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Homogeneous scalar field cosmologies with an exponential potential

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We shall study spatially homogeneous cosmological models containing a self-interacting scalar field with an exponential potential of the form $V(\phi)=\Lambda e^{k\phi}$. The asymptotic properties of these models are discussed. In particular, their possible isotropization and inflation are investigated for all values of the parameter k. A particular class of models is analyzed qualitatively using the theory of dynamical systems, illustrating the general asymptotic behavior. © *1997 American Institute of Physics.* [S0022-2488(97)01510-7]

I. INTRODUCTION

Scalar field cosmology is of importance in the study of inflation, an idea popularized by Guth,[1] during which the universe undergoes a period of accelerated expansion (see, for example, Olive[2]). One particular class of inflationary cosmological models are those with a scalar field and an exponential potential of the form $V(\phi)=\Lambda e^{k\phi}$, where Λ and k are non-negative constants. Models with an exponential scalar field potential arise naturally in alternative theories of gravity, such as, for example, theories based on the Brans–Dicke theory (for example, extended inflation,[3,4] and hyper-extended inflation[5]), in the Salam–Sezgin model of $N=2$ super-gravity coupled to matter,[6] and in theories undergoing dimensional reduction to an effective four-dimensional theory.[7] In addition, other theories of gravity, such as, for example, quadratic Lagrangian theories, are known to be conformally equivalent to general relativity plus a scalar field having exponential-like potentials.[8,9] Cosmologies of this type have been studied by a number of authors, including Halliwell,[7] Burd and Barrow,[10] Kitada and Maeda[11,12] and Feinstein and Ibáñez.[13]

Our aim here is to analyze Bianchi cosmologies containing a scalar field with an exponential potential. Since the potential is an exponential function the governing differential equations exhibit a symmetry,[14] and when appropriate expansion-normalized variables are defined, the governing equations reduce to a dynamical system with the following desirable properties:

(1) The resulting dynamical system is polynomial.
(2) The phase space is compact (except in the cases of Bianchi types VII_0, VIII and IX, in which the phase space is closed but unbounded).[15]
(3) The differential equation for the expansion decouples from the other equations, thereby allowing a reduced system of ordinary differential equations to be analyzed by standard geometric (dynamical systems) techniques.[15–18]
(4) In addition, all equilibrium points of the reduced system correspond to self-similar cosmological models.[19]

In particular, we wish to qualitatively study whether the spatially homogeneous models inflate

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and/or isotropize, thereby determining the applicability of the so-called cosmic no-hair conjecture in homogeneous scalar field cosmologies with an exponential potential. This latter aim is of relevance, in part, due to the fact that inflation in such models is of power-law type,[10] which is weaker than in conventional exponential inflation for which no-hair theorems exist.[20] Essentially the cosmic no-hair conjecture asserts that inflation is typical in a wide class of scalar field cosmologies. Another motivation for this work is to determine the relevance of the exact solutions (of Bianchi types III and VI) found by Feinstein and Ibáñez,[13] which neither inflate nor isotropize, and to investigate whether their qualitative properties are typical.

As noted earlier a number of authors have studied such cosmological models. Homogeneous and isotropic FRW (Friedmann–Robertson–Walker) models were studied by Halliwell[7] using phase-plane methods (see also, for example, Olive[2]). Homogeneous but anisotropic models of Bianchi types I and III (and Kantowski–Sachs models) have been studied by Burd and Barrow[10] in which they found exact solutions and discussed their stability. Lidsey[21] and Aguirregabiria *et al.*[22] found exact solutions for Bianchi type I models and Aguirregabiria *et al.*[22] also completed a qualitative analysis of these models. Bianchi models of types III and VI were studied by Feinstein and Ibáñez,[13] in which exact solutions were found. A qualitative analysis of all Bianchi models with $k^2<2$, including standard matter satisfying various energy conditions, was completed by Kitada and Maeda.[11,12] They found that the power-law inflationary solution is indeed an attractor for all initially expanding Bianchi models (except for a subclass of the Bianchi type IX models which will recollapse).

This paper is organized as follows. In Sec. II, we shall discuss general qualitative features of homogeneous scalar field cosmologies with an exponential potential, such as, for example, whether they isotropize or inflate, and we shall determine the relevance of the Feinstein–Ibáñez solutions.[13] In addition, we will show that all equilibrium points of the ''reduced'' dynamical system correspond to self-similar cosmological models. In Sec. III, we will perform a detailed qualitative analysis of a particular class of Bianchi models, which includes models of Bianchi types I, III, V and VI, and in so doing we will illustrate the general asymptotic properties of spatially homogeneous models discussed in Sec. II. We shall make some concluding remarks in Sec. IV.

II. ISOTROPIZATION AND THE COSMIC NO-HAIR THEOREM

A. Background

It was proven by Wald[20] that all initially expanding spatially homogeneous models with a positive cosmological constant (and ordinary matter satisfying both the strong and dominant energy conditions) asymptotically approach the isotropic de Sitter solution (except for a subclass of the Bianchi type IX models which recollapse). Following Wald's[20] result, a number of extended ''cosmic no-hair theorems'' have been proven for Bianchi models. In particular, and essentially using Wald's approach, Kitada and Maeda[11,12] have proven that for $k^2<2$, all initially expanding spatially homogeneous models containing a scalar field with an exponential potential (and ordinary matter satisfying the energy conditions) locally approach an isotropic, power-law inflationary solution (in the Bianchi type IX case the models must also satisfy the condition that the ratio of the effective vacuum energy to the maximum three curvature is larger than some critical value). In the special case $k=0$, the theorem essentially reduces to Wald's result,[20] and the unique attractor is the (exponential inflationary) de Sitter solution.

