$\mathscr{P}$-GENERATING POLYNOMIALS AND THE $\mathscr{P}$-FRACTAL OF A GRAPH

by

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Abstract

A graph property $\mathscr{P}$ is defined to be any subset of $\mathscr{G}$, the class of all finite graphs, such that $\mathscr{P}$ is closed under isomorphism and $\{K_0, K_1\} \subseteq \mathscr{P}$. We define a new polynomial which is a generating function for the number of $\mathscr{P}$-subgraphs of a graph G of size i. We call this polynomial the $\mathscr{P}$-generating polynomial of a graph G and it is defined by $\nu_{\mathscr{P}}(G, x) = \sum_{i=0}^{n} \beta_i x^i$ where β_i is the number of induced subgraphs of G of order i with property $\mathscr{P}$. We provide some results about computing $\nu_{\mathscr{P}}(G \circ H, x)$ for various graph operations $\circ$ and properties $\mathscr{P}$. For general properties $\mathscr{P}$ we consider the problem of determining the nature and location of the roots of $\nu_{\mathscr{P}}(G, x)$. We show that for a graph G there are many fractals that arise from studying the roots of $\mathscr{P}$-generating polynomials of certain supergraphs of G and properties $\mathscr{P}$ that behave similarly under substitution. These fractals are studied and shown to be the Julia set of $\nu_{\mathscr{P}}(G, x) - 1$. We conclude with a few open problems and possible future research directions.

List of Abbreviations and Symbols Used

$E(G)$ .... Edge set of a graph G

$V(G)$ .... Vertex set of a graph G

$\overline{G}$ .... Complement of a graph G

G^k .... k-fold substitution of a graph G with itself

P_n .... Path graph on n vertices

K_n .... Complete graph on n vertices

C_n .... Cycle graph on n vertices

$K_{n_1,n_2,\ldots,n_k}$ .... Complete multipartite graph

$G \cup H$ .... Disjoint union of graphs G and H

$G + H$ .... Graph join of graphs G and H

$G[H]$ .... Lexicographic product (graph substitution) of G and H

$I(G,x)$ .... Independence polynomial of a graph G

$\omega(G)$ .... Clique number of a graph G

$\alpha(G)$ .... Independence number of a graph G

$\nu_{\mathscr{P}}(G,x)$ .... $\mathscr{P}$-Generating polynomial of a graph G

$f_{\mathscr{P}}(G,x)$ .... Reduced $\mathscr{P}$-generating polynomial of a graph G

$\mathcal{F}(G,\mathscr{P})$ .... $\mathscr{P}$-Fractal of a graph G

$\mathcal{R}(G, \mathscr{P})$ .... $\lim_{k\to\infty} Roots(\nu_{\mathscr{P}}(G^k, x))$

$J(f)$ .... Julia set of a polynomial f

$F(f)$ .... Fatou set of a polynomial f

$\mathscr{G}$ .... The class of all finite graphs

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Chapter 1

Introduction

1.1 Background

For us, all graphs G are finite and simple. We will let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges respectively. For vertices $u, v \in V(G)$ we say u is adjacent to v and write $u \sim_G v$ or $v \sim_G u$ if they are joined by an edge, i.e. $uv \in E(G)$. We write $u \nsim_G v$ if there is no edge between u and v. For all $u \in V(G)$, $N_G(u) = \{v : u \sim_G v\}$ is the **open neighbourhood** of u and $N_G[u] = N_G(u) \cup \{u\}$ is the **closed neighbourhood** of u. Also for $u \in V(G)$, $\deg_G(u) = |N(u)|$ is the **degree** of u, $\delta(G) = \min\{\deg_G(u) : u \in V(G)\}$ and $\Delta(G) = \max\{\deg_G(u) : u \in V(G)\}$. If it is clear from context that the graph we are referring to is G then we drop the subscript "G" in the above notations. If $\deg(u) = k$ for all $u \in V(G)$ we say that G is k**-regular**.

We say that a graph H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph H is an **induced subgraph** of G if H is a subgraph of G and $\{u, v\} \subseteq V(H)$ and $uv \in E(G)$ implies $uv \in E(H)$. We write $H \trianglelefteq G$ for H an induced subgraph of G. The **clique number** of a graph G is the size of the largest complete subgraph of G, denoted $\omega(G)$. An **independent set** of a graph G is a subset S of $V(G)$ such that the graph induced on S has no edges. The **independence number** of a graph G, denoted $\alpha(G)$ is the size of the largest independent set of G. We will exclusively work with induced subgraphs in this work and so when we say subgraph from this point forward we mean induced subgraph. Given graphs G and H, the **disjoint union** of G and H, denoted $G \cup H$, is the graph on vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The **graph join** of G and H, denoted $G + H$, is the graph on vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. We let K_0 denote the **empty graph**, the graph such that $V(K_0) = \emptyset$. Any notation and other relevant background on basic graph theory can be found in West's book [43].

Graph properties will play a central role in this thesis so we take the time to develop the necessary background. Let $\mathscr{G}$ denote the class of all finite graphs. A **property** $\mathscr{P}$ is a subset of $\mathscr{G}$, closed under isomorphism, that contains the empty graph K_0 and the singleton K_1. A property $\mathscr{P}$ is a **nontrivial property** if $\mathscr{P} \neq \mathscr{G}$. A graph is said to be a **$\mathscr{P}$-graph** if G has property $\mathscr{P}$ and we write $G \in \mathscr{P}$. If $H \trianglelefteq G$ and $H \in \mathscr{P}$, then H is called a **$\mathscr{P}$-subgraph** of G. The **complement** of a property $\mathscr{P}$ is the set $\overline{\mathscr{P}} = \{\overline{G} : \ G \in \mathscr{P}\}$. It is very important to note that $\overline{\mathscr{P}}$ does not correspond to the class $\mathscr{G} \setminus \mathscr{P}$ as might be expected, indeed the latter is not even a property as it does not contain K_0 and K_1.

If $G \in \mathscr{P}$ implies $H \in \mathscr{P}$ for all $H \trianglelefteq G$, then $\mathscr{P}$ is called a **hereditary property**. An example of a property that is not hereditary is $\{G : \delta(G) = \Delta(G)\}$, i.e. regularity. We will exclusively work with hereditary properties in this thesis and so property and hereditary property are sometimes used interchangeably. For a fixed graph G of order at least 2, we define the set $-G$ to be the set of all graphs which do not contain an induced copy of G, i.e. $-G = \{H \in \mathscr{G} : G \ntrianglelefteq H\}$. Properties of the form $-G$ are called **elementary** and are easily seen to be hereditary. Graphs with property $-G$ are said to be G-free. We often describe properties as a finite or infinite intersection of elementary properties, for example, the hereditary property of bipartiteness can be described by $\bigcap\limits_{k \geq 1} -C_{2k+1}$. In fact, all hereditary properties $\mathscr{P}$ may be described as $\bigcap\limits_{G \notin \mathscr{P}} -G$ and uniquely so if the G's are minimal in the partial order $\trianglelefteq$ (these give a unique list of forbidden induced subgraphs for $\mathscr{P}$). It is for this reason that in this thesis, with respect to the partial order $\trianglelefteq$, the graphs used to describe intersections of elementary properties will always be assumed to be minimal and hence incomparable with respect to $\trianglelefteq$. For example, $\bigcap_{j\geq 2} -K_j$ would never be written since $K_2 \trianglelefteq K_i$ for all $i \geq 2$, this would be simply described as $-K_2$. This is mainly a technical detail but will be of some consequence later and ensures that forbidden subgraph properties are uniquely written. Finally, we say that a property $\bigcap_{i\in I} -G_i$ is **connected** if G_i is connected for each i.

1.2 Graph Polynomials

A variety of polynomials have arisen from the study of colourings in the literature. The study can be traced back to 1912 when Birkhoff first defined, for planar graphs, the chromatic polynomial [7] in an attempt to prove the Four Colour Conjecture. Of course, he was unsuccessful but his work paved the way for studying graphs by means of polynomials. There have been many graph polynomials that have arisen since then with some of the most widely studied being the chromatic, independence, matching, and matching-generating polynomials [7,25,27,31]. There are various other polynomials that are related to chromatic theory that are not as widely studied, for example the polynomials studied by Brenti [9]. The independence and matching-generating polynomials are generating polynomials for the number of independent sets and number of matchings of a graph, respectively, and it is from the former that we draw our inspiration for the polynomial defined in the next section and studied for the remainder of this thesis. We define the independence polynomial of a graph now, as we will often look at generalizing results proved for the independence polynomial.

Definition 1.2.1 *The* **independence polynomial** *of a graph G, $I(G,x)$ is defined as follows:*

$$I(G,x) = \sum_{i=0}^{|V(G)|} s_i x^i,$$

where s_i is the number of independent sets of G of size i.

Results that have been of interest for the independence polynomial of a graph include computing $I(G,x)$ for large families of graphs, computing $I(G \circ H, x)$ where $\circ$ is a graph operation such as union, join, Cartesian product, etc., finding recurrences, and locating the roots [12, 14–17, 21–25, 29–31, 39, 40]. The reader is directed to Levit's [31] survey on independence polynomials which provides an excellent introduction to the theory of independence polynomials despite being slightly dated. In fact, finding the roots of the independence polynomial of a graph and other graph polynomials has been of considerable interest over the past few decades with the following references containing results on locating roots of graph polynomials [9–12, 14–17, 27, 28, 31], just to name a few. It may appear, at least initially, that the roots of a graph polynomial do not contain important information about the graph but they do, and indeed the roots are of interest and certain families of graphs can yield interesting graph theoretic results. For example, the roots of the characteristic polynomial of a graph are the eigenvalues of its adjacency matrix, which contain lots of information about the graph [6]. We will not be studying characteristic polynomials in this work, but mention the roots as it motivates drawing the connection between the roots of other graph polynomial roots and the graph itself.

Now that we have briefly considered what directions have been explored when studying graph polynomials, we are ready to define a new polynomial and study it in a similar fashion to those previously defined.

1.3 $\mathscr{P}$ -Generating Polynomial Definition and Examples

We look at the problem of counting the number of subgraphs of a given graph G with a certain graph property $\mathscr{P}$. The following definition will be the central focus of the rest of the thesis.

Definition 1.3.1 *The $\mathscr{P}$**-generating polynomial** of a graph G, denoted, $\nu_{\mathscr{P}}(G,x)$, is defined as:*

$$\nu_{\mathscr{P}}(G,x) = \sum_{i=0}^{n} \alpha_i x^i$$

where $n = |V(G)|$ and α_i is the number of induced $\mathscr{P}$-subgraphs of G on i vertices.

Every graph has exactly one $\mathscr{P}$-subgraph of size 0, namely the empty graph K_0 and hence the $\mathscr{P}$-generating polynomial always has constant term 1. When $\mathscr{P} = -K_2$, $\nu_{\mathscr{P}}(G,x)$ is simply the independence polynomial of G; thus $\nu_{\mathscr{P}}(G,x)$ generalizes the independence polynomial. If $\mathscr{P}$ is a hereditary property and G has property $\mathscr{P}$, then by the definition of a hereditary property every induced subgraph of G is also a $\mathscr{P}$-graph and thus

$$\nu_{\mathscr{P}}(G,x) = \sum_{i=0}^{n} \binom{n}{i} x^i = (1+x)^n.$$

In fact the coefficient of x^k in $\nu_{\mathscr{P}}(G,x)$ is bounded above by $\binom{n}{k}$ and below by 0. We list a few examples with a brief justification for each to give the reader a feel for these polynomials.

$$\nu_{-P_4}(P_7, x) = x^6 + 12x^5 + 31x^4 + 35x^3 + 21x^2 + 7x + 1 \tag{1.1}$$

$$\nu_{-C_5}(P_4, x) = (1 + x)^4 \tag{1.2}$$

$$\nu_{-P_4}(P_4 \cup P_4, x) = 16x^6 + 48x^5 + 68x^4 + 56x^3 + 28x^2 + 8x + 1 \tag{1.3}$$

$$\nu_{-G}(G, x) = (1 + x)^n - x^n \tag{1.4}$$

$$\tag{1.5}$$

Refer to Figure 1.1 as we justify (1.1). The coefficient of x^i for $i \leq 3$ is simply $\binom{7}{i}$ as no graph on fewer than 4 vertices may contain P_4. To calculate the coefficient of x^4 we will count the number of induced P_4's in P_7 and subtract that number from $\binom{7}{4}$. The P_4's of P_7 are exactly the graphs induced on the vertex sets $\{i, i+1, i+2, i+3\}$ for $i = 1, 2, 3, 4$ and so there are $\binom{7}{4} - 4 = 21$ subgraphs of P_7 with property $-P_4$. To calculate the coefficient of x^5 we again count the subgraphs of P_7 are not $-P_4$-graphs and subtract the total from $\binom{7}{5}$. We know that there are exactly 4 distinct P_4's within P_7, and for each P_4 there are exactly 3 subgraphs of order 5 containing it. The subgraphs of order 5 isomorphic to $P_4 \cup K_1$ are all distinct but the 3 subgraphs isomorphic to P_5 are all counted twice which leaves $4 \cdot 3 - 3 = 9$ subgraphs of P_7 that contain a copy of P_4. Therefore the coefficient of x^5 in (1.1) is $\binom{7}{5} - 9 = 12$. The coefficient of x^6 is 1 since any subgraph of P_7 other than $\{1, 2, 3, 5, 6, 7\}$ must have at least 4 consecutive vertex labels and therefore a copy of P_4. The coefficient of x^7 is zero since P_7 is not a $-P_4$-graph.

Figure 1.1: P_7

The justification for (1.2) was mentioned in general above since $P_4 \in -C_5$. In fact, for any property $-G$ where $|V(G)| = n$, and for all graphs H such that $|V(H)| < n$, $\nu_{-G}(H,x) = (1+x)^{|V(H)|}$.

The calculation of (1.3) is more involved and we will do it now with combinatorics but it can be done in an easier way with results that we present later. The coefficient of x^i is calculated by counting the number of ways to choose $m \leq 3$ from one of the components and $l \leq 3$ from the other for all (l,m) such that $l+m=4$. The coefficient of x^4 is $\binom{4}{2}^2 + 2\binom{4}{3}\binom{4}{1} = 68$, of x^5 is $2\binom{4}{3}\binom{4}{2} = 48$, and of x^6 is $\binom{4}{3}^2 = 16$. Any subgraph of $P_4 \cup P_4$ on at least 7 vertices must contain one of the copies of P_4 leaving the remaining coefficients 0.

The calculation of (1.4) simply follows from the fact that the only subgraph of G that contains a copy of G is, of course, G itself. This specific example will be useful for our work on locating the roots of $\mathscr{P}$-generating polynomials.

For more examples, Appendix A contains $\nu_{\mathscr{P}}(G,x)$ for $\mathscr{P} = -K_3$ and $\mathscr{P} = -P_3$ for all connected graphs with $|V(G)| \leq 5$.

We feel that the study of $\mathscr{P}$-generating polynomials is important to the field of graph theory, in particular to the study of graphical enumeration and comparing graphs. Harary and Palmer [26] wrote an entire book on enumerating certain graphs which has always been an intriguing question, as graph theory and combinatorics are so tightly intertwined. With a better understanding of $\mathscr{P}$-generating polynomials we will be able to better understand the structure of many graphs by means of their $\mathscr{P}$-subgraphs. For example, we may compare two large networks by the ratio of $-P_4$-subgraphs or each size to the order of each network. The problem of comparing very large graphs has become increasingly important as modelling large real life or online networks has become very important [34]. Better understanding of $\mathscr{P}$-generating polynomials for different properties $\mathscr{P}$ will give a better idea of the substructures contained within very large networks and can be used to compute densities that can be compared with the observed network.

We conclude this introductory section with the following straightforward but useful result:

Observation 1.3.2 $\nu_{\mathscr{P}}(G,x) = \nu_{\overline{\mathscr{P}}}(\overline{G},x)$.

Proof. By the definition of the complement of a property, H is a $\mathscr{P}$-subgraph of G if and only if $\overline{H}$ is a $\overline{\mathscr{P}}$-subgraph of $\overline{G}$ and so for $n \in \mathbb{N}$ the number of $\mathscr{P}$-subgraphs of G of order n is equal to the number of $\overline{\mathscr{P}}$-subgraphs of $\overline{G}$ of order n. $\square$

Chapter 2

$\mathscr{P}$-Generating Polynomials of Products

When computing any graph polynomials of product graphs it is of interest to find the relationships with the graph polynomials of the smaller factor graphs. For example,

$$\nu_{-K_2}(G_1 \cup G_2, x) = \nu_{-K_2}(G_1, x) \cdot \nu_{-K_2}(G_2, x)$$

and

$$\nu_{-K_2}(G_1 + G_2, x) = \nu_{-K_2}(G_1, x) + \nu_{-K_2}(G_2, x) - 1$$

for vertex disjoint graphs G_1 and G_2 (see [31], for example). We will look at generalizing these two results to further properties in this section as well as generalizing a very interesting result about graph substitution and the independence polynomial due to Brown et al. [14].

We begin considering the disjoint union and graph join of graphs as they are the most elementary products.

2.1 Graph Join and Disjoint Union

We begin with an example. An easy calculation verifies that

$$\nu_{-\overline{K_2}}(K_2 \cup K_2, x) = 1 + 4x + 2x^2,$$

but

$$\nu_{-\overline{K_2}}(K_2, x) = (1+x)^2$$

so

$$\nu_{-\overline{K_2}}(K_2 \cup K_2, x) \neq \nu_{-\overline{K_2}}(K_2, x) \cdot \nu_{-\overline{K_2}}(K_2, x).$$

The next proposition answers the question: For which properties does the multiplication rule hold for all graphs with respect to disjoint union?

