ZEROS AND ASYMPTOTICS OF HOLONOMIC SEQUENCES

by

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To the memory of my father, Peers Noble.

Table of Contents

List of Tables

Abstract

In this thesis we study the zeros and asymptotics of sequences that satisfy linear recurrence relations with generally nonconstant coefficients.

By the theorem of Skolem-Mahler-Lech, the set of zero terms of a sequence that satisfies a linear recurrence relation with *constant* coefficients taken from a field of characteristic zero is comprised of the union of finitely many arithmetic progressions together with a finite exceptional set. Further, in the nondegenerate case, we can eliminate the possibility of arithmetic progressions and conclude that there are only finitely many zero terms. For generally nonconstant coefficients, there are generalizations of this theorem due to Bézivin and to Methfessel that imply, under fairly general conditions, that we obtain a finite union of arithmetic progressions together with an exceptional set of density zero. Further, a condition is given under which one can exclude the possibility of arithmetic progressions and obtain a set of zero terms of density zero. In this thesis, it is shown that this condition reduces to the nondegeneracy condition in the case of constant coefficients. This allows for a consistent definition of nondegeneracy valid for generally nonconstant coefficients and a unified result is obtained.

The asymptotic theory of sequences that satisfy linear recurrence relations with generally nonconstant coefficients begins with the basic theorems of Poincaré and Perron. There are some generalizations of these theorems that hold in greater generality, but if we restrict the coefficient sequences of our linear recurrences to be polynomials in the index, we obtain full asymptotic expansions of a predictable form for the solution sequences. These expansions can be obtained by applying a transfer method of Flajolet and Sedgewick or, in some cases, by applying a bivariate method of Pemantle and Wilson. In this thesis, these methods are applied to a family of binomial sums and full asymptotic expansions are obtained. The leading terms of the expansions are obtained explicitly in all cases, while in some cases a field containing the asymptotic coefficients is obtained and some divisibility properties for the asymptotic coefficients are obtained using a generalization of a method of Stoll and Haible.

List of Abbreviations and Symbols Used

$(\cdot, \cdot)$	The greatest common divisor
$(\cdot, \cdot)_r$	The greatest common right divisor. See Definition 2.4
$(c)_n$	The falling Pochhammer symbol given by $(c)_n = c(c-1)\ldots(c-n+1)$ or the rising factorial given by $(c)_n = c(c+1)\ldots(c+n-1)$
$[\cdot, \cdot]$	The least common multiple (except in Chapter 5), The commutator (Chapter 5)
$[\cdot, \cdot]_\ell$	The least common left multiple. See Definition 2.4
$\lfloor \cdot \rfloor$	The floor function that sends a given real number to the largest integer less than or equal to it
$\lvert \cdot \rvert_p$	The p-adic absolute value corresponding to the prime $p \in \mathbb{N}$. See Definition 3.1
$\sim$	Is asymptotic to
$_2F_1$	The hypergeometric function given by $$_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$ where $(\cdot)_k$ denotes the rising factorial
$\mathcal{A}_n$	The Weyl algebra of dimension n over K. See Definition 5.1
$\mathcal{B}$	The Bernstein filtration of $\mathcal{A}_n$. See Definition 5.6
$\mathfrak{B}$	A basis for the zero set of a linear recurrence operator
$B(x)$	$\Psi(F(x))$

β	The quantity given by

$$\beta = \frac{1}{z_0} \left(\frac{1 - \alpha z_0}{1 - z_0} \right)^a . \qquad \text{See (7.7)}$$

$\mathbb{C}$	The field of complex numbers
$\mathbb{C}_p$	The completion of $\overline{\mathbb{Q}_p}$ with respect to the p-adic absolute value corresponding to the prime $p \in \mathbb{N}$
$\chi(t)$	The Hilbert polynomial of the finitely generated graded module $\mathcal{M}$ over $K[x_1, \ldots, x_n]$. See Definition 5.4
χ_f	The characteristic polynomial of the linear recurrence operator f of Poincaré type. See (4.5)
$d(M)$	The Hilbert dimension of the finitely generated left $\mathcal{A}_n$-module M. See Definition 5.7
$D(n,n)$	The central Delannoy number. See (7.2)
D-**finite**	Differentially finite
D_t	The image of the closed disk of radius t centred at the origin under γ
$\deg(f)$	The degree of a polynomial or linear recurrence operator f
Δ	The difference operator given by $\Delta = T - 1$
δ	The quantity given by

$$\delta = \frac{1}{(1 - z_0) \sqrt[4]{\Delta_g}} . \qquad \text{See (7.7)}$$

$\Delta(y)$	The quantity given by (7.18)
Δ_g	The discriminant of $g(z)$. See (7.6)
δ_{ij}	The Kronecker delta equal to 1 if $i = j$ and 0 otherwise

∂_x	The standard partial differential operator $\partial/\partial x$
F	A field that is a K-subalgebra of $\mathcal{L}_K$ closed under the action of T (and also closed under $u \mapsto u^{(a,q)}$ for all $q \in \mathbb{N}$ and $0 \le a < q$ in Chapter 3)
$\mathcal{F}$	The $\mathbb{C}$-vector space of all generating functions $F(x) = \sum_{n=0}^{\infty} f_n x^n \in \mathbb{C}[\![x]\!]$ such that f_n admits a full asymptotic expansion. See Section 6.1
f	A linear recurrence operator in $F[T;T]$ for some field F that is a K-subalgebra of $\mathcal{L}_K$ closed under the elementary shift operator T
$F(\!(x_1,\ldots,x_n)\!)$	The field of finite-tailed Laurent series in the variable $x_1,\ldots,x_n$ with coefficients in a field F
$F(\!(z)\!)$	The field of finite-tailed Laurent series in the variable z with coefficients in a field F
$F(\alpha_1,\ldots,\alpha_r)$	The smallest subfield of $\overline{F}$ that contains the field F as well as each $\alpha_j \in \overline{F}$
$F(x)$	The ordinary generating function of the sequence f_n
$\overline{F}$	A fixed algebraic closure of a field F
f^-	The reciprocal of the polynomial f. See (5.8)
$F^{[\ell]}$	The function obtained from F by applying the differential operator θ^ℓ
$f_{mn}(z)$	An auxiliary polynomial. See Section 7.4
f_n	The sequence obtained from g_n by $f_n = \frac{g_n}{\gamma \rho^n}$
$f \mid_\ell g$	The linear recurrence operator f left divides the linear recurrence operator g. See Definition 2.3
$f \mid_r g$	The linear recurrence operator f right divides the linear recurrence operator g. See Definition 2.3
$\tilde{F}(z,w)$	The rational generating function of the bivariate sequence $\{a_{mn}\}_{m,n}$. See (5.10) and (7.8)

$G(x)$	The generating function of $\{u_n^{(\varepsilon,a,d)}\}_{n=0}^{\infty}$
$g(z)$	An auxiliary polynomial. See (7.5)
g_n	A sequence in K that admits an asymptotic expansion of the form $$g_n \sim \gamma \rho^n n^{-\varphi} \sum_{m=0}^{\infty} \frac{a_m}{n^{m/q}} \qquad (n \to \infty)$$ for some constants $\gamma, \rho, \varphi, a_0, a_1, a_2, \cdots \in \mathbb{C}$ and $q \in \mathbb{N}$ with $a_0 = 1$. See (6.8)
G_w	The decomposition subgroup of the Galois group of $\overline{\mathbb{Q}}$ over the number field K of the valuation w defined on K. See (5.5)
$\gamma(z)$	The Möbius transformation given by $$\gamma(z) = \frac{1}{\nu(z)} = \frac{1-z}{1-\alpha z}.$$
gcd	Greatest common divisor
gcrd	Greatest common right divisor
$gr^{\mathcal{F}} R$	The graded algebra associated to the filtered K-algebra R having filtration $\mathcal{F}$. See (5.1)
$gr^{\Gamma} M$	The graded module associated to the filtered left R-module M having filtration Γ. See (5.2)
$I(\{u_n\}_n)$	The left ideal of $F[T;T]$ consisting of the linear recurrence operators that annihilate the sequence u. See (2.2)
I_f	The annihilating ideal of $f \in K[\![x_1, \ldots, x_n]\!]$ given by the left ideal of $\mathcal{A}_n$ consisting of those differential operators that map f to zero. See Definition 5.2
$\Im$	The imaginary part
K	A field of characteristic 0

$K(n)$	The subfield of $\mathcal{L}_K$ consisting of the equivalence classes of sequences obtained by restricting rational functions over K to $\mathbb{N}_0$
$K(n)[T;T]$	The space of linear recurrence operators over K with polynomial coefficients. See (6.9)
$K(x)[\theta_x;\theta_x]$	The space of linear differential operators over K with polynomial coefficients. See (6.10)
$K[n,T;T]$	The set of all polynomials in n and T with coefficients in K. See (6.9)
$K[x,\theta_x;\theta_x]$	The set of all polynomials in θ_x and x with coefficients in K. See (6.10)
$K^{\mathbb{N}_0}$	The algebra of sequences of elements of a field K indexed by $\mathbb{N}_0$
K_v	The completion of the number field K with respect to the valuation v
$\mathcal{L}_K$	The sequence space equal to $K^{\mathbb{N}_0}/\sim$ where $\sim$ identifies sequences that agree eventually. See Definition 2.1
Λ	The union of all Λ_f as f ranges over $K(n)[T;T]$. See (5.4)
Λ_{const}	The union of all Λ_f as f ranges over $K[T;T]$. See (5.4)
Λ_f	The subset of K_v consisting of all limits of quotients of elements of $Z(f)$ for some $f \in K(n)[T;T]$. See (5.3)
lclm	Least common left multiple
lcm	Least common multiple
$\mathcal{M}$	The set of minimal points of $\tilde{F}(z,w)$
$\mathbb{N}$	The set of natural numbers

$\mathbb{N}_0$	The set of nonnegative integers
$\nu(z)$	The function given by $\nu(z) = \left(\frac{1-\alpha z}{1-z}\right)^a$. See (7.9)
$O(\cdot)$	Big Oh
$o(\cdot)$	Little Oh
$\mathcal{O}_K$	The ring of integers in the number field K
ODE	Ordinary differential equation
$\mathrm{ord}_F u$	The order of the sequence $u \in \mathcal{L}_K$ over F. See Definition 2.5
$\mathfrak{p}$	A nonzero prime ideal of the ring of integers $\mathcal{O}_K$ of a number field K
P-**recursive**	Polynomially recursive
$\mathbb{P}^1$	The projective line
PLID	Principal left ideal domain
$\Psi(F)$	A certain $\mathbb{C}$-linear transformation defined on $\mathcal{F}$. See Section 6.1
$\mathbb{Q}$	The field of rational numbers
$\overline{\mathbb{Q}}$	The field of algebraic numbers
$\mathbb{Q}_p$	The field of p-adic numbers. See Definition 3.1
$\mathbb{R}$	The field of real numbers
$R(x,y)$	The resultant of $p(z)y - (1-\alpha z)$ and $(1-\alpha z)^a x - z(1-z)^a$ with respect to z, where

$$p(z) = \alpha z^2 + (a\alpha - a - \alpha - 1)z + 1$$

$R[\![x_1, \ldots, x_n]\!]$	The ring of formal power series in the variables $x_1, \ldots, x_n$ with coefficients in a ring R
$R[\![z]\!]$	The ring of formal power series in the variable z with coefficients in a ring R
$R[T; T]$	The ring of recurrence operators over a subring R of $\mathcal{L}_K$. See Definition 2.2
$R[z]$	The ring of polynomials in the variable z with coefficients in the ring R
R^*	The set of nonzero elements in a ring R
$R^\times$	The group of units of a ring R
$\Re$	The real part
Res	Residue
$S(i, j)$	The (i, j)-th Stirling number of the second kind
$s(i, j)$	The (i, j)-th Stirling number of the first kind
S_{mn}	The subset of the set $\mathcal{M}$ of minimal points of $\tilde{F}$ defined by (5.12)
T	The elementary shift operator that shifts the indices of sequences by 1. See Definition 2.2. Also the bilinear map from X_2 to $\mathcal{A}_2$ given by (5.16)
θ_x	The differential operator given by $\theta_x = x\frac{d}{dx}$
$u^{(a,q)}$	The sequence in $\mathcal{L}_K$ having n-term given by $u_n^{(a,q)} = u_{qn+a}$. See (3.2)
$U_{i,j}(x)$	The polynomial given by

$$U_{i,j}(x) = \sum_{h=j}^{i} S(i, h)s(h, j)(x - 1)^h x^{i-h}$$

	defined for $j \geq i$. See (6.14)
$u, \{u_n\}_n, \{u_n\}_{n=0}^{\infty}$	A sequence in $\mathcal{L}_K$

$u_n = u_n^{(\varepsilon,a,d)}$ A binomial sum. See (7.1)

$\tilde{u}_{mn} = \tilde{u}_{mn}^{(\varepsilon,a,d)}$ A bivariate sequence. See (7.4)

$v_{m,n}$ Weighted Delannoy numbers. See (7.3)

$v_{\mathfrak{p}}$ The $\mathfrak{p}$-adic valuation corresponding to the prime $\mathfrak{p}$ of the number field K

$w \mid v \mid v_p$ A tower of valuations where w is defined on $\overline{\mathbb{Q}}$, v is defined on some number field K and v_p denotes the p-adic valuation associated to $p \in \mathbb{N} \cup \{\infty\}$ defined on $\mathbb{Q}$

X_2 A certain free $\mathbb{Z}$-algebra on $\mathcal{A}_2^2$. See (5.13), (5.14) and (5.15)

$\mathbb{Z}$ The ring of integers

$Z(f)$ The zero set of a linear recurrence operator f

$Z(G)$ The zero set of the linear recurrence or differential operator G

$Z(u)$ The set of indices n at which a sequence $u = \{u_n\}$ vanishes

$\mathbb{Z}_{<0}$ The set of negative integers

$\mathbb{Z}_{\leq 0}$ The set of nonpositive integers

$\mathbb{Z}_p$ The ring of p-adic integers. See Definition 3.1

$\zeta \Delta_0$ The Δ-domain at $\zeta \in \mathbb{C}^*$ associated to the Δ-domain at 1 Δ_0

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CHAPTER 1

Introduction

In this short preliminary chapter, we provide an informal account of the contents of this thesis. Throughout, we are interested in the theory of sequences that satisfy linear recurrence relations and keep an eye on what is known for the case of *constant* coefficients as we outline what can be said regarding the case of generally *nonconstant* coefficients. Since we will be using operator notation throughout this thesis, we start in Section 1.1 by illustrating how to view recurrence sequences as zeros of recurrence operators. We then turn to the first of our two main themes in Section 1.2, namely, the study of zeros of sequences that satisfy linear recurrence relations with nonconstant coefficients. In Section 1.3, we turn to the second main theme of this thesis by describing the asymptotic theory of sequences that satisfy linear recurrence relations with nonconstant coefficients.

1.1 Recurrence Sequences as Zeros of Recurrence Operators

Let K be a field of characteristic zero. For our purposes, K will be taken to be a number field, the field $\overline{\mathbb{Q}}$ of algebraic numbers, the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. The theory of sequences in such a field K that satisfy a linear recurrence relation with *constant* coefficients taken from K is well established. What we do here is investigate to what extent some well-known results from this theory generalize to linear recurrence relations over K having coefficients equal to (generally nonconstant) *sequences* in K. In the constant coefficient case, the

recurrence sequences $\{u_n\}_n$ are those that satisfy

$$\sum_{j=0}^{k} \pi_j u_{n+j} = 0 \qquad (n \geq 0), \tag{1.1}$$

for some constants $\pi_0, \pi_1, \ldots, \pi_k \in K$. Here we have the concept of characteristic polynomials f as well as the sets $Z(f)$ of all sequences that possess f as a characteristic polynomial. The *characteristic polynomial* of (1.1) is given by

$$f(z) = \sum_{j=0}^{k} \pi_j z^j \in K[z], \tag{1.2}$$

and $Z(f)$ consists of all sequences of elements in K which satisfy (1.1). Denoting the least common multiple and greatest common divisor functions by $[\cdot, \cdot]$ and $(\cdot, \cdot)$, respectively, and using the notation

$$X + Y = \{x + y \mid x \in X, \ y \in Y\}$$

for sets X and Y, we have the following result (see, [20, Section 1.1.4]).

Theorem 1.1. *Let f and g be monic polynomials defined over K having nonzero constant term. Then the following hold.*

(a) *$Z(f)$ is a K-vector space of dimension $\deg f$.*

(b) *$Z([f, g]) = Z(f) + Z(g)$.*

(c) *$Z((f, g)) = Z(f) \cap Z(g)$.*

(d) *$Z(f) \subseteq Z(g) \iff f \mid g$.*

In attempts to generalize Theorem 1.1 to accommodate nonconstant coefficients, the first thing we need to note is that our nonconstant coefficients may not be defined at all indices. For instance, this is the case when the coefficients are taken to be certain rational functions in the index. We therefore change the definition slightly and call a sequence a recurrence sequence if it satisfies a linear recurrence relation eventually. That is, the sequence $\{u_n\}_n \subseteq K$ satisfies

$$\sum_{j=0}^{k} p_j(n) u_{n+j} = 0 \qquad (n \geq N) \tag{1.3}$$

for some $k \in \mathbb{N}$, $N \in \mathbb{N}_0$ and suitable sequences $\{p_j(n)\}_n \subseteq K$. Even here, leaving the coefficient sequences undefined at certain indices seems unsatisfactory. This is remedied by identifying sequences that have the same tail so that we can redefine the sequences at any finitely many troublesome indices. We therefore work in the sequence space $\mathcal{L}_K$ defined by

$$\mathcal{L}_K = K^{\mathbb{N}_0} / \sim$$

where $\sim$ is the equivalence relation defined on $K^{\mathbb{N}_0}$ by

$$\{u_n\}_{n=0}^{\infty} \sim \{v_n\}_{n=0}^{\infty} \iff u_n = v_n \text{ for all sufficiently large } n.$$

We embed K in $\mathcal{L}_K$ along the diagonal thereby making the identification

$$a \in K \longleftrightarrow [\{a, a, a, \dots\}] \in \mathcal{L}_K,$$

where we have adopted the convention of enclosing elements of $K^{\mathbb{N}_0}$ in square brackets to denote the corresponding equivalence class in $\mathcal{L}_K$. The elements in K are then the equivalence classes in $\mathcal{L}_K$ of the constant sequences in $K^{\mathbb{N}_0}$. We now explain how to view (1.3) as expressing that the sequence $\{u_n\}_{n=0}^{\infty}$ is annihilated by a suitable linear operator. We define the elementary shift operator T on $\mathcal{L}_K$ by

$$T([\{u_0, u_1, u_2, \dots\}]) = [\{u_1, u_2, u_3, \dots\}].$$

The sequence space $\mathcal{L}_K$ is a K-algebra (with respect to componentwise operations) and the map $T : \mathcal{L}_K \to \mathcal{L}_K$ defined above is an automorphism with inverse given by

$$T^{-1}([\{u_0, u_1, \dots\}]) = [\{0, u_0, u_1, \dots\}].$$

For each $j \in \mathbb{N}_0$ we then see that the map $T^j : \mathcal{L}_K \to \mathcal{L}_K$ defined by

$$T^j([\{u_0, u_1, \dots\}]) = [\{u_j, u_{j+1}, \dots\}]$$

is an automorphism with inverse given by

$$T^{-j}([\{u_0, u_1, \dots\}]) = [\{0, 0, \dots, 0, u_0, u_1, \dots\}],$$

where here we have j initial zeros.

With this notation, and denoting $\{u_n\}_{n=0}^{\infty}$ simply by u, we can rewrite (1.3) as

$$f(T)([u]) = [0] \in \mathcal{L}_K, \tag{1.4}$$

for

$$f(T) = \sum_{j=0}^{k} [\{p_j(n)\}_n] T^j \in \mathcal{L}_K[T;T]. \tag{1.5}$$

Since $[0]$ is the zero element of $\mathcal{L}_K$, we can think of (1.4) as expressing that $[u] \in \mathcal{L}_K$ is a zero of $f(T)$. Here, we use the notation $\mathcal{L}_K[T;T]$ to remind us that the multiplication is given by the noncommutative operation of composition and so is generated by the commutativity law $[\{u_{n+1}\}_n]T = T[\{u_n\}_n]$. Throughout this thesis, it will be convenient to adopt the following notational convention. When no confusion is likely to result, we will use the same notation (e.g., u, $\{u_n\}_n$, $\{u_n\}_{n=0}^{\infty}$) to denote a sequence and its corresponding equivalence class in $\mathcal{L}_K$. We will also, when convenient, denote a sequence simply by providing its nth term. This "abuse of notation" proves especially useful when we consider linear recurrence operators such as f given by (1.5) since it allows us to write $p_j(n)$ in place of $[\{p_j(n)\}_n]$ without causing any ambiguity.

In case of constant coefficients, the operator given by (1.5) corresponds to the characteristic polynomial of the corresponding recurrence since $K[T;T] \cong K[T]$ as T acts trivially on the constant sequences. For nonconstant coefficients, we obtain a generalization of Theorem 1.1 by replacing polynomials with operators as defined by (1.5). In the general case, however, we will need to restrict the coefficients of our operators to lie in a field in order to obtain a suitable divisibility theory to define least common multiples and greatest common divisors. Since we will need to be able to compose our operators, and scalar multiply by elements of K, we require the field to be a K-subalgebra of $\mathcal{L}_K$ that is closed under the action of T. Also, as operator composition does not provide us with a commutative multiplication, we will need to be careful with regard to the order of factors in our definitions. We require left multiples and right divisors and use the symbols $[\cdot,\cdot]_\ell$, $(\cdot,\cdot)_r$ to represent the least common left multiple and greatest common right divisor accordingly. For constant coefficients, $Z(f)$ denotes the space of all sequences having f as a characteristic polynomial. In the general situation, we define $Z(f)$ to be the kernel of f. Denoting the least common left multiple and greatest common right divisor functions by $[\cdot,\cdot]_\ell$ and $(\cdot,\cdot)_r$, respectively, we have the following result, proved in Chapter 2.

Theorem 1.2. *Let the field F be a K-subalgebra of $\mathcal{L}_K$ closed under the action of T. Suppose that $f,g \in F[T;T]$ are monic and have nonzero constant terms. Then the following hold.*

(a) $Z(f)$ *is a K-vector space of dimension* $\deg f$.

(b) $Z([f,g]_\ell) = Z(f) + Z(g)$.

(c) $Z((f,g)_r) = Z(f) \cap Z(g)$.

(d) $Z(f) \subseteq Z(g) \iff f \mid_r g$.

1.2   Zero Terms in Linear Recurrence Sequences

The celebrated Skolem-Mahler-Lech Theorem describes the set of zero terms in sequences that satisfy linear recurrence relations with constant coefficients of the type given by (1.1). The version of this theorem stated below is taken from [20, Theorem 2.1].

Theorem 1.3 (Skolem-Mahler-Lech)**.** *The set of zeros of a linear recurrence sequence over a field of characteristic zero comprises a finite set together with a finite number of arithmetic progressions.*

In case the characteristic polynomial corresponding to a linear recurrence relation satisfied by the sequence fails to have two roots that share a common power, we call the recurrence sequence *nondegenerate*. In the nondegenerate case we can eliminate the possibility of arithmetic progressions in the set of zero terms. In this case, we obtain at most finitely many zero terms. Now, if $u \in \mathcal{L}_K$ satisfies a nonzero operator in $K[T;T]$, then it satisfies a unique monic operator in $K[T;T]$ of least degree which we call *the minimal operator of u over K*. If $u \in \mathcal{L}_K$ has minimal operator $f(T) \in K[T;T]$ over K, we define the *order of u over K*, denoted by $\mathrm{ord}_K u$, to be the degree of f. The typical argument that proves the existence of the minimal operator over K proceeds by defining the minimal operator to be the unique monic generator of the ideal of the principal ideal domain $K[T;T] \cong K[T]$ consisting of all operators that annihilate u. An important component of the proof of the theorem of Skolem-Mahler-Lech is the following result (see, e.g., [20, Theorem 1.3]).

Theorem 1.4. *Any subsequence $\{u_{qn+a}\}_{n=0}^{\infty}$, $q \in \mathbb{N}$, $0 \le a < q$, of a linear recurrence sequence u of order k is itself a linear recurrence sequence of order at most k.*

In order to obtain a unified result that accommodates nonconstant coefficients, we start by expressing these results in operator notation. Also, for $q \in \mathbb{N}$ and $0 \leq a < q$, we define the sequence $u^{(a,q)} \in \mathcal{L}_K$ to have n-th term given by $u_n^{(a,q)} = u_{qn+a}$.

If we refer to sequences that satisfy a nonzero linear recurrence operator with coefficients in a field L as *L-recurrent*, Theorem 1.4 can be expressed as follows. "If $u \in \mathcal{L}_K$ is K-recurrent of order k, then each $u^{(a,q)}$ is K-recurrent of order at most k." The theorem of Skolem-Mahler-Lech can be expressed as follows. "If $u \in \mathcal{L}_K$ is K-recurrent, then there exists $q \in \mathbb{N}$ such that each of the $u^{(a,q)}$ is either eventually zero or eventually nonzero, and in fact, in the nondegenerate case, each $u^{(a,q)}$ is eventually nonzero." In the general situation, we consider F-recurrent sequences where the field F is a K-subalgebra of $\mathcal{L}_K$ closed under the action of T and under taking arithmetic progressions (i.e., for all $u \in F$, $q \in \mathbb{N}$ and $0 \leq a < q$, we have $u^{(a,q)} \in F$). The argument used in the case of constant coefficients to establish the existence of the minimal operator can be extended to F since, as will be shown in Chapter 2, $F[T;T]$ is a principal left ideal domain and for elements $u \in \mathcal{L}_K$, $\{f(T) \in F[T;T] \mid f(T)(u) = 0\}$ is a left ideal in $F[T;T]$. We can therefore define the minimal operator over F to be the unique monic generator of this left ideal when it is nontrivial. The order of the sequence is then the degree of this minimal operator. We obtain the following result.

Theorem 1.5. *Let the field F be a K-subalgebra of $\mathcal{L}_K$ that is closed under the action of T as well as under taking arithmetic progressions. If u is F-recurrent then each $u^{(a,q)}$ is also F-recurrent with order at most $\operatorname{ord}_F u$.*

In [5], Bézivin showed that in the case $F = K(n)$, and under reasonable assumptions, the theorem of Skolem-Mahler-Lech still applies if we replace the finite exceptional set with an exceptional set of density zero. Then, Methfessel showed ([45, Corollary 1]) that we can obtain the same generalization for general fields F as above. In fact, the proof of this result shows that for F-recurrent sequences u of order k, we obtain an exceptional set of density zero that fails to possess any k-term arithmetic progressions. Further, if there does not exist $q \in \mathbb{N}$ such that each of the $u^{(a,q)}$ is F-recurrent of lower order than u, then we can eliminate the possibility of arithmetic progressions. We can therefore express Methfessel's generalization in our notation as "If $u \in \mathcal{L}_K$ is F-recurrent, then there exists $q \in \mathbb{N}$ such that the zero terms in every nonzero $u^{(a,q)}$ comprise a set of density zero." Now, as is shown in Chapter 3,

this condition that allows us to eliminate the possibility of arithmetic progressions in the set of zero terms for F-recurrent sequences reduces to the usual definition of nondegeneracy of K-recurrent sequences. We therefore obtain the following unified result.

Proposition 1.1. *Suppose that the field F is a K-subalgebra of $\mathcal{L}_K$ closed under the action of the elementary shift operator T, and closed under taking arithmetic progressions. Then, for $u \in F$ of order k, the set $Z(u)$ of zeros of u admits a decomposition*

$$Z(u) = S \cup U \cup V$$

where S is the union of finitely many infinite arithmetic progressions, U is a finite set and V is a set of density zero that fails to possess any k-term arithmetic progressions. Further, if $F = K$, then one can take V to be the empty set, and when u is nondegenerate, one can take S to be the empty set.

The question arises: "Under what conditions can we be guaranteed a finite exceptional set rather than an exceptional set of density zero so that we have a direct generalization of the theorem of Skolem-Mahler-Lech?" This question was studied by Laohakosol in [41] with the result that we can keep the same conclusion for several more general situations. Unfortunately, there was an error in one of the arguments, which was, however, corrected in the subsequent joint paper [6] of Bézivin and Laohakosol.

1.3 Asymptotics of Linear Recurrence Sequences

The asymptotics of sequences that satisfy linear recurrence relations with constant coefficients is complete since we have a closed form expression for such sequences given by Binet's formula. The precise statement appears below in Theorem 1.6. For a development in case K is algebraically closed, see, e.g., [20, Section 1.6].

Theorem 1.6. *Every K-recurrent sequence $u \in \mathcal{L}_K$ can be written in the form*

$$u_n = \sum_{j=1}^{m} P_j(n)\alpha_j^n$$

for suitable $m \in \mathbb{N}$ and polynomials $P_1, \ldots, P_m$ over $\overline{K}$, where the α_j are the distinct roots of the minimal polynomial of u over K.

When $K = \mathbb{C}$ and $f \in \mathbb{C}[T; T]$ is a recurrence operator, $Z(f)$ admits a standard basis defined as follows. If

$$f(T) = \sum_{j=0}^{k} \pi_j T^j \in \mathbb{C}[T; T], \qquad \pi_k \neq 0,$$

and the polynomial $f(x)$ factors as

$$f(x) = \pi_k \prod_{j=1}^{m} (x - \alpha_j)^{e_j}$$

for $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ distinct and $e_1, \ldots, e_m \in \mathbb{N}$ such that $e_1 + \cdots + e_m = k$, then the *standard basis for $Z(f)$* is given by

$$\mathfrak{B} = \{ n^{\rho_j} \alpha_j^n \mid 1 \leq j \leq m, \ 0 \leq \rho_j \leq e_j - 1 \}.$$

For linear recurrences with nonconstant coefficients, the absence of a universal closed form expression makes the study of asymptotics much more interesting and challenging.

The most general class of linear recurrence relations with nonconstant coefficients for which there are known asymptotic results is the class of linear recurrence relations of Poincaré type. An overview of this theory is provided in Chapter 4. Linear recurrences of this type have been studied extensively, yet the results remain partial, at least in comparison with what can be said regarding the asymptotics of the subclass of *holonomic sequences*. These are zeros of linear recurrence operators with rational functions as coefficients. The setting is as follows. We set $K = \mathbb{C}$ endowed with the usual absolute value. The linear recurrence operators of Poincaré type have *almost constant coefficients* in the sense that they can be written as

$$f(T) = \sum_{j=0}^{k} (\pi_j + \varepsilon_j(n)) T^j \in \mathcal{L}_{\mathbb{C}}[T; T],$$

where, for all j, $\pi_j \in \mathbb{C}$ and $\lim_{n \to \infty} \varepsilon_j(n) = 0$. We refer to the polynomial

$$\chi_f(x) = \sum_{j=0}^{k} \pi_j x^j \in \mathbb{C}[x]$$

as the *characteristic polynomial of f* and denote its roots in $\mathbb{C}$ by $\alpha_1, \ldots, \alpha_k$. We call one of its roots *simple* if it has absolute value distinct from the absolute values of each

of the other roots and call f *simple* in case every root of χ_f is simple. The starting point for the study of such sequences are the following basic theorems of Poincaré and Perron.

Theorem 1.7 (Poincaré, [54])**.** *With the above notation, suppose that f is simple and $\pi_k \neq 0$. Then, for all eventually nonzero $u \in Z(f)$, we have*

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \alpha_j \qquad \text{for some } 1 \leq j \leq k.$$

Theorem 1.8 (Perron, [51])**.** *With the above notation, suppose that f is simple and $\pi_0, \pi_k \neq 0$. Then, there exists a basis $\mathfrak{B} = \{u^{(1)}, \ldots, u^{(k)}\}$ for $Z(f)$ such that*

$$\lim_{n \to \infty} \frac{u^{(j)}_{n+1}}{u^{(j)}_n} = \alpha_j \qquad (1 \leq j \leq k).$$

Theorem 1.9 (Perron, [52])**.** *With the above notation, suppose that $\pi_0, \pi_k \neq 0$. Then, there exists a basis $\mathfrak{B} = \{u^{(1)}, \ldots, u^{(k)}\}$ for $Z(f)$ such that*

$$\limsup_{n \to \infty} \sqrt[n]{|u^{(j)}_n|} = |\alpha_j| \qquad (1 \leq j \leq k). \tag{1.6}$$

There are various generalizations of these results, a few of which we now mention. Using a result of Coffman that appears in [14], Pituk, in [53], proved that, under the same conditions as in Theorem 1.9, (1.6) holds for *every* nonzero solution in $Z(f)$ and not just for the basis elements. In what follows, we will say that two sets X and Y of sequences are asymptotic if we can match the sequences in X with sequences in Y to which they are asymptotic.

In 1958, Evgrafov (see [21]) proved that in case the α_j are distinct, $\pi_0, \pi_k \neq 0$ and

$$\sum_n |\varepsilon_j(n)| < \infty \qquad (0 \leq j \leq k),$$

there exists a basis for $Z(f)$ asymptotic to the standard basis for $Z(\chi_f)$. Kooman extended this result in his PhD thesis as follows.

Theorem 1.10 ([38, Corollary 4.2])**.** *With the above notation, assume that $\pi_0, \pi_k \neq 0$ and let L be the maximum algebraic multiplicity of the α_j as roots of χ_f. If*

$$\sum_n n^{L-1} |\varepsilon_j(n)| < \infty \qquad (0 \leq j \leq k)$$

then there exists a basis for $Z(f)$ asymptotic to the standard basis for $Z(\chi_f(T))$.

Kooman also generalized the first theorem of Perron by establishing the following result.

Proposition 1.2 ([38, Corollary 3.4]). *With the above notation, suppose that $\pi_0, \pi_k \neq 0$. Then, if α is a simple eigenvalue of f, there exists $u \in Z(f)$ such that*

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \alpha.$$

There are also some degree specific generalizations of the theorems of Poincaré and Perron. The degree one case can be solved explicitly, and both Coffman and Kooman have some results regarding the degree 2 case. (See, e.g., [14, §10],[38, Ch. 5,6],[39]).

If we now look at subclasses of linear recurrence sequences of Poincaré type, much more can be said. In this direction, the theorems of Poincaré and Perron can be generalized in case the $\varepsilon_j(n)$ are chosen to admit full asymptotic expansions (see, e.g., [19], [9]), and in case the $\varepsilon_j(n)$ are rational functions (the holonomic case), very strong asymptotic results can be obtained. Indeed, for subfields K of $\mathbb{C}$, $K(n)$-recurrent sequences admit full asymptotic expansions of a predictable form. This likely holds without restriction, and does hold in the case the generating function satisfies a linear ODE with polynomial coefficients with respect to which 0 is not an irregular singular point; see [24, Part B]. The $K(n)$-recurrent sequences are precisely those having generating functions that satisfy linear ODEs with polynomial coefficients over K and one can use the fact that the theory of series solutions of such ODEs is so well established to obtain meaningful results on the coefficient sequences of interest. In general, the method of Frobenius can be applied to obtain an asymptotic expansion of the generating function of the sequence about each of its singularities of least positive modulus, and then the transfer theory of Flajolet and Sedgewick can be applied to obtain the full asymptotic expansion of the sequence in question. Since every algebraic series satisfies a linear ODE with polynomial coefficients, the coefficient sequences of these types of generating functions can be obtained by analyzing a linear ODE as above but also by expanding the generating function in a Puiseux expansion about its singularities of least nonzero modulus. In some cases, full asymptotic expansions for $K(n)$-recurrent sequences can also be obtained using a multivariate method due to Pemantle and Wilson, developed in [50]. Once the existence of a full asymptotic expansion is determined, one can in certain cases apply a method of Stoll and Haible,

developed in [59] to discover divisibility properties for the asymptotic coefficients. An introduction to holonomic sequences is given in Chapter 5. The basic definitions are given, and it is shown how to generate fields using holonomic sequences. We also survey the literature on sequences that are known not to be holonomic. It is in that chapter as well that the methods of Flajolet and Sedgewick and of Pemantle and Wilson are described. In Chapter 6, a generalization of the method of Stoll and Haible is presented and made explicit.

By combining the method of Flajolet and Sedgewick with that of Pemantle and Wilson, we can determine full asymptotic expansions valid as $n \to \infty$ for the family of binomial sums given by

$$u_n = \sum_{k=0}^{n} (-1)^{\varepsilon k} \binom{n}{k} \binom{an}{k} d^k$$

where $\varepsilon \in \{0, 1\}$ and $a, d \in \mathbb{N}$. This family of binomial sums contains as examples the central binomial coefficients, the central Delannoy numbers as well as a sum related to Wolstenholme's Theorem that is considered in a paper of Chamberland and Dilcher [12]. The leading terms of the asymptotic expansion can be obtained explicitly in all cases, and a field containing the asymptotic coefficients can be determined in some subcases. The generalization of the method of Stoll and Haible from Chapter 6 can also be applied in certain cases to obtain divisibility properties for the asymptotic coefficients. This family of binomial sums will be studied in Chapter 7 and along the way, a conjecture of Chamberland and Dilcher will be proved.

