Dynamics of multi-scalar-field cosmological models and assisted inflation

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We investigate the dynamical properties of a class of spatially homogeneous and isotropic cosmological models containing a barotropic perfect fluid and multiple scalar fields with independent exponential potentials. We show that the assisted inflationary scaling solution is the global late-time attractor for the parameter values for which the model is inflationary, even when curvature and barotropic matter are included. For all other parameter values the multi-field curvature scaling solution is the global late-time attractor (in these asymptotically stable solutions the curvature is not dynamically negligible). Consequently, we find that in general all of the scalar fields in multi-field models with exponential potentials are non-negligible in late-time behavior, contrary to what is commonly believed. The early-time and intermediate behavior of the models is also studied. In particular, n-scalar field models are investigated and the structure of the saddle equilibrium points corresponding to inflationary m-field scaling solutions and non-inflationary m-field matter scaling solutions are also studied (where $m<n$), leading to interesting transient dynamical behavior perhaps associated with new physical scenarios of potential importance.

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I. INTRODUCTION

Inflation is generally considered to be a reasonable solution to many of the fundamental problems within the standard cosmological model. In original proposals [1], the early universe experiences a period of accelerated expansion and essentially expands at an exponential rate [i.e., $R(t)\propto e^{At}$, where $R(t)$ can be considered as the size of the universe and A is some positive constant]. Since these early proposals, there have been a variety of inflationary models that include scalar fields that have been proposed [2], and scalar fields have come to play an important role in determining the dynamics of the early universe. In one important class of inflationary models the condition of exponential expansion is relaxed, and the universe grows at a power-law rate, $R(t)\propto t^p$, where $p>1$ [3]. In particular, power-law inflationary models arise in models with a scalar field ϕ having an exponential potential $V(\phi)=V_0 e^{k\phi}$ [4]. Although power-law inflation is successful in solving the horizon and flatness problems, inflation in these models persists into the indefinite future and a phase transition is required to bring inflation to an end (however, see [5]).

Spatially homogeneous models containing a scalar field ϕ with an exponential potential have been analyzed extensively [6,7]. It is known that all ever-expanding scalar field models experience power-law inflation when the parameter $k^2<2$; i.e., when the potential is sufficiently flat. The models have also been studied when $k^2>2$ [7]. Recently cosmological models containing both a scalar field with an exponential potential and a perfect fluid with a linear barotropic equation of state have been studied. It is found that in the general class of Bianchi type B models that the power-law inflationary solution is still the global attractor in the physically realistic regime (i.e., when $\gamma>2/3$) if $k^2<2$ [8]. Interestingly, the addition of a barotropic perfect fluid creates the existence of a new type of solution, appropriately called a matter scaling solution [9–11], in which the energy density of the scalar field scales with that of the matter; the effective equation of state for the scalar field is the same as that of the perfect fluid. The stability of the matter scaling solution has been studied in [9,10].

Exponential potentials arise in many theories of the fundamental interactions including superstring and higher-dimensional theories [2,12]. Typically, ''realistic'' supergravity theories predict steep exponential potentials [12] (i.e., $k^2>2$), effectively eliminating the possibility of power-law inflation. However, dimensionally reduced higher-dimensional theories also predict numerous scalar fields, and so it is of interest to study models with multiple scalar fields.

In the recent work of Liddle, Mazumdar and Schunck [13] the effect of additional scalar fields with independent exponential potentials was considered. They assumed n scalar fields in a spatially flat Friedmann-Robertson-Walker (FRW) universe. They found that an arbitrary number of scalar fields with exponential potentials evolve towards a novel inflationary scaling solution, which they termed *assisted inflation*, in which all of the scalar fields scale with one another (and are hence non-negligible asymptotically) with the result that inflation occurs even if each of the individual potentials is too steep to support inflation on its own. The existence of multiple uncoupled scalar fields, each having an exponential potential, could therefore, through a combined (or *assisted*) effort, be a source for power-law inflation. This is true even though each individual scalar field need not be a source for inflation, and might therefore lead to compatibility with supergravity theory.

In a recent dynamical analysis [14] it was shown that this assisted inflationary solution is a late-time attractor in the class of zero-curvature FRW models. This was done by choosing a redefinition of the fields (a rotation in field space) which allows the effective potential for field variations orthogonal to this solution to be written down; in analogy with models of hybrid inflation [15] it was then shown that this potential has a global minimum along the attractor solution. Also, analytic solutions describing homogeneous and inhomogeneous perturbations about the attractor solution without resorting to slow-roll approximations were presented in [14], and curvature and isocurvature perturbation spectra produced from vacuum fluctuations during assisted inflation were discussed.

In this paper we shall present a qualitative analysis of models with the action

$$S=\int d^4x\sqrt{-g}\left[R-\frac{1}{2}\sum_{i=1}^{n}(\nabla\phi_i^2)-V_0\sum_{i=1}^{n}e^{k_i\phi_i}\right]+S_m, \tag{1.1}$$

where S_m is the matter contribution. Almost all previous analyses of multiple scalar field inflationary models have assumed zero-curvature FRW spacetimes with no matter; here we extend the analysis to include both curvature and matter. In Sec. II we shall present the governing equations for n scalar fields with exponential potentials and matter. In Sec. III we shall study the two-scalar field model with no matter, and, in particular, discuss the stability of the two-field assisted inflationary model. In Sec. IV we shall study the two-scalar field model with barotropic matter. In Sec. V we shall discuss three- and multi-scalar field models. In Sec. VI we present our conclusions.

Generalized assisted inflation

Recently, models with $n\times m$ scalar fields ϕ_{ij} and containing multiplicative exponential terms in the effective potential of the form

$$V_{eff}\equiv\sum_{i=1}^{n}\prod_{j=1}^{m}V_0\mathrm{e}^{k_{ij}\phi_{ij}}=\sum_{i=1}^{n}V_0^m\mathrm{e}^{\sum_{j=1}^{m}k_{ij}\phi_{ij}},$$

where $1\leqslant i\leqslant n$ and $1\leqslant j\leqslant m$ and k_{ij} are $n\times m$ real positive constants which are not zero, have also been studied. A qualitative analysis of the $m=1$ case has been given in [16], where an analogy was made with the dynamics of soft inflation [17].

In [18] a class of spatially flat FRW multi-scalar field models with multiplicative exponential potentials was studied. Potentials of this form are quite common in dimensionally reduced supergravity models [19,20]. Exact two-field and general n-field power-law scaling inflationary solutions were obtained, which were demonstrated to be late-time attractors, generalizing the assisted inflationary solutions previously obtained [13]; this behavior was dubbed ''generalized assisted inflation.'' It was shown that it is more difficult to obtain assisted inflation in these generalized models with cross couplings between the scalar fields in the potential; the fields in any one exponential term tend to conspire to act against one another rather than assist each other (a result also noticed in [20]). However, as with the original version of assisted inflation, this inhibiting affect can be compensated for if there are enough exponential terms present in the potential (i.e., if n is large enough) [13,18].

