SPECTRAL THEORY FOR BOUNDED OPERATORS ON HILBERT SPACE

by

Matthew A. Stephen

Submitted in partial fulfillment of the requirements
for the degree of Master of Science

at

Dalhousie University
Halifax, Nova Scotia
August 2013

Table of Contents

Abstract

This thesis is an exposition of spectral theory for bounded operators on Hilbert space. Detailed proofs are given for the functional calculus, the multiplication operator, and the projection-valued measure versions of the spectral theorem for self-adjoint bounded operators. These theorems are then generalized to finite sequences of self-adjoint and commuting bounded operators. Finally, normal bounded operators are discussed, as a particular case of the generalization.

List of Abbreviations and Symbols Used

$\|\cdot\|_X$ .... norm for the vector space X

$\mathcal{H}$ .... Hilbert space

$\langle\cdot,\cdot\rangle$ .... inner product

$\mathcal{L}(\mathcal{H})$ .... collection of bounded operators on $\mathcal{H}$

$\|\cdot\|$ .... operator norm

T^* .... adjoint for the operator T

$\sigma(T)$ .... spectrum for the operator T

$\sigma_p(T)$ .... point spectrum for the operator T

$\sigma_c(T)$ .... continuous spectrum for the operator T

$\sigma_r(T)$ .... residual spectrum for the operator T

$\sigma_{app}(T)$ .... approximate point spectrum for the operator T

$\rho(T)$ .... resolvent set for the operator T

$R_\lambda(T)$ .... resolvent for $\lambda \in \rho(T)$

$r(T)$ .... spectral radius for the operator T

LCH .... locally compact Hausdorff

$M(X)$ .... collection of complex Radon measures on X, an LCH space

$\mathcal{C}(X)$ .... collection of continuous functions $f : X \to \mathbb{C}$

$\|\cdot\|_u$ .... uniform norm

$M(X)$ .... collection of complex Radon measures on X, an LCH space

$\mu_{x,y}$ .... the measure associated with a self-adjoint $T \in \mathcal{L}(\mathcal{H})$, and the vectors $x, y \in \mathcal{H}$

$\mathcal{C}_0(X)$ .... collection of continuous functions $f : X \to \mathbb{C}$ which vanish at infinity

$\mathcal{C}_b(X)$ .... collection of bounded, continuous functions $f : X \to \mathbb{C}$

$\mathcal{B}_X$ .... Borel σ-algebra for the topological space X

$\mathcal{B}(X)$ .... collection of Borel measurable functions $f : X \to \mathbb{C}$

$\mathcal{B}_b(X)$ .... collection of bounded, Borel measurable functions $f : X \to \mathbb{C}$

PVM .... projection-valued measure

$p_{x,y}$ .... the measure associated with the PVM P, and the vectors $x, y \in \mathcal{H}$

$\operatorname{supp}(P)$ .... support of the PVM P

$\int f \, dP$ .... the element of $\mathcal{L}(\mathcal{H})$ associated with the Radon PVM P, and $f \in \mathcal{B}_b(\operatorname{supp}(P))$

T_R .... the real component of $T \in \mathcal{L}(\mathcal{H})$

T_I .... the imaginary component of $T \in \mathcal{L}(\mathcal{H})$

Chapter 1

Introduction

Spectral theory is a sophisticated area of mathematics, drawing on diverse concepts in algebra, functional analysis, measure theory, and complex analysis. The spectral theorems are fundamental to the mathematical modelling of quantum mechanics, which provided the initial inspiration for their inception. They play an important role in many other applied areas, including statistical mechanics, evolution equations, Brownian motion, financial mathematics, and recent work in image recognition. Applications within pure mathematics are also numerous, ranging from differential equations to harmonic analysis on manifolds and Lie groups.

In one of its most recognizable forms, the spectral theorem states that any self-adjoint operator is unitarily equivalent to a multiplication operator on some L^2 space. More specifically, if T is a self-adjoint operator on $\mathcal{H}$, then there is a unitary map $U : \mathcal{H} \to L^2(X, \Sigma, \mu)$ such that

$$U\,T\,U^{-1}(f) = M_F(f) := F \cdot f$$

for some function $F : X \to \mathbb{R}$. In the simplest case of $\mathbb{C}^n$, this *multiplication operator* version of the spectral theorem gives the familiar statement that every self-adjoint $n \times n$ matrix T is diagonalizable, i.e. there is a unitary matrix U and a diagonal matrix D such that $U\,T\,U^{-1} = D$. Here, D may be interpreted as a multiplication operator on the L^2 space $\mathbb{C}^n$.

The derivative operator $T = -i\,\frac{d}{dt}$ is an example of a self-adjoint, densely-defined operator on $L^2(\mathbb{R})$. Applying the Fourier transform $\mathcal{F}$ to T, and integrating by parts, gives

$$\mathcal{F}(Tf)(\xi) = \int_{-\infty}^{\infty} -i\,\frac{d}{dt} f(t) e^{-2\pi i \xi t}\,dt = (-2\pi i \xi) i \int_{-\infty}^{\infty} f(t) e^{-2\pi i \xi t}\,dt = 2\pi\xi\,(\mathcal{F}f)\,(\xi).$$

The Fourier transform is a unitary operator on $L^2(\mathbb{R})$, so this shows that T is unitarily equivalent to multiplication by $2\pi\xi$.

The real power of the spectral theorem described above is that it enables one to define "functions" of a self-adjoint operator T. For any bounded and measurable function $g : \mathbb{R} \to \mathbb{C}$, we obtain a bounded operator $g(T)$ on the Hilbert space $\mathcal{H}$, by setting

$$g(T) = U^{-1}\,M_{g \circ F}\,U.$$

The collection of all functions $g(T)$ defined in this way is referred to as the *functional calculus* for T. In the example of the derivative operator above, the functions $g\left(-i\,\frac{d}{dt}\right)$ are none other than the convolution operators, natural operators that are found throughout mathematics.

The functional calculus for a self-adjoint T is uniquely associated with its subcollection of characteristic functions $\chi_B(T)$. The functions $\chi_B(T)$ may be used to define a *projection-valued measure*, a map from the Borel σ-algebra for $\mathbb{R}$ to the set of orthogonal projections on $\mathcal{H}$, which behaves similarly as a measure.

This thesis is an exposition of spectral theory for bounded operators on Hilbert space. After discussing preliminary assumptions and results, we give detailed proofs of the functional calculus, the multiplication operator, and the projection-valued measure versions of the spectral theorem for self-adjoint bounded operators. These theorems are then generalized to finite sequences of self-adjoint and commuting bounded operators. As a particular case of this generalization, we discuss normal bounded operators.

The development largely follows that in [5] and [4], but expands on the presentations there, carefully filling in all the details. Any basic facts we use about Hilbert space come from [2], and standard results in analysis come from [3], unless otherwise stated.

Chapter 2

Preliminaries

This chapter provides, largely with proof, the background that is needed to discuss the spectral theorems. In the first section, we recall the definition of a bounded operator. We also describe $\mathcal{L}(\mathcal{H})$ as a C*-algebra, as it is a useful way of summarizing many facts that we will use later. The second section introduces positive bounded operators and orthogonal projections, which are related objects. In the final section, we define the spectrum and the resolvent of a bounded operator, and use the resolvent to prove various properties of the spectrum.

2.1 A C*-algebra of Bounded Operators

Definition 2.1. If X and Y are normed vector spaces, and $T : X \to Y$ is a linear map, then T is **bounded** if there exists a constant $C > 0$ such that $\|Tx\|_Y \leq C\,\|x\|_X$ for all $x \in X$. The collection of bounded linear maps from X to Y is denoted by $\mathcal{L}(X, Y)$. The **operator norm** is a function $\|\cdot\| : \mathcal{L}(X, Y) \to \mathbb{R}^{\geq 0}$ defined by

$$\|T\| = \inf\{C > 0 \mid \|Tx\|_Y \leq C\,\|x\|_X \;\forall\, x \in X\}.$$

If $\mathcal{H}$ is a Hilbert space, then a bounded linear map $T : \mathcal{H} \to \mathcal{H}$ is called a **bounded operator**. The space of bounded operators on $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$.

In this section, we show that $\mathcal{L}(\mathcal{H})$ satisfies the requirements for being a unital C*-algebra.

Definition 2.2. An **algebra** is a complex vector space $\mathcal{A}$ equipped with an additional multiplication operation which turns it into a ring, not necessarily with unity, and satisfies

$$\alpha\,(x\,y) = (\alpha\,x)\,y = x\,(\alpha\,y)$$

for all $\alpha \in \mathbb{C}$ and $x, y \in \mathcal{A}$. An **involution** for an algebra $\mathcal{A}$ is a function $^* : \mathcal{A} \to \mathcal{A}$ such that

$$\begin{aligned} &x^{**} = x \\ &(x\,y)^* = y^*\,x^* \\ &(\alpha\,x + y)^* = \overline{\alpha}\,x^* + y^* \end{aligned}$$

for all $\alpha \in \mathbb{C}$ and $x, y \in \mathcal{A}$. A **Banach algebra** is an algebra $\mathcal{A}$ with a norm $\|\cdot\|$, respect to which it is a Banach space, and such that $\|x\,y\| \leq \|x\| \cdot \|y\|$ for all $x, y \in \mathcal{A}$. A unital Banach algebra has the additional requirement that $\|I\| = 1$, where I is the multiplicative identity. A **C*-algebra** is a Banach algebra $\mathcal{A}$ with an involution such that $\|x^*\,x\| = \|x\|^2$ for all $x \in \mathcal{A}$.

The assertions made in the following theorem may be found, for example, in the comments on page 68 of [2].

Theorem 2.3. *If X and Y are normed vector spaces, then $\mathcal{L}(X, Y)$ is a normed vector space with respect to pointwise-defined operations, and the operator norm. If Y is Banach, then $\mathcal{L}(X, Y)$ is also Banach. In particular, $\mathcal{L}(\mathcal{H})$ is Banach.*

With the previous theorem in mind, it is easily verified that $\mathcal{L}(\mathcal{H})$ is a unital Banach algebra, with composition as its multiplication operation, and the identity map as its unit. Bounded linear functionals and bounded sesquilinear maps will allows us to construct an involution for $\mathcal{L}(\mathcal{H})$.

Definition 2.4. A **bounded linear functional** on a normed vector space X is a bounded linear map from X to the complex numbers. The collection of bounded linear functionals on X is denoted by X^*, rather than $\mathcal{L}(X, \mathbb{C})$.

Lemma 2.5 (Riesz Lemma)**.** *$l \in \mathcal{H}^*$ if and only if there exists $y \in \mathcal{H}$ such that $l(x) = \langle x, y\rangle$ for all $x \in \mathcal{H}$. When such a y exists, it is unique, and $\|l\| = \|y\|_{\mathcal{H}}$.*

Definition 2.6. A **sesquilinear form** on $\mathcal{H}$ is a sesquilinear map $\psi : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$. ψ is **bounded** if there exists a constant $C > 0$ such that $|\psi(x, y)| \leq C \cdot \|x\|_{\mathcal{H}} \cdot \|y\|_{\mathcal{H}}$ for all $x, y \in \mathcal{H}$. When ψ is bounded, we define

$$\|\psi\| = \inf\{C > 0 \mid |\psi(x, y)| \leq C \cdot \|x\|_{\mathcal{H}} \cdot \|y\|_{\mathcal{H}}\}.$$

Lemma 2.7. *A sesquilinear form $\psi : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is bounded if and only if there is a constant $C > 0$ such that $|\psi(x, x)| \leq C\,\|x\|^2_{\mathcal{H}}$ for all $x \in \mathcal{H}$.*

Proof. The forward implication is clear. Assuming there is a constant $C > 0$ such that $|\psi(x, x)| \leq C\|x\|^2_{\mathcal{H}}$ for all $x \in \mathcal{H}$, we obtain

$$\begin{aligned} |\psi(x, y)| &= \frac{1}{4}\,|\psi(x + y, x + y) - \psi(x - y, x - y) + i \cdot \psi(x + iy, x + iy) - i \cdot \psi(x - iy, x - iy)| \\ &\leq \frac{C}{4} \cdot \left(\|x + y\|^2_{\mathcal{H}} + \|x - y\|^2_{\mathcal{H}} + \|x + iy\|^2_{\mathcal{H}} + \|x - iy\|^2_{\mathcal{H}}\right) \\ &= C \cdot \left(\|x\|^2_{\mathcal{H}} + \|y\|^2_{\mathcal{H}}\right) \end{aligned}$$

for all $x, y \in \mathcal{H}$, using the polarization identity and parallelogram law. Therefore $|\psi(x, y)| \leq 2 \cdot C$ for all unit vectors $x, y \in \mathcal{H}$, which implies ψ is bounded. □

Theorem 2.8. *ψ is a bounded sesquilinear form on $\mathcal{H}$ if and only if there exists $T \in \mathcal{L}(\mathcal{H})$ such that $\psi(x, y) = \langle Tx, y\rangle$ for all $x, y \in \mathcal{H}$. When such a T exists, it is unique, and $\|T\| = \|\psi\|$.*

Proof. Given $T \in \mathcal{L}(\mathcal{H})$, consider the function

$$\begin{aligned} \psi : \mathcal{H} \times \mathcal{H} &\to \mathbb{C} \\ (x, y) &\mapsto \langle Tx, y\rangle. \end{aligned}$$

The linearity of T and the sesquilinearity of the inner product imply that ψ is sesquilinear. The Cauchy-Schwartz inequality and the boundedness of T then imply that ψ is a bounded sesquilinear form on $\mathcal{H}$, with $\|\psi\| \leq \|T\|$.

Now, suppose $\psi : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is a bounded sesquilinear form. Fixing $x \in \mathcal{H}$, consider the function

$$\begin{aligned} l_x : \mathcal{H} &\to \mathbb{C} \\ y &\mapsto \overline{\psi(x,y)}. \end{aligned}$$

l_x is linear because ψ is conjugate linear in its second term. It is also clear that l_x is bounded, with $\|l_x\| \leq \|\psi\| \cdot \|x\|$. Therefore, by the Riesz Lemma, there exists a unique vector $Tx \in \mathcal{H}$ such that

$$\psi(x,y) = \overline{l_x(y)} = \overline{\langle y, Tx \rangle} = \langle Tx, y \rangle$$

for all $y \in \mathcal{H}$, and $\|Tx\|_{\mathcal{H}} = \|l_x\|$. After repeating this process for each $x \in \mathcal{H}$, we have the well-defined function

$$\begin{aligned} T : \mathcal{H} &\to \mathcal{H} \\ x &\mapsto Tx. \end{aligned}$$

For any $\alpha \in \mathbb{C}$ and $x_1, x_2 \in \mathcal{H}$, we have

$$\begin{aligned} \langle T(\alpha\, x_1 + x_2), y \rangle &= \psi(\alpha\, x_1 + x_2, y) \\ &= \alpha\, \psi(x_1, y) + \psi(x_2, y) = \alpha\, \langle Tx_1, y \rangle + \langle Tx_2, y \rangle = \langle \alpha\, Tx_1 + Tx_2, y \rangle \end{aligned}$$

for all $y \in \mathcal{H}$. This implies the linearity of T. Recalling that $\|Tx\|_{\mathcal{H}} = \|l_x\| \leq \|\psi\| \cdot \|x\|_{\mathcal{H}}$, it is also clear that T is bounded, with $\|T\| \leq \|\psi\|$.

Taking together the reverse and forward implications proven above, it is clear that $\|T\| = \|\psi\|$. The uniqueness of T is necessary, given its construction using the Riesz Lemma in the forward implication. □

Theorem 2.9. *Let $T \in \mathcal{L}(\mathcal{H})$. There exists a unique $T^* \in \mathcal{L}(\mathcal{H})$ such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x, y \in \mathcal{H}$. Furthermore, $\|T\| = \|T^*\|$.*

Proof. Define the function

$$\begin{aligned} \psi : \mathcal{H} \times \mathcal{H} &\to \mathbb{C} \\ (y, x) &\mapsto \langle y, Tx \rangle. \end{aligned}$$

The linearity of T and the sesquilinearity of the inner product imply that ψ is sesquilinear. The Cauchy-Schwartz inequality and the boundedness of T then imply that ψ is a bounded sesquilinear form, with $\|\psi\| \leq \|T\|$. Therefore, by Theorem (2.8), there exists a unique $T^* \in \mathcal{L}(\mathcal{H})$ such that

$$\langle Tx, y \rangle = \overline{\langle y, Tx \rangle} = \overline{\psi(y,x)} = \overline{\langle T^* y, x \rangle} = \langle x, T^* y \rangle$$

for all $x, y \in \mathcal{H}$. Theorem (2.8) also implies that $\|T^*\| = \|\psi\| \leq \|T\|$. For all $x \in \mathcal{H}$, we have

$$\|Tx\|_{\mathcal{H}}^2 = \langle Tx, Tx \rangle = |\langle x, T^* Tx \rangle| \leq \|T^*\| \cdot \|x\|_{\mathcal{H}} \cdot \|Tx\|_{\mathcal{H}},$$

which implies $\|T\| \leq \|T^*\|$. □

Definition 2.10. Let $T \in \mathcal{L}(\mathcal{H})$. The unique operator associated with T in Theorem (2.9), denoted by T^*, is the **adjoint** of T. T is **normal** when it commutes with T^*, and **self-adjoint** when $T = T^*$.

The following lemma will be used later in this chapter.

Lemma 2.11. *If $T \in \mathcal{L}(\mathcal{H})$, then $\overline{T[\mathcal{H}]} = Ker(T^*)^\perp$. If T is normal, then $\overline{T[\mathcal{H}]} = Ker(T)^\perp$.*

Proof. $y \in T[\mathcal{H}]^\perp$ if and only if $\langle x, T^*y\rangle = \langle Tx, y\rangle = 0$ for all $x \in \mathcal{H}$, which is the case if and only if $T^*y = 0$. It follows that

$$\overline{T[\mathcal{H}]} = T[\mathcal{H}]^{\perp\perp} = Ker(T^*)^\perp.$$

If T is normal, then

$$\|Tx\|_{\mathcal{H}}^2 = \langle T^*\,Tx, x\rangle = \langle T\,T^*x, x\rangle = \|T^*x\|_{\mathcal{H}}^2$$

for any $x \in \mathcal{H}$. In that case,

$$\overline{T[\mathcal{H}]} = Ker(T^*)^\perp = Ker(T)^\perp.$$

□

The map $T \mapsto T^*$ is an involution for $\mathcal{L}(\mathcal{H})$. The following lemma confirms that this involution satisfies the necessary property for $\mathcal{L}(\mathcal{H})$ to be a C*-algebra.

Lemma 2.12. *If $T \in \mathcal{L}(\mathcal{H})$, then $\|T\|^{2^n} = \left\|(T^*T)^{2^{n-1}}\right\|$ for any $n \in \mathbb{N}$.*

Proof. The proof will be by induction. For the base step, consider $n = 1$. It is easily seen that $\|T^*T\| \le \|T\|^2$. Also,

$$\|Tx\|_{\mathcal{H}}^2 = |\langle T^*Tx, x\rangle| \le \|T^*T\| \cdot \|x\|_{\mathcal{H}}^2$$

for all $x \in \mathcal{H}$, which implies $\|T\|^2 \le \|T^*T\|$. The desired equality follows.

For the inductive step, consider $n \ge 1$, and assume $\|T\|^{2^m} = \left\|(T^*T)^{2^{m-1}}\right\|$ for $1 \le m \le n$. Because $(T^*T)^{2^{n-1}}$ is self-adjoint, we have

$$\|T\|^{2^{n+1}} = \left(\|T\|^{2^n}\right)^2 = \left\|(T^*T)^{2^{n-1}}\right\|^2 = \left\|(T^*T)^{2^{n-1}} \circ (T^*T)^{2^{n-1}}\right\| = \left\|(T^*T)^{2^n}\right\|,$$

using the stated assumption. This completes the proof. □

2.2 Positive Bounded Operators and Orthogonal Projections

Definition 2.13. If $T \in \mathcal{L}(\mathcal{H})$ is such that $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$, then T is **real**. If T is such that $\langle Tx, x\rangle \in \mathbb{R}^{\ge 0}$ for all $x \in \mathcal{H}$, then T is **positive**, which is denoted by $T \ge 0$.

Positive bounded operators are referred to in later chapters, in relation to the functional calculus.

Lemma 2.14. *$T \in \mathcal{L}(\mathcal{H})$ is real if and only if it is self-adjoint. If T is additionally idempotent, then it is positive.*

Proof. Assume T is real, and consider the sesquilinear forms $\psi_1(x,y) := \langle Tx, y\rangle$ and $\psi_2(x,y) := \langle x, Ty\rangle$ on $\mathcal{H}$. We necessarily have

$$\psi_1(x,x) = \langle Tx, x\rangle = \overline{\langle Tx, x\rangle} = \langle x, Tx\rangle = \psi_2(x,x)$$

for all $x \in \mathcal{H}$. Using the polarization identities for ψ_1 and ψ_2, it follows that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in \mathcal{H}$, so T is self-adjoint.

If T is self-adjoint, then

$$\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$$

for all $x \in \mathcal{H}$, which shows T is real. If T is additionally idempotent, then

$$\langle Tx, x\rangle = \|Tx\|_{\mathcal{H}}^2 \geq 0$$

for all $x \in \mathcal{H}$, which shows T is positive. □

Definition 2.15. Let M be a closed subspace of $\mathcal{H}$. The function

$$\begin{aligned} P : \mathcal{H} = M \oplus M^{\perp} &\to \mathcal{H} \\ x = x_1 + x_2 &\mapsto x_1 \end{aligned}$$

is called the **orthogonal projection** onto M.

Orthogonal projections are of particular importance. They are the basis for the projection-valued measure formulation of the spectral theorem. One consequence of the following theorem is that orthogonal projections are positive bounded operators.

Theorem 2.16. *$P : \mathcal{H} \to \mathcal{H}$ is an orthogonal projection if and only if it is an idempotent and self-adjoint element of $\mathcal{L}(\mathcal{H})$. If P is an orthogonal projection, then $P[\mathcal{H}] = \{x \in \mathcal{H} \mid Px = x\}$, and $\|P\| = 1$.*

Proof. Assume that P is an orthogonal projection onto the closed subspace M. Consider any $x, y \in \mathcal{H}$, with $x_1 + x_2$ and $y_1 + y_2$ as their respective representations in $M \oplus M^{\perp}$. If $\alpha \in \mathbb{C}$, then

$$P(\alpha x + y) = P\left((\alpha x_1 + y_1) + (\alpha x_2 + y_2)\right) = \alpha x_1 + y_1 = \alpha Px + Py$$

implies P is linear. Because $\langle x_1, x_2\rangle = \langle x_2, x_1\rangle = 0$, we have

$$\|Px\|_{\mathcal{H}}^2 = \langle x_1, x_1\rangle \leq \langle x_1, x_1\rangle + \langle x_1, x_2\rangle + \langle x_2, x_1\rangle + \langle x_2, x_2\rangle = \langle x_1 + x_2, x_1 + x_2\rangle = \|x\|_{\mathcal{H}}^2.$$

This implies P is bounded, with $\|P\| \leq 1$. Every element in $\mathcal{H}$ has a unique representation in $M \oplus M^{\perp}$, so $Px_1 = x_1$. It follows that

$$P\,Px = Px_1 = x_1 = Px,$$

which shows P is idempotent. $Px_1 = x_1$ also implies $\|P\| = 1$. Because $\langle x_1, y_2\rangle = \langle x_2, y_1\rangle = 0$, we have

$$\langle Px, y\rangle = \langle x_1, y_1 + y_2\rangle = \langle x_1, y_1\rangle = \langle x_1 + x_2, y_1\rangle = \langle x, Py\rangle.$$

Therefore, P is self-adjoint.

Now, assume P is an idempotent and self-adjoint element of $\mathcal{L}(\mathcal{H})$. The linearity of P implies that $P[\mathcal{H}]$ is a subspace of $\mathcal{H}$. If $x = Py \in P[\mathcal{H}]$, then

$$Px = P\,Py = Py = x,$$

because P is idempotent. It follows that $P[\mathcal{H}] = \{x \in \mathcal{H} \mid Px = x\}$. However, $Px = x$ if and only if $(P - I)x = 0$. As the preimage of a closed set, with respect to a continuous function, $P[\mathcal{H}]$ is closed. Therefore, we may consider the direct sum decomposition $\mathcal{H} = P[\mathcal{H}] \oplus P[\mathcal{H}]^{\perp}$. If $x_1 + x_2$ is the representation of $x \in \mathcal{H}$ in $P[\mathcal{H}] \oplus P[\mathcal{H}]^{\perp}$, then

$$Px = Px_1 + Px_2 = x_1 + Px_2.$$

To see that $Px_2 = 0$, note that

$$\langle y, Px_2\rangle = \langle Py, x_2\rangle = 0$$

for all $y \in \mathcal{H}$, because P is self-adjoint. We may conclude P is an orthogonal projection onto $P[\mathcal{H}]$. □

Lemma 2.17. *Let $P_1, P_2 \in \mathcal{L}(\mathcal{H})$ be orthogonal projections. $P_1\,P_2$ is an orthogonal projection if and only if P_1 and P_2 commute.*

Proof. Denote $P_1\,P_2 \in \mathcal{L}(\mathcal{H})$ by P. $P^* = P_2\,P_1$ because P_1 and P_2 are self-adjoint, so P is self-adjoint if and only if P_1 and P_2 commute. To see that P_1 and P_2 commuting is not only necessary, but sufficient, for P to be an orthogonal projection, note that

$$P^2 = (P_1\,P_2)\,(P_1\,P_2) = (P_1\,P_1)\,(P_2\,P_2) = P_1\,P_2$$

because P_1 and P_2 are idempotent. □

Lemma 2.18. *Let $P_1, P_2 \in \mathcal{L}(\mathcal{H})$ be orthogonal projections. $P_1 + P_2$ is an orthogonal projection if and only if $P_1\,P_2 = 0$ or $P_2\,P_1 = 0$.*

Proof. Denote $P_1 + P_2 \in \mathcal{L}(\mathcal{H})$ by P. As the sum of self-adjoint operators, P is also self-adjoint. Therefore, P will be an orthogonal projection if and only if it is idempotent. Because P_1 and P_2 are idempotent, we have

$$P^2 = (P_1 + P_2)\,(P_1 + P_2) = P_1 + P_1\,P_2 + P_2\,P_1 + P_2$$

which shows P is idempotent if and only if $P_1\,P_2 + P_2\,P_1 = 0$.