Proposition 2.1.1 *$\mathscr{P}$ is a connected hereditary property if and only if for all graphs G and H, $\nu_{\mathscr{P}}(G \cup H, x) = \nu_{\mathscr{P}}(G, x) \cdot \nu_{\mathscr{P}}(H, x)$.*

Proof. Suppose $\mathscr{P}$ is a connected hereditary property. Let H' be any $\mathscr{P}$-subgraph of H and G' be any $\mathscr{P}$-subgraph of G. Since $\mathscr{P}$ is a connected hereditary property it follows that $H' \cup G'$ is also a $\mathscr{P}$-subgraph of $G \cup H$. Also, for any $\mathscr{P}$-subgraph, F, of $G \cup H$, $H \cap F$ is a $\mathscr{P}$-graph and $G \cap F$ is a $\mathscr{P}$-graph since $\mathscr{P}$ is hereditary. Therefore all $\mathscr{P}$-subgraphs of $G \cup H$ are of the form $G' \cup H'$ where G' and H' are $\mathscr{P}$-subgraphs of G and H respectively. Therefore, the generating function, $\nu_{\mathscr{P}}(G \cup H)$, for the number of $\mathscr{P}$-subgraph of $G \cup H$ is given by the product of the generating function for G with the generating function for H. I.e: $\nu_{\mathscr{P}}(G \cup H, x) = \nu_{\mathscr{P}}(G, x) \cdot \nu_P(H, x)$.

Conversely, suppose $\mathscr{P}$ is not connected. Since $\mathscr{P}$ is hereditary, we express $\mathscr{P}$ as $\bigcap_{i=1}^{n} -F_i$ and since $\mathscr{P}$ is not connected by assumption there exists a F_i that is disconnected. Let F_i' and F_i'' be a partition of F_i such that there are no edges between vertices in F_i' and F_i''. Note that since $F_m \not\trianglelefteq F_n$ for $n \neq m$, (as all graphs in property expressions are assumed to be minimal), both F_i' and F_i'' are in $\mathscr{P}$, so

$\nu_{\mathscr{P}}(F_i', x) = (1+x)^{|V(F_i')|}$ and $\nu_{\mathscr{P}}(F_i'', x) = (1+x)^{|V(F_i'')|}$. Now,

$$\begin{aligned} \nu_{\mathscr{P}}(F_i' \cup F_i'', x) &= \nu_{\mathscr{P}}(F_i, x) \\ &\neq (1+x)^{|V(F_i)|} \\ &= (1+x)^{|V(F_i')|+|V(F_i'')|} \\ &= \nu_{\mathscr{P}}(F_i', x) \cdot \nu_{\mathscr{P}}(F_i'', x) \end{aligned}$$

□

Corollary 2.1.2 *$\mathscr{P}$ is a hereditary property such that $\overline{\mathscr{P}}$ is connected if and only if for all graphs G and H, $\nu_{\mathscr{P}}(G+H, x) = \nu_{\mathscr{P}}(G, x) \cdot \nu_{\mathscr{P}}(H, x)$.*

Proof. We know that $\overline{G+H} = \overline{G} \cup \overline{H}$, and from Proposition 2.1.1 that

$$\nu_{\mathscr{P}}(G+H, x) = \nu_{\overline{\mathscr{P}}}(\overline{G} \cup \overline{H}, x) = \nu_{\overline{\mathscr{P}}}(\overline{G}, x) \cdot \nu_{\overline{\mathscr{P}}}(\overline{H}, x).$$

The rest of the proof follows from Observation 1.3.2.

□

We see that Proposition 2.1.1 generalizes the result mentioned at the beginning of this section for the independence polynomial of a graph to all $\mathscr{P}$-generating polynomials for which $\mathscr{P}$ is a connected hereditary property. Corollary 2.1.2 gives us a result for properties $\mathscr{P}$ that are closed under the join operation, but there are other properties that are not closed with respect to the join operation that we wish to have results for. One class of properties of particular interest that Corollary 2.1.2 excludes is the properties $-K_n$ for $n \geq 3$. Although the simple result from Corollary 2.1.2 does not hold, we do have the following result:

Proposition 2.1.3 *If G and H are graphs, then*

$$\begin{aligned}\nu_{-K_l}(G+H,x) &= \nu_{-K_l}(G,x)+\nu_{-K_l}(H,x)-1+ \\ &\sum_{i=1}^{l-2}\Big((\nu_{-K_{l-i}}(G,x)-1)(\nu_{-K_{i+1}}(H,x)-\nu_{-K_i}(H,x))\Big)\end{aligned}$$

(It should be noted that we abuse notation here allowing ν_{-K_1} to be defined although all properties must contain K_1. This is simply for convenience here and $\nu_{-K_1}(G,x) = 1$ for all graphs G, which is what would be expected if $-K_1$ was indeed a property.)

Proof. The first three factors on the right hand side give the number of ways we may select a $-K_l$-subgraph with vertices from exactly one of G and H, subtracting 1 to account for counting the empty graph twice. What is left to count is the number of $-K_l$-subgraphs of G with vertices from both G and H. Since $\omega(G+H) = \omega(G)+\omega(H)$, for any $-K_l$-subgraph, $G'+H'$, of $G+H$ where $V(G')$ is a non-empty subset of $V(G)$ and $V(H')$ is a non-empty subset of $V(H)$, it must be the case that $\omega(G') + \omega(H') \leq l-1$. So we must count all non-empty $-K_{l-i}$-subgraphs of G and all non-empty subgraphs of H with clique number exactly i for $i \geq 1$. Now, the generating function for all non-empty $-K_{l-i}$-subgraphs of G is $\nu_{-K_{l-i}}(G,x)-1$ and the generating function for all non-empty subgraphs of H with clique number exactly i is $\nu_{-K_{i+1}}(H,x)-\nu_{-K_i}(H,x)$. Summing over the possible combinations gives the desired result.

□

To illustrate, we will now use Proposition 2.1.3 to calculate the $-K_4$-generating polynomial of $C_4 + P_3$:

$$\nu_{-K_4}(C_4 + P_3, x) = \nu_{-K_4}(C_4, x) + \nu_{-K_4}(P_3, x) - 1 + \sum_{i=1}^{2} \Big((\nu_{-K_{4-i}}(C_4, x) - 1)(\nu_{-K_{i+1}}(P_3, x) - \nu_{-K_i}(P_3, x)) \Big)$$

$$= (1+x)^4 + (1+x)^3 - 1 + (\nu_{-K_3}(C_4, x) - 1)(\nu_{-K_2}(P_3, x) - 1) + (\nu_{-K_2}(C_4, x) - 1)(\nu_{-K_3}(P_3, x) - \nu_{-K_2}(P_3, x))$$

$$= x^4 + 5x^3 + 9x^2 + 7x + 1 + ((1+x)^4 - 1)(x^2 + 3x) + (2x^2 + 4x)((1+x)^3 - (x^2 + 3x + 1))$$

$$= x^6 + 9x^5 + 27x^4 + 35x^3 + 21x^2 + 7x + 1.$$

Proposition 2.1.4 *$\nu_{\mathscr{P}}(G_1 + G_2, x) = \nu_{\mathscr{P}}(G_1, x) + \nu_{\mathscr{P}}(G_2, x) - 1$ for all graphs G_1 and G_2 if and only if $\mathscr{P} \subseteq -K_2$.*

Proof. Suppose $\nu_{\mathscr{P}}(G_1 + G_2, x) = \nu_{\mathscr{P}}(G_1, x) + \nu_{\mathscr{P}}(G_2, x) - 1$ for all graphs G_1 and G_2. Therefore,

$$\begin{aligned}
\nu_{\mathscr{P}}(K_2, x) &= \nu_{\mathscr{P}}(K_1 + K_1, x) \\
&= \nu_{\mathscr{P}}(K_1, x) + \nu_{\mathscr{P}}(K_1, x) - 1 \\
&= 1 + x + 1 + x - 1 \\
&= 1 + 2x.
\end{aligned}$$

Therefore, $K_2 \notin \mathscr{P}$ so $\mathscr{P} \subseteq -K_2$.

Conversely, suppose that $-K_2 \subseteq \mathscr{P}$. Therefore, $\mathscr{P} = -K_2 \cap -\overline{K_n}$ for some $n \geq 2$. Now, no subgraph of $G_1 + G_2$ which contains vertices from both G_1 and G_2 will be a $\mathscr{P}$-graph as it will contain a K_2. Therefore, the only $\mathscr{P}$-subgraphs of $G_1 + G_2$ are the $\mathscr{P}$-subgraphs of G_1 and the $\mathscr{P}$-subgraphs of G_2, which gives $\nu_{\mathscr{P}}(G_1 + G_2, x) = \nu_{\mathscr{P}}(G_1, x) + \nu_{\mathscr{P}}(G_2, x) - 1$.

□

2.2 Substitution

We now look at a more involved product known as the lexicographic product or graph substitution. Given graphs G and H such that $V(G) = \{v_1, v_2, ..., v_n\}$ and $V(H) = \{u_1, u_2, ..., u_k\}$, the **lexicographic product** which we will denoted $G[H]$ is defined as follows: $V(G[H]) = V(G) \times V(H)$ and $(v_i, u_l) \sim (v_j, u_m)$ if $v_i \sim_G v_j$ or $i = j$ and $u_l \sim_H u_m$. The lexicographic product $G[H]$ can be thought of as substituting all vertices of G with copies of H and joining the copies of H where edges where present in G. This intuitive way of understanding the product is why it is usually referred to as graph substitution. It should also be noted that the lexicographic product is also referred to as the composition of graphs by some authors and therefore denoted $G \circ H$. It is easily seen that graph substitution is associative and so we may write G^k for a k-fold substitution of G with itself. Substitution is not however commutative, consider $G = K_2[\overline{K_n}]$ and $H = \overline{K_n}[K_2]$. Noting that $K_2[F] = F + F$ for all graphs F, we see that G is the graph $K_{n,n}$ which is connected, but H is n disjoint K_2's which is disconnected and therefore the two graphs are not isomorphic. There is also the notation of a generalized graph substitution where for each vertex v_i $(i = 1, ..., n)$ of a graph G we substitute a graph H_i. The distinction here is that the H_i's are not necessarily isomorphic. We write this as $G[H_1, H_2, ..., H_n]$. To simplify this notation when we wish to only substitute a graph H for a single vertex v of G, we write $G[v \to H]$. Refer to Figure 2.1 for an example of graph substitution.

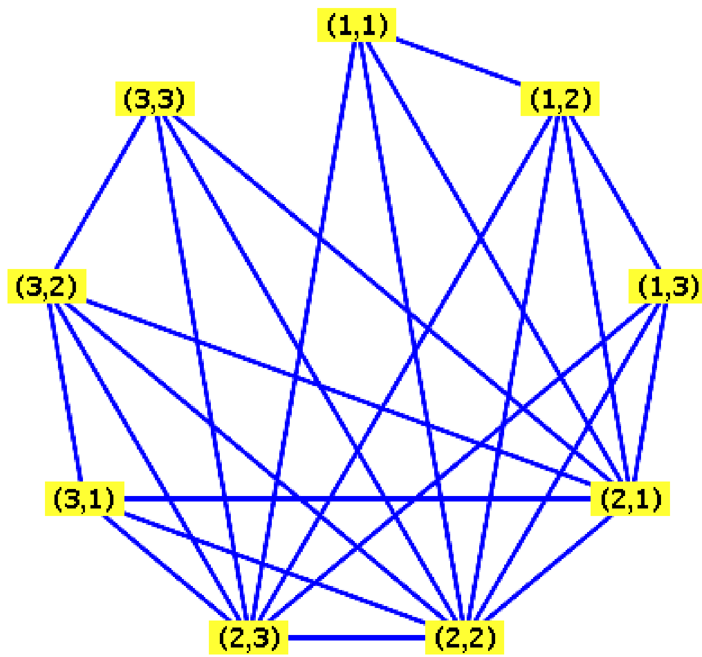

Figure 2.1: $P_3[P_3]$

We can now state the theorem by Brown et al. [14] that serves as motivation for a large portion of this section.

Theorem 2.2.1 *[14] For disjoint graphs G and H,*

$$\nu_{-K_2}(G[H], x) = \nu_{-K_2}(G, \nu_{-K_2}(H, x) - 1).$$

This theorem gives a recurrence for ν_{-K_2} that is of particular interest not only when computing $\nu_{-K_2}(G[H], x)$, as it simplifies the problem, but when determining the nature and location of the roots of $\nu_{-K_2}(G[H], x)$. We will talk about the roots of $\nu_{\mathscr{P}}(G, x)$ in Chapter 3 for various hereditary properties $\mathscr{P}$ and one of the most interesting topics will be the nature of the roots of the $\mathscr{P}$-generating polynomial of the k-fold substitution of G with itself.

We now turn our attention to certain properties that behave similarly under the substitution operation, i.e. we are looking for certain properties that will generalize Theorem 2.2.1. For general hereditary properties $-G$, Theorem 2.2.1 does not generalize, as the following example illustrates.

Let $\mathscr{P}$ be the elementary hereditary property $-K_3$ and we compute $\nu_{\mathscr{P}}(K_2[K_2], x)$ by Proposition 2.1.3 since $K_2[K_2] = K_2 + K_2 = K_4$.

$$\begin{aligned}\nu_{-K_3}(K_2[K_2], x) &= \nu_{-K_3}(K_2, x) + \nu_{-K_3}(K_2, x) - 1 + (\nu_{-K_2}(K_2, x) - 1)(\nu_{-K_2}(K_2, x) - 1)\Big) \\ &= (1 + x)^2 + (1 + x)^2 - 1 + (1 + 2x - 1)(1 + 2x - 1) \\ &= 1 + 4x + 6x^2\end{aligned}$$

However, $\nu_{-K_3}(K_2, \nu_{K_3}(K_2, x) - 1) = (1 + x)^4$ which overcounts significantly. Therefore Theorem 2.2.1 does not generalize for $\mathscr{P} = -K_3$. The reason is that $-K_3$ is not closed under substitution, K_2 has property $-K_3$ but $K_2[K_2]$ does not. So in generalizing Theorem 2.2.1, we will need to know which properties are closed under substitution. We formalize this with the following definition:

Definition 2.2.2 *For a hereditary property $\mathscr{P}$ we say that $\mathscr{P}$ is* **closed under substitution** *if the graph $A[B_1, B_2, ..., B_{|A|}]$ has property $\mathscr{P}$ whenever $A, B_1, B_2, ..., B_{|A|}$ are $\mathscr{P}$-graphs.*

Definition 2.2.2 also leads to the following Observation and Corollary:

Observation 2.2.3 *If $\mathscr{P}$ is a hereditary property, then every nonempty $\mathscr{P}$-subgraph of $G[H]$ may be expressed in the form $A[B_1, ..., B_{|G|}]$ where A is a nonempty $\mathscr{P}$-subgraph of G and the B_i's are all nonempty $\mathscr{P}$-subgraphs of H.*

Proof. Suppose $F \neq K_0$, $F \in \mathscr{P}$, and $F \trianglelefteq G[H]$. We know that $F = G'[H_1', ..., H_{|G'|}']$ for some nonempty $G' \trianglelefteq G$ and nonempty $H_i' \trianglelefteq H$, $i = 1, 2, ..., |G'|$. Now, $G' \trianglelefteq F$ since H_i's are nonempty, and since $\mathscr{P}$ is hereditary, $G' \in \mathscr{P}$. Also, $H_i' \trianglelefteq F$ for $i = 1, 2, ..., |G'|$ since G' is nonempty, so $H_i' \in \mathscr{P}$ for each i since $\mathscr{P}$ is hereditary. □

Note that the hypothesis for Observation 2.2.3 does not require $\mathscr{P}$ to be closed under substitution. Note also that we exclude the empty graph from Observation 2.2.3 because the empty graph may be expressed as $A[K_0]$ and $K_0[B]$, where neither A nor B have to be $\mathscr{P}$-graphs. Therefore, the extreme case is where all $\mathscr{P}$-graphs retain property $\mathscr{P}$ when subject to substitution with other $\mathscr{P}$ graphs.

Corollary 2.2.4 *Let $\mathscr{P}$ be closed under substitution and G and H be any nonempty graphs. If F is any nonempty induced subgraph of $G[H]$, then $F \in \mathscr{P}$ if and only if F may be expressed as $A[B_1, B_2, ..., B_{|A|}]$ where A is a nonempty induced $\mathscr{P}$-subgraph of G and the B_i's are all nonempty induced $\mathscr{P}$-subgraphs of H.*

Properties $\mathscr{P}$ that are closed under substitution lead to nice results for $\mathscr{P}$-generating polynomials, but checking for closure under substitution of a given property using only the definition is difficult. The following definition and lemma will allow us to use results from the literature to determine whether a finite property $\mathscr{P}$ is closed under substitution in polynomial time.

Definition 2.2.5 *[42] For a graph G we say that a set S is a* **module** *if $S \subset V(G)$, $|S| \geq 2$, and for all $u, v \in S$, $N(u) \setminus S = N(v) \setminus S$.*

We say a graph is **module-free** if it does not contain a module. An example of a module is any independent set with at least two vertices in the graph $K_{n,n}$ for $n \geq 2$.

Lemma 2.2.6 *$\mathscr{P} = \bigcap_{i \in I} -G_i$, $I \subseteq \mathbb{N}$ is closed under substitution if and only if each G_i is module-free for $i \in I$*

Proof. Suppose $\mathscr{P} = \bigcap_{i \in I} -G_i$ and suppose for some i, G_i contains a module. Let G be this graph containing a module S, and let H be the graph induced by S. Now, $H \lhd G$ since $S \neq V(G)$, so $H \in \mathscr{P}$ since the G_i's are minimal by definition, (any induced proper subgraph of G has property $\mathscr{P}$). Now, the induced proper subgraph of G constructed by replacing H with a single vertex v must be a $\mathscr{P}$-subgraph of G. But now, since $\mathscr{P}$ is closed under substitution, it follows that $G = G'[v \to H] \in \mathscr{P}$ which is a contradiction.

