DECISION UNDER COMPLETE UNCERTAINTY: BRIDGING ECONOMIC AND PHILOSOPHICAL RESEARCH

by

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Abstract

This thesis explores the topic of decision under conditions of complete uncertainty, advocating an interdisciplinary perspective that benefits from the insights of both economists and philosophers. Thus far, most of the results in the field have been the work of economists who have been responsible for important theorems and axiomatic characterizatoins of a variety of decision rules. While proceeding from a different methodology and focus, tantalizingly similar conjectures have been made by philosophical logicians. While the work of the latter has not (yet) become as advanced in deriving important theorems, I suggest that philosophers have something useful to offer in their method of analysis that would be useful in evaluating the different solutions to standard problems in the field. I attempt to provide a new solution motivated by both disciplines.

List of Abbreviations and Symbols Used

$\geq$	at least as good as (between states)
$>$	strict preference (between states)
$\sim$	indifference (between states)
$\succeq$	at least as good as (between sets)
$\succ$	strict preference (between sets)
$\simeq$	indifference (between sets)
Max	maximal element
Min	minimal element
$\cup$	union of sets
$\cap$	intersection of sets
$\&$	and-operator
$\vee$	or-operator
$\neg$	not-operator
$\Rightarrow$	if-then conditional
$\forall$	universal quantifier
$\exists$	existential quantifier
$\mathbb{C}$	the comparison of sets
$\rightarrow$	maps-to

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Chapter 1

Introduction

Despite the use of the phrase "like choosing between apples and oranges" as a means of indicating incommensurable and thus incomparable objects, the reality is that such comparisons are daily occurrences. By comparing apples and oranges on the basis of the strength of our preferences we are able to make a decision. In my view, the interesting question raised by the phrase is not so much how we choose between different types of goods, but rather how we choose between sets of outcomes that are indeterminate with respect to the specific outcome that obtains from each set. In being given a choice between an apple and an orange, what kind of apple or orange would I receive? What would be the specific properties of the apple should I choose 'apple' and what are the specific properties of the orange I would receive should I choose 'orange'? This is not a question that we typically consider, either because under normal conditions the degree to which we benefit from this exertion of cognitive effort is not worthwhile, or because perhaps our preferences are not sufficiently nuanced as to generate different preferences between different apple and orange outcomes. But suppose they were (and doubtless we could think of cases where we are faced with decisions where what specific outcome results is indeed relevant to what choice we make). How do we extend a preference between specific outcomes to sets of possible outcomes?

Let us add some additional details to our apple and orange problem. Each choice involves a range of possible properties and so a plethora of possible combinatorial outcomes. For the purposes of simplicity, let us confine ourselves to one additional property beyond type of fruit. Let us say that both apples and oranges may be sweet (or not sweet). Thus, we obtain the Apple and Orange choice as characterized by the following sets of outcomes:

$$\begin{aligned} Apple: &\quad \{sweet\,apple, not\,sweet\,apple\} \\ Orange: &\quad \{sweet\,orange, not\,sweet\,orange\} \end{aligned}$$

Suppose Jimmy most prefers sweet oranges, then sweet apples, then not-sweet apples, and least prefers not-sweet oranges. On the basis of this preference structure, on what basis should he choose ‘apple’ or ‘orange’? This will of course depend on his proclivity or adversity towards risk. This corresponds with a Maximax (maximize on the best possible outcome) versus Minimax (minimize the worse possible outcome) decision rule.

1.1 Two Related Problems

Notice, however, that while this answers the practical question of how Jimmy should choose, it does not answer the question of how he should choose in terms of the earlier question: How do we extend a preference between specific outcomes to sets of possible outcomes? This is a question about the specific logical relationship that exists between the two that allows us to generate a practical decision rule that will allow us to rank sets of possible outcomes. The search for the logical properties that would allow us to rank sets is a question of interest to both economics and philosophy.

1.1.1 Finding a Decision Rule for Conditions of Complete Uncertainty

In economics, the general problem described above is a category of decision theory known as decision under complete uncertainty (or complete ignorance). Decision theory is often divided under conditions of risk vs. conditions of uncertainty.

Conditions of risk involve situations in which we say that we have *objective* probabilities as given by knowledge of the number of possible outcomes and the probabilities directly associated with those outcomes. For example, in discussing a fair coin, the objective probability is 50% heads and 50% tails. A different example is one in which the probabilities of the outcomes are distinct but nevertheless objective, such as the probability of drawing a spade vs. a non-spade from a deck of cards (25% and 75% respectively). When it comes to decisions under risk, the standard solution is to make use of some form of expected utility obtained by taking the value of an outcome multiplied by the probability of that outcome obtaining. The action that should be taken is that which produces the highest expected utility.

In contrast, decisions under conditions of uncertainty concern cases where we possess only *subjective* probabilities. In these cases the probabilities directly associated with outcomes are unknown. Under standard conditions of uncertainty we must make use of our own subjective assessments of the probabilities associated with each outcome, that is, how likely we believe that any given outcome will obtain. Using this estimation we calculate expected utility as in conditions of risk and select the decision with highest expected utility. Of course we may be completely mistaken about our probability assessments, perhaps as established through repeated testing. Nevertheless, under conditions of uncertainty, it is by employing such estimations that we are able to come to a conclusion as to which decision to take.

Different types of uncertainty situations exist. Vercelli distinguishes three broad categories: soft uncertainty, hard uncertainty, and radical uncertainty. The above example would be a case of soft uncertainty in that our probability model is additive (i.e. that we assign specific probabilities that together add up to 1). By contrast, hard uncertainties are rather more complex. There exist non-additive probabilities, multiple probabilities, fuzzy measures, and belief functions. Decision under complete uncertainty falls within the category of what he identifies as radical uncertainty and constitutes an extreme case.(1999, 245-246) Whereas in standard cases of uncertainty we are able to at least employ estimates of probabilities in different forms, complete uncertainty rejects their use. A further challenge posed is that we are denied information on the utility of any particular outcome obtaining. However, we do maintain the ordinal utility of each outcome.

We may summarize the general problem of decision under complete uncertainty as follows:

1. No probabilities associated with each outcome, only possibilities

2. No utility information associated with each outcome, only relative preference

This has certain implications regarding approaches that we might be tempted to use to decide on the rational course of action:

1. *We cannot employ the principle of indifference.* In standard cases where we have $n > 1$ possible outcomes and these outcomes are indistinguishable, we assign a uniform probability to each outcome. This is the principle at work in assigning a 50% probability to any given side of a fair coin. On this approach, one might think that a set with a higher cardinality would be a riskier choice than one with fewer outcomes. However, complete uncertainty does not allow us to form this conclusion. Recall that no probabilities whatsoever may be attached to the outcomes. The only information attached to each outcome is the possibility of its obtaining.

2. *We cannot employ a weighted average.* Another approach that might be attempted is to take the average utility of the outcomes in each set. On this reasoning, we might at least get the approximate value of each set. This is somewhat different from utilizing the principle of indifference as its focus is more on getting a sense of the overall desirability of each set. However, under conditions of complete uncertainty we also lack information on the values of each outcomes. We are left with only information on how the outcomes compare to each other and so how they are ranked.

3. *We cannot employ a rank average.* In order to bypass the problem of lack of utility information for each outcome, we might attempt to take an average of the rankings present in each set and thus select the set that produces the lowest average. For example, consider the following two sets with the following outcomes $a_i$where i denotes rank: $\{a_1, a_3\}$ and $\{a_2\}$. Both the former and latter set would have a rank average of 2 (for the first: $(1+3)/2$); and the second: $2/1$). On this approach, we should be indifferent between the two sets. However, it seems to be unclear whether we can use this approach. After all, a rank is not in itself informative of the exact degree to which one thing is preferred over another. To treat each rank as imparting information seems to presuppose the principle of indifference in the case of ranks such that the difference between any one rank to any other is the same.

In a later result we shall see that if we accept certain basic axioms, then we are committed to a decision rule that focuses on the maximal and/or minimal element in each set under comparison - something that would not occur if using an averaging approach. We find then that we are left without a mathematical basis for calculating our decision. What we must use are relative preferences between outcomes. This is called the extension-rule[1] approach.

On this view, we "... assume that this ranking of sets of possible outcomes extends the agent's ranking of the outcomes themselves in the sense that the relative ranking of singletons (which represent certain outcomes) according to the set ranking is the same as the relative ranking of the corresponding (single) elements in the respective sets according to the ranking of alternatives. This property motivates the use of the term extension rule for set rankings with this interpretation."(Barberà et al., 2004, 900) The basis for decision under complete uncertainty is founded on this notion of extension. Taking this as a general principle, we seek to form other axioms that together give us decision rules.

1.1.2 Defining the Axiological 'Ought'

Both the framework of decision under complete uncertainty and the goal of finding a decision rule are paralleled in the work of philosophers. However, they approach it both from a different background and perspective.[2] While the economic work might be thought of as involving decision-making in general, philosophers have ventured into the field largely as a result of work in ethics and logic. While meta-ethicists have focused on what makes an act good, right, or morally acceptable, normative ethicists have been concerned with how a systematic moral theory might incorporate these definitions into a system that gives the moral status for any act α. This focus on the morality of certain actions has resulted in a more focused discussion regarding decision-making. Rather than considering arbitrary sets of outcomes, philosophers

[1] This term is employed in the economic literature. It should be distinguished from the philosophical concept of extensionality.

[2] One of the most obvious philosophers with work connecting to decision under complete uncertainty would be John Rawls, whom I do discuss in 2.3.2.1. However, as his work was influenced to a certain extent in the economic literature, for now I wish to consider decision under complete uncertainty as it emerged on completely independent, unintentionally connected grounds.

have tended to focus on sets of outcomes connected to doing either an act α or Not-α. And this is understandable given the focus of ethics on obligation. If an act's being obligatory is connected in any way to the outcomes that might result, then we will be concerned with the specific sets of states attached to that act.[3]

Some philosophers were more inclined towards a logic rather than just a system by making use of the framework of modal logic and attaching a semantics with moral content. Unfortunately, acts do not translate precisely into states of affairs. Instead, states of affairs are collected into sets on the basis of the propositions that are true in those states. It is propositions that are connected to states of affairs, that is, by finding the truth set of the propositions. Thus, for a proposition A, the truth set of A ($||A||$) is that set of states of affairs in which A is true. The relevance of acts is now related to acting in a way so as to bring about those states in which A is true.[4]

The general framework just given can be further specified to form two different systems: deontic logic and axiology (Van Fraassen, 1973, 5).[5] As might be guessed by its name, deontic logic is focused on the logic of duties as obligations. Having noticed an interesting relationship between obligation and necessity, and permissibility and possibility, logicians in this tradition have been interested in exposing the structural relationships between the moral status of different acts on the basis of existing relationships in modal logic. Axiology on the other hand, is concerned with what is valuable. (Schotch, 2006, 157-158) makes the helpful distinction between 'obligatory' and 'ought to be'. An act is obligatory just in cases its non-performance is impermissible. By contrast, what ought to be may cover the gamut from permissible to supererogatory (perhaps even including impermissible - if given a choice between two impermissible outcomes, if one is relatively better than the other, it may nevertheless

[3] This is not to say that we are necessarily concerned about the *value* of the outcome, but in some sense the outcome is related to the permissibility of the action. For example, the act of lying might be impermissible because the resultant states attached to that act A were all states in which the person being lied to was misinformed (and say the person has a right not to be misinformed).

[4] This is mostly a technical imposition; however, in the case of certain systems this is of importance to the moral theory under consideration. In the work of Jennings on a utilitarian semantics, for example, the moral theory is indifferent to specific intentions, and we might add acts, so long as a particular outcome is obtained. (Jennings, 1974, 446)

[5] Van Fraassen actually refers to the first as deontology. However, because I think this would lead to some equivocation between the normative system and the logics that are his focus, I will make use of the term 'deontic logic'.

be the outcome that we ought to pursue). As Bas Van Fraassen explains the axiology approach:

> [Axiology] deals with what ought to be because its being so would be good, or at least better than its alternatives. ... [T]he axiological thesis amounts to: there is some scale of values whereby what ought to be is exactly what is better on the whole. ... Whether an ought-statement is true depends on two factors: the set of alternative possibilities we are evaluating, and the scale of values by which we rate them. (1973, 6,7)

This sounds much like both the framework and challenge of decision under complete uncertainty: taking a preference structure and finding its relation to a set of possible outcomes.[6]

Ray Jennings takes an identical axiological approach from an explicitly utilitarian perspective:

> "An act is in conformance to the principle of utility if and only if its non-performance is not. But an act is in conformity to utility if and only if it is either permitted or obligatory. So a course of action may or ought to be pursued if and only if its consequences are better in point of advantages gained and disadvantages avoided than those of its non-performance." (Jennings, 1974, 448)

As well, we find him working within a similar framework:

> "A utilitarian theory assumes that we can (and possibly do) make comparative judgments about possible states of affairs and that these judgments can inform our judgments concerning what we ought to do. ... What is proposed is that we regard the moral agent as confronted by a number of different states of affairs any one of which he may bring about by some or other action. The differences between the contemplated states

[6]I hesitate however to say that this is completely what Van Fraassen had in mind. In giving his own equivalent of a decision rule, he states "To obviate an obvious objection: when a possible outcome is evaluated, its relative likelihood should be given due attention." (1973, 7) Nevertheless, the structure he proposes in the rest of his paper does not appear to have any way to include likelihoods; instead, he focuses merely on possibility.

> of affairs correspond to the differences between the different courses of action that are open to him. The situation of the moral agent is one which, in language recently fashionable, might be represented as a set of possible worlds. The utility judgments of the agent are judgments about the amount of happiness and unhappiness that there is to be had in the various kinds of world open to him."[7] (Jennings, 1974, 446-448)

What we find then is a significant overlap in the framework and goal for both philosophers and economists. Given a background in which we are faced with an act that may bring about a different range of possible outcomes, as well as a preference ordering amongst individual outcomes, how are we to decide between the acts when the outcome is uncertain?

1.2 Thesis Purpose and Direction

In light of the similarities observed in both the framework and goals formed independently by both economists and philosophers, I wish to bridge the gap between their work. The ultimate purpose of this thesis is to demonstrate the value of an interdisciplinary approach to decision under complete uncertainty that brings together economics and philosophy.[8]

This will be accomplished via the following sub-goals:

[7] Admittedly, this is a tad ambiguous as to the extent he conceives of sets of states of affairs of which one will obtain. However, it becomes evident upon an understanding of the logic he utilizes on which states of affairs are divided up on the basis of propositions in the manner outlined earlier in the section.

[8] Perhaps some might take this as a trivial goal. Of course considering this from the perspective of other related disciplines is useful; is it really advancing research to write an entire thesis on that topic? This would have been my original response to the idea of writing such a document. In fact, this project (which has been in development for a full year) began as an attempt to merely take the work that had been done by economists and apply a philosophical perspective to develop the field. However, the final product was much more modest in scope. Many of the 'insights' I believed were the results of a distinctly philosophical approach had not only been replicated by economists (in some cases decades earlier), but expounded in greater detail and precision. What original contributions I offer might be said to be the modest kernels that have remained after undergoing the threshing floor of interdisciplinary comparison. I have spared the reader (most) of the tedium I experienced with the back-and-forth of developing ideas only to find them later ridiculous (and I have spared myself the embarrassment of sharing what turned out to be many half-baked ideas). However, a not insignificant part of this thesis has been to present results demonstrating some of the overlap between economics and philosophy; after all, I take the optimistic view that showing a 'result' is not a result is itself a kind of result.

1. Demonstrating the value of the economics approach, as exemplified by:
    (a) The effectiveness of the language for its simplicity (Section 2.2, and in contrast to the language of philosophers in Section 2.3.2.2)
    (b) The discovery of impossibility results upon the acceptance of some basic, highly plausible axioms (Section 2.6)
    (c) Different axiomatization schemes that avoid impossibility and produce characterizations of various decision rules (Sections 3.1, 3.2)
2. Demonstrating the value of the philosophical approach, as exemplified by:
    (a) The existence of an overlap in both the framework and goals (As previously discussed in Section 1.1 and further elaborated in 2.1) and parallels between results and conjectures by philosophers with results by economists (Sections 2.3, 3.1)
    (b) A focus on the semantics of comparison to understand axioms and decision rules (Sections 4.1.2 and 4.2)
    (c) An alternative approach to respond to the impossibility theorems (Chapter 4)

Unfortunately, as shown by the scattered section references, accomplishing these tasks is not a straightforward matter that allows for linear arguments and conclusions. Instead, these points are brought out by a holistic study of the field. Thus, while I hope to accomplish all the above, stylistically at least, this thesis is exploratory with reminders of the overall goals at appropriate junctures along with overt gestures towards the instances in which I have attempted to provide what was promised.