CHAPTER 2

Linear Recurrence Operators

In this chapter, we develop the notation that will be used throughout the remainder of this thesis. We start, in Section 2.1, with the definition of the sequence space $\mathcal{L}_K$, the elementary shift operator T and the ring $F[T;T]$ of recurrence operators over fields F that are K-subalgebras of $\mathcal{L}_K$ preserved under the action of T. We then turn, in Section 2.2, to the study of divisibility in $F[T;T]$. It is there that the concepts of zeros, greatest common right divisors and least common left multiples of linear recurrence operators are defined. These are the concepts required for Theorem 1.2. Having developed all of the prerequisites at that point, we close that section by restating and proving Theorem 1.2. In particular, we will see in that section that $F[T;T]$ is a principal left ideal domain. This fact is used in the short Section 2.3 in order to define the minimal operator and order of certain elements of our sequence space $\mathcal{L}_K$. These definitions are essential to the study of zero terms in linear recurrence sequences undertaken in Chapter 3.

2.1 Definitions

Our study of linear recurrence sequences begins with the definition of the sequence space. This is the space in which all sequences considered in this thesis will lie. It is constructed by identifying sequences in K that eventually coincide.

Definition 2.1. We define the *sequence space $\mathcal{L}_K$ associated to K* to be the K-algebra obtained from the K-algebra $K^{\mathbb{N}_0}$ endowed with componentwise operations by identifying sequences that have the same tail. That is,

$$\mathcal{L}_K = K^{\mathbb{N}_0} / \sim$$

12

where $\sim$ is the equivalence relation defined on $K^{\mathbb{N}_0}$ by

$$\{u_n\}_{n=0}^{\infty} \sim \{v_n\}_{n=0}^{\infty} \qquad \Longleftrightarrow \qquad u_n = v_n \text{ for all sufficiently large } n.$$

Having defined the space $\mathcal{L}_K$ that contains all of the sequences to be considered in this thesis, we turn now to defining the operators whose zeros are the linear recurrence sequences of special interest.

Definition 2.2. We define the *elementary shift operator*

$$T : \mathcal{L}_K \to \mathcal{L}_K$$

to be the K-algebra automorphism of $\mathcal{L}_K$ given by

$$T(\{u_n\}_{n=0}^{\infty}) = \{u_{n+1}\}_{n=0}^{\infty} \quad (\{u_n\}_{n=0}^{\infty} \in \mathcal{L}_K).$$

Here the inverse map T^{-1} can be defined by

$$T^{-1}(\{u_n\}_{n=0}^{\infty}) = \{0, u_0, u_1. \ldots\} \quad (\{u_n\}_{n=0}^{\infty} \in \mathcal{L}_K).$$

and the $\mathcal{L}_K$-algebra of *recurrence operators* on $\mathcal{L}_K$, denoted by $\mathcal{L}_K[T;T]$, by

$$\mathcal{L}_K[T;T] = \{f(T) \mid f(x) \in \mathcal{L}_K[x]\},$$

endowed with the usual function addition, and multiplication given by function composition. For K-subalgebras R of $\mathcal{L}_K$ that are closed under the action of T, we define the R-algebra of *recurrence operators over R*, denoted $R[T;T]$, to be the R-subalgebra of $\mathcal{L}_K[T;T]$ consisting of those operators having coefficients in R.

Each nonzero element $f(T) \in \mathcal{L}_K[T;T]$ can be written uniquely as a finite left linear combination of powers of T and we define the *degree of f*, denoted $\deg f$, to be the largest power of T that appears in this expression with nonzero coefficient. Since the operator

$$\sum_{j=0}^{k} a_j(n)T^j \in \mathcal{L}_K[T;T]$$

can also be written as

$$\sum_{j=0}^{k} T^j a_j(n-j),$$

we see that each element of $\mathcal{L}_K[T; T]$ can also be written uniquely as a finite right linear combination of powers of T and that we obtain the same largest power of T that appears with nonzero coefficient. Consequently, the degree function can just as well be defined using right linear combinations. For completeness sake, we define $\deg 0 = -\infty$.

We will see that the ring of recurrence operators can be thought of as a noncommutative analogue of a polynomial ring. But first, we restrict the set of coefficients in such a way that some convenient properties enjoyed by *commutative* polynomial rings can be transferred over to our *noncommutative* situation. These properties include the usual degree formula for products, existence of a Euclidean algorithm and other properties as described in the following section.

In $\mathcal{L}_K$, we must deal with classes of sequences which are neither eventually zero nor eventually nonzero. For instance, the equivalence class of the sequence

$$\{1, 0, 1, 0, 1, 0, \ldots\}$$

consists entirely of sequences that fit this description. Classes of sequences such as these are the nonzero non units of the ring $\mathcal{L}_K$. By restricting our attention to $F[T; T]$ for a K-subalgebra F of $\mathcal{L}_K$ that is a field, we avoid complications similar to the complications that arise from the consideration of commutative polynomials over rings instead of fields. For instance, the nonzero non units of $\mathcal{L}_K$ are zero divisors, and so the familiar degree formula

$$\deg(fg) = \deg f + \deg g$$

does not hold for all f and g in $\mathcal{L}_K[T; T]$. Also, it will be beneficial to assume that the lead coefficient and constant term of our operators are, as sequences in K, eventually nonzero. If we restrict our coefficients to lie in a field, this condition is simply that these coefficients are nonzero as elements of the field. We will use the restriction on the lead coefficient to be able to assume our recurrence operators are monic, and we will show that the restriction on the constant term is necessary in order to obtain a space of zeros of the expected dimension. The requirements that F be both a field and a K-subalgebra of $\mathcal{L}_K$ that is closed under the action of T boil down to requiring F to be an extension field of K (viewed as consisting of the classes of constant sequences) contained in $\mathcal{L}_K$ for which the restriction of T to F is an embedding of F into itself

over K. Indeed, the fact that F has a 1 together with the fact that we can scalar multiply by K implies that F is a field extension of K and the restriction of T to F remains injective. We therefore consider only operators

$$f(T) = a_0(n) + a_1(n)T + \cdots + a_{k-1}(n)T^{k-1} + T^k \in F[T; T]$$

where F is both a field and a K-subalgebra of $\mathcal{L}_K$ that is closed under the action of T and $a_0 \neq 0$ as an element of F.

Suppose now that an element $\{u_n\}_{n=0}^{\infty}$ of $\mathcal{L}_K$ is algebraic over K. Then, there exists a nonzero polynomial $h(x) \in K[x]$ such that

$$h\left(\{u_n\}_{n=0}^{\infty}\right) = 0 \in \mathcal{L}_K.$$

It follows that the sequence $\{h(u_n)\}_n$ vanishes for all sufficiently large n. As h can have only finitely many zeros, we conclude that u takes on only finitely many values in K. We note that at least one of these values, say a, is taken on by u over and over again regardless of how far out we go in the sequence. We can then subtract the constant sequence $a = \{a\}_n$ to obtain a non unit in $\mathcal{L}_K$. Thus if both u and a lie in a subfield of $\mathcal{L}_K$, we can conclude that $u - a$ is eventually zero so that u is eventually constant. In particular, for subfields F of $\mathcal{L}_K$ containing K, such as those we are considering, we have

$$F \cap \overline{K} = K. \tag{2.1}$$

This implies that F is generated over K completely by elements transcendental over K. It is therefore natural to consider the simplest case for which this holds, namely, the case where F is isomorphic to a field of rational functions over K. In the transcendence degree 1 case, we do this by considering fields F whose elements are equivalence classes of sequences in $K^{\mathbb{N}_0}$ corresponding to rational functions on K with the understanding that we take a representative that agrees with our rational function wherever it is defined. This leads to $K(n)$-recurrent (or holonomic) sequences.

We close this section by explaining where the notation $F[T; T]$ comes from. In [15], Cohn (following Ore's work in [49]) defines the noncommutative polynomial ring (with trivial derivation 0)

$$k[x; \sigma, 0]$$

where k is a field, and σ is an endomorphism on k to be the set of all polynomials in x over k with multiplication given by

$$\left(\sum_j a_j x^j\right)\left(\sum_\ell b_\ell x^\ell\right) = \sum_{j,\ell} a_j \sigma^j(b_\ell) x^{j+\ell} \quad (a_j, b_\ell \in k).$$

By relabeling k as F, and σ as our elementary shift operator T, and removing the "0" from the notation, we arrive at the F-algebra

$$F[x; T],$$

which is isomorphic to our F-algebra of recurrence operators. Since the isomorphism is induced by the substitution $x \mapsto T$, we arrive at our notation $F[T; T]$.

2.2 Divisibility of Recurrence Operators

In order to prove Theorem 1.2, we need to develop a theory of divisibility for recurrence operators over F. We will accomplish this by utilizing the general theory of twisted polynomial rings. For a complete development of this theory, see, e.g., [10], [34, Chapter 1], or [16, Chapter 5]. Here, in order to avoid getting side-tracked, we will restrict ourselves to the special case of $F[T; T]$, where F is an extension field of K as well as a K-subalgebra of $\mathcal{L}_K$ preserved under the action of the elementary shift operator T. As remarked above, restricting the coefficients of our operators to lie in F allows us to obtain the expected additivity of degrees. It also allows us to rule out the existence of zero divisors in $F[T; T]$. This is the content of the following result.

Theorem 2.1. *With the above notation, the following hold.*

(a) *If $f, g \in F[T; T]$ then $\deg(fg) = \deg f + \deg g$.*

(b) *$F[T; T]$ is a domain. That is, $F[T; T]^*$ is closed under multiplication.*

Proof. (a) This is proved by considering the lead coefficient of the product. It is nonzero since F is a field and T is injective.

(b) This follows in the usual way by consideration of degrees. $\qquad\square$

Our next order of business is to study divisibility in $F[T; T]$. The starting point is the following definition.

Definition 2.3. For $f, g \in F[T; T]$ we say that f *left divides* g, written $f \mid_\ell g$, if $fF[T; T] \supseteq gF[T; T]$. We say that f *right divides* g, written $f \mid_r g$, if $F[T; T]f \supseteq F[T; T]g$. Here we call $F[T; T]f$ the *left principal ideal generated by f* and $fF[T; T]$ the *right principal ideal generated by f*.

The reason for defining right division in terms of left ideals and vice-versa is as follows. Having a containment, $F[T; T]f \supseteq F[T; T]g$ ($fF[T; T] \supseteq gF[T; T]$), of left (right) ideals means that $g = hf$ ($g = fh$) for some $h \in F[T; T]$ so that f is a right (left) factor of g. The following theorem collects some properties enjoyed by the ring $F[T; T]$ of recurrence operators.

Theorem 2.2. *With the above notation, the following statements hold.*

(a) $F[T; T]$ *is left euclidean with respect to* deg. *That is, for all $f, g \in F[T; T]^*$, there exist unique $q, r \in F[T; T]$ such that*

$$f = qg + r, \quad \text{where } r = 0 \text{ or } \deg r < \deg g.$$

(b) $F[T; T]$ *is a principal left ideal domain (PLID). That is, every left ideal in $F[T; T]$ is of the form $F[T; T]f$ for some f.*

(c) $F[T; T]$ *is left noetherian. That is, it satisfies the left ascending chain condition.*

(d) $F[T; T]$ *is left ore. That is, for any $f, g \in F[T; T]^*$, we have $F[T; T]f \cap F[T; T]g \neq \{0\}$.*

Proof. (a) Let $f = a_0 + a_1 T + \cdots + a_n T^n$, and $g = b_0 + b_1 T + \cdots + b_m T^m$, where $a_n, b_m \neq 0$. If $n < m$, then we have

$$f = 0g + f,$$

and otherwise, we can use the fact that we have inverses in order to find c such that $f - cT^{n-m}g$ is of degree at most $n - 1$. Indeed, defining c by

$$c = a_n \left(T^{n-m} b_m\right)^{-1}$$

does the trick. As in the classical case, we then obtain existence by induction and uniqueness by consideration of degrees. Therefore $F[T; T]$ is left euclidean.

(b) This follows from Part (a) exactly as in the commutative case by proving that an element of a given left ideal of least degree is a generator for the left ideal.

(c) Similarly to the commutative case, and by the same proof, the left ascending chain condition is equivalent to every left ideal being finitely generated and since every left ideal is principal, this condition is satisfied.

(d) Suppose that f and g are nonzero elements of $F[T;T]$ and consider the chain of left ideals given by

$$F[T;T]f \subseteq F[T;T]f + F[T;T]fg \subseteq F[T;T]f + F[T;T]fg + F[T;T]fg^2 \subseteq \ldots$$

By Part (c), this chain must terminate, so that for some ℓ we have

$$F[T;T]f + \cdots + F[T;T]fg^\ell = F[T;T]f + \cdots + F[T;T]fg^\ell + F[T;T]fg^{\ell+1}.$$

Consequently, we can write

$$fg^{\ell+1} = h_0 f + h_1 fg + \cdots + h_\ell fg^\ell$$

for some operators $h_0, \ldots, h_\ell \in F[T;T]$. Since $fg^{\ell+1} \neq 0$, (see Theorem 2.1, Part (b)), we know that at least one of the h_j is nonzero. Let s, where $0 \leq s \leq \ell$, be the least index such that $h_s \neq 0$. Then

$$fg^{\ell+1} = h_s fg^s + \cdots + h_\ell fg^\ell.$$

Cancelling g^s (on the right), and rearranging, we obtain the operator

$$(fg^{\ell-s} - h_{s+1}f - h_{s+2}fg - \cdots - h_\ell fg^{\ell-s-1})g = h_s f \in (F[T;T]f \cap F[T;T]g)^*$$

as required. □

Properties (a), (b) and (c) listed in Theorem 2.2 are natural analogues of important properties that hold in the commutative case. The ore condition (Property (d)), however, is not. This condition is only nontrivial in the noncommutative case, and is precisely what is needed to define least common left multiples. The definition, along with its dual, now follows.

Definition 2.4. Let f and g be elements of $F[T;T]$. A *greatest common right divisor (gcrd) of f and g*, denoted $(f,g)_r$, is an element d of $F[T;T]$ such that $F[T;T]d = F[T;T]f + F[T;T]g$. A *least common left multiple (lclm) of f and g*, denoted $[f,g]_\ell$, is an element m of $F[T;T]$ such that $F[T;T]m = F[T;T]f \cap F[T;T]g$.

We note that the gcrd and lclm of two recurrence operators in $F[T;T]$ are defined up to left multiplication by units, and satisfy the usual division properties. In the commutative case, we know that the product of a greatest common divisor of two polynomials with a least common multiple is equal to the product of the two polynomials. Consequently, the degrees of the greatest common divisor and least common multiple sum to the degree of the product. In the noncommutative case of $F[T;T]$, this degree condition remains valid.

Proposition 2.1. *Let $f, g \in F[T;T]$. Then $\deg(f,g)_r + \deg[f,g]_\ell = \deg(fg)$.*

Proof. We have the following two chains of containments of F-vector spaces, labelled with the dimensions of the corresponding quotients:

$$
\begin{array}{ccc}
F[T;T] & \qquad\qquad & F[T;T] \\
\Big| \deg f & & \Big| \deg(f,g)_r \\
F[T;T]f & & F[T;T](f,g)_r \\
\Big| a & & \Big| b \\
F[T;T][f,g]_\ell & & F[T;T]g
\end{array}
$$

From the third isomorphism theorem, we conclude that

$$\deg f + a = \deg[f,g]_\ell, \qquad \deg(f,g)_r + b = \deg g.$$

But we can apply the second isomorphism theorem to obtain the isomorphism of F-vector spaces:

$$
\begin{aligned}
F[T;T](f,g)_r/F[T;T]g &= (F[T;T]f + F[T;T]g)/F[T;T]g \\
&\cong F[T;T]f/(F[T;T]f \cap F[T;T]g) \\
&= F[T;T]f/F[T;T][f,g]_\ell,
\end{aligned}
$$

so that $a = b$. Therefore, we can subtract our equations to obtain

$$\deg f - \deg(f,g)_r = \deg[f,g]_\ell - \deg g,$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

For $f \in F[T;T]$ we denote the K-vector space of zeros of f in $\mathcal{L}_K$ by $Z(f)$. Then, $Z(0) = \mathcal{L}_K$ and for $f \in F[T;T]^*$, some properties of $Z(f)$ are given in Theorem 1.2. As remarked above, we restrict our attention to monic operators with nonvanishing constant term. We now have all that is required to restate and prove Theorem 1.2.

Theorem 2.3. *Let the field F be a K-subalgebra of $\mathcal{L}_K$ that is closed under the action of T and $f, g \in F[T; T]$ be monic and have nonzero constant terms. Then the following hold.*

(a) $Z(f)$ *is a K-vector space of dimension* $\deg f$.

(b) $Z([f, g]_\ell) = Z(f) + Z(g)$.

(c) $Z((f, g)_r) = Z(f) \cap Z(g)$.

(d) $Z(f) \subseteq Z(g) \iff f \mid_r g$.

Proof. (a) Let f be given by

$$f(T) = T^k - a_{k-1}T^{k-1} - \cdots - a_1 T - a_0,$$

where $a_0(n)$ is a representative for its class in $\mathcal{L}_K$ that is nonzero for all $n \in \mathbb{N}_0$. Define

$$\mathfrak{B} = \{\{u_n^{(1)}\}_n, \{u_n^{(2)}\}_n, \ldots, \{u_n^{(k)}\}_n\}$$

where for $0 \leq n < k$ and $1 \leq i \leq k$ we have $u_n^{(i)} = \delta_{i-1,n}$, and for $n = r + k \geq k$,

$$u_{r+k}^{(i)} = a_{k-1}(n)u_{r+k-1}^{(i)} + \cdots + a_0(n)u_r^{(i)}.$$

We therefore generate the members of $\mathfrak{B}$ by assigning the most natural initial conditions, and then use the recurrence associated to f to complete the sequences. We claim that $\mathfrak{B}$ is a K-basis for $Z(f)$. By construction, $\mathfrak{B} \subseteq Z(f)$. Suppose that $u \in Z(f)$. We can assume that u satisfies the recurrence corresponding to f for all values of the index by changing at most finitely many initial values. Here we use the fact that $a_0(n)$ is never equal to 0 in order to "run the recurrence backwards." We claim that

$$u = u_0 u^{(1)} + \cdots + u_{k-1} u^{(k)}.$$

Indeed, the two sides agree at each term from the 0-th to the $(k-1)$-th and then the rest of the components are generated by the same recurrence. We are left with showing independence. Towards a contradiction, suppose that $\mathfrak{B}$ is dependent. Then there exist constants $c_1, \ldots, c_k \in K$, not all zero, such that

$$c_1 u^{(1)} + \cdots + c_k u^{(k)} = 0.$$

But then, the sequence u generated by the recurrence associated to f with initial conditions $u_0 = c_1, \ldots, u_{k-1} = c_k$ is eventually zero. Since at least one of the c_j is nonzero, there is a last index $\ell \geq 0$ such that $u_\ell \neq 0$. But then, writing down the recurrence relation involving $u_\ell, \ldots, u_{\ell+k-1}$, the only term that survives is the term involving u_ℓ. Thus

$$a_0(\ell)u_\ell = 0,$$

contrary to the assumption that $a_0(n) \neq 0$ for $n \geq 0$. This completes the proof.

(b), (c) We know that $F[T;T][f,g]_\ell = F[T;T]f \cap F[T;T]g$ and $F[T;T](f,g)_r = F[T;T]f + F[T;T]g$. Therefore, we are reduced to proving that taking a zero set transforms "+" into "$\cap$" and vice-versa. But this follows as expected: the zero set of the *smallest* left ideal *containing* both f and g is equal to the *largest* set *contained in* both $Z(f)$ and $Z(g)$ and vice-versa.

(d) We have the implication

$$f \mid_r g \implies Z(f) \subseteq Z(g).$$

Conversely, suppose $Z(f) \subseteq Z(g)$ and write

$$g = qf + r$$

for $q, r \in F[T;T]$ where $r = 0$ or $\deg r < \deg f$. Then $Z(f) \subseteq Z(r)$ so that, by Part (a),

$$\deg f = \dim_K Z(f) \leq \dim_K Z(r) = \deg r.$$

It follows that $r = 0$ so that $f \mid_r g$, as required. $\square$

2.3   Minimal Operators and Orders

We say that a sequence $\{u_n\}_n \in \mathcal{L}_K$ is F-recurrent if it lies in $Z(f)$ for some nonzero $f \in F[T;T]$. The fact that $F[T;T]$ is a PLID allows us to define minimal operators and orders of F-recurrent sequences in the same way as is done with K-recurrent sequences. Indeed, given an F-recurrent sequence $\{u_n\}_{n=0}^\infty$, the set

$$I(\{u_n\}_n) = \{f \in F[T;T] \mid \{u_n\}_{n=0}^\infty \in Z(f)\} \tag{2.2}$$

is a nontrivial left ideal in $F[T;T]$ which must then be principal. We can therefore define the minimal operator and order of $\{u_n\}_{n=0}^\infty$ over F in the same way we do over K.

Definition 2.5. Let $\{u_n\}_n \in \mathcal{L}_K$ be F-recurrent. The *minimal operator of* $\{u_n\}_n$ *over* F is defined to be the unique monic generator of the left ideal $I(\{u_n\}_n)$ of $F[T; T]$ given by (2.2). The *order of* $\{u_n\}_n$ *over* F, denoted $\operatorname{ord}_F\{u_n\}_n$, is defined to be the degree of the minimal operator of $\{u_n\}_n$. In case $F = K$, so that the minimal operator corresponds to a commutative polynomial, we define the *eigenvalues* of $\{u_n\}_n$ to be the roots of the minimal polynomial of $\{u_n\}_n$.

CHAPTER 3

Zero Terms in Recurrence Sequences

3.1 The Skolem-Mahler-Lech Theorem

It is well-known that a sequence $\{u_n\}_{n=0}^{\infty}$ satisfies a linear recurrence relation with constant coefficients if and only if the ordinary generating function $\sum_{n=0}^{\infty} u_n z^n$ of $\{u_n\}_{n=0}^{\infty}$ represents a rational function. In this section, we are interested in the zero sets of such sequences, that is, the sets of indices n for which $u_n = 0$. Before stating the theorem of Skolem-Mahler-Lech that determines such sets, we illustrate the situation with an example.

Example 3.1. Let $q \in \mathbb{N}$, S be a subset of $\{0, 1, \ldots, q-1\}$ and $\{u_n\}_{n=0}^{\infty}$ be the sequence with ordinary generating function given by

$$\sum_{n=0}^{\infty} u_n z^n = \sum_{\substack{0 \le a < q \\ a \notin S}} \frac{z^a}{1 - z^q} = \sum_{\substack{0 \le a < q \\ a \notin S}} \sum_{n=0}^{\infty} z^{a+qn} = \sum_{\substack{0 \le a < q \\ a \notin S}} \sum_{\substack{n \ge 0 \\ n \equiv_q a}} z^n.$$

Then, for all $n \ge 0$,

$$u_n = \begin{cases} 0 & \text{if } n \equiv_q a \text{ for some } a \in S; \\ 1 & \text{otherwise.} \end{cases} \tag{3.1}$$

Since the ordinary generating function of $\{u_n\}_{n=0}^{\infty}$ is a rational function, $\{u_n\}_{n=0}^{\infty}$ is a sequence that satisfies a linear recurrence relation with constant coefficients. From (3.1) we see that the set of indices n for which $u_n = 0$ is equal to the union

$$\bigcup_{a \in S} (a + q\mathbb{N}_0)$$

of the infinite arithmetic progressions consisting of those nonnegative integers congruent to an element in S modulo q. If we now subtract from the generating function of u a polynomial consisting of finitely many of its terms, we will obtain a set of zero terms in the resulting coefficient sequence that consists of the union of a finite set with finitely many infinite arithmetic progressions.

The theorem of Skolem-Mahler-Lech states that in characteristic zero, no sets of any form other than those obtained in Example 3.1 can be realized as the zero set of a sequence that satisfies a linear recurrence relation with constant coefficients. A short statement of the result appears in Theorem 1.3. An expanded version now follows.

Theorem 3.1 (Skolem 1933; Mahler 1934, 1955; Lech 1953). *Let K be a field of characteristic zero. For all K-recurrent sequences u, there exists a modulus $q \in \mathbb{N}$ such that for all $a \in \mathbb{Z}$ with $0 \le a < q$, $u_{a+qn} = 0$ for all sufficiently large n or $u_{a+qn} \ne 0$ for all sufficiently large n. Further, in case no two distinct eigenvalues of u share a common power, $u_{a+qn} \ne 0$ for all sufficiently large n.*

Here, recall that K-recurrent sequences are the sequences that satisfy a nonzero linear recurrence operator with coefficients in K, and that the eigenvalues of such a K-recurrent sequence consist of the roots of its associated minimal polynomial.

It follows from Theorem 3.1 that the zero set $Z(u) := \{n \in \mathbb{N}_0 \mid u_n = 0\}$ of a K-recurrent sequence u is comprised of a finite set together with a finite number of infinite arithmetic progressions; and, if no two distinct eigenvalues of u share a common power then $Z(u)$ is finite.

We now sketch the proof of Theorem 3.1 in the case K is a number field and provide some remarks on the general case. If u is K-recurrent, then it admits the closed form expression given by Binet's formula:

$$u_n = \sum_{j=1}^{k} P_j(n) \alpha_j^n$$

for all sufficiently large n, and suitable $\alpha_j \in \overline{\mathbb{Q}}$ and $P_j(n) \in \overline{\mathbb{Q}}[n]$. If we define the functions $f_{a,q}$ for $q \in \mathbb{N}$ and $0 \le a < q$ by

$$f_{a,q}(z) = \sum_{j=1}^{k} P_j(a + qz) \alpha_j^{a+qz} = \sum_{j=1}^{k} P_j(a + qz) \alpha_j^a \exp((\log \alpha_j^q)z),$$

we see that $u_{a+qn} = f_{a,q}(n)$ for all sufficiently large n. It is therefore sufficient to prove that for some q, each of the $f_{a,q}$ is either identically zero or has only finitely many zeros in a superset of the integers. The general idea is to find a compact set containing the integers on which the $f_{a,q}$ are analytic, and then refer to the fact that nonzero analytic functions have isolated zeros and, consequently, only finitely many in a compact set. Now, over $\mathbb{C}$, we cannot find a compact set containing the integers since the set of integers is unbounded. However, for primes p, the ring of p-adic integers is such a compact set (with respect to the p-adic absolute value) containing the set of integers. The proof now proceeds by establishing that there exists a modulus q and a prime p such that each of the $f_{a,q}$ can be considered as a p-adic analytic function. We start by stating the relevant definitions and results needed for this purpose.

Definition 3.1. Let p be a prime. The p-adic absolute value $|\cdot|_p$ is defined on $\mathbb{Q}$ as follows. We define $|0|_p = 0$ and for $r \in \mathbb{Q}^*$, such that

$$r = p^\nu \frac{m}{n} \qquad (\nu, m, n \in \mathbb{Z}, \ p \nmid m, n),$$

we define

$$|r|_p = \frac{1}{p^\nu}.$$

The completion $\mathbb{Z}_p$ of $\mathbb{Z}$ with respect to this absolute value is called *the ring of p-adic integers*, denoted $\mathbb{Z}_p$, and its field of quotients is called *the field of p-adic numbers*, denoted $\mathbb{Q}_p$.

We collect together in the following lemma some properties of $\mathbb{Z}_p$ and $\mathbb{Q}_p$ that we require to complete the proof. (For a proof of these results, see, e.g., [31]. For a development also valid for finite extensions of $\mathbb{Q}_p$, see, e.g., [47, Ch. 2]).

Lemma 3.1. *The ring $\mathbb{Z}_p$ satisfies the following properties.*

1. *$\mathbb{Z}_p$ is a local ring with unique maximal ideal $p\mathbb{Z}_p$.*

2. *$\mathbb{Z}_p$ is compact with respect to the p-adic absolute value.*

3. *$\mathbb{Z}_p = \{z \in \mathbb{Q}_p \mid |z|_p \leq 1\}$*

4. *$p\mathbb{Z}_p = \{z \in \mathbb{Q}_p \mid |z|_p < 1\}$*

5. The group $\mathbb{Z}_p^{\times}$ of p-adic units consists of all elements of $\mathbb{Q}_p$ that have absolute value 1.

6. If u is a p-adic unit, then $u^{p-1} \in 1 + p\mathbb{Z}_p$.

7. The exponential and logarithm functions can be defined by the usual series on suitable subsets of $\mathbb{Q}_p$:

$$\exp(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \qquad and \qquad \log(z) = \sum_{m=1}^{\infty} \frac{(-1)^m(z-1)^m}{m}.$$

For each $n > \frac{1}{p-1}$, these define analytic inverse isomorphisms and homeomorphisms

$$\exp := p^n\mathbb{Z}_p \to 1 + p^n\mathbb{Z}_p \qquad and \qquad \log := 1 + p^n\mathbb{Z}_p \to p^n\mathbb{Z}_p.$$

We will also use the following result that can be proved by way of the Chebotarev Density Theorem.

Lemma 3.2. *Let L be a number field, and S be a finite subset of L^*. There exist infinitely many primes $p \in \mathbb{Z}$ such that we have an embedding of L into $\mathbb{Q}_p$ such that the images of the elements of S are all units.*

We will use Lemma 3.1 and Lemma 3.2 as follows to complete the proof of the theorem of Skolem-Mahler-Lech. Let L be a finite field extension of K that contains each of the α_j as well as the coefficients of each of the P_j. By Lemma 3.2 there are infinitely many primes p for which we can embed L into $\mathbb{Q}_p$ and therefore view each of the $f_{a,q}$ as functions on $\mathbb{Q}_p$. In fact, also by Lemma 3.2, we can choose a prime p from the infinitely many such primes in such a way that each of the α_j is a unit in $\mathbb{Z}_p$. By Lemma 3.1, Part (6), we then have

$$\alpha_j^{p-1} \in 1 + p\mathbb{Z}_p.$$

By Part (7) of Lemma 3.1, we have that

$$f_{a,p-1}(z) = \sum_{j=1}^{k} P_j(a + (p-1)z)\alpha_j^a \exp(\log(\alpha_j^{p-1})z)$$

is an analytic function on $\mathbb{Z}_p$. The general idea now is to refer to the compactness of $\mathbb{Z}_p$ to conclude that each $f_{a,p-1}$ is either identically zero or has only finitely many

zeros in $\mathbb{Z}_p$. However, to be rigorous, we should apply Strassmann's Theorem that states that nonzero power series with coefficients in $\mathbb{Z}_p$ that converge to zero have only finitely many zeros in $\mathbb{Z}_p$. Since the $f_{a,p-1}$ converge on $\mathbb{Z}_p$, their coefficients do indeed tend to zero so that Strassmann's Theorem applies. In particular, each $f_{a,p-1}$ is either identically zero or has only finitely many zeros in $\mathbb{Z}_p$. We now complete the proof by establishing that in case $u_n = 0$ for infinitely many n, two distinct eigenvalues of u must share a common power. Suppose then that u has infinitely many zeros. From what we just proved, there exists a prime p such that, for all n sufficiently large,

$$\sum_{j=1}^{k} P_j(a + (p-1)n)\alpha_j^a(\alpha_j^{p-1})^n = 0.$$

Since the $P_j(a + (p-1)z)\alpha_j^a$ are not all zero, and distinct power functions are independent over polynomial rings, we must have i and j for which $\alpha_i^{p-1} = \alpha_j^{p-1}$ as required.

In the general case, the proof goes through as above except that when L is finitely generated, but not necessarily algebraic, one must invoke a theorem of Cassels ([11]) to conclude that Lemma 3.2 still holds and, for arbitrary fields of characteristic zero, one must embed L into a suitable finite extension of $\mathbb{Q}_p$ and use the exponential and logarithm functions defined over this extension. The fact that one may do this follows from a result of Katz ([35, 5.9.3]).

3.2 Generalizations to Nonconstant Coefficients

In [5], Bézivin proves that the theorem of Skolem-Mahler Lech remains valid after replacing the word "finite" with "density zero" for $K(n)$-recurrent sequences, as long as the generating function of the sequence, which is known to satisfy a linear ODE with polynomial coefficients, satisfies such an ODE with respect to which neither 0 nor ∞ are irregular singular points. Specifically, the result is as follows.

Theorem 3.2 (Bézivin, [5]). *Suppose that the generating function of a $K(n)$-recurrent sequence u satisfies a linear ODE with polynomial coefficients with respect to which neither 0 nor ∞ are irregular singular points. Then, there exists $q \in \mathbb{N}$ such that for all $0 \le a < q$ one of the following holds:*

(a) *$u_{a+qn} = 0$ for all sufficiently large n;*

(b) $\{n \mid u_{a+qn} = 0\}$ *has density zero.*

Here and throughout this thesis, "density zero" denotes "asymptotic density equal to 0". Here, for a countable set $\{v_j\} \subseteq \mathbb{Z}$ this density is defined by

$$\lim_{n \to \infty} \frac{\# \left(\{v_j\} \cup \mathbb{N}_n \right)}{\# \mathbb{N}_n}.$$

Finally, we use the term "density zero" in case the above limit exists and converges to 0. Bézivin remarks that he is unaware of any examples that show that the conditions posed on 0 and ∞ are necessary or that we need to replace the word "finite" that appears in the statement of the theorem of Skolem-Mahler-Lech with "density zero."

In [45], Methfessel shows that the same conclusion (with "finite" replaced with "density zero") can be obtained for zeros of linear recurrence operators with coefficients taken from any ring R of sequences that is closed under taking arithmetic progressions and is such that each of its nonzero member sequences takes on the value zero only finitely often. This implies that one can eliminate the technical condition that appears in Theorem 3.2, but not necessarily the existence of a possibly infinite exceptional set of density zero. The conditions Methfessel places on the coefficient ring R are satisfied by the fields F of sequences we are considering as long as we insist on closure under taking arithmetic progressions. All in all, we require the field F to satisfy the following conditions. For all $u \in F$, $a \in \mathbb{Z}$ and $q \in \mathbb{N}$, we have $T(u), u^{(a,q)} \in F$, where, for all a and q, $u^{(a,q)}$ is the subsequence of u formed by the terms having indices congruent to a modulo q:

$$u_n^{(a,q)} = u_{a+qn} \qquad (n \in \mathbb{N}_0). \tag{3.2}$$

The sequences $u^{(a,q)}$ are referred to as *sections of u*. Recall that if $u \in \mathcal{L}_K$ is F-recurrent, then it satisfies a unique monic linear recurrence operator with coefficients in F of least degree, called the minimal operator of u over F. The degree of the minimal operator of u is then the order of u over F, denoted $\mathrm{ord}_F u$. It is equal to the least order of a recurrence over F satisfied by u. We will say that u *can be sectioned to obtain sequences of lower order* if there exists $q \in \mathbb{N}$ such that for all $0 \le a < q$, $\mathrm{ord}_F u^{(a,q)} < \mathrm{ord}_F u$.

Theorem 3.3 (Methfessel, [45]). *Let $u \in \mathcal{L}_K$ be F-recurrent. There exists $q \in \mathbb{N}$ such that for all $0 \le a < q$, $u^{(a,q)}$ is either eventually zero or has zero set of density*

zero. Further, in case u cannot be sectioned to obtain sequences of lower order, each $u^{(a,q)}$ has zero set of density zero. Consequently, the set of zero terms in u is comprised of a set of density zero together with finitely many infinite arithmetic progressions. Further, in case u cannot be sectioned to obtain sequences of lower order, the zero set of u has density zero.

Proof. The main component of the proof is Szemerédi's theorem on arithmetic progressions ([61]). It states that any set of integers of positive density must contain arithmetic progressions of arbitrary length. We will also require the following two lemmas. The first lemma appears in a slightly different form as Theorem 1.5 in Chapter 1.