Conversely, suppose that the $G_i's$ are module-free but $\mathscr{P}$ is not closed under substitution. So there exist graphs $H, S_1, S_2, ..., S_{|H|} \in \mathscr{P}$ such that $H[S_1, S_2, ..., S_{|H|}] \notin \mathscr{P}$. Thus there exists a G_i such that $G_i \cong H[S'_1, ..., S'_{|H|}]$ where $S'_j \unlhd S_j$ (possibly empty) for $j = 1, 2, ..., |H|$. If $|S'_j| \leq 1$ for $j = 1, 2, ..., |H|$, then $H[S'_1, ..., S'_{|H|}] \unlhd H$ and therefore has property $\mathscr{P}$ which is a contradiction. So for some j, $|S'_j| \geq 2$. But now S'_j is a module of G_i which contradicts our assumption.

□

By Lemma 2.2.6, to check that $\mathscr{P} = \bigcap_{i \in I} -G_i$ is closed under substitution, we only need to check that G_i is module-free for each i so $\mathscr{P}$ is closed under substitution if and only if each $-G_i$ is. Tedder, Habbib, and Paul [42] present a linear (in $|V(G_i)|$) time algorithm that determines whether each G_i is module-free. Some examples of properties that are closed under substitution (with proofs to follow) are $-C_n$ for

$n \geq 5$, $-P_n$ for $n = 2$ and $n \geq 4$, their complements, and intersections with each other.

We are now ready to prove the major theorem of this section, the generalization of Theorem 2.2.1.

Theorem 2.2.7 *$\mathscr{P}$ is closed under substitution if and only if*

$$\nu_{\mathscr{P}}(G[H], x) = \nu_{\mathscr{P}}(G, \nu_{\mathscr{P}}(H, x) - 1)$$

for all graphs G and H.

Proof. Suppose $\mathscr{P}$ is closed under substitution. By Corollary 2.2.4, the nonempty induced subgraphs of $G[H]$ of size l are all obtained by taking a nonempty subgraph of G, G', with property $\mathscr{P}$ and substituting for each vertex v_i of G' a nonempty subgraph of H, H_i', with property $\mathscr{P}$ such that $\sum_{i=1}^{|G'|} |H_i'| = l$. Therefore the number of $\mathscr{P}$-subgraphs of $G[H]$ of size l is given by the coefficient of x^l in

$$\sum_{k=0}^{|G|} g_k \Big(\sum_{j=1}^{|H|} h_j x_j \Big) \tag{2.1}$$

where g_k is the number of $\mathscr{P}$-subgraphs of G of size k and h_j is the number of $\mathscr{P}$-subgraphs of H of size j. But (2.1) is equal to $\nu_{\mathscr{P}}(G, \nu_{\mathscr{P}}(H, x) - 1)$ by definition.

Conversely, Observation 2.2.3 implies that for any positive integer l, the number of $\mathscr{P}$-subgraphs of $G[H]$ of order l is bounded above by the number of ways to make a subgraph of order l using graph substitution of $\mathscr{P}$-subgraphs of G and $\mathscr{P}$-subgraphs of H. But this number is the coefficient of x^l in $\nu_{\mathscr{P}}(G, \nu_{\mathscr{P}}(H, x) - 1)$ which is equal to $\nu_{\mathscr{P}}(G[H], x)$ by assumption. Therefore, $G'[H_1, H_2, ..., H_{|V(G)|}]$ is a $\mathscr{P}$-graph whenever

G' is a $\mathscr{P}$-subgraph of G and each H_i is a $\mathscr{P}$-subgraph of H. Thus, $\mathscr{P}$ is closed under substitution. □

Theorem 2.2.7 is of interest in its own right, but has applications to finding the roots of $\nu_{\mathscr{P}}(G[H], x)$ that are very elegant and will be discussed in detail in Chapter 3. Before moving to the roots, we will look further at module-free graphs. The following observation will be useful and follows directly from the definition of a module and the graph complement.

Observation 2.2.8 *G is module-free if and only if $\overline{G}$ is module-free.*

Lemma 2.2.9 *If G or $\overline{G}$ is a disconnected graph and $|V(G)| \geq 3$, then G contains a module.*

Proof. By Observation 2.2.8, we suppose without loss of generality that G is a disconnected graph on at least 3 vertices with connected components $G_1, G_2, ..., G_m$ $m \geq 2$. If $|V(G_i)| \geq 2$ for some $i \in \{1, 2, ..., m\}$, then for all $u, v \in V(G_i)$, $N(u) \setminus V(G_i) = N(u) \setminus V(G_i) = \emptyset$. Therefore G_i is a module of G. If $|V(G_i)| = 1$ for $i = 1, 2, ..., m$, then $m \geq 3$ and $S = V(G_1) \cup V(G_2)$ is a module of G. □

Lemma 2.2.10 *The properties $-C_n$ and $-\overline{C_n}$ are closed under substitution for $n \geq 5$.*

Proof. Suppose $n \geq 5$ and let $V(C_n) = \{v_0, v_1, ..., v_{n-1}\}$ such that $E(C_n) = \{v_iv_j : i = j \pm 1 \pmod{n}\}$. Suppose that there exists a set $S \subset V(C_n)$ such that S is a module. Since C_n is connected and $S \subset V(C_n)$, there must be a vertex $u \notin S$, such that u is adjacent to a vertex in S. Without loss of generality, suppose that $u = v_0$ and $v_1 \in S$. Since S is a module it must contain at least one more vertex, and since $v_0 \in N(v_1) \setminus S$, it follows that S also contains the vertex v_{n-1}. Now, if $v_2 \notin S$, then $(N(v_{n-1}) \setminus S) \neq (N(v_1) \setminus S)$ since $v_{n-1} \nsim v_2$ as $n \geq 5$ and if $v_2 \in S$, then $(N(v_2) \setminus S) \neq (N(v_1) \setminus S)$, which both contradict S being a module. Therefore C_n is module-free and so by Lemma 2.2.6 $-C_n$ is closed under substitution for $n \geq 5$. Finally, by Observation 2.2.8, $-\overline{C_n}$ is closed under substitution for $n \geq 5$.

□

A similar argument to Lemma 2.2.10 shows the following observation:

Observation 2.2.11 *The properties $-P_n$ and $-\overline{P_n}$ are closed under substitution for $n \geq 4$.*

We say that a graph is **chordal** if it contains no induced cycles of length 4 or greater, in our notation, $\bigcap_{n \geq 4} -C_n$. Chordal graphs are of significant interest for many problems like colouring and many graph algorithms can be made more efficient when the input is reduced to chordal graphs [35, 37]. The chordal property is not closed under substitution since $-C_4$ is not closed under substitution but a very similar property is, namely, forbidding all cycles on 5 or greater vertices. This class of graphs properly contains chordal graphs as well as the hereditary property perfection which has been the subject of much interest for graph theorists.

A perfect graph is defined as a graph for which every induced subgraph has chromatic number equal to its clique number. In 1960, Berge conjectured that a graph is perfect if and only if its complement is perfect and that a graph is perfect if and only if it does not contain C_{2n+1} or $\overline{C_{2n+1}}$ as an induced subgraph, the former known as the weak perfect graph conjecture and the latter known as the strong perfect graph conjecture [41]. The weak perfect graph conjecture was proved in 1972 by Lovász [32, 33, 41] but it wasn't until 2002 that the strong perfect graph conjecture was proved and even then was not finalized and published until 2006 by Chudnovsky, Robertson, Seymour, and Thomas [19, 41].

Corollary 2.2.12 *The perfection property,* $\mathscr{P} = \bigcap_{n \geq 2} \left(-C_{2n+1} \cap -\overline{C_{2n+1}} \right)$ *is closed under substitution.*

Proof. This Corollary follows directly from Lemma 2.2.10, Observation 2.2.8, and Lemma 2.2.6. □

Chapter 3

Roots of $\mathscr{P}$-Generating Polynomials

The problem of finding roots of graph polynomials has attracted considerable attention. In this chapter, we look at different methods for finding the roots of $\nu_{\mathscr{P}}(G, x)$ for different graphs G and properties $\mathscr{P}$ using some of the techniques from Chapter 2 to simplify the problem. In fact, some of the most interesting work of this chapter and thesis as a whole comes from Theorem 2.2.7 as it relates to finding the roots of graphs under substitution with themselves many times over. This process will lead to a fractal-like object and provides an elegant relation between the k-fold substitution of a graph with itself as k gets very large and the roots of its $\mathscr{P}$-generating polynomials for all properties $\mathscr{P}$ that are closed under substitution.

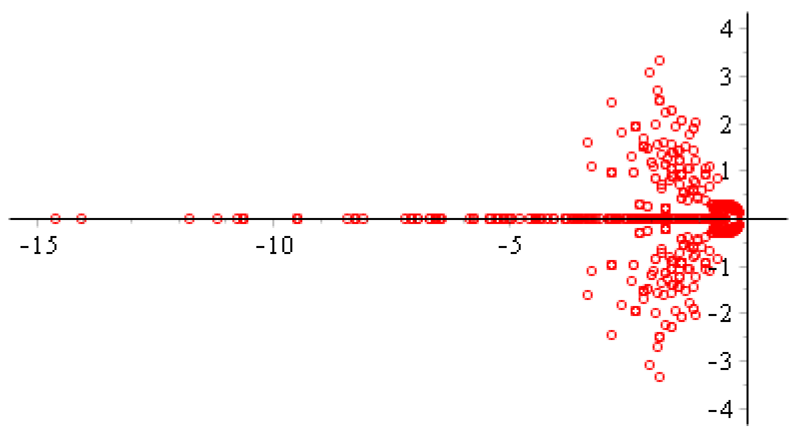

Figure 3.1: Roots of $\nu_{-K_3}(G, x)$ for all G such that $|V(G)| = 7$

We start this chapter with an example of locating the roots of $\mathscr{P}$-generating polynomials to give a feel of the problem and how limiting processes arise. Consider the graph $G = K_{n_1,n_2,\ldots,n_k}$ $k \geq 3$. Any subgraph of $K_{n_1,n_2,\ldots,n_k}$ that contains vertices from more than 2 of the partite sets will contain a triangle. So

$$\begin{aligned}\nu_{-K_3}(K_{n_1,n_2,\ldots,n_k}, x) &= 1 + \sum_{i=1}^{k}(\nu_{-K_3}(\overline{K_{n_i}}, x) - 1) + \sum_{i<j}(\nu_{-K_3}(K_{n_i,n_j}, x) - 1) \\ &= 1 + \sum_{i=1}^{k}((1+x)^{n_i} - 1) + \sum_{i<j}\Big(((1+x)^{n_i} - 1)((1+x)^{n_j} - 1)\Big)\end{aligned}$$

In general, the roots of $\nu_{-K_3}(K_{n_1,\ldots,n_k}, x)$ will be difficult to find, but in the case where $n_1 = n_2 = \ldots = n_k = n$ the roots are attainable by elementary methods.

Let G be the complete regular k-partite graph $(k \geq 3)$ $K_{n,n,\ldots,n}$. From the formula above, we obtain:

$$\nu_{-K_3}(K_{n,n,\ldots,n}, x) = 1 + k((1+x)^n - 1) + \binom{k}{2}((1+x)^n) - 1)^2 \qquad (3.1)$$

This is a quadratic in $((1+x)^n - 1)$, so applying the quadratic formula gives the following expression for the roots:

$$(1+x)^n = \frac{k-2}{k-1} \pm i\frac{\sqrt{k^2 - 2k}}{k(k-1)}.$$

Now, we find all possible values for x by taking the n-th roots which gives,

$$x = r^{\frac{1}{n}} \exp\left(i\left(\frac{\theta + 2l\pi}{n}\right)\right) - 1,$$

for $l = 0, 1, 2, ...n-1$, where $r = \sqrt{1-\frac{2}{k}}$ and $\theta = \arctan(\frac{\sqrt{1-\frac{2}{k}}}{k-2})$. These roots will be circles of radius $r^{\frac{1}{n}}$ centred at $Re(z) = -1$ in the complex plane. Moreover, we claim that if Roots(n) is the set of limit points of the roots of $\nu_{-K_3}(K_{n,n,\ldots,n}, x)$, then

$$\lim_{n\to\infty} \text{Roots}(n) = \{z \in \mathbb{C} : \ |z-1| = 1\}.$$

To prove this, we will make use of a powerful theorem due to Bereha et al. [5] which was later extended by Brown and Hickman [13] that will leave us with only elementary arguments to complete the proof.

Theorem 3.0.13 *[5, 13] For a family of functions $\{f_n(x) : n \in \mathbb{N}\}$ of the form $f_n(x) = \sum_{i=1}^{k} \alpha_i(x)\lambda_i(x)^n$ such that no $\alpha_i(x)$ is identically zero and for no pair $i \neq j$ does $\lambda_i(x) = \omega\lambda_i(x)$ for some $\omega \in \mathbb{C}$ with $|\omega| = 1$, then $z \in \mathbb{C}$ is a limit of the roots ($z \in \lim_{n\to\infty}$ Roots(f_n)) if and only if either:*

(i) two or more of the $\lambda_i(z)$ are of equal modulus, and strictly greater (in modulus) than the others; or

(ii) for some j , $\lambda_j(z)$ has modulus strictly greater than all the other $\lambda_i(z)$, and $\alpha_j(z) = 0$

In order to apply Theorem 3.0.13, we must return to (3.1) and simply the expression further to obtain:

$$\nu_{-K_3}(K_{n,n,\ldots,n}, x) = 1 + k(1+x)^n - k + \binom{k}{2}((1+x)^{2n} - 2(1+x)^n + 1) \tag{3.2}$$

$$= \left(1 + \binom{k}{2} - k\right)1^n + \left(k - 2\binom{k}{2}\right)(1+x)^n + \binom{k}{2}\left((1+x)^2\right)^n \tag{3.3}$$

From (3.3), we see that with $\alpha_1(x) = 1 + \binom{k}{2} - k$, $\lambda_1(x) = 1$, $\alpha_2(x) = k - 2\binom{k}{2}$, $\lambda_2(x) = 1+x$, $\alpha_3(x) = \binom{k}{2}$, and $\lambda_3(x) = (1+x)^2$, the polynomial is now in the form of the polynomial in the hypothesis of Theorem 3.0.13. To satisfy the hypothesis of the theorem, we must have $k \geq 3$ which it is assumed to be and is no loss of information since for $k \leq 2$, the polynomial is simply $(1+x)^{kn}$ which has no mystery surrounding its roots. Now, since $k \geq 3$, we see that

$$\begin{aligned}\alpha_1(x) &= 1 + \frac{k^2 - 3k}{2} \\ &\geq 1 + 0 = 1\end{aligned}$$

$$\begin{aligned}\alpha_2(x) &= k - k^2 - k \\ &= -k^2 \\ &\leq -9\end{aligned}$$

$$\begin{aligned}\alpha_3(x) &= \binom{k}{2} \\ &\geq 3.\end{aligned}$$

So no $\alpha_i(x)$ is identically zero. Also, there exists no $\omega \in \mathbb{C}$ with $|\omega| = 1$ such that for $i \neq j$, $\lambda_i(x) = \omega\lambda_j(x)$, therefore we may apply Theorem 3.0.13. Note that the $\alpha_i(x)$'s are all constant in our case, and we just showed that none are identically zero, thus $\alpha_i(z) \neq 0$ for all $z \in \mathbb{C}$ and $i = 1, 2, 3$. Therefore (ii) can never be satisfied, so we focus our efforts to determining when (i) is satisfied. This leads to 4 possible cases:

Case 1: $|1| = |1 + x| > |(1 + x)^2|$. This is never satisfied as $|1| = |1 + x|$ implies $|(1 + x)^2| = |1 + x|^2 = 1$.

Case 2: $|1| = |(1 + x)^2| > |1 + x|$. By the same argument as Case 1, this can never be satisfied.

Case 3: $|1+x| = |(1+x)^2| > 1$. Here, if $|1+x| = |(1+x)^2| = |1+x|^2 > 1$, then it follows by dividing $|1+x|$ from both sides that $|1+x| = 1$.

Case 4: $|1| = |1+x| = |(1+x)^2|$. This case is satisfied exactly when $1 = |1+x|$.

So by Theorem 3.0.13, $\lim_{n\to\infty} Roots(n) = \{z \in \mathbb{C} : |z+1| = 1\}$, the circle of unit radius centred at $z = -1$.

We now look at the specific example of complete tripartite graphs $G = K_{n,n,n}$. We see, from above, that

$$\nu_{-K_3}(K_{n,n,n}, x) = 1 + 3((1+x)^n - 1) + 3((1+x)^n - 1)^2.$$

By the above comments, the roots of $\nu_{-K_3}(K_{n,n,n}, x)$ are

$$x = \left(\frac{1}{\sqrt{3}}\right)^{\frac{1}{n}} \mathrm{e}^{\pm i\left(\frac{\pi}{6n} + \frac{2k\pi}{n}\right)} - 1 \quad \text{for } k = 0, 1, 2, ..., n-1.$$

So for every integer $n \geq 1$, the roots of $\nu_{-K_3}(K_{n,n,n}, x)$ lie on the circle in the complex plane

$$|x+1| = \left(\frac{1}{\sqrt{3}}\right)^{\frac{1}{n}}.$$

As $n \to \infty$, the radius increases to 1. Below is a plot of the roots of the $\nu_{-K_3}(K_{n,n,n}, x)$ for $n = 1, 2, ..., 20$ and the limiting circle:

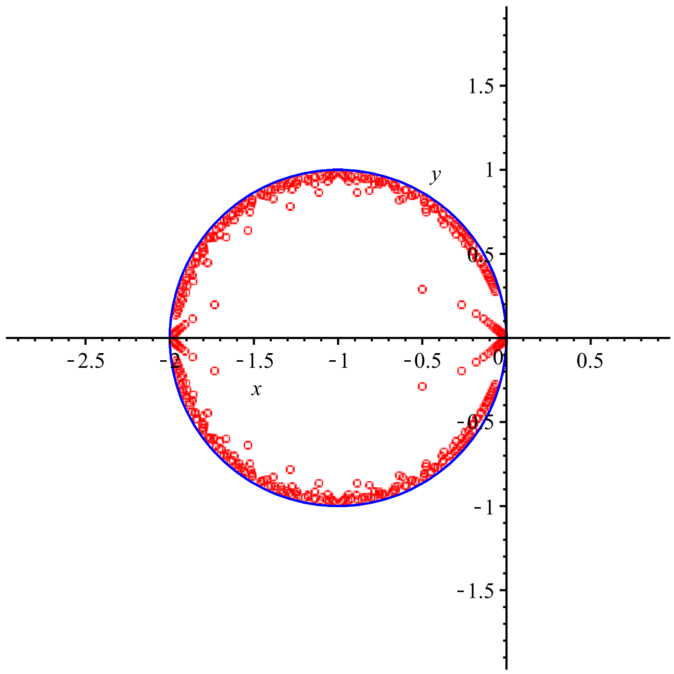

Figure 3.2: Roots of $\nu_{-K_3}(K_{n,n,n}, x)$ for $n = 1$ to $n = 20$

3.1 $\mathscr{P}$-generating Polynomials with Only Real Roots

We now look to make more general comments about the roots of $\mathscr{P}$-generating polynomials. If $\mathscr{P}$ is a hereditary property and G has property $\mathscr{P}$, then

$$\nu_{\mathscr{P}}(G, x) = \sum_{i=0}^{n} \binom{n}{i} x^i = (1 + x)^n$$

.