This thesis begins with a focus on the work done by economists that has framed the discussions. However, in doing so, I mention the preliminary work done by philosophers that parallels some of these ideas and results. This thesis is organized along the following lines:

Chapter 2 In this chapter I provide a survey of the development of the field and its foundations. Since decision under complete uncertainty as it is known today is understood through the lens of the economists, my focus is on their work. I review an important theorem with respect to the standard Maximax and Minimax decision rules. The chapter culminates in the presentation of several impossibility theorems that I take as being a defining problem for the field (even in spite of solutions that generate other productive results). This sets the ground for the discussion in the following chapter.

Chapter 3 This next chapter looks at responses to the impossibility theorems, focusing on three possible approaches. The first, an interpretation of the impossibility theorems as a rejection of completeness rather than the consistency of the basic axioms. The second, the avoidance of the problematic axioms and the selection of weaker axioms that generate the same desired results as the originals. The third, an abandonment of those axioms and the original useful theorem in favour of other decision rules that solve a problem for less robust rules.

Chapter 4 The penultimate chapter turns to offer a new response to impossibility that combines two standard and (initially) mutually exclusive solutions into one. This approach is motivated by a philosophical perspective into the very idea of comparison, leading to the development of an analytical tool to justify a rejection of wholesale set transitivity in favour of a qualified form. This allows us to unite the two standard solutions into one, leading to an interesting variation on two standard decision rules. The implications of this new approach are presented.

Chapter 5 This chapter recaps what has been accomplished and offers some closing remarks regarding topics of further study.

Chapter 2

Background, Goal, and Initial Impossibility Results

2.1 History of the Field

Before setting the technical groundwork for our discussion, it may be informative to get an overview of the development of the field by both economists and philosophers. In particular I wish to emphasize the development of the technical aspects of the framework.

While the general concept of 'complete uncertainty' dates at least as far back as the early 1950's (Luce and Raiffa, 1957), decision under complete uncertainty as understood with the absence of not only probability but value functions may quite plausibly be traced to Rawls (1971) (which might explain his fairly regular citation by economists in seminal works on the topic). Only a year later we find in (Arrow and Hurwicz, 1972) the first formal treatment of this framework so as to be consistent with the logics of existing forms of decision theory. Yet another year later we find the first venture by philosophers into utilizing the formal framework of modal logic to incorporate (though unintentionally) the aspect of complete uncertainty (Van Fraassen, 1973), and another year later, a further step in its formalization (Jennings, 1974).

It is not until the early 80's that we begin to see the first few important results being published, seemingly exclusively by economists. At this point, much of the focus of philosophers was on the standard-form deontic logic program with its problems and potential solutions.[1]

[1] Jennings being an example. See Schotch and Jennings (1981).

Economists	Philosophers
	Rawls (1971)
Arrow and Hurwicz (1972)	
	Van Fraassen (1973)
	Jennings (1974)
Pattanaik and Peleg (1984), Kannai and Peleg (1984), Barberà and Pattanaik (1984), Barberà et al. (1984)	
Bossert (1989)	

Interest in the field was revived in the early part of the year 2000 as the result of work by Bossert et al. (2000) and Arlegi (2003) which generated some noteworthy theorems characterizing important decision rules. The continued absence of philosophers in this field[2], however, is a conspicuous absence - one that I hope will be remedied over time. While much has been done in the way of working with axioms and proving theorems, it seems that conceptual analysis of comparisons that provide the foundations for these approaches is still necessary.

2.2 Defining the Language

Before delving into formulations of theories behind how one should choose between sets, we first need to provide the formal language describing comparisons. Since the issue is how to extend preferences between states into preferences between sets, we require a language that differentiates between the two in order to show how they relate with one another.

[2]Published work at any rate - it would be remiss to omit the work of Schotch (2006) that is responsible for driving interest in this topic.

2.2.1 The State-level Language

In defining our language we begin with the issue of the primary objects of our discussion. This comes in the form of states of affairs. Economists have taken a straightforward approach to these states of affairs, representing a state of affairs by an individual variable (e.g. x, y, z). The non-empty set of all possible states of affairs is set X, with a cardinality denoted by $|X|$.

We now turn to the operators giving the relations between the elements of X based on an agent's preference structure. They are all binary relations on X which involve comparisons between one state of affairs to another.

Definition 1. $\geq$ the 'at least as good as' state preference relation.[3] $x \geq y$ (alternatively written as $\geq (x, y)$) for an agent iff the state of affairs denoted by x is at least as good as the state of affairs y for that agent. For our purposes we assume it has the following properties:

reflexivity $(a \geq a)$ Everything is at least as good as itself;

anti-symmetry $((a \geq b)\&(b \geq a) \Rightarrow a \sim b)$ If both are at least as good as each other, then they are indifferent to each other;

transitivity $([(a \geq b)\&(b \geq c)] \Rightarrow (a \geq c))$ If one state of affairs is at least as good as a second, and that second at least as good as a third, then it is sufficient to guarantee that the first is at least as good as the third;

completeness $((a \geq b) \vee (b \geq a))$ For any two states of affairs, it is true that at least one is at least as good as the other, or alternatively, that any state is comparable with any other state.

Assuming these four properties generates a linear ordering.[4]

[3] For reading convenience, the reader might think of this as 'ALAGA'.

[4] The assumption of these properties is not uncontroversial, especially once one realizes that a total ordering is a rejection of circular preferences, and so, intransitivity. This is a discussion in its own right, and has been discussed on the basis of both empirical (e.g. Grether and Plott (1979), Korhonen et al. (1990)) and philosophical grounds (e.g. Schumm (1987),Anand (1993),MacIntosh (2010)). A thorough justification would take us too far afield, so I take it as granted simply on the basis that I consider it a desirable property for the system to possess. Another approach to this point might be to simply say that transitivity is not an empirical postulate and so not subject to empirical refutation. My thanks to Prof. Schotch for this point.

Definition 2. $\sim$ the 'indifference' state preference relation. $x \sim y$ (alternatively, $\sim (x, y)$) for an agent iff both x and y are at least as good as each other. It is reflexive and transitive.

We allow that one may be indifferent to completely different states of affairs.

Definition 3. $>$ the 'strict preference' relation. $x > y$ (alternatively, $> (x, y)$) for an agent iff x is at least as good as y, and it is not the case that y is at least as good as x. It is transitive, and asymmetric ($a > b \Rightarrow \neg(b > a)$); if one state is strictly preferred to another, then the reverse cannot be the case.

2.2.2 The Set-level Language

We now turn to define the language that exists at the level of sets. Recall that our set of states of affairs is X. We represent a set of states of affairs with a capital letter (e.g. A, B, C,...). For the power set of X, given by $\mathcal{P}(X)$ (which is the set of all combinations of states of X), $\mathcal{X}$ is a finite subset of $\mathcal{P}(X)$ where the empty set is removed. For a natural number n where $n \leq |X|$, $\mathcal{X}_n$ is the set of all members of $\mathcal{X}$ with a cardinality of n.

We now turn to the operators. Standard relations between sets and their symbols apply (e.g. unions, intersections, subsets). In addition, like with states of affairs, we have binary operators on sets of states of affairs that compare one set to another. They are analogous to their state operator counterparts. However, while comparisons between states are straight-forward (we simply consult our preference of one versus another), what defines one set existing in a particular relation to another is complex - such an account would be to give a decision rule. At this point, however, we may still discuss a few basic properties of set comparisons.

Definition 4. $\succeq$ the 'at least as good as' relation between two sets.[5] Like $\geq$, it is reflexive, anti-symmetric, transitive, and complete.

Definition 5. $\simeq$ the 'indifference' relation between two sets. Like $\sim$, it is reflexive, and transitive.

[5] For reading convenience, the reader might think of this as 'SALAGA' (set at least as good as).

Definition 6. $\succ$ is a 'strict preference' relation between two sets. Like $>$, it is asymmetric, and transitive.

2.2.2.1 Special Functions

We also define two well known special functions that take sets as their argument and return at least one state as its outcome.

Definition 7. $Max()$ takes a set as its argument and returns the member of the set that is maximal. An element x is maximal iff x is at least as good as all other elements in the set.

This is to be distinguished from a maximum, which is the unique maximal element. For example, consider the set $A : \{x, y, z\}$ where $x \sim y$. Here $Max(A)$ would return both x and y.

Definition 8. $Min()$ takes a set as its argument and returns the member of the set that is minimal. An element x is minimal iff all other elements in the set are at least as good as x.

We distinguish minimum and minimal similar to the way in which we distinguished maximal and maximum. For the set $A : \{x, y, z\}$ where $y \sim z$, $Min(A)$ would return y and z.

In sets consisting of one element, that element is both the maximal and minimal (as well as maximum and minimum).

2.3 Searching for a Decision Rule

With the above we can begin to focus on the goal of constructing a decision rule. While there are actually different approaches to decision under complete uncertainty, we will focus on the most natural approach of forming a decision rule: extension. The theory behind this approach is to take an agent's ranking of individual outcomes and extending this preference into a preference between singleton sets composed of those outcomes. In other words, "[the] [e]xtension rule requires the relative ranking of any two singleton sets according to $\succeq$ to be the same as the relative ranking of the corresponding alternatives themselves" according to $\geq$ (Barberà et al. (2004)). We may states this formally as:

Extension Rule: $x \geq y \Rightarrow \{x\} \succeq \{y\}$

On the basis of this principle that our decision rule be extension of a primary comparison between states, we can form a number of different decision rules. The most well known are Maximax, and Minimax.

Definition 9. Maximax

$$A \succeq_{Max} B \Longleftrightarrow Max(A) \geq Max(B)$$

$$A \succ_{Max} B \Longleftrightarrow Max(A) > Max(B)$$

On this rule, what is most important is maximizing the maximal outcome. The plausibility of this approach stems from Maximax as the implicit decision rule in other forms of decision theory. For example, in decision under risk and uncertainty, we are selecting the decision that maximizes expected utility.

In contrast to optimizing the maximal outcome, we might instead take the conservative approach offered by Minimax.

Definition 10. Minimax

$$A \succeq_{Min} B \Longleftrightarrow Min(A) \geq Min(B)$$

$$A \succ_{Min} B \Longleftrightarrow Min(A) > Min(B)$$

According to this rule, we should select the set which maximizes the minimal outcome. Thus, should the worst case occur, it is preferable that the worst outcome in our set of choice occur rather than the worst outcome in the set we rejected. While we do not have information on utilities, we can be assured of avoiding the absolute worst possible outcome between the choices.

On standard accounts, rationality is maximizing. However, what should be maximized in cases where we lack an assessment of the overall strength of a choice? One approach might be to say that maximizing requires us to maximize the maximum - this would be to extend the range of possible outcomes for the best. However, one might question whether this is the best approach on a decision under complete uncertainty framework. After all, it seems that the less information we have to work with, the more conservative our moves. Is it truly rational to hope for the best when there is no particular reason to think that the best will be the resulting outcome? This is particularly questionable in cases where the rest of the possibilities attached to the set are far worse than all those attached to the other set. But more fundamentally, why should we even be concerned about the maximum and minimum in particular? We shall see an argument for this in Section 2.5.

2.3.1 The Economic 'Axiomatic Characterization' Approach

From the economic analysis perspective, the decision rules just canvassed might be thought to be the final step of the process. What is needed, however, is a complete characterization of a rule according to smaller rules in the forms of axioms.

> "The methodology employed here is the axiomatic approach. We formulate criteria that can be considered desirable, ... and seek to identify the ranking satisfying (combinations of) them. For some collections of axioms, the set of rankings satisfying them is the empty set–that is, impossibility results emerge. In fact, particularly in the context of choice under complete uncertainty, it turns out that seemingly innocuous axioms can be mutually incompatible." (Barberà et al., 2004, 897-898)

We thus have two reasons to support a full axiomatic characterization of decision rules. First, it allows us to see what else we might be committed to in adopting a decision rule. While some implications are obvious, some are less so. Second, in light of the possibility of inconsistencies, this axiomatic approach certainly makes sense, especially when decision-making is considered in the context of rationality. It would be an undesirable result to find that basic axioms of comparing sets might lead us to be inconsistent.

For the moment we will pass over the details because this approach will be the primary focus of our analysis.

2.3.2 Inquiries by Philosophers into Decision Rules

Philosophers take a different approach towards locating a decision rule from that of the economists. Rather than begin with basic axioms and work up so as to show that only specific rules will possess the properties of a given set of axioms, philosophers could be said to have approached the problem from a more pragmatic perspective, and caring little for a set of axiomatic properties. At the moment I wish to focus on three such examples in the form of the moral and political philosopher John Rawls, and the philosophical logicians Bas Van Fraassen and Ray Jennings. What is notable about these three is that the date of their relevant publications on this topic appeared during the period in which decision under complete uncertainty developed. From what I can tell, these mainly developed independently of the framework developed by economic logicians. While Rawls' ideas certainly drew from existing ideas in decision theory and game theory and resulted in his advocacy of an existing decision rule, the background of complete uncertainty from which he developed it was his own. While eschewing the economic approach, he nonetheless settles on a decision rule on the basis of direct arguments.

2.3.2.1 Rawls (1971)

Rawls' concern in *A Theory of Justice* is with the question of how the benefits that come from co-operation in society should be divided up so as to bring about a morally fair division. One approach is to argue in favour of a meritocracy. However, the plausibility of such an answer is dependent upon the answer to another question: What constitutes a fair deal between those who are born into society advantaged/disadvantaged, either by nature or by birth? While a person born to a middle-class family, receiving a college education, and of above average intelligence might be more economically productive compared to someone from a less advantaged background, to what extent does the former deserve his/her benefits compared to the latter? To what extent is the former's success due to his/her own efforts rather than the good fortune of being born with those advantages?

In order to answer these questions, Rawls argues that any opinion of what constitutes fairness must be decided from 'the original position'. The original position is an abstraction away from anything other than agents as free and equal agents, allowing (in theory) impartial decisions in creating a hypothetical social contract for society. This impartiality stems from the veil of ignorance, which requires decisions to be made from the perspective of abstract individuals. In thinking from such a perspective, we are forced to make decisions apart from knowledge of our own ability to influence what outcomes will obtain or what benefits of social co-operation we might receive from our decision.

What these conditions amount to is complete uncertainty. Because they are not connected with their capacities and position in life, people are forced to make decisions without any reason to think that any particular outcome is likely to occur. Their decision is made knowing only which outcomes they would prefer.

Rawls believes that the conditions of this thought experiment are highly intuitive (perhaps for philosophers who are attempting to abstract away from immediate circumstances). "The veil of ignorance is so natural a condition that something like it must have occurred to many. The closest explicit statement of it known to me is found in J.C. Harsanyi, 'Cardinal Utility in Welfare Economics and in the Theory of Risk-Taking,' Journal of Political Economy, vol. 61 (1953)." (1971, 137, footnote 14) While Rawls cites Harsanyi, upon investigation it does not appear that the latter formulated complete uncertainty in the way used by Rawls.[6]

From this veil of ignorance, Rawls formulates an approach to decision-making he calls 'the difference principle'. On the difference principle, a decision is acceptable only if it is the benefit of the least advantaged members of society (75). The difference principle regulates what he calls the principle of efficiency. On the principle of efficiency, goods are divided so as to maximize utility. Unlike the difference principle, the maximization from the principle of efficiency is indifferent towards how benefits are divided up with respect to the different members of a society. However, from behind the veil of ignorance, individuals are unable to determine to what extent they would

[6]Luce and Raiffa (1957) make note of the idea of 'complete ignorance' but within a value-based framework, not the absolute lack of information that characterizes our form of complete uncertainty.

be on the preferred end of those benefits. Given the above potential distributions between most and least advantaged, all people will choose the point determined by the difference principle. The reasoning on the basis of this choice is that the conditions of the choice together with values of reciprocity and fairness (not a psychological aversion to uncertainty (2001, xvii)) will incline them towards this decision rule. As it so happens, the logical structure of the difference principle is Minimax, a point he recognizes.(1971, 152-153)

This kind of thinking also has parallels amongst economists (though not so much the emphasis on fairness and reciprocity). Vercelli also suggests a Minimax approach under conditions of complete uncertainty. He argues that as certainty decreases, we are forced to accept more conservative approaches to decision-making. Under conditions of certainty, we maximize utility. Under soft uncertainty, we maximize expected utility. Under conditions of hard uncertainty, he suggests we maximize Choquette expected utility (which is in between maximization of expected utility and Minimax). With even greater amounts of ignorance involved in complete uncertainty, we should then adopt the most conservative maximizing strategy: Minimax.(1999, 247-248)

2.3.2.2 Van Fraassen (1973) & Jennings (1974)

While Rawls' work is informed to a minor extent by the framework and results of economic decision theory, what is interesting about both Van Fraassen and Jennings is that their work appears to have developed completely independently from the conceptual scheme of the economists. Nevertheless, when parsing their conceptual language, we find the same framework developing from their axiological approach modifying deontic logic.

