ORBIFOLD ATLAS GROUPOIDS

by

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "ORBIFOLD ATLAS GROUPOIDS" by Alanod Sibih in partial fulfillment of the requirements for the degree of Master of Science.

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Abstract

We study orbifolds and strong maps of orbifolds. We begin with introducing a representation for orbifolds that consists of internal categories in the category of topological spaces. These categories are built from atlas charts and chart embeddings without equivalence relation. They represent orbifolds and atlas maps, but do not work well for general strong maps. We generalize the notion of category of fractions to internal categories in the category of topological spaces. We find its universal property for an internal category in the category of topological spaces. We apply this to the atlas category to obtain an atlas groupoid. We give a description of strong maps of orbifolds and the equivalence relation on them in terms of atlas groupoids. We define paths in orbifolds as strong maps. We use our construction to give an explicit description of the equivalence classes on such paths in terms of charts and chart embeddings.

List of Abbreviations and Symbols Used

Homeo(X) The homeomorphism group of X

TOP The category of topological spaces

$\mathcal{G} \ltimes X$ The translation groupoid for an action of a group $\mathcal{G}$ on a space X

csp(C) The space $C_1 \times_{t,C_0,t} C_1$, for an internal category C in *TOP*

allsq(C) The space of all not necessarily commutative squares,

$$(C_1 \times_{t,C_0,t} C_1) \times_{(s,s),C_0 \times C_0,(t,t)} (C_1 \times_{s,C_0,s} C_1)$$

for an internal category C in *TOP*

csq(C) The space of commutative squares for an internal category C in *TOP*, given by the following equalizer,

$$\text{csq(C)} \xrightarrow{j} \text{allsq(C)} \underset{m(\pi_1\pi_2,\pi_1\pi_1)}{\overset{m(\pi_2\pi_2,\pi_2\pi_1)}{\rightrightarrows}} C_1$$

pall(C) The space of the parallel arrows, $C_1 \times_{(s,t),C_0 \times C_0,(s,t)} C_1$, for an internal category C in *TOP*

CEq(C) The following equalizer,

$$\textit{CEq(C)} \longrightarrow \textit{pall(C)} \times_{t\pi_1,C_0,s} C_1 \underset{M_2}{\overset{M_1}{\rightrightarrows}} C_1$$

where $M_1 = m(\pi_3, \pi_1)$ and $M_2 = m(\pi_3, \pi_2)$.

Eq(C) The following equalizer,

$$\textit{Eq(C)} \longrightarrow C_1 \times_{t,C_0,s\pi_1} \textit{pall(C)} \underset{M_2'}{\overset{M_1'}{\rightrightarrows}} C_1$$

where $M_1' = m(\pi_2, \pi_1)$ and $M_2' = m(\pi_3, \pi_1)$.

P(C) The following equalizer,

$$P(C) \longrightarrow C_1 \times_{t,C_0,s} pall(C) \times_{t,C_0,s} C_1 \overset{M_1}{\underset{M_2}{\rightrightarrows}} C_1$$

where $M_1 = m(\pi_4, m(\pi_2, \pi_1))$ and $M_2 = m(\pi_4, m(\pi_3, \pi_1))$.

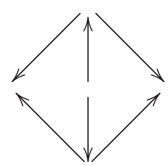
fork(C) The space, $C_1 \times_{s,C_0,s} C_1 \times_{s,C_0,t} C_1$, for an internal category C in TOP

spn(C) The space, $C_1 \times_{s,C_0,s} C_1$, for an internal category C in TOP

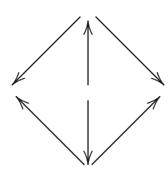
df(C) The space,

$$((C_1 \times_{s,C_0,s} C_1) \times_{s\pi_1,C_0,t} C_1) \times_{tts,(C_0 \times C_0) \times C_0,tts} ((C_1 \times_{s,C_0,s} C_1) \times_{s\pi_1,C_0,t} C_1)$$

for an internal category C in TOP, encoding not necessaily commutative diagrams of the form,



dblfork(C) The space encoding commutative diagrams of the form,



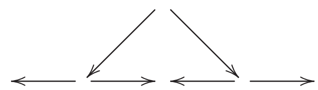
for an internal category C in TOP. I.e., the following equalizer,

$$dblfork(C) \longrightarrow df(C) \overset{M \circ \pi_1}{\underset{M \circ \pi_2}{\rightrightarrows}} spn(C)$$

spncsq(C) The space

$$C_1 \times_{s,C_0} csq(C) \times_{C_0,s} C_1$$

of commutative diagrams,



for an internal category C in *TOP*

CF1 - CF4 The classical calculus of fractions conditions

TOP- CF3 , TOP - CF4 The internal version in *TOP* of the calculus of fractions conditions

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I want to take this opportunity to thank all those who have contributed to help me to produce this thesis by giving either direct or indirect assistance to my research.
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Chapter 1

Introduction

1.1 The Notion of Orbifolds

The concept of orbifolds is very important in many fields of Mathematics. It plays a significant role in various areas such as geometry (moduli spaces), topology, algebraic geometry (Deligne - Mumford algebraic stacks with proper diagonal), crystallography (mathematics of crystals) as well as mathematical physics (moduli spaces in string theory). The orbifolds are needed in all these cases to model finite symmetry.
The concept of orbifolds was first introduced by Satake [21] in 1956. He called them V-manifolds. He described orbifolds as a generalization of manifolds. A manifold is a space which is locally homeomorphic to the Euclidean space $\mathbb{R}^n$, but an orbifold is locally homeomorphic to the quotient space of a Euclidean space $\mathbb{R}^n$ under the action of a finite group. Therefore, the quotient space of a manifold by a finite group action is an example of an orbifold. Manifolds share with orbifolds that they can be described by atlases and charts. However, a chart for an orbifold consists of an open subset $\tilde{U}$ of $\mathbb{R}^n$ with a finite group $\mathcal{G}$ which acts on $\tilde{U}$ by homeomorphisms such that we have a quotient space $\tilde{U}/\mathcal{G}$ which is homeomorphic to an open subset U of the underlying space M of this orbifold. This means that an orbifold is determined by a quotient space and an atlas. So, it is important to know the quotient space and the atlas together to describe the orbifold. The quotient space does not completely determine the orbifold because it does not have all the information. We can have many distinct orbifolds with the same quotient space. For instance, when Z_m acts on the unit disk in $\mathbb{R}^2$ by rotation, we get a cone shaped quotient. All these cones are homeomorphic to the disk as topological spaces.
We see that any manifold is an orbifold but the opposite is not true, see the list of examples in 2.5.5. The "V" in V-manifold is to suggest the possible presence of cone points. Thurston [23] introduced the term "orbifold" in his course in 1976-77 when he discussed 3-manifolds. He did not know about the concept of V-manifold at that

point in time. The word "orbifold" refers to the orbit of a group action on a manifold. The points of an orbifold correspond to the orbits of the group actions on the charts. After that, the word "orbifold" has been used instead of "V-manifold", especially because people thought that a V-manifold is a special kind of manifold. Thurston gave a great number of nice examples of orbifolds. We will talk about some of these. The first example is what we usually do when we want to make a heart shape from a piece of paper which is called a *Valentine heart*. We fold a paper in half and cut out half a heart and then we open the paper to have the heart. This is actually an orbifold. It consists of $\mathbb{R}^2$ as a space and $Z/2$ as a group action which is generated by reflection about the y- axis. Another example of an orbifold is the two parallel mirrors with repeated reflection in a barber shop. When you look at the mirrors, you see yourself from front and back. Also, you see reflections of your original pattern. This orbifold consists of $\mathbb{R}^3$ as a space and $Z/2 \star Z/2$ as a group whose action is generated by reflection about two parallel planes. From these examples we note that we encounter orbifolds in our daily life.

1.2 Orbifolds Maps and Groupoids

A smooth map between manifolds can be described as a continuous map between the underlying spaces such that for each point in the domain there is a lifting to a smooth map between charts for a chart neighborhood of that point.

Satake generalized this notion to obtain what we usually call smooth maps of orbifolds. We also call such maps weak maps of orbifolds. It was observed in [4] for instance that these maps do not carry enough information for orbifold homotopy theory. For example, if we have a unit disk in $\mathbb{R}^2$ with Z/n acting on it. We obtain a cone of order n. The weak maps give us the trivial group of homotopy classes of loops in the cone, but the homotopy classes of loops defined by strong maps give the group Z/n.

Maps between manifolds can also be described purely in terms of maps between atlas charts that agree on their overlaps. However, in that case we need to work with equivalence classes of maps due to the fact that a given manifold has many distinct atlases. When we generalize this notion to orbifolds, we obtain what are called strong maps or good maps of orbifolds.

A strong map can be described in terms of orbifold groupoids. These groupoids were introduced by Pronk in [18]. They represent orbifolds so well that some people suggest that the study of orbifolds is simply the study of these groupoids. However, the construction in [18] was rather ad hoc, and in this paper we want to construct them from a more naturally defined object which is the orbifold atlas category. Maps between orbifold atlases can be viewed as morphisms between these orbifold categories. However, they are not good enough to describe arbitrary strong maps between orbifolds. The problem is that refinements of atlases do not always give homomorphism of atlas categories and even when they do, these homomorphisms are not necessarily weak equivalences. To solve this problem, we need to find a groupoid that contains the atlas category. We construct this groupoid by introducing the notion of an internal category of fractions and developing the conditions for its construction and its universal property for an arbitrary internal category in *TOP*. We then apply this to atlas categories to obtain atlas groupoids. These are precisely the atlas groupoids introduced in [18], except that we do this here for not necessarily effective orbifolds. Then we will finish this paper by an application to morphisms of orbifolds that represent paths in orbifolds giving an explicit description of the equivalence relation between paths.

1.3 Overview

We have seen that an orbifold consists of a quotient space and an atlas. Therefore, we will first review the classical definition of orbifolds in term of charts and atlases. Subcharts and embeddings of charts play an important role. We will describe their properties in detail because they will be important later in this thesis. Then in Chapter 3, we begin with the definition of morphisms between orbifolds from Satake's view. We also give the description of strong maps in terms of atlases. We also introduce atlas categories and view maps of atlases as maps of atlas categories. However, in order to study strong maps and be able to describe them effectively, we need groupoids. In Chapter 4, we will have some background information about group actions. It will also introduce information about groupoids and topological groupoids, which are internal groupoids in the category *TOP* of topological spaces. We will see homomorphisms between groupoids as internal functors. We will review the definition of natural transformation in order to translate it in terms of internal groupoids. In this chapter, we will establish the relation between groupoids and individual orbifold charts. We will view orbifold charts as groupoids. Then we will review the definition of the calculus of fractions and the category of fractions which will be in Chapter 5. Then we will prove some important facts related to its construction which we will use later. The category of fractions is important for the purpose of this paper because we are going to internalize it to construct the atlas groupoids. After that, in the beginning of Chapter 6, we introduce surjective local homeomorphisms and prove some facts about them. They are important in the generalization of the calculus of fractions conditions. We will generalize the category of fractions conditions to internal categories in *TOP*. Finally, we will proceed to define the atlas groupoids. In the beginning of Chapter 7, we will give the definition for strong maps between orbifolds in terms of the language of groupoids. They correspond to certain spans of homomorphisms between atlas groupoids. Then we will talk about the equivalence relation on these maps. After that, we study strong maps and apply our construction to describe orbifold paths which are strong maps of orbifolds from the unit interval into an orbifold and find a simplified description, both of the paths and their equivalence relation.

Chapter 2

Orbifolds

In this chapter, we are going to introduce all notions that we need to give the definition of orbifolds. We will describe atlases, charts, and chart embeddings. In this chapter we have the results about chart embeddings from [18] and [22] for effective actions, but we generalize them to non-effective actions. However, before starting to talk about orbifolds, we first define some terms that we are going to use.

2.1 Group Actions

Definition 2.1.1. Let $\mathcal{G}$ be a group and X be a set. An *action of the group $\mathcal{G}$ on the set X* is given by a map $f : \mathcal{G} \times X \longrightarrow X$ defined as $(g, x) \mapsto gx$, i.e, $f(g, x) = gx$ which satisfies for all $x \in X$ and $g_1, g_2, e \in \mathcal{G}$,

- $g_1(g_2x) = (g_1g_2)x$.
- $ex = x$.

Definition 2.1.2. The action of a group $\mathcal{G}$ on a set X is called *effective* or *faithful* if whenever $gx = x$ for all $x \in X$ then $g = e$.

Definition 2.1.3. Let $\mathcal{G}$ be a group acting on a set X.

- For a given point $x \in X$, the set $G_x = \{g \in \mathcal{G} \mid gx = x\}$ is called the *isotropy group*.
- The set $X^{\mathcal{G}} = \{x \mid gx = x \text{ for all } g \in \mathcal{G}\}$ is called the *set of fixed points.* Moreover, if $\mathcal{H} \subseteq \mathcal{G}$, then $X^{\mathcal{H}} = \{x \mid gx = x \text{ for all } g \in \mathcal{H}\}$.
- The set $\ker \mathcal{G} = \{g \in \mathcal{G} \mid gx = x \text{ for all } x \in X\}$ is the *kernel of the action.*

Remark 2.1.4. Such a group action can be described as a homomorphism $\psi : \mathcal{G} \longrightarrow S_X$ where S_X is the group of permutations of X. Then $\ker \psi = \ker \mathcal{G}$.

Definition 2.1.5. Let X, Y be topological spaces. We say that X *is homeomorphic to* Y if there is a continuous map $\phi : X \longrightarrow Y$ such that ϕ has a continuous inverse. We denote a homeomorphism by $X \cong Y$.

Definition 2.1.6. Let X be a topological space. Then *the homeomorphism group of* X is a group which consists of all homeomorphisms $\phi : X \longrightarrow X$ with composition as multiplication. We denote the group of homeomorphisms of X by $Homeo(X)$.

Definition 2.1.7.
- *The category SET of sets* has all sets as objects and all functions between these sets as arrows.
- *The category of topological spaces TOP* has all topological spaces as objects and all continuous functions between these spaces as arrows.

Remarks 2.1.8.
- If $X \in TOP$ and $\mathcal{G}$ is a topological group, then we say that $\mathcal{G}$ acts continuously on X if the map f in Definition 2.1.1 is continuous.
- The action of an element $g \in \mathcal{G}$ on X gives a bijective map $\vartheta_g : X \longrightarrow X$ where $\vartheta_g(x) = g(x)$ and $(\vartheta_g)^{-1} = \vartheta_{g^{-1}}$. When there is no confusion possible, we will denote ϑ_g by g.
- If $\mathcal{G}$ acts continuously, then ϑ_g is a homeomorphism.

2.2 Charts

We are going to introduce the concept of a paracompact space because the underlying space of an orbifold is required to be paracompact and Hausdorff.

Definition 2.2.1. A Hausdorff space X is a *paracompact space* if every open cover has a locally finite open refinement that covers X.

We will first consider the structure of individual charts.

Definition 2.2.2. Let X be a paracompact Hausdorff space with fixed $n > 0$ and let $U \subset X$ be an open set. An *orbifold chart* for U is a triple $\{\tilde{U}, \mathcal{G}, \varphi\}$ such that:

- $\tilde{U}$ is a connected open subset of $\mathbb{R}^n$.

- $\mathcal{G}$ is a finite group which acts on $\tilde{U}$ by homeomorphisms, i.e., there is a group homomorphism $\theta : \mathcal{G} \longrightarrow \text{Homeo}(\tilde{U})$. We write $g : \tilde{U} \longrightarrow \tilde{U}$ for $\theta(g)$.

- $\varphi : \tilde{U} \longrightarrow U$ is a continuous map such that φ is $\mathcal{G}$-invariant. I.e., $\varphi \circ g = \varphi$ for all $g \in \mathcal{G}$. We require that the map φ gives rise to a homeomorphism $\phi : \tilde{U}/\mathcal{G} \longrightarrow U$ where $\tilde{U}/\mathcal{G}$ is the orbit space. So, $\tilde{U}/\mathcal{G}$ is the quotient space of $\tilde{U}$ where x is identified with x' if and only if there exists $g \in \mathcal{G}$ such that $gx = x'$. This space $\tilde{U}/\mathcal{G}$ has the quotient topology.

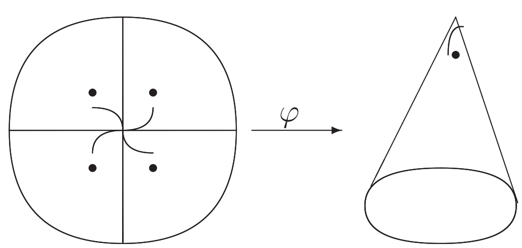
Assumption 2.2.3. Since in this paper all groups are finite, it follows that each element of $\mathcal{G}$ either

- acts like the identity, or

- has a set of fixed points of codimension greater than or equal to 1.

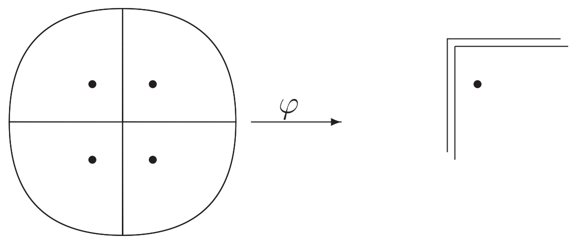
Examples 2.2.4.

- $\{D, \mathbb{Z}_4, \varphi\}$ is a chart where D is a disk and $\mathbb{Z}_4$ acts by rotation $\rho_{\frac{\pi}{2}}$. The quotient space is a *cone* of order 4.



For the quotient cone, any $x \in D$ will be identified with $\rho_{\frac{\pi}{2}}(x)$. We note that every orbit has four points except for the center. We see that the center of the disk has an isotropy group of order four. However, the other points in the disk have the trivial isotropy group, i.e., $\rho(x) \neq x$ for all $x \in D \setminus O$ where O is the center point.

- $\{\mathbb{R}^2, D_4, \varphi\}$ is a chart where $D_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$ such that D_4 acts on $\mathbb{R}^2$ by reflection about the X - axis and about the Y - axis. In the quotient, these axes will become a corner of order 2 and they become part of the boundary of the quotient space. Such a boundary is called a *silvered boundary (edge)*.



For the quotient, any $x \in D$ will be identified with its reflections about the X - axis and Y- axis. We note that every orbit has four points except for the points on the axes, which have orbits that contain two points, and the center which is in an orbit by itself. That means that all points in the disk have the trivial isotropy group except for the points on the axes, which have an isotropy group of order 2 and the center of the disk has an isotropy group of order four.

2.3 Subcharts

Definition 2.3.1. Let $\{\tilde{U}, \mathcal{G}, \varphi\}$ and $\{\tilde{U}', \mathcal{G}', \varphi'\}$ be two charts such that $U' \subseteq U$. Then a *chart embedding* $(\lambda, \ell) : \{\tilde{U}', \mathcal{G}', \varphi'\} \hookrightarrow \{\tilde{\mathcal{U}}, \mathcal{G}, \varphi\}$ is defined by:

- $\lambda : \tilde{U'} \hookrightarrow \tilde{U}$ is a topological embedding such that $\varphi' = \varphi \circ \lambda$.
- $\ell : \mathcal{G}' \longrightarrow \mathcal{G}$ is a group homomorphism such that:
  - $\lambda(g' \cdot x) = \ell(g') \cdot \lambda(x)$ for $x \in U'$ and $g' \in \mathcal{G}'$.
  - ℓ induces an isomorphism $\ker \mathcal{G}' \cong \ker \mathcal{G}$.

Definition 2.3.2. A chart $\{\tilde{U}', \mathcal{G}', \varphi'\}$ is a *subchart* of $\{\tilde{U}, \mathcal{G}, \varphi\}$ if there is a chart embedding $(\lambda, \ell) : \{\tilde{U}', \mathcal{G}', \varphi'\} \hookrightarrow \{\tilde{U}, \mathcal{G}, \varphi\}$. When there is no chance of confusion, we will write λ for a chart embedding instead of (λ, ℓ).

Remark 2.3.3. In Satake's definition of V-manifolds, he required the sets of fixed points $x \in \mathcal{G}$ to have dimension $\leq n - 2$. [21]

When we have an orbifold, we can always add smaller charts by the technique which will be described in the following proposition.

Proposition 2.3.4. *Let $\{\tilde{U}, \mathcal{G}, \varphi\}$ be a chart. Let $V \subseteq U$ be a connected open subset. Then any connected component $\tilde{V}$ of $\varphi^{-1}V$ forms a subchart $\{\tilde{V}, G_{\tilde{V}}, \psi\}$ of $\{\tilde{U}, \mathcal{G}, \varphi\}$ if we take $G_{\tilde{V}}$ to be the subgroup of $\mathcal{G}$ that keeps $\tilde{V}$ invariant, i.e., $g\tilde{V} = \tilde{V}$ for all $g \in G_{\tilde{V}}$, and $\psi = \varphi|_{\tilde{V}} : \tilde{V} \longrightarrow V$.*

Proof. Suppose $V \subseteq U$ is a connected open subset. Let $\tilde{V}$ be a connected component of $\varphi^{-1}(V) \subseteq \tilde{U}$. Since the map φ is $\mathcal{G}$ - invariant, i.e., $\varphi g = \varphi$ for $g \in \mathcal{G}$ and g acts as a homeomorphism, $g(\tilde{V})$ is also a connected component for all $g \in \mathcal{G}$. Therefore, we know from the definition of connected component that, either:

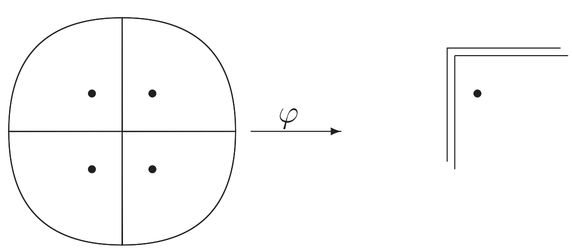
- $g(\tilde{V}) = \tilde{V}$, or
- $g(\tilde{V}) \cap \tilde{V} = \emptyset$.

So, we define $G_{\tilde{V}} = \{g \in \mathcal{G} \mid g(\tilde{V}) = \tilde{V}\}$, the subgroup of $\mathcal{G}$ that keeps $\tilde{V}$ invariant. As a result, we have a chart $\{\tilde{V}, G_{\tilde{V}}, \psi\}$ with $\psi = \varphi|_{\tilde{V}} : \tilde{V} \longrightarrow V$. □

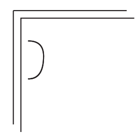
Proposition 2.3.5. *Let $\{\tilde{U}, \mathcal{G}, \varphi\}$ be a chart. Let $W \subseteq U$. Then we have a subchart $\tilde{W} = \varphi^{-1}W$ with $G_{\tilde{W}} = G_{\tilde{x}}$ for $\tilde{x} \in \tilde{W}$.*

Proof. Suppose $V \subseteq U$ is a connected open subset. Let $\tilde{V}$ be a connected component of $\varphi^{-1}(V) \subseteq \tilde{U}$. Let $x \in V$. Then we have a point $\tilde{x} \in \tilde{V}$ with $\varphi(\tilde{x}) = x$. Take $g \in \mathcal{G}$ such that $g\tilde{x} \neq \tilde{x}$. Since the space is Hausdorff, there are $\tilde{V}_g$ and $\tilde{U}_g$ such that $\tilde{V}_g \cap \tilde{U}_g = \emptyset$ with $\tilde{x} \in \tilde{U}_g$ and $g\tilde{x} \in \tilde{V}_g$. We have that $\tilde{x} \in \tilde{U}_g \cap g^{-1}\tilde{V}_g$. Take $\tilde{W}_g = \tilde{U}_g \cap g^{-1}\tilde{V}_g$. Then we get that $\tilde{W}_g \cap g\tilde{W}_g \subseteq \tilde{U}_g \cap g^{-1}\tilde{V}_g \cap g\tilde{U}_g \cap gg^{-1}\tilde{V}_g \subseteq \tilde{U}_g \cap \tilde{V}_g = \emptyset$. So, we find $\tilde{W}_g$ containing $\tilde{x}$. Let $\tilde{W}_{\tilde{x}} = \cap_{g \in \mathcal{G} - G_{\tilde{x}}} \tilde{W}_g$. Then we get that $\tilde{W} = \cap_{g \in G_{\tilde{x}}} g\tilde{W}_{\tilde{x}}$. Since g is arbitrary, we can conclude that $G_{\tilde{W}} = G_{\tilde{x}}$. □

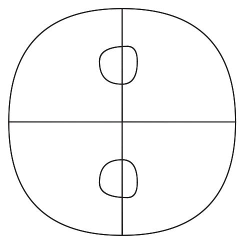
Example 2.3.6. We saw in Example 2.2.4 that a chart $\{D, \mathbb{Z}_2 \times \mathbb{Z}_2, \varphi\}$ where $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts on $\mathbb{R}^2$ by reflection about the X - axis and about the Y - axis gives us the quotient space U which is a *corner* of order 2 as follows.



If we take a subset V of U as,



Then its preimage would be in the chart $D = \tilde{U}$ as



Choose $\tilde{V}$ to be the top component for instance. As a result, $\tilde{V}$ contains orbits of order 2 with group $G_{\tilde{V}} = \{\rho \in \mathcal{G} \mid \rho(\tilde{V}) = \tilde{V}\} = \{id, \rho\} = Z_2$, which acts on $\mathbb{R}^2$ by reflection about the Y- axis, and $\psi = \varphi|_{\tilde{V}} : \tilde{V} \longrightarrow V$.

Remarks 2.3.7. These remarks can be found in [22]. Let $\{\tilde{U}, \mathcal{G}, \varphi\}$ be an orbifold chart. Then:

- For any $g \in \mathcal{G}$ there is a chart embedding from $\tilde{U}$ into itself since g is a homeomorphim. I.e., $g : \tilde{U} \hookrightarrow \tilde{U}$. That is as follows: for any $g \in \mathcal{G}$, we define $g : \tilde{U} \hookrightarrow \tilde{U}$ as $x \mapsto gx$ is a one to one map. This is a part of a chart embedding $(g, c_g) : \{\tilde{U}, \mathcal{G}, \varphi\} \hookrightarrow \{\tilde{U}, \mathcal{G}, \varphi\}$ where c_g is defined by $c_g(t) = gtg^{-1}$ for $t \in \mathcal{G}$. This is well defined because $g(t \cdot x) = c_g(t) \circ g(x)$ because $gt \cdot x = g(t \cdot x)$ and $c_g(t) \cdot g(x) = gtg^{-1} \cdot gx = gt \cdot x$.

- The composition of two chart embeddings is itself a chart embedding. Therefore, for any $\lambda : \{\tilde{U}, \mathcal{G}, \varphi\} \hookrightarrow \{\tilde{U}', \mathcal{G}', \varphi'\}$ we have $g' \circ \lambda$ is a chart embedding for any $g' \in \mathcal{G}$.

If all group actions are effective, then the embedding part of a chart embedding completely determines the group homomorphism.

Lemma 2.3.8. *Let $(\lambda, \ell) : \{\tilde{U}, \mathcal{G}, \varphi\} \hookrightarrow \{\tilde{U}', \mathcal{G}', \varphi'\}$ be a chart embedding. If we have $g' \in \mathcal{G}'$ with $\lambda(\tilde{U}) \cap (g'(\lambda(\tilde{U}))) \neq \emptyset$, then we have $g' \in Im\ \ell$. We can conclude that $\lambda(\tilde{U}) = g'(\lambda(\tilde{U}))$.*

Proof. Suppose $\lambda(\tilde{U}) \cap (g'(\lambda(\tilde{U}))) \neq \emptyset$. We want to show that $g' \in \mathrm{Im}\,\lambda$ and $\lambda(\tilde{U}) = g'(\lambda(\tilde{U}))$. Since $\lambda(\tilde{U}) \cap (g'(\lambda(\tilde{U})))$ is open, there is a $y \in \lambda(\tilde{U}) \cap (g'(\lambda(\tilde{U})))$ with $ker\ \mathcal{G}' = G'_y$. Then there are $x, x' \in \tilde{U}$ such that $\lambda(x) = y$ and $g'(\lambda(x')) = y$. So, $\varphi'(\lambda(x)) = \varphi'(\lambda(x'))$. We have that (λ, ℓ) is a chart embedding. That means that we have $\varphi(x) = \varphi(x')$. Then there is $g \in \mathcal{G}$ such that $gx' = x$. So, by definition of the chart embedding we have $\ell(g)(\lambda(x')) = \lambda(gx')$. As a result $\lambda(gx') = \lambda(x) = y = g'\lambda(x')$. Since $G'_{\lambda(x')} = ker\ \mathcal{G}'$, we obtain that $\ell(g)(g')^{-1} \in ker\ \mathcal{G}'$. Since ℓ gives the isomorphism $\ell|_{\neg\diagdown\,\mathcal{G}} : ker\ \mathcal{G} \longrightarrow ker\ \mathcal{G}'$, there is an element h such that $\ell(h) = \ell(g)(g')^{-1}$. As a result, $g' = \ell(h^{-1})\ell(g) = \ell(h^{-1}g)$. So we can conclude that $g' \in \mathrm{Im}\,\ell$. Now we want to show that $\lambda(\tilde{U}) = g'(\lambda(\tilde{U}))$. Let $k \in \mathcal{G}$ such that $\ell(k) = g'$. Then $\lambda(\tilde{U}) = \lambda g(\tilde{U}) = \ell(k)\lambda(\tilde{U}) = g'\lambda(\tilde{U})$. So, we can conclude that $\lambda(\tilde{U}) = g'\lambda(\tilde{U})$. □

Lemma 2.3.9. *Let $\{\tilde{U}, \mathcal{G}, \varphi\}$ and $\{\tilde{U}', \mathcal{G}', \varphi'\}$ be charts. Given two embeddings $\lambda, \mu : \tilde{U} \hookrightarrow \tilde{U}'$ such that $\varphi = \varphi' \circ \lambda$ and $\varphi = \varphi' \circ \mu$. Then:*

1. *There exists a $g' \in \mathcal{G}'$ such that $\mu = g' \circ \lambda$.*

2. *If $\mathcal{G}'$ acts effectively then there exists a unique $g' \in \mathcal{G}'$ such that $\mu = g' \circ \lambda$.*

Proof.

1. Let $x \in \tilde{U}$ such that $ker\ \mathcal{G} = G_x$. We have $\varphi'(\mu(x)) = \varphi(x) = \varphi'(\lambda(x))$. Then there is $g' \in \mathcal{G}'$ such that $\mu(x) = g'\lambda(x)$. Let $M \subseteq U$ be a connected open subset containing x such that there is not any $y \in M$ that has a higher isotropy group. Let N be a connected component of $\varphi^{-1}M$. By Proposition 2.3.4, we can have a subchart $\{N, \mathcal{G}_N, \psi\} \subseteq \{\tilde{U}, \mathcal{G}, \varphi\}$ such that $\mathcal{G}_N$ is a subgroup of $\mathcal{G}$ that keeps N invariant and $\psi = \varphi|_N : N \longrightarrow \varphi(N)$. So, ψ is a homeomorphism. So, we have λN is open in $\tilde{U}'$ containing λx and μN is open in $\tilde{U}'$ containing μx. Since $\mu(x) = g'\lambda(x)$, we obtain that $g'\lambda N$ contains μx. So, we have that $\mu N \cap g'\lambda N \neq \emptyset$. We have a homeomorphism $\varphi'|_{g'\lambda N} : g'\lambda N \longrightarrow \varphi N$. We have $\varphi'|_{\lambda N} : \lambda N \longrightarrow \varphi N$ is a homeomorphism. We also have $\varphi'|_{\mu N} : \mu N \longrightarrow \varphi N$ is a homeomorphism. So, we have that $g'\lambda N$ and μN are homeomorphic to φN. By Proposition 2.3.5, we obtain that $\varphi^{-1}M \cong \coprod g\tilde{N}$. So, $\varphi'^{-1}M \cong \coprod g'\lambda N$. Since μN is connected, we have that $\mu N \subseteq g'\lambda N$ for some $g' \in \mathcal{G}'$. Therefore, $g'^{-1}\mu N \subseteq \lambda N$. However, by Proposition 2.3.5, we also have that $\lambda N \subseteq h'\mu N$ for some $h' \in \mathcal{G}'$. We get that $g'^{-1}\mu N \subseteq h'\mu N$ So, $\mu N \subseteq g'h'\mu N$. Therefore,

$g'h' \in ker\ \mathcal{G}$ and $g'\lambda N = \mu N$. As a result, μ and $g'\lambda$ are equal pair wise on a neighborhood of x because $\varphi'|_{\mu N} \cong \varphi_N$. Since this collection is dense in $\tilde{U}$ this implies by continuity that $g'\lambda = \mu$.

2. Suppose $\mathcal{G}$ acts effectively. We have $\mu = g'\lambda$ for $g' \in \mathcal{G}$. We want to show that g' is unique. Suppose there is $h' \in \mathcal{G}'$ such that $h'\lambda = \mu$. We want to show that $h' = g'$. We have that $g'|_{\lambda\tilde{U}} = h'|_{\lambda\tilde{U}}$. However, $\lambda\tilde{U}$ is open. From our Assumption 2.2.3. It follows that $g' = h'$ So, we conclude that g' is unique.

$\square$

Corollary 2.3.10. *Let $\{\tilde{U}, \mathcal{G}, \varphi\}$ and $\{\tilde{U'}, \mathcal{G}', \varphi'\}$ be two charts such that $\varphi(\tilde{U}) \subseteq \varphi'(\tilde{U'})$. The number of chart embeddings $(\lambda, \ell) : \{\tilde{U}, \mathcal{G}, \varphi\} \hookrightarrow \{\tilde{U'}, \mathcal{G}', \varphi'\}$ is the same as the number of the elements of $\mathcal{G}'/ker\ \mathcal{G}'$. However, if $\mathcal{G}'$ acts effectively then the number of chart embeddings is precisely the number of elements of $\mathcal{G}'$.*

Proof. Fix a chart embedding $\lambda : \tilde{U} \hookrightarrow \tilde{U}'$. Let E be the set of all chart embeddings from $\tilde{U}$ to $\tilde{U}'$. Define $F : \mathcal{G}' \longrightarrow E$ by $g' \mapsto g'\lambda$. We want to prove the following

- Let $\mu \in E$ then from Lemma 2.3.9, there is $g' \in \mathcal{G}'$ such that $\mu = g'\lambda$. So, F is surjective.

- Suppose that $g'\lambda = h'\lambda$ for $g', h' \in \mathcal{G}'$. Then for all $x \in \tilde{U}$ we have $g'\lambda(x) = h'\lambda(x)$. That gives us $h'^{-1}g'\lambda(x) = \lambda(x)$. So, $h'^{-1}g'\lambda \in ker\ \mathcal{G}'$.

Since we have that the kernel of the action is ker $\mathcal{G}' = \{g' \in \mathcal{G}' \mid g'x = x \text{ for all } x \in X\}$ then we will have the number of the embeddings in E is equal to the number of the elements in $\mathcal{G}/\ ker\mathcal{G}$. However, if $\mathcal{G}'$ acts effectively then we have ker $\mathcal{G}' = \{\text{id}\}$. As a result, we will have the number of the embeddings in E is equal to the number of the elements in $\mathcal{G}'$. $\square$

Corollary 2.3.11. *Let $\lambda : \{\tilde{U}, \mathcal{G}, \varphi\} \hookrightarrow \{\tilde{U'}, \mathcal{G}', \varphi'\}$ be an embedding making the following diagram commutative*

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\lambda} & \tilde{U}' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ U & \xrightarrow[i]{} & U' \end{array},$$

where i is an inclusion map. Then for each $g \in \mathcal{G}$, there is a $g' \in \mathcal{G}'$ such that $\lambda \circ g = g' \circ \lambda$. Moreover, if $\mathcal{G}'$ acts effectively, g' is unique and there is precisely *one group homomorphism $\ell : \mathcal{G} \longrightarrow \mathcal{G}'$ such that $(\lambda, \ell) : \{\tilde{U}, \mathcal{G}, \varphi\} \longrightarrow \{\tilde{U}', \mathcal{G}', \varphi'\}$ forms (λ, ℓ) a chart embedding. I.e., the map λ gives us a unique injective group homeomorphism $\ell : \mathcal{G} \longrightarrow \mathcal{G}'$.*

Proof. Let $\lambda : \{\tilde{U}, \mathcal{G}, \varphi\} \hookrightarrow \{\tilde{U}', \mathcal{G}', \varphi'\}$ be an embedding. Let $g \in \mathcal{G}$. Since $g : \tilde{U} \hookrightarrow \tilde{U}$ is an embedding of $\{\tilde{U}, \mathcal{G}, \varphi\}$ into itself, λg is an embedding. By Lemma 2.3.9, there is $g' \in \mathcal{G}'$ such that $\lambda g = g' \lambda$.
Now suppose $\mathcal{G}'$ acts effectively. Let $g \in \mathcal{G}$. We have λg is an embedding. By Lemma 2.3.9, there is a unique $g' \in \mathcal{G}'$ such that $\lambda g = g' \lambda$. Now we show that λ gives us an injective group homeomorphism. Let $g_1, g_2 \in \mathcal{G}$. We want to show that $\lambda(g_1 g_2) = \lambda(g_1)\lambda(g_2)$. We have that $\varphi' \lambda = \varphi$. Then we get $\varphi' \lambda(g_1 g_2) = \varphi(g_1 g_2)$. Howevere, φ is a homeomorphism. So, we get that $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$. So, we have that $\varphi(g_1)\varphi(g_2) = \varphi' \lambda(g_1)\varphi' \lambda(g_2)$. We have that $\varphi' \lambda(g_1 g_2) = \varphi' \lambda(g_1)\varphi' \lambda(g_2)$. By taking φ'^{-1} , we conclude that $\lambda(g_1 g_2) = \lambda(g_1)\lambda(g_2)$. So, λ gives us a group homeomorphism. Since the action is effective, we obtain that the group homeomorphism is injective. □

2.4 Atlases

Definition 2.4.1. Let $\mathcal{U}$ be a collection of charts $\{\tilde{U}, \mathcal{G}, \varphi\}$ for a paracompact space X, i.e., $\mathcal{U} = \{\{\tilde{U}, \mathcal{G}, \varphi\}\}$. Then $\mathcal{U}$ is an *orbifold atlas* if

- $\mathcal{U}$ is a cover for the space X. I.e.,

$$X = \bigcup_{\{\tilde{U}, \mathcal{G}, \varphi\} \in \mathcal{U}} \varphi(\tilde{U}).$$

- For any two charts $\{\tilde{U}, \mathcal{G}, \varphi\}$ and $\{\tilde{U}', \mathcal{G}', \varphi'\}$ with $x \in \varphi(\tilde{U}) \cap \varphi'(\tilde{U}')$, there exists an open subset $Z \subseteq U \cap U'$ with a smaller chart $\{\tilde{Z}, \mathcal{K}, \rho\}$ such that $x \in \rho(\tilde{Z}) = Z$.

- For any two charts $\{\tilde{U}, \mathcal{G}, \varphi\}$ and $\{\tilde{U}', \mathcal{G}', \varphi'\}$ if $\varphi'(\tilde{U}') \subseteq \varphi(\tilde{U})$ then there exists a chart embedding $\lambda : \tilde{U}' \hookrightarrow \tilde{U}$.

Remark 2.4.2. For any two charts $\{\tilde{U}, \mathcal{G}, \varphi\}$ and $\{\tilde{U}', \mathcal{G}', \varphi'\}$ with $x \in U \cap U'$ there is an open set $W \subseteq U \cap U'$ such that $x \in W$and a chart $\{\tilde{W}, \mathcal{H}, \psi\}$ with two chart embeddings $\lambda_1 : \tilde{W} \hookrightarrow \tilde{U}$ and $\lambda_2 : \tilde{W} \hookrightarrow \tilde{U}'$.

Definition 2.4.3. Given two atlases $\mathcal{U}$ and $\mathcal{U}'$, then $\mathcal{U}$ is a *refinement* of $\mathcal{U}'$ if for every chart $\{\tilde{U}, \mathcal{G}, \varphi\}$ in $\mathcal{U}$ there is a chart embedding (λ, ℓ) into a chart $\{\tilde{U}', \mathcal{G}', \varphi'\}$ in $\mathcal{U}'$. I.e., $(\lambda, \ell) : \{\tilde{U}, \mathcal{G}, \varphi\} \longrightarrow \{\tilde{U}', \mathcal{G}', \varphi'\}$

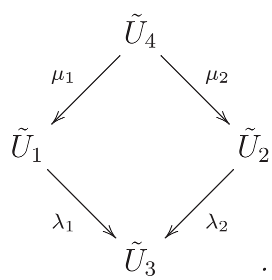
Definition 2.4.4. Two atlases are *equivalent* if they have a common refinement.

Remarks 2.4.5.

- For any orbifold atlas $\mathcal{U}$ on X there is a unique maximal atlas which contains $\mathcal{U}$.
- Let $\mathcal{U}$ and $\mathcal{U}'$ be two atlases then $\mathcal{U}$ is equivalent to $\mathcal{U}'$ if and only if they are in the same maximal atlas.

The next lemma is generalization of Lemma 4.4.1 in [18]. Pronk in [18] gives this lemma for effective group actions but, in this paper, we will prove it in general. The lemmas in the previous chapter help us to adjust the proof for the non-effective group actions.

