}_x$ is finite dimensional this means that as a C*-algebra $\mathcal{A}_x$ is simple. Also, by Theorem IV 8.14 [T] we may assume that $\mathcal{H}_x = \mathbb{C}^{n(x)}$. Since $n(x)$ is a measurable function, we assume that $n(x) = n$ a.e. (Otherwise $X = \dot{\bigcup_n} X_n$, $X_n = \{x : n(x) = n\}$). Let $s(x) = \text{rank}(1_x)$, $m(x) =$ the multiplicity of $\mathcal{A}_x$ and $j(x) = (\dim \mathcal{A}_x)^{1/2}$. From Lemma 3.2 these are all measurable functions, and from the classical theory of matrix algebras, we have, for almost each x in X, a unitary equivalence between the two algebras

$$\mathcal{A}_x \cong \begin{pmatrix} \mathcal{M}_{j(x)} & & & & \\ & \ddots & & & \\ (m(x)) & & \mathcal{M}_{j(x)} & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} = \mathcal{B}_x$$

and

$$m(x)j(x) = s(x) \le n \quad \text{a.e.}$$

Since j and m are measurable functions, we obtain a well-defined direct integral

$$\mathcal{B} = \int_X^{\oplus} \mathcal{B}_x d\mu$$

with the property that

$$\mathcal{A}_x \cong \mathcal{B}_x \text{ (unitarily equivalent) a.e.}$$

We now invoke Theorem IV 8.28 [Tak] to conclude that we have a direct-integral unitary equivalence

$$\mathcal{A} \cong \mathcal{B} .$$

The Theorem now follows.

Recall that our object of study is a direct limit of a system $(\mathcal{A}_n, \phi_{mn})$ where for each $n = 1,2,3,\ldots$, $\mathcal{A}_n$ is a direct integral of finite dimensional von Neumann algebras acting on a direct integral of finite dimensional Hilbert spaces. We wish to characterize the maps $\phi_{nn+1} : \mathcal{A}_n \to \mathcal{A}_{n+1}$, and in view of Theorem 3.3, it is enough to consider *-monomorphisms ϕ

$$\phi : \mathcal{M}_n (\mathcal{L}^\infty(Y,\eta)) \to \mathcal{M}_n (\mathcal{L}^\infty(X,\mu)) .$$

As mentioned earlier, we are looking for a "canonical" map. Thinking of ϕ as a *-homomorphism into the bounded operators on $(\mathcal{L}^2(X,\mu)^{(n)}$, established theorems from representation theory would give us a canonical decomposition (see e.g. [Arv] Theorem 2.1.8). The problem with that approach is that the direct integral structure is not necessarily respected. We require a "canonical decomposition" which is both compatible with and reveals the fibre structure of the direct integrals. To accomphish this we need three lemmas.

Lemma 3.4: Let (Y,η) and (X,μ) be standard σ-finite measure spaces and suppose that

$$\phi : \mathcal{L}^\infty(Y,\eta) \to \mathcal{L}^\infty(X,\mu)$$

is a *-isomorphism of von Neumann algebras. Then there exist Borel null sets $N \subseteq Y$, $M \subseteq X$ and a Borel isomorphism Φ of $X \setminus M$ onto $Y \setminus N$ such that $\Phi(\mu)$ and η are equivalent in the sense of absolute continuity and, for every B in $\mathcal{L}^\infty(Y,\eta)$

$$\phi(B)(x) = B(\Phi(x))$$

for almost all x in $X \setminus N$.

Proof: See [Tak] Lemma IV 8.22.

Lemma 3.5: Let (Y,η) and (X,μ) be as in the previous lemma and suppose that

$$\phi : \mathcal{L}^\infty(Y,\eta) \to \mathcal{L}^\infty(X,\mu)$$

is a *-monomorphism of von Neumann algebras. Then there exists a Borel set $X_1 \subseteq X$ such that

$$\phi : \mathcal{L}^\infty(Y,\eta) \to \mathcal{L}^\infty(X_1,\mu)$$

is a *-isomorphism of von Neumann algebras.

Proof: Let 1_Y be the identity of $\mathcal{L}^\infty(Y,\eta)$. The set X_1 is then obtained from the projection $\phi(1_Y)$ in $\mathcal{L}^\infty(Y,\mu)$.

Lemma 3.6: Suppose ϕ_0 is a *-monomorphism of von Neumann algebras

$$\phi_0 : \mathcal{M}_k(\mathbb{C}) \to \mathcal{M}_n(\mathcal{L}^\infty(X,\mu)) .$$

Then there exists an essentially bounded measurable map

$$m : X \to \mathbb{N} \ , \ m(x) \cdot k \le n$$

such that ϕ_0 is direct integral unitarily equivalent to the map ψ_0 defined by

$$\psi_0(A_0)(x) = \begin{pmatrix} A_0 & \overset{(m(x))}{\ddots} & & & & 0 & \\ & & A_0 & & & & \\ & & & 0 & & & \\ & 0 & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

for almost all x in X.

Proof: By Theorem IV 8.25 of [Tak], there is a measurable field of *-homomorphisms

$$\phi_0(x) : \mathcal{M}_k(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$$

essentially unique such that

$$\phi_0 = \int_X^{\oplus} \phi_0(x) d\mu \quad .$$

As in the proof of Lemma 3.1,

$$x \mapsto \dim \mathfrak{H}(\text{rg } \phi_0(x)) = m(x)$$

is a measurable function; and by the classical theory (see [Tak], Sec. I.11),. $\phi_0(x)$ is unitarily equivalent to the map

$$\psi_0(x) : A \mapsto \begin{pmatrix} A & \overset{(m(x))}{\ddots} & & & 0 & \\ & & A & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix} .$$

But now $x \mapsto \psi_0(x)$ defines a measurable field of *-homomorphisms. So by Theorem IV 8.28 [Tak], there exists a measurable field of unitary operators

$$x \mapsto U(x) \in \mathcal{M}_n(\mathbb{C}) \quad \text{a.e.}$$

which implements the required equivalence, that is,

$$\phi_0 = \int_X^{\oplus} \phi_0(x)d\mu \cong \int_X^{\oplus} \psi_0(x)d\mu \quad \text{via} \quad U = \int_X^{\oplus} U(x)d\mu \ .$$

This completes the proof.

The next theorem is the decomposition theorem we've been heading for, and tells us that if ϕ is a *-monomorphism between operator algebras $\mathcal{M}_m(\mathcal{L}^\infty(Y,\eta))$ and $\mathcal{M}_n(\mathcal{L}^\infty(Y,\mu))$, then except for "missing parts of X", and certain Borel isomorphisms of the underlying measure spaces, ϕ looks just like a canonical monomorphism of matrix algebras.

<u>Theorem 3.7</u>: Let (Y,η) and (X,μ) be as before and

$$\phi : \mathcal{M}_m(\mathcal{L}^\infty(Y,\eta)) \to \mathcal{M}_n(\mathcal{L}^\infty(X,\mu))$$

be a *-monomorphism of von Neumann algebras. Then there exists a measurable function $m : X \to \mathbb{N}$ such that for each k, $1 \le k \le n$ and $X_k = \{x : m(x) = k\}$ there exist (not necessarily distinct) Borel subsets

$$X_1^k, \ldots, X_k^k \subseteq X_k$$

and Borel isomorphisms

$$\Phi_j^k : (X_j^k, \mu) \to (Y, \eta) \ , \text{ as in Lemma 3.5 },$$

such that if χ_j^k is the characteristic function for X_j^k, then ϕ is direct integral unitarily equivalent to the map χ defined by

$$\chi(A)(x) = \begin{pmatrix} \chi_1^k(x)A(\Phi_1^k(x)) & & & & 0 \\ & \ddots & & & \\ & & \chi_k^k(x)A(\Phi_k^k(x)) & & \\ & & & 0 & \\ & & & & \ddots \\ 0 & & & & 0 \end{pmatrix}$$

for almost all x in X_k, $k = 1,\ldots,n$.

Proof: Let ϕ_0 be the restriction of ϕ to the subalgebra of matrices with constant entries. Then by Lemma 3.6 there exists a measurable function $m : X \to \mathbb{N}$ such that ϕ_0 is direct integral unitarily equivalent to the map given by

$$\chi_0(A_0)(x) = \begin{pmatrix} A_0 & \overset{(m(x))}{\ddots} & & & & 0 \\ & & A_0 & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix} .$$

Therefore without loss of generality we assume that ϕ_0 is already of this form. Also since $m : X \to \mathbb{N}$ is measurable, $X = \bigcup_{k=1}^{n} X_k$, $X_k = \{x : m(x) = k\}$ is a disjoint union of measurable sets; so we assume that $m(x) = k = \text{constant}$.

Now consider ϕ_1 , the restriction of ϕ to the center $\mathcal{Z}_Y$ which, as a von Neumann algebra, is *-isomorphic to $\mathcal{L}^\infty(Y,\eta)$. (In fact, by Proposition IV 8.29 [Tak], this is a direct integral isomorphism.) We then have the inclusions:

$$\phi(\mathcal{Z}_Y) \subseteq \mathcal{Z}(\text{rg}\,\phi) \subseteq \mathcal{Z}(\text{rg}\,\phi_0) \subseteq \mathcal{M}_n(\mathcal{L}^\infty(X,\mu)) .$$

From the preceeding paragraph, it follows that

$$\phi(\mathcal{Z}_Y) \subseteq \sum_{j=1}^{k}{}^{\oplus} \mathcal{L}^\infty(X,\mu)$$

and that

$$\phi = \sum_{j=1}^{k}{}^{\oplus} \phi_j \quad ,$$

where

$$\phi_j : \mathcal{Z}_Y \to \mathcal{L}^\infty(X,\mu)$$

is a *-monomorphism.

From Lemmas 3.4 and 3.5 we obtain Borel sets

$$X_1,\ldots,X_k \subsetneq X\ ,$$

and Borel isomorphisms

$$\Phi_j : (X_j,\mu) \to (Y,\eta)$$

for each ϕ_j , $j = 1,\ldots,k$.

Therefore,. with χ_j = the characteristic function for X_j ,

$$\phi_1(f\cdot I)(x) = \begin{pmatrix} \chi_1(x)f(\Phi_1(x))\cdot I & & & & & 0 \\ & \ddots & & & & \\ & & \chi_k(x)f(\Phi_k(x))\cdot I & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{pmatrix}$$

where I = the $m\times m$ identity matrix, $f \in \mathcal{L}^\infty(Y,\eta)$. But now the result follows. For if $\{E_{rs}\}_{r,s=1,\ldots,m}$ is the canonical system of matrix units, then given A in $\mathcal{M}_m(\mathcal{L}^\infty(Y,\eta))$

$$A = \sum_{r,s=1}^{m} (f_{rs}\cdot I)E_{rs}\ ,\ f_{rs} \in \mathcal{L}^\infty(Y,\eta)$$

so that

$$\phi(A) = \sum_{r,s=1}^{m} \phi_1(f_{rs}\cdot I)\phi_0(E_{rs})\ .$$

We now turn our attention back to the directed sequences $(\mathcal{Q}_n,\phi_{mn})$

$$\mathcal{Q}_n = \int_{X_n}^{\oplus} \mathcal{Q}_n(x)d\mu_n\ ,\ n = 1,2,3,\ldots$$

as defined at the beginning of this chapter. We impose certain conditions:

(i) The measure spaces (X_n,μ_n) are all the same (X,μ) ;

(ii) With the notation of Theorem 3.3, the multiplicities $r_k(x) = 1$ = constant;

(iii) For each n , the measurable function $x \mapsto \dim \mathcal{Q}_n(x)$ is essentially bounded; and

(iv) For each of the system maps $\phi_{nn+1} : \mathcal{A}_n \to \mathcal{A}_{n+1}$, all of the associated Borel isomorphisms (Theorem 3.7) are the identity. So there is no "shuffling" of the measure space as we pass from one algebra in the system to the next. More precisely, each $\phi_{nn+1} : \mathcal{A}_n \to \mathcal{A}_{n+1}$ preserves fibres and is a direct integral *-monomorphism. So

$$\phi_{nn+1} = \int_X^{\oplus} \phi_{nn+1}(x) d\mu$$

where

$$\phi_{nn+1}(x) : \mathcal{A}_n(x) \to \mathcal{A}_{n+1}(x) \ .$$

Now note that it is straightforward to check that the lemma on inner equivalence (quoted from [Eff]) behaves well with respect to direct integrals (the proof is algebraic). We therefore may assume that our system $(\mathcal{A}_n, \phi_{mn})$ is "canonical" in the sense given by Theorem 3.3 and Theorem 3.7. And since we have an at most countable sequence of morphisms ϕ_{nn+1}, we obtain an almost everywhere defined family of canonical directed systems $(\mathcal{A}_n(x), \phi_{mn}(x))$ of finite dimensional C*-algebras, and so a field of Bratteli diagrams $\mathcal{D}(x) = \mathcal{D}(\mathcal{A}(x))$, where $\mathcal{A}(x) = \lim\limits_{\rightarrow} (\mathcal{A}_n(x), \phi_{mn}(x))$. The weights of the edges are given by measurable multiplicity functions as in Theorem 3.2. (See also Definitions 4.6 and 4.7.)

Nomenclature:

We call a direct limit C*-algebra satisfying conditions (i) - (iv) an approximately poly-normal C*-algebra, and abbreviate this by APN algebra.

A poly-normal algebra is one of the form $\bigoplus_{k=1}^{m} \mathcal{M}_{j_k}(\mathcal{L}^\infty(X,\mu))$; and elements in such an algebra are called poly-normal operators.

Remark: Condition (iii) is very important at this stage. For in studying $\mathcal{Q}_4$ of an APN algebra we will use our "Ballantine-type" factorization theorem for polynomial operators

(Theorem 1.4). As it stands this theorem clearly requires that the poly-normal operators be of (almost everywhere) bounded rank. The problem encountered, if this is not so, is drawn out in the following example:

Let $X = \mathbb{N}$, μ be counting measure, $\mathcal{A}(n) = \mathcal{M}_n(\mathbb{C})$ and

$$A \in \int_{\mathbb{N}}^{\oplus} \mathcal{A}(n)d\mu = \mathcal{A} .$$

(In this case the direct integral corresponds to the usual direct sum of operators.) Suppose now that A is as "nice" as possible, i.e.

$$\inf\{\|A(n) - \lambda_n I_n\| : n \in \mathbb{N}, \lambda_n \in \mathbb{C}\} = \varepsilon > 0$$

and

$$\det A(n) > 0 \quad \text{for all} \quad n .$$

Then by Theorem 1.8, for each n there exist positive invertible matrices $P_1(n),\ldots,P_4(n)$ such that

$$A(n) = P_1(n)P_2(n)P_3(n)P_4(n) .$$

To obtain the factorization

$$A = P_1 P_2 P_3 P_4$$

within the direct integral algebra $\mathcal{A}$, we would need $\|P_i(n)\|$ to be bounded, for $i = 1,2,3,4$. But this question takes us out of the category of measurable fields of algebras and into that of continuous fields of algebras. This is part of upcoming research (see Chapter 5 and Concluding Remarks).

Going back to our example at the beginning of the section, we find that $\mathcal{L}^\infty(X,\mu) \otimes \mathcal{K}$ is an algebra of the kind just defined. To see this, let

$$\mathcal{A}_n = \mathcal{M}_n(\mathcal{L}^\infty(X,\mu))$$

and

$$\phi_{nn+1} : \mathcal{A}_n \to \mathcal{A}_{n+1}$$

by

$$\phi_{nn+1} : A \mapsto \begin{pmatrix} A & & 0 \\ & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in \mathcal{M}_{n+1}(\mathcal{L}^\infty(X,\mu)) \ .$$

Here the multiplicity function $m(x)$ is almost everywhere 1. So the "Bratteli diagrams" for this part of the system are

$$n \xrightarrow{m(x)=1} n+1 \ .$$

For another example let $X = \mathbb{N}$ and μ be the counting measure. We give the field of diagrams below. Note that the rows are indexed by x in $X = \mathbb{N}$ and that each is a Bratteli diagram corresponding to that index.

$x = 1$:	$1 — 2 — 3 — 4 — \cdots$	$\mathcal{A}(1)$
$x = 2$:	$1 — 1 — 2 — 3 — \cdots$	$\mathcal{A}(2)$
$x = 3$:	$1 — 1 — 1 — 2 — \cdots$	$\mathcal{A}(3)$
etc.	etc. etc.	

Observe that for each x,

$$\mathcal{A}(x) = \varinjlim(\mathcal{A}_n(x), \phi_{mn}(x)) = \mathcal{K} \ .$$

But $\mathcal{A} = \varinjlim(\mathcal{A}_n, \phi_{mn})$ is not isomorphic to $\mathcal{L}^\infty(X,\mu) \otimes \mathcal{K}$. For as we will show

$$\overline{\mathcal{Q}}_4(\mathcal{L}^\infty(X,\mu) \otimes \mathcal{K}) = \mathcal{L}^\infty(X,\mu) \otimes \mathcal{K}$$

whereas for $\mathcal{A}$

$$\overline{\mathcal{Q}}_4(\mathcal{A}) \subsetneq \mathcal{A} \ .$$

Chapter 4

Factorization in APN-algebras

This chapter is devoted to the characterization of those APN-algebras $\mathcal{A} = \mathcal{A} = \varinjlim(\mathcal{A}_n, \phi_{mn})$ for which $\overline{\mathcal{Q}_4}(\mathcal{A}) = \mathcal{A}$. Note that because of condition (iii) in the defiition of APN-algebras (which follows the proof of Theorem 3.7), Theorem 1.8 can be applied to the subalgebras $\mathcal{A}_n$ of the directed system. Consequently,

$$\overline{\mathcal{Q}_4}(\mathcal{A}_n) = \overline{\mathcal{P}}_4(\mathcal{A}_n) = \overline{\mathcal{P}}_\infty(\mathcal{A}_n) \quad , n = 1,2,3,...$$

and therefore

$$\overline{\mathcal{Q}_4}(\mathcal{A}) = \overline{\mathcal{Q}_\infty}(\mathcal{A})$$

as in the proof of Lemma 2.4

Before proceeding with the general theory, we present four examples which help illustrate typical behaviour in these algebras. We number them and will refer to them later.

Examples: For each example $X = \mathbb{N}$ and μ is counting measure. The diagrams are the associated fields of Bratteli diagrams (see the paragraph preceeding the definition of APN-algebras, in the last chapter).

Example 1:

	n=1		n=2		n=3		
x=1	1	—	2	—	3	—···	$\mathcal{A}(1) = \mathcal{K}$
x=2	1	—	2	—	3	—···	$\mathcal{A}(2) = \mathcal{K}$
x=3	1	—	2	—	3	—···	$\mathcal{A}(3) = \mathcal{K}$

...

$$\overline{\mathcal{Q}_4}(\mathcal{A}) = \mathcal{A}$$

Here $\mathcal{A} \cong \ell^\infty(\mathbb{N}) \otimes \mathcal{K}$.

Example 2: n=1 n=2 n=3

x=1 0 — 2 — 3 —··· $\mathcal{A}(1) = \mathcal{K}$

x=2 0 — 0 — 3 —··· $\mathcal{A}(2) = \mathcal{K}$

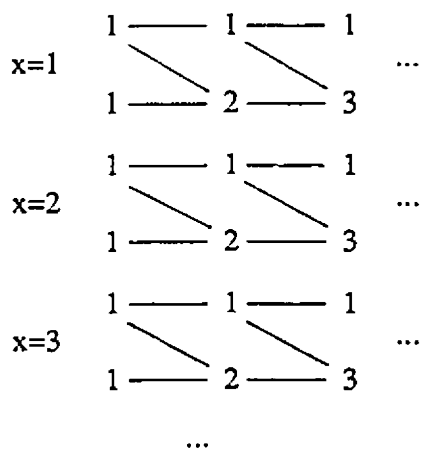
x=3 0 — 0 — 0 —··· $\mathcal{A}(3) = \mathcal{K}$

...

$$\overline{Q}_4(\mathcal{A}) = \mathcal{A}$$

This is an example of what will be called a diagram which is attracted to zero (see Definition 4.11). In this particular case $\mathcal{A} \cong c_0(\mathbb{N}) \otimes \mathcal{K}$.

Example 3: n=1 n=2 n=3



$\mathcal{A}(1) = \mathcal{K}^\sim$

$\mathcal{A}(2) = \mathcal{K}^\sim$

$\mathcal{A}(3) = \mathcal{K}^\sim$

$$\overline{Q}_4(\mathcal{A}) \subsetneqq \mathcal{A} \ .$$

Here $\mathcal{A} \cong \ell^\infty(\mathbb{N}) \otimes \mathcal{K}^\sim$ and has an ideal $\mathcal{J}$ such that $\mathcal{A}/\mathcal{J} \cong \ell^\infty(\mathbb{N})$.