The dynamics of ''generalized assisted inflation'' was investigated in more detail in [21]. By introducing field rotations, which results in the introduction of two orthogonal fields one of which is massless and the other possesses an exponential potential [14,21], the nature of the late-time attractor solution in a particular class of models was determined. A dimensionally reduced action resulting from within the context of a generalized toroidal compactification of higher-dimensional fields in Einstein gravity minimally coupled to massless scalar fields was shown to give rise to a model of the form under investigation, and it was shown how the addition of interactions between the fields impede inflation in this model.

Similar behavior was also noted by Kanti and Olive [22] in multi-field assisted inflationary models with standard chaotic polynomial (rather than exponential) potentials, which can arise in modern Kaluza-Klein theories (and are a natural outcome of the compactification of higher dimensional theories down to four dimensions). Indeed, Kanti and Olive [23] have recently proposed a possible realization of assisted inflation based on the compactification of a five-dimensional Kaluza-Klein model, and have shown how the additional fields of the assisted sector actually impede inflation (they also showed that the assisted sector, coming from a Kaluza-Klein compactification, eliminates the need for a fine-tuned quartic coupling to drive chaotic inflation). In Kaloper and Liddle [24] the dynamics of a simple implementation of the idea in Kanti and Olive [23] was analyzed in more detail. Since assisted inflation no longer corresponds to an asymptotic attractor, they found that as inflation proceeds the number of fields participating in the assisted behavior decreases resulting in the interesting novel feature that the density perturbations generated retain some information about the initial conditions.

II. THE MODEL

We shall assume that the spacetime is spatially homogeneous and isotropic. The line element for such a spacetime has the form

$$ds^2=-dt^2+R^2(t)\left[\frac{dr^2}{1-kr^2}+r^2(d\theta^2+\sin^2(\theta)d\phi^2)\right]$$

where $k=+1,-1,0$ determines whether the model is closed (positive-curvature), open (negative-curvature), or flat (zero-curvature).

We shall consider n scalar fields ϕ_i, where $1\leqslant i\leqslant n$, in which the effective potential has the form

$$V_{eff}\equiv\sum_{i=1}^{n}V_0\mathrm{e}^{k_i\phi_i},$$

where the k_i are real non-zero positive constants. We also assume that there exists a non-interacting perfect fluid with density ρ and pressure

$$p=(\gamma-1)\rho, \tag{2.1}$$

and we shall assume that $1\leqslant\gamma<2$. The Einstein field equations, the conservation equations, together with the Klein-Gordon equations for the scalar fields, yield the following autonomous system of ordinary differential equations:

$$H^2-\frac{1}{6}\left(\sum_{i=1}^{n}\dot{\phi}_i^2\right)-\frac{1}{3}V_{eff}-\frac{1}{3}\rho=-\frac{{}^3R}{6},$$

$$\dot{H}=-H^2-\frac{1}{3}\left(\sum_{i=1}^{n}\dot{\phi}_i^2\right)+\frac{1}{3}V_{eff}-\frac{1}{6}(3\gamma-2)\rho, \tag{2.2}$$

$$\dot{\rho}=-3\gamma H\rho,$$

$$\ddot{\phi}_i+3H\dot{\phi}_i+k_iV_0e^{k_i\phi_i}=0,$$

where ${}^3R=k/R^2$ is the curvature of the spacelike hypersurfaces, $H=\dot{R}/R$ is the Hubble expansion, and an overdot represents differentiation with respect to coordinate time t. Units have been chosen so that $8\pi G=c=1$.

To analyze the system given by Eq. (2.2) we transform to expansion-normalized variables. Expansion-normalized variables have proven to be very useful in analysis of the asymptotic behavior of many cosmological models. See [7,25] for arguments in support of using dimensionless expansion-normalized variables. One primary reason is the decoupling of one of the differential equations, which effectively reduces the dimension of the system by one, and, in some cases, leads to the compactification of the phase space. We choose expansion-normalized variables of the form

$$\Omega=\frac{\rho}{3H^2},\quad \Phi_i=\frac{\sqrt{V_0}e^{k_i\phi_i/2}}{\sqrt{3}H},\quad \Psi_i=\frac{\dot{\phi}_i}{\sqrt{6}H},\quad \frac{dt}{d\tau}=\frac{1}{H}. \tag{2.3}$$

The resulting dynamical system describing these perfect fluid multiple scalar field models becomes

$$\frac{d\Omega}{d\tau}=\Omega(2q-3\gamma+2),$$

$$\frac{d\Psi_i}{d\tau}=\Psi_i(q-2)-\frac{\sqrt{6}}{2}k_i\Phi_i^2, \tag{2.4}$$

$$\frac{d\Phi_i}{d\tau}=\Phi_i\left(q+1+\frac{\sqrt{6}}{2}k_i\Psi_i\right),$$

for $(1\leqslant i\leqslant n)$, where the deceleration parameter has the following form:

$$q=\frac{(3\gamma-2)}{2}\Omega+2\sum_{i=1}^{n}\Psi_i^2-\sum_{i=1}^{n}\Phi_i^2,$$

and

$$\frac{{}^3R}{6H^2}=-1+\Omega+\sum_{i=1}^{n}\Psi_i^2+\sum_{i=1}^{n}\Phi_i^2.$$

Assuming a non-negative energy density (i.e., $\Omega\geqslant0$) and if ${}^3R\leqslant0$, (i.e., in the negative and zero-curvature cases) the phase space for the dynamical system in the expansion normalized variables (Ω,Φ_i,Ψ_i) is compact. If ${}^3R>0$ (i.e., in the positive curvature case) then the transformation given by Eq. (2.3) becomes singular when $H=0$. Here we shall only make some partial comments with regard to the asymptotic behavior of the positive curvature models. All of the equilibrium points correspond to self-similar cosmological models and hence to power-law solutions [8].