Van Fraassen Van Fraassen develops his view on the basis of an axiological approach to the concepts of obligation and 'ought'. The latter is a more general version of the former, recommending one act over another on the basis of the desirability of the consequences associated with each act rather than on the basis of the permissibility of the act itself. However, because the axiological approach defines the former on the basis of the latter, any definitions chosen will need to be relateable. Van Fraassen finds just such a definition from Moore: "[T]o assert that a certain line of conduct is, at a given time, absolutely right or obligatory, is obviously to assert that more good or less evil will exist in the world, if it is adopted, than if anything else be done instead." (1973, cited on p.6) Using this definition of obligation, he works backwards to a definition of 'ought'.

But, before coming ot this definition, another issue must be addressed, namely that actions do not cleanly map to desired states. After all, "the alternatives considered are the possible outcomes of our action" (1973, 7) and not the certain outcomes of our actions. Accounting for this through his set theoretic framework, Van Fraassen suggests that we are to bring about that proposition (understood as the truth set) which produces more good or less evil. But, what constitutes 'more good or less evil' for a set where only one of those outcomes will obtain? He suggests the following:

Van Fraassen's Theory: "For any sentence A, let $H(A)$ be those alternatives which make A true.[7] Then we say: 'It ought to be the case that A' is true exactly if some value attaching to some outcome in $H(A)$ is higher than any attaching to any outcome in $H(\textit{not}\, A)$. Intuitively phrased: we ought to opt for the realization of the highest possible values, and, more generally, for any state of affairs that is a necessary condition for the realization of the highest attainable values." (1973, 7)

Unfortunately, little is given in the way of justification for why we should favour the set with that one highest outcome while ignoring the other states associated with that choice. Perhaps we might offer the justification that all we are morally required to do is choose the set with the best possible outcome - whether it is what obtains is beyond our control. In simply choosing the set that contains this outcome, we have done our duty. Leaving the absence of his own logical or moral justification aside, we may form a technical definition of his view.

[7]Note that unlike the language of economists which makes use of general states of affairs and sets as any combination of states, the philosophical approach makes use of propositions to form truth sets, i.e. the collection of states of affairs in which the proposition is true. While this offers many benefits from the perspective of precision, it can become fairly unwieldy by comparison to the language of economists that we will be employing.

Definition 11. $\succeq_V$

$$A \succeq_V B \Leftrightarrow (\exists a \in A)(\forall b \in B)(a \geq b)$$

Colloquially, a set A is at least as good as a set B iff there is some state in A that is at least as good as all states in B. We can next find the strict preference relation by application of our definition of $\succ$as the asymmetric version of $\succeq$. We obtain:

Definition 12. $\succ_V$

$$A \succ_V B \Leftrightarrow [(\exists a \in A)(\forall b \in B)(a \geq b)] \& \neg [(\exists b \in B)(\forall a \in A)(b \geq a)]$$

Jennings Unlike Van Fraassen, Jennings had a more specific intention in mind, namely, a framework for a utilitarian approach to deontic logic. This may sound rather bizarre, since deontic logic brings to mind deontology which is typically contrasted with utilitarianism. However, what Jennings has in mind is to appropriate the logical framework of deontic logic which itself makes use of modal logic. As he sees it, the utilitarian approach to ethics may be conceived of as a moral theory with one duty: to maximize benefit for greatest number. The issue becomes how to express the heart of the utilitarian argument in the language of possible worlds.

The difficulty with using the framework of standard deontic logic is that it it is focused only on obligations and ignores the way in which some outcomes are at least relatively better than others. By contrast, utilitarianism can incorporate and distinguish between the two.Jennings (1974, 447-448) The question however, is how to incorporate these ideas into the framework of the former. Perhaps, he suggests, like this: "A Utility Structure is an ordered pair $< \mathcal{D}, B >$ where $\mathcal{D}$ is a non-empty set and B is a function from $\mathcal{D}$ into $\mathcal{P}(\mathcal{D}^2)$ which determines for each point $u \in \mathcal{D}$ a transitive and irreflexive relation B^u. The expression xB^uy is read 'x is better than y for u'."[8] Jennings (1974, 449). In plainer terms, what he suggests is to combine two things, sets of possible outcomes (rather than just outcomes themselves) and

[8]Note, however, that this is stated in terms of what we call strict preference rather than the 'at least as good as' relation.

a preference relation between two any two outcomes. This setup for his language is strikingly familiar to the setup that was provided for the language that we are utilizing. In fact, what he is suggesting is the need for some kind of extension rule that takes preferences between outcomes and extends it to preferences between sets of outcomes.

Jennings arrives at his rule by first considering a stronger form and rejecting it: "It cannot mean that every conceivable state of affairs in which the act is performed is better than every conceivable state of affairs in which it remains unperformed." Jennings (1974, 448) This would be far too strong a requirement. However, we do want to guarantee that every outcome from the set we reject is beaten by at least one outcome from the set we choose. Nothing then is superior to every one of the points within our set of choice. This ensures that for every outcome in the rejected set, our set possesses an outcome that makes the world better.

Jennings' Theory: "An action is obligatory if and only if for every way of refraining from performing it, there is a way of performing it which will make the world happier." (1974, 445)

Jennings goes so far as to give us a precise formulation of how this parses out in his logic (1974, 450):

Definition 13. $\succeq_J$

$$A \succeq_J B \Leftrightarrow (\forall b \in B)(\exists a \in A)(a \geq b)$$

This states that for every element in B, there is at least one element in A that is at least as good as that element in B. Again, we find the strict preference counterpart by our general definition of strict preference as the asymmetric form of the 'at least as good as' relation.

Definition 14. $\succ_J$

$$A \succ_J B \Leftrightarrow [(\forall b \in B)(\exists a \in A)(a \geq b)]\&\neg[(\forall A \in A)(\exists b \in B)(b \geq a)]$$

We now have two different complex decision rules for decision under complete uncertainty. Or do we? At first glance, it seems to be the case. Whereas Jennings is concerned with ensuring that everything in choice set B is beaten by at least something in choice set A, Van Fraassen seems intent on everything in B being beaten by a single point - some kind of super point. It appears then that Jennings' rule is a more conservative choice than Van Fraassen's approach of hoping for the best. Schotch (2006, 178) makes this very point:

> "[Van Fraassen's rule] is clearly the logically stronger, so strong that it seems to be wrong in a certain sense. This one perhaps. Suppose we had two propositions (sets of alternative states), say $||\alpha||$ and $||\beta||$. ... $||\beta||$ contains nothing but very attractive (for us) states. $||\alpha||$, on the other hand, contains really awful hellishly unattractive states, millions upon millions of them, except for one only exception, which is virtually perfect. Now since the nearly perfect state in $||\alpha||$ beats all the extremely nice states in $||\beta||$ it follows on [Van Fraassen's rule], that $||\alpha||$ is [at least as good as] $||\beta||$. 'But wait!' we want to object. 'What about all those terrible states in $||\alpha||$?' This is obviously a legitimate worry since we are choosing sets of states (propositions) rather than individual states. If we choose $||\alpha||$, how do we know we'll end up in heaven (which is just one state after all) rather than hell (lots and lots of states)?"

However, he continues:

> "This is a tricky issue, and one which isn't improved very much by moving to the type-raising scheme [given by Jennings' rule] even though it may initially appear less headlong than [Van Fraassen's rule]."

The reason for these doubts arises from an interesting result Schotch discovers upon translating the predicates into a different form. On doing so, he finds that $\succ_V$ is equivalent to $\succ_J$.

$$\begin{aligned} A \succ_V B \quad &\Leftrightarrow \quad [(\exists a \in A)(\forall b \in B)(a \geq b)] \& \neg[(\exists b \in B)(\forall a \in A)(b \geq a)] \\ &\Leftrightarrow \quad [(\exists a \in A)(\forall b \in B)(a \geq b)] \& [(\forall b \in B)(\exists a \in A)(\neg(b \geq a))] \end{aligned}$$

$$\begin{aligned} A \succ_J B &\Leftrightarrow [(\forall b \in B)(\exists a \in A)(a \geq b)] \& \neg[(\forall A \in A)(\exists b \in B)(b \geq a)] \\ &\Leftrightarrow [(\forall b \in B)(\exists a \in A)(a \geq b)] \& [(\exists a \in A)(\forall b \in B)(\neg(b \geq a))] \end{aligned}$$

Since $\succeq$ is complete, he sees $\succ_V$ as the conjuncts of $\succeq_V$ and $\succeq_J$, and $\succ_J$ as the conjuncts of $\succeq_J$ and $\succeq_V$. This shows that the two are equivalent.

However, while Schotch is correct that there is an equivalence, the proof is not quite this straightforward - or clear for that matter. In order to accomplish this, completeness must be such that $\neg(b \geq a)$ is equivalent to $a \geq b$. If so, we obtain

$$\begin{aligned} A \succ_V B &\Leftrightarrow [(\exists a \in A)(\forall b \in B)(a \geq b)] \& [(\forall b \in B)(\exists a \in A)(a \geq b)] \\ A \succ_J B &\Leftrightarrow [[(\forall b \in B)(\exists a \in A)(a \geq b)] \& [(\exists a \in A)(\forall b \in B)((a \geq b))] \end{aligned}$$

which obtain the exact results described. But, what completeness properly gives us from $\neg(b \geq a)$ is actually $a > b$. By definition of $>$ this implies $a \geq b$, but also $\neg(b \geq a)$. So what we actually have are:

$$\begin{aligned} A \succ_V B &\Leftrightarrow [(\exists a \in A)(\forall b \in B)(a \geq b)] \& [(\forall b \in B)(\exists a \in A)(a > b)] \\ A \succ_J B &\Leftrightarrow [(\forall b \in B)(\exists a \in A)(a \geq b)] \& [(\exists a \in A)(\forall b \in B)(a > b)] \end{aligned}$$

While each rule is not simply the conjunct of itself and the other, clearly on the basis of the above we can see a deep relationship between the two rules. However, trying to mentally compute every relation between one set of points to another makes it difficult to see how they are related. Moreover, from an explanatory perspective, it is difficult to understand their relationship by fiddling with the logic. But we can in fact simplify these logics into a much more transparent form - namely, that of economics.

Proposition 15. *$\succeq_V$ is equivalent to $\succeq_{Max}$.*

Proof. Suppose $A \succeq_V B$. Therefore, there is an arbitrary $a \in A$ such that it is at least as good as every member of B, including $Max(B)$. Either a is $Max(A)$ or it is not. If it is, then $Max(A) \geq Max(B)$. If it is not, then since $\geq$ generates a linear ordering, $Max(A) \geq a$. Therefore, by transitivity, $Max(A) \geq Max(B)$.

In reverse, if $Max(A) \geq Max(B)$, then since $Max(B)$ is at least as good as every b, by transitivity $Max(A)$ is at least as good as every b.

Therefore, $\succeq_V$ is equivalent to $\succeq_{Max}$. □

While $\succeq_V$ is quite clearly related to $\succeq_{Max}$, it is less evident how $\succeq_J$ is. The reason why this may be the case is that we are inclined to think that $\succeq_J$ implies that for every element in B, there is some element in A that beats it - *but not necessarily the same element*. But, this is mistaken. Correcting this idea, the following proof becomes clear:

Proposition 16. *$\succeq_J$ is equivalent to $\succeq_{Max}$*

Proof. Suppose $A \succeq_J B$. Therefore, for every b there is an a which is at least as good as that b. Since $Max(A)$ is at least as good as any other a, by transitivity, $Max(A)$ is also at least as good as every b.

In reverse, if $Max(A) \geq Max(B)$, then since $Max(B)$ is at least as good as every other b, by transitivity for each element in B there will minimally be one element in A that beats it, namely, $Max(A)$.

Therefore, $\succeq_J$ is equivalent to $\succeq_{Max}$. □

The following important result alluded to by Schotch follows:

Theorem 17. *$\succeq_V$ is equivalent to $\succeq_J$.*

Proof. This follows by transitivity from the equivalence of both$\succeq_V$ and $\succeq_J$ to $\succeq_{Max}$ □

Corollary 18. *$\succ_V$ is equivalent to $\succ_J$.*

Proof. This follows immediately from Theorem 17 and the definition of $\succ$. □

A little may be said to explain the reason why this equivalence is not immediately apparent. In many cases, the order in which predicates are nested matters. However, the reason it does not in this case is that $\geq$ is complete and linear, ensuring that every element is placed on a 'line' and related to every other element.

So what has been accomplished? After all, Maximax as a decision rule most certainly preceded the work of Van Fraassen and Jennings. I see this discussion as accomplishing three things:

1. In Van Fraassen and Jennings' logical framework we saw an overlap between economics and philosophy in the form of ideas highly reminiscent of nascent work on decision under complete uncertainty.

2. Propositions 15 and 16 further evince the overlap in the form of proposed decision rules. While not characterized in the way of the economists, it is noteworthy that they independently reached similar results.

3. In trying to work and derive with the complex language utilized by philosophers, we find value in employing some aspects of the language used by economists so as to simplify and add a new form of transparency to set comparisons. This suggests a benefit to philosophers from adopting an interdisciplinary approach to the field.

2.4 Basic Axioms

I now turn to present some of the basic axioms used by economists that begin the project.

2.4.1 Dominance Axioms

Dominance: For all $A \in \mathcal{X}$, and for any $x \in X$,

$$\begin{array}{ll} \text{(i)} & [\text{if } x > y \text{ for all } y \in A] \Rightarrow A \cup \{x\} \succ A \\ \text{(ii)} & [\text{if } y > x \text{ for all } y \in A] \Rightarrow A \succ A \cup \{x\} \end{array}$$

According to the first part of this axiom, if we take a state of affairs that is strictly preferred to every state of affairs in a set, then this new set must be better than the old one. The same holds in the opposite direction, that adding a state worse than all those in an existing set will make that set worse than the previous one. The intuition behind this axiom is that the addition of a new element that is in no ways worse will certainly be an improvement, and adding a worse option certainly makes things worse.

Simple Dominance: For all $x, y \in X$,

$$x > y \Rightarrow [\{x\} \succ \{x, y\} \succ \{y\}]$$

This axiom is a more modest version of the former, but makes use of the same intuition but within a restricted case. Certainly anyone who holds to the more general DOMINANCE condition will hold to this weaker form (the former clearly implying the latter). Even on its own, we can see its merits. If choosing between my preferred option as a certainty and merely the possibility of my preferred option along with its weaker counter-part, then I would rather take the certain outcome. If faced with choosing between the certainty of my lesser preferred outcome and even the hope of my more preferred outcome, I would prefer the latter.

2.4.2 Independence Axioms

Strict Independence: For all $A, B \in \mathcal{X}$, for all $x \in X \setminus (A \cup B)$,

$$A \succ B \Rightarrow A \cup \{x\} \succ B \cup \{x\}$$

The principle behind this axiom is that where one set is strictly preferred to another, the addition of a completely new element to both sets should have no material effect on set rankings. After all, if this addition exists in both sets, then one would be inclined to say that this addition helps neither of the sets relative to each other. By a related line of reasoning, this new alternative is all but irrelevant.

Independence: For all $A, B \in \mathcal{X}$, for all $x \in X \setminus (A \cup B)$,

$$A \succ B \Rightarrow A \cup \{x\} \succeq B \cup \{x\}$$

This is a weaker form of independence that does not require that this new element preserve strict preference. However, it does guarantee that the addition of the same element would not *reverse* the set rankings.