In related work, Heusler[23] proved that all Bianchi models with ordinary matter satisfying the usual energy conditions and containing a scalar field with a positive, convex potential [with a local minimum such that $V(\phi_0)=0$; for example, $V(\phi)=\frac{1}{2}m\phi^2$], can only approach isotropy at infinite times if the underlying Lie group is admitted by a FRW model. This work partially extends (by including scalar fields) the famous result of Collins and Hawking[24] that only a subclass of measure zero in the space of all homogeneous models can asymptotically approach isotropy. Here we shall extend Heusler's result to the case of a scalar field with an exponential potential with $k^2>2$ (see also Ref. 25). In this case the scalar field ϕ is generally not bounded and $\phi V'(\phi)\geqslant V(\phi)$ is only satisfied when ϕ is positive; therefore the conditions in Heusler's main theorem are not met. However, Heusler's Proposition 1 (where now $\theta\rightarrow 0$ and $V\rightarrow 0$ as $t\rightarrow\infty$ if there exists a time t_0 with $\theta(t_0)\geqslant 0$) and Proposition 2 (which gives necessary conditions in order for a homogeneous model which is not among the Bianchi types admitted by a FRW model to isotropize), are both true in the case of an exponential potential. Consequently in our calculation below we effectively replace Heusler's Proposition 3 with an analogous result on the behavior of V/E in the case of an exponential potential.

B. Equations

Cosmological models with a minimally coupled scalar field have a stress-energy tensor given by

$$T_{ab}=\phi_{;a}\phi_{;b}-g_{ab}\left(\tfrac{1}{2}\phi_{;c}\phi^{;c}+V(\phi)\right), \tag{2.1}$$

where for a homogeneous scalar field $\phi=\phi(t)$, so that $\phi_{;c}\phi^{;c}=-\dot{\phi}^2$ (where an over-dot denotes differentiation with respect to the proper time). In this case we can formally treat the stress-energy tensor as a perfect fluid with velocity vector $u^a=\phi^{;a}/\sqrt{-\phi_{;b}\phi^{;b}}$, where the energy density and the pressure are given by

$$\rho_\phi\equiv E=\tfrac{1}{2}\dot{\phi}^2+V(\phi), \tag{2.2a}$$

$$p_\phi=\tfrac{1}{2}\dot{\phi}^2-V(\phi). \tag{2.2b}$$

In the models under consideration, the potential of the scalar field is given by

$$V(\phi)=\Lambda e^{k\phi}, \tag{2.3}$$

where Λ (>0) and k are constants.

From the Einstein field equations we have the Raychaudhuri equation governing the evolution of the expansion

$$\dot{\theta}=-2\sigma^2-\tfrac{1}{3}\theta^2-\dot{\phi}^2+V(\phi), \tag{2.4}$$

and the generalized Friedmann equation

$$\theta^2=3\sigma^2+\tfrac{3}{2}\dot{\phi}^2+3V(\phi)-\tfrac{3}{2}P, \tag{2.5}$$

where σ is the shear scalar, P is the scalar curvature of the homogeneous hypersurfaces, which is always negative except in the Bianchi IX case,[20] and $V(\phi)$ is given by Eq. (2.3). The Klein–Gordon equation for the scalar field with an exponential potential is then

$$\ddot{\phi}+\theta\dot{\phi}+kV(\phi)=0. \tag{2.6}$$

Defining ψ by

$$\psi=\dot{\phi}+\frac{k}{3}\theta, \tag{2.7}$$

and using Eqs. (2.4) and (2.5), the Klein–Gordon equation can be written as

$$\dot{\psi}+\theta\psi+\frac{k}{3}P=0. \tag{2.8}$$

We now introduce new expansion-normalized variables and a new time variable as follows:

$$\beta=\sqrt{3}\ \frac{\sigma}{\theta},\quad \frac{dt}{d\Omega}=\frac{3}{\theta},\quad \Psi=\frac{\sqrt{6}}{2}\ \frac{\dot{\phi}}{\theta},\quad \Phi=\sqrt{3\Lambda}\ \frac{e^{k\phi/2}}{\theta}. \tag{2.9}$$

With these definitions, Eqs. (2.4)–(2.6) can be rewritten as

$$\Psi'=-\Psi(2-2\beta^2-2\Psi^2+\Phi^2)-\frac{\sqrt{6}k}{2}\Phi^2, \tag{2.10a}$$

$$\Phi'=-\Phi\left(-1-2\beta^2-2\Psi^2+\Phi^2-\frac{\sqrt{6}k}{2}\Psi\right), \tag{2.10b}$$

where a prime denotes differentiation with respect to the new time Ω. The equilibrium points of the system have either $\Phi=\Psi=0$, which corresponds to the massless scalar field case, or $\beta^2+\Psi^2=1,\Phi=0$, which represents the Kasner-like initial (line) singularity, or else (and in all cases of interest here) obey the following relation:

$$\Phi^2+\Psi^2=-\frac{\sqrt{6}}{k}\Psi. \tag{2.11}$$

In terms of these new expansion-normalized variables the energy density of the scalar field (2.2a) can be written as

$$\frac{E}{\theta^2}=\frac{1}{3}(\Psi^2+\Phi^2), \tag{2.12}$$

and we have that

$$\Psi=-\frac{k}{\sqrt{6}}+\frac{\sqrt{3}}{\sqrt{2}}\ \frac{\psi}{\theta}. \tag{2.13}$$