Lemma 3.3. *If $u \in \mathcal{L}_K$ is F-recurrent, $q \in \mathbb{N}$ and $a \in \mathbb{Z}$, then $u^{(a,q)}$ is F-recurrent and $\mathrm{ord}_F u^{(a,q)} \leq \mathrm{ord}_F u$.*

Proof. Let $f(T)$ be the minimal operator of u over F. We prove that there exists $h \in F[T;T]$ such that

$$h(T)f(T) = g(T^q) \tag{3.3}$$

for some $g \in F[T;T]$ of order k. It will then follow that u is a zero of $g(T^q)$ which implies that $u^{(a,q)}$ is a zero of an operator related to g. Since the order of g is equal to k, we will be able to conclude that $u^{(a,q)}$ is F-recurrent of order at most k as required. Writing

$$h(T) = \sum_{j=0}^{(q-1)k} d_j T^j, \qquad f(T) = \sum_{j=0}^{k} c_j T^j,$$

we find that

$$h(T)f(T) = \sum_{j} \left(\sum_{\ell=0}^{j} d_\ell T^\ell (c_{j-\ell}) \right) T^j.$$

This product will be of the form $g(T^q)$ for an operator g of degree k if and only if the F-linear system consisting of the $(q-1)k$ equations

$$\sum_{\ell=0}^{j} d_\ell T^\ell (c_{j-\ell}) = 0 \qquad (0 \leq j \leq qk, \ j \not\equiv_q 0)$$

in the $(q-1)k + 1$ unknowns

$$d_0, \ldots, d_{(q-1)k} \in F$$

has a nontrivial solution. Since there are fewer equations than unknowns, this is indeed the case. □

Lemma 3.4. *Let u be F-recurrent of order k. If, for some $q \in \mathbb{N}$ and $0 \leq a < q$, we have $u_n^{(a,q)} = 0$ for all sufficiently large n, then, for all $0 \leq a' < q$, we have $\mathrm{ord}_F u^{(a',q)} < k$.*

Proof. This follows similarly to the proof of Lemma 3.3. In this case, we seek an operator $h(T) \in F[T;T]$ such that

$$h(T)f(T) = c_0(n) + T^\alpha g(T^q) \tag{3.4}$$

for $\alpha := \varepsilon q + a' - a$ and $g(T) \in F[T;T]$ of degree at most $k - 1$, where $\varepsilon = 0$ in case $a' \geq a$ and $\varepsilon = 1$ otherwise. If we write out h, f and g as explicit left linear combinations of T and then compare coefficients, we obtain $(q-1)k - q + \alpha$ equations in $(q-1)k - q + \alpha + 1$ unknowns. We therefore obtain a nontrivial solution to this system of equations over F. If we now evaluate both sides of (3.4) at u, we obtain that, for all n sufficiently large and suitable sequences $d_0, d_1, \ldots, d_{k-1}$,

$$c_0(n)u_n + \sum_{j=0}^{k-1} d_j(n)u_{n+\varepsilon q+a'-a+qj} = 0.$$

If we now restrict our attention to indices of the form $a + qn$, and use the fact that $u_{a+qn} = 0$ for all sufficiently large n, we obtain

$$\sum_{j=0}^{k-1} d_j(a + qn)u_{a'+q(\varepsilon+j+n)} = 0.$$

We conclude that $u^{(a',q)}$ is a zero of the operator

$$\sum_{j=0}^{k-1} d_j(a + q(n - \varepsilon))T^j$$

of order at most $k - 1$. □

Armed with these results, we proceed as follows. Assume that u is F-recurrent of order k and the set $Z(u)$ of zero terms in u has positive density. Then, by Szemerédi's Theorem, it must contain a k-term arithmetic progression. That is, there exists $q \in \mathbb{N}$, $N \in \mathbb{N}_0$ and $a \in \mathbb{Z}$ with $0 \leq a < q$ such that

$$u_{a+Nq} = u_{a+(N+1)q} = \cdots = u_{a+(N+k-1)q} = 0. \tag{3.5}$$

By Lemma 3.3, we know that $u^{(a,q)}$ is F-recurrent of order at most k. It follows from (3.5) that $u_{a+nq} = 0$ for all $n \geq N$. We now apply Lemma 3.4 to conclude that each of $u^{(0,q)}, u^{(1,q)}, \ldots, u^{(q-1,q)}$ has order strictly less than k. We have therefore proved the second part of the theorem. Namely, we have shown that if u cannot be sectioned to obtain sequences of lower order, then the set of zero terms in u has density zero. We now complete the proof by showing that in general we obtain a finite union of arithmetic progressions together with a set of density zero. We proceed by contradiction. Suppose then that there does not exist a modulus q for which each $u^{(a,q)}$ is either eventually zero or has a set of zero terms of density zero. The collection of all F-recurrent sequences for which the result fails is then nonempty and the set of orders of such sequences is then a nonempty subset of the natural numbers. By the least integer principle, we have a least such order k and a representative sequence v of that order. By hypothesis, the set of zero terms in v has positive density and so by what we just proved, we can section v to obtain sequences of lower order. But then, since each of these sections has lower order, by minimality, their sets of zero terms are comprised of sets of density zero and finitely many infinite arithmetic progressions. This implies that the set of zero terms in v is of the same form which provides us with the contradiction we were after. As remarked above, this contradiction completes the proof of Theorem 3.3. □

3.3 Nondegeneracy and a Unified Result

In the theorem of Skolem-Mahler-Lech, if we restrict our attention to sequences for which no two distinct eigenvalues share a common power, then we can eliminate the possibility of arithmetic progressions and consequently obtain that there are only finitely many zero terms in the sequence. This condition is referred to as *nondegeneracy*. Further, in the general case, if we restrict to sequences that cannot be sectioned to obtain sequences of lower order, then we can eliminate the possibility of arithmetic progressions and consequently be left with a zero set of density zero. We now show that this latter condition reduces to the former in case of constant coefficients, thereby allowing for a uniform result. The first step is to define degeneracy for general F-recurrent sequences.

Definition 3.2. A sequence $u \in \mathcal{L}_K$ is said to be *F-degenerate with modulus q* if

$$\operatorname{ord}_F u^{(a,q)} < \operatorname{ord}_F u \qquad (0 \le a < q).$$

We are now ready to prove that Definition 3.2 agrees with the usual definition of degeneracy of K-recurrent sequences.

Proposition 3.1. *Let $u \in \mathcal{L}_K$ be K-recurrent of order k and $q \in \mathbb{N}$. The following are equivalent:*

  (a) *Two distinct eigenvalues of u share a common q-th power.*

  (b) *u is K-degenerate with modulus q.*

  (c) *At least one of the $u^{(a,q)}$ is K-recurrent of order at most $k - 1$.*

Proof. Suppose that u is K-recurrent of order k, and that the minimal operator, f, of u has distinct eigenvalues $\alpha_1, \ldots, \alpha_m$. Thus

$$f(x) = \prod_{j=1}^{m} (x - \alpha_j)^{e_j},$$

where the positive integers $e_1, \ldots, e_m$ satisfy

$$\sum_{j=1}^{m} e_j = k.$$

Then, we have constants $c_{i,j} \in \overline{K}$ such that, for all n sufficiently large,

$$u_n = \sum_{j=1}^{m} \sum_{i=0}^{e_j-1} c_{i,j} n^i \alpha_j^n. \tag{3.6}$$

Here, for all j, $c_{e_j-1,j} \neq 0$. This is due to the fact that f is the operator in $K[T]$ of least degree having u as a root. Indeed, if $c_{e_j-1,j} = 0$ for some $1 \le j \le m$, then u would satisfy the operator $g(T) \in K[T]$ given by

$$g(T) = (T - \alpha_j)^{e_j-1} \prod_{1 \le i \neq j \le m} (T - \alpha_i)^{e_i},$$

having degree one less than the degree of f. For the same reason, we see that the α_j are nonzero. From (3.6), we see that, for all n sufficiently large,

$$u_n^{(a,q)} = u_{a+qn} = \sum_{j=1}^{m} \sum_{i=0}^{e_j-1} c_{i,j}(a+qn)^i \alpha_j^{a+qn} = \sum_{j=1}^{m} \sum_{i=0}^{e_j-1} d_{i,j}^{(a,q)} n^i \left(\alpha_j^q\right)^n, \tag{3.7}$$

where the coefficients $d_{i,j}^{(a,q)}$ are determined by the binomial theorem and, in particular, for any j we have

$$d_{e_j-1,j}^{(a,q)} = c_{e_j-1,j}\alpha_j^a q^{e_j-1} \neq 0.$$

Consequently, $u^{(a,q)}$ is a zero of the operator

$$h_q(T) = \prod_{j=1}^{m}(T - \alpha_j^q)^{e_j}. \tag{3.8}$$

Now, since each of the $d_{e_j-1,j}^{(a,q)}$ is nonzero, we see that the following conditions are equivalent.

- The operator $h_q(T)$ given by (3.8) is the minimal operator of $u^{(a,q)}$ for *some* integer a with $0 \leq a < q$.

- The operator $h_q(T)$ given by (3.8) is the minimal operator of $u^{(a,q)}$ for *every* integer a with $0 \leq a < q$.

- The powers α_j^q, for $1 \leq j \leq m$ are distinct.

The result now follows readily from this equivalence together with the fact that the operator h_q given by (3.8) has degree k. $\qquad\square$

Having shown that the sectioning condition is equivalent to degeneracy in the case of constant coefficients, we are now prepared to state a unified result regarding the sets of zero terms in F-recurrent sequences that generalizes the theorem of Skolem-Mahler-Lech.

Proposition 3.2. *Suppose that the field F is a K-subalgebra of $\mathcal{L}_K$ preserved under the action of the elementary shift operator T as well as under taking arithmetic progressions ($u \in F \implies u^{(a,q)} \in F$ for all $a \in \mathbb{Z}$, $q \in \mathbb{N}$). Then, for $u \in F$ of order k, the zero set, $Z(u)$, of u admits the decomposition*

$$Z(u) = S \cup U \cup V$$

where S is the union of finitely many infinite arithmetic progressions, U is a finite set and V is a set of density zero that fails to possess any k-term arithmetic progressions. Further, if u is nondegenerate then $S = \emptyset$, and if $F = K$ then one can take $V = \emptyset$.

In particular, if u is of order 1 or 2, then the conclusion of Skolem-Mahler-Lech holds.

3.4 Sequences Satisfying the Conclusion of Skolem-Mahler-Lech

For the purposes of this section, we will say that a sequence u *has the SML property* if there exists $q \in \mathbb{N}$ such that for all $0 \leq a < q$, $u^{(a,q)}$ is eventually zero or eventually nonzero. The SML property is then, by the theorem of Skolem-Mahler-Lech, a property that is enjoyed by K-recurrent sequences. We now describe some other sequences that are known to have the SML property. The most general result is due to Bézivin and Laohakosol (see [6]). To state it, we first require a definition.

Definition 3.3. Let $s, t, m \in \mathbb{N}$ and $\{\psi_1(z), \ldots, \psi_s(z)\} \subseteq \overline{\mathbb{Q}}[\![z]\!]$. We say that this set is (m,t)-*proper* if there exist infinitely many primes p such that for all embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}_p$, the radii of convergence of each of the $\psi_i(z)$ in $\mathbb{C}_p$ is greater than $p^{-m/t(p-1)}$.

For $m \in \mathbb{N}$, we denote by $J_m(z)$ the series

$$J_m(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!^m}.$$

The first two values of m give rise to well-known functions: For $m = 1$, we have the exponential function, and for $m = 2$ we have a series related to the zeroth Bessel J function. With this notation, we have the following result.

Theorem 3.4 ([6]). *Let $s, t, m \in \mathbb{N}$. If $F(z) = \sum_{n=0}^{\infty} u_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$ is of the form*

$$F(z) = P(z) + \sum_{j=1}^{s} \psi_j(z) J_m(\beta_j z^t), \tag{3.9}$$

where $\{\psi_1(z), \ldots, \psi_s(z)\} \subseteq \overline{\mathbb{Q}}[\![z]\!]$ is (m,t)-proper, $\beta_1, \ldots, \beta_s \in \overline{\mathbb{Q}}^*$, and $P(z) \in \overline{\mathbb{Q}}[z]$, then u has the SML property. Further, in case $s = t = 1$, u has only finitely many zeros.

As a corollary, Bézivin and Laohakosol prove that Theorem 3.4 always holds for power series ψ_j that lie in the algebra generated by the algebraic, binomial and logarithmic series.

Corollary 3.1. *Let E be the algebra of formal power series generated by the algebraic series, series of the form $(1+az)^b$ and series of the form $\log(1+cz)$ where $a, b, c \in \overline{\mathbb{Q}}$. Then all series of the form (3.9) with $\psi_j \in E$ have the SML property.*

Now, we can think of the theorem of Skolem-Mahler-Lech, by way of Binet's formula, as describing the set of integer zeros of exponential polynomials. If we instead consider sequences having generating function that is an exponential polynomial, we obtain functions that satisfy linear ordinary differential equations with constant coefficients. Laohakosol has shown that, in this case, the sequence has the SML property. In fact, he proved the following more general result.

Theorem 3.5 ([41, Theorem 2.1.1]). *Let u have generating function F given by*

$$F(z) = \sum_{j=1}^{s} \psi_j(z) e^{\varphi_j(z)}$$

where the ψ_j and φ_j are polynomials. Then u has the SML property.

A similar result is [6, Corollary 1]. It reads as follows.

Proposition 3.3. *Let $s \in \mathbb{N}$ and ψ_j, φ_j for $1 \leq j \leq s$ be nonzero algebraic power series over $\overline{\mathbb{Q}}$ such that $\mathrm{ord}_{z=0}(\varphi_j(z)) = 1$ for all j. Then the coefficient sequence u of the generating function*

$$F(z) = \sum_{j=1}^{s} \psi_j(z) e^{\varphi_j(z)}$$

has the SML property.

Laohakosol, in [41], along with Bézivin in [6], also prove some results that show that one can replace "density zero" with "finite" in Theorem 3.2 in certain cases. We've already seen that this is the case for linear ODEs with constant coefficients, but it also holds under some more general conditions. The setup is as follows. We are given a sequence u with ordinary generating function $F(z) = \sum_{n=0}^{\infty} u_n z^n$ that satisfies a linear ODE of the form

$$\sum_{j=0}^{m} A_j(z) F^{(j)}(z) = 0,$$

for polynomials $A_0, \ldots, A_m$. Then, if any of the following conditions are satisfied, u has the SML property. In fact, for condition 4 below, we can conclude that the zero set of u is finite.

1. Each of the A_j is constant. ([41, Theorem 4.4.1])

2. Each of the A_j is of degree 1 and a technical condition is satisfied. ([6, Corollary 3])

3. $A_m(0) \neq 0$, $\deg A_m \geq \deg A_j$ for all j, the finite singularities are all regular and the general solution is single-valued. ([41, Theorem 4.4.7])

4. $\deg A_m > \deg A_j + m - j - 1$ for all j, A_m is nonconstant and $\mathrm{ord}_{z=0}(A_m(z)) \leq 1$ and A_m has roots with distinct absolute values. ([41, Theorem 3.4.1])

Laohakosol remarks that the conditions placed on the linear ODE satisfied by the generating function of u in 4 above guarantee that u is of "Poincaré type." We will investigate this type of sequence further in the next chapter in the context of asymptotics.

CHAPTER 4

Asymptotics of Sequences of Poincaré Type

Before we turn, in Chapter 5, to the study of the asymptotics of holonomic sequences, where one can obtain full asymptotic expansions, we provide, in this chapter, an overview of what can be said regarding the asymptotics of recurrence sequences in the most general situation for which meaningful results are known. These sequences are said to be of Poincaré type. We start off, in Section 4.1, with a description of the basic theorems of Poincaré and Perron. The rest of the chapter consists of the description of various generalizations of these results. In Section 4.2 we describe various degree independent generalizations of the theorems of Poincaré and Perron. In that section, in particular, the notion of "fast convergence" is explored. In Section 4.3, we provide the first of two sections on generalizations of the theorems of Poincaré and Perron specific to degree two. In this first section we describe how to reduce the study to that of the difference operator $\Delta^2 - C_n$ where $\Delta = T - 1$ is the standard difference operator and $\{C_n\}_n$ is a suitable sequence. Finally, we close the chapter with Section 4.4 where we describe the specific generalizations of the theorems of Poincaré and Perron due to R. J. Kooman.

4.1   The Theorems of Poincaré and Perron

As remarked in Section 1.3, the asymptotic theory of K-recurrent sequences is complete since we have a closed form expression for such sequences given by Binet's

formula. By Theorem 1.6, every K-recurrent sequence $u \in \mathcal{L}_K$ can be written as

$$u_n = \sum_{j=1}^{m} P_j(n)\alpha_j^n \tag{4.1}$$

for all sufficiently large n and suitable $m \in \mathbb{N}$ and polynomials $P_1(n), \ldots, P_m(n) \in \overline{K}[n]$ where the α_j denote the distinct eigenvalues of u over K. In fact, for $\mathbb{C}$-recurrent operators f, $Z(f)$ admits a standard $\mathbb{C}$-basis given by

$$\mathfrak{B} = \{n^{\rho_j}\alpha_j^n \mid 1 \leq j \leq m, \ 0 \leq \rho_j \leq e_j - 1\}, \tag{4.2}$$

if

$$f(T) = \sum_{j=0}^{k} \pi_j T^j \in \mathbb{C}[T;T], \qquad \pi_k \neq 0,$$

and the polynomial $f(x)$ factors as

$$f(x) = \pi_k \prod_{j=1}^{m}(x - \alpha_j)^{e_j} \tag{4.3}$$

for $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ distinct and $e_1, \ldots, e_m \in \mathbb{N}$ such that $e_1 + \cdots + e_m = k$.

The most general class of linear recurrence operators with nonconstant coefficients for which there are known asymptotic results is the class of linear recurrence operators of Poincaré type. Recall the following notation, set out in Chapter 1. Recurrence operators of Poincaré type have almost-constant coefficients in the sense that they can be written as

$$f(T) = \sum_{j=0}^{k} (\pi_j + \varepsilon_j(n))T^j \in \mathcal{L}_{\mathbb{C}}[T;T], \tag{4.4}$$

where, for all j, $\pi_j \in K$ and $\lim_{n\to\infty} \varepsilon_j(n) = 0$. We refer to the polynomial

$$\chi_f(z) = \sum_{j=0}^{k} \pi_j z^j \in \mathbb{C}[z] \tag{4.5}$$

as the characteristic polynomial of f and call its roots $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ the *eigenvalues* of f. We call an eigenvalue *simple* if it has absolute value distinct from the absolute values of each of the other roots, and call f *simple* in the case that each of its eigenvalues is simple. The starting point for the study of such sequences is the theorem of Poincaré along with that of Perron. In each case, we will motivate the statement of the result in question by illustrating its validity for the subcase of $\mathbb{C}$-recurrent operators.

Suppose that the operator f, given by (4.4), has constant coefficients (so that each $\varepsilon_j(n) = 0$) and is simple. Then, $m = k$ in (4.3) so that the standard $\mathbb{C}$-basis for $Z(f)$, given by (4.2), becomes

$$\mathfrak{B} = \{\alpha_j^n \mid 1 \le j \le k\}. \tag{4.6}$$

Consequently, each eventually nonzero sequence $u \in Z(f)$ satisfies

$$\frac{u_{n+1}}{u_n} = \frac{\sum_{j=1}^k c_j \alpha_j^{n+1}}{\sum_{j=1}^k c_j \alpha_j^n},$$

for constants $c_1, \ldots, c_k \in \mathbb{C}$ not all zero. It follows from this that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \alpha_\ell$$

where α_ℓ is the eigenvalue of greatest absolute value that appears in (4.1) with nonzero coefficient. Poincaré's theorem, proved in 1885, states that this holds, in the case f is simple, even if the $\varepsilon_j(n)$ are nonzero.

Theorem 4.1 (Poincaré, [54]). *Let f, given by (4.4), be simple and $\pi_k \neq 0$. Then, for all eventually nonzero $u \in Z(f)$, we have*

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \alpha_j \qquad \text{for some } 1 \le j \le k,$$

where $\alpha_1, \ldots, \alpha_k$ are the eigenvalues of f.

Now, the standard $\mathbb{C}$-basis given by (4.2) consists of sequences $u^{(1)}, \ldots, u^{(k)} \in Z(f)$ such that $\lim_{n\to\infty} u_{n+1}^{(j)}/u_n^{(j)} = \alpha_j$ for $1 \le j \le k$. Here, we have relabeled the eigenvalues to account for multiplicities. The first theorem of Perron, proved in 1909, shows that this phenomenon holds true in the case f is simple, even when the $\varepsilon_j(n)$ are nonzero.

Theorem 4.2 (Perron, [51]). *Let f, given by (4.4), be simple and $\pi_0, \pi_k \neq 0$. Then there exists a $\mathbb{C}$-basis $\mathfrak{B} = \{u^{(1)}, \ldots, u^{(k)}\}$ for $Z(f)$ such that*

$$\lim_{n\to\infty} \frac{u_{n+1}^{(j)}}{u_n^{(j)}} = \alpha_j \qquad (1 \le j \le k).$$

where $\alpha_1, \ldots, \alpha_k$ are the eigenvalues of f.

We note also that in the standard $\mathbb{C}$-basis given by (4.2), the constituent sequences $u^{(1)}, \ldots, u^{(k)}$ satisfy

$$\lim_{n \to \infty} \sqrt[n]{|u_n^{(j)}|} = \alpha_j \qquad (1 \le j \le k).$$

Here, once again, we have relabeled the eigenvalues to account for multiplicities. The second theorem of Perron, proved in 1921, states that this remains true even when the $\varepsilon_j(n)$ are nonzero, if we replace the limit with a limit superior.

Theorem 4.3 (Perron, [52]). *Let f be given by (4.4) and $\pi_0, \pi_k \ne 0$. Then, there exists a $\mathbb{C}$-basis $\mathfrak{B} = \{u^{(1)}, \ldots, u^{(k)}\}$ for $Z(f)$ such that*

$$\limsup_{n \to \infty} \sqrt[n]{|u_n^{(j)}|} = |\alpha_j| \qquad (1 \le j \le k). \tag{4.7}$$

4.2 Some Generalizations of the Theorems of Poincaré and Perron

There are various generalizations of the theorems of Poincaré and Perron, a few of which we now mention. Using a result of Coffman that appears in [14], Pituk, in [53], proved that, under the same conditions as in Theorem 4.3, (4.7) holds for *every* eventually nonzero solution in $Z(f)$ and not just for the basis elements. In what follows, we will say that two sets X and Y of sequences are *asymptotic* if we can match up the sequences in X with sequences in Y to which they are asymptotic.

In 1958, Evgrafov proved that in case the α_j are distinct, $\pi_0, \pi_k \ne 0$ and

$$\sum_n |\varepsilon_j(n)| < \infty \qquad (0 \le j \le k),$$

there exists a $\mathbb{C}$-basis for $Z(f)$ asymptotic to the standard $\mathbb{C}$-basis for $Z(\chi_f)$. (See [21]). Kooman then extended this result in his PhD thesis as follows.

Theorem 4.4 ([38, Corollary 4.2]). *With the above notation, assume that $\pi_0, \pi_k \ne 0$ and let L be the maximum algebraic multiplicity of the α_j as roots of χ_f. If*

$$\sum_n n^{L-1} |\varepsilon_j(n)| < \infty \qquad (0 \le j \le k) \tag{4.8}$$

then there exists a $\mathbb{C}$-basis for $Z(f)$ asymptotic to the standard $\mathbb{C}$-basis for $Z\left(\chi_f(T)\right)$.

We now provide some examples that illustrate the concept of fast convergence defined by (4.8). These examples are taken from Kooman's thesis ([38]), and they appear as well in the paper [40] that provides an overview of that thesis.

Example 4.1 ([38, Proposition 5.3]). Consider the recurrence

$$u_{n+2} - 2u_{n+1} + \left(1 + \frac{1}{n^2}\right) u_n = 0.$$

The characteristic polynomial is $(z - 1)^2$ and so we obtain one eigenvalue $\alpha = 1$ with algebraic multiplicity 2. Therefore, condition (4.8) cannot hold, for otherwise we'd be able to find a solution asymptotic to 1 and a solution asymptotic to n. The condition (4.8) for fast convergence is that the harmonic series

$$\sum_n \frac{1}{n}$$

converges. As will be shown in Section 4.4, however, one can find a $\mathbb{C}$-basis for the solution space that is asymptotic to $\{n^\alpha, n^\beta\}$ for

$$\alpha = \frac{1 + i\sqrt{3}}{2}, \qquad \beta = \frac{1 - i\sqrt{3}}{2}.$$

Example 4.2 ([38, p. 88]). Consider the recurrence

$$u_{n+2} - \left(1 + \frac{(-1)^n}{n}\right) u_n = 0.$$

If $\{u_n\}_{n=0}^{\infty}$ is a solution for $n \geq 2N$ then

$$u_{n+2} = \begin{cases} \left(1 + \frac{1}{n}\right) u_n & \text{if } n \text{ is even;} \\ \left(1 - \frac{1}{n}\right) u_n & \text{if } n \text{ is odd.} \end{cases} \tag{4.9}$$

Thus, for some nonzero constants λ and μ determined by Stirling's formula, we have

$$u_{2n} = u_{2N} \prod_{j=N}^{n-1} \left(1 + \frac{1}{2j}\right) \sim \lambda\sqrt{n} \to \infty, \quad u_{2n+1} = u_{2N+1} \prod_{j=N}^{n-1} \left(1 - \frac{1}{2j+1}\right) \sim \frac{\mu}{\sqrt{n}} \to 0$$

as $n \to \infty$. It is therefore impossible to find solutions asymptotic to 1 or $(-1)^n$ even though the characteristic polynomial of our recurrence is

$$z^2 - 1 = (z - 1)(z + 1).$$

It follows that condition (4.8) cannot be satisfied. Here the sum occurring in condition (4.8) is again the harmonic series.

Now, the first theorem of Perron (Theorem 4.2) states that if *every* eigenvalue has an absolute value distinct from the others, then *every* eigenvalue is realized as the limit of the quotient of successive values of a zero of the operator in question. Kooman also generalized this result by showing that if *one* eigenvalue has absolute value distinct from the absolute values of the other eigenvalues, then *it* can be realized as the limit of the quotient of successive values of a zero of the operator in question. The precise statement, which is stated in Chapter 1, is as follows.

Proposition 4.1 ([38, Corollary 3.4]). *With the above notation, suppose that $\pi_0, \pi_k \neq 0$. Then, if α is a simple eigenvalue of f, there exists $u \in Z(f)$ such that*

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \alpha.$$

As we have mentioned, there are also some degree specific generalizations of the theorems of Poincaré and Perron. The degree one case can be solved explicitly, and both Coffman and Kooman have some results regarding the degree two case. (See, e.g., [14, §10], [38, Ch. 5,6], [39]). We will explore the degree two case in Sections 4.3 and 4.4.

If we restrict our attention to subclasses of linear recurrence sequences of Poincaré type, much more can be said. In this direction, the theorems of Poincaré and Perron can be generalized in case the $\varepsilon_j(n)$ are chosen to admit full asymptotic expansions (see, e.g., [19], [9]), and in case the $\varepsilon_j(n)$ are rational functions, the theory is essentially complete. Indeed, as will be shown in Chapter 5, holonomic sequences admit full asymptotic expansions of a predictable form (likely without restriction, but at least in case the generating function satisfies a linear ODE with polynomial coefficients with respect to which 0 is not an irregular singular point. See [24, Part B]).

4.3   The Degree Two Case: a Reduction

In this section and Section 4.4, we are interested in the asymptotics of zeros of degree two recurrence operators of Poincaré type. We will therefore restrict our attention to operators $f(T)$ of the form

$$f(T) = T^2 + (\pi_1 + \varepsilon_1(n))T + (\pi_0 + \varepsilon_0(n)) \in \mathcal{L}_{\mathbb{C}}[T; T],$$

where $\pi_0, \pi_1 \in \mathbb{C}$ and the sequences $\{\varepsilon_0(n)\}_{n=0}^{\infty}, \{\varepsilon_1(n)\}_{n=0}^{\infty} \subseteq \mathbb{C}$ are such that $\lim_{n \to \infty} \varepsilon_0(n) = \lim_{n \to \infty} \varepsilon_1(n) = 0$.

We start by noticing that there are only two types of operators that need to be considered. Indeed, if $\pi_1 + \varepsilon_1(n) \neq 0$, then the zero set of our operator is in one-to-one correspondence with the zero set of the operator h given by

$$h(T) = \Delta^2 - \left(1 - \frac{4\pi_0 + 4\varepsilon_0(n)}{(\pi_1 + \varepsilon_1(n))(\pi_1 + \varepsilon_1(n-1))}\right)$$

where $\Delta = T - 1$ is the standard difference operator on $\mathcal{L}_{\mathbb{C}}$. To see this, it is enough to note that the invertible change of variables given by

$$v_n = (-1)^{n-1} u_n \prod_{j=0}^{n-2} \frac{2}{\pi_1 + \varepsilon_1(j)}$$

maps the zero $u \in Z(f)$ to the zero $v \in Z(h)$. It follows that the only two operators we need to consider are the ones of the form $T^2 - C_n$ and $\Delta^2 - C_n$ for sequences $\{C_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$. Now, for the operator $T^2 - C_n$, we can solve for the zero set explicitly by separating out the even terms from the odd terms. Indeed, if

$$u_{n+2} = C_n u_n \qquad (n \geq 0),$$

then we have

$$\frac{u_{2n}}{u_{2n-2}} = C_{2n-2}, \qquad \frac{u_{2n+1}}{u_{2n-1}} = C_{2n-1} \qquad (n \geq 1).$$

We therefore have

$$u_{2n} = u_0 \prod_{j=1}^{n} C_{2j-2}, \qquad u_{2n+1} = u_1 \prod_{j=1}^{n} C_{2j-1} \qquad (n \geq 0).$$

We are therefore reduced to the study of operators of the form $\Delta^2 - C_n$ for sequences $\{C_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$. Coffman, in [14], used this reduction, together with a degree two subcase of Theorem 4.4 in order to obtain a few results that are valid for the case that $\{C_n\}_{n=0}^{\infty}$ is a convergent sequence. The results obtained by Kooman in his thesis and subsequent 2007 paper are more extensive, and will be described below in Section 4.4.

4.4   The Degree Two Case: Kooman's Results

The degree two results of Kooman are all for the case that $C_n \sim \frac{\gamma}{n^a}$ for some nonzero constant γ and $a \in \mathbb{R}$. Although the results are valid under these conditions, we will

motivate the statements of Kooman's results by providing an informal outline of the subcase obtained by assuming the sequence C_n is obtained by restricting a complex function $C(z)$ that is meromorphic at infinity to the set of nonnegative integers. We will also assume that $a = 2$. In this case, we have the differential operator

$$\frac{d^2}{dz^2} - C(z),$$

related to the difference operator $\Delta^2 - C_n$ of interest, and, provided C has at worst a regular singularity at infinity, the asymptotics of solutions to our difference equation will mimic the asymptotics of solutions to the related differential equation. In fact, even in the case of an irregular singularity at infinity, as long as the growth of solutions is sub-exponential, we still obtain the same behaviour for solutions of the discrete problem as we do for solutions of the corresponding continuous problem (see [4, Chapter 5]).

First of all, if we make the substitution $w = 1/z$ so that we can transfer neighbourhoods of infinity to the origin, we obtain

$$\frac{d}{dz} = -w^2 \frac{d}{dw}, \qquad \frac{d^2}{dz^2} = 2w^3 \frac{d}{dw} + w^4 \frac{d^2}{dw^2}.$$

The local behaviour of our differential operator near infinity is then the local behaviour of the operator

$$\frac{d^2}{dw^2} + \frac{2}{w} \frac{d}{dw} - \frac{1}{w^4} C\left(\frac{1}{w}\right)$$

near the origin. It follows from the fact that $C(z)$ has a zero at infinity of order 2 that infinity is a regular singular point of our ODE. Furthermore, looking for a solution in the form of a Frobenius series at infinity, namely

$$z^\lambda \left[1 + \sum_{n=1}^\infty \frac{a_n}{z^n} \right],$$

leads us to the indicial equation given by

$$\lambda^2 - \lambda - \gamma = -\lambda(-\lambda - 1) - 2\lambda - \gamma = 0. \tag{4.10}$$

We let α denote a root of (4.10) having real part greater than or equal to $1/2$. The other root of (4.10) is then $1 - \alpha$. By the Frobenius theory of linear differential

operators with meromorphic coefficients (see, e.g., [13, §4.8]), we conclude that, in case $\alpha \neq 1 - \alpha$, we obtain a basis of zeros $\{w^{(1)}(z), w^{(2)}(z)\}$ for which

$$w^{(1)}(z) \sim z^{\alpha}, \qquad w^{(2)}(z) \sim z^{1-\alpha} \qquad (z \to \infty),$$

while for $\alpha = 1 - \alpha = 1/2$,

$$w^{(1)}(z) \sim z^{1/2}, \qquad w^{(2)}(z) \sim z^{1/2} \log z \qquad (z \to \infty).$$

We should say a little more regarding the lack of a logarithmic term in $w^{(2)}$ when the indicial roots differ by a nonzero integer. The theory tells us to expect a solution of the form $w^{(2)}(z) = w^{(3)}(z) + g \log(z) w^{(1)}(z)$ where $w^{(3)}$ is a solution to the ODE of the form $z^{1-\alpha} \sum_{n=0}^{\infty} \frac{b_n}{z^n}$ and g is a constant. However, upon substitution of $w^{(2)}$ into the ODE, we find that the only way g can be nonzero is to have $\alpha = \frac{1}{2}$. Returning to the discrete situation, we see that we obtain solutions asymptotic to n^{α} and $n^{1-\alpha}$ in the case $\alpha \neq 1 - \alpha$ and obtain solutions asymptotic to $n^{\alpha} = \sqrt{n}$ and $n^{\alpha} \log n = \sqrt{n} \log n$ otherwise. Also, since we are in the meromorphic case, the fact that $C_n = \frac{\gamma}{n^2} + O(\frac{1}{n^3})$ as $n \to \infty$ implies that $(nC_n - \frac{\gamma}{n}) \log n$ is absolutely summable. Kooman showed, in [39], that under this condition, the same behaviour holds for all values of $a \geq 2$, even in the absence of meromorphicity.

Theorem 4.5 ([39]). *With the notation introduced at the beginning of this section, let $a \geq 2$, $\gamma \neq -1/4$ and suppose that $nC_n - \gamma/n$ is absolutely summable. Then there exists a basis $\{u^{(1)}, u^{(2)}\}$ for $Z(f)$ such that*

$$u_n^{(1)} \sim n^{\alpha}, \qquad u_n^{(2)} \sim n^{1-\alpha} \qquad (n \to \infty).$$

Theorem 4.6 ([39]). *With the above notation, let $a = 2$, $\gamma = -1/4$ and suppose that $(nC_n - \gamma/n) \log n$ is absolutely summable. Then there exists a basis $\{u^{(1)}, u^{(2)}\}$ for $Z(f)$ such that*

$$u_n^{(1)} \sim \sqrt{n}, \qquad u_n^{(2)} \sim \sqrt{n} \log n \qquad (n \to \infty).$$

We now make a few remarks related to Theorems 4.5 and 4.6. First of all, Kooman obtained the same conclusions for $a = 2$ in his thesis ([38, Chapter 5]) under a more complicated condition. Also in [38, Chapter 5], Kooman gives conditions under which one can obtain the weaker result that there exist $u^{(1)}, u^{(2)} \in Z(f)$ such that

$$n \left(\frac{u_{n+1}^{(1)}}{u_n^{(1)}} - 1 \right) \to \alpha, \qquad n \left(\frac{u_{n+1}^{(2)}}{u_n^{(2)}} - 1 \right) \to 1 - \alpha.$$

Now, in [39], Kooman shows that one can choose $u^{(1)}$ and $u^{(2)}$ such that

$$\frac{u_{n+1}^{(1)}}{u_n^{(1)}} - 1 = \frac{1}{n}\left(\alpha + o(1)\right), \qquad \frac{u_{n+1}^{(2)}}{u_n^{(2)}} - 1 = \frac{1}{n}\left(1 - \alpha + o(1)\right) \qquad (n \to \infty)$$

in Theorem 4.5 and

$$\frac{u_{n+1}^{(1)}}{u_n^{(1)}} - 1 = \frac{1}{n}\left(\frac{1}{2} + \frac{o(1)}{\log n}\right), \qquad \frac{u_{n+1}^{(2)}}{u_n^{(2)}} - 1 = \frac{1}{n}\left(\frac{1}{2} + \frac{1 + o(1)}{\log n}\right) \qquad (n \to \infty)$$

in Theorem 4.6.

We now close this chapter by turning to the case $a < 2$. In this case, the ODE related to our difference equation has an irregular singularity at infinity and is therefore more difficult to analyze. The result proved by Kooman in [39] that is valid for this situation reads as follows.

Theorem 4.7 ([39] Theorem 1 part 1). *With the above notation, suppose that $a < 2$ and the following two conditions hold:*

1. *$n\left(\frac{C_{n+1}}{C_n} - 1\right) + a$ is absolutely summable,*

2. *If $C < 0$ then the products*
$$\prod_{k=p}^{q}\left|\frac{1 + \sqrt{C_k}}{1 - \sqrt{C_k}}\right|$$
   are all bounded from above or all bounded from below.

Then there exists a basis $\{u^{(1)}, u^{(2)}\}$ for $Z(f)$ such that

$$u_n^{(1)} \sim n^{a/4}\prod_{k=1}^{n-1}\left(1 + \sqrt{C_k}\right), \qquad u_n^{(2)} \sim n^{a/4}\prod_{k=1}^{n-1}\left(1 - \sqrt{C_k}\right),$$

and

$$\frac{u_{n+1}^{(1)}}{u_n^{(1)}} - 1 \sim \sqrt{C_n}, \qquad \frac{u_{n+1}^{(2)}}{u_n^{(2)}} - 1 \sim -\sqrt{C_n}.$$

Here we take the principal branch of the square root.

CHAPTER 5

Asymptotics of Holonomic Sequences

In this chapter we describe the asymptotic theory for $K(n)$-recurrent sequences. When $K = \mathbb{C}$, these sequences are called *P-recursive* or *holonomic*. When working over a field K that is not necessarily equal to $\mathbb{C}$, we will use the terms *P-recursive over K* or *K-holonomic* to distinguish from the classical case. In the setting of holonomic sequences, one can obtain fairly complete asymptotic results. Indeed, for all practical purposes, holonomic sequences admit full asymptotic expansions of a predictable form. The theory developed in this chapter will be used extensively in Chapter 7 where we study a class of binomial sums, and, in particular, prove a conjecture of Chamberland and Dilcher regarding the existence of a full asymptotic expansion for a particular case related to Wolstenholme's Theorem. In Chapter 4, we saw that the asymptotic theory in the constant coefficient case is complete and that the general case of sequences of Poincaré type is very difficult. Holonomic sequences provide us with a reasonable compromise; these sequences are general enough to admit an interesting asymptotic theory, yet specific enough to allow for the use of powerful tools in their study.