Example 4: n=1 n=2 n=3

x=1 1 — 2 — 3 —··· $\mathcal{A}(1) = \mathcal{K}$

x=2 1 — 1 — 2 —··· $\mathcal{A}(2) = \mathcal{K}$

x=3 1 — 1 — 1 —··· $\mathcal{A}(3) = \mathcal{K}$

...

Here $\mathcal{A}$ has an ideal $\mathcal{J}$ such that $\mathcal{A}/\mathcal{J} \cong \ell^{\infty}(\mathbb{N})/c_0(\mathbb{N})$. The ideal $\mathcal{J} = \overline{\bigcup_n \mathcal{J}_n}$, where $\mathcal{J}_n$ is the ideal of $\mathcal{A}_n$ obtained by replacing the 1's in the diagram by 0's. (See Corollary 4.18.)

In order to make this rigorous, the first thing we do is obtain a new definition of constant edges in an AF-algebra, one which will lend itself more easily to our new setting. We require a definition in terms of "measurable" quantities.

Definition 4.1: Let $\mathcal{A}$ be a separable non-zero unital C*-algebra and $\mathcal{Z}(\mathcal{A})$ be its center. For each non-zero projection $P \in \mathcal{Z}(\mathcal{A})$, $P\mathcal{A} \subseteq \mathcal{A}$ is a C*-algebra and therefore the dimension of $P\mathcal{A}$ is well-defined, denoted $\dim P\mathcal{A}$. The minimal central dimension of $\mathcal{A}$, denoted $\mathrm{mcd}(\mathcal{A})$, is defined to be

$$\mathrm{mcd}(\mathcal{A}) = \inf \dim\{P\mathcal{A} : P \in \mathcal{Z}(\mathcal{A}) \text{, a non-zero projection}\} .$$

Note: When $\mathcal{A} = \bigoplus_{j=1}^{r} \mathcal{M}_{n_j}$ is a direct sum of matrix algebras, then

$$\mathrm{mcd}(\mathcal{A}) = (\min_j n_j)^2 .$$

Note: For von Neumann algebras it might be tempting to define a similar quantity in terms of ranks of projections. This however would depend on the action on the Hilbert space. We require that $\mathrm{mcd}(\mathcal{A})$ be independent of representation.

Now suppose $\mathcal{B} \subseteq \mathcal{A}$ are two C*-algebras, $\mathcal{A}$ separable and unital.

Definition 4.2: The C*-central support of $\mathcal{B}$ in $\mathcal{A}$, denoted $[\mathcal{B}]_{\mathcal{A}}$, is defined to be

$$[\mathcal{B}]_{\mathcal{A}} = \bigcap \{P\mathcal{A} : P \in \mathcal{Z}(\mathcal{A}),\ P\mathcal{A} \supseteq \mathcal{B}\} .$$

Note: When $\mathcal{B}$ is a direct summand, $\mathcal{B}$ is unital and $[\mathcal{B}]_{\mathcal{A}} = \mathcal{B}$.

When $\mathcal{A}$ has trivial center, $[\mathcal{B}]_{\mathcal{A}} = \mathcal{A}$.

Now, for an AF-algebra $\mathcal{A} = \varinjlim (\mathcal{A}_n, \phi_{mn})$ with diagram $\mathcal{D}(\mathcal{A})$, $\mathcal{D}(\mathcal{A}_m)$ has an obvious meaning, namely it is the diagram of weighted vertices corresponding to the m^{th} algebra $\mathcal{A}_m$. For $n \geq m$, $\mathcal{D}(\mathcal{A}_m)_n$ is defined by

$$\mathcal{D}(\mathcal{A}_m)_n = \mathcal{D}([\phi_{mn}(\mathcal{A}_m)]_{\mathcal{A}_n}) \ .$$

Since $\mathcal{A}_n$ is a finite dimensional C*-algebra, $[\phi_{mn}(\mathcal{A}_m)]_{\mathcal{A}_n}$ consists exactly of those summands of $\mathcal{A}_n$ which have a "non-zero intersection" with $\phi_{mn}(\mathcal{A}_m)$.

Example: Define

$$\phi : \mathcal{M}_2 \to \mathcal{M}_4 \oplus \mathcal{M}_4 \oplus \mathcal{M}_4 = \mathcal{A}$$

by

$$A : \mapsto \begin{pmatrix} A & \\ & 0 \end{pmatrix} \oplus \begin{pmatrix} A & \\ & A \end{pmatrix} \oplus \begin{pmatrix} 0 & \\ & 0 \end{pmatrix} .$$

Then

$$[\phi(\mathcal{M}_2)]_{\mathcal{A}} = \mathcal{M}_4 \oplus \mathcal{M}_4 \oplus 0 \ .$$

Note: To stress the rôle of the Bratteli diagram we often write $\mathrm{mcd}\ \mathcal{D}(\mathcal{A}_m)_n$ for the quantity $\mathrm{mcd}(\mathcal{A}_m)_n$.

Definition 4.3: Let $\mathcal{A} = \varinjlim (\mathcal{A}_n, \phi_{mn})$ be an AF-algebra with diagram $\mathcal{D}(\mathcal{A})$.

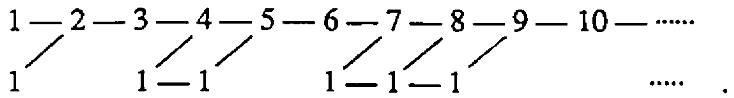
(i) $\mathcal{D}(\mathcal{A})$ is said to be eventually increasing from $\mathcal{D}(\mathcal{A}_m)$ if for each $N \in \mathbb{N}$ there exists $n \geq m$ such that $\mathrm{mcd}\ \mathcal{D}(\mathcal{A}_m)_n \geq N$.

(ii) $\mathcal{D}(\mathcal{A})$ is said to be eventually increasing if it is eventually increasing from $\mathcal{D}(\mathcal{A}_m)$ for $m = 1,2,3,\ldots$.

Lemma 4.4: A diagram $\mathcal{D}(\mathcal{A})$ for an AF-algebra $\mathcal{A}$ is eventually increasing if and only if there exist no constant edges.

Proof: Suppose $\mathcal{D}(\mathcal{A})$ has a constant edge E, emanating from $\mathcal{A}_{m_0}$. Then clearly $\mathcal{D}(\mathcal{A})$ is not eventually increasing. Conversely, suppose $\mathcal{D}(\mathcal{A})$ is not eventually increasing. Then there exists m_0 such that $\mathcal{D}(\mathcal{A})$ is not eventually increasing from $\mathcal{D}(\mathcal{A}_{m_0})$. Therefore there exists N_0 such that mcd $\mathcal{D}(\mathcal{A}_{m_0})_n \leq N_0$ for all $n > m_0$. Hence, we obtain an infinite edge each of whose vertices has weight w_n, $1 \leq w_n \leq N\text{-}1$. The weights must therefore "stabilize" (for they are non-decreasing). We get an infinite constant edge. This proves the lemma.

Example: This diagram is eventually increasing:



Here the limit $\mathcal{A}$ is isomorphic to $\mathcal{K}$, so $\overline{\mathcal{Q}_4}(\mathcal{A}) = \mathcal{A}$. So for a single AF-algebra we're allowed to have "long constant edges" without disturbing $\overline{\mathcal{Q}_4}(\mathcal{A}) = \mathcal{A}$ - as long as they are eventually "absorbed" by the system. This can be a subtle point for APN-algebras. (See Example 4 and Theorem 2.3). Of course by Theorem 2.5 we have an immediate corollary.

Corollary 4.5: If $\mathcal{A}$ is an AF-algebra with diagram $\mathcal{D}(\mathcal{A})$, then

$$\overline{\mathcal{Q}_4}(\mathcal{A}) = \mathcal{A} \Leftrightarrow \mathcal{D}(\mathcal{A}) \text{ is eventually increasing.}$$

To deal with fields of AF-algebras we introduce the following definition.

Definition 4.6: Let (X,μ) be a standard σ-finite measure space, and

$$x \mapsto (\mathcal{A}_n(x), \mathcal{H}_n(x))$$

$$x \mapsto \phi_{mn}(x) \qquad m \le n$$

be a sequence of fields of von Neumann algebras and a double sequence of WOT continuous *-monomorphisms respectively.

Then $(\mathcal{A}_n(x), \phi_{mn}(x))$ is called a measurable field of coherent systems (of von Neumann algebras) if for each $m \le n$

$$x \mapsto (\mathcal{A}_n(x), \mathcal{H}_n(x))$$

is a measurable field of von Neumann algebras,

$$x \mapsto \phi_{mn}(x) : \mathcal{A}_m(x) \to \mathcal{A}_n(x)$$

is a measurable field of *-monomorphisms and with

$$\mathcal{A}_n = \int_X^{\oplus} \mathcal{A}_n(x) d\mu \quad \text{and} \quad \phi_{mn} = \int_X^{\oplus} \phi_{mn}(x) d\mu$$

$(\mathcal{A}_n, \phi_{mn})$ is a coherent system of C*-algebras.

Note: If conditions (ii) and (iii) in the definition of APN-algebras are satisfied then we retrieve the measurable field of systems for an APN-algebra.

Definition 4.7: A measurable field of Bratteli diagrams $\mathcal{D}(x)$ is the field of diagrams associated to a canonical measurable field of coherent systems $(\mathcal{A}_n(x), \phi_{mn}(x))$ of essentially bounded (in dimension) finite dimensional von Neumann algebras.

For a measurable field of Bratteli diagrams $\mathcal{D}(x)$, we form the direct integral diagram

$$\mathcal{D} = \int_X^{\oplus} \mathcal{D}(x) d\mu .$$

This is just the disjoint union of all the diagrams $\mathcal{D}(x)$.

Definition 4.8: Let $\mathcal{A} = \lim\limits_{\rightarrow} (\mathcal{A}_n, \phi_{mn})$ be an APN-algebra and $\mathcal{D}(\mathcal{A})$ be the associated direct integral of Bratteli diagrams, so

$$\mathcal{A}(x) = \varinjlim(\mathcal{A}_n(x), \phi_{mn}(x)) \quad \text{and} \quad \mathcal{D}(\mathcal{A}) = \int_X^{\oplus} \mathcal{D}(\mathcal{A}(x))d\mu \ .$$

Then we say that $\mathcal{D}(\mathcal{A})$ is <u>eventually uniformly increasing from $\mathcal{D}(\mathcal{A}_m)$</u>, where

$$\mathcal{D}(\mathcal{A}_m) = \int_X^{\oplus} \mathcal{D}(\mathcal{A}_m(x))d\mu \ ,$$

if for each N there exists $n \geq m$ such that

$$\text{mcd } \mathcal{D}(\mathcal{A}_m(x))_n \geq N$$

for almost all x in X.

<u>Lemma 4.9</u>: The function defined by

$$x \mapsto \text{mcd } \mathcal{D}(\mathcal{A}_m(x))_n \qquad m \leq n$$

is measurable. In particular,

$$x \mapsto \text{mcd } \mathcal{D}(\mathcal{A}_m(x))_m = \text{mcd } \mathcal{A}_m(x)$$

is measurable.

<u>Proof</u>: Let $\mathcal{B}$ and $\mathcal{A}$ be two canonical polynormal algebras and suppose that $\phi : \mathcal{B} \to \mathcal{A}$ is a direct integral *-monomorphism. By Theorems 3.3 and 3.7, we have that $\mathcal{B}$, $\mathcal{A}$ and ϕ are of the following form:

$$\mathcal{B} = \bigoplus_{j=1}^{s} \mathcal{M}_{m_j}(\mathcal{L}^\infty(X,\mu)) \ , \quad \mathcal{A} = \bigoplus_{j=1}^{t} \mathcal{M}_{m_i}(\mathcal{L}^\infty(X,\mu))$$

$$\phi = (\phi_1, \ldots, \phi_t)$$

and

$$\phi_i : \mathcal{B} \to \mathcal{M}_{n_i}(\mathcal{L}^\infty(X,\mu))$$

by

$$\phi_i : (B_1,\dots,B_s) \mapsto \begin{pmatrix} B_1^{(r_{i1})} & & & & & \\ & \ddots & & & & \\ & & B^{(r_{is})} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

where

$$x \mapsto r_{ij}(x)$$

is a measurable function for each $i = 1,\dots,t$, $j = 1,\dots,s$. For each r_{ij} consider the measurable set

$$R_{ij} = \{x : r_{ij}(x) \neq 0\}$$

and let

$$n_{i_0} \in \{n_1,\dots,n_t\} .$$

Then

$$R_0 = \bigcup_{j=1}^{s} \{R_{i_0 j}\}$$
$$= \{x : r_{i_0 j}(x) \neq 0 \quad \text{for some } j\}$$

is a measurable set. It follows that

$$\{x : \mathrm{mcd}[\phi(x)B(x)]_{\mathcal{Q}(x)} = n_{i_0}^2\}$$

is a measurable set and hence that

$$x \to \mathrm{mcd}[\phi(x)B(x)]_{\mathcal{Q}(x)}$$

is a measurable function. Since every system map

$$\phi_{mn} : \mathcal{Q}_m \to \mathcal{Q}_n$$

is of the form just considered, the lemma is proven.

Lemma 4.10: Suppose $d_n : (X,\mu) \to \mathbb{N}$ is a sequence of measurable functions for which

$$d_1(x) \leq d_2(x) \leq d_3(x) \leq \dots \qquad \text{a.e.}$$

Suppose also that there exists $N \in \mathbb{N}$ such that

$$1 \leq d_n(x) \leq N-1 \quad \text{for all } n, \text{ a.e.}$$

Then the functions f, ℓ defined respectively by

$$x \mapsto \sup_n d_n(x) = f(x) \leq N - 1$$

$$x \mapsto \min\{n : d_n(x) = f(x)\} = \ell(x) < \infty$$

are measurable.

Proof: That f is measurable is a basic fact about sequences of measurable functions.

For the function ℓ, let $n \in \mathbb{N}$. Then

$$\{x : \ell(x) = n\} = \{x : d_n(x) = f(x)\} \cap \{x : d_{n-1}(x) < f(x)\} .$$

Therefore ℓ is also measurable.

Definition 4.11: Let $(\mathcal{A}_n(x), \phi_{mn}(x))$ be a measurable field of coherent systems of von Neumann algebras.

We say that the field is attracted to zero if there exists a partition of X into measurable sets $\{X_i\}_{i=1}^{\infty}$, each of positive measure, and an increasing sequence of positive integers $n_1 < n_2 < n_3 < \ldots$ such that for almost all $x \in X_i$

$$\mathcal{A}_n(x) = 0 \Leftrightarrow 1 \leq n \leq n_i .$$

Example: $X = \mathbb{N}$, μ = discrete measure. The following field is attracted to zero.

	n=1		n=2		n=3		
x=1	0	—	1	—	2	—···	$\mathcal{A}(1) = \mathcal{K}$
x=2	0	—	0	—	1	—···	$\mathcal{A}(2) = \mathcal{K}$
x=3	0	—	0	—	0	—···	$\mathcal{A}(3) = \mathcal{K}$
			. . .				

Here $X_i = \{i\}$ and $n_i = i$.

We say that the field of systems $(\mathcal{A}_n(x), \phi_{mn}(x))$ is attracted to infinity if it is eventually uniformly increasing.

We say that the field of systems is of infinite type if there exists a partition of X into a countably infinite sequence of measurable subsets $\{X_i\}$ each of positive measure such that restricted to X_i (see Definition 4.13) $\mathcal{A}|_{X_i}$ is eventually uniformly increasing, $i = 1,2,3,\ldots$.

Finally, we say that the field is jointly attracted to zero and of infinite type if the partitions for each of these properties can be chosen to coincide (at least up to measure zero).

Note: The previous example is jointly attracted to zero and of infinite type.

Example: $X = \mathbb{N}$, μ = discrete measure. The system is of infinite type, but not attracted to zero.

$$
\begin{array}{lccccccc}
 & n=1 & & n=2 & & n=3 & & \\
x=1 & 1 & - & 2 & - & 3 & - \cdots & \quad \mathcal{A}(1) = \mathcal{K} \\
x=2 & 1 & - & 1 & - & 2 & - \cdots & \quad \mathcal{A}(2) = \mathcal{K} \\
x=3 & 1 & - & 1 & - & 1 & - \cdots & \quad \mathcal{A}(3) = \mathcal{K} \\
 & & & \cdots & & & &
\end{array}
$$

We can now state our main characterization theorem.

Theorem 4.12: Let $\mathcal{A}$ be an APN-algebra with measurable field of systems $(\mathcal{A}_n(x),\phi_{mn}(x))$. Then $\overline{\mathcal{Q}_4}(\mathcal{A}) = \mathcal{A}$ if and only if one of the following three conditions is satisfied:

(i) The field of systems is attracted to infinity.

(ii) The field of systems is jointly attracted to zero and of infinite type.

(iii) The underlying measure space (X,μ) is partitioned into the disjoint union of two measurable subsets Z and Y , each of positive measure, such that over Y (i) is satisfied and over Z (ii) is satisfied.

We will present the proof in two parts. The first part will be concerned with showing that if none of the three conditions are satisfied we obtain $\overline{\mathcal{Q}_4}(\mathcal{A}) \subsetneq \mathcal{A}$. Once this is established, we will then show that each of the conditions implies that $\overline{\mathcal{Q}_4}(\mathcal{A}) = \mathcal{A}$.

Before this, however, we require some lemmas.

Example: Let (X,μ) be standard and $\mathcal{K}$ be the algebra of compact operators. Then

$$\overline{\mathcal{Q}_4}(\mathcal{L}^\infty(X,\mu) \otimes \mathcal{K}) = \mathcal{L}^\infty(X,\mu) \otimes \mathcal{K} .$$

See Example 1 at the beginning of this chapter.

Definition 4.13: Let $\mathcal{A}$ be an APN-algebra with the field of systems $(\mathcal{A}_n(x),\phi_{mn}(x))$. Suppose $Y \subseteq X$ is a measurable subset of X . Then

$$\mathcal{A}(Y) = \mathcal{A}|_Y = \lim_{\rightarrow}\left(\int_Y^{\oplus} \mathcal{A}_n(x)d\mu, \phi_{mn}(x)\right)$$

is the APN-algebra obtained by restricting the field to Y .

Lemma 4.14: In the situation of Definition 4.13,

$$\mathcal{A}(Y) = \mathcal{A}(Y) \oplus \mathcal{A}(X\backslash Y) .$$

Proof: Obvious.

Lemma 4.15: Let $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3 \subseteq \ldots \subseteq \mathcal{A}$ be an ascending chain of C*-algebras and

$$\mathcal{A} = \lim_{\rightarrow} \mathcal{A}_n = \overline{\bigcup_n \mathcal{A}_n}$$

be the direct limit of the sequence.

(i) If $\mathcal{J}$ is an ideal in $\mathcal{A}$ and $\mathcal{J}_n = \mathcal{A}_n \cap \mathcal{J}$, then $\mathcal{J}_n$ is an ideal in $\mathcal{A}_n$ and

$$\mathcal{J} = \overline{\bigcup_n \mathcal{J}_n} .$$

(ii) If $\mathcal{J}_n$ is a closed two-sided ideal in $\mathcal{A}_n$, $n = 1,2,3,\ldots$ and $\mathcal{J}_1 \subsetneq \mathcal{J}_2 \subsetneq \mathcal{J}_3 \subsetneq \ldots$ and $\mathcal{J}_n = \mathcal{J}_{n+1} \cap \mathcal{A}_n$ then

$$\overline{\bigcup_n \mathcal{J}} = \mathcal{J}$$

is a closed two-sided ideal in $\mathcal{A}$.

(iii) If $\mathcal{J}$ is as in (ii) then $\mathcal{A}/\mathcal{J}$ is isomorphic to the direct limit $\varinjlim \mathcal{A}_n/\mathcal{J}_n$.

Proof: See Lemma 3.1 of [Br].

For (iii) we use the usual universal property of direct limits.

The key fact for (i) and (ii) is that if $\mathcal{B}$ and $\mathcal{A}$ are C*-algebras with ideals $\mathcal{J}(\mathcal{B})$ and $\mathcal{J}(\mathcal{A})$, and if $\mathcal{B}$ is contained in $\mathcal{A}$ with $\mathcal{J}(\mathcal{B}) = \mathcal{B} \cap \mathcal{J}(\mathcal{A})$, then the canonical map

$$\mathcal{B}/\mathcal{J}(\mathcal{B}) \to \mathcal{A}/\mathcal{J}(\mathcal{A})$$

is one-one, hence isometric.

Definition 4.16: Let $\mathcal{B}(1), \mathcal{B}(2), \mathcal{B}(3), \ldots$ be a sequence of C*-algebras. We define the c_0-direct sum by

$$c_0 - \bigoplus_{i=1}^{\infty} \mathcal{B}(i) = \{(B_i)_{i=1}^{\infty} : ||B_i|| \to 0\}$$

that is, $c_0 - \bigoplus_{i=1}^{\infty} \mathcal{B}(i)$ is the C*algebra of sequences which converge to zero.