III. QUALITATIVE ANALYSIS OF TWO-SCALAR FIELD MODEL

We shall first discuss the dynamics of the model with only two minimally coupled scalar fields and with no matter. We obtain this model by setting $n=2$ and $\Omega=0$ in Eq. (2.4). In this case we obtain the four-dimensional dynamical system given by

$$\frac{d\Psi_1}{d\tau}=\Psi_1(q-2)-\frac{\sqrt{6}}{2}k_1{\Phi_1}^2$$

$$\frac{d\Psi_2}{d\tau}=\Psi_2(q-2)-\frac{\sqrt{6}}{2}k_2{\Phi_2}^2$$

$$\frac{d\Phi_1}{d\tau}=\Phi_1\left(q+1+\frac{\sqrt{6}}{2}k_1\Psi_1\right) \tag{3.1}$$

$$\frac{d\Phi_2}{d\tau}=\Phi_2\left(q+1+\frac{\sqrt{6}}{2}k_2\Psi_2\right)$$

where

$$q=2\Psi_1^2+2\Psi_2^2-\Phi_1^2-\Phi_2^2$$

and

$$\frac{{}^3R}{6H^2}=-1+\Psi_1^2+\Psi_2^2+\Phi_1^2+\Phi_2^2.$$

It is possible to choose simplified variables as in [26] via a rotation in field space; although this would simplify the analysis of the assisted inflationary solution, it would perhaps be more difficult to describe all of the qualitative properties of the models and relate this analysis to previous work.

A. Assisted inflation

The flat *Assisted Inflation* model [13] corresponds to the equilibrium point A of the system (3.1) given by

$$\{\Psi_1,\Psi_2,\Phi_1,\Phi_2\}^A \equiv \left\{-\frac{k_1 {k_2}^2}{\sqrt{6}({k_1}^2+{k_2}^2)},-\frac{{k_1}^2 k_2}{\sqrt{6}({k_1}^2+{k_2}^2)},k_2\frac{\sqrt{6({k_1}^2+{k_2}^2)-{k_1}^2{k_2}^2}}{\sqrt{6}({k_1}^2+{k_2}^2)},k_1\frac{\sqrt{6({k_1}^2+{k_2}^2)-{k_1}^2{k_2}^2}}{\sqrt{6}({k_1}^2+{k_2}^2)}\right\}, \tag{3.2}$$

which is equivalent to

$$\left\{-\frac{K^2}{\sqrt{6}k_1},-\frac{K^2}{\sqrt{6}k_2},\frac{\sqrt{K^2(6-K^2)}}{\sqrt{6}k_1},\frac{\sqrt{K^2(6-K^2)}}{\sqrt{6}k_2}\right\}$$

where

$$K^{-2}\equiv\frac{1}{{k_1}^2}+\frac{1}{{k_2}^2}.$$

The deceleration parameter for this solution is given by

$$q_A\equiv\frac{{k_1}^2{k_2}^2-2({k_1}^2+{k_2}^2)}{2({k_1}^2+{k_2}^2)}=\frac{K^2-2}{2} \tag{3.3}$$

and hence this solution, with

$$R(t)\propto t^p$$

and

$$k_1\phi_1=k_2\phi_2,$$

is inflationary ($q_A<0$) if

$$p\equiv 2\sum_{i=1}^{2}\frac{1}{{k_i}^2}=2K^{-2}=\frac{1}{1+q_A}>1;\quad 2\geqslant K^2. \tag{3.4}$$

Since a single scalar field can only give rise to an inflationary power-law solution if $1/{k_i}^2>\frac{1}{2}$ for $i=1$ or 2 [4,8], this means that the two-scalar field model can be inflationary even when each of the individual potentials is too steep for the corresponding single scalar field model to inflate (and hence the terminology *assisted* inflation). The eigenvalues corresponding to the equilibrium point A are given by (see Appendix for details)

$$K^2-2,\ \ \frac{K^2-6}{2},\ \ \frac{1}{4}((K^2-6)\pm\sqrt{(K^2-6)^2+8K^2(K^2-6)}). \tag{3.5}$$

Hence this equilibrium point is stable when Eq. (3.4) is satisfied, and so the corresponding assisted inflationary solution is a late-time attractor [14].

B. Stability of equilibria

We note that several of the equilibrium points occur in the three-dimensional invariant set corresponding to the zero-curvature models defined by

$$1=\Psi_1^2+\Psi_2^2+\Phi_1^2+\Phi_2^2.$$

When matter is included, there exists a monotonic function so that in the full dynamical phase space there can be no periodic or recurrent orbits and the global dynamics can be determined. This implies that the qualitative features described in this section can be more rigorously proven. All of the equilibrium points and their corresponding eigenvalues are listed in Table I. Using this table let us discuss the local stability of these equilibrium points.

As noted above the equilibrium point A, given by Eq. (3.2), corresponds to the assisted inflationary solution. It exists for all parameter values satisfying

$$\frac{1}{6}<\frac{1}{{k_1}^2}+\frac{1}{{k_2}^2}, \tag{3.6}$$

and is a sink (late-time attractor) for all parameter values satisfying Eq. (3.4) (else it is a saddle).

There are two equilibrium points, denoted by P_1 and P_2, whose coordinate values and associated eigenvalues are given in Table I, which correspond to zero-curvature power-law solutions in which one scalar field (either ϕ_1 or ϕ_2, respectively) is negligible; these solutions exist if $\frac{1}{6}<1/k_i^2$ and are inflationary if, in addition, $\frac{1}{2}<1/k_i^2$ (for each $i=1,2$, respectively) and correspond to the well-known single scalar field power-law solutions [3,4]. From Table I we see that each P_i has two negative eigenvalues and one positive eigenvalue for all relevant parameter values and an additional eigenvalue which is negative if $k_i^2<2$ (and positive for $2<k_i^2<6$); hence these points are saddles and have a one- or two-dimensional unstable manifold depending upon whether $k_i^2<2$ or $k_i^2>2$, respectively.

There also exist equilibrium points, denoted by CS_1, CS_2 and CS, whose coordinate values and the associated eigenvalues are given in Table I. The solutions correspond to power-law solutions in which the curvature scales with the first scalar field, the second scalar field or both, respectively. The single-field curvature scaling equilibrium points CS_1 and CS_2 are both saddles. The two-field curvature scaling equilibrium point CS is a sink whenever $\frac{1}{2}>1/k_1^2+1/k_2^2$ (otherwise a saddle). Whenever the two-field curvature scaling solution is stable, it necessarily has negative curvature.

There is an equilibrium point, denoted by M, corresponding to the Milne form of flat spacetime, which is always a saddle.

Finally, there is a one-dimensional set of equilibrium points parametrized by Ψ_0, denoted by MSF, corresponding to zero-curvature massless scalar field models (in which both potentials are zero). There is one zero eigenvalue corresponding to the fact that there is a one-dimensional set of equilibrium points. There are values for Ψ_0 for which the remaining three eigenvalues are positive and hence a subset

TABLE I. Equilibrium points in the two-scalar field model with no matter. In the table, A and CS correspond to the *two-field* assisted inflationary solution and the *two-field* curvature scaling solution, respectively.