Lemma 2.4.6. *Suppose there are charts* $\{\tilde{U}_1, \mathcal{G}_1, \varphi_1\}, \{\tilde{U}_2, \mathcal{G}_2, \varphi_2\},$ *and* $\{\tilde{U}_3, \mathcal{G}_3, \varphi_3\}$ *such that* $\tilde{U}_1$ *and* $\tilde{U}_2$ *are subcharts of* $\tilde{U}_3$ *with chart embeddings* $\lambda_1 : \tilde{U}_1 \hookrightarrow \tilde{U}_3$ *and* $\lambda_2 : \tilde{U}_2 \hookrightarrow \tilde{U}_3$ *such that* $\lambda_1(\tilde{U}_1) \cap \lambda_2(\tilde{U}_2) \neq \emptyset$*. Then there is a chart* $\tilde{U}_4 \subseteq \tilde{U}_1 \cap \tilde{U}_2$ *with chart embeddings* $\mu_1 : \tilde{U}_4 \hookrightarrow \tilde{U}_1$ *and* $\mu_2 : \tilde{U}_4 \hookrightarrow \tilde{U}_2$ *such that* $\lambda_1\mu_1 = \lambda_2\mu_2$*.*



.

Proof. Suppose there are charts $\tilde{U}_1, \tilde{U}_2$, and $\tilde{U}_3$ such that $\tilde{U}_1$ and $\tilde{U}_2 \subseteq \tilde{U}_3$ with chart embeddings $\lambda_1 : \tilde{U}_1 \hookrightarrow \tilde{U}_3$ and $\lambda_2 : \tilde{U}_2 \hookrightarrow \tilde{U}_3$. Take $x \in \lambda_1(\tilde{U}_1) \cap \lambda_2(\tilde{U}_2)$. Then there are x_1 and x_2 such that $x_i = \lambda_i^{-1}x \in \tilde{U}_i$ for $i = \{1, 2\}$. Then $\varphi_3(x) \in U_1 \cap U_2$. Then there exists $U_4 \subseteq U_1 \cap U_2$ with $y \in \tilde{U}_4$ and chart embeddings $\mu_1 : \tilde{U}_4 \hookrightarrow \tilde{U}_1$ and $\mu_2 : \tilde{U}_4 \hookrightarrow \tilde{U}_2$ such that $\mu_1(y) = x_1$ and $\mu_2(y) = x_2$. Therefore, $\lambda_1\mu_1(y) = x = \lambda_2\mu_2(y)$. So, by Lemma 2.3.9, there exists $g_3 \in \mathcal{G}_3$ such that $g_3\lambda_1\mu_1 = \lambda_2\mu_2$. We have that $\lambda_1\mu_1(y) = x = \lambda_2\mu_2(y)$. So, $\lambda_1\mu_1(\tilde{U}_4) \cap \lambda_2\mu_2(\tilde{U}_4) \neq \emptyset$. However, we have that

$g_3\lambda_1\mu_1 = \lambda_2\mu_2$. So, we obtain that $\lambda_1\mu_1(\tilde{U}_4) \cap g_3\lambda_1\mu_1(\tilde{U}_4) \neq \emptyset$. By Lemma 2.3.8, we have that $\lambda_1\mu_1 = g_3\lambda_1\mu_1$. However, $g_3\lambda_1\mu_1 = \lambda_2\mu_2$. As a result, $\lambda_1\mu_1 = \lambda_2\mu_2$.

□

2.5 Orbifolds

Definition 2.5.1. As in [21], an *orbifold* $\mathcal{Q}$ is a composite concept formed by paracompact Hausdorff space X together with an equivalence class of atlases.

Notation 2.5.2. We will write $\mathcal{Q} = (Q, \mathcal{U})$ for an orbifold $\mathcal{Q}$ which consists of the underlying space Q and an atlas $\mathcal{U}$.

Lemma 2.5.3. *Let $\mathcal{Q}$ be an orbifold with $x \in Q$. Then x has a well -defined isotropy group up to isomorphism.*

Proof. Let $\mathcal{Q}$ be an orbifold with $x \in \mathcal{Q}$. Let $\{\tilde{U}_1, \mathcal{G}_1, \varphi_1\}$ and $\{\tilde{U}_2, \mathcal{G}_2, \varphi_2\}$ be two charts such that $x \in U_1 \cap U_2$. Take $\tilde{x}_i \in \tilde{U}_i$ for $i = \{1, 2\}$ such that $\varphi(\tilde{x}_i) = x$ with isotropy groups $G_{\tilde{x}_i} = \{g_i \in \mathcal{G}_i \mid g_i(\tilde{x}_i) = \tilde{x}_i\}$. Then by Lemma 2.4.6 there is a chart $\{\tilde{W}, \mathcal{H}, \psi\}$ with $z \in \tilde{W} \subseteq \tilde{U}_1 \cap \tilde{U}_2$ and chart embeddings $(\lambda_i, \ell_i) : \tilde{W} \hookrightarrow \tilde{U}_i$ such that $\lambda_i(z) = \tilde{x}_i$. We want to show that

- if $h \in H_z$ with $H_z = \{h \in \mathcal{H} \mid h(z) = z\}$, then $\ell_i(h) \in G_{\tilde{x}_i}$.
  Let $h \in H_z$. Then $\ell_i(h)(\tilde{x}_i) = \ell_i(h)(\lambda_i(z)) = \lambda_i(h(z)) = \lambda_i(z) = \tilde{x}_i$. So, ℓ_i send H_z to $G_{\tilde{x}_i}$.

- The map $\ell_i|_{H_z} : H_z \longrightarrow G_{\tilde{x}_i}$ is surjective.
  Let $g_i \in G_{\tilde{x}_i}$. We want to show that there ia an $h \in H_z$ such that $\ell_i|_{H_z}(h) = g_i$ for $g_i \in \mathcal{G}_i$. Since $\lambda_i(\tilde{W})$ is a connected component of $\varphi_i^{-1}(\tilde{W})$ in $\tilde{U}_i$ and since $\tilde{x}_i \in g_i(\lambda_i(\tilde{W})) \cap \lambda_i(\tilde{W})$ we have $g_i(\lambda_i(\tilde{W})) = \lambda_i(\tilde{W})$. Since the following diagram commutes,

$$\begin{array}{ccc} \tilde{W} & \xrightarrow{\lambda_i} & \tilde{U}_i \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi_i} \\ W & \xrightarrow[i]{} & U \end{array},$$

  by Corollary 2.3.11, there is an $h \in \mathcal{H}_{\ddagger}$ such that $\lambda_i(h) = g_i\lambda_i$. If the action is effective this implies that $\ell(h) = g_i$. If it is not, then $\ell(h)g^{-1} \in ker\ (\mathcal{G}_i)$.

We have that ℓ induces an isomorphism. Then we have an isomorphism from *ker* $(\mathcal{H})$ to *ker* $(\mathcal{G}_i)$. It follows that there is an element $h' \in$ *ker* $\mathcal{H}$ such that $\ell(h') = g_i$. As a result, the map $\ell_i|_{H_z} : H_z \longrightarrow G_{\tilde{x}_i}$ is surjective.

□

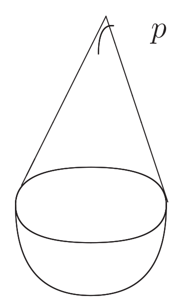
Definition 2.5.4. For an orbifold $\mathcal{Q} = (Q,\mathcal{U})$ the *singular set* is

$$\sum(\mathcal{Q}) = \{x \in Q \mid G_x \neq 1\}$$

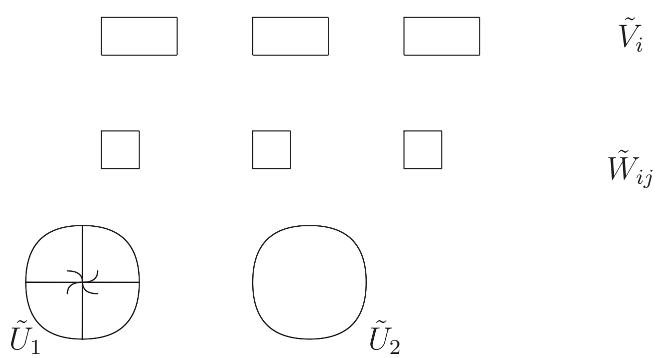
.

Examples 2.5.5. • The orbifold $\mathcal{Q} = Z_p$-*teardrop* or p-*teardrop*

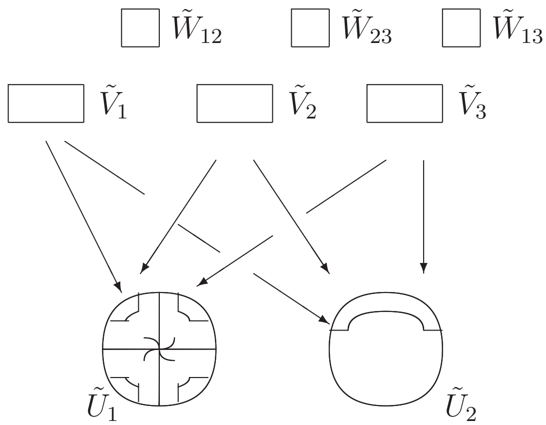
The underlying space is the sphere S^2, and there is a chart with $\mathcal{G} = Z_p$ which acts by rotation over $2\pi/p$ around the north pole. Then the underlying space is S^2 with a singular point at the north pole of order p.



If we choose $p = 4$ we will have the following charts: $\tilde{U}_1$ with the group action $\mathcal{G}_1 = Z_4$, $\tilde{U}_2$ with $\mathcal{G}_2 = \{\text{id}\}$. Also we have $\tilde{V}_i$ for $i = \{1, 2, 3\}$ with trivial group actions. Finally, we have $\tilde{W}_{ij}$ which is the intersection of $\tilde{V}_i$ with $\tilde{V}_j$. The following figure will have these charts.



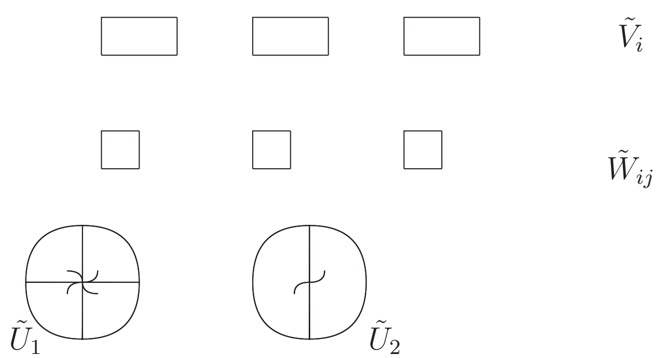
Now we will see the chart embeddings from $\tilde{V}_1$ into $\tilde{U}_1$ and $\tilde{U}_2$. It will be the same idea for $\tilde{V}_2$ and $\tilde{V}_3$. There will be four copies of $\tilde{V}_1$ in $\tilde{U}_1$ since $\mathcal{G}_1 = \mathbb{Z}_4$ and one copy of $\tilde{V}_1$ in $\tilde{U}_2$ because $\mathcal{G}_2 = \{id\}$. There are a certain number of chart embeddings. So, we get the following:



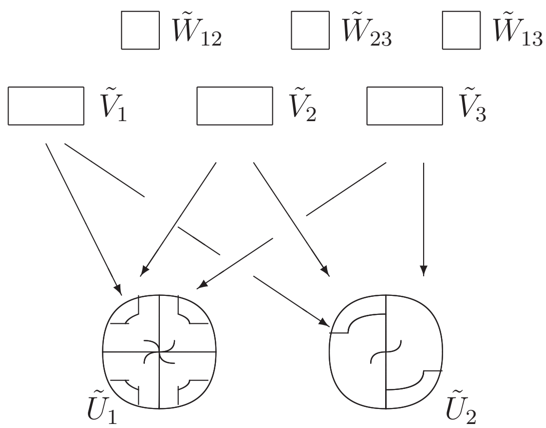
- The orbifold $\mathcal{Q} = Z_p$-Z_q-*football* or (p,q)-*spindle*

The space is S^2 and there is a chart with $\mathcal{G}_1 = Z_p$ which acts by rotation over $\frac{2\pi}{p}$ around the north pole. Also a chart with $\mathcal{G}_2 = Z_q$ which acts around the south pole by rotation $\rho = \frac{2\pi}{q}$. The quotient space is S^2 with two cone points. One cone point of order p and the other of order q.

If we have $p = 4$, and $q = 2$ we will have the following charts: $\tilde{U}_1$ with the group $\mathcal{G}_1 = Z_4$, $\tilde{U}_2$ with $\mathcal{G}_2 = Z_2$. Also we have $\tilde{V}_i$ for $i = \{1, 2, 3\}$ with trivial groups. Finally, we have $\tilde{W}_{ij}$ which is the intersection of $\tilde{V}_i$ with $\tilde{V}_j$. The following figure will have these charts.

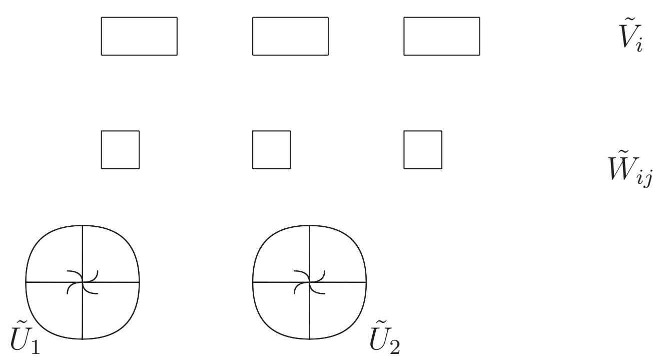


Now we will see the chart embeddings from $\tilde{V}_1$ into $\tilde{U}_1$ and $\tilde{U}_2$. It will be the same idea for $\tilde{V}_2$ and $\tilde{V}_3$. There will be four copies of $\tilde{V}_1$ in $\tilde{U}_1$ since $\mathcal{G}_1 = \mathbb{Z}_4$ and two copies of $\tilde{V}_1$ in $\tilde{U}_2$ because $\mathcal{G}_2 = \mathbb{Z}_2$. We get the following:

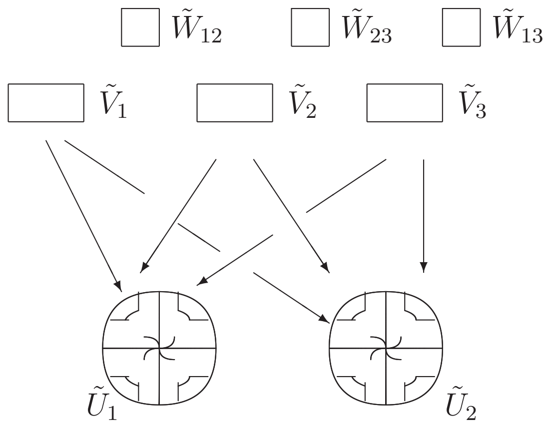


- The orbifold $\mathcal{Q} = Z_p$-*football* or p-*spindle* which is a special case of the previous example when $q = p$.

  If we have $p = q = 4$, we will have the following charts $\tilde{U}_1$ and $\tilde{U}_2$ with the groups $\mathcal{G}_1 = \mathcal{G}_2 = Z_4$. Also we have $\tilde{V}_i$ for $i = \{1, 2, 3\}$ with trivial groups. Finally, we have $\tilde{W}_{ij}$ which is the intersection of $\tilde{V}_i$ with $\tilde{V}_j$. The following figure will have these charts.



Now we will see the chart embeddings from $\tilde{V}_1$ into $\tilde{U}_1$ and $\tilde{U}_2$. It will be the same idea for $\tilde{V}_2$ and $\tilde{V}_3$. There will be four copies of $\tilde{V}_1$ in $\tilde{U}_1$ and $\tilde{U}_2$ since $\mathcal{G}_1 = \mathcal{G}_2 = \mathbb{Z}_4$. We get the following:



Note: these examples are orbifolds but not manifolds because $\sum(\mathcal{Q})$ consists of two points.

Chapter 3

Morphisms Between Orbifolds

There are many ways to define maps between orbifolds. In [21], Satake defined the first notion of orbifold map. However, there are other definitions for the maps between orbifolds which have stronger properties. People have called Satake's maps smooth maps. We will study these maps at the beginning of this section. Satake's notion of maps is a straightforward generalization of the notion of smooth maps between manifolds in that a smooth map is a map between quotient spaces with some lifting properties with respect to the charts. For manifolds, one can also describe smooth maps purely in terms of maps between atlases, but in that case one needs to talk about equivalence classes of such maps. When we generalize the second description to orbifolds, we obtain a stronger notion of morphisms called strong maps or good maps, which we will introduce here. It turns out that for orbifold homotopy theory, we need this stronger notion of maps as was obtained in [12] and [4] for instance. Smooth maps do not carry enough information to distinguish between for instance paths in fine moduli spaces as we will see in Chapter 7. That means we need to consider a notion of morphisms that carries more information. It turns out that the strong maps do carry enough information. In order to study these better we will introduce orbifold atlas categories, and show how strong maps can be viewed as maps between them.

3.1 Weak maps

Definition 3.1.1. Let $\mathcal{Q} = (Q, \mathcal{U})$ and $\mathcal{R} = (R, \mathcal{V})$ be two orbifolds. Then a continuous map $f : Q \longrightarrow R$ is a *weak orbifold map* if for all $x \in Q$ there are charts $\{\tilde{U}, \mathcal{G}, \varphi\} \in \mathcal{U}$, and $\{\tilde{V}, \mathcal{H}, \psi\} \in \mathcal{V}$ such that $x \in \varphi(\tilde{U})$ and $f(x) \in \psi(\tilde{V})$ and there is a continuous lifting map $\tilde{f}$ from $\tilde{U}$ into $\tilde{V}$ with $\psi \circ \tilde{f} = f \circ \varphi$, as in the following commutative square,

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{f}} & \tilde{V} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ U & \xrightarrow[f]{} & V\,. \end{array}$$

Remarks 3.1.2.
- This kind of map is usually called a *smooth map* between orbifolds as in [4] when we require that $\tilde{f} : \tilde{U} \longrightarrow \tilde{V}$ is smooth.
- We will call this type of map a weak map.
- Since we do not require that the orbifold atlas forms a basis for the topology of Q, we may need to take a refinement of $\mathcal{U}$ to obtain small enough charts for $\mathcal{Q}$ that will map into the charts in $\mathcal{V}$.

Definition 3.1.3. Let $\mathcal{Q} = (Q, \mathcal{U})$ and $\mathcal{R} = (R, \mathcal{V})$ be two orbifolds. Then $\mathcal{Q}$ and $\mathcal{R}$ are *homeomorphic*, if there are weak maps $f : Q \longrightarrow R$ and $g : R \longrightarrow Q$ such that $f \circ g = \mathrm{id}_R$ and $g \circ f = \mathrm{id}_Q$.

Satake's definition of maps between orbifolds does not necessarily give rise to a map between atlases. Our notion of strong maps below will do that. Then we will prove that a strong map is also a weak map, but the converse is not true in general.

3.2 Strong Maps

Definition 3.2.1. Let $\mathcal{Q} = (Q, \mathcal{U})$ and $\mathcal{R} = (R, \mathcal{V})$ be two orbifolds. A *map of atlases* $f : \mathcal{U} \longrightarrow \mathcal{V}$ consists of

1. For any chart $\{\tilde{U}, \mathcal{G}, \varphi\} \in \mathcal{U}$ there is a continuous map $f_U : \tilde{U} \longrightarrow \tilde{V}$ into a chart $\{\tilde{V}, \mathcal{H}, \psi\} \in \mathcal{V}$.
2. For any chart embedding $\lambda : \tilde{U}_1 \longrightarrow \tilde{U}_2 \in \mathcal{U}$ there is a chart embedding $f(\lambda) \in \mathcal{V}$ such that the following square commutes,

$$\begin{array}{ccc} \tilde{U}_1 & \xrightarrow{f_{U_1}} & \tilde{V}_1 \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle f(\lambda)} \\ \tilde{U}_2 & \xrightarrow[f_{U_2}]{} & \tilde{V}_2\,, \end{array}$$

   and $f(\lambda_1) \circ f(\lambda_2) = f(\lambda_1 \circ \lambda_2)$ and $f(\mathrm{id}) = \mathrm{id}_f$.

Definition 3.2.2. Let $\mathcal{Q} = (Q, \mathcal{U})$ and $\mathcal{R} = (R, \mathcal{V})$ be two orbifolds. A *strong map of orbifolds* $f : \mathcal{Q} \longrightarrow \mathcal{R}$ is represented by a refinement $\mathcal{U}'$ of $\mathcal{U}$ with a map of atlases $f' : \mathcal{U}' \longrightarrow \mathcal{V}$.

Note that a weak map of orbifolds with a weak inverse can be turned into a strong map with a strong inverse. We will define strong maps in terms of atlas gropuoids and define the equivalence relation on these maps in Chapter 6.

Proposition 3.2.3. *Let $\mathcal{Q} = (Q, \mathcal{U})$ and $\mathcal{R} = (R, \mathcal{V})$ be orbifolds. Any map of atlases $f = (f_U, f(\lambda)) : \mathcal{U} \longrightarrow \mathcal{V}$ induces a weak map of orbifolds.*

Proof. The only thing we need to prove is that there is a well defined continuous map $\bar{f} : Q \longrightarrow R$ such that for each $U \in \mathcal{U}$ the following square commutes.

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{f_U} & \tilde{V} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ Q & \xrightarrow[\bar{f}]{} & R\,, \end{array}$$

where f_U is a component of f. Let $x \in U \subseteq Q$ and let $\tilde{x} \in \varphi^{-1}(x)$. We define $\bar{f}(x) = \psi f_U(\tilde{x})$. We want to show that $\bar{f}(x)$ does not depend on the choice of $\tilde{U}$ and $\tilde{x}$. Take $\tilde{U}'$ another chart with $\tilde{x}' \in \tilde{U}'$ with $f_{U'} : \tilde{U}' \longrightarrow \tilde{V}'$ where $\psi' : \tilde{V}' \longrightarrow R$. Since we have $x \in U \cap U'$ there is $W \subseteq U \cap U'$ with a chart $\{\tilde{W}, \mathcal{H}, \tau\}$ and a point $\tilde{y} \in \tilde{W}$ with chart embeddings $\lambda_1 : \tilde{W} \longrightarrow \tilde{U}$ and $\lambda_2 : \tilde{W} \longrightarrow \tilde{U}'$ such that $\lambda_1(\tilde{y}) = \tilde{x}$ and $\lambda_2(\tilde{y}) = \tilde{x}'$. Let $\{\tilde{S}, \mathcal{K}, \sigma\}$ be the chart in $\mathcal{V}$ with continuous map $f_W : \tilde{W} \longrightarrow \tilde{S}$. As a result from Condition 2 in the definition of map of atlases there is an embedding $f(\lambda_1) : \tilde{S} \longrightarrow \tilde{V}$ for which the following square commutes.

$$\begin{array}{ccc} \tilde{W} & \xrightarrow{f_W} & \tilde{S} \\ {\scriptstyle\lambda_1}\downarrow & & \downarrow{\scriptstyle f(\lambda_1)} \\ \tilde{U} & \xrightarrow[f_U]{} & \tilde{V}\,. \end{array}$$

Consequently, we have $f(\lambda_1) f_W(\tilde{y}) = f_U(\tilde{x})$ because $f_U(\lambda_1(\tilde{y})) = f_U(\tilde{x})$. From the definition of an orbifold embedding we get

$$\sigma(f_W(\tilde{y})) = \psi(f_U(\tilde{x})). \tag{3.1}$$

Likewise for λ_2 we will have $f(\lambda_2) f_W(\tilde{y}) = f_{U'}(\tilde{x}')$ and

$$\sigma(f_W(\tilde{y})) = \psi'(f_{U'}(\tilde{x}')). \tag{3.2}$$

From the equations (3.1) and (3.2) we get that $\psi(f_U(\tilde{x})) = \psi'(f_{U'}(\tilde{x}'))$. So, $\bar{f}$ is well defined and it makes the following diagram commute,

$$\begin{array}{ccc} \tilde{W} & \xrightarrow{f_W} & \tilde{S} \\ {\scriptstyle \lambda_1}\downarrow & & \downarrow{\scriptstyle f(\lambda_1)} \\ \tilde{U} & \xrightarrow[f_U]{} & \tilde{V}\,. \end{array}$$

□

Remark 3.2.4. For atlas maps of manifolds, there is a notion of equivalence which roughly says that two atlas maps are equivalent if they agree on a common refinement. We want to generalize this to orbifolds and would call two morphisms *equivalent* if there is a common refinement so that the maps agree on the refinement. Generally, we will use a slightly weaker notion of equivalence. To make the notion of equivalence precise we will need a description of strong maps in terms of morphisms between atlas groupoids and the natural transformations between maps of atlas groupoids. We will give a precise definition of this notion in Chapter 7.

3.3 Orbifold Atlas Categories

We are going to recall the notion of an internal category and then we will use it to define orbifold atlas categories.

Definition 3.3.1. Let $\mathcal{A}$ to be a category. An *internal category* $\mathcal{C}$ in $\mathcal{A}$ is given by:

- An object $C_0 \in \mathcal{A}$ (the object of objects).
- An object $C_1 \in \mathcal{A}$ (the object of arrows).
- Two morphisms, source and target, which are: $s, t : C_1 \longrightarrow C_0 \in \mathcal{A}$.
- A morphism $u : C_0 \longrightarrow C_1 \in \mathcal{A}$ which represents the identity arrows.
- A composition morphism $m : C_1 \times_{C_0} C_1 \longrightarrow C_1 \in \mathcal{A}$ where $C_1 \times_{C_0} C_1$ is the pullback $\{(g, f) \in C_1 \times_{C_0} C_1 \mid s(g) = t(f)\}$,

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{\pi_1} & C_1 \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle s} \\ C_1 & \xrightarrow[t]{} & C_0\,. \end{array}$$

We often write $m(g, f) = g \circ f$.

These functions need to satisfy the following conditions:

- $s \circ u = \mathrm{id}_{C_0} = t \circ u$,

$$\begin{array}{ccc} C_0 & \xrightarrow{u} & C_1 \\ & \searrow^{id} & \downarrow{s} \\ & & C_0\,, \end{array} \qquad \begin{array}{ccc} C_0 & \xrightarrow{u} & C_1 \\ & \searrow^{id} & \downarrow{t} \\ & & C_0\,. \end{array}$$

- $s \circ m = s \circ \pi_2$ and $t \circ m = t \circ \pi_1$,

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{m} & C_1 \\ \downarrow{\pi_2} & & \downarrow{s} \\ C_1 & \xrightarrow[s]{} & C_0\,, \end{array}$$

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{m} & C_1 \\ \downarrow{\pi_1} & & \downarrow{t} \\ C_1 & \xrightarrow[t]{} & C_0\,. \end{array}$$

- $m \circ (u \times \mathrm{id}) = \pi_2$ and $m \circ (\mathrm{id} \times u) = \pi_1$,

$$\begin{array}{ccc} C_0 \times_{C_0} C_1 & \xrightarrow{u \times id} & C_1 \times_{C_0} C_1 \\ & \searrow^{\pi_2} & \downarrow{m} \\ & & C_1\,, \end{array}$$

$$\begin{array}{ccc} C_1 \times_{C_0} C_0 & \xrightarrow{id \times u} & C_1 \times_{C_0} C_1 \\ & \searrow^{\pi_1} & \downarrow{m} \\ & & C_1\,. \end{array}$$

- $m \circ (m \times \mathrm{id}) = m \circ (\mathrm{id} \times m)$,

$$\begin{array}{ccc} G_1 \times_{G_0} G_1 \times_{G_0} G_1 & \xrightarrow{m \times id_{G_1}} & G_1 \times_{G_0} G_1 \\ \downarrow{id_{G_1} \times m} & & \downarrow{m} \\ G_1 \times_{G_0} G_1 & \xrightarrow[m]{} & G_1\,. \end{array}$$

Note that if $\mathcal{A} = \mathrm{SET}$, the category of sets and functions, we get our usual definition of a (small) category. Therefore, the notion of an internal category is a generalization of the notion of a category.

Definition 3.3.2. Let $\mathcal{Q} = (Q, \mathcal{U})$ be an orbifold. We define the *orbifold atlas category* $C(\mathcal{U})$ as an internal category in TOP as follows:

- The space of objects is
$$C(\mathcal{U})_0 = \coprod_{\tilde{U} \in \mathcal{U}} \tilde{U}.$$

- The space of arrows is
$$C(\mathcal{U})_1 = \coprod_{\lambda:\tilde{U}_i \to \tilde{U}_j} \tilde{U}_i = \bigcup_{\lambda:\tilde{U}_i \to \tilde{U}_j} \{\lambda\} \times \tilde{U}_i = \bigcup_{\lambda:\tilde{U}_i \to \tilde{U}_j} \{(\lambda, x) | x \in \tilde{U}_i\}.$$

  With the disjoint union topology, where (λ, x) denotes the point $x \in \tilde{U}_i$ in the copy of $\lambda : \tilde{U}_i \to \tilde{U}_j$ and $\{\lambda\} \times \tilde{U}_i$ denotes the part for $\lambda : \tilde{U}_i \to \tilde{U}_j$.

For $\lambda : \tilde{U}_i \to \tilde{U}_j$ and $x \in \tilde{U}_i$, the structure maps are defined as follows:

- The source map $s : C(\mathcal{U})_1 \longrightarrow C(\mathcal{U})_0$ is defined as $s(\lambda, x) = x \in \tilde{U}_i$.

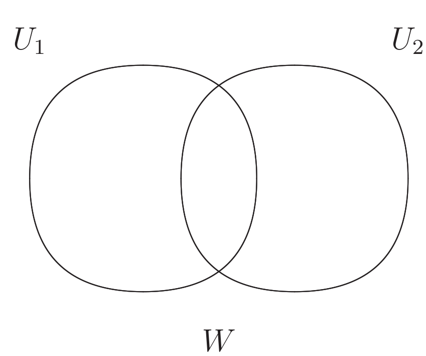
- The target map $t : C(\mathcal{U})_1 \longrightarrow C(\mathcal{U})_0$ is defined as $t(\lambda, x) = \lambda(x) \in \tilde{U}_j$.

- The identity arrow $u : C(\mathcal{U})_0 \longrightarrow C(\mathcal{U})_1$ is defined as $u(x) = (\mathrm{id}_{\tilde{U}_i}, x)$ for $x \in \tilde{U}_i$.

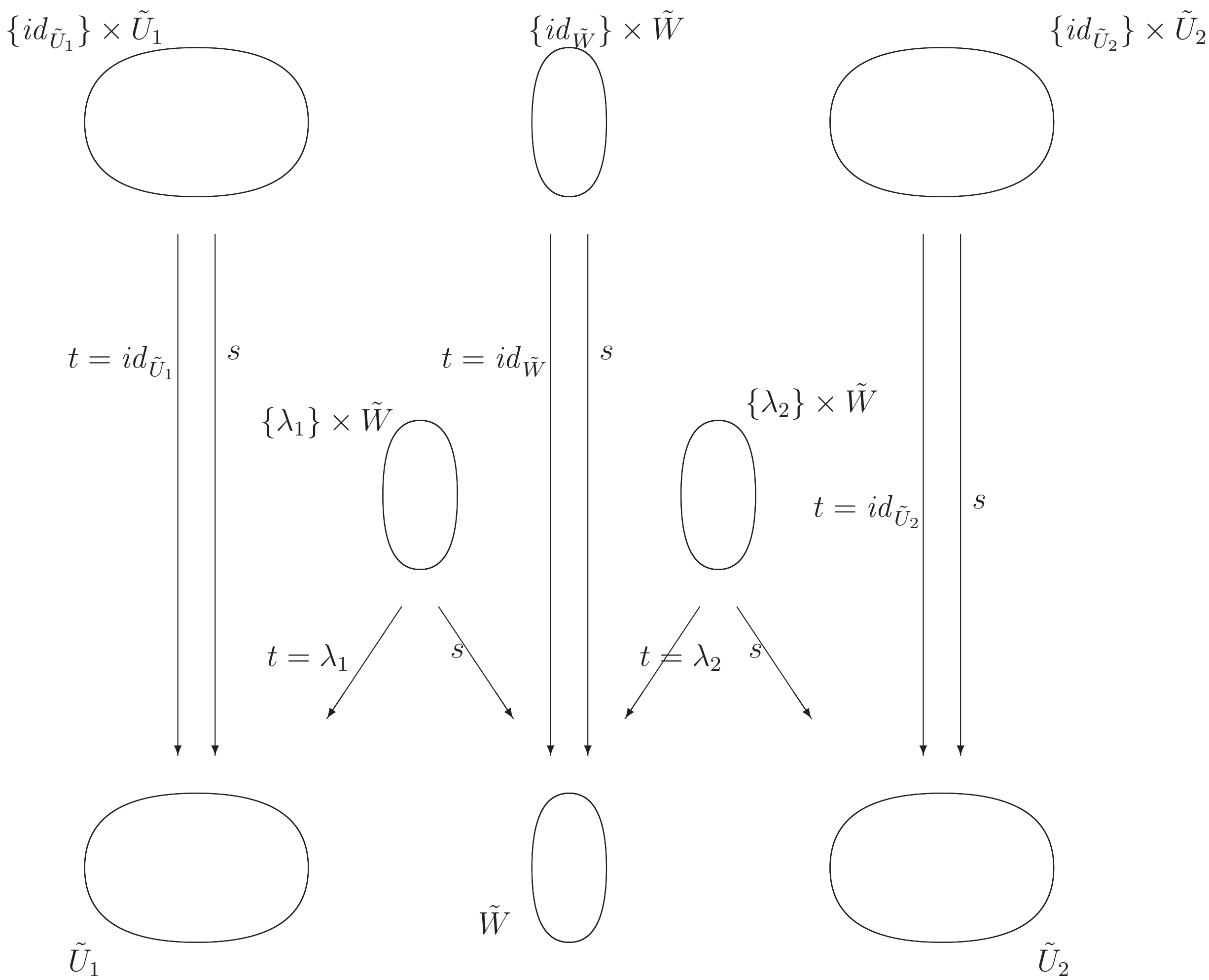
- Composition $m : C(\mathcal{U})_1 \times_{C\mathcal{U}_0} C(\mathcal{U})_1 \longrightarrow C(\mathcal{U}_1)$ is defined as: for any $\lambda : \tilde{U}_i \to \tilde{U}_j$ and $\mu : \tilde{U}_j \to \tilde{U}_k$ where $t(\lambda, x) = s(\mu, y)$ we have that $\lambda(x) = y$ and hence we define $m((\mu, y), (\lambda, x)) = (\mu\lambda, x)$.

It is clear that this satisfies the internal category conditions from Definition 3.3.1.

Example 3.3.3. Let $\mathcal{Q}$ be an orbifold with an atlas covering of three charts and trivial groups: $U_1, U_2 \subseteq \mathcal{Q}$ opens with $W = U_1 \cap U_2$ as follows



Then we have the atlas $\mathcal{U} = \{\tilde{U}_1, \tilde{U}_2, \tilde{W}\}$. Then we have $\tilde{U}_1 \coprod \tilde{U}_2 \coprod \tilde{W}$ as a space of objects and $(\{\lambda_1\} \times \tilde{W}) \cup (\{\lambda_2\} \times \tilde{W}) \cup (\{\mathrm{id}_{\tilde{W}}\} \times \tilde{W}) \cup (\{\mathrm{id}_{\tilde{U}_1}\} \times \tilde{U}_1) \cup (\{\mathrm{id}_{\tilde{U}_2}\} \times \tilde{U}_2)$ as a space of arrows. We get the following diagram,



Example 3.3.4. For a single chart orbifold $\{\tilde{U}, \mathcal{G}, \varphi\}$ we have for each $g \in \mathcal{G}$ a chart embedding $g : \tilde{U} \longrightarrow \tilde{U}$. So, the space of objects of the atlas category is $\tilde{U}$ and the space of arrows is

$$\coprod_{g:\tilde{U}\to\tilde{U}} \tilde{U} = \mathcal{G} \times \tilde{U},$$

where $\mathcal{G}$ has the discrete topology. This internal category has the following structure maps,

- The source map $s : \mathcal{G} \times \tilde{U} \longrightarrow \tilde{U}$ where $s(g, x) = x$.

- The target map $t : \mathcal{G} \times \tilde{U} \longrightarrow \tilde{U}$ where $t(g, x) = gx$.

- The identity map $u : \tilde{U} \longrightarrow \mathcal{G} \times \tilde{U}$ where $u(x) = (e, x)$.

- The composition is defined as $m : (\mathcal{G} \times \tilde{U}) \times_{\tilde{U}} (\mathcal{G} \times \tilde{U}) \longrightarrow (\mathcal{G} \times \tilde{U})$ where $m((g_2, y), (g_1, x)) = (g_2 g_1, x)$ with $g_1(x) = y$ such that $((g_2, y), (g_1, x)) \in (\mathcal{G} \times \tilde{U}) \times_{\tilde{U}} (\mathcal{G} \times \tilde{U})$ with $s(g_2, y) = t(g_1, x)$.

Note that every arrow in this category has an inverse because $\mathcal{G}$ is a group. Moreover, we have the inverse map which is defined as $i : \mathcal{G} \times \tilde{U} \longrightarrow \mathcal{G} \times \tilde{U}$ where $i(g, x) = (g^{-1}, gx)$,

$$x \underset{(g^{-1},gx)}{\overset{(g,x)}{\rightleftarrows}} gx \ .$$

This satisfies the conditions to make $\mathcal{C}(\mathcal{U})$ an internal groupoid as we will see in the next chapter. $\mathcal{C}(\mathcal{U})$ is called the *translation groupoid of the action of* $\mathcal{G}$ *on* $\tilde{U}$.

We are going to recall the definition of a functor between categories. Then we will define the homomorphisms of internal categories which is similar to the functors between categories but with the continuity condition.

Definition 3.3.5. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* F from $\mathcal{C}$ to $\mathcal{D}$ is a map that satisfies the following:

- For each object $c \in \mathcal{C}$ there is an object $F(c) \in \mathcal{D}$.

- For each map $f : c_1 \longrightarrow c_2 \in \mathcal{C}$ there is a map $F(f) : F(c_1) \longrightarrow F(c_2) \in \mathcal{D}$ such that the following two conditions hold for all $c \in \mathcal{C}$:

  - $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$.
  - $F(f \circ g) = F(f) \circ F(g)$.

Definition 3.3.6. Let $\mathcal{C}$ and $\mathcal{D}$ be internal categories in *TOP*. A *homomorphism of internal categories* or an *internal functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ consists of a pair of continuous functions $F_0 : C_0 \longrightarrow D_0$ and $F_1 : C_1 \longrightarrow D_1$ such that the following diagrams commute,

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{F_1 \times F_1} & D_1 \times_{D_0} D_1 \\ {\scriptstyle m_C}\downarrow & & \downarrow{\scriptstyle m_D} \\ C_1 & \xrightarrow{F_1} & D_1\,, \end{array}$$

$$\begin{array}{ccc} C_1 & \xrightarrow{F_1} & D_1 \\ {\scriptstyle t_C}\downarrow \uparrow{\scriptstyle u_C} \downarrow{\scriptstyle s_C} & & {\scriptstyle t_D}\downarrow \uparrow{\scriptstyle u_D} \downarrow{\scriptstyle s_D} \\ C_0 & \xrightarrow[F_0]{} & D_0\,. \end{array}$$

I.e., $F_0 \circ s_C = s_D \circ F_1$, $t_D \circ F_1 = F_0 \circ t_C$, and $u_D \circ F_0 = F_1 \circ u_C$.

Example 3.3.7. A strong map between atlases $\mathcal{U} \longrightarrow \mathcal{V}$ corresponds precisely to a morphism $\mathcal{C}(\mathcal{U}) \longrightarrow \mathcal{C}(\mathcal{V})$. If a refinement $\mathcal{U}'$ of $\mathcal{U}$ is such that we have embeddings $\mu_U : \tilde{U} \hookrightarrow \tilde{V}$ such that for each chart embedding $\lambda : \tilde{U}_1 \hookrightarrow \tilde{U}_2 \in \mathcal{U}'$ that is a corresponding chart embedding $f(\lambda) : \tilde{V}_1 \hookrightarrow \tilde{V}_2 \in \mathcal{V}$ such that the following square commutes

$$\begin{array}{ccc} \tilde{U}_1 & \xrightarrow{\mu_{U_1}} & \tilde{V}_1 \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle f(\lambda)} \\ \tilde{U}_2 & \xrightarrow[\mu_{U_2}]{} & \tilde{V}_2\,, \end{array}$$

then this refinement induces a morphism $\mathcal{C}(\mathcal{U}') \longrightarrow \mathcal{C}(\mathcal{U})$ which is faithful but not an essentially surjective homomorphism (see Definition 7.1.1), and not necessarily full. We would like refinements to correspond to weak equivalences of internal categories. However, in order to accomplish that, we need to work with groupoids instead of categories. We will get an atlas groupoid out of an atlas category by applying the internal category of fractions' construction. We will then see that a general strong map of orbifolds will correspond to a diagram,

$$\mathcal{G}(\mathcal{U}) \longleftarrow \mathcal{G}(\mathcal{U}') \longrightarrow \mathcal{G}(\mathcal{V}).$$

We will see more in Chapter 7.

Chapter 4

Groupoids

Although groups and group actions form the building blocks to define the orbifold charts for instance, we will need a more general concept to describe a whole orbifold atlas. So, in this chapter we will talk about groupoids. We will define topological groupoids which are a special kind of internal categories in the category *TOP*. We will also define homomorphism between groupoids. While doing that, we will consider some examples of groupoids that represent orbifolds.

4.1 Groupoids

Definition 4.1.1. A groupoid $\mathcal{G}$ is a category where all arrows are invertible. I.e., for each $f : c \longrightarrow c' \in \mathcal{G}$ there exists an arrow $f^{-1} : c' \longrightarrow c \in \mathcal{G}$ with $c, c' \in \mathcal{G}$ such that $f \circ f^{-1} = id_c$ and $f^{-1} \circ f = id_{c'}$.

The following example is the simplest example of a groupoid. It shows that we can construct a groupoid from one object and the elements of a group.