The asymptotic theory of holonomic sequences is facilitated by the fact that these sequences have generating functions that satisfy linear ODEs with polynomial coefficients. Using the fact that the asymptotic theory of these types of ODEs is well established, together with the fact that one can generally transfer asymptotic information of generating functions to their coefficient sequences, one can obtain quite complete asymptotic results. In general, the method of Frobenius can be applied to

47

obtain asymptotic expansions of the generating function of the sequence about each of its singularities of least nonzero modulus, and then a transfer method of Flajolet and Sedgewick can be applied to obtain a full asymptotic expansion of the sequence in question. Since every algebraic series satisfies a linear ODE with polynomial coefficients, the coefficient sequences of these types of generating functions generally admit full asymptotic expansions that can be obtained by analyzing a linear ODE as above, but also by expanding the generating function in a Puiseux expansion about its singularities of least nonzero modulus. Another possibility is to use a general bivariate method of Pemantle and Wilson that applies to this case.

We start off the chapter with Section 5.1, where an introduction to holonomic sequences is given. In Section 5.2, we illustrate how to generate fields using holonomic sequences. We then provide in Section 5.3 an overview of some sequences that are known to be non-holonomic. After this, we turn, in Section 5.4, to the transfer method of Flajolet and Sedgewick. In particular, we will describe how to apply this method to obtain full asymptotic expansions of holonomic sequences. We close the chapter with Section 5.5, where we describe the bivariate method of Pemantle and Wilson.

5.1   Properties of Holonomic Sequences

There are two alternate names for $\mathbb{C}(n)$-recurrent sequences found in the literature. These sequences are usually referred to as *polynomially recursive sequences* (*P-recursive sequences* for short) or as *holonomic sequences.* When we are working over a field K that is not necessarily equal to $\mathbb{C}$, we will use the terms polynomially recursive over K (*P*-recursive over K for short) or K-holonomic. Referring to $K(n)$-recurrent sequences as polynomially recursive over K is quite natural, since these sequences satisfy linear recurrence relations with polynomial coefficients in $K[n]$. This language was introduced (for $K = \mathbb{C}$) by Stanley in [57]. In that paper, Stanley also introduced *differentially finite* (*D*-finite for short) generating functions. These generating functions are defined to be those that satisfy linear ordinary differential equations of finite order with polynomial coefficients in $\mathbb{C}[x]$ and are shown to be precisely those generating functions having P-recursive coefficient sequences. Similarly to the above, when we are working over a field K that is not necessarily equal to $\mathbb{C}$, we will use the term differentially finite over K (*D*-finite over K for short).

The more modern language is that of holonomy. Here, the language comes from the algebraic study of differential equations. In this setting, there is a notion of holonomy, defined in terms of the Hilbert polynomial, for finite-dimensional left modules over the Weyl algebra. Generating functions are shown to be D-finite if and only if they lie in such a left module. For this reason, such generating functions, as well as their coefficient sequences, are called holonomic. We will ultimately settle on using the term holonomic (or K-holonomic when K is not necessarily equal to $\mathbb{C}$) to refer to the sequences and generating functions of interest, but first, until the equivalence of language is established, we will provide a brief outline of some basic results using the original terminology.

The first result provides the correspondence between P-recursiveness and D-finiteness mentioned above. For $K = \mathbb{C}$, it can be found in [57], and for general K it can be found in [58, Chapter 6].

Theorem 5.1. *A sequence is P-recursive over K if and only if its generating function is D-finite over K.*

We will also need to know that algebraic series are D-finite. Recall that a generating function $f(x) \in K[\![x]\!]$ is said to be *algebraic* if it is algebraic over the ground field $K(x)$ consisting of rational functions with coefficients in K. Equivalently, the algebraic series f are those that satisfy $P(x, f(x)) = 0$ for some nonzero polynomial $P(x, y) \in K[x, y]$.

Theorem 5.2. *Every algebraic series over K is D-finite over K. In fact, an algebraic series of order d over $K(x)$ satisfies a linear homogeneous ODE of order d over $K(x)$.*

Proof. It is sufficient to verify the second part. To this end, note that for an algebraic series $f(x) \in K[\![x]\!]$, its derivative f' lies in $K(x)(f)$. Therefore $f, f', \ldots, f^{(d)}$ are $d + 1$ elements in the d-dimensional $K(x)$-vector space $K(x)(f)$. These functions must then be linearly dependent so that some nontrivial $K(x)$-linear combination of these functions is equal to zero. $\qquad\square$

In [57] for $K = \mathbb{C}$ and in [58, Chapter 6] for general K, Stanley shows that the set of D-finite power series over K forms a K-subalgebra of $K[\![x]\!]$ that is closed under Hadamard product (term by term multiplication) and algebraic substitution. This

fact provides us with methods to generate new P-recursive sequences from known ones. In contrast, several authors have been interested in finding sequences that fail to be P-recursive. We will discuss this further in Section 5.3.

We now turn to the description of the algebraic definition of holonomy. We will follow the exposition given in [46, Chapter III] which, in turn, is based on [18]. Although we are interested primarily in the univariate case, the bivariate Weyl algebra will come up also in our discussion of the method of Pemantle and Wilson in Section 5.5, and so we will develop the notion of holonomy in the general multivariate setting. We start with the definition of the Weyl algebra over K. In what follows, for variables x, we use the notation ∂_x to denote the partial differential operator $\partial/\partial x$.

Definition 5.1. The Weyl algebra $\mathcal{A}_n$ of dimension n over K is the K-algebra

$$\mathcal{A}_n = K[x_1, \ldots, x_n, \partial_{x_1}, \ldots, \partial_{x_n}]$$

with the usual addition but with multiplication induced by the commutation law

$$[\partial_{x_i}, x_j] = \delta_{ij}$$

where $[a, b] = ab - ba$ denotes the commutator and δ_{ij} is the Kronecker delta equal to 1 if $i = j$ and 0 otherwise.

The Weyl algebra consists of all noncommutative polynomials in $x_1, \ldots, x_n$ and $\partial_{x_1}, \ldots, \partial_{x_n}$ with commutation law given by the commutator. We can identify $\mathcal{A}_n$ with the set of linear differential operators in $x_1, \ldots, x_n$ with polynomial coefficients. Every element of $\mathcal{A}_n$ can be written uniquely in the form

$$\sum_{(\alpha,\beta)\in(\mathbb{N}_0^n)^2} a_{\alpha,\beta} x^\alpha \partial^\beta, \qquad a_{\alpha,\beta} \in K$$

where we have used the standard multivariate notation

$$x^\alpha \partial^\beta = x_1^{\alpha_1} \ldots x_n^{\alpha_n} \partial_{x_1}^{\beta_1} \ldots \partial_{x_n}^{\beta_n}.$$

This is called *standard form.*

We have a natural action of $\mathcal{A}_n$ on the ring $K[\![x_1, \ldots, x_n]\!]$ of formal power series in $x_1, \ldots, x_n$ defined by letting the x_j act by multiplication and the ∂_{x_j} act by differentiation. The collection of operators that annihilate a given formal power series with respect to this action is of particular interest.

Definition 5.2. For $f \in K[\![x_1, \ldots, x_n]\!]$, we define the *annihilating ideal of f*, denoted by I_f, to be the left ideal of $\mathcal{A}_n$ consisting of those differential operators that map f to zero.

The notion of holonomy is related to the degree of the so-called Hilbert polynomial of a finitely generated module over a polynomial ring. In order to define this polynomial, we require the notion of a grading.

Definition 5.3.

(a) Let R be a K-algebra. We say that R is *graded* if there are K-subspaces R_i of R for $i \geq 0$ such that R is the direct sum of the R_i and $R_i \cdot R_j \subseteq R_{i+j}$ for all i and j. In this case, $\{R_i\}_{i \geq 0}$ is called a *grading* of R.

(b) Given a graded K-algebra R with grading $\{R_i\}_{i \geq 0}$ and a left R-module M, we say that M is *graded* if there exist K-subspaces M_i of M for $i \geq 0$ such that M is the direct sum of the M_i and $R_i \cdot M_j \subseteq M_{i+j}$ for all i and j. In this case, $\{M_i\}_{i \geq 0}$ is referred to as a *grading* of M.

The following result ([18, Chapter 9, Theorem 1.1]) allows us to define the Hilbert polynomial.

Theorem 5.3. *Let $M = \bigoplus_{i \geq 0} M_i$ be a finitely generated graded module over the polynomial ring $K[x_1, \ldots, x_n]$ with grading $\{M_i\}_{i \geq 0}$. There exists a polynomial $\chi(t) \in \mathbb{Q}[t]$ such that for all sufficiently large s we have*

$$\sum_{i \leq s} \dim_K M_i = \chi(s).$$

Definition 5.4. The polynomial χ that appears in Theorem 5.3 is referred to as the *Hilbert polynomial of M*.

Now, in the polynomial ring $K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ in commutative variables, we have a natural grading obtained by expressing polynomials as a sum of homogeneous components. That is, if, for $i \geq 0$, we define G_i to be the K-subspace consisting of all homogeneous polynomials of total degree i, we see that

$$K[x_1, \ldots, x_n, y_1, \ldots, y_n] = \bigoplus_{i \geq 0} G_i, \qquad G_i G_j \subseteq G_{i+j} \qquad (i, j \geq 0).$$

One might expect to have an analogous grading on the Weyl algebra $\mathcal{A}_n$, determined by homogeneous noncommutative polynomials in the variables $x_1, \ldots, x_n, \partial_{x_1}, \ldots, \partial_{x_n}$. But without modification, the notion of homogeneity is not well-defined. Indeed, one would expect the $x_j \partial_{x_j}$ to be homogeneous of degree two in $\mathcal{A}_n$. However, we have

$$x_j \partial_{x_j} = \partial_{x_j} x_j + 1,$$

and the right-hand side doesn't appear to be homogeneous at all. To remedy this, one might hope that we can define homogeneity only for standard forms. Even then, we run into trouble since we would expect the product of two homogeneous elements to remain homogeneous (with degree equal to the sum of the degrees of the factors). However, if we consider $x_j \partial_{x_j}$ to be homogeneous of degree two, then, since the standard form of $(x_j \partial_{x_j})(x_j \partial_{x_j})$ is given by $x_j(x_j \partial_{x_j} + 1)\partial_{x_j} = x_j^2 \partial_{x_j}^2 + x_j \partial_{x_j}$, homogeneity is not preserved in the product.

In order to work around this, we start by using the concept of a filtration and then associate a grading to this filtration. We will see that we can obtain a polynomial ring in this fashion using a particular filtration on $\mathcal{A}_n$ and so be able to use the concept of Hilbert polynomials to define holonomy.

Definition 5.5.

(a) Let R be a K-algebra. We say that R is *filtered* if there are K-subspaces F_i of R for $i \geq 0$ such that R is the increasing union of the F_i ($F_i \subseteq F_{i+1}$ for all i) and $F_i \cdot F_j \subseteq F_{i+j}$ for all i and j. In this case, $\{F_i\}_{i \geq 0}$ is called a *filtration* of R.

(b) Given a filtered K-algebra R with filtration $\{F_i\}_{i \geq 0}$ and a left R-module M, we say that M is *filtered* if there exist K-subspaces Γ_i of M for $i \geq 0$ such that M is the increasing union ($\Gamma_i \subseteq \Gamma_{i+1}$ for all i) of the Γ_i and $F_i \cdot \Gamma_j \subseteq \Gamma_{i+j}$ for all i and j. In this case, $\Gamma = \{\Gamma_i\}_{i \geq 0}$ is referred to as a *filtration* of M.

As remarked above, we can associate a grading to a filtration. The construction is as follows. Given a filtration $\mathcal{F} = \{F_i\}_{i \geq 0}$ of a K-algebra R, we define the associated graded algebra

$$gr^{\mathcal{F}} R = \bigoplus_{i \geq 0} F_i / F_{i-1} \tag{5.1}$$

with grading $\{F_i/F_{i-1}\}_{i\geq 0}$. Here, we define $F_{-1} = 0$ and, if $\sigma_i : F_i \to F_i/F_{i-1}$ is the canonical projection, then multiplication is given by $\sigma_i(a)\sigma_j(b) = \sigma_{i+j}(ab)$. Similarly to the construction of $gr^{\mathcal{F}}R$, to each filtered left R-module M with filtration $\Gamma = \{\Gamma_i\}_{i\geq 0}$ we can associate a graded module

$$gr^{\Gamma}M = \bigoplus_{i\geq 0}\Gamma_i/\Gamma_{i-1} \tag{5.2}$$

with grading $\{\Gamma_i/\Gamma_{i-1}\}_{i\geq 0}$. A filtration of M is said to be *good* if $gr^{\Gamma}M$ is finitely generated. This condition will be required in order to be able to define the Hilbert polynomial for left modules M over $\mathcal{A}_n$. To define holonomy, we will be interested in the graded modules associated to filtrations with respect to the Bernstein filtration of $\mathcal{A}_n$. This specific filtration is defined as follows.

Definition 5.6. The *Bernstein filtration* of $\mathcal{A}_n$ is the family $\mathcal{B} = \{B_i\}_{i\geq 0}$ of K-subspaces of $\mathcal{A}_n$ where, for $i \geq 0$, B_i denotes the space of all operators of total degree at most i when written in standard form.

The following result ([18, Chapter 7, Theorem 3.1]) allows us to define the Hilbert polynomial over $gr^{\mathcal{B}}\mathcal{A}_n$.

Theorem 5.4. $gr^{\mathcal{B}}\mathcal{A}_n$ *is isomorphic to the polynomial ring in $2n$ variables.*

We now have all that is required to define the Hilbert dimension of finitely generated left $\mathcal{A}_n$-modules. This dimension is minimized in case of holonomicity.

Definition 5.7. Let M be a finitely generated left $\mathcal{A}_n$-module. Choose a good filtration Γ with respect to the Bernstein filtration. The *Hilbert dimension $d(M)$* of M is the degree of the Hilbert polynomial χ_M of the graded module $gr^{\Gamma}M$ over the polynomial ring $gr^{\mathcal{B}}\mathcal{A}_n$ in $2n$-variables.

This is well-defined (independent of Γ). Also, for t sufficiently large, we have

$$\chi_M(t) = \sum_{i\leq t}\dim_K(\Gamma_i/\Gamma_{i-1}) = \dim_K(\Gamma_t).$$

(See [18, Chapter 9, Section 2]).

By the following result ([18, Chapter 9, Theorem 4.2]), the smallest possible degree for left modules over $\mathcal{A}_n$ is n.

Theorem 5.5 (Bernstein's Inequality)**.** *Let M be a finitely generated left $\mathcal{A}_n$-module. Then either $M = 0$ or $d(M) \geq n$.*

Holonomic modules are the ones that realize this minimum degree.

Definition 5.8. Let M be a finitely generated left $\mathcal{A}_n$-module. Then M is *holonomic over K* if it is either zero or if it has dimension n. A formal power series $f(x_1, \ldots, x_n) \in K[\![x_1, \ldots, x_n]\!]$ is *holonomic over K* if $\mathcal{A}_n/I_f$ is holonomic over K, viewed as a finitely generated left $\mathcal{A}_n$-module.

We have finally arrived at the equivalence of D-finiteness and holonomy for univariate generating functions.

Theorem 5.6. *A formal power series $f(x) \in K[\![x]\!]$ is D-finite over K if and only if it is holonomic over K.*

Proof. If $f(x) \in K[\![x]\!]$ is D-finite, then its annihilating ideal I_f is a nontrivial left ideal of $\mathcal{A}_1$. If $I_f = \mathcal{A}_1$, then $\mathcal{A}_1/I_f = 0$ so that $\mathcal{A}_1/I_f$ is holonomic. On the other hand, if I_f is proper, then $d(\mathcal{A}_1/I_f) = 1$ due to Bernstein's inequality. In any case, we conclude that the $\mathcal{A}_1$-module $\mathcal{A}_1/I_f$ is holonomic. Thus, f is holonomic as required. Conversely, if $f \in K[\![x]\!]$ is holonomic, then $\mathcal{A}_n/I_f$ is holonomic. We must then have $I_f \neq 0$. Indeed, $\mathcal{A}_1$ is not a holonomic $\mathcal{A}_1$-module (its dimension is equal to 2). Therefore f is D-finite. $\qquad\square$

Having established the equivalence of language, we will now refer to the sequences and generating functions of interest as holonomic (or K-holonomic in the case K is not necessarily equal to $\mathbb{C}$).

5.2   Generating Fields with Holonomic Sequences

In [38], Kooman develops the following theory in the case $K = \mathbb{Q}$ endowed with the usual absolute value. Here we outline Kooman's result, and state it in greater generality. Studying his arguments reveals that they can be applied to an arbitrary number field K endowed with an arbitrary absolute value. Let v be a valuation on K so that v extends the p-adic valuation v_p on $\mathbb{Q}$ for p a prime or ∞. We fix an extension w of v to $\overline{\mathbb{Q}}$ and denote the valuations obtained on algebraic extensions of

K by restriction also by w. We are interested in studying all of the elements in the completion K_v that one can obtain as limits (with respect to v) of quotients of zeros of $K(n)$-recurrence operators. That is, for $f \in K(n)[T;T]$, we are interested in the subset Λ_f of K_v given by

$$\Lambda_f = \lim Z(f)Z(f)^{-1} = \left\{ \lim_{n \to \infty} \frac{u_n}{v_n} \mid u, v \in Z(f) \right\}. \tag{5.3}$$

We consider the union of all such sets of numbers as well as the subset that ignores all operators with nonconstant coefficients. These are the sets Λ and Λ_{const} given by

$$\Lambda = \bigcup \{ \Lambda_f \mid f \in K(n)[T;T] \} \qquad \Lambda_{const} = \bigcup \{ \Lambda_f \mid f \in K[T;T] \}. \tag{5.4}$$

Before stating the main result, we provide an example that illustrates how one can obtain transcendental elements in Λ.

Example 5.1. In this example, we show how we can obtain the transcendental elements $\exp(k)$ and $\ln k$ as limits of quotients of zeros of $K(n)$-recurrence operators for suitable k. This will show, in particular, that Λ contains transcendental elements. It turns out, however, that we can only obtain transcendental elements if we use $K(n)$-recurrence operators with nonconstant coefficients. In fact, we will see that Λ_{const} consists precisely of the algebraic elements in Λ. Consider the element $u \in \mathcal{L}_K$ given by

$$u_n = \sum_{j=0}^{n-1} \prod_{i=0}^{j-1} q(i),$$

where $q(z) \in K(z)$. As long as the series converges, the limit will be an element of Λ_f for

$$f(T) = (T - q(n))(T - 1).$$

Indeed, for this f,

$$Z(f) = \left\{ y \in \mathcal{L}_K \mid \frac{y_{n+2} - y_{n+1}}{y_{n+1} - y_n} = q(n) \right\}.$$

We compute

$$\frac{u_{n+2} - u_{n+1}}{u_{n+1} - u_n} = \frac{\sum_{j=0}^{n+1} \prod_{i=0}^{j-1} q(i) - \sum_{j=0}^{n} \prod_{i=0}^{j-1} q(i)}{\sum_{j=0}^{n} \prod_{i=0}^{j-1} q(i) - \sum_{j=0}^{n-1} \prod_{i=0}^{j-1} q(i)} = \frac{\prod_{i=0}^{n} q(i)}{\prod_{i=0}^{n-1} q(i)} = q(n).$$

Thus $u \in Z(f)$. Also, the constant sequence v given by $v_n = 1$ for all n is a member of $Z(f)$. Thus, assuming that $\lim_{n\to\infty} u_n$ exists and is equal to β, we obtain

$$\beta = \lim_{n\to\infty} \frac{u_n}{v_n} \in \Lambda_f.$$

If we choose

$$q(n) = \frac{k}{n+1},$$

we obtain

$$u_n = \sum_{j=0}^{n-1} \prod_{i=0}^{j-1} \frac{k}{i+1} = \sum_{j=0}^{n-1} \frac{k^j}{j!}.$$

Therefore, in this case we obtain the element

$$\beta = \exp(k) \in \Lambda_f \subseteq \Lambda$$

as long as the series converges. There is no issue in the archimedean case and in the nonarchimedean case, we require $v(k) > \frac{e}{p-1}$ if v extends the p-adic absolute value and e is the ramification index of p in $K_v/\mathbb{Q}_p$.

On the other hand, if we choose

$$q(n) = \frac{k-1}{k} \frac{n+1}{n+2},$$

we obtain

$$u_n = \sum_{j=0}^{n-1} \prod_{i=0}^{j-1} \frac{k-1}{k} \frac{i+1}{i+2} = \sum_{j=0}^{n-1} \left(\frac{k-1}{k}\right)^j \frac{1}{j+1}$$

$$= \frac{k}{k-1} \sum_{j=0}^{n-1} \left(\frac{k-1}{k}\right)^{j+1} \frac{1}{j+1} = \frac{k}{k-1} \sum_{j=0}^{n-1} \left(1 - \frac{1}{k}\right)^{j+1} \frac{1}{j+1}.$$

So if the series converges to β we obtain

$$\beta = -\frac{k}{k-1} \ln\left(\frac{1}{k}\right) = \frac{k}{k-1} \ln k.$$

We therefore also obtain a multiple of $\ln k$ in Λ for suitable k.

We now turn to the statement of the main result of this section.

Theorem 5.7. *The following three statements hold:*

1. Λ is a field.

2. $\Lambda_{const} = \overline{\mathbb{Q}} \cap K_v$.

3. $\overline{\mathbb{Q}} \cap K_v \subsetneq \Lambda \subsetneq K_v$.

This tells us in particular that $\overline{\mathbb{Q}} \cap \Lambda = \overline{\mathbb{Q}} \cap K_v = \Lambda_{const}$. That is, we generate the field Λ of all limits of quotients of zeros of $K(n)$-recurrence operators, and the algebraic elements of this field are precisely the elements of the field we obtain by using only the K-recurrence operators. In the non-archimedean case, the equality $\Lambda_{const} = \overline{\mathbb{Q}} \cap K_v$ shows that Λ_{const} is the decomposition field of $(\overline{\mathbb{Q}}, w)/(K, v)$ (the henselization of (K, v), or the minimal extension of K that admits a unique extension of valuations to $\overline{\mathbb{Q}}$) and so is the fixed field of the group of all K-automorphisms of $\overline{\mathbb{Q}}/K$ that are continuous with respect to w. This group is the decomposition group

$$G_w = \left\{ \sigma \in Gal(\overline{\mathbb{Q}}/K) \mid w \circ \sigma = w \right\}. \tag{5.5}$$

In the next section, we survey the literature on nonholonomicity.

5.3 Some Non-holonomic Sequences

We can see from the definition (of D-finiteness) that the following functions are holonomic: exp, sin, cos, the Bessel functions. By Theorem 5.2, we also know that the algebraic functions are holonomic and so, in particular, every rational function is holonomic. By looking at the coefficient sequences, together with Theorem 5.1, the fact that rational functions are holonomic can be seen from the fact that $\mathbb{C}$-recurrent sequences are holonomic. Extracting the coefficient sequences from the above examples of holonomic functions provides us with examples of holonomic sequences. Further, using the closure properties mentioned in Section 5.1, we can generate new examples of holonomic sequences from known holonomic sequences.

In this section, we provide a survey of the present knowledge of non-holonomic sequences and functions. Since every algebraic series is holonomic, these results can be seen as strong transcendence results.

Some General Criteria

We start by listing some criteria that can be used to rule out holonomy for sequences and functions, as well as some examples that are ruled out by the criteria:

1. Holonomic sequences can't grow too fast. Indeed, as is shown in [30], for every holonomic sequence u, there exists a constant α such that $u_n = O(n!^\alpha)$. This is a special case of work of Mahler ([42]). In particular, this rules out from contention the sequences with n-th term given by 2^{2^n} and 2^{n^2}. We note in passing that the sequence $\{n!^\alpha\}_n$ that appears in the above bound was shown in [3] to be holonomic only for integer values of α.

2. Holonomic functions can't have infinitely many singularities, since each singularity must be a root of the leading polynomial that appears in an ODE satisfied by the function. In particular, this rules out the functions

$$\tan z, \qquad \frac{z}{e^z - 1}, \qquad \text{and} \qquad \prod_n (1 - z^n)^{-1}$$

from contention. Looking at the coefficient sequences allows us to rule out Bernoulli numbers and the partition sequence from contention.

3. In [32], it is shown that the reciprocal $1/f$ of a holonomic function f is holonomic if and only if f'/f is algebraic. In [56], it is shown that the the function $e^{\int f}$ is holonomic if and only if f is algebraic. These criteria can be used to rule out the Bernoulli numbers and the Bell numbers with exponential generating functions $z/(e^z - 1)$ and $e^{e^z - 1}$ respectively.

4. For $\mathbb{C}$-recurrent sequences, it is known that the reciprocal sequence is itself $\mathbb{C}$-recurrent if and only if the sequence is an interlacing of geometric sequences. For holonomic sequences a similar result holds: The reciprocal of a holonomic sequence is holonomic if and only if the sequence is an interlacing of hypergeometric sequences. See Chapter 4 of [62] for a proof via difference Galois Theory. This criterion rules out all other reciprocals of holonomic sequences. For instance, if $\alpha, \beta \in \mathbb{C}$ with $|\alpha| > |\beta| > 0$, and $a_n = \alpha^n + \beta^n$, then a_n is holonomic but its reciprocal is not. (See [30]).

Some Algebraic Sequences

There are several other sequences that are known to be non-holonomic. For instance, in [29], Gerhold shows that certain powers of hypergeometric sequences are non-holonomic. The specific statement is given in the following result.

Theorem 5.8. *Let $a_1, \ldots, a_p, b_1, \ldots, b_q$ denote distinct positive integers, where at least one of p, q is positive. Define the sequence v by*

$$v_n = \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \qquad (n \geq 0),$$

where $(\cdot)_n$ denotes the rising factorial given by $(c)_n = c(c+1) \ldots (c+n-1)$ and let $r \in \mathbb{Q} \setminus \mathbb{Z}$. Then $\{v_n^r\}_n$ is non-holonomic.

If we set $p = q = 1$, $a_1 = 2$, $b_1 = 1$, and $r = 1/2$ in Theorem 5.8, we see that the sequence $\{\sqrt{n+1}\}_n$ and consequently the sequence $\{\sqrt{n}\}_n$ is non-holonomic. The result that algebraic *functions* are holonomic therefore does not carry over to *sequences*. The non-holonomicity of $\{\sqrt{n}\}_n$ was generalized first in [22] where it was established that the sequence $\{n^\alpha\}_n$ for $\alpha \in \mathbb{C}$ is holonomic if and only if $\alpha \in \mathbb{Z}$. Then, in [3], it was established that, in fact, for an algebraic function $f(z)$, that is analytic in a neighbourhood of $[1, \infty)$, the sequence $\{f(n)\}_n$ is holonomic if and only if f is a rational function.

Some Further Examples

In [29], the non-holonomicity of the sequence $\{\log n\}_n$ is established under the assumption of a weak form of Schanuel's Conjecture. The dependence on this conjecture was later removed in [22] where the authors provide a proof based on complex analysis. A simple proof of the non-holonomicity of $\{\log n\}_n$ was provided later in [36].

In [29], Gerhold also established the non-holonomicity of the sequence $\{n^n\}_n$ as well as the Lambert W function. In fact, the non-holonomicity of the sequences $\{(a+n)^{bn}\}_n$ was established for all rational numbers a and b with $b \neq 0$. This was later generalized, in [3], to sequences of the form $\{n^{\alpha n}\}_n$ for $\alpha \in \mathbb{C} \setminus \{0\}$.

There are a couple of sequences related to number theory that have been proved to be non-holonomic. First, the non-holonomicity of the sequence of primes was established in [22]. Then, the sequence $\{\zeta(n)\}_n$ where ζ denotes the Riemann Zeta Function was shown to be non-holonomic in [3].

Also found to be non-holonomic in [3] are the sequences of integral values of rational functions in log and arctan, as well as the sequences $\{e^{\sqrt{n}}\}_n$, $\{e^{e^{1/n}}\}_n$. The non-holonomicity of $\{e^{\sqrt{n}}\}_n$ was generalized in [23], a follow-up paper to [22]. In

that paper, the authors established by way of Lindelöf Representations the non-holonomicity of the sequences $\{e^{cn^\theta}\}_n$ for $c, \theta \in \mathbb{R}$ and $c \neq 0$, $\theta \notin \{0, 1\}$.

Several other examples of non-holonomic sequences and functions can be found by asymptotics. Indeed, functions satisfying linear ODEs with polynomial coefficients have a fairly restricted asymptotic form. Any function that does not comply to this asymptotic form must be non-holonomic. More on the asymptotic theory of holonomic sequences and functions will be presented throughout the remainder of this chapter.

5.4 The Transfer Method of Flajolet and Sedgewick

As already remarked on several occasions, K-recurrent sequences admit a closed form expression given by Binet's formula (4.1); such sequences u have n-th term u_n for which

$$u_n = \sum_{j=1}^{m} P_j(n) \alpha_j^n \tag{5.6}$$

for all sufficiently large n, and suitable polynomials $P_j(n)$ defined over $\overline{K}$ where the α_j denote the distinct eigenvalues of u. If we write the polynomial $P_j(n)$ that appears in (5.6) as

$$P_j(n) = a_j \left(n^{d_j} + c_{d_j-1,j} n^{d_j-1} + \cdots + c_{1j} n + c_{0j} \right),$$

for suitable constants $a_j, c_{\ell,j} \in \overline{K}$ ($1 \leq j \leq m$, $0 \leq \ell < d_j$), then we can rewrite (5.6) as

$$u_n = \sum_{j=1}^{m} a_j n^{d_j} \alpha_j^n \left(1 + \sum_{\ell=1}^{d_j} \frac{c_{d_j-\ell,j}}{n^\ell} \right).$$

Since this holds for all sufficiently large n, we see that, in particular, u_n admits a full asymptotic expansion of the form

$$u_n \sim \sum_{\alpha \in S} a_\alpha n^{d_\alpha} \alpha^n \left(1 + \sum_{\ell=1}^{\infty} \frac{c_{\ell,\alpha}}{n^\ell} \right) \qquad (n \to \infty) \tag{5.7}$$

for suitable constants $d_\alpha \in \mathbb{N}_0$, $a_\alpha, c_{\ell,\alpha} \in \overline{K}$ for $\ell \geq 1$ where $c_{\ell,\alpha} = 0$ for all $\ell > d_\alpha$ and S denotes the finite set consisting of the *dominant eigenvalues* of u, where these are the eigenvalues with largest absolute value. Here, in order for this to be well-defined, we must take K to be a valued field.

Now, it is well-known that the generating function of a K-recurrent sequence u having minimal operator $f(T) \in K[T; T]$ can be written in the form $p(z)/f^-(z)$ for

some polynomial p of degree at most $\deg f$ determined by the initial conditions where f^- denotes the reciprocal polynomial of f given by

$$f^-(z) = z^{\deg f} f(1/z). \tag{5.8}$$

Therefore, since the nonzero roots of f^- are the reciprocals of the nonzero roots of f, we see that the dominant eigenvalues of u are the singularities of the generating function of u having least nonzero absolute value. We will refer to these singularities as the *dominant singularities* of the generating function of u. Extending the definition of dominant singularities and eigenvalues in the natural way, the purpose of this section is to show that one can obtain full asymptotic expansions for $K(n)$-recurrent sequences of a similar form to (5.7) when $K \subseteq \mathbb{C}$. We need to restrict to subfields of the field of complex numbers since complex analysis is used to obtain the results. It may be possible to obtain similar results over p-adic fields for primes $p < \infty$, but we will not pursue this here. We follow the work of Flajolet and Sedgewick that appears in [24, Part B].

Let ϕ and R be real numbers with $R > 1$ and $0 < \phi < \pi/2$. The open domain $\Delta(\phi, R)$ is defined as

$$\Delta(\phi, R) = \{z \in \mathbb{C} \mid |z| < R, \ z \neq 1, \ |\mathrm{Arg}(z - 1)| > \phi\}. \tag{5.9}$$

A domain is a Δ-*domain at 1* if it is equal to some $\Delta(\phi, R)$. For general nonzero $\zeta \in \mathbb{C}$, a Δ-*domain at ζ* is defined to be a set of the form $\zeta \Delta_0$ where Δ_0 is a Δ-domain at 1. The following result follows from the theory developed in Chapter VI of [24].

Proposition 5.1. *Suppose that $\zeta_1, \ldots, \zeta_r$ are the dominant singularities of the ordinary generating function F of the sequence $\{u_n\}_n$. Suppose that F is analytic at the origin and that Δ_0 is a Δ-domain at 1 such that F is analytic in the domain*

$$D = \bigcap_{j=1}^{r} (\zeta_j \Delta_0).$$

If, for each j, F admits an expansion of the form

$$F(z) \sim \sum_{k \geq k_j} c_{j,k} (\zeta_j - z)^{\gamma_k} \qquad (z \to \zeta_j, \ z \in D),$$

then

$$u_n \sim \sum_{j=1}^{n} \sum_{k,m} \frac{c_{j,k} e_m(-\gamma_k) \zeta_j^{\gamma_k - n}}{\Gamma(-\gamma_k)} n^{-m-\gamma_k-1} \qquad (n \to \infty),$$

where the $e_m(-\gamma_k) \in \mathbb{Q}(\gamma_k)$.

Proof. From the theory developed in Chapter VI of [24], we obtain

$$u_n \sim \sum_{j=1}^{n} \sum_{k \geq k_j} c_{j,k} \zeta_j^{\gamma_k - n} \binom{n - \gamma_k - 1}{n} \qquad (n \to \infty).$$

Now, from [24, Theorem VI.1, p. 381 and Note VI.3, p. 384], we have

$$\binom{n - \gamma_k - 1}{n} \sim \frac{n^{-\gamma_k - 1}}{\Gamma(-\gamma_k)} \sum_{m=0}^{\infty} \frac{e_m(-\gamma_k)}{n^m} \qquad (n \to \infty)$$

where $e_0(x) = 1$ and $e_j(x) \in \mathbb{Q}[x]$ is of degree $2j$ and divisible by $x(x-1)\ldots(x-j)$. In fact, we have

$$e_m(x) = \sum_{\ell=m}^{2m} \mu_{\ell m}(x-1)(x-2)\ldots(x-\ell),$$

where $\mu_{\ell m} \in \mathbb{Q}$ is the coefficient of $v^m t^\ell$ in the power series expansion of $e^t(1 + vt)^{-1-1/v}$. Putting this together yields

$$u_n \sim \sum_{j=1}^{n} \sum_{k,m} \frac{c_{j,k} \zeta_j^{\gamma_k - n}}{\Gamma(-\gamma_k)} \frac{e_m(-\gamma_k)}{n^{m+\gamma_k+1}} \qquad (n \to \infty).$$

$\square$

Now, for $K(n)$-recurrent sequences, the exponents γ_k that appear in Proposition 5.1 turn out to be of the the form $\gamma_k = k - \theta$ for some θ algebraic over K. We state the corresponding special case of Proposition 5.1 as a corollary.

Corollary 5.1. *With the same notation as in Proposition 5.1, set* $\gamma_k = k - \theta$. *Then*

$$u_n \sim \sum_{j=1}^{r} \frac{c_{j,k_j} n^{\theta-1} \zeta_j^{k_j - \theta - n}}{\Gamma(\theta - k_j)} \left(1 + \sum_{\ell=k_j+1}^{\infty} \frac{\mu_{j,\ell}}{n^\ell} \right)$$

where $\mu_{j,\ell} \in \mathbb{Q}(\theta, \zeta_j, c_{j,k_j+1}/c_{j,k_j}, \ldots, c_{j,\ell}/c_{j,k_j})$ *for each* j *and* ℓ.