Hence, at the equilibrium points we obtain

$$\frac{E}{\theta^2}=-\frac{\sqrt{6}}{3k}\ \Psi=\frac{1}{3}\left(1-\frac{3}{k}\ \frac{\psi}{\theta}\right), \tag{2.14a}$$

$$\frac{V}{E}=\frac{\Phi^2}{\Psi^2+\Phi^2}=1-\frac{k^2}{6}+\frac{k}{2}\ \frac{\psi}{\theta}. \tag{2.14b}$$

C. Isotropization

Following Heusler,[23] the necessary conditions for an anisotropic and homogeneous solution to isotropize are:

$$\beta=0, \tag{2.15}$$

and (Heusler's Proposition 2[23])

$$\frac{E}{\theta^2} \to \frac{1}{3}, \tag{2.16a}$$

$$\left\langle \frac{V}{E} \right\rangle \geq \frac{2}{3}, \tag{2.16b}$$

where $\langle\ \rangle$ denotes an appropriate time average [Heusler,[23] Eq. (20)].

Now, using Eq. (2.14a), Eq. (2.16a) implies that

$$\frac{\psi}{\theta} \to 0. \tag{2.17}$$

Using Eq. (2.17) we can now compute $\langle V/E \rangle$ viz.,

$$\left\langle \frac{V}{E} \right\rangle = 1 - \frac{k^2}{6} \tag{2.18}$$

(this replaces Heusler's Proposition 3[23]). Hence Eq. (2.16b) implies that

$$1 - \frac{k^2}{6} \geq \frac{2}{3} \Rightarrow k^2 \leq 2. \tag{2.19}$$

Therefore, we have shown that if the model is not of Bianchi types I, V, VII, or IX (i.e., is not one which is admitted by the FRW model), then $k^2 \leq 2$ is a necessary condition for these models to isotropize. Like Heusler,[23] we have not completely generalized the Collins and Hawking[24] result that only a subclass of homogeneous models of measure zero can isotropize since we have not explicitly investigated Bianchi models of types VII_h and IX.

The following questions consequently arise concerning the future asymptotic behavior of the models when $k^2 > 2$:

(1) For those models that may isotropize (namely Bianchi types I, V, VII, and IX), do they indeed isotropize?
(2) For those models which cannot isotropize, what is the role of the Feinstein–Ibáñez solutions[13] (since for $k^2 > 2$ these solutions are neither isotropic nor inflationary)?

The first question is answered in Sec. II E. The second question is addressed in Sec. III.

D. Inflation

For inflation to occur we must have that

$$2\beta^2 + 2\Psi^2 - \Phi^2 < 0, \tag{2.20}$$

so that, using Eqs. (2.11), (2.13) and (2.20), at the equilibrium points the solution will inflate if

$$(k^2 - 2) - 3k\frac{\psi}{\theta} < 0. \tag{2.21}$$

Therefore, from Eqs. (2.15) and (2.17), for models to inflate and isotropize k^2 must be less than two, a well known result.[7,11,12]

We have shown that $k^2 \leq 2$ is a necessary condition for the homogeneous models under consideration to isotropize, and for $k^2 < 2$ these models will inflate. However, we have not proven that all such models with $k^2 \leq 2$ do isotropize (although we shall explicitly demonstrate that this is

TABLE I. The isotropic equilibrium points of the Bianchi type VII_h models and their stability.[a]

Equilibrium point (β,Ψ,Φ) [a]	Values of k^2	Description	Stability [b]	Corresponding Eq. in Ref. 26[a]
$(0,0,0)$	$0<k^2$	Milne	Unstable	(2.16)
$(0,\pm 1,0)$	$0<k^2$	Flat FRW	Unstable	(2.18),(2.20)
$\left(0,-\frac{\sqrt{6}}{6}k,\sqrt{1-\frac{k^2}{6}}\right)$	$0<k^2<2$	Power-law inflation	Stable	(2.22)
	$2<k^2<6$	Flat FRW	Unstable	
	$6<k^2$		DNE	
$\left(0,-\frac{\sqrt{6}}{3k},\frac{2\sqrt{3}}{3k}\right)$	$2<k^2$	Open FRW	Stable	(2.24)
	$0<k^2<2$		DNE	

[a]The information given here utilizes the variables defined in Sec. II. Note that different variables were used in Ref. 26.
[b]DNE means that the equilibrium point does not exist in this case.

the case for a subclass of Bianchi models in Sec. III). The no-hair theorem of Kitada and Maeda,[11,12] described in Sec. II A, does show that for $k^2<2$ the isotropic, power-law inflationary FRW solution is the unique attractor for any initially expanding Bianchi model. In addition, these authors also showed[12] that in these models anisotropies always enhance inflation in models with non-positive spatial curvature (over their isotropic counterparts) and generally enhance inflation in models of Bianchi type IX (however; see the detailed discussion in Kitada and Maeda,[12] pp. 720–721).

E. The Bianchi VII_h case

To determine if there exist any spatially homogeneous spacetimes which isotropize when $k^2>2$, we need to consider Bianchi models of type I, V, VII and IX. (See Sec. III for details of the Bianchi type I and V models.) In the case of the Bianchi type IX models, Kitada and Maeda[12] showed that for the case $k^2<2$ any initially expanding model will isotropize toward the power-law solution provided that the ratio of the effective cosmological constant to the maximum three-curvature is larger than some critical value (and that the time derivative of this ratio be positive). However, their analysis is incomplete. For $k^2>2$, it is apparent that there exists an open set of Bianchi IX initial data such that these models isotropize and an open set of initial data such that these models recollapse. Henceforth, since the Bianchi types VII_0, V and I are special classes of Bianchi models, we shall concentrate on whether the Bianchi VII_h models isotropize.