Example 4.1.2. Let $\{\star\}$ be an object and $\mathcal{G}$ be a group. Then we can form a groupoid by making the arrows as $g : \star \longrightarrow \star$ for all $g \in \mathcal{G}$. I.e., $\star \circlearrowright g$. The composition is defined by $g \circ h = g * h$ where $*$ is the multiplication in $\mathcal{G}$. We write $G_0 = \star$ and $G_1 = \mathcal{G}$.

Example 4.1.3. Let $\Re$ be an equivalence relation and X be a set. We can make a groupoid $\mathcal{G}(\Re)$ from $\Re$ on X as follows:

1. The set of objects is X.

2. The set of arrows is $\Re$ considered as a subset of $X \times X$. There is a unique arrow $(x, x') : x \longrightarrow x'$ if and only if $(x, x') \in \Re$ with the first projection as the source map and the second projection as the target map.

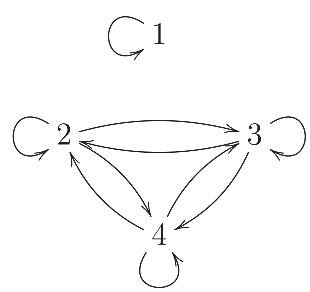
Note that $\mathcal{G}(\Re)$ is a groupoid for the following reasons:

- There are identity arrows since the relation is reflexive.
- The arrows are invertible since the relation is symmetric.
- There is a composition map since the relation is transitive.

Example 4.1.4. Let $X = \{1, 2, 3, 4\}$, and let

$$\Re = \{(1,1), (2,2), (3,3), (4,4), (2,3), (3,2), (3,4), (4,3), (2,4), (4,2)\}.$$

Then the corresponding groupoid is:



Example 4.1.5. Let $\mathcal{G}$ be a group which acts on a set X. We can form a category from this action as follows:

- The set of objects is X. I.e., we write $G_0 = X$.
- The set of arrows is the product, i.e., $G_1 = \mathcal{G} \times X$. So, for any $g \in \mathcal{G}$ we have $(g, x) : x \longrightarrow y$ where $y = g(x)$.

with the following structure maps

- The source map $s : \mathcal{G} \times X \longrightarrow X$ where $s(g, x) = x$. That means that $s = \pi_2$.
- The target map $t : \mathcal{G} \times X \longrightarrow X$ where $t(g, x) = gx$. That means that $t = f$ where f is the group action map which is defined in 2.1.1. So, we can write (g, x) as an arrow from x to gx.
- The identity map $u : X \longrightarrow \mathcal{G} \times X$ where $u(x) = (e, x)$.

- The inverse map $i : \mathcal{G} \times X \longrightarrow \mathcal{G} \times X$ where $i(g,x) = (g^{-1}, gx)$,

$$x \underset{(g^{-1},gx)}{\overset{(g,x)}{\rightleftarrows}} gx \ .$$

- Let $(G_1 \times X) \times_X (G_1 \times X)$ be the following pullback,

$$\begin{array}{ccc} (G_1 \times X) \times_X (G_1 \times X) & \xrightarrow{\pi_2} & G_1 \times X \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle t} \\ G_1 \times X & \xrightarrow[s]{} & X \end{array} \tag{4.1}$$

  Then composition is defined as $m : (\mathcal{G} \times X) \times_X (\mathcal{G} \times X) \longrightarrow (\mathcal{G} \times X)$ where $m((g_2, y), (g_1, x)) = (g_2 g_1, x)$ since $g_1(x) = y$ as $((g_2, y), (g_1, x)) \in (\mathcal{G} \times X) \times_X (\mathcal{G} \times X)$ with $s(g_2, y) = t(g_1, x)$. This satisfies:

  - Let $(f, x), (g, f(x)), (h, g(f(x))) \in G_1$ for $x \in X$. Then the composition $m((h, g(f(x)), m((g, f(x)), (f, x))) = m((h, g(f(x)), (gf, x)) = (h(gf), x) = ((hg)f, x) = m((hg, f(x)), (f, x)) = m(m(h, gf(x)), (g, f(x))), (f, x))$. As a result, $m((h, g(f(x)), m((g, f(x)), (f, x))) = m(m(h, gf(x)), (g, f(x))), (f, x))$.
  - Let $(f, x) : x \longrightarrow y$ be an arrow in G_1 with $y = f(x)$. Then $m((f, x), u(x)) = m((f, x), (e, x)) = (fe, x) = (f, x)$ and $m(u(y), (f, x)) = m((e, y), (f, x)) = (e(f), x) = (f, x)$.

This groupoid is called *the action groupoid of the action of* $\mathcal{G}$ *on a set* X . It is also called the *translation groupoid* which is the groupoid associated to the action of a single group $\mathcal{G}$ on X. We denote it by $\mathcal{G} \ltimes X$. In the next section we will introduce topological groupoids by translating the previous example in terms of topological spaces.

Proposition 4.1.6. *$(G_1 \times X) \times_X (G_1 \times X) \cong G_1 \times G_1 \times X$ and m factors through m' where $m' : G_1 \times G_1 \times X \longrightarrow G_1 \times X$ is a composition map which is defined by $(g_2, g_1, x) \mapsto (g_2 g_1, x)$.*

Proof. Let us consider the following square,

$$\begin{array}{ccc} G_1 \times G_1 \times X & \xrightarrow{(\pi_1, t(\pi_2, \pi_1))} & G_1 \times X \\ \downarrow{\scriptstyle (\pi_2, \pi_3)} & & \downarrow{\scriptstyle s} \\ G_1 \times X & \xrightarrow[t]{} & X \ . \end{array} \tag{4.2}$$

We want to prove that this square commutes. Let (g_2, g_1, x) be an element in $G_1 \times G_1 \times X$. When we calculate both sides of the composition in the square (4.2), we have that $s((\pi_1, t(\pi_2, \pi_1))(g_2, g_1, x)) = s(g_2, t(g_1, x)) = s(g_2, g_1 x) = g_1 x$. Also, we have that $t((\pi_2, \pi_3)(g_2, g_1, x)) = t(g_1, x) = g_1 x$. So, $s((\pi_1, t(\pi_2, \pi_1)) = t((\pi_2, \pi_3)$ and the square (4.2) commutes. Now we want to show that $(G_1 \times X) \times_X (G_1 \times X) \cong G_1 \times G_1 \times X$. We need to define an isomorphism φ from $(G_1 \times X) \times_X (G_1 \times X)$ to $G_1 \times G_1 \times X$. Let $\varphi : (G_1 \times X) \times_X (G_1 \times X) \longrightarrow G_1 \times G_1 \times X$ be defined as $\varphi((g_2, y), (g_1, x)) = (g_2, g_1, x)$. Let $\varphi' : G_1 \times G_1 \times X \longrightarrow (G_1 \times X) \times_X (G_1 \times X)$ be the map where $\varphi'(g_2, g_1, x) = ((g_2, g_1 x), (g_1, x))$. Now we need to see that φ and φ' are inverses to each other. Let $((g_2, y), (g_1, x)) \in (G_1 \times X) \times_X (G_1 \times X)$. Therefore, $\varphi' \circ \varphi((g_2, y), (g_1, x)) = \varphi'(g_2, g_1, x) = ((g_2, g_1 x), (g_1, x))$. Since $g_1 x = y$, we get $\varphi' \circ \varphi((g_2, y), (g_1, x)) = ((g_2, y), (g_1, x))$. Then $\varphi' \varphi = \mathrm{id}_{(G_1 \times X) \times_X (G_1 \times X)}$. Suppose $(g_2, g_1, x) \in G_1 \times G_1 \times X$. Then, $\varphi \circ \varphi'(g_2, g_1, x) = \varphi((g_2, g_1 x), (g_1, x)) = (g_2, g_1, x)$. So, $\varphi \circ \varphi' = \mathrm{id}_{G_1 \times G_1 \times X}$. We can conclude that $(G_1 \times X) \times_X (G_1 \times X) \cong G_1 \times G_1 \times X$. Now we need to check that $m = m' \varphi$. Let $((g_2, y), (g_1, x)) \in (G_1 \times X) \times_X (G_1 \times X)$. Therefore, $m' \circ \varphi((g_2, y), (g_1, x)) = m'(g_2, g_1, x) = (g_2 g_1, x)$. We can conclude that $m = m' \varphi$ which means that the composition is well defined,

$$
\begin{array}{ccc}
(G_1 \times X) \times_X (G_1 \times X) & \xrightarrow{\cong} & G_1 \times G_1 \times X \\
 & \searrow^{m} & \downarrow{\scriptstyle m'} \\
 & & G_1 \times G_1 .
\end{array}
$$

□

4.2 Topological Groupoids

A topological groupoid is an internal groupoid category in the category of topological spaces. So, we will use Definition 3.3.1 to define topological groupoids.

Definition 4.2.1. A *topological groupoid* $\mathcal{G}$ is an internal groupoid category in the category TOP with,

- A space of objects G_0.
- A space of arrows G_1.

together with the following continuous maps:

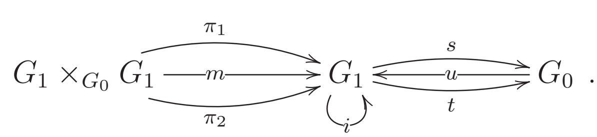
- $s : G_1 \longrightarrow G_0$ represents the source and $t : G_1 \longrightarrow G_0$ represents the target of $g \in G_1$ such that $\forall x, y \in G_0$. I.e., if $g : x \longrightarrow y$ then $s(g) = x$ and $t(g) = y$.

- $m : G_1 \times_{G_0} G_1 \longrightarrow G_1$ where $G_1 \times_{G_0} G_1 = \{(g, f) \in G_1 \times_{G_0} G_1 | s(g) = t(f)\}$ such that $m(g, f) = g \circ f$ is the composition of elements in G_1 with the following condition. For all $g_1, g_2, g_3 \in G_1$ we have $m(g_1, m(g_2, g_3)) = m(m(g_1, g_2), g_3)$ where $m(g_1, g_2) = g_1 \circ g_2$.

- $u : G_0 \longrightarrow G_1$ represents the identity arrows and $i : G_1 \longrightarrow G_1$ represents the inverse arrows which satisfy the following,

  - $s(u(x)) = x = t(u(x))$.
  - $m(g, u(s(g))) = m(u(t(g)), g)$.
  - $m(g, i(g)) = u(t(g))$.
  - $m(i(g), g) = u(s(g))$.
  - $t(i(x)) = s(x)$.
  - $s(i(x)) = t(x)$.

  So, we can form the following diagram,



Example 4.2.2. A topological group is a topological groupoid with only one object $G_0 = \{\star\}$. The arrows are $g : \star \longrightarrow \star$ for all $g \in G_1$. I.e., $\star \circlearrowright g$. The composition is defined by $g \circ h = g * h$ where $*$ is the multiplication in $\mathcal{G}$.

Example 4.2.3. Let X be a topological space. We can say that X is a topological groupoid as follows. The space of objects is X. The space of arrows is also X. Consequently, the space of composition pairs of arrows is $X \times_X X \cong X$. All the structure maps s, t, u, i, m are the identities.

Example 4.2.4. Let X be a topological space and $\mathcal{G}$ be a topological group with a continuous action of $\mathcal{G}$ on X. If we look again at the example 4.1.5, we see that the structure maps are continuous. That is because of the following:

- The map t is continuous since it is defined to be the group action f and f is continuous.
- m is continuous because $\mathcal{G}$ is a topological group and it corresponds to multiplication in $\mathcal{G}$.
- u, i are continuous because $\mathcal{G}$ is a topological group and i corresponds to the inverse in $\mathcal{G}$.

As a result, the product, identity, and inverse arrows are continuous.

4.3 Homomorphisms Between Groupoids

We will use the definition of an internal functor to define the homomorphisms between groupoids which is similar to the functors between categories but with the continuity condition.

Definition 4.3.1. Let $\mathcal{G}$ and $\mathcal{H}$ be topological groupoids. A *homomorphism* $F : \mathcal{G} \longrightarrow \mathcal{H}$ consists of a pair of continuous functions $F_0 : G_0 \longrightarrow H_0$ and $F_1 : G_1 \longrightarrow H_1$ such that the following diagram commutes. I. e., $F_0 \circ s_G = s_H \circ F_1$, $t_H \circ F_1 = F_0 \circ t_G$, and $u_H \circ F_0 = F_1 \circ u_G$,

$$\begin{array}{ccc}
G_1 \times G_1 & \xrightarrow{F_1 \times F_1} & H_1 \times H_1 \\
\downarrow{\scriptstyle m_G} & & \downarrow{\scriptstyle m_H} \\
G_1 & \xrightarrow{F_1} & H_1 \\
{\scriptstyle t_G}\downarrow \ \uparrow{\scriptstyle u_G} \ \downarrow{\scriptstyle s_G} & & {\scriptstyle t_H}\downarrow \ \uparrow{\scriptstyle u_H} \ \downarrow{\scriptstyle s_H} \\
G_0 & \xrightarrow[F_0]{} & H_0 .
\end{array}$$

The next examples show the relation between orbifold charts and homomorphisms between groupoids.

Examples 4.3.2. 1. Suppose that U and V are $\mathcal{G}$-spaces. Let $\chi : U \longrightarrow V$ be a map such that $\chi(g \cdot x) = g \cdot \chi(x)$ for all $x \in U$. This gives rise to a homomorphism of action groupoids, $\text{id} \ltimes \chi : \mathcal{G} \ltimes U \longrightarrow \mathcal{G} \ltimes V$ where $\chi : U \longrightarrow V$ acts on the objects and $\text{id} \times \chi : \mathcal{G} \times U \longrightarrow \mathcal{G} \times V$ acts on the arrows.

2. Suppose $\{\tilde{U}, \mathcal{G}, \phi\}$ and $\{\tilde{V}, \mathcal{H}, \psi\}$ are charts with a chart embedding (λ, ℓ) : $\{\tilde{U}, \mathcal{G}, \phi\} \hookrightarrow \{\tilde{V}, \mathcal{H}, \psi\}$. Then we have $\lambda(g \cdot x) = \ell(g) \cdot \lambda(x)$ for all $x \in \tilde{U}$. This chart embedding gives rise to a homomorphism of action groupoids, $\ell \ltimes \lambda$: $\mathcal{G} \ltimes U \longrightarrow \mathcal{H} \ltimes V$ which is λ on the objects and $\ell \times \lambda$ on the arrows.

We are going to define a natural transformation between topological groupoid homomorphisms, but let us first recall the following definition from category theory.

Definition 4.3.3. If $F, G : C \longrightarrow D$ are functors then, a *natural transformation* $\theta : F \longrightarrow G$ is defined by:

- For all $c \in C$ there is $\theta_c : Fc \longrightarrow Gc$, an arrow in D.

- For all $f : c \longrightarrow c' \in C$, there is a commutative square,

$$\begin{array}{ccc} Fc & \xrightarrow{\theta_c} & Gc \\ {\scriptstyle F_f}\downarrow & & \downarrow{\scriptstyle G_f} \\ Fc' & \xrightarrow[\theta_{c'}]{} & Gc' \end{array} .$$

We will now translate this definition into the language of internal groupoids in the category of topological spaces, TOP.

Definition 4.3.4. Let $F, F' : \mathcal{G} \longrightarrow \mathcal{H}$ be two homomorphisms between topological groupoids,

$$\begin{array}{ccc} G_1 & \overset{F_1}{\underset{F_1'}{\rightrightarrows}} & H_1 \\ {\scriptstyle t_G}\downarrow\uparrow{\scriptstyle u}\downarrow{\scriptstyle s_G} & & {\scriptstyle t_H}\downarrow\uparrow{\scriptstyle u}\downarrow{\scriptstyle s_H} \\ G_0 & \overset{F_0}{\underset{F_0'}{\rightrightarrows}} & H_0 \end{array} .$$

A *natural transformation* $\theta : F \longrightarrow F'$ is given by a continuous function $\theta : G_0 \longrightarrow H_1$ such that $s_H \circ \theta = F_0$ and $t_H \circ \theta = F_0'$,

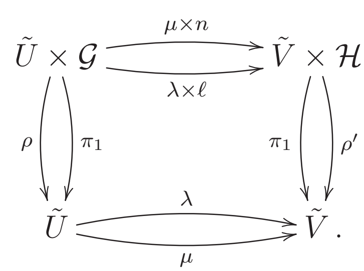
$$\begin{array}{ccc} C_0 & \xrightarrow{\theta} & H_1 \\ & {\scriptstyle F_0}\searrow & \downarrow{\scriptstyle s} \\ & & H_0 \end{array} , \qquad \begin{array}{ccc} C_0 & \xrightarrow{\theta} & H_1 \\ & {\scriptstyle F_0'}\searrow & \downarrow{\scriptstyle t} \\ & & H_0 \end{array} ,$$

with the following naturality condition $m(\theta t_G, F_1) = m(F_1', \theta s_G) : G_1 \longrightarrow H_1$. I.e., the following square commutes in TOP,

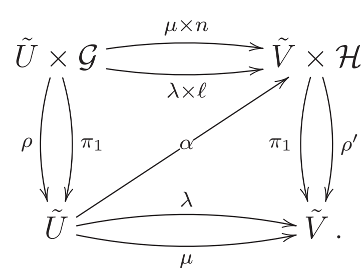
$$\begin{array}{ccc} G_1 & \xrightarrow{(\theta t_G, F_1)} & H_1 \times_{H_0} H_1 \\ {\scriptstyle (F_1', \theta s_G)} \downarrow & & \downarrow {\scriptstyle m} \\ H_1 \times_{H_0} H_1 & \xrightarrow[m]{} & H_1 , \end{array}$$

where $m : H_1 \times_{H_0} H_1 \longrightarrow H_1$ is the composition map.

Example 4.3.5. Suppose that $\{\tilde{U}, \mathcal{G}, \phi\}$ and $\{\tilde{V}, \mathcal{H}, \psi\}$ are charts such that $\mathcal{G}$ and $\mathcal{H}$ are discrete groups with chart embeddings $(\lambda, \ell), (\mu, n) : \{\tilde{U}, \mathcal{G}, \phi\} \hookrightarrow \{\tilde{V}, \mathcal{H}, \psi\}$. Then we have the following square,



Where ρ and ρ' are the group actions. Then if $h \in \mathcal{H}$ is an element such that $\mu = h \circ \lambda$, we can define a natural transformation $\alpha = (\lambda, h) : \tilde{U} \longrightarrow \tilde{V} \times \mathcal{H}$ with $\mu = h \circ \lambda$ where $h \in \mathcal{H}$. We obtain the following square,



Now we want to check the commutativity. Take $x \in \tilde{U}$ then $\rho'\alpha(x) = \rho'(\lambda, h)(x) = \rho'(\lambda(x), h) = h \cdot (\lambda(x)) = \mu(x)$. Also, $\pi_1\alpha(x) = \pi_1(\lambda, h)(x) = \pi_1(\lambda(x), h) = \lambda(x)$. We have proved that there is $h \in \mathcal{H}$ such that $\mu = h \circ \lambda$ gives rise to a natural transformation. Now we want to check the naturality. Let $x \in \tilde{U}$ and $g \in \mathcal{G}$. We want to check that the following diagram commutes in $\tilde{V} \ltimes \mathcal{H}$.

$$\begin{array}{ccc} \lambda(x) & \xrightarrow{\alpha(x)} & \mu(x) \\ {\scriptstyle (\lambda(x), \ell(g))} \downarrow & & \downarrow {\scriptstyle (\mu(x), n(g))} \\ \lambda(gx) & \xrightarrow[\alpha(gx)]{} & \mu(gx) . \end{array}$$

We have

$$\begin{aligned} m((\mu(x), n(g)), \alpha(x)) &= m((\mu(x), n(g)), (\lambda(x), h)) \\ &= n(g) \cdot \mu(x) \\ &= \mu(gx). \end{aligned}$$

and we have that

$$\begin{aligned} m(\alpha(gx), (\lambda(x), \ell(g))) &= m((\lambda(gx), h), (\lambda(x), \ell(g))) \\ &= (\lambda(gx), h) \\ &= \mu(gx). \end{aligned}$$

So, $m(\mu(x), n(g), \alpha(x)) = m(\alpha(gx), (\lambda(x), \ell(g)))$.

As a result, the previous square commutes. We have that $\tilde{U}$ and $\tilde{V}$ are connected and $\mathcal{H}$ is discrete group. Since the space $\tilde{V} \times \mathcal{H}$ has disconnected copies of $\tilde{V}$ for all $h \in \mathcal{H}$, and since the space is continuous, we can not jump from one copy to another. So, for any λ, we need an $h \in \mathcal{H}$ to defined the corresponding copy in $\tilde{V} \times \mathcal{H}$. We can conclude that each natural transformation between $\ell \ltimes \lambda$ and $m \ltimes \mu$ corresponds to precisely one $h \in \mathcal{H}$ such that $h\lambda = \mu$.

Remark 4.3.6. 1. Suppose we have two natural transformations $\theta : F \longrightarrow F'$ and $\theta' : F' \longrightarrow F''$ such that F, F', and F'' are arrows from $\mathcal{G}$ to $\mathcal{H}$. Then there are families of arrows $\theta_c : F_c \longrightarrow F'_c$ and $\theta'_c : F'_c \longrightarrow F''_c \in H_1$ such that $m(\theta_c, \theta'_c) = (\theta \circ \theta')_c$. So, they give a natural transformation from F to F''.

2. By using (1) we can conclude that every natural transformation between groupoids is an isomorphism. Namely, let $\theta : F \longrightarrow F'$ with $\theta : G_0 \longrightarrow H_1$ be a natural transformation. Then $i\theta : F \longrightarrow F'$ with $i\theta : G_0 \longrightarrow H_1$ is another natural transformation such that their composites are $m(i\theta, \theta) = \mathrm{id}_F$, and $m(\theta, i\theta) = \mathrm{id}_{F'}$.

Chapter 5

The Category of Fractions

We have already seen that we need an extra condition on an atlas refinement to obtain a morphism of atlas categories, and in general, this morphism is not full. We will solve this issue by replacing the atlas categories by atlas groupoids. We obtain these groupoids by an internal version of the category of fractions construction. In this chapter, we will review the classical conditions of the category of fractions which have been introduced by Gabriel and Zisman [7]. Then we will review the classical construction and spell out the details of the proofs for some of the lemmas involved because they will help us in internalizing the process in the next chapter.

5.1 Calculus of Fractions

Definition 5.1.1. Let $\mathcal{C}$ be a category, and $W \subseteq C_1$ be a class of arrows. We say that W admits a *calculus of fractions* if it satisfies the following conditions.

CF1 W contains all isomorphisms.

CF2 W is closed under composition. I.e., if $w : A \longrightarrow B$ and $w' : B \longrightarrow E$ are arrows in W then $w'w : A \longrightarrow E \in W$.

CF3 If there is a pair $(w, f) \in W \times_{t,C_0,t} C_1$, then there is a pair $(u, g) \in W \times_{s,C_0,s} C_1$ such that we have a commutative square. I.e., if there are $f : E \longrightarrow B \in C_1$, and $w : A \longrightarrow B \in W$ then there are $g : D \longrightarrow A \in C_1$, and $u : D \longrightarrow E \in W$ such that the following square commutes,

$$\begin{array}{ccc} D & \xrightarrow{g} & A \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle w} \\ E & \xrightarrow[f]{} & B\,. \end{array}$$

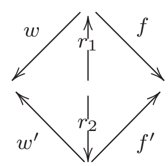
CF4 If there are $f, g : A \longrightarrow B$ and $w : B \longrightarrow E \in W$ such that $wf = wg$ then there exists $v : D \longrightarrow A \in W$ with $fv = gv$.

5.2 The Category of Fractions

Definition 5.2.1. Let $\mathcal{C}$ be a category. Let W be a class of arrows that satisfies the conditions of calculus of fractions. The *category of fractions* $C[W^{-1}]$ is defined by:

- The class of objects $C[W^{-1}]_0$ is C_0.

- The class of arrows $C[W^{-1}]_1$ consists of equivalence classes of pairs $(w, f) : A \longrightarrow B$ where $w : E \longrightarrow A$ and $f : E \longrightarrow B$ with $w \in W$ and $f \in C_1$, A, B, and $E \in C_0$, i.e., $A \xleftarrow{w} E \xrightarrow{f} B$. We say that any two pairs, (w, f) and (w', f'), are equivalent if there is a pair (r_1, r_2) such that $wr_1 = w'r_2 \in W$ with the following commutative diagram.



- To define composition of arrows in $C[W^{-1}]$ we need to make some choices. For each pair (w, f) where $f : E \longrightarrow B \in C_1$, and $w : A \longrightarrow B \in W$ choose a pair $(u, g) : A \longrightarrow E$ with $g : D \longrightarrow A \in C_1$, and $u : D \longrightarrow E \in W$ such that the following square commutes,

$$\begin{array}{ccc} D & \xrightarrow{g} & A \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle w} \\ E & \xrightarrow[f]{} & B\,. \end{array} \tag{5.1}$$

  For the pair (w, id), we choose $g = \text{id}$ and $u = w$. Similarly for the pair (id, f), we choose $u = \text{id}$ and $g = f$.

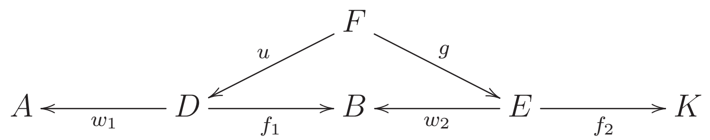
Now we can define the composition of two pairs (w_1, f_1) and (w_2, f_2) which fit in a diagram as,

$$A \xleftarrow[w_1]{} D \xrightarrow[f_1]{} B \xleftarrow[w_2]{} E \xrightarrow[f_2]{} K\ .$$

Let the following square be a chosen square as in (5.1) above,

$$\begin{array}{ccc} F & \xrightarrow{g} & E \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle w_2} \\ D & \xrightarrow[f_1]{} & B\,, \end{array}$$

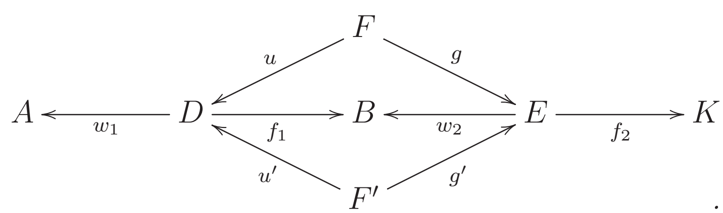
where $g : F \longrightarrow E$ and $u : F \longrightarrow D$ for $F \in C_0$. Then we have that the following diagram commutes.



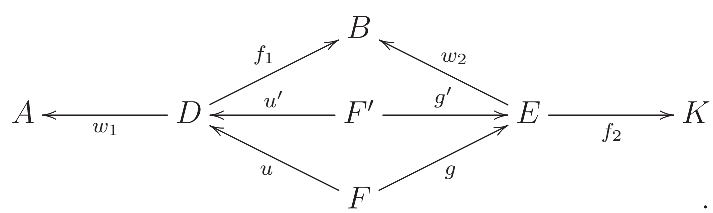
(5.2)

We can form the following definition, $(w_2, f_2) \circ (w_1, f_1) := (w_1 \circ u, f_2 \circ g)$.

Lemma 5.2.2. *This composition does not depend on the choice of the commutative square. I.e., if we have two pairs* (u, g) *and* (u', g') *with* $f_1 u = w_2 g$ *and* $f_1 u' = w_2 g'$ *then* $(w_1 \circ u, f_2 \circ g)$ *is equivalent to* $(w_1 \circ u', f_2 \circ g')$.



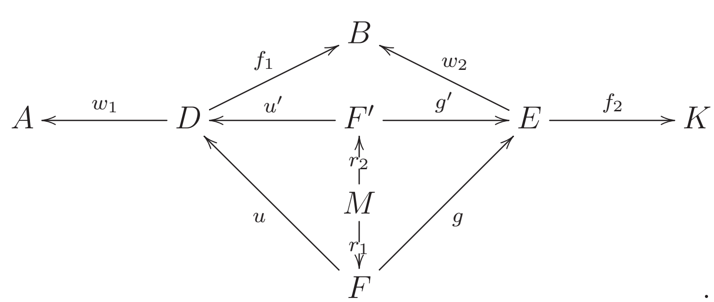
Proof. Suppose we have two pairs (u, g) and (u', g') with $f_1 u = w_2 g$ and $f_1 u' = w_2 g'$. Note that we can draw the diagram as follows,



By condition **CF3** there exists an object $M \in C_0$ with $r_1 : M \longrightarrow F$ and $r_2 : M \longrightarrow F'$ such that

$$ur_1 = u' r_2 \in W, \tag{5.3}$$

as in the following diagram,



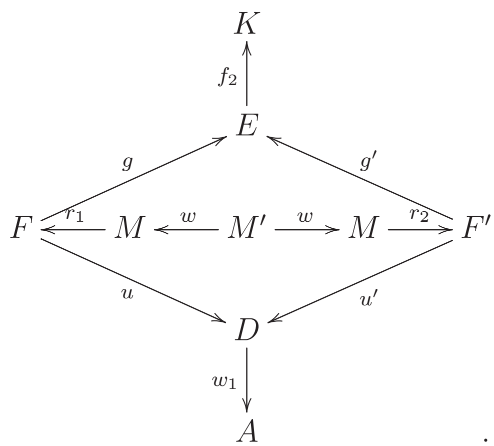
Also, since we have $w_2 g = f_1 u$ we get

$$w_2 g r_1 = f_1 u r_1. \tag{5.4}$$

From (5.3) we have that $f_1 u r_1 = f_1 u' r_2$. Since we have $w_2 g' = f_1 u'$ we get

$$w_2 g' r_2 = f_1 u' r_2. \tag{5.5}$$

Now, from (5.4) and (5.5) we have arrows $gr_1, g'r_2 : M \longrightarrow E$ and $w_2 : E \longrightarrow B \in W$ with $w_2 g r_1 = w_2 g' r_2$. By condition **CF4** there exists an arrow $w : M' \longrightarrow M \in W$ such that $g r_1 w = g' r_2 w$. As a result, we have the following commutative diagram,

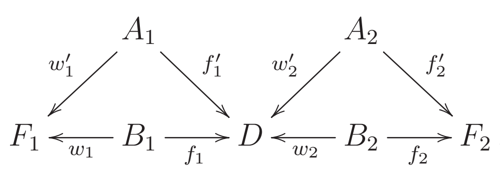


Since $w \in W$ and ur_1, $u'r_2 \in W$ also we have $ur_1 w$, $u'r_2 w \in W$, we can conclude that $(w_1 \circ u, f_2 \circ g)$ is equivalent to $(w_1 \circ u', f_2 \circ g')$. □

Composition lemma 5.2.3. The composition is well-defined on equivalence classes.

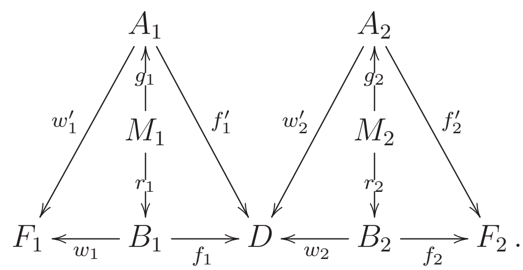
Proof. Suppose we have two pairs $(w_1, f_1) : F_1 \longrightarrow D$ and $(w_2, f_2) : D \longrightarrow F_2 \in W \times_{s,C_0,s} C_1$ where $f_1 : B_1 \longrightarrow D$, $w_1 : B_1 \longrightarrow F_1$, $f_2 : B_2 \longrightarrow F_2$, and $w_2 : B_2 \longrightarrow D$. with B_1 and B_2 objects in C_0 such that $t(w_2) = t(f_1)$ Also, we have A_1 and A_2 with $w_1' : A_1 \longrightarrow F_1$, $f_1' : A_1 \longrightarrow D$, $w_2' : A_2 \longrightarrow D$, and $f_2' : A_2 \longrightarrow F_2$. We will have the following diagram,

(5.6)



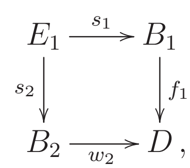
We assume that $(w_1, f_1) \sim (w_1', f_1')$ and $(w_2, f_2) \sim (w_2', f_2')$. Therefore, there exists a pair $(r_i, g_i) : A_i \longrightarrow B_i \in W \times_{s,C_0,s} C_1$ such that $r_i : M_i \longrightarrow B_i$ and $g_i : M_i \longrightarrow A_i$

with $M_i \in C_0$ where $i = \{1, 2\}$ which gives us the following commutative diagram.

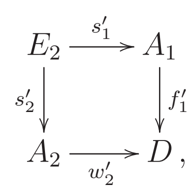


(5.7)

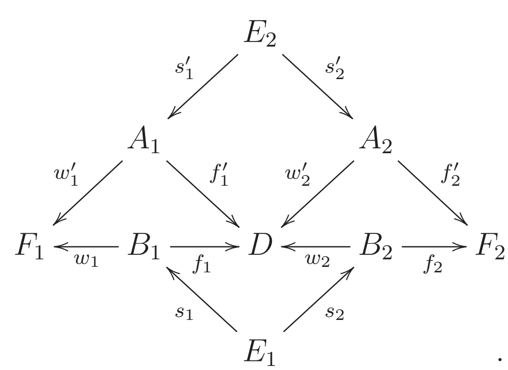
So, we have $w_i r_i = w'_i g_i$ and $f_i r_i = f'_i g_i$ where $i = \{1, 2\}$. From this diagram,we have two pairs of arrows (w_2, f_1), and (w'_2, f'_1) in $W \times_{t,C_0,t} C_1$. Let



and

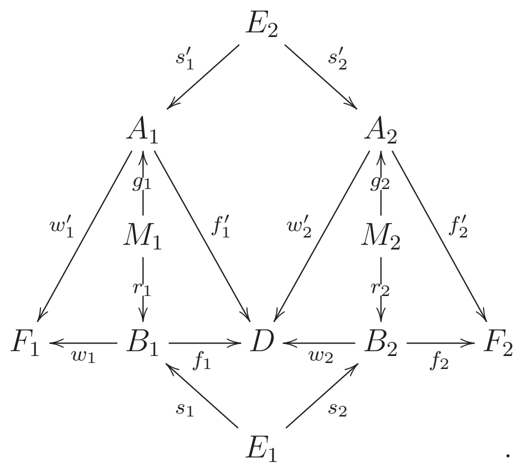


be the chosen squares such that with E_1 and $E_2 \in C_0$. So, we have that $f_1 s_1 = w_2 s_2$ and $f'_1 s'_1 = w'_2 s'_2$. We will have the following diagram,



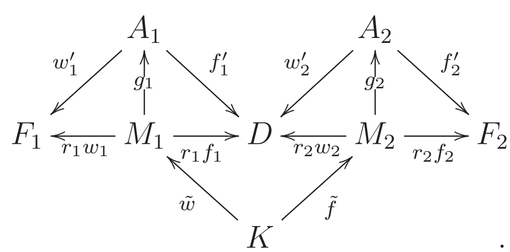
(5.8)

If we add (5.7) to (5.8), we will have the following diagram,



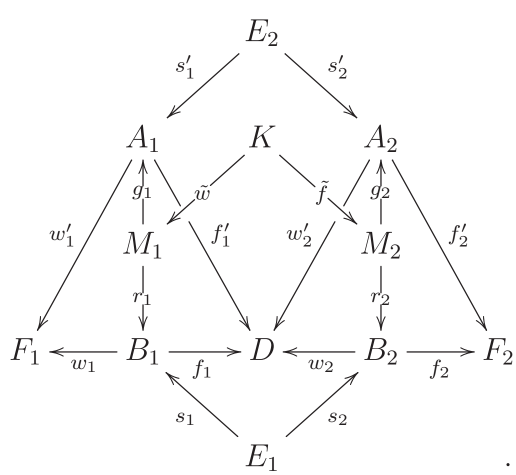
(5.9)

Then we find a new pair of arrows $(f_1r_1, w_2r_2) \in W \times_{t,C_0,t} C_1$. Therefore, by applying condition **CF3**, we get a pair $(\tilde{w}, \tilde{f})$ with $K \in C_0$, $\tilde{w} : K \longrightarrow M_1 \in W$ and $\tilde{f} : K \longrightarrow M_2$ such that $f_1r_1\tilde{w} = w_2r_2\tilde{f}$, and $f'_1g_1\tilde{w} = w'_2g_2\tilde{f}$,



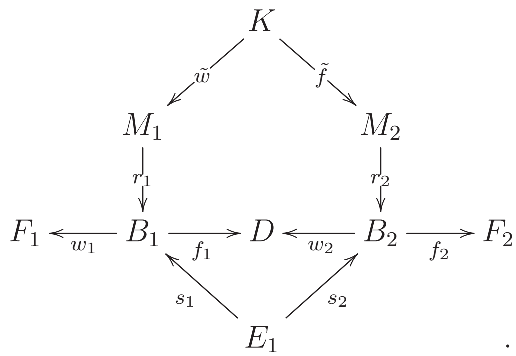
(5.10)

If we combine (5.9) and (5.10), we will have the following diagram,



(5.11)

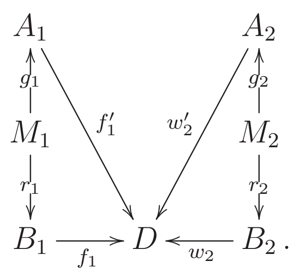
From the diagram (5.11), we obtain the following shape,



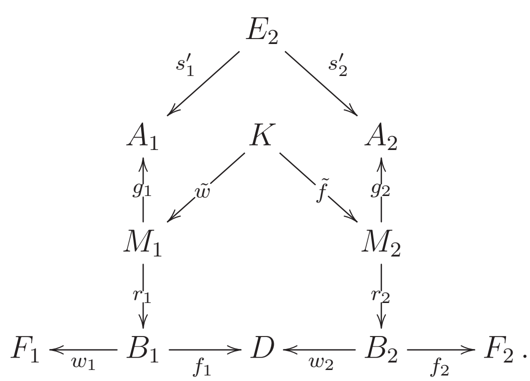
We proved in Lemma 5.2.2 that the composition does not depend on the choice of the square. If we apply this lemma in the previous diagram, we will have the following equivalence relation,

$$(w_1 s_1, f_2 s_2) \sim (w_1 r_1 \tilde{w}, f_2 r_2 \tilde{f}) \,. \tag{5.12}$$

We can see from diagram (5.11) that we have $f_i' g_i = f_i r_i$ and $w_i' g_i = w_i r_i$ which are in the following shape,



Then we get $(w_2' r_2, f_2' r_2) \circ (w_1' r_1, f_1' r_1) := (\tilde{f} w_1' g_1, \tilde{w} f_2' g_2)$. We also can see from the diagram (5.11) the following,



Then by Lemma 5.2.2 we get the following equivalence relation,

$$(w_1 r_1 \tilde{w}, f_2 r_2 \tilde{f}) \sim (w_1' s_1', f_2' s_2') \,. \tag{5.13}$$

From (5.12) and (5.13), we have $(w_1 s_1, f_2 s_2) \sim (w_1' s_1', f_2' s_2')$ as required. □

The following proposition expresses the universal property of $C[W^{-1}]$ and is immediate consequence of Lemma 1.2 and Proposition 2.4 in [7]. We will internalize it in the next chapter.

Proposition 5.2.4. *Composition with the inclusion function $J_c : C \longrightarrow C[W^{-1}]$ induses an isomorphism of categories $hom_W\ (C, D) \cong hom\ (C[W^{-1}], D)$ where $hom_W\ (C, D) \subseteq hom\ (C, D)$ is the full subcategory on the functors $C \longrightarrow D$ which send arrows in W to isomorphisms.*

Chapter 6

Topological Categories of Fractions

This chapter will generalize the category of fractions conditions and construction to internal categories in the category *TOP*. We will restrict ourselves to the case where $W = C_1$, the object of all arrows of $\mathcal{C}$. We will see that surjective local homeomorphisms play an important role in the new category of fractions' conditions. So, we will discuss their properties first.

6.1 Surjective Local Homeomorphisms

Definition 6.1.1. A function $f : X \longrightarrow Y$ is a *surjective local homeomorphism* if

- f is surjective.
- For each $x \in X$ there is an open subset U_x with $x \in U_x$ such that $f|_{U_x} : U_x \longrightarrow f(U_x)$ is a homeomorphism.

Note that every surjective local homeomorphism is a continuous open map. The following results are standard.

Proposition 6.1.2. *Surjective local homeomorphisms are closed under composition.*

Proof. Suppose $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are surjective local homeomorphisms. We want to show that $gf : X \longrightarrow Z$ is a surjective local homeomorphism.

- Since f and g are surjective, gf is surjective.
- Take $x \in X$. Since f is a surjective local homeomorphism, there exists an open subset U_x with $x \in U_x$ such that $f|_{U_x} : U_x \longrightarrow f(U_x)$ is a homeomorphism. g is also a surjective local homeomorphism. So, there is an open set $V_{f(x)}$ with $f(x) \in V_{f(x)}$ such that $g|_{V_{f(x)}} : V_{f(x)} \longrightarrow g(V_{f(x)})$ is a homeomorphism. We have $f(x) \in f(U_x) \cap V_{f(x)}$ which is open in $f(U_x)$ and in Y. Then $g|_{f(U_x)\cap V_{f(x)}} : f(U_x) \cap V_{f(x)} \longrightarrow g(f(U_x) \cap V_{f(x)})$ is a homeomorphism. Since

$f(U_x) \cap V_{f(x)} \subseteq V_{f(x)}$, we have that $f^{-1}(f(U_x) \cap V_{f(x)}) \cap U_x = U_x \cap f^{-1}V_{f(x)}$ is open in X which contains x. Since $U_x \cap f^{-1}V_{f(x)} \subseteq U_x$, we have that $f|_{U_x \cap f^{-1}V_{f(x)}} : U_x \cap f^{-1}V_{f(x)} \longrightarrow f(U_x) \cap V_{f(x)}$ is a homeomorphism. Therefore,
we get $g|_{f(U_x) \cap V_{f(x)}} \circ f|_{U_x \cap f^{-1}V_{f(x)}} = gf|_{U_x \cap f^{-1}V_{f(x)}} : U_x \cap f^{-1}V_{f(x)} \longrightarrow g(V_{f(x)} \cap f(U_x))$ is a homeomorphism because it is a composition of two homeomorphisms.