If $P_1 P_2 = 0$, then

$$0 = 0^* = (P_1 P_2)^* = P_2 P_1.$$

It follows that $P_1 P_2 = 0$ if and only if $P_2 P_1 = 0$. Therefore, if $P_1 P_2 = 0$ or $P_2 P_1 = 0$, P will be idempotent.

Now, suppose P is idempotent. $P_1 P_2 + P_2 P_1 = 0$ implies $P_2 P_1 P_2 = 0$, so the range of $P_1 P_2$ is contained in the kernel of P_2. If $M_2 \subset \mathcal{H}$ is the closed subspace onto which P_2 projects, then the kernel of P_2 is $M_2^{\perp}$. In this case, $P_1 P_2[\mathcal{H}] \subset M_2^{\perp}$. However, $P_1 P_2 + P_2 P_1 = 0$ also implies $P_1 P_2 = -P_2 P_1$, so $P_1 P_2[\mathcal{H}] \subset M_2$. Therefore, $P_1 P_2 = 0$, because $M_2 \cap M_2^{\perp} = \{0\}$. $\square$

2.3 The Spectrum and the Resolvent

Definition 2.19. Let $T \in \mathcal{L}(\mathcal{H})$. The **spectrum** of T, denoted $\sigma(T)$, is the set of all $\lambda \in \mathbb{C}$ such that $\lambda - T \in \mathcal{L}(\mathcal{H})$ is not bijective. The **point spectrum** of T, denoted $\sigma_p(T)$, is the set of all $\lambda \in \sigma(T)$ such that $\lambda - T$ is not injective. The **continuous spectrum** of T, denoted $\sigma_c(T)$, is the set of all $\lambda \in \sigma(T)$ such that $\lambda - T$ is injective with a dense range, but not surjective. The **residual spectrum** of T, denoted $\sigma_r(T)$, is the set of all $\lambda \in \sigma(T)$ such that $\lambda - T$ is injective, but without a dense range. $\sigma(T)$ is the disjoint union of $\sigma_p(T)$, $\sigma_c(T)$, and $\sigma_r(T)$.

Theorem 2.20 (Inverse Mapping Theorem)**.** *Let X and Y be Banach spaces, and let $T \in \mathcal{L}(X, Y)$. If T is bijective, then $T^{-1} \in \mathcal{L}(Y, X)$.*

The previous theorem may be found, for example, in [2] (Theorem 12.5). The following theorem, whose proof depends on the Inverse Mapping Theorem, allows us to define another subset of the spectrum.

Theorem 2.21. *$T \in \mathcal{L}(\mathcal{H})$ is bijective if and only if the range of T is dense in $\mathcal{H}$, and there exists $\epsilon > 0$ such that $\|Tx\|_{\mathcal{H}} \geq \epsilon \cdot \|x\|_{\mathcal{H}}$ for all $x \in \mathcal{H}$.*

Proof. Assume T is bijective. The range of T is clearly dense in $\mathcal{H}$. From the boundedness of T^{-1}, we have

$$\|x\|_{\mathcal{H}} = \|T^{-1} Tx\|_{\mathcal{H}} \leq \|T^{-1}\| \cdot \|Tx\|_{\mathcal{H}}$$

for all $x \in \mathcal{H}$. Taking ϵ to be the inverse of $\|T^{-1}\|$, we obtain the desired inequality.

Now, assume the range of T is dense in $\mathcal{H}$, and that there exists $\epsilon > 0$ such that $\|Tx\|_{\mathcal{H}} \geq \epsilon \cdot \|x\|_{\mathcal{H}}$ for all $x \in \mathcal{H}$. The given inequality immediately implies that $Ker(T) = \{0\}$, so T is injective. If $\{Tx_n\} \subset \mathcal{H}$ is a Cauchy sequence, then the same inequality also implies that $\{x_n\}$ is Cauchy. The completeness of $\mathcal{H}$ and the continuity of T then imply that there exists $x \in \mathcal{H}$ such that $\lim_n Tx_n = Tx$, so $T[\mathcal{H}]$ is closed. As a closed and dense subset of $\mathcal{H}$, it follows that $T[\mathcal{H}] = \mathcal{H}$. Therefore, T is bijective. $\square$

Definition 2.22. Let $T \in \mathcal{L}(\mathcal{H})$. $\lambda \in \mathbb{C}$ is in the **approximate point spectrum** of T if for every $\epsilon > 0$, there exists $x \in \mathcal{H}$ such that $\|(\lambda - T)x\|_{\mathcal{H}} < \epsilon \cdot \|x\|_{\mathcal{H}}$.

Theorem 2.23. *For any $T \in \mathcal{L}(\mathcal{H})$, $\sigma_p(T)$ and $\sigma_c(T)$ are subsets of $\sigma_{app}(T)$. If T is normal, then $\sigma_r(T) = \emptyset$ and $\sigma_{app}(T) = \sigma(T)$.*

Proof. Consider any $\lambda \in \sigma_p(T)$. There exists a nonzero $x \in \mathcal{H}$ such that $(\lambda - T)x = 0$, because $\lambda - T$ is linear and not injective. Clearly $\|(\lambda - T)x\|_{\mathcal{H}} = 0$, and $0 < \epsilon \cdot \|x\|_{\mathcal{H}}$ for all $\epsilon > 0$, so $\lambda \in \sigma_{app}(T)$. Now, consider any $\lambda \in \sigma_c(T)$. $\lambda - T$ has a dense range, but it is not invertible. In view of Theorem (2.21), we must have $\lambda \in \sigma_{app}(T)$.

Assume T is normal, and let $\lambda \in \mathbb{C}$ be such that $(\lambda - T) \in \mathcal{L}(\mathcal{H})$ is injective, i.e. $Ker(\lambda - T) = \{0\}$. T is normal, and I is the multiplicative identity in $\mathcal{L}(\mathcal{H})$, so $\lambda - T$ is normal. Therefore,

$$\overline{(\lambda - T)[\mathcal{H}]} = Ker(\lambda - T)^{\perp} = \{0\}^{\perp} = \mathcal{H}$$

by Lemma (2.11). This shows that $\lambda - T$ has a dense range, so $\lambda \notin \sigma_r(T)$. As desired, $\sigma_r(T)$ is empty. It follows that $\sigma_{app}(T) = \sigma(T)$. $\square$

Theorem 2.24. *If $T \in \mathcal{L}(\mathcal{H})$ is self-adjoint, then $\sigma(T) \subset \mathbb{R}$.*

Proof. Consider any $\lambda \in \mathbb{C}$. If $\lambda_r, \lambda_i \in \mathbb{R}$ are such that $\lambda = \lambda_r + i \cdot \lambda_i$, then we have

$$\begin{aligned}\|(\lambda - T)x\|_{\mathcal{H}}^2 &= \langle (\lambda_r - i\lambda_i - T)(\lambda_r + i\lambda_i - T)x, x\rangle \\ &= \langle (\lambda_r - T)^2 x, x\rangle + \langle \lambda_i^2 x, x\rangle = \|(\lambda_r - T)x\|_{\mathcal{H}}^2 + \lambda_i^2 \cdot \|x\|_{\mathcal{H}}^2 \geq \lambda_i^2 \cdot \|x\|_{\mathcal{H}}^2,\end{aligned}$$

for all $x \in \mathcal{H}$, because T is self-adjoint. This implies $\lambda \notin \sigma_{app}(T)$ when $\lambda_i \neq 0$. Noting that $\sigma_{app}(T) = \sigma(T)$, the desired result follows. $\square$

Definition 2.25. Consider $T \in \mathcal{L}(\mathcal{H})$. The **resolvent set** of T, denoted $\rho(T)$, is the set of all $\lambda \in \mathbb{C}$ such that $\lambda - T \in \mathcal{L}(\mathcal{H})$ is bijective. For $\lambda \in \rho(T)$, $R_\lambda(T) \equiv (\lambda - T)^{-1} \in \mathcal{L}(\mathcal{H})$ is called the **resolvent** of T at λ.

Lemma 2.26. *Let $T \in \mathcal{L}(\mathcal{H})$. If $\|T\| < 1$, then $\sum_{n=0}^{\infty} T^n \in \mathcal{L}(\mathcal{H})$. If $\sum_{n=0}^{\infty} T^n \in \mathcal{L}(\mathcal{H})$, then $I - T$ is invertible, with $\sum_{n=0}^{\infty} T^n$ as its inverse.*

Proof. If $\|T\| < 1$, then we have

$$\sum_{n=0}^{\infty} \|T^n\| \leq \sum_{n=0}^{\infty} \|T\|^n = \frac{1}{1 - \|T\|},$$

which shows $\sum_{n=0}^{\infty} T^n$ converges absolutely. $\mathcal{L}(\mathcal{H})$ is Banach, so every series in $\mathcal{L}(\mathcal{H})$ that converges absolutely also converges with respect to the norm topology. Therefore, $\sum_{n=0}^{\infty} T^n \in \mathcal{L}(\mathcal{H})$.

If $\sum_{n=0}^{\infty} T^n \in \mathcal{L}(\mathcal{H})$, then

$$(I - T)\left(\sum_{n=0}^{\infty} T^n\right) = \sum_{n=0}^{\infty} (I\,T^n) - \sum_{n=0}^{\infty} (T\,T^n) = \sum_{n=0}^{\infty} T^n - \sum_{n=0}^{\infty} T^{n+1} = \sum_{n=0}^{\infty} T^n - \sum_{n=1}^{\infty} T^n = I,$$

using the continuity of composition. The continuity of composition can also be used to show that $I - T$ commutes with $\sum_{n=0}^{\infty} T^n$. Therefore, $I - T$ is invertible, and $\sum_{n=0}^{\infty} T^n$ is its inverse. $\square$

Theorem 2.27. *Let $T \in \mathcal{L}(\mathcal{H})$. The series $\sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n$ has an annulus of convergence which contains $\{\lambda \in \mathbb{C} \mid |\lambda| > \|T\|\}$, and is contained in $\rho(T)$. If $\lambda \in \rho(T)$ is such that $\sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n \in \mathcal{L}(\mathcal{H})$, then $R_\lambda(T) = \sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n$.*

Proof. Consider any $\lambda_0 \in \mathbb{C}$ such that $|\lambda_0| > \|T\| \geq 0$. It is then clear $\lambda_0^{-1} T \in \mathcal{L}(\mathcal{H})$, with $\|\lambda_0^{-1} T\| < 1$. Therefore, $\sum_{n=0}^{\infty} \left(\lambda_0^{-1} T\right)^n$ is convergent, by Lemma (2.26).

Now, consider any $\lambda \in \mathbb{C}$ for which $\sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n$ is convergent. Note that λ_0 satisfies this additional property, because $\sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n \in \mathcal{L}(\mathcal{H})$ if and only if $\sum_{n=0}^{\infty} \left(\lambda_0^{-1} T\right)^n \in \mathcal{L}(\mathcal{H})$. By Lemma (2.26),

$$\left(\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}} \right) = \left(\frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{T}{\lambda} \right)^n \right)^{-1} = \lambda \left(I - \frac{T}{\lambda} \right) = \lambda - T.$$

Therefore, $\lambda \in \rho(T)$, with $R_\lambda(T) = \sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n$. □

Corollary 2.28. *If $T \in \mathcal{L}(\mathcal{H})$, then $\lim_{\lambda \to \infty} R_\lambda(T) = 0$.*

Proof. By Theorem (2.27), we have $R_\lambda(T) = \sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n$ for $|\lambda| > \|T\|$. Letting $\psi = \lambda^{-1}$, we then have $R_\lambda(T) = \sum_{n=0}^{\infty} \psi^{n+1} T^n$ for $0 < |\psi| < \|T\|$. This implies that the power series $\sum_{n=0}^{\infty} \psi^{n+1} T^n$ is valid for $|\psi| < \|T\|$, and hence continuous at $\psi = 0$. Therefore,

$$\lim_{\lambda \to \infty} R_\lambda(T) = \lim_{\psi \to 0} \sum_{n=0}^{\infty} \psi^{n+1} T^n = \sum_{n=0}^{\infty} 0^{n+1} T^n = 0$$

as desired. □

Theorem 2.29. *Let $T \in \mathcal{L}(\mathcal{H})$. $\rho(T)$ is an open subset of $\mathbb{C}$, on which $R_\lambda(T)$ is an $\mathcal{L}(\mathcal{H})$-valued analytic function.*

Proof. Fix $\lambda_0 \in \rho(T)$, and let $\delta = \|R_{\lambda_0}(T)\|^{-1} > 0$. If $\lambda \in B_\delta(\lambda_0) \subset \mathbb{C}$, then clearly

$$\|(\lambda_0 - \lambda) R_{\lambda_0}(T)\| < \|R_{\lambda_0}(T)\|^{-1} \cdot \|R_{\lambda_0}(T)\| = 1.$$

In that case, $\sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n \left[R_{\lambda_0}(T)\right]^n \in \mathcal{L}(\mathcal{H})$ is the inverse of $I - (\lambda_0 - \lambda) R_{\lambda_0}(T)$, by Lemma (2.26). This implies $R_{\lambda_0}(T) \circ \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n \left[R_{\lambda_0}(T)\right]^n$ is invertible, with

$$\left(R_{\lambda_0}(T) \circ \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n \left[R_{\lambda_0}(T)\right]^n \right)^{-1} = \left[I - (\lambda_0 - \lambda) R_{\lambda_0}(T)\right] \circ (\lambda_0 - T) = \lambda - T$$

as its inverse. Finally, note that

$$R_{\lambda_0}(T) \circ \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n \left[R_{\lambda_0}(T)\right]^n = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n \left[R_{\lambda_0}(T)\right]^{n+1},$$

by the continuity of composition.

For an arbitrary $\lambda_0 \in \rho(T)$, we have shown there exists $\delta > 0$ such that $B_\delta(\lambda_0) \subset \rho(T)$. Furthermore, $R_\lambda(T) : \rho(T) \to \mathcal{L}(\mathcal{H})$ has a power series representation on $B_\delta(\lambda_0)$, namely $R_\lambda(T) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n \left[R_{\lambda_0}(T)\right]^{n+1}$. Therefore, $\rho(T) \subset \mathbb{C}$ is open, and $R_\lambda(T)$ is analytic on its domain. □

Corollary 2.30. *Let $T \in \mathcal{L}(\mathcal{H})$. $\sigma(T)$ is a nonempty, compact subset of $\mathbb{C}$.*

Proof. $\sigma(T) = \rho(T)^c$ is a closed and bounded subset of $\mathbb{C}$, so it is compact. We are left with the more difficult task of showing $\sigma(T)$ is nonempty. Towards this, assume $\rho(T) = \mathbb{C}$.

By Corollary (2.28), $\lim_{\lambda\to\infty} R_\lambda(T) = 0$. Therefore, there exists $\delta > 0$ such that $\|R_\lambda(T)\| < 1$ for $|\lambda| > \delta$. For such a δ, consider the closed ball $\overline{B_\delta(0)} \subset \mathbb{C}$. By the continuity of $R_\lambda(T)$, the image of $\overline{B_\delta(0)}$ under $R_\lambda(T)$ is compact in $\mathcal{L}(\mathcal{H})$, and hence bounded. Therefore, there exists $C_0 > 0$ such that $\|R_\lambda(T)\| < C_0$ for $|\lambda| \leq \delta$. Letting $C = \max(1, C_0)$, it follows that $R_\lambda(T)$ is bounded.

$R_\lambda(T)$ is a bounded and entire function on $\mathbb{C}$, so the vector-valued version of Liouville's theorem implies the existence of some $S \in \mathcal{L}(\mathcal{H})$ such that $R_\lambda(T) \equiv S$. However, because $\lim_{\lambda\to\infty} R_\lambda(T) = 0$, we must have $R_\lambda(T) \equiv 0 \in \mathcal{L}(\mathcal{H})$. $0 \in \mathcal{L}(\mathcal{H})$ is not invertible, so this is a contradiction. It follows that $\sigma(T)$ is nonempty, as desired. $\square$

Definition 2.31. Let $T \in \mathcal{L}(\mathcal{H})$. We define $r(T) \equiv \sup_{\lambda\in\sigma(T)} |\lambda|$. $r(T)$ is called the **spectral radius** of T.

Theorem 2.32. *If $T \in \mathcal{L}(\mathcal{H})$, then $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$. If T is normal, then $r(T) = \|T\|$.*

Proof. We will first show that $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ exists. If $T \equiv 0$, then existence of the limit is trivial. Assuming $\|T\| > 0$, define $a_n = \log \|T^n\|$ for $n \in \mathbb{Z}^{\geq 0}$. If $m, n \in \mathbb{Z}^{\geq 0}$, then

$$a_{m+n} = \log \|T^{m+n}\| \leq \log\left(\|T^m\| \cdot \|T^n\|\right) = \log \|T^m\| + \log \|T^n\| = a_m + a_n$$

and

$$a_{n\cdot m} = \log \|T^{n\cdot m}\| \leq \log\left(\|T^m\|^n\right) = n \cdot \log \|T^m\| = n \cdot a_m.$$

Now, fix $m_0 \in \mathbb{Z}^{>0}$. For any $n \in \mathbb{Z}^{\geq m_0}$, there exist unique $q_n, r_n \in \mathbb{Z}$ such that $0 \leq r_n < m_0$ and $n = q_n \cdot m_0 + r_n$. Noting that $\frac{q_n}{n} = \left(1 - \frac{r_n}{n}\right)\frac{1}{m_0}$, we have

$$\frac{a_n}{n} = \frac{a_{q_n\cdot m_0 + r_n}}{n} \leq \frac{q_n}{n} \cdot a_{m_0} + \frac{a_{r_n}}{n} = \left(1 - \frac{r_n}{n}\right)\frac{a_{m_0}}{m_0} + \frac{a_{r_n}}{n}.$$

$\{r_n\}$ and $\{a_{r_n}\}$ are finite, and hence bounded, sets of real numbers, so

$$\limsup_n \frac{a_n}{n} \leq \limsup_n \left(\left(1 - \frac{r_n}{n}\right)\frac{a_{m_0}}{m_0} + \frac{a_{r_n}}{n}\right) = \lim_{n\to\infty} \left(\left(1 - \frac{r_n}{n}\right)\frac{a_{m_0}}{m_0} + \frac{a_{r_n}}{n}\right) = \frac{a_{m_0}}{m_0}.$$

Because m_0 was an arbitrary positive integer, this shows $\limsup_n \frac{a_n}{n}$ is a lower bound for $\{\frac{a_n}{n}\}_{n\in\mathbb{Z}^{>0}}$. Therefore,

$$\limsup_n \frac{a_n}{n} \leq \inf\left\{\frac{a_n}{n}\right\}_{n\in\mathbb{Z}^{>0}} \leq \liminf_n \frac{a_n}{n},$$

which implies

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf\left\{\frac{a_n}{n}\right\}_{n\in\mathbb{Z}^{>0}} \in \mathbb{R}.$$

Because e^x is a continuous, strictly increasing function from $\mathbb{R}$ onto $\mathbb{R}^{>0}$, we have

$$\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{a_n}{n}} = e^{\inf\{\frac{a_n}{n}\}_{n\in\mathbb{Z}^{>0}}} = \inf\{e^{\frac{a_n}{n}}\}_{n\in\mathbb{Z}^{>0}} = \inf\{\|T^n\|^{\frac{1}{n}}\}_{n\in\mathbb{Z}^{>0}} \in \mathbb{R}.$$

This shows that $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ exists, as desired.

$R_\lambda(T)$ is an analytic, $\mathcal{L}(\mathcal{H})$-valued function on the open set $\rho(T)$, so it is clear from Theorem (2.27) that $\sum_{n=0}^{\infty} \lambda^{-(n+1)}\, T^n$ is the unique Laurent series representation for $R_\lambda(T)$, centred at $0 \in \mathbb{C}$, and valid for $|\lambda| > \|T\|$. Let r be the inner radius of convergence of $\sum_{n=0}^{\infty} \lambda^{-(n+1)}\, T^n$. Because the annulus $\{\lambda \in \mathbb{C} \mid |\lambda| > r(T)\}$ is contained in $\rho(T)$, we must have $r \leq r(T)$. However, if $r < r(T)$, then the annulus of convergence for $\sum_{n=0}^{\infty} \lambda^{-(n+1)}\, T^n$ will have a nonempty intersection with $\sigma(T)$. This is a contradiction, by Theorem (2.27), so $r = r(T)$. Using the vector-valued version of Hadamard's formula, we get

$$r(T) = r = \limsup_{n} \|T^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}.$$

Finally, if $T \in \mathcal{L}(\mathcal{H})$ is normal, then we have

$$\begin{aligned} r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \left\| T^{2^{n-1}} \right\|^{2\cdot 2^{-n}} &= \lim_{n\to\infty} \left\| \left(T^{2^{n-1}}\right)^* T^{2^{n-1}} \right\|^{2^{-n}} \\ &= \lim_{n\to\infty} \left\| (T^*T)^{2^{n-1}} \right\|^{2^{-n}} = \lim_{n\to\infty} \|T\|^{2^n \cdot 2^{-n}} = \|T\|, \end{aligned}$$

using two applications of Lemma (2.12). $\square$

Chapter 3

Spectral Theory for Self-Adjoint Bounded Operators

In this chapter, we formulate and prove three versions of the spectral theorem for self-adjoint, bounded operators. We introduce the functional calculus in the first section, and the multiplication operator version in the second. The main reference for both of these sections is [5], but we provide an expanded presentation. In the last section, we develop the projection-valued measure version, using both [4] and [5] as references.

3.1 The Functional Calculus

Definition 3.1. For $T \in \mathcal{L}(\mathcal{H})$ and $p(z) = \sum_{i=0}^{n} \alpha_i \cdot z^i$, a complex polynomial of one variable, we define $p(T) := \sum_{i=0}^{n} \alpha_i \cdot T^i \in \mathcal{L}(\mathcal{H})$, where $T^0 = I$.

The proof of the following lemma is omitted, but it is trivial.

Lemma 3.2. *Let $T \in \mathcal{L}(\mathcal{H})$. If $\mathcal{Q}$ denotes the algebra of complex polynomials of one variable, then the map $p \mapsto p(T)$ is a unital algebraic homomorphism from $\mathcal{Q}$ to $\mathcal{L}(\mathcal{H})$.*

Lemma 3.3. *If $T \in \mathcal{L}(\mathcal{H})$, and $p(z)$ is a complex polynomial of one variable, then $\sigma\left[p(T)\right] = p[\sigma(T)]$. In particular, if $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$ and $x \in \mathcal{H}$, then $p(T)x = p(\lambda)x$.*

Proof. Consider $\lambda \in \sigma(T)$. Because $p(\lambda) - p(z)$ has λ as a root, there is a complex polynomial $q(z)$ such that

$$p(\lambda) - p(z) = (\lambda - z) \cdot q(z) = q(z) \cdot (\lambda - z),$$

which implies

$$p(\lambda) - p(T) = (\lambda - T)\, q(T) = q(T)\, (\lambda - T) \in \mathcal{L}(\mathcal{H}).$$

Because $\lambda - T$ is not invertible, it is not bijective. If $\lambda - T$ is not surjective, $p(\lambda) - p(T) = (\lambda - T)\, q(T)$ shows that $p(\lambda) - p(T)$ is not surjective. If $\lambda - T$ is not injective, i.e. $\lambda x - Tx = 0$ for some nonzero $x \in \mathcal{H}$, then $p(\lambda)x - p(T)x = q(T)\,(\lambda x - Tx) = 0$ shows that $p(\lambda) - p(T)$ is not injective. Therefore, $p(\lambda) - p(T)$ is not invertible. We may conclude $\sigma\left[p(T)\right] \supset p[\sigma(T)]$.