Solution/Label	Coordinates $\{\Psi_1,\Psi_2,\Phi_1,\Phi_2\}$	Deceleration Parameter, q	Curvature ${}^3R/H^2$	Eigenvalues
Assisted Inflation, A	$\{\Psi_1,\Psi_2,\Phi_1,\Phi_2\}^A$ [see Eq. (3.2)]	q_A [see Eq. (3.3)]	0	[see Eq. (3.5)]
Power Law, P_1	$\left\{-\frac{k_1}{\sqrt{6}},0,\sqrt{1-\frac{k_1{}^2}{6}},0\right\}$	$\frac{k_1{}^2-2}{2}$	0	$k_1{}^2-2,\frac{k_1{}^2}{2},\frac{k_1{}^2-6}{2},\frac{k_1{}^2-6}{2}$
Power-Law, P_2	$\left\{0,-\frac{k_2}{\sqrt{6}},0,\sqrt{1-\frac{k_2{}^2}{6}}\right\}$	$\frac{k_2{}^2-2}{2}$	0	$k_2{}^2-2,\frac{k_2{}^2}{2},\frac{k_2{}^2-6}{2},\frac{k_2{}^2-6}{2}$
Curvature Scaling, CS	$\left\{-\frac{\sqrt{6}}{3k_1},-\frac{\sqrt{6}}{3k_2},\frac{2}{\sqrt{3}k_1},\frac{2}{\sqrt{3}k_2}\right\}$	0	$\frac{2(k_1{}^2+k_2{}^2)-k_1{}^2k_2{}^2}{k_1{}^2k_2{}^2}$	$-1\pm\sqrt{3}i$, $-1\pm\sqrt{1+4[2(k_1^{-2}+k_2^{-2})-1]}$
Curvature Scaling, CS_1	$\left\{-\frac{\sqrt{6}}{3k_1},0,\frac{2}{\sqrt{3}k_1},0\right\}$	0	$\frac{2-k_1{}^2}{k_1{}^2}$	$-2,1,-1\pm\sqrt{1+4k_1^{-2}(2-k_1{}^2)}$
Curvature Scaling, CS_2	$\left\{0,-\frac{\sqrt{6}}{3k_2},0,\frac{2}{\sqrt{3}k_2}\right\}$	0	$\frac{2-k_2{}^2}{k_2{}^2}$	$-2,1,-1\pm\sqrt{1+4k_2^{-2}(2-k_2{}^2)}$
Milne, M	$\{0,0,0,0\}$	0	-1	$-2,-2,1,1$
Massless Scalar Field, MSF	$\{\Psi_0,\epsilon\sqrt{1-\Psi_0{}^2},0,0\}$ where $0\leqslant\Psi_0{}^2\leqslant1$	2	0	$0,4,3+\frac{\sqrt{6}}{2}k_1\Psi_0,3+\frac{\sqrt{6}}{2}\epsilon k_2\sqrt{1-\Psi_0{}^2}$ where $\epsilon=\pm1$

of MSF are sources (the remainder are saddles). These correspond to well-known early-time attracting massless scalar field models [8].

C. Discussion

From the analysis above we conclude that the two-field assisted inflationary solution A is the global attractor when $\Sigma_{i=1}^{2}k_i^{-2}>\frac{1}{2}$ and the two-field curvature scaling solution CS is the global attractor when $\Sigma_{i=1}^{2}k_i^{-2}<\frac{1}{2}$. The massless scalar field solutions MSF are always the early-time attractors.

In all cases both scalar fields are non-negligible in generic late-time behavior. This is contrary to the commonly held belief that in multi-field models with exponential potentials the scalar field with the shallowest potential (i.e., smallest value of k) would dominate at late times. Indeed, we have shown that the single field power-law inflationary models always correspond to saddles, so that we have the rather surprising result that generically a single scalar field model *never* dominates at late times.

We note that both the assisted inflationary solution and the massless scalar field early-time attractors correspond to zero-curvature models. However, the curvature is not always dynamically negligible asymptotically because the two-field curvature scaling solution has non-zero curvature.

There is a range of parameter values for which the assisted inflationary solution is the global late-time attractor (when the solution is non-inflationary it corresponds to a saddle). For all of these parameter values the single field power-law solutions P_1 and P_2 are saddles. However, there are allowable parameter values for which either P_1 and P_2 are both inflationary, or one is inflationary while the other is not, or both are non-inflationary. This might give rise to some new interesting physical scenarios. For example, a model could asymptote towards an inflationary single field solution P_i, stay close to P_i for an arbitrarily long period of time (since P_i is an equilibrium point) inflating all the time, and then eventually leave P_i and evolve towards the stable attracting inflationary solution A. [Note that if either of P_1 or P_2 are inflationary, then A is necessarily inflationary—see Eq. (3.4)]. This is akin to a double-inflationary model [27] in which the density fluctuations on large and small scales decouple (i.e., the scale invariance of the spectrum is broken) thereby allowing the possibility of more power on large scales which is in better accord with observations.

IV. QUALITATIVE ANALYSIS OF TWO-SCALAR FIELD MODEL WITH MATTER

To understand the underlying dynamics of the model with matter (i.e., with $\Omega\neq0$) we shall shall study the model with two minimally coupled scalar fields together with matter having energy density ρ with the barotropic equation of state given by Eq. (2.1). This model is obtained by setting $n=2$ in Eq. (2.4), whence we obtain the five-dimensional dynamical system given by

$$\frac{d\Omega}{d\tau}=\Omega(2q-3\gamma+2)$$

$$\frac{d\Psi_1}{d\tau}=\Psi_1(q-2)-\frac{\sqrt{6}}{2}k_1\Phi_1{}^2$$

TABLE II. Equilibrium points with $\Omega=0$ in the two-scalar field model with matter. Each equilibrium point has $\Omega=0$ and the coordinates given in Table I. The additional fifth eigenvalue is displayed.

Label	A	P_1	P_2	CS	CS_1	CS_2	M	MSF
5^{th} Eigenvalue	$\frac{k_1{}^2k_2{}^2}{k_1{}^2+k_2{}^2}-3\gamma$	$k_1{}^2-3\gamma$	$k_2{}^2-3\gamma$	$2-3\gamma$	$2-3\gamma$	$2-3\gamma$	$2-3\gamma$	$3(2-\gamma)$

$$\frac{d\Psi_2}{d\tau}=\Psi_2(q-2)-\frac{\sqrt{6}}{2}k_2\Phi_2{}^2$$

$$\frac{d\Phi_1}{d\tau}=\Phi_1\left(q+1+\frac{\sqrt{6}}{2}k_1\Psi_1\right)$$

$$\frac{d\Phi_2}{d\tau}=\Phi_2\left(q+1+\frac{\sqrt{6}}{2}k_2\Psi_2\right) \tag{4.1}$$

where

$$q=\frac{3\gamma-2}{2}\Omega+2\Psi_1^2+2\Psi_2^2-\Phi_1^2-\Phi_2^2$$

and

$$\frac{{}^3R}{6H^2}=-1+\Omega+\Psi_1^2+\Psi_2^2+\Phi_1^2+\Phi_2^2.$$

A. Invariant sets and monotonic functions

The zero-curvature models constitute a four-dimensional invariant set. The models with no matter also constitute a four-dimensional invariant set.