Therefore, gf is a surjective local homeomorphism. □

Proposition 6.1.3. *Surjective local homeomorphisms are stable under pullback.*

Proof. Suppose we have the following pullback with g a surjective local homeomorphism.

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{k} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

We want to show that h is a surjective local homeomorphism.

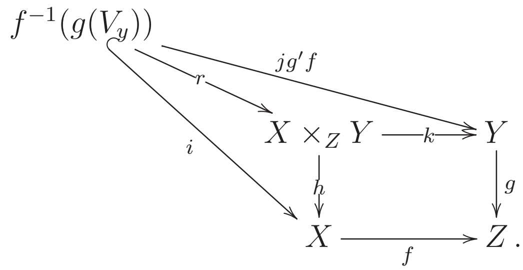
1. Let $x \in X$ then $f(x) \in Z$. However, g is surjective. Therefore, there is $y \in Y$ such that $g(y) = f(x)$. As a result, we have $(x, y) \in X \times_Z Y$ such that $h(x, y) = x$. Thus, h is surjective.

2. Let $(x, y) \in X \times_Z Y$. Then $k(x, y) = y$. The map g is a surjective local homeomorphism so, there is $V_y \subseteq Y$ an open subset containing y such that $g|_{V_y} : V_y \longrightarrow g(V_y)$ is homeomorphism. We write $g' : g(V_y) \longrightarrow V_y$ for its inverse. We have that $g(V_y)$ is open in Z and contains $g(y) = f(x)$. This means that $x \in f^{-1}(g(V_y))$. There is a continuous map $g'f : f^{-1}(g(V_y)) \longrightarrow V_y$. So, we have the following commutative square,

$$\begin{array}{ccc} f^{-1}(g(V_y)) & \xrightarrow{g'f} & V_y \\ {\scriptstyle i}\big\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z, \end{array}$$

where i is the inclusion map. Let $j : V_y \hookrightarrow Y$ be an inclusion map. However, since $X \times_Z Y$ is a pullback, there is a unique map $r : f^{-1}(g(V_y)) \longrightarrow X \times_Z Y$

such that $hr = i$ and $kr = jg'f$. I.e.,



(6.1)

We have $W_y = f^{-1}(g(V_y)) \times_Z V_y = h^{-1}(f^{-1}(g(V_y))) \cap k^{-1}(V_y) \subseteq X \times_Z Y$ is an open subset. We want to show that h restricted to this subset is a homeomorphism with image $f^{-1}(g(V_y))$. We have from the Diagram (6.1) that $hr = i$, we want to show that $rh|_{W_y} = id$. Note that $r : f^{-1}(g(V_y)) \longrightarrow h^{-1}|_{W_y}(f^{-1}(g(V_y))) \cap k^{-1}(V_y)) \subseteq X \times_Z Y$ is defined as $r(x') = (x', y'')$ where $y'' = jg'f(x')$. Let $(x', y') \in h^{-1}|_{W_y}(f^{-1}(g(V_y))) \cap k^{-1}(V_y))$. We have $rh|_{W_y}(x', y') = r(x') = (x', y'')$. However, y' and $y'' \in V_y$ with $g(y') = x' = g(y'')$. So, $y' = y''$. Therefore, $h|_{W_y}$ is a homeomorphism.

We can conclude that h is a surjective local homeomorphism. □

Proposition 6.1.4. *Each surjective local homeomorphism is the coequalizer of its kernel pair.*

Proof. Suppose $f : X \longrightarrow Y$ is a surjective local homeomorphism. Let $\ker(f) = \{(x, x') \mid f(x) = f(x')\} \subseteq X \times X$, and let $r_1, r_2 : \ker(f) \longrightarrow X$ such that $r_1(x, x') = x$ and $r_2(x, x') = x'$. We want to show that f is the coequalizer of r_1 and r_2.

- We want to show that $fr_1 = fr_2$. Take $(x, x') \in \ker(f)$. Then $f(r_1(x, x')) = f(x) = f(x') = f(r_2(x, x'))$ for $(x, x') \in \ker(f)$. So, we have $fr_1 = fr_2$.
- We are going to check the universal property of the coequalizer. We will define a map $h : Y \longrightarrow Z$ and check that it is well defined as well as that h is continuous.
  - Suppose there is a map $g : X \longrightarrow Z$ such that $gr_1 = gr_2$. We want to define $h : Y \longrightarrow Z$ such that $g = hf$. Take $y \in Y$. We have f is a surjective, so we choose $x \in X$ such that $f(x) = y$. Now we take $h(y) = g(x)$.

– We want to show that h is well defined. Suppose there are x and x' such that $x, x' \in f^{-1}(y)$. Then $(x, x') \in ker\ (f)$, so $g(x) = g(x')$. So, h well defined.

– Now we want to check that h is continuous. Take U an open subset in Z. Since g is continuous, $g^{-1}(U)$ is open in X. However, f is a surjective local homeomorphism, so $f(g^{-1}(U))$ is open in Y. We claim that $h^{-1}(U) = f(g^{-1}(U))$. Let $y \in h^{-1}(U)$. Then there is $x \in X$ such that $f(x) = y$ and $g(x) \in U$. So, $x \in g^{-1}(U)$ and $y \in f(g^{-1}(U))$. As a result, $h^{-1}(U) \subseteq f(g^{-1}(U))$. Suppose that there is $y \in f(g^{-1}(U))$. Then there is $x \in X$ such that $f(x) = y$ and $x \in g^{-1}(U)$. By the definition of h, we have that $h(y) \in U$. As a result, $f(g^{-1}(U)) \subseteq h^{-1}(U)$. Therefore, $h^{-1}(U) = f(g^{-1}(U))$. So, h is continuous.

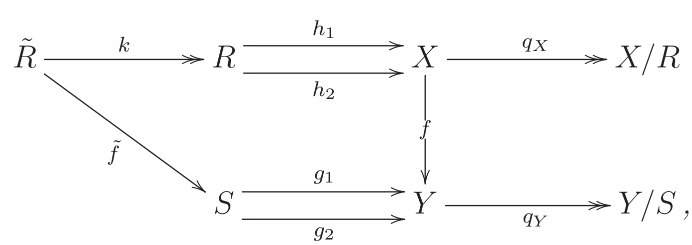
– Finally, we need to check that $h : Y \longrightarrow Z$ is unique and $g = hf$.

  * First, we want to check the equality. We have defined that $h(f(x)) = g(x)$for $x \in X$. Then we can conclude that $hf = g$.

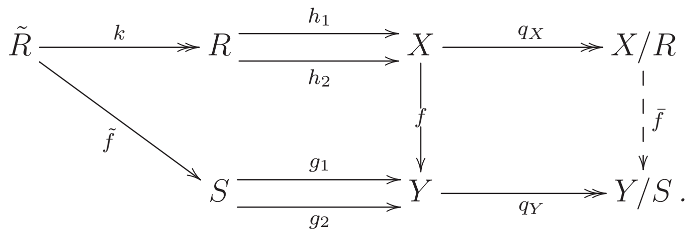
  * We need to see the uniqueness of h. Suppose there is $h' : Y \longrightarrow Z$ such that $h' \neq h$ with $g = h'f$. We have $g = hf$ so, $hf = h'f$. However, f is an epimorphism. Hence, $h = h'$.

□

Lemma 6.1.5. *In the diagram below, suppose that k is a surjective local homeomorphism, with q_X and q_Y are quotient maps. Furthermore, if there are $f : X \longrightarrow Y$ and $\tilde{f} : \tilde{R} \longrightarrow S$ continuous maps such that $fh_ik = g_i\tilde{f}$ for $i = \{1, 2\}$,*



then there is a unique continuous map $\bar{f} : X/R \longrightarrow Y/S$ such that the following diagram commutes,



Proof. Take $r \in R$. We want to show that $q_Y f h_1(r) = q_Y f h_2(r)$. Since k is surjective, there is $x \in \tilde{R}$ such that $k(x) = r$. We have that k is a surjective local homeomorphism. Then there is U_x an open subset in $\tilde{R}$ containing x such that $k|_{U_x} : U_x \longrightarrow k(U_x)$ is a homeomorphism. As a result, there is $(k|_{U_x})^{-1} : k(U_x) \longrightarrow U_x$. In addition, we have that $f h_i k(x) = g_i \tilde{f}(x)$, so $f h_i(r) = g_i \tilde{f}(k|_{U_x})^{-1}(r)$. That implies that $q_Y f h_i(r) = q_Y g_i \tilde{f}(k|_{U_x})^{-1}(r)$. However, q_Y is the coequalizer of the g_i. Therefore, $q_Y g_1 \tilde{f}(k|_{U_x})^{-1}(r) = q_Y g_2 \tilde{f}(k|_{U_x})^{-1}(r)$. We have $q_Y f h_i(r) = q_Y g_i \tilde{f}(k|_{U_x})^{-1}(r)$. Then $q_Y f h_1(r) = q_Y f h_2(r)$. So, $q_Y f h_1 = q_Y f h_2$ since r was arbitrary. Since q_X is the coequalizer of h_i, there is a unique map $\bar{f} : X/R \longrightarrow Y/S$ such that $\bar{f} q_X = q_Y f$. □

Notation 6.1.6. We will also call a surjective local homeomorphism an *étale surjection.*

6.2 The Internal Category of Fractions

We have defined an orbifold atlas category in Chapter 3. However, we want the maps induced by refinements to become weak equivalences of categories. We want a refinement to give us weak equivalences between the atlas groupoids. This can be obtained if all arrows in the atlas category are invertible. Therefore, we need an atlas groupoid instead of just an atlas category. In order to obtain a topological groupoid for an orbifold atlas, we need to add inverse maps for the arrows in $C(\mathcal{U})_1$. We will work to use the category of fractions conditions from Section 5.1 to obtain these maps in a universal way. We will need to generalize the conditions to obtain an internal version of the category of fractions which we will call the internal category of fractions, and then we need to prove that our atlas category satisfies the conditions of the internal calculus of fractions.

Since we want every arrow in C_1 to be invertible we will consider $W = C_1$ as a special case in this paper. We can denote $C[W^{-1}]$ in this case by $\mathcal{G}(C)$ to indicate that this is a groupoid, i.e., the internal groupoid in TOP such that there is an inclusion $\mathcal{C} \longrightarrow \mathcal{G}(\mathcal{C})$ which has the following universal property which we will prove at the end of this section.

Theorem 6.2.1. *Composition with the inclusion map $j_c : C \longrightarrow \mathcal{G}(C)$ gives an isomorphism $hom(\mathcal{G}(C), \mathcal{K}) \longrightarrow hom(C, \mathcal{K})$ for any topological groupoid $\mathcal{K}$.*

$$\begin{array}{ccc} C & \xrightarrow{j_c} & \mathcal{G}(C) \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle F'} \\ & & \mathcal{K}\,. \end{array}$$

Now we will generalize the conditions of the calculus of fractions for an internal category $\mathcal{C}$ in TOP. Since we take $W = C_1$ we obtain conditions **CF1** and **CF2** for free because C_1 contains all isomorphisms and is closed under composition. We only need to translate conditions **CF3** and **CF4** to construct *a groupoid of fractions for an internal category in TOP*.

6.2.1 The Topological Calculus of Fractions Conditions

TOP - CF3

Before we translate condition **CF3**, we introduce some notation which we will use later. Denote the space $C_1 \times_{t,C_0,t} C_1$ by *csp(C)*, i.e., the space encoding diagrams in C of the form

$$\longrightarrow \longleftarrow ,$$

and the space of all not necessarily commutative squares $(C_1 \times_{t,C_0,t} C_1) \times_{(s,s),C_0 \times C_0,(t,t)} (C_1 \times_{s,C_0,s} C_1)$, be *allsq(C)*. I.e., this space encodes diagrams in C of the form

$$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \cdot \end{array},$$

the subset of commutative squares of *allsq(C)* by *csq(C)*. We obtain *csq(C)* as the following equalizer diagram,

$$\text{csq(C)} \xrightarrow{j} \text{allsq(C)} \underset{m(\pi_1\pi_2,\pi_1\pi_1)}{\overset{m(\pi_2\pi_2,\pi_2\pi_1)}{\rightrightarrows}} C_1 \qquad .$$

We get that j is a topological embedding. Note that *allsq(C), csq(C)*, and *csp(C)* will be objects in *TOP*. Now recall the condition **CF3** which is that for any pair $(u, f) \in W \times_{t,C_0,t} C_1$ there exists a pair $(v, g) \in W \times_{s,C_0,s} C_1$ such that the following square commutes,

$$\begin{array}{ccc} & \xrightarrow{\;g\;} & \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ & \xrightarrow[\;f\;]{} & . \end{array}$$

Now we will generalize condition **CF3**. In the category *SET*, this is equivalent to requiring the existence of a map $\varphi' : csp(C) \longrightarrow csq(C)$ such that the following diagram commutes,

$$\begin{array}{ccc} & & csq(C) \\ & {\scriptstyle \varphi'}\nearrow & \big\downarrow{\scriptstyle j} \\ csp(C) & \xrightarrow{\;\varphi\;} & allsq(C) \\ & {\scriptstyle id}\searrow & \big\downarrow{\scriptstyle \pi_1} \\ & & csp(C)\,. \end{array}$$

However, in *TOP* to require the existence of φ' is too strong a condition which we will find is not satisfied by our atlas category. Therefore, we will require a weaker condition which says that the map $\phi = \pi_1 \circ j : csq(C) \longrightarrow csp(C)$ is an étale surjection so, there are local sections of ϕ but, we do not require a global map. We have the following commutative diagram.

$$\begin{array}{ccccc} csq(C) & \xrightarrow{\;j\;} & allsq(C) & \overset{m(\pi_2\pi_2,\pi_2\pi_1)}{\underset{m(\pi_1\pi_2,\pi_1\pi_1)}{\rightrightarrows}} & C_1 \\ {\scriptstyle \phi}\big\downarrow & \swarrow{\scriptstyle \pi_1} & & & \\ csp(C) & & & & . \end{array} \tag{6.2}$$

From here we can define condition **CF3** for the internal calculus of fractions as follows.

Top-CF3 The map $\phi : csq(C) \longrightarrow csp(C)$ is a surjective local homeomorphism.

TOP - CF4

Before we translate condition **CF4**, we introduce some notation which we will use. We will denote the space $C_1 \times_{(s,t),C_0 \times C_0,(s,t)} C_1$ that encodes pairs of parallel arrows by *pall(C)*. I.e., the space encoding diagrams of the form,

$$\rightrightarrows .$$

Note that $pall(C)\times_{t\pi_1,C_0,s}C_1$ is the space encoding diagrams of the form,

$$\rightrightarrows \longrightarrow .$$

So, we have two composition maps M_1 and M_2 from $pall(C)\times_{t\pi_1,C_0,s}C_1$ to C_1 as follows,

$$pall(C)\times_{t\pi_1,C_0,s}C_1 \underset{M_2}{\overset{M_1}{\rightrightarrows}} C_1 .$$

We define the space $CEq(C)$ to be the equalizer of M_1 and M_2. I.e.,

$$CEq(C) \longrightarrow pall(C)\times_{t\pi_1,C_0,s}C_1 \underset{M_2}{\overset{M_1}{\rightrightarrows}} C_1 .$$

We also have the space of the form $C_1 \times_{t,C_0,s\pi_1} pall(C)$, i.e., the space encoding diagrams of the form,

$$\longrightarrow \rightrightarrows .$$

Then we have two composition maps M_1' and M_2' from $C_1 \times_{t,C_0,s\pi_1} pall(C)$ as follows,

$$C_1 \times_{t,C_0,s\pi_1} pall(C) \underset{M_2'}{\overset{M_1'}{\rightrightarrows}} C_1 .$$

We define the space $Eq(C)$ to be the equalizer of M_1' and M_2'. I.e.,

$$Eq(C) \longrightarrow C_1 \times_{t,C_0,s\pi_1} pall(C) \underset{M_2'}{\overset{M_1'}{\rightrightarrows}} C_1 .$$

Note that $CEq(C)$, $Eq(C)$, and *pall(C)* are objects in *TOP*. Let us recall the classical condition **CF4**. For any two pairs $(u, f), (u, g) \in W \times_{s,C_0,t} C_1$ with $s(f) = s(g)$ such that $m(u, f) = m(u, g)$ there exists an arrow $v \in W$ with $(v, f), (v, g) \in W \times_{t,C_0,s} C_1$ such that $m(f, v) = m(g, v)$.

Now we will generalize the condition **CF4**. Let C be an internal category in *TOP*. For *SET* this is equivalent to require the existence of a map $\varphi : CEq(C) \longrightarrow Eq(C)$ such that the following diagram commutes,

$$\begin{array}{ccc} CEq(C) & \xrightarrow{\quad\varphi\quad} & Eq(C) \\ & {\scriptstyle\pi_1}\searrow \quad \swarrow{\scriptstyle\pi_2} & \\ & pall(C) & \end{array} .$$

However, this condition is too strong and we need to have a weaker condition which requires a surjective local homeomorphism. We have the space $C_1 \times_{t,C_0,s} pall(C) \times_{t,C_0,s} C_1$, of not necessarily commutative diagrams of the form,

$$\longrightarrow \rightrightarrows \longrightarrow \; .$$

There are two ways of taking a triple composite here. Therefore, we will define *P(C)* to be the following equalizer,

$$P(C) \longrightarrow C_1 \times_{t,C_0,s} pall(C) \times_{t,C_0,s} C_1 \underset{M_2}{\overset{M_1}{\rightrightarrows}} C_1 \, .$$

Then we require that we have a surjective local homeomorphism φ' from *P(C)* to *CEq(C)* which is defined by $\varphi' = (\pi_2, \pi_3)$. From here we can define the condition **CF4** as follows. **Top-CF4** The map $\varphi' : P(C) \longrightarrow CEq(C)$ in the following pullback is a surjective local homeomorphism,

$$\begin{array}{ccc} P(C) & \xrightarrow{\varphi'} & CEq(C) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ Eq(C) & \xrightarrow[\pi_1]{} & pall(C) \, . \end{array}$$

where $\phi = (\pi_1, \pi_2)$.

We have generalized the calculus of fractions' conditions to obtain the internal version of them. We have that **CF1** and **CF2** are satisfied because we took $W = C_1$. We generalize **CF3**, and **CF4** to **Top - CF3** and **Top - CF4**, the internal version of the calculus of fractions' conditions. As a result, we have the following definition.

Definition 6.2.2. Let $\mathcal{C}$ be an internal category in the category TOP, and $W = C_1$ be the space of arrows. We say that W admits an *internal calculus of fractions* if it satisfies the following conditions.

- **Top - CF3** The map ϕ in the following diagram is a surjective local homeomorphism,

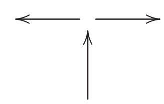
$$\begin{array}{ccccc} csq(C) & \xrightarrow{j} & allsq(C) & \underset{m(\pi_1\pi_2,\pi_1\pi_1)}{\overset{m(\pi_2\pi_2,\pi_2\pi_1)}{\rightrightarrows}} & C_1 \\ {\scriptstyle \phi}\downarrow & \swarrow{\scriptstyle \pi_1} & & & \\ csp(C) & & & & . \end{array}$$

- **Top - CF4** The map $\varphi' : P(C) \longrightarrow CEq(C)$ in the following pullback is a surjective local homeomorphism,

$$\begin{array}{ccc} P(C) & \xrightarrow{\varphi'} & CEq(C) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ Eq(C) & \xrightarrow[\pi_1]{} & pall(C)\,, \end{array}$$

  where $\phi = (\pi_1, \pi_2)$.

Let *fork(C)* be the space $C_1 \times_{s,C_0,s} C_1 \times_{s,C_0,t} C_1$ encoding diagrams of the form,



,

and *spn(C)* be the space $C_1 \times_{s,C_0,s} C_1$ encoding diagrams of the form,

$$\longleftarrow \quad \longrightarrow \;.$$

Then we obtain the following coequalizer diagram which will give us the object of arrows $G(C)_1$,

$$fork(C) \underset{(m(\pi_1,\pi_3),m(\pi_2,\pi_3))}{\overset{(\pi_1,\pi_2)}{\rightrightarrows}} spn(C) \xrightarrow{q} G(C)_1\,.$$

Definition 6.2.3. Let C be a topological category which satisfies the internal calculus of fractions conditions. Define the *topological groupoid* $\mathcal{G}(C)$ as follows:

- The space of objects $\mathcal{G}(C)_0 = C_0$.

- Space of arrows $G(C)_1$ which is the coequalizer of M and N in the following diagram,

$$fork(C) \underset{N}{\overset{M}{\rightrightarrows}} spn(C) \xrightarrow{q} G(C)_1\,,$$

  where

$$M = (m(\pi_1, \pi_3), m(\pi_2, \pi_3))$$

  and

$$N = (\pi_1, \pi_2).$$

Such that for $(\lambda_1, \lambda_2) \in spn(C)$ the structure maps are defined as follows:

- The source map $s : \mathcal{G}(C)_1 \longrightarrow \mathcal{G}(C)_0$ is defined as $s(\lambda_1, \lambda_2) = t(\lambda_1)$.
- The target map $t : \mathcal{G}(C)_1 \longrightarrow \mathcal{G}(C)_0$ is defined as $t(\lambda_1, \lambda_2) = t(\lambda_2)$.
- Since $C_0 = G(C)_0$ and $G(C)_1 = C_1 \times_{s,C_0,s} C_1/\sim$, we will use the identity map $u : C_0 \longrightarrow C_1$ in the internal category $\mathcal{C}$ to define the identity map $u' : G(C)_0 \longrightarrow G(C)_1$. We have $u : C_0 \longrightarrow C_1$ is the identity map in the internal category. This gives us a map $(u, u) : C_0 \longrightarrow C_1 \times_{s,C_0,s} C_1$ which is the identity map in *spn(C)*. Since $G(C)_1 = spn(C)/\sim$ and q is the coequalizer, we can define the identity map in the topological groupoid as

$$u' = q(u, u),$$

  where u is the identity map in the internal category
- The inverse map $i : \mathcal{G}(C)_1 \longrightarrow \mathcal{G}(C)_1$ is defined as $i(\lambda_1, \lambda_2) = (\lambda_2, \lambda_1)$.

We need to check that the structure maps that we have defined are well defined. We need to see that for any $(\lambda_1, \lambda_2, \lambda_3) \in fork(C)$, we obtain the following

- We need to see that $sM = sN$.
  - $sM(\lambda_1, \lambda_2, \lambda_3) = s(m(\lambda_1, \lambda_3), m(\lambda_2, \lambda_3)) = s(\lambda_1\lambda_3, \lambda_2\lambda_3) = t(\lambda_1\lambda_3) = t(\lambda_1)$.
  - $sN(\lambda_1, \lambda_2, \lambda_3) = s(\lambda_1, \lambda_2) = t(\lambda_1)$.

  As a result, $sM = sN$.
- We need to see that $tM = tN$.
  - $tM(\lambda_1, \lambda_2, \lambda_3) = t(m(\lambda_1, \lambda_3), m(\lambda_2, \lambda_3)) = t(\lambda_1\lambda_3, \lambda_2\lambda_3) = t(\lambda_2\lambda_3) = t(\lambda_2)$.
  - $tN(\lambda_1, \lambda_2, \lambda_3) = t(\lambda_1, \lambda_2) = t(\lambda_2)$.

  So, we have that $tM = tN$.
- We need to see that $iM = iN$.
  - $iM(\lambda_1, \lambda_2, \lambda_3) = i(m(\lambda_1, \lambda_3), m(\lambda_2, \lambda_3)) = i(\lambda_1\lambda_3, \lambda_2\lambda_3) = (\lambda_2\lambda_3, \lambda_1\lambda_3)$.

– $iN(\lambda_1, \lambda_2, \lambda_3) = i(\lambda_1, \lambda_2) = (\lambda_2, \lambda_1)$.

We know that in $\mathcal{G}(C)_1$, we have that $[\lambda_2\lambda_3, \lambda_1\lambda_3] \sim [\lambda_2, \lambda_1]$. As a result, $iM = iN$.

We need to define the composition for $\mathcal{G}(\mathcal{C})$ to have the complete conditions for an internal category. Therefore, in the next subsection, we will define the composition in the internal category of fractions. After that we will check that $\mathcal{G}(\mathcal{C})$ satisfies the conditions for an internal category and prove its universal property.

6.2.2 The Composition in the Internal Category of Fractions

Before starting to talk about the composition we note that any coequalizer in *TOP* is a quotient map which we will use when we define the composition. The following proposition is stated in [14].

Proposition 6.2.4. *In TOP, the coequalizer of any pair of arrows is a quotient map.*

Let $\mathcal{C}$ be a topological category. We want to talk about composition in $\mathcal{G}(\mathcal{C})$, but we need to define the composition as in Section 5.2 in terms of continuous functions. Recall that we define *fork(C)* and *spn(C)* as

$$fork(C) = (C_1 \times_{s,C_0,s} C_1) \times_{s,C_0,t} C_1\,,$$

$$spn(C) = C_1 \times_{s,C_0,s} C_1\,.$$

We have defined that $G(C)_1 = spn(C)/fork(C)$, the coequalizer of N and M as in the following diagram,

$$fork(C) \underset{N}{\overset{M}{\rightrightarrows}} spn(C) \xrightarrow{q} G(C)_1\,.$$

where N, M from *fork(C)* to *spn(C)* are defined as:

$$N = (\pi_1, \pi_2)$$

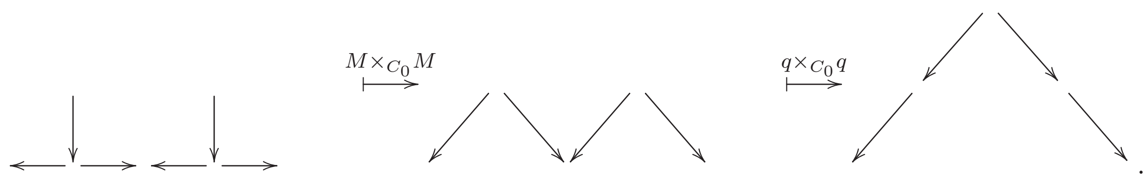
and

$$M = (m(\pi_1, \pi_3), m(\pi_2, \pi_3)).$$

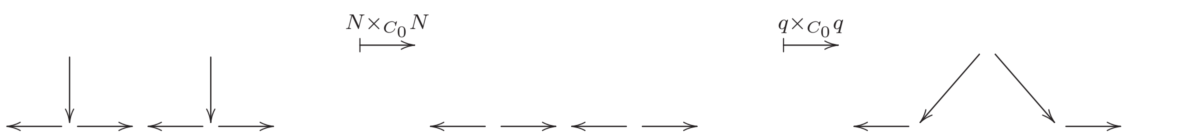
The domain of the composition map is $G(C)_1 \times_{s,C_0,t} G(C)_1$. We can obtain this space as part of the following commutative diagram,

$$fork(C) \times_{s,C_0,t} fork(C) \underset{N \times_{C_0} N}{\overset{M \times_{C_0} M}{\rightrightarrows}} spn(C) \times_{s,C_0,t} spn(C) \xrightarrow{q \times_{C_0} q} G(C)_1 \times_{s,C_0,t} G(C)_1 \ .$$

The maps in this diagram act on diagrams of arrows in $\mathcal{C}$ as follows,



Also,



We claim that this diagram is again a coequalizer. To prove this, we need to define the notion of a reflexive coequalizer and we are going to prove that $\mathcal{G}(C)_1$ is the reflexive coequalizer for M and N.

Definition 6.2.5. Two parallel arrows $f, g : A \rightrightarrows B$ form a *reflexive pair* if they have a common section. I.e., there exists an arrow $\omega : B \longrightarrow A$ such that $f\omega = g\omega = \mathrm{id}_B$. A coequalizer of a reflexive pair is a *reflexive coequalizer.*

Lemma 6.2.6. *The product of reflexive coequalizers is a coequalizer.*

Proof. See for instance, [2]. For more a detailed proof see [5] which gives a proof in the special case of tensor products. This proof generalizes in a straightforward way to our case. □

Proposition 6.2.7. $G(C)_1 \times_{s,C_0,t} G(C)_1$ *is the coequalizer of* $M \times_{C_0} M$ *and* $N \times_{C_0} N$,

$$fork(C) \times_{s,C_0,t} fork(C) \underset{N \times_{C_0} N}{\overset{M \times_{C_0} M}{\rightrightarrows}} spn(C) \times_{s,C_0,t} spn(C) \xrightarrow{q \times_{C_0} q} G(C)_1 \times_{s,C_0,t} G(C)_1 \ .$$

Proof. Recall that we have defined $G(C)_1$ as the coequalizer of N and M from *fork(C)* to *spn(C)* as in the following diagram,

$$fork(C) \underset{N}{\overset{M}{\rightrightarrows}} spn(C) \xrightarrow{q} G(C)_1 \ .$$

We want to prove that $G(C)_1 \times_{s,C_0,t} G(C)_1$ is the coequalizer of $M \times_{C_0} M$ and $N \times_{C_0} N$. We will show that $G(C)_1$ is a reflexive coequalizer. We need to define a map $\omega : spn(C) \longrightarrow fork(C)$ such that $M\omega = N\omega = id_{spn(C)}$. So, we define $\omega : C_1 \times_{s,C_0,s} C_1 \longrightarrow (C_1 \times_{s,C_0,s} C_1) \times_{s,C_0,t} C_1$ by $\omega = (u, u \circ s \circ \pi_1)$.

Now we want to see that $M\omega = N\omega = \mathrm{id}_{spn(C)}$. Take $(f,g) \in C_1 \times_{s,C_0,s} C_1$. We have $M\omega(f,g) = M(u, u \circ s(f)) = M(f, g, u(s(f))) = (m(f, u(s(f))), m(g, u(s(f))))$. Since $u(s(f)) = u(s(g))$, we get $(m(f, u(s(f))), m(g, u(s(f)))) = (f,g)$. Also, we have that $N\omega(f,g) = N(u, u \circ s(f)) = N(f, g, u(s(f))) = (f,g)$ We conclude that $N\omega = M\omega = \mathrm{id}_{spn(C)}$.

As a result, $G(C)_1$ is a reflexive coequalizer. So, $G(C)_1 \times_{s,C_0,t} G(C)_1$ is the coequalizer of $M \times_{C_0} M$ and $N \times_{C_0} N$. Therefore, by Lemma 6.2.6 we have the following coequalizer,

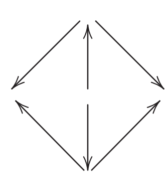
$$fork(C) \times_{s,C_0,t} fork(C) \underset{N \times_{C_0} N}{\overset{M \times_{C_0} M}{\rightrightarrows}} spn(C) \times_{s,C_0,t} spn(C) \xrightarrow{q \times_{C_0} q} G(C)_1 \times_{s,C_0,t} G(C)_1 \ .$$

□

The Extension of the Composition map

In order to be able to talk about composition in the internal category of fractions, we define the space df(C) to be the space encoding non commutative diagrams of the form,

(6.3)



.

Let $tts : (C_1 \times_{C_0} C_1) \times_{C_0} C_1 \longrightarrow (C_0 \times C_0) \times C_0$ be the map defined as

$$tts = ((t \times t) \circ \pi_1, s \circ \pi_3) = (t \circ \pi_1, t \circ \pi_1, s \circ \pi_3).$$

So, $df(C) = ((C_1 \times_{s,C_0,s} C_1) \times_{s\pi_1,C_0,t} C_1) \times_{tts,(C_0 \times C_0)\times C_0,tts} ((C_1 \times_{s,C_0,s} C_1) \times_{s\pi_1,C_0,t} C_1)\,.$
Now we can define *dblfork(C)* to be the following equalizer,

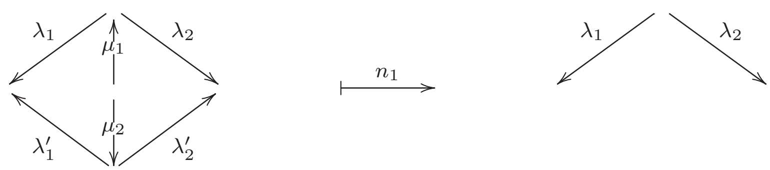
$$dblfork(C) \longrightarrow df(C) \underset{M\circ\pi_2}{\overset{M\circ\pi_1}{\rightrightarrows}} spn(C)\,.$$

So, *dblfork(C)* is the subspace of *df(C)* encoding commutative diagrams of the form (6.3).

We have three maps from *dblfork(C)* to *spn(C)* which are:

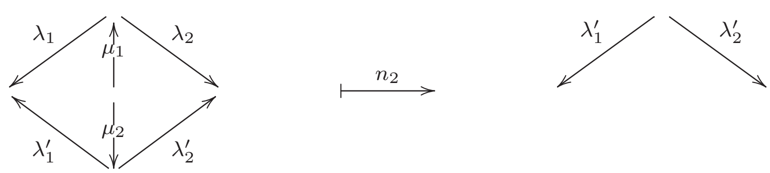
$$n_1 = (\pi_1\pi_1, \pi_2\pi_1)\,,$$

I.e.,



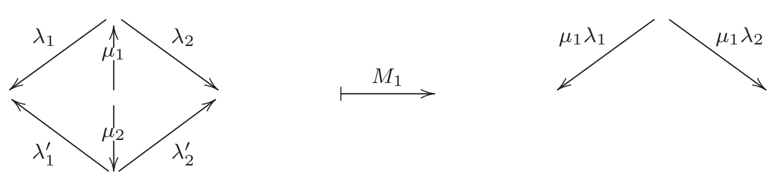
$$n_2 = (\pi_1\pi_2, \pi_2\pi_2)\,,$$

I.e.,

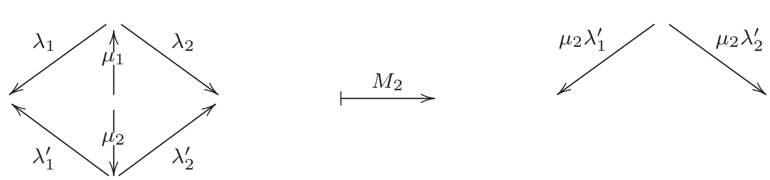


$$M_1 = (m(\pi_1\pi_1, \pi_3\pi_1), m(\pi_2\pi_1, \pi_3\pi_1)) = (m(\pi_1\pi_2, \pi_3\pi_2), m(\pi_2\pi_2, \pi_3\pi_2)) = M_2\,.$$

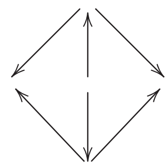
I.e.,



Also we have,



Since we have that *dblfork(C)* is the space of commutative diagrams of the form

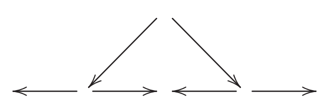


,

we obtain that $\mu_1\lambda_1 = \mu_2\lambda'_1$ and $\mu_1\lambda_2 = \mu_2\lambda'_2$. So, $M_1 = M_2$.
In Section 5.2, we used chosen squares to define composition in the category of fractions. This means for us that we will need to use the étale surjection

$$spncsq(C) \xrightarrow{g} spn(C) \times_{s,C_0,t} spn(C)\,,$$

which can be obtained from condition **TOP-CF3**, and the fact that étale surjections are stable under pullbacks. We have that *spncsq(C)* is a space of the form $C_1 \times_{s,C_0} csq(C) \times_{C_0,s} C_1$. I.e., this encodes diagrams of arrows in C of the form,



.

Defining the composition map h will involve an extension of the following diagram.

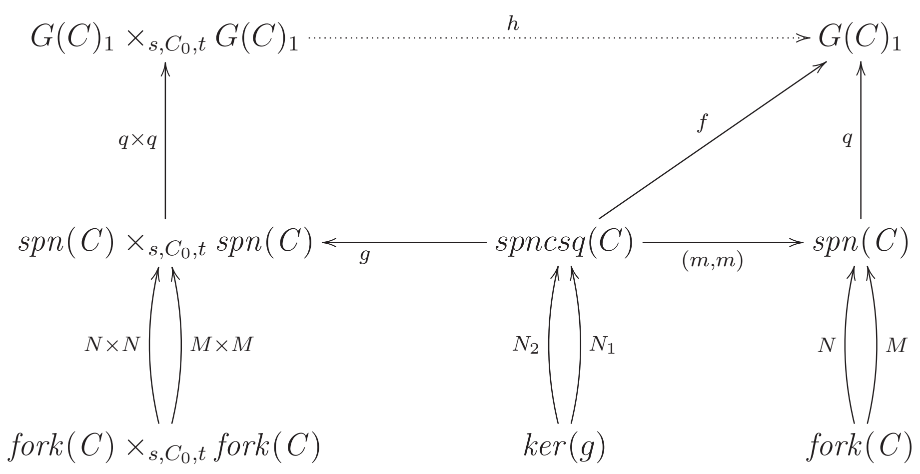


Figure 6.1: The composition in terms of coequalizers

However, we would like to change the codomain coequalizer because *fork(C)* is a relation, but not an equivalence relation. We need an equivalence relation that generate the relation in the codomain. Then we will use the following fact to define the composition in 6.2.12.

Proposition 6.2.8. *The coequalizers of*

$$\mathit{dblfork}(C) \underset{n_2}{\overset{n_1}{\rightrightarrows}} \mathit{spn}(C)$$

and

$$\mathit{fork}(C) \underset{N}{\overset{M}{\rightrightarrows}} \mathit{spn}(C)$$

are homeomorphic.

Proof. Suppose there are the following coequalizers,

$$\mathit{fork}(C) \underset{N}{\overset{M}{\rightrightarrows}} \mathit{spn}(C) \xrightarrow{q} G(C)_1\,,$$

and

$$\mathit{dblfork}(C) \underset{n_2}{\overset{n_1}{\rightrightarrows}} \mathit{spn}(C) \xrightarrow{q'} G(C)'_1\,.$$

We want to show that $G(C)_1$ and $G(C)'_1$ are homeomorphic and that this homeomorphism fits in a diagram,

$$\begin{array}{ccc} \mathit{spn}(C) & \xrightarrow{q} & G(C)_1 \\ & \searrow^{q'} & \downarrow{\cong}\,\chi \\ & & G(C)'_1\,, \end{array}$$

We will find a map from $G(C)_1$ to $G(C)'_1$ and a map from $G(C)'_1$ to $G(C)_1$.

- We need to find a map from $G(C)_1$ to $G(C)'_1$. We define $\omega : \mathit{fork}(C) \longrightarrow \mathit{dblfork}(C)$ by
  $\omega = (u, u \circ s \circ \pi_1, m(\pi_1, \pi_3), m(\pi_2, \pi_3))$. Take $(f, g, h) \in \mathit{fork}(C)$. We have

  $$\omega(f, g, h) = (u, u \circ s(f), m(f, g), m(g, h)) = (f, g, h, u(s(f)), fh, gh).$$

  Then we have the following diagram,

  $$\begin{array}{ccccc} \mathit{fork}(C) & \underset{N}{\overset{M}{\rightrightarrows}} & \mathit{spn}(C) & \xrightarrow{q} & G(C)_1 \\ {\scriptstyle\omega}\downarrow & & \| & & \\ \mathit{dblfork}(C) & \underset{n_2}{\overset{n_1}{\rightrightarrows}} & \mathit{spn}(C) & \xrightarrow[q']{} & G(C)'_1\,. \end{array}$$

We want to show that $n_1\omega = N$ and $n_2\omega = M$. Take $(f, g, h) \in \mathit{fork}(C)$ Then we have

$$n_1\omega(f, g, h) = n_1(f, g, h, u(s(f)), fh, gh) = (f, g) = N(f, g, h).$$

Also we have that

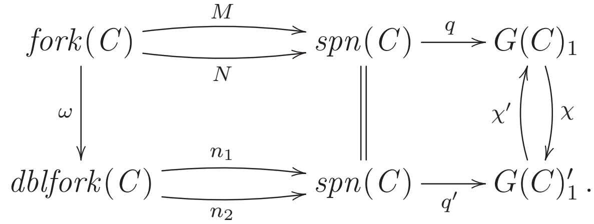
$$n_2\omega(f, g, h) = n_2(f, g, h, u(s(f)), fh, gh) = (fh, gh) = M(f, g, h).$$

As a result $n_1\omega = N$ and $n_2\omega = M$. So, $q'N = q'M$. By the universal property of the coequalizer, there is a unique map $\chi : G(C)_1 \longrightarrow G(C)'_1$ such that $q' = \chi q$.

- We need to find a map from $G(C)'_1$ to $G(C)_1$. Note that we can define *dblfork(C)* as $\mathit{fork}(C) \times_{M,spn(C),M} \mathit{fork}(C)$. Take $(F_1, F_2) \in \mathit{fork}(C) \times_{M,spn(C),M} \mathit{fork}(C)$ where $F_i = (f_i, g_i, h_i)$ with $i = \{1, 2\}$. We want to show that $qn_1 = qn_2$. We have $n_i = N\pi_i$ for $i = \{1, 2\}$. Now we have $qn_1(F_1, F_2) = qN\pi_1(F_1, F_2) = qN(F_1)$. However, the map q is the coequalizer of N and M so, $qN(F_1) = qM(F_1)$. Since $(F_1, F_2) \in \mathit{fork}(C) \times_{M,spn(C),M} \mathit{fork}(C)$, we have $M(F_1) = M(F_2)$. So, $qM(F_1) = qM(F_2)$ but, $qM(F_2) = qN(F_2)$. So, $qN(F_1) = qN(F_2)$. We can conclude that $qn_1 = qn_2$. As a result, since q' is the coequalizer of n_1 and n_2, there is a unique map $\chi' : G(C)'_1 \longrightarrow G(C)_1$ such that $q = \chi' q'$.

