XII.—On the Definition of Work Done.—By Prof. J. G. MacGregor, *Dalhousie College, Halifax, N. S.*

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The usual definition of work done is the product of the magnitude of a force into the component in its action line of the displacement of its point of application. Positive work is said to be done *by* the force *on* the body on which it acts, if the force and the competent displacement have the same direction; negative work, if they have opposite directions. Positive work is said to be done *by* the body *against* the force if the force and the component displacement have opposite directions; negative work, if they have the same direction. Thus work done by the body is just work done by the force with the sign changed.

Prof. Simon Newcomb has pointed out* that as no axes of reference are specified in the above definition, the displacement referred to in it, and therefore the work done, as determined by it, are quite arbitrary. He has therefore proposed to give the definition the following form :—

"The work done by a force is the product of the intensity of the force into the amount by which the two material points between which it acts approach to or recede from each other; the work being positive when the approach or recession is in the direction of the force, negative in the opposite case."

It has been held by many writers† that the Laws of Motion require, for their complete enunciation, the specification of the axes of co-ordinates, by reference to which they hold, any such system of axes being called a dynamical reference system. If in the definition of work done also, the axes of reference which are to be employed, be specified, its arbitrary character will disappear; for by reference to given axes, any displacement will

*Philosophical Magazine, Ser. 5, Vol. xxvii (1889), p. 115.

†See Phil. Mag, Ser. 5, Vol. xxxvi (1893), p. 233.

have a definite magnitude and direction. That the definition may be adapted for use in reasoning based on the Laws of Motion, the axes employed in the definition must be the same as those by reference to which the Laws of Motion hold. Hence the usual definition will be freed from its arbitrary character and rendered capable of convenient employment in dynamical reasoning, by the following modification:—Work done is the product of the magnitude of a force into the component displacement of its point of application, in its action line, relatively to any dynamical reference system.

It will be obvious that from this modified definition, the statement which Prof. Newcomb suggests as a definition, may be deduced by the aid of the Third Law of Motion.

It will also be obvious that in the elementary study of dynamics, in which motions of small duration and extent on the earth's surface are considered, and for which lines fixed in the earth are a sufficient reference system, the young student need not be asked to employ so general a definition. For him the ordinary definition will be quite definite, as, in the first stages of study, all displacements, velocities, etc., are specified relatively to lines fixed in the earth at his place of observation, *e. g.*, the North-South line, the East-West line and the vertical.

The arbitrary character of the ordinary definition being thus removed, Prof. Newcomb's suggested modification loses its *raison d'être*. There are moreover three objections which may be urged against it.—(1.) It is not a definition merely, but embodies a dynamical hypothesis as well, viz., the Third Law of Motion; and for the sake of clearness, definitions should be kept quite distinct from hypotheses. (2.) It is not a definition of the work done by a force, as it purports to be, but of the work done by a stress; and the work done by a force as distinct from the work done by a stress has been found to be a convenient conception in dynamical reasoning. (3.) It is applicable only in cases of action at a distance. In all cases of contact action, it would make the work done by a force, zero. In treating of elastic solids and fluids therefore by the contact action method, and in treating cases of apparent action at a distance on the

assumption that the distance action is due to contact action through an elastic medium, we would require to re-define work done for the purpose. It would surely be more convenient to define work done in such a way that the same conception of it might be employed in both classes of problems.

Prof. O. J. Lodge has proposed* to define work done in the following way :—

"Whenever a body exerting a force moves in the sense of the force it exerts, it is said to do work; and whenever a body exerting a force moves in the sense opposite to that of the force it exerts, it is said to have work done upon it or to do anti-work, the quantity of work being measured in each case by the product of the force into the distance moved through in its own direction."

The definition is not quite precise; for it is not clear whether the "distance moved through" is by the body or by the point of application of the force it exerts. In cases of contact action with a view to which Dr. Lodge proposed this definition, the distance moved through by both would be the same. For his purpose, therefore, it was not necessary to be more precise. If, however, we are to form a jndgment as to the relation of this definition to the one ordinarily used, greater precision is necessary. That it was the distance moved through by the place of application of the force that was meant seems clear from the definition of working power which follows that quoted above, viz.: "The working power of a body is measured by the average force it can exert, multiplied by the range or distance through which it can exert it." The distance contemplated in the definition is thus the distance through which the force is exerted, *i.e.*, through which its point of application moves. And, indeed, had the distance contemplated been that moved through by the body exerting the force the proposed definition would have been practically equivalent to the one it was intended to displace.

To compare this definition with the usual one, let A and B be

* Philosophical Magazine, Ser. 5, Vol. VIII (1879), p. 278.

two bodies between which a stress acts, let F_{AB} and F_{BA} be the forces exerted by A on B and by B on A respectively, andlet S_A and S_B be the distances moved through in the line of the stress by A and B respectively; then the work done by A and by B respectively would be, according to Lodge's definition, $F_{AB}\ S_B$ and and $F_{BA}\ S_A$ and according to the ordinary definition, $-F_{BA}\ S_A$ and $-F_{AB}\ S_B$. Since in all dynamical problems the Third Law of Motion holds, we may put $F_{AB} = -F_{BA}$. Hence the work done by A and B respectively is, according to Lodge's definition, $F_{AB}\ S_B$ and $-F_{AB}\ S_A$ and according to the ordinary definition $F_{AE}\ S_A$ and $-F_{AB}\ S_B$. The term work done will therefore, in general, have different denotations according to the two definitions, and consequently theorems involving work done and working-power or energy which have been established in terms of the old definition would not hold in terms of the new one. The law of the conservation of energy would no longer be generally true.

In the particular case of contact action, $S_A = S_B$. Hence in this case the work done by A and B respectively would be the same according to both definitions, and both definitions would thus have the same denotation. For cases of contact action therefore, established theorems involving work done and energy would still hold notwithstanding the change of definition.

The advantage of Lodge's definition is that in cases of contact action it indicates directly that a body in doing positive or negative work respectively, loses or gains working power, or as he puts it, that it is the thing which does the work which possesses energy. This result, however, follows from the ordinary definition by a single step. For as already seen, by putting S_A equal to S_B the work done by A and B respectively is seen to be the same according to both definitions. Any advantage which Lodge's definition offers in dealing with cases of contact action is therefore afforded by the ordinary definition as well.

If we always employed the assumption of contact action in dynamical reasoning it might be worth while to change the definition of work done in the way suggested. But whatever may be the future of this mode of treatment, we cannot at present apply it in all cases, and even in cases in which it

can be applied, it is frequently more convenient to make use of the assumption of action at a distance. Should Lodge's definition be adopted as a general definition, then in all cases in which we either must employ the action-at a-distance method or find it more convenient to do so, the simple law of the conservation of enery must be replaced by a more complex law of energy, and the labor involved in the solution of problems must be largely increased. Should it be adopted as a special definition applicable in cases of contact action, it would be necessary to employ a different definition in dealing with cases of distance action and of mixed contact and distance action, a necessity which would give rise to new difficulty in the elementary teaching of the subject and even to confusion in more advanced work.

It seems clear, therefore, with regard to both of the proposed modifications referred to above, that the better course is to retain the old definition of work done, with the dynamical reference system specified, as a general definition applicable in all cases whether of contact action, of action at a distance, or of mixed contact and distance action; and in dealing with cases of wholly distance action or wholly contact action respectively, to prove at the outset, if it be considered desirable, that the statements which Prof. Newcomb and Prof. Lodge wish to use as definitions are particular cases of the general definition, applicable, the one in cases of action at a distance, the other in cases of contact action.