Note: If $\mathcal{B}(1) = \mathcal{B}(2) = \mathcal{B}(3) = \ldots = \mathcal{B}$, then

$$c_0 - \bigoplus_{i=1}^{\infty} \mathcal{B}(i) = c_0(\mathbb{N}) \otimes \mathcal{B}.$$

Note: This construction can be generalized to apply to "continuous" fields of C*-algebras. See [Dix] Chapter 10.

Note: Letting $\bigoplus_{i=1}^{\infty} \mathfrak{B}(i)$ denote the usual direct sum, then $c_0 - \bigoplus_{i=1}^{\infty} \mathfrak{B}(i)$ is a norm closed two-sided ideal in $\bigoplus_{i=1}^{\infty} \mathfrak{B}(i)$; and the ideal is non-trivial if infinitely many of the algebras $\mathfrak{B}(i)$ are non-zero.

Lemma 4.17: Let $\mathfrak{N}(i)$ and $\mathfrak{F}_n(i)$, i,n=1,2,3,... be sequences of C*-algebras. Consider the following (discrete) field of coherent systems:

$$
\begin{array}{lllllllll}
 & n=1 & & & n=n_1 & & n=n_2 & & \\
 & \mathfrak{N}(1) \to & \mathfrak{N}(1) \to & & \to \mathfrak{N}(1) & & & & \\
x=1 & \searrow & \searrow & \cdots \searrow & & \searrow \delta_1 & & & \mathfrak{B}(1) \\
 & \mathfrak{F}_1(1) \to & \mathfrak{F}_2(1) \to & & \to \mathfrak{F}_{n_1}(1) \to & \mathfrak{F}_{n_1+1}(1) & \to \cdots & & \\
 & & & & & & & & \\
 & \mathfrak{N}(2) \to & \mathfrak{N}(2) \to & & \to & & \mathfrak{N}(2) & & \\
x=2 & \searrow & \searrow & \cdots \searrow & & & & \searrow \delta_2 & \mathfrak{B}(2) \\
 & \mathfrak{F}_1(2) \to & \mathfrak{F}_2(2) \to & & \to & & \mathfrak{F}_{n_2}(2) \to & \mathfrak{F}_{n_2+1}(2) \to \cdots & \\
 & & & \cdots & & & & &
\end{array}
$$

where for i = 1,2,3,...

(i) $\mathfrak{N}(i) \to \mathfrak{N}(i)$ is the identity

(ii) $\mathfrak{F}_n(i) \to \mathfrak{F}_{n+1}(i)$ is one-one

(iii) $\delta_i : \mathfrak{N}(i) \to \mathfrak{F}_{n_i+1}(i)$ is one-one and not onto

and $n_1 < n_2 < n_3 < \ldots$ is an increasing sequence.

(Note: It may be that $\mathfrak{F}_1(i),\ldots,\mathfrak{F}_{n_i}(i)$ are zero.)

Set

$$\mathcal{B}_n(i) = \begin{cases} \mathcal{N}(i) \oplus \mathcal{J}_n(i) \ , & 1 \le n \le n_i \\ \mathcal{J}_n(i) & n > n_i \end{cases}$$

and

$$\mathcal{B}_n = \bigoplus_{i=1}^{\infty} \mathcal{B}_n(i) \ .$$

By using the maps in the diagram we obtain a coherent system of C*-algebras $(\mathcal{B}_n, \psi_{mn})$. It then follows that the direct limit

$$\mathcal{B} = \varinjlim(\mathcal{B}_n, \psi_{mn})$$

has a non-trivial closed two-sided ideal $\mathcal{J}$ such that

$$\mathcal{B}/\mathcal{J} \cong \bigoplus_{i=1}^{\infty} \mathcal{N}(i)/c_0 - \bigoplus_{i=1}^{\infty} \mathcal{N}(i) = \mathcal{S} \ .$$

Proof: Let

$$\mathcal{J}_n = \bigoplus_{i=1}^{\infty} \mathcal{J}_n(i)$$

which is an ideal in $\mathcal{B}_n$.

Observe that for all n

$$\mathcal{J}_n \subseteq \mathcal{J}_{n+1}$$

and for the subsequence defined by $n_1 < n_2 < n_3 < \ldots$

$$\mathcal{B}_{n_i}/\mathcal{J}_{n_i} = \left(\bigoplus_{k=1}^{i-1} 0\right) \oplus \left(\bigoplus_{k=i}^{\infty} \mathcal{N}(k)\right) = \mathcal{R}_i \ .$$

The induced natural map

$$\mathcal{B}_{n_i}/\mathcal{J}_{n_i} \to \mathcal{B}_{n_{i+1}}/\mathcal{J}_{n_{i+1}}$$

is given by

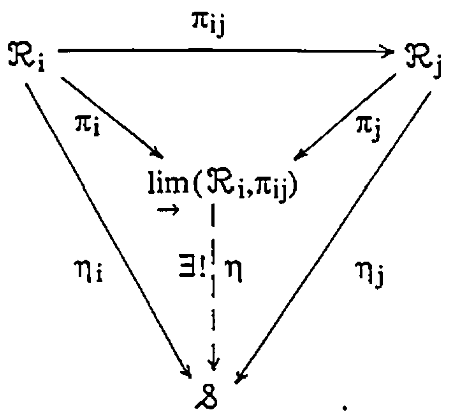
$$\pi_{i,i+1} : \mathcal{R}_i \to \mathcal{R}_{i+1}$$

$$\pi_{i,i+1} : (0,0,\ldots,0,N_i,N_{i+1},\ldots) \mapsto (0,0,\ldots,0,0,N_{i+1},N_{i+2},\ldots) \ .$$

From Lemma 4.15

$$\mathfrak{B}/\mathfrak{J} \cong \lim_{\rightarrow}(\mathfrak{R}_i, \pi_{ij}) .$$

But now we may use the universal property for direct limits. The argument is summarized by the following commutative diagram:



$i \leq j$

.

Indeed, for each i, η_i is the natural quotient map, and for all $i \leq j$, $\eta_j \pi_{ij} = \eta_i$. Therefore there exists a unique $\eta : \lim_{\rightarrow} \mathfrak{R}_i \to \mathcal{S}$ satisfying $\eta \pi_i = \eta_i$ for all i. It is easy to see that η is onto (each η_i is onto). We show that η is one-one.

Let $R = \lim_i \pi_i(R_i)$, $R_i = (0,\ldots,0,N_j^i,N_{i+1}^i) \in \mathfrak{R}_i$ be in $\lim_{\rightarrow} \mathfrak{R}_i$, and suppose $\eta(R) = 0$. Then $\eta\pi_i(R_i)$ converges to zero and so $\eta_i(R_i)$ converges to zero as well. Thus, for each $\varepsilon > 0$ there exists i_0 such that $i \geq i_0$ implies $\|\eta_i(R_i)\| < \varepsilon$. Hence $\|N_j^i\| < \varepsilon$ for all j sufficiently large, $j \geq j_0$ say. But then $\|\pi_{ij_0}(R_i)\| < \varepsilon$, which gives that

$$\|\pi_{j_0}\pi_{ij_0}(R_i)\| = \|\pi_i(R_i)\| < \varepsilon .$$

Therefore $\pi_i(R_i)$ converges to zero and η is one-one. This completes the proof.

<u>Corollary 4.18</u>: Let $\mathfrak{a}$ be an APN-algebra with measurable field of systems $(\mathfrak{a}_n(x),\phi_{mn}(x))$. Fix $r_0 \in \mathbb{N}$, $r_0 > 0$. Suppose that X is partitioned by a sequence of

measurable sets L_i, each of positive measure, and that the ensuing partition of the field of systems is of the form given in Lemma 4.17, where the i^{th} row corresponds to $\mathcal{A}|_{L_i}$ and $\mathcal{N}(i) = \mathcal{M}_{r_0}(\mathcal{L}_\infty(L_i,\mu))$. Then $\mathcal{A}$ has a non-trivial ideal $\mathcal{J}$ such that

$$\mathcal{A}/\mathcal{J} \cong \overset{\infty}{\underset{i=1}{\oplus}} \mathcal{M}_{r_0}(\mathcal{L}^\infty(L_i,\mu)) / c_0 - \overset{\infty}{\underset{i=1}{\oplus}} \mathcal{M}_{r_0}(\mathcal{L}^\infty(L_i,\mu)) .$$

Now let $(\omega_i)_{i=1}^\infty$ be a sequence of non-zero multiplicative linear functionals, $\omega_i : \mathcal{L}^\infty(L_i,\mu) \to \mathbb{C}$. This gives rise to a *-epimorphism ω

$$\omega : \mathcal{A}/\mathcal{J} \to \overset{\infty}{\underset{i=1}{\oplus}} \mathcal{M}_{r_0}(\mathbb{C}) / c_0 - \overset{\infty}{\underset{i=1}{\oplus}} \mathcal{M}_{r_0}(\mathbb{C}) ;$$

and the range algebra is isomorphic to

$$\frac{\ell^\infty(\mathbb{N})\otimes\mathcal{M}_{r_0}}{c_0(\mathbb{N})\otimes\mathcal{M}_{r_0}} \cong \mathcal{M}_{r_0}\left(\frac{\ell^\infty(\mathbb{N})}{c_0(\mathbb{N})}\right) .$$

Proof: The first claim is a simple application of the lemma. The second part is straightforward, relying on the fact that if $\mathcal{I}$ is an ideal in a C*-algebra $\mathcal{B}$, then

$$\frac{\mathcal{B}\otimes\mathcal{M}_{r_0}}{\mathcal{I}\otimes\mathcal{M}_{r_0}} \cong \mathcal{B}/\mathcal{I} \otimes \mathcal{M}_{r_0} .$$

Corollary 4.19: For an algebra $\mathcal{A}$ as in Corollary 4.18,

$$\overline{\mathcal{Q}_4}(\mathcal{A}) \subsetneq \mathcal{A} .$$

Proof: First, $\mathcal{A}$ has a quotient isomorphic to

$$\frac{\ell^\infty(\mathbb{N})}{c_0(\mathbb{N})} \otimes \mathcal{M}_{r_0} \to \mathcal{M}_{r_0} .$$

But $\dfrac{\ell^\infty(\mathbb{N})}{c_0(\mathbb{N})}$ is a commutative C*-algebra and is isomorphic to $C(\beta\mathbb{N}\backslash\mathbb{N})$, where $\beta\mathbb{N}$ is the Stone-Cech compactification of $\mathbb{N}$. Therefore we obtain a map

$$\frac{\ell^\infty(\mathbb{N})}{c_0(\mathbb{N})} \otimes \mathcal{M}_{r_0} \to \mathcal{M}_{r_0}$$

by evaluation at a point. By composing this with the quotient map of $\mathcal{A}$, we get a *-epimorphism of $\mathcal{A}$ onto $\mathcal{M}_{r_0}$. From Ballantine's theorem for matrices the result follows.

Note: For an instance of this behaviour see Example 4 at the beginning of the chapter.

We now begin our proof of the theorem (Theorem 4.12). Suppose then that $\mathcal{A}$ is an APN-algebra which does not satisfy the conditions in the statement of Theorem 4.12. We first show that we can reduce to the generic case of a field of systems $(\mathcal{A}_n(x),\phi_{mn}(x))$ such that $\mathcal{A}_1(x) \neq 0$ a.e. and the system is not attracted to infinity. Let

$$Y = \{x : \mathcal{A}_1(x) \neq 0\}$$

$$Z = \{x : \mathcal{A}_1(x) = 0\}$$

$$Z_n = \{x : \mathcal{A}_n(x) = 0\} \quad , \quad n = 1,2,3,\ldots \quad .$$

By our structure theorem (Theorem 3.3) each of these sets is measurable. Suppose $\mu(Y) > 0$.

If $\mathcal{A}|_Y$ is not attracted to infinity then the reduction is accomplished, for by Lemma 4.14

$$\mathcal{A} \cong \mathcal{A}|_Y \oplus \mathcal{A}|_Z$$

and $\overline{\mathcal{Q}_4}(\mathcal{A}|_Y) \subsetneq \mathcal{A}|_Y$ implies $\overline{\mathcal{Q}_4}(\mathcal{A}) \subsetneq \mathcal{A}$.

If $\mathcal{A}|_Y$ is attracted to infinity (or if $\mu(Y) = 0$), then by hypothesis, $\mu(Z) > 0$, and $\mathcal{A}|_Z$ is not jointly attracted to zero and of infinite type. So now consider $\mathcal{A}|_Z$. Notice that since for almost all x in X, $\phi_{mn}(x)$ is one-one for all $m \leq n$, we have the inclusions (up to measure zero)

$$Z_1 \geq Z_2 \geq Z_3 \geq \ldots \quad .$$

<u>Case 1</u>: Suppose the sequence stabilizes and let n_0 be the first integer satisfying

$$Z_{n_0} = Z_{n_0+1} = \dots \ .$$

<u>Case 1(a)</u>: $\mu(Z_{n_0-1}) > 0$, $\mu(Z_{n_0}) = 0$. Then $\mathcal{A}_{n_0}(x) \neq 0$ a.e. and $\varinjlim(\mathcal{A}_n, \phi_{mn})$ is

isomorphic to $\varinjlim_{n \geq n_0} (\mathcal{A}_n, \phi_{mn})$. Letting

$$\mathcal{B}_1 = \mathcal{A}_{n_0} \ , \ \ \mathcal{B}_2 = \mathcal{A}_{n_0+1} \ , \dots$$

we obtain a new directed system which is not attracted to infinity and for which $\mathcal{B}_1(x) \neq 0$ a.e.

<u>Case 1(b)</u>: $\mu(Z_{n_0}) > 0$. Then for x in Z_{n_0} , $\mathcal{A}(x) = 0$. Therefore

$$\mathcal{A}\big|_Z \cong \mathcal{A}\big|_{Z \setminus Z_{n_0}} \quad \text{and} \quad \mu(Z \setminus Z_{n_0}) > 0 \ .$$

Let $\mathcal{B}$ be the system obtained from $\mathcal{A}\big|_{Z \setminus Z_{n_0}}$ by starting at $n = n_0$, as in 1(a) . Then

$$\mathcal{B} \cong \mathcal{A}|_Z \ , \ \ \mathcal{B}_1(x) \neq 0 \quad (x \in Z)$$

and

$\mathcal{B}$ is not attracted to infinity.

<u>Case 2</u>: We obtain a "best" subsequence of measurably distinct sets

$$Z_1 = Z_{n_1} \supsetneqq Z_{n_2} \supsetneqq Z_{n_3} \supsetneqq \dots$$

$$\mu(Z_{n_i} \setminus Z_{n_i+1}) > 0 \ .$$

(where by "best" we mean $\mathcal{A}_j(x) \neq 0 \Leftrightarrow j > n_i$ for almost all $x \in Z_{n_i}$) .

Then $\mathcal{A}|_{Z_1}$ is attracted to zero and by hypothesis cannot be jointly of infinite type.

Therefore there exists i such that $\mathcal{A}\big|_{Z_{n_i} \setminus Z_{n_{i+1}}}$ is not attracted to infinity and

$\mathcal{A}_{n_{i+1}}(x) \neq 0$ for all x in $Z_{n_i} \setminus Z_{n_{i+1}}$.

Hence, invoking Lemma 4.14 again, we are reduced in all cases to the situation where

$$\mathcal{G}_1(x) \neq 0 \text{ a.e.}$$

and

$$\mathcal{G}_n(x) \text{ is not attracted to infinity.}$$

Let $\mathcal{D}$ be the field of diagrams for the systems, and suppose then that $\mathcal{D}$ is not eventually uniformly increasing. Then there exists m_0 such that $\mathcal{D}$ is not eventually uniformly increasing from $\mathcal{D}(\mathcal{G}_{m_0})$. Therefore there exists $N \in \mathbb{N}$ such that for each $n \geq m_0$, mcd $\mathcal{D}(\mathcal{G}_{m_0}(x))_n \leq N\text{-}1$ on a measurable set J_n of positive measure. For by Lemma 4.9 $x \mapsto$ mcd $\mathcal{D}(\mathcal{G}_{m_0}(x))_n$ is a measurable function which means that $J_n = \{x : \text{mcd } \mathcal{D}(\mathcal{G}_{m_0}(x))_n \leq N\text{-}1\}$ is a measurable set. Without loss of generality $m_0 = 1$. Note also that since $\mathcal{G}_1(x) \neq 0$ a.e., $0 < \text{mcd}(\mathcal{G}_1(x))_n$ a.e. for $n=1,2,3,\ldots$. Furthermore we have the following includions:

$$J_1 \geq J_2 \geq J_3 \geq \ldots$$

<u>Case 1</u>: The sequence of sets stabilizes (up to measure zero).

Let n_0 be the first integer such that $J_{n_0} = J_{n_0+1} = \ldots$ and let $J = J_{n_0}$. Then for almost all x in J $\text{mcd}(\mathcal{G}_1(x))_n \leq N\text{-}1$, $n \geq n_0$. Observe that by defiition 4.8, $\mu(J_{n_0}) > 0$. Now, let

$$d_n(x) = \text{mcd } \mathcal{D}(\mathcal{G}_1(x))_n \ .$$

By Lemma 4.10

$$f(x) = \sup_n d_n(x)$$

and

$$\ell(x) = \min_n \{n : d_n(x) = f(x)\}$$

are measurable functions. Let

$$F_r = \{x : f(x) = r\} \ , \ 1 \leq r \leq N\text{-}1 \ .$$

At least one of the sets F_r has positive measure, $r = r_0$ say. Let

$$L_s = \{x \in F_{r_0} : \ell(x) = s\} .$$

Since ℓ is measurable there exists s_0 such that $\mu(L_{s_0}) > 0$. So for almost all x in L_{s_0}, $\ell(x) = s_0$, $f(x) = r_0$. It follows that, restricted to L_{s_0}, the field of systems has the form (starting from $s_0 + 1$)

$$\begin{array}{ccccc} \mathfrak{M}_{r_0}(\mathcal{L}^\infty(L_{s_0}, \mu)) & \longrightarrow & \mathfrak{M}_{r_0}(\mathcal{L}^\infty(L_{s_0}, \mu)) & \longrightarrow & \cdots \\ & \searrow & & & \\ \mathfrak{J}_{s_0+1} & \longrightarrow & \mathfrak{J}_{s_0+2} & \longrightarrow & \cdots \end{array}$$

where the maps

$$\mathfrak{M}_{r_0}(\mathcal{L}^\infty(L_{s_0}, \mu)) \to \mathfrak{M}_{r_0}(\mathcal{L}^\infty(L_{s_0}, \mu))$$

are of multiplicity 1, a.e. Therefore, $\mathfrak{A}(L_{s_0})$ has an ideal $\mathfrak{J} = \varinjlim \mathfrak{J}_{s_0+n}$ and

$$\mathfrak{A}/\mathfrak{J} \cong \mathfrak{M}_{r_0}(\mathcal{L}^\infty(L_{s_0}, \mu)) .$$

We conclude that

$$\overline{\mathfrak{Q}}_4(\mathfrak{A}) \subsetneqq \mathfrak{A} .$$

<u>Case 2</u>: We obtain a best subsequence of measurably distinct sets

$$J_{n_1} \supsetneqq J_{n_2} \supsetneqq J_{n_3} \supsetneqq \cdots$$

$$\mu(J_{n_i} \backslash J_{n_{i+1}}) > 0 .$$

Let

$$K_{n_i} = J_{n_i} \backslash J_{n_{i+1}} .$$

Since $x \mapsto \mathrm{mcd}(\mathfrak{A}_1(x))_n$ is measurable for each n, and for each n_i

$$1 \leq \mathrm{mcd}\ \mathfrak{A}_1(x)_1 \leq \ldots \leq \mathrm{mcd}\ \mathfrak{A}_1(x)_{n_i} = f(x) \leq N\text{-}1$$

for almost all x in K_{n_i},

there exists r_0, $1 \leq r_0 \leq N\text{-}1$ and (by passing to a subsequence of the K_{n_i}'s if necessary) a sequence of measurable sets L_i,

$$L_i \leq K_{n_i}$$

$$\mu(L_i) > 0$$

so that restricted to $L = \bigcup_i L_i$, the system for $\mathcal{A}(L)$ is of the form given in Corollary 4.18. Therefore

$$\overline{Q}_4(\mathcal{A}(L)) \subsetneq \mathcal{A}(L)$$

from which it follows that

$$\overline{Q}_4(\mathcal{A}) \subsetneq \mathcal{A} \ .$$

This completes the proof of the necessity of the conditions in Theorem 4.12.

