# TOWARDS A THEORY OF QUANTUM DOMAINS

by

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Dalhousie University is located in Mi'kma'ki,
the ancestral and unceded territory of the Mi'kmaq.
We are all Treaty people.

*To the great people whom I have the privilege to have as family*

# Table of Contents

# List of Figures

## Abstract

In this work, we lay the foundation for defining a category of quantum domains and developing a model for quantum programming language. We give a definition of quantum domains, which act as objects of the category. In this definition, we consider systems which deal with 'progressive information'. Also, we give a definition of progressive superoperators for finite domains. In the category of quantum domains, the progressive superoperators act as the morphisms of the category.

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# Chapter 1

## Introduction

Quantum computers are a promising technology with the potential to revolutionize the future of computation and communication, as they have more computational power than the classical computers when it comes to certain problems [6, 17]. Therefore, it is very important that we develop 'suitable' quantum programming languages. The first step would be to determine what are appropriate conditions for a suitable programming language. For example, due to the *no-cloning* rule of quantum physics [18], debugging a quantum program at runtime is not possible [7]. Therefore, it would be useful to be able to give a formal proof of correctness for certain quantum programs. In order to assist in the development of such quantum programming languages, it is important to study their semantics [5, 16].

Domain theory is an indispensable tool for studying semantics of programming languages, as it allow us to find mathematical objects that represent the behavior of computer programs [1]. Domain theory was developed by Dana Scott and others in the 1960's. There have been a number of attempts to define a theory of quantum domains. Some of these focused on quantifying entropy in a quantum setting [4, 9]. Also, in [14], Selinger has proposed a simple first-order quantum programming language by developing a categorical model involving directed-complete partial orders of superoperators.

In this thesis, we develop a notion of quantum domains that deals with progressive information, i.e., partial information which might become more complete over time, which is something that has not been considered in [14]. As an example of partial information, consider a system built from a pair of quantum devices. The first device can output zero, one, or two qubits. The second device receives the output of the first one and after processing it, outputs a result.

If the first device outputs, say, one qubit, it is still possible that later on, it will output a second qubit. So the second device needs to perform its function without complete information about the number of qubits it will receive in total. Note that the second device cannot undo the result after it has been output. When we speak of "progressive information", what we mean in this case is that the behavior of the second device when it receives two qubits of input, must be consistent with its behavior when it receives just one qubit.

In this thesis, we give a general definition of quantum domains suitable for modelling progressive information. The ultimate goal of this line of research is to define a category of quantum domains (similar to the model given in [14]), where the objects are quantum domains and the morphisms are "progressive superoperators". We achieve part of this goal: we give a general definition of quantum domains, and we define progressive superoperators only in the case of *finite* quantum domains. Our definition of progressive superoperator has the potential to be extended to infinite domains in future work.

### Related Work

The work of this thesis was inspired by the proposal [15] and builds on the earlier work of summer students Yoann Le Montagner [11] and Alain Patey [13]. In [11], Le Montagner defined a quantum domain and also gave a definition of progressive superoperators for finite domains. However, the cumbersome notation he used made it difficult to show that his definition of progressive superoperator has the necessary conditions, i.e., is a generalization of special cases such as states, observations, superoperators, and stochastic maps. Also, the definition he had given for progressive superoperators could not be easily generalized to the infinite domains.

In [13], Patey defined probabilistic domains (which are a special case of quantum domains) and gave a definition of stochastic maps. Also, using Le Montagner's work as the basis of his work, he proposed conditions for the implementability of progressive superoperators for finite domains, but showing the correctness of these conditions was left for future work.

### Contribution

In this thesis, we propose a new definition of quantum domains, which is much simpler than previous definitions. We define notions of observation and state which are fundamental notions in quantum computing. Then we consider the problem of physical realizability and define finitely compactly supported states, a class of states that are physically realizable. Then we show that all states can be achieved as limits of finitely compactly supported states. Also, we give two equivalent definitions of progressive superoperators between finite quantum domains. The second of these definitions is stated in such a way that it can potentially be extended to infinite domains in the future.

**Outline**

In Chapter 2, we give a brief overview of the basic concepts of quantum computing, domain theory, and probabilistic domains. In Chapter 3, we state our results. In Section 3.1, we give our definition of quantum domains. Sections 3.2, 3.3, and 3.4 address the concepts of observations, states, and finitely compactly supported states, respectively. In the last two sections we give two definitions of progressive superoperators for finite domains and we discuss why only one of them has the potential to be generalized to the infinite case.

# Chapter 2

# Background

In this chapter, we will briefly cover some basic notions used in this thesis, as well as some related background. In the first two sections, we cover some basics of Quantum Computing and Domain Theory. The last section will address a simple case of quantum domains and progressive superoperators, called probabilistic domains and stochastic maps.

## 2.1 Quantum Computing

Quantum computers are computers governed by the laws of quantum physics and quantum computing is using these computers for processing information (by taking advantage of some of their properties like superposition and entanglement). In this section, we will briefly cover the basics of quantum computing which are used in this thesis. For a more detailed introduction to quantum computing, see [12].

### 2.1.1 Qubits

A *quantum bit* or *qubit* is the foundation of quantum computers. It is a unit of information which generalizes classical bit. While a classical bit can take only the states 0 or 1, a qubit can take a non-zero linear complex combination of these two states. States are defined up to scalar multiples. Therefore, we can assume without loss of generality that states are normalized, i.e., described by vectors of unit length. More generally, a state $q$ is *normalized* if $\|q\| = 1$, and *subnormalized* if $\|q\| \leq 1$. While it is possible to normalize the state after each operation, it is sometimes useful not to do so. Here, we consider the states to be subnormalized.

For the states of the qubits, we sometimes use the Dirac notation $|\cdot\rangle$. The states $|0\rangle$ and $|1\rangle$ (the column vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$, respectively) are called the computational basis states, while all the other states $q = \alpha |0\rangle + \beta |1\rangle$, where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$, are called superpositions of these basis states.