If the Bianchi VII_h models are to isotropize then they must approach a FRW model as they evolve to the future. If we consider the system of ordinary differential equations describing the evolution of the Bianchi type VII_h models as a dynamical system, then we are able to determine whether the models isotropize by examining the stability of the isotropic equilibrium points. This has been done in a companion paper by the authors[26] in which the particular details of the analysis of the Bianchi type VII_h models can be found. The results are summarized in Table I.

We observe that for $k^2<2$ the zero-curvature, power-law inflationary FRW model is an attractor for the Bianchi VII_h models. On the other hand, if $k^2>2$, then we find that the attractor is a negatively curved FRW model. Since the Bianchi VII_h model represents a general class of spatially homogeneous models, we can now assert that (with respect to scalar field cosmological models with an exponential potential) there exists a set of initial data (Bianchi VII_h initial data in particular) of non-zero measure in the space of all spatially homogeneous initial data which will evolve toward an isotropic FRW model to the future.

We note that each of the equilibrium points in Table I also exist as equilibrium points in the Bianchi V phase space (see Sec. III).

F. Self-similarity

For an exponential potential the equation for the evolution of the expansion (2.4) decouples from the ''reduced dynamical system'' in the new expansion-normalized variables (2.9) [Refs. 15 and 19; see also Eqs. (3.11) in Sec. III], and consequently at the equilibrium points we must have that

$$\theta=\theta_0 t^{-1}, \tag{2.22}$$

hence the corresponding cosmological models are necessarily self-similar in that they admit a homothetic vector[27] (except in the degenerate case $k=0$ in which the right-hand side of Eq. (2.4) can be zero and the corresponding model is the de Sitter space–time which does not admit a homothetic vector). In particular, the isotropic, power-law inflationary (FRW) attracting solutions (in the case $k^2<2$) are self-similar models and the Feinstein–Ibáñez[13] solutions (in the case $k^2>2$) are also self-similar.

III. A CLASS OF ANISOTROPIC COSMOLOGICAL MODELS

A. Equations

The diagonal form of the Bianchi type VI_h metric is given by

$$ds^2=-dt^2+a(t)^2dx^2+b(t)^2e^{2mx}dy^2+c(t)^2e^{2x}dz^2, \tag{3.1}$$

where $m=h-1$. If $m=1$ then the metric is of Bianchi type V, if $m=0$ then the metric is of Bianchi type III, and if $m=-1$ then the metric is of Bianchi type VI_0. Thus we are considering a one-parameter (m) class of Bianchi models which include Bianchi types III $(m=0)$, V $(m=1)$, VI_0 $(m=-1)$, and VI_h (all other m).

The expansion scalar, which determines the volume behavior of the fluid, is given by

$$\theta=\frac{\dot a}{a}+\frac{\dot b}{b}+\frac{\dot c}{c} \tag{3.2}$$

(where an over-dot denotes differentiation with respect to the proper time). The shear tensor, σ_{ab}, determines the distortion arising in the fluid flow leaving the volume invariant. The non-zero components of the shear tensor are

$$\begin{aligned}\sigma_{11}&=\frac{a^2}{3}\left(2\frac{\dot a}{a}-\frac{\dot b}{b}-\frac{\dot c}{c}\right),\\ \sigma_{22}&=\frac{b^2e^{2mx}}{3}\left(2\frac{\dot b}{b}-\frac{\dot a}{a}-\frac{\dot c}{c}\right),\\ \sigma_{33}&=\frac{c^2e^{2x}}{3}\left(2\frac{\dot c}{c}-\frac{\dot a}{a}-\frac{\dot b}{b}\right),\end{aligned} \tag{3.3}$$

and the shear scalar, $\sigma^2\equiv\frac{1}{2}\sigma^{ab}\sigma_{ab}$, is given by

$$\sigma^2=\frac{1}{3}\left[\left(\frac{\dot a}{a}\right)^2+\left(\frac{\dot b}{b}\right)^2+\left(\frac{\dot c}{c}\right)^2-\frac{\dot a\dot b}{ab}-\frac{\dot a\dot c}{ac}-\frac{\dot b\dot c}{bc}\right]. \tag{3.4}$$

In the case under consideration here, there is no rotation and no acceleration.

For a scalar field with an exponential potential, the Einstein field equations are

$$\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}=-\dot{\phi}^2+\Lambda e^{k\phi}, \tag{3.5a}$$

$$\frac{\dot{a}}{a}(1+m)-m\frac{\dot{b}}{b}-\frac{\dot{c}}{c}=0, \tag{3.5b}$$

$$\frac{\ddot{a}}{a}+\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}-\frac{m^2+1}{a^2}=\Lambda e^{k\phi}, \tag{3.5c}$$

$$\frac{\ddot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}-\frac{m^2+m}{a^2}=\Lambda e^{k\phi}, \tag{3.5d}$$

$$\frac{\ddot{c}}{c}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}-\frac{m+1}{a^2}=\Lambda e^{k\phi}. \tag{3.5e}$$

From the above equations one obtains the generalized Friedmann equation [see Eq. (2.5)]

$$\theta^2=3\sigma^2+\frac{3}{2}\dot{\phi}^2+3\Lambda e^{k\phi}+\frac{3}{a^2}(m^2+m+1). \tag{3.6}$$

Note that the quantity $m^2+m+1\geqslant 3/4>0$. The Raychaudhuri equation is [see Eq. (2.4)]

$$\dot{\theta}=-2\sigma^2-\tfrac{1}{3}\theta^2-\dot{\phi}^2+\Lambda \mathrm{e}^{k\phi}. \tag{3.7}$$