The function

$$W=\frac{\Omega^2}{(\Omega+\Psi_1^2+\Psi_2^2+\Phi_1^2+\Phi_2^2-1)^2} \tag{4.2}$$

has derivative

$$\frac{dW}{d\tau}=2(2-3\gamma)W. \tag{4.3}$$

We observe that this function is monotonic when $\Omega\neq0$ (i.e., non-zero matter) and $(\Omega+\Psi_1^2+\Psi_2^2+\Phi_1^2+\Phi_2^2-1)\neq0$ (i.e., non-zero curvature). We also observe that the sign of $3\gamma-2$ signifigantly changes the dynamics of these models. For example, in the case of interest here $3\gamma-2>0$, whence W is a decreasing function of time τ. This immediately implies that

> There exist no periodic or recurrent orbits in the full five-dimensional phase space (this does not preclude the existence of closed orbits in the invariant sets $\Omega=0$ and ${}^3R=0$; however, we shall be primarily concerned with the dynamics of the models in the complete phase space with matter and non-zero curvature).
>
> The future asymptotic state lies within the invariant set $\Omega=0$. Matter becomes dynamically unimportant to the future.
>
> The past asymptotic state lies within the set of zero-curvature models.

B. Stability of equilibria

The equilibrium points can be classisfied into two sets; those with $\Omega=0$ and those with $\Omega\neq0$. All equilibrium points listed in Table I exist in this case with $\Omega=0$, and Table II lists the equilibrium points with $\Omega=0$ together with the additional eigenvalue due to the addition of matter. Using the function W above, we can further conclude that those equilibrium points in the set $\Omega\neq0$ necessarily must have zero curvature. Table III lists the eigenvalues found in the invariant set $\Omega\neq0$.

Let us focus on the stability of the attractors in the full physical phase space. All late-time attractors (sinks) occur in the invariant set $\Omega=0$. In the previous section we found that A and CS are the only sinks in the invariant set $\Omega=0$ (clearly, all of the saddles remain saddles in the full five-dimensional phase space). The additional eigenvalue for the equilibrium point A in the full physical phase space is given in Table II and is negative if $\Sigma_{i=1}^2 k_i^{-2}>1/3\gamma$. But this is always satisfied when $\Sigma_{i=1}^2 k_i^{-2}>\frac{1}{2}$ and $\gamma>\frac{2}{3}$, and hence A is a sink and assisted inflation is a global attractor. Similarly, from Table II the equilibrium point CS is always a sink for $\Sigma_{i=1}^2 k_i^{-2}<\frac{1}{2}$ and hence the two-field curvature scaling solution remains the global attractor in this case.

The early-time attractors lie in the zero-curvature invariant set and consist of massless scalar field models. From Table II we see that the massless scalar field models corresponding to the repelling equilibrium points MSF are always sources (for $\gamma<2$).

C. Matter scaling solutions

In the case of a single scalar field there exist zero-curvature FRW ‘‘matter scaling’’ solutions when the exponential potential is too steep to drive inflation, in which the scalar field energy density tracks that of the perfect fluid (so that at late times neither field is negligible) [4]. In [9] it was shown that whenever these matter scaling solutions exist they are the unique late-time attractors within the class of flat FRW models. The cosmological consequences of these scaling models have been further studied in [28]. For example, in these models the scalar field energy density tracks that of the perfect fluid and a significant fraction of the current energy

TABLE III. Equilibrium points with $\Omega\neq 0$ in the two-scalar field model with matter. Note that in each case ${}^3R=0$ and $q=(3\gamma-2)/2$.

Solution/Label	Coordinates $\{\Omega,\Psi_1,\Psi_2,\Phi_1,\Phi_2\}$	Eigenvalues
FRW, F	$\{1,0,0,0,0\}$	$3\gamma-2,\frac{3}{2}(\gamma-2),\frac{3}{2}(\gamma-2),\frac{3}{2}\gamma,\frac{3}{2}\gamma$
Matter Scaling, MS_1	$\left\{1-\frac{3\gamma}{k_1{}^2},-\frac{\sqrt{6}\gamma}{2k_1},0,\frac{\sqrt{6\gamma(2-\gamma)}}{2k_1},0\right\}$	$\frac{3}{2}(\gamma-2),\frac{3}{2}\gamma,3\gamma-2,$ $\frac{3}{4}((\gamma-2)\pm\sqrt{(\gamma-2)^2+8\gamma(\gamma-2)[1-3\gamma k_1{}^{-2}]})$
Matter Scaling, MS_2	$\left\{1-\frac{3\gamma}{k_2{}^2},0,-\frac{\sqrt{6}\gamma}{2k_2},0,\frac{\sqrt{6\gamma(2-\gamma)}}{2k_2}\right\}$	$\frac{3}{2}(\gamma-2),\frac{3}{2}\gamma,3\gamma-2,$ $\frac{3}{4}((\gamma-2)\pm\sqrt{(\gamma-2)^2+8\gamma(\gamma-2)[1-3\gamma k_2{}^{-2}]})$
Matter Scaling, MS	$\left\{1-3\gamma(k_1{}^{-2}+k_2{}^{-2}),-\frac{\sqrt{6}\gamma}{2k_1},-\frac{\sqrt{6}\gamma}{2k_2},\right.$ $\left.\frac{\sqrt{6\gamma(2-\gamma)}}{2k_1},\frac{\sqrt{6\gamma(2-\gamma)}}{2k_1}\right\}$	$3\gamma-2,\frac{3}{4}((\gamma-2)\pm\sqrt{(\gamma-2)^2+8\gamma(\gamma-2)})$ $\frac{3}{4}((\gamma-2)\pm\sqrt{(\gamma-2)^2+8\gamma(\gamma-2)[1-3\gamma(k_1{}^{-2}+k_2{}^{-2})]})$

density of the Universe may be contained in the homogeneous scalar field whose dynamical effects mimic cold dark matter; the tightest constraint on these cosmological models comes from primordial nucleosynthesis bounds on any such relic density [4,9,28]. The stability of these flat, isotropic matter scaling solutions was studied within the class of spatially homogeneous cosmological models with a barotropic perfect fluid and a scalar field with an exponential potential in [10]. It was found that while the matter scaling solutions are stable to shear perturbations, for realistic matter with $\gamma\geqslant 1$ they are unstable to curvature perturbations.