In this case it is obvious that -1 is the only root. For example, the triangle-free subgraph generating polynomial of the complete bipartite graph, $K_{n,m}$ is simply $(1 + x)^{m+n}$. So for every hereditary property $\mathscr{P}$, there exists a graph G such that $\nu_{\mathscr{P}}(G, x)$ has only real roots. A natural question that arises from this observation is, for which hereditary properties $\mathscr{P}$ does $\nu_{\mathscr{P}}(G, x)$ have <u>all</u> real roots for <u>all</u> graphs G? We can completely answer this question.

Theorem 3.1.1 *For a hereditary property $\mathscr{P}$, $\nu_{\mathscr{P}}(G,x)$ has only real roots for all graphs G if and only if $\mathscr{P} = \mathscr{G}$ or $\mathscr{P} = -K_2 \cap -\overline{K_2}$.*

Proof. For $\mathscr{P} = -K_2 \cap -\overline{K_2}$, $\nu_{\mathscr{P}}(G,x) = 1 + (|V(G)|)\, x$ where G is any graph. In this case all roots of $\nu_{\mathscr{P}}(G,x)$ are real. Now for $\mathscr{P} = \mathscr{G}$, $\nu_{\mathscr{P}}(G,x) = (1+x)^{|V(G)|}$ for all graphs G, which has all real roots.

Now suppose $\mathscr{P} \neq -K_2 \cap -\overline{K_2}$ and $\mathscr{P} \neq \mathscr{G}$. Let $\mathscr{P} = \bigcap_{i\in I} -G_i$ where $I \subseteq \mathbb{N}$. Let $-G_k \in \mathscr{P}$ such that $|V(G_k)|$ is minimum.

Since $|V(G_k)|$ is minimum with respect to $\mathscr{P}$, every proper subgraph of G_k has property $\mathscr{P}$. Let $|V(G_k)| = n$ and consider $\nu_{\mathscr{P}}(G_k,x)$.

$$\nu_{\mathscr{P}}(G_k,x) = \sum_{i=0}^{n-1} \binom{n}{i} x^i = (1+x)^n - x^n.$$

Case 1: $|V(G_k)| \geq 3$

Now,

$$\begin{aligned}
(1+x)^n - x^n &= 0 \\
\left(\frac{x}{1+x}\right)^n &= 1 \\
\frac{x}{1+x} &= \omega \qquad\qquad \text{where } \omega \text{ is an } n\text{th-root of unity } (\omega \neq 1) \\
\frac{\omega}{1-\omega} &= x. \quad (*)
\end{aligned}$$

Let $a,\ b \in \mathbb{R}$ such that $\omega = a + ib$. Therefore $a^2 + b^2 = 1$. So from (*), we obtain:

$$\begin{aligned} x &= \frac{a+ib}{(1-a)-ib} \\ &= \frac{a+ib}{(1-a)-ib} \cdot \frac{(1-a)+ib}{(1-a)+ib} \\ &= \frac{a-a^2-b^2+ib(a+(1-a))}{a^2-2a+1+b^2} \\ &= -\frac{1}{2} + \frac{b}{2(1-a)}i \end{aligned}$$

So all roots of $\nu_{\mathscr{P}}(G_k, x)$ lie on the line $Re(x) = -\frac{1}{2}$ in the complex plane. The roots of $\nu_{\mathscr{P}}(G_k, x)$ are all real if and only if $b = 0$ for all choices of ω, but for $n \geq 3$, there are choices of ω for which $b = Im(\omega) \neq 0$. We have shown that if $|V(G_k)| \geq 3$, then $\nu_{\mathscr{P}}(G_k, x)$ has a nonreal root.

Case 2: $G_k = K_2$ or $G_k = \overline{K_2}$ and $\mathscr{P} \neq -K_2 \cap -\overline{K_2}$.

Suppose $G_k = K_2$. If $\mathscr{P} = -K_2$, then, $\overline{\mathscr{P}} = -\overline{K_2}$. Consider the graph $K_{3,3}$, we see that $\nu_{\mathscr{P}}(K_{3,3}, x) = \nu_{-K_2}(\overline{K_3}+\overline{K_3}, x) = 2((1+x)^3)-1$ by Proposition 2.1.4, which has nonreal roots. And by Observation 1.3.2, $\nu_{\overline{\mathscr{P}}}(\overline{K_{3,3}}, x) = \nu_{\mathscr{P}}(K_{3,3}, x)$ which was just shown to have a non-real root.

Now, if $\mathscr{P} \neq -K_2$, then it must be the case that $\mathscr{P} = -K_2 \cap -\overline{K_j}$, $j \in \mathbb{N}$, since all G_i's containing an edge will be forbidden by $-K_2$. Since $\mathscr{P} \neq -K_2 \cap -\overline{K_2}$, it follows that $j \geq 3$. But now,

$$\nu_{\mathscr{P}}(\overline{K_j}, x) = \sum_{i=0}^{j-1} \binom{j}{i} x^i = (1+x)^j - x^j,$$

which has a non-real root by the argument for Case 1. And similarly, if $\mathscr{P} \neq -\overline{K_2}$, then $\mathscr{P} = -\overline{K_2} \cap K_j$ for some $j \geq 3$ and we obtain $\nu_{\mathscr{P}}(K_j, x)$ having a non-real root.

Case 2 ensures that if $G_k = K_2$ or $G_k = \overline{K_2}$ and $\mathscr{P} \neq -K_2 \cap -\overline{K_2}$, then there exists a graph such that its $\mathscr{P}$-generating polynomial has a nonreal root.

Since $\{K_0, K_1\} \subseteq \mathscr{P}$ for all graph properties by definition, we have considered all hereditary properties and thus the proof is complete.

□

3.2 Background on Iteration Theory

Now that we have answered some general questions about the nature of the roots of $\nu_{\mathscr{P}}(G,x)$ we will delve deeper into the study of the roots for certain properties. The properties that we will consider are properties that are closed under substitution and the graphs that we will consider are k-fold lexicographic products of graphs with themselves. In doing this we may use Theorem 2.2.7 to describe what happens to the roots as k gets large. This will involve extensive use of results from iteration theory, so we give some background on the notation and theory.

We will be working exclusively with polynomials, so we consider the field of complex numbers $\mathbb{C}$ together with the usual modulus metric, $|\cdot|$. While we develop most of the necessary theory here, any other relevant background can be found in [4, 8]. For us, f will always denote a polynomial with real (integer, in fact) coefficients, but much of the theory is not limited to polynomials. We let $f^{\circ k}$ denote the map obtained by the k-fold composition of f with itself, $f^{\circ 0}$ be the identity map, and $f^{\circ(-1)}$ be the set-valued inverse of f, i.e. $f^{\circ(-1)}(z) = \{w \in \mathbb{C} : f(w) = z\}$, with $f^{\circ(-k)}$ being the k-fold composition of $f^{\circ(-1)}$ with itself. For a set $S \subseteq \mathbb{C}$, we let $f(S) = \{f(s) : s \in S\}$

and $f^{\circ(-1)}(S) = \{w \in \mathbb{C} : f(w) \in S\}$.

For a point $z_0 \in \mathbb{C}$, the **forward orbit** with respect to a function f is the set, $\mathcal{O}^+(z_0) = \{f^{\circ k}(z_0)\}_{k=0}^{\infty}$ and the **backward orbit** is the set $\mathcal{O}^-(z_0) = \bigcup_{k=0}^{\infty} f^{\circ -k}(z_0)$. Much of iteration theory, and almost all of what we will require, arises from studying the forward and backward orbits of complex numbers. The study of forward orbits of different points for specific polynomials leads to the following very important sets.

Definition 3.2.1 *[4] If f is a polynomial, the **filled Julia set**, denoted $K(f)$ is the set of all points in $\mathbb{C}$ that have bounded (in modulus) forward orbits. The **Julia set** of f, denoted $J(f)$, is the boundary of $K(f)$, i.e. $J(f) = \partial K(f)$. The **Fatou set** of f, denoted $F(f)$ is the complement of $J(f)$ in $\mathbb{C}$.*

It is important to note that $F(f)$ is an open subset of $(\mathbb{C}, |\cdot|)$ and $J(f)$ is compact [4].

An **exceptional point** of a function is a point in which the backward orbit is finite. Polynomials have at most one exceptional point and if one exists it must lie in the Fatou set. The following result presented by Beardon [4] shows that the backward orbits of non-exceptional points contain the Julia set and will be useful later on.

Theorem 3.2.2 *[4] If f is a polynomial of degree at least 2, then*
(i) if z_0 is non-exceptional, then $J(f) \subseteq Cl(\mathcal{O}^-(z_0))$, and
(ii) if $z_0 \in J(f)$, then $J(f) = Cl(\mathcal{O}^-(z_0))$.

($Cl(A)$ denotes the topological closure of the set A, i.e. A together with its limit points.)

A point $z_0 \in \mathbb{C}$ is called a **periodic point** of a polynomial f if there exists a positive integer k such that $f^{\circ k}(z_0) = z_0$. If $k = 1$, z_0 is called a **fixed point** of

f. The least such k, provided one exists, is the **period** of z_0. Forward orbits of periodic points are called **cycles** of f and properties of cycles will play an important role in developing our theory. For a periodic point z_0 with period k, the **multiplier** of the cycle is the number $\lambda = \left(f^{\circ k}\right)'(z_0)$. The multiplier of the cycle provides a characterization of cycles into four types:

(i) **Attracting** cycles have $0 < |\lambda| < 1$ (if $|\lambda| = 0$ the cycle is called **super attracting**),

(ii) **Repelling** cycles have $|\lambda| > 1$,

(iii) **Rationally indifferent** cycles have λ a root of unity,

(iv) **Irrationally indifferent** cycles have $|\lambda| = 1$ but not a root of unity.

Attracting cycles are contained in $F(f)$, rationally indifferent and repelling cycles are contained in $J(f)$, the latter being dense in $J(f)$, and irrationally indifferent cycles may be contained in either $F(f)$ or $J(f)$. These facts are basic but nontrivial and the details may be found in [4].

We will need to determine whether the sets $f^{\circ -k}(z_0)$ converge. For convergence we will use the Hausdorff metric which measures the distance between two compact subsets of a metric space (M, d) in the following manner, if A and B are compact subsets of (M, d), then the Hausdorff distance between A and B is given by

$$h(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\}.$$

As we will only be working with the metric space $(\mathbb{C}, |\cdot|)$,

$$h(A, B) = \max\{\max_{a \in A} \min_{b \in B} |a - b|, \max_{b \in B} \min_{a \in A} |a - b|\}.$$

Note that we may use max and min rather than sup and inf since the sets are compact. Using the Hausdorff metric will allow us to take advantage of a result due to Hickman [28] that will be valuable to us, but we must first present one more definition.

Definition 3.2.3 *[4] A **Siegel disk** is the maximal open connected subset of $F(f)$ containing a fixed point z_0 with an irrationally indifferent cycle.*

Siegel disks lie in the Fatou set of f and do not need to be discussed beyond this fact for our work. As in [15], we note that since attracting cycles are also contained in $F(f)$ which is disjoint from $J(f)$, $f^{\circ(-k)}(z_0) \to J(F)$ as $k \to \infty$ for all $z_0 \in J(f)$.

Theorem 3.2.4 *[28] Let f be a polynomial and z_0 be a point which does not lie in any attracting cycle or Siegel disk of f. Then*

$$\lim_{k\to\infty} f^{\circ -k}(z_0) = J(f),$$

where the limit is taken with respect to the Hausdorff metric on compact subsets of $(\mathbb{C}, |\cdot|)$.

Note that $f^{\circ(-k)}(z_0)$ is finite for all k and therefore compact.

3.3 The $\mathscr{P}$-fractal of a Graph

In this section we look at the roots of $\nu_{\mathscr{P}}(G, x)$ where G can be obtained by means of graph substitution of smaller graphs. We will rely heavily on Theorem 2.2.7 and are motivated and guided by the work of Brown, Hickman, and Nowakowski [14, 15]. In [15], the independence fractal of a graph was introduced and much of the work needed to establish the theory behind the independence fractal will generalize to $\mathscr{P}$-generating polynomials for properties $\mathscr{P}$ closed under substitution. This leads to the $\mathscr{P}$-fractal of a graph, and hence associates with each graph many fractals. We define the *reduced $\mathscr{P}$-generating polynomial* of a graph G, denoted $f_{\mathscr{P}}(G,x)$, by $f_{\mathscr{P}}(G,x) = \nu_{\mathscr{P}}(G,x) - 1$. As each graph has only one subgraph on 0 vertices, namely the empty graph, the reduced $\mathscr{P}$-generating polynomial really has the same amount of information as the $\mathscr{P}$-generating polynomial. The advantage of the reduced version is that $f_{\mathscr{P}}(G^k,x) = f_{\mathscr{P}}^{\circ k}(G,x)$ for $\mathscr{P}$ closed under substitution, where $f_{\mathscr{P}}^{\circ k}$ denotes the k-fold composition of $f_{\mathscr{P}}$ with itself (recall that G^k is the k-fold lexicographic product of G with itself). This follows easily from Theorem 2.2.7 and gives us an easier way to characterize the roots of $f_{\mathscr{P}}(G^k,x)$ as $k \to \infty$. We will see later that the limiting root set of $\nu_{\mathscr{P}}(G^k,x)$ will usually be the same as that of $f_{\mathscr{P}}(G^k,x)$ and when it is not, the limiting root set of $f_{\mathscr{P}}(G^k,x)$ is the more interesting object anyway.

We are now ready to develop the theory of the $\mathscr{P}$-fractal of a graph for properties $\mathscr{P}$ closed under substitution. It will be given as a limit of the roots of $f_{\mathscr{P}}(G^k,x)$ as k tends to infinity. Let the set $\text{Roots}(f_{\mathscr{P}}(G,x))$ denote the set of all roots of $f_{\mathscr{P}}(G,x)$. We wish to show that $\text{Roots}\big(f_{\mathscr{P}}^{\circ k}(G,x)\big) \subseteq \text{Roots}\big(f_{\mathscr{P}}^{\circ k+1}(G,x)\big)$ for all $k \geq 1$. Note that $f_{\mathscr{P}}(G,0) = 0$ and $f_{\mathscr{P}}(G^2,x) = f_{\mathscr{P}}(G, f_{\mathscr{P}}(G,x))$ and therefore $\text{Roots}(f_{\mathscr{P}}(G,x)) \subseteq \text{Roots}(f_{\mathscr{P}}^{\circ 2}(G,x))$. Now suppose

$$\text{Roots}\left(f_{\mathscr{P}}^{\circ k}(G,x)\right) \subseteq \text{Roots}\left(f_{\mathscr{P}}^{\circ k+1}(G,x)\right)$$

for all $k \leq n-1$. Recall $f_{\mathscr{P}}^{\circ n+1}(G,x) = f_{\mathscr{P}}(G, f_{\mathscr{P}}^{\circ n}(G,x))$. If $r \in \text{Roots}(f_{\mathscr{P}}^{\circ n}(G,x))$, then

$$\begin{aligned} f_{\mathscr{P}}^{\circ n+1}(G,r) &= f_{\mathscr{P}}(G, f_{\mathscr{P}}^{\circ n}(G,r)) \\ &= f_{\mathscr{P}}(G,0) \\ &= 0. \end{aligned}$$

So $r \in \text{Roots}(f_{\mathscr{P}}^{\circ n+1}(G,x))$. It follows that $\text{Roots}(f_{\mathscr{P}}^{\circ n}(G,x)) \subseteq \text{Roots}\big(f_{\mathscr{P}}^{\circ n+1}(G,x)\big)$ and so by induction, $\text{Roots}\big(f_{\mathscr{P}}^{\circ k}(G,x)\big) \subseteq \text{Roots}\big(f_{\mathscr{P}}^{\circ k+1}(G,x)\big)$ for all $k \geq 1$.

Definition 3.3.1 *For a property $\mathscr{P}$ that is closed under substitution, the* ***$\mathscr{P}$-fractal*** *of a graph G is the set,*

$$\mathcal{F}(G,\mathscr{P}) = \lim_{k\to\infty} \textit{Roots}(f_{\mathscr{P}}(G^k,x)).$$

It is reasonable to suspect that Definition 3.3.1 may not be meaningful as the convergence of the limit is not obvious. Theorem 3.3.2 ensures that the definition is meaningful and that the object defined is indeed fractal-like. From Theorem 2.2.7 and the fact that $\text{Roots}\big(f_{\mathscr{P}}^{\circ k}(G,x)\big) \subseteq \text{Roots}\big(f_{\mathscr{P}}^{\circ k+1}(G,x)\big)$ for all $k \geq 1$, it follows that

$$\mathcal{F}(G,\mathscr{P}) = \lim_{k\to\infty} f_{\mathscr{P}}^{\circ(-k)}(G,0).$$

For K_1 the situation is that, $f_{\mathscr{P}}(K_1,x) = x$ for all properties $\mathscr{P}$,(recall $\{K_0,K_1\} \subseteq \mathscr{P}$ by definition), and therefore, $\mathcal{F}(K_1,\mathscr{P}) = \{0\}$.