The evolution equation for the shear is

$$\dot{\sigma}=-\sigma\theta+\frac{(1-m)}{3\sqrt{3}\sqrt{m^2+m+1}}\left(\theta^2-3\sigma^2-\frac{3}{2}\dot{\phi}^2-3\Lambda e^{k\phi}\right). \tag{3.8}$$

The Klein–Gordon equation for the scalar field is [see Eq. (2.6)]

$$\ddot{\phi}=-\theta\dot{\phi}-k\Lambda e^{k\phi}. \tag{3.9}$$

The above system of Eqs. (3.6)–(3.9) is invariant under the transformation (see Coley and van den Hoogen[19]),

$$\theta\to\lambda\theta,\quad \dot{\phi}\to\lambda\dot{\phi},\quad \phi\to\phi+\frac{2}{k}\ln\lambda$$
$$\sigma\to\lambda\sigma,\quad t\to\lambda^{-1}t. \tag{3.10}$$

This invariance implies that there exists a symmetry in the dynamical system (3.6)–(3.9).[14] With the change of variables given by Eq. (2.9), the evolution equations for β, Ψ and Φ become independent of the variable θ. That is, θ decouples from the dynamical system describing the evolution of β, Ψ and Φ. The dynamical system can be considered as a reduced dynamical system for β, Ψ and Φ together with an evolution equation for θ (see the equations below).

The system of differential equations in the expansion-normalized variables becomes:

$$\frac{d\beta}{d\Omega}=\beta(q-2)+\frac{1-m}{\sqrt{m^2+m+1}}(1-\beta^2-\Psi^2-\Phi^2), \tag{3.11a}$$

$$\frac{d\Psi}{d\Omega}=\Psi(q-2)-\frac{\sqrt{6}k}{2}\Phi^2, \tag{3.11b}$$

$$\frac{d\Phi}{d\Omega}=\Phi(1+q)+\frac{\sqrt{6}k}{2}\Psi\Phi, \tag{3.11c}$$

and the decoupled evolution equation for the expansion

$$\frac{d\theta}{d\Omega}=-\theta(1+q), \tag{3.11d}$$

where the deceleration parameter, q, is defined by

$$q=2\beta^2+2\Psi^2-\Phi^2. \tag{3.12}$$

The domain of interest [determined by Eq. (3.6)] is the region defined by

$$\beta^2+\Psi^2+\Phi^2\leqslant 1, \tag{3.13}$$

which describes the surface and interior of a sphere in the (reduced) phase space (β,Ψ,Φ). We also note that the above system is invariant under the transformation $\Phi\to-\Phi$, and hence without loss of generality we restrict ourselves to the set Eq. (3.13) and $\Phi\geqslant 0$; i.e., the upper hemisphere of the sphere defined by Eq. (3.13).

Inflation in the context of this paper is defined to occur whenever the deceleration parameter is negative, i.e., $q<0$. We easily see from Eq. (3.12) that the inflationary regime describes the interior of a cone inside the sphere defined by Eq. (3.13).

B. Qualitative Behavior

1. Equilibrium points

The equilibrium point

$$\left\{\beta=\frac{1-m}{2\sqrt{m^2+m+1}},\ \ \Psi=0,\ \ \Phi=0\right\}, \tag{3.14}$$

satisfies the boundary condition, Eq. (3.13), for all m, and when $m=-1$ the point is part of the non-isolated line of equilibrium points $\beta^2+\Psi^2=1$ (which will be discussed later). The inflationary condition $q<0$ is never satisfied and hence this point is non-inflationary. The linearized system in a neighborhood of the equilibrium point has eigenvalues

$$\lambda_1=\frac{-3(m+1)^2}{2(m^2+m+1)},\quad \lambda_2=\frac{-3(m+1)^2}{2(m^2+m+1)},\quad \lambda_3=\frac{3(m^2+1)}{2(m^2+m+1)}. \tag{3.15}$$

It is easily seen that this point is a saddle point with a two-dimensional stable manifold. The exact solution corresponding to this point is that of a vacuum Bianchi type VI_h model or one of its degeneracies (i.e., if $m=0$ it is type III, and if $m=1$ it is an isotropic Milne model), with line element (after a re-coordinatization)

$$ds^2 = -dt^2 + a_0{}^2(t^{2p_1}dx^2 + t^{2p_2}e^{2mx}dy^2 + t^{2p_3}e^{2x}dz^2), \tag{3.16}$$

where

$$p_1 = 1, \quad p_2 = \frac{m^2+m}{m^2+1}, \quad p_3 = \frac{m+1}{m^2+1}, \tag{3.17}$$

so that $p_1+p_2+p_3 = p_1^2+p_2^2+p_3^2$.

The equilibrium point

$$\left\{\beta = 0, \quad \Psi = -\frac{\sqrt{6}k}{6}, \quad \Phi = \frac{\sqrt{6}}{6}\sqrt{6-k^2}\right\}, \tag{3.18}$$

does not exist if $k^2 > 6$ and is part of the non-isolated line of equilibrium points $\beta^2 + \Psi^2 = 1$ when $k^2 = 6$. The point lies on the boundary of the phase space $\beta^2 + \Psi^2 + \Phi^2 = 1$ and hence it corresponds to a model with zero curvature. The point is inflationary if

$$q = \frac{k^2-2}{2} < 0; \tag{3.19}$$

that is, the point represents an inflationary model if $k^2 < 2$. The linearized system in a neighborhood of the equilibrium point has eigenvalues

$$\lambda_1 = \frac{k^2-6}{2}, \quad \lambda_2 = \frac{k^2-6}{2}, \quad \lambda_3 = k^2 - 2. \tag{3.20}$$