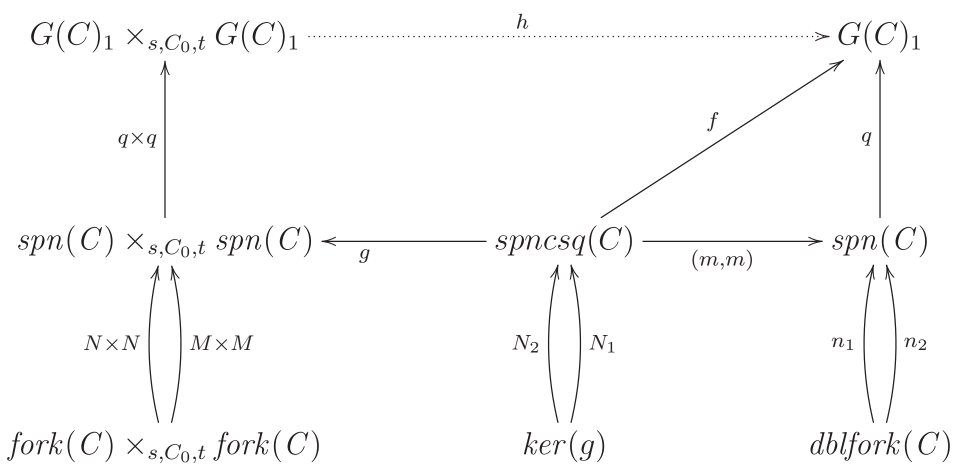
- We have that $\chi \circ \chi' = id_{G(C)'_1}$ and $\chi' \circ \chi = id_{G(C)_1}$ by the universal property of the coequalizers.

As a result we obtain the following diagram,

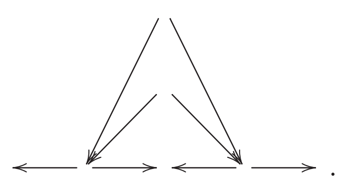


We can conclude that $G(C)_1$ and $G(C)'_1$ are homeomorphic. □

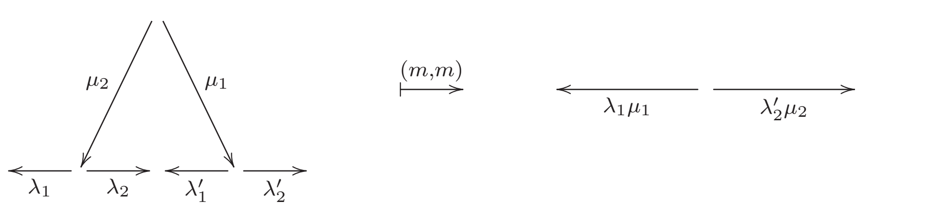
Remark 6.2.9. In this paper, we will mostly use *dblfork(C)* instead of *fork*. However, we will still use *fork(C)* × *fork(C)* in Diagram (6.1). This is just because we want to define a map between coequalizers. We want to use the universal property for the coequalizer and it is easier to map out of *fork(C)* $\times$ *fork(C)* and map into *dblfork(C)*. Therefore, we want to use a smaller space on the domain side which has less information and encodes a relation that generates the equivalence relation rather than the full equivalence relation and we need a space which has more information and the full equivalence relation on the side of the codomain. Then we adjust (6.1) as follows.



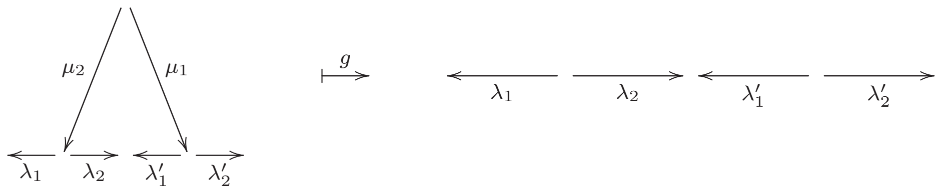
We are now going to define the remaining spaces and maps. First, we define the spaces *ker* $(g) = C_1 \times_{s,C_0} (csq(C) \times_{(\pi_3\pi_1,\pi_4\pi_1),(C_1\times_{C_0}C_1),(\pi_3\pi_2,\pi_4\pi_2)} csq(C)) \times_{C_0,s} C_1$. I.e., it is the space that encodes commutative diagrams of arrow in C of the form,



Now we are going to define the arrows. We have $(m,m) : spncsq(C) \longrightarrow spn(C)$ which is defined as $(m,m) = (m(\pi_1,\pi_3), m(\pi_5,\pi_6))$. I.e.,

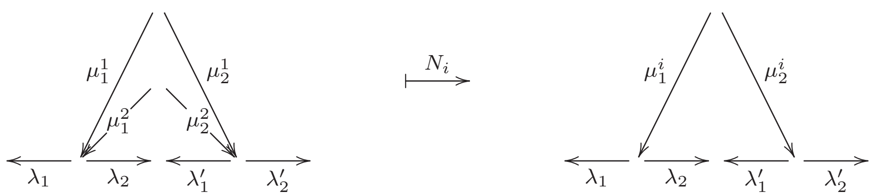


Also, we have $g : spncsq(C) \longrightarrow spn(C) \times_{s,C_0,t}$ *spn(C)* which is defined as $g = ((\pi_1,\pi_2),(\pi_4,\pi_5))$. I.e.,



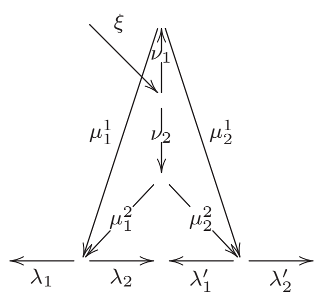
.

There is also $f = q(m,m)$ and $N_1, N_2 : \ker(g) \longrightarrow$ *spncsq(C)* which are defined by $N_i = (\pi_1, \pi_2, \pi_3\pi_i, \pi_4, \pi_5, \pi_6\pi_i)$ where $i = \{1,2\}$.



.

Now we want to define *cker(g)*, which encodes commutative diagrams of arrows in C of the form,



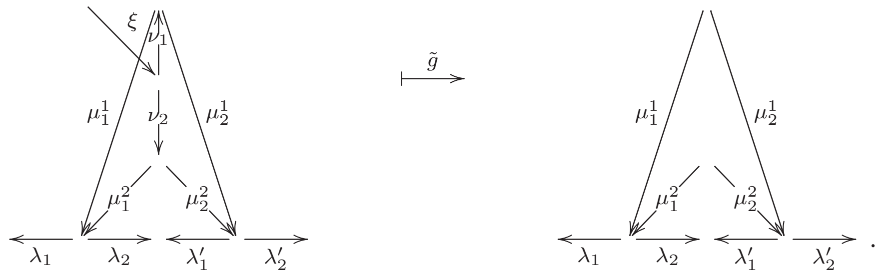
,

where $\mu_1^1\nu_1 = \mu_1^2\nu_2$ and $\lambda_2'\mu_2^1\nu_1\xi = \lambda_2'\mu_2^2\nu_2\xi$ as well as $\lambda_2\mu_1^i = \lambda_1'\mu_2^i$ for $i = \{1,2\}$.

Then we will have a projection map $\tilde{g} : cker(g) \longrightarrow ker\ (g)$ which is defined by $\tilde{g} = ((n(\pi_1,\pi_2), \pi_3\pi_1, n(\pi_4,\pi_5), \pi_6\pi_1), (n(\pi_1,\pi_2), \pi_3\pi_2, n(\pi_4,\pi_5), \pi_6\pi_2))$. We can see it as follows : let $(\lambda_1, \lambda_2, \mu_1^1, \mu_2^1, \lambda_1', \lambda_2', \mu_1^2, \mu_2^2, \nu_1, \nu_2, \xi) \in cker(g)$. Then we have

$$\tilde{g}((\lambda_1, \lambda_2, \mu_1^1, \mu_2^1, \lambda_1', \lambda_2', \mu_1^2, \mu_2^2, \nu_1, \nu_2, \xi)) = ((\lambda_1, \lambda_2), \mu_1^1, (\lambda_1', \lambda_2'), \mu_2^1), (\lambda_1, \lambda_2), \mu_1^2, (\lambda_1', \lambda_2'), \mu_2^2)).$$

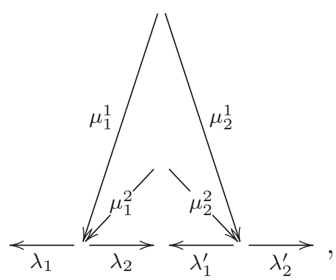
I.e.,



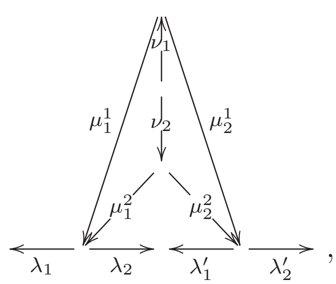
Now we want to check that $\tilde{g}$ is an étale surjection.

Lemma 6.2.10. *The map* $\tilde{g} : cker(g) \longrightarrow ker\ (g)$ *is an étale surjection.*

Proof. We have that *ker(g)* encodes commutative diagrams of arrows in C of the form,



we obtain a projection map from $ker(g)$ to $csp(C) = \xrightarrow{\mu_i^1}\ \xleftarrow{\mu_i^2}$. By **TOP-CF3**, we have an étale surjection map from *csq(C)* to *csp(C)*. Define *dker(g)* encodes commutative diagrams of arrows in C of the form,



where $\mu_1^1\nu_1 = \mu_1^2\nu_2$ and $\lambda'_2\mu_2^1\nu_1 = \lambda'_2\mu_2^2\nu_2$ as well as $\lambda_2\mu_1^i = \lambda'_1\mu_2^i$ for $i = \{1, 2\}$. As a result, *dker(g)* is the following pullback,

$$\begin{array}{ccc} dker(g) & \longrightarrow & csq(C) \\ \downarrow & & \downarrow \\ ker(g) & \longrightarrow & csp(C)\,. \end{array}$$

We have a projection map from *dker(g)* to *CEq(C)* where

$$CEq(C) = \;\; \underset{\mu_2^2\nu_2}{\overset{\mu_2^1\nu_1}{\rightrightarrows}} \xrightarrow{\lambda_2'} .$$

By **TOP-CF4**, we have an étale surjection map for a space *P(C)* which encodes diagrams of arrows in C of the form,

$$\xrightarrow{\xi} \underset{\mu_2^2\nu_2}{\overset{\mu_2^1\nu_1}{\rightrightarrows}} \xrightarrow{\lambda_2'} ,$$

to *CEq(C)*. Then we obtain that *cker(g)* is the following pullback,

$$\begin{array}{ccc} cker(g) & \longrightarrow & P(C) \\ \downarrow & & \downarrow \\ dker(g) & \longrightarrow & CEq(C)\,. \end{array}$$

Therefore, $\tilde{g}$ is an étale surjection.

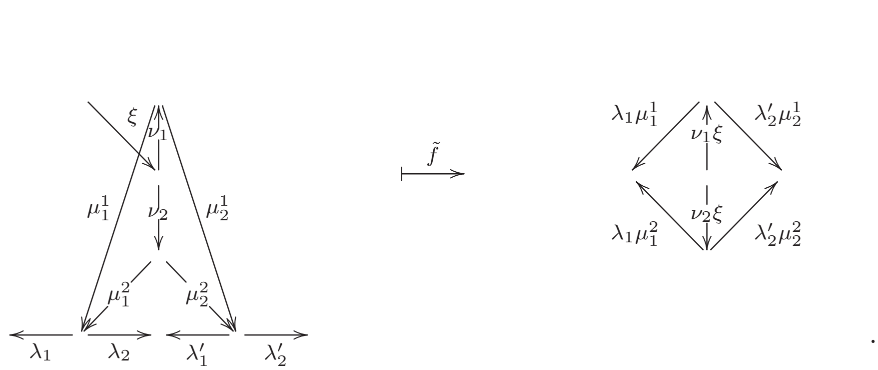
□

Now we define a map $\tilde{f} : cker(g) \longrightarrow dblfork(C)$ as

$$\tilde{f} = ((m(\pi_1, \pi_3\pi_1), (m(\pi_1, \pi_3\pi_2), m(\pi_4, \pi_6\pi_1), m(\pi_4, \pi_6\pi_2), m(\pi_7, \pi_9), m(\pi_8, \pi_9)).$$

Therefore, take $(\lambda_1, \lambda_2, \mu_1^1, \mu_1^2, \lambda_1'\lambda_2', \mu_2^1, \mu_2^2, \nu_1, \nu_2, \xi) \in cker(g)$. Then $\tilde{f}$ is defined as

$$\tilde{f}(\lambda_1, \lambda_2, \mu_1^1, \mu_1^2, \lambda_1'\lambda_2', \mu_2^1, \mu_2^2, \nu_1, \nu_2, \xi) = (\lambda_1\mu_1^1, \lambda_1\mu_1^2, \lambda_2, \lambda_2'\mu_2^1, \lambda_2'\mu_2^2, \nu_1\xi, \nu_2\xi).$$



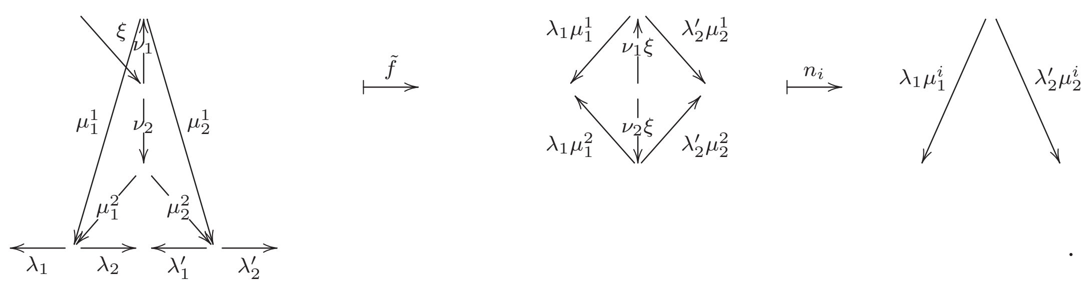
We also have $\tilde{\pi}_1, \tilde{\pi}_2 : \ker(g) \longrightarrow spn(C)$ defined by $\tilde{\pi}_i = (m,m)N_i$. This can be written as,

$$\begin{aligned}
\tilde{\pi}_i(((\lambda_1,\lambda_2),\mu_1^1,(\lambda_1',\lambda_2'),\mu_2^1),(\lambda_1,\lambda_2),\mu_1^2,(\lambda_1',\lambda_2'),\mu_2^2))) &= \\
(m,m)N_i(((\lambda_1,\lambda_2),\mu_1^1,(\lambda_1',\lambda_2'),\mu_2^1),(\lambda_1,\lambda_2),\mu_1^2,(\lambda_1',\lambda_2'),\mu_2^2))) &= \\
(m,m)(\lambda_1,\lambda_2,\mu_1^i,\lambda_1',\lambda_2',\mu_2^i) &= \\
(\lambda_1\mu_1^i, \lambda_2'\mu_2^i)\,,
\end{aligned}$$

where $i = \{1,2\}$. Now we want to check that $n_i\tilde{f} = \tilde{\pi}_i\tilde{g}$ and for $i = \{1,2\}$. Take $(\lambda_1,\lambda_2,\mu_1^1,\mu_1^2,\lambda_1'\lambda_2',\mu_2^1,\mu_2^2,\nu_1,\nu_2,\xi) \in cker(g)$. Then

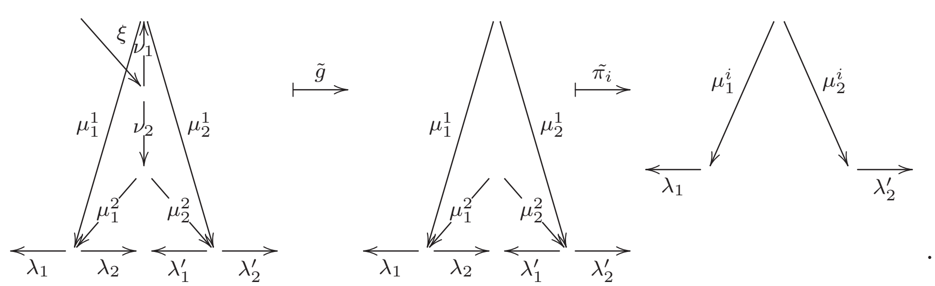
$$\begin{aligned}
n_i\tilde{f}(\lambda_1,\lambda_2,\mu_1^1,\mu_1^2,\lambda_1'\lambda_2',\mu_2^1,\mu_2^2,\nu_1,\nu_2,\xi) &= n_i(\lambda_1\mu_1^1,\lambda_1\mu_1^2,\lambda_2,\lambda_2'\mu_2^1,\lambda_2'\mu_2^2,\nu_1\xi,\nu_2\xi) \\
&= (\lambda_1\mu_1^i, \lambda_2'\mu_2^i)\,,
\end{aligned}$$

i.e.,



Also we have,

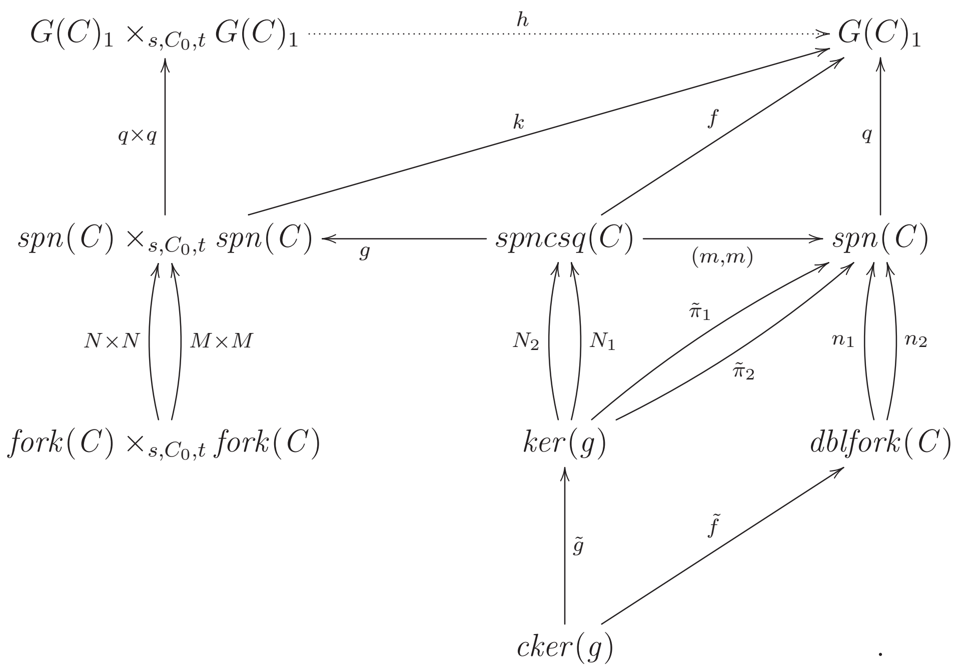
$$\begin{aligned}
\tilde{\pi}_i\tilde{g}(\lambda_1,\lambda_2,\mu_1^1,\mu_1^2,\lambda_1'\lambda_2',\mu_2^1,\mu_2^2,\nu_1,\nu_2,\xi) &= \\
\tilde{\pi}_i((\lambda_1,\lambda_2),\mu_1^1,(\lambda_1',\lambda_2'),\mu_2^1,(\lambda_1,\lambda_2),\mu_1^2,(\lambda_1',\lambda_2'),\mu_2^2) &= \\
(\lambda_1\mu_1^i, \lambda_2'\mu_2^i)\,,
\end{aligned}$$



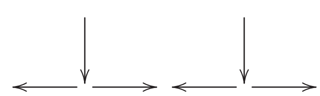
As a result, $n_i\tilde{f} = \tilde{\pi}_i\tilde{g}$. By Lemma 6.1.5 and the fact that $\tilde{g}$ is a surjective local homeomorphism, there is a unique continuous map

$$k : spn(C) \times spn(C) \longrightarrow G(U)_1$$

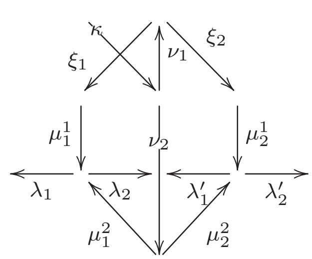
with $kg = f = q(m,m)$. Consequently, we get the following diagram.



Now we are going to do with $fork(C) \times_{s,C_0,t} fork(C)$ what we have we done above with *ker* (g). We will define an étale surjection for $fork(C) \times_{s,C_0,t} fork(C)$ by **CF3** and Lemma 5.2.3. Then we will use Lemma 6.1.5 to find $h : \mathcal{G}(\mathcal{C})_1 \times_{s,C_0,t} \mathcal{G}(\mathcal{C})_1 \longrightarrow \mathcal{G}(\mathcal{C})_1$ which gives us the composition map. Recall that $fork(C) \times_{s,C_0,t} fork(C)$ is the space encoding diagrams of the form,



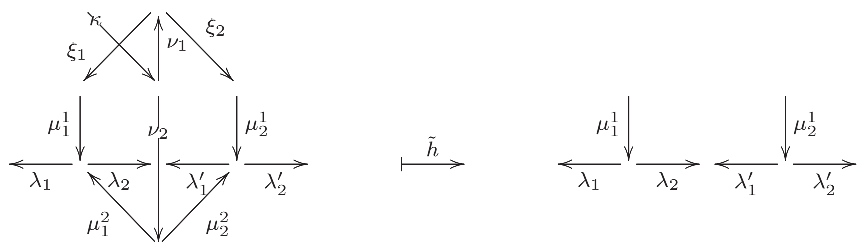
Let *cf(C)* be the space that encodes diagrams of arrows in C of the following form,



where $\mu_1^1\xi_1\nu_1 = \mu_1^2\nu_2$ and $\lambda_2'\mu_2^1\xi_2\nu_1\kappa = \lambda_2'\mu_2^2\nu_2\kappa$ as well as $\lambda_2\mu_1^1\xi 1 = \lambda_1'\mu_2^1\xi_2$ and $\lambda_2\mu_1^2 = \lambda_1'\mu_2^2$. Let define a map $\tilde{h} : cf(C) \longrightarrow fork(C) \times_{s,C_0,t} fork(C)$ as $\tilde{h} = ((\pi_1, \pi_2, \pi_3\pi_1), (\pi_4, \pi_5, \pi_6\pi_1))$.
Take $(\lambda_1, \lambda_2, \mu_1^1, \mu_1^2, \lambda_1', \lambda_2', \mu_2^1, \mu_2^2, \nu_1, \nu_2, \xi_1, \xi_2, \kappa) \in cf(C)$ then

$$\tilde{h}(\lambda_1, \lambda_2, \mu_1^1, \mu_1^2, \lambda_1', \lambda_2', \mu_2^1, \mu_2^2, \nu_1, \nu_2, \xi_1, \xi_2, \kappa) = ((\lambda_1, \lambda_2, \mu_1^1), (\lambda_1', \lambda_2', \mu_2^1)).$$

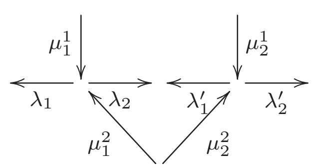
In term of diagrams of arrows in C we have that $\tilde{h}$ is given by:



.

Lemma 6.2.11. *The map $\tilde{h} : cf(C) \longrightarrow fork(C) \times_{s,C_0,t} fork(C)$ is an étale surjection.*

Proof. We have a projection map from $fork(C) \times_{s,C_0,s} fork(C)$ to $csp(C)$. By **TOP-CF3**, we have an étale surjection map from *csq(C)* to *csp(C)*. Define *dfork(C)* encodes commutative diagrams of arrows in C of the form,



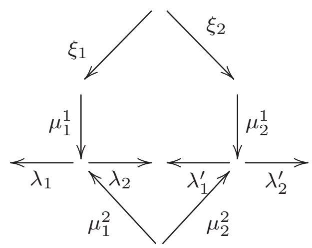
,

where $\lambda_2\mu_1^2 = \lambda_1'\mu_2^2$.

As a result, *dfork(C)* is the following pullback,

$$\begin{array}{ccc} dfork(C) & \longrightarrow & csq(C) \\ \downarrow & & \downarrow \\ fork(C) \times_{s,C_0,s} fork(C) & \longrightarrow & csp(C)\,. \end{array}$$

We have a projection map from *dfork(C)* to *csp(C)* By **TOP-CF3**, we have an étale surjection map from *csq(C)* to *csp(C)*. Define *cfork(C)* encodes commutative diagrams of arrows in C of the form,

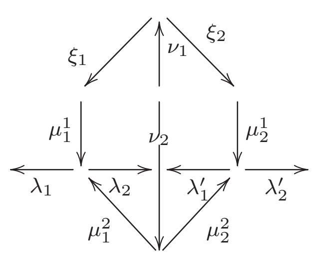


,

where $\lambda_2\mu_1^1\xi 1 = \lambda_1'\mu_2^1\xi_2$ and $\lambda_2\mu_1^2 = \lambda_1'\mu_2^2$. As a result, *dfork(C)* is the following pullback,

$$\begin{array}{ccc} cfork(C) & \longrightarrow & csq(C) \\ \downarrow & & \downarrow \\ dfork(C) & \longrightarrow & csp(C)\,. \end{array}$$

We have a projection map from *cfork(C)* to *csp(C)* By **TOP-CF3**, we have an étale surjection map from *csq(C)* to *csp(C)*. Define *efork(C)* encodes commutative diagrams of arrows in C of the form,



,

where $\mu_1^1\xi_1\nu_1 = \mu_1^2\nu_2$ and $\lambda_2\mu_1^1\xi 1 = \lambda_1'\mu_2^1\xi_2$ and $\lambda_2\mu_1^2 = \lambda_1'\mu_2^2$. As a result, *efork(C)* is the following pullback,

$$\begin{array}{ccc} efork(C) & \longrightarrow & csq(C) \\ \downarrow & & \downarrow \\ cfork(C) & \longrightarrow & csp(C)\,. \end{array}$$

We have a projection map from *efork(C)* to *CEq(C)* By **TOP-CF4**, we have an étale surjection map from *P(C)* to *CEq(C)*. Then we obtain that *cf(C)* is the following pullback,

$$\begin{array}{ccc} cf(C) & \longrightarrow & P(C) \\ \downarrow & & \downarrow \\ efork(C) & \longrightarrow & CEq(C)\,. \end{array}$$

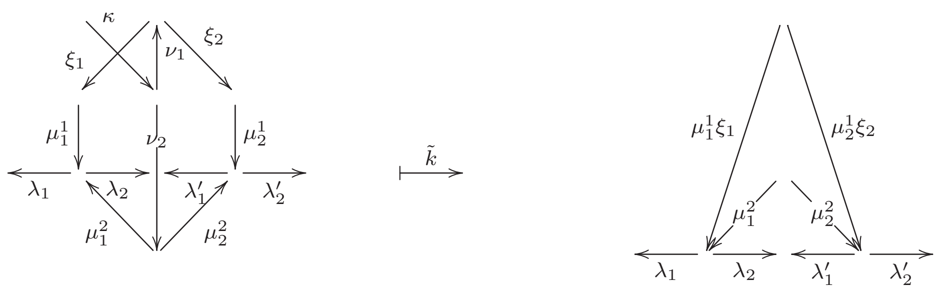
Therefore, $\tilde{h}$ is an étale surjection. $\square$

Now we are going to define a map $\tilde{k} : cf(C) \longrightarrow \ker(g)$ which is

$$\tilde{k} = ((\pi_1, \pi_2, m(\pi_3\pi_1, \pi_9), \pi_4, \pi_5, m(\pi_6\pi_1, \pi_{10})), (\pi_1, \pi_2, \pi_3\pi_2, \pi_4, \pi_5, \pi_6\pi_2)).$$

Take $(\lambda_1, \lambda_2, \mu_1^1, \mu_1^2, \lambda'_1, \lambda'_2, \mu_2^1, \mu_2^2, \nu_1, \nu_2, \xi_1, \xi_2, \kappa) \in cf(C)$. Then $\tilde{k}$ is defined as,

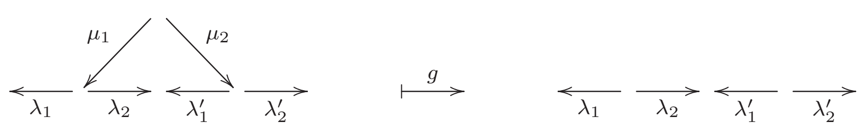
$$\begin{aligned} \tilde{k}(\lambda_1, \lambda_2, \mu_1^1, \mu_1^2, \lambda'_1, \lambda'_2, \mu_2^1, \mu_2^2, \nu_1, \nu_2, \xi_1, \xi_2, \kappa) &= \\ ((\lambda_1, \lambda_2, \mu_1^1\xi_1, \lambda'_1, \lambda'_2, \mu_2^1\xi_2), (\lambda_1, \lambda_2, \mu_1^2, \lambda'_1, \lambda'_2, \mu_2^2)). \end{aligned}$$



We also have $g : spncsq(C) \longrightarrow spn(C) \times_{s,C_0,t} spn(C)$ which is defined by, $g = (n(\pi_1, \pi_2), n(\pi_4, \pi_5)$. Then if we take $(\lambda_1, \lambda_2, \mu_1, \lambda'_1, \lambda'_2, \mu_2) \in spncsq(C)$ we have

$$g(\lambda_1, \lambda_2, \mu_1, \lambda'_1, \lambda'_2, \mu_2) = ((\lambda_1, \lambda_2), (\lambda'_1, \lambda'_2)).$$

I.e.,

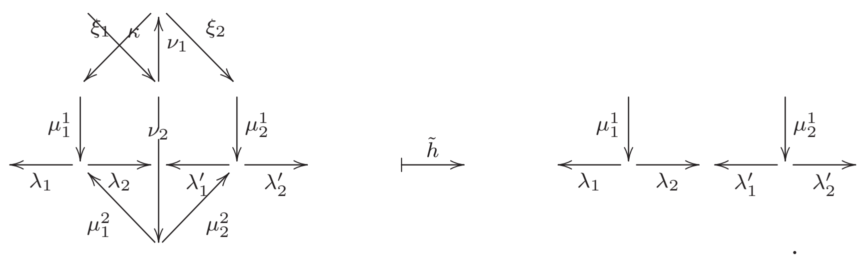


Now we want to check that $(M \times_{C_0} M)\tilde{h} = gN_1\tilde{k}$ and $(N \times_{C_0} N)\tilde{h} = gN_2\tilde{k}$. Take $(\lambda_1, \lambda_2, \mu_1^1, \mu_1^2, \lambda'_1, \lambda'_2, \mu_2^1, \mu_2^2, \nu_1, \nu_2, \xi_1, \xi_2, \kappa) \in cf(C)$ then

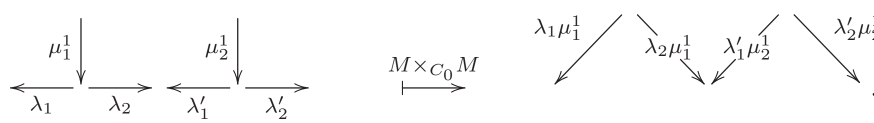
$$\begin{aligned} (M \times_{C_0} M)\tilde{h}(\lambda_1, \lambda_2, \mu_1^1, \mu_1^2, \lambda'_1, \lambda'_2, \mu_2^1, \mu_2^2, \nu_1, \nu_2, \xi_1, \xi_2, \kappa) &= \\ (M \times_{C_0} M)((\lambda_1, \lambda_2, \mu_1^1), (\lambda'_1, \lambda'_2, \mu_2^1)) &= \\ ((\lambda_1\mu_1^1, \lambda_2\mu_1^1), (\lambda'_1\mu_2^1, \lambda'_2\mu_2^1)). \end{aligned}$$

Now we see the composition in the diagram language. We have $(\lambda_1, \lambda_2, \mu_1^1, \mu_1^2, \lambda_1', \lambda_2', \mu_2^1, \mu_2^2, \nu_1, \nu_2, \xi_1, \xi_2, \kappa) \in cf(C)$. Then we apply $\tilde{h}$ to get the follwing diagram,

(6.4)



Then we apply $M \times_{C_0} M$ and it gives us the following,

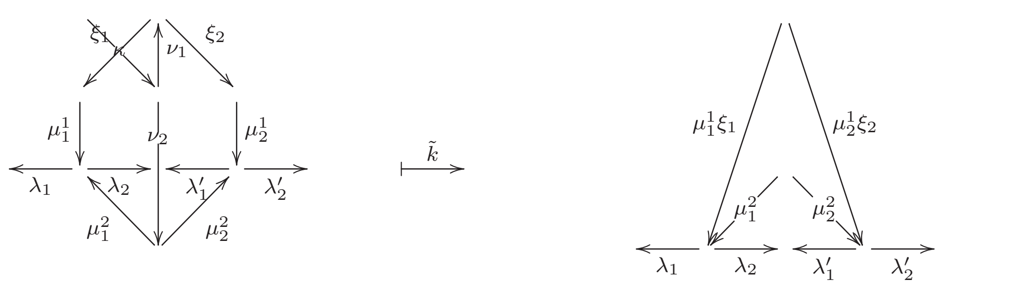


and

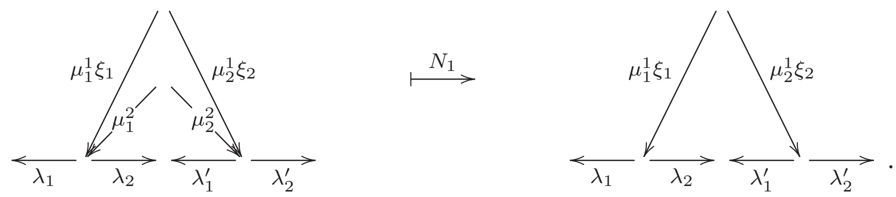
$$\begin{aligned}
gN_1\tilde{k}(\lambda_1, \lambda_2, \mu_1^1, \mu_1^2, \lambda_1', \lambda_2', \mu_2^1, \mu_2^2, \nu_1, \nu_2, \xi_1, \xi_2, \kappa) &= \\
gN_1((\lambda_1, \lambda_2, \mu_1^1\xi_1), (\lambda_1', \lambda_2', \mu_2^1\xi_2)), (\lambda_1, \lambda_2, \mu_1^2, \lambda_1', \lambda_2', \mu_2^2)) &= \\
g(\lambda_1, \lambda_2, \mu_1^1\xi_1), \lambda_1', \lambda_2', \mu_2^1\xi_2) &= \\
((\lambda_1, \lambda_2), (\lambda_1', \lambda_2')).
\end{aligned}$$

Now we see the composition in the diagram language. We have $(\lambda_1, \lambda_2, \mu_1^1, \mu_1^2, \lambda_1', \lambda_2', \mu_2^1, \mu_2^2, \nu_1, \nu_2, \xi_1, \xi_2, \kappa) \in cf(C)$. Then we apply $\tilde{k}$ to get the following diagram,

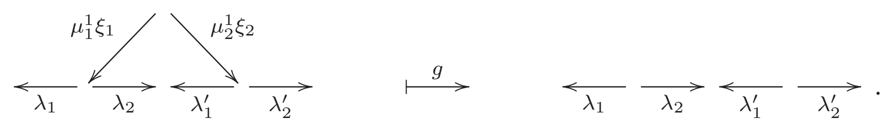
(6.5)



Then we apply N_1 and it gives us the following,



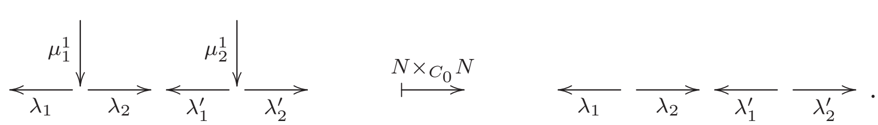
Finally, we apply g to obtain,



We have that $(\lambda_1, \lambda_2) = (\lambda_1\mu_1^1, \lambda_2\mu_1^1) \in \mathcal{G}(C)_1 \times_{s,C_0,t} \mathcal{G}(C)_1$, as well as $(\lambda'_1, \lambda'_2) = (\lambda'_1\mu_2^1, \lambda'_2\mu_2^1)$. Then we can conclude that $(M \times_{C_0} M)\tilde{h} = gN_1\tilde{k}$. Similarly,

$$
\begin{aligned}
(N \times_{C_0} N)\tilde{h}(\lambda_1, \lambda_2, \mu_1^1, \mu_1^2, \lambda'_1, \lambda'_2, \mu_2^1, \mu_2^2, \nu_1, \nu_2, \xi_1, \xi_2, \kappa) &= \\
(N \times_{C_0} N)((\lambda_1, \lambda_2, \mu_1^1), (\lambda'_1, \lambda'_2, \mu_2^1)) &= \\
((\lambda_1, \lambda_2), (\lambda'_1, \lambda'_2)).&
\end{aligned}
$$

Now we see the composition in the diagram language. We have applied $\tilde{h}$ in (6.4) and now we apply $N \times_{C_0} N$ to the result to get the following,

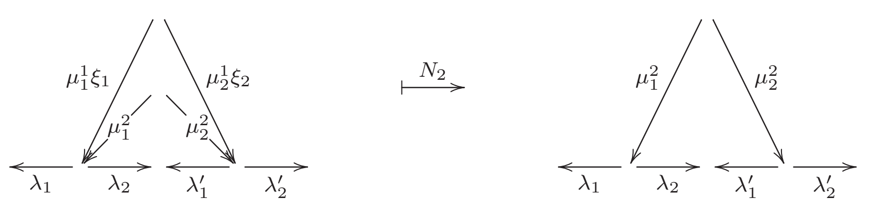


Also we have

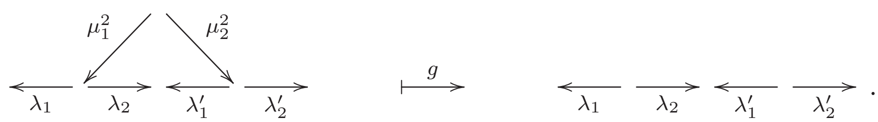
$$
\begin{aligned}
gN_2\tilde{k}(\lambda_1, \lambda_2, \mu_1^1, \mu_1^2, \lambda'_1, \lambda'_2, \mu_2^1, \mu_2^2, \nu_1, \nu_2, \xi_1, \xi_2, \kappa) &= \\
gN_2((\lambda_1, \lambda_2, \mu_1^1\xi_1), (\lambda'_1, \lambda'_2, \mu_2^1\xi_2)), (\lambda_1, \lambda_2, \mu_1^2, \lambda'_1, \lambda'_2, \mu_2^2)) &= \\
g(\lambda_1, \lambda_2, \mu_1^2, \lambda'_1, \lambda'_2, \mu_2^2) &= \\
((\lambda_1, \lambda_2), (\lambda'_1, \lambda'_2)).&
\end{aligned}
$$

Now we see the composition in the diagram language. We have applied $\tilde{k}$ in (6.5)

Then we apply N_2 and it gives us the following,



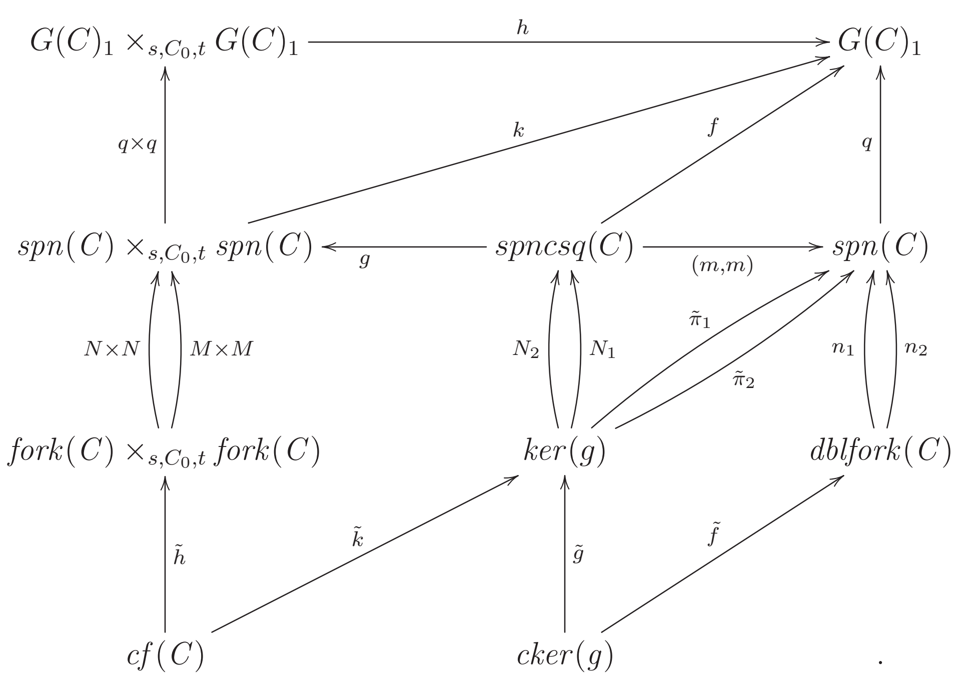
finally, we apply g to obtain that,



We conclude that $(N \times_{C_0} N)\tilde{h} = gN_2\tilde{k}$. By Lemma 6.1.5, and the fact that $\tilde{h}$ is a surjective local homeomorphism, there is a unique continuous map

$$h : G(C)_1 \times_{s,C_0,t} G(C)_1 \longrightarrow G(C)_1$$

with $h(q \times_{C_0} q) = k$. Consequently, we get the following diagram.