Theorem 3.3.2 *For a graph $G \neq K_1$ and property $\mathscr{P}$ that is closed under substitution, the $\mathscr{P}$-fractal of G is exactly $J(f_{\mathscr{P}}(G,x))$.*

Proof. If $\deg(f_{\mathscr{P}}(G,x)) = 1$, then $f_{\mathscr{P}}(G,x) = nx$ where $n = |V(G)| \geq 2$, since $K_1 \in \mathscr{P}$. By Theorem 2.2.7, $f_{\mathscr{P}}(G^k,x) = n^k x$. Now all nonzero numbers have unbounded forward orbit, so $J(f_{\mathscr{P}}(G,x)) = \{0\}$ which is clearly equal to $\mathcal{F}(G,\mathscr{P})$.

If $\deg(f_{\mathscr{P}}(G,x)) \geq 2$, then 0 is a fixed point of $f_{\mathscr{P}}(G,x)$ with multiplier $f'_{\mathscr{P}}(G,0) = n \geq 2$ and so 0 is a repelling fixed point of $f_{\mathscr{P}}(G,x)$. Hence $0 \in J(f_{\mathscr{P}}(G,x))$ by the comments following the definition of a cycle and so we may apply Theorem 3.2.4 to obtain

$$\lim_{k\to\infty} f_{\mathscr{P}}^{\circ(-k)}(G,0) = J(f_{\mathscr{P}}(G,x)).$$

From earlier comments, $\lim_{k\to\infty} f_{\mathscr{P}}^{\circ(-k)}(G,0) = \mathcal{F}(G,\mathscr{P})$ which completes the proof. □

What we have been concerned with for the majority of this work is $\nu_{\mathscr{P}}(G,x)$ and so we draw the connection between the $\mathscr{P}$-fractal of a graph and the set

$$\mathcal{R}(G,\mathscr{P}) = \lim_{k\to\infty} \text{Roots}\left(\nu_{\mathscr{P}}(G^k,x)\right).$$

Note that $\nu_{\mathscr{P}}(G,x) = f_{\mathscr{P}}(G,x) + 1$ and so

$$\lim_{k\to\infty} \text{Roots}\left(\nu_{\mathscr{P}}(G^k,x)\right) = \lim_{k\to\infty} f_{\mathscr{P}}^{\circ(-k)}(G,-1).$$

We state Theorem 3.2.14 in Hickman's PhD. Thesis [28] regarding the sets $\mathscr{F}(G) = \lim\limits_{k\to\infty} \text{Roots}(I(G,x))$ and $\mathcal{I}(G) = \lim\limits_{k\to\infty} \text{Roots}(I(G,x)-1)$ which are referred to as the independence attractor and independence fractal respectively, as it will help us bridge the gap.

Theorem 3.3.3 *[28] Let G be a graph with at least one edge, and denote by $\eta(G)$ the multiplicity of -1 as a root of $I(G,x)$. Set $f_G = I(G,x) - 1$.*

(i) If $\eta(G) \leq 1$, then $\mathcal{I}(G) = J(f_G)$, the Julia set of f_G.

(ii) If $\eta(G) \geq 1$, then

$$\mathcal{I}(G) = Cl\left(\bigcup_{k\geq 1} Roots((I(G^k,x))\right) = Cl\left(\bigcup_{k\geq 1} f_G^{\circ(-k)}(-1)\right).$$

In case (ii), $I(G^k,x)$ is divisible by $\left(I(G^{k-1},x)\right)^{\eta(G)}$ for each $k \geq 2$, and

$$\lim_{k\to\infty}\left((Roots(I(G^k,x)) \setminus Roots(I(G^{k-1},x))\right) = \lim_{k\to\infty} Roots\left(\frac{I(G^k,x)}{\left(I(G^{k-1},x)\right)^{\eta(G)}}\right) = J(f_G).$$

Further, for $\eta(G) > 1$, $\mathcal{I}(G)$ is partitioned by the set, $\bigcup_{k\geq 1} Roots(I(G^k,x))$, and its accumulation points, $J(f_G)$.

This Theorem is proved in enough generality to hold for $\mathcal{R}(G,\mathscr{P})$ for any property $\mathscr{P}$ by simply changing the terms specific to independence. What this theorem tells us, in our terminology, is that for a graph G such that $G \notin \mathscr{P}$, if η is the multiplicity of -1 as a root of $\nu_{\mathscr{P}}(G,x)$, then $\mathcal{R}(G,\mathscr{P}) = J(f_{\mathscr{P}}(G,x))$ if $\eta \leq 1$ and $J(f_{\mathscr{P}}(G,x)) \subseteq \mathcal{R}(G,\mathscr{P})$ if $\eta \geq 2$. It further says that in the case where $\eta \geq 2$, Roots$(\nu_{\mathscr{P}}(G^k,x))$ $\subseteq$Roots$(\nu_{\mathscr{P}}(G^{k+1},x))$ and that as $k \to \infty$,

$$\text{Roots}(\nu_{\mathscr{P}}(G^{k+1},x)) \setminus \text{Roots}(\nu_{\mathscr{P}}(G^k,x)) \to J(f_{\mathscr{P}}(G,x)).$$

In fact, $\mathcal{R}(G,\mathscr{P}) = Cl(\bigcup_{k\geq 1}\text{Roots}(\nu_{\mathscr{P}}(G^k,x)))$. Hickman's proof of the theorem involves some technical details which we will not include since the Julia set of $f_{\mathscr{P}}(G,x)$ is what we end up interested in and we have found a connection with the roots of the reduced $\mathscr{P}$-generating polynomial and this set.

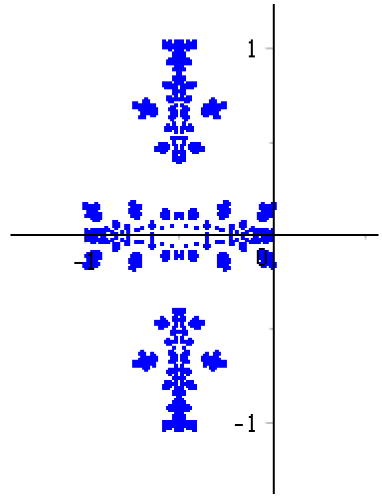

Figure 3.3: Approximation of $\mathcal{F}(P_7, -P_7)$

As Brown et al. [15] consider for $\mathscr{P} = -K_2$, we wish to know for which graphs and properties is $\mathcal{F}(G, \mathscr{P})$ connected and for which is it disconnected. Other questions that arise are about the relations between $\mathscr{P}$-fractals for a graph G for different properties $\mathscr{P}$. Are there different properties, $\mathscr{P}$ and $\mathscr{Q}$, for which $\mathcal{F}(G, \mathscr{P}) = \mathcal{F}(G, \mathscr{Q})$? Are there different graphs G and H for which $\mathcal{F}(G, \mathscr{P}) = \mathcal{F}(H, \mathscr{P})$ for certain properties $\mathscr{P}$?

3.3.1 Connectivity of $\mathscr{P}$-fractal

A **separation** of a set S is a pair R, T of disjoint nonempty open subsets of S such that $R \cup T = S$ [36]. If there exists a separation the set is **disconnected** and **connected** if no separation exists [36]. We say S is **totally disconnected** if the only connected subsets of S are one-point sets [36]. Beardon [4] gives a complete characterization of when the Julia set of a polynomial of degree at least 2 is connected and when it is totally disconnected.

Theorem 3.3.4 *[4] If f is a polynomial of degree at least 2, then*

(i) $J(f)$ is connected if and only if $\mathcal{O}^+(z_0)$ is bounded in modulus for all critical points z_0,

(ii) $J(f)$ is totally disconnected if $\mathcal{O}^+(z_0)$ is unbounded in modulus for each of its critical points z_0.

For example, $f_{-C_5}(C_5, x) = 5x + 10x^2 + 10x^3 + 5x^4$ is an unbounded increasing function that lies above the x-axis (when plotted on $\mathbb{R} \times \mathbb{R}$) from $[1, +\infty)$, so to show that the forward orbit of a point z_0 is unbounded we must only show that for some $k \in \mathbb{N}$, $f^{\circ k}_{-C_5}(C_5, z_0)$ is a positive real number greater than 1. The critical points of $f_{-C_5}(C_5, x)$ can be easily verified to be $-\frac{1}{2} + \frac{i}{2}$, $-\frac{1}{2} - \frac{i}{2}$, and $-\frac{1}{2}$. Now, $f^{\circ 2}_{-C_5}\left(C_5, -\dfrac{1}{2} - \dfrac{i}{2}\right) = f^{\circ 2}_{-C_5}\left(C_5, -\dfrac{1}{2} + \dfrac{i}{2}\right) = \dfrac{525}{256}$ which is a rational number larger than 1, so $O^+(-\frac{1}{2} + \frac{i}{2})$ and $O^+(-\frac{1}{2} - \frac{i}{2})$ are unbounded. Therefore, Theorem 3.3.4 assures us that $\mathcal{F}(C_5, -C_5)$ is disconnected. In fact, it is totally disconnected as $f^{\circ 5}_{-C_5}(C_5, -\frac{1}{2}) \approx 9013$ and so the forward orbits of all critical points of $f_{-C_5}(C_5, x)$ are unbounded. $\mathcal{F}(C_5, -C_5)$ can be seen in Figure 3.4, it is easy to see that it is disconnected, but without Theorem 3.3.4, it would be difficult to say for certain that it is actually totally disconnected.

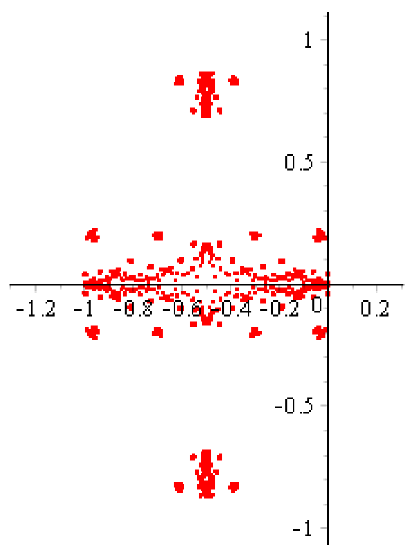

Figure 3.4: Approximation of $\mathcal{F}(C_5, -C_5)$ in $\mathbb{C}$.

On the other hand, $\mathcal{F}(P_3, -K_2)$ was shown in [15] to be connected as can be seen in Figure 3.5.

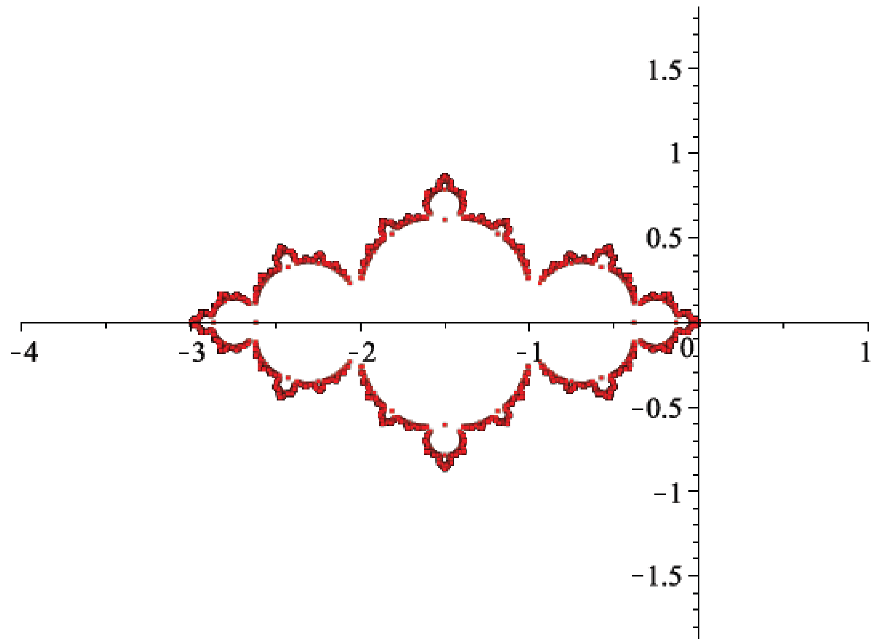

Figure 3.5: Approximation of $\mathcal{F}(P_3, -K_2)$ in $\mathbb{C}$.

3.3.2 Comparing $\mathscr{P}$-fractals

In this section we wish to compare $\mathscr{P}$-fractals for different graphs, and for different properties the different fractals that can be generated by the same graph. We know that for independence polynomials, nonisomorphic graphs can have equal independence polynomials [18]. For example, the graphs in Figure 3.6 are nonisomorphic but both have independence polynomial $1 + 4x + 2x^2$.

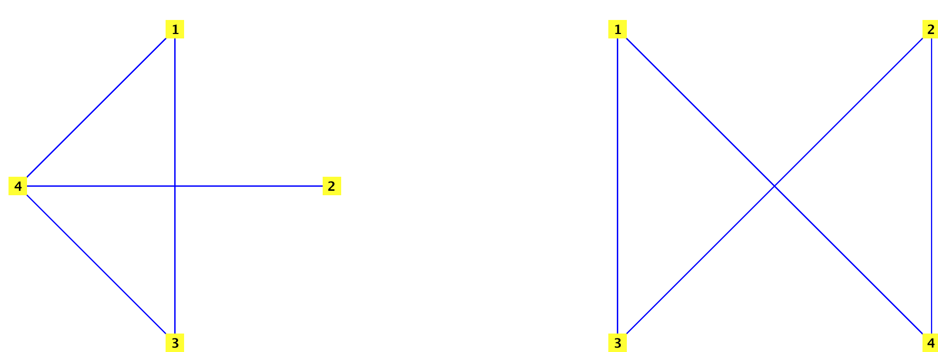

Figure 3.6: Graphs with equal independence polynomial

In our general case, for most properties $\mathscr{P}$ there are nonisomorphic graphs G and H on the same number of vertices such that $\nu_{\mathscr{P}}(G,x) = \nu_{\mathscr{P}}(H,x) = (1+x)^n$. For example, the graphs C_4 and K_4 are both $-K_5$-graphs and so both have $-K_5$-generating polynomial equal to $(1+x)^4$. However, this yields a very uninteresting $\mathscr{P}$-fractal. Recalling Observation 1.3.2, we can remark that for every graph G and property $\mathscr{P} = \bigcap_{n \in I} -G_n$ such that G_n is module-free for each n, then $\mathcal{F}(G,\mathscr{P}) = \mathcal{F}(\overline{G},\overline{\mathscr{P}})$. Moreover, if G_n is self-complementary for each n, then $\mathscr{P} = \overline{\mathscr{P}}$ and so $\mathcal{F}(G,\mathscr{P}) = \mathcal{F}(\overline{G},\mathscr{P})$. Thus, for every graph G, there exists a property $\mathscr{P}$ such that the $\mathscr{P}$-fractal is the same as the $\mathscr{P}$-fractal of $\overline{G}$. Some properties that satisfy being closed under substitution and self-complementary are $-P_4$, $-C_5$, and $-G$ where G is the graph in Figure 3.7.

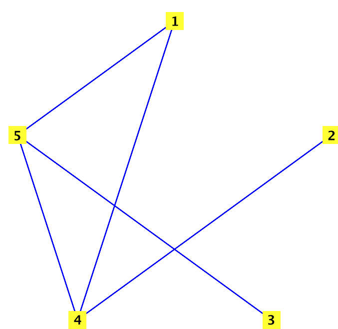

Figure 3.7: A self-complementary and module-free graph on 5 vertices

Consider the graph on 5 vertices, pictured in Figure 3.8.

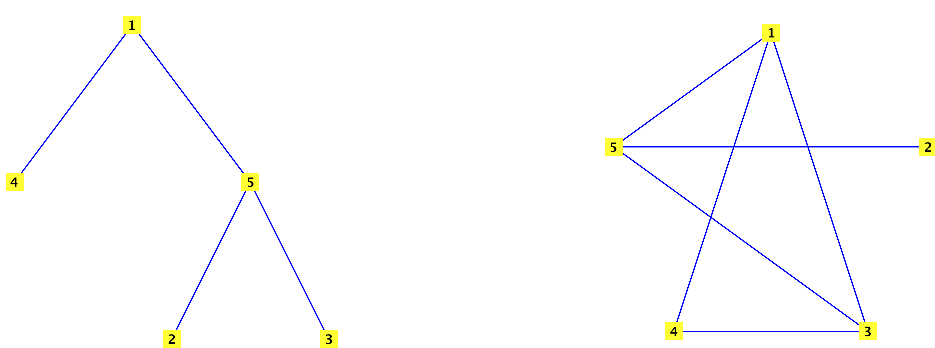

Figure 3.8: A graph and its complement on 5 vertices

If G is either graph in Figure 3.8, then $\nu_{-P_4}(G,x) = 1 + 5x + 10x^2 + 10x^3 + 3x^4$ which gives the $-P_4$-fractal that is pictured in Figure 3.9

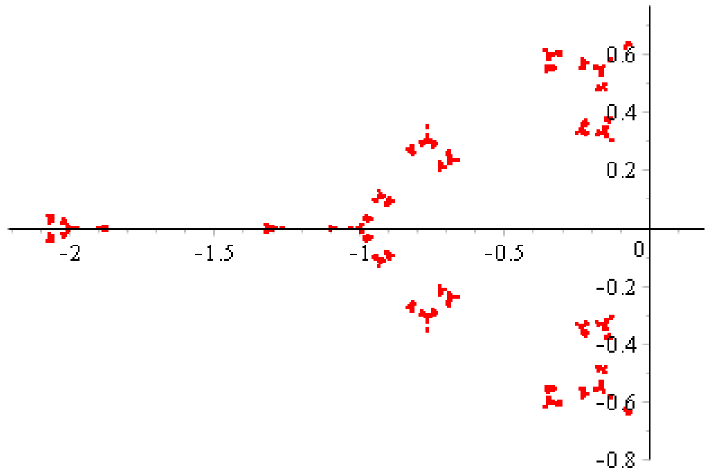

Figure 3.9: Approximation of $\mathcal{F}(G, -P_4)$ where G is either graph in Figure 3.8

The benefit of the $\mathscr{P}$-fractal theory is that for each graph we have associated many fractals. We consider the graph C_6 and the fractals associated with it for properties $-K_2$, $-\overline{K_2}$, $-P_4$, $-P_5$ and $-C_6$. The following figures illustrate these fractals.