The state of a pair of qubits is described by a normalized vector in $\mathbb{C}^2 \otimes \mathbb{C}^2$, and can be written as

$$|q\rangle = \alpha |00\rangle + \beta |01\rangle + \gamma |10\rangle + \delta |11\rangle,$$

where $|00\rangle = |0\rangle \otimes |0\rangle$, etc. If $|q\rangle$ is of the form $|q\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$, where $a|0\rangle + b|1\rangle$ and $c|0\rangle + d|1\rangle$ are single-qubit states, we say that $|q\rangle$ is *separable*, otherwise we call $|q\rangle$ *entangled*. For example, the state $|01\rangle = |0\rangle \otimes |1\rangle$ is a separable state, while the state $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ is an example of an entangled state.

Similarly, we can write the state of a $n$-tuple of qubits as follows:

$$|q\rangle = \alpha_{0\cdots 00} |\underbrace{0\cdots 00}_{n}\rangle + \alpha_{0\cdots 01} |\underbrace{0\cdots 01}_{n}\rangle + \cdots + \alpha_{1\cdots 11} |\underbrace{1\cdots 11}_{n}\rangle$$

where $|0\cdots 00\rangle, |0\cdots 01\rangle, \ldots, |1\cdots 11\rangle$ are the $2^n$ computational basis states and $\alpha_{0\cdots 00}, \ldots, \alpha_{1\cdots 11} \in \mathbb{C}$.

### 2.1.2 Unitary Transformations and Measurements

In a physical experiment, there are two types of operation we can perform on a qubit: Unitary transformations and measurement. A *unitary transformation* is an operation analogous to a classical logic gate which evolves the state of the input qubit(s). In a quantum circuit, unitary transformations are considered to be quantum gates. A unitary transformation of states (considered as column vectors) is represented by a unitary square matrix $U$ (i.e., $UU^\dagger = I$ and $U^\dagger U = I$) which maps a system in state $|q_1\rangle = \alpha_1 |0\rangle + \beta_1 |1\rangle$ to a state $|q_2\rangle = \alpha_2 |0\rangle + \beta_2 |1\rangle$ where

$$\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = U \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}.$$

An example of a quantum gate is the *Hadamard gate* $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ which acts as follows

$$|0\rangle \mapsto \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \qquad |1\rangle \mapsto \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle.$$

This quantum gate acts on a single qubit. Another example of single quantum gate is the *not-gate* $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which switches the basis states (sends $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$).

Some unitary transformations act on more than one qubit. An example of a gate which operates on two qubits is a controlled gate. For example, the *controlled not gate* given by the following $4 \times 4$ matrix,

$$N_c = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & N \end{array}\right),$$

is a unitary transformation. When applied to a basis state, it does not affect the first qubit and performs the *not gate* on the second qubit only if the first qubit is in state $|1\rangle$. In the other words, it acts as follows.

$$|00\rangle \mapsto |00\rangle \qquad |01\rangle \mapsto |01\rangle$$

$$|10\rangle \mapsto |11\rangle \qquad |11\rangle \mapsto |10\rangle$$

Measurement is a probabilistic operation which maps a state $|q\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ to state $|0\rangle$, with probability $|\alpha|^2$ and maps it to state $|1\rangle$, with probability $|\beta|^2$. Like a unitary transformation, we can measure the state of a system containing more than one qubit. Let $|q\rangle = \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle$ be the state of a system of two entangled qubits. If the left qubit is measured, then the outcome of the measurement is either

$$\frac{\alpha\,|00\rangle + \beta\,|01\rangle}{\sqrt{|\alpha|^2 + |\beta|^2}} \tag{2.1}$$

with probability $\sqrt{|\alpha|^2 + |\beta|^2}$ or the outcome is

$$\frac{\gamma\,|10\rangle + \delta\,|11\rangle}{\sqrt{|\gamma|^2 + |\delta|^2}} \tag{2.2}$$

with probability $\sqrt{|\gamma|^2 + |\delta|^2}$. Now suppose that after the first measurement, the qubits are in state (2.1). Then if the right qubit is measured, the outcome will be either $|00\rangle$ with probability $|\alpha|^2$ or it is $|01\rangle$ with probability $|\beta|^2$. On the other hand if after the first measurement the qubits are in state (2.2), then the result of the second measurement is either $|10\rangle$ with probability $|\gamma|^2$ or it is $|11\rangle$ with probability $|\delta|^2$.

### 2.1.3 Mixed States and Density Matrices

In classical computation, it is possible to examine bits to determine whether they are in state 0 or 1. In quantum computation it is not possible to determine the state of a qubit by only examining it, i.e., finding the values of the coefficients $\alpha$ and $\beta$ in the state $\alpha\,|0\rangle + \beta\,|1\rangle$. Instead, we have the operation of *measurement*. When a qubit in state $\alpha\,|0\rangle + \beta\,|1\rangle$ is measured, the output always will be either 0 or 1 (with probability $|\alpha|^2$, the output is 0 and with probability $|\beta|^2$, the output is 1).

Because of this nature of quantum physics, sometimes we have incomplete information about the state. So we are unable to exactly determine the state of a qubit, but we can present a set of states that are the possible states for the qubit, together with the probability of each them happening. This ensemble of states and their probabilities is called a *mixed state*. In general a mixed state can be presented as a subconvex linear combination $\sum_{i\in I} \lambda_i\{|q_i\rangle\}$, i.e., a linear combination where $\{\lambda_i\}_{i\in I}$ are real numbers such that $\lambda_i \geq 0$, for all $i \in I$, and $\sum_{i\in I} \lambda_i \leq 1$. A mixed state $\sum_{i\in I} \lambda_i\{|q_i\rangle\}$, where $\lambda_i = 0$ for all $i \neq j$, for some $j \in I$, is called a pure state. As an example of mixed state, consider the state

$$|q\rangle = \frac{1}{2}\{|0\rangle\} + \frac{1}{2}\{|1\rangle\}.$$

This mixed state represents a qubit which is in the state $|0\rangle$ with probability $\frac{1}{2}$ and is in the state $|1\rangle$ with the same probability. Note that this state is different from $|q'\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle$ which a pure state (a superposition of states $|0\rangle$ and $|1\rangle$). As another example, consider the state

$$|q\rangle = \frac{4}{9}\left\{\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right\} + \frac{5}{9}\{|1\rangle\}.$$

The qubit is in the state $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$ with probability $\frac{4}{9}$ and is in the state $|1\rangle$ with probability $\frac{5}{9}$.