Therefore, we have defined the composition map of the internal category of fractions $\mathcal{G}(\mathcal{C})$ which is the map

$$h : G(C)_1 \times_{s,C_0,t} G(C)_1 \longrightarrow G(C)_1$$

in this diagram.

6.2.3 The Internal Category of Fractions

We have started to define the *topological groupoid* $\mathcal{G}(\mathcal{C})$ in 6.2.3. However, we needed to find the composition map $h : G(C)_1 \times_{s,C_0,t} G(C)_1 \longrightarrow G(C)_1$. Now we will recall the definition of the topological groupoid and add the composition map.

Definition 6.2.12. Let C be a topological category which satisfies the internal calculus of fractions conditions. Define the *topological groupoid* $\mathcal{G}(C)$ as follows:

- The space of objects $\mathcal{G}(C)_0 = C_0$.
- Space of arrows $\mathcal{G}(C)_1$ which is the coequalizer of M and N as follows

$$fork(C) \underset{N}{\overset{M}{\rightrightarrows}} spn(C) \xrightarrow{q} G(C)_1 ,$$

  where is defined as $M = (m(\pi_1, \pi_3), m(\pi_2, \pi_3))$ and $N = (\pi_1, \pi_2)$

such that for $(\lambda_1, \lambda_2) \in spn(C)$ the structure maps are defined as follows:

- The source map is $s : \mathcal{G}(C)_1 \longrightarrow \mathcal{G}(C)_0$ defined as $s(\lambda_1, \lambda_2) = t(\lambda_1)$.
- The target map is $t : \mathcal{G}(C)_1 \longrightarrow \mathcal{G}(C)_0$ defined as $t(\lambda_1, \lambda_2) = t(\lambda_2)$.
- The identity arrow is $u' : \mathcal{G}(C)_0 \longrightarrow \mathcal{G}(C)_1$ defined as $u' = q(u, u)$, where u is the identity map in the internal category $\mathcal{C}$.
- The inverse arrow is $i : \mathcal{G}(C)_1 \longrightarrow \mathcal{G}(C)_1$ defined as $i(\lambda_1, \lambda_2) = (\lambda_2, \lambda_1)$.
- The composition map is $h : \mathcal{G}(C)_1 \times_{C_0} \mathcal{G}(C)_1 \longrightarrow \mathcal{G}(C)_1$.

Remark 6.2.13. If we want to calculate the composition for a specific pair of spans. I.e., $h([\lambda_1, \lambda_2], [\mu_1, \mu_2])$ where $[\lambda_1, \lambda_2]$ and $[\mu_1, \mu_2] \in \mathcal{G}(C)_1$. It is sufficient to find an element $\rho \in spncsq(C)$ such that $g(\rho) = ((\lambda_1, \lambda_2), (\mu_1, \mu_2)) \in spn(C) \times_{s,C_0,s} spn(C)$ and calculate $f(\rho)$ since $f(\rho) = f(\rho')$ for any $\rho' \in g^{-1}((\lambda_1, \lambda_2), (\mu_1, \mu_2))$.

Now we want to check that $\mathcal{G}(C)$ satisfies the conditions to be an internal category.

- We want to show that $su' = id = tu'$. We have that
  - $su' = sq(u, u) = tu = id$.

– $tu' = tq(u,u) = tu = id.$

Then $su' = tu'$.

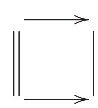
- We want to show that

$$h([\lambda_1,\lambda_2], u's[\lambda_1,\lambda_2]) = h(u't[\lambda_1,\lambda_2],[\lambda_1,\lambda_2]).$$

First we want to define the composition on the left side, $h([\lambda_1,\lambda_2], u's[\lambda_1,\lambda_2])$. We find that $((\lambda_1,\lambda_2), u's(\lambda_1,\lambda_2))$ encoding diagrams of the form,

$$\longleftarrow \; \longrightarrow \; = \!= \; = \!= \; .$$

The square ρ, we choose is



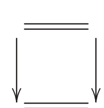
As a result, we have that,

$$\begin{aligned} h([\lambda_1,\lambda_2], u's[\lambda_1,\lambda_2]) &= h([\lambda_1,\lambda_2],[ut(\lambda_1), ut(\lambda_1)]) \\ &= f(\rho) \\ &= q \circ (m,m)(\lambda_1,\lambda_2, us\lambda_2, ut\lambda_2, ut\lambda_2, \lambda_2) \\ &= q(\lambda_1 us\lambda_2, ut\lambda_2\lambda_2) \\ &= q(\lambda_1,\lambda_2) \\ &= ([\lambda_1,\lambda_2]). \end{aligned}$$

Now we want to define the composition on the right side, $h(u't[\lambda_1,\lambda_2],[\lambda_1,\lambda_2])$. We find that $(u't(\lambda_1,\lambda_2),(\lambda_1,\lambda_2))$ encoding diagrams of the form,

$$= \!= \; = \!= \; \longleftarrow \; \longrightarrow \; .$$

The square ρ, we choose is



As a result, we have that,

$$
\begin{aligned}
h(u't[\lambda_1, \lambda_2], [\lambda_1, \lambda_2]) & \\
&= h([ut(\lambda_2), ut(\lambda_2)], [\lambda_1, \lambda_2]) \\
&= f(\rho) \\
&= q \circ (m, m)(ut\lambda_1, ut\lambda_1, \lambda_1, \lambda_1, \lambda_2, us\lambda_1) \\
&= q(ut\lambda_1\lambda_1, \lambda_2 us\lambda_1) \\
&= q(\lambda_1, \lambda_2) \\
&= ([\lambda_1, \lambda_2]).
\end{aligned}
$$

Therefore, $h([\lambda_1, \lambda_2], u's[\lambda_1, \lambda_2]) = [\lambda_1, \lambda_2] = h(u't[\lambda_1, \lambda_2], [\lambda_1, \lambda_2])$.

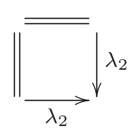
- We want to show that

$$h([\lambda_1, \lambda_2], i[\lambda_1, \lambda_2]) = (u't[\lambda_1, \lambda_2]).$$

First we want to define the composition on the left side, $h([\lambda_1, \lambda_2], i[\lambda_1, \lambda_2])$. We find that $((\lambda_1, \lambda_2), i(\lambda_1, \lambda_2))$ encoding diagrams of the form,

$$\xleftarrow{\lambda_1} \xrightarrow{\lambda_2} \xleftarrow{\lambda_2} \xrightarrow{\lambda_1} .$$

The square ρ, we choose is



As a result, we have that,

$$
\begin{aligned}
h([\lambda_1, \lambda_2], i[\lambda_1, \lambda_2]) &= h([\lambda_1, \lambda_2], [\lambda_2, \lambda_1])) \\
&= f(\rho) \\
&= q \circ (m, m)(\lambda_1, \lambda_2, us\lambda_2, \lambda_2, \lambda_1, us\lambda_2) \\
&= q(\lambda_1 us\lambda_2, \lambda_1 us\lambda_2) \\
&= q(\lambda_1, \lambda_1) \\
&= ([\lambda_1, \lambda_1]) \\
&= id.
\end{aligned}
$$

Also, we have that,

$$\begin{aligned} u't[\lambda_1, \lambda_2)] &= \\ ([ut(\lambda_2), ut(\lambda_2)]) &= \\ q((ut(\lambda_2), ut(\lambda_2)) &= \\ id. \end{aligned}$$

Therefore, $h([\lambda_1, \lambda_2], i[\lambda_1, \lambda_2]) = (u't[\lambda_1, \lambda_2])$.

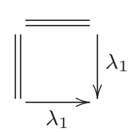
- We want to show that

$$h(i[\lambda_1, \lambda_2], [\lambda_1, \lambda_2]) = (u's[\lambda_1, \lambda_2]).$$

First we want to define the composition on the left side $h(i[\lambda_1, \lambda_2], [\lambda_1, \lambda_2])$. We find that $(i(\lambda_1, \lambda_2), (\lambda_1, \lambda_2))$ encoding diagrams of the form,

$$\xleftarrow{\lambda_2} \; \xrightarrow{\lambda_1} \; \xleftarrow{\lambda_1} \; \xrightarrow{\lambda_2} \; .$$

The square ρ, we choose is



As a result, we have that,

$$\begin{aligned} h(i[\lambda_1, \lambda_2], [\lambda_1, \lambda_2]) &= h([\lambda_2, \lambda_1], [\lambda_1, \lambda_2])) \\ &= f(\rho) \\ &= q \circ (m, m)(\lambda_2, \lambda_1, us\lambda_1, \lambda_1, \lambda_2, us\lambda_1) \\ &= q(\lambda_2 us\lambda_1, \lambda_2 us\lambda_1) \\ &= q(\lambda_2, \lambda_2) \\ &= ([\lambda_2, \lambda_2]) \\ &= id. \end{aligned}$$

Also, we have that,

$$\begin{aligned} u's[\lambda_1, \lambda_2)] &= \\ ([ut(\lambda_1), ut(\lambda_1)]) &= \\ q((ut(\lambda_1), ut(\lambda_1)) &= \\ id. & \end{aligned}$$

Therefore, $h(i[\lambda_1, \lambda_2], [\lambda_1, \lambda_2]) = (u's[\lambda_1, \lambda_2])$.

- $t(i[\lambda_1, \lambda_2]) = t[\lambda_2, \lambda_1] = t\lambda_(1) = s[\lambda_1, \lambda_2]$.
- $s(i[\lambda_1, \lambda_2] = s[\lambda_2, \lambda_1] = t\lambda_2 = t[\lambda_1, \lambda_2]$.
- The associativity can be checked by a rather long but relatively straightforwards calculation. We omit it here.

After checking that $\mathcal{G}(\mathcal{C})$ satisfies the conditions of an internal groupoid, we have the following diagram,

$$G(C)_1 \times_{s,C_0,t} G(C)_1 \overset{\pi_1,\ h,\ \pi_2}{\rightrightarrows} G(C)_1 \underset{t}{\overset{s}{\rightleftarrows}}\!\!{}^{u}\ G(C)_0 \quad (i : G(C)_1 \to G(C)_1) .$$

Proposition 6.2.14. *There inclusion map $J_c : C \longrightarrow \mathcal{G}(C)$ is a well-defined homomorphism.*

Proof. Recall that J_c is the following map of internal categories

$$\begin{array}{ccc} C_1 \times C_1 & \xrightarrow{j \times j} & G(C)_1 \times G(C)_1 \\ \downarrow m & & \downarrow h \\ C_1 & \xrightarrow{j} & G(C)_1 \\ t_C \downarrow\downarrow s_C & & s_{G(C)} \downarrow\downarrow t_{G(C)} \\ C_0 & \xrightarrow{id} & G(C)_0 , \end{array} \tag{6.6}$$

where j is defined as $j(\lambda) = (id, \lambda)$ for all $\lambda \in C_1$. We want to check the commutativity of the previous diagram and then show that $J_c^* : hom(\mathcal{G}(\mathcal{C}), \mathcal{K}) \longrightarrow hom(\mathcal{C}, \mathcal{K})$ defined as $F \longmapsto F \circ J_c$ for all $F \in hom(\mathcal{G}(\mathcal{C}), \mathcal{K})$ is an isomorphism. Now we want to check the commutativity of Diagram (6.6). Suppose $\lambda \in \mathcal{C}$ is defined as $\lambda : x \to y$.

So, we have that $ids_C(\lambda) = id(x) = x$ and we have that $s_{G(C)}j(\lambda) = s_{G(C)}(id, \lambda) = x$ so, we can conclude that $ids_C = s_{G(C)}j$. Also, we have that $idt_C(\lambda) = id(y) = y$ and we have that $t_{G(C)}j(\lambda) = t_{G(C)}(id, \lambda) = t_{G(C)}(\lambda) = y$ so, we can conclude that $idt_C = t_{G(C)}j$. Also, let $x \in C_0$, we have that $ju(x) = j(id_x) = (id_x, id_x)$. So, $ju = (id, id)$. We have that $u' = q(u, u) = (id, id)$. So, we can conclude that $ju = u'$. Now we need to check that the composition commutes. I.e., $h(j \times j) = j \circ m$. Let $(\lambda_1, \lambda_2) \in C_1 \times C_1$. We have that $h(j \times j)(\lambda_1, \lambda_2) = h((id, \lambda_1), (id, \lambda_2))$. By Remark 6.2.13, we find that $((id, \lambda_1), (id, \lambda_2))$ encoding diagrams of the form,

$$= \xrightarrow[\lambda_1]{} = \xrightarrow[\lambda_2]{} .$$

The square ρ, we choose is

$$\begin{array}{ccc} \cdot & \xrightarrow{\lambda_1} & \cdot \\ \| & & \| \\ \cdot & \xrightarrow[\lambda_1]{} & \cdot \end{array}$$

As a result, we have that,

$$\begin{aligned} h([id, \lambda_1], [id, \lambda_2]) &= f(\rho) \\ &= q \circ (m, m)(id, \lambda_1, id, id, \lambda_1, \lambda_2) \\ &= q(id, id, \lambda_1, \lambda_2) \\ &= q(id, \lambda_1 \lambda_2) \\ &= ([id, \lambda_1 \lambda_2]). \end{aligned}$$

Also, we have $j \circ m(\lambda_1, \lambda_2) = j(\lambda_1 \lambda_2) = (id, \lambda_1 \lambda_2)$. As a result, the Diagram (6.6) commutes. □

Now we want to prove the universal property which we have stated in Theorem 6.2.1.

Theorem 6.2.15. *Composition with the inclusion map $J_c : C \longrightarrow \mathcal{G}(C)$ gives an isomorphism $J_c^* : hom(\mathcal{G}(C), \mathcal{K}) \longrightarrow hom(C, \mathcal{K})$ for any internal groupoid $\mathcal{K}$ in* TOP.

$$\begin{array}{ccc} C & \xrightarrow{J_c} & \mathcal{G}(C) \\ & \searrow^{F'} & \downarrow F \\ & & \mathcal{K}\,. \end{array}$$

Proof. • We want to show that J_c^* is surjective.

Suppose we have $F' : \mathcal{C} \longrightarrow \mathcal{K}$. We want to construct $F : \mathcal{G}(\mathcal{C}) \longrightarrow \mathcal{K}$ such that $F' = F \circ J_c$. We can define F' on objects as $F'_0 = F_0$ since $J_{c0} = id$. Now to define $F_1 : G(C)_1 \longrightarrow K_1$, we first define $\omega : spn(C) \longrightarrow K_1$ as $\omega(\lambda_1, \lambda_2) = m(F'_1(\lambda_2), iF'_1(\lambda_1))$, for all $(\lambda_1, \lambda_2) \in spn(C)$. Then we obtain the following diagram,

$$fork(C) \underset{N}{\overset{M}{\rightrightarrows}} spn(C) \xrightarrow{\omega} K_1 .$$

Now we need to check that $\omega M = \omega N$. Let $(\lambda_1, \lambda_2, \lambda_3) \in fork(C)$. Then we have

$$\begin{aligned} \omega M(\lambda_1, \lambda_2, \lambda_3) = \omega(m(\lambda_1, \lambda_3), m(\lambda_2, \lambda_3)) = \\ \omega(\lambda_1\lambda_3, \lambda_2\lambda_3) = m(F'_1(\lambda_2\lambda_3), iF'_1(\lambda_1\lambda_3)) = \\ m(m(F'_1(\lambda_2), F'_1(\lambda_3)), m(iF'_1(\lambda_3), iF'_1(\lambda_1))) = \\ m(F'_1(\lambda_2), iF'_1(\lambda_1) \end{aligned}$$

We also have,

$$\omega N(\lambda_1, \lambda_2, \lambda_3) = \omega(\lambda_1, \lambda_2) = m(F'_1(\lambda_2), iF'_1(\lambda_1))$$

So, $\omega M = \omega N$. Therefore, this gives rise to a unique map $F_1 : G(C)_1 \longrightarrow K_1$ because of the universal property of the coequalizer. We need to show that $F_1 \circ h = m \circ (F_1 \times F_1)$ as in the following diagram,

$$\begin{array}{ccc} G(C)_1 \times_{s,C_0,s} G(C)_1 & \xrightarrow{h} & G(C)_1 \\ \downarrow{\scriptstyle F_1 \times F_1} & & \downarrow{\scriptstyle F_1} \\ K_1 \times_{s,K_0,s} K_1 & \xrightarrow[m]{} & K_1 . \end{array}$$

Since we have that $q : spn(C) \longrightarrow G(C)_1$ is a coequalizer map, it is enough to check the equality on *spn(C)* by using the map $\omega : spn(C) \longrightarrow K_1$. So, we need to prove commutitive of the following diagram,

$$\begin{array}{ccccc} spn(C) \times_{s,C_0,s} spn(C) & \xleftarrow{g} & spncsq(C) & \xrightarrow{(m,m)} & spn(C) \\ \downarrow{\scriptstyle \omega \times \omega} & & & & \downarrow{\scriptstyle \omega} \\ K_1 \times_{s,K_0,s} K_1 & & \xrightarrow[m]{} & & K_1 . \end{array}$$

Since g is a coequalizer of its kernel pair, it is also enough to check the equality on *spncsq(C)*. So, we need to check commutitive of the following diagram,

$$\begin{array}{ccc} spncsq(C) & \xrightarrow{(m,m)} & spn(C) \\ {\scriptstyle (\omega\times\omega)g}\downarrow & & \downarrow{\scriptstyle \omega} \\ K_1 \times_{s,K_0,s} K_1 & \xrightarrow[m]{} & K_1\,. \end{array}$$

Now let $(\lambda_1, \lambda_2, \mu_1, \lambda_1', \lambda_2', \mu_2) \in spncsq(C)$. Then we have the folowing,

$$\begin{aligned} \omega \circ (m,m)(\lambda_1, \lambda_2, \mu_1, \lambda_1', \lambda_2', \mu_2) &= \omega(m(\lambda_1, \mu_1), m(\lambda_2', \mu_2)) \\ &= \omega(\lambda_1\mu_1, \lambda_2'\mu_2) \\ &= m(F_1'\lambda_2'\mu_2, iF_1'\lambda_1\mu_1) \\ &= m(F_1(id, \lambda_2'\mu_2), iF_1(id, \lambda_1\mu_1)) \\ &= F_1 m((id, \lambda_2'\mu_2), i(id, \lambda_1\mu_1)) \\ &= F_1 m((id, \lambda_2'\mu_2), (\lambda_1\mu_1, id)) \\ &= F_1(\lambda_1\mu_1, \lambda_2'\mu_2). \end{aligned}$$

Also, we have that

$$\begin{aligned} m(\omega\times\omega)g(\lambda_1, \lambda_2, \mu_1, \lambda_1', \lambda_2', \mu_2) &= \\ m(\omega\times\omega)((\lambda_1, \lambda_2), (\lambda_1', \lambda_2')) &= \\ m(m(F_1'\lambda_2, iF_1'\lambda_1), m(F_1'\lambda_2', iF_1'\lambda_1')) &= \\ m(m(F_1(id, \lambda_2), iF_1(id, \lambda_1)), m(F_1(id, \lambda_2'), iF_1(id, \lambda_1'))) &= \\ m(F_1 m((id, \lambda_2), (\lambda_1, id)), F_1 m((id, \lambda_2'), (\lambda_1', id))) &= \\ mF_1((\lambda_1, \lambda_2), (\lambda_1', \lambda_2')) &= \\ F_1 m((\lambda_1, \lambda_2), (\lambda_1', \lambda_2')) &= \\ F_1(\lambda_1\mu_1, \lambda_2'\mu_2). \end{aligned}$$

We conclude that $\omega(m,m) = m(\omega\times\omega)g$.

Now we need to check that the following diagram commutes,

$$\begin{array}{ccc} C & \xrightarrow{J_c} & \mathcal{G}(C) \\ & {\scriptstyle F'}\searrow & \downarrow{\scriptstyle F} \\ & & \mathcal{K}\,. \end{array}$$

We have that $F_0 J_{c0} = F_0 = F'_0$. Let $\lambda \in C_1$. We have that $F_1 J_{c1}(\lambda) = F_1[id, \lambda] = \omega(id, \lambda) = m(F'_1(\lambda), iF'_1(id)) = F'_1(\lambda)$. So, $FJ_c = F'$. We can conclude $s \circ F'_1 = F'_0 \circ s$ and $t \circ F'_1 = F'_0 \circ t$ as well as $F'_1 \circ m = h(F'_1 \times F'_1)$. So, J_c^* is surjective.

- We want to show that J_c^* is injective
  Suppose there is F and $H : \mathcal{G}(\mathcal{C}) \longrightarrow \mathcal{K}$ such that $J_c^*(F) = J_c^*(H)$. So, we have that $F_1(id, \lambda) = H_1(id, \lambda)$ for all $\lambda \in \mathcal{C}_1$. Take $(\lambda, \mu) \in G(C)_1$. We want to show that $F_1(\lambda, \mu) = H_1(\lambda, \mu)$. We have that $F_1(\lambda, \mu) \in K_1$. Then we have that

$$\begin{aligned}
F_1(\lambda, \mu) &= m(F'_1\mu, iF'_1\lambda) \\
&= m(F_1(id, \mu), iF_1(id, \lambda)) \\
&= m(H_1(id, \mu), iH_1(id, \lambda)) \\
&= H_1(m((id, \mu), (\lambda, id))) \\
&= H_1(\lambda, \mu).
\end{aligned}$$

  We can conclude that $F = H$. Therefore, J_c^* is injective.

We can conclude that J_c^* is an isomorphism on objects.

Now we need to check that J_c^* is an isomorphism of categories.

Let $F, H : \mathcal{G}(\mathcal{C}) \longrightarrow K$.

- Let $\alpha' : F' \Rightarrow H'$ be a natural transformation. Then $\alpha' : F \circ J_c \Rightarrow H \circ J_c$. We need to find a natural transformation $\alpha : F \Rightarrow H$ such that $\alpha' = \alpha \circ J_c$. Since we have $\alpha' : C_0 \longrightarrow K_1$ and we have that $\alpha : G(C)_0 \longrightarrow K_1$ and $\alpha \circ J_c = \alpha \circ (J_c)_0$. Since we have that $(J_c)_0 = id$, we get $\alpha = \alpha'$. We can conclude that $\alpha \circ J_c = \alpha'$.

- Let $\alpha' = \beta' : F' \Rightarrow H'$. So, we have that $\alpha \circ J_c = \beta \circ J_c$. We want to show that $\alpha = \beta$. We have that $\alpha \circ J_c, \beta \circ J_c : C_0 \longrightarrow K_1$. We also have that $\alpha \circ J_c = \alpha \circ (J_c)_0 = \alpha$ and $\beta \circ J_c = \beta \circ (J_c)_0 = \beta$. Since $\alpha \circ J_c = \beta \circ J_c$, we have that $\alpha = \beta$.

We can conclude that J_c^* is an isomorphism in terms of natural transformations. □

6.3 Atlas Groupoids

We have defined the internal category of fractions for any topological category. I.e., an internal category in *TOP*. Now we apply this to the special case of the atlas category of an orbifold and construct the atlas groupoid as an internal category of fractions. So, we will describe the spaces and maps in term of charts and chart embeddings. First, we want to check that the atlas category of an orbifold satisfies the internal calculus of fraction conditions as defined in 6.2.2.

1. Now we check the condition **TOP - CF3**

- The space $allsq(\mathcal{U})$ is defined by

$$allsq(\mathcal{U}) = \coprod_{\substack{\tilde{W} \hookrightarrow \tilde{V}_1 \hookrightarrow \tilde{U}, \\ \tilde{W} \hookrightarrow \tilde{V}_2 \hookrightarrow \tilde{U}}} \tilde{W} = \coprod [\mu_1, \lambda_1, \tilde{W}, \mu_2, \lambda_2]$$

- The equalizer,

$$csq(\mathcal{U}) \xrightarrow{j} allsq(\mathcal{U}) \overset{\lambda_1\mu_1}{\underset{\lambda_2\mu_2}{\rightrightarrows}} ,$$

is defined by

$$csq(\mathcal{U}) = \coprod_{\substack{\tilde{W} \hookrightarrow \tilde{V}_1 \hookrightarrow \tilde{U}, \tilde{W} \hookrightarrow \tilde{V}_2 \hookrightarrow \tilde{U}, \\ \lambda_1\mu_1 = \lambda_2\mu_2}} \tilde{W} = \coprod_{\substack{\tilde{W} \hookrightarrow \tilde{V}_1 \hookrightarrow \tilde{U}, \tilde{W} \hookrightarrow \tilde{V}_2 \hookrightarrow \tilde{U}, \\ \lambda_1\mu_1 = \lambda_2\mu_2}} [\mu_1, \lambda_1, \tilde{W}, \mu_2, \lambda_2]$$

- The space $csp(\mathcal{U})$ is defined by

$$csp(\mathcal{U}) = \coprod_{\substack{\lambda_1 : \tilde{V}_1 \hookrightarrow \tilde{U}, \\ \lambda_2 : \tilde{V}_2 \hookrightarrow \tilde{U}}} \lambda_1(\tilde{V}_1) \cap \lambda_2(\tilde{V}_2) = \coprod [\lambda_1, \lambda_1(\tilde{V}_1) \cap \lambda_2(\tilde{V}_2), \lambda_2]$$

- The map $\phi : csq(\mathcal{U}) \longrightarrow csp(\mathcal{U})$. I.e.,

$$\phi : \coprod_{\substack{\tilde{W} \hookrightarrow \tilde{V}_1 \hookrightarrow \tilde{U}, \tilde{W} \hookrightarrow \tilde{V}_2 \hookrightarrow \tilde{U}, \\ \lambda_1\mu_1 = \lambda_2\mu_2}} [\mu_1, \lambda_1, \tilde{W}, \mu_2, \lambda_2] \longrightarrow \coprod_{\substack{\lambda_1 : \tilde{V}_1 \hookrightarrow \tilde{U}, \\ \lambda_2 : \tilde{V}_2 \hookrightarrow \tilde{U}}} [\lambda_1, \lambda_1(\tilde{V}_1) \cap \lambda_2(\tilde{V}_2), \lambda_2]$$

is defined by $\phi([\mu_1, \lambda_1, \tilde{W}, x, \mu_2, \lambda_2]) = [\lambda_1, \lambda_1(\tilde{V}_1) \cap \lambda_2(\tilde{V}_2), \lambda_1\mu_1(x), \lambda_2]$ where $x \in \tilde{W}$.

We need to check that ϕ is an étale surjection. We need to see that,

- ϕ is surjective
Let $[\lambda_1, \lambda_1(\tilde{V_1}) \cap \lambda_2(\tilde{V_2}), y, \lambda_2] \in csp(\mathcal{U})$. Then $y \in \tilde{U}$ with $\lambda_1(\tilde{V_1}) \cap \lambda_2(\tilde{V_2}) \in \tilde{U}$. By Lemma 2.4.6, there is a smaller chart $\tilde{W}$ with $x \in \tilde{W}$ and chart embeddings $\mu_i : \tilde{W} \hookrightarrow \tilde{V_i}$ where $i = \{1, 2\}$ such that $\lambda_1\mu_1 = \lambda_2\mu_2$ and $\lambda_i\mu_i(x) = y$ for $i = \{1, 2\}$. Therefore, there is $[\mu_1, \lambda_1, \tilde{W}, x, \mu_2, \lambda_2]$ with $\phi([\mu_1, \lambda_1, \tilde{W}, x, \mu_2, \lambda_2]) = [\lambda_1, \lambda_1(\tilde{V_1}) \cap \lambda_2(\tilde{V_2}), \lambda_1\mu_1(x), \lambda_2]$. So, we conclude that ϕ is surjective.

- We want to show that for each $x \in \mathcal{U}$ there is an open subset $\tilde{U}$ with $x \in \tilde{U}$ such that $\phi|_{\tilde{U}} : \tilde{U} \longrightarrow \phi(\tilde{U})$ is a homeomorphism.
We have that $\phi([\mu_1, \lambda_1, \tilde{W}, x, \mu_2, \lambda_2]) = [\lambda_1, \lambda_1(\tilde{V_1}) \cap \lambda_2(\tilde{V_2}), \lambda_1\mu_1(x), \lambda_2]$ where $x \in \tilde{W}$. By Lemma 2.4.6, we can conclude that $\phi|_{\tilde{W}} : \tilde{W} \longrightarrow \phi(\tilde{W})$ is homeomorphism.

2. Now we check condition **TOP - CF4**

- The space $CEq(\mathcal{U})$ is defined by

$$CEq(\mathcal{U}) = \coprod_{\substack{\mu_1,\mu_2:\tilde{W}\hookrightarrow\tilde{V},\lambda:\tilde{V}\hookrightarrow\tilde{U},\\ \lambda\mu_1=\lambda\mu_2}} \tilde{W} = \coprod_{\substack{\mu_1,\mu_2:\tilde{W}\hookrightarrow\tilde{V},\lambda:\tilde{V}\hookrightarrow\tilde{U},\\ \lambda\mu_1=\lambda\mu_2}} [\mu_1, \mu_2, \tilde{W}, \lambda]$$

- The space $Eq(\mathcal{U})$ is defined by

$$Eq(\mathcal{U}) = \coprod_{\substack{\lambda:\tilde{W}\hookrightarrow\tilde{U},\mu_1,\mu_2:\tilde{U}\hookrightarrow\tilde{V},\\ \mu_1\lambda=\mu_2\lambda}} \tilde{W} = \coprod_{\substack{\lambda:\tilde{W}\hookrightarrow\tilde{U},\mu_1,\mu_2:\tilde{U}\hookrightarrow\tilde{V},\\ \mu_1\lambda=\mu_2\lambda}} [\lambda, \tilde{W}, \mu_1, \mu_2]$$

- The space $P(\mathcal{U})$ which is the pullback defined by

$$P(\mathcal{U}) = \coprod_{\substack{\lambda:\tilde{W}\hookrightarrow\tilde{U},\mu_1,\mu_2:\tilde{U}\hookrightarrow\tilde{V},\\ \nu:\tilde{V}\hookrightarrow\tilde{V}',\nu\mu_1\lambda=\nu\mu_2\lambda}} \tilde{W} = \coprod_{\substack{\lambda:\tilde{W}\hookrightarrow\tilde{U},\mu_1,\mu_2:\tilde{U}\hookrightarrow\tilde{V},\\ \nu:\tilde{V}\hookrightarrow\tilde{V}',\nu\mu_1\lambda=\nu\mu_2\lambda}} [\lambda, \tilde{W}, \mu_1, \mu_2, \nu]$$

- The map $\varphi' : P(\mathcal{U}) \longrightarrow CEq(\mathcal{U})$. I.e.,

$$\varphi' : \coprod_{\substack{\lambda:\tilde{W}\hookrightarrow\tilde{U},\mu_1,\mu_2:\tilde{U}\hookrightarrow\tilde{V},\\ \nu:\tilde{V}\hookrightarrow\tilde{V}',\nu\mu_1\lambda=\nu\mu_2\lambda}} [\lambda, \tilde{W}, \mu_1, \mu_2, \nu] \longrightarrow \coprod_{\substack{\mu_1,\mu_2:\tilde{W}\hookrightarrow\tilde{V},\lambda:\tilde{V}\hookrightarrow\tilde{U},\\ \lambda\mu_1=\lambda\mu_2}} [\mu_1, \mu_2, \tilde{W}, \lambda].$$

is defined as $\varphi'([\lambda, \tilde{W}, x, \mu_1, \mu_2, \nu]) = [\mu_1, \mu_2, \lambda(\tilde{W}), \lambda(x), \nu]$. By using the same argument to prove that ϕ is étale surjection, we can conclude that φ' is étale surjection.

Before defining the orbifold atlas groupoid, we need to define some terms in the language of charts.

$$spn(\mathcal{U}) = \coprod_{\substack{\lambda_1:\tilde{U}\hookrightarrow\tilde{V}_1,\\ \lambda_2:\tilde{U}\hookrightarrow\tilde{V}_2}} \tilde{U} = \coprod [\lambda_1, \tilde{U}_1, \lambda_2]\,,$$

$$fork(\mathcal{U}) = \coprod_{\substack{\lambda_1:\tilde{U}\hookrightarrow\tilde{V}_1,\lambda_2:\tilde{U}\hookrightarrow\tilde{V}_2,\\ \nu:\tilde{W}\hookrightarrow\tilde{U}}} \tilde{W} = \coprod [\lambda_1, \lambda_2, \nu, \tilde{W}]\,.$$

Then we have two maps N, M from $fork(\mathcal{U})$ to $spn(\mathcal{U})$ which are defined as follows: for $[\lambda_1, \lambda_2, \nu, \tilde{W}] \in fork(\mathcal{U})$ we have $N[\lambda_1, \lambda_2, \nu, \tilde{W}] = [\lambda_1, \nu(\tilde{W}), \lambda_2] \in spn(\mathcal{U})$ and $M[\lambda_1, \lambda_2, \nu, \tilde{W}] = [\lambda_1\nu, \tilde{W}, \lambda_2\nu] \in spn(\mathcal{U})$. Now we can define the orbifold atlas groupoid.

Definition 6.3.1. Let $\mathcal{Q} = (Q, \mathcal{U})$ be an orbifold. Define the *orbifold atlas groupoid* $\mathcal{G}(\mathcal{U})$ as follows:

- The space of objects is

$$\mathcal{G}(\mathcal{U})_0 = C(\mathcal{U})_0 = \coprod_{\tilde{U}\in\mathcal{U}} \tilde{U}$$

- The space of arrows is

$$\mathcal{G}(\mathcal{U})_1 = [\coprod_{\substack{\lambda_1:\tilde{V}\to\tilde{U}_1,\\ \lambda_2:\tilde{V}\to\tilde{U}_2}} \tilde{V}]/\sim$$

  We denote the span of $\mathcal{G}(\mathcal{U})_1$ as $[\lambda_1, \tilde{V}, \lambda_2]$ and individual points of this space are denoted by $[\lambda_1, \tilde{V}, x, \lambda_2]$ where $x \in \tilde{V}$. I.e., $\mathcal{G}(\mathcal{U})_1$ is the coequalizer of M and N as follows

$$fork(\mathcal{U}) \underset{N}{\overset{M}{\rightrightarrows}} spn(\mathcal{U}) \xrightarrow{q} G(U)_1$$

  Therefore, we will define the equivalence relation on $G(\mathcal{U}) = spn(U)/fork(U)$ which is defined by $[\lambda_1, \tilde{V}, \mu(x), \lambda_2] \sim [\lambda_1\mu, \tilde{W}, x, \lambda_2\mu]$ for $[\lambda_1, \lambda_2, \mu, \tilde{W}, x] \in fork(\mathcal{U})$.

such that for $[\lambda_1, \tilde{V}, x, \lambda_2] \in spn(\mathcal{U})$ the structure maps defined as follows:

- The source map $s : \mathcal{G}(\mathcal{U})_1 \longrightarrow \mathcal{G}(\mathcal{U})_0$ is defined as $s([\lambda_1, \tilde{V}, x, \lambda_2]) = \lambda_1(x)$.
- The target map $t : \mathcal{G}(\mathcal{U})_1 \longrightarrow \mathcal{G}(\mathcal{U})_0$ is defined as $t([\lambda_1, \tilde{V}, x, \lambda_2]) = \lambda_2(x)$.

- The identity map $u : \mathcal{G}(\mathcal{U})_0 \longrightarrow \mathcal{G}(\mathcal{U})_1$ is defined as for $x \in \tilde{V}$ we have $u(x) = [\mathrm{id}_{\tilde{V}}, \tilde{V}, x, \mathrm{id}_{\tilde{V}}]$.

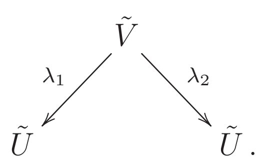
- The inverse map $i : \mathcal{G}(\mathcal{U})_1 \longrightarrow \mathcal{G}(\mathcal{U})_1$ is defined as $i([\lambda_1, \tilde{V}, x, \lambda_2]) = [\lambda_2, \tilde{V}, x, \lambda_1]$.

- Before defining the composition, note that $s[\lambda_1, \tilde{V}, x, \lambda_2] = t[\lambda_1', \tilde{V}', x', \lambda_2']$ if and only if $\lambda_1(x) = \lambda_2'(x')$ for $x \in \tilde{V}$ and $x' \in \tilde{V}'$.
  Then the composition map $h : \mathcal{G}(\mathcal{U})_1 \times_{C_0} \mathcal{G}(\mathcal{U})_1 \longrightarrow \mathcal{G}(\mathcal{U})_1$ is defined as: $h([\lambda_1, \tilde{V}, x, \lambda_2], [\lambda_1', \tilde{V}', x', \lambda_2']) = [\lambda_1\mu, \tilde{W}, y, \lambda_2'\mu']$ where $\tilde{V} \xleftarrow{\mu} \tilde{W} \xrightarrow{\mu'} \tilde{V}'$ with $\mu(y) = x$, $\mu'(y) = x'$ and $\lambda_2\mu = \lambda_1'\mu'$.

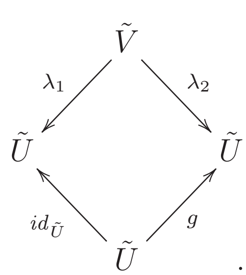
Proposition 6.3.2. *Let $\tilde{U}$ be a chart in an orbifold atlas $\mathcal{U}$ with a group $\mathcal{G}$. Then $s^{-1}(\tilde{U}) \cap t^{-1}(\tilde{U}) \subseteq \mathcal{G}(\mathcal{U})_1$, is homeomorphic to $\mathcal{G} \times \tilde{U}$ and each element in this space is of the form $[id_{\tilde{U}}, \tilde{U}, x, g]$ for some $g \in \mathcal{G}$.*

Proof. Let E be the space $s^{-1}(\tilde{U}) \cap t^{-1}(\tilde{U})$. We want to show that $\mathcal{G} \times \tilde{U} \cong E$. Define a map $\omega : \mathcal{G} \times \tilde{U} \longrightarrow E$ as follows, $\omega(g, \tilde{U}) = [id_{\tilde{U}}, \tilde{U}, x, g]$. We want to show that ω is surjective and injective.

- ω is surjective.
  Let $[\lambda_1, \tilde{V}, x, \lambda_2] \in E$. Then we get the following diagram,



  Since $\lambda_1, \lambda_2 : \tilde{V} \hookrightarrow \tilde{U}$, by Lemma 2.3.9, there is an element $g \in \mathcal{G}$ such that $g\lambda_1 = \lambda_2$. So, in particular, $g\lambda_1(x) = \lambda_2(x)$. We want to show that $[\lambda_1, \tilde{V}, x, \lambda_2] = [id_{\tilde{U}}, \tilde{U}, \lambda_1(x), g]$. We have the following diagram,



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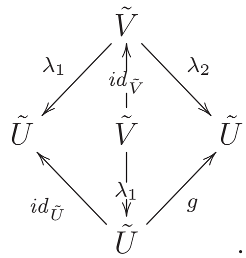
Since $\lambda_2 = g\lambda_1$, we can define $(id_{\tilde{V}}, \lambda_1) \in spn(C)$ such that following commutative diagram,



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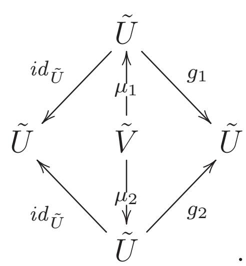
Therefore, $[\lambda_1, \tilde{V}, x, \lambda_2] = [id_{\tilde{U}}, \tilde{U}, \lambda_1(x), g]$. So, we can conclude that $\omega(g, \tilde{U}) = [\lambda_1, \tilde{U}, x, \lambda_2]$. Therefore, ω is surjective.

- ω is injective

Suppose that (g_1, x_1) and $(g_2, x_2) \in \mathcal{G} \times \tilde{U}$ such that $\omega(g_1, x_1) = \omega(g_2, x_2)$. So, we have that $[id_{\tilde{U}}, \tilde{U}, x_1, g_1] = [id_{\tilde{U}}, \tilde{U}, x_2, g_2]$. This means that, we have (μ_1, μ_2) such that $\mu_1, \mu_2 : \tilde{V} \hookrightarrow \tilde{U}$ with $y \in \tilde{V}$ such that $\mu_1(y) = x_1$ and $\mu_2(y) = x_2$ such that the following diagram commutes,



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We have from the diagram that $\mu_1 = \mu_2$. So, we have that $\mu_1(y) = \mu_2(y)$. Therefore, $x_1 = x_2$. Also, we have that $g_1\mu_1 = g_2\mu_2 = g_2\mu_1$. Therefore, $g_1 = g_2$. So, we conclude that $(g_1, x_1) = (g_2, x_2)$. Therefore, ω is injective.

$\square$

Chapter 7

Paths in Orbifolds

We have defined a weak map between orbifolds in Chapter 3, but it does not carry enough information. So, in the same chapter, we have defined a strong map between atlases. However, we do not want the notion of strong map between orbifolds to be dependent on the choice of the atlas. In order to solve this problem, we will define a strong map of orbifolds in terms of groupoids and weak equivalences. We will see that strong maps between orbifolds correspond to certain spans of maps between atlas groupoids. Then we are going to look at paths and equivalence classes of paths, and we will see the equivalence relation between them.
We will see that atlas refinements for orbifolds correspond to weak equivalences between the corresponding atlas groupoids. So we begin by defining what weak equivalences are.