Consider $\lambda \in \sigma\left[p(T)\right]$, and assume $p(z)$ has degree zero, i.e. $p(z) \equiv \alpha$ for some $\alpha \in \mathbb{C}$. Because $\lambda - p(T) = (\lambda - \alpha)I$ is not invertible, we must have $\lambda = \alpha$. Noting that $p[\sigma(T)] = \{\alpha\}$ because $\sigma(T)$ is nonempty, it is then clear $\sigma\left[p(T)\right] \subset p[\sigma(T)]$.

Now, assume $p(z)$ has degree $n \geq 1$. If $\{\lambda_i\}_{i=1}^{n}$ are the roots of $\lambda - p(z)$, then we have

$$\lambda - p(z) = \beta(\lambda_1 - z) \cdots (\lambda_n - z)$$

for some necessarily nonzero $\beta \in \mathbb{C}$. Given this factorization,

$$\lambda - p(T) = \beta(\lambda_1 - T) \cdots (\lambda_n - T).$$

Because $\beta \neq 0$, $\lambda - p(T)$ would be invertible if each $\lambda_i - T$ was invertible. However, $\lambda - p(T)$ is not invertible, so there is some $\lambda_0 \in \{\lambda_i\}_{i=1}^n$ for which $\lambda_0 - T$ is not invertible. Because $p(\lambda_0) = \lambda$, we may conclude $\sigma\,[p(T)] \subset p[\sigma(T)]$. $\square$

Lemma 3.4. *Let $T \in \mathcal{L}(\mathcal{H})$, and let p be a complex polynomial of one variable.*
(a) If T is normal, then $\|p(T)\| = \sup_{\lambda \in \sigma(T)} |p(\lambda)|$.
(b) If p is a polynomial of a real variable, then so is $\overline{p}$, and $\overline{p}(T) = p(T)^*$.

Proof. Consider the complex polynomial $p(z) = \sum_{i=0}^n a_i z^i$. The adjoint of $p(T)$ is given by

$$p(T)^* = \left(\sum_{i=0}^n a_i T^i \right)^* = \sum_{i=0}^n \left(a_i T^i\right)^* = \sum_{i=0}^n \overline{a_i}\left(T^*\right)^i.$$

From this, two facts are now clear. First, if $\overline{p}$ denotes the pointwise complex conjugation of p, then part (b) is immediate. Second, if T is normal, then $p(T)$ is normal. In that case,

$$\|p(T)\| = r[p(T)] = \sup_{\lambda \in \sigma[p(T)]} |\lambda| = \sup_{\lambda \in \sigma(T)} |p(\lambda)|,$$

using Theorem (2.32), and Lemma (3.3). This proves part (a). $\square$

Consider the following version of the Stone-Weierstrass Theorem, which may be found in [3] (Theorem 4.51).

Theorem 3.5 (Complex Stone-Weierstrass Theorem)**.** *Let X be a compact Hausdorff space. If $\mathcal{A}$ is a closed, complex *-subalgebra of $\mathcal{C}(X)$ that separates points, then either $\mathcal{A} = \mathcal{C}(X)$ or $\mathcal{A} = \{f \in \mathcal{C}(X) \mid f(x_0) = 0\}$ for some $x_0 \in X$.*

Corollary 3.6. *Let X be a compact subset of $\mathbb{R}^n$, where $n \in \mathbb{N}$. $\mathcal{C}(X)$ is the completion of the collection of complex polynomials of n variables, with respect to the uniform norm.*

Proof. Define $\mathcal{Q}$ to be the collection of complex polynomials of n variables. Because X is compact, $\mathcal{C}(X)$ is a Banach space under the uniform norm. Therefore, the closure of $\mathcal{Q}$ in $\mathcal{C}(X)$ will also be the completion of $\mathcal{Q}$ with respect to the uniform norm.

It is easily seen $\mathcal{Q}$ is a complex subalgebra of $\mathcal{C}(X)$. $\mathcal{Q}$ will also be closed under complex conjugation, because the variables for any $p \in \mathcal{Q}$ are necessarily real-valued. By the continuity of the algebra and complex conjugation operations, $\overline{\mathcal{Q}}$ will be a closed, complex *-subalgebra of $\mathcal{C}(X)$. The coordinate projection maps of $\mathbb{R}^n$ to $\mathbb{C}$ are polynomials contained in $\overline{\mathcal{Q}}$, so $\overline{\mathcal{Q}}$ will necessarily separate points of X.

The Stone-Weierstrass Theorem implies that either $\overline{\mathcal{Q}} = \mathcal{C}(X)$ or

$$\overline{\mathcal{Q}} = \{f \in \mathcal{C}(X) \mid f(a_1, \ldots, a_n) = 0\}$$

for some $[a_1, \ldots, a_n] \in X$. However, the constant function $q(x_1, \ldots, x_n) = 1$ is nonzero on X and contained in $\overline{\mathcal{Q}}$, ruling out the second possibility. Therefore, $\mathcal{C}(X)$ is the completion of $\mathcal{Q}$. $\square$

Theorem 3.7 (Bounded Linear Transformation Theorem)**.** *Let X be a normed vector space, and let Y be a Banach space. Each $T \in \mathcal{L}(X, Y)$ has a unique continuous extension to $\overline{T} \in \mathcal{L}\left(\overline{X}, Y\right)$, where $\overline{X}$ is the completion of X. Furthermore, $\left\|\overline{T}\right\| = \|T\|$.*

Proof. Given any $x \in \overline{X}$, there is a sequence $\{x_n\} \subset X$ which converges to x. The boundedness of T and the completeness of Y imply that $\{Tx_n\} \subset Y$ is convergent. With this in mind, we define $\overline{T} : \overline{X} \to Y$ by letting $\overline{T}x = \lim_n Tx_n \in Y$. However, for $\overline{T}$ to be well-defined, $\overline{T}x$ must be independent of the choice of sequence used in its construction.

Let $\{y_n\} \subset X$ be a sequence converging to $y \in \overline{X}$. If $x = y$, then $\{x_n - y_n\} \subset X$ converges to $0 \in X$, by the continuity of vector subtraction in $\overline{X}$. The continuity of T then implies

$$0 = \lim_n T(x_n - y_n) = \lim_n Tx_n - \lim_n Ty_n,$$

and we may conclude $\overline{T}$ is well-defined. For any $\alpha \in \mathbb{C}$, $\{x_n + \alpha y_n\} \subset X$ converges to $x + \alpha y \in \overline{X}$, by the continuity of the vector space operations on $\overline{X}$. Therefore,

$$\overline{T}(x + \alpha y) = \lim_n T(x_n + \alpha y_n) = \lim_n (Tx_n + \alpha T y_n) = \lim_n Tx_n + \alpha \lim_n Ty_n = \overline{T}x + \alpha \overline{T}y,$$

using the linearity of T, and the continuity of the vector space operations on Y. This proves the linearity of $\overline{T}$.

Using the continuity of the norms on Y and X, and the boundedness of T on X, we obtain

$$\left\|\overline{T}x\right\|_Y = \lim_n \|Tx_n\|_Y \leq \lim_n \|T\| \cdot \|x_n\|_X = \lim_n \|T\| \cdot \|x_n\|_{\overline{X}} = \|T\| \cdot \|x\|_{\overline{X}}.$$

This implies $\overline{T}$ is bounded with $\left\|\overline{T}\right\| \leq \|T\|$. It is clear that $\left\|\overline{T}\right\| \geq \|T\|$, because $\overline{X}$ contains X.

Finally, suppose $S : \overline{X} \to Y$ is another continuous function extending T. S and $\overline{T}$ are continuous functions agreeing on a dense subset of their domain, so $S \equiv \overline{T}$. $\square$

Theorem 3.8. *Let X and Y be normed vector spaces, and consider $T \in \mathcal{L}(X, Y)$. If $\|Tv\|_Y = \|v\|_X$ for all $v \in V$, where V is a dense subset of X, then T is a linear isometry. If, additionally, X is complete and $T[X]$ is a dense subset of Y, then T is surjective.*

Proof. Given any $x \in X$, there is a sequence $\{x_n\} \subset V$ which converges to x. We then have

$$\|Tx\|_Y = \lim_{n\to\infty} \|Tx_n\|_Y = \lim_{n\to\infty} \|x_n\|_X = \|x\|_X,$$

using the continuity of T and of the norms on X and Y. It follows that T is a linear isometry.

Because T is a linear isometry, it has a well-defined inverse, $T^{-1} : T[X] \to X$, which is a linear isometry on the subspace $T[X] \subset Y$. If X is complete, and $T[X]$ is dense in Y, then T^{-1} has an extension to $\overline{T^{-1}} \in \mathcal{L}(Y, X)$, by the Bounded Linear Transformation Theorem. The first part of this theorem then implies $\overline{T^{-1}}$ is injective. Therefore, for any $y \in Y$ with $\overline{T^{-1}}\, y = x \in X$,

$$\overline{T^{-1}}\, y = x = T^{-1}\, Tx = \overline{T^{-1}}\, Tx$$

implies $y = Tx$. It follows that T is surjective. $\square$

The map ϕ in the following theorem will be referred to as the continuous functional calculus for T.

Theorem 3.9 (Spectral Theorem)**.** *Let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint. There is a unique map $\phi : \mathcal{C}(\sigma(T)) \to \mathcal{L}(\mathcal{H})$ such that:*
(a) ϕ is continuous.
(b) If $Id \in \mathcal{C}(\sigma(T))$ is the identity function, i.e. $Id(z) = z$, then $\phi(Id) = T$.
(c) ϕ is a unital algebraic homomorphism.
In addition, ϕ has the following properties:
(d) ϕ is an isometry.
*(e) ϕ is a *-homomorphism.*
(f) If $f \in \mathcal{C}(\sigma(T))$ is such that $f \geq 0$, then $\phi(f) \geq 0$.
(g) If $S \in \mathcal{L}(\mathcal{H})$ commutes with T, then $\phi(f)\, S = S\, \phi(f)$ for all $f \in \mathcal{C}(\sigma(T))$.
(h) If $Tx = \lambda x$ for some $\lambda \in \sigma(T)$ and $x \in \mathcal{H}$, then $\phi(f)x = f(\lambda)x$ for all $f \in \mathcal{C}(\sigma(T))$.
(i) $\sigma\left[\phi(f)\right] = f\left[\sigma(T)\right]$ for all $f \in \mathcal{C}(\sigma(T))$.

Proof. Let $\mathcal{Q}$ be the collection of complex polynomials of one variable, but with their domain restricted to $\sigma(T) \subset \mathbb{R}$. $\mathcal{Q}$ is a normed vector space with respect to the uniform norm, because of the continuity of polynomials and the compactness of $\sigma(T)$. Let $\hat{\phi} : \mathcal{Q} \to \mathcal{L}(\mathcal{H})$ be the map described in Lemma (3.2), i.e. $\hat{\phi}(p) = p(T)$. $\hat{\phi}$ is linear, and

$$\|\hat{\phi}(p)\|_{\mathcal{L}(\mathcal{H})} = \|p(T)\|_{\mathcal{L}(\mathcal{H})} = \sup_{\lambda \in \sigma(T)} |p(\lambda)| = \|p\|_u$$

for all $p \in \mathcal{Q}$, by Lemma (3.4). Therefore, $\hat{\phi}$ is a bounded linear transformation from $\mathcal{Q}$ to $\mathcal{L}(\mathcal{H})$. Noting that $\mathcal{L}(\mathcal{H})$ is complete, the Bounded Linear Transformation Theorem implies the existence of a unique bounded linear transformation $\phi : \overline{\mathcal{Q}} \to \mathcal{L}(\mathcal{H})$ which extends $\hat{\phi}$, where $\overline{\mathcal{Q}}$ is the completion of $\mathcal{Q}$. However, $\overline{\mathcal{Q}} = \mathcal{C}(\sigma(T))$ by Corollary (3.6).

We claim this ϕ is the desired map. Because ϕ agrees with $\hat{\phi}$ on polynomials, it must preserve the multiplicative identity and map $Id(z) = z$ to T. If ϕ preserves the multiplication operation, it will be a unital algebraic homomorphism. Let $f, g \in \mathcal{C}(\sigma(T))$. $\mathcal{Q}$ is dense in $\mathcal{C}(\sigma(T))$, so there exist sequences $\{p_n\}, \{q_n\} \subset \mathcal{Q}$ that converge uniformly to f and g, respectively. By continuity of the pointwise product, $\{p_n \cdot q_n\} \subset \mathcal{Q}$ converges uniformly to $f \cdot g \in \mathcal{C}(\sigma(T))$, and by Lemma (3.2), ϕ preserves the multiplication operation when restricted to $\mathcal{Q}$. Therefore,

$$\phi(f \cdot g) = \lim_{n \to \infty} \phi(p_n \cdot q_n) = \lim_{n \to \infty} \left(\phi(p_n) \circ \phi(q_n)\right) = \phi(f) \circ \phi(g)$$

using the continuity of ϕ and composition.

Towards proving the uniqueness of ϕ, consider any map ψ that satisfies properties (a), (b), and (c) of the theorem. For any polynomial $p(z) = \sum_{i=0}^{n} \alpha_i \cdot z^i$,

$$\psi(p) = \sum_{i=0}^{n} \alpha_i \cdot T^i = \hat{\phi}(p),$$

since ψ is a unital algebraic homomorphism and $\psi(Id) = T$. Because ψ and ϕ agree on a dense subset, their continuity implies $\psi \equiv \phi$.

We will now prove the additional properties of ϕ. First, recall $\|\phi(p)\|_{\mathcal{L}(\mathcal{H})} = \|p\|_u$ for all $p \in \mathcal{Q}$. $\mathcal{Q}$ is dense in $\mathcal{C}(\sigma(T))$, and $\phi \in \mathcal{L}(\mathcal{C}(\sigma(T)), \mathcal{H})$, so Theorem (3.8) implies ϕ is an isometry.

For the remainder of the proof, let $f \in \mathcal{C}(\sigma(T))$ and $\{p_n\}$ be as before. By Lemma (3.4), ϕ will preserve the star operation on $\mathcal{Q}$, which is pointwise complex conjugation. Also, by the continuity of complex conjugation, $\{\overline{p_n}\} \subset \mathcal{Q}$ converges uniformly to $\overline{f} \in \mathcal{C}(\sigma(T))$. Therefore,

$$\phi(\overline{f}) = \lim_{n\to\infty} \phi(\overline{p_n}) = \lim_{n\to\infty} \phi(p_n)^* = \phi(f)^*$$

using the continuity of ϕ and *. This shows ϕ is a *-homomorphism.

Suppose $f \geq 0$, and let $g = \sqrt{f}$. We then have $g \in \mathcal{C}(\sigma(T))$ because f is nonnegative. Because ϕ is an algebraic *-homomorphism,

$$\langle \phi(f)x, x\rangle = \langle \phi(g) \circ \phi(g)x, x\rangle = \langle \phi(g)x, \phi\left(\overline{g}\right)x\rangle = \langle \phi(g)x, \phi(g)x\rangle \geq 0$$

for all $x \in \mathcal{H}$. Therefore, $\phi(f) \geq 0$.

Suppose $T\,S = S\,T$ for some $S \in \mathcal{L}(\mathcal{H})$. It is clear that $p(T)\,S = S\,p(T)$ for any $p \in \mathcal{P}$. Therefore,

$$\phi(f)\,S = \lim_{n\to\infty} (\phi(p_n)\,S) = \lim_{n\to\infty} (S\,\phi(p_n)) = S\,\phi(f)$$

using the continuity of ϕ, right composition, and left composition. In particular, this implies that any two bounded operators in the range of ϕ will commute.

Suppose $Tx = \lambda x$ for some $\lambda \in \sigma(T)$ and $x \in \mathcal{H}$. By Lemma (3.3), $\phi(p_n)x = p_n(\lambda)x$. Noting that convergence with respect to the operator norm implies strong convergence, and uniform convergence implies pointwise convergence, we have

$$\phi(f)x = \lim_{n\to\infty} (\phi(p_n)x) = \lim_{n\to\infty} (p_n(\lambda)x) = f(\lambda)x$$

from the continuity of ϕ and the line path $l_x(\alpha) = \alpha \cdot x$.

Consider $\lambda \notin f\,[\sigma(T)]$. $\frac{1}{\lambda - f} \in \mathcal{C}(\sigma(T))$ because $\lambda - f \neq 0$ on $\sigma(T)$. ϕ is an algebraic homomorphism, so

$$\phi\left(\frac{1}{\lambda - f}\right) \circ (\lambda - \phi(f)) = \phi\left(\frac{1}{\lambda - f} \cdot (\lambda - f)\right) = \phi(1) = I$$

This shows $\lambda - \phi(f)$ is invertible, because all elements of $\mathcal{L}(\mathcal{H})$ in the range of ϕ commute. We may conclude $\sigma\,[\phi(f)] \subset f\,[\sigma(T)]$.

Finally, consider $\lambda \in f[\sigma(T)]$. In this case, there exists $\lambda_0 \in \sigma(T)$ such that $f(\lambda_0) = \lambda$. Because T is self-adjoint, λ_0 is in the approximate point spectrum of T. In other words, there is a sequence of unit vectors $\{x_n\} \subset \mathcal{H}$ such that $\lim_{n\to\infty} \|(\lambda_0 - T)x_n\|_{\mathcal{H}} = 0$. We would like to use these vectors to show $\lambda \in \sigma_{app}(\phi(f))$. Towards this, fix $\epsilon > 0$. There is a polynomial $p \in \{p_n\}$ such that $\|f - p\|_u < \frac{\epsilon}{3}$. Note that $p(\lambda_0) - p(z)$ has λ_0 as a root, implying $p(\lambda_0) - p(z) = q(z) \cdot (\lambda_0 - z)$ for some $q \in \mathcal{Q}$. For this q, there is an $x \in \{x_n\}$ such that $\|(\lambda_0 - T)x\|_{\mathcal{H}} < \frac{\epsilon}{3 \cdot \|q\|_u + 1}$. With these selections, we obtain

$$\begin{aligned}\|(\lambda - \phi(f))x\|_{\mathcal{H}} &= \|(f(\lambda_0) - p(\lambda_0) + p(\lambda_0) - \phi(p) + \phi(p) - \phi(f))x\|_{\mathcal{H}} \\ &\leq 2 \cdot \|f - p\|_u + \|q\|_u \cdot \|(\lambda_0 - T)x\|_{\mathcal{H}} < \epsilon.\end{aligned}$$

Because $\epsilon > 0$ was arbitrary, $\lambda \in \sigma_{app}(\phi(f))$. We may conclude $\sigma[\phi(f)] = f[\sigma(T)]$. $\square$

Definition 3.10. Let X be a locally compact Hausdorff space. $M(X)$ denotes the normed vector space of complex Radon measures on X. The norm is given by

$$\begin{aligned} \|\cdot\|_{M(X)} : M(X) &\to \mathbb{R}^{\geq 0} \\ \mu &\mapsto |\mu|(X), \end{aligned}$$

where $|\mu|$ is the total variation of μ.

We will now use the continuous functional calculus to construct a family of complex Radon measures with certain properties. These measures will reappear throughout the thesis. For the construction, we require the following theorem, which may be found in [3] (Theorem 7.17).

Theorem 3.11 (Riesz Representation Theorem)**.** *Let X be a locally compact Hausdorff space. For $\mu \in M(X)$ and $f \in C_0(X)$, define $I_\mu(f) = \int f\ d\mu$. The map $\mu \to I_\mu$ is an isometric isomorphism from $M(X)$ to $C_0(X)^*$.*

Theorem 3.12. *Let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint. For every pair of vectors $x, y \in \mathcal{H}$, there exists a unique, complex measure $\mu_{x,y}$ on $\left(\sigma(T), \mathcal{B}_{\sigma(T)}\right)$ such that*

$$\langle \phi(f)x, y\rangle = \int_{\sigma(T)} f\ d\mu_{x,y}$$

for all $f \in \mathcal{C}(\sigma(T))$. The family $\{\mu_{x,y}\}_{x,y\in\mathcal{H}}$ has the following properties:
(a) $\|\mu_{x,y}\|_{M(\sigma(T))} \leq \|x\|_{\mathcal{H}} \cdot \|y\|_{\mathcal{H}}$.
(b) Each $\mu_{x,x}$ is a finite positive measure, with $\mu_{x,x}(\sigma(T)) = \|x\|^2_{\mathcal{H}}$.
(c) $(x, y) \mapsto \mu_{x,y}$ is a sesquilinear map from $\mathcal{H} \times \mathcal{H}$ to $M(\sigma(T))$.
(d) $\mu_{x,y} = \overline{\mu_{y,x}}$.
(e) For any $f \in \mathcal{C}(\sigma(T))$, $d\mu_{\phi(f)x,y} = f\ d\mu_{x,y} = d\mu_{x,\phi(\overline{f})y}$.

Proof. Fix $x, y \in \mathcal{H}$, and define the map

$$\begin{aligned} l_{x,y} : \mathcal{C}_0(\sigma(T)) &\to \mathbb{C} \\ f &\mapsto \langle \phi(f)x, y\rangle \end{aligned}$$

where ϕ gives the continuous functional calculus. $l_{x,y}$ is linear because of the linearity of ϕ and the inner product. Using the fact that ϕ is an isometry,

$$|l_{x,y}(f)| = |\langle \phi(f)x, y\rangle| \leq \|\phi(f)\|_{\mathcal{L}(\mathcal{H})} \cdot \|x\|_{\mathcal{H}} \cdot \|y\|_{\mathcal{H}} = \|f\|_u \cdot \|x\|_{\mathcal{H}} \cdot \|y\|_{\mathcal{H}}$$

for all $f \in \mathcal{C}_0(\sigma(T))$. Therefore, $l_{x,y} \in \mathcal{C}_0(\sigma(T))^*$.

$\sigma(T)$ is a compact subset of $\mathbb{R}$. This implies $\sigma(T)$ is a second countable, compact Hausdorff space. Therefore, every complex measure on $\left(\sigma(T), \mathcal{B}_{\sigma(T)}\right)$ is Radon, and $\mathcal{C}_0(\sigma(T)) = \mathcal{C}(\sigma(T))$. With this in mind, the Riesz Representation Theorem implies the unique existence of a complex measure $\mu_{x,y}$ on $\left(\sigma(T), \mathcal{B}_{\sigma(T)}\right)$ such that

$$\langle \phi(f)x, y\rangle = \int_{\sigma(T)} f\ d\mu_{x,y}$$

for all $f \in \mathcal{C}(\sigma(T))$. By repeating this process for every pair of vectors in $\mathcal{H}$, we create the family $\{\mu_{x,y}\}_{x,y\in\mathcal{H}}$.

The Riesz Representation Theorem also implies

$$\|\mu_{x,y}\|_{M(\sigma(T))} = \|l_{x,y}\|_{\mathcal{C}_0(\sigma(T))^*} \leq \|x\|_{\mathcal{H}} \cdot \|y\|_{\mathcal{H}}$$

for all $x, y \in \mathcal{H}$. If $x = y$, then

$$|\mu_{x,x}|(\sigma(T)) \leq \|x\|^2_{\mathcal{H}} = \langle \phi(1)x, x\rangle = \int_{\sigma(T)} 1 \; d\mu_{x,x} = \mu_{x,x}(\sigma(T)) \leq |\mu_{x,x}|(\sigma(T))$$

because $\phi(1) = I$. This implies $\mu_{x,x}(\sigma(T)) = \|x\|^2_{\mathcal{H}} = |\mu_{x,x}|(\sigma(T))$. Since μ and its total variation agree on $\sigma(T)$, $\mu_{x,x}$ is a finite positive measure.

To see that $(x, y) \mapsto \mu_{x,y}$ is a sesquilinear map from $\mathcal{H} \times \mathcal{H}$ to $M(\sigma(T))$, consider any $\alpha \in \mathbb{C}$ and $x_1, x_2 \in \mathcal{H}$. For all $f \in \mathcal{C}(\sigma(T))$,

$$\begin{aligned}
&\int_{\sigma(T)} f \; d\mu_{\alpha x_1 + x_2, y} \\
&= \langle \phi(f)(\alpha x_1 + x_2), y\rangle = \alpha\langle \phi(f)x_1, y\rangle + \langle \phi(f)x_2, y\rangle = \alpha \int_{\sigma(T)} f \; d\mu_{x_1,y} + \int_{\sigma(T)} f \; d\mu_{x_2,y} \\
&= \int_{\sigma(T)} f \; d(\alpha\mu_{x_1,y} + \mu_{x_2,y}).
\end{aligned}$$

By uniqueness, $\mu_{\alpha x_1 + x_2, y} = \alpha\mu_{x_1,y} + \mu_{x_2,y}$. Similarly, it can be shown $\mu_{x,\alpha y_1 + y_2} = \overline{\alpha}\mu_{x,y_1} + \mu_{x,y_2}$ for $y_1, y_2 \in \mathcal{H}$.

Because ϕ preserves the star operation,

$$\int_{\sigma(T)} f \; d\mu_{x,y} = \langle \phi(f)x, y\rangle = \overline{\langle \phi(\overline{f})y, x\rangle} = \overline{\int_{\sigma(T)} \overline{f} \; d\mu_{y,x}} = \int_{\sigma(T)} f \; d\overline{\mu_{y,x}}$$

for all $f \in \mathcal{C}(\sigma(T))$. By uniqueness, $\mu_{x,y} = \overline{\mu_{y,x}}$.