To prove the sufficiency of each of the conditions, we prepare the way with a general proposition on certain direct limits.

<u>Proposition 4.20</u>: Let $X = \mathbb{N}$ and μ be the counting measure on X. Let $\mathcal{A}_n(i)$ be a double sequence of C*-algebras such that

$$\mathcal{A}_1(i) \subseteq \mathcal{A}_2(i) \subseteq \ldots \subseteq \mathcal{A}(i) = \varinjlim \mathcal{A}_n(i) = \overline{\bigcup_n \mathcal{A}_n(i)} \ .$$

Let $1 \leq n_1 < n_2 < n_3 < \ldots$ be an increasing sequence of positive integers and consider the field of systems given by the following diagram:

$$\begin{array}{lllll}
x=1 & 0 \to \cdots \to 0 \to \mathcal{A}_{n_1}(1) \to \cdots \to \mathcal{A}_{n_2}(1) \to \cdots \to \mathcal{A}_{n_3}(1) \to \cdots & \to & \mathcal{A}(1) \\
x=2 & 0 \to \cdots\cdots\cdots\cdots \to 0 \to \mathcal{A}_{n_2}(2) \to \cdots \to \mathcal{A}_{n_3}(2) \to \cdots & \to & \mathcal{A}(2) \\
x=3 & 0 \to \cdots\cdots\cdots\cdots\cdots\cdots \to 0 \to \mathcal{A}_{n_3}(3) \to \cdots & \to & \mathcal{A}(3) \\
\vdots & \qquad\qquad \cdots & & \vdots
\end{array}$$

Then with

$$\mathcal{A} = \varinjlim \left(\int_{\mathbb{N}}^{\oplus} \mathcal{A}_n(i) d\mu \right)$$

$$\overline{\mathcal{Q}}_4(\mathcal{A}) = \mathcal{A} \Leftrightarrow \overline{\mathcal{Q}}_4(\mathcal{A}(i)) = \mathcal{A}(i) \quad \text{for each } i = 1,2,3,\ldots \ .$$

Example: $\mathcal{A}(i) = \mathcal{B}$, a fixed algebra. Then

$$\mathcal{A} \cong c_0(\mathbb{N}) \otimes \mathcal{B}$$

and

$$\overline{\mathcal{Q}}_4(\mathcal{A}) = \mathcal{A} \Leftrightarrow \overline{\mathcal{Q}}_4 \mathcal{B} = \mathcal{B} \ .$$

See also Example 2 from the beginning of the chapter.

Proof: First notice that for each i, $\mathcal{A}(i)$ is a direct summand of $\mathcal{A}$. This follows from Lemma 4.14 and the fact that $\mathbb{N} = (\mathbb{N}\backslash\{i\}) \cup \{i\}$.

Suppose $\overline{\mathcal{Q}}_4(\mathcal{A}) = \mathcal{A}$, let $A(i) \in \mathcal{A}(i)$ and suppose $\varepsilon > 0$. Choose $A_\varepsilon \in \mathcal{Q}_4(\mathcal{A})$ such that

$$\|A_\varepsilon - 0 \oplus A(i)\| < \varepsilon$$

$$A_\varepsilon = A_\varepsilon(\mathbb{N}\backslash\{i\}) \oplus A_\varepsilon(i) \ .$$

It follows that $A_\varepsilon(i) \in \mathcal{Q}_4(\mathcal{A}(i))$ and

$$\|A_\varepsilon(i) - A(i)\| < \varepsilon \ .$$

Therefore

$$\overline{\mathcal{Q}}_4(\mathcal{A}(i)) = \mathcal{A}(i) \ .$$

Conversely, suppose that

$$\overline{\mathcal{Q}}_4(\mathcal{A}(i)) = \mathcal{A}(i) \ , \quad i = 1,2,3,\ldots \ .$$

Let $A \in \mathcal{A}$ and $\varepsilon > 0$. Choose $A_n \in \mathcal{A}_n$ such that

$$\|A_n - A\| < \varepsilon/2 \ .$$

By definition,

$$\mathcal{A}_n = \int_{\mathbb{N}}^{\oplus} \mathcal{A}_n(i)d\mu = \bigoplus_i \mathcal{A}_n(i) \ .$$

From the condition on the sequence $\{n_i\}$, as a member of the cdirect sum $\bigoplus_i \mathcal{Q}_n(i)$, A_n has only finitely many non-zero components. Push A_n through the direct system so that each of these is approximated within $\varepsilon/2$ by an element of $\mathcal{Q}_4$ - this is possible since $\overline{\mathcal{Q}_4}(\mathcal{Q}(i)) = \mathcal{Q}(i)$ for all i. We obtain an element

$$A_{\varepsilon/2} \in \mathcal{Q}_4(\mathcal{Q})$$

such that

$$\|A_{\varepsilon/2} - A_n\| < \varepsilon/2 .$$

But then

$$\|A - A_{\varepsilon/2}\| < \varepsilon$$

and we conclude that

$$\overline{\mathcal{Q}_4}(\mathcal{Q}) = \mathcal{Q} ,$$

as claimed.

Because of the last Proposition, to complete the proof of Theorem 4.12, it is enough to show that if $\mathcal{Q}$ is an APN-algebra which is attracted to infinity, then $\overline{\mathcal{Q}_4}(\mathcal{Q}) = \mathcal{Q}$. To deal with such an algebra we require a generalization of Lemma 2.8.

<u>Lemma 4.21</u>: Suppose that $\phi_1,\ldots,\phi_n$ are bounded measurable functions such that $\prod_{i=1}^{n} \phi_i$ is essentially invertible. Then for each $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that the following is true:

If $m_1,\ldots,m_n,z$ are non-negative integers such that $m_1 + \ldots + m_n + z \geq N$. Then there exist measurable functions $\mu_1,\ldots,\mu_n$, ζ with

(i) $|\mu_i(x) - \phi_i(x)| < \varepsilon$ a.e., for all i

(ii) $\frac{\varepsilon}{2} \leq |\zeta(x)| < \varepsilon$ a.e., and

(iii) $\mu_1^{m_1} \ldots \mu_n^{m_n} \zeta^z >> 0$.

Fundamental to our proof of this Lemma is the notion of "measurable selection". Endow the set $\mathfrak{C}$ of compact subsets of $\mathbb{C}$ with the Hausdorff metric and so obtain the

Borel structure subordinate to this topology (sometimes called the finite topology). In this way, we can talk about measurable functions $F : (x,\mu) \to \mathfrak{C}$. We call such functions multi-functions, and in keeping with convention we often write $F : X \to \mathbb{C}$. A function $\phi : X \to \mathbb{C}$ is a selection for a multi-function F if $\phi(x) \in F(x)$ for all $x \in X$.

Lemma 4.22: Suppose that $x \mapsto I(x)$ defines a measurable multi-function of closed real intervals. Then there exists a measurable function $\phi : X \to \mathbb{C}$ such that $\phi(x) \in I(x)$ a.e.

Proof: In this simple case where $I(x) \subseteq \mathbb{R}$, we need only define $\phi(x) = \min\{\lambda : \lambda \in I(x)\}$. Or we may invoke Aumann's theorem (Theorem 5.2 in [Hi]).

Lemma 4.23: Suppose that $x \mapsto G(x)$ and $x \mapsto K(x)$ define measurable multi-functions with values closed intervals in $\mathbb{R}$, and that $x \mapsto \sigma(x) \in G(x) + K(x)$ a.e. determines a measurable selection of the measurable multi-function $x \mapsto G(x) + K(x)$. Then there are measurable functions

$$x \mapsto \gamma(x) \in G(x) \quad \text{a.e.}$$

$$x \mapsto \kappa(x) \in K(x) \quad \text{a.e.}$$

such that

$$\sigma(x) = \gamma(x) + \kappa(x) \quad \text{a.e.}$$

Proof: Let

$$S_G(x) = \{g \in G(x) : g + k = \sigma(x) \text{ for some } k \in K(x)\}$$

$$= G(x) \cap [\sigma(x) - K(x)] .$$

By Theorem 4.1 [Hi] $x \mapsto S_G(x)$ is a measurable multi-function. So by Theorem 5.2 of [Hi] there exists a measurable function

$$x \mapsto \gamma(x) \in S_G(x) \quad \text{a.e.}$$

Now let $\kappa(x) = \sigma(x) - \gamma(x)$.

Lemma 4.24: Let Z be a closed subset of $\mathbb{R}$ and suppose that $x \mapsto I(x)$ is a measurable multi-function of intervals. Then

$$x \mapsto I_Z(x) = I(x) \cap Z$$

is a measurable multi-function.

Proof: This is just a special case of Proposition 2.4 in [Hi].

Proof of Lemma 4.21: Since each ϕ_j is essentially bounded, there exists $R > 0$ such that $|\phi_j(x)| \leq R$ a.e., for all j. Let $\theta : \mathbb{C} \to [0,2\pi)$ be the principal branch of the argument function. Since $|\phi_j(x)|$ is uniformly bounded, there exists δ, $0 < \delta < \varepsilon$, $\delta < \frac{\pi}{4}$, such that when $|\xi| = |\eta| \leq R$ and $|\theta(\xi) - \theta(\eta)| \leq \delta \bmod 2\pi$ then $|\xi - \eta| < \varepsilon$. (Just find the δ which works on the circle $\{\lambda : |\lambda| = R\}$). (*)

Since θ is a measurable function, so is $\theta \circ \phi_j$ for $j = 1,\dots,n$. Therefore, we get n measurable multi-functions

$$S_j(x) = [\theta(\phi_j(x)) - \tfrac{\delta}{2}, \theta(\phi_j(x)) + \tfrac{\delta}{2}] \bmod 2\pi .$$

Define

$$S_{n+1}(x) = [-\tfrac{\delta}{2}, \tfrac{\delta}{2}] \bmod 2\pi .$$

There exists $n_0 \in \mathbb{N}$ such that $n_0 \cdot \delta > 2\pi$. Let $N = (n+1)n_0$. Then $m_1 + \dots + m_n + z \geq (n+1)n_0 = N$ implies there exists j_0 such that $m_{j_0} \geq n_0$ or $z \geq n_0$. Therefore, the measurable field

$$I \mapsto I(x) = m_1 S_1(x) + \dots + m_n S_n(x) + z S_{n+1}(x)$$

of intervals is almost everywhere of length greater than 2π. By letting

$$Z = \{2\pi n : n \in \mathbb{Z}\} \subseteq \mathbb{R}$$

we obtain a measurable multi-function

$$x \mapsto I_Z(x) = I(x) \cap Z \neq \phi \quad \text{a.e.}$$

By Theorem 5.2 [Hi] we obtain a measurable map

$$x \mapsto \sigma(x) \in I_Z(x) \qquad \text{a.e.}$$

But

$$I_Z(x) \subsetneq I(x) = m_1S_1(x) + \ldots + m_nS_n(x) + zS_{n+1}(x) \ .$$

Therefore, by induction applied to Lemma 4.23 we obtain measurable functions

$$x \mapsto \alpha_j(x) \in m_jS_j(x) \qquad \text{a.e.} \qquad j = 1,\ldots,n$$

and

$$x \mapsto \beta(x) \in zS_{n+1}(x) \qquad \text{a.e.}$$

such that

$$\sigma(x) = \alpha_1(x) + \ldots + \alpha_n(x) + \beta(x) \quad \text{a.e.}$$

To complete the proof, let

$$\theta_j = \begin{cases} \frac{1}{m_j}\alpha_j \ , & m_j > 0 \\ 0 \ , & m_j = 0 \end{cases}$$

$$\theta_{n+1} = \begin{cases} \frac{1}{z}\beta \ , & z > 0 \\ 0 \ , & z = 0 \end{cases}$$

and set $\quad \mu_j = |\phi_j|\, e^{i\theta_j} \ , \ j = 1,\ldots,n$

and $\quad \zeta = \frac{\varepsilon}{2}\, e^{i\theta_{n+1}} \ .$

From (*) we have $\|\mu_j - \phi_j\| < \varepsilon$ and by construction

$$\mu_1^{m_1} \ldots \mu_n^{m_n}\rho^{z} >> 0 \ .$$

This completes the proof of Lemma 4.21.

Finally, using Lemma 4.21, the proof that $\overline{\mathfrak{Q}_4}(\mathfrak{A}) = \mathfrak{A}$ (for an APN-algebra attracted to infinity) is formally the same as the proof of Theorem 2.5, the case of AF-algebras, and our result is established.

Remarks: 1) Originally, our proof of Lemma 4.21 was more elementary, constructing explicit measurable selection functions. But selection theorems touch deep into the heart of Direct Integral Theory, so employing Himmelberg's results seemed appropriate (as well as convenient). Furthermore, one of our interests is to develop a factorization theory for the topological case, e.g. $\mathfrak{M}_n(C(\mathbb{T}))$ (see Concluding Remarks) and our use of the selection theorems is therefore suggestive. For by doing so the possible requirement of "continuous selection" theorems is brought to attention as are the kind of difficulties we could expect.

2) It would seem that using Lemma 4.15 it would not be too difficult to obtain a characterization of the ideal structure of APN-algebras, including a statement in terms of diagrams. One significant difference between APN-algebras and AF-algebras is exhibited in the existence of ideals arising from attraction of the algebra to zero.

3) The proof of Theorem 4.12 depended on Lemma 4.17. While the exact result required the structure of certain polynormal subalgebras involved, one of the key features of the direct limit algebra $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ (for which $\mathfrak{Q}_4(\mathfrak{A})$ was not dense in $\mathfrak{A}$) was the existence of an ideal $\mathfrak{J} = \overline{\bigcup_n \mathfrak{J}_n}$, $\mathfrak{J}_n = \mathfrak{A}_n \cap \mathfrak{J}$ such that $\overline{\mathfrak{Q}_4}(\mathfrak{A}_n/\mathfrak{J}_n) \subsetneq \mathfrak{A}_n/\mathfrak{J}_n$. Can this be generalized nicely? So if $\mathfrak{A}$ is a direct limit of not necessarily polynormal subalgebras, when does this condition imply that $\overline{\mathfrak{Q}_4}(\mathfrak{A}) \subsetneq \mathfrak{A}$. In addition, could this approach be used to find examples of algebras for which $\overline{\mathfrak{Q}_4}(\mathfrak{A}) \subsetneq \overline{\mathfrak{Q}_k}(\mathfrak{A}) \subsetneq \mathfrak{A}$ for some k , $5 \leq k \leq \infty$? On this question see also Chapter 5.

4) Note that an APN-algebra is a direct limit of nuclear C*-algebras and so is itself nuclear ([KR], Proposition 11.3.12). Therefore, as for AF-algebras, we obtain a unique C*-algebra tensor product $\mathfrak{A} \otimes \mathfrak{B}$, for APN-algebras $\mathfrak{A}$ and $\mathfrak{B}$. We wish to obtain a generalization of Proposition 2.11. As it turns out, there are some subtleties not prsent in the case of AF-algebras. And to realize this tensor product in a way which

is compatible wih the measurable fibre-structures, we show that (in a sense to be made precise) every APN-algebra $\mathcal{A} = \varinjlim \mathcal{A}_n$ is naturally a subalgebra of the "direct integral of canonically represented C*-algebras"

$$\int_X^{\oplus} \mathcal{A}(x)d\mu \ , \ \mathcal{A}(x) = \varinjlim \mathcal{A}_n(x) \ .$$

We will treat these matters in detail in Chapter 5 where we introduce various direct integral constructions for fields of C*-algebras.

Our last result of this chapter is a straightforward generalization of Proposition 2.15, which we include for completeness.

Proposition 4.25: Let $\mathcal{A}$ be an APN-algebra. If $A \in \mathcal{A}$ and $\sigma(A) \geq 0$ then $A \in \overline{\mathcal{Q}_2}(\mathcal{A})$.

Proof: The proof is essentially the same as that of Proposition 2.15. Use the fact that an n-normal operator may be unitarily triangularized ([R&R], Theorem 7.20) and that for

$$A = \begin{pmatrix} \phi_n & & * \\ & \ddots & \\ 0 & & \phi_n \end{pmatrix} \quad \text{in} \quad \mathcal{M}_n(\mathcal{L}^\infty(X,\mu)) \ ,$$

$$\sigma(A) = \bigcup_{i=1}^{n} \text{ess rg } \phi_i \ .$$

Chapter 5

Direct Integral Constructions

As we have discussed in Chapter Three, to a measurable field of coherent systems $(\mathcal{A}_n(x), \phi_{mn}(x))$ of von Neumann algebras we can associate the direct limit C*-algebra

$$\mathcal{A} = \lim_{\rightarrow} \int_X^{\oplus} \mathcal{A}_n(x) d\mu \ .$$

When certain additional conditions are satisfied we obtain our so-called APN-algebras (see Chapter 3). There are in fact two other algebras which in a natural way may be constructed from the field of systems. One of these we call the compact direct integral, and is denoted

$$\oint_X^{\oplus} \mathcal{A}(x) d\mu \text{ , where } \mathcal{A}(x) = \lim_{\rightarrow} \mathcal{A}_n(x) \ .$$

The second of these which contains both the direct limit algebra and the compact direct integral, and can be thought of as the ambient space, is called simply the direct integral (of the "measurable field of C*-algebras $\mathcal{A}(x)$ ") and is denoted by

$$\int_X^{\oplus} \mathcal{A}(x) d\mu \ .$$

Precise definitions will be given later in the chapter. To relate these alagebras, note that we usually have the following inclusions

$$\lim_{\rightarrow} \int_X^{\oplus} \mathcal{A}_n(x) d\mu \subseteq \oint_X^{\oplus} \mathcal{A}(x) d\mu \subseteq \int_X^{\oplus} \mathcal{A}(x) d\mu$$

where in general the inclusions can be proper all at once. Although, relaxing condition (iii) in the definition of APN-algebras gives examples of the first inclusion on the left being broken.

Using the notion of compact direct integral we obtain a definition of $\mathcal{L}^\infty(X,\mu)\otimes\mathcal{A}$ as a certain subalgebra of the direct integral $\int_X^\oplus \mathcal{A}\, d\mu$. As a consequence we have a new proof that $\overline{\mathcal{Q}}_4(\mathcal{L}^\infty(X,\mu)\otimes\mathcal{K}) = \mathcal{L}^\infty(X,\mu)\otimes\mathcal{K}$. Furthermore, the concepts promise to generalize to yield a characterization of those compact direct integrals $\mathcal{A}$ for which $\mathcal{Q}_4(\mathcal{A})$ is dense.

For an example of a field of systems where we have proper inclusions between our three algebras, let $X = \mathbb{N}$, μ be a counting measure and consider the following field of Bratteli diagrams:

	n=1	n=2	n=3		
x=1	1 —	2 —	3 —	···	$\mathcal{A}(1) = \mathcal{K}$
x=2	0 —	1 —	2 —	···	$\mathcal{A}(2) = \mathcal{K}$
x=3	0 —	0 —	1 —	···	$\mathcal{A}(3) = \mathcal{K}$
		...			...

In this case

$$\varinjlim \int_{\mathbb{N}}^\oplus \mathcal{A}_n(x)d\mu \equiv c_0(\mathbb{N})\otimes\mathcal{K}$$

$$\oint_X^\oplus \mathcal{A}(x)d\mu \cong \ell^\infty(\mathbb{N})\otimes\mathcal{K}$$

and
$$\int_X^\oplus \mathcal{A}(x)d\mu \cong \ell^\infty(\mathbb{N},\mathcal{K}) \ .$$

That the first algebra is properly contained in the second is clear. The second algebra, we will show, can be identified with the C*-algebra of functions $\mathbb{N}\to\mathcal{K}$ with compact essential range (compact as a subset of the Banach space $\mathcal{K}$). The third algebra is the C*-algebra of all bounded measurable maps $\mathbb{N}\to\mathcal{K}$. So it is clear that we have the second inclusion proper as well.

We begin by addressing the question of "measurable fields of C*-algebras", and defining the direct integral of such a field.

Definition 5.1. Let (X,μ) be a standard σ-finite measure space and suppose that

$$x \mapsto \mathcal{A}(x) \subseteq \mathcal{B}(\mathcal{H}(x))$$

defines a field of separable C*-algebras of operators. The field is called measurable if

(i) the field of Hilbert spaces $x \mapsto \mathcal{H}(x)$ is measurable, and

(ii) there exists a countable sequence of measurable fields of operators

$$x \mapsto A_n(x) \quad , \quad n = 1,2,3,\ldots$$

such that

$$C^*\{A_n(x) : n = 1,2,3,\ldots\} = \mathcal{A}(x) \quad \text{a.e.}$$

We call the sequence $(A_n)_{n=1}^{\infty}$ a fundamental sequence (of μ-measurable operator fields). By considering the new field $x \mapsto A_n(x)/\|A_n(x)\|$ if necessary, we may assume that each A_n is a bounded decomposable operator.