Extended Independence: For all $A, B \in \mathcal{X}$, for all $C \subseteq X \setminus (A \cup B)$,

$$A \succ B \Rightarrow A \cup C \succeq B \cup C$$

Extended independence simply allows INDEPENDENCE to apply to sets of new elements. Barberà et al. (1984) prove a biconditional relation between INDEPENDENCE and EXTENDED INDEPENDENCE given one or the other and SIMPLE DOMINANCE.

2.5 An Important Initial Result

We now have the axioms needed for an important result for a first step towards a decision rule. The following proof is owed to Kannai and Peleg (1984).

Theorem 19. *Suppose $A \in \mathcal{X}$. If $\succeq$satisfies Dominance and Independence, then $A \simeq \{Max(A), Min(A)\}$.*

Proof. Let $A \in X$. If $|A| \leq 2$, then the theorem holds immediately. Suppose $n > 2$. A may be defined as the set $\{x_1, ..., x_n\}$, and let us further suppose that the subscripts also give the ordinal ranking of the state of affairs.

By dominance, $\{x_1\} \succeq \{x_1, x_2\}$. By dominance, $\{x_1, x_2\} \succ \{x_1, x_2, x_3\}$. By transitivity, $\{x1\} \succ \{x_1, x_2, x_3\}$. By repeated applications of dominance and transitivity until the inclusion of x_{n-1}, we come to $\{x_1\} \succ \{x_1, ..., x_{n-1}\}$. By independence to include x_n, $\{x_1, x_n\} \geq \{x_1, ..., x_n\}$.

We now turn to prove the reverse, beginning with $\{x_2, x_3\} \succ \{x_n\}$, we apply dominance and transitivity until $\{x_2, ..., x_n\} \succ \{x_n\}$. Applying independence, $\{x_1, ..., x_n\} \succeq \{x_1, x_n\}$.

By definition of indifference, $\{x_1, x_n\} \simeq \{x_1, ..., x_n\}$. Since $x_1$and $x_n$give us $Max(A)$ and $Min(A)$ respectively, therefore $A \simeq \{Max(A), Min(A)\}$. □

Bossert et al. (2000) prove that the same result as Theorem 19 can be obtained even when the dominance property is weakened to SIMPLE DOMINANCE.

Theorem 20. *Suppose $A \in \mathcal{X}$. If $\succeq$satisfies Simple Dominance and Independence, then $A \simeq \{Max(A), Min(A)\}$.*

Proof. We make use of proof by induction. Once again, if $|A| \leq 2$, then the theorem holds immediately. This gives us our base case. Suppose $n > 2$ and $A \simeq$

$\{Max(A), Min(A)\}$ where $|A| < n$. Define $A : \{x_1, ..., x_n\}$ where subscript indicates ordinal ranking.

By SIMPLE DOMINANCE, $\{x_1\} \succ \{x_1, x_{n-1}\}$. Since the sets just mentioned have a cardinality less than A, we can apply our assumption; this leads to $\{x_1, x_{n-1}\} \sim \{x_1, ..., x_{n-1}\}$. By applying transitivity to this formula and the one just prior, we obtain $\{x_1\} \succ \{x_1, ..., x_{n-1}\}$. Applying INDEPENDENCE to add x_n, we obtain $\{x_1 x_n\} \succeq \{x_1, ..., x_n\}$.

Similarly, to prove the reverse, begin again by SIMPLE DOMINANCE, $\{x_2, x_n\} \succ \{x_n\}$. By our assumption, $\{x_2, x_n\} \sim \{x_2, ..., x_n\}$. By transitivity, $\{x_2, ..., x_n\} \succ \{x_n\}$, and by INDEPENDENCE to add x_1, we obtain $\{x_1, ..., x_n\} \succeq \{x_1, x_n\}$.

By definition of indifference, $\{x_1, x_n\} \simeq \{x_1, ..., x_n\}$. Since $x_1$and x_n give us $Max(A)$ and $Min(A)$ respectively, therefore $A \simeq \{Max(A), Min(A)\}$. $\square$

This theorem is significant in that it shows that middle elements are irrelevant against its Max and Min. We find that a set is ultimately characterized by its maximal and minimal elements - a result consistent with the thinking associated with either a Maximax or Minimax rule.[9]

2.6 The Initial Problem: Impossibility Results

Despite our initial success in showing that the Max and Min of a set are central to a decision rule given plausible combinations of basic axioms, the field runs into a significant problem in the form of impossibility theorems. What these theorems suggest is that an inconsistency exists between our axioms. If this is so, then we cannot rely upon decision rules that rely upon those combinations of axioms.

2.6.1 From Dominance and Independence

Kannai and Peleg (1984) show that under certain conditions, any characterization that meets DOMINANCE and INDEPENDENCE will lead to a contradiction.

[9]Though of course our axioms would not generate either Maximax or Minimax. After all, our dominance-type conditions rule out such approaches since the addition of a weaker element to a set should not make a difference to Maximax, and vice versa in the case of Minimax.

Theorem 21. *Suppose $|X| \geq 6$. There exists no characterization of $\succeq$satisfying Dominance and Independence.*[10]

Proof. We accomplish this proof by contradiction. Let $|X| \geq 6$ and $\succeq$satisfying Dominance and Independence. Let X consist of states $x_1, ..., x_n$ where subscript indicates ordinal ranking.

Suppose $\{x_3\} \succ \{x_2, x_5\}$. Applying Independence, $\{x_3, x_6\} \succeq \{x_2, x_5, x_6\}$. Applying Theorem 19 to $\{x_3, x_6\}$ gives $\{x_3, x_6\} \simeq \{x_3, x_4, x_5, x_6\}$. Applying Theorem 19 to $\{x_2, x_5, x_6\}$ gives $\{x_2, x_5, x_6\} \simeq \{x_2, x_6\}$ and again to give $\{x_2, x_6\} \simeq \{x_2, x_3 x_4, x_5, x_6\}$. By transitivity, $\{x_3, x_4, x_5, x_6\} \succeq \{x_2, x_3 x_4, x_5, x_6\}$. But this would contradict Dominance, which holds that since x_2 is greater than any element in $\{x_3 x_4, x_5, x_6\}$, it should make the set $\{x_2, x_3 x_4, x_5, x_6\}$ strictly preferred to $\{x_3 x_4, x_5, x_6\}$.

Suppose $\{x_2, x_5\} \succeq \{x_3\}$. Since $\{x_3\} \succ \{x_4\}$, by transitivity, $\{x_2, x_5\} \succ \{x_4\}$ (since $\{x_2, x_5\}$ is at least as good as, then minimally it is indifferent to $\{x_3\}$, so if $\{x_3\} \succ \{x_4\}$, we know that $\{x_2, x_5\}$ must be as well). By Independence, we add $x_1$to get $\{x_1, x_2, x_5\} \succeq \{x_1, x_4\}$. By 19 $\{x_1, x_5\} \simeq \{x_1, x_2, x_3, x_4, x_5\}$ and $\{x_1, x_4\} \simeq \{x_1, x_2, x_3, x_4\}$. By transitivity, $\{x_1, x_2, x_3, x_4, x_5\} \succeq \{x_1, x_2, x_3, x_4\}$. But, this violates Dominance, according to which, when $\{x_5\}$ is added to $\{x_1, x_2, x_3, x_4\}$, $\{x_1, x_2, x_3, x_4\} \succ \{x_1, x_2, x_3, x_4, x_5\}$.

Notice that the two suppositions are exhaustive under completeness. Since both suppositions lead to contradictions, it follows that our assumptions, that a set of states greater than or equal to six with$\succeq$ satisfies Dominance and Independence, is inconsistent. □

2.6.2 From Simple Dominance and Strict Independence

Barberà and Pattanaik (1984) prove a similar result when Dominance is weakened to Simple Dominance and Independence is strengthened to Strict Independence.

[10] As a passing note for interest's sake, this proof can be extended to the case where the number of elements is greater than or equal to 4 if a neutrality property is added to a characterization (Barberà et al., 2004). I omit this proof, however, since I later argue that the particular number of elements needed to generate these impossibility theorems is not pertinent to the philosophical implications of impossibility that I wish to bring out (see 2.6.3).

Theorem 22. *Suppose $|X| \geq 3$. There exists no characterization of $\succeq$satisfying Simple Dominance and Strict Independence.*

Proof. We prove this by contradiction. Suppose $|X| \geq 3$ and that there exists a characterization of $\succeq$satisfying Simple Dominance and Strict Independence. Let $X : \{x_1, x_2, x_3\}$where subscript denotes ordinal ranking.

By Simple Dominance, $\{x_1\} \succ \{x_1, x_2\}$. Applying Strict Independence, $\{x_1, x_3\} \succ \{x_1, x_2, x_3\}$.

Simple Dominance also implies $\{x_2, x_3\} \succ \{x_3\}$. Applying Strict Independence, $\{x_1, x_2, x_3\} \succ \{x_1, x_3\}$.

Since $\succ$is asymmetric, we have a contradiction. Therefore, for a set of states greater than or equal to three, there is no consistent characterization of $\succeq$that satisfies Simple Dominance and Strict Independence. □

2.6.3 Some Comments on the Impossibility Results

What significance should we attach to the condition that the number of elements is greater than or equal to 6? Perhaps one might wish to argue that because these impossibility theorems only hold in cases with a given number of elements then it is not strictly a problem for the relevant axioms.

This reply, however, seems to be a reply of desperation. If these axioms are taken on the grounds that rationality demands that we accept them, then it is of cold comfort to our expectations of rationality to say that they are at least consistent when only working with a few states of affairs while more complex comparisons be damned.[11] I take these impossibility results, then, to be the central problem of decision under complete uncertainty. Given the intuitive plausibility of our basic axioms, I take a satisfactory solution to involve not merely the provision of a consistent set of axioms, but also an explanation of what about these initial axioms generates

[11] A further objection pointed out to me by Prof. Hymers is that perhaps we need not be concerned because we will seldom ever be required to compare more than six elements. I reply that such a situation seems plausible when it comes to making policy decisions. Such decisions are often made in light of several goals. If we conceive of each goal as a proposition, then each proposition carries with it at least two possible states (if it was certain then we would not even need to consider it as a goal). It seems to me that the conjunction of a few goals will quickly give us a decision where more than six elements are involved.

an impossibility. After all, if we reject them, then we assent to their negation. But if we assent to their negation, then it behooves us to provide just a justification.

There is another reason why we should be concerned about a system with even the constrained inconsistencies mentioned above. If our ethics is modeled on an axiological approach, then the ability to generate inconsistencies implies the existence of conflicting 'oughts' and obligations (since obligations are a subset of 'oughts'). And unless we adopt a paraconsistent model of inference, inconsistent moral judgments allow us to prove any moral claim by *reductio ad absurdum*.

In the next chapter I canvass different solutions to the impossibility theorems.

Chapter 3

Responses to Impossibility

We now look at two solutions to the impossibility problems previously discussed. We also consider a third 'solution': reject all the axioms that generate impossibility and start with fresh or modified axioms.

3.1 Solution 1: Reject Completeness

When drawing implications from impossibility theorems, there are two different conclusions we can reach. Thus far, we have interpreted the impossibility results as showing that the axioms involved are inconsistent. However, an alternate approach is to maintain consistency and instead to reject completeness. This is the interpretation taken towards Gödel's incompleteness theorem regarding axiomatized forms of arithmetic. We do not take arithmetic as inconsistent (we just assume that they are consistent), so we reject completeness. Analogously, we might take this approach towards the sets of axioms we select in our context of decision under complete uncertainty. What this would look like is a decision rule that would tell us that a set A is non-comparable to a set B. In order to do this, we would be rejecting completeness at the set level of $\succeq$. How does this address the impossibility theorems? Recall that in the first impossibility theorem, the proof utilized the rule of disjunction elimination by supposing both sides of a mutually exclusive and exhaustive relation, specifically, $A \succ B$ or $B \succeq A$. In order to generate impossibility, both disjuncts were shown to produce a contradiction. But, without completeness, then $A \succ B$ together with $B \succeq A$ are not exhaustive - there remains $A \bowtie B$ (where '$\bowtie$' is the non-comparable relation). But we cannot prove a contradiction where A is non-comparable to B, since non-comparability means no relation by 'at least as good as' (hence no conflicting 'at least as good as' relations). Thus, the proof showing a contradiction breaks down.

What would an incomplete decision rule look like? Interestingly, in Schotch (2006), we find an example of such a rule.

3.1.1 Schotch (2006)'s 'Onto' Decision Rule

In order to move away from the headlong attitude taken by Van Fraassen and Jennings, Schotch formulates his own theory he calls 'Onto', adopting an additional constraint on a plausible decision rule in the form of a second conjunct. According to this conjunct, if a set A is at least as good as a set B, then it should also be the case that every A-state is minimally at least as good as one B-state. This would make it such that there is no A state so bad that every B state would be better. This would prevent the cases he takes as problematic for Van Fraassen, since a set containing a heaven state and many hellish states would not be at least as good as another unless even the first set's most hellish state is not quite so bad as one in the other. Let us look at the formal form of 'Onto':

Definition 23. $\succeq_O$

$$A \succeq_O B \Longleftrightarrow [(\forall b \in B)(\exists a \in A)(a \geq b)]\&[(\forall a \in A)(\exists b \in B)(a \geq b)]$$

Thinking about the relationships between the two sets, we can see why the theory is called 'Onto'. The Onto function is a mapping such that every member of a set A corresponds to a member of a set B. In this case, every A element is connected to a B element by being at least as good as that B element.

We now apply the definition of $\succ$ to obtain:

Definition 24. $\succ_O$

$$\begin{aligned} A \succ_O B \quad \Longleftrightarrow \quad & [(\forall b \in B)(\exists a \in A)(a \geq b)\&(\forall a \in A)(\exists b \in B)(a \geq b)]\& \\ & [\neg[(\forall a \in A)(\exists b \in B)(b \geq a)\&(\forall b \in B)(\exists a \in A)(b \geq a)] \\ \Longleftrightarrow \quad & [(\forall b \in B)(\exists a \in A)(a \geq b)\&(\forall a \in A)(\exists b \in B)(a \geq b)]\& \\ & [[(\exists a \in A)(\forall b \in B)\neg(b \geq a)] \vee [(\exists b \in B)(\forall a \in A)\neg(b \geq a)]] \end{aligned}$$

Schotch notes that this rule is not found in the literature, but seems conceptually natural in spite of that. While we can simply apply the definition of $\succ$ in order to find the predicate logic version of this decision rule, let us first see if we can simplify. Because the first conjunct of this formula comes from [J], we can immediately simplify it to Maximax. What about the second conjunct? It turns out that this is Minimax, making the economic version of Onto be the conjunct of Maximax & Minimax. Let us call '$\succeq_D$' the version of $\succeq$ that requires both $\succeq_{Max}$ and $\succeq_{Min}$.

Definition 25. $\succeq_D$

$$A \succeq_D B \Longleftrightarrow (Max(A) \geq Max(B))\&(Min(A) \geq Min(B))$$

Proposition 26. $\succeq_O$ *is equivalent to* $\succeq_D$

Proof. As previous stated, since the first conjunct of $\succeq_O$ is $\succeq_J$, and $\succeq_J$ is equivalent to $\succeq_{Max}$, this proves the first conjunct of $\succeq_D$. Next we need to prove that the second conjunct is $\succeq_{Min}$. If every element of A beats some element of B, then this includes $Min(A)$. If $Min(A)$ beats some element b of B, then either b is $Min(B)$ or it is not. If b is $Min(B)$, then $Min(A) \geq Min(B)$. If b is not Min(B), then it follows by the linear ordering of $\geq$that $b \geq Min(B)$. By transitivity, if $Min(A) \geq b$, and $b \geq Min(B)$, then $Min(A) \geq Min(B)$ ($\succeq_{Min}$). □

Definition 27. $\succ_D$

$$\begin{aligned} A \succ_D B \iff & [(Max(A) \geq Max(B))\,\&\,(Min(A) \geq Min(B))]\& \\ & \neg[(Max(B) \geq Max(A))\&(Min(B) \geq Min(A))] \\ \iff & [(Max(A) \geq Max(B))\,\&\,(Min(A) \geq Min(B))]\& \\ & [\neg((Max(B) \geq Max(A)) \vee \neg((Min(B) \geq Min(A))] \\ \iff & [(Max(A) \geq Max(B))\&(Min(A) \geq Min(B))]\& \\ & [(Max(A) > Max(B)) \vee (Min(A) > Min(B))] \end{aligned}$$

This rule is much clearer and easier to apply than its Onto formulations. According to this rule, A is at least as good as B iff both of A's Max and Min are at least as good as those of A. For the symmetry of indifference, we would require the same in reverse. Thus, A and B are indifferent iff both extremes are indifferent. This aligns perfectly with Theorem 20. However, we now have room for non-comparability, namely, when one of the Max or Min differs. Strict preference is obtained when one extreme is equal between both sets[1], but the other extreme is greater. This almost has the plausibility of a dominance-type property in the sense that anyone who accepts either a Maximax or Minimax rule would agree that if one extreme is equal between both sets and one set is greater in the other extreme, then surely the set with one greater extreme is preferable.