To demonstrate the final property, temporarily fix $f \in \mathcal{C}(\sigma(T))$. For all $g \in \mathcal{C}(\sigma(T))$,

$$\int_{\sigma(T)} g \; d\mu_{\phi(f)x,y} = \langle \phi(g) \circ \phi(f)x, y\rangle = \langle \phi(g \cdot f)x, y\rangle = \int_{\sigma(T)} g \cdot f \; d\mu_{x,y}$$

because ϕ preserves the multiplication operation. By uniqueness, $d\mu_{\phi(f)x,y} = f \; d\mu_{x,y}$. From this, and the conjugate symmetry of the measures, it follows that

$$d\mu_{x,\phi(\overline{f})y} = \overline{d\mu_{\phi(\overline{f})y,x}} = \overline{\overline{f} \; d\mu_{y,x}} = f \; d\mu_{x,y}$$

which completes the proof. □

These measures will soon be used to extend the domain of the continuous functional calculus to bounded, Borel measurable functions. This extension will be referred to as the functional calculus.

Definition 3.13. If M is a metric space, then $\mathcal{B}_b(M)$ will denote the *-algebra of bounded, complex-valued, Borel functions on M. $\mathcal{C}_b(M)$ will denote the *-algebra of bounded, complex-valued, continuous functions on M.

Before constructing the functional calculus, we prove $\mathcal{B}_b(M)$ is the smallest vector space containing $\mathcal{C}_b(M)$ and closed under pointwise limits of bounded sequences of functions. This fact will be used to show that most of the properties of the continuous functional calculus transfer to the functional calculus. We require the use of the Monotone Class Theorem, which may be found, for example, in [1] (Theorem 1.9.3).

Theorem 3.14 (Monotone Class Theorem)**.** *(i) If $\mathcal{A}$ is an algebra of subsets of X, then the σ-algebra generated by $\mathcal{A}$ is the same as the monotone class generated by $\mathcal{A}$. (ii) If $\mathcal{E}$ is a collection of subsets of X that is closed with respect to finite intersections, then the σ-algebra generated by $\mathcal{E}$ is the same as the σ-additive class generated by $\mathcal{E}$.*

Theorem 3.15. *If M is a metric space, then $\mathcal{B}_b(M)$ is the smallest vector space containing $\mathcal{C}_b(M)$ and closed with respect to pointwise limits of bounded sequences of functions.*

Proof. It is clear that $\mathcal{B}_b(M)$ is a vector space containing $\mathcal{C}_b(M)$. The pointwise limit of a bounded sequence of functions will also be bounded, so $\mathcal{B}_b(M)$ is closed with respect to such limits. Let $\mathcal{V}$ be any subspace of $\mathcal{B}_b(M)$ that also has these properties. If we can show $\mathcal{B}_b(M) \subset \mathcal{V}$, the theorem will follow.

Let $\mathcal{S}$ be the collection of all $B \in \mathcal{B}_M$ such that $\chi_B \in \mathcal{V}$. We claim $\mathcal{S}$ is a σ-additive class. Towards proving this claim, first note that $M \in \mathcal{S}$, because $\chi_M \in \mathcal{C}_b(M) \subset \mathcal{V}$. Now, consider $A, B \in \mathcal{S}$ such that $B \subset A$. $A - B$ is also a Borel set, and $\chi_{A-B} = \chi_A - \chi_B \in \mathcal{V}$, so $A - B \in \mathcal{S}$. Finally, consider a pairwise-disjoint and countable collection $\{A_n\} \subset \mathcal{S}$. $\{\chi_{\bigcup_{n=1}^{j} A_n}\}_j \subset \mathcal{V}$ because each $\chi_{\bigcup_{n=1}^{j} A_n}$ is equivalent to the sum $\sum_{n=1}^{j} \chi_{A_n} \in \mathcal{V}$. Also, all characteristic functions are bounded by 1, and $\chi_{\bigcup_{n=1}^{j} A_n} \to \chi_{\bigcup A_n}$ pointwise. Therefore, $\chi_{\bigcup A_n} \in \mathcal{V}$ and $\bigcup A_n \in \mathcal{S}$. As claimed, $\mathcal{S}$ is a σ-additive class.

Letting $\mathcal{E}$ be the collection of closed subsets of M, we will show $\mathcal{E} \subset \mathcal{S}$. Consider an arbitrary $C \in \mathcal{E}$. We define a monotonically decreasing sequence of functions $\{f_n\}$ by $f_n = g_n \circ d(\cdot, C)$, where

$$g_n(y) = \begin{cases} 1 \;\text{ if }\; y = 0 \\ 1 - n \cdot y \;\text{ if }\; 0 < y < n^{-1} \\ 0 \;\text{ if }\; y \geq n^{-1} \end{cases}$$

is a complex-valued function on $[0, \infty)$, and $d(\cdot, C)$ is the function on M giving the distance from C. Each f_n is continuous because it is a composition of continuous functions. It is also clear that each f_n is bounded by 1, and that $f_n \to \chi_{C_1}$ pointwise, where $C_1 = \{x \in M \mid d(x, C) = 0\}$. However, $C_1 = C$ because C is closed. Therefore, $\{f_n\} \subset \mathcal{C}_b(M)$ and $\chi_C \in \mathcal{V}$. As desired, $\mathcal{E} \subset \mathcal{S}$.

$\mathcal{E}$ is closed with respect to finite intersections, so the σ-additive class it generates is the same as $\mathcal{B}_M$, by part (ii) of the Monotone Class Theorem. $\mathcal{S}$ is a σ-additive class containing $\mathcal{E}$ and contained in $\mathcal{B}_M$, so $\mathcal{S} = \mathcal{B}_M$. In other words, $\mathcal{V}$ contains all characteristic functions. It follows that $\mathcal{V}$ also contains all simple functions, because $\mathcal{V}$ is a vector space. This implies $\mathcal{B}_b(M) \subset \mathcal{V}$, because every bounded Borel function is the pointwise limit of a bounded sequence of simple functions, and $\mathcal{V}$ is closed with respect to such limits. $\square$

In the case where M is a compact subset of $\mathbb{R}^n$, we have the following corollary. Its proof is immediate from Theorem (3.15) and Corollary (3.6).

Corollary 3.16. *Let M be a compact subset of $\mathbb{R}^n$. $\mathcal{B}_b(M)$ is the smallest vector space containing all complex polynomials of n variables, and closed with respect to pointwise limits of bounded sequences of functions.*

The map Φ in the following theorem will be referred to as the functional calculus for T.

Theorem 3.17 (Spectral Theorem)**.** *Let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint. There is a unique map $\Phi : \mathcal{B}_b(\sigma(T)) \to \mathcal{L}(\mathcal{H})$ such that:*
(a) If $Id \in \mathcal{B}_b(\sigma(T))$ is the identity function, i.e. $Id(z) = z$, then $\Phi(Id) = T$.
(b) Φ is a unital algebraic homomorphism.
(c) If $\{f_n\} \subset \mathcal{B}_b(\sigma(T))$ is a bounded sequence which converges to f pointwise, then $\{\Phi(f_n)\}$ converges to $\Phi(f)$ strongly.
In addition, Φ has the following properties:
(d) Φ is continuous, with operator norm $\|\Phi\| = 1$.
(e) Φ is a *-homomorphism.
(f) If $f \in \mathcal{B}_b(\sigma(T))$ is such that $f \geq 0$, then $\Phi(f) \geq 0$.
(g) If $S \in \mathcal{L}(\mathcal{H})$ commutes with T, then $\Phi(f)\, S = S\, \Phi(f)$ for all $f \in \mathcal{B}_b(\sigma(T))$.
(h) If $Tx = \lambda x$ for some $x \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, then $\Phi(f)x = f(\lambda)x$ for all $f \in \mathcal{B}_b(\sigma(T))$.

Proof. Letting $\{\mu_{x,y}\}_{x,y\in\mathcal{H}}$ be the family of complex measures from Theorem (3.12), consider the map

$$\begin{aligned} \psi_f : \mathcal{H} \times \mathcal{H} &\to \mathbb{C} \\ (x, y) &\mapsto \int_{\sigma(T)} f \; d\mu_{x,y} \end{aligned}$$

for any fixed $f \in \mathcal{B}_b(\sigma(T))$. ψ_f is sesquilinear because $(x, y) \mapsto \mu_{x,y}$ is sesquilinear. Furthermore,

$$|\psi_f| \leq \|f\|_u \cdot \|\mu_{x,y}\|_{M(\sigma(T))} \leq \|f\|_u \cdot \|x\|_{\mathcal{H}} \cdot \|y\|_{\mathcal{H}}$$

for all $x, y \in \mathcal{H}$, which shows $\|f\|_u$ is a bound for ψ_f. Therefore, ψ_f is a bounded sesquilinear form on $\mathcal{H}$. By Theorem (2.8), there exists a unique element $\Phi(f) \in \mathcal{L}(\mathcal{H})$ such that $\|\Phi(f)\|_{\mathcal{L}(\mathcal{H})} \leq \|f\|_u$ and

$$\langle \Phi(f)x, y\rangle = \int_{\sigma(T)} f \; d\mu_{x,y} = \psi_f(x, y)$$

for all $x, y \in \mathcal{H}$. This allows us to unambiguously define the map $\Phi : \mathcal{B}_b(\sigma(T)) \to \mathcal{L}(\mathcal{H})$.

We claim Φ is the desired map. Recalling that $\langle \phi(f)x, y\rangle = \psi_f(x, y)$ for all $f \in \mathcal{C}(\sigma(T)) \subset \mathcal{B}_b(\sigma(T))$ and $x, y \in \mathcal{H}$, we see Φ is an extension of the continuous functional calculus. In particular, because ϕ maps $Id(z) = z$ to T and preserves the multiplicative identity, Φ does also.

We will now show that Φ is an algebraic homomorphism. For any $\alpha \in \mathbb{C}$ and $f, g \in \mathcal{B}_b(\sigma(T))$, we have

$$\begin{aligned}\langle \Phi(\alpha f + g)x, y\rangle = \int_{\sigma(T)} \alpha f + g \; d\mu_{x,y} = \alpha \int_{\sigma(T)} f \; d\mu_{x,y} + \int_{\sigma(T)} g \; d\mu_{x,y} \\ = \alpha\langle \Phi(f)x, y\rangle + \langle \Phi(g)x, y\rangle = \langle [\alpha\Phi(f) + \Phi(g)]x, y\rangle\end{aligned}$$

for all $x, y \in \mathcal{H}$, which implies Φ is linear. Temporarily fix $g \in \mathcal{B}_b(\sigma(T))$. Using part (e) of Theorem (3.12), and the fact that ϕ preserves the star operation, we have

$$\int_{\sigma(T)} f \; d\mu_{\Phi(g)x,y} = \langle \phi(f)\,\Phi(g)x, y\rangle = \langle \Phi(g)x, \phi(\overline{f})y\rangle = \int_{\sigma(T)} g \; d\mu_{x,\Phi(\overline{f})y} = \int_{\sigma(T)} f \cdot g \; d\mu_{x,y}$$

for all $f \in \mathcal{C}(\sigma(T))$. The uniqueness of the measures $\{\mu_{x,y}\}_{x,y\in\mathcal{H}}$ then implies $d\mu_{\Phi(g)x,y} = g \; d\mu_{x,y}$. Therefore, for any $f, g \in \mathcal{B}_b(\sigma(T))$,

$$\langle \Phi(f \cdot g)x, y\rangle = \int_{\sigma(T)} f \cdot g \; d\mu_{x,y} = \int_{\sigma(T)} f \; d\mu_{\Phi(g)x,y} = \langle \Phi(f) \circ \Phi(g)x, y\rangle$$

for all $x, y \in \mathcal{H}$, implying $\Phi(f \cdot g) = \Phi(f)\,\Phi(g)$.

To see that Φ preserves the star operation, note that

$$\langle \Phi(f)^* x, y\rangle = \overline{\langle \Phi(f)y, x\rangle} = \overline{\int_{\sigma(T)} f \; d\mu_{y,x}} = \int_{\sigma(T)} \overline{f} \; d\mu_{x,y} = \langle \Phi(\overline{f})x, y\rangle$$

for all $f \in \mathcal{B}_b(\sigma(T))$ and $x, y \in \mathcal{H}$. The fact that Φ is a *-homomorphism is not part of the uniqueness requirement. However, we use this fact in showing that Φ satisfies property (c).

Suppose $\{f_n\} \subset \mathcal{B}_b(\sigma(T))$ is a sequence of functions such that $f_n \to f$ pointwise and $\{\|f_n\|_u\}$ is bounded. In this case, $f \in \mathcal{B}_b(\sigma(T))$. For any $x \in \mathcal{H}$,

$$\begin{aligned}\|[\Phi(f_n) - \Phi(f)]x\|^2_{\mathcal{H}} &= \|\Phi(f_n - f)\|^2_{\mathcal{H}} \\ &= \langle \Phi(f_n - f)^*\Phi(f_n - f)x, x\rangle = \langle \Phi(|f_n - f|^2)x, x\rangle = \int_{\sigma(T)} |f_n - f|^2 \; d\mu_{x,x}\end{aligned}$$

because Φ preserves the star and multiplication operations. Noting that $\{f_n - f\}$ is dominated by $2M \cdot \chi_{\sigma(T)} \in L^2(\sigma(T), \mu_{x,x})$, where $M = \sup \|f_n\|_u$, the dominated convergence theorem for L^p spaces implies

$$\lim_{n\to\infty} \|[\Phi(f_n) - \Phi(f)]x\|_{\mathcal{H}} = 0.$$

This is true for every $x \in \mathcal{H}$, so $\{\Phi(f_n)\}$ converges to $\Phi(f)$ with respect to the strong operator topology.

We are ready to prove the uniqueness of Φ. Consider another map $\Psi : \mathcal{B}_b(\sigma(T)) \to \mathcal{L}(\mathcal{H})$ that satisfies properties (a), (b), and (c) of the theorem, and let $\mathcal{V}$ be the collection of all $f \in \mathcal{B}_b(\sigma(T))$ such that $\Psi(f) = \Phi(f)$. As in the proof of the continuous functional calculus, properties (a) and (b) imply that Ψ and Φ agree on all polynomials. It is also clear that $\mathcal{V}$ is a vector space, because of the linearity of Ψ and Φ. If $\{f_n\} \subset \mathcal{V}$ is a bounded sequence which converges to f pointwise, then we have

$$\Psi(f) = \lim_{n\to\infty} \Psi(f_n) = \lim_{n\to\infty} \Phi(f_n) = \Phi(f),$$

using property (c). Therefore, by Corollary (3.16), $\mathcal{V} = \mathcal{B}_b(\sigma(T))$, i.e. $\Psi \equiv \Phi$.

We will now show that Φ satisfies the remaining properties. Recalling that $\|\Phi(f)\|_{\mathcal{L}(\mathcal{H})} \leq \|f\|_u$ for all $f \in \mathcal{B}_b(\sigma(T))$, it is then clear Φ is continuous. Because Φ preserves the multiplicative identity with

$$\|\Phi(1)\|_{\mathcal{L}(\mathcal{H})} = \|I\|_{\mathcal{L}(\mathcal{H})} = 1 = \|1\|_u,$$

$\Phi : \mathcal{B}_b(\sigma(T)) \to \mathcal{L}(\mathcal{H})$ has $\|\Phi\| = 1$ as its bound.

Suppose $f \in \mathcal{B}_b(\sigma(T))$ is such that $f \geq 0$. Using the same method as in the proof of the continuous functional calculus, we may show $\Phi(f) \geq 0$. However, this may also be proven directly from the construction of Φ. Indeed,

$$\langle \Phi(f)x, x\rangle = \int_{\sigma(T)} f \ d\mu_{x,x} \geq 0$$

for all $x \in \mathcal{H}$, because each $\mu_{x,x}$ is positive.

Consider any $S \in \mathcal{L}(\mathcal{H})$ which commutes with T, and let $\mathcal{V}$ be the collection of all $f \in \mathcal{B}_b(\sigma(T))$ such that $\Phi(f)$ commutes with S. By Theorem (3.9), $\mathcal{V}$ contains $\mathcal{C}(\sigma(T))$. Also, $\mathcal{V}$ is a vector space, because Φ and composition are linear. If $\{f_n\} \subset \mathcal{V}$ is a bounded sequence which converges to f pointwise, then we have

$$\Phi(f) \circ Sx = \lim_{n\to\infty} \Phi(f_n) \circ Sx = \lim_{n\to\infty} S \circ \Phi(f_n)x = S \circ \Phi(f)x$$

using the strong convergence of $\{\Phi(f_n)\}$, and the continuity of S. Therefore, by Theorem (3.15), $\mathcal{V} = \mathcal{B}_b(\sigma(T))$, i.e. Φ satisfies property (g).

Suppose that $Tx = \lambda x$ for some $x \in \mathcal{H}$ and $\lambda \in \sigma(T)$, and let $\mathcal{V}$ be the collection of all $f \in \mathcal{B}_b(\sigma(T))$ such that $\Phi(f)x = f(\lambda)x$. By Theorem (3.9), $\mathcal{V}$ contains $C(\sigma(T))$. The linearity of Φ and the evaluation map at x, and the distributivity of scalar multiplication, imply that $\mathcal{V}$ is a vector space. If $\{f_n\} \subset \mathcal{V}$ is a bounded sequence which converges to f pointwise, then we have

$$\Phi(f)x = \lim_{n\to\infty} \Phi(f_n)x = \lim_{n\to\infty} f_n(\lambda)x = f(\lambda)x$$

using the strong convergence of $\{\Phi(f_n)\}$, and the continuity of the line path $l_x(\alpha) = \alpha \cdot x$. Therefore, by Theorem (3.15), $\mathcal{V} = \mathcal{B}_b(\sigma(T))$, i.e. Φ satisfies property (h). □

Corollary 3.18. *If $T \in \mathcal{L}(\mathcal{H})$ is self-adjoint, then*

$$\sigma(T) = \overline{\bigcup_{x,y\in\mathcal{H}} \text{supp } \mu_{x,y}}$$

Proof. Let A be the closure of $\bigcup_{x,y\in\mathcal{H}}$ supp $\mu_{x,y}$, and assume A is a strict subset of $\sigma(T)$. A^c is outside of the support of each $\mu_{x,y}$, so

$$\langle \Phi\left(\chi_{A^c}\right) x, y\rangle = \int_{\sigma(T)} \chi_{A^c} \ d\mu_{x,y} = \mu_{x,y}\left(A^c\right) = 0$$

for all $x, y \in \mathcal{H}$. This implies $\Phi(\chi_{A^c}) = 0$. Recalling that Φ is an algebraic homomorphism extending the continuous functional calculus, we then have

$$\phi(f) = \Phi(f) = \Phi(f \cdot \chi_{A^c}) = \Phi(f) \circ \Phi(\chi_{A^c}) = 0$$

for any $f \in \mathcal{C}(\sigma(T))$ whose support is contained in A^c. If there exists such an f that is also not identically nonzero, we will have arrived at a contradiction, because ϕ is injective as a linear isometry.

A is closed, so there is some $r > 0$ and $a \in \sigma(T)$ such that the closure of the ball $B_r(a)$ is contained in A^c. Using this r and a, we define $f := g \circ d(\cdot, a)$, where

$$g(z) = \begin{cases} 1 & \text{if } \ z = 0 \\ 1 - \frac{z}{r} & \text{if } \ 0 < z < r \\ 0 & \text{if } \ z \geq r \end{cases}$$

is a complex-valued function on $[0, \infty)$, and $d(\cdot, a)$ is the function on $\sigma(T)$ giving the distance from a. f is continuous on $\sigma(T)$, because it is the composition of continuous functions. It is also clear that f is not identically zero, with the closure of $B_r(a)$ as its support. Having constructed a function with the desired properties, the aforementioned contradiction follows. We may conclude $A = \sigma(T)$. ☐

3.2 The Associated Multiplication Operator

Definition 3.19. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. A surjective linear map $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\langle Ux, Uy \rangle_{\mathcal{H}_2} = \langle x, y \rangle_{\mathcal{H}_1}$$

for all $x, y \in \mathcal{H}_1$ is called a **unitary map**.

Lemma 3.20. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. $U : \mathcal{H}_1 \to \mathcal{H}_2$ is a unitary map if and only if it is a surjective linear isometry.*

Proof. If U is unitary, then, in particular, $\langle Ux, Ux \rangle_{\mathcal{H}_2} = \langle x, x \rangle_{\mathcal{H}_1}$ for all $x \in \mathcal{H}_1$. Therefore, the forward implication is clear. If U is a linear isometry, then

$$\begin{aligned} \langle x, y \rangle_{\mathcal{H}_1} &= \frac{1}{4} \left(\|x + y\|^2_{\mathcal{H}_1} - \|x - y\|^2_{\mathcal{H}_1} + i \cdot \|x + iy\|^2_{\mathcal{H}_1} - i \cdot \|x - iy\|^2_{\mathcal{H}_1} \right) \\ &= \frac{1}{4} \left(\|Ux + Uy\|^2_{\mathcal{H}_2} - \|Ux - Uy\|^2_{\mathcal{H}_2} + i \cdot \|Ux + i \cdot Uy\|^2_{\mathcal{H}_2} - i \cdot \|Ux - i \cdot Uy\|^2_{\mathcal{H}_2} \right) \\ &= \langle Ux, Uy \rangle_{\mathcal{H}_2}. \end{aligned}$$

The reverse implication follows. ☐

Definition 3.21. $x \in \mathcal{H}$ is a **cyclic vector** for $T \in \mathcal{L}(\mathcal{H})$ if $\{p(T)x \mid \text{p is a complex polynomial}\}$ is a dense subset of $\mathcal{H}$.

A self-adjoint bounded operator T with a cyclic vector is unitarily equivalent to a multiplication operator. In proving this statement, we use the following theorem, which can be found, for example, in [3] (Proposition 7.9).

Theorem 3.22. *If μ is a positive Radon measure on the locally compact Hausdorff space X, then $\mathcal{C}_c(X)$ is dense in $L^p(X,\mu)$, for $1 \leq p < \infty$.*

Lemma 3.23. *Let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint with cyclic vector $x \in \mathcal{H}$. There is a positive Radon measure μ on $\sigma(T)$ and a unitary map $U : \mathcal{H} \to L^2(\sigma(T),\mu)$ such that*

$$\begin{aligned} UTU^{-1} : L^2(\sigma(T),\mu) &\to L^2(\sigma(T),\mu) \\ f(z) &\mapsto z \cdot f(z) \end{aligned}$$

Proof. Given the cyclic vector $x \in \mathcal{H}$, let $\mu = \mu_{x,x}$. μ is a positive Radon measure on the compact Hausdorff space $\sigma(T) \subset \mathbb{R}$. By Theorem (3.22), $\mathcal{C}(\sigma(T))\ /\ (f = g \text{ a.e.})$ is a dense subspace of $L^2(\sigma(T),\mu)$. We wish to define a suitable map on $\mathcal{C}(\sigma(T))\ /\ (f = g \text{ a.e.})$ that we may then extend to a unitary transformation between $L^2(\sigma(T),\mu)$ and $\mathcal{H}$.

Towards this end, define

$$\begin{aligned} \widehat{U}_0 : \mathcal{C}(\sigma(T))\ /\ (f = g \text{ a.e.}) &\to \mathcal{H} \\ f &\mapsto \Phi(f)x, \end{aligned}$$

where Φ gives the functional calculus. It is not immediately clear that this map is well-defined. Using the algebraic properties of Φ, and its relationship with $\mu_{x,x}$, we have

$$\|\Phi(f)x\|_{\mathcal{H}}^2 = \langle \Phi(|f|^2)x, x\rangle = \int_{\sigma(T)} |f|^2\ d\mu = \|f\|^2_{L^2(\sigma(T),\mu)}$$

for all $f \in \mathcal{C}(\sigma(T))$. This implies

$$\|\Phi(f)x - \Phi(g)x\|_{\mathcal{H}} = \|\Phi(f-g)x\|_{\mathcal{H}} = \|f-g\|_{L^2(\sigma(T),\mu)} = 0$$

when f and g are a.e. equal elements of $\mathcal{C}(\sigma(T))$, confirming that $\widehat{U}_0$ is well-defined.

$\widehat{U}_0$ is linear because both Φ and the evaluation map at $x \in \mathcal{H}$ are linear. $\|\widehat{U}_0(f)\|_{\mathcal{H}} = \|f\|_{L^2(\sigma(T),\mu)}$ then shows that $\widehat{U}_0$ is a linear isometry. Furthermore, the range of $\widehat{U}_0$ will be dense in $\mathcal{H}$, because x is cyclic for T and $\mathcal{C}(\sigma(T))$ contains all complex polynomials. By the Bounded Linear Transformation Theorem, $\widehat{U}_0$ has a unique continuous extension to a linear map U_0 from $L^2(\sigma(T),\mu)$ to $\mathcal{H}$. By Theorem (3.8), U_0 is a surjective linear isometry. Noting that $L^2(\sigma(T),\mu)$ and $\mathcal{H}$ are Hilbert spaces, it follows that U_0 is a unitary map.