Now it is immediate that if $\overline{\mathcal{A}}(x)$ is the WOT closure of $\mathcal{A}(x)$, then the field

$$x \mapsto (\overline{\mathcal{A}}(x), \mathcal{H}(x))$$

of von Neumann algebras is measurable (see Definition IV 8.17 [Tak]). Therefore the direct integral

$$\int_X^{\oplus} \overline{\mathcal{A}(x)}\, d\mu \quad \text{on} \quad \int_X^{\oplus} \mathcal{H}(x) d\mu$$

is well-defined.

Definition 5.2: We now define the direct integral of the measurable field of C*-algebras $x \mapsto \mathcal{A}(x)$ to be

$$\left\{ A \in \int_X^{\oplus} \overline{\mathcal{A}(x)}\, d\mu \; : \; A(x) \in \mathcal{A}(x) \;\; \text{a.e.} \right\}$$

and denote this set by

$$\int_X^{\oplus} \mathcal{A}(x) d\mu \; .$$

This direct integral is actually a C*-algebra. The proof of this we leave to the reader.

Definition 5.3: A field of *-homomorphisms $\pi(x) : \mathcal{A}(x) \to \mathcal{B}(x)$ is said to be measurable if for every measurable field $x \mapsto A(x) \in \mathcal{A}(x)$, the field $x \mapsto \pi(x)A(x)$ is also measurable. Such a field induces a *-homomorphism

$$\pi = \int_X^{\oplus} \pi(x)d\mu : \int_X^{\oplus} \mathcal{A}(x)d\mu \to \int_X^{\oplus} \mathcal{B}(x)d\mu .$$

The map π is said to be a direct integral homomorphism. We also say that π is a direct integral isomorphism if $\pi(x)$ is an isomorphism a.e. Finally, two direct integral algebras are direct integral unitarily equivalent if there exists a measurable field of unitary operators which implements a direct integral isomorphism.

Examples: 1) Let n be a finite positive integer and let $\mathcal{A}(x)$ be the C*-algebra of $n\times n$ matrices acting on $\mathcal{H}(x) = \mathbb{C}^n$ in the canonical way. Then $\overline{\mathcal{A}}(x) = \mathcal{A}(x)$ and

$$\int_X^{\oplus} \mathcal{A}(x)d\mu = \int_X^{\oplus} \overline{\mathcal{A}}(x) = \mathcal{M}_n(\mathcal{L}^{\infty}(X,\mu)) .$$

2) Let $\mathcal{A}(x) = \mathcal{K}$, the C*-algebra of compact operators acting on $\mathcal{H}(x) = \ell^2(\mathbb{N})$. Then

$$\int_X^{\oplus} \mathcal{K}d\mu = \mathcal{L}^{\infty}(X,\mathcal{K}) = \text{decomposable operators which are almost everywhere compact,}$$

while

$$\int_X^{\oplus} \overline{\mathcal{K}}d\mu = \mathcal{L}^{\infty}(X,\mathcal{B}(\mathcal{H})) = \text{all decomposable operators.}$$

3) Let $\mathcal{A} = \mathcal{M}_m(\mathcal{L}^{\infty}(X,\mu))$ and $\mathcal{B} = \mathcal{M}_n(\mathcal{L}^{\infty}(X,\mu))$, and suppose that $\pi : \mathcal{A} \to \mathcal{B}$ is a "canonical *-homomorphism", as defined in Chapter 3. Then π is a direct integral homomorphism.

We now claim that if $\mathcal{A}$ is an APN-algebra,

$$\mathfrak{A} = \varinjlim \int_X^{\oplus} \mathfrak{A}_n(x)d\mu$$

then there exists a canonical measurable field of Hilbert spaces $x \mapsto \mathcal{H}(x)$ on which we may canonically represent each AF-algebra

$$\mathfrak{A}(x) = \varinjlim \mathfrak{A}_n(x)$$

in such a way that the field

$$x \mapsto \mathfrak{A}(x)$$

of C*-algebras is measurable. So we will make sense of

$$\int_X^{\oplus} \mathfrak{A}(x)d\mu$$

and by this make possible the definition of the direct limit and the compact direct integral as subalgebras of the direct integral.

For an indication of how we obtain the canonical Hilbert spaces and consequently the direct integral of algebras, consider the case where X is a singleton and our (single) AF-algebra is the algebra of compact operators

$$\mathcal{K} = \overline{\cup \mathfrak{M}_n} \ .$$

Each subalgebra $\mathfrak{M}_n$ in the generating nest acts naturally on the Hilbert space $\mathbb{C}^n$. Moreover, defining a partial isometry

$$V_{mn} : \mathbb{C}^m \to \mathbb{C}^n$$

by

$$V_{mn} : f \mapsto \begin{pmatrix} f \\ 0 \end{pmatrix} \ , \ m \le n$$

and letting

$$\phi_{mn} : \mathfrak{M}_m \to \mathfrak{M}_n$$

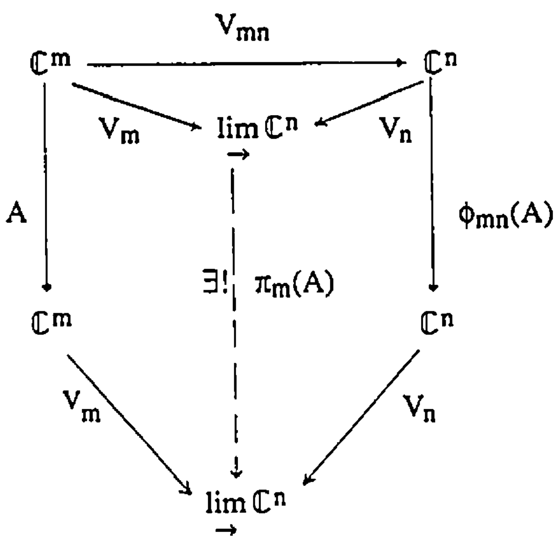
be the usual inclusion

$$\phi_{mn} : A \mapsto \begin{pmatrix} A & \\ & 0 \end{pmatrix}$$

we obtain a system of commuting diagrams

$$\begin{array}{ccc} \mathbb{C}^m & \xrightarrow{V_{mn}} & \mathbb{C}^n \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle \phi_{mn}(A)} \\ \mathbb{C}^m & \xrightarrow[V_{mn}]{} & \mathbb{C}^n \end{array} \qquad \text{for all } A \in \mathfrak{M}_m \, .$$

In other words the action of $\mathfrak{M}_n$ on $\mathbb{C}^n$ carries forward through the system in a way which respects the direct limit structure. We therefore obtain a unique operator $\pi_m(A)$, $A \in \mathfrak{M}_m$, acting on the "direct limit of the Hilbert spaces": assuming for now the existence of the Hilbert space $\varinjlim(\mathbb{C}^n, V_{mn})$, we have for each $m \le n$ a diagram



which commutes and thererfore by the universal property a unique bounded operator $\pi_m(A)$. Of course in this example $\varinjlim \mathbb{C}^n = \ell^2(\mathbb{N})$ and

$$\pi_m(A) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

For the general case, we require a sequence of preliminary results.

Lemma 5.4: Let $\mathcal{H}_n = \int_X^\oplus \mathcal{H}_n(x)d\mu$ be a direct integral of separable Hilbert spaces (finite or infinite dimensional) and suppose that for $m \le n$

$$V_{mn} : \mathcal{H}_m \to \mathcal{H}_n$$

$$V_{mn} = \int_X^\oplus V_{mn}(x)d\mu$$

is a decomposable partial isometry satisfying

$$V_{np}V_{mn} = V_{mp} \quad \text{for all} \quad m \le n \le p .$$

Then the direct limit

$$\mathcal{H} = \lim_{\rightarrow}(\mathcal{H}_n, V_{mn})$$

exists. Moreover

$$\mathcal{H} \cong \int_X^\oplus \lim_{\rightarrow}(\mathcal{H}_n(x), V_{mn}(x))d\mu \ .$$

Proof: For the existence of $\mathcal{H}$, see [KR] Ex. 11.5.26. That $\mathcal{H}$ is the direct integral of the individual direct limits follows from the fact that for each n, $\mathcal{H}_n$ is naturally identified with a subspace of $\mathcal{H}$ and that if $\{e_k^{(n)}\}_{k=1}^\infty$ is a fundamental sequence for $\mathcal{H}_n$ then $\{e_k^{(n)} : n, k = 1,2,3,\dots\}$ is a fundamental sequence for $\mathcal{H}$.

Note: Each canonical

$$V_n : \mathcal{H}_n \to \mathcal{H}$$

is a decomposable operator,

$$V_n = \int_X^\oplus V_n(x)d\mu \, , \; V_n(x) : \mathcal{H}_n(x) \to \mathcal{H}(x) \ .$$

This is because the set $\{e_k^{(n)} : n, k = 1,2,3,\dots\}$ is a fundamental sequence and

$$x \mapsto \left\langle V_{mn}e_k^{(m)}, e_\ell^{(n)} \right\rangle(x)$$

is measurable for all k, ℓ and $m \le n$.

Note: Within the category of direct integrals, $\mathcal{H}$ is unique up to direct integral unitary equivalence.

Lemma 5.5: Let $\mathcal{H}_n = \int_X^{\oplus} \mathcal{H}_n(x)d\mu$, $V_{mn} : \mathcal{H}_m \to \mathcal{H}_n$, $V_n : \mathcal{H}_n \to \mathcal{H}$, $\mathcal{H} = \lim\limits_{\to} \mathcal{H}_n$ be as in the previous lemma. Suppose that we have a bounded sequence $(\sup\limits_n \|A_n\| < \infty)$ of decomposable operators

$$A_n : \mathcal{H}_n \to \mathcal{H}_n, \; A_n = \int_X^{\oplus} A_n(x)d\mu$$

such that for each $m \leq n$ the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{H}_m & \xrightarrow{V_{mn}} & \mathcal{H}_n \\ {\scriptstyle A_m}\downarrow & & \downarrow{\scriptstyle A_n} \\ \mathcal{H}_m & \xrightarrow[V_{mn}]{} & \mathcal{H}_n \end{array} .$$

Then there exists a unique decomposable operator

$$A : \mathcal{H} \to \mathcal{H}$$

such that

$$AV_n = V_nA_n$$

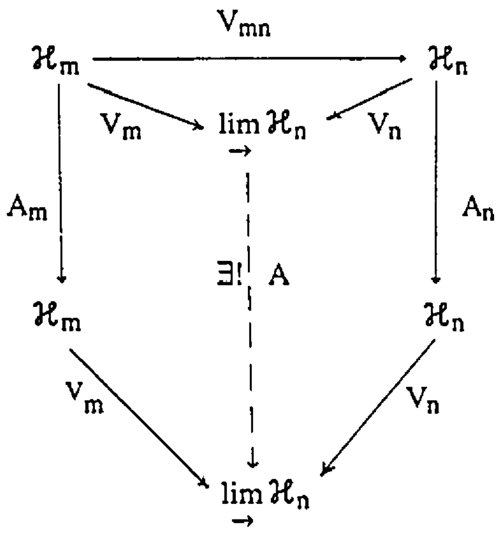
for all $n = 1,2,3,\ldots$.

The operator A is called the inductive limit of the bounded family $\{A_n\}$ of operators.

Proof: For each $m \leq n$

$$V_mA_m = V_nA_nV_{mn}$$

that is, the diagram



commutes. Therefore, by the universal property for direct limits, there is a unique operator A satisfying the required properties. Since $AV_n = V_nA_n$ and $\{V_ne_k^{(n)} : n,k \in \mathbb{N}\}$ is a fundamental set for $\mathcal{H}$, it follows that A is decomposable as claimed.

Note: Let $A(x)$ be the direct limit of the family $\{A_n(x)\}$. Then by uniqueness $A = \int_X^{\oplus} A(x)d\mu$, since $x \mapsto A(x)$ is easily shown to be a measurable field of operators.

Note: Here are two alternate proofs that A is decomposable:

1) Via the partial isometries $V_m : \mathcal{H}_m \to \mathcal{H}$, each A_m determines a decomposable operator A'_m on $\mathcal{H}$. The operator A is the norm limit of the sequence A'_m, and the decomposable operators are norm closed.

2) We show that A commutes with the diagonal algebra $\mathcal{L}$: Let $A_0 = A$ restricted to the linear space $\mathcal{H}_0 = \bigcup_n V_n\mathcal{H}_n$. Then A_0 is a bounded linear map on $\mathcal{H}_0$ and extends uniquely to A on $\mathcal{H}$. Moreover, for each $L \in \mathcal{L}$ and $f \in \mathcal{H}_0$, $A_0Lf = LA_0f$, since $AV_n = V_nA_n$ and A_n is decomposable. Therefore A_0 commutes with $\mathcal{L}$, hence A commutes with $\mathcal{L}$ and A is decomposable ([Tak], Corollary IV.8.16).

Proposition 5.6: For each $m,n \in \mathbb{N}$ let V_{mn}, V_n, $\mathcal{H}_n$ and $\mathcal{H}$ be as in the previous lemma, and

$$\mathcal{A}_n = \int_X^{\oplus} \mathcal{A}_n(x)d\mu$$

be the direct integral of a measurable field of von Neumann algebras on $\mathcal{H}_n(x)$ which is also measurable as a field of separable C*-algebras. For each $m \leq n$ let

$$\phi_{mn} = \int_X^{\oplus} \phi_{mn}(x)d\mu$$

be the direct integral of a measurable field of faithful WOT continuous *-homomorphisms

$$\phi_{mn}(x) : \mathcal{A}_m(x) \rightarrow \mathcal{A}_n(x) .$$

Suppose also that the C*-algebras $\mathcal{A}_n$ together with the *-monomorphisms $\phi_{mn} : \mathcal{A}_m \rightarrow \mathcal{A}_n$ form a coherent system, with direct limit C*-algebra $\mathcal{A}$ and canonical injections $\phi_n : \mathcal{A}_n \rightarrow \mathcal{A}$. Finally, suppose that for each $A_m \in \mathcal{A}_m$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}_m & \xrightarrow{V_{mn}} & \mathcal{H}_n \\ {\scriptstyle A_m}\downarrow & & \downarrow{\scriptstyle \phi_{mn}(A_m)} \\ \mathcal{H}_m & \xrightarrow[V_{mn}]{} & \mathcal{H}_n \end{array} .$$

Then the following are true:

(i) There exists a unique (faithful) representation

$$\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$$

whose range consists of decomposable operators and such that for each $A_m \in \mathcal{A}_m$

$$\pi(\phi_m(A_m)) = V_m A_m . \qquad (*)$$

(ii) If

$$\mathcal{A}(x) = \varinjlim(\mathcal{A}_n(x), \phi_{mn}(x)) \; ,$$

then there exists an almost everywhere unique field of faithful representations

$$\pi(x) : \mathcal{A}(x) \to \mathcal{B}(\mathcal{H}(x))$$

such that for each $A_m(x) \in \mathcal{A}_m(x)$

$$\pi(x)(\phi_m(x)A_m(x))V_m(x) = V_m(x)A_m(x) \quad \text{a.e.} \qquad (**)$$

(iii) The field of C*-algebras

$$x \mapsto \pi(x)\mathcal{A}(x) \subseteq \mathcal{B}(\mathcal{H}(x))$$

is measurable.

(iv) For each $A_m \in \mathcal{A}_m$

$$\pi(\phi_m(A)) = \int_X^{\oplus} \pi(x)(\phi_m(x)A_m(x))d\mu$$

and $\pi \circ \phi_m$ is WOT continuous.

We conclude that there is a natural inclusion

$$\mathcal{A} \cong \pi(\mathcal{A}) \subseteq \int_X^{\oplus} \pi(x)\mathcal{A}(x)d\mu \; .$$

<u>Proof</u>: (i) The direct limit $\mathcal{A}$ is equal to $\overline{\bigcup_n \phi_n(\mathcal{A}_n)}$. For each A_m in $\mathcal{A}_m$ we obtain a unique (decomposable) operator $\pi(\phi_m(A_m))$ satisfying (*) (Lemma 5.5). Moreover π is a *-homomorphism on $\phi_m(\mathcal{A}_m)$. Therefore π extends uniquely to a representation of $\mathcal{A}$. The map π is faithful because restricted to $\phi_m(\mathcal{A}_m)$ it is faithful.

(ii) For almost all x we have a well-defined coherent system $(\mathcal{A}_n(x), \phi_{mn}(x))$. Therefore the same proof as for (i) applies.

(iii) For each n

$$\mathcal{A}_n = \int_X^{\oplus} \mathcal{A}_n(x)d\mu$$

is a measurable field of C*-algebras and therefore has a fundamental sequence $A_n^{(k)}$,

$k \in \mathbb{N}$. This makes the double sequence $\{\pi\phi_n A_n^{(k)} : k,n \in \mathbb{N}\}$ a fundamental sequence for the field of C*-algebras

$$x \mapsto \pi(x)\mathcal{A}(x) \ .$$

(iv) Let $A_m = \int_X^{\oplus} A_m(x)d\mu \in \mathcal{A}_m$.

Then $\pi(\phi_m(A_m))$ is decomposable and satisfies (*). Therefore $\pi(\phi_m(A_m))(x)$ satisfies (**) a.e. But $\pi(x)(\phi_m(x)A_m(x))$ satisfies (**) a.e. The result follows.

Note: Implicit in (iv) is the fact that for each m , $x \mapsto \pi(x)\circ\phi_m(x)$ defines a measurable field of WOT continuous *-homomorphisms.

Lemma 5.7: Let $\mathcal{A}$ be an APN-algebra with canonical measurable field of systems

$$(\mathcal{A}_n(x),\phi_{mn}(x)) , \ \mathcal{H}_n = \int_X^{\oplus} \mathbb{C}^{d_n(x)}d\mu \ .$$

Then there exist decomposable partial isometries

$$V_{mn} : \mathcal{H}_m \to \mathcal{H}_n$$

which satisfy the following two conditions:

(i) $\operatorname{rg} V_{mn} = \operatorname{rg} \phi_{mn}(1_m)$, 1_m = identity of $\mathcal{A}_m$

and (ii) the conditions in Proposition 5.6.

Proof: We first consider a special case. Let $m_1,\ldots,m_k,r_1,\ldots,r_k,n$ be fixed positive integers, and suppose

$$\phi : \bigoplus_{i=1}^{k} \mathcal{M}_{m_i}(\mathcal{L}^\infty(X,\mu)) \to \mathcal{M}_n(\mathcal{L}^\infty(X,\mu))$$

is defined by

$$\phi : A = (A_1,\ldots,A_k) \mapsto \sum_{i=1}^{k}{}^{\oplus} A_i^{(r_i)} = \begin{pmatrix} A_1^{(r_1)} & & & & & \\ & \ddots & & & & \\ & & A_k^{(r_k)} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

Define the partial isometry

$$V : \bigoplus_{i=1}^{k} \mathcal{L}^2(X,\mu)\otimes\mathbb{C}^{m_i} \to \mathcal{L}^2(X,\mu)\otimes\mathbb{C}^{n}$$

by

$$V : f = (f_1,\ldots,f_k) \mapsto \Big(\underbrace{\frac{f_1}{\sqrt{r_1}},\ldots,\frac{f_1}{\sqrt{r_1}}}_{r_1 \text{ times}},\ldots,\underbrace{\frac{f_k}{\sqrt{r_k}},\ldots,\frac{f_k}{\sqrt{r_k}}}_{r_k \text{ times}},0,\ldots,0\Big).$$

Then

$$\phi(A)Vf = VAf$$

for all A and f. Because of the structure theorems 3.3 and 3.7 the general case follows.

Corollary 5.8: Every APN-algebra has a canonical faithful representation as a subalgebra of a direct integral of a measurable field of represented AF-algebras such that all of the conditions of Proposition 5.6 are satisfied. Also, by standard direct integral theory, we may assume that there is a fixed separable Hilbert space, $\mathcal{H}_0$ say, such that we have the inclusions

$$\int_X^{\oplus} \mathfrak{A}(x)d\mu \subseteq \int_X^{\oplus} \overline{\mathfrak{A}}(x)d\mu \subseteq \int_X^{\oplus} \mathcal{B}(\mathcal{H}_0)d\mu\,,\ \ \mathcal{H}(x) \subsetneq \mathcal{H}_0 \ \text{a.e.}$$

Proof: This follows immediately from Lemma 5.7.