If $k^2 < 2$ the point is therefore a sink, and if $2 < k^2 < 6$ then the point is a saddle point. (The nature of this point when $k^2 = 2$ or $k^2 = 6$, the bifurcation values, will be discussed later.) For $k \neq 0$ the exact solution corresponding to this equilibrium point is that of a flat FRW model with line element given by (after a re-coordinatization)

$$ds^2 = -dt^2 + t^{4/k^2}(dx^2 + dy^2 + dz^2), \tag{3.21}$$

and if $k = 0$ (the scalar field potential is equivalent to a positive cosmological constant) then the exact solution is the de Sitter model. The scalar field for $k \neq 0$ is given by

$$\phi = \phi_0 - \frac{2}{k} \ln t. \tag{3.22}$$

The equilibrium point

$$\left\{\beta = -\frac{(k^2-2)}{2}\frac{(m-1)\sqrt{m^2+m+1}}{[(k^2-2)(m^2+m+1)+3(m^2+1)]},\right.$$

$$\Psi = -\frac{\sqrt{6}k}{2}\frac{(m^2+1)}{[(k^2-2)(m^2+m+1)+3(m^2+1)]},$$

$$\left.\Phi = \frac{\sqrt{6}}{2}\frac{\sqrt{m^2+1}\sqrt{[(k^2-2)(m+1)^2+4(m^2+1)]}}{[(k^2-2)(m^2+m+1)+3(m^2+1)]}\right\}, \tag{3.23}$$

can be shown (after much algebra) to satisfy the boundary condition, Eq. (3.13), if $k^2 \geqslant 2$ and satisfies the inflationary condition $q<0$ if $k^2<2$. This implies that the corresponding solution is non-inflationary when the point exists inside the physical phase space given by Eq. (3.13). The linearized part of the system in a neighborhood of the equilibrium point has eigenvalues

$$\lambda_1 = -\frac{3}{2}\left\{ \frac{4(m^2+1)+(k^2-2)(m+1)^2}{(k^2-2)(m^2+m+1)+3(m^2+1)} \right\},$$

$$\lambda_2 = -\frac{3}{4}\left\{ \frac{(k^2-2)(m+1)^2+4(m^2+1)}{(k^2-2)(m^2+m+1)+3(m^2+1)} \right]$$
$$+\frac{3}{4}\left\{ \frac{\sqrt{[(k^2-2)(m+1)^2+4(m^2+1)][4(m^2+1)-(k^2-2)(7m^2-2m+7)]}}{(k^2-2)(m^2+m+1)+3(m^2+1)} \right\}, \quad (3.24)$$

$$\lambda_3 = -\frac{3}{4}\left\{ \frac{(k^2-2)(m+1)^2+4(m^2+1)}{(k^2-2)(m^2+m+1)+3(m^2+1)} \right\}$$
$$-\frac{3}{4}\left\{ \frac{\sqrt{[(k^2-2)(m+1)^2+4(m^2+1)][4(m^2+1)-(k^2-2)(7m^2-2m+7)]}}{(k^2-2)(m^2+m+1)+3(m^2+1)} \right\}.$$

It can be shown that if $k^2>2$, then all three eigenvalues are negative and hence the equilibrium point is a stable node. It is also interesting to note that if $k^2>2+4(m^2+1)/(7m^2-2m+7)$, then the equilibrium point is a focus (i.e., the solution oscillates in a neighborhood of the equilibrium point as it approaches the equilibrium point). The behavior of the system at the bifurcation value $k^2=2$ will be discussed later. The exact solution corresponding to this point is that of a Bianchi type VI_h model or one of its degeneracies (i.e., if $m=0$ it is of type III and if $m=1$ it is a negatively curved FRW model), with line element (after a re-coordinatization)

$$ds^2 = -dt^2 + a_0{}^2(t^{2p_1}dx^2 + t^{2p_2}e^{2mx}dy^2 + t^{2p_3}e^{2x}dz^2), \quad (3.25)$$

where

$$p_1 = 1,$$
$$p_2 = \frac{2}{k^2}\left(1+\frac{(k^2-2)(m^2+m)}{2(m^2+1)}\right), \quad (3.26)$$
$$p_3 = \frac{2}{k^2}\left(1+\frac{(k^2-2)(m+1)}{2(m^2+1)}\right).$$

The scalar field in this case is given by

$$\phi = \phi_0 - \frac{2}{k}\ln t. \quad (3.27)$$

For $m \neq 1$, the solution given by Eqs. (3.25)–(3.27) is the exact solution found by Feinstein and Ibáñez.[13] Thus we see that if $k^2>2$ then the non-isotropic and non-inflationary Feinstein and

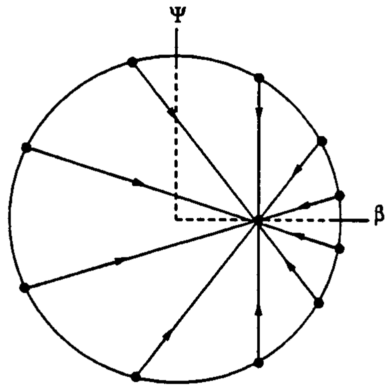

FIG. 1. Phase portrait in the invariant set $\Phi=0$.

Ibáñez[13] solution is a stable attractor for the type III and VI_h cases. When $m=1$, the corresponding isotropic solution represents the future asymptotic attractor for the Bianchi type VII_h models as well as the asymptotic attractor for the Bianchi type V models.