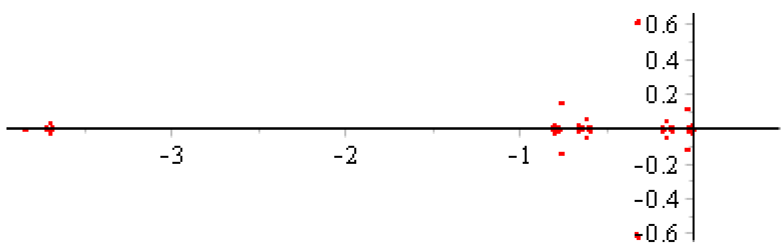

Figure 3.10: Approximation of $\mathcal{F}(C_6, -K_2)$ in $\mathbb{C}$

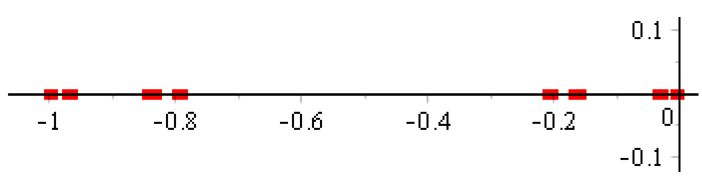

Figure 3.11: Approximation of $\mathcal{F}(C_6, -\overline{K_2})$ in $\mathbb{C}$

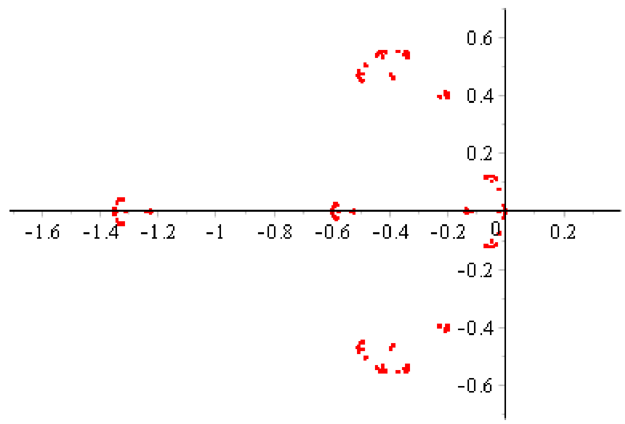

Figure 3.12: Approximation of $\mathcal{F}(C_6, -P_4)$ in $\mathbb{C}$

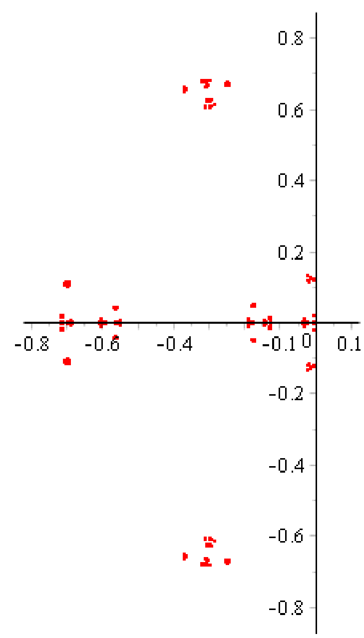

Figure 3.13: Approximation of $\mathcal{F}(C_6, -P_5)$ in $\mathbb{C}$

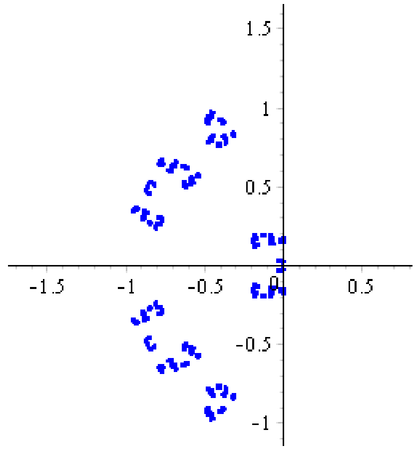

Figure 3.14: Approximation of $\mathcal{F}(C_6, -C_6)$ in $\mathbb{C}$

We have seen that the roots of $\nu_{\mathscr{P}}(G,x)$ can have nonzero imaginary part, but we wish to know if for certain properties closed under substitution $\mathscr{P}$ and graphs G, $\mathcal{F}(G,\mathscr{P}) \subseteq \mathbb{R}$. Since the coefficients of $\nu_{\mathscr{P}}(G,x)$ are all nonnegative integers, if $\mathcal{F}(G,\mathscr{P})$ is real, it will lie on the ray $(-\infty, 0]$. For the property $\mathscr{P} = -K_2 \cap -\overline{K_2}$, we have $\nu_{\mathscr{P}}(G,x) = 1 + nx$ where $n = |V(G)|$ for all graphs G. In this case, $\mathcal{F}(G,\mathscr{P}) = \{0\} = \mathcal{R}(G,\mathscr{P})$ and is of little interest. On the other hand, we know that for a graph with property $\mathscr{P}$, $\nu_{\mathscr{P}}(G,x) = (1+x)^n$ and so $\mathcal{R}(G,\mathscr{P}) = \{-1\}$ which is not interesting, but $f_{\mathscr{P}}(G^k,x) = (1+x)^{n^k} - 1$ and in [15, 28] it is shown that $\mathcal{F}(G,\mathscr{P})$ is the circle $|z+1| = 1$. We show that for each graph G on at least 4 vertices such that $\alpha(G) = 2$ or $\alpha(\overline{G}) = 2$ that there exists a property $\mathscr{P}$ such that $\mathcal{F}(G,\mathscr{P})$ lies entirely on the real line.

Proposition 3.3.5 *The Julia set of a quadratic polynomial $f = mx^2 + nx$, $m \geq 1$ and $n \geq 4$ is contained in the interval $\left[-\dfrac{n}{m}, 0\right]$.*

Proof. We prove that $f^{\circ(-k)}(0) = \text{Roots}\big(f^{\circ k}(0)\big) \subseteq \left[-\frac{n}{m}, 0\right]$ by induction on k and then the result will follow from Theorem 3.2.4 that $J(f) \subseteq \left[-\frac{n}{m}, 0\right]$ since $f'(0) = n > 1$. Now for $k = 1$, $\text{Roots}(f) = \left\{0, -\frac{n}{m}\right\} \subseteq \left[-\frac{n}{m}, 0\right]$. Suppose $\text{Roots}(f^{\circ k}(x)) \subseteq \left[-\frac{n}{m}, 0\right]$ for some $k \geq 1$. Now since $f^{\circ(k+1)} = f^{\circ k}(f)$, it follows that

$$\text{Roots}\left(f^{\circ k+1}\right) = \bigcup_{r\in\ \text{Roots}\left(f^{\circ k}\right)} \{s \in \mathbb{C} : f(s) - r = 0\}.$$

Now for all $r \in \text{Roots}(f^{\circ k})$, the roots of $f(x) - r$ are given by the quadratic formula:

$$s_1 = \frac{-n + \sqrt{n^2 + 4mr}}{2m} \text{ and } s_2 = \frac{-n - \sqrt{n^2 + 4mr}}{2m}.$$

We first note that the discriminant of $f(x) - r$ is $n^2 + 4(m)(r)$ and since $r \geq -\frac{n}{m}$ by assumption, the discriminant is at least 0 and so s_1 and s_2 are real for all such r. Also, we have that $s_2 \leq s_1$, so we show that $-\frac{n}{m} \leq s_2$ and $s_1 \leq 0$ to complete the proof. Now,

$$\begin{aligned} n^2 &\geq n^2 + 4mr \quad \text{(since } r \leq 0) \\ \implies n &\geq \sqrt{n^2 + 4mr} \\ \implies 0 &\geq \frac{-n + \sqrt{n^2 + 4mr}}{2m} = s_1 \end{aligned}$$

Also,

$$\begin{aligned} & n^2 \geq n^2 + 4mr \quad \text{(since } r \leq 0) \\ \implies & -n \leq -\sqrt{n^2 + 4mr} \\ \implies & -2n \leq -n - \sqrt{n^2 + 4mr} \\ \implies & -\frac{n}{m} \leq \frac{-n - \sqrt{n^2 + 4mr}}{2m} = s_2 \end{aligned}$$

Therefore, $f^{\circ(-k)}(0) \subseteq \left[-\frac{n}{m}, 0\right]$ for all $k \geq 1$ and since $\left[-\frac{n}{m}, 0\right]$ is a compact subset of $(\mathbb{C}, |\cdot|)$, it follows that

$$J(f) = \lim_{k\to\infty} f^{\circ(-k)}(0) \subseteq \left[-\frac{n}{m}, 0\right].$$

□

Corollary 3.3.6 *For a graph G on at least 4 vertices, if there exists a property $\mathscr{P}$ that is closed under substitution such that $\nu_{\mathscr{P}}(G,x) = 1 + nx + mx^2$, $m \neq 0$, then* $\mathcal{F}(G,\mathscr{P}) \subseteq \left[-\dfrac{n}{m}, 0\right]$.

Proof. The proof follows directly from the fact that $\mathcal{F}(G,\mathscr{P}) = J(f)$ and Proposition 3.3.5. □

A specific case of Corollary 3.3.6 was considered for the independence fractal of a graph in [15] which showed that the independence fractal of a graph with independence number 2, $n \neq 3$ vertices, and m non-edges has independence fractal in the line segment $\left[-\frac{n}{m}, 0\right]$. In fact they show that for $n = 3$ the real part of the independence fractal is also contained in $\left[-\frac{n}{m}, 0\right]$ while the imaginary part of the independence fractal is contained in $\left[-\frac{\sqrt{3}}{m}, \frac{\sqrt{3}}{m}\right]$. While the property $\mathscr{P} = \bigcap_{|V(G)|=3} -G$ gives a quadratic reduced $\mathscr{P}$-generating polynomial satisfying the hypothesis of Proposition 3.3.5, direct checking shows that no graphs on 3 vertices are module-free and so by Lemma 2.2.6, $\mathscr{P}$ is not closed under substitution. Hence, Theorem 2.2.7 cannot be applied to generate a $\mathscr{P}$-fractal. In fact, since no graphs on 3 vertices are module-free, it follows that for every graph G on at least 3 vertices and property $\mathscr{P}$ that is closed under substitution, the coefficient of x^3 in $\nu_{\mathscr{P}}(G,x)$ is at least 1 if $\mathscr{P} \notin \{-K_2, -\overline{K_2}, -K_2 \cap -\overline{K_2}\}$. Therefore, the only new property and graphs that Corollary 3.3.6 gives us real $\mathscr{P}$-fractals for is the property $\mathscr{P} = -\overline{K_2}$ and graphs G, $|V(G)| \neq 3$ such that $\alpha(\overline{G}) = 2$, i.e. graphs for which every induced subgraph on more than 2 vertices contains a pair of vertices that are not adjacent.

Theorem 3.3.7 *For a graph G on $n \neq 3$ vertices with m edges such that $\alpha(\overline{G}) = 2$ and property $\mathscr{P} = -\overline{K_2}$, the $\mathscr{P}$-fractal of G is contained entirely in the real line segment* $\left[-\frac{n}{m}, 0\right]$

Proof. The proof follows from Theorem 4.1 in [15] whose result is stated in the previous paragraph and the fact that $\nu_{\mathscr{P}}(G,x) = \nu_{\overline{\mathscr{P}}}(\overline{G},x)$. □

Beyond the properties $-K_2$ and $-\overline{K_2}$ determining whether the $\mathscr{P}$-fractal is a subset of $\mathbb{R}$ is difficult as we cannot exploit the quadratic formula.

3.4 A Word on Properties Not Closed Under Substitution

We have seen that when $\mathscr{P}$ is closed under substitution that $\mathcal{R}(G,\mathscr{P})$ is usually a fractal-like object, but what does $\mathcal{R}(G,\mathscr{P})$ look like when $\mathscr{P}$ is not closed under substitution? This is a more difficult problem to answer because we cannot exploit Theorem 2.2.7 to calculate the $\mathscr{P}$-generating polynomial for G^k and results for the reduced $\mathscr{P}$-generating polynomial to find the roots. We suspect that there exists a property $\mathscr{P}$ not closed under substitution and a graph G such that $\mathcal{R}(G,\mathscr{P})$ is not a fractal although the only results we present here do lead to fractals. We will not spend much time on this question as it becomes very difficult to calculate $\nu_{\mathscr{P}}(G^k,x)$ for a general k. We consider properties $-K_m$ for $m \geq 3$ and the graphs K_n because we can calculate $\nu_{-K_m}(K_n^k,x)$ for a general k with little effort. Note that any proper subset of $V(K_m)$ with at least 2 vertices forms a module and so by Lemma 2.2.6, $-K_m$ is not closed under substitution. If $n=1$, then $\nu_{-K_m}(K_1^k,x) = \nu_{-K_m}(K_1,x) = 1+x$ and so we suppose that $n \geq 2$ for the remainder of this section. We will require the following result due from Barbeau [2] to show what $\mathcal{R}(K_n,-K_m)$ will look like.

Theorem 3.4.1 *[2] If $n \geq 1$ and a_i is a positive real number for $0 \leq i \leq n$, then*

(i) The polynomial $g(x) = -a_0 - a_1x - ... - a_{n-1}x^{n-1} + a_nx^n$ has a unique positive real zero r, and

(ii) every root w of the polynomial $f(x) = a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1} + a_nx^n$ satisfies $|w| \leq r$.

Theorem 3.4.2 *For all $n \geq 2$ and $m \geq 3$, $\mathcal{R}(K_n, -K_m) = \{0\}$.*

Proof. Suppose $n \geq 2$ and $m \geq 3$. Note that $K_n^k = K_{n^k}$ and so for each $n \geq 2$ and $m \geq 3$, there exists a k sufficiently large such that $n^k \geq m$ and so $\nu_{-K_m}(K_n^l, x) = \sum_{i=0}^{m-1} \binom{n^l}{i} x^i$ for all $l \geq k$. Let $g_l(x) = \binom{n^l}{m-1} x^{m-1} - \sum_{i=0}^{m-2} \binom{n^l}{i} x^i$. Now suppose l is sufficiently large so that $n^l > m$ and thus,

$$g_l\left(\frac{1}{\sqrt{n^l}}\right) = \binom{n^l}{m-1}\left(\frac{1}{\sqrt{n^l}}\right)^{m-1} - \binom{n^l}{m-2}\left(\frac{1}{\sqrt{n}^l}\right)^{m-2} - \ldots - n^l\left(\frac{1}{\sqrt{n}^l}\right) - 1$$

The right hand side of the equation above is a polynomial of degree $m-1$ in $\sqrt{n^l}$ and as $l \to \infty$, $\sqrt{n^l} \to \infty$ which dominates the rest ensuring that $g_l\left(\frac{1}{\sqrt{n^l}}\right) > 0$. Now since $g_l(x)$ is continuous for all l, $g_l(0) = -1$ and $g_l\left(\frac{1}{\sqrt{n}^l}\right) > 0$ for sufficiently large l, $g_l(r) = 0$ for some $0 < r < \frac{1}{\sqrt{n}^l}$ by the Intermediate Value Theorem. Now by Theorem 3.4.1, we know that $g_l(x)$ has a unique positive root. Finally by Theorem 3.4.1, all roots w of $\nu_{-K_m}(K_n^l, x)$ satisfy $|w| < r$ and so for sufficiently large l, $|w| < r < \frac{1}{\sqrt{n}^l} \to 0$. Therefore $\mathcal{R}(K_n, -K_m) = \{0\}$. $\square$

Chapter 4

Conclusion/Open Problems

We have introduced the $\mathscr{P}$-generating polynomial of a graph and studied the behaviour of certain properties under graph products. We considered the problem of finding the roots of $\mathscr{P}$-generating polynomials which led to the $\mathscr{P}$-fractal of a graph for properties that are closed under substitution. There are many questions and directions that remain open for future study. We present a brief discussion of these here.

Problem 1: We have only considered graph join, union, and substitution as they relate to $\mathscr{P}$-generating polynomials of graphs but there are many other products that would be of interest to consider in the light of $\mathscr{P}$-generating polynomials. Some in particular would be the Cartesian, Categorical, Kronecker, and Strong product.

Problem 2: We have looked at many properties in an attempt to provide some general results, but an extensive study of specific properties would be beneficial. The study of the independence polynomial has lead to many interesting results and so a look at $\nu_{\mathscr{P}}(G,x)$ for other specific properties $\mathscr{P}$ would likely be interesting. Elementary properties that could be of interest are the properties $-C_n$ and $-P_k$ for $n \geq 5$ and $k \geq 4$ as we have shown that these properties are closed under substitution and therefore can be used to generate fractals. We also showed that the property of perfection $\mathscr{P} = \bigcap_{n\geq 2}^{\infty} \left(-C_{2n+1} \cap -\overline{C_{2n+1}}\right)$ is closed under substitution and can therefore be used to generate $\mathscr{P}$-fractals. Other properties that are not closed under substitution would also be of interest and could lead to nice results other than the

$\mathscr{P}$-fractals. A good starting place would be properties $-G$ where $|V(G)| = 3$. In Appendix A we have provided a table that explicitly gives $\nu_{-K_3}(G, x)$ and $\nu_{-P_3}(G, x)$ for all connected graphs on 5 and fewer vertices.