Returning to the models under investigation here, none of the equilibrium points with $\Omega\neq 0$ can be late-time attractors for $\gamma>\frac{2}{3}$. Indeed, from Table III all such equilibrium points are seen to be saddles. In particular, the two-field matter scaling solution corresponding to the equilibrium point MS, which exists for $\Sigma_{i=1}^{2}k_i^{-2}<1/3\gamma$, is a saddle. From Table III we see that the first eigenvalue associated with MS is positive, while the real parts of the remaining four eigenvlaues are all negative. This is consistent with the stability analysis of matter scaling solutions in models with a single scalar field which found that the models were unstable to curvature perturbations when $\gamma>\frac{2}{3}$ [10]. However, these two-field matter scaling solutions may still be of physical import. We note that when the curvature is zero, the two-field matter scaling solution is an attractor (all four eigenvalues of MS in the four-dimensional zero-curvature invariant set have negative real parts—so that MS is a sink in this invariant set), as in the case for the matter scaling solution in a single field model. Note also from Table III that both of the single-field matter scaling solutions, corresponding to the equilibrium points MS_1 and MS_2, have two positive eigenvalues, so that again the solution with multiple scalar fields is the ''stronger'' attractor.

V. QUALITATIVE ANALYSIS OF THREE-SCALAR FIELD MODEL

Let us now consider models with more than two scalar fields. For simplicity, we shall exclude a matter term here. However, from the previous section we can easily determine the essential properties resulting from the inclusion of a matter field. In particular, in this case a monotonic function exists and this enables us to prove the qualitative results outlined below. Let us begin with the three-scalar-field model, obtained by setting $n=3$ and $\Omega=0$ in Eq. (2.4). In this case the resulting six-dimensional dynamical system is given by

$$\begin{aligned}
\frac{d\Psi_1}{d\tau}&=\Psi_1(q-2)-\frac{\sqrt{6}}{2}k_1\Phi_1{}^2\\
\frac{d\Psi_2}{d\tau}&=\Psi_2(q-2)-\frac{\sqrt{6}}{2}k_2\Phi_2{}^2\\
\frac{d\Psi_3}{d\tau}&=\Psi_3(q-2)-\frac{\sqrt{6}}{2}k_2\Phi_3{}^2\\
\frac{d\Phi_1}{d\tau}&=\Phi_1\left(q+1+\frac{\sqrt{6}}{2}k_1\Psi_1\right)\\
\frac{d\Phi_2}{d\tau}&=\Phi_2\left(q+1+\frac{\sqrt{6}}{2}k_2\Psi_2\right)\\
\frac{d\Phi_3}{d\tau}&=\Phi_3\left(q+1+\frac{\sqrt{6}}{2}k_3\Psi_3\right)
\end{aligned}\tag{5.1}$$

where

$$q=2\Psi_1^2+2\Psi_2^2+2\Psi_3^2-\Phi_1^2-\Phi_2^2-\Phi_3^2$$

and

$$\frac{^3R}{6H^2}=-1+\Psi_1^2+\Psi_2^2+\Psi_3^2+\Phi_1^2+\Phi_2^2+\Phi_3^2.$$

Again it would possible to choose simplified variables as in [26] via a rotation in field space as was done in recent work [14,21]. However, we shall not do this here. Indeed, we shall not present a complete qualitative analysis similar to that done in Sec. III, since the essential features are similar and the detailed analysis would be long and painful. Rather, let us describe the main effects of including a third scalar field on the inflationary solutions.

There exists a zero-curvature assisted inflationary solution which now corresponds to the equilibrium point given by

$$\left\{\Psi_i=-\frac{K^2}{\sqrt{6}k_i},\Phi_i=\frac{\sqrt{K^2(6-K^2)}}{\sqrt{6}k_i}\right\}$$

where

$$K^{-2}\equiv k_1^{-2}+k_2^{-2}+k_3^{-2}.$$

In this solution all of the three scalar fields scale together at late times. The corresponding eigenvalues are

$$K^2-2,\frac{K^2-6}{2},\frac{1}{4}((K^2-6)\pm\sqrt{(K^2-6)^2+8K^2(K^2-6)}),$$

$$\frac{1}{4}((K^2-6)\pm\sqrt{(K^2-6)^2+8K^2(K^2-6)}).$$

It is known [13] to be a stable late-time attractor for all parameter values for which the solution is inflationary (i.e., $K^2<2$; recall the point does not exist if $K^2>6$).

There are three solutions in which two scalar fields scale together asymptotically and the third is negligible. Assuming that the third scalar field is zero ($\Psi_3=\Phi_3=0$), the coordinates of the corresponding equilibrium point, denoted by P_{120}, are given by

$$\left\{-\frac{k_1k_2{}^2}{\sqrt{6}(k_1{}^2+k_2{}^2)},-\frac{k_1{}^2k_2}{\sqrt{6}(k_1{}^2+k_2{}^2)},0,k_2\frac{\sqrt{6(k_1{}^2+k_2{}^2)-k_1{}^2k_2{}^2}}{\sqrt{6}(k_1{}^2+k_2{}^2)},k_1\frac{\sqrt{6(k_1{}^2+k_2{}^2)-k_1{}^2k_2{}^2}}{\sqrt{6}(k_1{}^2+k_2{}^2)},0\right\}. \tag{5.2}$$

Four of the eigenvalues are given by Eq. (3.5), which all have negative real parts.

There are three solutions in which one scalar field scale dominates asymptotically and the remaining two are negligible. Assuming that the first scalar field is non-zero ($\Psi_1\neq0\neq\Phi_1$), the coordinates of the corresponding equilibrium point, denoted by P_{100}, are given by

$$\left\{-\frac{k_1}{\sqrt{6}},0,0,\sqrt{1-\frac{k_1{}^2}{6}},0,0\right\}.$$

Two of the eigenvalues are negative, one is positive, and there is an additional eigenvalue which is negative if $k_1^2<2$ and positive if $2<k_1^2<6$.

In both of these cases the additional (remaining) two eigenvlaues can be calculated and are given by

$$\{q-2<0,q+1>0\},$$

where q is the deceleration parameter evaluated at the equilibrium point. Hence, the point P_{120} is a saddle with one eigenvalue with positive real part. The equilibrium points denoted by P_{103} and P_{023} are also saddles with one eigenvalue with positive real part. In addition, the point P_{100} is a saddle with two eigenvalues with positive real parts (if $k_i^2<2$, and three eigenvalues with positive real parts if $k_i^2>2$). The same is true for the equilibrium points denoted by P_{020} and P_{003}.