Proof. Setting $\gamma_k = k - \theta$ yields

$$u_n \sim \sum_{j=1}^{n} \sum_{k,m} \frac{c_{j,k} e_m(\theta - k) \zeta_j^{k-\theta-n}}{\Gamma(\theta - k)} n^{-m-k+\theta-1} \qquad (n \to \infty)$$

But this sum can be re-written as

$$\sum_{j=1}^{n} n^{\theta-1}\zeta_j^{-\theta-n} \sum_{k,m} \frac{c_{j,k}e_m(\theta-k)\zeta_j^k}{\Gamma(\theta-k)n^{k+m}} = \sum_{j=1}^{n} n^{\theta-1}\zeta_j^{-\theta-n} \sum_{\ell=k_j}^{\infty} \frac{h_{j,\ell}}{n^\ell},$$

where

$$h_{j,\ell} = \sum_{k=k_j}^{\ell} \frac{c_{j,k}\zeta_j^k e_{\ell-k}(\theta-k)}{\Gamma(\theta-k)} \qquad (1 \le j \le r, \ \ell \ge k_j).$$

Taking out the leading terms

$$h_{j,k_j} = \frac{c_{j,k_j}\zeta_j^{k_j}}{\Gamma(\theta-k_j)},$$

and defining

$$\mu_{j,\ell} := \frac{h_{j,\ell}}{h_{j,k_j}} = \sum_{k=k_j}^{\ell} \frac{c_{j,k}\zeta_j^k e_{\ell-k}(\theta-k)}{\Gamma(\theta-k)} \frac{\Gamma(\theta-k_j)}{c_{j,k_j}\zeta_j^{k_j}} \in \mathbb{Q}(\theta,\zeta_j,c_{j,k_j+1}/c_{j,k_j},\ldots,c_{j,\ell}/c_{j,k_j}),$$

we have

$$u_n \sim \sum_{j=1}^{r} \frac{c_{j,k_j}n^{\theta-1}\zeta_j^{k_j-\theta-n}}{\Gamma(\theta-k_j)} \left(1 + \sum_{\ell=k_j+1}^{\infty} \frac{\mu_{j,\ell}}{n^\ell}\right)$$

as required. $\qquad\qquad\square$

We now close this section by describing how to expand $K(n)$-recurrent sequences for subfields K of $\mathbb{C}$ into full asymptotic series using the results above. By Theorem 5.1, we know that the generating function of our sequence of interest is holonomic. If the generating function satisfies a linear ODE with polynomial coefficients with respect to which 0 is a regular singularity and no two indicial roots differ by an integer, then the method of Frobenius (see, e.g., [13, §4.8]) implies that Corollary 5.1 can be applied to obtain the asymptotic expansion we seek for our sequence. In general, even if two indicial roots differ by an integer, we can still obtain asymptotic expansions, but we now require potential logarithmic terms (see [24, § VII.9.1]). Finally, if the singularity is irregular, then one can obtain full asymptotic expansions for the generating function, but it is still unknown whether or not one can transfer the asymptotics to the coefficient sequence. It is expected that one can do so, and it was claimed that the sequence always admits a full asymptotic expansion of the expected form by Birkhoff and Trjitzinsky (see [7, 8]), but this is not accepted as a theorem by experts. (See the remarks following [24, Theorem VIII.7], where the

authors refer to discussions provided by Odlyzko [48, p. 1135–1138], Wimp [64, p. 64], and Wimp-Zeilberger [65] on this question.) If the generating function is algebraic, then we could proceed as above via singularity analysis of a suitable ODE, but also by expanding a suitable algebraic function into a Puiseux expansion. We will then be able to apply Corollary 5.1, with $\theta \in \mathbb{Q}$. An alternative is to apply the bivariate method of Pemantle and Wilson that applies in this situation. This method will be outlined in the next section.

5.5   The Bivariate Method of Pemantle and Wilson

Although the method of Pemantle and Wilson applies in the general multivariate case with meromorphic generating functions, we will restrict ourselves to the bivariate case with rational generating functions as this is all that will be required in the sequel. We will also restrict ourselves to the case $K \subseteq \mathbb{C}$ to accommodate analytic methods. The objects of study are bivariate sequences $\{a_{mn}\}_{m,n}$ having rational generating functions

$$\tilde{F}(z,w) = \sum_{m,n \geq 0} a_{mn} z^n w^m = \frac{G(z,w)}{H(z,w)} \tag{5.10}$$

where G and H are relatively prime polynomials in $\mathbb{C}[z,w]$. Using the multivariate methods developed by Pemantle and Wilson in [50], we can obtain a full asymptotic expansion for such sequences, valid in suitable directions determined by the simple poles of $\tilde{F}$ that are minimal in a sense to be described below. Before stating the relevant results, we need to define the set S_{mn} of points that determine the directions of expansion. First of all, we say that a pole (z_0, w_0) of $\tilde{F}$ (so that $H(z_0, w_0) = 0$) is *minimal* if every pole that lies in the closed bi-disk determined by (z_0, w_0) in fact lies in the torus determined by (z_0, w_0). That is, a pole (z_0, w_0) of $\tilde{F}$ is minimal provided that for all poles (z, w) of $\tilde{F}$, we have

$$|z| \leq |z_0| \text{ and } |w| \leq |w_0| \implies |z| = |z_0| \text{ and } |w| = |w_0|.$$

With

$$Q(z,w) = -w^2 H_w^2 z H_z - w H_w z^2 H_z^2 - w^2 z^2 (H_w^2 H_{zz} + H_z^2 H_{ww} - 2H_z H_w H_{zw}), \tag{5.11}$$

the set S_{mn} is given by

$$S_{mn} = \{z \in \mathbb{C} \mid (z, w(z)) \text{ is a minimal simple pole of } \tilde{F}, \; G(z, w(z)) \neq 0, \quad (5.12)$$
$$mw(z)H_w(z, w(z)) = nzH_z(z, w(z)) \text{ and } Q(z, w(z)) \neq 0\}.$$

The condition $mw(z)H_w(z, w(z)) = nzH_z(z, w(z))$ comes from the requirement that

$$[m, n] = [zH_z(z, w(z)), w(z)H_w(z, w(z))] \in \mathbb{P}^1.$$

This is the direction along which we obtain our asymptotic expansion for $m, n \to \infty$. The first result combines Theorems 3.1, 3.3, and Corollary 3.7 of [50].

Proposition 5.2. *Let $\{a_{mn}\}_{m,n=0}^{\infty}$ denote a bivariate sequence of complex numbers with ordinary generating function $\tilde{F}$ given by*

$$\tilde{F}(z, w) = \sum_{m,n \geq 0} a_{mn} z^n w^m = \frac{G(z, w)}{H(z, w)},$$

for some relatively prime polynomials G and H. Let S_{mn} be defined by (5.12) and suppose that S_{mn} is finite and nonempty. Then there exist constants $c_{\ell}^{(z_{mn})}$ for $\ell \in \mathbb{N}$ and $z_{mn} \in S_{mn}$ such that, with $w_{mn} = w(z_{mn})$,

$$a_{mn} \sim \sum_{z_{mn} \in S_{mn}} L(z_{mn}, w_{mn}) \left(1 + \sum_{\ell=1}^{\infty} \frac{c_{\ell}^{(z_{mn}, w_{mn})}}{m^{\ell}}\right)$$

as $m, n \to \infty$ (with m/n, n/m remaining bounded) for

$$L(z, w) = -\frac{1}{\sqrt{2\pi}} \frac{G(z, w)}{z^n w^{m+1} H_w(z, w)} \sqrt{\frac{-w^3 H_w(z, w)^3}{mQ(z, w)}},$$

where $\sqrt{\cdot}$ denotes the principal branch of the square root and Q is given by (5.11).

We now revisit the Weyl algebra defined in Definition 5.1 in order to derive a simple algebraic formula for the quantity Q defined by (5.11). We know that $\mathcal{A}_2$ has a natural action on $\mathbb{C}[\![z, w]\!]$ defined by letting z and w act by multiplication and ∂_z, ∂_w act by differentiation. The resulting action applies the operator to the given formal power series. We define X_2 to be the free $\mathbb{Z}$-algebra on $\mathcal{A}_2^2$ with product given by

$$[f_1, f_2][g_1, g_2] = [f_1 \times g_1, f_2 \circ g_2], \quad (5.13)$$

where

$$(f \times g)(H) = f(H)g(H) \qquad (f, g \in \mathcal{A}_2, \; H \in \mathbb{C}[\![z, w]\!]) \quad (5.14)$$
$$(f \circ g)(H) = f(g(H)) \qquad (f, g \in \mathcal{A}_2, \; H \in \mathbb{C}[\![z, w]\!]) \quad (5.15)$$

Definition 5.9. Let $T : X_2 \to \mathcal{A}_2$ be defined via

$$T([f, g]) = f \times g \qquad ([f, g] \in X_2). \tag{5.16}$$

Then T is bilinear, and provides us with a way to consider the elements of X_2 as functions from $C[\![z, w]\!]$ to itself:

$$[f, g](H) = T([f, g])(H) = f(H)g(H) \qquad (H \in \mathbb{C}[\![z, w]\!]).$$

Proposition 5.3. *For a variable t, define $\theta_t = t\partial_t$. Then, with the above notation we have*

$$Q(z, w) = ([\theta_z, \theta_w] - [\theta_w, \theta_z])^2(H)(z, w).$$

We can therefore make the identification $Q \equiv ([\theta_z, \theta_w] - [\theta_w, \theta_z])^2 \in X_2$.

We now restrict our attention to a subcase that will be used in Chapter 7. Bivariate sequences $\{a_{mn}\}_{m,n}$ having generating function $\tilde{F}(z, w)$ of the form

$$\tilde{F}(z, w) = \sum_{m,n \geq 0} a_{mn} z^n w^m = \frac{\varphi(z)}{1 - w\nu(z)}$$

for meromorphic functions φ and ν that are analytic at $z = 0$ are called *generalized Riordan arrays* (see, e.g., [63]). Using Proposition 5.2, we can obtain a full asymptotic expansion for such sequences, valid in suitable directions determined by the set S_{mn} defined by (5.12). In [63], Wilson determined the leading terms of an expansion in case there exists one, and showed that if the sequence consists entirely of non-negative numbers, then there is a unique simple pole determining a direction in which we obtain an asymptotic expansion. In the subcase of interest, the set S_{mn} becomes

$$S_{mn} = \{z \in \mathbb{C} \mid (z, \nu(z)^{-1}) \text{ is minimal, } \varphi(z) \neq 0, \tag{5.17}$$
$$mz\nu'(z) = n\nu(z) \text{ and } mz\nu''(z) \neq (n - m)\nu'(z)\}.$$

In the particular case studied in Chapter 7, we will have $\nu(0) \neq 0$ and ν not equal to a polynomial. In order to simplify the statement of the special case of Proposition 5.2 corresponding to generalized Riordan arrays, as well as the relevant results from [63], we will add these hypotheses. The first result combines Proposition 5.2 to obtain the existence of the expansion with [63] to determine the leading terms.

Proposition 5.4. *Let $\{a_{mn}\}_{m,n=0}^{\infty}$ denote a bivariate sequence of complex numbers with ordinary generating function $\tilde{F}$ given by*

$$\tilde{F}(z,w) = \sum_{m,n \geq 0} a_{mn} z^n w^m = \frac{\varphi(z)}{1 - w\nu(z)},$$

for some meromorphic functions φ and ν that are analytic at $z = 0$. Suppose further that ν is not a polynomial and $\nu(0) \neq 0$. Let S_{mn} be defined by (5.17) and suppose that S_{mn} is finite and nonempty. Then there exist constants $c_{\ell}^{(z_{mn})}$ for $\ell \in \mathbb{N}$ and $z_{mn} \in S_{mn}$ such that

$$a_{mn} \sim \sum_{z_{mn} \in S_{mn}} \frac{\varphi(z_{mn})\nu(z_{mn})^m}{z_{mn}^n \sqrt{2\pi m Q_{mn}(z_{mn})}} \left(1 + \sum_{\ell=1}^{\infty} \frac{c_{\ell}^{(z_{mn})}}{m^{\ell}}\right)$$

as $m,n \to \infty$ (with m/n, n/m remaining bounded), where $\sqrt{\cdot}$ denotes the principal branch of the square root and

$$Q_{mn}(z) = \frac{z^2 \nu''(z)}{\nu(z)} - \frac{n(n-m)}{m^2}.$$

In [63], Wilson shows that in case $a_{mn} \geq 0$ for all m and n, S_{mn} is a singleton, consisting of a single positive real number less than the radius of convergence ρ of ν. To close this chapter, we provide the resulting corollary of Proposition 5.4.

Corollary 5.2. *With notation as in Proposition 5.4, let $\rho > 0$ denote the radius of convergence of ν and suppose further that ν is not a polynomial and $\nu(0) \neq 0$. Let S_{mn} be defined by (5.12). Then $S_{mn} = \{x_{mn}\}$ for some $0 < x_{mn} < \rho$ and there exist constants $c_{\ell}^{(m,n)}$ for $\ell \in \mathbb{N}$ such that*

$$a_{mn} \sim \frac{\varphi(x_{mn})\nu(x_{mn})^m}{x_{mn}^n \sqrt{2\pi m Q_{mn}(x_{mn})}} \left(1 + \sum_{\ell=1}^{\infty} \frac{c_{\ell}^{(m,n)}}{m^{\ell}}\right)$$

as $m,n \to \infty$ (with m/n, n/m remaining bounded), where $\sqrt{\cdot}$ denotes the principal branch of the square root and

$$Q_{mn}(z) = \frac{z^2 \nu''(z)}{\nu(z)} - \frac{n(n-m)}{m^2}.$$

CHAPTER 6

Properties of the Asymptotic Parameters

Let K be a number field. We have seen that many K-holonomic sequences admit full asymptotic expansions of the form

$$g_n \sim \gamma \rho^n n^{-\varphi} \sum_{m=0}^{\infty} \frac{a_m}{n^{m/q}} \qquad (n \to \infty) \tag{6.1}$$

for some constants $\gamma, \rho, \varphi, a_0, a_1, a_2, \cdots \in \mathbb{C}$ and $q \in \mathbb{N}$ with $a_0 = 1$. The purpose of this chapter is to investigate what can be said regarding the parameters involved in this expansion. Both ρ and φ are algebraic numbers, and given a linear recurrence operator satisfied by $\{g_n\}_n$, it is possible in both cases to write down an explicit polynomial that is satisfied by the parameter. For ρ, the polynomial will be the characteristic polynomial of the recurrence, and for φ, the polynomial will be related to the indicial equation of a particular differential equation. The former polynomial will have coefficients in K and the latter will have coefficients in $K(\rho)$. We obtain the properties mentioned above regarding ρ and φ, in passing, as we generalize and make explicit the method of Stoll and Haible from [59], that uses an auxiliary function B, the coefficients of which determine the a_m. In Section 6.1 we generalize the method of Stoll and Haible. We then turn to making the method explicit in Section 6.2 where we derive an ODE satisfied by a function related to B. The hope is that this ODE will provide enough information regarding the coefficients of B to obtain meaningful results regarding the original asymptotic coefficients. We then turn, in Section 6.3 to a situation where enough information can be obtained. In that section we consider the case where the ODE can be solved explicitly.

6.1 A Generalization of the Method of Stoll and Haible

The Transformation

Fix $\varphi \in \mathbb{C}$ and $q \in \mathbb{N}$. Let $\mathcal{F}$ denote the $\mathbb{C}$-vector space of all generating functions $F(x) = \sum_{n=0}^{\infty} f_n x^n \in \mathbb{C}[\![x]\!]$ such that f_n admits a full asymptotic expansion of the form

$$f_n \sim n^{-\varphi} \sum_{m=N}^{\infty} \frac{a_m}{n^{m/q}} \qquad (n \to \infty) \tag{6.2}$$

for some integer N and sequence $\{a_m\}_{m \geq N} \subseteq \mathbb{C}$. With this notation, and denoting the space of all finite-tailed Laurent series by the usual $\mathbb{C}(\!(x)\!)$, we define a $\mathbb{C}$-linear transformation $\Psi : \mathcal{F} \to x^{\varphi}\mathbb{C}(\!(x^{1/q})\!)$ as follows. Given $F \in \mathcal{F}$ with coefficient sequence $\{f_n\}_{n \geq 0}$ satisfying (6.2), we set

$$\Psi(F) = \sum_{k=N}^{\infty} \frac{a_k}{\Gamma(\varphi + k/q)} \log(1 + x)^{\varphi + k/q - 1} \in x^{\varphi}\mathbb{C}(\!(x^{1/q})\!).$$

Here we are considering division by Γ as being defined to be multiplication by the entire function $1/\Gamma$. Our transformation Ψ is therefore well-defined. As is shown in [59, Theorem 2] for $\varphi = 0$, we have the following result.

Proposition 6.1. *With the above notation, the linear transformation Ψ satisfies the following properties.*

(a) $\Psi(xF(x)) = (x + 1)\Psi(F(x))$.

(b) $\Psi\left(\frac{d}{dx}F(x)\right) = \frac{d}{dx}\Psi(F(x))$.

(c) *If F is a polynomial then $\Psi(F(x)) = 0$.*

Proof. Let $F(x)$ have coefficient sequence $\{f_n\}_{n \geq 0}$ that satisfies (6.2).

(a) Defining $f_{-1} := 0$, the generating function $xF(x)$ has coefficient sequence $\{f_{n-1}\}_n$ satisfying

$$f_{n-1} \sim (n - 1)^{-\varphi} \sum_{m=N}^{\infty} \frac{a_m}{(n - 1)^{m/q}} \qquad (n \to \infty).$$

We now re-write this asymptotic series in terms of n rather than $n-1$. We find that

$$(n-1)^{-\varphi} \sum_{m=N}^{\infty} \frac{a_m}{(n-1)^{m/q}} = \sum_{m=N}^{\infty} a_m n^{-\varphi-m/q} \left(1-\frac{1}{n}\right)^{-\varphi-m/q}$$

$$= \sum_{m=N}^{\infty} a_m n^{-\varphi-m/q} \sum_{j=0}^{\infty} \binom{-\varphi-m/q}{j} (-1)^j n^{-j}$$

$$= n^{-\varphi} \sum_{m=N}^{\infty} \sum_{j=0}^{\lfloor \frac{m-N}{q} \rfloor} a_{m-qj} \binom{\varphi+m/q-1}{j} n^{-m/q}.$$

Therefore

$$\Psi(xF(x)) = \sum_{k=N}^{\infty} \sum_{j=0}^{\lfloor \frac{k-N}{q} \rfloor} a_{k-qj} \binom{\varphi+k/q-1}{j} \frac{1}{\Gamma(\varphi+k/q)} \log(1+x)^{\varphi+k/q-1}.$$

On the other hand, we have

$$(x+1)\Psi(F(x)) = \exp(\log(1+x))\Psi(F(x))$$

$$= \left(\sum_{k=0}^{\infty} \frac{\log(1+x)^k}{k!}\right) \left(\sum_{k=N}^{\infty} \frac{a_k}{\Gamma(\varphi+k/q)} \log(1+x)^{\varphi+k/q-1}\right)$$

$$= \sum_{k=N}^{\infty} \sum_{j=0}^{\lfloor \frac{k-N}{q} \rfloor} \frac{a_{k-qj}}{j!\Gamma(\varphi+k/q-j)} \log(1+x)^{\varphi+k/q-1}.$$

The proof is then completed by noticing that

$$\binom{\varphi+k/q-1}{j} \frac{1}{\Gamma(\varphi+k/q)} = \frac{1}{j!\Gamma(\varphi+k/q-j)}.$$

(b) From (a), it is sufficient to verify that $\Psi\left(x\frac{d}{dx}F(x)\right) = (x+1)\frac{d}{dx}\Psi(F(x))$. To this end, we start by noticing that the coefficient sequence of $x\frac{d}{dx}F(x)$ is $\{nf_n\}_n$, having asymptotic expansion

$$nf_n \sim n^{-\varphi} \sum_{m=N-q}^{\infty} \frac{a_{m+q}}{n^{m/q}} \qquad (n \to \infty).$$

Therefore

$$\Psi\left(x\frac{d}{dx}F(x)\right) = \sum_{k=N-q}^{\infty} \frac{a_{k+q}}{\Gamma(\varphi+k/q)} \log(1+x)^{\varphi+k/q-1}.$$

On the other hand, we have

$$(x+1)\frac{d}{dx}\Psi(F(x)) = (x+1) \sum_{k=N}^{\infty} \frac{a_k(\varphi+k/q-1)}{\Gamma(\varphi+k/q)} \frac{\log(1+x)^{\varphi+k/q-2}}{1+x}$$

$$= \sum_{k=N-q}^{\infty} \frac{a_{k+q}}{\Gamma(\varphi+k/q)} \log(1+x)^{\varphi+k/q-1},$$

where for the last equality we used the basic identity $\Gamma(z+1) = z\Gamma(z)$.

(c) If F is a polynomial, then the sequence $\{f_n\}$ eventually consists of all zero terms and so each of the a_m is equal to zero. $\qquad\square$

By induction we can conclude from Proposition 6.1 that if $F(x)$ is such that $L_x(F)$ is a polynomial for some linear differential operator L_x with polynomial coefficients, then $\Psi(F(x))$ satisfies the linear differential operator L_{x+1}.

A Generalization of the Main Result of Stoll and Haible

In order to state the results, we will need to make clear what is meant by prime divisors and the denominator of an element lying in a number field. So let K be a number field with ring of integers $\mathcal{O}_K$ and $\delta \in K$. We have

$$\delta\mathcal{O}_K = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(\delta)}$$

where the product is over all nonzero prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ (the primes of K) and the uniquely determined exponents $v_{\mathfrak{p}}(\delta)$ are integers, all but finitely many of which are equal to zero. The prime divisors of δ are the primes $\mathfrak{p}$ of K for which $v_{\mathfrak{p}}(\delta) > 0$ and by the denominator of δ, we mean the product of $\mathfrak{p}^{-v_{\mathfrak{p}}(\delta)}$ over all primes $\mathfrak{p}$ of K for which $v_{\mathfrak{p}}(\delta) < 0$.

Lemma 6.1. *Let $r \in \mathbb{Q}$, K be a number field and $\mathfrak{p}$ be a prime of K. If $v_{\mathfrak{p}}(r) \geq 0$ then $v_{\mathfrak{p}}(\binom{r}{n}) \geq 0$ for all $n \in \mathbb{N}_0$.*

Proof. Let p be the prime lying below $\mathfrak{p}$. Since $v_{\mathfrak{p}}(r) \geq 0$, we have also $v_p(r) \geq 0$. Consequently, r is a p-adic integer. It follows that for any $n \in \mathbb{N}_0$, $\binom{r}{n}$ is also a p-adic integer (see, e.g., [31, Lemma 4.3.9]). Since $v_p(\binom{r}{n}) \geq 0$, we conclude that $v_{\mathfrak{p}}(\binom{r}{n}) \geq 0$ as well. $\qquad\square$

The following lemma is a generalization of [59, Lemma 4] and follows from an entirely analogous argument.

Lemma 6.2. *Let $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{<0}$ and K be a number field. Then*

$$(e^x - 1)^\alpha = \sum_{n=0}^{\infty} \frac{s_n(\alpha)}{(\alpha+1)\dots(\alpha+n)} x^{\alpha+n}$$

with

$$s_n(x) = \sum_{0 \le k \le m \le n} \binom{x+n}{m+n}\binom{m+n}{k+n}(-1)^{m-k}S(n+k,k) \in \mathbb{Q}[x],$$

where $S(a,b)$ denotes the appropriate Stirling number of the second kind, and for all primes $\mathfrak{p}$ of K we have

$$v_{\mathfrak{p}}(s_n(\alpha)) \ge \begin{cases} 0 & \text{if } v_{\mathfrak{p}}(\alpha) \ge 0; \\ 2nv_{\mathfrak{p}}(\alpha) - v_{\mathfrak{p}}((2n)!) & \text{if } v_{\mathfrak{p}}(\alpha) < 0. \end{cases}$$

We note from the definition of Ψ that

$$\Psi(F(x)) = \frac{x^{\varphi+N/q-1}}{\Gamma(\varphi+N/q)}\sum_{n=0}^{\infty}b_n x^{n/q} \tag{6.3}$$

where, in particular,

$$b_\ell = \frac{\Gamma(\varphi+N/q)}{\Gamma(\varphi+(N+\ell)/q)}a_{\ell+N} \qquad (0 \le \ell < q). \tag{6.4}$$

Here we make the assumption

$$\varphi + \frac{N+\ell}{q} \notin \mathbb{Z}_{\le 0} \qquad (0 \le \ell < q) \tag{6.5}$$

in order to have well-defined and nonzero quotients appearing in (6.4).

The following result is based on [59, Corollary 5].

Proposition 6.2. *With the above notation, suppose that $\varphi \in \mathbb{Q}$, K is a number field and $0 \le \ell < q$. If $b_\ell \ne 0$ then $a_{N+\ell} \ne 0$. In this case, if $b_{qn+\ell}/b_\ell \in K$ for all n then $a_{qk+N+\ell}/a_{N+\ell} \in K$ for all k and we have for primes $\mathfrak{p}$ of K and $k \ge 0$,*

(a) *if $v_{\mathfrak{p}}(\varphi + N/q + \ell/q) \ge 0$ then*

$$v_{\mathfrak{p}}(a_{kq+N+\ell}/a_{N+\ell}) \ge \min\{v_{\mathfrak{p}}(n!b_{nq+\ell}/b_\ell) \mid 0 \le n \le k\};$$

(b) *if $v_{\mathfrak{p}}(\varphi + N/q + \ell/q) < 0$ then*

$$v_{\mathfrak{p}}(a_{kq+N+\ell}/a_{N+\ell}) \ge \min\{v_{\mathfrak{p}}(b_{nq+\ell}/b_\ell) + (2k-n)v_{\mathfrak{p}}(\varphi + N/q + \ell/q)$$

$$-v_{\mathfrak{p}}((2k-2n)!) \mid 0 \le n \le k\}.$$

We note in particular that for a given ℓ such that $0 \leq \ell < q$, the only primes of K that can divide the denominator of the coefficients $a_{kq+N+\ell}/a_{N+\ell}$ are the primes that divide the denominator of $\varphi + N/q + \ell/q$ and the primes that divide the denominator of $n! b_{nq+\ell}/b_\ell$ for some $0 \leq n \leq k$.

Proof of Proposition 6.2. Define $B(x) = \Psi(F(x))$ and $A(x) = B(e^x - 1)$. Then

$$A(x) = \sum_{k=0}^{\infty} \frac{a_{k+N}}{\Gamma(\varphi + k/q + N/q)} x^{\varphi+k/q+N/q-1}. \tag{6.6}$$

By (6.3) we have

$$A(x) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(\varphi + N/q)} (e^x - 1)^{n/q+\varphi+N/q-1}.$$

By (6.5) we can apply Lemma 6.2 to obtain the following expression for $A(x)$:

$$\sum_{n=0}^{\infty} \frac{b_n}{\Gamma(\varphi + N/q)} \sum_{m=0}^{\infty} \frac{s_m(n/q + \varphi + N/q - 1) x^{n/q+\varphi+N/q-1+m}}{(n/q + \varphi + N/q) \ldots (n/q + \varphi + N/q - 1 + m)}. \tag{6.7}$$

Comparing the coefficient of $x^{\varphi+k/q+N/q-1}$ in (6.6) with that of (6.7) we find that the quotient $\frac{a_{k+N}}{\Gamma(\varphi+k/q+N/q)}$ is given by

$$\sum_{n+mq=k} \frac{b_n}{\Gamma(\varphi + N/q)} \frac{s_m(n/q + \varphi + N/q - 1)}{(n/q + \varphi + N/q) \ldots (n/q + \varphi + N/q - 1 + m)}.$$

Thus, for $k \geq 0$,

$$a_{k+N} = \sum_{n+mq=k} \frac{\Gamma(\varphi + N/q + n/q)}{\Gamma(\varphi + N/q)} b_n s_m(\varphi + N/q + n/q - 1)$$

$$= \sum_{m=0}^{\lfloor k/q \rfloor} \frac{\Gamma(\varphi + N/q + k/q - m)}{\Gamma(\varphi + N/q)} b_{k-mq} s_m(\varphi + N/q + k/q - m - 1).$$

Replacing k with $kq + \ell$ we see that the quotient $a_{kq+\ell+N}/a_{\ell+N}$ is given by

$$\frac{\Gamma(\varphi + N/q)}{\Gamma(\varphi + N/q + \ell/q) b_\ell} \sum_{m=0}^{k} \frac{\Gamma(\varphi + N/q + k + \ell/q - m)}{\Gamma(\varphi + N/q)}$$

$$\times b_{(k-m)q+\ell} s_m(\varphi + N/q + k + \ell/q - m - 1)$$

$$= \sum_{m=0}^{k} \frac{\Gamma(\varphi + N/q + \ell/q + m)}{\Gamma(\varphi + N/q + \ell/q)} \frac{b_{mq+\ell}}{b_\ell} s_{k-m}(\varphi + N/q + \ell/q + m - 1)$$

$$= \sum_{m=0}^{k} \binom{\varphi + N/q + \ell/q + m - 1}{m} m! \frac{b_{mq+\ell}}{b_\ell} s_{k-m}(\varphi + N/q + \ell/q + m - 1).$$

The proof now follows from Lemmas 6.1 and 6.2. □

Applying the Main Result

We start with a number field K and a sequence $\{g_n\}_n \subseteq K$ that admits an asymptotic expansion of the form

$$g_n \sim \gamma \rho^n n^{-\varphi} \sum_{m=0}^{\infty} \frac{a_m}{n^{m/q}} \qquad (n \to \infty) \tag{6.8}$$

for some constants $\gamma, \rho, \varphi, a_0, a_1, a_2, \cdots \in \mathbb{C}$ and $q \in \mathbb{N}$ with $a_0 = 1$. As we have seen, many holonomic sequences are of this sort. We then define

$$f_n = \frac{g_n}{\gamma \rho^n} \sim n^{-\varphi} \sum_{m=0}^{\infty} \frac{a_m}{n^{m/q}} \qquad (n \to \infty)$$

and let $F(x)$ be the ordinary generating function of $\{f_n\}_n$. With the above notation, we then define

$$B(x) := \Psi(F(x)) = \frac{x^{\varphi-1}}{\Gamma(\varphi)} \sum_{n=0}^{\infty} b_n x^{n/q},$$

for some constants b_n ($n \geq 0$). By Proposition 6.2, if $\varphi + \ell/q \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ for all $0 \leq \ell < q$, $b_0, b_1, b_2, \cdots \in K$, and we understand the divisibility properties of the b_n, then we can obtain information regarding the divisibility properties of our original asymptotic coefficients a_m ($m \geq 0$).

When $\{g_n\}_n$ is holonomic, $\{f_n\}_n$ is as well, and so $F(x)$ satisfies a linear ODE with polynomial coefficients. We then know that $B(x)$ satisfies the ODE obtained by replacing x with $x+1$. The hope is that one can use this differential equation to obtain enough information regarding B and its coefficient sequence to obtain meaningful divisibility properties for the original asymptotic coefficient sequence. The following example illustrates the simplest case in which we can solve for the b_n explicitly. Taking $a = 4$, $b = -2$, $c = 1$, $d = 0$ gives the case of the central binomial coefficients considered in [59, §5.2]. In Section 6.3 we will provide the general case where one can solve for B explicitly.

Example 6.1. Let K be a number field and λ, a, b, c, $d \in K$ with $ac\lambda \neq 0$ and $d/c - b/a \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Define

$$g_n = \lambda \prod_{j=1}^{n} \frac{aj + b}{cj + d} = \lambda \frac{\Gamma(1 + d/c)a^n \Gamma(n + 1 + b/a)}{\Gamma(1 + b/a)c^n \Gamma(n + 1 + d/c)} \in K.$$

Stirling's formula can be used to obtain a sequence $\{a_m\}_m$ such that

$$g_n \sim \frac{\lambda\Gamma(1+d/c)}{\Gamma(1+b/a)}\left(\frac{a}{c}\right)^n n^{b/a-d/c}\left(1+\sum_{m=1}^{\infty}\frac{a_m}{n^m}\right) \qquad (n \to \infty).$$

Note that

$$\frac{g_n}{g_{n-1}} = \frac{an+b}{cn+d}.$$

Define

$$f_n = \frac{g_n\Gamma(1+b/a)c^n}{\lambda\Gamma(1+d/c)a^n} = \frac{\Gamma(n+1+b/a)}{\Gamma(n+1+d/c)} \sim n^{b/a-d/c}\left(1+\sum_{m=1}^{\infty}\frac{a_m}{n^m}\right) \qquad (n \to \infty).$$

Then

$$\frac{f_{n+1}}{f_n} = \frac{n+1+b/a}{n+1+d/c}.$$

Equivalently,

$$(n+1+d/c)f_{n+1} = (n+1+b/a)f_n.$$

Summing both sides over n against x^n yields

$$F'(x) + \frac{d}{c}\left(\frac{F(x)-f_0}{x}\right) = xF'(x) + \left(1+\frac{b}{a}\right)F(x),$$

where $F(x) = \sum_{n=0}^{\infty} f_n x^n$. Equivalently,

$$x(x-1)F'(x) + \left[\left(1+\frac{b}{a}\right)x - \frac{d}{c}\right]F(x) = -\frac{df_0}{c}.$$

It follows that $B(x)$ satisfies

$$(x+1)xB'(x) + \left[\left(1+\frac{b}{a}\right)(x+1) - \frac{d}{c}\right]B(x) = 0.$$

Consequently

$$B(x) = C(x+1)^{-d/c}x^{-1-b/a+d/c}$$

for some constant C. But from (6.3) we know that

$$B(x) = \frac{x^{d/c-b/a-1}}{\Gamma(d/c-b/a)}\sum_{m=0}^{\infty}b_m x^m.$$

Therefore

$$C = \frac{b_0}{\Gamma(d/c-b/a)} = \frac{1}{\Gamma(d/c-b/a)},$$

and

$$b_n = \binom{-d/c}{n}.$$

We can now apply Proposition 6.2 to transfer the divisibility properties of the b_m to the a_m. We consider two special cases to illustrate how this is done. First, if $d = 0$ then $b_0 = 1$ and $b_m = 0$ for all $m > 0$. We conclude that for primes $\mathfrak{p}$ of K,

(a) If $v_{\mathfrak{p}}(b/a) \geq 0$ then $v_{\mathfrak{p}}(a_k) \geq 0$ for all k.

(b) If $v_{\mathfrak{p}}(b/a) < 0$ then $v_{\mathfrak{p}}(a_k) \geq 2k v_{\mathfrak{p}}(b/a) - v_{\mathfrak{p}}((2k)!)$ for all k.

In particular, the only primes that can divide the denominator of the asymptotic coefficients are the prime divisors of the denominator of b/a. We also obtain information in the case $b = 0$. In this case we find that

(a) If $v_{\mathfrak{p}}(d/c) \geq 0$ then $v_{\mathfrak{p}}(a_k) \geq 0$ for all k.

(b) If $v_{\mathfrak{p}}(d/c) < 0$ then $v_{\mathfrak{p}}(a_k) \geq 2k v_{\mathfrak{p}}(d/c) - v_{\mathfrak{p}}(k!) - v_{\mathfrak{p}}((2k)!)$ for all k.

In particular, the only primes that can divide the denominator of the asymptotic coefficients are the prime divisors of the denominator of d/c.

In [59], the authors provide examples illustrating two different situations in which enough information about the coefficients of the function B can be obtained in order to say something meaningful about the original asymptotic coefficients. The first situation occurs when one can solve the ODE satisfied by B explicitly, and the other occurs when the ODE for B can be mapped back to the ODE satisfied by F by means of a sequence of Möbius transformations. In order to not have to work on a case by case basis, we now derive an explicit ODE satisfied by a function related to B in full generality. We will then look at what can be said regarding the first of these two situations.

6.2 An Explicit ODE

The Holonomic Correspondence

In order to derive an explicit ODE satisfied by a function related to B, we start by making explicit the correspondence between the linear recurrences satisfied by a

holonomic sequence and the linear ODEs satisfied by its generating function. In what follows, it will often be more convenient to think of $K(n)[T;T]$ as consisting of all polynomials in n and T with commutation law given by

$$Tn = (n+1)T. \tag{6.9}$$

In this case, we will denote $K(n)[T;T]$ by $K[n,T;T]$ instead.

We define *the space of linear differential operators over K with polynomial coefficients*, denoted by $K(x)[\theta_x;\theta_x]$ to be the set of all polynomials in θ_x with coefficients in $K(x)$ where multiplication is generated by the commutation law

$$\theta_x x = x(\theta_x + 1). \tag{6.10}$$

Here, θ_x is the differential operator $x\frac{d}{dx}$. Similarly to the discrete case, it will often be more convenient to consider this space as consisting of all polynomials in x and θ_x with commutation law given by (6.10) and, when this is the case, our space will be denoted by $K[x,\theta_x;\theta_x]$.

Note that although not every linear differential equation with polynomial coefficients corresponds to a polynomial in x and θ_x, every one is equivalent, by multiplying by a sufficiently high power of x, to a differential equation corresponding to a polynomial in x and θ_x.

Now, for a linear (differential or recurrence) operator G, we denote its zero set by $Z(G)$. Thus, in the recurrence case, this consists of sequences, and in the differential case it consists of generating functions. The substitutions in the following result correspond to the formal Mellin transform. For more on this, see [33].

Lemma 6.3. *We have the following isomorphisms of K-algebras:*

$$K[n,T;T] \cong K[x,\theta_x;\theta_x] \qquad\qquad K[x,\theta_x;\theta_x] \cong K\left[\tfrac{1}{x},\theta_x;\theta_x\right]$$

$$n,T \mapsto -\theta_x, x \qquad\qquad\qquad \theta_x, x \mapsto -\theta_x, \tfrac{1}{x}$$

Further,

(a) *For a sequence, $\{u_n\}_{n=-\infty}^{\infty} \subseteq K$, defined for all integer indices, and an element*

$G(n,T) \in K[n,T;T]$, we have

$$\{u_n\}_{n=-\infty}^{\infty} \in Z(G(n,T)) \iff \sum_{n=-\infty}^{\infty} \frac{u_n}{x^n} \in Z(G(-\theta_x, x))$$

$$\iff \sum_{n=-\infty}^{\infty} u_n x^n \in Z\left(G\left(\theta_x, \tfrac{1}{x}\right)\right).$$

(b) *For a sequence $\{u_n\}_{n=0}^{\infty} \subseteq K$, defined only for non-negative integer indices, and an element $G(n,T) \in K[n,T;T]$, we have*

$$\{u_n\}_{n=0}^{\infty} \in Z(G(n,T)) \implies G(-\theta_x, x)\left(\sum_{n=0}^{\infty} \frac{u_n}{x^n}\right) \in K[x]$$

$$\implies G\left(\theta_x, \tfrac{1}{x}\right)\left(\sum_{n=0}^{\infty} u_n x^n\right) \in K\left[\tfrac{1}{x}\right].$$

Given a linear operator in $K[n,T;T]$, we can express it uniquely in the form

$$\sum_{i=0}^{k} P_i(n)T^i,$$

for polynomials $P_0, P_1, \ldots, P_k$ with $P_k \neq 0$. We then call k the *order* of the operator and the maximum of the degrees of the P_i the *degree* of the operator. Similarly, given a linear operator in $K[x, \theta_x; \theta_x]$, we can express it uniquely in the form

$$\sum_{i=0}^{k} P_i(x)\theta_x^i,$$

for polynomials $P_0, P_1, \ldots, P_k$ with $P_k \neq 0$. We then call k the *order* of the operator and the maximum of the degrees of the P_i the *degree* of the operator.