We need to show $U_0^{-1}TU_0 f = z \cdot f(z)$ for all $f \in L^2(\sigma(T),\mu)$. Towards this, define

$$\begin{aligned} V : L^2(\sigma(T),\mu) &\to L^2(\sigma(T),\mu) \\ f(z) &\to z \cdot f(z) \end{aligned}$$

It is clear that V is a linear map, provided that it is well-defined. Because $\sup_{z\in\sigma(T)} |z|^2$ is finite,

$$\|z \cdot f(z)\|^2_{L^2(\sigma(T),\mu)} = \int_{\sigma(T)} |z \cdot f(z)|^2\ d\mu \leq \left(\sup_{z\in\sigma(T)} |z|^2\right) \cdot \|f(z)\|^2_{L^2(\sigma(T),\mu)}$$

implies $z \cdot f(z) \in L^2(\sigma(T),\mu)$ when $f \in L^2(\sigma(T),\mu)$. Therefore, V is a well-defined, bounded linear operator on $L^2(\sigma(T),\mu)$.

Recalling that U_0 extends $\widehat{U}_0$, and $\Phi(z) = T$, we have

$$U_0^{-1}\ T\ U_0 f = U_0^{-1}\ T\ \Phi(f)x = U_0^{-1}\ \Phi(z \cdot f(z))x = U_0^{-1}\ U_0(z \cdot f(z)) = z \cdot f(z) = Vf$$

for all $f \in \mathcal{C}(\sigma(T))\ /\ (f = g$ a.e). $U_0^{-1}TU_0$ and V are bounded linear operators agreeing on a dense subset of $L^2(\sigma(T),\mu)$, so continuity implies $U_0^{-1}TU_0 \equiv V$.

Letting $U = U_0^{-1}$, we obtain the desired statement of the theorem. □

Lemma 3.24. *Let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint. There is a collection $\{\mathcal{H}_i\}_{i\in I}$ of pairwise orthogonal, closed subspaces of $\mathcal{H}$ such that:*
(a) For each $i \in I$, $T|_{\mathcal{H}_i} \in \mathcal{L}(\mathcal{H}_i)$.
(b) For each $i \in I$, there exists $x_i \in \mathcal{H}_i$ such that x_i is a cyclic vector for $T|_{\mathcal{H}_i}$.
(c) $\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i$.

Proof. If $\mathcal{H}$ is trivial, then the results are immediate. Assuming $\mathcal{H}$ is non-trivial, we will use Zorn's Lemma to construct the collection $\{\mathcal{H}_i\}_{i\in I}$. However, before giving the partial order, we define a family of closed subspaces of $\mathcal{H}$ that are not necessarily pairwise orthogonal.

For every nonzero $x \in \mathcal{H}$, let

$$P_x = \{\Phi(p)x \mid p \text{ is a complex polynomial}\} \subset \mathcal{H},$$

where Φ gives the functional calculus. Each P_x is a subspace of $\mathcal{H}$, because both Φ and the evaluation map at x are linear. Furthermore, T is invariant on each P_x, because

$$T \circ \Phi(p)x = \Phi(z \cdot p(z))x \in P_x$$

for every polynomial p. It follows that each $\overline{P_x}$ will be a closed subspace of $\mathcal{H}$ and invariant under T, by the continuity of vector addition, scalar multiplication, and $T \in \mathcal{L}(\mathcal{H})$. We may then view each $\overline{P_x}$ as a Hilbert space with the inherited inner product, and we will have $T|_{\overline{P_x}} \in \mathcal{L}(\overline{P_x})$. It is obvious that $x \in \overline{P_x}$ will be a cyclic vector for $T|_{\overline{P_x}}$.

We construct our partial order by first defining

$$\mathcal{S} = \{A \subset \mathcal{H}\backslash\{0\} \mid \overline{P_x} \perp \overline{P_y}\ \forall\ x, y \in A \text{ s.t. } x \neq y\}.$$

It is clear $(\mathcal{S}, \subset)$ is a partial ordering, where $\subset$ is inclusion. Assuming $\mathcal{H}$ is non-trivial, i.e. $\mathcal{H}\backslash\{0\}$ is non-empty, then $\mathcal{S}$ is non-empty. Let $\mathcal{C}$ be an arbitrary, non-empty chain in $\mathcal{S}$. If we can show $\bigcup_{B\in\mathcal{C}} B \in \mathcal{S}$, it will immediately follow that $\bigcup_{B\in\mathcal{C}} B$ is an upper bound for $\mathcal{C}$. First, we have $\bigcup_{B\in\mathcal{C}} B \subset \mathcal{H}\backslash\{0\}$, because $0 \notin B$ for all $B \in \mathcal{C}$. Now, consider distinct $x, y \in \bigcup_{B\in\mathcal{C}} B$. There exists some $B_{x,y} \in \mathcal{C}$ such that $x, y \in B_{x,y}$, because $\mathcal{C}$ is a chain. Because $B_{x,y} \in \mathcal{S}$, we have $\overline{P_x} \perp \overline{P_y}$. It follows that $\bigcup_{B\in\mathcal{C}} B \in \mathcal{S}$. Therefore, every non-empty chain in $\mathcal{S}$ will have an upper bound. Zorn's Lemma implies $\mathcal{S}$ contains a maximal element M.

We claim $\{\overline{P_x}\}_{x\in M}$ is our desired collection $\{\mathcal{H}_i\}_{i\in I}$. By construction, the elements of $\{\overline{P_x}\}_{x\in M}$ are pairwise orthogonal, closed subspaces of $\mathcal{H}$ for which properties (a) and (b) of the theorem hold. We may then take the internal direct sum $\mathcal{K} := \bigoplus_{x\in M} \overline{P_x} \subset \mathcal{H}$. Before proving property (c) of the theorem, we note that T is invariant on $\mathcal{K}$, because T is continuous and invariant on each $\overline{P_x}$. This then implies that T is invariant on $\mathcal{K}^\perp$, because T is self-adjoint.

Assume that $\mathcal{K}$ is a strict subset of $\mathcal{H}$. Because $\mathcal{K}$ is closed, an equivalent assumption is that $\mathcal{K}$ is not dense in $\mathcal{H}$, i.e. $\mathcal{K}^\perp \neq \{0\}$. Let $x_0 \in \mathcal{K}^\perp$ be non-zero. $\overline{P_{x_0}} \subset \mathcal{K}^\perp$ because T is invariant on $\mathcal{K}^\perp$, and $\mathcal{K}^\perp$ is a closed subspace. Therefore, $M \bigcup \{x_0\}$ is an element of $\mathcal{S}$. The maximality of M implies $x_0 \in M$, but this is a contradiction because $\mathcal{K} \cap \mathcal{K}^\perp = \{0\}$. We must have $\bigoplus_{x\in M} \overline{P_x} = \mathcal{H}$. □

Theorem 3.25. *Let $\{\mathcal{H}_i, \mathcal{K}_i\}_{i\in I}$ be a collection of Hilbert spaces. If $\{U_i : \mathcal{H}_i \to \mathcal{K}_i\}_{i\in I}$ is a family of unitary maps, then the map*

$$\begin{aligned} U : \bigoplus_{i\in I} \mathcal{H}_i &\to \bigoplus_{i\in I} \mathcal{K}_i \\ \sum_{i\in I} x_i &\mapsto \sum_{i\in I} U_i(x_i) \end{aligned}$$

is well-defined and unitary.

Proof. Let $\mathcal{H} := \bigoplus_{i\in I} \mathcal{H}_i$ and $\mathcal{K} = \bigoplus_{i\in I} \mathcal{K}_i$. Each $x \in \mathcal{H}$ has a unique representation as a formal sum $\sum_{i\in I} x_i$. From the definition of the norm on the direct sum of Hilbert spaces, we have

$$\|U(x)\|_{\mathcal{K}}^2 = \sum_{i\in I} \|U_i(x_i)\|_{\mathcal{K}_i}^2 = \sum_{i\in I} \|x_i\|_{\mathcal{H}_i}^2 = \|x\|_{\mathcal{H}}^2 < \infty,$$

because each U_i is unitary. Therefore, U is well-defined and norm-preserving. The linearity of the U_i implies that U is linear. Noting that each U_i is surjective, it follows that U is a surjective linear isometry, and hence unitary. □

Theorem 3.26 (Spectral Theorem)**.** *Let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint. There is a collection $\{\mu_i\}_{i\in I}$ of finite Radon measures on $\mathbb{R}$, and a unitary map $U : \mathcal{H} \to \bigoplus_{i\in I} L^2(\mathbb{R}, \mu_i)$ such that*

$$\begin{aligned} UTU^{-1} : \bigoplus_{i\in I} L^2(\mathbb{R}, \mu_i) &\to \bigoplus_{i\in I} L^2(\mathbb{R}, \mu_i) \\ \sum_{i\in I} f_i(z) &\mapsto \sum_{i\in I} z \cdot f_i(z). \end{aligned}$$

Proof. Let $\bigoplus_{i\in I} \mathcal{H}_i$ be the direct sum decomposition for $\mathcal{H}$ that is guaranteed by Lemma (3.24). For each $i \in I$, we have $T|_{\mathcal{H}_i} \in \mathcal{L}(\mathcal{H}_i)$, and there exists a cyclic vector $x_i \in \mathcal{H}_i$ for $T|_{\mathcal{H}_i}$. Because T is self-adjoint, $T|_{\mathcal{H}_i}$ will be self-adjoint with respect to the inherited inner product. Therefore, we may apply Lemma (3.23) to each $T|_{\mathcal{H}_i}$. Let $\{\mu_i\}_{i\in I}$ and $\{U_i\}_{i\in I}$ be the resulting collection of finite Radon measures and unitary mappings.

The map

$$\begin{aligned} U : \mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i &\to \bigoplus_{i\in I} L^2(\sigma(T|_{\mathcal{H}_i}), \mu_i) \\ y = \sum_{i\in I} y_i &\mapsto \sum_{i\in I} U_i(y_i) \end{aligned}$$

is well-defined and unitary, by Theorem (3.25). For any $\sum_{i\in I} f_i \in \bigoplus_{i\in I} L^2(\sigma(T|_{\mathcal{H}_i}),\mu_i)$, we have

$$UTU^{-1}\left(\sum_{i\in I} f_i\right) = UT\left(\sum_{i\in I} U_i^{-1} f_i\right) = U\left(\sum_{i\in I} TU_i^{-1} f_i\right) = \sum_{i\in I} U_i T U_i^{-1} f_i = \sum_{i\in I} z\cdot f_i(z)$$

by the definition of U, the continuity of T, and Lemma (3.23).

For each $i \in I$, we may consider μ_i as a finite Radon measure on $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$ with its support contained in $\sigma(T|_{\mathcal{H}_i})$, because $\sigma(T|_{\mathcal{H}_i})$ is a Borel subset of $\mathbb{R}$. With this identification,

$$\bigoplus_{i\in I} L^2(\sigma(T|_{\mathcal{H}_i}),\mu_i) = \bigoplus_{i\in I} L^2(\mathbb{R},\mu_i).$$

It is finally clear that U is the desired map. □

Theorem 3.27. *For $\{(X_n,\Sigma_n,\mu_n)\}_{n=1}^{\infty}$, a countable family of positive measure spaces such that $\sum_n \mu_n(X_n)$ is finite, define $X := \bigsqcup_n X_n$, $\Sigma := \{\bigsqcup_n A_n \mid A_n \in \Sigma_n\}$, and*

$$\begin{aligned} \mu : \Sigma &\to \mathbb{R}^{\geq 0} \\ \bigsqcup_n A_n &\mapsto \sum_n \mu_n(A_n), \end{aligned}$$

where $\bigsqcup$ denotes the disjoint union. (X,Σ,μ) is a finite measure space, and

$$\begin{aligned} V : \bigoplus_n L^2(X_n,\mu_n) &\to L^2(X,\mu) \\ \sum_n f_n &\mapsto f \text{ such that } f|_{X_n} \equiv f_n \end{aligned}$$

is unitary.

Proof. It is easily seen that Σ is a σ-algebra for the set X. Because $\{\mu_n\}$ is a family of positive measures such that $\sum_n \mu_n(X_n) < \infty$, μ is well-defined, and is such that $\mu(\emptyset) = 0$. μ will also be countably additive, because any partitioning of the terms of a convergent series of positive numbers will converge to the same sum. Therefore, (X,Σ,μ) is a finite measure space.

f is a complex valued function on X if and only if there exists a unique family of functions $\{f_n : X_n \to \mathbb{C}\}$ such that $f|_{X_n} \equiv f_n$ for each n. It is also clear that f is measurable with respect to Σ if and only if each $f|_{X_n}$ is measurable with respect to Σ_n. In the case f is measurable, and considering each $f|_{X_n}$ as a function on X that vanishes outside of $X_n \subset X$, we have

$$\begin{aligned} \int_X |f|^2\, d\mu = \int_X \sum_n |f|_{X_n}|^2\, d\mu &= \lim_{N\to\infty} \int_X \sum_{n=1}^{N} |f|_{X_n}|^2\, d\mu \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \int_X |f|_{X_n}|^2\, d\mu = \sum_n \int_{X_n} |f|_{X_n}|^2\, d\mu \end{aligned}$$

using the Monotone Convergence Theorem. With these comments in mind, it follows that V is a surjective linear isometry, and hence unitary. □

The following is a corollary to the multiplication operator version of the spectral theorem. In this corollary, the unitary equivalence of T to a multiplication operator is clear.

Corollary 3.28. *Let $\mathcal{H}$ be a separable Hilbert space, and let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint. There exists a finite measure space (X, Σ, μ), a unitary map $U : \mathcal{H} \to L^2(X, \mu)$, and a function $F : X \to \mathbb{R}$ such that*

$$\begin{aligned} UTU^{-1} : L^2(X, \mu) &\to L^2(X, \mu) \\ f &\mapsto F \cdot f. \end{aligned}$$

Proof. Let $U_0 : \mathcal{H} \to \bigoplus_{i \in I} L^2(\mathbb{R}, \mu_i)$ be the map from Theorem (3.26). $\mathcal{H}$ has a countable basis, because it is separable. Noting that U_0 is unitary, we may then assume I is a countable index. From the construction of each μ_i, we have

$$\mu_i(\mathbb{R}) = \mu_{x_i, x_i}(\sigma(T|_{\mathcal{H}_i})) = \|x_i\|^2_{\mathcal{H}},$$

where $\mathcal{H}_i$ is a closed subspace of $\mathcal{H}$ on which T is invariant, and $x_i \in \mathcal{H}_i$ is cyclic for $T|_{\mathcal{H}_i}$. For any nonzero $\alpha \in \mathbb{C}$, $\alpha \cdot x_i$ will also be cyclic for $T|_{\mathcal{H}_i}$. Therefore, we may assume that the family of measures $\{\mu_i\}$ was constructed so that $\sum_{i \in I} \mu_i(\mathbb{R})$ is finite.

Let $\left(\bigsqcup_{i \in I} \mathbb{R}, \Sigma, \mu\right)$ be the measure space, and $V : \bigoplus_{i \in I} L^2(\mathbb{R}, \mu) \to L^2\left(\bigsqcup_{i \in I} \mathbb{R}, \mu\right)$ the unitary map, from Theorem (3.27). As the composition of unitary maps, $U := V \circ U_0$ is unitary. Also, the properties of U and the definition of V imply that $UTU^{-1}(f) = F \cdot f$ for all $f \in L^2\left(\bigsqcup_{i \in I} \mathbb{R}, \mu\right)$, where $F(z, i) := z$ is a real-valued function on $\bigsqcup_{i \in I} \mathbb{R}$. ☐

3.3 Projection-Valued Measures

Definition 3.29. A **projection-valued measure** on a measurable space (X, Σ) is a function $P : \Sigma \to \mathcal{L}(\mathcal{H})$ such that:

(i) For each $E \in \Sigma$, $P(E)$ is an orthogonal projection on $\mathcal{H}$.

(ii) $P(\emptyset) = 0$ and $P(X) = I$.

(iii) If $\{E_n\}_{n \in \mathbb{N}} \subset \Sigma$ is a sequence of pairwise-disjoint sets, then $\sum_{n=1}^{N} P(E_n) \to P\left(\bigcup_{n=1}^{\infty} E_n\right)$ strongly.

Lemma 3.30. *If P is a PVM on the measurable space (X, Σ), then*

$$P\left(E_1 \cap E_2\right) = P(E_1)\, P(E_2)$$

for all $E_1, E_2 \in \Sigma$.

Proof. First, consider disjoint sets $A, B \in \Sigma$. P is a PVM, so $P\left(A \cup B\right), P(A)$, and $P(B)$ are orthogonal projections such that $P\left(A \cup B\right) = P(A) + P(B)$. Therefore,

$$P(A)\, P(B) = P(B)\, P(A) = 0$$

using Lemma (2.18), and the fact that $(P(A)\, P(B))^* = P(B)\, P(A)$.

Now, consider arbitrary sets $E_1, E_2 \in \Sigma$. The behaviour of P on disjoint sets implies

$$P(E_1)\, P(E_2) = \left(P(E_1 \backslash E_2) + P\left(E_1 \cap E_2\right)\right) \left(P(E_2 \backslash E_1) + P\left(E_1 \cap E_2\right)\right) = P\left(E_1 \cap E_2\right)$$

because $E_1 \backslash E_2,\ E_2 \backslash E_1,\ E_1 \cap E_2 \in \Sigma$ are pairwise disjoint. ☐

Theorem 3.31. *Let P be a PVM on the measurable space (X, Σ). For any $x, y \in \mathcal{H}$, the function*

$$\begin{aligned} p_{x,y} &: \Sigma \to \mathbb{C} \\ E &\mapsto \langle P(E)x, y\rangle \end{aligned}$$

is a complex measure on (X, Σ). The family $\{p_{x,y}\}_{x,y\in\mathcal{H}}$ has the following properties:
(a) Each $p_{x,x}$ is a finite positive measure, with $p_{x,x}(X) = \|x\|^2_{\mathcal{H}}$.
(b) $(x, y) \mapsto p_{x,y}$ is a sesquilinear map from $\mathcal{H} \times \mathcal{H}$ to the space of complex measures on (X, Σ).
(c) $p_{x,y} = \overline{p_{y,x}}$.
(d) For any $E \in \Sigma$, $dp_{P(E)x,y} = \chi_E \; dp_{x,y} = dp_{x,P(E)y}$.

Proof. Fix $x, y \in \mathcal{H}$. $p_{x,y}$, as given in the theorem, is clearly a well-defined, complex-valued function on the σ-algebra Σ. $P(\emptyset) = 0 \in \mathcal{L}(\mathcal{H})$ because P is a PVM, so

$$p_{x,y}(\emptyset) = \langle P(\emptyset)x, y\rangle = \langle 0, y\rangle = 0$$

All that remains in proving $p_{x,y}$ is a complex measure is showing that it is countably additive.

Let $\{E_n\}_{n\in\mathbb{N}} \subset \Sigma$ be a sequence of pairwise-disjoint sets. Because P is a PVM, $\{\sum_{n=1}^N P(E_n)\}$ converges to $P(E)$ strongly, where $E = \bigcup_{n=1}^\infty E_n$. In particular, $\{\sum_{n=1}^N (P(E_n)x)\}$ converges to $P(E)x$ in $\mathcal{H}$. Noting that $\langle \cdot, y\rangle$ is a continuous functional on $\mathcal{H}$, we have

$$\langle P(E)x, y\rangle = \lim_{N\to\infty} \left\langle \sum_{n=1}^N (P(E_n)x), y \right\rangle = \lim_{N\to\infty} \sum_{n=1}^N \langle P(E_n)x, y\rangle = \sum_{n=1}^\infty \langle P(E_n)x, y\rangle.$$

As desired, $p_{x,y}(E) = \sum_{n=1}^\infty p_{x,y}(E_n)$. Therefore, $\{p_{x,y}\}_{x,y\in\mathcal{H}}$ is a family of complex measures.

We will proceed with proving the properties (a)-(d). Consider $p_{x,x}$. For any $E \in \Sigma$, we have

$$0 \le \langle P(E)x, P(E)x\rangle = \langle P(E)x, x\rangle = p_{x,x}(E),$$

because $P(E)$ is self-adjoint and idempotent. We also have $p_{x,x}(X) = \|x\|^2_{\mathcal{H}}$, because $P(X) = I$. Therefore, $p_{x,x}$ is a finite positive measure.

It is clear that $(x, y) \mapsto p_{x,y}$ is a well-defined map from $\mathcal{H} \times \mathcal{H}$ to the vector space of complex measures on (X, Σ). The range of P is contained in $\mathcal{L}(\mathcal{H})$, and the inner product is sesquilinear, so this map will be sesquilinear. For every $E \in \Sigma$, we have

$$p_{x,y}(E) = \langle P(E)x, y\rangle = \overline{\langle y, P(E)x\rangle} = \overline{\langle P(E)y, x\rangle} = \overline{p_{y,x}(E)}$$

because each $P(E)$ is self-adjoint. Therefore, $p_{x,y} = \overline{p_{y,x}}$.

Finally, fix $E \in \Sigma$. Using Lemma (3.30), we have

$$p_{P(E)x,y}(F) = \langle P(F)P(E)x, y\rangle = \langle P(E \cap F)x, y\rangle = p_{x,y}(E \cap F) = \int_F \chi_E \; dp_{x,y},$$

for all $F \in \Sigma$. Therefore, $dp_{P(E)x,y} = \chi_E \; dp_{x,y}$. Property (c) then implies

$$dp_{x,P(E)y} = \overline{dp_{P(E)y,x}} = \overline{\chi_E \; dp_{y,x}} = \chi_E \; dp_{x,y},$$

which completes the proof. $\square$

Theorem 3.32. *Let P be a PVM on the measurable space (X, Σ). For each $f \in \mathcal{M}_b(X, \Sigma)$, there exists a unique $T \in \mathcal{L}(\mathcal{H})$ such that*

$$\langle Tx, y \rangle = \int f \, dp_{x,y}$$

for all $x, y \in \mathcal{H}$.

Proof. Fix $f \in \mathcal{M}_b(X, \Sigma)$, and consider the well-defined map

$$\begin{aligned} \psi : \mathcal{H} \times \mathcal{H} &\to \mathbb{C} \\ (x, y) &\mapsto \int f \, dp_{x,y} \end{aligned}$$

Integration with respect to the sum and scaling of measures is linear, and we have already shown that the map $(x, y) \mapsto p_{x,y}$ is sesquilinear. Therefore, ψ is sesquilinear. For all $x \in \mathcal{H}$, we have

$$|\psi(x, x)| = \left| \int f \, dp_{x,x} \right| \leq \|f\|_u \cdot \|x\|_{\mathcal{H}}^2,$$

recalling that $p_{x,x}(X) = \|x\|_{\mathcal{H}}^2$. Therefore, by Lemma (2.7), ψ is bounded. Because ψ is a bounded sesquilinear form on $\mathcal{H}$, there exists a unique $T \in \mathcal{L}(\mathcal{H})$ such that $\langle Tx, y \rangle = \psi(x, y)$ for all $x, y \in \mathcal{H}$. □

Definition 3.33. Suppose P is a PVM on the measurable space (X, Σ). For any $f \in \mathcal{M}_b(X, \Sigma)$, $\int f \, dP$ will denote the unique bounded operator on $\mathcal{H}$, guaranteed by Theorem (3.32).

Corollary 3.34. *If P is a PVM on the measurable space (X, Σ), then the map*

$$\begin{aligned} \int \cdot \, dP : \mathcal{M}_b(X, \Sigma) &\to \mathcal{L}(\mathcal{H}) \\ f &\mapsto \int f \, dP \end{aligned}$$

is well defined. In addition:
*(a) $\int \cdot \, dP$ is a unital, algebraic *-homomorphism.*
(b) $\int \cdot \, dP$ is continuous, with operator norm $\left\| \int \cdot \, dP \right\| = 1$.
(c) If $\{f_n\} \subset \mathcal{M}_b(X, \Sigma)$ is a bounded sequence of functions which converges to f pointwise, then $\left\{ \int f_n \, dP \right\}$ converges to $\int f \, dP$ with respect to the strong operator topology.

Proof. It is clear that $\int \cdot \, dP$ is well-defined, so we will proceed with proving property (a). Let $x, y \in \mathcal{H}$ be arbitrary. To see that $\int \cdot \, dP$ is unital, note that

$$\langle Ix, y \rangle = \langle P(X)x, y \rangle = p_{x,y}(X) = \int 1 \, dp_{x,y}$$

Uniqueness implies $I = \int 1 \, dP$. Now, let $f, g \in \mathcal{M}_b(X, \Sigma)$, and let $S = \int f \, dP$ and $T = \int g \, dP$. For any $\alpha \in \mathbb{C}$, we have

$$\int \alpha \cdot f + g \, dp_{x,y} = \alpha \cdot \int f \, dp_{x,y} + \int g \, dp_{x,y} = \langle (\alpha \cdot S + T)x, y \rangle$$

using the linearity of integration. Furthermore,

$$\langle S^*y, x\rangle = \overline{\langle Sx, y\rangle} = \overline{\int f \; dp_{x,y}} = \int \overline{f} \; d\overline{p_{x,y}} = \int \overline{f}(z) \; dp_{y,x},$$

because $\overline{p_{x,y}} = p_{y,x}$. Uniqueness implies $\int \alpha \cdot f + g \; dP = \alpha \cdot T + S$ and $T^* = \int \overline{f} \; dP$, so $\int \cdot \; dP$ is linear and preserves the star operation. We are left with showing $\int \cdot \; dP$ preserves the multiplication operation. For all $E \in \Sigma$, we have

$$p_{Tx,y}(E) = \langle P(E) \circ Tx, y\rangle = \langle Tx, P(E)y\rangle = \int g \; dp_{x,P(E)y} = \int_E g \; dp_{x,y}$$

because $P(E)$ is self-adjoint, and $dp_{x,P(E)y} = \chi_E \; dp_{x,y}$. This implies $dp_{Tx,y} = g \; dp_{x,y}$, so

$$\langle S(Tx), y\rangle = \int f \; dp_{Tx,y} = \int f \cdot g \; dp_{x,y}.$$

Uniqueness implies $S \circ T = \int f \cdot g \; dP$, completing the proof of property (a).