3.1.2 Bossert (1989)'s Max and Min-based Dominance Rule

Unfortunately, while Schotch's rule has not been seen in the philosophical literature, it has been conceived and characterized in the economic literature. Its creation was for the very purpose of finding a solution to the impossibility theorems by rejecting completeness. This gives us yet another example of the benefit of bridging the gap between the work of economics and philosophy on decision under complete uncertainty.

We need one more axiom in order to give Bossert's characterization.

Neutrality: For all $A, B \in \mathcal{X}$, for all one-to-one mappings $\varphi : A \cup B \to X$,

$$([x \geq y \Leftrightarrow \varphi(x) \geq \varphi(y) \& y \geq x \Leftrightarrow \varphi(y) \geq \varphi(x)]$$

The neutrality axiom allows the narrowing (including closing) or expanding of ranking gaps between states if irrelevant states exist (i.e. states that exist in X but are not involved in the comparison. For this to be done properly, there are three necessary conditions: (1) It must be done consistently - each element may be mapped onto at most one other element and no two elements may share the same mapping element; (2) The gap being narrowed/closed must be a true gap without any relevant state between the original state and the state onto which we wish to map it (e.g. x_1 may be mapped to x_5 so long as none of $x_2, ..., x_4$ are in use).; (3) All rankings between all relevant states must be preserved, so one cannot map x_1 to x_5 while mapping x_2 to x_4.

[1]Technically, at least as good as, but because we forbid its being strictly dispreferred, by implication it must be at least equal.

It may be worthwhile to make a few comments about NEUTRALITY. One interesting aspect of it is that it does not tell us what is strictly preferred in particular cases, but rather tells us something about the structure of comparisons. Second, while at first glance it seems to be an intuitive logical principle (Barberà et al. (2004) for example suggest that: "the labeling of alternatives is irrelevant to establishing the ranking $\succeq$" (911)), if we explore it a bit, then the very idea seems philosophically contentious as it implies that if something is not an option in a given comparison, then it should be irrelevant with respect to that comparison. In the event that a state of affairs is not present in the comparison, then NEUTRALITY would allow us to pretend that it did not exist for the sake of the comparison. But that seems a bit odd given our framework. Recall that with respect to a set of comparisons, we identified X as the set of all possible states for the comparisons in question (i.e. a state is included iff it is used in one of the comparisons). We should also note that X is a finite set; in other words, we do not make X consist of the set of all *logically possible* states, but rather those states that we deem *relevant* to a discussion. But, this being the case, is it reasonable to discount the information given by the presence of a state within X even if that state does not occur in one specific stage of comparison? Its inclusion within X might give us a good reason to make a decision while being cognizant of its presence, such that ignoring its absence in one specific comparison (as is implied by NEUTRALITY) suggests a loss of information. For example: is comparing $\{x_5\}$ to $\{x_1, x_2, x_4\}$ really the same as comparing it to$\{x_1, x_2, x_3\}$? I think that one could plausibly maintain that there is a distinction between the two comparisons that would be collapsed upon accepting NEUTRALITY.

Leaving aside the philosophical issues involved with NEUTRALITY, we now have the axioms necessary for Bossert's theorem. As it will not be pertinent to the rest of our discussion, I give the the characterization without the proof.

Theorem 28. *Suppose $|X| > 3$. $\succeq$ (without completeness) satisfies Simple Dominance, Independence, and Neutrality iff $\succeq_D$.*

3.1.3 Some Comments

This rule seems to me highly plausible despite lack of discussion regarding its acceptance in the economic literature. I suspect that one might be averse towards it because of its rejection of the idea that any two sets may be ranked by preference; however, I take this as a merit when our purpose is to select a decision rule on the basis of a non-personal conception of rationality. When faced with a comparison that might fall either way between two sets depending on whether one accepts Maximax or Minimax, there seems to be no ideal answer correct for everyone. We typically would say that it depends on one's preferences with respect to uncertainty. This might suggest that if trying to suggest what the idealized agent would rationally prefer, we must be indifferent. It is only once we use actual agents with their preferences, not just pure undifferentiated rationality, that we may answer - but then only for that actual agent.

But, it seems to me that the rejection of completeness advocated by Bossert and Schotch's rule gives us a way to say a bit more without taking a stance towards either Max or Min. What we say on their answer is that the idealized agent's decision rule is incomplete, *but* that this does not leave us in the quandary that we can no longer say what decisions a rational agent would make under any situation. Rather, we may only say what any rational agent would make in certain limited situations, namely, those situations where both a Maximax and Minimax agent would agree! These circumstances are simply those where at least one extreme is equal between two sets. In such a scenario we look to the other extreme for guidance.[2] However, in cases which would require us to place a priority on either extreme, we deny our ability to choose between the two by declaring that the two sets are incomparable (rather than indifferent[3]).

[2]This is actually a bit quick. According to a certain class of Maximax and Minimax rules, rather than move to the other extreme, we should consider the next element in the series (i.e. the next Max element or the next Min element). We will see this in the next section.

[3]It would be a fairly simple thing to form a decision rule that is actually indifferent between Max and Min. Rather than make the decision rule's condition a conjunct of both maximizing Max and Min for the at least as good as relation, we create the condition that it maximizes $Max\ or\ Min$, thus treating the two extremes as equally important. If A maximizes exactly one of the two, then it follows that B maximizes exactly one, making the sets symmetrically at least as good as (i.e. indifferent). If A maximizes both, we have our strict preference for A. If it maximizes neither, then

3.2 Solution 2: Use Weaker Axioms

The most straightforward form of this solution is simply to avoid any problematic combinations. This solution has been the focus of the work by economists, and from the perspective of finding a consistent approach with at least minimally plausible axioms, has been successful. In this section I review the different constructions of decision rules based on axioms that do not generate the above impossibility theorems.

The impossibility theorem that arises from Dominance and Independence involves the strongest form of dominance and the weakest form of independence. In reverse, the impossibility arising from Simple Dominance and Strict Independence relies on the weakest form of dominance and strongest form of independence. This being the case, it makes sense to rely on weaker versions of both axioms in the forms of Simple Dominance and Independence or Extended Independence. In fact, it does not matter which independence we take. Barberà et al. (2004) show that given Simple Dominance and a linear ordering on X and at least a quasi-ordering on $\mathcal{X}$, Independence and Extended Independence are equivalent.

Fortunately, these weakened axioms are sufficient. The attentive reader will have noticed that Theorem 20 made use of exactly these axioms. With our important theorem recovered, we also recover the legitimacy of Max and Min-based rules.

But consider the following sets while applying Maximax. Assuming subscript indicates ordinal ranking, $A : \{a_1, a_2\}$ and $B : \{a_1, a_3\}$ are indifferent. Or suppose the sets $A : \{a_1, a_3\}$ and $B : \{a_1, a_2\}$ on Minimax. By this decision rule the two sets are equivalent. In fact, even our axiom of Simple Dominance violates Maximax and Minimax! According to Maximax, even if $x_1 > x_2$, $\{x_1\}$ should be indifferent to $\{x1, x2\}$. But, this would be ridiculous - the certainty of a strictly preferred outcome should be preferred over the possibility of that outcome as well as the possibility of the strictly dispreferred outcome! This has motivated the development of two new rules.

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we have a strict preference for B. This rule would be a more stringent rule than either Maximax or Minimax.

3.2.1 Max-Min and Min-Max Rules

The following two decision rules are based on initial work by Bossert et al. (2000) subsequently corrected by Arlegi (2003). I will not be focusing on either the formal presentation of the rule or the characterization proofs involved with each, as this is not my primary concern. My purpose is instead to colloquially explain the decision rules and the axioms required for each.

We can understand these rules as having a primary and secondary point, each being either the Max or Min. The rule checks for which set maximizes the primary point, and failing this (i.e. in cases of indifference), moves on to the secondary point. Whichever maximizes the secondary point (given indifference of the primary point) is the preferred set. If both the Max and Min of a set are indifferent to the Max and Min or another set, the two sets are declared indifferent.[4]

Definition 29. Max-Min: the Max point is the primary focus, and the Min point is the secondary. This may be considered a development of the Maximax rule. If the two sets have different Max points, we prefer the set with the greater Max. If the Max points are equal, then we prefer the set with a higher Min. If these are also equal, then we are indifferent between the sets.

Definition 30. Min-Max: the Min point is the primary focus, and the Max point is the secondary. This may be considered a development of the Minimax rule. If the two sets have different Min points, we prefer the set with the greater Min. If the Min points are equal, then we prefer the set with a higher Max. If these are also equal, then we are indifferent between the sets.

Let us briefly cover the necessary axioms to provide characterizations of these rules.

Simple Uncertainty Appeal: For all $x, y, z \in X$,

$$x > y > z \Rightarrow \{x, z\} \succ \{y\}$$

Simple Uncertainty Aversion: For all $x, y, z \in X$,

[4]The motivating philosophy behind this rule is similar to that of Bossert and Schotch's rule in the case of what should be done if two sets share the same extreme(s).

$$x > y > z \Rightarrow \{y\} \succ \{x, z\}$$

Simple uncertainty aversion and appeal give us our Max and Min foci, respectively. They are obviously mutually exclusive, so any rule will choose one over the other.

Simple Top Monotonicity: For all $x, y, z \in X$,

$$x > y > z \Rightarrow \{x, z\} \succ \{y, z\}$$

Simple Bottom Monotonicity: For all $x, y, z \in X$,

$$x > y > z \Rightarrow \{x, y\} \succ \{x, z\}$$

Both SIMPLE TOP and BOTTOM MONOTONICITY are intuitively plausible axioms reminiscent of the dominance type axioms. In the former case, with the bottom fixed, the substitution of the top element for a weaker one surely generates a weaker set. Likewise, if the top is fixed, then the substitution of the bottom element for a weaker one generates a weaker set. While they are consistent, Max-Min and Min-Max each require only one. Combining one with the rest of the axioms for the relevant rule is sufficient to generate both these forms of monotonicity. We see this in how both Max-Min and Min-Max will generate this ordering. However, from the logical perspective of characterizing by a decision rule by the weakest possible axioms, it is preferable to only require one of these two axioms.

Monotone Consistency: For all $A, B \in \mathcal{X}$,

$$A \succeq B \Rightarrow A \cup B \succeq B$$

This axiom is similar to INDEPENDENCE. The idea behind this axiom is that if a set A is even at least as good as a set B, then the addition of the elements of B to A is insufficient to reverse the ordering.

Robustness: For all $A, B, C \in \mathcal{X}$,

$$(A \succeq B)\&(A \succeq C) \Rightarrow A \succeq B \cup C$$

ROBUSTNESS ensures that if A is at least as good as the sets B and C separately, then A is at least as good as the set containing the union of the points in both B and

C. The plausibility lies in the idea that joining the points of B and C together into a single set should not somehow make that set stronger than the individual sets when individually compared to A.

The combination of the above axioms is necessary and sufficient to characterize our two rules.

Theorem 31. *$\succeq$ satisfies Simple Dominance, Simple Uncertainty Appeal, Simple Bottom Monotonicity, Monotone Consistency, and Robustness iff $\succeq = \succeq_{Mxn}$.*

Theorem 32. *$\succeq$ satisfies Simple Dominance, Simple Uncertainty Aversion, Simple Top Monotonicity, Monotone Consistency, and Robustness iff $\succeq = \succeq_{Mnx}$.*

3.3 Another Problem and the Need for Better Decision Rules

But suppose that *both* Max(A) and Min(A) were identical to Max(B) and Min(B), respectively. Are they really indifferent? Following the result that $A \simeq \{Max(A), Min(A)\}$, they should be. But, to extend the reasoning that was used in developing Max-Min and Min-Max, if our main points of comparison are indifferent to each other, why should we stop there if there are other factors? Particularly, if the non-extreme elements give us a very clear ranking? While the success of Theorem 20 in recapturing an important result is hailed as an achievement, most have moved on to another goal in light of another problem that could be considered an impossibility result. Let us suppose a third plausible axiom:

Optimality: For all $A \in \mathcal{X}$, and for any $x, y \in X$ such that $x, y \notin A$,

$$x > y \Rightarrow A \cup \{x\} \succ A \cup \{y\}$$

What this tells us is that if two sets overlap except for a single element in each, and if the former element is strictly preferred to the latter, then the set containing the former will be strictly preferred to the set containing the latter. It is a stronger version of a joint form of Simple Top and Simple Bottom Monotonicity, and a weaker form of of Strict Independence that does not allow for the same impossibility theorem as above. But, it does give us a different problem.

Theorem 33. *Suppose $|X| \geq 4$. There exists no characterization of $\succeq$satisfying optimality and any set of axioms strong enough to generate $A \sim \{Max(A), Min(A)\}$.*

Proof. Let X be a set composed of $\{s_1, s_2, s_3, s_4\}$ such that subscript indicates ordinal ranking. Let $A : \{s_1, s_4\}$.

By OPTIMALITY, since $s_2 > s_3$ and neither are in A, $\{s_1, s_2, s_4\} \succ \{s_1, s_3, s_4\}$. By $A \simeq \{Max(A), Min(A)\}$, $\{s_1, s_2, s_4\} \simeq \{s_1, s_4\}$ and $\{s_1, s_3, s_4\} \simeq \{s_1, s_4\}$. By transitivity, $\{s_1, s_2, s_4\} \simeq \{s_1, s_3, s_4\}$. But by $\succ$ being the asymmetric form of $\succeq$, this would contradict our earlier ranking. Therefore, we obtain a contradiction. □

Corollary 34. *Suppose $|X| \geq 4$. There exists no characterization of $\succeq$satisfying simple dominance, independence, and optimality.*

Proof. Theorem 20 makes use of simple dominance, and independence. Therefore, by our previous theorem, adding OPTIMALITY is sufficient to generate a contradiction. □

It is also worth noting that if we accept OPTIMALITY as a basic axiom, then it allows for very direct proofs of the original impossibility results in 2.6 along the same lines as the above proof. From a logical perspective, this is rather trivial because we can already generate impossibility without this extra axiom. However, from a more broadly philosophical perspective, I take it as useful in revealing problems with the standard economic approach, thus suggesting a philosophical resolution.

By approaching impossibility from the direction of optimality, we see that what is required is any set of axioms that allow the generation of the $A \simeq \{Max(A), Min(A)\}$ theorem. The source of the problem comes from an unqualified form of this theorem which is indifferent to whatever elements are ranked in the middle of the set. This generates a problem when we compare sets with the same extremes but different middle elements. In some cases, this is just fine. For example, consider$\{x_1, x_2, x_4, x_5\}$ and $\{x_1, x_3, x_5\}$. It seems fairly plausible to be indifferent between these two sets. However, as in the above theorem, in cases where all middle elements of one are strictly preferred to all the middle elements of another set, then intuition (I think rightly) rejects the plausibility of this setup.

Despite the absence of the above proof in the literature, it is likely that this is the main concern motivating a rejection of the basic axioms. Though Simple Dominance and Independence themselves are consistent with each other, the result that $A \simeq \{Max(A), Min(A)\}$ is met with suspicion since it completely ignores middle elements when they might be relevant (such as in the case of the above proof).

Rejecting the basic axioms, then, brings us back to square one. We need a set/sets of plausible axioms by which we can generate consistent decision rules. We now turn to canvass these improved decision rules. Again, my purpose is not to provide the proofs for each, but to focus on the general nature of the rules and the axioms used.