Also, mixed states are used for situations where two quantum system are entangled and we want to describe the state of only one of these entangled systems. In this case, its state cannot be described by a pure state. Since a mixed state can be a mix of any number of states, working with Dirac notation of states for performing computation can become cumbersome. A solution for this problem is to use density matrices. A density matrix can be used to represent both pure and mixed states. The density matrix of a pure state $|q\rangle$ is $|q\rangle\langle q|$ (the outer product). For example, the density matrix of the pure state $|0\rangle$ is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

The notion of density matrices is especially helpful with mixed states. Let

$$\sum_{i \in I} \lambda_i \{|q_i\rangle\}$$

be the ensemble of pure states corresponding to a mixed state $|q\rangle$. Then $|q\rangle$ can be represented by density matrix $\sum_{i \in I} \lambda_i |q_i\rangle\langle q_i|$. For example, consider the mixed state $\frac{1}{2}\left\{\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right\} + \frac{1}{2}\left\{\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right\}$. The density matrix of this state is

$$\frac{1}{2}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

One of the benefits of using density matrices is that the density representation of states is unique. In other words, (pure and mixed) states with same density matrix are indistinguishable by a process allowed by quantum mechanics. For example, the state

$$\frac{1}{2}\left\{\frac{|0\rangle + |1\rangle}{2}\right\} + \frac{1}{2}\left\{\frac{|0\rangle - |1\rangle}{2}\right\}$$

given above is indistinguishable from the mixed state $\frac{1}{2}\{|0\rangle\} + \frac{1}{2}\{|1\rangle\}$, as both have the same density matrix $\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Using the density matrices, we can determine that the pure superposition state $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ is different than the mixed state $\frac{1}{2}\{|0\rangle\} + \frac{1}{2}\{|1\rangle\}$, since the former has a density matrix of $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, which is not equal to the density matrix of $\frac{1}{2}\{|0\rangle\} + \frac{1}{2}\{|1\rangle\}$, given above.

Now, we will address the necessary and sufficient conditions for density matrices, but before doing so, we need to define positive matrices.

**Definition 2.1.** Using physics terminology, a matrix $A$ is called *positive*, if it is hermitian and positive semi-definite. We write $0 \sqsubseteq A$ to indicate $A$ is positive.

**Definition 2.2.** For two matrices $A$ and $B$, we have $A \sqsubseteq B$ if and only if $0 \sqsubseteq B - A$.

**Proposition 2.3.** *A matrix $A$ is the density matrix of some mixed state if and only if $A$ is positive and has a trace less than* 1.

Note that equivalently, we can define a unitary transformation as a map on density matrices. In our previous definition, a unitary transformation $U$ mapped a state $q$ to the state $Uq$. By expanding this definition, we have that a unitary transformation $U$ maps a density matrix $M$ to the density matrix $UMU^\dagger$.

Similar to unitary transformations, we can define the measurement operation on density matrices. The density matrix of a pure state $|q\rangle = \alpha |0\rangle + \beta |1\rangle$ is

$$\begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix}.$$

So measurement maps the above density matrix to density matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ with probability $\alpha\alpha^*$ and maps it to the density matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ with probability $\beta\beta^*$. More generally, the density matrix of a mixed state is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a + d = 1$. Measurement maps the density matrix of such mixed state to density matrix $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ with probability $a$ and maps it to the density matrix $\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$ with probability $d$.

### 2.1.4 Superoperators

For modelling the general evolution of quantum systems, we need to define an operator which maps the initial state (density matrix) to the new state (density matrix). Also, this operator needs to preserve the properties of density matrices. This operator is called a superoperator. Before giving a formal definition of superoperator, we first need to define convex spaces and linear maps on convex spaces.

*Notation* 2.4. Let $H$ be a finite dimensional Hilbert space. $\mathcal{B}(H)$ denotes the set of (bounded) linear maps on $H$.

**Definition 2.5.** A subset $X$ of a real vector space is called *pointed* if $0 \in X$, and it is called *convex* if for all $v, w \in X$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$, we have $\lambda_1 v + \lambda_2 w \in X$. The expression $\lambda_1 v + \lambda_2 w$ is called a *convex combination* of $v$ and $w$.

Note that if $X$ is pointed and convex, we can also form *subconvex combinations*, namely $\lambda_1 v + \lambda_2 w$ where $\lambda_1 + \lambda_2 \leq 1$, or more generally, $\lambda_1 v_1 + \cdots + \lambda_n v_n$, where $\lambda_1 + \cdots + \lambda_n \leq 1$.

**Definition 2.6.** A *pointed convex space* is a pointed convex subset of a real vector space.

**Example 2.7.** The following are some examples of pointed convex spaces:

- any real vector space
- $[0, 1]$
- $\prod_{i \in I} \mathcal{B}(H_i)^+$, where $\mathcal{B}(H_i)^+$ is the set of positive bounded linear maps on Hilbert spaces
- $\left\{T \in \prod_{i \in I} \mathcal{B}(H_i)^+ \mid T \sqsubseteq I\right\}$
- $\left\{T \in \prod_{i \in I} \mathcal{B}(H_i)^+ \mid tr(T) \leq 1\right\}$

**Definition 2.8.** Let $X$ and $Y$ be pointed convex spaces. We say that a map $F : X \to Y$ is *linear* if it preserves subconvex combinations, i.e., for all $v, w \in X$ and $\lambda_1, \lambda_2 \in [0, 1]$ with $\lambda_1 + \lambda_2 \leq 1$,

$$F(\lambda_1 M_1 + \lambda_2 M_2) = \lambda_1 F(M_1) + \lambda_2 F(M_2).$$

**Definition 2.9.** [2] Let $\mathbb{C}^{n \times n}$ denote the vector space of complex $n \times n$-matrices. A *superoperator* $f : \mathbb{C}^{n \times n} \to \mathbb{C}^{m \times m}$ is an operator which has the following properties:

- $f$ is linear
- $f$ is positive, i.e., for all $A \in \mathbb{C}^{n\times n}$ such that $A$ is positive, $f(A)$ is positive
- $f$ is trace non-increasing, i.e., for all $A \in \mathbb{C}^{n\times n}$ such that $0 \sqsubseteq A$, $tr(f(A)) \leq tr(A)$
- $f$ is completely positive, i.e., for all $k \geq 1$ and for all $A \in \mathbb{C}^{kn\times kn}$ such that $0 \sqsubseteq A$, $0 \sqsubseteq (id_k \otimes f)(A)$.