Consequently there is a ''nested'' set of equilibrium points. At the top is the stable three-scalar field assisted inflationary solution. In the next layer there are three two-scalar field models which are saddles with one eigenvalue with positive real part. In the final layer there are three one-scalar field models which are saddles with two eigenvalues with positive real parts (or three eigenvalues with positive real parts). Associated with this dynamical nesting are cosmological models with very interesting physical properties.

This will follow through in the case of n scalar fields. There will be a unique stable n-scalar field assisted inflationary solution. There will then be n of the $(n-1)$-scalar field models which are saddles with one eigenvalue with positive real part. There will be $\frac{1}{2}n(n-1)$ of the $(n-2)$-scalar field models which are saddles with two eigenvalues with positive real parts. And so on. Finally, there will be n of the (1)-scalar field models which are saddles with $n-1$ (or $n-2$) eigenvalues with positive real parts. As one ''goes up'' the nested structure the equilibrium points respectively become ''stronger attractors'' (i.e., the stable manifold of the equilibrium points increases in dimension).

There is also a three-field curvature scaling solution corresponding to the equilibrium point given by

$$\left\{\Psi_i=-\frac{2}{\sqrt{6}k_i},\Phi_i=\frac{2}{\sqrt{3}k_i}\right\}$$

whose associated eigenvalues are given by

$$-1\pm\sqrt{1+4K^{-2}(2-K^2)},-1\pm\sqrt{3}i,-1\pm\sqrt{3}i.$$

This equilibrium point is a sink whenever $K^2>2$, in which case it represents an FRW model with negative curvature $(2-K^2)/K^2$ (else it is a saddle and represents a positive curvature model).

Finally, there are saddle equilibrium points corresponding to the Milne model and the one- and two-field curvature scaling solutions, and a set of equilibrium points with $\{\Sigma_{i=1}^{3}\Psi_i{}^2=1,\Phi_i=0\}$ corresponding to massless scalar field models, a subset of which are sources.

A complete qualitative analysis can be done for n-scalar field models. All of these results can be proven by induction (see, for example, [14]). The n-scalar field assisted inflationary solution is given by [13]

$$R(t)\propto t^p$$

and

$$k_i\phi_i=k_j\phi_j;\forall 1\leq i\neq j\leq n,$$

and

$$p\equiv 2\sum_{i=1}^{n}\frac{1}{k_i{}^2}>1.$$

We note that in the two-scalar field model, although inflation can occur for potentials that are steeper than in the single-field case, it cannot occur for arbitrarily steep potentials. For example, if $k_1=k_2\equiv k$, then inflation occurs if $k^2<4$. However, for n-fields, if $k_i=k$ for all i, then inflation occurs if $k^2<2n$; e.g., $k^2<8$ for four scalar field models.

VI. CONCLUSIONS

We have studied multi-scalar-field FRW cosmological models with exponential potentials, extending previous analysis by including non-zero curvature and barotropic matter. We have used dynamical systems techniques, and by establishing a monotonic function in the complete dynamical phase space (which includes both matter and curvature), we have been able to deduce global results.

In Sec. III a comprehensive qualitative analysis was presented in the case of two scalar fields with no matter. We concluded that the two-field assisted inflationary solution A is the global attractor when $\Sigma_{i=1}^{2}k_i^{-2}>\frac{1}{2}$ and the two-field curvature scaling solution CS is the global attractor when $\Sigma_{i=1}^{2}k_i^{-2}<\frac{1}{2}$. A subset of the massless scalar field solutions MSF are always the early-time attractors. Consequently, we found that in all cases both scalar fields are non-negligible in generic late-time behavior; this is an interesting and unexpected result and is contrary to the commonly held belief that in multi-field models with exponential potentials the scalar field with the shallowest potential would dominate at late times (indeed, we have shown that the single field power-law inflationary models always correspond to saddles). We note that both the assisted inflationary solution and the massless scalar field early-time attractors correspond to zero-curvature models. However, the curvature is not always dynamically negligible asymptotically because the two-field curvature scaling solution has non-zero curvature.

The zero-curvature assisted inflationary FRW scaling solutions [13] are of particular importance since, through the combined effect of multiple uncoupled scalar fields each having an exponential potential, power-law inflation is possible even when each individual scalar field need not be a source for inflation. We have discussed the stability of the two-field assisted inflationary model, and generalized previous results by including non-zero curvature to show that for an appropriate range of parameter values the assisted inflationary solution is the global late-time attractor. For these parameter values the single field power-law solutions P_1 and P_2 were shown to be saddles, and we showed that there are allowable parameter values for which either P_1 and P_2 are both inflationary, or one is inflationary while the other is not, or both are non-inflationary, perhaps leading to new interesting physical scenarios.

In Sec. IV we studied the two-scalar field model with barotropic matter. A monotonic function was established in the resulting phase space. This proved that the matter must be negligible at late times and we found that A and CS are the only global sinks and that consequently assisted inflation and the two-field curvature scaling solution are the global late-time attractors in their appropriate respective parameter ranges. This confirmed the result that both scalar fields must be dynamically non-negligible in generic late-time behavior, and establishes the stability of the two-field assisted inflationary model when matter is included. The monotonic function also shows that the early-time attractors lie in the zero-curvature invariant set, and we showed that they consist of a subset of the massless scalar field models.

For $\gamma>\frac{2}{3}$, all of the equilibrium points with $\Omega\neq 0$ were shown to be saddles (see Table III). The two-field matter scaling solution corresponding to the equilibrium point MS was shown to have a single positive eigenvalue. Both of the single-field matter scaling solutions, corresponding to the equilibrium points MS_1 and MS_2, were shown to have two positive eigenvalues, so that again the solution with multiple scalar fields is the "stronger" attractor. We note that when the curvature is zero, the two-field matter scaling solution is the late-time attractor, consistent with the stability analysis in [10]. These matter scaling solutions, and particularly the two-field matter scaling solutions, give rise to new transient dynamical behavior and may be of physical import. For example, there are solutions which spend a period of time with the scalar field mimicking the barotropic fluid in which there is a non-negligible scalar field (dark matter) energy density (corresponding to a matter scaling saddle equilibrium point) and subsequently evolve towards a scalar-field dominated power-law inflationary epoch (corresponding to a single-field saddle equilibrium point or a two-field assisted inflationary attractor) with an accelerated expansion, perhaps explaining current high redshift data.