7.1 The Strong Maps Between Orbifolds

7.1.1 Weak Equivalences

Definition 7.1.1. Let $\mathcal{G}$ and $\mathcal{H}$ be topological groupoids. Let $F : \mathcal{G} \longrightarrow \mathcal{H}$ be a map between them. Then we say that

- F is *open* if F_1 is open
- F is *essentially surjective on objects* if $t\pi_2$ in the following diagram is open and surjective,

$$\begin{array}{ccccc} G_0 \times_{H_0} H_1 & \xrightarrow{\pi_2} & H_1 & \xrightarrow{t} & H_0 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle s} & & \\ G_0 & \xrightarrow[F_0]{} & H_0 & & \end{array} \quad .$$

- F is *fully faithful* if the following square is a pullback,

$$\begin{array}{ccc} G_1 & \xrightarrow{F_1} & H_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ G_0 \times G_0 & \xrightarrow[F_0 \times F_0]{} & H_0 \times H_0 \,. \end{array}$$

Note that if F is essentially surjective, then for each object $d \in H_0$ there is an object $c \in G_0$ such that $F(c) \cong d \in H_0$. So, for all $d \in H_0$ there is $(c, h) \in G_0 \times_{H_0} H_1$ such that $t\pi_2(c, h) = t(h) = d$.

Definition 7.1.2. Let $\mathcal{G}$ and $\mathcal{H}$ be topological groupoids. A homomorphism $F : \mathcal{G} \longrightarrow \mathcal{H}$ is a *weak equivalence* if it is essentially surjective and fully faithful.

Proposition 7.1.3. *Let $\mathcal{U}$ and $\mathcal{W}$ be atlases such that $\mathcal{U} \subseteq \mathcal{W}$. Then the inclusion map $\mathcal{U} \hookrightarrow \mathcal{W}$ gives rise to a weak equivalence $\mathcal{G}(\mathcal{U}) \longrightarrow \mathcal{G}(\mathcal{W})$.*

Proof. Suppose we have that $\mathcal{U}$ and $\mathcal{W}$ are atlases such that $\mathcal{U} \subseteq \mathcal{W}$. Let $\omega : \mathcal{G}(\mathcal{U}) \longrightarrow \mathcal{G}(\mathcal{W})$ be a map defined on objects and on arrows by inclusion. Then ω is well defined on equivalence relation because the equivalence relation on $\mathcal{G}(\mathcal{U})$ is a subset of the equivalence classes on $\mathcal{G}(\mathcal{W})$. Now we want to show that ω is a weak equivalence. We need to show that it is essentially surjective and fully faithful.

- ω is fully faithful

  - We want to show that ω is faithful.
    Let $\tilde{U}_1$ and $\tilde{U}_2$ be charts in $\mathcal{U}$ such that $[\lambda_1, \tilde{U}_1, x_1, \lambda_2] = [\mu_1, \tilde{U}_2, x_2, \mu_2] \in \mathcal{G}(\mathcal{W})_1$. We want to show that $[\lambda_1, \tilde{U}_1, x_1, \lambda_2] = [\mu_1, \tilde{U}_2, x_2, \mu_2] \in \mathcal{G}(\mathcal{U})_1$. Since $[\lambda_1, \tilde{U}_1, x_1, \lambda_2] = [\mu_1, \tilde{U}_2, x_2, \mu_2] \in \mathcal{W}$, there is a smaller chart $\tilde{W} \in \mathcal{W}$ with $y \in \tilde{W}$ and chart embeddings $\nu_1 : \tilde{W} \hookrightarrow \tilde{U}_1$ and $\nu_2 : \tilde{W} \hookrightarrow \tilde{U}_2$ such that $\nu_i(y) = x_i$ and $\lambda_i\nu_1 = \mu_i\nu_2$ for $i = \{1, 2\}$. This implies that $\varphi_1(x_1) = \varphi_2(x_2) \in U_1 \cap U_2$ where $\varphi_i : \tilde{U}_i \longrightarrow U_i$ for $i = \{1, 2\}$. Since $\tilde{U}_1, \tilde{U}_2 \in \mathcal{U}$ with $U_1 \cap U_2 \neq \emptyset$, there is a chart $\tilde{U}_3 \in \mathcal{U}$ with $x_3 \in \tilde{U}_3$ and chart embeddings $\xi_1 : \tilde{U}_3 \hookrightarrow \tilde{U}_1$ and $\xi_2 : \tilde{U}_3 \hookrightarrow \tilde{U}_2$ such that $\xi_i(x_3) = x_i$ and $\lambda_1\xi_1 = \mu_1\xi_2$ by Lemma 2.4.6. We want to show that $\lambda_2\xi_1 = \mu_2\xi_2$.

Since $\tilde{U}_3, \tilde{W} \in \mathcal{W}$, there is a smaller chart $\tilde{W}' \in \mathcal{W}$ with $y' \in \tilde{W}'$ and chart embeddings $\nu_1' : \tilde{W}' \hookrightarrow \tilde{W}$ and $\nu_2' : \tilde{W}' \hookrightarrow \tilde{U}_3$ such that $\nu_1'(y') = y$, $\nu_2'(y') = x_3$ and $\nu_i \nu_1' = \xi_i \nu_2'$ for $i = \{1, 2\}$. We have that $\lambda_2 \nu_1 = \mu_2 \nu_2$, So, we obtain that $\lambda_2 \nu_1 \nu_1' = \mu_2 \nu_2 \nu_1'$. We also have that $\nu_i \nu_1' = \xi_i \nu_2'$. Then we get that $\lambda_2 \xi_1 \nu_2' = \mu_2 \xi_2 \nu_2'$. Therefore, $\lambda_2 \xi_1 = \mu_2 \xi_2$ since they agree on an open subset. So, $[\lambda_1, \tilde{U}_1, x_1, \lambda_2] = [\mu_1, \tilde{U}_2, x_2, \mu_2] \in \mathcal{G}(\mathcal{U})_1$. We can conclude that ω is faithful.

– We want to show that ω is full.
Let $[\lambda_1, \tilde{V}, x, \lambda_2] \in \mathcal{G}(\mathcal{W})_1$ where $\lambda_i : \tilde{V} \hookrightarrow \tilde{U}_i$ for $i = \{1, 2\}$ and $\tilde{U}_i \in \mathcal{U}$. Since $U_1 \cap U_2 \neq \emptyset$, there is $\tilde{U} \in \mathcal{U}$ with $y \in \tilde{U}$ and chart embeddings $\mu_1 : \tilde{U} \hookrightarrow \tilde{U}_1$ and $\mu_2 : \tilde{U} \hookrightarrow \tilde{U}_2$ such that $\mu_i(y) = \lambda_i(x)$ for $i = \{1, 2\}$ by Lemma 2.4.6. Since $\tilde{U}, \tilde{V} \in \mathcal{W}$ with $U \cap V \neq \emptyset$, there is a chart $\tilde{W} \in \mathcal{W}$ with $z \in \tilde{W}$ and chart embeddings $\nu_1 : \tilde{W} \hookrightarrow \tilde{V}$ and $\nu_2 : \tilde{W} \hookrightarrow \tilde{U}$ such that $\nu_1(z) = x$, $\nu_2(z) = y$ and $\lambda_1 \nu_1 = \mu_1 \nu_2$. We want to adjust μ_2 to make the other side of the diagram commutative.

$$\begin{array}{ccccc} & & \tilde{V} & & \\ & \swarrow^{\lambda_1} & \uparrow{\scriptstyle \nu_1} & \searrow^{\lambda_2} & \\ \tilde{U}_1 & = & & & \tilde{U}_2 \\ & \nwarrow_{\mu_1} & \downarrow{\scriptstyle \nu_2} & \nearrow_{\mu_2} & \\ & & \tilde{U} & & \end{array} .$$

We have $\lambda_2 \nu_1, \mu_2 \nu_2 : \tilde{W} \hookrightarrow \tilde{U}_2$. So. by Lemma 2.3.9, there is $g \in \mathcal{G}_{\tilde{U}_2}$, where $\mathcal{G}_{\tilde{U}_2}$ is the group that acts on $\tilde{U}_2$, such that $\lambda_2 \nu_1 = g \mu_2 \nu_2$. So, $[\lambda_1, \tilde{V}, x, \lambda_2] = [\mu_1, \tilde{U}, y, g\mu_2]$. As a result, ω is full.

So, we can conclude that ω is fully faithful

- We want to show that ω is essentially surjective.
We have the following diagram,

$$\begin{array}{ccccc} G(\mathcal{U})_0 \times_{G(\mathcal{W})_0} G(\mathcal{W})_1 & \xrightarrow{\pi_2} & G(\mathcal{W})_1 & \xrightarrow{t} & G(\mathcal{W})_0 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle s} & & \\ G(\mathcal{U})_0 & \xrightarrow[\omega_0]{} & G(\mathcal{W})_0 & & \end{array} .$$

Since $\mathcal{U} \subseteq \mathcal{W}$ and from the definition of ω, we have that $G(\mathcal{U})_0 \subseteq G(\mathcal{W})_0$. So, we have that ω_0 is an inclusion map. So, we have that $t\pi_2$ is open because all the maps in the previous diagram are open. Now we want to show that $t\pi_2$ is surjective. Let $\tilde{W} \in \mathcal{W}$. with $x \in \tilde{W}$. Let $\tilde{U} \in \mathcal{U}$ with $y \in \tilde{U}$ such that $\varphi_W(x) = \varphi_u(y)$. Then $\tilde{U} \in \mathcal{W}$. So, there is $[\lambda_1, \tilde{V}, z, \lambda_2] \in \mathcal{G}(\mathcal{W})_1$ such that $\lambda_1(z) = y$ and $\lambda_2(z) = x$. As a result, $t\pi_2$ is surjective. So, ω is essentially surjective.

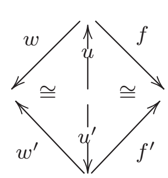
We can conclude that ω is a weak equivalence. □

7.1.2 The Strong Maps

If $\mathcal{U}$ and $\mathcal{W}$ are atlases such that $\mathcal{U} \subseteq \mathcal{W}$, then there is an induced strong map of atlases. This gives a morphism $\mathcal{C}(\mathcal{U}) \longrightarrow \mathcal{C}(\mathcal{W})$ which is in general not essentially surjective. However, we have shown that the induced morphism $\omega : \mathcal{G}(\mathcal{U}) \longrightarrow \mathcal{G}(\mathcal{W})$ is essentially surjective which implies that ω is a weak equivalence $\mathcal{G}(\mathcal{U}) \longrightarrow \mathcal{G}(\mathcal{W})$. However, weak equivalences do not necessarily have inverses. To solve this, we use the fact that the class of weak equivalences satisfies the conditions to form a bicategory of fractions by Theorm 6.2.1. We can invert them by making a bicategory with arrows of the form

$$\xleftarrow{w} \xrightarrow{f} , \tag{7.1}$$

where w is a weak equivalence. A category of fractions takes equivalence classes of them. The spans $\xleftarrow{w} \xrightarrow{f}$ and $\xleftarrow{w'} \xrightarrow{f'}$ are equivalent if there are u and u' such that we the following diagram commutes up to isomorphism,



,

and such that wu and $w'u'$ are weak equivalences. We will work with these equivalence classes. Since we have defined a weak equivalence, we can define a strong map between orbifolds as an equivalence class of spans of groupoid homomorphisms as in (7.1).

Definition 7.1.4. • Let $\mathcal{Q} = (Q, \mathcal{U})$ and $\mathcal{R} = (R, \mathcal{V})$ be two orbifolds. A *strong map of atlas groupoids* $f : \mathcal{Q} \longrightarrow \mathcal{R}$ is an equivalence class represented by a pair of groupoid homomorphisms (w, f'), also called a *generalized map*

$$\mathcal{G}(\mathcal{U}) \xleftarrow{\ w\ } \mathcal{K} \xrightarrow{\ f'\ } \mathcal{G}(\mathcal{V}) \tag{7.2}$$

where w is a weak equivalence and $\mathcal{K}$ is an étale groupoid. I.e., the source and target maps in $\mathcal{K}$ are local homeomorphisms.

- Let (w, g) and (w', g') be spans of groupoid homeomorphisms with w and w' weak equivalences. Then these two spans are *equivalent* if there exist a groupoid $\mathcal{M}$ with weak equivalences u and u', and natural transformations α and β as in the following diagram,

$$\begin{array}{ccccc} & & \mathcal{K} & & \\ & \swarrow^{w} & \uparrow u & \searrow^{f} & \\ \mathcal{G}(\mathcal{U}) & \cong^{\beta} & \mathcal{M} & \cong^{\alpha} & \mathcal{G}(\mathcal{V}) \\ & \nwarrow_{w'} & \downarrow u' & \nearrow_{f'} & \\ & & \ell & & \end{array} \tag{7.3}$$

Proposition 7.1.5. *Each equivalence class of generalized maps contains a representative consisting of maps of atlas groupoids,*

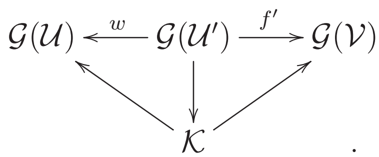
$$\mathcal{G}(\mathcal{U}) \xleftarrow{\ w\ } \mathcal{G}(\mathcal{U}') \xrightarrow{\ f'\ } \mathcal{G}(\mathcal{V}),$$

where w is weak equivalence. Also, two such representatives are equivalent if there exists a grupoid $\mathcal{G}(\mathcal{U}'')$ fitting in a diagram of the form,

$$\begin{array}{ccccc} & & \mathcal{G}(\mathcal{U}'_1) & & \\ & \swarrow & \uparrow & \searrow & \\ \mathcal{G}(\mathcal{U}) & \cong & \mathcal{G}(\mathcal{U}'') & \cong & \mathcal{G}(\mathcal{V}) \\ & \nwarrow & \downarrow & \nearrow & \\ & & \mathcal{G}(\mathcal{U}'_2) & & \end{array},$$

where $\mathcal{U}'_i$ for $i = \{1, 2\}$ are refinements of $\mathcal{U}$ and $\mathcal{U}''$ is a common refinement of $\mathcal{U}'_i$.

Proof. Pronk, in [18], showed that for any étale groupoid $\mathcal{K}$ with a weak equivalence $\mathcal{K} \longrightarrow \mathcal{G}(\mathcal{U})$, there is an atlas $\mathcal{U}'$ with a weak equivalence $\mathcal{G}(\mathcal{U}') \longrightarrow \mathcal{K}$. This gives us the following diagram,

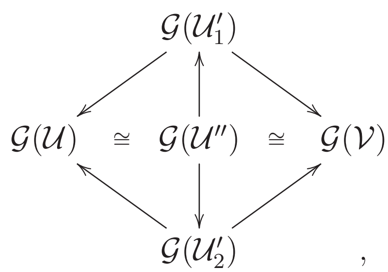


We have that w is weak equivalence. So, we can replace in (7.3) K by $\mathcal{G}(\mathcal{U})'_1$ and tℓ by $\mathcal{G}(\mathcal{U})'_2$ as well as M by $\mathcal{G}(\mathcal{U})''$. □

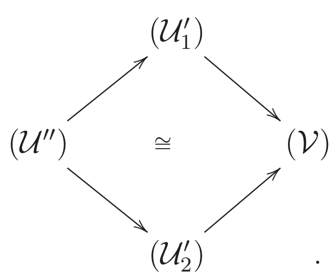
Remark 7.1.6. In this remark, we discuss the relation between the strong maps of atlas groupoids and the strong maps of atlases defined in 3.2.2.

1. Since $w : \mathcal{G}(\mathcal{U}') \longrightarrow \mathcal{G}(\mathcal{U})$ is a weak equivalence, it gives us that $\mathcal{U}'$ is a refinement of $\mathcal{U}$.

2. We have from the previous proposition that two strong maps of atlas groupoids are equivalent if there is $\mathcal{G}(\mathcal{U}'')$ with maps into $\mathcal{G}(\mathcal{U}')_\infty$ and $\mathcal{G}(\mathcal{U}'')_\in$ as in the following diagram,



where $\mathcal{U}'_i$ for $i = \{1, 2\}$ are refinements of $\mathcal{U}$ and $\mathcal{U}''$ is a common refinement of the $\mathcal{U}'_i$. This gives us the equivalence relation between two strong maps of atlases since $\mathcal{U}''$ is a common refinement of the $\mathcal{U}'_i$ as follows,



Definition 7.1.7. Let $\mathcal{Q} = (Q, \mathcal{U})$, $\mathcal{R} = (R, \mathcal{V})$, and $\mathcal{S} = (S, \mathcal{W})$ be orbifolds. Let $f : \mathcal{Q} \longrightarrow \mathcal{R}$ and $g : \mathcal{R} \longrightarrow \mathcal{S}$ be strong maps of orbifolds which correspond to pairs of groupoid homomorphisms (w, f') and (w', g'). I.e.,

Then the *composition of strong maps* is defined by choosing a pair of groupoid homomorphisms (u, h) where u is weak equivalence such that the following diagram commutes up to isomorphism,

$$\begin{array}{ccccccc} & & & \overset{u}{\swarrow} \quad \overset{h}{\searrow} & & & \\ \xleftarrow{\;w\;} & & \xrightarrow{\;f'\;} & & \xleftarrow{\;w'\;} & & \xrightarrow{\;g'\;} \end{array},$$

so, the composition is $(wu, g'h)$.

7.2 Paths in an Orbifold Atlas

A path in the space is a map from the unit interval to that space. The unit interval I is a manifold with boundary. Therefore, it is an orbifold with boundary. We may represent it by its unit groupoid

$$I \times I \xrightarrow{\;m\;} I \underset{t}{\overset{s}{\rightrightarrows}} I \;, \quad u: I \to I, \quad i: I \to I$$

where the soure and the target maps are the identity maps. This implies that paths in an orbifold $\mathcal{Q} = (Q, \mathcal{U})$ can be represented as a strong map of atlas groupoids $I \longrightarrow \mathcal{G}(\mathcal{U})$. This can be represented by spans of groupoid homomorphisms

$$I \xleftarrow{\;w\;} I' \xrightarrow{\;\alpha\;} \mathcal{G}(\mathcal{U})\,,$$

where w is a weak equivalence. It follows that we may assume that I'_0 consists of a collection of open subsets of I which covers I and $I'_1 = I \times_{I\times I} (I_{\sigma 0} \times I_{\sigma 0})$, together with the necessary maps to make this groupoid weakly equivalent to I. A path in an orbifold can be represented by a sequence of paths in charts connected by jumps. This implies that we get a continuous path in the quotient space. However, there may be more than one orbifold path corresponding to the same path in the quotient space.

7.2.1 Paths Consisting of Open Intervals

We translate this to groupoid language $I \xleftarrow{\;w\;} I_\sigma \xrightarrow{\;\alpha\;} \mathcal{G}(\mathcal{U})$ where the space of objects of the unit interval I_0 = the space of arrows I_1 = the unit interval I where the source and the target are the identity maps. We may assume that $I_{\sigma 0} = \coprod u_i$ where $u_i \subseteq I$ are open such that $\cup u_i = I$. Since we take disjoint unions, we assume that each u_i is connected, i.e., an interval of the form $(a, b), (a, b]$, or $[a, b)$. Since I is compact, we may assume that we have just finitely many u_i, and that they are ordered as below,

$$I_{\sigma 0} = [0, a_1) \coprod (b_1, a_2) \coprod (b_2, a_3) \coprod \ldots \coprod (b_n, 1]$$

such that $0 < b_1 < a_1 < \ldots < b_n < a_n < 1$. This makes that $I_{\sigma 1} = I \times_{I \times I} (I_{\sigma 0} \times I_{\sigma 0})$ because it is a weak equivalence since it is fully faithful. So, we can define $I_{\sigma 1}$ as follows

$$I_{\sigma 1} = I_{\sigma 0} \coprod \coprod_{i=1}^{n} (b_i, a_i).$$

Now we can say that a path consists of

$$\alpha_i : (b_i, a_{i+1}) \longrightarrow \tilde{U}_i \,,$$

$$\alpha_0 : [0, a_1) \longrightarrow \tilde{U}_0 \,,$$

and

$$\alpha_n : (b_n, 1] \longrightarrow \tilde{U}_n \,,$$

together with jump maps $j_i : (b_i, a_{i+1}) \longrightarrow (\lambda_1^i, \tilde{W}_i, \lambda_2^i)$.

7.2.2 Paths Consisting of Closed intervals

Observation, if we take points $t_i \in (b_i, a_i)$ in the overlaps, since the paths consisting of open intervals are one to one correspondence, we can make paths in closed interval. Also, we can make paths in open interval from paths in closed interval because the atlas groupoids are étale. We will define paths in closed interval in the following definition.

Definition 7.2.1. An *atlas path* in an orbifold $(Q, \mathcal{U})$ is represented by:

- A subdivision of the unit interval I into $[t_{i-1}, t_i]$, $1 \leq i \leq n$,

Let $\sigma = (0, t_1, t_2, \ldots, t_n)$; we write I_σ for the unit interval with this subdivision.

- A family of paths $\alpha_i : [t_{i-1}, t_i] \longrightarrow \tilde{U}_i$ with $\tilde{U}_i \in \mathcal{U}$.
- For each $i \in \{1, 2, \ldots, n-1\}$, a *jump*

$$j_{t_i} = [\lambda_1^i, \tilde{W}_i, x_i, \lambda_2^i] \in G(\mathcal{U})_1$$

  defined by:

  - $W_i \subseteq U_i \cap U_{i+1} \subseteq Q$
  - Chart embeddings $\lambda_1^i : \tilde{W}_i \hookrightarrow \tilde{U}_i$ and $\lambda_2^i : \tilde{W}_i \hookrightarrow \tilde{U}_{i+1}$ such that there is a point $x_i \in \tilde{W}_i$ with chart embeddings $\lambda_1^i(x_i) = \alpha_i(t_i)$ and $\lambda_2^i(x_i) = \alpha_{i+1}(t_i)$.

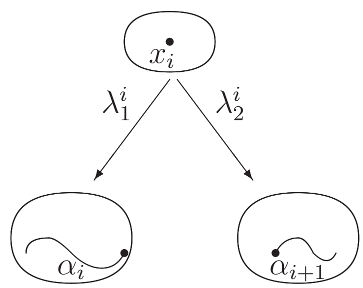


Figure 7.1: Atlas path with one jump.

When we take a point $x \in [t_{i-1}, t_i]$, we can form $\tilde{I}_\sigma = \coprod_{i=1}^n [t_{i-1}, t_i] \longrightarrow I_{\sigma 0}$. We have a pullback,

$$\begin{array}{ccc} \tilde{I}_{\sigma 1} & \longrightarrow & I_{\sigma 1} \\ \downarrow & & \downarrow \\ \tilde{I}_{\sigma 0} \times \tilde{I}_{\sigma 0} & \hookrightarrow & I_{\sigma 0} \times I_{\sigma 0} \end{array}$$

Therefore, $\tilde{I}_{\sigma 1} = \tilde{I}_{\sigma 0} \coprod_{i=1}^n j_i \coprod_{i=1}^n j_i^{-1}$

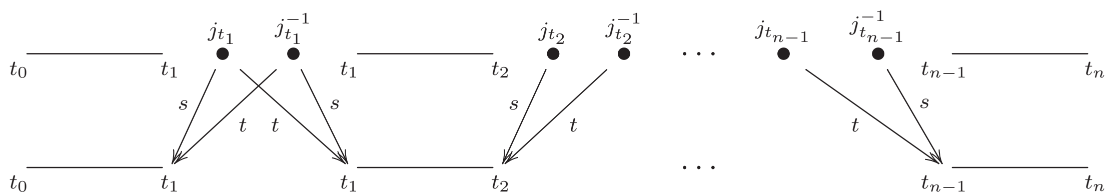
$$\begin{array}{ccccc} \tilde{I}_{\sigma 0} \times I_{\sigma 0} I_{\sigma 1} & \longrightarrow & I_{\sigma 1} & \xrightarrow{t} & I_{\sigma 0} \\ \downarrow & & \downarrow{\scriptstyle s} & & \\ \tilde{I}_{\sigma 0} & \hookrightarrow & I_{\sigma 0} & & \end{array}$$

The map $\tilde{I}_\sigma \longrightarrow I_\sigma$ is fully faithful because we we choose $\tilde{I}_{\sigma 1}$ to be the pullback. If we have $\tilde{I}_\sigma \longrightarrow \mathcal{G}$, we can extend the map $I_\sigma \longrightarrow \mathcal{G}$. We use $\tilde{I}_\sigma$ instead of I_σ. So, a path $\alpha : I \longrightarrow \mathcal{G}$ corresponds to a groupoid homomorphism $\tilde{I}_\sigma \longrightarrow \mathcal{G}$ for some subdivision σ of I, where $\tilde{I}_\sigma$ is the groupoid of the subdivision. This has

- Space of objects : $(\tilde{I}_\sigma)_0 = \coprod_{i=1}^{n}[t_{i-1}, t_i]$

- Space of arrows : $(\tilde{I}_\sigma)_1 = \coprod_{i=1}^{n}[t_{i-1}, t_i]$ as identity arrows with jumps $\{j_{t_1}, j_{t_2}, \ldots, j_{t_{n-1}}\}$ and their inverses $\{j_{t_1}^{-1}, j_{t_2}^{-1}, \ldots, j_{t_{n-1}}^{-1}\}$ such that $s(j_{t_i}) = t_i \in [t_{i-1}, t_i]$ and $t(j_{t_i}) = t_i \in [t_i, t_{i+1}]$. We will denote the identity arrow on x by id_x. When there is potential confusion, we write t_i^l for $t_i \in [t_{i-1}, t_i]$ and t_i^r for $t_i \in [t_i, t_{i+1}]$. For instance, the identity arrow on t_i^r in the space of arrows will be $\mathrm{id}_{t_i^r} = t_i \in [t_i, t_{i+1}] \subseteq (I_\sigma)_1$.

- Composition : since all arrows are invertible, $j_{t_i} \circ j_{t_i}^{-1} = \mathrm{id}_{t_i^l}$ and $j_{t_i}^{-1} \circ j_{t_i} = \mathrm{id}_{t_i^r}$. Also, there are $\mathrm{id}_{t_i^r} \circ j_{t_i} = j_{t_i}$ and $j_{t_i} \circ \mathrm{id}_{t_i^l} = j_{t_i}$.



This means that a path in a groupoid is defined by a map $\tilde{I}_\sigma \longrightarrow \mathcal{G}$ for some subdivision σ. This consists of

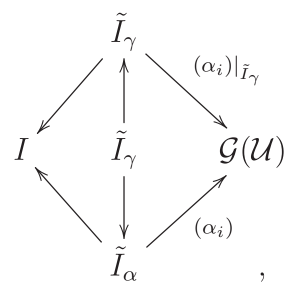
- On objects by $\alpha_i : [t_{i-1}, t_i] \longrightarrow \mathcal{G}$ for $i = 1, 2, \ldots, n$.

- On arrows by a sequence of paths with jumps $(\tilde{\alpha_1}, j_{t_1}^{\alpha}, \tilde{\alpha_2}, j_{t_2}^{\alpha} \ldots, j_{t_{n-1}}^{\alpha}, \tilde{\alpha_n})$ such that:

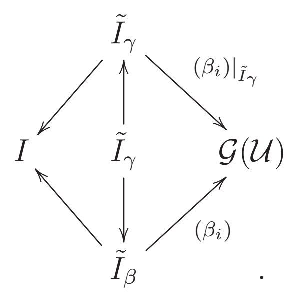
  - On the identity arrows α_i agrees with the α_i on the objects.
  - For each jump j_{t_i} we obtain $j_{t_i}^{\alpha} \in G_1$ with $s(j_{t_i}^{\alpha}) = \alpha_{i-1}(t_i)$ and $t(j_{t_i}^{\alpha}) = \alpha_i(t_i)$.

We translate the equivalence relation on the strong maps to an equivalence relation on orbifold paths which is defined as follows. If we have subdivisions $\tilde{I}_\sigma = \coprod_{i=1}^{n}[t_{i-1}, t_i]$ with α_i and $\tilde{I}_s = \coprod_{j=1}^{m}[r_{j-1}, j_i]$ with β_i, then we can find another subdivision $\tilde{I}_\gamma =$

$\coprod_{i=1}^{n}[t_{i-1}, t_i] \coprod \coprod_{j=1}^{m}[r_{j-1}, j_i]$ such that $\alpha_i|_{\tilde{I}_\gamma} = \beta_i|_{\tilde{I}_\gamma}$. So, we have the following compositions,



and



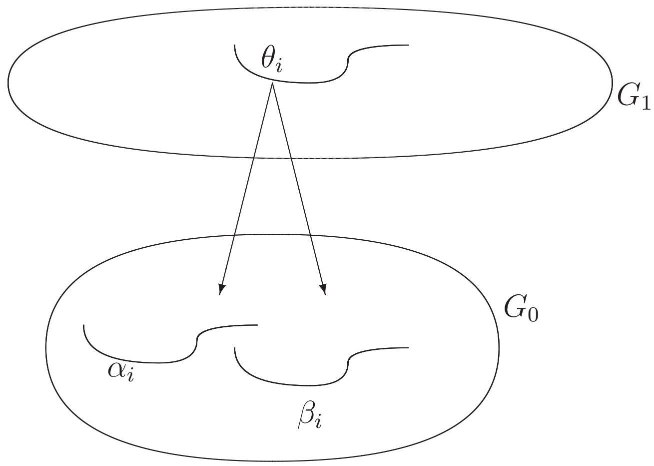
So, we have that $\alpha \cong \alpha|_{\tilde{I}_\gamma}$ and $\beta \cong \beta|_{\tilde{I}_\gamma}$. As a result, it is enough to study when two paths with the same subdivision are equivalent.

7.3 Natural Transformations Between Paths

Suppose there is a second path $\beta : \tilde{I}_s \longrightarrow \mathcal{G}$. If there is a natural transformation θ from α to β, then we can consider α and β as equivalent paths because θ is invertible since $\mathcal{G}$ is a groupoid. Therefore, α and β are isomorphic. Recall that, a natural transformation θ is given by a function from the space of objects of $\tilde{I}_s$, $\coprod_{i=1}^{n}[t_{i-1}, t_i]$, to G_1. I.e.,

$$\theta : \coprod_{i=1}^{n}[t_{i-1}, t_i] \longrightarrow G_1.$$

This means that θ consists of a family of functions, $\theta_i : [t_{i-1}, t_i] \longrightarrow G_1$ such that $s \circ \theta_i = \alpha_i$ and $t \circ \theta_i = \beta_i$.



Also, we need to check the naturality. So, for each $y \in (\tilde{I}_s)_1$, we need to have that $m(\theta t, \beta)(y) = m(\alpha, \theta s)(y)$. This is trivially true whenever $y = u(x)$ for some $x \in (\tilde{I}_s)_0$. So, we just need to check the naturality at the jumping arrows $j^k_{t_i} : k_i(t_i) \longrightarrow k_{i+1}(t_i)$ where k is α or β. This means that we require that $m(\theta_{i+1}(t_i), j^\alpha_{t_i}) = m(j^\beta_{t_i}, \theta_i(t_i))$, i.e.

$$\begin{array}{ccc} \alpha_i(t_i) & \xrightarrow{\theta_i(t_i)} & \beta_i(t_i) \\ {\scriptstyle j^\alpha_{t_i}}\downarrow & & \downarrow{\scriptstyle j^\beta_{t_i}} \\ \alpha_{i+1}(t_i) & \xrightarrow[\theta_{i+1}(t_i)]{} & \beta_{i+1}(t_i) \end{array} \tag{7.4}$$

commutes in $\mathcal{G}$

Example 7.3.1. Let $\mathcal{G} = \mathbb{Z}_4$ and let $\tilde{U} = D$ be the unit disk. Suppose $\mathcal{G}$ acts on $\tilde{U}$ by rotation over $\frac{\pi}{2}$. Recall that we write $\mathcal{G} \ltimes \tilde{\mathcal{U}}$ for the translation groupoid of the action. The space of objects is $\tilde{U}$ and space of arrows is $\mathcal{G} \times \tilde{\mathcal{U}}$. We will denote the different copies of $\tilde{U}$ in G_1 by $G_1 = \coprod_{\tau \in G}\{(x, id, \tau) | x \in \tilde{U}\}$. For each $\tau \in G$, define the source map by $s(x, id, \tau) = x$, and the target map by $t(x, id, \tau) = \tau(x)$. Let $I_s = [0, \frac{1}{2}], [\frac{1}{2}, 1]$, and let $(\alpha_1, j^\alpha_{\frac{1}{2}}, \alpha_2)$ and $(\beta_1, j^\beta_{\frac{1}{2}}, \beta_2)$ be paths on this subdivision. I.e, $\alpha_1, \beta_1 : [0, \frac{1}{2}] \longrightarrow \tilde{U}$ and $\alpha_2, \beta_2 : [\frac{1}{2}, 1] \longrightarrow \tilde{U}$. Now let $(\alpha_1, j^\alpha_{\frac{1}{2}}, \alpha_2) : [0, \frac{1}{2}] \coprod [\frac{1}{2}, 1] \longrightarrow G_0$ with $j^\alpha_{\frac{1}{2}} = (\alpha_1(\frac{1}{2}), id, id)$ be the path in Figure (7.2), and $(\beta_1, j^\beta_{\frac{1}{2}}, \beta_2) : [0, \frac{1}{2}] \coprod [\frac{1}{2}, 1] \longrightarrow G_0$ with $j^\beta_{\frac{1}{2}} = (\alpha_1(\frac{1}{2}), id, \rho)$ be the path in Figure (7.3).

Note that $\alpha_1 = \beta_1$ but $\beta_2 = \rho\alpha_2$. By the previous discussion, $(\alpha_1, j^\alpha_{\frac{1}{2}}, \alpha_2)$ is equivalent to $(\beta_1, j^\beta_{\frac{1}{2}}, \beta_2)$ when there is a natural transformation represented by a continuous

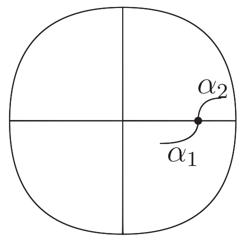


Figure 7.2: The paths α_1 and α_2

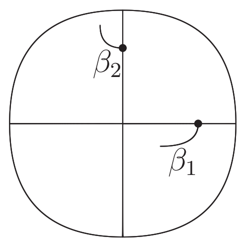


Figure 7.3: The paths β_1 and β_2

function $\theta : [0, \frac{1}{2}] \coprod [\frac{1}{2}, 1] \longrightarrow G_1$ which is defined by $\theta_1(t) = (\alpha_1(t), id, id)$, $\theta_2(t) = (\alpha_2(t), id, \rho)$.

Since, $s(\theta_1(t)) = s(\alpha_1(t), id, id) = \alpha_1(t)$ and $s(\theta_2(t)) = s(\alpha_2(t), id, \rho) = \alpha_2(t)$,

$$s(\theta(t)) = \begin{cases} \alpha_1(t) & \text{if} \quad 0 \leq t \leq \frac{1}{2} \\ \\ \alpha_2(t) & \text{if} \quad \frac{1}{2} \leq t \leq 1. \end{cases}$$

Analogously,

$$t(\theta(t)) = \begin{cases} \alpha_1(t) = \beta_1(t) & \text{if} \quad 0 \leq t \leq \frac{1}{2} \\ \\ \rho\alpha_2(t) = \beta_2(t) & \text{if} \quad \frac{1}{2} \leq t \leq 1. \end{cases}$$

Finally we check naturality at $j_{\frac{1}{2}}$

since we have $\theta_2 \circ j^{\alpha}_{\frac{1}{2}} = \rho \circ id$ and $j^{\beta}_{\frac{1}{2}} \circ \theta_1 = \rho \circ id$, the following square commutes.

$$\begin{array}{ccc} \alpha_1(\frac{1}{2}) & \xrightarrow{\theta_1(\frac{1}{2})} & \beta_1(\frac{1}{2}) \\ {\scriptstyle j^{\alpha}_{\frac{1}{2}}}\downarrow & & \downarrow{\scriptstyle j^{\beta}_{\frac{1}{2}}} \\ \alpha_2(\frac{1}{2}) & \xrightarrow[\theta_2(\frac{1}{2})]{} & \beta_2(\frac{1}{2}) \end{array}$$

In general: let $(\alpha_1, j^{\alpha}_{\frac{1}{2}}, \alpha_2)$ and $(\beta_1, j^{\beta}_{\frac{1}{2}}, \beta_2)$ be paths on the subdivision $\tilde{I}_s = [0, \frac{1}{2}]\coprod[\frac{1}{2}, 1]$ such that $\alpha_1, \beta_1 : [0, \frac{1}{2}] \longrightarrow \tilde{U}$ and $\alpha_2, \beta_2 : [\frac{1}{2}, 1] \longrightarrow \tilde{U}$ with jump $j_{\frac{1}{2}}$ such that $j^{\alpha}_{\frac{1}{2}} = (\alpha_1(\frac{1}{2}), id, \tau_\alpha)$ with $\tau_\alpha \in \mathcal{G}$ where $\tau_\alpha(\alpha_1(\frac{1}{2})) = \alpha_2(\frac{1}{2})$ and $j^{\beta}_{\frac{1}{2}} = (\beta_1(\frac{1}{2}), id, \tau_\beta)$ with $\tau_\beta \in \mathcal{G}$ where $\tau_\beta(\beta_1(\frac{1}{2})) = \beta_2(\frac{1}{2})$. Then, $(\alpha_1, j^{\alpha}_{\frac{1}{2}}, \alpha_2)$ and $(\beta_1, j^{\beta}_{\frac{1}{2}}, \beta_2)$ are equivalent if $\exists g_1, g_2 \in \mathcal{G}$ such that $g_1(\alpha_1(t)) = \beta_1(t)$ and $g_2(\alpha_2(t)) = \beta_2(t)$. This gives us a natural transformation represented by $\theta_1(t) = (\alpha_1(t), id, g_1)$ if $0 \leq t \leq \frac{1}{2}$ and $\theta_2(t) = (\alpha_2(t), id, g_2)$ if $\frac{1}{2} \leq t \leq 1$ such that there is naturality at jump $j_{\frac{1}{2}}$ which gives a commutative square

$$\begin{array}{ccc} \alpha_1(\frac{1}{2}) & \xrightarrow{\theta_1(\frac{1}{2})} & \beta_1(\frac{1}{2}) \\ {\scriptstyle j^{\alpha}_{\frac{1}{2}}}\downarrow & & \downarrow{\scriptstyle j^{\beta}_{\frac{1}{2}}} \\ \alpha_2(\frac{1}{2}) & \xrightarrow[\theta_2(\frac{1}{2})]{} & \beta_2(\frac{1}{2}) \end{array}$$

So, we need that $g_1 \circ \tau_\beta = \tau_\alpha \circ g_2$ in $\mathcal{G}$.

Remark 7.3.2. We can always find such a natural transformation when $\alpha_1(t_1)$ is not a fixed point. For example, the center O of the disk is not a good point because in this case there is only one arrow from $\alpha_1(\frac{1}{2})$ to $\beta_2(\frac{1}{2})$ in the groupoid $\mathcal{G} \ltimes X$. If $\alpha_1(\frac{1}{2}) = \beta_2(\frac{1}{2}) = O$, then it is possible to have non equivalent paths that map to the same path in the quotient orbifold as the following example shows.

Example 7.3.3. Let $\mathcal{G} = \mathbb{Z}_4$ and let $\tilde{U} = D$ be the unit disk. Suppose $\mathcal{G}$ acts on $\tilde{U}$ by rotation over $\frac{\pi}{2}$. Let $I_s = [0, \frac{1}{2}], [\frac{1}{2}, 1]$, and let $(\alpha_1, j^{\alpha}_{\frac{1}{2}}, \alpha_2)$ and $(\beta_1, j^{\beta}_{\frac{1}{2}}, \beta_2)$ be paths on this subdivision.
I.e, $\alpha_1, \beta_1 : [0, \frac{1}{2}] \longrightarrow \tilde{U}$ and $\alpha_2, \beta_2 : [\frac{1}{2}, 1] \longrightarrow \tilde{U}$. Let $(\alpha_1, j^{\alpha}_{\frac{1}{2}}, \alpha_2) : [0, \frac{1}{2}]\coprod[\frac{1}{2}, 1] \longrightarrow G_0$ with $j^{\alpha}_{\frac{1}{2}} = (\alpha_1(\frac{1}{2}), id, \rho)$ be the path in Figure 7.4. $(\beta_1, j^{\beta}_{\frac{1}{2}}, \beta_2) : [0, \frac{1}{2}]\coprod[\frac{1}{2}, 1] \longrightarrow G_0$ with $j^{\beta}_{\frac{1}{2}} = (\alpha_1(\frac{1}{2}), id, \rho^2)$ be the path in Figure (7.5). Note that $\alpha_1(\frac{1}{2}) = \beta_1(\frac{1}{2}) = 0$ and $\alpha_1 = \beta_1$ but $\beta_2 = \rho\alpha_2$.

Let $\theta : [0, \frac{1}{2}]\coprod[\frac{1}{2}, 1] \longrightarrow G_1$ represent a natural transformation which is defined by $\theta_1(t) = (\alpha_1(t), id, id)$, $\theta_2(t) = (\alpha_2(t), id, \rho)$.