**Theorem 2.10.** *[10] For any completely positive operator $f : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$, there exist operators $S_j \in \mathbb{C}^{m\times n}$, $j \in J$ such that $\sum_{j\in J} S_j^\dagger S_j \sqsubseteq I$ and for all $A \in \mathbb{C}^{n\times n}$,*

$$f(A) = \sum_{j\in J} S_j A S_j^\dagger.$$

**Definition 2.11.** [10] The sum defined in the previous theorem is called a *Kraus representation* or *operator-sum representation* of the operator $f$. Note that Kraus representations are not unique.

Next we will present a simplified version of Choi's Theorem on completely positive maps [3].

**Theorem 2.12.** *[Simplified Choi's Theorem] Let $H$ be a Hilbert Space. A map $F : \mathbb{C} \to \mathcal{B}(H)$ is completely positive if and only if it is positive. Similarly, a map $F : \mathcal{B}(H) \to \mathbb{C}$ is completely positive if and only if it is positive.*

## 2.2 Domain Theory

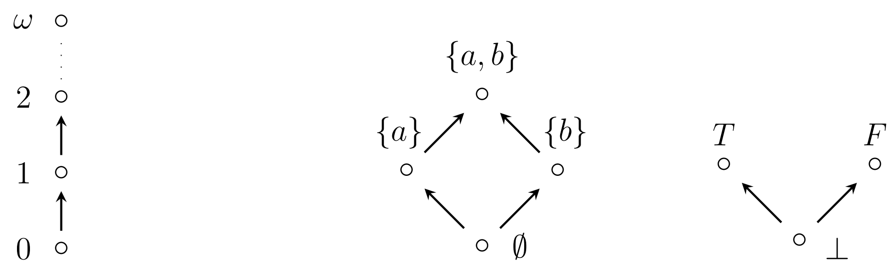
In this section, we present some basic concepts of domain theory.

**Definition 2.13.** A *partially ordered set* or *poset* is a set $P$ together with a binary relation $\leq$ such that the following properties hold for all $x, y, z \in P$:

- Reflexivity: $x \leq x$
- Antisymmetry: $x \leq y \wedge y \leq x \implies x = y$
- Transitivity: $x \leq y \wedge y \leq z \implies x \leq z$.

Finite (and some infinite) posets can be visualized by their Hasse diagram, which shows each element of $P$ as a vertex and uses directed edges to indicate $x \leq y$ relation.

**Example 2.14.** The diagram representation of three different posets is given in Figure 2.1.




(a) Natural numbers with infinity (b) Subsets of $\{a, b\}$ (c) Flat boolean

Figure 2.1: Three examples of posets

Figure 2.1(a), shows the diagram of the poset of the ordinal natural numbers with 'infinity'. It is an example of an infinite poset. Figure 2.1(b) shows the set of subsets of $\{a, b\}$ ordered by the inclusion relation. Figure 2.1(c) shows the flat boolean poset. We will later return to this poset as it is used for explaining the behavior of a system with partial information. Figures 2.1(b) and 2.1(c) are examples of finite posets.

**Definition 2.15.** Let $(P, \leq)$ be a partially ordered set and $A \subseteq P$. An *upper bound* for $A$ is an element $u \in P$ such that $u$ is above every element of A $(\forall x \in A,\ x \leq u)$. Similarly, a lower bound of A is an element $l \in P$ such that $l$ is below all the elements of $A$ $(\forall x \in A,\ l \leq x)$.

As an example, consider the subset $A = \{\emptyset, \{a\}\}$ in Figure 2.1(b). The elements $\{a\}$ and $\{a, b\}$ are upper bounds of $A$ and $\emptyset$ is the lower bound of $A$. In the third figure (Figure 2.1(c)), the subset $A = \{F, T\}$ does not have an upper bound. The lower bound of $A$ is the element $\perp$.

**Definition 2.16.** Let $(P, \leq)$ be a partially ordered set and $A \subseteq P$. The *supremum* (*infimum*) of $A$ is an element $x \in P$ which is the smallest upper bound (greatest lower bound) of $A$. The supremum of $A$ is denoted by $\sup_{a \in A} a$. The element $x$ is called the *maximum* (*minimum*) element of $A$ if $x \in A$ and $x$ is the supremum (infimum) of $A$.

Consider the subset $A = \{\emptyset, \{a\}, \{a, b\}\}$, in Figure 2.1(b). The maximum element of $A$ is $\{a, b\}$ and its minimum element is $\emptyset$. The subset $A = \{\bot, F, T\}$ of the poset given in Figure 2.1(c) does not have a maximum element, but it has a minimum element $\bot$.

**Definition 2.17.** Let $(P, \leq)$ be an ordered set. If an element $x \in P$ is below all the elements of $P$, then $x$ is called the *least element* or *bottom element* and is denoted by $\bot$. Dually, an element $x \in P$ is called the *top element* if it is above every element of $P$.

**Definition 2.18.** Let $P$ be a poset. A subset $A \subseteq P$ is called *directed*, if $A$ is nonempty and each pair of elements of $A$ has an upper bound in $A$.

As an example, consider the subset $A = \{1, 2, 3\}$ of the poset in Figure 2.1(a). This totally ordered subset (chain) is directed. The subset $A = \{\emptyset, \{a\}, \{b\}\}$ is not directed, since there is not upper bound of $\{a\}$ and $\{b\}$ in $A$.

**Definition 2.19.** A *directed-complete partial order* (*dcpo*) (sometimes *domain*) is a poset in which all directed subsets have a supremum.