Properties of the Differential Operator θ

We now provide some notation and properties of the differential operator θ that will be needed in the sequel. We start by adopting the following notation. Given a differentiable function F, we denote by $F^{[\ell]}$ the function obtained from F by applying the differential operator θ^ℓ.

By induction, the commutation law implies that for polynomials P we have $\theta P = P\theta + P^{[1]}$. This is a special case of what can be said for general powers of θ. For any

$\ell \geq 0$ we have

$$\theta^\ell F = \sum_{j=0}^{\ell} \binom{\ell}{j} F^{[\ell-j]} \theta^j.$$

This equation will allow us to write differential operators in standard form. We will also need to know how θ transforms under Möbius transformations. The result is as follows.

$$y = \frac{ex+f}{gx+h} \implies \theta_y = y\frac{d}{dy} = \frac{(ex+f)(gx+h)}{eh-fg}\frac{d}{dx} = \frac{(ex+f)(gx+h)}{(eh-fg)x}\theta_x.$$

In particular, we find that $\theta_{ex} = \theta_x$, and $\theta_{f/x} = -\theta_x$.

Finally, we will need to know how powers of θ transform under linear combinations. In this case, the Stirling numbers come into play. Indeed, assuming $ey = x + c$ for some constants c, e, we have

$$\theta_y^j = \left[(x+c)\frac{d}{dx}\right]^j = \sum_{i=0}^{j} S(j,i)(x+c)^i \frac{d^i}{dx^i}, \tag{6.11}$$

where $S(j,i)$ denotes the appropriate Stirling number of the second kind (see, e.g.,[17, p. 220]). Now, we have the following property satisfied by the Stirling numbers $s(n,k)$ and $S(n,k)$ of the first and second kind, respectively (see, e.g., [17, p. 144])

$$f_n = \sum_k S(n,k)g_k \iff g_n = \sum_k s(n,k)f_k.$$

We conclude from the $c = 0$ case of (6.11) that

$$\frac{d^i}{dx^i} = x^{-i} \sum_{t=0}^{i} s(i,t)\theta_x^t,$$

where $s(i,t)$ denotes the appropriate Stirling number of the first kind. Thus

$$\theta_y^j = \sum_{i=0}^{j} S(j,i)(x+c)^i x^{-i} \sum_{t=0}^{i} s(i,t)\theta_x^t = \sum_{0 \leq i \leq h \leq j} S(j,h)s(h,i)\left(\frac{x+c}{x}\right)^h \theta_x^i. \tag{6.12}$$

We now have all that is required to derive an explicit ODE for a function related to B.

The ODE and Its Indicial Equation

Let K be a number field and $Q_0(z)$, $Q_1(z)$, $\ldots$, $Q_k(z) \in K[z]$. Suppose that the sequence $\{g_n\} \subseteq K$ is a zero of the linear recurrence operator

$$\sum_{j=0}^{k} n^j Q_j(T) \in K[n, T; T],$$

and has an asymptotic expansion

$$g_n \sim \gamma \rho^n n^{-\varphi} \sum_{m=0}^{\infty} \frac{a_m}{n^{m/q}} \qquad (n \to \infty)$$

for some constants $\rho \in K$, $q \in \mathbb{N}$, $\gamma, a_0, a_1, a_2, \cdots \in \mathbb{C}$, $\varphi + \ell/q \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ for all $0 \leq \ell < q$ where $a_0 = 1$. We may suppose that Q_k is monic. Defining

$$f_n = \frac{g_n}{\gamma \rho^n},$$

we obtain a zero of the linear recurrence operator

$$\sum_{j=0}^{k} n^j P_j(T) \in K[n, T; T],$$

with asymptotic expansion

$$f_n \sim n^{-\varphi} \sum_{m=0}^{\infty} \frac{a_m}{n^{m/q}}, \qquad (n \to \infty)$$

where

$$P_j(x) = Q_j(\rho x), \qquad (0 \leq j \leq k).$$

With the above notation, we now define $F(x) = \sum_{n=0}^{\infty} f_n x^n$ to be the ordinary generating function of $\{f_n\}_n$, $B(x) = \Psi(F(x))$ and $C(x) = B(\frac{x}{1-x}) = \frac{x^{\varphi-1}}{\Gamma(\varphi)} \sum_{n=0}^{\infty} c_n x^{n/q}$. The following result provides an explicit linear differential operator satisfied by C.

Proposition 6.3. *With the above notation, the function $C(x)$ satisfies the linear ordinary differential operator*

$$\sum_{p=0}^{k} F_p(x) \theta_x^p \in K[x, \theta_x; \theta_x], \tag{6.13}$$

where

$$F_p(x) = \sum_{p \leq j \leq q \leq i \leq k} (-1)^{i+q-j} \binom{i}{j} x^{k-i+q-j} U_{i-j,q-j}(x) U_{j,p}(x) P_i^{[q-j]}(1-x) \qquad (0 \leq p \leq k)$$

and

$$U_{i,j}(x) = \sum_{h=j}^{i} S(i,h)s(h,j)(x-1)^h x^{i-h} \qquad (j \leq i), \tag{6.14}$$

that has indicial equation

$$I(\lambda) = \sum_{\Delta \leq j \leq i \leq k} (-1)^i \binom{i}{j} P_i^{(i-\Delta)}(1)(k-\Delta)_{k-j}(\lambda)_j \tag{6.15}$$

where $\Delta = \max\{j - \mathrm{ord}_{x=1}P_j(x) \mid 0 \leq j \leq k\}$ *and* $(c)_n = c(c-1)\dots(c-n+1)$ *denotes the falling Pochhammer symbol.*

Proof. From Lemma 6.3, we know that F satisfies $\left(\sum_{j=0}^{k} \theta_x^j P_j\left(\frac{1}{x}\right)\right)(F(x)) \in K\left[\frac{1}{x}\right]$. Multiplying by x^l for l sufficiently large gives $x^l\left(\sum_{j=0}^{k} \theta_x^j P_j\left(\frac{1}{x}\right)\right)(F(x)) \in K[x]$. We then have that B is a zero of the linear differential operator $\sum_{j=0}^{k} \theta_{x+1}^j P_j\left(\frac{1}{x+1}\right)$. If we change variables using $z = \frac{1}{x+1}$, and recall that $\theta_{1/z} = -\theta_z$, we obtain that

$$\begin{aligned}
B(x) &= \frac{x^{\varphi-1}}{\Gamma(\varphi)}\sum_{m=0}^{\infty} b_m x^{m/q} = \frac{\left(\frac{1-z}{z}\right)^{\varphi-1}}{\Gamma(\varphi)}\sum_{m=0}^{\infty} b_m \left(\frac{1-z}{z}\right)^{m/q} \\
&= \frac{(1-z)^{\varphi-1}}{\Gamma(\varphi)}\sum_{m,j=0}^{\infty} b_m \binom{m/q+\varphi+j-2}{j}(1-z)^{m/q+j} \\
&= \frac{(1-z)^{\varphi-1}}{\Gamma(\varphi)}\sum_{m=0}^{\infty} c_m(1-z)^{m/q}, \tag{6.16}
\end{aligned}$$

satisfies

$$\sum_{j=0}^{k}(-1)^j\theta_z^j P_j(z) = \sum_{j=0}^{k}(-1)^j\sum_{l=0}^{j}\binom{j}{l}P_j^{[j-l]}(z)\theta_z^l = \sum_{0 \leq j \leq i \leq k}(-1)^i\binom{i}{j}P_i^{[i-j]}(z)\theta_z^j,$$

where, for all $m \geq 0$, $c_m = \sum_{i=0}^{\lfloor m/q \rfloor} b_{m-qi}\binom{m/q+\varphi-2}{i}$. *Changing variables one more time, we find that with* $w = 1-z$, *we have* $\theta_z = \left(\frac{w-1}{w}\right)\theta_w$, *and the series*

$$C(w) = \frac{w^{\varphi-1}}{\Gamma(\varphi)}\sum_{m=0}^{\infty} c_m w^{m/q}$$

is a zero of the differential operator

$$\sum_{0 \leq j \leq i \leq k}(-1)^i\binom{i}{j}\theta_z^{i-j}\left(P_i(1-w)\right)\theta_z^j.$$

From (6.12) we have

$$w^j\theta_z^j = w^j\sum_{i=0}^{j} S(j,i)(w-1)^i w^{-i}\sum_{t=0}^{i} s(i,t)\theta_w^t = \sum_{0 \leq i \leq h \leq j} S(j,h)s(h,i)(w-1)^h w^{j-h}\theta_w^i.$$

Let $U_{j,i}(w) = \sum_{h=i}^{j} S(j,h)s(h,i)(w-1)^h w^{j-h}$, so that $w^j \theta_z^j = \sum_{i=0}^{j} U_{j,i}(w)\theta_w^i$ for all $j \geq 0$. Our differential operator is then

$$\sum_{0 \leq t \leq j \leq i \leq k} (-1)^i \binom{i}{j} w^{-j} \theta_z^{i-j} \left(P_i(1-w)\right) U_{j,t}(w)\theta_w^t.$$

Here, for all i and j, we have

$$\theta_z^{i-j}\left(P_i(1-w)\right) = w^{j-i} \sum_{q=j}^{i} U_{i-j,q-j}(w)\theta_w^{q-j}(P_i(1-w)).$$

We can therefore express our differential operator as

$$\sum_{0 \leq p \leq j \leq q \leq i \leq k} (-1)^i \binom{i}{j} w^{-i} U_{i-j,q-j}(w) U_{j,p}(w)\theta_w^{q-j}(P_i(1-w))\theta_w^p.$$

Scaling by w^k, we get the linear differential operator

$$\sum_{p=0}^{k} F_p(w)\theta_w^p \in K[w,\theta_w],$$

where, for $0 \leq p \leq k$,

$$F_p(w) = \sum_{p \leq j \leq q \leq i \leq k} (-1)^i \binom{i}{j} w^{k-i} U_{i-j,q-j}(w) U_{j,p}(w)\theta_w^{q-j}(P_i(1-w))$$

$$= \sum_{p \leq j \leq q \leq i \leq k} (-1)^{i+q-j} \binom{i}{j} w^{k-i+q-j} U_{i-j,q-j}(w) U_{j,p}(w) P_i^{[q-j]}(1-w).$$

We now derive the indicial equation for this differential operator. Let $\nu = \min\{\mathrm{ord}_{w=0}(F_p(w)) \mid 0 \leq p \leq k\}$. If r_j denotes the order of vanishing of $P_j(z)$ at $z = 1$, and $\Delta_j = j - r_j$, then $\nu = \min\{k - \Delta_j \mid 1 \leq j \leq k\}$. Let $\Delta = \max\{\Delta_j \mid 0 \leq j \leq k\}$, so that $\nu = k - \Delta$. The indicial polynomial is given by

$$I(\lambda) := (-1)^{\Delta}(k-\Delta)! \sum_{p=0}^{k} \left[\lim_{w \to 0} w^{-(k-\Delta)} F_p(w)\right] \lambda^p = (-1)^{\Delta} \sum_{p=0}^{k} F_p^{(k-\Delta)}(0)\lambda^p.$$

It has degree equal to the largest index i such that $\Delta_i = \Delta$. The quantity $F_p^{(k-\Delta)}(0)$ is given by

$$\sum_{p \leq j \leq q \leq i \leq k} (-1)^i \binom{i}{j}\binom{k-\Delta}{k-i}(k-i)! \frac{d^{i-\Delta}}{dw^{i-\Delta}} \left[U_{i-j,q-j}(w)U_{j,p}(w)\theta_w^{q-j}(P_i(1-w))\right]_{w=0}$$

$$= \sum_{p \leq j \leq q \leq i \leq k} (-1)^i \binom{i}{j}\binom{k-\Delta}{k-i}(k-i)! U_{i-j,q-j}(0)U_{j,p}(0)[\theta_w^{q-j}(P_i(1-w))]^{(i-\Delta)}\Big|_{w=0}.$$

Recalling that $\theta_w^j = \sum_{i=0}^j S(j,i)w^i \frac{d^i}{dw^i}$, where the $S(j,i)$ denote the Stirling numbers of the second kind, we see that the quantity $F_p^{(k-\Delta)}(0)$ can be expressed as

$$\sum_{p \le j \le q \le i \le k} (-1)^{r_i-i}\binom{i}{j}\binom{k-i+r_i}{k-i}(k-i)!U_{i-j,q-j}(0)U_{j,p}(0)r_i^{q-j}\delta_{\Delta,\Delta_i}P_i^{(r_i)}(1). \quad (6.17)$$

Since

$$U_{j,i}(w) = \sum_{h=i}^{j} S(j,h)s(h,i)(w-1)^h w^{j-h},$$

we find that

$$U_{n,m}(0) = \sum_{h=m}^{n} S(n,h)s(h,m)(-1)^h \delta_{h,n} = S(n,n)s(n,m)(-1)^n = (-1)^n s(n,m).$$

Substituting this into (6.17) we find that $F_p^{(k-\Delta)}(0)$ is given by the following sum:

$$\sum_{p \le j \le q \le i \le k} (-1)^{r_i-i}\binom{i}{j}\binom{k-i+r_i}{k-i}(k-i)!(-1)^{i-j}$$

$$\times s(i-j,q-j)(-1)^j s(j,p)r_i^{q-j}\delta_{\Delta,\Delta_i}P_i^{(r_i)}(1).$$

But we have the basic identity

$$\sum_{q=j}^{i} s(i-j,q-j)r_i^{q-j} = (r_i)_{i-j},$$

(see [17, p. 213]) and so we obtain,

$$F_p^{(k-\Delta)}(0) = \sum_{p \le j \le i \le k} (-1)^{r_i}\binom{i}{j}(k-i+r_i)_{k-i}(r_i)_{i-j}s(j,p)\delta_{\Delta,\Delta_i}P_i^{(r_i)}(1).$$

Finally, since $(k-i+r_i)_{k-i}(r_i)_{i-j} = (k-i+r_i)_{k-j}$, and the effect of multiplying by δ_{Δ,Δ_i} is the same as replacing r_i by $i-\Delta$ (since this derivative of P_i would vanish at 1 in the case $\Delta \ne \Delta_i$ anyway), we can write, for all $0 \le p \le k$,

$$F_p^{(k-\Delta)}(0) = (-1)^\Delta \sum_{p \le j \le i \le k} (-1)^i\binom{i}{j}(k-\Delta)_{k-j}s(j,p)P_i^{(i-\Delta)}(1).$$

Now, we need only sum over $j \ge \Delta$ since $(k-\Delta)_{k-j}$ vanishes otherwise. Also, we can start our sum at any value of $j \le p$ since $s(j,p)$ will vanish for all of these other additional terms. Therefore

$$F_p^{(k-\Delta)}(0) = (-1)^\Delta \sum_{\Delta \le j \le i \le k} (-1)^i \binom{i}{j} (k-\Delta)_{k-j} s(j,p) P_i^{(i-\Delta)}(1).$$

Now, summing this times λ^p over p and recalling that $\sum_{p=0}^{j} s(j,p)\lambda^p = (\lambda)_j$, the indicial equation can be written as

$$I(\lambda) = \sum_{\Delta \le j \le i \le k} (-1)^i \binom{i}{j} P_i^{(i-\Delta)}(1)(k-\Delta)_{k-j}(\lambda)_j$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

For $q = 1$, $C(x)$ has the form of a Frobenius series, and so the exponent $\varphi - 1$ must be a root of the indicial equation given by (6.15). As for the parameter ρ that appears in the asymptotic expansion, the observation that

$$\lim_{n \to \infty} \frac{g_{n+1}}{g_n} = \rho$$

and that this limit must be a root of the characteristic polynomial Q_k of the recurrence operator satisfied by $\{g_n\}_n$ provides us with a polynomial satisfied by ρ. These arguments do not require any assumptions to be placed on ρ or φ and so we obtain the following corollary to Proposition 6.3.

Corollary 6.1. *Suppose that the holonomic sequence $\{g_n\}_n$ satisfies the linear recurrence operator*

$$\sum_{j=0}^{k} n^j Q_j(T)$$

for polynomials $Q_0, \ldots, Q_k \in K[x]$ and admits the asymptotic expansion $g_n \sim \gamma \rho^n n^{-\varphi} \sum_{m=0}^{\infty} \frac{a_m}{n^m}$ as $n \to \infty$ for constants $\gamma, \rho, \varphi, a_0, a_1, a_2, \ldots$. Then ρ is a root of Q_k and $\varphi - 1$ is a root of the polynomial

$$I(\lambda) = \sum_{\Delta \le j \le i \le k} (-1)^i \binom{i}{j} \rho^{i-\Delta} Q_i^{(i-\Delta)}(\rho)(k-\Delta)_{k-j}(\lambda)_j,$$

where $\Delta = \max\{j - \mathrm{ord}_{x=\rho} Q_j(x) \mid 0 \le j \le k\}$.

6.3   The Case Where the ODE Can Be Solved Explicitly

We now investigate what can be said in the case our linear recurrence operator is of degree one. In the above notation, $k = 1$, and the operator satisfied by $\{f_n\}_n$ takes the form $nP_1(T) + P_0(T)$. We find that the series

$$C(x) = \frac{x^{\varphi-1}}{\Gamma(\varphi)} \sum_{m=0}^{\infty} c_m x^{m/q}$$

is a zero of the linear differential operator $F_1(x)\theta_x + F_0(x) \in K[x, \theta_x; \theta_x]$ given by (6.13). Here, F_0 and F_1 simplify to

$$F_1(x) = (1-x)P_1(1-x), \qquad F_0(x) = xP_0(1-x) + (1-x)P_1^{[1]}(1-x).$$

Also, we have $\Delta = 1 - \mathrm{ord}_{x=1}P_1(x) \le 0$ since otherwise the indicial polynomial would be constant. Therefore, the indicial polynomial given by (6.15) becomes

$$\sum_{\Delta \le j \le i \le 1} (-1)^i P_i^{(i-\Delta)}(1)(1-\Delta)_{1-j}(\lambda)_j = \sum_{0 \le j \le i \le 1} (-1)^i P_i^{(i-\Delta)}(1)(1-\Delta)_{1-j}(\lambda)_j.$$

If $-\mathrm{ord}_{x=1}P_0(x) < \Delta$, then our indicial polynomial would become $-P_1^{(1-\Delta)}(1)(\lambda+1-\Delta)$. But then $\varphi-1$ could not be a root since $\varphi-1 \notin \mathbb{Z}^{\le -1}$. Therefore, $-\mathrm{ord}_{x=1}P_0(x) = \Delta$, and we have

$$P_1^{(r_1)}(1)\lambda + r_1 \left[P_1^{(r_1)}(1) - P_0^{(r_1-1)}(1) \right] = 0,$$

where $r_1 := \mathrm{ord}_{x=1}P_1(x)$. Solving this, we obtain

$$\varphi - 1 = r_1 \left[\frac{P_0^{(r_1-1)}(1)}{P_1^{(r_1)}(1)} - 1 \right].$$

Now $\frac{P_0}{P_1}$ has a simple pole at 1, and

$$r_1 \frac{P_0^{(r_1-1)}(1)}{P_1^{(r_1)}(1)} = \mathrm{Res}_{z=1}\left(\frac{P_0(z)}{P_1(z)} \right).$$

Therefore,

$$\varphi - 1 = \mathrm{Res}_{z=1}\left(\frac{P_0(z)}{P_1(z)} \right) - r_1.$$

We obtain the following result.

Proposition 6.4. *Suppose that the holonomic sequence $\{g_n\}_n$ satisfies the linear recurrence operator $nQ_1(T) + Q_0(T)$ for polynomials Q_0, Q_1 and has the asymptotic expansion*

$$g_n \sim \gamma \rho^n n^{-\varphi} \sum_{m=0}^{\infty} \frac{a_m}{n^{m/q}} \qquad (n \to \infty)$$

for some $q \in \mathbb{N}$ and constants $\rho, \gamma, a_0, a_1, a_2, \cdots \in \mathbb{C}$, $\varphi \notin \mathbb{Z}_{\leq 0}$ where $a_0 = 1$. Then ρ is a root of Q_1 and

$$\varphi = 1 + \frac{1}{\rho} \text{Res}_{z=\rho} \left(\frac{Q_0(z)}{Q_1(z)} \right) - \text{ord}_{z=\rho} Q_1(z).$$

We can solve the differential equation explicitly to obtain

$$C(x) = \frac{x^{\varphi-1}}{\Gamma(\varphi)} \sum_{m=0}^{\infty} c_m x^{m/q} = \frac{\mu}{P_1(1-x)} \exp \left(\int^{1-x} \frac{P_0(z)}{zP_1(z)} \, dz \right),$$

for some constant μ that depends on the lower limit of integration that we leave undetermined for now. Therefore

$$\sum_{m=0}^{\infty} c_m x^{m/q} = \frac{\mu \Gamma(\varphi)}{P_1(1-x)} x^{1-\varphi} \exp \left(\int^{1-x} \frac{P_0(z)}{zP_1(z)} \, dz \right).$$

Now, let $\delta = \text{Res}_{z=1} \frac{P_0(z)}{P_1(z)} = \text{Res}_{z=1} \frac{P_0(z)}{zP_1(z)}$. If we write

$$\frac{P_0(z)}{zP_1(z)} = \frac{\delta}{z-1} + L(z), \qquad P_1(z) = (z-1)^{r_1} S(z),$$

where L is analytic at $z = 1$, and $S(z) \in K[z]$ does not vanish at $z = 1$, we obtain

$$\sum_{m=0}^{\infty} c_m x^{m/q} = \frac{\mu'}{S(1-x)} x^{1-\varphi+\delta-r_1} \exp \left(\int^{1-x} L(z) \, dz \right),$$

for some constant μ'. We note that

$$1 - \varphi + \delta - r_1 = 0. \tag{6.18}$$

Therefore, we can write our equation as

$$\sum_{m=0}^{\infty} c_m x^{m/q} = \frac{\mu'}{S(1-x)} \exp \left(\int^{1-x} L(z) \, dz \right).$$

Now, since L is analytic at $z = 1$, we can make the lower limit of integration 1, and then determine the resulting constant. We obtain

$$\sum_{m=0}^{\infty} c_m x^{m/q} = \frac{S(1)}{S(1-x)} \exp \left(\int_{1}^{1-x} L(z) \, dz \right). \tag{6.19}$$

We now provide an example where enough information regarding the c_m can be found to obtain meaningful divisibility properties for the original asymptotic coefficients a_m.

Example 6.2. Let K be a number field and $q = 1$. Suppose that Q_1 has only one root ρ of multiplicity $r \geq 1$ and that $\deg Q_0 \leq \deg Q_1$. The $r = 1$ case has been considered already in Example 6.1. We are assuming, as above, that Q_1 is monic. Now, we know that Q_1 must vanish at ρ, and that $\frac{Q_0}{Q_1}$ must have a simple pole at $z = \rho$. Thus

$$Q_1(z) = (z - \rho)^r, \qquad Q_0(z) = d(z - \rho)^{r-1}(z - \rho\beta), \text{ or } Q_0(z) = d(z - \rho)^{r-1}$$

for some $d, \beta \in K$, $\beta \neq 1$. We take cases according to whether or not Q_0 has the same degree as Q_1. Suppose first that this holds. Then in the notation above, $P_1(z) = Q_1(\rho z) = \rho^r(z - 1)^r$, and $P_0(z) = Q_0(\rho z) = d\rho^r(z - 1)^{r-1}(z - \beta)$. We use partial fractions to obtain

$$\frac{P_0(z)}{zP_1(z)} = \frac{d(z - \beta)}{z(z - 1)} = \frac{d\beta}{z} + \frac{d - d\beta}{z - 1}.$$

In the notation above, we have $S(z) = \rho^r$, $L(z) = \frac{d\beta}{z}$. Therefore, (6.19) now reads

$$\sum_{m=0}^{\infty} c_m x^m = \frac{\rho^r}{\rho^r} \exp\left(\int_1^{1-x} \frac{d\beta}{z} \, dz\right),$$

$$= (1 - x)^{d\beta}.$$

Consequently, we can combine (6.16) and (6.18) to obtain

$$\sum_{m=0}^{\infty} b_m x^m = (x + 1)^{r-\delta} \sum_{m=0}^{\infty} c_m \left(\frac{x}{x + 1}\right)^m = (x + 1)^{r-\delta} \left(\frac{1}{1 + x}\right)^{d\beta}$$

$$= \sum_{m=0}^{\infty} \binom{r - \delta - d\beta}{m} = \sum_{m=0}^{\infty} \binom{r - d}{m} x^m.$$

Therefore $b_m = \binom{r-d}{m}$. Now, in the case $Q_0(z) = d(z - \rho)^{r-1}$, we have $P_0(z) = d\rho^{r-1}(z - 1)^{r-1}$. Therefore

$$\frac{P_0(z)}{zP_1(z)} = \frac{d/\rho}{z - 1} - \frac{d/\rho}{z},$$

so that $L(z) = -\frac{d}{\rho z}$, and $S(z) = \rho^r$. Therefore, (6.19) reads

$$\sum_{m=0}^{\infty} c_m x^m = \frac{\rho^r}{\rho^r} \exp\left(\int_1^{1-x} \frac{-d/\rho}{z} \, dz\right),$$

$$= (1 - x)^{-d/\rho}.$$

Consequently, we can combine (6.16) and (6.18) to obtain

$$\sum_{m=0}^{\infty} b_m x^m = (x+1)^{r-\delta} \sum_{m=0}^{\infty} c_m \left(\frac{x}{x+1}\right)^m = (x+1)^{r-\delta} \left(\frac{1}{1+x}\right)^{-d/\rho}$$

$$= \sum_{m=0}^{\infty} \binom{r-\delta+d/\rho}{m} x^m = \sum_{m=0}^{\infty} \binom{r}{m} x^m.$$

Therefore $b_m = \binom{r}{m}$. Defining

$$\psi = \begin{cases} 1 & \text{if } \deg Q_1 = r; \\ 0 & \text{if } \deg Q_1 = r-1, \end{cases}$$

we obtain

$$b_m = \binom{r-d\psi}{m}.$$

Let $\mathfrak{p}$ be a prime of K and $d \in \mathbb{Q}$. Since, for $v_\mathfrak{p}(d\psi) \geq 0$ we have $v_\mathfrak{p}\binom{r-d\psi}{m} \geq 0$ (see Lemma 6.1), we conclude that both $v_\mathfrak{p}(m!b_m)$ and $v_\mathfrak{p}(b_m)$ are at least zero. On the other hand, if $v_\mathfrak{p}(d\psi) < 0$, then $v_\mathfrak{p}(j - d\psi) = v_\mathfrak{p}(d\psi)$ for all $j \in \mathbb{Z}$. Thus

$$v_\mathfrak{p}(m!b_m) = \sum_{j=0}^{m-1} v_\mathfrak{p}(r - d\psi - j) = m v_\mathfrak{p}(d\psi).$$

Let

$$\eta = \begin{cases} 1-\beta & \text{if } \deg Q_1 = r; \\ 1/\rho & \text{if } \deg Q_1 = r-1. \end{cases}$$

Then, we have $\varphi = 1 - r + d\eta$ and if $\varphi \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ then

$$v_\mathfrak{p}(a_m) \geq \begin{cases} 0 & \text{if } v_\mathfrak{p}(\varphi) \geq 0, \ v_\mathfrak{p}(d\psi) \geq 0; \\ m v_\mathfrak{p}(d\psi) & \text{if } v_\mathfrak{p}(\varphi) \geq 0, \ v_\mathfrak{p}(d\psi) < 0; \\ \min\{S_{n,m} \mid 0 \leq n \leq m\} & \text{if } v_\mathfrak{p}(\varphi) < 0, \end{cases} \qquad (6.20)$$

where $S_{n,m} = v_\mathfrak{p}\binom{r-d\psi}{n} + (2m-n)v_\mathfrak{p}(\varphi) - v_\mathfrak{p}((2m-2n)!)$. We have

$$v_\mathfrak{p}(\varphi) \geq 0 \iff v_\mathfrak{p}(1 - r + d\eta) \geq 0 \iff v_\mathfrak{p}(d\eta) \geq 0.$$

Therefore, we can rewrite (6.20) as

$$v_\mathfrak{p}(a_m) \geq \begin{cases} 0 & \text{if } v_\mathfrak{p}(d\eta) \geq 0, \ v_\mathfrak{p}(d\psi) \geq 0; \\ m v_\mathfrak{p}(d\psi) & \text{if } v_\mathfrak{p}(d\eta) \geq 0, \ v_\mathfrak{p}(d\psi) < 0; \\ \min\{S_{n,m} \mid 0 \leq n \leq m\} & \text{if } v_\mathfrak{p}(d\eta) < 0. \end{cases}$$

where $S_{n,m} = v_{\mathfrak{p}}\binom{r-d\psi}{n} + (2m-n)v_{\mathfrak{p}}(d\eta) - v_{\mathfrak{p}}((2m-2n)!)$. Here, we have

$$\psi = \begin{cases} 1 & \text{if } \deg Q_1 = r; \\ 0 & \text{if } \deg Q_1 = r - 1. \end{cases} \qquad \eta = \begin{cases} 1 - \beta & \text{if } \deg Q_1 = r; \\ 1/\rho & \text{if } \deg Q_1 = r - 1. \end{cases}$$

In particular, the only primes that can divide the denominators of the a_m are the primes dividing the denominator of one of $d\eta$, $d\psi$.

CHAPTER 7

Asymptotics of a Family of Binomial Sums

In this chapter, we apply the asymptotic methods described in Chapter 5 to obtain full asymptotic expansions for a family of binomial sums. We will also apply the generalization of the method of Stoll and Haible from Chapter 6 (Proposition 6.2) to say something regarding the divisibility properties of the asymptotic coefficients for a special case. Section 7.1 provides an introduction. In Section 7.2 we give some preliminaries that set the stage for the remainder of the chapter. After a few auxiliary results in Section 7.3, we deal, in Section 7.4, with the cases covered by the method of Pemantle and Wilson introduced in Section 5.5. In Section 7.5 we use the method of Flajolet and Sedgewick introduced in Section 5.4 to consider the remaining cases. At that point, having established our main result in all cases, we conclude with some examples in Section 7.6. Throughout this chapter, we make extensive use of Maple [43] using the gfun strategy explained in the article [55] due to Salvy and Zimmerman.

7.1   Introduction

Some combinatorial sequences of interest can be written as binomial sums of the form

$$u_n^{(\varepsilon,a,d)} = \sum_{k=0}^{n} (-1)^{\varepsilon k} \binom{n}{k} \binom{an}{k} d^k \tag{7.1}$$

for $\varepsilon \in \{0, 1\}$ and $a, d \in \mathbb{N}$. For instance, the central binomial coefficients are given by

$$\binom{2n}{n} = u_n^{(0,1,1)} = \sum_{k=0}^{n} \binom{n}{k}^2$$

and the central Delannoy numbers $D(n, n)$ that count the number of paths from the origin $(0, 0)$ to the point (n, n) using steps $(1, 0)$, $(0, 1)$ and $(1, 1)$ are given by

$$D(n, n) = u_n^{(0,1,2)} = \sum_{k=0}^{n} \binom{n}{k}^2 2^k \tag{7.2}$$

(see, e.g., [17, p. 81], [58, p. 185]). These two examples illustrate the general fact that all of the sums given by (7.1) can be considered as certain diagonals of suitable weighted Delannoy numbers. These are defined as follows. Fix $\alpha, \beta, \gamma \in \mathbb{C}$. We consider paths that start at the origin, remain in the first quadrant and use only the steps $(1, 0)$ with weight α, $(0, 1)$ with weight β and $(1, 1)$ with weight γ. The weight of a path is then the product of the weights of the individual steps that comprise the path. For $m, n \in \mathbb{N}_0$, let $v_{m,n}$ denote the total of all of the weights of paths that connect the origin to the point (m, n). The $v_{m,n}$ are known as the weighted Delannoy numbers and are given by the recurrence relation

$$v_{m+1,n+1} = \alpha v_{m,n+1} + \beta v_{m+1,n} + \gamma v_{m,n} \qquad (m, n \geq 0) \tag{7.3}$$

subject to the initial conditions

$$v_{m,0} = \alpha^m \ (m \geq 0), \qquad v_{0,n} = \beta^n \ (n \geq 0).$$

We have the closed form expression

$$v_{m,n} = \sum_{k=0}^{n} \binom{n}{k} \binom{m}{k} \alpha^{m-k} \beta^{n-k} (\alpha\beta + \gamma)^k$$

(see [25, p. 87]). We therefore obtain our binomial sums of interest by setting $m = an$, $\alpha^a \beta = 1$ and $\gamma = \alpha\beta((-1)^\varepsilon d - 1)$. For a general discussion of asymptotics of lattice paths see [1]. For more on the Delannoy numbers see [2, 17, 58, 60] and for more on weighted lattice paths see [25, 26].

Another sequence of interest, having $\varepsilon = 1$, is given by

$$u_n^{(1,2,1)} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n}{k}.$$

The divisibility properties of this sequence are studied by Chamberland and Dilcher in [12] where it is shown that it behaves in many ways like a single binomial coefficient and, in particular, satisfies a version of Wolstenholme's Theorem. In [12], it is conjectured that this sequence possesses a full asymptotic expansion of a particular form as n tends to infinity. Here, we prove this conjecture and provide similar asymptotic expansions for the case of arbitrary ε, a and d in (7.1). Our approach will be to view the univariate sequence $\{u_n^{(\varepsilon,a,d)}\}_n$ as the diagonal of the bivariate sequence $\{\tilde{u}_{mn}^{(\varepsilon,a,d)}\}_{m,n}$ given by

$$\tilde{u}_{mn}^{(\varepsilon,a,d)} = \sum_{k=0}^{n} (-1)^{\varepsilon k} \binom{n}{k} \binom{am}{k} d^k. \tag{7.4}$$

It will turn out that the method of Pemantle and Wilson, described in Section 5.5, can accommodate our sequence for all but finitely many values of a. We will then deal with the remaining cases by applying the transfer method of Flajolet and Sedgewick, described in Section 5.4. For ease of notation, when the superscripts ε, a, d are understood, they will be omitted from the notation and we will write u_n and $\tilde{u}_{mn}$ instead of the more cumbersome $u_n^{(\varepsilon,a,d)}$ and $\tilde{u}_{mn}^{(\varepsilon,a,d)}$, respectively. Our main result relies on the following notation.

Let $a, d \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$. Set $\alpha = 1 - (-1)^\varepsilon d$, and define the polynomial g by

$$a(\alpha - 1)g(z) = \alpha z^2 + (a\alpha - a - \alpha - 1)z + 1. \tag{7.5}$$

Let

$$\Delta_g = \frac{(a-1)^2\alpha - (a+1)^2}{(\alpha - 1)a^2} \tag{7.6}$$

denote the discriminant of g, and z_0 be the root of g for which

$$\frac{2\alpha z_0 + a\alpha - a - \alpha - 1}{a(\alpha - 1)} = \sqrt{\Delta_g}$$

where $\sqrt{\cdot}$ denotes the principal branch of the square root. Further, define

$$\delta = \frac{1}{(1 - z_0)\sqrt[4]{\Delta_g}} \qquad \text{and} \qquad \beta = \frac{1}{z_0}\left(\frac{1 - \alpha z_0}{1 - z_0}\right)^a, \tag{7.7}$$

where $\sqrt[4]{\cdot}$ denotes the principal branch of the fourth root. The case when g has repeated real roots yields cube root asymptotics for u_n, while the other cases yield square root asymptotics for u_n. This gives rise to our main result which is split into two theorems to accommodate this distinction.

Theorem 7.1 ($\Delta_g \neq 0$ Case). *With the above notation, there exist constants μ_ℓ for $\ell \in \mathbb{N}$ such that*

$$\sum_{k=0}^{n} (-1)^{\varepsilon k} \binom{n}{k} \binom{an}{k} d^k \sim \frac{\delta \beta^n}{\sqrt{2\pi n}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{n^\ell}\right) \qquad (n \to \infty),$$

in case $\Delta_g > 0$ and

$$\sum_{k=0}^{n} (-1)^{\varepsilon k} \binom{n}{k} \binom{an}{k} d^k \sim \frac{\delta \beta^n}{\sqrt{2\pi n}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{n^\ell}\right) + \frac{\overline{\delta \beta}^n}{\sqrt{2\pi n}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\overline{\mu_\ell}}{n^\ell}\right) \qquad (n \to \infty),$$

in case $\Delta_g < 0$.

A calculation shows that $\Delta_g = 0$ only for $(\varepsilon, a, d) \in \{(1, 2, 8), (1, 3, 3)\}$, which accounts for the two cases in the following theorem.