Problem 3: Studying the behaviour of $\mathscr{P}$-fractals for certain families of graphs and properties that have similar structures is open. Specific families of $\mathscr{P}$-fractals and graphs that we have observed to have similar structures are the families, $\mathcal{F}(P_{2n}, -P_{2n})$ and $\mathcal{F}(P_{2n+1}, -P_{2n+1})$ for $n \geq 2$. Refer to Figure 4.1 and note the similarity in the fractals and Figure 4.2 to see them all plotted together.

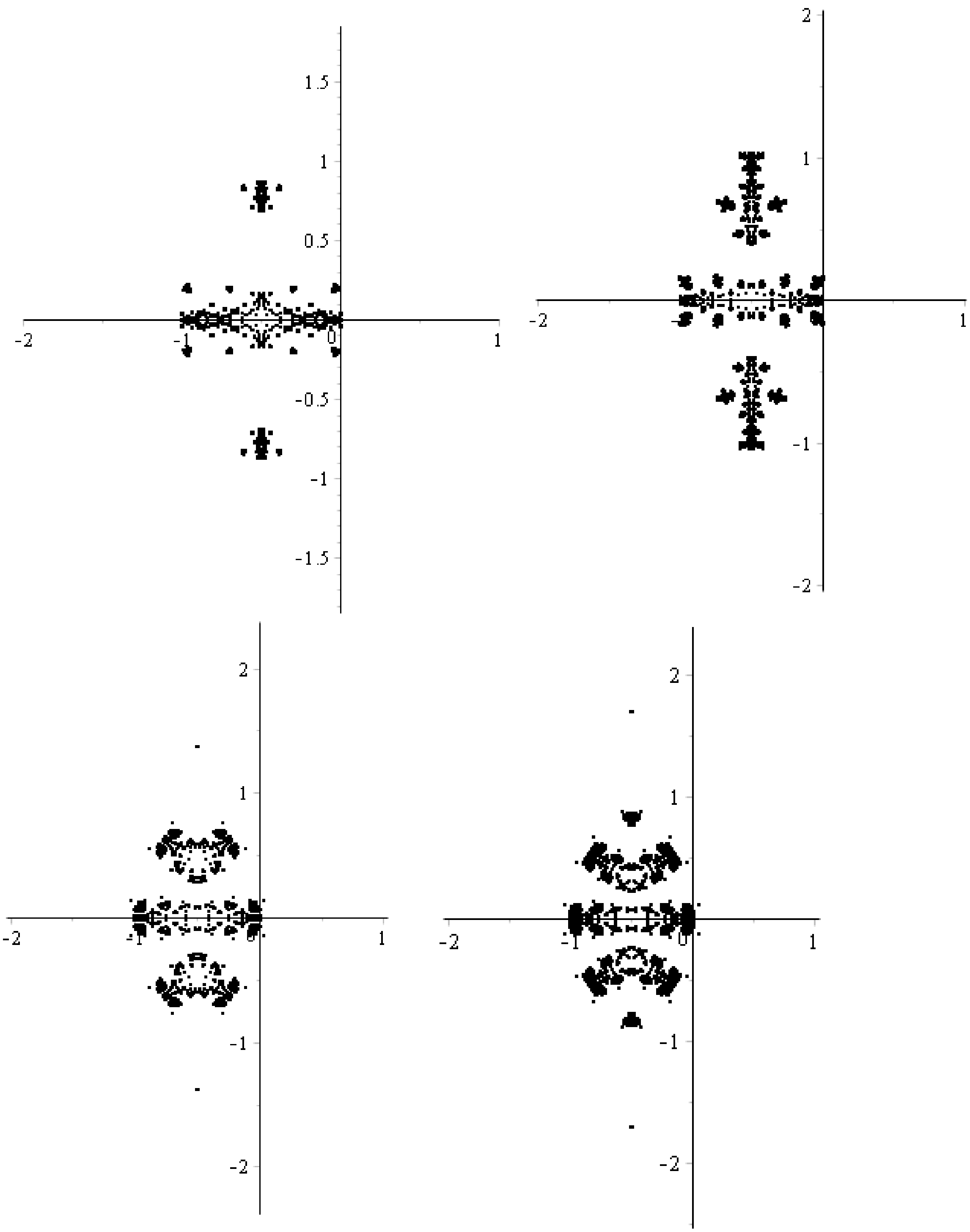

Figure 4.1: Approximations of $\mathcal{F}(P_5, -P_5)$ top left, $\mathcal{F}(P_7, -P_7)$ top right, $\mathcal{F}(P_9, -P_9)$ bottom left, $\mathcal{F}(P_{11}, -P_{11})$ bottom right.

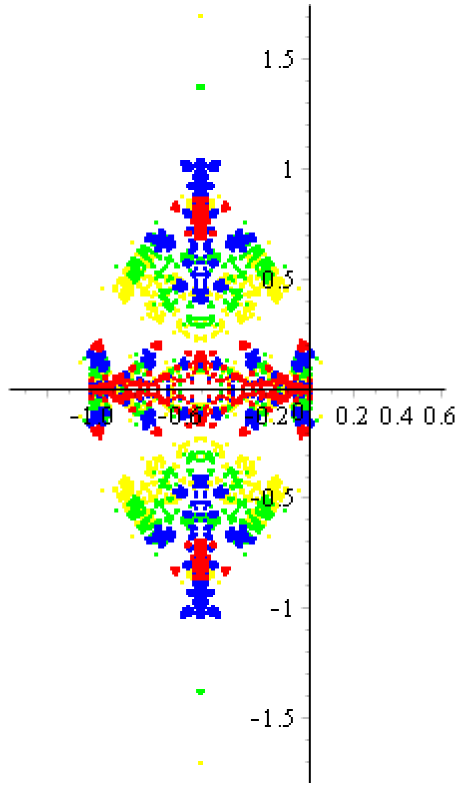

Figure 4.2: Approximations of $\mathcal{F}(P_{2n+1}, -P_{2n+1})$ $n = 2, 3, 4, 5$

Figure 4.3 shows the even paths and Figure 4.4 shows them plotted together. Again note the similar structure.

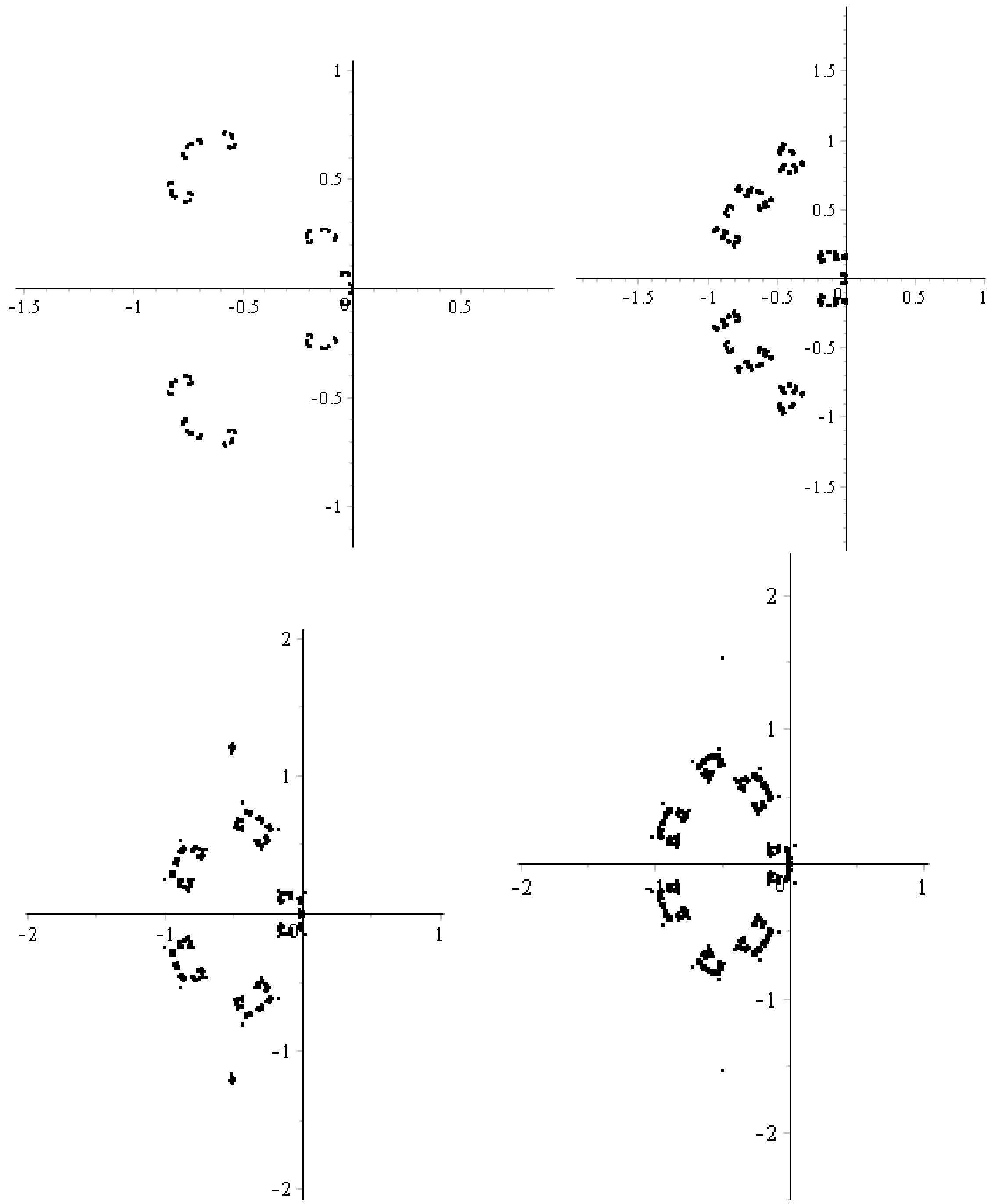

Figure 4.3: Approximations of $\mathcal{F}(P_4, -P_4)$ top left, $\mathcal{F}(P_6, -P_6)$ top right, $\mathcal{F}(P_8, -P_8)$ bottom left, $\mathcal{F}(P_{10}, -P_{10})$ bottom right

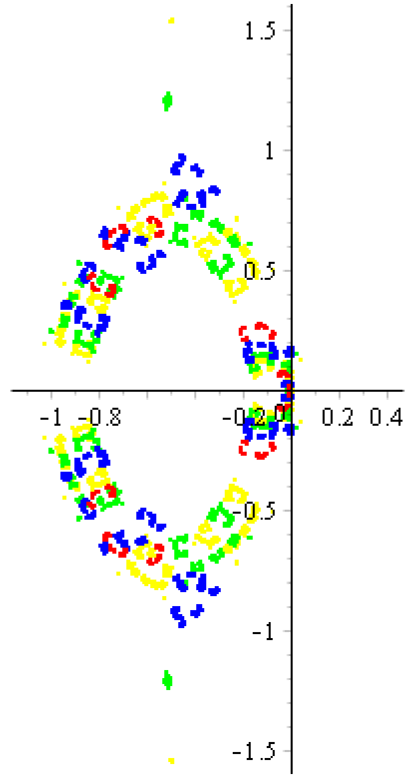

Figure 4.4: Approximations of $\mathcal{F}(P_{2n}, -P_{2n})$ $n = 2, 3, 4, 5$

Problem 4: A multivariate and more general version of the $\mathscr{P}$-generating polynomial may be defined for labelled graphs and has potential to lead to an interesting and deeper theory of counting all subgraphs with a certain property. Consider the following definition.

Definition 4.0.3 *The **labelled $\mathscr{P}$-generating polynomial** of a graph G with vertex set $\{v_1, ..., v_n\}$ is denoted $\nu_{\mathscr{P}}(G; x_1, x_2, ..., x_n)$ and defined by:*

$$\nu_{\mathscr{P}}(G; x_1, x_2, ..., x_n) = \sum_{i=1}^{n} \prod_{j=i}^{n} x_j^{\varepsilon_j}$$

where $\varepsilon \in \{0, 1\}$ such that $\sum_{j=i}^{n} \varepsilon_j = i$ and the graph induced on $\{v_i : \varepsilon_j = 1\}$ is a $\mathscr{P}$-graph.

The advantage of studying the labelled $\mathscr{P}$-generating polynomial over the $\mathscr{P}$-generating polynomial is that the labelled $\mathscr{P}$-generating polynomial keeps track of which vertices belong to the $\mathscr{P}$-subgraphs of the graph. This is important information that could lead to extended results for $\mathscr{P}$-generating polynomials of product graphs for properties $\mathscr{P}$ that are not closed with respect to the product. By setting $x_1 = x_2 = ... = x_n = x$ in the labelled $\mathscr{P}$-generating polynomial, it is easy to see that we receive the $\mathscr{P}$-generating polynomial and so results for the labelled $\mathscr{P}$-generating polynomial can easily be applied to the special case of the $\mathscr{P}$-generating polynomial.

One disadvantage of the labelled $\mathscr{P}$-generating polynomial is that it is not invariant under isomorphism. This is not a significant problem as the labelled $\mathscr{P}$-generating polynomials of isomorphic graphs will differ by at most a permutation of the indices on the variables. Another disadvantage is that calculating the labelled $\mathscr{P}$-generating polynomial is a more complicated problem than calculating the $\mathscr{P}$-generating polynomial. Again, the difference in difficulties is not great enough to dismiss the labelled $\mathscr{P}$-generating polynomial. The most significant disadvantage of the labelled $\mathscr{P}$-generating polynomial is that the problem of finding the zeros is far more "complex" than finding the roots of the $\mathscr{P}$-generating polynomial. To solve for the zeros of the labelled $\mathscr{P}$-generating polynomials would require results on monomial ideals and extensive use of Gröbner basis theory. As a good portion of our work focused on finding the roots of $\mathscr{P}$-generating polynomials which we saw is a difficult problem in and of itself, we decided to leave a study of the labelled $\mathscr{P}$-generating polynomial for a future work.

Problem 5: For which properties $\mathscr{P}$ are the roots of $\nu_{\mathscr{P}}(G,x)$ bounded? Certainly for $\mathscr{P} = \mathscr{G}$, in this case, the roots of $\nu_{\mathscr{P}}(G,x)$ all belong to the set $\{-1\}$. Also, for $\mathscr{P} = -K_2 \cap -\overline{K_2}$, all roots lie on the line segment $[0,1]$. We provide one more class of properties for which the roots of the $\mathscr{P}$-generating polynomial are bounded. For this we will require the well known Eneström-Kakeya Theorem.

Theorem 4.0.4 *(Eneström-Kakeya) [1] If* $f(x) = \sum_{i=0}^{n} a_i z^i$ *such that* $0 < a_0 \leq a_1 \leq ... \leq a_n$*, then the roots of* $f(x)$ *lie in the disk* $|z| \leq 1$.

Consider the property $\mathscr{P} = \{G \in \mathscr{G} : |V(G)| < k\}$ for a fixed $k \geq 3$. Now for all graphs G such that $n = |V(G)| \geq 2k$, we know that $\nu_{\mathscr{P}}(G, x) = \sum_{i=1}^{k-1} \binom{n}{i} x^i$ and since $n \geq 2k$, the coefficients are non-decreasing with constant term 1. Thus, by Theorem 4.0.4, the roots of $\nu_{\mathscr{P}}(G, x)$ are bounded by 1 for graphs on at least $2k$ vertices. There are a finite number of roots that arise from $\nu_{\mathscr{P}}(H, x)$ for graphs H on less than $2k$ vertices, and so the roots of the $\mathscr{P}$-generating polynomial are bounded. This leads to infinitely many properties for which the roots are bounded, however, for elementary properties the problem remains open. Brown et al. [14] show that for $\mathscr{P} = -K_2$, the roots of $\nu_{\mathscr{P}}(G, x)$ are unbounded and their closure is in fact the entire complex plane.

Problem 6: For which properties $\mathscr{P}$ is the closure of the roots of $\nu_{\mathscr{P}}(G, x)$ the entire plane? As we mentioned above, Brown et al. [14] show that for $\mathscr{P} = -K_2$ that is indeed the case. It is certainly not the case for properties $\mathscr{P}$ for which the roots are bounded and so this question is closely related to Problem 5. Appendix B contains a table that shows the roots of $\nu_{\mathscr{P}}(G, x)$ for all graphs G of order at most 7 with respect to properties $-K_3$ and $-P_3$.