3.3.1 Lexi-Max and Lexi-Min Rules

This first alternative set of rules is owed to Pattanaik and Peleg (1984). These can be understood as modifications of Maximax and Minimax to deal with cases where the Max or Min elements in both sets are equal. The definitions of the lexi-rules are complex to read but simple to understand. Each rule takes either the Max or Min as priority and operates the same as the Maximax and Minimax rules, respectively. However, on encountering an indifferent element Max or Min in both sets, the decision rule moves on to the next Max or Min element, proceeding through the set until reaching elements related by strict preference. If one set's elements are completely bypassed without reaching a strict preference relation, then we treat the situation differently depending on whether the rule emphasizes Max or Min. In the case where Lexi-Max exhausts all elements of one set, call it C, then C is strictly preferred to the alternative set. The reason for this is that the exhausted set did not have any weaker elements. In comparison, the non-exhausted set still had progressively weaker elements than the prior matching ones. In the case of Lexi-Min, the situation is in reverse. If C is exhausted, then the other set is to be preferred, since any remaining elements would be higher than the exhausted points.

Definition 35. Lexi-Max: compare the Max elements. If one is strictly preferred, then choose the set containing that element. If not, continue to the next highest element of both sets until a strict preference is found. If one set is exhausted, choose the exhausted set.

Definition 36. Lexi-Min: compare the Min elements. If one is strictly preferred, then choose the set containing that element. If not, continue to the next lowest element of both sets until a strict preference is found. If one set is exhausted, choose the non-exhausted set.

I now turn to the relevant axioms:

Top Independence: For all $A, B \in \mathcal{X}$, for all $x \in X$ and for all $y \in A \cup B$ such that $x > y$,

$$A \succ B \Rightarrow A \cup \{x\} \succ B \cup \{x\}$$

Bottom Independence: For all $A, B \in \mathcal{X}$, for all $x \in X$ and for all $y \in A \cup B$ such that $y > x$,

$$A \succ B \Rightarrow A \cup \{x\} \succ B \cup \{x\}$$

The TOP and BOTTOM INDEPENDENCE axioms are weakened versions of STRICT INDEPENDENCE. In the case of TOP INDEPENDENCE, the difference is that it constrains the additional element to one which is strictly preferred to anything within the existing sets. BOTTOM INDEPENDENCE constrains the additional element to any which is strictly less preferred to one within the existing sets. While consistent with each other (in fact, these together make STRICT INDEPENDENCE), only one is required for each.

Disjoint Independence: For all $A, B \in \mathcal{X}$ such that $A \cap B = \emptyset$, for all $x \in X \setminus (A \cup B)$,

$$A \succ B \Leftrightarrow A \cup \{x\} \succ B \cup \{x\}$$

DISJOINT INDEPENDENCE is also closely related to STRICT INDEPENDENCE. It is STRICT INDEPENDENCE with the stipulation that there is no overlap between the original sets under comparison (i.e. it is disjoint).

Theorem 37. *Suppose $|X| \geq 4$. $\succeq$ satisfies Dominance, Neutrality, Top Independence, and Disjoint Independence iff $\succeq = \succeq_{L-Max}$.*

Theorem 38. *Suppose $|X| \geq 4$. $\succeq$ satisfies Dominance, Neutrality, Bottom Independence, and Disjoint Independence iff $\succeq = \succeq_{L-Min}$.*

3.3.2 Lexi-Max-Min and Lexi-Min-Max

An alternative set of decision rules comes in the lexicographical versions of Max-Min and Min-Max, due to (Bossert et al., 2000). Rather than continuing straight on from either the Max or Min, these rules alternate according to the order of priority. If a set C's elements have been exhausted in the case of Max-Min, then C is to be preferred; if Min-Max, then the other set is preferred.

Definition 39. Lexi-Max-Min: compare the Maxelement of the two sets. If one is strictly preferred, then that set is strictly preferred. If not, compare the Min element and if one is strictly preferred to the other, choose that set. If not, alternate between the next Max and Min. If one set no longer has elements for a comparison, then the empty set is preferred.

Definition 40. Lexi-Min-Max: compare the Min element of the two sets. If one is strictly preferred, then that set is strictly preferred. If not, compare the Max element and if one is strictly preferred to the other, choose that set. If not, alternate between the next Min and Max. If one set no longer has elements for a comparison, then the non-empty set is preferred.

We now consider the relevant axioms:

Top Dominance: For all $A \in \mathcal{X}$, and for all $x, y \in X$ such that $(\forall a \in A)(\forall x)(x > a)\&(\forall y)(a > y)$,

$$\{x, y\} \succ A \cup \{y\}$$

Bottom Dominance: For all $A \in \mathcal{X}$, and for all $x, y \in X$ such that $(\forall a \in A)(\forall x)(x > a)\&(\forall y)(a > y)$,

$$A \cup \{x\} \succ \{x, y\}$$

These two dominance-type axioms suppose an x and y that are strictly preferred and strictly dispreferred, respectively, to all elements in a set A. According to Top Dominance, if a two-element set possesses the same worst element as another arbitrary set, but the former possesses an element greater than all those in the other set, then we should prefer the two-element set. The idea here is that {x,y} gives us the same worst case, and a better best case. Bottom Dominance holds that if a set contains the same best element as a two-element set, but the two-element set's worst element is worse than any in the former, then the former should be preferred over the latter. Note that the comparison to a two-element set is essential - this is not generalized to a set of arbitrary length.

Top Dominance Extension: For all $A \in \mathcal{X}$, for all $x, y \in X$ such that $x, y \notin A$, and for all $a \in A$,

$$[(\{a\} \succ \{x, y\})\&(A \succ \{x, y\})] \Rightarrow [A \cup \{x, y\} \succ \{x, y\}]$$

Bottom Dominance Extension: For all $A \in \mathcal{X}$, for all $x, y \in X$ such that $x, y \notin A$, and for all $a \in A$,

$$[(\{x, y\} \succ \{a\})\&(A \succ \{x, y\})) \Rightarrow [\{x, y\} \succ A \cup \{x, y\}]$$

These two axioms are specific variations on the Extension Rule axiom combined with a kind of dominance approach. According to the first alternative, if it is both the case that the set composed of each individual element in A is strictly preferred to a set composed of any two elements not in A, and the set A is strictly preferred to the set composed of those two non-A points, then the set formed by adding those two points to A will be strictly preferred to those two points alone. While this looks and sounds rather complex, the idea here is that the antecedent requires that A is better than $\{x, y\}$ in every way - both in the form of the set as a whole, and each individual element. If this is the case, then it is sufficient for the union of A and $\{x, y\}$ to be an improvement over $\{x, y\}$. The second axiom holds for the opposite case where $\{x, y\}$ is completely superior to A. This guarantees that the addition of A would be to weaken $\{x, y\}$.

Top Joint Monotonicity: For all $A, B \in \mathcal{X}$, and for all $x \in X$,

$$[(\{x\} \succ A)\&(\{x\} \succ B)] \Rightarrow \{x\} \succ A \cup B$$

Bottom Joint Monotonicity: For all $A, B \in \mathcal{X}$, and for all $x \in X$,

$$[(A \succ \{x\})\&(B \succ \{x\})] \Rightarrow A \cup B \succ \{x\}$$

These two monotonicity conditions are related to Robustness. If $\{x\}$ is independently strictly preferred to both sets A and B individually, then it is preferred to the union of A and B. If $\{x\}$ is independently strictly dispreferred to A and B, this it will also be strictly dispreferred to the union of A and B.

Extension Independence: For all $A \in \mathcal{X}$, and for all $x, y \in X$ such that $x, y \notin (A \cup B)$ and for all $a \in A \cup B$ such that $x > a$ and $a > y$

$$A \succeq B \Leftrightarrow [A \cup \{x, y\} \succeq B \cup \{x, y\}]$$

On this axiom, if A is at least as good as B, then the addition of an element strictly preferred to any in A and an element strictly dispreferred to any in A to both sets will not alter the rankings.

We now have the axioms required to characterize our two decision rules:

Theorem 41. $\succeq$ *satisfies Simple Dominance, Simple Uncertainty Appeal, Bottom Dominance Extension, Bottom Joint Monotonicity, and Extension Independence,iff* $\succeq = \succeq_{L-Mxn}$.

Theorem 42. $\succeq$ *satisfies Simple Dominance, Simple Uncertainty Aversion, Top Dominance Extension, Top Joint Monotonicity, and Extension Independence,iff* $\succeq = \succeq_{L-Mnx}$.

Chapter 4

A Mixed-Response to Impossibility

This chapter attempts to offer a philosophical perspective[1] on the field. I return to the two more well-developed responses to the impossibility problems: use weaker axioms or accept the impossibility and replace the axioms. In the section on replacing the axioms (3.3) I gave a justification for why one might prefer that response in light of a problem with OPTIMALITY. This move to a new set of axioms led towards rules that would not only consider the Max and/or Min of a set (as in the case of the Max-Min and Min-Max rules), but subsequent rounds of next Max and/or Min elements in cases where the preceding sets were equal (as in Lexi-Max-Min and Lexi-Min-Max).

While this latter class of rules constitutes a more robust form of the original rules, I argue that it is not necessary to give up the important result of Theorem 20 which we obtain from the weaker axioms that generate Max-Min and Min-Max. While avoiding the optimality problem, by neglecting certain distinctions we consequently produce a different class of outcomes that conflict with some plausible intuitions.

I attempt to remedy this problem by making what I see as the necessary distinction between different applications of set transitivity. This distinction will be justified on the the basis of both a conceptual analysis of transitivity, and on the basis of a conceptual tool that I develop to make rigorous what I see as a plausible intuition towards comparison. Based on this distinction, I then show how we are able to both maintain Theorem 20 while avoiding the OPTIMALITY problem. This will turn out to have some rather interesting implications for the field as a whole and produce a modified form of Lexi-Max-Min and Lexi-Min-Max.

[1] I must admit to being rather sloppy with this description. While I do touch on a few conceptual issues that might be called philosophical that motivate my opinions, these are not particularly front and centre within the discussion. At the very least, as one trained in philosophy rather than economics, I offer the perspective of an outsider with a philosophical outlook.

4.1 One-stage Vs. Multi-stage Lexicographical Rules

In responding to impossibility, the choice of weaker axioms than required to generate the impossibility theorems led to the creation of two rules: Max-Min and Min-Max. These rules are considered lexicographical rules, i.e. rules that take into account the equivalent of an alphabetical ordering by moving on to a next highest element. Since Max-Min and Min-Max move onto only the next highest element, let us call them one-stage lexi-rules. By contrast, Lexi-Max-Min and Lexi-Min-Max continue to compare next highest elements until one element is strictly preferred to another, or one set is exhausted. Let us call these multi-stage lexi-rules.

Recall that we generated our optimality problem by using Theorem 20 (which shows that $A \simeq \{Max(A), Min(A)\}$) and the OPTIMALITY axiom which states that if two sets are differentiated by a single element, and that element of one set is strictly preferred to that of the other, then the former set is strictly preferred to the latter. This suggested that, such as it was, Theorem 20 was false, encouraging us to move towards decision rules that would consider more than the first set of Max and/or Min elements. To do so, however, was not without sacrifice. The first casualty was the replacement of a set of extremely plausible and intuitive axioms by sets of plausible but less intuitive axioms. As well, we lost Theorem 20.[2] But should we not think of this as an improvement? After all, it was this theorem that played the key role in our optimality problem. While I do not deny that an inconsistency is a serious problem, I take the theorem as having an intuitive plausibility. Let us consider an analysis in favour of this intuition.

4.1.1 Set Expansions of Middle Elements

What makes $A \simeq \{Max(A), Min(A)\}$ plausible to me is that if I begin with a set with two elements, the addition of a middle element does not seem to be either an improvement or a worsening of my original set.

Consider a set of one element. When we add a new element to that set, my evaluation of the relational structure between these two sets is straightforward - is

[2] We see this is the case on the very basis of the multi-stage lexi-rules which suggest that in cases where the $|A| > 2$, we have another element to consider before coming to indifference.

the element better or worse than my original? I need only make a single comparison. If the new element is better than the old, then I rank the sets in the way given by SIMPLE DOMINANCE. Next, consider an expansion of a two-element set in the form of a new middle element (i.e. strictly dispreferred to the Max, and strictly preferred to the Min). Let us ask again: is this better or worse? If we follow the same line of thinking as in our first case, we can compare it to our original set. Well, it certainly gives us a new possibility against the worst possible outcome, but by that same token it gives us a new possibility against the best possible outcome. On this basis, a Maximaxer will disprefer the expanded set and a Minimaxer will prefer the it. But what if we were to try to think from a non-subjective and purely structural perspective? From a purely structural perspective, since every middle expansion is both a gain in one direction and a loss in another, I suggest we should be indifferent between the two. For an expansion *of our original two-element set* with a set of elements of any cardinality, we will see this exact same balanced structural change. This seems to suggest intuitive grounds that $A \simeq \{Max(A), Min(A)\}$.

But, what of our optimality problem? Recall that in our proof, we made use of the transitivity of $\simeq$ to make the two expansions of A indifferent to each other. The solution I propose to develop is a rejection of the transitivity that would allow us to make this move.

4.1.2 Proposed Solution: Reject Set Transitivity in Favour of a Qualified Set Transitivity

What would be the result of rejecting set transitivity? It would imply that, if X is composed s_1, s_2, s_3, s_4 (where subscript indicates ordinal ranking) and $A : \{s_1, s_4\}$, $B : \{s_1, s_2, s_4\}$, and $C : \{s_1, s_3, s_4\}$: $A \simeq B$, $A \simeq C$, but it is not the case that $B \simeq C$. Instead, $B \succ C$. This would give us exactly the results I suggest are intuitively plausible. There is a catch. $A \simeq \{Max(A), Min(A)\}$, when generated under weakened axioms, required transitivity! If we want to maintain $A \simeq \{Max(A), Min(A)\}$ but in a way that maintains OPTIMALITY, then we must have some form of transitivity (or discover an innovative new proof). An alternate form of transitivity used in social choice is quasi-transitivity, which is a weakening of transitivity to only apply to strict preference. However, the proof of our theorem required transitivity of $\succeq$, not just $\succ$. What I propose then is a rejection of set transitivity (but *not* transitivity of states) with a replacement by a qualified form of set transitivity, one that holds only under certain conditions (such as is required to generate $A \simeq \{Max(A), Min(A)\}$). Let us call this property 'semi-transitivity'.

Semi-transitivity thesis: $\succeq$ is semi-transitive if and only if there exists some condition on which transitivity holds and some on which it does not.

The next step is to ascertain on what condition transitivity does or does not hold. But, what reason do we have for even thinking that there might be any reason to deny set transitivity in any cases?

In an earlier footnote (footnote 4 on page 13) I stated that I took the transitivity of states of affairs as being a properly basic assumption and ruled out any discrepancies as 'experimental noise'; so I should begin by giving some kind of account for why I take it as plausible to reject set transitivity. The reason for my distinction between the two is that we are talking about the transitivity of two forms of preference, each with its own object. With respect to comparison between states of affairs, I see comparison as a (relatively) straightforward matter. I have a holistic preference for each object. In contrast, sets of states within a complete-uncertainty framework create rather odd objects for comparison. If only one will occur, and I have no idea which (at least when considering non-singletons), I am unable to make use of a holistic assessment to tell me about the preferrability of any given set as a whole to any other.[3] But, for transitivity to work, we must see whether a feature is preserved through the chain. In order to show that these sets of completely uncertain outcomes are odd objects, and in order to ground a distinction between proper and improper applications of transitivity, we first need a means of analysing the structure of comparison.

4.2 A Metalogic of Comparison

In my initial view of the expansion of a set of two elements (where one is strictly preferred to another), I made reference to the relational structure that existed between

[3]This might be contrasted with the use of hard/soft uncertainty where we also work with sets of outcomes but can use expected utility to give us a holistic assessment of one set relative to another.

two sets to justify my intuitions as to their indifference. In this section I attempt to expand on this idea of a relational structure for comparison by constructing an analytical method to explain my intuitions. I see the result of this project as providing a kind of metalogic for comparisons, a means of detecting structures in a comparison, but not (in itself) an alternative decision rule.

4.2.1 The Concept of Comparison

> "..if they are not equal, they will not have what is equal, but this is the origin of quarrels and complaints–when either equals have and are awarded unequal shares, or unequals equal shares." -Aristotle, Nichomachean Ethics Part V, Ch. 3

'Treat like cases alike', says the dictum attributed to Aristotle. I take as a corollary the claim 'treat unalike cases differently'. This latter idea emphasizes that in deciding between options, our decision of one option over another should ultimately be based upon the differences between the two.