2. Boundaries

The qualitative behavior of the system on the boundaries can also help to determine the behavior in the interior of the phase space. Each of the boundary sets $\Phi=0$ and $\beta^2+\Psi^2+\Phi^2=1$ is an invariant set. The invariant set $\Phi=0$ represents models with a massless scalar field with a zero potential. This invariant set will represent the behavior of the system as the scalar field ϕ tends to minus infinity. The remaining system of equations for β and Ψ can be directly integrated to yield

$$\Psi=C\left(2\beta-\frac{(1-m)}{\sqrt{m^2+m+1}}\right). \tag{3.28}$$

These are straight lines emanating from the equilibrium point Eq. (3.14) directed inwards, and thus in the two-dimensional invariant set $\Phi=0$ the point is a sink. However, in the full three-dimensional phase space the point is a saddle point, and thus we can conclude that the invariant set $\Phi=0$ is the two-dimensional stable manifold. Also, it is easy to see that the outer ring described by $\beta^2+\Psi^2=1$ is a source (see Figure 1).

We can also analyze the invariant set $\beta^2+\Psi^2+\Phi^2=1$, which represents Bianchi type I models with a scalar field and an exponential potential. Again the system of equations can be integrated and the solutions are found to be straight lines emanating from the ring of non-isolated equilibrium points given by $\beta^2+\Psi^2=1$ and evolving to the equilibrium point Eq. (3.18) if $k^2<6$. In the full three-dimensional phase space this equilibrium point Eq. (3.18) is a saddle when $2<k^2<6$, and consequently in this case the invariant set $\beta^2+\Psi^2+\Phi^2=1$ is the two-dimensional stable manifold.

In the full three-dimensional phase space the ring of equilibrium points ($\beta^2+\Psi^2=1$, $\Phi=0$) for $k^2<6$ is a global source, and for $k^2>6$ we find that some part of the ring acts like a source and the remaining part of the ring acts like a saddle (see Figures 2 and 3). The exact solution corresponding to the equilibrium points $(\beta_0, \pm\sqrt{1-\beta_0^2}, 0)$ has the form

$$ds^2=-dt^2+t^{2p_1}dx^2+t^{2p_2}dy^2+t^{2p_3}dz^2, \tag{3.29}$$

where

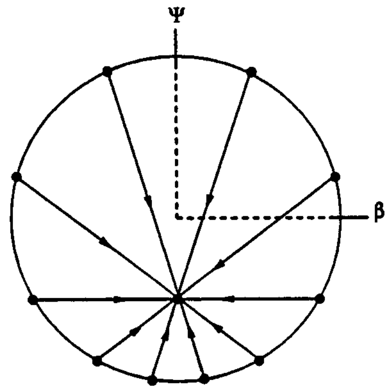

FIG. 2. Projection of the phase portrait in the invariant set $\beta^2+\Psi^2+\Phi^2=1$ with $k^2<6$.

$$p_1=\frac{1}{3}\left(1+\frac{(1-m)\beta_0}{\sqrt{m^2+m+1}}\right),$$
$$p_2=\frac{1}{3}\left(1-\frac{(2+m)\beta_0}{\sqrt{m^2+m+1}}\right), \qquad (3.30)$$
$$p_3=\frac{1}{3}\left(1+\frac{(1+2m)\beta_0}{\sqrt{m^2+m+1}}\right),$$

where $-1\leqslant\beta_0\leqslant 1$. Note that $p_1+p_2+p_3=1$ but $p_1^2+p_2^2+p_3^2=\frac{1}{3}(1+2\beta_0)$; hence in general $(\beta_0\neq 0)$ these Kasner-like points do not correspond to exact Kasner models.

3. Closed orbits

It is very difficult to prove or disprove the existence of periodic and/or recurrent orbits in the phase space of any of the dynamical systems corresponding to the general Bianchi models. However, in the Bianchi V case $(m=1)$, for example, in which the phase space can be described by a number of invariant sets, some results are possible. Recall that the phase space is a hemisphere described by $\beta^2+\Psi^2+\Phi^2\leqslant 1$ and $\Phi\geqslant 0$. The invariant sets and their dimension, as well as a

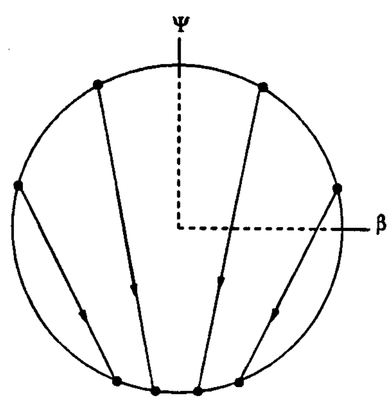

FIG. 3. Projection of the phase portrait in the invariant set $\beta^2+\Psi^2+\Phi^2=1$ with $k^2>6$.

TABLE II. The invariant sets in the Bianchi V phase space.

Label	Variables	Dimension	Description
A	$\Phi=0,\beta^2+\Psi^2=1$	1	Equator of hemisphere (Kasner ring)
B	$\Phi=0,\beta^2+\Psi^2<1$	2	Bottom disk of hemisphere (massless scalar field)
C	$\Phi>0,\beta^2+\Psi^2+\Phi^2=1$	2	Surface of hemisphere (Bianchi I)
D	$\Phi>0,\beta^2+\Psi^2+\Phi^2<1,\beta=0$	2	Half disk in interior of hemisphere (FRW)
E–	$\Phi>0,\beta^2+\Psi^2+\Phi^2<1,\beta<0$	3	Half of interior of hemisphere (Bianchi V)
E+	$\Phi>0,\beta^2+\Psi^2+\Phi^2<1,\beta>0$	3	Half of interior of hemisphere (Bianchi V)

brief description, is given in Table II. Since the dimension of set A is 1, closed orbits cannot exist. In sets B and C, we are able to find the equations of the orbits explicitly and we find that there are no closed orbits [see the solution given by Eq. (3.28) and Figure 1 and Figures 2 and 3, respectively]. The set D which represents the FRW models contain no closed orbits. In the sets E$-$ and E$+$ we have that $d\beta/d\Omega>0$ and $d\beta/d\Omega<0$, respectively, and consequently, since β is a monotonic function in each invariant set, there do not exist any closed orbits in the interior of the hemisphere.[15] Summarizing, there do not exist any closed periodic orbits in the case of the Bianchi V $(m=1)$ models.