Every finite poset is a dcpo. So the posets in Figures 2.1(b) and 2.1(c) are both dcpo's. The poset in Figure 2.1(a), is an example of an infinite dcpo. Note that the poset of ordinal natural numbers (without infinity) is not a dcpo, since the poset itself is directed, but does not have a supremum.

**Definition 2.20.** Let $P$ and $Q$ be two posets. A map $f : P \to Q$ is *monotone*, if for all $x, y \in P$ such that $x \leq y$, $f(x) \leq f(y)$.

**Definition 2.21.** A map $f$ between two dcpo's which is both monotone and preserves suprema of directed sets is called *Scott-continuous.*

**Proposition 2.22.** *Let $P$ be a dcpo. Let $I$ and $J$ be directed posets and let $\{a_{i,j}\}_{i\in I, j\in J}$ be a monotone family of elements of $P$, i.e., $i \leq i'$ and $j \leq j'$ implies $a_{ij} \leq a_{i'j'}$. Then the following suprema exist and are equal:*

$$\sup_{i\in I}\left(\sup_{j\in J}(a_{i,j})\right) = \sup_{j\in J}\left(\sup_{i\in I}(a_{i,j})\right).$$

### 2.2.1 Approximation

**Definition 2.23.** [1] Let $D$ be a dcpo and $x, y \in D$. We say that $x$ *approximates* $y$ if for all directed subsets $A \subseteq D$, $y \leq \sup_{z \in A} z$ implies $x \leq a$, for some $a \in A$. An element is called *compact* if it approximates itself.

*Notation* 2.24. In this thesis, some of the notations were adapted from [1]. Let $D$ be a dcpo and $x, y \in D$:

- $x \ll y \;\Leftrightarrow\; x$ approximates $y$
- $\twoheaddownarrow x = \{y \in D \mid y \ll x\}$
- $\mathcal{K}(D) = \{x \in D \mid x \text{ is compact}\}$

**Example 2.25.** Consider the poset given in Figure 2.1(a). Then $n \ll n+1$, for all $n \in \mathbb{N}$. Now consider the poset $P$, given in Figure 2.2 and $a_1, a_2 \in P$. It is clear that $a_1$ does not approximate $a_2$, since for the subset $B = \{b_0, b_1, \ldots, \omega\}$, we have $a_2 \leq \omega$, but there does not exists any $b_i \in B$ such that $a_1 \leq b_i$. As a matter of fact, for all $x, y \in P$, $x$ does not approximate $y$ ($x \not\ll y$).

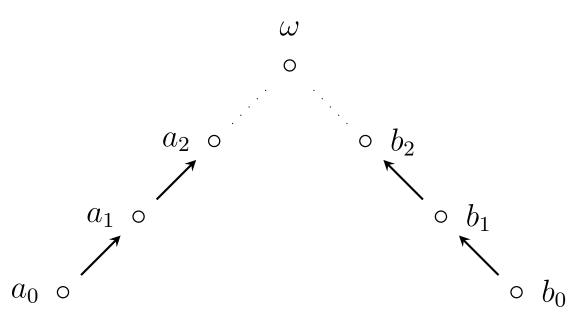



Figure 2.2: Two copies of natural numbers with infinity, connected at the top element

**Proposition 2.26.** *Let $D$ be a dcpo and let $x, y \in D$. Then the following holds:*

- $x \ll y \Rightarrow x \leq y$
- *If $x \in \mathcal{K}(D)$, then $x \leq y$ implies $x \ll y$ ($x$ approximates all the elements above it).*

### 2.2.2 Algebraic Domains

**Definition 2.27.** [1] A dcpo $D$ is called an *algebraic domain* if for all $x \in D$, the set $\mathcal{K}(D) \cap {\downarrow} x$ of compact elements approximating $x$ is directed, and $x = \sup(\mathcal{K}(D) \cap {\downarrow} x)$.

**Example 2.28.** Figure 2.3, presents an example of an algebraic domain, while the dcpo given in Figure 2.2, is an example of a domain which is not algebraic.

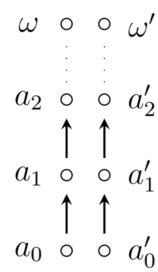



Figure 2.3: Two parallel copies of natural numbers with infinity

*Remark* 2.29. Note that every finite poset is an algebraic domain.

**Definition 2.30.** [1] Let $S$ be a poset (we can think of $S$ as being the set of compact elements of some algebraic domain). $S$ is *mub-complete*, if for every finite subset $M \subseteq S$, below each upper bound $u \in S$ of $M$, there is at least one minimal upper bound of $M$ in $S$. For a finite set $M \subseteq S$, the *mub-closure of* $M$, denoted $mc(M)$, is the smallest set that contains all minimal upper bounds (in $S$) of finite subsets of $mc(M)$.

*Remark* 2.31. Let $S$ be a poset with bottom element $\perp$ and let $M \subset S$ be a finite set. Then $\emptyset \subset mc(M)$ and $\perp$ is the minimal upper bound for $\emptyset$. So $\perp \in mc(M)$.

**Definition 2.32.** [1] A dcpo $D$ is *bifinite* when:

1. $D$ has a least element (denoted by $\perp$),

2. $D$ is algebraic,

3. $\mathcal{K}(D)$ is mub-complete,

4. given a finite set $M \subseteq K(D)$, the mub-closure of $M$ $(mc(M))$ is finite.

## 2.3 Probabilistic Domains

In this section, we cover a special case of quantum domains which has been addressed in the literature [8](see also [13, 15]), namely probabilistic domains. For simplicity, we first consider the finite case.

**Definition 2.33.** A *finite probabilistic domain* $P_D$ is a finite domain $D$, together with a tuple of probabilities $p_x$, where $x \in D$ and $\sum_{x \in D} p_x = 1$. The values of the tuple $(p_x)_{x \in D}$ are called the *probabilistic states* of the domain.