In the same vein, one of the insights I take from Van Fraassen, Jennings, and Schotch is that a decision rule should be founded on the comparisons that may be made between all the different elements between any two sets by the $\geq$ relation.[4] Their purpose, however, is to find a decision rule in virtue of a very general property that exists between the two comparisons. We see this in the expression of their decision rules via predicate logic to express a relation of some to all or of all to some.

I wish to use their analytical method differently. My goal is not to use this to define a decision rule. Instead, I want to use this idea to reduce a comparison between sets to the collection of comparisons between the states of each set. By means of this reduction, we are able to see the full list of differences between the two sets. By consulting important types of comparisons and listing their differences in this manner, we will, I hope, be able to perceive important patterns which will indicate a structure to the comparison. Let us call this method 'comparison structure analysis' (CSA).

[4] We could use strict preference, but it makes things rather complicated in the same way first-order logic can be done with fewer than the standard five relations.

4.2.2 Comparison Structure Analysis

This analysis is founded on the philosophical observation that a decision is ultimately based in some way upon the comparative relations that exist between the exhaustive combinations of state relations. Let us begin with two basic cases that are uncontroversial to both intuition and all the lexi-rules canvassed above: Extension Axiom and Simple Dominance.

Let us use $\mathbb{C}$ to indicate a comparison relation between two sets. Let us call the listing and organization of the elements of each set: 'listing the relative merits'. Suppose a set of states $s_1, ..., s_n$ where subscript indicates ordinal ranking. We exhaustively list the combinations of state relations from one set to the other, then group these relations according to which set the left-hand side of the 'at least as good as' relation belongs to.[5]

For the comparison $\{s_1\}\mathbb{C}\{s_2\}$, we can list the relative merits as follows:

$\{s_1\}$	$\{s_2\}$
$s_1 \geq s_2$	

An obvious thing to notice is that strict preference is shown by a lack of a corresponding symmetrical relative merit for $\{s_2\}$ in the form of $s_2 \geq s_1$. On this basis, we see why $\{s_1\}$ is strictly preferred over $\{s_2\}$ in that the latter has no relative merits to speak of.

For the comparison $\{s_1\}\mathbb{C}\{s_1, s_2\}$ (which is the first part of Simple Dominance), we obtain the following relative merits:

$\{s_1\}$	$\{s_1, s_2\}$
$s_1 \geq s_1$	$s_1 \geq s_1$
$s_1 \geq s_2$	

Here we also see indifference represented by symmetry. Going by the idea that any decision should be on the basis of the differences between two sets, we should consider symmetries as irrelevant. By ignoring these, we are left with:

[5] The number of relative merits prior to grouping will be equal to $|A| \times |B|$.

$\{s_1\}$	$\{s_1, s_2\}$
$s_1 \geq s_2$	

This is like our EXTENSION AXIOM case. What I see as interesting about comparison structure analysis is that in reducing a comparison to the differences between sets, it does not obscure relevant aspects of overlapping elements. Rather than simply reducing the sets themselves to non-overlapping elements, CSA preserves the way in which overlapping elements relate to the other elements. Whereas a reduction to $\{\emptyset\}$ and $\{\emptyset, s_2\}$ would create a kind of nonsense (how could we compare a state to nothing?), CSA is a useful way of bringing out the intuition that what makes $\{s_1\}$ better is that it does not possess the worsening that is present when s_2 is added. We see something similar in the second portion of SIMPLE DOMINANCE:

$\{s_1, s_2\}$	$\{s_2\}$
$s_1 \geq s_2$	

At this point we can see a pattern emerging. Axioms that work on the basis of dominance involve the one set possessing at least one relative merit and another possessing none (or effectively none by reduction). Let us give a name to this pattern:

Dominance Structure: A dominance structure exists iff on CSA, one set possesses at least one relative merit and the other possesses none.

We can prove that this is the case for the DOMINANCE axiom despite its set being unspecified.

Proposition 43. *The Dominance axiom exhibits the Dominance Structure.*

Proof. Suppose $A : \{s_1, ..., s_m\}$. In comparing $A \cup \{x\}$ with A, we know that the only difference in relative merits will be in comparisons of x with $a_1, ..., a_m$. DOMINANCE may be used in two ways: if x is strictly preferred or dispreferred to all elements of A.

In the case where x is strictly preferred to all elements in A, since we know that comparison between identical sets produces an empty set of full merits (since they would be completely symmetrical), we therefore know that all generated merits will belong on the side of $A \cup \{x\}$. In the latter case it follows in reverse - all merits will exist on the side of A. $\square$

The same can be shown of the various dominance axioms described in preceding chapters. And this is all rather unsurprising. After all, that is what we think dominance is - when one choice is strictly preferred to another choice under every circumstance. Let us note that the Dominance Structure does not (in itself) tell us which set should be strictly preferred. To do that, we also require the following preference condition:

Dominance Preference: A preference relation shows Dominance Preference iff on encountering a Dominance Structure, the preference favours the set with un-eliminated relative merits.

So far so good. Now let us turn to a different type of comparison: $\{s_1, s_3\}\mathbb{C}\{s_2\}$. This represents the fundamental Maximax and Minimax divergence. Let us, however, be agnostic with respect to the comparison and focus only on its structural analysis.

$\{s_1, s_3\}$	$\{s_2\}$
$s_1 \geq s_2$	$s_2 \geq s_3$

This gives us a different structure to dominance. It is easy to see that an expansion of the first set in the former of increasingly preferred and and increasingly dispreferred members will perpetuate this mirror. This follows since we know that all greater preferred states will form a new relative merit for the first set, and all new lesser preferred states will form a new merit for the second set.

Mirror Structure: A mirror structure exists iff we see the relative merits sharing a common central point around which there exists a one-one correspondence of $\geq$ relations.

Just like Dominance Structure, Mirror Structure in itself tells us nothing about the preference relation without the following:

Max Priority Preference: A preference relation shows Max Priority Preference iff on encountering a Mirror Structure, the preference favours the set with the highest Max un-eliminated $\geq$ relation.

Min Priority Preference: A preference relation shows Min Priority Preference iff on encountering a Mirror Structure, the preference favours the set with the highest Min un-eliminated $\geq$ relation.

For now we will concern ourselves only with Dominance Preference for two reasons: (1) only dominance type axioms are used to generate $A \simeq \{Max(A), Min(A)\}$ and the optimality problem; and, (2) all rules capture the dominance axioms in some form.

4.2.3 CSA and Transitivity

So how does this relate to the semi-transitivity thesis? Recall that I suggested we think of transitivity as ensuring that a feature is preserved through a chain. I propose that this feature that distinguishes valid from invalid applications of transitivity lies in the structure revealed by CSA. This leads to the following:

CSA Semi-Transitivity Thesis: An application of set transitivity is valid if and only if the preference relation that is carried over to the consequent exhibits the same structure as it does in those instances of the preference relation present in the antecedent.

So, if $A\mathbb{C}B$ produces $A \succ B$ and exhibits a structure Σ, and $B\mathbb{C}C$ produces $B \succ C$ and also exhibits structure Σ, then the application of transitivity to form the conclusion $A \succ C$ must also exhibit Σ. If $B\mathbb{C}C$ produces $B \simeq C$ with a structure Σ', then we will only be concerned with $A \succ C$ possessing the structure Σ since it is the structure of $\succ$that is being carried over to the consequent.

Note that the semi-transitivity thesis and the CSA semi-transitivity thesis are independent; one may hold to the former without agreeing to the latter. Indeed, one may agree that there is something that distinguishes valid applications of set transitivity from invalid ones without agreeing that it is connected to CSA or the use of CSA that I have suggested.

I think we also have good reasons in favour of semi-transitivity on purely philosophical grounds. The rejection of wholesale set transitivity is highly plausible in light of just how far we are moving from the ordinary objects of state transitivity.

Sets under complete uncertainty are very unusual objects, and it is not clear that our intuitions towards the transitivity of states extends to the transitivity of sets. Further, we are motivated to maintain some form of transitivity since there do appear to be very clear cases where transitivity should hold, such as Dominance Structure (as we shall see in the next section). What is most interesting about these cases is that the reason why we are inclined to accept these cases is that they avoid the philosophical grounds for rejecting others; the objects used in cases of dominance are simple objects - perhaps even as simple as state comparisons. In such cases the conditions of complete uncertainty do not create the kind of complexity that cause us to be suspicious of transitivity. It is not difficult to see why. If every member of a set A is strictly preferred to every member of a set B, then complete uncertainty becomes irrelevant because no matter what state obtains, A will always be the better choice between the two sets. Since complete uncertainty is not a factor in such comparisons, these cases of set transitivity are almost as straightforward (if not as straightforward) as state transitivity. In the next section we will see such cases where set transitivity plausibly *should* hold and where set transitivity and should *not* hold.

4.3 Applying CSA to the Optimality Problem

Let us attempt to apply CSA to evaluating the uses of transitivity in generating the optimality problem. First we must show that the use of transitivity in generating $A \simeq \{Max(A), Min(A)\}$ is legitimate.

Proposition 44. *The proof of Theorem 20 makes valid use of set transitivity.*

Proof. Theorem 20 uses transitivity twice. In the first instance, from $\{s_1\} \succ \{s_1, s_{n-1}\}$, to $\{s_1, s_{n-1}\} \sim \{s_1, ..., s_{n-1}\}$ to $\{s_1\} \succ \{s_1, ..., s_{n-1}\}$. Since it is strict preference that is preserved through transitivity, we need only investigate $\{s_1\}\mathbb{C}\{s_1, s_{n-1}\}$ and $\{s_1\}\mathbb{C}\{s_1, ..., s_{n-1}\}$ to see if the same structure is preserved.

The first comparison comes from Simple Dominance, so from our earlier investigation we know that it has a Dominance Structure.

CSA of $\{s_1\}\mathbb{C}\{s_1, ..., s_{n-1}\}$ also shows a Dominance Structure. We can determine this easily since we know that no relative merit exists on the right side of the comparison (other than with s_1, which is ignored by symmetry), and s_1 is strictly preferred to every item on the right side, giving us $n-1$ relative merits for the left side. Since both the relevant parts of the antecedent and consequent exhibit the same structure, this use of transitivity is valid.

Since the second use of transitivity is the same in reverse, it will evidently exhibit the same Dominance Structures and so the same valid uses of transitivity.

Since all applications of transitivity in the proof of Theorem 20 are valid, the proof is not undermined □

Having obtained the theorem required to generate the optimality problem, we turn to the problem itself.

Theorem 45. *On CSA, the optimality problem does not make use of a valid application transitivity.*

Proof. Let $A : \{s_1, s_2, s_4\}$, $B : \{s_1, s_4\}$, and $C : \{s_1, s_3, s_4\}$. To recap, by Theorem 20, $A \simeq B$. By the same, $B \simeq C$, and by transitivity, $A \simeq C$. But, by optimality, $A \succ C$. Let us apply CSA to see what is going on. Since the final preference relation as a result of transitivity is set indifference and the same relation is in both conjuncts of the antecedent, we must check that the final structure matches the structures of both conjuncts.

$\{s_1, s_2, s_4\}$	$\{s_1, s_4\}$
$s_2 \geq s_4$	$s_1 \geq s_2$

$\{s_1, s_4\}$	$\{s_1, s_3, s_4\}$
$s_1 \geq s_3$	$s_3 \geq s_4$

In both cases we see a Mirror Structure.[6] However, the structure that appears in

[6]It might be objected that this is not a genuine case of Mirror Structure, and even if so, that it does not match. The first might be raised since a Mirror Structure as shown above showed a one-state difference around the centre point (i.e. if $s_m$is the centre point, then a Mirror Structure requires the points surrounding it be s_{m-1} and s_{m+1}, or s_{m-2} and s_{m+2}, or s_{m-3} and s_{m+3}. Even if we accept that this still counts as a Mirror Structure, it is reversed between the first and second comparison. $\{s_1, s_2, s_4\}$ possess the the gappy merit in the first case, but the Max and Minset possess the gappy merit in the second case. One way of solving this is simply by accepting the Neutrality axiom. What is interesting about Neutrality is that it seems closer to a property that belongs in the metalogic than as a standard axiom. After all, it tells us something about the structure of comparisons, not about what we should decide in a given comparison. If we accept

a comparison of $\{s_1, s_2, s_4\}$ and $\{s_1, s_3, s_4\}$ reveals something very different:

$\{s_1, s_2, s_4\}$	$\{s_1, s_3, s_4\}$
$s_1 \geq s_3$	$s_1 \geq s_2$
$s_2 \geq s_3$	
$s_2 \geq s_4$	$s_3 \geq s_4$

We do not see *any* kind of Mirror Structure. In fact, we can do a bit more to parse the data. If we allow the cross-elimination of $s_1 \geq s_3$ and $s_1 \geq s_2$, then the only remaining relative merit is $s_2 \geq s_3$ on the left and nothing on the right. Such a result is a Dominance Structure - exactly what one would expect from OPTIMALITY (which actually is a dominance axiom).

Since the Mirror Structure was not preserved by transitivity, according to the CSA semi-transitivity thesis, $A \simeq C$ cannot be validly reached by transitivity. □

Since we have grounds for rejecting set transitivity, we are allowed to keep both Theorem 20 *and* OPTIMALITY.

4.4 Wider Implications

If we reject set transitivity in the cases I have suggested (for whatever reasons) while accepting those that I proposed to be valid, we obtain some interesting results. Before getting to those however, we need one more important result:

Corollary 46. *If* $(\forall a \in A)(\forall b \in B)(\forall c \in C)[(a > b)\&(b > c)]$ *and* $|A| = |C|$*:*

$$(A \cup B \cup C) \simeq (A \cup C)$$

Proof. By Theorem 20, $(A \cup C) \simeq \{Max(A), Min(C)\}$. By the same theorem, $\{Max(A), Min(C)\} \simeq (A \cup B \cup C)$. By set transitivity, $(A \cup C) \simeq (A \cup B \cup C)$. It is easy to see that this application of set transitivity is valid on CSA, producing a

NEUTRALITY, then we eliminate the gaps and deal with both objections. But perhaps this is not even necessary for a Mirror Structure. This is not an issue I can adequately resolve for the moment, but we have an escape by accepting NEUTRALITY (though this will require us to accept some of the philosophical questions about it that I raised earlier when first presenting it in 3.1.2).

Mirror Structure in all cases. For the first comparison, the relative merits of $(A \cup C)$ will be its middle elements as preferred to $Min(C)$, and for $\{Max(A), Min(C)\}$ its relative merits will consist of $Max(A)$ being at least as good as all the middle elements of $(A \cup C)$. In the second comparison, we have the same relative merits on both sides as before, plus those that would relate to B. Finally, because of the significant overlap, we can see that the merits of $(A \cup C)$ will be all the points of A being at least as good as those of B and C; on the other side, the relative merits will be all the points of A being at least as good as those of C and B being at least as good as those of C.[7] $\square$

What this corollary shows is that a set is not only indifferent to its Max and Min, but also to a set composed of any number of elements from its upper and lower end so long as both are taken in equal number. In the case of Theorem 20, we described this as middle elements being irrelevant (where 'middle elements' refer to the collection of points within a set that are neither a Max or Min).

We can now say something similar with respect to our new corollary by introducing some new terms. Let the 'extended extremes' refer to the collection of grouped points as $A \cup C$ in our above proof. We can be more specific by referring to the set A as the 'extended Max elements' and the set C as the 'extended Min elements'. Let 'reduced middle elements' refer to the collection of points within a set that are not members of the extended extremes. We may now say that our corollary shows that any set is indifferent to a set composed of its extended extremes, making its reduced middle elements irrelevant in such a comparison.

In one sense, this can be seen as a more general form of Theorem 20, but from the perspective of our one-stage lexi-rules, it is trivial. After all, if we accept that indifference between the extremes of two sets is sufficient for indifference, why should anything else matter? But, given our rejection of certain forms of set transitivity, this has some interesting implications that we develop in the next sections.

[7]It is also worthwhile to note the reason for the equal cardinality requirement. If they are not equal in cardinality, we will not generate the Mirror Structure, as one extreme will have one extra relative merit.