As the examples on page 136 illustrate, the APN-algebra $\mathcal{A}$ is in general properly contained in the canonical direct integral algebra associated to the systems. In fact, let $\mathcal{B}$ be a single represented AF-algebra,

$$\mathcal{B} = \varinjlim(\mathcal{B}_n, \psi_{mn}) .$$

Let (X,μ) be standard and define

$$\mathcal{A}_n = \int_X^{\oplus} \mathcal{B}_n d\mu$$

and

$$\phi_{mn} : \mathcal{A}_m \to \mathcal{A}_n$$

by

$$\phi_{mn} = \int_X^{\oplus} \psi_{mn} d\mu .$$

Then the APN-algebra obtained is isomorphic to

$$\mathcal{L}^{\infty}(X,\mu) \otimes \mathcal{B}$$

while the direct integral algebra is

$$\int_X^{\oplus} \mathcal{B} d\mu$$

which by definition can be identified with $\mathcal{L}^{\infty}(X,\mathcal{B})$. Therefore

$$\varinjlim \mathcal{A}_n \subsetneq \int_X^{\oplus} \mathcal{B} d\mu \Leftrightarrow \mathcal{B} \text{ is not finite dimensional.}$$

More generally, if $\mathcal{A}$ is an APN-algebra, then $\mathcal{A}$ consists of (equivalence classes of) measurable functions $A : X \to \bigcup_x \mathcal{A}(x)$ such that

(i) $x \mapsto \|A(x)\|$ is essentially bounded

(ii) $A(x) \in \mathcal{A}(x)$ a.e. , and

(iii) $\int_X^{\oplus} A(x) d\mu$ satisfies whatever conditions are imposed by the generating nest of subalgebras, that is $A = A_n$, $\lim_n A_n \in \mathcal{A}_n$. On the other hand, operators in the direct integral algebra need only satisfy the first two conditions.

Now that we have defined the direct integral algebras, our questions of factorization can be asked. One of the features of these algebras is a structure which is less stringent than that of the APN-algebras. So, there is an APN-algebra $\mathcal{A}$ isomorphic to $\ell^\infty(\mathbb{N})\otimes\mathcal{K}$ whose associated direct integral algebra is therefore $\ell^\infty(\mathbb{N},\mathcal{K})$. Contrary to our original expectations, we now suspect that the algebra $\ell^\infty(\mathbb{N},\mathcal{K})$ is spatious enough to yield interesting operators but restricted enough to make the behaviour of the sets $\mathcal{Q}_k(\ell^\infty(\mathbb{N},\mathcal{K}))$ non-trivial. (See the discussion preceeding Proposition 5.14.) However, before pursuing these matters, we attend to some unfinished work of the last chapter. There we posed the following question: Is there a theorem for tensor products of APN-algebras which is analogous to Proposition 2.11 and which can be expressed in a way which respects the underlying fibre structure? As it happens we can answer this in the affirmative. Our formulation of the theorem (on APN-algebras) requires the concept of spatial tensor product, along with the canonical representation just established. We therefore take a detour to settle this question. Once this is done we will take up the direct integral algebras again, and in more detail.

Tensor Products of APN-algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be two represented C*-algebras acting on $\mathcal{H}$ and $\mathcal{W}$ (see [KR] 11.1). We can then form the (represented C*-algebra) tensor product, which we denote $\mathcal{A}\otimes\mathcal{B}$.

If $\mathcal{A}$ and $\mathcal{B}$ happen to be von Neumann algebras, the categorically appropriate object is the (von Neumann algebra) tensor product denoted $\mathcal{A}\overline{\otimes}\mathcal{B}$. It is defined to be the weak operator topology closure of $\mathcal{A}\otimes\mathcal{B}$ and so is again a von Neumann algebra.

Suppose now that $\mathcal{A}$ and $\mathcal{B}$ are APN-algebras with systems $(\mathcal{A}_n(x),\phi_{mn}(x))$ and $(\mathcal{B}_n(x),\psi_{mn}(x))$. So for each n

$$\mathcal{A}_n = \int_X^\oplus \mathcal{A}_n(x)d\mu \quad \text{and} \quad \mathcal{B}_n = \int_X^\oplus \mathcal{B}_n(x)d\mu$$

are von Neumann algebras and we may form the tensor product $\mathcal{A}_n \overline{\otimes} \mathcal{B}_n$. By Theorem 11.2.9 [KR] we obtain a coherent sequence of von Neumann algebras

$$(\mathcal{A}_n \overline{\otimes} \mathcal{B}_n , \phi_{mn} \overline{\otimes} \psi_{mn}) .$$

In fact, we obtain another APN-algebra $\mathcal{C}$, defined by

$$\mathcal{C} = \varinjlim(\mathcal{A}_n \overline{\otimes} \mathcal{B}_n , \phi_{mn} \overline{\otimes} \psi_{mn}) .$$

Moreover, letting $\mathcal{C}$ be canonically represented (as in Corollary 5.8) we obtain a (direct integral) C*-isomorphism between $\mathcal{C}$ and the spatial tensor product of $\mathcal{A}$ and $\mathcal{B}$. It is via this isomorphism that we are able to obtain our theorem (Theorem 5.13) on Q_4 for tensor products of APN-algebras. But we have made several claims which require proof, and these we now provide.

<u>Lemma 5.9</u>: Let

$$\mathcal{M} = \int_X^{\oplus} \mathcal{M}(x)d\mu \quad \text{and} \quad \mathcal{N} = \int_X^{\oplus} \mathcal{N}(x)d\mu$$

be direct integrals of von Neumann algebras acting on

$$\int_X^{\oplus} \mathcal{H}(x)d\mu \quad \text{and} \quad \int_X^{\oplus} \mathcal{W}(x)d\mu \quad \text{respectively.}$$

Then

$$\mathcal{M} \overline{\otimes} \mathcal{N} \cong \int_X^{\oplus} (\mathcal{M}(x) \overline{\otimes} \mathcal{N}(x))d\mu \quad \text{on} \quad \int_X^{\oplus} (\mathcal{H}(x) \otimes \mathcal{W}(x))d\mu \ .$$

<u>Example</u>: $\mathcal{M} = \mathcal{M}_2(\mathcal{L}^\infty(X,\mu)) = \mathcal{N}$

$\mathcal{M} \overline{\otimes} \mathcal{N} = \mathcal{M}_4(\mathcal{L}^\infty(X,\mu))$.

In this particular example we can "see" that we get equality by considering elementary tensors. Let

$$S = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}, \ T = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}, \ \phi_{ij} , \psi_{ij} \in \mathcal{L}^\infty(X,\mu) .$$

Then $S \otimes T$ has representation

$$\begin{pmatrix} \phi_{11}T & \phi_{12}T \\ \phi_{21}T & \phi_{22}T \end{pmatrix}.$$

Alternately letting S and T be the identity, we obtain all elements of the form

$$\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \text{ and } \begin{pmatrix} \phi_{11}I & \phi_{12}I \\ \phi_{21}I & \phi_{22}I \end{pmatrix}$$

and hence all elements of the form

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \; T_{ij} \in \mathcal{M}_2(\mathcal{L}^\infty(X,\mu)),$$

and therefore all of $\mathcal{M}_4(\mathcal{L}^\infty(X,\mu))$.

Proof of Lemma 5.9: Let $\{A_k : k \in \mathbb{N}\}$ be a fundamental sequence for $\mathcal{M}$ and $\{\mathcal{B}_\ell : \ell \in \mathbb{N}\}$ be a fundamental sequence for $\mathcal{N}$. Then it is straightforward to verify that

$$\mathcal{H} \otimes \mathcal{W} \equiv \int_X^\oplus \mathcal{H}(x) \otimes \mathcal{W}(x) d\mu$$

and that $A_k \otimes B_\ell$ is a decomposable operator on $\mathcal{H} \otimes \mathcal{W}$ with $(A_k \otimes B_\ell)(x) = A_k(x) \otimes B_\ell(x)$ a.e. Furthermore, by paragraph 3, page 812 of [KR], the set $\{A_k(x) \otimes B_\ell(x) : k, \ell \in \mathbb{N}\}$ generates $\mathcal{M}(x) \overline{\otimes} \mathcal{N}(x)$ a.e. Therefore $x \mapsto \mathcal{M}(x) \overline{\otimes} \mathcal{N}(x)$ defines a measurable field of von Neumann algebras, and since $\overline{\mathcal{M} \otimes \mathcal{N}} = \mathcal{M} \overline{\otimes} \mathcal{N}$ we have that

$$\mathcal{M} \overline{\otimes} \mathcal{N} \subseteq \int_X^\oplus (\mathcal{M}(x) \overline{\otimes} \mathcal{N}(x)) d\mu \;.$$

To obtain equality we call on von Neumann's double commutant theorem. We therefore show that

$$(\mathcal{M} \overline{\otimes} \mathcal{N})' \subseteq \left(\int_X^\oplus \mathcal{M}(x) \overline{\otimes} \mathcal{N}(x) d\mu \right)' .$$

To see this let $C \in (\mathcal{M} \overline{\otimes} \mathcal{N})'$. Then since the diagonal algebra for

$$\mathcal{H} \otimes \mathcal{W} = \int_X^\oplus \mathcal{H}(x) \otimes \mathcal{W}(x) d\mu$$

is contained in $\mathcal{M} \overline{\otimes} \mathcal{N}$, we have that C is decomposable. If

$$T = \int_X^{\oplus} T(x)d\mu \in \int_X^{\oplus} \mathcal{M}(x) \overline{\otimes} \mathcal{N}(x)d\mu$$

then since

$$C(x)(A_k(x) \otimes B_\ell(x)) = (A_k(x) \otimes B_\ell(x))C(x) \quad \text{a.e.}$$

it follows that

$$C(x)T(x) = T(x)C(x) \quad \text{a.e.}$$

from which we obtain

$$CT = TC \ .$$

Therefore

$$C \in \left(\int_X^{\oplus} \mathcal{M}(x) \overline{\otimes} \mathcal{N}(x)d\mu \right)' \ .$$

The commutant theorem now reverses the inclusion in question and equality of the algebras is established.

Corollary 5.10: If $(\mathcal{A}_n, \phi_{mn})$, $(\mathcal{B}_n, \psi_{mn})$ are coherent systems corresponding to the APN-algebras $\mathcal{A}$ and $\mathcal{B}$, then the new system $(\mathcal{A}_n \overline{\otimes} \mathcal{B}_n, \phi_{mn} \overline{\otimes} \psi_{mn})$ also corresponds to an APN-algebra.

Proof: By hypothesis, we have direct integrals of von Neumann algebras

$$\mathcal{A}_m = \int_X^{\oplus} \mathcal{A}_m(x)d\mu \quad , \quad \mathcal{B}_m = \int_X^{\oplus} \mathcal{B}_m(x)d\mu$$

and

$$\phi_{mn} = \int_X^{\oplus} \phi_{mn}(x)d\mu \quad , \quad \psi_{mn} = \int_X^{\oplus} \psi_{mn}(x)d\mu \ .$$

From the preceeding lemma, for each $m \in \mathbb{N}$

$$\mathcal{A}_m \overline{\otimes} \mathcal{B}_m = \int_X^{\oplus} \mathcal{A}_m(x) \overline{\otimes} \mathcal{B}_m(x)d\mu$$

and by theorems 11.2.9 and 11.2.10 [KR], for each $m \leq n$ there is a unique WOT continuous (*-monomorphism)

$$\phi_{mn} \overline{\otimes} \psi_{mn} : \mathcal{A}_m \overline{\otimes} \mathcal{B}_m \rightarrow \mathcal{A}_n \overline{\otimes} \mathcal{B}_n$$

which respects elementary tensors. But the measurable field of WOT continuous *-monomorphisms

$$x \mapsto \phi_{mn} \overline{\otimes} \psi_{mn}(x)$$

determines such a map τ_{mn}

$$\tau_{mn} = \int_X^{\oplus} \phi_{mn}(x) \overline{\otimes} \psi_{mn}(x) d\mu \ .$$

Therefore by uniqueness

$$\tau_{mn} = \phi_{mn} \overline{\otimes} \psi_{mn}$$

and the corollary follows.

Proposition 5.11. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}_n(x)$, $\mathcal{B}_n(x)$, $\phi_{mn}(x)$, $\psi_{mn}(x)$ be as in the above corollary. Then the spatial tensor product of the canonical representations $\pi(\mathcal{A})$, $\rho(\mathcal{B})$ is C*-isomorphic to the APN-algebra

$$\mathcal{C} = \varinjlim \left(\int_X^{\oplus} \mathcal{A}_n(x) \overline{\otimes} \mathcal{B}_n(x) d\mu, \ \phi_{mn} \overline{\otimes} \psi_{mn} \right) .$$

Note: By canonically representing $\mathcal{C}$ we could make sense of the correspondence as a direct integral isomorphism, but don't require this here.

Proof: Let $\tau_{mn} = \phi_{mn} \overline{\otimes} \psi_{mn}$.

By Theorems 11.2.9 and 11.2.10 of [KR], there exists a unique weak operator topology continuous *-monomorphism

$$(\pi \circ \phi_m) \overline{\otimes} (\rho \circ \psi_m) : \mathcal{A}_m \overline{\otimes} \mathcal{B}_m \rightarrow \overline{\pi(\mathcal{A})} \overline{\otimes} \overline{\rho(\mathcal{B})}$$

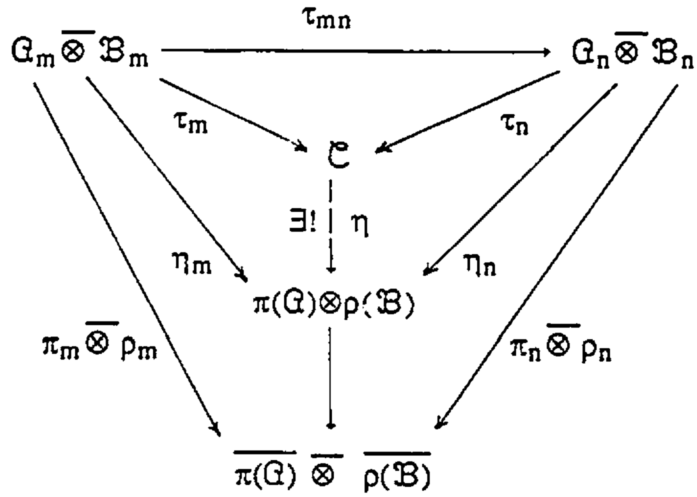
where $\overline{\pi(\mathcal{A})}$, $\overline{\rho(\mathcal{B})}$ are the closures of $\pi(\mathcal{A})$, $\rho(\mathcal{B})$ in the weak operator topology. We simplify the notation by setting

$$\pi_m = \pi \circ \phi_m \ \text{and} \ \rho_m = \rho \circ \psi_m \ , m \in \mathbb{N} \ .$$

Then it is easy to check that for $m \leq n$

$$(\pi_n \overline{\otimes} \rho_n) \circ (\phi_{mn} \overline{\otimes} \psi_{mn}) = \pi_m \overline{\otimes} \rho_m$$

and that the range of $\pi_m \overline{\otimes} \rho_m$ is contained in $\pi(\mathcal{A}) \otimes \rho(\mathcal{B})$ (spatial tensor product). We therefore obtain the following commutative diagram:



where η_m is defined to be $\pi_m \overline{\otimes} \rho_m : \mathcal{A}_m \overline{\otimes} \mathcal{B}_m \to \pi(\mathcal{A}) \otimes \rho(\mathcal{B})$. By the universal property for direct limits of C*-algebras there exists a unique *-homomorphism $\eta : \mathcal{C} \to \pi(\mathcal{A}) \otimes \rho(\mathcal{B})$ which completes the commutative diagram. The map η is onto because $\pi(\mathcal{A}) \otimes \rho(\mathcal{B})$ is the norm completion of the algebraic tensor product and every elementary tensor $\pi_m(A_m) \otimes \rho_m(B_m) \in \pi_m(\mathcal{A}_m) \otimes \rho_m(\mathcal{B}_m)$ belongs to the range of η. To show that η is one-one, suppose that $T = \lim_m T_m$, $T_m \in \mathcal{A}_m \overline{\otimes} \mathcal{B}_m$ and $\eta(T) = 0$. Then $\lim_m \eta_m(T_m) = 0$. But each η_m is a one-one *-homomorphism. Therefore $\lim_m \|T_m\| = 0$ and $T = 0$.

<u>Corollary 5.12</u>: Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be the APN-algebras in Proposition 5.11, and $\mathcal{A} \otimes \mathcal{B}$ the unique C*-tensor product of $\mathcal{A}$ and $\mathcal{B}$. Then

$$\overline{\mathcal{Q}}_4(\mathcal{A} \otimes \mathcal{B}) = \mathcal{A} \otimes \mathcal{B} \Leftrightarrow \overline{\mathcal{Q}}_4(\mathcal{C}) = \mathcal{C}.$$

Proof: Since $\mathfrak{A}$ and $\mathfrak{B}$ are nuclear we have

$$\mathfrak{A}\otimes\mathfrak{B} \equiv \pi(\mathfrak{A}) \otimes \rho(\mathfrak{B}) \equiv \mathfrak{C} .$$

This corollary generalizes Lemma 2.12 and makes possible the following result which partially extends Proposition 2.11.

Theorem 5.13: Let $\mathfrak{A}$ and $\mathfrak{B}$ be APN-algebras and suppose there exist measurable set $X_{\mathfrak{A}}, X_{\mathfrak{B}} \subseteq X$ such that $X = X_{\mathfrak{A}} \cup X_{\mathfrak{B}}$

$$\overline{\mathfrak{Q}}_4(\mathfrak{A}|_{X_{\mathfrak{A}}}) = \mathfrak{A}|_{X_{\mathfrak{A}}} \quad \text{and} \quad \overline{\mathfrak{Q}}_4((\mathfrak{B}|_{X_{\mathfrak{B}}})) = \mathfrak{B}|_{X_{\mathfrak{B}}} .$$

Then

$$\overline{\mathfrak{Q}}_4(\mathfrak{A}\otimes\mathfrak{B}) = \mathfrak{A}\otimes\mathfrak{B} .$$

Remark: It is not necessary that $\mu(X_{\mathfrak{A}} \cap X_{\mathfrak{B}}) = 0$. For an easy example, let $X = \{1,2,3\}$ with μ discrete measure. Let $\mathfrak{A}$ and $\mathfrak{B}$ have the following fields of Bratteli diagrams:

		n=1		n=2		n=3		
	x=1	1	—	2	—	3	—···	$\mathfrak{A}(1) = \mathfrak{K}$
$\mathfrak{A}$:	x=2	1	—	2	—	3	—···	$\mathfrak{A}(2) = \mathfrak{K}$
	x=3	1	—	1	—	1	—···	$\mathfrak{A}(1) = \mathfrak{M}_1$

		n=1		n=2		n=3		
	x=1	1	—	1	—	1	—···	$\mathfrak{B}(1) = \mathfrak{M}_1$
$\mathfrak{B}$:	x=2	1	—	2	—	3	—···	$\mathfrak{B}(2) = \mathfrak{K}$
	x=3	1	—	2	—	3	—···	$\mathfrak{B}(3) = \mathfrak{K}$.

Then

$$\overline{\mathfrak{Q}}_4(\mathfrak{A}((\{1,2\})) = \mathfrak{A}(\{1,2\})$$

and

$$\overline{Q}_4(\mathcal{B}((\{1,2\})) = \mathcal{B}(\{1,2\})$$

So $X_{\mathcal{A}} = \{1,2\}$, $X_{\mathcal{B}} = \{2,3\}$ and $\overline{Q}_4(\mathcal{A}\otimes\mathcal{B}) = \mathcal{A}\otimes\mathcal{B}$.

Proof: Without loss of generality $\mathcal{A}(x) \neq 0$ a.e. and $\mathcal{B}(x) \neq 0$ a.e.

Suppose $\overline{Q}_4(\mathcal{A}) = \mathcal{A}$ and consider $\mathcal{A}\otimes\mathcal{B}$.

Let

$$Z_{n_0} = \{x : \mathcal{B}_{n_0}(x) = 0\} .$$

If there exists n_0 such that $\mu(Z_{n_0}) = 0$, then by considering the new system obtained by starting at the n_0^{th} algebra, we may assume $\mathcal{B}_1(x) \neq 0$ a.e. This situation we call Case I. Case II is where we obtain a properly decreasing (up to measure zero) chain of measurable subsets

$$Z_1 = Z_{n_1} \supsetneq Z_{n_2} \supsetneq Z_{n_3} \supsetneq \cdots$$

such that $\mu(Z_{n_j} \setminus Z_{n_{j+1}}) > 0$. Note that throughout the proof we rely on Theorem 4.12.

Case I(a): The field of systems $(\mathcal{A}_n(x), \phi_{mn}(x))$ is attracted to infinity. Then the field $(\mathcal{A}_m(x) \overline{\otimes} \mathcal{B}_n(x), \phi_{mn}(x) \overline{\otimes} \psi_{mn}(x))$ is clearly attracted to infinity as well.

Case I(b): The field $(\mathcal{A}_n(x), \phi_{mn}(x))$ is jointly attracted to zero and of infinite type. Let $\{X_{n_j}\}_{j=1}^{\infty}$ be the corresponding partition for X so that

(i) $\mathcal{A}(X_{n_j})$ is attracted to infinity, and

(ii) $\mathcal{A}_{n_j-k}(x) = 0$ for $0 < k \leq n_j - 1$ a.e. on X_{n_j} .