For a better understanding of this definition, consider a system which can produce 0, 1, or 2 tokens with probability $p_0$, $p_1$, and $p_2$, respectively, where $p_0 + p_1 + p_2 = 1$. This system corresponds to a 3-element domain $D = \{0, 1, 2\}$, where a probability $p_i$ is assigned to each element $i = 0, 1, 2$ of $D$. The domain is presented in Figure 2.4. The tuple $(1/3, 1/3, 1/3)$ is a probabilistic state of this system.

$$\begin{array}{ll} 2 & p_2 \\ \uparrow & \\ 1 & p_1 \\ \uparrow & \\ 0 & p_0 \end{array}$$

Figure 2.4: A probabilistic system which might output 0, 1, or 2 tokens

If we consider the amount of information we have about the behavior of the system, then we can define an information order on the probabilistic states of this system. Figure 2.5 shows three probabilistic states of this system and the order on them.

$$\begin{array}{ll} 2 & p_2 = 0 \\ \uparrow & \\ 1 & p_1 = 0 \\ \uparrow & \\ 0 & p_0 = 1 \end{array} \quad \preceq \quad \begin{array}{ll} 2 & p_2 = 0 \\ \uparrow & \\ 1 & p_1 = 1 \\ \uparrow & \\ 0 & p_0 = 0 \end{array} \quad \preceq \quad \begin{array}{ll} 2 & p_2 = 1 \\ \uparrow & \\ 1 & p_1 = 0 \\ \uparrow & \\ 0 & p_0 = 0 \end{array}$$

Figure 2.5: Information order on the probabilistic states

The intuition behind this order is that as we move up in the domain, we have more information about the behavior of the system. For example, when the system has output one token, we know for certain that the system will not output zero tokens.

In the same way, when the system has output two tokens, we know that it will not output zero or one token (or in other words, will not stay in the positions 0 or 1).

In order to formally define this order, first the concept of observations needs to be defined.

**Definition 2.34.** Let $P_D$ be a finite probabilistic domain. An *observation* on $D$ is a monotone function $q : D \to [0, 1]$.

Now, a pairing function can be defined on states and observations, by assigning a probability to each pair of a state and an observation. This will be used to define the information order.

**Definition 2.35.** Let $P_D$ be a finite probabilistic domain, let $p = (p_x)_{x \in D}$ be a state, and let $q : D \to [0, 1]$ be an observation. The pairing of state and observation is defined by

$$\langle p \mid q \rangle = \sum_{x \in D} p_x q(x).$$

**Definition 2.36.** Let $P_D$ be a finite probabilistic domain and let $p, p'$ be two states of the domain. The *information order* on states is defined by

$$p \preceq p' \quad \Leftrightarrow \quad \text{for all observations } q,\ \langle p \mid q \rangle \leq \langle p' \mid q \rangle.$$

Note that the 3-elements states in Figure 2.5 are ordered as shown here.

While the given definition of states works nicely in the finite case, in order to work with infinite domains, the definition of probabilistic domains, their observations and states needs to be refined.

Note that the first definition of observations can be easily extended to non-finite domains by defining the observations only on compact elements, i.e., for a non-finite quantum domain $D$, an observation on $D$ is a monotone function $B : \mathcal{K}(D) \to [0, 1]$.

**Definition 2.37.** A *probabilistic domain* is an algebraic domain $D$. A *probabilistic state* on $D$ is a linear Scott-continuous function $p$ from observations to $[0, 1]$.

*Remark* 2.38. In case $D$ is finite, this definition agrees with our previous definition, as it can be proved that the state $p$ is uniquely defined by the linear function $q \mapsto \langle p \mid q \rangle$ and vice versa.

Note that in an infinite domain, the non-compact elements can only be reached as limits of compact ones. So, it would be ideal to prove that under suitable conditions a state can be represented as a limit of compact ones. This goal is achieved through compactly-supported states.

**Definition 2.39.** A *compactly-supported state* is a state which can be represented as an assignment of probabilities to the set of compact elements of the domain, such that the probabilities sum to 1 (similar to the finite case).

**Theorem 2.40.** *[13, 15] (Fundamental theorem of probabilistic powerdomains) Let $D$ be a bifinite domain. Every state on $D$ is a supremum of a directed set of compactly-supported states.*

In order to give a general model for the behavior and evolution of probabilistic systems (modelled by probabilistic domains), an operator needs to be defined on the states of the system. In [13], Patey has given two definitions for such operators (called *stochastic maps*) between two bifinite probabilistic domains $P$ and $Q$:

1. a Scott-continuous function from $P$ to $States(Q)$

2. a Scott-continuous linear function from $States(P)$ to $States(Q)$

where $States(P)$ denotes the set of all probabilistic states of $P$. Then the author has proved that these definitions are equivalent, i.e., the set of all Scott-continuous functions from $P$ to $States(Q)$ is isomorphic to the set of all Scott-continuous linear functions from $States(P)$ to $States(Q)$. For proving this isomorphism, Patey has used the results of Claire Jones on probabilistic non-determinism [8].

In her thesis, Jones defined the notion of evaluations on topological spaces. An *evaluation* is a map from the set of open sets to the interval $[0, \infty]$. But equivalently, evaluations can be defined from the set of open sets to the interval $[0, 1]$. Next, Jones defined a functor $\mathcal{V}$ on the category **DCPO**, which is a category with dcpo's as its objects and continuous functions as its morphisms. This functor takes every dcpo $D$ to the set of its evaluations $\mathcal{V}(D)$, which itself is also a dcpo. For every two evaluations $\mu$ and $\nu$, the order on evaluations is defined by

$$\mu \leq \nu \ \Leftrightarrow \ \mu(O) \leq \nu(O), \text{ for all open sets } O.$$

Lastly, Jones has proved that for two dcpo's $P$ and $Q$, the set of continuous functions $f : P \to \mathcal{V}(Q)$ is the same as the set of continuous so-called super-linear functions $f : P \to Q$.

In [13], the author has discussed that based on the definition of open sets and evaluations, the open sets can be viewed as observations and the evaluations can be viewed as states. In this way, evaluating an open set is equivalent to the inner product of states and observations and the information order is same as the order on the evaluations. Therefore, using the result obtained by Jones, for bifinite domains $P$ and $Q$, the set of all Scott-continuous functions $f : P \to States(Q)$ is isomorphic to the set of Scott-continuous linear functions $f : States(P) \to States(Q)$. Note that for using this result, it is important that the domains are bifinite (and thus algebraic). This result is formulated in the following proposition.