Using the homomorphism properties of $\int \cdot \; dP$, and recalling that $p_{x,x}(X) = \|x\|^2_{\mathcal{H}}$, we get

$$\|Sx\|^2_{\mathcal{H}} = \langle S^* \circ Sx, x\rangle = \int f \cdot \overline{f} \; dp_{x,x} = \int |f|^2 \; dp_{x,x} \leq \|f\|^2_u \cdot \|x\|^2_{\mathcal{H}}.$$

This implies $\|S\|_{\mathcal{L}(\mathcal{H})} \leq \|f\|_u$, so $\int \cdot \; dP$ is continuous, as desired. Noting that $\|I\|_{\mathcal{L}(\mathcal{H})} = 1 = \|1\|_u$, the remainder of property (b) follows.

Finally, suppose $\{f_n\} \subset \mathcal{M}_b(X, \Sigma)$ is a bounded sequence of functions converging to f pointwise. f is measurable, because each f_n is measurable, and bounded, because $\sup \|f_n\|_u < \infty$. If $S_n = \int f_n \; dP$, then we have

$$\|(S - S_n)x\|^2_{\mathcal{H}} = \langle (S - S_n)^*(S - S_n)x, x\rangle = \int \overline{(f - f_n)}(f - f_n) \; dp_{x,x} = \int |f - f_n|^2 \; dp_{x,x},$$

again using the homomorphism properties of $\int \cdot \; dP$. Noting that $\{f - f_n\}$ is dominated by $2M \in L^2(X, p_{x,x})$, where $M = \sup \|f_n\|_u$, the dominated convergence theorem for L^p spaces implies that S_n converges to S with respect to the strong operator topology. □

Definition 3.35. Suppose X is a locally compact Hausdorff space, and that P is a PVM on $(X, \mathcal{B}_X)$. P is a **Radon PVM** if $\{p_{x,y}\}_{x,y \in \mathcal{H}}$ is a family of complex Radon measures. The **support** of P is

$$\mathrm{supp}(P) \equiv \overline{\bigcup_{x,y \in \mathcal{H}} \mathrm{supp}(p_{x,y})}$$

P is **compact** if its support is compact in X.

The above definitions differ from those given in [4]. Rather than defining the regularity and support of a PVM P directly, we define these concepts in terms of the complex measures generated by P. This was done so that we may avoid discussing the supremum and infimum of a family of orthogonal projections. Even with these non-standard definitions, the support of a Radon PVM is what one would expect.

Lemma 3.36. *If X is a locally compact Hausdorff space, and P is a Radon PVM on $(X, \mathcal{B}_X)$, then:*
(a) $P(B) = 0$, for any $B \in \mathcal{B}_X$ contained in the complement of supp(P).
(b) $P\left(\text{supp}(P)\right) = I$.
(c) supp$(P) = N^c$, where N is the union of all open sets $B \in \mathcal{B}_X$ such that $P(B) = 0$.

Proof. Towards proving the first part of the lemma, consider any $B \in \mathcal{B}_X$ that is contained in the complement of supp(P), and let $x, y \in \mathcal{H}$ be arbitrary. By the definition of supp(P), B is contained in the complement of supp$(p_{x,y})$. $p_{x,y}$ is a complex Radon measure, so $p_{x,y}(B) = \langle P(B)x, y\rangle = 0$. Theorem (2.8) then implies $P(B) = 0$, as desired. In particular, part (a) implies $P(\text{supp}(P)^c) = 0$. Part (b) immediately follows.

We will prove the last part of the lemma by showing that N, as defined, is equal to the complement of supp(P). It is clear N contains supp$(P)^c$, because supp$(P)^c$ is an open set such that $P(\text{supp}(P)^c) = 0$. Now, consider any open $B \in \mathcal{B}_X$ such that $P(B) = 0$. Clearly, $p_{x,x}(B) = \langle P(B)x, x\rangle = 0$ for all $x \in \mathcal{H}$. This shows B is null for every element of $\{p_{x,x}\}_{x\in\mathcal{H}}$, a family of positive measures. $(x, y) \mapsto p_{x,y}(\cdot)$ is a sesquilinear map from $\mathcal{H} \times \mathcal{H}$ to the vector space of complex measures on $(X, \mathcal{B}_X)$, so we may consider the polarization identity

$$p_{x,y}(\cdot) = \frac{1}{4}\left[p_{x+y,x+y}(\cdot) - p_{x-y,x-y}(\cdot) + i \cdot p_{x+iy,x+iy}(\cdot) - i \cdot p_{x-iy,x-iy}(\cdot)\right]$$

for any $x, y \in \mathcal{H}$. B is null for the positive measures on the right-hand side of the above equality, so $p_{x,y}(A \cap B) = 0$ for any $A \in \mathcal{B}_X$. In other words, B is null for each $p_{x,y}$, and hence contained in the complement of $\bigcup_{x,y\in\mathcal{H}} \text{supp}(p_{x,y})$. Because B is open, it is contained in supp$(P)^c$. Therefore, N is contained in supp$(P)^c$, completing the proof. $\square$

If P is a Radon PVM on X, Theorem (3.32) may be proven for any measurable function $f : X \to \mathbb{C}$ which is bounded on the support of P, rather than its entire domain. Corollary (3.34) may also be altered accordingly. These generalizations are important because the PVM's we consider from this point onwards will be Radon.

Theorem 3.37. *If X is a locally compact Hausdorff space that is second countable, then every PVM on $(X, \mathcal{B}_X)$ is Radon. In particular, every PVM on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ is Radon.*

Proof. Every complex measure on a second countable, locally compact Hausdorff space is Radon (see, for example, the comments on page 222 of [3]). The proof is then immediate from the definition of a Radon PVM. $\square$

Lemma 3.38. *If P is a compact PVM on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$, and $\Pi_i : \mathbb{R}^n \to \mathbb{R}$ is the projection map for the i-th coordinate, then $\left\{\int \Pi_i \; dP\right\}_{i=1}^n$ is a finite collection of self-adjoint and commuting operators in $\mathcal{L}(\mathcal{H})$.*

Proof. Fix the index $i \in \{1, \ldots, n\}$. Because supp(P) is compact and Π_i is continuous, $\Pi_i \in \mathcal{B}(\mathbb{R}^n)$ is bounded on supp(P). Therefore, $\int \Pi_i \; dP$ is well-defined. Using the homomorphism properties of $\int \cdot \; dP$, and the fact that Π_i is real-valued, we have

$$\left(\int \Pi_i \; dP\right)^* = \int \overline{\Pi_i} \; dP = \int \Pi_i \; dP.$$

The multiplication operation in $\mathcal{B}_b(\text{supp}(P))$ is commutative, and $\int \cdot \, dP$ is an algebraic homomorphism, so $\left\{\int \Pi_i \; dP\right\}_{i=1}^{n} \subset \mathcal{L}(\mathcal{H})$ is a family of commuting operators. $\square$

Theorem 3.39. *If P and P_0 are compact PVM's on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ such that*

$$\int \Pi_i \; dP = \int \Pi_i \; dP_0$$

for $i \in \{1, \ldots, n\}$, then $P \equiv P_0$.

Proof. Assume that $\int \cdot \, dP$ and $\int \cdot \, dP_0$ agree on the coordinate projection maps. Recalling that $\int \cdot \, dP$ and $\int \cdot \, dP_0$ are unital algebraic homomorphisms, it follows that they agree on all complex polynomials of n real variables. Now, let K be the union of the supports for P and P_0, and consider $f \in \mathcal{C}_0(\mathbb{R}^n) \subset \mathcal{B}_b(K)$. K is compact, so Corollary (3.6) implies there is a sequence $\{q_m\}$ of complex polynomials of n real variables which converges to f uniformly on K. Therefore,

$$\int f \; dP = \lim_m \int q_m \; dP = \lim_m \int q_m \; dP_0 = \int f \; dP_0,$$

using the continuity of $\int \cdot \, dP$ and $\int \cdot \, dP_0$, and the definition of K.

It follows that $\int f \; dp_{x,y} = \int f \; dp_{0\,(x,y)}$ for all $x, y \in \mathcal{H}$, and all $f \in \mathcal{C}_0(\mathbb{R}^n)$. Noting that $p_{x,y}$ and $p_{0\,(x,y)}$ are Radon, the Riesz Representation Theorem implies

$$\langle P(\cdot)x, y\rangle = p_{x,y} = p_{0\,(x,y)} = \langle P_0(\cdot)x, y\rangle$$

for all $x, y \in \mathcal{H}$. We may conclude $P \equiv P_0$. $\square$

Theorem 3.40. *Let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint. The function*

$$\begin{aligned} P : \mathcal{B}_{\mathbb{R}} &\to \mathcal{L}(\mathcal{H}) \\ B &\mapsto \chi_B(T) \equiv \Phi(\chi_B) \end{aligned}$$

is a PVM on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

Proof. Characteristic functions of Borel sets are Borel measurable and bounded, so P is well-defined, using the functional calculus for T. We may proceed with demonstrating P is a PVM. For a fixed $B \in \mathcal{B}_{\mathbb{R}}$, consider $P(B) \in \mathcal{L}(\mathcal{H})$. Using the homomorphism properties of the functional calculus, we have

$$P(B)^* = \chi_B(T)^* = \overline{\chi_B}(T) = \chi_B(T) = P(B)$$

and

$$P(B)^2 = \chi_B(T)^2 = \chi_B^2(T) = \chi_B(T) = P(B)$$

because χ_B is real-valued and idempotent. Therefore, $P(B) \in \mathcal{L}(\mathcal{H})$ is an orthogonal projection.

When restricted to $\sigma(T)$, $\chi_\emptyset$ and $\chi_{\mathbb{R}}$ are, respectively, the zero element and multiplicative identity in $\mathcal{B}_b(\sigma(T))$. Therefore, $P(\emptyset) = \chi_\emptyset(T) = 0$ and $P(\mathbb{R}) = \chi_{\mathbb{R}}(T) = I$.

Finally, suppose $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_{\mathbb{R}}$ is a sequence of pairwise-disjoint subsets. Letting $B_N = \bigcup_{n=1}^N B_n$ and $B = \bigcup_{n=1}^\infty B_n$, it is clear $\{\chi_{B_N}\} \subset \mathcal{B}_b(\sigma(T))$ is a bounded sequence of functions converging to χ_B pointwise. The convergence property of the functional calculus then implies $\chi_{B_N}(T) \to \chi_B(T)$ strongly in $\mathcal{L}(\mathcal{H})$. However, $\chi_{B_N} = \sum_{n=1}^N \chi_{B_n}$ because the elements of $\{B_n\}$ are pairwise disjoint, so

$$\sum_{n=1}^N P(B_n) = \sum_{n=1}^N \chi_{B_n}(T) = \chi_{B_N}(T).$$

Therefore, $\sum_{n=1}^N P(B_n) \to P(B)$ strongly, as desired. □

Corollary 3.41. *Let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint, let $\{\mu_{x,y}\}_{x,y\in\mathcal{H}}$ be the complex Radon measures on $\sigma(T)$ used in the construction of the functional calculus for T, and let P be the PVM on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ constructed from T. If $\mu_{x,y}$ is considered as a measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, then $p_{x,y} \equiv \mu_{x,y}$. Furthermore, supp$(P) = \sigma(T)$ is compact.*

Proof. From the definition of P and the construction of the functional calculus for T,

$$\langle P(B)x, y\rangle = \langle \chi_B(T)x, y\rangle = \int \chi_B \; d\mu_{x,y} = \mu_{x,y}(B)$$

for all $B \in \mathcal{B}_{\mathbb{R}}$ and $x, y \in \mathcal{H}$. Corollary (3.18) then implies $\text{supp}(P) = \sigma(T)$. □

Lemma 3.42. *Let $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint, let Φ be the functional calculus for T, and let P be the PVM on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ constructed from T. Then $\Phi(f) = \int f \; dP$ for all $f \in \mathcal{B}(\sigma(T))$, and $T = \int z \; dP(z)$, in particular.*

Proof. If $f \in \mathcal{B}_b(\sigma(T))$, then $\Phi(f)$ is a well-defined element of $\mathcal{L}(\mathcal{H})$. From the construction of the functional calculus, and using Corollary (3.41),

$$\langle \Phi(f)x, y\rangle = \int f \; d\mu_{x,y} = \int f \; dp_{x,y}$$

for all $x, y \in \mathcal{H}$. Noting that $\text{supp}(P) = \sigma(T)$, uniqueness implies $\Phi(f) = \int f \; dP$. □

Theorem 3.43 (Spectral Theorem)**.** *There is a bijection between the compact PVM's on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ and the self-adjoint elements of $\mathcal{L}(\mathcal{H})$.*

Proof. If $\mathcal{P}$ represents the collection of compact PVM's on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, and $\mathcal{T}$ represents the collection of self-adjoint elements in $\mathcal{L}(\mathcal{H})$, then consider the map

$$\begin{aligned} F : \mathcal{P} &\to \mathcal{T} \\ P &\mapsto \int z \; dP(z). \end{aligned}$$

Lemma (3.38) ensures that F is well-defined, and Theorem (3.39) implies that F is injective. If P is the PVM on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ constructed from $T \in \mathcal{T}$, as in Theorem (3.40), then it has compact support, by Corollary (3.41). Therefore, $P \in \mathcal{P}$, and

$$F(P) = \int z \; dP(z) = T$$

by Lemma (3.42). This shows that F is also surjective. □

Chapter 4

Generalizing to Finite Sequences

In this chapter, we generalize the spectral theorems from Chapter 3 to finite sequences $\{T_i\}_{i=1}^n \subset \mathcal{L}(\mathcal{H})$ of self-adjoint and commuting operators. The idea for this generalization comes from the proof of Theorem (44.1) in [4], the projection-valued measure version of the spectral theorem for normal bounded operators. The first section defines and proves the existence of a decomposing PVM for $\{T_i\}_{i=1}^n$, the second section presents a generalized functional calculus, and the third section generalizes the multiplication operator version of the spectral theorem. In the final section, we consider the particular case where $n = 2$, so that we may develop the spectral theory for normal bounded operators.

4.1 Decomposing Projection-Valued Measures

Lemma 4.1. *Let $\{T_i\}_{i=1}^n \subset \mathcal{L}(\mathcal{H})$ be a finite sequence of self-adjoint and commuting operators, with $\{P_i\}_{i=1}^n$ as their respective PVM's on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. For any collection $\{B_i\}_{i=1}^n \subset \mathcal{B}_{\mathbb{R}}$, $\{P_i(B_i)\}_{i=1}^n$ commute, and $P_1(B_1) \cdots P_n(B_n) \in \mathcal{L}(\mathcal{H})$ is an orthogonal projection.*

Proof. Consider any indices $i, j \in \{1, \ldots, n\}$. Because T_i commutes with T_j, the functional calculus for T_i has the property that $P_i(B_i) = \chi_{B_i}(T_i)$ commutes with T_j. The functional calculus for T_j then has the property that $P_j(B_j) = \chi_{B_j}(T_j)$ commutes with $P_i(B_i)$. It follows that $\{P_i(B_i)\}_{i=1}^n$ is a family of commuting orthogonal projections, which implies $P_1(B_1) \cdots P_n(B_n) \in \mathcal{L}(\mathcal{H})$ is an orthogonal projection. $\square$

The following theorem may be found, for example, in [3] (Theorem 1.14).

Theorem 4.2. *A σ-finite premeasure on an algebra $\mathcal{A}$ of sets has a unique extension to a positive measure on the σ-algebra generated by $\mathcal{A}$.*

Theorem 4.3. *Let $\{T_i\}_{i=1}^n \subset \mathcal{L}(\mathcal{H})$ be a finite sequence of self-adjoint and commuting operators, with $\{P_i\}_{i=1}^n$ as their respective PVM's on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. For each $x \in \mathcal{H}$, there is a unique positive measure ν_x on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ such that*

$$\nu_x(B_1 \times \cdots \times B_n) = \langle P_1(B_1) \cdots P_n(B_n)x, x\rangle$$

for every measurable rectangle $B_1 \times \cdots \times B_n \in \mathcal{B}_{\mathbb{R}^n}$. This family $\{\nu_x\}_{x \in \mathcal{H}}$ has the following properties:
(a) $\nu_x(\mathbb{R}^n) = \|x\|_{\mathcal{H}}^2$.
(b) $\nu_{\alpha \cdot x} = |\alpha|^2 \cdot \nu_x$ for any $\alpha \in \mathbb{C}$.

Proof. Let $\mathcal{A}$ be the collection of all finite, disjoint unions of measurable rectangles in $\mathcal{B}_{\mathbb{R}^n}$. $\mathcal{A}$ is an algebra of sets which generates the σ-algebra $\mathcal{B}_{\mathbb{R}^n}$. We will first create a family of premeasures on

$\mathcal{A}$ with the desired property on measurable rectangles, and then extend them uniquely to a family of positive measures on $\mathcal{B}_{\mathbb{R}^n}$.

For each $x \in \mathcal{H}$, define the map

$$\begin{aligned} \hat{\nu}_x : \mathcal{A} &\longrightarrow \mathbb{R}^{\geq 0} \\ \bigcup_{j=1}^{k} B_{1,j} \times \cdots \times B_{n,j} &\mapsto \sum_{j=1}^{k} \langle P_1(B_{1,j}) \cdots P_n(B_{n,j})x, x \rangle. \end{aligned}$$

It is not immediately clear that $\hat{\nu}_x$ is well-defined, for two reasons. First, the range of $\hat{\nu}_x$ may not be $\mathbb{R}^{\geq 0}$. However, each $P_1(B_{1,j}) \cdots P_n(B_{n,j})$ is idempotent and self-adjoint, as an orthogonal projection, so

$$\sum_{j=1}^{k} \langle P_1(B_{1,j}) \cdots P_n(B_{n,j})x, x \rangle = \sum_{j=1}^{k} \|P_1(B_{1,j}) \cdots P_n(B_{n,j})x\|_{\mathcal{H}}^2 \geq 0.$$

Second, a set in $\mathcal{A}$ does not in general have a unique representation as a finite, disjoint union of measurable rectangles. $\hat{\nu}_x$ may give different values for the same set, depending upon the representation.

Towards showing this is not the case, consider countable collections $\{B_{1,j_1}\}_{j_1 \in \mathbb{N}}, \ldots, \{B_{n,j_n}\}_{j_n \in \mathbb{N}}$ of disjoint sets in $\mathcal{B}_{\mathbb{R}}$. Letting $B_i = \bigcup_{j_i=1}^{\infty} B_{i,j_i}$, then

$$B_1 \times \cdots \times B_n = \bigcup_{i=1}^{n} \bigcup_{j_i=1}^{\infty} B_{1,j_1} \times \cdots \times B_{n,j_n} \in \mathcal{A}.$$

We will refer to $\{B_{1,j_1} \times \cdots \times B_{n,j_n}\}_{j_1,\ldots,j_n \in \mathbb{N}} \subset \mathcal{A}$ as a grid for $B_1 \times \cdots \times B_n$. The countable additivity of P_1 implies that

$$P_1(B_1) \cdots P_n(B_n) = \sum_{j_1=1}^{\infty} P_1(B_{1,j_1}) P_2(B_2) \cdots P_n(B_n)$$

with respect to the strong operator topology. Also, for any $i \in \{2, \ldots, n\}$, the countable additivity of P_i, and the continuity of $P_1(B_{1,j_1}) \cdots P_{i-1}(B_{i-1,j_{i-1}})$, implies that

$$P_1(B_{1,j_1}) \cdots P_{i-1}(B_{i-1,j_{i-1}}) P_i(B_i) \cdots P_n(B_n) = \sum_{j_i=1}^{\infty} P_1(B_{1,j_1}) \cdots P_i(B_{i,j_i}) P_{i+1}(B_{i+1}) \cdots P_n(B_n)$$

with respect to the strong operator topology. Noting that strong convergence implies weak convergence, it follows that

$$\langle P_1(B_1) \cdots P_n(B_n)x, x \rangle = \sum_{i=1}^{n} \sum_{j_i=1}^{\infty} \langle P_1(B_{1,j_1}) \cdots P_n(B_{n,j_n})x, x \rangle$$

which implies

$$\hat{\nu}_x(B_1 \times \cdots \times B_n) = \sum_{i=1}^{n} \sum_{j_i=1}^{\infty} \hat{\nu}_x(B_{1,j_1} \times \cdots \times B_{n,j_n}) < \infty.$$

Any two representations for the same set in $\mathcal{A}$ will have a common refinement that partitions both into grids. Reordering the terms of an absolutely convergent series of real numbers will not change the sum, so $\hat{\nu}_x$ will give the same value for both representations. We may now conclude $\hat{\nu}_x$ is well-defined.

We claim $\hat{\nu}_x$ is a finite premeasure on $\mathcal{A}$. Using the properties of $\{P_i\}_{i=1}^n$,

$$\hat{\nu}_x(\emptyset) = \langle P_1(\emptyset) \cdots P_n(\emptyset)x, x\rangle = 0$$

and

$$\hat{\nu}_x(\mathbb{R}^n) = \langle P_1(\mathbb{R}) \cdots P_n(\mathbb{R})x, x\rangle = \|x\|_{\mathcal{H}}^2.$$

To show that $\hat{\nu}_x$ is countably additive, it is enough to consider a collection $\{B_{1,j} \times \cdots \times B_{n,j}\}_{j\in\mathbb{N}}$ of disjoint measurable rectangles in $\mathcal{B}_{\mathbb{R}^n}$ whose union is in $\mathcal{A}$, i.e.

$$\bigcup_{j=1}^{\infty} B_{1,j} \times \cdots \times B_{n,j} = \bigcup_{j=1}^{k} A_{1,j} \times \cdots \times A_{n,j} \in \mathcal{A}.$$

From the same argument used to prove $\hat{\nu}_x$ respects different representations of the same set, it follows that

$$\hat{\nu}_x\left(\bigcup_{j=1}^{\infty} B_{1,j} \times \cdots \times B_{n,j}\right) = \sum_{j=1}^{\infty} \hat{\nu}_x(B_{1,j} \times \cdots \times B_{n,j}).$$

As claimed, $\hat{\nu}_x$ is a finite premeasure on $\mathcal{A}$. Theorem (4.2) ensures the existence of a unique positive measure ν_x on $\mathcal{B}_{\mathbb{R}^n}$ which extends $\hat{\nu}_x$. Suppose ν_x' is a positive measure on $\mathcal{B}_{\mathbb{R}^n}$ such that

$$\nu_x'(B_1 \times \cdots \times B_n) = \langle P_1(B_1) \cdots P_n(B_n)x, x\rangle$$

for all measurable rectangles $B_1 \times \cdots \times B_n$. The finite additivity of measures implies ν_x' agrees with $\hat{\nu}_x$ on $\mathcal{A}$, so $\nu_x' \equiv \nu_x$ by uniqueness.

All that remains of the proof is to show the family $\{\nu_x\}_{x\in\mathcal{H}}$ has the additional properties specified in the theorem. It is clear that $\nu_x(\mathbb{R}^n) = \|x\|_{\mathcal{H}}^2$. Given any $\alpha \in \mathbb{C}$,

$$\begin{aligned}\nu_{\alpha\cdot x}(B_1 \times \cdots \times B_n) &= \langle P_1(B_1) \cdots P_n(B_n)(\alpha \cdot x), \alpha \cdot x\rangle \\ &= \alpha \cdot \overline{\alpha}\langle P_1(B_1) \cdots P_n(B_n)x, x\rangle = |\alpha|^2 \cdot \nu_x(B_1 \times \cdots \times B_n).\end{aligned}$$

for all measurable rectangles $B_1 \times \cdots \times B_n$. Uniqueness implies $\nu_{\alpha\cdot x} = |\alpha|^2 \cdot \nu_x$. $\square$

Theorem 4.4. *Let $\{T_i\}_{i=1}^n \subset \mathcal{L}(\mathcal{H})$ be a finite sequence of self-adjoint and commuting operators, with $\{P_i\}_{i=1}^n$ as their respective PVM's on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. There is a unique PVM P on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ such that*

$$P(B_1 \times \cdots \times B_n) = P_1(B_1) \cdots P_n(B_n)$$

for all measurable rectangles $B_1 \times \cdots \times B_n \in \mathcal{B}_{\mathbb{R}^n}$.