Theorem 7.2 ($\Delta_g = 0$ Case). *There exist constants μ_ℓ, η_ℓ, $\tilde{\mu}_\ell$, $\tilde{\eta}_\ell \in \mathbb{Q}$ for $\ell \in \mathbb{N}$ such that, as $n \to \infty$,*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n}{k} 8^k \sim \frac{(-27)^n}{2^{2/3} \Gamma(2/3) n^{1/3}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{n^\ell}\right) + \frac{(-27)^n}{2^{4/3} \Gamma(1/3) n^{2/3}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\eta_\ell}{n^\ell}\right)$$

and

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{3n}{k} 3^k \sim \frac{2^{2/3}(-16)^n}{3 \Gamma(2/3) n^{1/3}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\tilde{\mu}_\ell}{n^\ell}\right) + \frac{2^{1/3}(-16)^n}{3 \Gamma(1/3) n^{2/3}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\tilde{\eta}_\ell}{n^\ell}\right).$$

Further, the constants μ_ℓ, η_ℓ have denominators that are divisible only by the primes 2 and 3.

Asymptotics of binomial sums have been studied before. For instance, in [44], McIntosh established asymptotic expansions for sums of the form

$$\sum_{k=0}^{n} \binom{n}{k}^{r_0} \binom{n+k}{k}^{r_1} \binom{n+2k}{k}^{r_2} \cdots \binom{n+mk}{k}^{r_m}$$

as $n \to \infty$ for non-negative integers $r_0, r_1, r_2, \ldots, r_m$.

7.2 Preliminaries

Both the method of Flajolet and Sedgewick as well as the method of Pemantle and Wilson will proceed by analysis of the bivariate ordinary generating function

$$\tilde{F}(z, w) := \sum_{m,n \geq 0} \tilde{u}_{mn} z^n w^m. \tag{7.8}$$

Recall that we are setting $\alpha = 1 - (-1)^\varepsilon d$. If $\alpha = 0$, so that $\varepsilon = 0$ and $d = 1$, our sum is given by

$$\tilde{u}_{mn} = \sum_{k=0}^{n} \binom{n}{k} \binom{am}{k} = \binom{am+n}{n},$$

as a result of the Vandermonde convolution (see, e.g., [17, p. 44]). Since this case can be dealt with by way of Stirling's formula, we may suppose that $\alpha \neq 0$. Furthermore, as $d \neq 0$, we also have $\alpha \neq 1$. Our generating function is rational, as is shown by the following lemma.

Lemma 7.1. *Let $\varepsilon \in \{0, 1\}$, $a, d \in \mathbb{N}$ and define $\alpha = 1 - (-1)^\varepsilon d$. With*

$$\tilde{u}_{mn} = \sum_{k=0}^{n} (-1)^{\varepsilon k} \binom{n}{k} \binom{am}{k} d^k \qquad and \qquad \tilde{F}(z, w) = \sum_{m,n \geq 0} \tilde{u}_{mn} z^n w^m,$$

we have

$$\tilde{F}(z, w) = \frac{\varphi(z)}{1 - w\nu(z)}$$

for

$$\varphi(z) = \frac{1}{1 - z}, \qquad \nu(z) = \left(\frac{1 - \alpha z}{1 - z} \right)^a. \tag{7.9}$$

Proof. In order to compute the bivariate generating function $\tilde{F}$ of $\{\tilde{u}_{mn}\}_{m,n}$ given by $\tilde{F}(z, w) = \sum_{m,n \geq 0} \tilde{u}_{mn} z^n w^m$, observe that for sequences $\{a_n\}_n$ and $\{b_n\}_n$ such that

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \qquad (n \geq 0),$$

the ordinary generating functions $P(z) = \sum_{n=0}^{\infty} a_n z^n$ of $\{a_n\}_n$ and $Q(z) = \sum_{n=0}^{\infty} b_n z^n$ of $\{b_n\}_n$ are related by

$$Q(z) = \frac{1}{1 - z} P\left(\frac{z}{1 - z} \right).$$

This is related to Knuth's concept of (inverse) binomial transform (see [37]) as well as Flajolet's concept of binomial convolution (see [24, §II.2]). In our case, we find that

$$\sum_{n=0}^{\infty} \tilde{u}_{mn} z^n = \frac{1}{1 - z} P\left(\frac{z}{1 - z} \right),$$

where P is the ordinary generating function of

$$\left\{ \binom{am}{n} (1 - \alpha)^n \right\}_n.$$

Since P is given by

$$P(z) = \sum_{n=0}^{\infty} \binom{am}{n} ((1-\alpha)z)^n = (1 + (1-\alpha)z)^{am},$$

we find that

$$\sum_{n=0}^{\infty} \tilde{u}_{mn} z^n = \frac{1}{1-z} P\left(\frac{z}{1-z}\right)$$

$$= \frac{1}{1-z}\left(1 + (1-\alpha)\frac{z}{1-z}\right)^{am}$$

$$= \frac{1}{1-z}\left(\frac{1-\alpha z}{1-z}\right)^{am}.$$

Summing over m against w^m yields

$$\tilde{F}(z,w) = \sum_{m,n\geq 0} \tilde{u}_{mn} z^n w^m = \frac{1}{1-z}\sum_{m\geq 0}\left[\left(\frac{1-\alpha z}{1-z}\right)^a w\right]^m$$

$$= \frac{\frac{1}{1-z}}{1 - \left(\frac{1-\alpha z}{1-z}\right)^a w} = \frac{\varphi(z)}{1 - w\nu(z)}, \tag{7.10}$$

where

$$\varphi(z) = \frac{1}{1-z}, \qquad \nu(z) = \left(\frac{1-\alpha z}{1-z}\right)^a.$$

This is as claimed. $\qquad\square$

Since $\{u_n\}_n$ is the diagonal of a bivariate sequence having rational generating function, $G(x) = \sum_{n=0}^{\infty} u_n x^n$ is algebraic. This was first proved by Furstenberg in [27]. In order to compute G, we will use the method given by Stanley ([58, p. 179]).

We rewrite $\tilde{F}(z,w)$ as

$$\tilde{F}(z,w) = \sum_{m,n\geq 0} \tilde{u}_{mn} z^n w^m = \frac{(1-z)^{a-1}}{(1-z)^a - w\,(1-\alpha z)^a};$$

then we substitute $w = x/z$ and divide by z to obtain

$$\frac{1}{z}\tilde{F}(z,x/z) = \sum_{m,n\geq 0} \tilde{u}_{mn} z^{n-m-1} x^m = \frac{(1-z)^{a-1}}{z(1-z)^a - x\,(1-\alpha z)^a}.$$

We see from here that

$$G(x) = \sum_{m=0}^{\infty} \tilde{u}_{mm} x^m$$

is the coefficient of z^{-1} in $\frac{1}{z}\tilde{F}(z, x/z)$. This is equal to the residue of $\frac{1}{z}\tilde{F}(z, x/z)$ at its unique pole $z(x)$ that tends to zero as x tends to zero. It is a simple pole and so the residue is obtained by evaluating the numerator at $z = z(x)$ and dividing by the derivative of the denominator evaluated at $z = z(x)$. This gives

$$G(x) = \frac{(1 - z(x))^{a-1}}{(1 - z(x))^{a-1}(1 - (a+1)z(x)) + a\alpha x(1 - \alpha z(x))^{a-1}}. \tag{7.11}$$

Once we find a polynomial $P(x, y)$ such that $P(x, G(x)) = 0$, we can use P to expand G into a Puiseux series about any chosen value of x. In particular, if we expand about the singularities of G having least nonzero modulus (the dominant singularities of G) then we can transfer the data appearing in these expansions by way of the singularity analysis of Flajolet and Sedgewick to obtain a full asymptotic expansion for u_n, valid as $n \to \infty$.

In our case, we will show that $G(x)$ admits an asymptotic expansion near each of its dominant singularities ζ that involves sums of the form

$$a_0(\zeta - x)^{-p/q}\left(1 + \sum_{k=1}^{\infty} c_k(\zeta - x)^k\right)$$

for suitable $p, q \in \mathbb{N}$. By Corollary 5.1, we obtain the asymptotic term

$$\frac{a_0\zeta^{-p/q-n}n^{p/q-1}}{\Gamma(p/q)}\left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{n^\ell}\right), \qquad (\mu_\ell \in \mathbb{Q}(\zeta, c_1, \ldots, c_\ell)) \tag{7.12}$$

in the asymptotic expansion of u_n. We will then show that $y = G(x)$ satisfies a polynomial $P(x, y) \in \mathbb{Q}[x, y]$ of degree $a + 1$ in y. It will follow from Theorem 5.2 that $G(x)$ satisfies a linear ordinary differential operator with coefficients in $\mathbb{Q}[x]$ of order $a + 1$. By the method of Frobenius, the expression

$$(\zeta - x)^{-p/q}\left(1 + \sum_{k=1}^{\infty} c_k(\zeta - x)^k\right)$$

will be a series solution to the corresponding ordinary differential equation which will lead to a linear recurrence relation for the c_k of the form

$$\sum_{j=0}^{s} Q_j(k)c_{k+j} = 0 \qquad (k \geq 0),$$

for some s and suitable polynomials $Q_0(x), Q_1(x), \ldots, Q_s(x) \in \mathbb{Q}(\zeta)[x]$ with $Q_s \neq 0$. From this we conclude that all of the c_k lie in $\mathbb{Q}(\zeta)$ provided that $c_1, c_2, \ldots, c_{s-1}$ lie

in $\mathbb{Q}(\zeta)$. For each of the cases that remain after applying the methods of Pemantle and Wilson, we show that this is indeed the case and conclude that $c_k \in \mathbb{Q}(\zeta)$ for all k. Since ζ will lie in $\mathbb{Q}(\sqrt{\Delta_g})$, we conclude ultimately that $c_k \in \mathbb{Q}(\sqrt{\Delta_g})$ for all k. From (7.12), we then have $\mu_\ell \in \mathbb{Q}(\sqrt{\Delta_g})$ for all ℓ as well.

7.3 Some Auxiliary Results

Our simple pole $z = z(x)$ satisfies

$$x = \frac{z(1-z)^a}{(1-\alpha z)^a}.$$

Also, from (7.11) we see that for this value of z, we have

$$G(x) = \frac{(1 - z(x))^{a-1}}{(1 - z(x))^{a-1}(1 - (a+1)z(x)) + a\alpha x(1 - \alpha z(x))^{a-1}}.$$

If we eliminate x, with $y = G(x)$, we have the parametric equations

$$x = \frac{z(1-z)^a}{(1-\alpha z)^a}, \qquad y = \frac{1 - \alpha z}{p(z)}, \qquad p(z) = \alpha z^2 + (a\alpha - a - \alpha - 1)z + 1. \quad (7.13)$$

We can therefore determine the singularities of G by computing $\frac{dy}{dx}$ implicitly. We obtain

$$G'(x) = \frac{-(1 - \alpha z)^{a+1} q(z)}{(1 - z)^a p(z)^3}$$

where

$$q(z) = \alpha^2 z^3 - \alpha(\alpha + 2)z^2 + (a + 1 - a\alpha + 2\alpha)z + a\alpha - a - 1.$$

Now, since we seek the singularities of least nonzero modulus and $x = 0$ when $z = 1$ and $x \to \infty$ as $z \to \frac{1}{\alpha}$, we can exclude these values of z from contention. Also, if p and q share a root then their resultant, given by

$$a^2 \alpha^2 (1 - \alpha)^3 ((a - 1)^2 \alpha - (a + 1)^2)$$

would have to vanish. Since $a \in \mathbb{N}$ and $\alpha \notin \{0, 1\}$, this would force $(a - 1)^2 \alpha - (a + 1)^2 = 0$, so that

$$\alpha = \left(\frac{a+1}{a-1}\right)^2.$$

But this forces p to have a double root at $z = \frac{1-a}{1+a}$ which then appears in the denominator with multiplicity 6. Since it appears as a root of q with multiplicity at most

3, it follows that, in any case, the roots of $p(z) = \alpha z^2 + (a\alpha - a - \alpha - 1)z + 1$ are singularities. In the case this polynomial has complex conjugate roots, both roots correspond to dominant singularities while in the case this polynomial has real roots, the corresponding value of x having smaller absolute value is the unique dominant singularity.

In order to find $P(x, y)$, we eliminate z from the parametric equations given by (7.13). This is done by calculating the resultant of

$$p(z)y - (1 - \alpha z) \qquad \text{and} \qquad (1 - \alpha z)^a x - z(1 - z)^a \qquad (7.14)$$

with respect to z, where

$$p(z) = \alpha z^2 + (a\alpha - a - \alpha - 1)z + 1 = (1 - z)(1 - \alpha z) - a(1 - \alpha)z.$$

This resultant is given by

$$R(x, y) = (\alpha y)^{a+1} \prod_{j=1}^{2} [(1 - \alpha z_j(y))^a x - z_j(y)(1 - z_j(y))^a]$$

where $z_1(y)$ and $z_2(y)$ are the roots of $p(z)y - (1 - \alpha z)$ (see, e.g., [28, Ch. 12]). A calculation using the computer algebra system Maple 11 (see [43]) determines that

$$R(x, y) = a^a \alpha^{a+1} (\alpha - 1)^a y^{a+1} x^2 + S(y)x + (\alpha - 1)^a (y - 1)(ay + 1)^a. \qquad (7.15)$$

where

$$S(y) = \frac{(\alpha - 1)^a}{2^{a+1}} \left((L^-(y) - \sqrt{\Delta(y)})(L^+(y) + \sqrt{\Delta(y)})^a \right.$$
$$\left. + (L^-(y) + \sqrt{\Delta(y)})(L^+(y) - \sqrt{\Delta(y)})^a \right)$$
$$= \frac{(\alpha - 1)^a}{2^a} \left(L^-(y) \sum_k \binom{a}{2k} L^+(y)^{a-2k} \Delta(y)^k \right.$$
$$\left. - \Delta(y) \sum_k \binom{a}{2k+1} L^+(y)^{a-2k-1} \Delta(y)^k \right), \qquad (7.16)$$

$$L^+(y) = (\alpha(a - 1) + (a + 1))y + \alpha, \qquad L^-(y) = (\alpha(a - 1) - (a + 1))y + \alpha, \qquad (7.17)$$

and

$$\Delta(y) = (\alpha - 1)((a - 1)^2 \alpha - (a + 1)^2)y^2 + 2\alpha(a - 1)(\alpha - 1)y + \alpha^2$$
$$= L^+(y)^2 - 4a\alpha y(ay + 1) = L^-(y)^2 - 4\alpha y(y - 1). \qquad (7.18)$$

We then have $P(x, G(x)) = 0$, where we set

$$P(x, y) = \frac{R(x, y)}{(\alpha - 1)^a} = a^a \alpha^{a+1} y^{a+1} x^2 + \frac{S(y)}{(\alpha - 1)^a} x + (y - 1)(ay + 1)^a.$$

In particular, the dominant singularities satisfy the resultant of $p(z)$ and $(1 - \alpha z)^a x - z(1 - z)^a$, which is the leading term of $R(x, y)$ as a polynomial in y. Using Maple to compute the coefficient of y in $y^{a+1} R(x, 1/y)$ we find that the coefficient of y^a in $R(x, y)$ equals 0. Also, we have

$$R(x, 0) = -(\alpha - 1)^a.$$

Therefore, with

$$\{\zeta_1, \zeta_2\} = \left\{ \frac{z(1 - z)^a}{(1 - \alpha z)^a} \;\middle|\; p(z) = 0 \right\},$$

we have

$$P(x, y) = \frac{R(x, y)}{(\alpha - 1)^a} = a^a \alpha^{a+1} y^{a+1} x^2 + \frac{S(y)}{(\alpha - 1)^a} x + (y - 1)(ay + 1)^a$$

$$= a^a \alpha^{a+1} (x - \zeta_1)(x - \zeta_2) y^{a+1} - \sum_{k=1}^{a-1} L_k^{(a,\alpha)}(x) y^k - 1 \tag{7.19}$$

for suitable linear polynomials $L_k^{(a,\alpha)}(x) \in \mathbb{Q}[x]$. A further calculation shows that $L_{a-1}^{(a,\alpha)}(x) \neq 0$. Being unable to explicitly determine the $L_k^{(a,\alpha)}(x)$ for general $a \in \mathbb{N}$, we turn to the method of Pemantle and Wilson in order to reduce the problem to finitely many values of a. We will then compute $P(x, y)$ explicitly, on an individual basis, for the cases that remain.

7.4 The Cases Covered by Pemantle and Wilson

We are interested in the asymptotics of the binomial sums

$$\tilde{u}_{mn} = \sum_{k=0}^{n} \binom{n}{k} \binom{am}{k} (1 - \alpha)^k,$$

as m and n tend to infinity in a suitable direction. By setting $m = an$ in the bivariate asymptotic expansions obtained, we may suppose that $a = 1$. We then have

$$\varphi(z) = \frac{1}{1 - z}, \qquad \nu(z) = \frac{1 - \alpha z}{1 - z}.$$

Since $\varphi(z) \neq 0$ for any z satisfying $1 - w\nu(z) = 0$, the set S_{mn} defined by (5.12) is given by

$$S_{mn} = \{z \in \mathbb{C} \mid (z, \nu(z)^{-1}) \text{ is minimal}, \ mz\nu'(z) = n\nu(z), \ mz\nu''(z) \neq (n-m)\nu'(z)\}.$$

But

$$\frac{z\nu'(z)}{\nu(z)} = \frac{1}{1-z} - \frac{1}{1-\alpha z}, \qquad \frac{z\nu''(z)}{\nu'(z)} = \frac{2z}{1-z}.$$

Denoting the set of minimal points by $\mathcal{M}$, we can therefore rewrite the conditions of membership in the set S_{mn} as $(z, \nu(z)^{-1}) \in \mathcal{M}$ and

$$\mu(z) := \frac{1}{1-z} - \frac{1}{1-\alpha z} = \frac{n}{m}, \qquad \frac{2z}{1-z} \neq \frac{n}{m} - 1.$$

The second condition is equivalent to $z \neq (n-m)/(n+m)$, but this follows from the first equation since if $z = (n-m)/(n+m)$, the first equation forces $\alpha = (n+m)^2/(n-m)^2$ which fails to be a constant. Defining f_{mn} by

$$r\alpha f_{mn}(z) = (1-z)(1-\alpha z)(n - m\mu(z)) = n\alpha z^2 - ((1+\alpha)n + (1-\alpha)m)z + n,$$

we can rewrite S_{mn} as

$$S_{mn} = \{z \in \mathbb{C} \mid (z, \nu(z)^{-1}) \in \mathcal{M} \text{ and } f_{mn}(z) = 0\}. \tag{7.20}$$

From now on, we will denote the roots of f_{mn} by z_{mn}^+ and z_{mn}^-, where we have labelled the roots so that $z_{mn}^+ - z_{mn}^- = \sqrt{\Delta_{f_{mn}}}$ where $\Delta_{f_{mn}}$ denotes the discriminant of f_{mn} and $\sqrt{\cdot}$ denotes the principal branch of the square root. Also, the main terms of the asymptotic expansions appearing in Proposition 5.4 are given by

$$\frac{\varphi(z_{mn}^\pm)\nu(z_{mn}^\pm)^m}{(z_{mn}^\pm)^n \sqrt{2\pi m Q_{mn}(z_{mn}^\pm)}}, \qquad \text{where} \qquad Q_{mn}(z) = \frac{z^2\nu''(z)}{\nu(z)} - \frac{n(n-m)}{m^2}.$$

A calculation shows that

$$mQ_{mn}(z_{mn}^\pm) = m\left[\frac{(z_{mn}^\pm)^2\nu''(z_{mn}^\pm)}{\nu(z_{mn}^\pm)} + \frac{z_{mn}^\pm\nu'(z_{mn}^\pm)}{\nu(z_{mn}^\pm)} - \left(\frac{z_{mn}^\pm\nu'(z_{mn}^\pm)}{\nu(z_{mn}^\pm)}\right)^2\right]$$

$$= \frac{m(1-\alpha)z_{mn}^\pm(1-\alpha(z_{mn}^\pm)^2)}{(1-z_{mn}^\pm)^2(1-\alpha z_{mn}^\pm)^2} = \frac{n^2}{m(1-\alpha)}\left[\frac{1-\alpha(z_{mn}^\pm)^2}{z_{mn}^\pm}\right].$$

But the product of the roots of f_{mn} is equal to $1/\alpha$ and so

$$z_{mn}^\pm(z_{mn}^\pm \mp \sqrt{\Delta_{f_{mn}}}) = \frac{1}{\alpha} \qquad \text{or} \qquad \frac{1-\alpha(z_{mn}^\pm)^2}{z_{mn}^\pm} = \mp\alpha\sqrt{\Delta_{f_{mn}}}.$$

Therefore, we have

$$mQ_{mn}(z_{mn}^{\pm}) = \frac{n^2}{m(1-\alpha)}\left[\frac{1-\alpha(z_{mn}^{\pm})^2}{z_{mn}^{\pm}}\right] = \pm\frac{n^2\alpha\sqrt{\Delta_{fmn}}}{m(\alpha-1)}.$$

The leading terms of the expansion then become

$$\frac{\varphi(z_{mn}^{\pm})\nu(z_{mn}^{\pm})^m}{(z_{mn}^{\pm})^n\sqrt{\pm 2\pi\frac{n^2\alpha\sqrt{\Delta_{fmn}}}{m(\alpha-1)}}} = \frac{(1-\alpha z_{mn}^{\pm})^m}{n(z_{mn}^{\pm})^n(1-z_{mn}^{\pm})^{m+1}}\sqrt{\pm\frac{(\alpha-1)m}{2\pi\alpha\sqrt{\Delta_{fmn}}}}. \tag{7.21}$$

Finally, we need to determine the set S_{mn}. If $\varepsilon = 0$ so that $\alpha < 0$, then $\tilde{u}_{mn} \geq 0$ for all m and n and so Corollary 5.2 applies and we can conclude that S_{mn} is a singleton, consisting of a single positive real number less than one. By graphing the curve

$$\mu(x) = \frac{1}{1-x} - \frac{1}{1-\alpha x},$$

it is seen that for any $m, n > 0$, $\mu(x) = n/m$ has two solutions, one lying between 0 and 1 and the other being negative and less than $1/\alpha$. It follows that $S_{mn} = \{x_{mn}\}$ where $x_{mn} = z_{mn}^+$. Replacing α with $1 - d$ yields the following result.

Proposition 7.1. *Let $d \in \mathbb{N}$. The polynomials f_{mn} given by*

$$n(1-d)f_{mn}(z) = (1-d)nz^2 + ((d-2)n - dm)z + n,$$

have distinct real roots $x_{mn}^+ > x_{mn}^-$. Define $x_{mn} = x_{mn}^+$. Then $0 < x_{mn} < 1$ and there exist constants $c_\ell^{(m,n)}$ for $\ell \in \mathbb{N}$ such that

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}d^k \sim \frac{(1-(1-d)x_{mn})^m}{nx_{mn}^n(1-x_{mn})^{m+1}}\sqrt{\frac{dm}{2\pi(d-1)\sqrt{\Delta_{fmn}}}}\left(1+\sum_{\ell=1}^{\infty}\frac{c_\ell^{(m,n)}}{m^\ell}\right)$$

as $n, m \to \infty$ (with m/n, n/m remaining bounded), where $\sqrt{\cdot}$ denotes the principal branch of the square root.

We now turn to the alternating case given by $\varepsilon = 1$. This corresponds to the case $\alpha > 1$. We need to determine whether 0, 1 or 2 of the roots of f_{mn} give rise to minimal points. Define

$$\gamma(z) = \frac{1}{\nu(z)} = \frac{1-z}{1-\alpha z}.$$

Every point of $\mathcal{M}$ has first coordinate z such that $\gamma(z)$ realizes the minimum modulus of the points in the image of the closed disk determined by z under γ. That is, if

$(z, w(z))$ is minimal and we define D_t for $t > 0$ to be the image of the closed disk of radius t centred at the origin, we have

$$|\gamma(z)| = \min\{|w| : w \in D_{|z|}\}.$$

We now turn to the determination of such points. We will use the fact that γ is a Möbius transformation defined on the extended complex plane $\mathbb{P}^1(\mathbb{C})$ and as such sends disks to disks, preserving their boundary circles. Let $t > 0$, and consider the circle centred at the origin with radius t. Since

$$\gamma(t) = \frac{1-t}{1-\alpha t}, \tag{7.22}$$

$$\gamma(ti) = \frac{(1+\alpha t^2) + i(\alpha - 1)t}{1 + \alpha^2 t^2}, \tag{7.23}$$

$$\gamma(-t) = \frac{1+t}{1+\alpha t}, \tag{7.24}$$

we see that the image of the circle in question is the unique circle in $\mathbb{P}^1(\mathbb{C})$ passing through the points (7.22), (7.23) and (7.24). This is easily seen to be the unique circle C_t in $\mathbb{P}^1(\mathbb{C})$ having centre lying on the extended real axis $\mathbb{P}^1(\mathbb{R})$ for which $C_t \cap \mathbb{P}^1(\mathbb{R}) = \{\gamma(-t), \gamma(t)\}$. In case $t = 1/\alpha$, this circle is given by $C_{1/\alpha} = \{z \in \mathbb{C} \mid \Re(z) = \frac{\alpha+1}{2\alpha}\} \cup \{\infty\} \subseteq \mathbb{P}^1(\mathbb{C})$. Now, each circle in $\mathbb{P}^1(\mathbb{C})$ is the boundary circle of two disks in $\mathbb{P}^1(\mathbb{C})$. Indeed, the exterior of any disk is itself a disk having the same boundary circle. The image of the open disk centred at the origin with radius t will be the open disk in $\mathbb{P}^1(\mathbb{C})$ with boundary circle C_t that contains $\gamma(0) = 1$. Its closure will be the previously defined closed disk D_t. Suppose that $(z, w(z))$ is minimal. Since $1 - w\nu(z) = 0$ we see that $z \neq 1$ so that $\gamma(z) \neq 0$. Letting $|z| = t$, we see that

$$0 \neq |\gamma(z)| = \min\{|w| : w \in D_t\},$$

so that $0 \notin D_t$. Since $1 \in D_t$, we conclude that in order to obtain a minimal point having first coordinate z with modulus t, we require exactly one of 0, 1 to lie between $\gamma(-t)$ and $\gamma(t)$. Also, when this is the case, $z = \pm t$ unless C_t is centred at the origin and has radius less than 1. Indeed, since C_t is centred on the real axis, we see that the minimum modulus of points on C_t occurs at one of $\gamma(t)$, $\gamma(-t)$ and only occurs at additional points if C_t is centred at the origin. This latter case occurs when $\gamma(t) = -\gamma(-t)$ which a calculation shows to occur when $t = 1/\sqrt{\alpha}$. Since $Q_{rs}(z) \neq 0$,

we are excluding $\pm\frac{1}{\sqrt{\alpha}}$, and so we obtain in this case that $|z| = 1/\sqrt{\alpha}$, $z \in \mathbb{C} \setminus \mathbb{R}$. A calculation provides us with the information found in Table 7.1. An inspection of Table 7.1 shows that we fail to obtain minimal points when $t > 1$ and obtain minimal points otherwise. Finally, we need to determine, for $t < 1$, which of $\gamma(t)$, $\gamma(-t)$ is closer to the origin. If $\gamma(-t) \neq -\gamma(t)$ then we obtain a unique minimal point. We obtain the possible minimal points described in Table 7.2. Also, in the limiting case $t \to \frac{1}{\alpha}$, the image of $|z| = t$ under γ is equal to $\Re(z) = \frac{\alpha+1}{2\alpha}$. We therefore obtain minimal points for this modulus since $0 < \frac{\alpha+1}{2\alpha} < 1$ when $\alpha > 1$. The minimal point obtained in this case is given by $\left(-\frac{1}{\alpha}, \frac{\alpha+1}{2\alpha}\right)$. Putting this all together gives the following characterization of the set $\mathcal{M}$ of minimal points:

Proposition 7.2. *For $\alpha > 1$ and excluding $\pm 1/\sqrt{\alpha}$, the set of minimal points is given by*

$$\left\{ (x, \gamma(x)) \;\Big|\; -\frac{1}{\sqrt{\alpha}} < x < 0 \; or \; \frac{1}{\sqrt{\alpha}} < x < 1 \right\} \cup \left\{ (z, \gamma(z)) \;\Big|\; |z| = \frac{1}{\sqrt{\alpha}}, \; z \in \mathbb{C} \setminus \mathbb{R} \right\}.$$

Proof. We showed above that these are the only possibilities for minimal points. What needs to be shown here is that each of these candidates is in fact minimal. In each case, we know that for our candidate $(z, w(z))$, we have

$$|\gamma(z)| = \min\{|\gamma(z')| : |z'| \leq |z|\}. \tag{7.25}$$

Now, if $|z'| \leq |z|$ and $|w(z')| \leq |w(z)|$, we obtain $|\gamma(z')| \leq |\gamma(z)|$. By (7.25) we conclude that $|\gamma(z')| = |\gamma(z)|$ so that $|w(z')| = |w(z)|$. We have therefore reduced the proof that $(z, w(z))$ is minimal to the verification that $|z'| = |z|$. For $z = x \in \mathbb{R}$, $\gamma(x)$ is the unique point of $D_{|x|}$ of least modulus, and so we can conclude from $|\gamma(z')| = |\gamma(x)|$ that $\gamma(z') = \gamma(x)$. By applying γ^{-1}, we obtain that $z' = x$ so that $|z'| = |x|$, as required. The remaining case is given by $|z| = \frac{1}{\sqrt{\alpha}}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. In this case, $D_{|z|}$ consists precisely of the complex numbers with modulus at least $|\gamma(z)|$, and for $|z'| < |z|$ we have $|\gamma(z')| > |\gamma(z)|$. We conclude that $|z'| = |z|$ in this case as well. $\square$

With the above notation, we have

$$S_{mn} = \{z \in \mathbb{C} \mid (z, \nu(z)^{-1}) \in \mathcal{M} \text{ and } f_{mn}(z) = 0\}.$$

Range for t	Ordering of 0, 1, $\gamma(t)$, $\gamma(-t)$
$0 < t < \frac{1}{\alpha}$	$0 < \gamma(-t) < 1 < \gamma(t)$
$\frac{1}{\alpha} < t < 1$	$\gamma(t) < 0 < \gamma(-t) < 1$
$t > 1$	$0 < \gamma(t) < \gamma(-t) < 1$

Table 7.1: Ordering of γ values.

Range for t	Ordering of 0, 1, $\gamma(t)$, $\gamma(-t)$	Possible minimal point(s)
$0 < t < \frac{1}{\alpha}$	$0 < \gamma(-t) < 1 < \gamma(t)$	$\{(-t, \gamma(-t))\}$
$\frac{1}{\alpha} < t < \frac{1}{\sqrt{\alpha}}$	$\gamma(t) < 0 < \gamma(-t) < 1,\ |\gamma(-t)| < |\gamma(t)|$	$\{(-t, \gamma(-t))\}$
$t = \frac{1}{\sqrt{\alpha}}$	$\gamma(t) < 0 < \gamma(-t) < 1,\ \gamma(-t) = -\gamma(t)$	$\{(z, \gamma(z)) :\ |z| = t,\ z \notin \mathbb{R}\}$
$\frac{1}{\sqrt{\alpha}} < t < 1$	$\gamma(t) < 0 < \gamma(-t) < 1,\ |\gamma(t)| < |\gamma(-t)|$	$\{(t, \gamma(t))\}$

Table 7.2: Possible minimal points for $0 < t < 1$.

Range for n/m	Values of x_1 and x_2
$0 < \frac{n}{m} < \mu\left(-\frac{1}{\sqrt{\alpha}}\right)$	$x_1 < -\frac{1}{\sqrt{\alpha}} < x_2 < 0$
$\frac{n}{m} = \mu\left(-\frac{1}{\sqrt{\alpha}}\right)$	$x_1 = x_2 = -\frac{1}{\sqrt{\alpha}}$
$\frac{n}{m} = \mu\left(\frac{1}{\sqrt{\alpha}}\right)$	$x_1 = x_2 = \frac{1}{\sqrt{\alpha}}$
$\frac{n}{m} > \mu\left(\frac{1}{\sqrt{\alpha}}\right)$	$\frac{1}{\alpha} < x_1 < \frac{1}{\sqrt{\alpha}} < x_2 < 1$

Table 7.3: Location of the roots of $\mu(x) = n/m$.

A calculation shows that for $|z| = 1/\sqrt{\alpha}$, in order for $f_{mn}(z) = 0$, we require $z \in \mathbb{R}$. Since this case is being excluded, we may suppose that $|z| \neq \frac{1}{\sqrt{\alpha}}$. Then every minimal point has real coordinates. We wish to locate the real roots x of f_{mn} that lie in suitable intervals determined by $\mathcal{M}$. By sketching the graph of

$$\mu(x) = \frac{1}{1 - x} - \frac{1}{1 - \alpha x},$$

we find that for $\mu(-1/\sqrt{\alpha}) < \frac{n}{m} < \mu(1/\sqrt{\alpha})$ we have no real solutions to $\mu(x) = \frac{n}{m}$, and otherwise, we have real solutions $x_1 \leq x_2$ to $\mu(x) = \frac{n}{m}$ determined as in Table 7.3. Here, we have

$$\mu\left(-\frac{1}{\sqrt{\alpha}}\right) = \frac{\sqrt{\alpha} - 1}{\sqrt{\alpha} + 1}, \qquad \mu\left(\frac{1}{\sqrt{\alpha}}\right) = \frac{\sqrt{\alpha} + 1}{\sqrt{\alpha} - 1}.$$

Since $\frac{n}{m} = \mu\left(\pm\frac{1}{\sqrt{\alpha}}\right)$ results in roots having modulus $1/\sqrt{\alpha}$, this possibility has been excluded. We have therefore determined that for $\alpha > 1$ we have

$$S_{mn} = \begin{cases} \emptyset & \text{if } \frac{\sqrt{\alpha}-1}{\sqrt{\alpha}+1} \le \frac{n}{m} \le \frac{\sqrt{\alpha}+1}{\sqrt{\alpha}-1}; \\ \{(z_{mn}^+, \gamma(z_{mn}^+))\} & \text{otherwise.} \end{cases}$$

We note that the condition that n/m not lie in the above interval is precisely the condition that f_{mn} have distinct real roots. Replacing α with $d+1$ yields

$$n(d+1)f_{mn}(z) = (d+1)nz^2 - ((d+2)n - dm)z + n.$$

The polynomials f_{mn} have distinct real roots $x_{mn}^+ > x_{mn}^-$ whenever

$$\frac{n}{m} \notin \left[\frac{\sqrt{d+1}-1}{\sqrt{d+1}+1}, \frac{\sqrt{d+1}+1}{\sqrt{d+1}-1}\right].$$

Putting this all together yields the following result.

Proposition 7.3. *With the above notation, define $x_{mn} = x_{mn}^+$. Then there exist constants $c_\ell^{(m,n)}$ for $\ell \in \mathbb{N}$ such that*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m}{k}d^k \sim \frac{(1-(d+1)x_{mn})^m}{nx_{mn}^n(1-x_{mn})^{m+1}}\sqrt{\frac{dm}{2\pi(d+1)\sqrt{\Delta_{f_{mn}}}}}\left(1+\sum_{\ell=1}^{\infty}\frac{c_\ell^{(m,n)}}{m^\ell}\right)$$

as $m, n \to \infty$ $\left(\text{with } m/n, n/m \text{ remaining bounded and } n/m \notin \left[\frac{\sqrt{d+1}-1}{\sqrt{d+1}+1}, \frac{\sqrt{d+1}+1}{\sqrt{d+1}-1}\right]\right)$, where $\sqrt{\cdot}$ denotes the principal branch of the square root.

If we now look in the direction given by $m = an$, Proposition 7.1 and Proposition 7.3 provide us with a proof of Theorem 7.1 in case $\Delta_g > 0$. We are therefore reduced to proving Theorem 7.1 in case $\Delta_g < 0$ and proving Theorem 7.2.

7.5 The Remaining Cases

The cases not covered by Section 7.4 all have $\varepsilon = 1$ so that our sequence of interest is given by

$$u_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{an}{k}d^k.$$

The remaining cases correspond to $a, d \in \mathbb{N}$ such that $(a-1)^2 d \le 4a$. These values of a and d are given in Table 7.4.

a	1	2	3	4	5
d	$1 \leq d$	$1 \leq d \leq 8$	$1 \leq d \leq 3$	1	1

Table 7.4: Values of a and d for which $(a-1)^2 d \leq 4a$.

Recall that our plan is to calculate the polynomial $P(x, y)$ given by (7.19) that is satisfied by $y = G(x)$. We then use $P(x, y)$ to compute the Puiseux expansion for $G(x)$ about its dominant singularities which occur at values of x that correspond to roots z of $p(z)$. We then obtain full asymptotic expansions for u_n valid as $n \to \infty$ by applying Proposition 5.1. Recall further that from Corollary 5.1, the transfer of asymptotics for G to asymptotics for u_n can be expressed as

$$a_0(\zeta - x)^{-p/q}\left(1 + \sum_{k=1}^{\infty} c_k(\zeta - x)^k\right) \mapsto \frac{a_0 \zeta^{-p/q-n} n^{p/q-1}}{\Gamma(p/q)}\left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{n^\ell}\right). \quad (7.26)$$

where the constants $\mu_\ell \in \mathbb{Q}(\sqrt{\Delta_g}, c_1, \ldots, c_\ell)$. Finally, we use a linear ODE satisfied by G to obtain a linear recurrence relation satisfied by the c_k. The recurrence obtained will be used to show that all of the c_k lie in $\mathbb{Q}(\sqrt{\Delta_g})$, where g and its discriminant Δ_g are given by (7.5) and (7.6) respectively. We will then have that all of the μ_ℓ lie in $\mathbb{Q}(\sqrt{\Delta_g})$ as well. We start with the case $a = 1$.