**Proposition 2.41.** *[13] Let $P$ and $Q$ be bifinite probabilistic domains. The set of Scott-continuous functions $f : P \to States(Q)$ is isomorphic to the set of Scott-continuous linear functions $f : States(P) \to States(Q)$.*

Using this result, the author in [13] defined the stochastic maps as follows:

**Definition 2.42.** [13] Let $\mathcal{D}$ and $\mathcal{E}$ be two probabilistic bifinite domains. A *stochastic map* is a Scott-continuous function from $\mathcal{D}$ to $States(\mathcal{E})$. Equivalently, a stochastic map can also be defined as a Scott-continuous linear map from $States(\mathcal{D})$ to $States(\mathcal{E})$, and we will use the two definitions interchangeably.

# Chapter 3

# Quantum Domains

In this chapter, we define quantum domains. A quantum domain is a domain which can be used to describe the behavior of a quantum system. In the first section, we will give a formal definition for the quantum domains. In Section 3.2, we will give a definition for observations and will show that the set of all observations forms a dcpo. In Section 3.3, we will define states and show that the set of all states together with information order will form a dcpo. In Section 3.4, we will introduce the notion of finitely compactly supported states (states which are physically realizable) and will show that every state can be viewed as a limit of finitely compactly supported ones. In Section 3.5, we will give a definition for progressive superoperator for finite quantum domains and discuss how this definition will extend special cases like states, observations, superoperators, and stochastic maps. In Section 3.6, we will discuss why the definition of progressive superoperator given in the previous section was not suitable for extension to non-finite domains. Then we will give the dual definition of progressive superoperators which addresses those problems.

Before giving a definition for quantum domains, let us consider an example of a quantum domain. Assume that we have a quantum system which might output 0, 1, or 2 qubits. The behavior of this system can be described using the domain given in Figure 3.1.

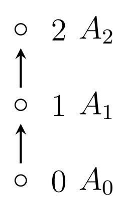



Figure 3.1: A quantum domain which might outputs 0, 1, or 2 qubits

Here, similar to the probabilistic domains, the arrows indicate an increase in the information about the behavior of the system. However, unlike the probabilistic domain, $A_0$, $A_1$, and $A_2$ are not all probabilities. While $A_0 \in \mathbb{C}^{1\times 1}$ indicates the chance of system outputting no qubits, $A_1 \in \mathbb{C}^{2\times 2}$ and $A_2 \in \mathbb{C}^{4\times 4}$ are matrices which give more information about the output of the system than just the probability with which the system outputs one and two qubits, respectively. The system outputs one qubit (resp. two qubits) in state $A_1$ (resp. $A_2$) with probability $tr\,(A_1)$ (resp. $tr\,(A_2)$).

This means that for giving a definition of quantum domains, in addition to considering the structure of the positions of the system (given by the domain), we need to consider the space of density matrices corresponding to each position. We can use this information to give a formal definition of quantum domains, but before doing so, we first will mention a definition and a notation used in this thesis.

**Definition 3.1.** Let $H$ and $K$ be finite dimensional Hilbert spaces. A linear multiplicative involution preserving unital map (*miu-map*) $f : \mathcal{B}(H) \to \mathcal{B}(K)$ is a map with following properties:

- $f(AB) = f(A)f(B)$
- $f(A^\dagger) = f(A)^\dagger$
- $f(I) = I$

*Notation* 3.2. Let $FHilb_{imiu}$ be the category which has

- as objects, the finite dimensional Hilbert spaces,
- as morphisms, the injective $miu$-maps ($imiu$-maps) of the form $f : \mathcal{B}(H_1) \to \mathcal{B}(H_2)$, where $H_1$ and $H_2$ are finite dimensional Hilbert spaces.

*Remark* 3.3. The morphisms of this category are Scott-continuous maps.

### 3.1 Definition of Quantum Domain

Now we can define quantum domains formally:

**Definition 3.4.** A *quantum domain* $\mathcal{D} = (D, H)$ consists of:

- a bifinite domain $D$,

- a functor $H\colon \mathcal{K}(D) \to FHilb_{imiu}$ where
  - On objects: For all $x \in \mathcal{K}(D)$, $x \mapsto H(x) = H_x$
  - On morphisms: For all $x, y \in \mathcal{K}(D)$ with $x \leq y$, $H_{x,y}$ is an injective miu-map $\mathcal{B}(H_x) \to \mathcal{B}(H_y)$:

$$\begin{array}{ccc} y & & \mathcal{B}(H_y) \\ \uparrow & \longmapsto & \uparrow H_{x,y} \\ x & & \mathcal{B}(H_x) \end{array}$$

**Example 3.5.** Consider the diagram in Figure 3.2. The diagram represents a quantum system which has two channels and might output zero qubit or one qubit (from one of the channels). The nodes indicate the number of qubits that have been output and their label shows the Hilbert space assigned to that compact element.

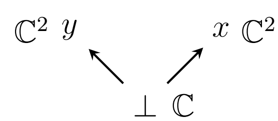



Figure 3.2: A quantum system with two channels

**Definition 3.6.** For all $x, y \in \mathcal{K}(D)$ with $x \leq y$, the adjoint of the function $H_{x,y}$ (also called the *partial trace*) is the map $H^{\dagger}_{x,y} : \mathcal{B}(H_y) \to \mathcal{B}(H_x)$. It is a linear function, but not necessarily injective nor miu-map.

## 3.2 Observations

Quantum systems are usually described by their states and the observations on them, so in this section, we will give a definition for observations in our setting. In order to do so, we first will give a definition of the space of matrix tuples and the operations over them.

**Definition 3.7.** Let $\mathcal{D} = (D, H)$ be a quantum domain. Every element

$$T \in \prod_{x \in \mathcal{K}(D)} \mathcal{B}(H_x)$$

is called a *matrix tuple*. The space of all matrix tuples over $\mathcal{D}$ is denoted by

$$\mathcal{V}_{\mathcal{D}} = \prod_{x \in \mathcal{K}(D)} \mathcal{B}\left(H_x\right).$$

A matrix tuple is called *positive* if $A_x$ is positive (i.e., positive semi-definite and hermitian) for all $x \in \mathcal{K}(D)$. The set of positive matrix tuples is denoted by ${\mathcal{V}_{\mathcal{D}}}^+$.