Proof. Let $\{\nu_x\}_{x\in\mathcal{H}}$ be the family of finite positive measures on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ from Theorem (4.3). With the polarization identity in mind,

$$\nu_{x,y} \equiv \frac{1}{4}\left(\nu_{x+y} - \nu_{x-y} + i\cdot\nu_{x+i\cdot y} - i\cdot\nu_{x-i\cdot y}\right)$$

creates a family $\{\nu_{x,y}\}_{x,y\in\mathcal{H}}$ of complex measures. It is easily shown $\nu_{x,x} = \nu_x$, using the fact that $\nu_{\alpha\cdot x} = |\alpha|^2\cdot\nu_x$ for any $\alpha\in\mathbb{C}$. These complex measures will be used to construct the desired PVM.

For any $B\in\mathcal{B}_{\mathbb{R}^n}$, the map $(x,y)\mapsto\nu_{x,y}(B)$ is a conjugate symmetric, sesquilinear form on $\mathcal{H}$. We will first verify this for the collection of measurable rectangles in $\mathcal{B}_{\mathbb{R}^n}$, which will be denoted by $\mathcal{R}$. For each $B_1\times\cdots\times B_n\in\mathcal{R}$, we have

$$\nu_{x,y}(B_1\times\cdots\times B_n) = \langle P_1(B_1)\cdots P_n(B_n)x, y\rangle,$$

using the definition of $\nu_{x,y}$, the behaviour of the measures $\{\nu_x\}_{x\in\mathcal{H}}$ on $B_1\times\cdots\times B_n$, and the polarization identity. Noting that $P_1(B_1)\cdots P_n(B_n)$ is self-adjoint, $(x,y)\mapsto\nu_{x,y}(B_1\times\cdots\times B_n)$ is as claimed.

Now, let $\mathcal{S}$ be the collection of all $B\in\mathcal{B}_{\mathbb{R}^n}$ such that $(x,y)\mapsto\nu_{x,y}(B)$ is conjugate symmetric and sesquilinear. We claim $\mathcal{S}$ is a σ-additive class. It is already clear that $\mathbb{R}^n\in\mathcal{S}$, because $\mathcal{R}\subset\mathcal{S}$. If $A, B\in\mathcal{S}$ are such that $B\subset A$, then

$$\nu_{x,y}(A\backslash B) = \nu_{x,y}(A) - \nu_{x,y}(B)$$

implies $(x,y)\mapsto\nu_{x,y}(A\backslash B)$ will also have the desired properties, so $A\backslash B\in\mathcal{S}$. If $\{B_j\}_{j\in\mathbb{N}}\subset\mathcal{S}$ is a disjoint collection of sets, then

$$\nu_{x,y}\left(\bigcup_{j=1}^{\infty} B_j\right) = \sum_{j=1}^{\infty}\nu_{x,y}(B_j) < \infty$$

implies $(x,y)\mapsto\nu_{x,y}\left(\bigcup_{j=1}^{\infty} B_j\right)$ will also have the desired properties, so $\bigcup_{j=1}^{\infty} B_j\in\mathcal{S}$.

The intersection of two measurable rectangles is again a measurable rectangle, so $\mathcal{R}$ is closed with respect to finite intersections. Noting that $\mathcal{B}_{\mathbb{R}^n}$ is the σ-algebra generated by $\mathcal{R}$, part (ii) of the Monotone Class Theorem then implies $\mathcal{B}_{\mathbb{R}^n}$ will also be the σ-additive class generated by $\mathcal{R}$. It follows that $\mathcal{S} = \mathcal{B}_{\mathbb{R}^n}$, because $\mathcal{S}$ is a σ-additive class and $\mathcal{R}\subset\mathcal{S}\subset\mathcal{B}_{\mathbb{R}^n}$. This confirms $(x,y)\mapsto\nu_{x,y}(B)$ is a conjugate symmetric, sesquilinear form on $\mathcal{H}$ for any $B\in\mathcal{B}_{\mathbb{R}^n}$.

Furthermore, each map $(x,y)\mapsto\nu_{x,y}(B)$ is bounded. $\nu_{x,x} = \nu_x$ is a positive measure for each $x\in\mathcal{H}$, so

$$\nu_{x,x}(B) \le \nu_x(\mathbb{R}^n) = \|x\|^2_{\mathcal{H}}.$$

The boundedness then follows from Lemma (2.7).

For each $B\in\mathcal{B}_{\mathbb{R}^n}$, there exists a unique $P(B)\in\mathcal{L}(\mathcal{H})$ such that

$$\nu_{x,y}(B) = \langle P(B)x, y\rangle$$

for all $x, y \in \mathcal{H}$, because $(x, y) \mapsto \nu_{x,y}(B)$ is a bounded sesquilinear form. The uniqueness of $P(B)$ implies that $P(B_1 \times \cdots \times B_n) = P_1(B_1) \cdots P_n(B_n)$ for measurable rectangles $B_1 \times \cdots \times B_n$. Therefore, the map

$$\begin{aligned} P : \mathcal{B}_{\mathbb{R}^n} &\to \mathcal{L}(\mathcal{H}) \\ B &\mapsto P(B) \end{aligned}$$

has the desired decomposition on rectangles. However, it still needs to be shown that P is a PVM.

First, we will show $P(B)$ is an orthogonal projection for every $B \in \mathcal{B}_{\mathbb{R}^n}$. Using the previously demonstrated conjugate symmetry,

$$\langle P(B)x, y \rangle = \nu_{x,y}(B) = \overline{\nu_{y,x}(B)} = \overline{\langle P(B)y, x \rangle} = \langle P(B)^* x, y \rangle$$

for all $x, y \in \mathcal{H}$. The uniqueness of $P(B)$ implies that it is self-adjoint. Therefore, if $P(B)$ is idempotent, it is an orthogonal projection. The idempotentcy will be demonstrated indirectly.

We claim $P(A \cap B) = P(A)P(B)$ for all $A, B \in \mathcal{B}_{\mathbb{R}^n}$. For measurable rectangles $A_1 \times \cdots \times A_n$ and $B_1 \times \cdots \times B_n$ in $\mathcal{R}$,

$$\begin{aligned} P\left((A_1 \times \cdots \times A_n) \cap (B_1 \times \cdots \times B_n)\right) &= P_1\left(A_1 \cap B_1\right) \cdots P_n\left(A_n \cap B_n\right) \\ &= P_1(A_1)P_1(B_1) \cdots P_n(A_n)P_n(B_n) \\ &= P(A_1 \times \cdots \times A_n)P(B_1 \times \cdots \times B_n) \end{aligned}$$

using Lemma (3.30) and Lemma (4.1). Therefore, the claim holds for pairs of measurable rectangles.

Fixing $B_0 \in \mathcal{R}$, redefine $\mathcal{S}$ to be the collection of all $A \in \mathcal{B}_{\mathbb{R}^n}$ such that $P(A \cap B_0) = P(A)P(B_0)$. As before, we will show $\mathcal{S}$ is a σ-additive class. Clearly, $\mathbb{R}^n \in \mathcal{S}$ because $\mathcal{R} \subset \mathcal{S}$. If $A, B \in \mathcal{S}$ are such that $B \subset A$, then

$$\begin{aligned} \nu_{x,y}\left((A \backslash B) \cap B_0\right) &= \nu_{x,y}\left(A \cap B_0\right) - \nu_{x,y}\left(B \cap B_0\right) = \langle P(A)P(B_0)x, y \rangle - \langle P(B)P(B_0)x, y \rangle \\ &= \nu_{P(B_0)x,y}(A) - \nu_{P(B_0)x,y}(B) = \nu_{P(B_0)x,y}(A \backslash B) = \langle P(A \backslash B)P(B_0)x, y \rangle \end{aligned}$$

for all $x, y \in \mathcal{H}$. The uniqueness of $P\left((A \backslash B) \cap B_0\right)$ implies $P\left((A \backslash B) \cap B_0\right) = P(A \backslash B)P(B_0)$, so $A \backslash B \in \mathcal{S}$. If $\{A_j\}_{j\mathbb{N}} \subset \mathcal{S}$ is a disjoint collection of sets, and $A = \bigcup_{j=1}^{\infty} A_j$, then

$$\begin{aligned} \nu_{x,y}\left(A \cap B_0\right) = \sum_{j=1}^{\infty} \nu_{x,y}\left(A_j \cap B_0\right) = \sum_{j=1}^{\infty} \langle P(A_j)P(B_0)x, y \rangle &= \sum_{j=1}^{\infty} \nu_{P(B_0)x,y}(A_j) \\ &= \nu_{P(B_0)x,y}(A) = \langle P(A)P(B_0)x, y \rangle \end{aligned}$$

for all $x, y \in \mathcal{H}$. The uniqueness of $P(A \cap B_0)$ implies $P(A \cap B_0) = P(A)P(B_0)$, so $A \in \mathcal{S}$.

$\mathcal{S}$ is a σ-additive class containing $\mathcal{R}$ and contained in $\mathcal{B}_{\mathbb{R}^n}$. Using the Monotone Class Theorem as before, $\mathcal{S} = \mathcal{B}_{\mathbb{R}^n}$. It follows that $P(A \cap B) = P(A)P(B)$ for all $A \in \mathcal{B}_{\mathbb{R}^n}$ and $B \in \mathcal{R}$. This process can be repeated, by first fixing $A_0 \in \mathcal{B}_{\mathbb{R}^n}$, and once more redefining $\mathcal{S}$ to be the collection of all $B \in \mathcal{B}_{\mathbb{R}^n}$ such that $P(A_0 \cap B) = P(A_0)P(B)$. $\mathcal{S}$ is again a σ-additive class containing $\mathcal{R}$, so $\mathcal{S} = \mathcal{B}_{\mathbb{R}^n}$. It follows that $P(A \cap B) = P(A)P(B)$ for all $A, B \in \mathcal{B}_{\mathbb{R}^n}$. In particular, $P(B) = P(B \cap B) = P(B)P(B)$ demonstrates the idempotentcy of $P(B)$.

We have established that P is projection-valued. Clearly, $P(\emptyset) = P_1(\emptyset)\cdots P_n(\emptyset) = 0$ and $P(\mathbb{R}^n) = P_1(\mathbb{R})\cdots P_n(\mathbb{R}) = I$. Now, let $\{B_j\}_{j\in\mathbb{N}} \subset \mathcal{B}_{\mathbb{R}^n}$ be a collection of disjoint sets. Fixing $x \in \mathcal{H}$, consider the sequence $\{P(B_j)x\}_{j\in\mathbb{N}} \subset \mathcal{H}$. For indices $i \neq j$,

$$\langle P(B_i)x, P(B_j)x\rangle = \langle P(B_j)P(B_i)x, x\rangle = \langle P\left(B_j \cap B_i\right)x, x\rangle = \langle P(\emptyset)x, x\rangle = 0,$$

so $\{P(B_1)x\}_{i\in\mathbb{N}}$ is an orthogonal sequence. If $B = \bigcup_{j=1}^{\infty} B_j$, then

$$\|P(B)x\|_{\mathcal{H}}^2 = \langle P(B)x, x\rangle = \nu_{x,x}(B) = \sum_{j=1}^{\infty} \nu_{x,x}(B_j) = \sum_{j=1}^{\infty}\langle P(B_j)x, x\rangle = \sum_{j=1}^{\infty} \|P(B_j)x\|_{\mathcal{H}}^2.$$

The generalized Pythagorean Theorem implies $\sum_{i=1}^{\infty} P(B_i)x \in \mathcal{H}$. Therefore,

$$\langle P(B)x, y\rangle = \nu_{x,y}(B) = \sum_{j=1}^{\infty} \nu_{x,y}(B_j) = \sum_{j=1}^{\infty}\langle P(B_j)x, y\rangle = \left\langle \sum_{j=1}^{\infty} P(B_j)x, y \right\rangle$$

for all $y \in \mathcal{H}$. This implies $P(B)x = \sum_{j=1}^{\infty} P(B_j)x$. $x \in \mathcal{H}$ was arbitrary, so $P(B) = \sum_{j=1}^{\infty} P(B_j)$ with respect to the strong operator topology. We conclude that P is a PVM on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$.

Finally, suppose P_0 is another PVM on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ with the desired decomposition property on measurable rectangles. For each $x \in \mathcal{H}$, $p_{0\,(x,x)}$ is a positive measure on $\mathcal{B}_{\mathbb{R}^n}$ such that

$$p_{0\,(x,x)}(B_1 \times \cdots \times B_n) = \langle P_0(B_1 \times \cdots \times B_n)x, x\rangle = \langle P_1(B_1)\cdots P_n(B_n)x, x\rangle$$

for all $B_1 \times \cdots \times B_n \in \mathcal{R}$. The uniqueness of ν_x implies $p_{0\,(x,x)} \equiv \nu_x \equiv p_{x,x}$. The polarization identities for $(x, y) \mapsto p_{0\,(x,y)}$ and $(x, y) \mapsto p_{x,x}$ then imply

$$\langle P_0(\cdot)x, y\rangle = p_{0\,(x,y)} = p_{x,y} = \langle P(\cdot)x, y\rangle$$

for all $x, y \in \mathcal{H}$. We conclude $P_0 \equiv P$. □

Definition 4.5. Let $\{T_i\}_{i=1}^n \subset \mathcal{L}(\mathcal{H})$ be a finite sequence of self-adjoint and commuting operators. The unique PVM on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ guaranteed by Theorem (4.4) will be called the **decomposing PVM** for $\{T_i\}_{i=1}^n$. If P is the decomposing PVM for $\{T_i\}_{i=1}^n$, then P_i will be assumed to be the PVM on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ associated with T_i.

Corollary 4.6. *Let $\{T_i\}_{i=1}^n \subset \mathcal{L}(\mathcal{H})$ be a finite sequence of self-adjoint and commuting operators, and let P be its decomposing PVM. P is a compact PVM whose support is contained in $\sigma(T_1) \times \cdots \times \sigma(T_n)$. Furthermore, if Π_i is the the function projecting the i-th coordinate of $\mathbb{R}^n$ onto $\mathbb{R}$, then $\sigma(T_i) = \Pi_i\left[supp(P)\right]$.*

Proof. Let $E = \sigma(T_1) \times \cdots \times \sigma(T_n)$. Recalling that $\text{supp}(P_i) = \sigma(T_i)$, part (a) of Lemma (3.36) implies $P_i\left(\sigma(T_i)^c\right) = 0$. Because E^c is the union of the disjoint rectangles

$$\{\sigma(T_1) \times \cdots \times \sigma(T_{i-1}) \times \sigma(T_i)^c \times \mathbb{R} \times \cdots \times \mathbb{R} \mid 1 \leq i \leq n\} \subset \mathcal{B}_{\mathbb{R}^n},$$

the decomposition property of P implies $P(E^c) = 0$. Noting that E^c is open, part (c) of Lemma (3.36) implies that $\text{supp}(P)$ is contained in E. As a closed subset of a compact set, $\text{supp}(P)$ is compact.

For some fixed index $i \in \{1, \dots, n\}$, let $E_i = \Pi_i\,[\text{supp}(P)]$. It is clear $E_i \subset \sigma(T_i)$; however, we will assume the inclusion is strict. E_i is closed because continuous functions carry compactness forwards. Therefore, E_i^c is an open set having a nonempty intersection with $\sigma(T_i) = \text{supp}(P_i)$. Part (c) of Lemma (3.36) implies $P_i(E_i^c) \neq 0 \in \mathcal{L}(\mathcal{H})$. For the measurable rectangle $\mathbb{R} \times \cdots \times E_i^c \times \cdots \times \mathbb{R} \in \mathcal{B}_{\mathbb{R}^n}$, we then have

$$P(\mathbb{R} \times \cdots \times E_i^c \times \cdots \times \mathbb{R}) = P_1(\mathbb{R}) \cdots P_i(E_i^c) \cdots P_n(\mathbb{R}) = P_i(E_i^c) \neq 0 \in \mathcal{L}(\mathcal{H}),$$

using the decomposition property of P. This contradicts part (a) of Lemma (3.36), because $\mathbb{R} \times \cdots \times E_i^c \times \cdots \times \mathbb{R}$ is disjoint from $\text{supp}(P)$. We may conclude $E_i = \sigma(T_i)$. $\square$

We require the following lemma, which may be found in [3] (Proposition 2.34).

Lemma 4.7. *Let $\{(X_i, \mathcal{X}_i)\}_{i=1}^n$ be a collection of measurable spaces, and let $f : \prod_{i=1}^n X_i \to \mathbb{C}$ be a function that is dependent only on its i-th variable, i.e.*

$$f : \prod_{i=1}^n X_i \to \mathbb{C}$$
$$(z_1, \dots, z_i, \dots, z_n) \mapsto g(z_i)$$

for some function $g : X_i \to \mathbb{C}$. f is $(\bigotimes_{i=1}^n \mathcal{X}_i, \mathcal{B}_{\mathbb{C}})$-measurable if and only if g is $(\mathcal{X}_i, \mathcal{B}_{\mathbb{C}})$-measurable.

Lemma 4.8. *Let $\{T_i\}_{i=1}^n \subset \mathcal{L}(\mathcal{H})$ be a finite sequence of self-adjoint and commuting operators, let P be its decomposing PVM, and let $f \in \mathcal{B}(\mathbb{R}^n)$ be bounded on supp(P). If f is dependent only on its i-th variable, i.e.*

$$f : \mathbb{R}^n \to \mathbb{C}$$
$$(z_1, \dots, z_n) \mapsto g(z_i)$$

for some function $g : \mathbb{R} \to \mathbb{C}$, then

$$\int_{\mathbb{R}^n} f \; dP = \int_{\mathbb{R}} g \; dP_i.$$

In particular, $\int_{\mathbb{R}^n} \Pi_i \; dP = T_i$, where Π_i is the i-th coordinate projection.

Proof. Suppose $g \in \mathcal{B}_{\mathbb{R}}$ is a simple function. If $\sum_{j=1}^M \alpha_j \cdot \chi_{B_j}$ is the standard representation for g, then we necessarily have $f = \sum_{j=1}^M \alpha_j \cdot \chi_{\Pi_i^{-1}[B_j]}$, where $\Pi_i : \mathbb{R}^n \to \mathbb{R}$ is the projection map for the i-th coordinate. $f \in \mathcal{B}(\mathbb{R}^n)$ because Π_i is continuous. Also,

$$\int_{\mathbb{R}^n} f \; dp_{x,y} = \sum_{j=1}^M \alpha_j \langle P_1(\mathbb{R}) \cdots P_i(B_j) \cdots P_n(\mathbb{R}) x, y \rangle = \sum_{j=1}^M \alpha_j \langle P_i(B_j) x, y \rangle = \int_{\mathbb{R}} g \; dp_{i\,(x,y)}$$

for all $x, y \in \mathcal{H}$. Therefore, $\int_{\mathbb{R}^n} f \; dP = \int_{\mathbb{R}} g \; dP_i$, by uniqueness.

Now, consider an arbitrary f with the hypothesised properties. The boundedness of f on supp(P) implies g is bounded on $\sigma(T_i)$, because $\sigma(T_i) = \Pi_i\,[\text{supp}(P)]$. Also, by Lemma (4.7), g is measurable with respect to $\mathcal{B}_{\mathbb{R}}$, because f is measurable with respect to $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{i=1}^n \mathcal{B}_{\mathbb{R}}$. Therefore, there exists a sequence $\{t_m\} \subset \mathcal{B}(\mathbb{R})$ of simple functions which converges to g uniformly on the compact set $\sigma(T_i) = \text{supp}(P_i)$. As was done above, we may use $\{t_m\}$ to construct another sequence $\{s_m\} \subset \mathcal{B}(\mathbb{R}^n)$ of simple functions. From its construction, $\{s_m\}$ will converge to f uniformly on supp(P). Therefore,

$$\int_{\mathbb{R}^n} f \; dP = \lim_m \int_{\mathbb{R}^n} s_m \; dP = \lim_m \int t_m \; dP_i = \int_{\mathbb{R}} g \; dP_i$$

using the continuity of the maps $\int_{\mathbb{R}^n} \cdot \, dP$ and $\int_{\mathbb{R}} \cdot \, dP_i$ on $\mathcal{B}_b(\text{supp}(P))$ and $\mathcal{B}_b(\text{supp}(P_i))$, respectively.

Finally, consider $\Pi_i : \mathbb{R}^n \to \mathbb{R}$. Because supp$(P)$ is compact, and continuous functions preserve compactness, we have $\Pi_i \in \mathcal{B}_b(\text{supp}(P))$. It is clear Π_i is only dependent on its i-th coordinate, so

$$\int_{\mathbb{R}^n} \Pi_i \; dP = \int_{\mathbb{R}} z \; dP_i(z) = T_i$$

by Lemma (3.42). □

Theorem 4.9 (Spectral Theorem)**.** *Fix $n \in \mathbb{N}$. There is a bijection between the compact PVM's on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ and sequences $\{T_i\}_{i=1}^n \subset \mathcal{L}(\mathcal{H})$ of self-adjoint and commuting operators.*

Proof. Let $\mathcal{P}$ represent the compact PVM's on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$, and let $\mathcal{T}$ represent the sequences $\{T_i\}_{i=1}^n$ of self-adjoint and commuting operators in $\mathcal{L}(\mathcal{H})$. Consider the map

$$\begin{aligned} F : \mathcal{P} &\to \mathcal{T} \\ P &\mapsto \left\{ \int \Pi_i \; dP \right\}_{i=1}^n , \end{aligned}$$

where Π_i is the i-th coordinate projection. Lemma (3.38) ensures that F is well-defined, and Theorem (3.39) implies that F is injective. If P is the decomposing PVM for $\{T_i\}_{i=1}^n \subset \mathcal{T}$, then it has compact support, by Corollary (4.6). Therefore, $P \in \mathcal{P}$, and

$$F(P) = \left\{ \int_{\mathbb{R}^n} \Pi_i \; dP \right\}_{i=1}^n = \{T_i\}_{i=1}^n,$$

by Lemma (4.8). This shows that F is also surjective. □

4.2 The Functional Calculus

Theorem 4.10 (Spectral Theorem)**.** *Let $\{T_i\}_{i=1}^n \subset \mathcal{L}(\mathcal{H})$ be a finite sequence of self-adjoint and commuting operators, and let P be its decomposing PVM. There is a unique map $\Phi : \mathcal{B}_b(supp(P)) \to \mathcal{L}(\mathcal{H})$ such that:*

(a) $\Phi(\Pi_i) = T_i$, where $\Pi_i : \mathbb{R}^n \to \mathbb{R}$ is the i-th coordinate projection.

(b) Φ is a unital algebraic homomorphism.

(c) If $\{f_m\} \subset \mathcal{B}_b(supp(P))$ is a bounded sequence which converges to f pointwise, then $\{\Phi(f_m)\}$ converges to $\Phi(f)$ strongly.

In addition, Φ has the following properties:

(d) Φ is continuous, with operator norm $\|\Phi\| = 1$.

*(e) Φ is a *-homomorphism.*

(f) If $f \in \mathcal{B}_b(supp(P))$ is such that $f \geq 0$, then $\Phi(f) \geq 0$.

(g) If $S \in \mathcal{L}(\mathcal{H})$ commutes with each T_i, then $\Phi(f)\, S = S\, \Phi(f)$ for all $f \in \mathcal{B}_b(supp(P))$.

(h) If for some $x \in \mathcal{H}$ there exist $\{\lambda_i\}_{i=1}^n \subset \mathbb{C}$ such that $T_i x = \lambda_i x$, then $\Phi(f)x = f(\lambda_1, \ldots, \lambda_n)x$ for all $f \in \mathcal{B}_b(supp(P))$.

Proof. P is a PVM on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$, so we may define the function

$$\begin{aligned} \Phi : \mathcal{B}_b(\mathrm{supp}(P)) &\to \mathcal{L}(\mathcal{H}) \\ f &\mapsto \int f\ dP. \end{aligned}$$

By Lemma (4.8), we have that $\Phi(\Pi_i) = T_i$. Φ also will inherit properties (b), (c), (d), and (e) from $\int \cdot\, dP$. Noting that $p_{x,x}$ is a positive measure for every $x \in \mathcal{H}$, property (f) may also be verified as in Theorem (3.17).