Therefore by I(a)

(i) $\mathcal{A}(X_{n_j}) \otimes \mathcal{B}(X_{n_j}) = \mathcal{C}(X_{n_j})$ is attracted to infinity, and

(ii) $\mathcal{C}_{n_j-k}(x) = 0$ for $0 < k \leq n_j - 1$ a.e. on X_j .

Hence $\mathcal{C}_n(x) = (\mathcal{A}\otimes\mathcal{B})_n(x)$ is jointly attracted to zero and of infinite type.

Case I(c): The measure space decomposes into a disjoint union

$$X = X_\infty \dot{\cup} X_0$$

where $\mathcal{A}(X_\infty)$ is attracted to infinity and $\mathcal{A}(X_0)$ is jointly attracted to zero and of infinite type. But now

$$\mathcal{A}\otimes\mathcal{B} \equiv [\mathcal{A}(X_\infty) \otimes \mathcal{B}(X_\infty)] \oplus [\mathcal{A}(X_0) \otimes \mathcal{B}(X_0)]$$

to which cases I(a) and I(b) apply.

Case II: Here X is partitioned into $(X \setminus Z_{n_1}) \dot{\cup} Z_{n_1}$, where $\mathcal{B}(X \setminus Z_{n_1})$ is an algebra of the kind considered in case I. Therefore, suppose $X = Z_1 = Z_{n_1}$.

Case II(a): Suppose that the field $(\mathcal{A}_n(x), \phi_{mn}(x))$ is attracted to infinity and without loss of generality $\mathcal{A}_1(x) \neq 0$ a.e. Then $\mathcal{A}(Z_{n_j}) \otimes \mathcal{B}(Z_{n_j})$ is attracted to zero and of infinite type.

Case II(b): Suppose that $(\mathcal{A}_n(x), \phi_{mn}(x))$ is jointly attracted to zero and of infinite type. Let $\{X_{m_i}\}_{i=1}^{\infty}$ be the corresponding partition for X. Now, for each j

$$Z_{n_j} = (X_{m_1} \cap Z_{n_j}) \cup (X_{m_2} \cap Z_{n_j}) \cup \cdots .$$

But since m_i is a strictly increasing sequence, this disjoint union is actually finite (since $\mu(X_{m_i} \cap Z_{n_j}) = 0$ for $m_i > n_j$) and of length $k(j)$. For $1 \leq i \leq k(j)$ and $x \in X_{m_i} \cap Z_{n_j}$, $(\mathcal{A}\otimes\mathcal{B})_n(x) \neq 0 \Leftrightarrow n \geq \max\{m_i, n_j\} := \langle m_i, n_j \rangle$. Let $W_\ell = \cup(X_{m_i} \cap Z_{n_j} : \langle m_i, n_j \rangle = \ell\}$. Since both sequences m_i and n_j are strictly increasing W_ℓ is a finite union. Therefore $(\mathcal{A}\otimes\mathcal{B})(W_\ell)$ is attracted to infinity and since $X = \dot{\bigcup}_\ell W_\ell$ it follows that $\overline{Q_4}(\mathcal{A}\otimes\mathcal{B}) = \mathcal{A}\otimes\mathcal{B}$.

Case II(c): Here X is a disjoint union of measurable sets X_∞, X_0, as in case I(c). But now II(a) and II(b) apply to each part and the result follows.

Concerning the sufficiency of the conditions in Theorem 5.13, we were able to find counter-examples, only when we dropped the condition on the field of systems that for each n,

$$x \mapsto \dim \mathcal{A}_n(x)$$

is essentially bounded (Condition (iii), following Theorem 3.7). We therefore have the following:

<u>Conjecture</u>: The conditions in Theorem 5.13 are necessary and sufficient to yield

$$\overline{\mathcal{Q}}_4(\mathcal{A}\otimes\mathcal{B}) = \mathcal{A}\otimes\mathcal{B} \ .$$

This finishes our discussion of tensor products and we now return to direct integrals of measurable fields of AF-algebras. We begin by considering the algebra $\mathcal{A} = \int_X^\oplus \mathcal{K}\, d\mu$, where as usual $\mathcal{K}$ is the C*-algebra of compact operators on $\ell^2(\mathbb{N})$. We will show how this algebra, and so others of this kind, seem to be in a fundamental way out of reach of the techniques developed so far in our present work. The reason for this difficulty is the large size of the algebra (which is "very much" non-finite dimensional). By the same token we obtain the motivation for the definition of the "compact direct integral", which may yield to our techniques and would appear to be the largest algebra which would do so (at least amongst those obtained from the various constructions so far encountered in this thesis, that is, within the direct integral ambient space.)

To be more precise, let $T \in \mathcal{A}$ with measurable map $x \mapsto T(x) \in \mathcal{K}$ a.e., $T = \int_X^\oplus T(x)\, d\mu$. For certain operators and questions of approximation one strategy might be to obtain an "upper-triangular" form for T, as in the case of n-normal operators (see Theorem 7.20 in [R&R]). Supposing, for example, that T is even self-adjoint, then we can mimick the proof (in [Con]) of the spectral theorem for compact operators. Being just a little careful about the measure theory, we obtain a sequence $(\lambda_n)_{n=1}^\infty$ of measurable real-valued functions and a sequence $\mathcal{E}_n(x)$ of measurable fields of finite dimensional Hilbert spaces satisfying the following conditions almost everywhere:

(i) $\lambda_n(x)$ is an eigenvalue for $T(x)$

(ii) $|\lambda_1(x)| \geq |\lambda_2(x)| \geq \ldots$

(iii) $\mathcal{E}_n(x) = \ker(T(x)-\lambda_n(x))$; $|\lambda_{n+1}(x)| = \|T \mid (\mathcal{E}_1(x)\oplus\ldots\oplus\mathcal{E}_n(x))^{\perp}\|$

(iv) $\lambda_n(x) \to 0$.

In the classical setting, X = a singleton = $\{x_0\}$. So with

$$P_n = \text{the orthogonal projection onto } \mathcal{E}_n(x_0)$$

$$\|T - \sum_{j=1}^{n} \lambda_j P_j\| = |\lambda_{n+1}(x_0)| \underset{n}{\to} 0$$

from which we conclude that

$$T = \sum_{j=1}^{\infty} \lambda_j P_j \ .$$

In our setting, however, there is an obstruction to this formula. It certainly is true that with

$$P_n(x) = \text{the orthogonal projection onto } \mathcal{E}_n(x)$$

$$T(x) = \sum_{j=1}^{\infty} \lambda_j(x) P_j(x) \quad \text{a.e.} \ .$$

But with

$$P_j = \int_X^{\oplus} P_j(x)d\mu \quad \text{and} \quad \mathcal{E}_n = \int_X^{\oplus} \mathcal{E}_n(x)d\mu$$

$$\|T - \sum_{j=1}^{n} \lambda_j P_j\| = \text{ess sup}\,|\lambda_{n+1}|$$

and it may happen that $\|\lambda_{n+1}\|_\infty$ does not converge to zero.

For an example let $X = (0,1)$, μ = Lebesgue measure, and

$$T(x) = \begin{pmatrix} x & & & & 0 \\ & x^2 & & & \\ & & x^3 & & \\ & & & x^4 & \\ & 0 & & & \ddots \end{pmatrix} .$$

Then $T(x)$ is compact a.e. and $x \mapsto T(x)$ is measurable.

What happens for the general self-adjoint operator in $\mathcal{Q}$ is that for each $\varepsilon > 0$ there exists $n(\varepsilon)$ and a measurable subset $X_n \subseteq X$ such that X is a disjoint union $\dot{\cup} X_n = X$, and restricted to each X_n

$$\|(T - \sum_{j=1}^{n} \lambda_j P_j)\| < \frac{\varepsilon}{2} .$$

Therefore on each X_n we can use the approximation techniques of [KLMR] to obtain the sequence of operators

$$S_n \in \mathcal{Q}_4 (\int_{X_n}^{\oplus} \mathcal{K} \, d\mu)$$

$$S_n = Q_{n,1} Q_{n,2} Q_{n,3} Q_{n,4}$$

satisfying

$$\|(S_n - \sum_{j=1}^{n} \lambda_j P_j)\| < \frac{\varepsilon}{2} .$$

But now the problem becomes evident. The hoped-for approximant would be S, defined by

$$S(x) = S_n(x) \quad \text{for} \quad x \in X_n$$

where

$$S_n(x) = Q_{n,1}(x) \, Q_{n,2}(x) \, Q_{n,3}(x) \, Q_{n,4}(x) .$$

We might then define

$$Q_j(x) = Q_{n,j}(x) \quad \text{for} \quad x \in X_n , \ j = 1,2,3,4 .$$

This produces a measurable field of operators, to be sure, but we have no guarantee that the field is essentially bounded, with

$$\sup_n(\operatorname{ess\,sup}_{X_n} \|Q_{n,j}(x)\|) < \infty,\ j = 1,2,3,4 .$$

This is really a question of continuity. If $T(x)$ is a bounded field of operators such that $T(x) \in \mathcal{Q}_4(\mathcal{A}(x))$ a.e. then can we obtain the factors $Q_i(x) \in \mathcal{Q}_1(\mathcal{A}(x))$ in such a way that we control the norms $\|Q_i(x)\|$ as functions of $\|T(x)\|$? (See also Chapter 1 where we discuss $\mathcal{Q}_4$ for the algebra of n-normal operators. And this seems to be the generic obstruction to our techniques developed so far. In regards to the algebra $\mathcal{A} = \int_X^{\oplus} \mathcal{K}\, d\mu$, two possibilities come to mind. The first is that for suitable (X,μ) (e.g. $X = \mathbb{N}$, μ discrete)

$$\overline{\mathcal{Q}_1}(\mathcal{A}) \subsetneq \overline{\mathcal{Q}_2}(\mathcal{A}) \subsetneq \dots \subsetneq \overline{\mathcal{Q}_\infty}(\mathcal{A}) \subsetneq \mathcal{A} .$$

The second is that $\overline{\mathcal{Q}_4}(\mathcal{A})$ is indeed equal to $\mathcal{A}$, and that for the general direct integral of separable C*-algebras $\mathcal{A} = \int_X^{\oplus} \mathcal{A}(x)\, d\mu$,

$$\overline{\mathcal{Q}_k}(\mathcal{A}) = \mathcal{A} \Leftrightarrow \overline{\mathcal{Q}_k}(\mathcal{A}(x)) = \mathcal{A}(x) \quad \text{a.e. , } k \in \mathbb{N} .$$

But if this second relation is true we conjecture that the proof would require sophisticated selection theorems beyond the scope of this paper.

Note that $\int_X^{\oplus} \mathcal{K}\, d\mu$ is a subalgebra of $\int_X^{\oplus} \overline{\mathcal{K}}\, d\mu = \int_X^{\oplus} \mathcal{B}(\mathcal{H})d\mu = \mathcal{B}$; and in forthcoming work we have characterized $\overline{\mathcal{Q}_\infty}(\mathcal{B}) = \overline{\mathcal{Q}_{17}}(\mathcal{B})$. Dependent on our generalization of Wu's theorem [Wu], this extends a result of [KLMR] (Theorem 3) and shows that in this more spatious environment $\mathcal{A} \subsetneq \overline{\mathcal{Q}_{17}}(\mathcal{B})$. Also, we at least have the following partial result which seems to be worth including.

<u>Proposition 5.14</u>: Let $x \mapsto \mathcal{A}(x)$ be a measurable field of C*-algebras and $\mathcal{A} = \int_X^{\oplus} \mathcal{A}(x)\, d\mu$. Then

$$\overline{\mathcal{Q}_4}(\mathcal{Q}) = \mathcal{Q} \Rightarrow \overline{\mathcal{Q}_4}(\mathcal{Q}(x)) = \mathcal{Q}(x) \quad \text{a.e.} \ .$$

Proof: By definition, there is a fundamental sequence of measurable fields of operators $(A_n)_{n=1}^{\infty}$ such that

$$C^*\{A_n(x) : n \in \mathbb{N}\} = \mathcal{Q}(x) \quad \text{a.e.} \ .$$

By considering polynomials of several variables with coefficients of the form $r + is$, r, s rational, we find that we may assume $\{A_n(x) : n \in \mathbb{N}\}$ is actually dense a.e. Now, since $\overline{\mathcal{Q}_4}(\mathcal{Q}) = \mathcal{Q}$, it follows that for each A_n and for each $\varepsilon(n,m) = \frac{1}{2^{n+m}}$ there exists $S(n,m) \in \mathcal{Q}_4(\mathcal{Q})$ such that $\|S(n,m) - A_n\| < \frac{1}{2^{n+m}}$. Hence $\|S(n,m)(x) - A_n(x)\| < \frac{1}{2^{n+m}}$ and $S(n,m)(x) \in \mathcal{Q}_4(\mathcal{Q}(x))$ a.e. Let

$$Y(n,m) = \{x : \|S(n,m) - A_n(x)\| < \frac{1}{2^{n+m}} \text{ and } S(n,m)(x) \in \mathcal{Q}_4(\mathcal{Q}(x))\}$$

and

$$Y = \bigcap_{n,m} Y(n,m) \ .$$

Now let $N = X \setminus Y$, $\mu(Y) = \mu(X)$, $\mu(N) = 0$. It now follows that for each $x \in Y$, $\overline{\mathcal{Q}_4}(\mathcal{Q}(x)) = \mathcal{Q}(x)$. For let $x_0 \in Y$, $T_0 \in \mathcal{Q}(x_0)$ and let $\varepsilon > 0$. Choose n so that $\|A_n(x_0) - T_0\| < \frac{\varepsilon}{2}$. Go to the direct integral algebra and consider $A_n = \int_X^{\oplus} A_n(x)\, d\mu$.

There exists $S(n,m) \in \mathcal{Q}_4(\mathcal{Q})$ satisfying $\|S(n,m)(x) - A_n(x)\| < \frac{\varepsilon}{2}$ for all $x \in Y = X \setminus N$. Therefore $\|S(n,m)(x_0) - A_n(x_0)\| < \frac{\varepsilon}{2}$ and we conclude that $\|S(n,m)(x_0) - T_0\| < \varepsilon$. Hence $\overline{\mathcal{Q}_4}(\mathcal{Q}(x_0)) = \mathcal{Q}(x_0)$, as claimed.

Remark: Notice how the separability of the C*-algebras was needed. The W*-algebras $\overline{\mathcal{A}}(x)$ are not in general separable, so that this argument would fail. See however Remark 2 at the end of this chapter.

We have given evidence to suggest that our techniques have definite limitations, with an "upper-bound", as it were, being direct integrals of measurable fields of C*-algebras. Is there a "least upper bound"? Can we extend our results and methods to algebras larger than APN-algebras, and in doing so will we observe new phenomena? Implicit in our Conjecture at the end of this chapter is that the answer to the second question is "Yes". We in fact produce a family of algebras which are between the APN-algebras and the direct integral algebras and which appear to be tractable. In response to the first question, as mentioned earlier, these algebras seem to be optimal with respect to inclusion in the direct integrals.

We now give the motivating result, which we will prove shortly.

Proposition 5.15: For a separable C*-subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, the spatial tensor product $\mathcal{L}^\infty(X,\mu) \otimes \mathcal{A}$ can be identified with (equivalence classes of) bounded measurable functions $F : X \to \mathcal{A}$ for which the essential range, ess rg F, is sequentially compact.

With this proposition in mind, let $x \mapsto \mathcal{A}(x)$ be a measurable field of canonically represented AF-algebras, so that

$$\int_X^\oplus \mathcal{A}(x)\, d\mu \subseteq \int_X^\oplus \mathcal{B}(\mathcal{H}_0)\, d\mu\,,\ \mathcal{H}(x) \subseteq \mathcal{H}_0 \quad \text{a.e.}$$

We then define the compact direct integral to be

$$\oint_X^\oplus \mathcal{A}(x) d\mu = \{F \in \int_X^\oplus \mathcal{A}(x)\, d\mu : \text{ess rg } F \text{ is sequentially compact in } \mathcal{B}(\mathcal{H}_0)\}\,.$$

Our conjecture is that the set $\mathcal{Q}_4$ of such an algebra is dense if and only if $\mathcal{Q}_4(\mathcal{A}(x))$ is dense almost everywhere. Note that if $\mathcal{A}(x) = \mathcal{A}_0$, a fixed AF-algebra, then the conjecture holds by Proposition 5.15 and Theorem 4.12. So if $\mathcal{A}(x) = \mathcal{K}$ a.e. then

$$\int_X^{\oplus} \mathcal{K}\, d\mu \cong \mathcal{L}^\infty(X,\mu) \otimes \mathcal{A}$$

for which we already know that $\overline{\mathcal{Q}_4}(\mathcal{A}) = \mathcal{A}$. Using the alternate characterization of $\mathcal{A}$, as afforded by Proposition 5.15, we give a new proof of this fact (Corollary 5.20). It is this proof which we hope will extend to the general compact direct integral of AF-algebras. To present our results, we begin by establishing Proposition 5.15.

Lemma 5.16: Let $A_1,\dots,A_n \in \mathcal{B}(\mathcal{H})$, $\phi_1,\dots,\phi_n \in \mathcal{L}^\infty(X,\mu)$ and $\varepsilon > 0$. Suppose there exists $T \in \mathcal{B}(\mathcal{H})$ such that

$$\|\phi_1 \otimes A_1 + \dots + \phi_n \otimes A_n - \int_X^{\oplus} T\, d\mu\| < \varepsilon .$$

Then there exists $\lambda_1,\dots,\lambda_n \in \mathbb{C}$ such that

$$\|\lambda_1 A_1 + \dots + \lambda_n A_n - T\| < \varepsilon .$$

Proof: The conditions holds if and only if

$$\|\phi_1(x)A_1 + \dots + \phi_n(x)A_n - T\| < \varepsilon \text{ a.e.}$$

Let $\mathcal{V}$ be the linear span of $\{A_1,\dots,A_n\}$, and $F : (X,\mu) \to \mathcal{V}$ by $F(x) = \sum_{i=1}^{n} \phi_i(x)A_i$. We then have ess rg $F \subseteq \mathcal{V}$ and for all $B \in$ ess rg F, $\|B-T\| < \varepsilon$. The result follows.

Proof of Proposition 5.15: First, suppose $F \in \mathcal{L}^\infty(X,\mathcal{A}) = \int_X^{\oplus} \mathcal{A}\, d\mu$ with ess rg F sequentially compact in $\mathcal{A}$. Without loss of generality, $\|F(x)\| \le 1$ a.e., hence ess rg F $\subsetneq$ ball $\mathcal{A}$. Let $\varepsilon > 0$ and $\mathcal{O} = \{B_\varepsilon(A_n) : n \in \mathbb{N}\}$ where $\{A_n : n \in \mathbb{N}\}$ is a countable dense subset of ball $\mathcal{A}$. The family $\mathcal{O}$ is a countable open cover for ball $\mathcal{A}$ and hence for ess rg F $\subseteq$ ball $\mathcal{A}$. Therefore, there exists $n_1,\dots,n_k$ such that

$$\text{ess rg } F \subsetneq B_\varepsilon(A_{n_1}) \cup \dots \cup B_\varepsilon(A_{n_k}) \ .$$

Therefore

$$F(x) \in B_\varepsilon(A_{n_1}) \cup \dots \cup B_\varepsilon(A_{n_k}) \text{ a.e. } .$$

But F is measurable, therefore $F^{-1}(B_\varepsilon(A_{n_i}))$ is measurable in X , $i = 1,\dots,k$. Let $Y_i = F^{-1}(B_\varepsilon(A_{n_i}))$ and define F_ε by

$$F_\varepsilon(x) = A_1 \ , \ x \in Y_1$$

$$F_\varepsilon(x) = A_2 \ , \ x \in Y_2 \setminus Y_1 \ .$$

Let χ_i be the characteristic function for $Y_i \setminus \left(\bigcup_{j=1}^{i-1} Y_i \right)$. Then

$$F_\varepsilon = \chi_1 A_1 + \dots + \chi_n A_n \in \mathcal{L}^\infty(X,\mu) \odot \mathcal{A} \ ,$$

the algebraic tensor product, and by construction

$$\|F_\varepsilon - F\| < \varepsilon \ .$$

Therefore the set $\{F \in \int_X^\oplus \mathcal{A}\, d\mu : \text{ess rg } F \text{ is sequentially compact}\}$ is contained in $\mathcal{L}^\infty(X,\mu) \otimes \mathcal{A}$, the spatial tensor product.