In Sec. V we discussed three- and multi-scalar field models (where, for simplicity, a matter term was excluded). In the three-scalar field model we again established the assisted inflationary solution and three-field curvature scaling solution as the stable late-time attractors. We then considered n-scalar field models, and established a nested structure for the m-field scaling (assisted inflationary) solutions. The n-scalar field assisted inflationary solution is again the late-time attractor. All of the m-field (with $m<n$) scaling solutions are saddles and in general the equilibrium points corresponding to the m-field scaling solutions will have $n-m$ eigenvalues with positive real parts so that the equilibrium points corresponding to the greater number of non-negligible scalar fields are, respectively, the "stronger attractors." Again we should emphasize that Malik and Wands [14] showed that the multi-scalar field assisted inflationary solution is a late time attractor by utilizing a rotation in field space; indeed, the stable modes in a general stability analysis of this solution are presumably related to the isocurvature perturbations orthogonal to the attractor trajectory in field space obtained in their analysis.

Finally, from previous investigations [8] of spatially homogeneous scalar field cosmological models with an exponential potential and barotropic matter and from the above analysis, we can conclude that the assisted inflationary solution is a global attractor for all ever-expanding spatially homogeneous multi-field cosmological models with exponential potentials provided $\Sigma_{i=1}^{n} k_i^{-2} > 1/2$. We can also conclude that the multi-field curvature scaling solution is a global attractor for models of Bianchi types V and VII_h provided $\Sigma_{i=1}^{n} k_i^{-2} < 1/2$ [29] and a multi-field generalization of the Feinstein-Ibanez anisotropic single-field solution [30] is the global attractor for models of Bianchi types III and VI_h if $\Sigma_{i=1}^{n} k_i^{-2} < 1/2$. Indeed, there will be n-field generalizations corresponding to all equilibrium points of the single-field Bianchi type B models (cf. [8]).

In closing, we note that spatially flat FRW matter scaling solutions also exist in the context of generalized assisted inflation. In [21] it was shown that in the higher-dimensional context, in the six-dimensional model the assisted dynamics between the scalar fields mimics the behavior of a relativistic fluid ($\gamma = 4/3$), while for higher dimensions the scalar fields dominate the radiation component, perhaps leading to a "moduli" problem for the early universe.

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APPENDIX: DETERMINATION OF THE EIGENVALUES FOR THE EQUILIBRIUM POINT A

If we let $\mathbf{X} = (\Psi_1, \Psi_2, \Phi_1, \Phi_2)$, then we are able to write the dynamical system (3.1) as $d\mathbf{X}/d\tau = \mathbf{F}(\mathbf{X})$ where $\mathbf{F}$ is an analytic function from $\mathbb{R}^4 \to \mathbb{R}^4$. Standard results from dynamical systems theory state that the local behavior near an equilibrium point, $\mathbf{X}_0$, of a system of non-linear autonomous differential equations of the form $d\mathbf{X}/d\tau = \mathbf{F}(\mathbf{X})$ is determined by that of the corresponding linearized system $d\mathbf{X}/d\tau = D\mathbf{F}(\mathbf{X}_0)(\mathbf{X} - \mathbf{X}_0)$ in a neighborhood of the equilibrium point, provided the eigenvalues of the derivative matrix $D\mathbf{F}(\mathbf{X}_0)$ have non-zero real parts [31]. For the system given by Eq. (3.1) the derivative matrix has the form

$$D\mathbf{F}(\mathbf{X}) = \begin{bmatrix} q-2+4\Psi_1^2 & 4\Psi_1\Psi_2 & -2\Psi_1\Phi_1 - \sqrt{6}k_1\Phi_1 & -2\Psi_1\Phi_2 \\ 4\Psi_1\Psi_2 & q-2+4\Psi_2^2 & -2\Psi_2\Phi_1 & -2\Psi_2\Phi_2 - \sqrt{6}k_2\Phi_2 \\ \Phi_1\left(4\Psi_1 + \frac{\sqrt{6}}{2}k_1\right) & 4\Psi_2\Phi_1 & q-2\Phi_1^2+1+\frac{\sqrt{6}}{2}k_1\Psi_1 & -2\Phi_1\Phi_2 \\ 4\Psi_1\Phi_2 & \Phi_2\left(4\Psi_2 + \frac{\sqrt{6}}{2}k_2\right) & -2\Phi_1\Phi_2 & q-2\Phi_2^2+1+\frac{\sqrt{6}}{2}k_2\Psi_2 \end{bmatrix}.$$

As an example, let us evaluate $D\mathbf{F}(\mathbf{X})$ at the assisted inflationary equilibrium point A given by Eq. (3.2):

$$
D\mathbf{F}(A)=\begin{bmatrix}
\frac{1}{2}K^2+\frac{2}{3}\frac{K^4}{k_1^2}-3 & \frac{2}{3}\frac{K^4}{k_1k_2} & \frac{1}{3}\frac{\overline{K}(K^2-3k_1^2)}{k_1^2} & \frac{1}{3}\frac{K^2\overline{K}}{k_1k_2} \\
\frac{2}{3}\frac{K^4}{k_1k_2} & \frac{1}{2}K^2+\frac{2}{3}\frac{K^4}{k_2^2}-3 & \frac{1}{3}\frac{K^2\overline{K}}{k_1k_2} & \frac{1}{3}\frac{\overline{K}(K^2-3k_2^2)}{k_2^2} \\
\frac{1}{6}\frac{\overline{K}(-4K^2+3k_1^2)}{k_1^2} & -\frac{2}{3}\frac{K^2\overline{K}}{k_1k_2} & -\frac{1}{3}\frac{K^2(6-K^2)}{k_1^2} & -\frac{1}{3}\frac{K^2(6-K^2)}{k_1k_2} \\
-\frac{2}{3}\frac{K^2\overline{K}}{k_1k_2} & \frac{1}{6}\frac{\overline{K}(-4K^2+3k_2^2)}{k_2^2} & -\frac{1}{3}\frac{K^2(6-K^2)}{k_1k_2} & -\frac{1}{3}\frac{K^2(6-K^2)}{k_2^2}
\end{bmatrix}
$$

where $\overline{K}\equiv\sqrt{K^2(6-K^2)}$.

The characteristic polynomial of the matrix $D\mathbf{F}(A)$ can be simplified to the form

$$
c_A(\lambda)=\lambda^4+(8-2K^2)\lambda^3+\left(\frac{3}{4}K^4+21-8K^2\right)\lambda^2+\left(\frac{1}{2}K^6-\frac{7}{2}K^4+18\right)\lambda-\frac{1}{4}K^8+18K^2-15K^4+\frac{7}{2}K^6.
$$

The roots of $c_A(\lambda)$ yield the eigenvalues displayed in Eq. (3.5). The eigenvalues associated with all of the other equilibrium points can be computed in a similar fashion.

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