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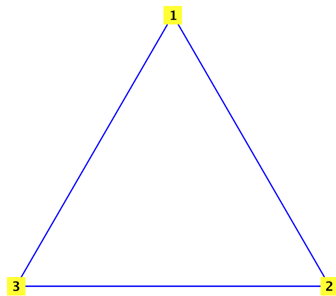
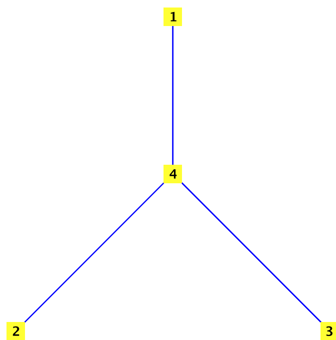
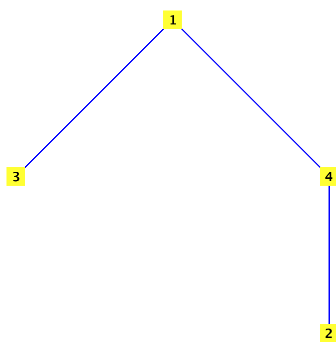
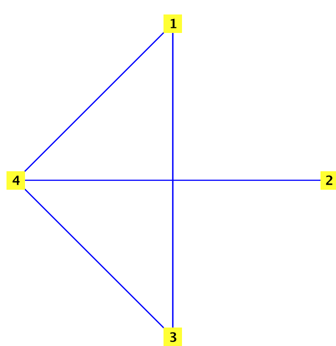
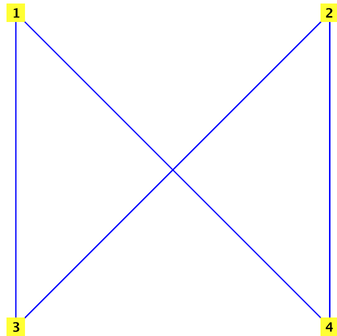
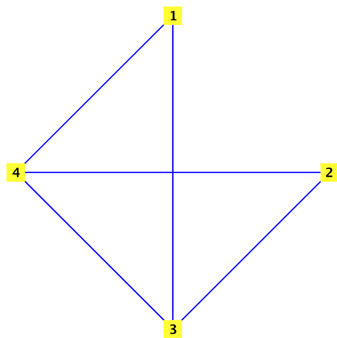
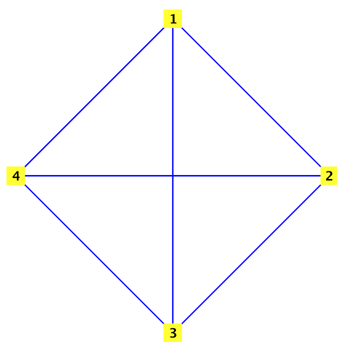
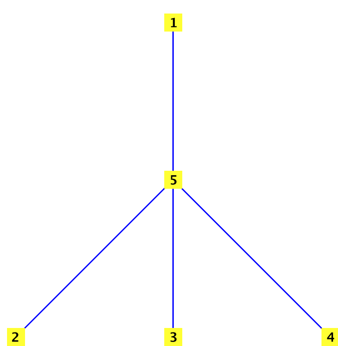
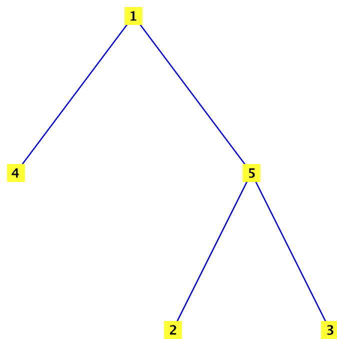
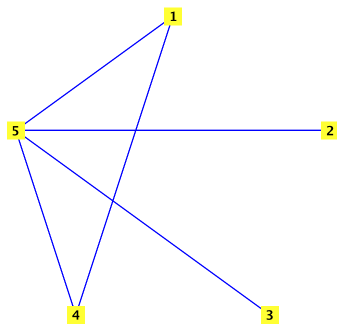
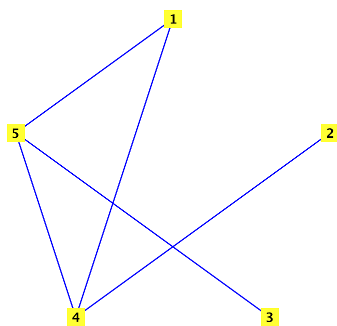
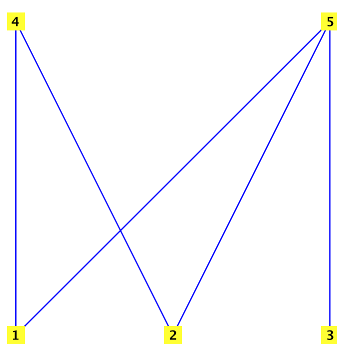
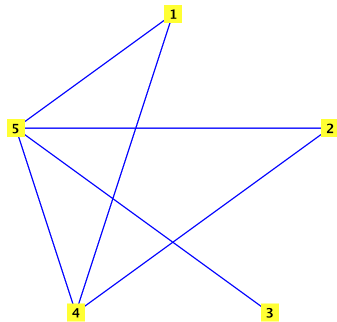
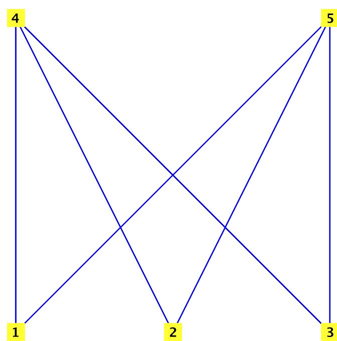
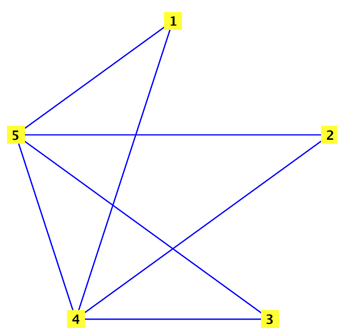
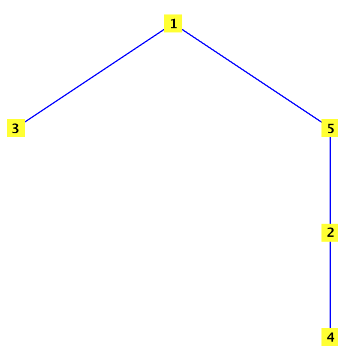
Appendix A

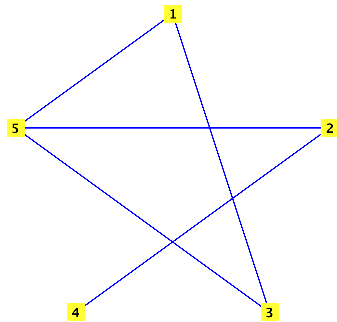
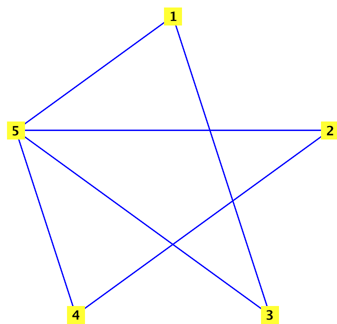
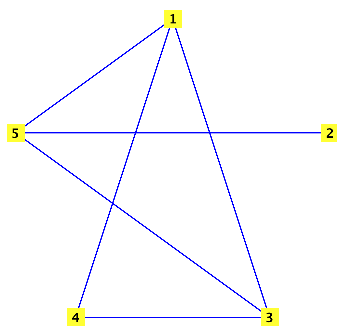
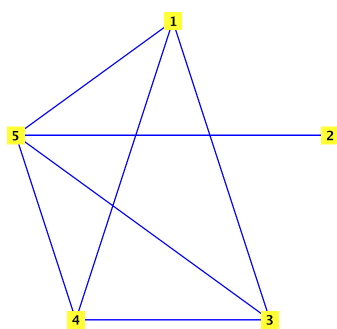
G	$\nu_{-K_3}(G,x)$	$\nu_{-P_3}(G,x)$
	$1+x$	$1+x$
	$(1+x)^2$	$(1+x)^2$
	$(1+x)^3$	$3x^2+3x+1$

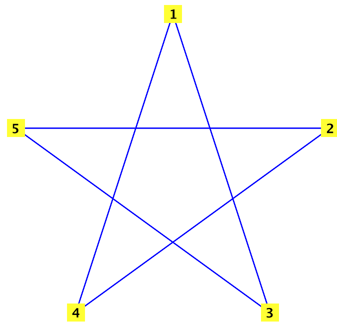
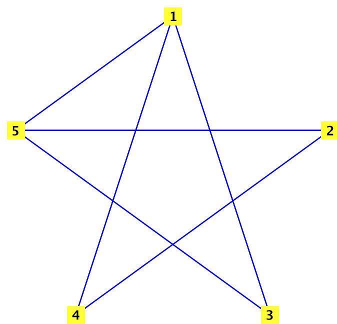
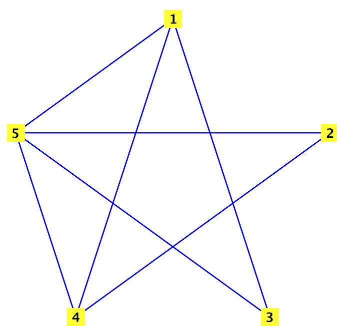
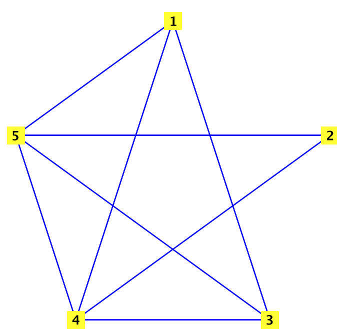
	$3x^2 + 3x + 1$	$(1+x)^3$
	$(1+x)^4$	$x^3 + 6x^2 + 4x + 1$
	$(1+x)^4$	$2x^3 + 6x^2 + 4x + 1$
	$3x^3 + 6x^2 + 4x + 1$	$2x^3 + 6x^2 + 4x + 1$

	$(1+x)^4$	$6x^2+4x+1$
	$2x^3+6x^2+4x+1$	$2x^3+6x^2+4x+1$
	$6x^2+4x+1$	$(1+x)^4$
	$(1+x)^5$	$x^4+4x^3+10x^2+5x+1$

	$(1+x)^5$	$x^4+6x^3+10x^2+5x+1$
	$3x^4+9x^3+10x^2+5x+1$	$x^4+5x^3+10x^2+5x+1$
	$3x^4+9x^3+10x^2+5x+1$	$6x^3+10x^2+5x+1$
	$(1+x)^5$	$4x^3+10x^2+5x+1$

	$2x^4 + 8x^3 + 10x^2 + 5x + 1$	$5x^3 + 10x^2 + 5x + 1$
	$(1 + x)^5$	$x^3 + 10x^2 + 5x + 1$
	$2x^4 + 7x^3 + 10x^2 + 5x + 1$	$4x^3 + 10x^2 + 5x + 1$
	$(1 + x)^5$	$x^4 + 7x^3 + 10x^2 + 5x + 1$

	$3x^4 + 9x^3 + 10x^2 + 5x + 1$	$2x^4 + 7x^3 + 10x^2 + 5x + 1$
	$x^4 + 8x^3 + 10x^2 + 5x + 1$	$x^4 + 6x^3 + 10x^2 + 5x + 1$
	$2x^4 + 8x^3 + 10x^2 + 5x + 1$	$x^4 + 6x^3 + 10x^2 + 5x + 1$
	$6x^3 + 10x^2 + 5x + 1$	$2x^4 + 7x^3 + 10x^2 + 5x + 1$

	$(1+x)^5$	$5x^3+10x^2+5x+1$
	$3x^4+9x^3+10x^2+5x+1$	$4x^3+10x^2+5x+1$
	$x^4+7x^3+10x^2+5x+1$	$5x^3+10x^2+5x+1$
	$5x^3+10x^2+5x+1$	$x^4+6x^3+10x^2+5x+1$

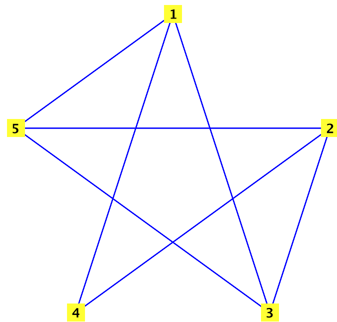
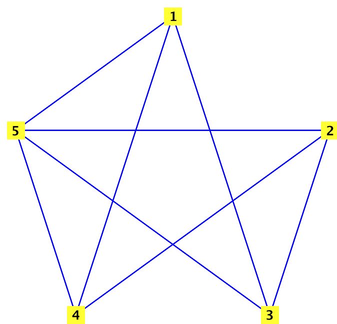
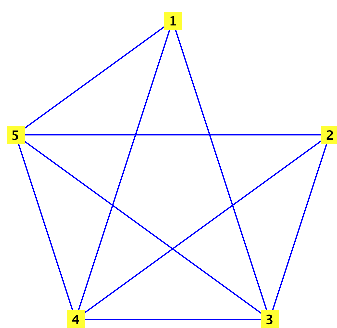
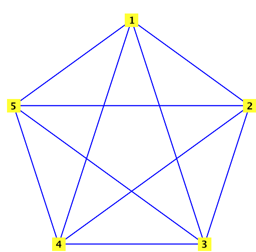
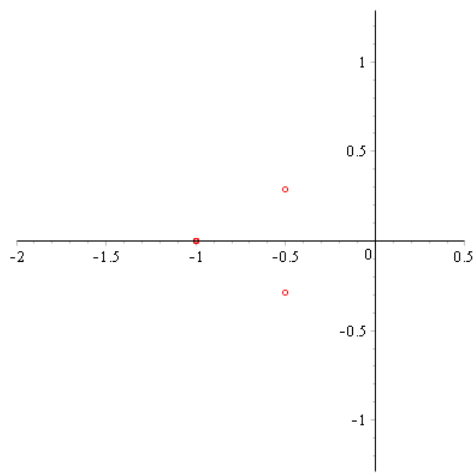
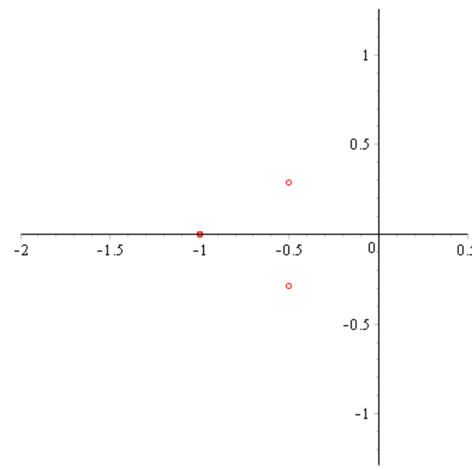
	$2x^4 + 8x^3 + 10x^2 + 5x + 1$	$3x^3 + 10x^2 + 5x + 1$
	$x^4 + 6x^3 + 10x^2 + 5x + 1$	$4x^3 + 10x^2 + 5x + 1$
	$3x^3 + 10x^2 + 5x + 1$	$2x^4 + 7x^3 + 10x^2 + 5x + 1$
	$10x^2 + 5x + 1$	$(1 + x)^5$

Table A.1: $\nu_{\mathscr{P}}(G, x)$ for $\mathscr{P} = -K_3$, $\mathscr{P} = -P_3$, G connected order at most 5

Appendix B

$\lvert V(G)\rvert$	Roots of $\nu_{-K_3}(G,x)$	Roots of $\nu_{-P_3}(G,x)$
3		

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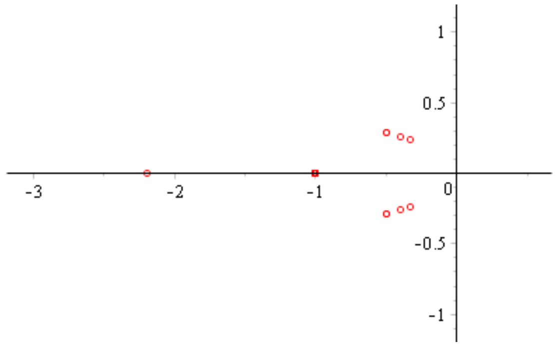
<td>
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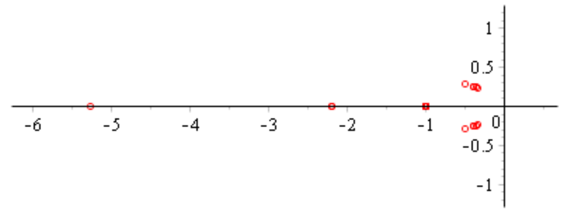
<td>
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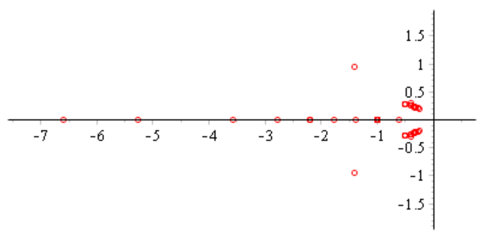
<td>
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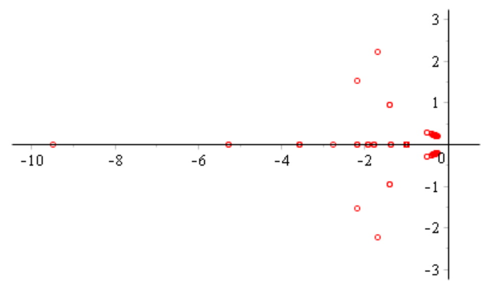
<td>
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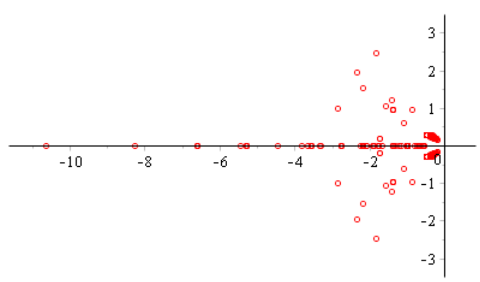
<td>
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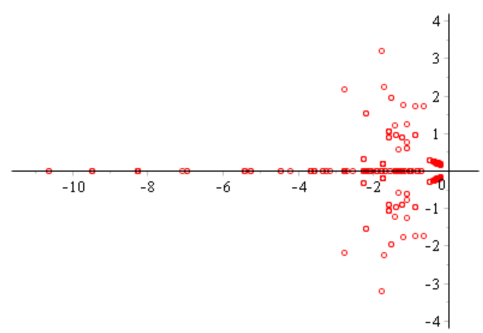
<td>
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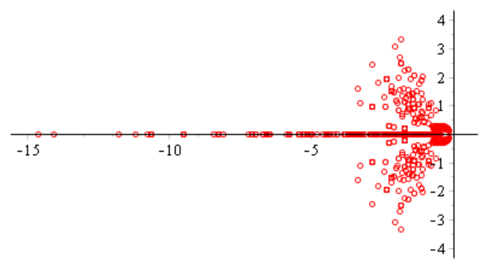
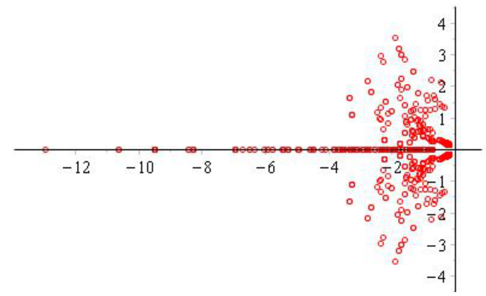
7*		

Table B.1: Roots of $\nu_{\mathscr{P}}(G,x)$ for $\mathscr{P} = -K_3$, $\mathscr{P} = -P_3$, and $|V(G)| \leq 7$

* The image for $\nu_{-P_3}(G,x)$, $|V(G)| = 7$ is approximate