Suppose there is some $S \in \mathcal{L}(\mathcal{H})$ which commutes with each T_i, and suppose there is some $x \in \mathcal{H}$ and $\{\lambda_i\}_{i=1}^n$ such that $T_i x = \lambda_i x$. Let $\mathcal{V}$ be the collection of all $f \in \mathcal{B}_b(\mathrm{supp}(P))$ such that $\Phi(f)$ commutes with S, and $\Phi(f)x = f(\lambda_1, \ldots, \lambda_n)x$. First, note that $\mathcal{B}_b(\mathrm{supp}(P))$ contains all complex polynomials of n real variables. Given any polynomial $p(z_1, \ldots, z_n) = \sum_{j=0}^k \alpha_j \cdot z_1^{m_{1,j}} \cdots z_n^{m_{n,j}}$, we have

$$\Phi(p) = \sum_{j=0}^k \alpha_j \cdot \Phi(\Pi_1)^{m_{1,j}} \cdots \Phi(\Pi_n)^{m_{n,j}} = \sum_{j=0}^k \alpha_j \cdot T_1^{m_{1,j}} \cdots T_n^{m_{n,j}},$$

because Φ is an algebraic homomorphism. It follows that

$$\Phi(p)\, S = \left(\sum_{j=0}^k \alpha_j \cdot T_1^{m_{1,j}} \cdots T_n^{m_{n,j}} \right) S = S \left(\sum_{j=0}^k \alpha_j \cdot T_1^{m_{1,j}} \cdots T_n^{m_{n,j}} \right) = S\, \Phi(p).$$

It is easily seen by induction that $T_i^m x = \lambda_i^m x$ for any $m \in \mathbb{N}$, and clearly $x = T_i^0 x = \lambda_i^0 x$. Therefore,

$$\begin{aligned} \Phi(p)x = \sum_{j=0}^k \alpha_j \cdot T_1^{m_{1,j}} \cdots T_n^{m_{n,j}} x &= \sum_{j=0}^k \alpha_j \lambda_{i-1}^{m_{i-1,j}} \cdots \lambda_n^{m_{n,j}} T_1^{m_{1,j}} \cdots T_i^{m_{i,j}} x \\ &= \sum_{j=1}^k \alpha_j \lambda_1^{m_{1,j}} \cdots \lambda_n^{m_{n,j}} x = p(\lambda_1, \ldots, \lambda_n)x. \end{aligned}$$

This shows that all complex polynomials of n real variables are contained in $\mathcal{V}$. $\mathcal{V}$ is a vector space, because Φ, composition, and the evaluation map at x are linear, and because of the distributivity of scalar multiplication. If $\{f_m\} \subset \mathcal{V}$ is a bounded sequence converging to f pointwise, we may use the strong convergence of $\{\Phi(f_m)\}$ to show $f \in \mathcal{V}$, exactly as was done in the proof Theorem (3.17). Therefore, by Corollary (3.16), $\mathcal{V} = \mathcal{B}_b(\mathrm{supp}(P))$, i.e. Φ satisfies properties (g) and (h).

Let $\Psi : \mathcal{B}_b(\mathrm{supp}(P)) \to \mathcal{L}(\mathcal{H})$ be another function satisfying properties (a), (b), and (c), and let $\mathcal{V}$ be the collection of all $f \in \mathcal{B}_b(\mathrm{supp}(P))$ such that $\Psi(f) = \Phi(f)$. Properties (a) and (b) imply that $\mathcal{V}$ contains all complex polynomials of n real variables. Also, the linearity of Ψ and Φ implies that $\mathcal{V}$ is a vector space. If $\{f_m\} \subset \mathcal{V}$ is a bounded sequence which converges to f pointwise, then we necessarily have $f \in \mathcal{V}$, because of property (c). Therefore, by Corollary (3.16), $\mathcal{V} = \mathcal{B}_b(\mathrm{supp}(P))$, i.e. $\Psi \equiv \Phi$. $\square$

4.3 The Associated Multiplication Operator

Definition 4.11. Let $\{T_i\}_{i=1}^n \subset \mathcal{L}(\mathcal{H})$ be a finite sequence of commuting operators. $x \in \mathcal{H}$ is a **cyclic vector** for $\{T_i\}_{i=1}^n$ if

$$\{p(T_1, \ldots, T_n)x \mid p \text{ is a complex polynomial of } n \text{ variables}\}$$

is a dense subset of $\mathcal{H}$.

Lemma 4.12. *Let $\{T_i\}_{i=1}^n \subset \mathcal{L}(\mathcal{H})$ be a finite sequence of self-adjoint and commuting operators, with a cyclic vector $x \in \mathcal{H}$, and P as its decomposing PVM. There is a positive Radon measure μ on supp$(P) \subset \mathbb{R}^n$, and a unitary map $U : \mathcal{H} \to L^2(\mathit{supp}(P), \mu)$, such that*

$$\begin{aligned} UT_iU^{-1} : L^2(\mathit{supp}(P), \mu) &\to L^2(\mathit{supp}(P), \mu) \\ f(z_1, \ldots, z_n) &\mapsto z_i \cdot f(z_1, \ldots, z_n) \end{aligned}$$

for each $i \in \{1, \ldots, n\}$.

Proof. The proof of this lemma is very similar to that of Lemma (3.23). We will only provide the outline, with the necessary alterations.

Given the cyclic vector $x \in \mathcal{H}$ and the decomposing PVM P, let $\mu = p_{x,x}$. μ is a positive Radon measure on $\mathbb{R}^n$, with its support contained in the compact set supp(P). By Theorem (3.22), $\mathcal{C}(\text{supp}(P))\ /\ (f = g \text{ a.e.})$ is a dense subspace of $L^2(\text{supp}(P), \mu)$.

Define the map

$$\begin{aligned} \widehat{U}_0 : \mathcal{C}(\text{supp}(P))\ /\ (f = g \text{ a.e.}) &\to \mathcal{H} \\ f &\mapsto \Phi(f)x, \end{aligned}$$

where Φ gives the functional calculus for $\{T_i\}_{i=1}^n$. $\widehat{U}_0$ is a well-defined linear isometry. Also, its range will be dense in $\mathcal{H}$, because $\mathcal{C}(\text{supp}(P))\ /\ (f = g \text{ a.e.})$ contains all complex polynomials of n variables, and x is cyclic. Since $\widehat{U}_0$ has these properties, it may be extended to a unitary map U_0 from $L^2(\text{supp}(P)), \mu)$ to $\mathcal{H}$, using the Bounded Linear Transformation Theorem, and Theorem (3.8).

For $i \in \{1, \ldots, n\}$, define

$$\begin{aligned} V : L^2(\text{supp}(P), \mu) &\to L^2(\text{supp}(P), \mu) \\ f(z_1, \ldots, z_n) &\mapsto z_i \cdot f(z_1, \ldots, z_n). \end{aligned}$$

V is a well-defined, bounded linear operator on $L^2(\text{supp}(P), \mu)$ satisfying $U_0^{-1}T_iU_0 f = Vf$ for all $f \in \mathcal{C}(\text{supp}(P))\ /\ (f = g \text{ a.e.})$. Continuity implies $U_0^{-1}T_jU_0 \equiv V$.

Letting $U = U_0^{-1}$, we obtain the desired statement of the theorem. □

Lemma 4.13. *Let $\{T_i\}_{i=1}^n \subset \mathcal{L}(\mathcal{H})$ be a finite sequence of self-adjoint and commuting operators. There is a collection $\{\mathcal{H}_j\}_{j\in J}$ of pairwise orthogonal, closed subspaces of $\mathcal{H}$ such that*

(a) For each $j \in J$, $\left\{T_i|_{\mathcal{H}_j}\right\}_{i=1}^n \subset \mathcal{L}(\mathcal{H}_j)$.
(b) For each $j \in J$, there exists $x_j \in \mathcal{H}_j$ such that x_j is a cyclic vector for $\left\{T_i|_{\mathcal{H}_j}\right\}_{i=1}^n$.
(c) $\mathcal{H} = \bigoplus_{j\in J} \mathcal{H}_j$.

Proof. The proof of this lemma is very similar to that of Lemma (3.24). We will only provide the outline, with the necessary alterations.

If $\mathcal{H}$ is trivial, then the desired results are immediate. We will assume $\mathcal{H}$ is non-trivial. For every nonzero $x \in \mathcal{H}$, let

$$P_x = \{\Phi(p)x \mid p \text{ is a complex polynomial of } n \text{ variables}\} \subset \mathcal{H},$$

where Φ gives the functional calculus for $\{T_i\}_{i=1}^n$. Each P_x is a subspace of $\mathcal{H}$ such that

$$T_i\ \Phi(p)x = \Phi(z_i \cdot p(z_1, \ldots, z_n))x \in P_x$$

for every polynomial of n variables. Therefore, each $\overline{P_x}$ will be a closed subspace of $\mathcal{H}$ that is invariant under $\{T_i\}_{i=1}^n$. It is then obvious that $\left\{T_i|_{\overline{P_x}}\right\}_{i=1}^n \subset \mathcal{L}(\overline{P_x})$ is a finite sequence of self-adjoint and commuting operators which also has x as a cyclic vector.

An application of Zorn's Lemma guarantees there is a maximal pairwise orthogonal subcollection of $\{\overline{P_x}\}_{x\in\mathcal{H}}$. Denote this subcollection by $\{\overline{P_x}\}_{x\in M}$, where $M \subset \mathcal{H}$. We claim $\{\overline{P_x}\}_{x\in M}$ is our desired collection $\{\mathcal{H}_j\}_{j\in J}$. It is already clear that properties (a) and (b) of the theorem hold.

Let $\mathcal{K} := \bigoplus_{x\in M} \overline{P_x} \subset \mathcal{H}$. $\{T_i\}_{i=1}^n$ are invariant on $\mathcal{K}$, and hence invariant on $\mathcal{K}^\perp$, because they are self-adjoint. If $\mathcal{K}$ is a strict subset of $\mathcal{H}$, then there exists a nonzero $x_0 \in \mathcal{K}^\perp$. In that case, $\overline{P_{x_0}} \subset \mathcal{K}^\perp$, because $\{T_i\}_{i=1}^n$ are invariant on $\mathcal{K}^\perp$. This is a contradiction, proving that property (c) holds. □

Theorem 4.14 (Spectral Theorem)**.** *Let $\{T_i\}_{i=1}^n$ be a finite sequence of self-adjoint and commuting operators. There is a collection $\{\mu_j\}_{j\in J}$ of finite, positive measures on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$, and a unitary map $U : \mathcal{H} \to \bigoplus_{j\in J} L^2(\mathbb{R}^n, \mu_j)$, such that*

$$\begin{aligned} UT_iU^{-1} : \bigoplus_{j\in J} L^2(\mathbb{R}^n, \mu_j) &\to \bigoplus_{j\in J} L^2(\mathbb{R}^n, \mu_j) \\ \sum_{j\in J} f_j(z_1, \ldots, z_n) &\mapsto \sum_{j\in J} z_i \cdot f_j(z_1, \ldots, z_n) \end{aligned}$$

for each $i \in \{1, \ldots, n\}$.

Proof. The proof is nearly identical to that of Theorem (3.26). Lemma (4.13) gives the direct sum decomposition $\mathcal{H} = \bigoplus_{j\in J} \mathcal{H}_j$, where $\left\{T_i|_{\mathcal{H}_j}\right\}_{i=1}^n \subset \mathcal{L}(\mathcal{H}_j)$ is a sequence of self-adjoint and commuting operators, with cyclic vector $x_j \in \mathcal{H}_j$, for each $j \in J$. Letting P_j be the decomposing PVM for $\left\{T_i|_{\mathcal{H}_j}\right\}_{i=1}^n$, we use Lemma (4.12) to obtain the collections $\{\mu_j\}_{j\in J}$ and $\{U_j\}_{j\in J}$ of measures and unitary maps. U is constructed, and shown to have the desired properties, as in Theorem (3.26). Noting that $L^2(\text{supp}(P_j), \mu_j) = L^2(\mathbb{R}^n, \mu_j)$ for each $j \in J$, the proof is complete. □

4.4 Normal Bounded Operators

Lemma 4.15. *For every $T \in \mathcal{L}(\mathcal{H})$, there exist unique self-adjoint operators $T_R, T_I \in \mathcal{L}(\mathcal{H})$ such that $T = T_R + i \cdot T_I$. T is normal if and only if T_R and T_I commute.*

Proof. For an arbitrary $T \in \mathcal{L}(\mathcal{H})$, let $T_R = \frac{1}{2}(T^* + T)$ and $T_I = \frac{i}{2}(T^* - T)$. It is easily seen that T_R and T_I are self-adjoint, with $T = T_R + i\,T_I$. Now, suppose there are additional self-adjoint operators S_R, $S_I \in \mathcal{L}(\mathcal{H})$ which also decompose T in this way. It follows that

$$(T_R - S_R) + i\,(T_I - S_I) = 0 = 0^* = (T_R - S_R) - i\,(T_I - S_I).$$

This implies $T_I = S_I$, so we must also have $T_R = S_R$.

If T_R and T_I commute, then clearly $T = T_R + i\,T_I$ will commute with $T^* = T_R - i\,T_I$. In the case that T is normal,

$$T_R^2 + i\,(T_I\,T_R - T_R\,T_I) + T_I^2 = T\,T^* = T^*\,T = T_R^2 + i\,(T_R\,T_I - T_I\,T_R) + T_I^2$$

implies $T_R\,T_I = T_I\,T_R$. □

Definition 4.16. For $T \in \mathcal{L}(\mathcal{H})$, let $T_R, T_I \in \mathcal{L}(\mathcal{H})$ be as given in the previous lemma. T_R will be called the **real component** of T, and T_I will be called the **imaginary component** of T_I.

Theorem 4.17 (Spectral Theorem)**.** *There is a bijection between the compact PVM's on $(\mathbb{C}, \mathcal{B}_{\mathbb{C}})$ and the normal operators in $\mathcal{L}(\mathcal{H})$.*

Proof. Let $\mathcal{P}$ represent the compact PVM's on $(\mathbb{C}, \mathcal{B}_{\mathbb{C}})$, let $\mathcal{T}$ represent the ordered pairs of self-adjoint and commuting operators in $\mathcal{L}(\mathcal{H})$, and let $\mathcal{N}$ represent the normal operators in $\mathcal{L}(\mathcal{H})$. Consider the maps

$$\begin{aligned} F : \mathcal{P} &\to \mathcal{T} \\ P &\mapsto \left\{ \int_{\mathbb{C}} Re(z)\; dP(z),\; \int_{\mathbb{C}} Im(z)\; dP(z) \right\} \end{aligned}$$

and

$$\begin{aligned} G : \mathcal{T} &\to \mathcal{N} \\ \{T_R,\, T_I\} &\mapsto T_R + i\,T_I. \end{aligned}$$

F is the bijection from Theorem (4.9), once $(\mathbb{C}, \mathcal{B}_{\mathbb{C}})$ is identified with $(\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2})$. In view of Lemma (4.15), G is also a bijection. Therefore, $G \circ F$ is a bijection. □

Definition 4.18. Let T be normal, and let T_R and T_I be its real and imaginary components. The decomposing PVM for T_R and T_I on $(\mathbb{C}, \mathcal{B}_{\mathbb{C}})$ will also be called the **decomposing PVM** for T.

Lemma 4.19. *$S \in \mathcal{L}(\mathcal{H})$ commutes with the real and imaginary components of $T \in \mathcal{L}(\mathcal{H})$ if and only if S commutes with both T and T^*.*

Proof. Let T_R and T_I be the real and imaginary components of T. T_R and T_I are self-adjoint, so $T^* = T_R - iT_I$. In the case that S commutes with T_R and T_I, it is clear S also commutes with T and T^*. If S commutes with T and T^*, then

$$\begin{aligned} 2\,S\,T_R = S\,T_R + i\,S\,T_I + S\,T_R - i\,S\,T_I = S\,T + S\,T^* \\ = T\,S + T^*\,S = T_R\,S + i\,T_I\,S + T_R\,S - i\,T_I\,S = 2\,T_R\,S \end{aligned}$$

implies that S commutes with T_R; similarly, the identity $S\,T - S\,T^* = T\,S - T^*\,S$ implies that S commutes with T_I. ☐

Lemma 4.20. *If $T \in \mathcal{L}(\mathcal{H})$ is normal, with T_R and T_I as its real and imaginary components, then $\lambda \in \sigma_p(T)$ if and only if $Re(\lambda) \in \sigma_p(T_R)$ and $Im(\lambda) \in \sigma_p(T_I)$ have a common eigenvector. The eigenvectors for $\lambda \in \sigma_p(T)$ will be the same as the shared eigenvectors for $Re(\lambda) \in \sigma_p(T_R)$ and $Im(\lambda) \in \sigma_p(T_I)$.*

Proof. If $\lambda_R \in \sigma_p(T_R)$ and $\lambda_I \in \sigma_p(T_I)$ have $x \in \mathcal{H}\backslash\{0\}$ as a common eigenvector, then

$$Tx = T_R x + i\,T_I x = (\lambda_R + i\lambda_I)x.$$

This demonstrates that $\lambda = \lambda_R + i\lambda_I$ is in the point spectrum of T, with x as an eigenvector. Because T_R and T_I are self-adjoint, their spectrums are subsets of $\mathbb{R}$. Therefore, $Re(\lambda) = \lambda_R$ and $Im(\lambda) = \lambda_I$.

Now, consider any $\lambda \in \sigma_p(T)$. If $x \in \mathcal{H}\backslash\{0\}$ is an eigenvector for λ, and $\lambda_R = Re(\lambda)$ and $\lambda_I = Im(\lambda)$, then we have

$$(T_R + i\,T_I)x = (\lambda_R + i\lambda_I)x,$$

which implies

$$(T_R - \lambda_R)x = i(\lambda_I - T_I)x.$$

With this identity in mind,

$$\begin{aligned} i\,\|(T_R - \lambda_R)x\|_{\mathcal{H}}^2 &= i\,\langle (T_R - \lambda_R)x, i(\lambda_I - T_I)x\rangle \\ &= \lambda_I\langle T_R x, x\rangle - \langle T_R x, T_I x\rangle - \lambda_R\lambda_I\|x\|_{\mathcal{H}} + \lambda_R\langle x, T_I x\rangle \\ &= \lambda_I\langle T_R x, x\rangle - \langle T_I\,T_R x, x\rangle - \lambda_R\lambda_I\|x\|_{\mathcal{H}} + \lambda_R\langle T_I x, x\rangle. \end{aligned}$$

Because T is normal, T_R and T_I are self-adjoint and commuting operators, so $T_I\,T_R$ is self-adjoint. Therefore, the right hand side of the above equality is real, while the left hand side is imaginary. This implies $(T_R - \lambda_R)x = 0$, and, with the same argument, $i(\lambda_I - T_I)x = 0$. It follows that $T_R x = \lambda_R x$ and $T_I x = \lambda_I x$. We may conclude λ_R is in $\sigma_p(T_R)$, λ_I is in $\sigma_p(T_I)$, and that they have x as a common eigenvector. ☐

Theorem 4.21 (Spectral Theorem)**.** *Let $T \in \mathcal{L}(\mathcal{H})$ be normal, and let P be its decomposing PVM on $(\mathbb{C}, \mathcal{B}_{\mathbb{C}})$. There exists a unique map $\Phi : \mathcal{B}_b(supp(P)) \to \mathcal{L}(\mathcal{H})$ such that:*

(a) If $Id \in \mathcal{B}_b(supp(P))$ is the identity function, i.e. $Id(z) = z$, then $\Phi(Id) = T$.

*(b) Φ is a unital, algebraic *-homomorphism.*

(c) If $\{f_n\} \subset \mathcal{B}_b(supp(P))$ is a bounded sequence which converges to f pointwise, then $\{\Phi(f_n)\}$ converges to $\Phi(f)$ strongly.

In addition, Φ has the following properties:

(d) Φ is continuous, with operator norm $\|\Phi\| = 1$.

(e) If $f \in \mathcal{B}_b(supp(P))$ is such that $f \geq 0$, then $\Phi(f) \geq 0$.

(f) If $S \in \mathcal{L}(\mathcal{H})$ commutes with both T and T^*, then $\Phi(f)\, S = S\, \Phi(f)$ for all $f \in \mathcal{B}_b(supp(P))$.

(g) If $Tx = \lambda x$ for some $x \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, then $\Phi(f)x = f(\lambda)x$ for all $f \in \mathcal{B}_b(supp(P))$.

Proof. Let T_R and T_I be the real and imaginary components for T. Let Φ be the functional calculus for $\{T_R, T_I\}$, an ordered pair of self-adjoint and commuting operators. Noting that

$$Id(z) = Re(z) + i\, Im(z),$$

and recalling Lemma (4.19) and Lemma (4.20), it is clear that Φ will satisfy properties (a) - (g).

The uniqueness conditions differ from those in Theorem (4.10). By requiring Φ to be a *-homomorphism in addition to being a unital algebraic homomorphism, it follows that $\Phi(Re(z))$ and $\Phi(Im(z))$ are self-adjoint. In this case,

$$\Phi(Re(z)) + i\, \Phi(Im(z)) = \Phi(Id(z)) = T = T_R + i\, T_I$$

implies $\Phi(Re(z)) = T_R$ and $\Phi(Im(z)) = T_I$, because the real and imaginary components of T are unique. The uniqueness of Φ now follows from Theorem (4.10). $\square$

Theorem 4.22 (Spectral Theorem)**.** *Let $T \in \mathcal{L}(\mathcal{H})$ be normal. There is a collection $\{\mu_j\}_{j\in J}$ of finite, positive measures on $(\mathbb{C}, \mathcal{B}_{\mathbb{C}})$, and a unitary map $U : \mathcal{H} \to \bigoplus_{j\in J} L^2(\mathbb{C}, \mu_j)$, such that*

$$\begin{aligned} UTU^{-1} : \bigoplus_{j\in J} L^2(\mathbb{C}, \mu_j) &\to \bigoplus_{j\in J} L^2(\mathbb{C}, \mu_j) \\ \sum_{j\in J} f_j(z) &\mapsto \sum_{j\in J} z \cdot f_j(z). \end{aligned}$$

Proof. Applying Theorem (4.14) to the real and imaginary components of T, let $\{\mu_j\}_{j\in J}$ be the resulting measures on $(\mathbb{C}, \mathcal{B}_{\mathbb{C}})$, and $U : \mathcal{H} \to \bigoplus_{j\in J} L^2(\mathbb{C}, \mu_j)$ the resulting unitary map. If T_R and T_I are the real and imaginary components for T, then

$$UTU^{-1} = U(T_R + i\, T_I)U^{-1} = UT_RU^{-1} + i\, UT_IU^{-1}.$$

Therefore, UTU^{-1} maps each $\sum_{j\in J} f_j(z) \in \bigoplus_{j\in J} L^2(\mathbb{C}, \mu_j)$ to

$$\left(\sum_{j\in J} Re(z) \cdot f_j(z)\right) + i \left(\sum_{j\in J} Im(z) \cdot f_j(z)\right) = \sum_{j\in J} z \cdot f_j(z).$$

$\square$

Chapter 5

Conclusion

Spectral theory is an important area of mathematics with many applications; however, many of the proofs provided in standard texts on spectral theory are terse. In this thesis, we carefully developed the background necessary to rigorously prove several versions of the spectral theorem.

We began by constructing the functional calculus $\Phi : \mathcal{B}_b(\sigma(T)) \to \mathcal{L}(\mathcal{H})$ for a self-adjoint $T \in \mathcal{L}(\mathcal{H})$. This was done by considering polynomials of T, and then constructing the continuous functional calculus $\phi : \mathcal{C}(\sigma(T)) \to \mathcal{L}(\mathcal{H})$. This ϕ was used to create a family of complex measures $\{\mu_{x,y}\}_{x,y\in\mathcal{H}}$, which allowed us to define Φ. Thus defined, Φ is the unique algebra homomorphism which maps each polynomial function p on $\sigma(T)$ to $p(T)$.

Turning our attention to the multiplication operator version of the spectral theorem, we showed that T is unitarily equivalent to multiplication by $Id(z) := z$ on $L^2\left(\sigma(T), \mu_{x,x}\right)$ when it has a cyclic vector. The functional calculus was essential to this proof. By finding a suitable direct sum decomposition of $\mathcal{H}$, we were able to prove a similar result for the case where T does not have a cyclic vector.

We then considered the projection-valued measure version of the spectral theorem. For an arbitrary compact PVM P on $(\mathcal{B}_{\mathbb{R}}, \mathbb{R})$, we constructed a family of complex measures $\{p_{x,y}\}_{x,y\in\mathcal{H}}$. These measures enabled us to define the map $\int \cdot \, dP : \mathcal{B}_{\mathbb{R}} \to \mathcal{L}(\mathcal{H})$, and we saw that $\int z \, dP(z)$ would be self-adjoint. Conversely, the functional calculus for T allowed us to construct a compact PVM on $(\mathcal{B}_{\mathbb{R}}, \mathbb{R})$. In summary, we showed that there is a bijection between such PVM's and self-adjoint bounded operators.

Finally, we generalized the aforementioned spectral theorems to finite sequences $\{T_i\}_{i=1}^n \subset \mathcal{L}(\mathcal{H})$ of self-adjoint and commuting operators. Once we had proved the existence of a "decomposing" PVM for $\{T_i\}_{i=1}^n$, the proofs of the generalized spectral theorems largely followed those from before. By considering the particular case where $n = 2$, we obtained the spectral theorems for normal bounded operators.

Spectral theory for densely-defined operators on Hilbert space is the natural extension of the material presented in this thesis. In certain applications, the operators that arise tend to be densely-defined and unbounded, so their associated theory is of particular importance. Furthermore, the spectral theorems for collections of such operators is poorly documented; it would be worthwhile to provide a rigorous presentation.

Bibliography

[1] V. I. Bogachev. *Measure theory*, volume one. Springer-Verlag, Berlin, 2007.

[2] John B. Conway. *A course in functional analysis*. Springer-Verlag, New York, second edition, 1990.

[3] Gerald B. Folland. *Real analysis*. John Wiley & Sons Inc., New York, second edition, 1999.

[4] Paul R. Halmos. *Introduction to Hilbert Space and the theory of Spectral Multiplicity*. Chelsea Publishing Company, New York, 1951.

[5] Michael Reed and Barry Simon. *Methods of modern mathematical physics*, volume one. Academic Press, New York, 1972.