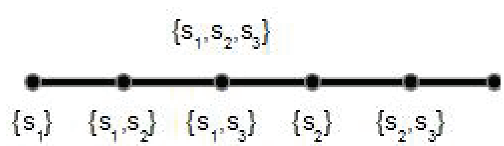
4.4.1 The Shape of the $\succeq$ Ordering

The main result involves literally changing the shape of the field. $\succeq$ no longer generates a simple linear ordering, but a hierarchy of classes of linear orderings - a main class with embedded subclass upon subclass depending on the cardinality of X. To understand this it helps to begin visualizing the regular linear ordering generated by our other lexi-rules.

4.4.1.1 The Traditional Picture

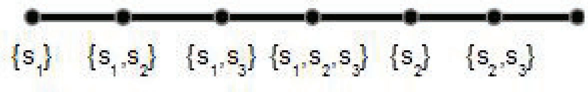
The first, and simplest, are the linear orderings generated by the one-stage lexi-rules: Max-Min and Min-Max. On these rules, we can imagine a horizontal line where the left end of the line gives us a singleton of the highest element within X. Proceeding along the line we can place the other singletons according to descending preference. Depending on whether we adopt Max-Min or Min-Max, we can insert the two-elements sets within this line. Any set with a cardinality greater than two will occupy the same position as the point consisting of the set's Max and Min. In other words, the set of sets with a cardinality less than or equal to 2 is sufficient to give us the entire ordering of $\mathcal{X}$. What happens then is that sets of cardinality greater than 2 are in a sense irrelevant/invisible.

Figure 4.1: The horizontal line created by Max-Min where $|X| = 3$



In contrast, we have the two multi-stage lexi-rules. Lexi-Max-Min and Lexi-Min-Max are similar to the linear shape we have just described. However, sets of cardinality greater than 2 are not hidden. Instead, they are either improvements or worsened forms of the sets composed of their Max and Min, thus they will stand to its left or right.

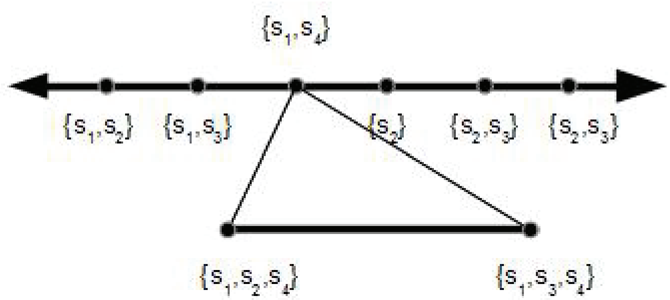
Figure 4.2: The horizontal line created by Lexi-Max-Min where $|X| = 3$



4.4.1.2 The Modified Picture

What a rejection of wholesale set transitivity has done is maintain linear orderings, but (in some sense) not on the same line. In addition to the line similar to the one-stage Max-Min and Min-Max rules, we now have a number of sub-lines. Let us define our main line as the linear ordering created by singleton and two-element sets. For every two-element sets with a gap between the Max and Min, that single point maps into an entire sub-line. Graphically, this may be represented thusly:

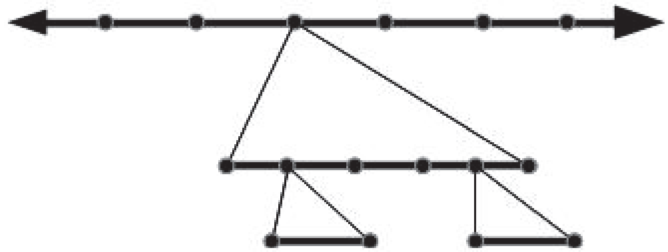
Figure 4.3: The main line and sub-line created by Bounded Lexi-Max-Min where $|X| = 4$



A sub-line is created at any point where there is a gap between middle elements, including points on sub-lines. Thus, with increasing cardinality we have increasing sub-lines. Letting $\mathcal{L}$ represent the number of total lines (main line and sub-lines),

$$\mathcal{L} = \begin{cases} |X|\, is\, odd & (|X|+1)/2 \\ |X|\, is\, even & (|X|/2) \end{cases}.$$

Figure 4.4: A main line and increasingly embedded sub-lines



4.4.1.3 Transitivity Further Explained

This graphical representation along with our alternative representation of sets may help us intuitively identify cases in which transitivity fails. If set transitivity relies on points that lie on different line classes, then transitivity will not hold. In the case of our optimality problem, we end up comparing two different points on the sub-line of the main line. But, because the two points themselves lie on the same class of linear ordering, they should be directly compared to each other rather than indirectly through a point on another line.

However, we see that transitivity works in cases that begin on sub-lines and end on lines higher up in the class hierarchy. Because each sub-line is contained within a point of a higher line, it follows that if a point on a sub-line relates to any point on a higher line, it will minimally be through points to which it is indifferent. If it ends in a point to which it is indifferent, we know that this is valid per Corollary 46. If it ends on a point to which it is not indifferent, then the intermediate point to which it is indifferent will relate to that other point in virtue of being strictly preferred or dispreferred. Since the relationships between any two points on a single line are by dominance, we know that it will be preserved through transitivity.

All this put simply is the rule that transitivity holds so long as there is no movement back and forth between different classes. A comparison between two points ultimately on the same line should be made directly.

4.4.2 A Plausible Variation and Interpretation of Existing Decision Rules

Despite the complexity involved in describing different representations of the effects of rejecting set transitivity, its application to a decision rule is fairly straightforward and intuitive. We can call this the idea of 'boundedness', which produces a set of rules that identify as indifferent a comparison of a set to another set composed of only its extended extremes, but can simultaneously compare sets with the same boundaries but different reduced middle elements .

When applied to Lexi-Max-Min and Lexi-Min-Max, it results in just a minor variation relative to the originals. These two rules behave as they regularly do *except* for comparisons of a set A with another set composed of A's extended extremes, in which case this results in indifference. We can define these bounded versions thusly:

Bounded Lexi-Max-Min: If a set A is compared to a set composed of only A's extended extremes, then the two sets are equal, otherwise proceed by Lexi-Max-Min.

Bounded Lexi-Min-Max: If a set A is compared to a set composed of only A's extended extremes, then the two sets are equal, otherwise proceed by Lexi-Min-Max.

But why is this minor variation significant? What is the benefit? One of the aspects of the original rules that I find questionable is that they allow certain problematic set preferences merely on the basis of cardinality. In some cases this is completely reasonable. For example, consider SIMPLE DOMINANCE. This is a case of preference by cardinality, but this constitutes an unquestionable case. In both these cases, cardinality is significant because it differentiates between certainty and uncertainty, and it occurs at extremes. To make this clear, consider the Lexi-Max and Lexi-Min rule. While their headlong preference for a Max or a Min might not be a palatable approach in practice, the attitude of these rules towards comparisons of proper subsets makes complete sense. Suppose $A \mathbb{C} A \cup C$ where every point in C is dispreferred to those in A. When approached from a Lexi-Max perspective, a victory by cardinality makes sense (in this case where the set with lower cardinality is strictly preferred) since an approach from the more greatly preferred side implies that the difference in cardinality is to be attributed to strictly dispreferred elements, in which case cardinality is connected to dominance.

But cardinality does not have the same significance when the difference in cardinality is not due to extremes; in other words, when the higher cardinality is due purely to a difference in middle elements. In such a situation I take it that cardinality should not matter in the same way as highlighted above. The fact that there are more middle elements gives us nothing related to dominance - it simply gives us a Mirror Structure. While it makes sense to have a preference when extremes differ (e.g. where $x > y > z$, $\{x, z\}\mathbb{C}\{y\}$), where they share the same extremes it does not. From a Maximaxer perspective, it it makes sense in the former case to side with one set because the maximal element of X does lie in $\{y\}$. However, in the latter case, the maximal element already belongs in both sets and a middle element is not a dominance-type weakening.

What we have in Bounded Lexi-Max-Min and Bounded Lexi-Min-Max is a way of rejecting undesirable instances of preference by cardinality while retaining desirable instances. For this reason, it seems to me that these bounded rules are an improvement over their standard versions.

4.4.3 A Method for Constructing Axioms

In order to create these modified decision rules it will be necessary to form a new set of axioms that characterize these rules. One of the benefits of our shape analysis, however, is that it reveals that axioms must be sufficiently nuanced to treat cases involving extremes differently from those involving middle elements.

For example, our Independence axiom might be modified as follows:

Max/Min Independence: For all $A, B \in \mathcal{X}$, for all $x \in X \setminus (A \cup B)$ where $(\forall s \in A \cup B)((x > s) \vee (s > a)$:

$$A \succ B \Rightarrow A \cup \{x\} \succeq B \cup \{x\}$$

This restricts our original independence condition to only cases of extremes. Note that Max/Min Independence is still enough to generate Theorem 20 as the proof made use of a new element x where x was beyond the extremes of the two sets under comparison.

But what of non-extreme elements? While it holds more generally, it is too weak with respect to middle elements. In such cases, Strict Independence is preserved in some form (such as the variations used in some of the lexi-rules). For each decision rule, a suitable modification of its axioms is required to accommodate this new independence rule by restricting its own independence condition to non-extreme elements.[8]

I suspect that by making the appropriate modifications we will be able to retain in large part the plausibility of our basic axioms while avoiding their more troublesome aspects. While changes are necessary, we no longer lose the intuitive nature of our axioms as I believe we have now a good reason for creating our axioms according to certain constraints.

[8]Unfortunately, Max/Min Independence runs into the same problem as regular Independence as identified by Arlegi (2003) - at least with the axioms as they are. However, I suspect that modifying the other axioms to take into account the distinction between extreme and middle elements will rectify this problem.

Chapter 5

Conclusion

In this thesis, I have argued for the value of an interdisciplinary approach between economics and philosophy on the topic of decision under complete uncertainty. I have attempted to demonstrate that the value is not trivial. I attempted to argue on the basis of the following sub-goals:

1. Demonstrating the value of economics approach, as exemplified by:
   (a) The effectiveness of the language for its simplicity (Section 2.2, and in contrast to the language of philosophers in Section 2.3.2.2)
   (b) The discovery of impossibility results upon the acceptance of some basic, highly plausible axioms (Section 2.6)
   (c) Different axiomatization schemes that avoid impossibility and produce characterizations of various decision rules (Sections 3.1, 3.2)
2. Demonstrating the value of philosophical approach, as exemplified by:
   (a) The existence of an overlap in both the framework and goals (As previously discussed in Section 1.1 and further elaborated in 2.1) and parallels between results and conjectures by philosophers with results by economists (Sections 2.3, 3.1)
   (b) A focus on the semantics of comparison to understand axioms and decision rules (Sections 4.1.2 and 4.2)
   (c) An alternative approach to respond to the impossibility theorems (Chapter 4)

As stated in the introduction, because of the need to develop the topic in a way that would progress naturally, it would be difficult to present my contributions in a direct and linear fashion. However, having reached the end after presenting a big picture of the field and contextualizing my ideas within that picture, I am now in a position to provide a meaningful recap in list form of my results and contributions. I have:

1. Shown the deep similarities between the framework used by axiological approaches to formalizing moral philosophy with decision under complete uncertainty.

2. Evidenced the benefits of using the economic language (over the philosophical one) due to its simplicity in representing the essential aspects of a comparison by means of Max and Min functions.

3. Proved the equivalence of $\succeq_V$ to the economic $\succeq_{Max}$.

4. Proved the equivalence of $\succeq_J$ to the economic $\succeq_{Max}$.

5. Improved on the existing approach to proving the equivalence of $\succeq_V$ to $\succeq_J$ by the indirect means of showing their equivalence of $\succeq_{Max}$.

6. Proved $\succeq_O$ as a solution to impossibility by rejecting completeness.

7. Proved $\succeq_O$ as equivalent to the economic $\succeq_D$ proposed earlier as a solution to impossibility.

8. Proved a means of retaining both $A \simeq \{Max(A), Min(A)\}$ and Optimality by rejecting a wholesale form of set transitivity and replacing it with a qualified version.

9. Given a philosophical argument for rejecting the wholesale form of set transitivity.

10. Proposed CSA as a means of analysis to determine valid and invalid applications of set transitivity.

11. Created a variation of linear ordering by allowing both $A \simeq \{Max(A), Min(A)\}$ and Optimality and showed how this produced variations of existing lexicographic rules with a property I identified as 'boundedness'.

12. Proposed a methodology of axiom construction to characterize bounded lexi-rules.

5.1 Further Work

While accomplishing the above has sketched out the basis for a new approach to decision under complete uncertainty that is informed by both economic and philosophical insights, much of the implementation of this approach remains to be done. In particular, what is needed is a complete axiomatic characterization of Bounded Lexi-Max-Min and Bounded Lexi-Min-Max. It is my hope that philosophers will engage with the work and methodology of economists and that economists will consider philosophical perspectives in order to develop new approaches.

Bibliography

Anand, P. (1993). The Philosophy of Intransitive Preference. *The Economic Journal*, 103(417):337–346.

Arlegi, R. (2003). A note on Bossert , Pattanaik and Xu's "Choice under complete uncertainty : axiomatic characterization of some decision rules". *Economic Theory*, 225:219–225.

Arrow, K. J. and Hurwicz, L. (1972). An Optimality Criterion for Decision-making under Ignorance. In Carter, C. and Ford, J., editors, *Uncertainty and Expectations in Economics*, pages 1–11. Basil Blackwell, Oxford.

Barberà, S., Barrett, C., and Pattanaik, P. (1984). On some axioms for ranking sets of alternatives. *Journal of Economic Theory*, 33:301–308.

Barberà, S., Bossert, W., and Pattanaik, P. (2004). Ranking sets of objects. In Barberà, S., Hammond, P., and Seidl, C., editors, *Handbook of Utility Theory*, volume 2, chapter 17, pages 893–977. Kluwer Academic Publishers.

Barberà, S. and Pattanaik, P. K. (1984). Extending an Order on a Set to the Power Set: Some Remarks on Kannai and Peleg's Approach. *Journal of Economic Theory*, 32(1):185–191.

Bossert, W. (1989). On the extension of preferences over a set to the power set: An axiomatic characterization of a quasi-ordering. *Journal of Economic Theory*, 49(1):84–92.

Bossert, W., Pattanaik, P. K., and Xu, Y. (2000). Choice under complete uncertainty: axiomatic characterizations of some decision rules. *Economic Theory*, 16(2):295–312.

Grether, D. M. and Plott, C. R. (1979). Economic Theory of Choice and the Preference Reversal Phenomenon. *The American Economic Review*, 69(4):623–638.

Jennings, R. E. (1974). A Utilitarian Semantics for Deontic Logic. *Journal of Philosophical Logic*, 3(4):445–456.

Kannai, Y. and Peleg, B. (1984). A Note on the Extension of an Order on a Set to the Power Set. 175:172–175.

Korhonen, P., Moskowitz, H., and WalleniusJ. (1990). Changing behaviour in interactive multiple-criteria decision making. *Annals of Operations Research*, 23:161–179.

Luce, R. D. and Raiffa, H. (1957). *Games and Decisions*. John Wiley and Sons, Inc., New York.

MacIntosh, D. (2010). Intransitive Preferences, Vagueness, and the Structure of Procrastination. In Andreou, C. and White, M. D., editors, *Procrastination: The Thief of Time*. Oxford University Press, Oxford.

Pattanaik, P. and Peleg, B. (1984). An Axiomatic Characterization of the Lexicographic Maximin Extension of an Ordering Over a Set to the Power Set. *Social Choice and Welfare*, 1:113–122.

Rawls, J. (1971). *A Theory of Justice*. Harvard University Press, Cambridge.

Rawls, J. (2001). *Justice as Fairness*. Harvard University Press, Cambridge.

Schotch, P. K. (2006). *Excursions in Philosophical Logic*. Unpublished manuscript., Halifax.

Schotch, P. K. and Jennings, R. E. (1981). Non-Kripkean Deontic Logic. In Hilpinen, R., editor, *New Studies in Deontic Logic*, pages 149–162. Kluwer Academic Publishers, Dordrecht.

Schumm, G. F. (1987). Transitivity, Preference, and Indifference. *Philosophical Studies*, 52(3):435–437.

Van Fraassen, B. C. (1973). Values and the Heart's Command. *The Journal of Philosophy*, 70(1):5–19.

Vercelli, A. (1999). The Recent Advances in Decision Theory Under Uncertainty. In Luini, L., editor, *Uncertain Decisions: Bridging Theory and Experiments*, chapter 11, pages 237–260. Kluwer Academic Publishers, Boston.