4. Bifurcation values

We shall now concern ourselves with the bifurcation values.[28] If $k^2=0$, it is easily determined that the critical points and the qualitative behavior is the same as in the case $0<k^2<2$. However, the corresponding exact solutions are different. (Note that the $k^2=0$ case corresponds to the case of a positive cosmological constant.) At the bifurcation value $k^2=2$, we find that the equilibrium points Eq. (3.18) and Eq. (3.24) coalesce to become a single equilibrium point. The linearized system at this point has a zero eigenvalue. However, by using polar coordinates we find that the point is a sink and hence the qualitative behavior of the system is the same as for the case $0<k^2<2$. We thus conclude that the equilibrium point Eq. (3.18) undergoes a *trans*-critical bifurcation at $k^2=2$. At the bifurcation value of $k^2=6$, the equilibrium point Eq. (3.18) now becomes part of the ring of equilibrium points $(\beta^2+\Psi^2=1,\ \Phi=0)$. This particular point remains a saddle point and the rest of the ring of equilibrium points remain sources; however, as the value of k^2 is increased past 6 more and more of the ring starts to behave like saddle points. Thus in some extended sense of the definition, the ring of equilibrium points $(\beta^2+\Psi^2=1,\ \Psi=0)$ undergoes a *trans*-critical-like bifurcation at $k^2=6$.

C. Discussion

The qualitative behavior of the class of cosmological models under consideration depends critically on the value of k and somewhat on the parameter m. The parameter m determines which Bianchi model we are considering and consequently determines if the model will isotropize to the future. However, the parameter k has a profound affect on the qualitative behavior of the models. For $0\leqslant k^2<2$ all trajectories (that is, all models of Bianchi types I, III, V and the VI_h), except for a set of measure zero, evolve from the ring of equilibrium points $\beta^2+\Psi^2=1$, $\Phi=0$ toward the isotropic and inflationary model corresponding to the equilibrium point given by Eq. (3.18). For $k^2=2$, all trajectories evolve from the ring of equilibrium points $\beta^2+\Psi^2=1$, $\Phi=0$ toward the isotropic model given by Eq. (3.18); however these models need not inflate. For $2<k^2$, all

trajectories in the Bianchi III and VI_h phase spaces evolve from some portion of the ring $\beta^2+\Psi^2=1$ toward the equilibrium point given by Eq. (3.24), which is neither isotropic nor inflationary. However, in the Bianchi I and V cases for $2<k^2<6$ all trajectories evolve from some portion of the ring and isotropize to the future, but they need not inflate. When $6<k^2$, the Bianchi V models continue to isotropize to the future while the Bianchi I models fail to do so.

IV. CONCLUSION

In Sec. II, we described a number of results concerning the behavior of spatially homogeneous cosmological models with a scalar field and an exponential potential of the form $V(\phi)=\Lambda e^{k\phi}$ (where Λ is a positive constant). Summarizing, we found that:

(1) If $k=0$, then all initially expanding Bianchi models (except a subclass of the Bianchi type IX models) will isotropize to the future toward the de Sitter solution.[20]
(2) If $0<k^2<2$, then all initially expanding Bianchi models (except for a subclass of the Bianchi type IX models) isotropize to the future toward a power-law inflationary solution.[11,12]
(3) If $0<k^2<2$, then a subclass of the Bianchi type IX models will recollapse.[11,12]
(4) If $2<k^2$, then the only Bianchi models that can possibly isotropize to the future are those of Bianchi types I, V, VII and IX.[25]
(5) If $2<k^2$, then the Bianchi VII_h models do indeed isotropize; and therefore, there exists an open set of initial conditions in the space of all spatially homogeneous initial data for which the models isotropize to the future.[26]

In the remainder of the paper a detailed qualitative analysis of a one-parameter family of Bianchi models (which includes the Bianchi models of types I, III, V, and VI_h) was presented, illustrating the validity of the points above. In particular, it was shown that the future asymptotic behavior of the Bianchi type III and VI_h models is represented by the Feinstein–Ibáñez solution.[13] We note that the Feinstein–Ibáñez solution[13] is a self-similar, non-isotropic and non-inflationary solution that is stable when $k^2>2$; hence the cosmic no-hair conjecture is clearly not satisfied in this case. In addition, it was shown that the Bianchi type V models when $k^2>2$ asymptotically tend to an isotropic but non-inflationary open FRW model. This does not mean that the models do not experience inflation, it is the final equilibrium point which is marginally non-inflationary. Note that if $k^2>8/3$, then these Bianchi V models can be shown to experience periods of inflation as they evolve toward the isotropic ''marginally'' non-inflationary solution.

We note that in our investigation we have not included ordinary matter (satisfying the usual energy conditions). Matter can be included in precisely the same way as in Heusler[23] and Kitada and Maeda.[11,12] However, the addition of ordinary matter is not expected to change the primary results.

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