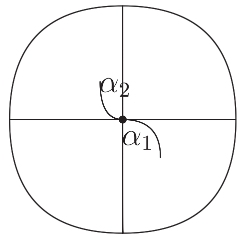


Figure 7.4: The paths α_1 and α_2

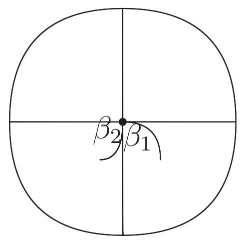


Figure 7.5: The paths β_1 and β_2

Since, $s(\theta_1(t)) = s(\alpha_1(t), id, id) = \alpha_1(t)$ and $s(\theta_2(t)) = s(\alpha_2(t), id, \rho) = \alpha_2(t)$,

$$s(\theta(t)) = \begin{cases} \alpha_1(t) & \text{if} \quad 0 \leq t \leq \frac{1}{2} \\ \\ \alpha_2(t) & \text{if} \quad \frac{1}{2} \leq t \leq 1. \end{cases}$$

Similarity,

$$t(\theta(t)) = \begin{cases} \alpha_1(t) = \beta_1(t) & \text{if} \quad 0 \leq t \leq \frac{1}{2} \\ \\ \rho\alpha_2(t) = \beta_2(t) & \text{if} \quad \frac{1}{2} \leq t \leq 1. \end{cases}$$

now we check naturality at $j_{\frac{1}{2}}$

since we have $\theta_2 \circ j^{\alpha}_{\frac{1}{2}}(t_1) = \rho \circ id$ and $j^{\beta}_{\frac{1}{2}} \circ \theta_1(t_1) = \rho^2 \circ id$, the following square does not commute.

$$\begin{array}{ccc} \alpha_1(\frac{1}{2}) & \xrightarrow{\theta_1(\frac{1}{2})} & \beta_1(\frac{1}{2}) \\ {\scriptstyle j^{\alpha}_{\frac{1}{2}}}\downarrow & & \downarrow{\scriptstyle j^{\beta}_{\frac{1}{2}}} \\ \alpha_2(\frac{1}{2}) & \xrightarrow[\theta_2(\frac{1}{2})]{} & \beta_2(\frac{1}{2}) \end{array}$$

So that the paths α and β are not equivalent.

7.4 The Equivalence Relation on Paths

Let $\alpha = (\alpha_1, j^{\alpha}_{t_1}, \alpha_2, \ldots, \alpha_n) = ([\alpha_1, \lambda^1_1, \tilde{W}_1, x_1, \lambda^1_2, \ldots, \alpha_i, \lambda^i_1, \tilde{W}_i, x_i, \lambda^i_2, \alpha_{i+1}, \lambda^{i+1}_1, \ldots, \alpha_n])$ where
$\alpha_i : [t_{i-1}, t_i] \longrightarrow \tilde{U}_i$ for $\tilde{U}_i \in \mathcal{U}$, and
$\beta = (\beta_1, j^{\beta}_{t_1}, \beta_2, \ldots, \beta_n) = ([\beta_1, \mu^1_1, \tilde{W}'_1, y_1, \mu^1_2, \ldots, \beta_i, \mu^i_1, \tilde{W}'_i, y_i, \mu^i_2, \beta_{i+1}, \mu^{i+1}_1, \ldots, \beta_n])$ where $\beta_i : [t_{i-1}, t_i] \longrightarrow \tilde{V}_i$ for $\tilde{V}_i \in \mathcal{U}$ be paths in the atlas groupoid $\mathcal{G}(\mathcal{U})$. We want to discuss what a natural transformation θ from α to β looks like. The simplest way to see a natural transformation consists of map. As we discussed before, there is a chart $\tilde{S}_i$ and a natural transformation $\theta_i : [t_{i-1}, t_i] \longrightarrow S_i$ with embeddings $\nu^i_1 : \tilde{S}_i \longrightarrow \tilde{U}_i$ and $\nu^i_2 : \tilde{S}_i \longrightarrow \tilde{V}_i$ such that $\nu^i_1 \circ \theta_i = \alpha_i$ and $\nu^i_2 \circ \theta_i = \beta_i$.

Remark 7.4.1. If α', β' are open paths and α, β are the corresponding closed paths. Then the natural transformations $\alpha' \Rightarrow \beta'$ are in one to one correspondent with the natural transformation $\alpha \Rightarrow \beta$.

Lemma 7.4.2. *Let α and $\beta : I \longrightarrow \tilde{U}$ be paths in one chart. Then α and β are equivalent if and only if there is a group element $g \in \mathcal{G}$ such that $g\alpha = \beta$.*

Proof. Suppose there is a group element $g \in \mathcal{G}$ such that $g\alpha = \beta$. Let $\theta : I \longrightarrow \mathcal{G} \times \tilde{U}$ be a map defined as $\theta(x) = [id_{\tilde{U}}, \tilde{U}, \alpha(x), g]$ for $x \in I$. We want to show that θ is a natural transformation from α to β. We have that $s\theta(x) = s[id_{\tilde{U}}, \tilde{U}, \alpha(x), g] = t(id_{\tilde{U}}(\alpha(x))) = \alpha(x)$ So, we can conclude that $s\theta = \alpha$. Also, we have that $t\theta(x) = t[id_{\tilde{U}}, \tilde{U}, \alpha(x), g] = t(g(\alpha(x))) = \beta(x)$ So, we can conclude that $t\theta = \beta$. Since there are not jumps, we have the naturality. Now suppose that α and β are equivalent. We need to show that there is a group element $g \in \mathcal{G}$ such that $g\alpha = \beta$. Since we have that α and β are equivalent, there is a natural transformation $\theta : I \longrightarrow G \times \tilde{U}$. Since I is connected, there is an element $g \in \mathcal{G}$ such that $\theta(I) \subseteq \{g\} \times \tilde{U}$. So, *theta* is of the form $\theta(x) = [id_{\tilde{U}}, \tilde{U}, \alpha(s), g]$. We have $t\theta = g\alpha$. Since $t\theta = \beta$, we can conclude that $\beta = g\alpha$. □

Proposition 7.4.3. *Let α and β be paths such that $\alpha = (\alpha_1, j^{\alpha}_{t_1}, \alpha_2, \ldots, \alpha_n) = ([\alpha_1, \lambda^1_1, \tilde{W}_1, x_1, \lambda^1_2, \ldots, \alpha_i \lambda^i_1, \tilde{W}_i, x_i, \lambda^i_2, \alpha_{i+1}, \lambda^{i+1}_1, \ldots, \alpha_n])$ where $\alpha_i : [t_{i-1}, t_i] \longrightarrow \tilde{U}_i$ for $\tilde{U}_i \in \mathcal{U}$, and*
$\beta = (\beta_1, j^{\beta}_{t_1}, \beta_2, \ldots, \beta_n) = ([\beta_1, \mu^1_1, \tilde{W}'_1, y_1, \mu^1_2, \ldots, \beta_i, \mu^i_1, \tilde{W}'_i, y_i, \mu^i_2, \beta_{i+1}, \mu^{i+1}_1, \ldots, \beta_n])$

where $\beta_i : [t_{i-1}, t_i] \longrightarrow \tilde{V}_i$ for $\tilde{V}_i \in \mathcal{U}$. α and β are equivalent if and only if for all $1 \leq i \leq n$, there exist charts $\tilde{T}_i$ with embeddings $\gamma_1^i : \tilde{T}_i \longrightarrow \tilde{S}_i$ and $\gamma_2^i : \tilde{T}_i \longrightarrow \tilde{W}_i$ such that the diagram of chart embeddings in (7.6) commutes.

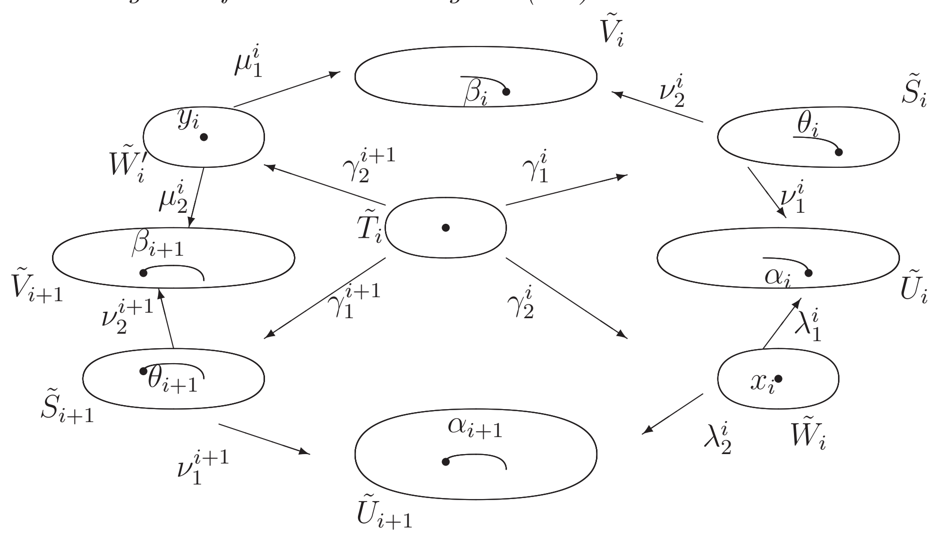
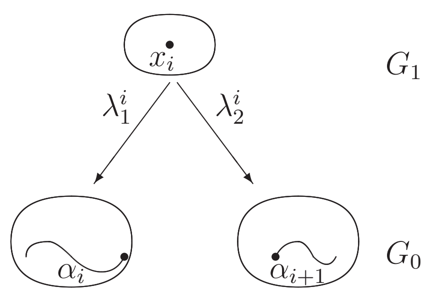
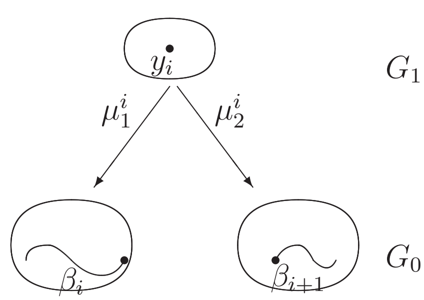


Figure 7.6: equivalent paths

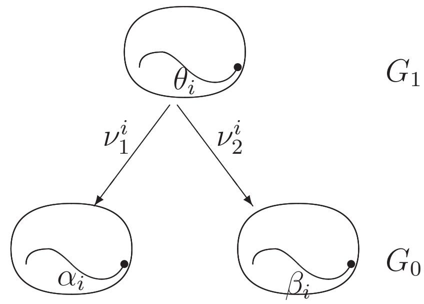
Proof. We will use the atlas path and the natural transformation definitions to prove this lemma. Since we have that α is a path such that $\alpha = ([\alpha_1, \lambda_1^1, \tilde{W}_1, x_1, \lambda_2^1, \alpha_2, \ldots, \alpha_n])$ where $\alpha_i : [t_{i-1}, t_i] \longrightarrow \tilde{U}_i \subseteq G_0$ and $\tilde{W}_i \subseteq G_1$ with embeddings $\lambda_1^i : \tilde{W}_i \longrightarrow \tilde{U}_i$ and $\lambda_2^i : \tilde{W}_i \longrightarrow \tilde{U}_{i+1}$. From the definition of an atlas path we get the following diagram.



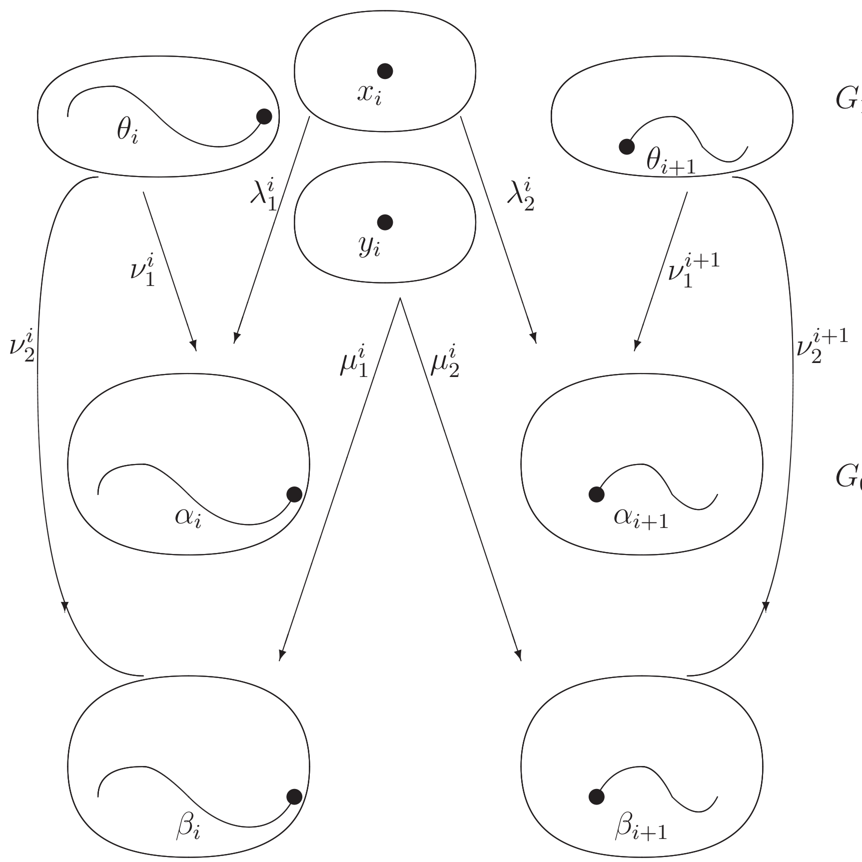
Similarly with $\beta = (\beta_1, \mu_1^i, \tilde{W}'_i, y_i, \mu_2^i, \beta_2, \ldots, \beta_n)$ we have the following diagram



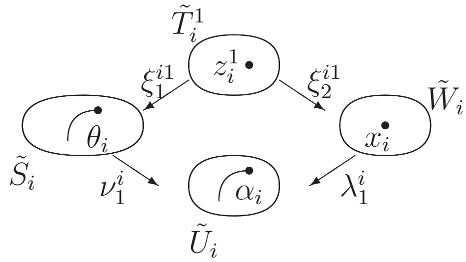
We have paths α_i and β_i for each $1 \le i \le n$, so we can define a natural transformation $\theta_i : [t_{i-1}, t_i] \longrightarrow G_1$ with chart $\tilde{S}_i$ and embeddings $\nu_1^i : \tilde{S}_i \longrightarrow \tilde{U}_i$ and $\nu_2^i : \tilde{S}_i \longrightarrow \tilde{V}_i$ such that $\nu_1^i(\theta_i)\theta = \alpha_i$ and $\nu_2^i(\theta_i)\theta = \beta_i$. Therefore, we get the following diagram.



From these three diagrams we get the following figure



From this diagram there exist $\tilde{U}_i^1$ and $\tilde{U}_i^2$ in $\tilde{U}_i$ with $\tilde{U}_i^1 \cap \tilde{U}_i^2 \neq \emptyset$. Then we need to check the naturality. We need to use Lemma 2.4.6. Therefore, there exist $\tilde{T}_i^1$ with point $z_i^1 \in \tilde{T}_i^1$ and embeddings $\xi_1^{i1} : \tilde{T}_i^1 \longrightarrow \tilde{S}_i$ and $\xi_2^{i1} : \tilde{T}_i^1 \longrightarrow \tilde{W}_i$ such that $\lambda_1^i \xi_2^{i1} = \nu_1^i \xi_1^{i1}$



Similarly with each pair of charts $(\tilde{S}_i, \tilde{W'}_i)$, $(\tilde{S}_{i+1}, \tilde{W}_i)$, $(\tilde{S}_{i+1}, \tilde{W'}_i)$ for each $1 \leq i \leq n$. Therefore, we find four atlas charts with points and embeddings which are

$$(\tilde{S}_i, \xi_1^{i1}, \tilde{T}_i^1, z_i^1, \xi_2^{i1}, \tilde{W}_i),$$

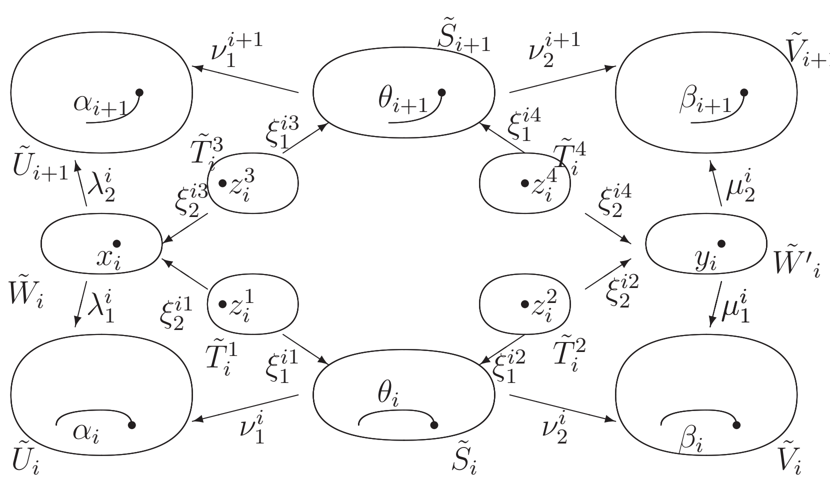
$$(\tilde{S}_i, \xi_1^{i2}, \tilde{T}_i^2, z_i^2, \xi_2^{i2}, \tilde{W'}_i),$$

$$(\tilde{S}_{i+1}, \xi_1^{i3}, \tilde{T}_i^3, z_i^3, \xi_2^{i3}, \tilde{W}_i),$$

and

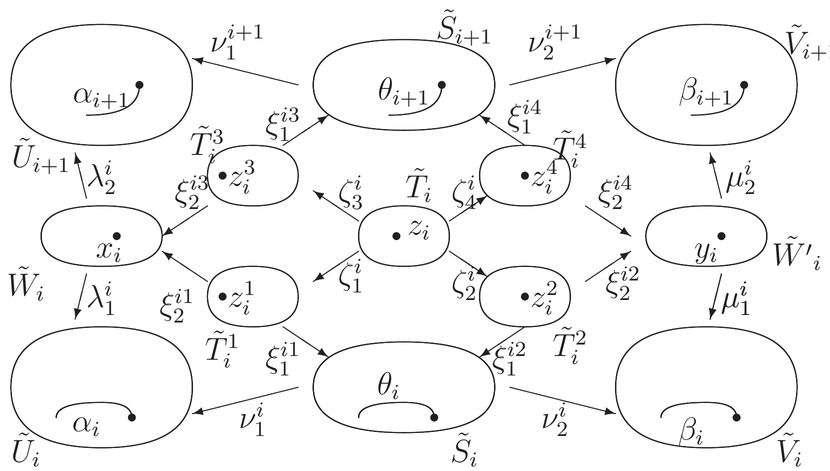
$$(\tilde{S}_{i+1}, \xi_1^{i4}, \tilde{T}_i^4, z_i^4, \xi_2^{i4}, \tilde{W'}_i)$$

such that
$\lambda_2^i \xi_2^{i4} = \nu_1^{i+1} \xi_1^{i4}$, $\mu_1^i \xi_2^{i2} = \nu_1^i \xi_1^{i2}$, and $\mu_3^i \xi_2^{i3} = \nu_2^{i+1} \xi_1^{i3}$.
These points represent the components.



From the same Lemma 2.4.6 we can find $\tilde{T}_i$ with embeddings $\zeta_1^i : \tilde{T}_i \longrightarrow \tilde{T}_{i1}$,

$\zeta_2^i : \tilde{T}_i \longrightarrow \tilde{T}_{i2}$, $\zeta_3^i : \tilde{T}_i \longrightarrow \tilde{T}_{i3}$ and $\zeta_4^i : \tilde{T}_i \longrightarrow \tilde{T}_{i4}$



If we define $\gamma_1^i = \xi_1^{i1}\zeta_1^i$, $\gamma_2^i = \xi_2^{i4}\zeta_4^i$, $\gamma_1^{i+1} = \xi_1^{i3}\zeta_3^i$ and $\gamma_2^{i+1} = \xi_2^{i2}\zeta_2^i$. We can have a chart $\tilde{T}_i$ with embeddings $\gamma_1^i : \tilde{T}_i \longrightarrow \tilde{S}_i$ and $\gamma_2^i : \tilde{T}_i \longrightarrow \tilde{W}_i$ such that the diagram 7.6 commutes with $\nu_1^{i+1}\gamma_1^{i+1} = \lambda_2^i\gamma_2^i$, $\nu_2^i\gamma_1^i = \mu_1^i\gamma_2^i$, $\nu_2^{i+1}\gamma_1^{i+1} = \mu_2^i\gamma_2^{i+1}$ and $\nu_1^i\gamma_1^i = \lambda_1^i\gamma_2^i$.

□

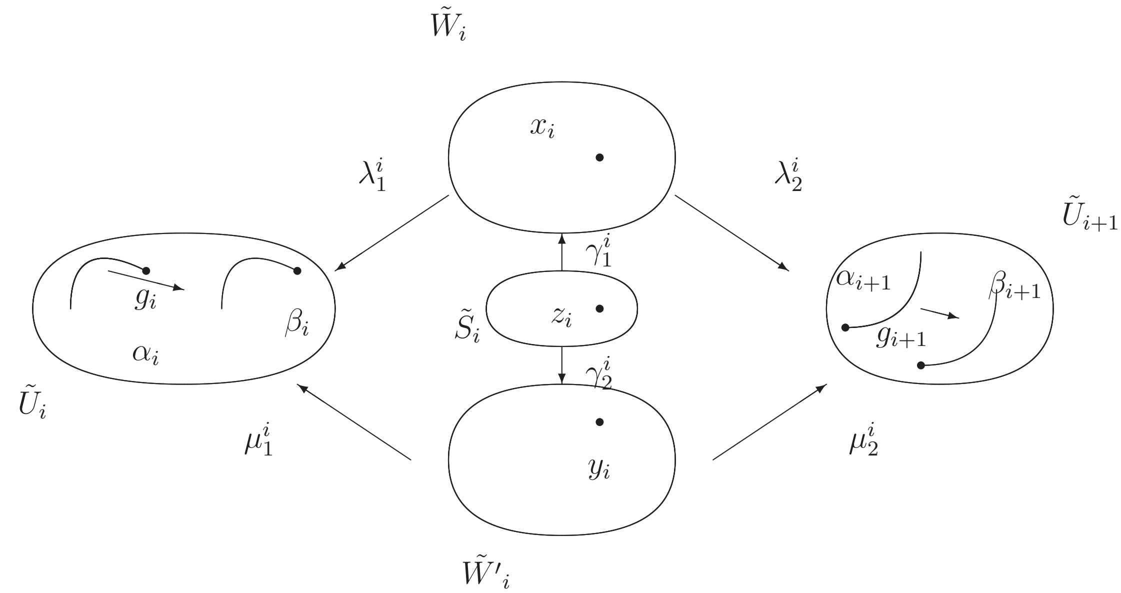
Let α and β be paths in the atlas groupoid $\mathcal{G}(\mathcal{U})$. In the following proposition, we will show the equivalence relation between α and β where α_i and $\beta_i \in \tilde{U}_i$.

Proposition 7.4.4. *If* $\alpha = ([\alpha_1, \lambda_1^1, \tilde{W}_1, x_1, \lambda_2^1, \ldots, \alpha_i\lambda_1^i, \tilde{W}_i, x_i, \lambda_2^i, \alpha_{i+1}, \lambda_1^{i+1}, \ldots, \alpha_n])$ *and*

$\beta = (\beta_1, j_{t_1}^{\beta}, \beta_2, \ldots, \beta_n) = ([\beta_1, \mu_1^1, \tilde{W}_1', y_1, \mu_2^1, \ldots, \beta_i, \mu_1^i, \tilde{W}_i', y_i, \mu_2^i, \beta_{i+1}, \mu_1^{i+1}, \ldots, \beta_n])$ *where*

$\alpha_i, \beta_i : [t_{i-1}, t_i] \longrightarrow \tilde{U}_i$ *where* $\{\tilde{U}_i, \mathcal{G}_i, \varphi_i\} \in \mathcal{U}$, *are paths in the atlas groupoid* $\mathcal{G}(\mathcal{U})$, *they are equivalent if and only if there exists a family of* $g_i \in \mathcal{G}_i$ *such that* $g_i\alpha_i = \beta_i$ *and there is a family of charts* $\tilde{S}_i$ *with* $z_i \in \tilde{S}_i$ *and chart embeddings* $\gamma_1^i : \tilde{S}_i \hookrightarrow \tilde{W}_i$

and $\gamma_2^i : \tilde{S}_i \hookrightarrow \tilde{W}_i'$ *such that the following diagram commutes,*

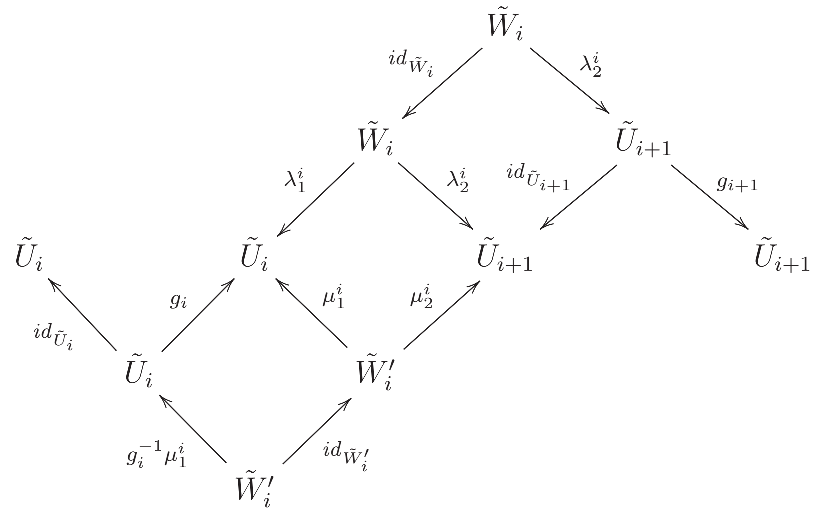


(7.5)

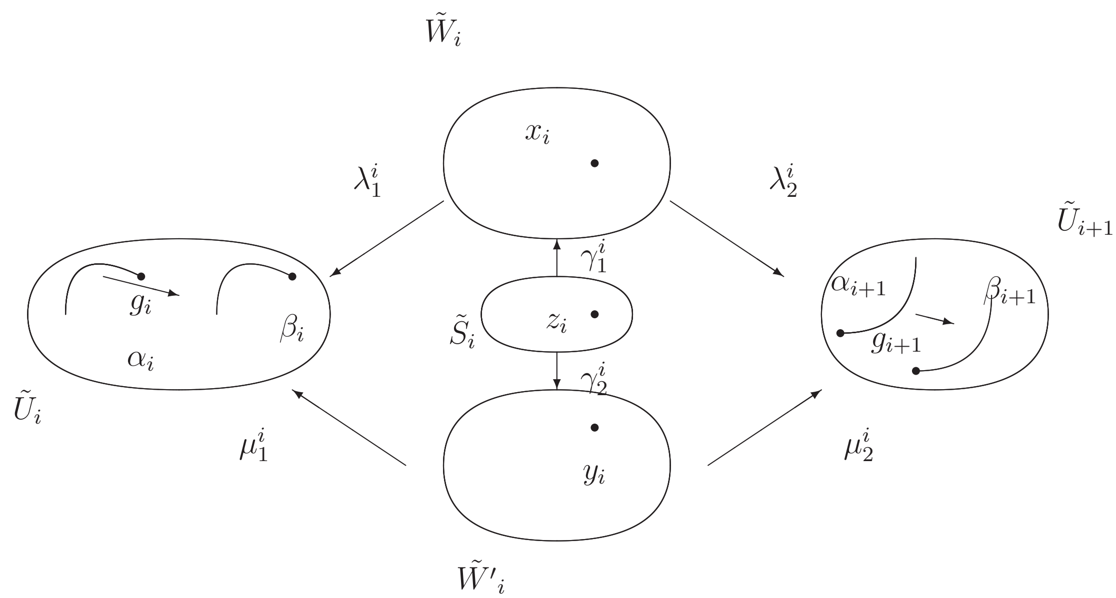
I.e., $\gamma_1^i(z_i) = x_i$ *and* $\gamma_2^i(z_i) = y_i$ *as well as* $g_i\lambda_1^i\gamma_1^i = \mu_1^i\gamma_2^i$ *and* $g_{i+1}\lambda_2^i\gamma_1^i = \mu_2^i\gamma_2^i$.

Proof. Suppose that α and β are equivalent. So, there is a natural transformaion $\theta : I \longrightarrow \mathcal{G}(\mathcal{U})$. We want to show that there are $g_i \in \mathcal{G}_i$ such that the previous diagram commutes. Since $I = \coprod_{i=1}^{n}[t_{i-1}, t_i]$, there is a family of natural trasformation $\theta_i : [t_{i-1}, t_i] \longrightarrow \tilde{U}_i$. Now we have two paths $\alpha_i, \beta_i : [t_{i-1}, t_i] \longrightarrow \tilde{U}_i$, in the same chart, Since we have that by Lemma 7.4.2 there is a $g_i \in \mathcal{G}_i$ such that $\beta_i = g_i\alpha_i$ for all $i \in \mathbb{N}$. So $\theta_i(t_i) = (g_i, \alpha_i(t_i))$. Then we have the following diagram,

Now the naturality of θ gives us $m(\theta_{i+1}(t_i), j_i^{\alpha}) = m(j_i^{\beta}, \theta_i(t_i))$. We calculate these composites with chosen squares in the following diagram,



Therefore, we have that $[\lambda_1^i, \tilde{W}_i, x_i, g_{i+1}\lambda_2^i] = [g_i^{-1}\mu_1^i, \tilde{W}_i^1, y_i, \mu_2^1]$. So, there is achart $\tilde{S}$ with $z_i \in \tilde{S}$ and chart embeddings $\gamma_1^i : \tilde{S} \hookrightarrow \tilde{W}_i$ and $\gamma_2^i : \tilde{S} \hookrightarrow \tilde{W}_i'$ such that $\lambda_1^i\gamma_1^i = g_i^{-1}\mu_1^i\gamma_2^i$ and $g_{i+1}\lambda_2^i\gamma_1^i = \mu_2^i\gamma_2^i$. So, we get the following,



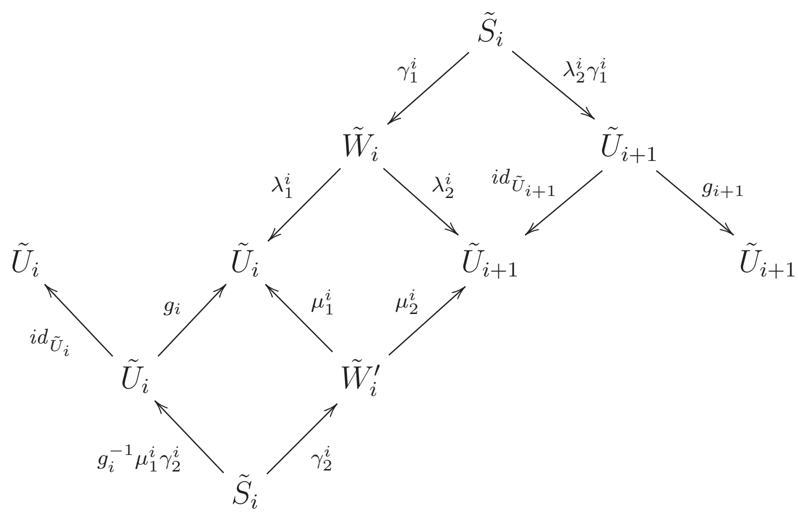
I.e., $\gamma_1^i(z_i) = x_i$ and $\gamma_2^i(z_i) = y_i$

$g_i\lambda_1^i\gamma_1^i = \mu_1^i\gamma_2^i$ and $g_{i+1}\lambda_2^i\gamma_1^i = \mu_2^i\gamma_2^i$.

Suppose the Diagram (7.5) commutes, we define $\theta : \coprod_{i=1}^n [t_{i-1}, t_i] \longrightarrow G(\mathcal{U})_1$ for $x \in [t_{i-1}t_i]$ as $\theta_i(x) = [id_{\tilde{U}_i}, \tilde{U}_i, \alpha_i(x), g_i]$. We have proved in Lemma 7.4.2 that

$s\theta_i = \alpha_i$ and $t\theta_i = \beta_i$. So, we can conclude that $s\theta = \alpha$ and $t\theta = \beta$.

Now we need to check the naturality at the jumps. We want to show that $m(\theta_{i+1}(t_i), j^{\alpha}_{t_i}) = m(j^{\beta}_{t_i}, \theta_i(t_i))$. By the Diagram (7.5), we get that $g_i\lambda^i_1\gamma^i_1 = \mu^i_1\gamma^i_2$ and $g_{i+1}\lambda^i_2\gamma^i_1 = \mu^i_2\gamma^i_2$. We can have the following diagram,



This means that $m(j^{\beta}_{t_i}, \theta_i(t_i)) = [g_i^{-1}\mu^i_1\gamma^i_2, \tilde{S}, z_i, \mu^i_2\gamma^i_2]$. Also, we have that $m(\theta_{i+1}(t_i), j^{\alpha}_{t_i}) = [\lambda^i_1\gamma^i_1, \tilde{S}, z_i, g_{i+1}\lambda^i_2\gamma^i_1]$. Since $[g_i\lambda^i_1\gamma^i_1, \tilde{S}, z_i, \mu^i_2\gamma^i_2] \sim [\mu^i_1\gamma^i_2, \tilde{S}, z_i, g_{i+1}\lambda^i_2\gamma^i_1]$, we can conclude that $m(\theta_{i+1}(t_i), j^{\alpha}_{t_i}) = m(j^{\beta}_{t_i}, \theta_i(t_i))$.

□

In the Proposition 7.4.3, we have characterized when two paths α and β in the atlas groupoid $\mathcal{G}(\mathcal{U})$ are equivalent in general. Then, we discussed in Proposition 7.4.4, the special case when the components of α and β are paths in the same chart. Now we will derive a more specific case, namely when we have that $\tilde{W}_i = \tilde{W}'_i$.

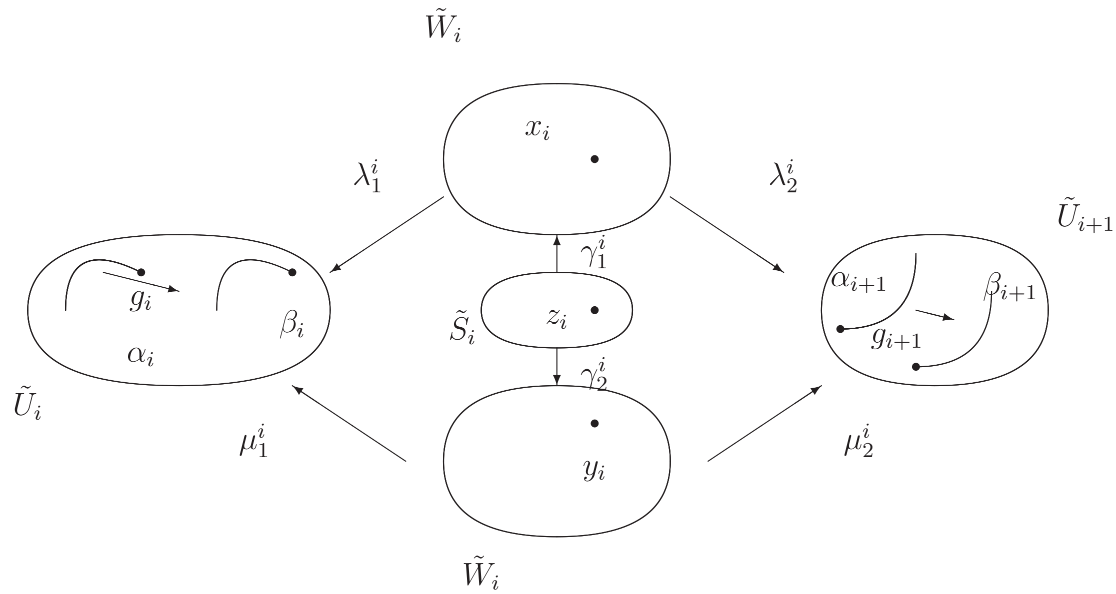
Corollary 7.4.5. *If* $\alpha = ([\alpha_1, \lambda^1_1, \tilde{W}_1, x_1, \lambda^1_2, \ldots, \alpha_i\lambda^i_1, \tilde{W}_i, x_i, \lambda^i_2, \alpha_{i+1}, \lambda^{i+1}_1, \ldots, \alpha_n])$ *and*

$\beta = (\beta_1, j^{\beta}_{t_1}, \beta_2, \ldots, \beta_n) = ([\beta_1, \mu^1_1, \tilde{W}_1, y_1, \mu^2_2, \beta_2 \ldots, \beta_i, \mu^i_1, \tilde{W}_i, y_i, \mu^i_2, \beta_{i+1}, \mu^{i+1}_1, \ldots, \beta_n])$ *where*

$\alpha_i, \beta_i : [t_{i-1}, t_i] \longrightarrow \{\tilde{U}_i, \mathcal{G}_i, \varphi_i\}$ *for* $\tilde{U}_i \in \mathcal{U}$, *are paths in the atlas groupoid* $\mathcal{G}(\mathcal{U})$, *they are equivalent if and only if there exists a group element* $k_i \in \mathcal{G}$ *such that* $g_i\lambda^i_1 = \mu^i_1 k_i$ *and* $g_{i+1}\lambda^i_2 = \mu^i_2 k_i$.

Proof. Suppose α and β are two equivalent atlas paths. We want to show that there is a group element $k_i \in \mathcal{G}$ such that $g_i\lambda^i_1 = \mu^i_1 k_i$ and $g_{i+1}\lambda^i_2 = \mu^i_2 k_i$.

We have $g_i\lambda_1^i, \mu_1^i : \tilde{W}_i \hookrightarrow \tilde{U}_i$. Then by Lemma 2.3.9, there exists $k_i \in \mathcal{G}$ such that $\mu_1^i k_i = g_i\lambda_1^i$. We want to show that $g_{i+1}\lambda_2^i = \mu_2^i k_i$. Since α and β are equivalent, we get the following commutative diagram,

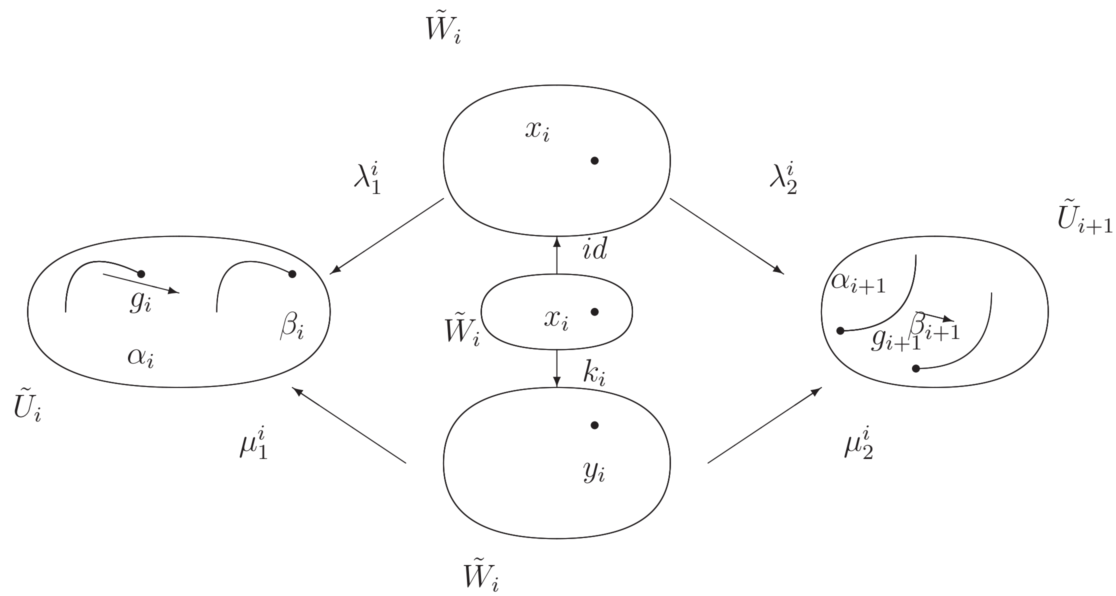


(7.6)

We have that $g_i\lambda_1^i\gamma_1 = \mu_1\gamma_2$ and $g_i\lambda_1^i = \mu_1^i k_i$. Therefore, $\mu_1^i k_i\gamma_1 = \mu_1\gamma_2$. Since μ_1^i is an embedding, we have that $k_i\gamma_1 = \gamma_2$. Therefore, we have that

$$g_{i+1}\lambda_2^i\gamma_1 = \mu_2^i\gamma_2 = \mu_2^i k_i\gamma_1.$$

Since γ_1 is monomorphism, we can conclude that $g_{i+1}\lambda_2^i = \mu_2^i k_i$. As a result, if α and β are equivalent, then there exists $k_i \in \mathcal{G}$ such that $\mu_1^i k_i = g_i\lambda_1^i$ and $g_{i+1}\lambda_2^i = \mu_2^i k_i$. Now we want to show that if there exists $k_i \in \mathcal{G}$ such that $\mu_1^i k_i = g_i\lambda_1^i$ and $g_{i+1}\lambda_2^i = \mu_2^i k_i$, then α and β are equivalent. Since $\mu_1^i k_i = g_i\lambda_1^i$ and $g_{i+1}\lambda_2^i = \mu_2^i k_i$, we can form the following commutative diagram,



We have the naturality at jumps since this corollary is a special case of Propositian 7.4.4. □

Note that from this proposition, we can conclude that if we have two paths in the same chart such that the jumps are in the same chart, we can find a natural transformation between the paths from a group element.

Chapter 8

Conclusion

We have defined two kinds of maps for orbifolds. A weak notion of map which does not give us a map between atlases. So, we defined a strong map between atlases corresponding to a morphism between these orbifold categories but this map is not essentially surjective. So, we have defined atlas groupoids and give a useful tool to define a strong maps of orbifolds corresponding to weak equivalences. This is important since the maps between orbifolds play a main role in orbifold homotopy theory. We also constructed the notion of an internal category of fractions and developed the conditions for its construction and its universal property for an arbitrary internal category in TOP. Hopefully it can be generalized to other internal categories in other categories, such atlas categories for different type of geometric orbifolds. In this thesis we just focused on topological orbifolds. Finally, we finished this paper by using the definition of morphisms between orbifolds to define paths and obtained a nice description of the equivalence relation on paths. Hopefully this result can be generalized to other types of orbifolds, for example differentiable orbifolds.

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