In this case, with $\alpha = d + 1$, we are considering the sequence

$$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 (1 - \alpha)^k,$$

which is the diagonal of the bivariate sequence given by

$$\tilde{u}_{mn} = \sum_{k=0}^{n} \binom{n}{k}\binom{m}{k}(1 - \alpha)^k.$$

We find that

$$G(x) = \frac{1}{1 - 2z(x) + \alpha x},$$

where $z(x)$ is the unique root of $z(1 - z) - x(1 - \alpha z)$ that tends to 0 as x tends to 0. The two roots of this polynomial are given by

$$\frac{\alpha x + 1 \pm \sqrt{\alpha^2 x^2 + 2(\alpha - 2)x + 1}}{2},$$

and the sign that gives the root that tends to 0 as x tends to zero is the $-$ sign. We conclude that

$$z(x) = \frac{\alpha x + 1 - \sqrt{\alpha^2 x^2 + 2(\alpha - 2)x + 1}}{2},$$

so that

$$G(x) = \frac{1}{\sqrt{\alpha^2 x^2 + 2(\alpha - 2)x + 1}}.$$

We see from this that the dominant singularities of G are given by the roots ζ and $\overline{\zeta}$ of

$$\alpha^2 x^2 + 2(\alpha - 2)x + 1.$$

These roots are

$$\zeta = \frac{2 - \alpha - 2i\sqrt{\alpha - 1}}{\alpha^2}, \qquad \overline{\zeta} = \frac{2 - \alpha + 2i\sqrt{\alpha - 1}}{\alpha^2}.$$

We now expand $G(x)$ into a Puiseux expansion about ζ and $\overline{\zeta}$ and then transfer by way of (7.26) to obtain our asymptotic expansion for u_n. We find that $G(x)$ admits the following expansions in suitable neighbourhoods of ζ and $\overline{\zeta}$:

$$G(x) = a_0(\zeta - x)^{-1/2}\left(1 + \sum_{k=1}^{\infty} c_k(\zeta - x)^k\right),$$

$$G(x) = \overline{a_0}(\overline{\zeta} - x)^{-1/2}\left(1 + \sum_{k=1}^{\infty} \overline{c_k}(\overline{\zeta} - x)^k\right)$$

for constants $c_1, c_2, c_3, \ldots$ and

$$a_0 = \frac{1 + i}{2^{3/2} d^{1/4}}.$$

Further, $G(x)$ satisfies the linear ODE given by

$$(\alpha + \alpha^2 x - 2)y(x) + (1 + 2\alpha x + \alpha^2 x^2 - 4x)y'(x) = 0. \qquad (7.27)$$

Substituting in

$$G(x) = (\zeta - x)^{-1/2}\sum_{k=0}^{\infty} c_k(\zeta - x)^k$$

leads to the recurrence relation $c_0 = 1$ and

$$\frac{c_k}{c_{k-1}} = \frac{\alpha^2}{4\sqrt{1 - \alpha}}\left(1 - \frac{1}{2k}\right) \qquad (k \geq 1).$$

We obtain

$$c_k = \frac{c_k}{c_{k-1}} \frac{c_{k-1}}{c_{k-2}} \cdots \frac{c_1}{c_0} c_0 = \frac{\alpha^{2k}}{4^k(1-\alpha)^{k/2}} \prod_{j=1}^{k} \left(1 - \frac{1}{2j}\right) = \binom{k-1/2}{k} \frac{\alpha^{2k}}{4^k(1-\alpha)^{k/2}}.$$

Since each of the $c_k \in \mathbb{Q}(i\sqrt{d})$, we obtain that for $d \in \mathbb{N}$ there exists an asymptotic expansion

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 d^k \sim \frac{(1+i)(1-i\sqrt{d})^{2n+1}}{2^{3/2}d^{1/4}\sqrt{\pi n}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{n^\ell}\right) +$$

$$\frac{(1-i)(1+i\sqrt{d})^{2n+1}}{2^{3/2}d^{1/4}\sqrt{\pi n}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\overline{\mu_\ell}}{n^\ell}\right) \qquad (n \to \infty)$$

where the constants $\mu_\ell \in \mathbb{Q}(i\sqrt{d})$.

A calculation shows that this agrees with Theorem 7.1. Since the above calculations did not require $\varepsilon = 1$, we also obtain that for $d \in \mathbb{N}$, there exists an asymptotic expansion

$$\sum_{k=0}^{n} \binom{n}{k}^2 d^k \sim \frac{(\sqrt{d}+1)^{2n+1}}{2d^{1/4}\sqrt{\pi n}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{n^\ell}\right) \qquad (n \to \infty),$$

where the constants $\mu_\ell \in \mathbb{Q}(\sqrt{d})$. In this case, the method of Stoll and Haible described in Chapter 6 applies and we can say something regarding the divisibility properties of the asymptotic coefficients. Using the notation of Chapter 6, we set $g_n = \sum_{k=0}^{n} \binom{n}{k}^2 d^k$ and

$$\gamma = \frac{\sqrt{d}+1}{2d^{1/4}\sqrt{\pi}}, \qquad \rho = (\sqrt{d}+1)^2, \qquad \varphi = 1/2, \qquad q = 1.$$

With these values of the parameters, Proposition 6.2 implies the following result.

Proposition 7.4. *With the above notation, let K be a number field. Suppose further that $\{b_n\}_n$ is defined by*

$$\Psi(F(x)) = \frac{1}{\sqrt{\pi x}} \left(1 + \sum_{n=1}^{\infty} b_n x^n\right),$$

where each $b_n \in K$. Then the coefficients μ_ℓ for $\ell \geq 0$ all lie in K and the only primes that can divide their denominators are the primes dividing 2 and the primes dividing the denominator of some $n!b_n$.

In order to apply Proposition 7.4, we proceed as follows. We define G to be the generating function of $\{g_n\}_n$ and F to be the generating function of

$$f_n := \frac{g_n}{\gamma \rho^n}.$$

The generating function G satisfies the linear ODE given by (7.27) and so since the generating function for $\{f_n\}_n$ is given by $F(x) = \frac{1}{\gamma} G(\frac{x}{\rho})$ we see that it satisfies the ODE

$$\left(\alpha^2 (x/\rho)^2 + 2(\alpha - 2)(x/\rho) + 1 \right) \rho F'(x) + \left(\alpha^2 (x/\rho) + \alpha - 2 \right) F(x) = 0.$$

It follows that $B(x) := \Psi(F(x))$ defined in Chapter 6 satisfies the ODE

$$\left(\alpha^2 \left(\frac{x+1}{\rho} \right)^2 + 2(\alpha - 2) \left(\frac{x+1}{\rho} \right) + 1 \right) \rho B'(x) + \left(\alpha^2 \left(\frac{x+1}{\rho} \right) + \alpha - 2 \right) B(x) = 0.$$

Solving this ODE for B and substituting in the value for ρ we see that $B(x)$ is given by

$$\frac{\eta}{\sqrt{\alpha^2 x^2 + 8\alpha x + 4\alpha x \sqrt{1-\alpha} - 8x - 8x\sqrt{1-\alpha}}}$$

for some constant η. But by (6.3) and (6.4) we know that

$$B(x) = \frac{1}{\sqrt{\pi x}} \left(1 + \sum_{n=1}^{\infty} b_n x^n \right). \tag{7.28}$$

We conclude that

$$\eta = \frac{2}{\sqrt{\pi}} \sqrt{-2\sqrt{1-\alpha} + 2\alpha + \alpha\sqrt{1-\alpha} - 2}.$$

Substituting in this value for η, replacing α with $1 - d$ and simplifying yields

$$B(x) = \frac{2d^{1/4}}{\sqrt{\pi x} \sqrt{4\sqrt{d} - (\sqrt{d}-1)^2 x}} = \frac{1}{\sqrt{\pi x}} \left(1 - \frac{(\sqrt{d}-1)^2}{4\sqrt{d}} x \right)^{-1/2}. \tag{7.29}$$

Comparing the right-hand sides of (7.28) and (7.29) yields

$$b_k = \binom{-1/2}{k} (-1)^k \delta^k = \binom{k-1/2}{k} \delta^k$$

where

$$\delta = \frac{(\sqrt{d}-1)^2}{4\sqrt{d}}.$$

We conclude from Proposition 7.4 that the only primes that can divide the denominators of the asymptotic coefficients are the prime divisors of 2 and $\sqrt{d}$. Since we can once again replace d with $-d$, we obtain the following result.

a	$P(x,y)$
2	$(4(d+1)^3x^2 + (d^2+20d-8)x+4)y^3 - ((d+1)^2x+3)y - 1$
3	$(27(d+1)^4x^2 + (4d^3+18d^2+216d-54)x+27)y^4$
	$-(3(d+3)(d+1)^2x+18)y^2 - ((d+1)^3x+8)y - 1$
4	$(256(d+1)^5x^2 + (27d^4+144d^3+320d^2+2816d-512)x+256)y^5$
	$-(2(9d^2+32d+48)(d+1)^2x+160)y^3 - (8(d+2)(d+1)^3x+80)y^2$
	$-((d+1)^4x+15)y - 1$
5	$(3125(d+1)^6x^2 + (256d^5+1600d^4+4250d^3+6250d^2+43750d-6250)x$
	$+3125)y^6 - (10(2d+5)(8d^2+15d+25)(d+1)^2x+1875)y^4$
	$-(10(8d^2+25d+25)(d+1)^3x+1000)y^3 - (5(3d+5)(d+1)^4x+225)y^2$
	$-((d+1)^5x+24)y - 1$

Table 7.5: The polynomials $P(x,y)$ for $2 \le a \le 5$.

Proposition 7.5. *Let $d \in \mathbb{Z}$ be nonzero. There exists an asymptotic expansion*

$$\sum_{k=0}^{n} \binom{n}{k}^2 d^k \sim \frac{(1+\sqrt{d})^{2n+1}}{2\sqrt[4]{d}\sqrt{\pi n}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{n^\ell}\right) \qquad (n \to \infty)$$

if $d > 0$, and

$$\sum_{k=0}^{n} \binom{n}{k}^2 d^k \sim \frac{(1+\sqrt{d})^{2n+1}}{2\sqrt[4]{d}\sqrt{\pi n}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{n^\ell}\right) + \frac{(1-\sqrt{d})^{2n+1}}{2\sqrt[4]{d}\sqrt{\pi n}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\overline{\mu_\ell}}{n^\ell}\right) \qquad (n \to \infty)$$

if $d < 0$, where the constants $\mu_\ell \in \mathbb{Q}(\sqrt{d})$ and $\sqrt[4]{\cdot}$ denotes the principal branch of the fourth root. Further, the only primes of $\mathbb{Q}(\sqrt{d})$ that can divide the denominators of the μ_ℓ are the prime divisors of 2 and the prime divisors of $\sqrt{d}$.

We now turn to the other remaining cases.

Our sequence is given by

$$u_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{an}{k} d^k.$$

The cases $2 \le a \le 5$ in Table 7.4 remain to be determined. The polynomials $P(x,y)$ given by (7.19) are as given in Table 7.5.

In case $(a,d) \in \{(2,8),(3,3)\}$, we have a unique dominant singularity equal to

$$\zeta = \frac{1-a}{1+a} \left(\frac{2a}{2a+(a-1)d}\right)^a.$$

Now, according to Maple, in every case we obtain only one form of a Puiseux expansion that fails to be analytic at ζ and so since we know that $G(x)$ fails to be analytic at ζ, the Puiseux expansion of G at ζ must be of this form. Further, if we use Maple to compute the Puiseux expansions of the branches of the roots of $P(x, y)$, we can conclude that the leading term of the expansion for G is off from the leading term obtained by our calculation by at worst a suitable root of unity. The correct root of unity can then be determined numerically. Also, applying the method of Frobenius to a linear ordinary differential operator with coefficients in $\mathbb{Q}[x]$ satisfied by our asymptotic series leads to a linear recurrence relation for the coefficients involved in the expansions. By checking sufficiently many of the terms in the sequence, this recurrence proves that all of the coefficients in question lie in $\mathbb{Q}(\sqrt{\Delta_g})$. We end up with the following propositions.

Proposition 7.6. *For* $(a, d) \in \{(2, 8), (3, 3)\}$, $G(x)$ *admits a Puiseux expansion of the following form in a suitable neighbourhood of* $\zeta = \frac{1-a}{1+a}\left(\frac{2a}{2a+(a-1)d}\right)^a$:

$$G(x) = \frac{a_0}{(\zeta - x)^{2/3}}\left(1 + \sum_{k=1}^{\infty} c_k(\zeta - x)^k\right) + \frac{b_0}{(\zeta - x)^{1/3}}\left(1 + \sum_{k=1}^{\infty} d_k(\zeta - x)^k\right)$$

in case $(a, d) = (2, 8)$ *and*

$$G(x) = \frac{a_0}{(\zeta - x)^{2/3}}\left(1 + \sum_{k=1}^{\infty} c_k(\zeta - x)^k\right) + \frac{b_0}{(\zeta - x)^{1/3}}\left(1 + \sum_{k=1}^{\infty} d_k(\zeta - x)^k\right)$$

$$+ \sum_{k=0}^{\infty} e_k(\zeta - x)^k$$

in case $(a, d) = (3, 3)$. *Here, the constants* c_k *and* d_k *lie in* $\mathbb{Q}$.

Proposition 7.7. *Suppose that*

$$(a, d) \in \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (3, 1), (3, 2), (4, 1), (5, 1)\}.$$

Then, with the above notation, $G(x)$ *admits a Puiseux expansion of the following form in suitable neighbourhoods of* ζ *and* $\overline{\zeta}$ *respectively:*

$$G(x) = \frac{a_0}{\sqrt{\zeta - x}}\left(1 + \sum_{k=1}^{\infty} c_k(\zeta - x)^k\right) + \sum_{k=0}^{\infty} b_k(\zeta - x)^k$$

and

$$G(x) = \frac{\overline{a_0}}{\sqrt{\zeta - x}}\left(1 + \sum_{k=1}^{\infty}\overline{c_k}(\overline{\zeta} - x)^k\right) + \sum_{k=0}^{\infty}\overline{b_k}(\overline{\zeta} - x)^k$$

where each of the c_k lies in $\mathbb{Q}(\sqrt{\Delta_g})$.

Using the transfer method of Flajolet and Sedgewick, we obtain the following asymptotics for our sequence $\{u_n\}_n$.

Proposition 7.8. *Let $(a,d) \in \{(2,8),(3,3)\}$, $\zeta = \frac{1-a}{1+a}\left(\frac{2a}{2a+(a-1)d}\right)^a$, and*

$$u_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{an}{k}d^k.$$

There exist constants a_0, b_0 and $\mu_\ell, \eta_\ell \in \mathbb{Q}$ for $\ell \geq 1$ such that

$$u_n \sim \frac{a_0\zeta^{-n}}{\Gamma(2/3)\zeta^{2/3}n^{1/3}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\mu_\ell}{n^\ell}\right) + \frac{b_0\zeta^{-n}}{\Gamma(1/3)\zeta^{1/3}n^{2/3}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\eta_\ell}{n^\ell}\right) \quad (n \to \infty)$$

Proposition 7.9. *Suppose that*

$$(a,d) \in \{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(2,7),(3,1),(3,2),(4,1),(5,1)\}.$$

Then, with the above notation, there exists an asymptotic expansion

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{an}{k}d^k \sim \frac{a_0\zeta^{-n}}{\sqrt{\pi\zeta n}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\mu_\ell}{n^\ell}\right) + \frac{\overline{a_0}\,\overline{\zeta}^{-n}}{\sqrt{\pi\overline{\zeta} n}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\overline{\mu_\ell}}{n^\ell}\right) \quad (n \to \infty)$$

for some constants $\mu_\ell \in \mathbb{Q}(\sqrt{\Delta_g})$.

In each case, a calculation using Maple shows that we obtain the same leading term as is given in Theorem 7.1 and Theorem 7.2. The only thing left is to establish the divisibility properties of the asymptotic coefficients in the case $(a,d) = (2,8)$. In this case, we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n}{k}8^k \sim \frac{(-27)^n}{2^{2/3}\Gamma(2/3)n^{1/3}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\mu_\ell}{n^\ell}\right) + \frac{(-27)^n}{2^{4/3}\Gamma(1/3)n^{2/3}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\eta_\ell}{n^\ell}\right)$$

as $n \to \infty$. In the notation of Chapter 6, we then set $g_n = u_n^{(1,2,8)}$ and define

$$f_n = \frac{2^{2/3}\Gamma(2/3)g_n}{(-27)^n}$$

so that

$$f_n \sim \frac{1}{n^{1/3}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{n^\ell} \right) + \frac{\Gamma(2/3)}{2^{2/3}\Gamma(1/3)n^{2/3}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\eta_\ell}{n^\ell} \right).$$

We therefore set the parameters $\varphi = 1/3$, $N = 0$, $q = 3$. With these values of the parameters, Proposition 6.2 implies the following result.

Proposition 7.10. *With the above notation, let K be a number field. Suppose further that $\{c_n\}_n$ and $\{d_n\}_n$ are defined by*

$$\Psi(F(x)) = \frac{x^{-2/3}}{\Gamma(1/3)} \left(1 + \sum_{n=1}^{\infty} c_n x^n \right) + bx^{-1/3} \left(1 + \sum_{n=1}^{\infty} d_n x^n \right),$$

where each $c_n, d_n \in K$ and $b \in \mathbb{C}$ is nonzero. Then the coefficients μ_ℓ, η_ℓ for $\ell \geq 0$ all lie in K and the only primes that can divide their denominators are the primes dividing 3 or the denominator of some $n!c_n$ or the denominator of some $n!d_n$.

In order to apply Proposition 7.10, we proceed as follows. A calculation using Maple shows that the generating function $F(x) = \sum_{n=0}^{\infty} f_n x^n$ satisfies the linear ODE

$$(18x^3 - 36x^2 + 18x)F''(x) + (45x^2 - 54x + 9)F'(x) + (9x - 5)F(x) = 0.$$

We conclude that the function $B(x) := \Psi(F(x))$ satisfies the ODE obtained by replacing x with $x + 1$. That is,

$$18x^2(x + 1)B''(x) + 9x(5x + 4)B'(x) + (9x + 4)B(x) = 0.$$

Solving this ODE with Maple yields

$$B(x) = C_1 x^{-1/3} {}_2F_1 \left(\frac{1}{6}, \frac{2}{3}; \frac{4}{3}; -x \right) + C_2 x^{-2/3} {}_2F_1 \left(-\frac{1}{6}, \frac{1}{3}; \frac{2}{3}; -x \right)$$

for some constants C_1, C_2. But from (6.3) and (6.4) we know that

$$B(x) = \frac{x^{-2/3}}{\Gamma(1/3)} \left(1 + \sum_{n=1}^{\infty} c_n x^n \right) + \frac{x^{-1/3}}{2^{2/3}\Gamma(1/3)} \left(1 + \sum_{n=1}^{\infty} d_n x^n \right)$$

for certain sequences $\{c_n\}_n$ and $\{d_n\}_n$. It follows that

$$c_n = [x^n] {}_2F_1 \left(-\frac{1}{6}, \frac{1}{3}; \frac{2}{3}; -x \right) = \frac{(-1)^n(-1/6)_n(1/3)_n}{n!(2/3)_n},$$

$$d_n = [x^n] {}_2F_1 \left(\frac{1}{6}, \frac{2}{3}; \frac{4}{3}; -x \right) = \frac{(-1)^n(1/6)_n(2/3)_n}{n!(4/3)_n},$$

where $(c)_n = c(c+1)\ldots(c+n-1)$ denotes the rising Pochhammer symbol. Since $n!c_n$ and $n!d_n$ have nonnegative valuations at each prime except 2 and 3, we obtain from Proposition 7.10 the divisibility properties stated for the μ_ℓ and η_ℓ. We now explain why it is not as easy to deal with the other exceptional case given by $(a, d) = (3, 3)$. In this case, we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{3n}{k}3^k \sim \frac{2^{2/3}(-16)^n}{3\Gamma(2/3)n^{1/3}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\tilde{\mu}_\ell}{n^\ell}\right) + \frac{2^{1/3}(-16)^n}{3\Gamma(1/3)n^{2/3}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\tilde{\eta}_\ell}{n^\ell}\right)$$

as $n \to \infty$. We set $g_n = u_n^{(1,3,3)}$ and

$$f_n = \frac{3\Gamma(2/3)g_n}{2^{2/3}(-16)^n},$$

so that

$$f_n \sim \frac{1}{n^{1/3}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\tilde{\mu}_\ell}{n^\ell}\right) + \frac{\Gamma(2/3)}{2^{1/3}\Gamma(1/3)n^{2/3}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\tilde{\eta}_\ell}{n^\ell}\right).$$

We therefore set the parameters $\varphi = 1/3$, $N = 0$, $q = 3$. A calculation using Maple shows that the generating function $F(x) = \sum_{n=0}^{\infty} f_n x^n$ satisfies the linear ODE

$$(-4x - 1)F(x) + (-76x^2 + 34x + 2)F'(x) + (-90x^3 + 108x^2 - 18x)F''(x) +$$

$$(-18x^4 + 36x^3 - 18x^2)F'''(x) = 0.$$

It follows that $B(x)$ satisfies the linear differential equation obtained by replacing x with $x + 1$. After simplification, we obtain

$$-18x^2(x+1)^2 B'''(x) - 18x(x+1)(5x+4)B''(x)$$

$$-2(19x + 20)(2x + 1)B'(x) - (4x + 5)B(x) = 0.$$

Solving this ODE using Maple yields

$$B(x) = C_1\,{}_2F_1\left(\frac{1}{6}, \frac{1}{2}; \frac{1}{3}; x+1\right)^2 + C_2(x+1)^{4/3}{}_2F_1\left(\frac{5}{6}, \frac{7}{6}; \frac{5}{3}; x+1\right)^2 +$$

$$C_3(x+1)^{2/3}{}_2F_1\left(\frac{1}{6}, \frac{1}{2}; \frac{1}{3}; x+1\right){}_2F_1\left(\frac{5}{6}, \frac{7}{6}; \frac{5}{3}; x+1\right),$$

for some constants C_1, C_2, C_3. Using the identity

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a, b; a+b+1-c; 1-z) +$$

$$\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}{}_2F_1(c-a,c-b;1+c-a-b;1-z)$$

we can rewrite the expression for B in terms of a series in x but the expression we obtain is very complicated. Therefore we cannot find the coefficients of B as easily as in the $(a,d) = (2,8)$ case.

7.6 Examples

Having proved our main result, we now conclude this chapter with some examples.

Example 7.1. In the limiting case $\varepsilon = 0$ and $d \to 1^+$ we obtain the asymptotic expansion of the binomial coefficients given by Stirling's formula. Let $a \in \mathbb{N}$. There exist constants $\mu_\ell(a)$ for $\ell \in \mathbb{N}$ such that

$$\binom{(a+1)n}{n} = \sum_{k=0}^{n}\binom{n}{k}\binom{an}{k} \sim \frac{\delta\beta^n}{\sqrt{2\pi n}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\mu_\ell(a)}{n^\ell}\right) \qquad (n \to \infty)$$

where

$$\delta = \sqrt{\frac{a+1}{a}}, \qquad \beta = \frac{(a+1)^{a+1}}{a^a}.$$

In particular, the central binomial coefficients satisfy

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\mu_\ell(1)}{n^\ell}\right) \qquad (n \to \infty),$$

and the Catalan numbers satisfy

$$\frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{(n+1)\sqrt{\pi n}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\mu_\ell(1)}{n^\ell}\right) \qquad (n \to \infty).$$

In these special cases, we can conclude further that the $\mu_\ell(1) \in \mathbb{Q}$ and have denominators that are all powers of 2.

Example 7.2. Proposition 7.5 provides us with an asymptotic expansion for generalizations of the central Delannoy numbers. For $d \in \mathbb{Z}$ nonzero, we have constants $\mu_\ell(d) \in \mathbb{Q}(\sqrt{d})$ having denominators divisible only by the prime divisors of 2 and $\sqrt{d}$ such that

$$\sum_{k=0}^{n}\binom{n}{k}^2 d^k \sim \frac{(1+\sqrt{d})^{2n+1}}{2\sqrt[4]{d}\sqrt{\pi n}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\mu_\ell(d)}{n^\ell}\right) \qquad (n \to \infty)$$

if $d > 0$, and

$$\sum_{k=0}^{n}\binom{n}{k}^2 d^k \sim \frac{(1+\sqrt{d})^{2n+1}}{2\sqrt[4]{d}\sqrt{\pi n}}\left(1+\sum_{\ell=1}^{\infty}\frac{\mu_\ell(d)}{n^\ell}\right)+\frac{(1-\sqrt{d})^{2n+1}}{2\sqrt[4]{d}\sqrt{\pi n}}\left(1+\sum_{\ell=1}^{\infty}\frac{\overline{\mu_\ell(d)}}{n^\ell}\right) \quad (n\to\infty)$$

if $d < 0$. In particular, the central Delannoy numbers satisfy

$$\sum_{k=0}^{n}\binom{n}{k}^2 2^k \sim \frac{(2^{1/4}+2^{-1/4})}{2\sqrt{\pi n}}(3+2\sqrt{2})^n\left(1+\sum_{\ell=1}^{\infty}\frac{\mu_\ell(2)}{n^\ell}\right) \quad (n\to\infty),$$

where the $\mu_\ell(2)$ lie in $\mathbb{Q}(\sqrt{2})$ and have denominators divisible only by the prime $\sqrt{2}\mathbb{Z}[\sqrt{2}]$.

Example 7.3 (The Conjecture of Chamberland and Dilcher). The special case given by $\varepsilon = 1$, $a = 2$, $d = 1$ yields

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n}{k} \sim \frac{\delta\beta^n}{\sqrt{2\pi n}}\left(1+\sum_{\ell=1}^{\infty}\frac{\mu_\ell}{n^\ell}\right)+\frac{\overline{\delta}\overline{\beta}^n}{\sqrt{2\pi n}}\left(1+\sum_{\ell=1}^{\infty}\frac{\overline{\mu_\ell}}{n^\ell}\right) \quad (n\to\infty),$$

where

$$\delta = \frac{1}{\sqrt{-\sqrt{-7}}}\left(\frac{-31-3\sqrt{-7}}{8}\right)^{1/4}, \qquad \beta = \frac{-13+7\sqrt{-7}}{8}$$

and the μ_ℓ lie in $\mathbb{Q}(\sqrt{-7})$. In [12] the authors conjecture that the coefficient of $\beta^n/\sqrt{n}$ is very close to

$$0.3468\exp\left(i\pi\frac{20}{1001}\right) \approx .3461170356 + 0.02175402677i.$$

The correct value of of this coefficient evaluates to

$$\frac{\delta}{\sqrt{2\pi}} = \frac{1}{\sqrt{-2\pi\sqrt{-7}}}\left(\frac{-31-3\sqrt{-7}}{8}\right)^{1/4} \approx .3461762814 + 0.02172120012i.$$

Chapter 8

Discussion

8.1 Summary

Sequences that satisfy linear recurrence relations with constant coefficients have been studied extensively. Such sequences admit a closed form expression given by Binet's formula that facilitates their study. This closed form expression can be used to derive the Skolem-Mahler-Lech Theorem that describes the set of zero terms in such sequences. This closed form also determines the asymptotics of such sequences completely. In this thesis, the objects of study are sequences that satisfy linear recurrence relations with generally nonconstant coefficients and the goal is to explore what can be said regarding the set of zero terms and the asymptotics of such sequences in this more general setting.

By the theorem of Skolem-Mahler-Lech, the set of zero terms of a sequence that satisfies a linear recurrence relation with constant coefficients taken from a field of characteristic zero is comprised of the union of finitely many arithmetic progressions together with a finite exceptional set. Further, in the nondegenerate case where no two eigenvalues of the sequence share a common power, we can eliminate the possibility of arithmetic progressions and conclude that there are only finitely many zero terms. For generally nonconstant coefficients, the first generalization of the theorem of Skolem-Mahler-Lech is due to Bézivin. In [5], Bézivin shows that under suitable conditions, one can extend the theorem of Skolem-Mahler-Lech to sequences that satisfy linear recurrence relations with coefficients that are polynomials in the index as long as the finite exceptional set is replaced with an exceptional set of density zero. This is generalized in [45], where Methfessel shows that for more general

coefficient sequences, the same conclusion holds; the set of zero terms is comprised of finitely many arithmetic progressions together with an exceptional set of density zero. In that paper, similarly to the constant coefficient subcase, a condition is provided that allows for the elimination of the possibility of arithmetic progressions in the sets of zero terms of such sequences. The sequences in question satisfy a recurrence of least positive order, and it is shown that in the case a given sequence cannot be sectioned into sequences satisfying recurrences of lower order, one is left with a set of zero terms of density zero. It can then be shown that this sectioning condition that eliminates the possibility of arithmetic progressions being present in the set of zero terms reduces to the eigenvalue condition given in the theorem of Skolem-Mahler-Lech. It is then possible to consistently define nondegeneracy in the general case (Definition 3.2) and obtain a unified statement for a generalization of the theorem of Skolem-Mahler-Lech (Proposition 3.2). Laohakosol in [41] and Bézivin-Laohakosol in [6] generalize the theorem of Skolem-Mahler-Lech in a different direction. They find more general sequences that satisfy the conclusion of the theorem of Skolem-Mahler-Lech.

The asymptotic theory of sequences that satisfy linear recurrence relations with generally nonconstant coefficients begins with the basic theorems of Poincaré and Perron from [54] and [51]. Together, they state that for linear recurrence relations with coefficients that converge as the index tends to infinity (recurrences of Poincaré type), if the characteristic equation has roots with distinct moduli, then for every non-trivial solution sequence, the quotient of successive terms converges to a root of the characteristic equation, and conversely, every root is realized in such a fashion. There are several generalizations of the theorems of Poincaré and Perron that hold for sequences of Poincaré type. Notably, [14, 53, 21, 38, 39] deal with such generalizations. If we restrict the coefficient sequences of our linear recurrences to be polynomials in the index (the holonomic case), much more can be said about asymptotics. Under quite general conditions, we obtain full asymptotic expansions of a predictable form for holonomic sequences. These expansions can be obtained by applying the transfer method of Flajolet and Sedgewick (see [24]) or, in some cases, by applying the bivariate method of Pemantle and Wilson (see [50]). In particular, these methods can be applied to a family of binomial sums coming from certain weighted lattice paths and full asymptotic expansions are obtained. See (Theorems 7.1, 7.2). The leading terms

of the expansions are obtained explicitly in all cases, a field containing the asymptotic coefficients is obtained in several subcases, and, it is possible in some cases to provide divisibility properties for the asymptotic coefficients using a generalized version of a method of Stoll and Haible's from [59].

8.2   Original Material in This Thesis

In this section I describe my original contributions to this thesis.

Chapter 2

As one peruses the literature on linear recurrence sequences, one finds that the focus lies heavily on sequences that satisfy linear recurrence relations with polynomial coefficients. Even when authors consider a more general setting, the examples they provide typically involve polynomial coefficients. In Chapter 2, I provide an explanation for why this setting comes up so naturally. I start by explaining why it is convenient to take the coefficients of our recurrence operators to lie in a subfield F of the sequence space $\mathcal{L}_K$, and then show that for such fields F we have $F \cap \overline{K} = K$ (see (2.1)). This shows that F is generated over K entirely by elements transcendental over K. The simplest transcendence degree 1 case is given by rational functions in one variable. This corresponds to polynomial coefficients.

Chapter 3

The theorem of Skolem-Mahler-Lech explains the structure of the set of zero terms in sequences that satisfy linear recurrence relations with constant coefficients. They are comprised of finitely many arithmetic progressions together with a finite exceptional set. For nonconstant coefficients, Methfessel proved that one obtains finitely many arithmetic progressions together with an exceptional set of density zero. In both cases, a condition is given to eliminate the possibility of arithmetic progressions, and I showed in Chapter 3 that the condition given by Methfessel reduces to the condition given for the constant coefficients case. It was then possible for me to provide a consistent definition of degeneracy for recurrence sequences with generally nonconstant coefficients and obtain a unified result.

Chapter 4

In [39] Kooman obtains asymptotic formulae for zeros of second order linear recurrence relations with nonconstant coefficients. In Section 4.4 I motivate these results using a heuristic argument involving Frobenius series and indicial equations.

Chapter 5

In [38, Chapter 2], Kooman generates field extensions of $\mathbb{Q}$ by taking limits of quotients of zeros of $\mathbb{Q}(n)$-recurrence operators. In particular, Kooman shows how to generate the field of real algebraic numbers. Studying the proofs from that chapter of [38] shows that the arguments go through for a general number field endowed with an arbitrary valuation. In Section 5.2, I remark that if the results are applied in the nonarchimedean case, one can generate decomposition fields for the absolute Galois group of number fields.

In Section 5.5, the bivariate method of Pemantle and Wilson is described. Along the way, a quantity Q (given by (5.11)) is defined. This quantity is defined by a fairly complicated expression involving partial derivatives. In Section 5.5, I derive a simple algebraic expression for this quantity. This expression shows that Q can be considered, in some sense, as a measure of the noncommutativity of the operator θ in the Weyl algebra.

Chapter 6

In [59], Stoll and Haible develop a method that can, in some cases, determine divisibility properties of the asymptotic coefficients of sequences known to admit full asymptotic expansions of a particular form. Their method applies to the case of rational asymptotic coefficients in the case the quantity $\Psi(F(x))$ is a rational power of x times a power series. I showed in Proposition 6.2 how one can accommodate fractional power series expressions for $\Psi(F(x))$ as well. I also showed how to use prime ideals in number fields to deal with general algebraic asymptotic coefficients. I also made the method of Stoll and Haible explicit in the derivation of a certain differential equation satisfied by an auxiliary function that determines the asymptotic coefficients. Along

the way, I derived an explicit polynomial satisfied by the parameter φ. I also provided the general case where one can solve for the auxiliary function explicitly.

Chapter 7

In this chapter, I used the asymptotic methods from Chapter 5 to obtain the existence of full asymptotic expansions for a family of binomial sums. I provide the leading terms explicitly in all cases, and give fields containing the asymptotic coefficients for several subcases. As one of these subcases, I prove a conjecture of Chamberland and Dilcher. For the binomial sums of the form

$$\sum_{k=0}^{n}(-1)^{\varepsilon k}\binom{n}{k}^{2}d^{k}$$

for $\varepsilon \in \{0,1\}$ and $d \in \mathbb{N}$ my generalization of the method of Stoll and Haible from Chapter 6 (Proposition 6.2) also applies and I prove that the asymptotic coefficients all lie in $\mathbb{Q}(\sqrt{d})$ and have denominators divisible only by the primes dividing 2 or $\sqrt{d}$. In particular, I obtain divisibility properties for the asymptotic coefficients of the central Delannoy numbers. Further, using this generalization, I obtain divisibility properties for the asymptotic coefficients in one of the two exceptional cases given by Theorem 7.2.

8.3   Future Research

Distinguishing Transcendental Holonomic from Algebraic

A sequence is holonomic if and only if its generating function satisfies a linear ODE with polynomial coefficients. Since every algebraic power series satisfies such an ODE, and the rational power series are the algebraic series of degree one, we see that holonomic sequences can be classified according to the nature of their generating function as rational, algebraic or transcendental. We can distinguish the holonomic sequences having rational generating functions from the others since these are the sequences that satisfy a linear recurrence relation with constant coefficients. However, it is unknown how to distinguish the nontrivial algebraic case from the transcendental case by inspection of the linear recurrences satisfied by the sequence in question. Although

a complete classification of the linear recurrence relations satisfied by sequences with algebraic generating functions may be too ambitious, I hope to obtain at least partial results in this direction.

The Study of the Zeros of Holonomic Sequences

As shown in Chapter 3, the set of zero terms in a holonomic sequence is composed of a set of density zero together with finitely many infinite arithmetic progressions and in the constant coefficient case we can replace "density zero" with "finite." Also, I developed a unified notion of nondegeneracy valid for all holonomic sequences that allows us to eliminate the possibility of arithmetic progressions. Work of Bézivin and Laohakosol shows that we can replace "density zero" with "finite" for a larger class of holonomic sequences than those satisfying recurrences with constant coefficients. This class, in particular, contains all sequences with generating functions equal to an algebraic, logarithmic or binomial series multiplied by a hypergeometric series $_0F_m$ for some integer $m \geq 0$. The question I wish to investigate is whether we in fact obtain only finitely many zero terms for all holonomic sequences. If this turns out to be false, then I'd like to investigate to what extent the largest field of coefficients for which we maintain finiteness can be described.

The Study of the Asymptotics of Holonomic Sequences

As described in Chapter 5, holonomic sequences typically admit complete asymptotic expansions. In Chapter 7 I obtained complete asymptotic expansions for a particular family of binomial sums and in certain cases was able to determine fields that contained the asymptotic coefficients as well as some divisibility properties of these coefficients. I would like to investigate, using these binomial sums as a starting point, which holonomic sequences admit asymptotic expansions for which divisibility properties of the asymptotic coefficients can be determined.

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