To prove the reverse inclusion is a little more complicated. Suppose $F \in \mathcal{L}^\infty(X,\mu) \otimes \mathcal{A}$ and let $\varepsilon > 0$. Then, by definition, there exists $F_\varepsilon = \phi_1 \otimes A_1 + \dots + \phi_n \otimes A_n$ such that $\|F - F_\varepsilon\| < \varepsilon$. If $T \in$ ess rg F then the measurable set $\{x : \|F(x) - T\| < \varepsilon\}$ has positive measure. Therefore, $\{x : \|F_\varepsilon(x) - T\| < 2\varepsilon\}$ also has positive measure. By the previous lemma, there exists $\lambda_1,\dots,\lambda_n \in \mathbb{C}$ such that $\|\lambda_1 A_1 + \dots + \lambda_n A_n - T\| < 2\varepsilon$. But since $\varepsilon > 0$ was arbitrary, it follows that for each $\varepsilon > 0$ there exists a finite dimensional linear subspace $\mathcal{V}_\varepsilon \subsetneq \mathcal{A}$ such that $d(T, \mathcal{V}_\varepsilon) < \varepsilon$ for all $T \in$ ess rg F .

Now, let $\{T_n : n \in \mathbb{N}\}$ be a sequence in the essential range of F, and let $\varepsilon_k = \frac{1}{k}$. From the above discussion, for each $k \in \mathbb{N}$ there is a finite-dimensional linear subspace $\mathcal{V}_k \subseteq \mathcal{Q}$ and a sequence $A_1^k, A_2^k, A_3^k, \ldots$ in $\mathcal{V}_k$ such that

$$\|A_1^k - T_n\| < \frac{1}{k}, \quad n = 1,2,3,\ldots \quad (*)$$

By forming the linear join of successive spaces, we assume that $\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \mathcal{V}_3 \subseteq \ldots$. Now consider the following diagram:

$$\begin{array}{llll l}
A_1^1 & A_2^1 & A_3^1 & \ldots & \in \mathcal{V}_1 \\
A_1^2 & A_2^2 & A_3^2 & \ldots & \in \mathcal{V}_2 \\
A_1^3 & A_2^3 & A_3^3 & \ldots & \in \mathcal{V}_3 \\
 & & & \vdots &
\end{array}$$

where for each k, the column $A_n^k \xrightarrow[k]{} T_n$ uniformly (see (*)).

Since the sequence $A_1^1, A_2^1, A_3^1, \ldots$ is contained in the finite-dimensional vector space $\mathcal{V}_1$, it must have a convergent subsequence $A_{n_1}^1, A_{n_2}^1, A_{n_3}^1, \ldots$. Similarly, the sequence $A_{n_1}^2, A_{n_2}^2, A_{n_3}^2, \ldots$ has a convergent subsequence $A_{n_{j,1}}^2, A_{n_{j,2}}^2, A_{n_{j,3}}^2, \ldots$. Iterating this process, we obtain a sequence of convergent subsequences

$$A_{n_{j_1 \cdots j_r}}^k \xrightarrow[j_r]{} S^k \in \mathcal{V}_k .$$

We claim that the sequence $\{S^k : k \in \mathbb{N}\}$ is convergent. To prove the claim let $\varepsilon > 0$. Then by (*) there exists m such that

$$\frac{1}{m} < \frac{\varepsilon}{3} \text{ and } k, \ell \geq m \Rightarrow \|A_n^k - A_n^\ell\| < \frac{\varepsilon}{3} \text{ for all } n .$$

But

$$\|S^k - S^\ell\| = \|S^k - A_n^k + A_n^k - A_n^\ell + A_n^\ell - S^\ell\|$$

$$\leq \|S^k - A_n^k\| + \|A_n^k - A_n^\ell\| + \|A_n^\ell - S^\ell\|$$

$$\leq \|S^k - A_n^k\| + \frac{\varepsilon}{3} + \|A_n^\ell - S^\ell\| \ .$$

But n can be freely chosen. Choose n such that

$$\|S^k - A_n^k\| < \frac{\varepsilon}{3} \text{ and } \|S^\ell - A_n^\ell\| < \frac{\varepsilon}{3} \ .$$

(This is possible because of how the convergent subsequences have nested indices.) Hence, the sequence $\{S^k : k \in \mathbb{N}\}$ is Cauchy and therefore convergent, with limit S. Taking the diagonal terms it follows that

$$A^k_{n_{j_1 \cdots j_{k-1} k}} \xrightarrow[k]{} S$$

and hence that

$$T_{n_{j_1 \cdots j_{k-1} k}} \xrightarrow[k]{} S \ .$$

So the sequence $\{T_n : n \in \mathbb{N}\}$ has a convergent subsequence and ess rg F is sequentially compact. The claim is proved and the theorem established.

Corollary 5.17: (This is really a corollary to the proof.) Let $\mathcal{Q}$ be any Banach space. Then $\mathcal{S} \subseteq$ ball $\mathcal{Q}$ is sequentially compact if and only if for each $\varepsilon > 0$ there exists a finite dimensional linear subspace $\mathcal{V}_\varepsilon \subseteq \mathcal{Q}$ such that

$$d(T, \text{ball}\,\mathcal{V}_\varepsilon) < \varepsilon \quad \text{for all} \quad T \in \mathcal{S} \ .$$

Corollary 5.18: The algebra $\mathcal{L}^\infty(x) \otimes \mathcal{K}$ can be identified with the (equivalence classes of) bounded measurable functions $F : X \to \mathcal{K}$ such that for each $\varepsilon > 0$ there exists $n_0(\varepsilon)$ so that $n \geq n_0$ implies $\|P_n F(x) P_n - F(x)\| < \varepsilon$ a.e., where P_n is the canonical corner projection.

Proof: If $F : X \to \mathcal{K}$ is measurable and has the given property then by the proposition, $F \in \mathcal{L}^\infty(X) \otimes \mathcal{K}$. On the other hand, if $F \in \mathcal{L}^\infty(X) \otimes \mathcal{K}$ then ess rg F is sequentially compact. Let $\varepsilon > 0$ and $\mathcal{O}_n = \{T \in \mathcal{K} : \|P_nTP_n - T\| < \varepsilon\}$. Then $\bigcup_n \mathcal{O}_n = \mathcal{K}$. Therefore there exists n_0 such that ess rg $F \subsetneq \mathcal{O}_{n_0}$ and the result follows.

Corollary 5.19: As sets, $\mathcal{L}^\infty(x) \otimes \mathcal{M}_n = \mathcal{L}^\infty(x) \overline{\otimes} \mathcal{M}_n$.

Corollary 5.20: Let $\mathcal{K}$ be the algebra of compact operators. Then $\overline{\mathcal{Q}}_4(\mathcal{L}^\infty(X) \otimes \mathcal{K}) = \mathcal{L}^\infty(X) \otimes \mathcal{K}$.

Proof: Let $F \in \mathcal{L}^\infty(X) \otimes \mathcal{K}$ and $\varepsilon > 0$. Choose n such that

$$\|P_nF(x)P_n - F(x)\| < \tfrac{\varepsilon}{3} \quad \text{a.e.}$$

and set

$$G = P_nFP_n \ .$$

By Theorem 7.20 [R&R] there exists a unitary U_n in $P_n(\mathcal{L}^\infty(X) \otimes \mathcal{K})P_n$ such that

$$G' = U_n^* G U_n = \left(\begin{array}{ccc|c} \phi_1 & & * & \\ & \ddots & & 0 \\ 0 & & \phi_n & \\ \hline & 0 & & 0 \end{array}\right) .$$

Perturb the diagonals ϕ_i, $i = 1,\dots,n$, to obtain G" such that

$$\text{(i)} \quad G'' = \left(\begin{array}{c|c} \begin{matrix} \psi_1 & & * \\ & \ddots & \\ 0 & & \psi_n \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right)$$

(ii) G'' is invertible in $P_n(\mathcal{L}^\infty(X) \otimes \mathcal{K})P_n$

and (iii) $\|G' - G''\| < \frac{\varepsilon}{3}$.

Now choose $m \in \mathbb{N}$ and a measurable function ζ satisfying $\frac{\varepsilon}{6} < |\zeta(x)| < \frac{\varepsilon}{3}$ a.e. and $\psi_n(x),\ldots,\psi_n(x)\zeta(x)^m > 0$ a.e. Then let

$$H'' = \left(\begin{array}{c|c|c} \begin{matrix} \psi_1 & & * \\ & \ddots & \\ 0 & & \psi_n \end{matrix} & 0 & 0 \\ \hline 0 & \begin{matrix} \zeta & & \\ & \ddots^{(m)} & \\ & & \zeta \end{matrix} & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

so that by Theorem 1.8

(i) $H'' \in \overline{\mathcal{Q}}_4(P_{n+m}\,(\mathcal{L}^\infty(x) \otimes \mathcal{K})P_{n+m}) \subseteq \overline{\mathcal{Q}}_4(\mathcal{L}^\infty(x) \otimes \mathcal{K})$

and (ii) $\|H'' - G''\| < \frac{\varepsilon}{3}$.

But now $U_nH''U^*_n \in \overline{\mathcal{Q}}_4(\mathcal{L}^\infty(X) \otimes \mathcal{K})$ and

$$\begin{aligned} \|F - U_nH''U^*_n\| &\le \|F - G\| + \|G - U_nH''U^*_n\| \\ &< \tfrac{\varepsilon}{3} + \|U^*_nGU_n - H''\| \\ &\le \tfrac{\varepsilon}{3} + \|G' - H''\| \\ &\le \tfrac{\varepsilon}{3} + \|G' - G''\| + \|G'' - H''\| \\ &\le \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} \\ &= \varepsilon \ . \end{aligned}$$

We conclude that $\overline{\mathcal{Q}}_4(\mathcal{L}^\infty(x) \otimes \mathcal{K}) = \mathcal{L}^\infty(x) \otimes \mathcal{K}$.

We conclude Chapter 5 with the formal statement of our conjecture on compact direct integrals. A special case of this is when the measurable field of AF-algebras comes from a field of coherent systems, or an APN-algebra.

Conjecture: Let (X,μ) be a standard σ-finite measure space and suppose that $x \mapsto (\mathcal{A}(x),\mathcal{H}(x))$ defines a measurable field of represented AF-algebras. Then with

$$\mathcal{A} = \int_X^\oplus \mathcal{A}(x)d\mu$$

$\overline{\mathcal{Q}}_4(\mathcal{A}) = \mathcal{A} \Leftrightarrow \overline{\mathcal{Q}}_4(\mathcal{A}(x)) = \mathcal{A}(x)$ a.e.

Remarks: (1) Suppose that $x \mapsto \mathcal{A}(x)$ arises from a measurable field of coherent systems. Then there is a fundamental sequence $(A_j)_{j=1}^\infty$ such that $A_j \in \int_X^\oplus \mathcal{A}(x)d\mu$, for all j. This is because the sequence can be chosen from the polynormal subalgebras. But now the same proof as for Proposition 5.14 shows that if $\overline{\mathcal{Q}}_4(\mathcal{A}) = \mathcal{A}$ then $\overline{\mathcal{Q}}_4(\mathcal{A}(x)) = \mathcal{A}(x)$ a.e.

(2) Another proof that $\overline{\mathcal{Q}}_4(\mathcal{A}) = \mathcal{A} \Rightarrow \overline{\mathcal{Q}}_4(\mathcal{A}(x)) = \mathcal{A}(x)$ a.e. is based on the theory of lifting:

Let (X,μ) be standard and $\mathcal{H}$ be a separable Hilbert space. Let $L^\infty(X,\mu)$ be the bounded measurable functions on (X,μ) and $\mathcal{L}^\infty(X,\mu)$ be the canonical quotient space. By Theorem 3 (Chapter 4) of [I] there exists a lifting $\rho : \mathcal{L}^\infty(X,\mu) \to L^\infty(X,\mu)$, a *-isomorphism such that $\phi \sim \psi \Rightarrow \rho(\phi) = \rho(\psi)$.

Now recall that in regards to decomposable operators, a map $T : (X,\mu) \to \mathcal{B}(\mathcal{H})$ is a measurable field of operators if the map $x \mapsto \langle T(x)\xi,\eta\rangle$ is a measurable map of $(X,\mu) \to \mathbb{C}$ for all $\xi,\eta \in \mathcal{H}$. In [I], a map $T : (X,\mu) \to \mathcal{B}(\mathcal{H})$ is said to be weakly measurable if for each continuous function $h : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$, $h \circ T : X \to \mathbb{C}$ is measurable. Since for each $\xi,\eta \in \mathcal{H}$ the map $\mathcal{B}(\mathcal{H}) \to \mathbb{C}$ defined by $S \mapsto \langle S\xi,\eta\rangle$

is continuous, and for a measurable field of operators, $x \mapsto \|T(x)\|$ is measurable, it follows that these two notions coincide here. Furthermore, in the norm topology $\mathcal{B}(\mathcal{H})$ is a completely regular metric space. Therefore, with

$$\mathcal{C} = \{ \int_X^{\oplus} T(x)d\mu \mid \text{ess rg } T \text{ is compact}\}$$

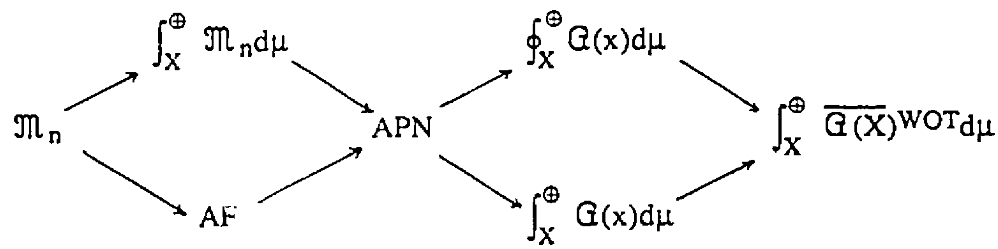
and

$$L_{\mathcal{C}}^{\infty}(X, \mathcal{B}(\mathcal{H})) = \{T : (X,\mu) \to \mathcal{B}(\mathcal{H}) \mid T \text{ is measurable and } \overline{\text{rg } T} \text{ is compact}\}$$

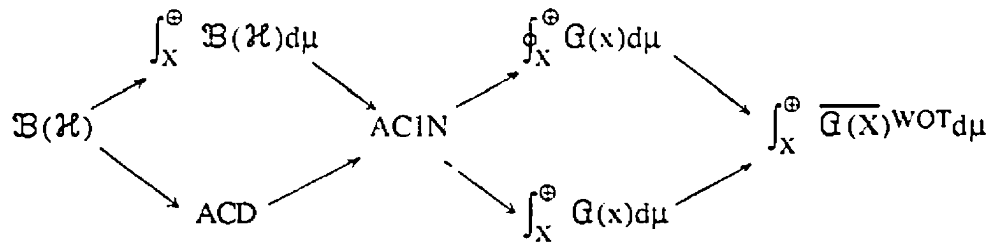
we obtain, by Theorem 7 of [I], a unique lifting $\rho' : \mathcal{C} \to L_{\mathcal{C}}^{\infty}(X, \mathcal{B}(\mathcal{H}))$ associated with ρ. Using the field of a.e. defined *-homomorphisms $\mathcal{C} \xrightarrow{\rho'} L_{\mathcal{C}}^{\infty}(X, \mathcal{B}(\mathcal{H})) \xrightarrow{ev(x)} \mathcal{B}(\mathcal{H})$ together with the fundamental sequence $(A_n)_{n=1}^{\infty}$ we can now show that $\overline{\mathcal{Q}}_4(\mathcal{A}) = \mathcal{A} \Rightarrow \overline{\mathcal{Q}}_4(\mathcal{A}(x)) = \mathcal{A}(x)$ a.e.

Concluding Remarks

In this thesis we have been primarily concerned with C*-algebras built from finite dimensional C*-algebras. The constructions have included tensor products, direct integrals and direct limits. There is a convenient diagram which relates the various algebras:



In fact we did not concern ourselves with the last algebra in the diagram ($\int_X^{\oplus} \overline{\mathcal{A}(X)}^{WOT} d\mu$), for it is intrinsically infinite dimensional whenever $\mathcal{A}(x)$ is not finite dimensional. For instance when $\mathcal{A}(x) = \mathcal{K}$, $\overline{\mathcal{A}(X)}^{WOT}$ is the non-separable space $\mathcal{B}(\mathcal{H})$ and we obtain the decomposable operators on $\mathcal{L}^2(X,\mu) \otimes \mathcal{H}$ ($\mathcal{H}$ separable and infinite dimensional). What about these "larger" algebras, where "atoms" such as $\mathcal{B}(\mathcal{H})$ are permitted? In the case where X is a singleton, we obtain just $\mathcal{B}(\mathcal{H})$, for which Wu characterized $\mathcal{Q}_\infty(\mathcal{B}(\mathcal{H})) = \mathcal{Q}_{18}(\mathcal{B}(\mathcal{H}))$ and showed that if $\mathcal{G}$ is the group of invertible operators then $\mathcal{Q}_\infty = \overline{\mathcal{G}}$. (See [Wu,2]). In [KLMR] we find a proof that $\overline{\mathcal{G}} = \overline{\mathcal{P}}_\infty$, where it is also shown that $\overline{\mathcal{P}}_\infty$ coincides with $\overline{\mathcal{P}}_5$. Extending these results to decomposable operators and larger classes of algebras is the content of our forthcoming research. Some of the algebras under consideration can be represented by a diagram much as for the present thesis:



<u>Definition</u>: (a) We call a type I von Neumann algebra <u>centrally discrete</u> (CD) if it is isomorphic to a (possibly infinite) direct sum $\overset{n}{\underset{1}{\oplus}} \mathcal{B}(\mathcal{H})$, $1 \le n \le \infty$.

(b) We call a type I von Neumann algebra <u>centrally 1-normal</u> (C1N) if it is isomorphic to a (possibly infinite) direct sum $\overset{n}{\underset{1}{\oplus}} \int_X^\oplus \mathcal{B}(\mathcal{H})d\mu$, $1 \le n \le \infty$.

<u>Note</u>: In the diagram above, the "A" in the acronyms stands for "approximately". So an ACD algebra is a direct limit of CD algebras, etc.

In this context, where $\mathcal{H}$ is infinite dimensional, we observe new phenomena, and the proofs usually require different techniques. For example, to generalize Wu's theorem and so obtain a characterization of $\mathcal{P}_\infty$ for the decomposable operators we consider measurable fields of unbounded operators. Our proof yields spectral information for measurable fields of unitary operators and is related to the results of Fillmore [F] and Azoff and Clancey [AC]. (Fillmore extended the Halmos-Kakutani factorization theorem to properly infinite von Neumann algebras, while Azoff and Clancey dealt with direct integrals of normal operators in the algebra $\mathcal{B}(\mathcal{H})$.)

Note that in considering infinite direct sums, such as $\overset{\infty}{\underset{1}{\oplus}} \mathcal{B}(\mathcal{H})$, certain subtleties arise, such as how a given operator relates to the Calkin algebra of $\mathcal{B}(\sum_1^{\infty \oplus} \mathcal{H})$. To deal with these algebras and to obtain generalizations of Theorem 3 in [KLMR] to other C*-algebras, we suggest a generalized index map will be relevant.

In view of our results so far, further algebras for study would be von Neumann algebras of types II and III. However, important C*-algebras noticeably absent from our paper are those of the form $\mathcal{M}_n(C(X))$, e.g. $\mathcal{M}_n(C(S^1))$ or $\mathcal{M}_n(C[0,1])$. As already pointed out in several remarks, there are crucial features of continuity which require attention. For these questions we have begun to sketch out possible (geometric) techniques which could be interesting in their own right. As well, we are on the way to obtaining some continuous selection theorems, of which one application would be a topological explanation of Sourour's factorization theorem [S].

We are hoping that the geometric approach will allow precise formulation of a new invariant for C*-algebras. For a unital C*-algebra $\mathcal{A}$, this will involve a length function for the multiplicative group $\mathcal{P}_\infty(\mathcal{A})$. (Recall that $\mathcal{P}_\infty(\mathcal{M}_1) = \mathcal{P}_1$, $\mathcal{P}_\infty(\mathcal{M}_n) = \mathcal{P}_5$, $n \geq 2$ and $\mathcal{P}_\infty(\mathcal{B}(\mathcal{H})) = \mathcal{P}_{17}$, although "17" may not be optimal. The K-groups give $K_0(\mathcal{M}_1) \cong K_0(\mathcal{M}_n) \cong \mathbb{Z}$, $K_1(\mathcal{M}_1) \cong K_1(\mathcal{M}_n) \cong 0$, $K_0(\mathcal{B}(\mathcal{H})) \cong K_1(\mathcal{B}(\mathcal{H})) \cong 0$.) Basically, since products of positive invertible operators form a normal subgroup of the connected component of the identity, the theory of Lie groups and Lie algebras will be appropriate when the algebra (or dense subalgebra) is sufficiently smooth. We hope that this will be a rich line of enquiry.

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