**Definition 3.8.** For $A = (A_x)_{x \in \mathcal{K}(D)} \in \mathcal{V}_{\mathcal{D}}$, the *support of* $A$ is $Supp(A) = \{x \mid A_x \neq 0\}$.

**Definition 3.9.** For all positive $A = (A_x) \in \mathcal{V}_{\mathcal{D}}$, the *trace* is defined as:

$$tr(A) = \sum_{x \in \mathcal{K}(D)} tr(A_x) \in [0, \infty]$$

Now we will define the concept of observation in our setting of quantum domains:

**Definition 3.10.** $B = (B_x) \in \mathcal{V}_{\mathcal{D}}$ is an *observation* if:

- $\forall x \in \mathcal{K}(D)$, $0 \sqsubseteq B_x$
- $\forall x \in \mathcal{K}(D)$, $B_x \sqsubseteq I$
- $\forall x, y \in \mathcal{K}(D),\ x \leq y,\ \ H_{x,y}\left(B_x\right) \sqsubseteq B_y$

The set of all observations is denoted by $\mathcal{O}_{\mathcal{D}}$.

Note that the observations are only defined at compact elements. For example, consider a system which outputs an infinite stream of qubits. Such a system corresponds to the diagram in Figure 3.3:

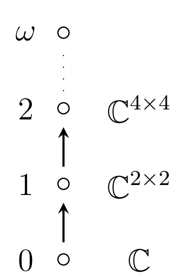



Figure 3.3: The domain representing a stream of qubits

The element $\omega$ cannot be achieved except as a limit of finite elements. This notion was considered in our definition of observation and is the reason that the observation was defined only for compact elements. Next we will show that the set of all observations forms a dcpo.

**Theorem 3.11.** *The partially ordered set $(\mathcal{O}_{\mathcal{D}}, \sqsubseteq)$ is a dcpo.*

*Proof.* Let $\{B^j\}_{j\in J}$ be a directed family of observations. Define $B = \sup_{j\in J}\left(B^j\right)$ be the pointwise supremum of $B^j$'s. We want to show that $B$ is an observation, as well. In order to do so we need to show

1. $\forall x \in \mathcal{K}(D),\ 0 \sqsubseteq B_x$
2. $\forall x \in \mathcal{K}(D),\ B_x \sqsubseteq I$
3. $\forall x, y \in \mathcal{K}(D),\ x \leq y,\ \ H_{x,y}(B_x) \sqsubseteq B_y$

The proof of the required conditions is as follows:

1. We know $\forall x \in \mathcal{K}(D),\ 0 \sqsubseteq B^j_x$, so $0 \sqsubseteq B^j_x \sqsubseteq \sup_{j\in J}\left(B^j_x\right) = B_x$. This implies that $\forall x \in \mathcal{K}(D),\ 0 \sqsubseteq B_x$.
2. We know $\forall x \in \mathcal{K}(D),\ B^j_x \sqsubseteq I$. So $I$ is an upper bound for $B^j_x$'s. So as $B_x$ is the least upper bound of $\{B^j_x\}$, $\forall x \in \mathcal{K}(D),\ B_x \sqsubseteq I$.
3. We know $\forall x, y \in \mathcal{K}(D),\ x \leq y,\ \ H_{x,y}(B^j_x) \sqsubseteq B^j_y$. So

$$H_{x,y}(B^j_x) \sqsubseteq B^j_y \sqsubseteq \sup_{j\in J}\left(B^j_y\right) = B_y.$$

   Also, $H_{x,y}$ is Scott-continuous since it is an injective miu-map. This implies that $H_{x,y}\left(B_x\right)$ is the least upper bound of $\left\{H_{x,y}(B^j_x)\right\}_{j\in J}$. So $H_{x,y}\left(B_x\right) \sqsubseteq B_y$ and thus we have $\forall x, y \in \mathcal{K}(D),\ x \leq y,\ \ H_{x,y}(B_x) \sqsubseteq B_y$.

Therefore, $B$ is an observation. This implies that the set $(\mathcal{O}_{\mathcal{D}}, \sqsubseteq)$ is a dcpo and we are done. $\square$

### 3.3 States

In the context of quantum computing, the concepts of states and observation can be considered as dual, because on one hand, the observation we make is completely dependent on the state in which the system is in. On the other hand, the only way we can determine the system's state is by making an observation. Therefore, in order to make our definitions compatible with actual systems and experiments, we will define the state as a function of observations.

**Definition 3.12.** Let $\mathcal{D}$ and $\mathcal{E}$ be two quantum domains. A function $F : \mathcal{V}_\mathcal{D} \to \mathcal{V}_\mathcal{E}$ is called *linear* if it preserves the subconvex combination. In other words, for all $M_1, M_2 \in \mathcal{V}_\mathcal{D}$ and $\lambda_1, \lambda_2 \in [0,1]$ with $\lambda_1 + \lambda_2 \leq 1$,

$$F(\lambda_1 M_1 + \lambda_2 M_2) = \lambda_1 F(M_1) + \lambda_2 F(M_2)$$

**Definition 3.13.** A *state* of $\mathcal{D}$ is a linear Scott-continuous function $A : \mathcal{O}_\mathcal{D} \to [0,1]$.

The set of all states is denoted as $\mathcal{S}_\mathcal{D}$. From now on, $A(B)$ will be denoted $\langle A, B \rangle$. Next we need to define an order on the states.

**Proposition 3.14.** *If $B_1 \sqsubseteq B_2 \in \mathcal{O}_\mathcal{D}$ and $A \in \mathcal{S}_\mathcal{D}$, then $\langle A, B_1 \rangle \leq \langle A, B_2 \rangle$.*

*Proof.* The result follows from monotonicity of $A$. □

**Definition 3.15.** The information order $\preceq$ over $\mathcal{S}_\mathcal{D}$ is defined as follows. Let $A_1$ and $A_2$ be two states on a quantum domain $\mathcal{D}$, then we say $A_1 \preceq A_2$ if

$$\forall B \in \mathcal{O}_\mathcal{D},\ \langle A_1, B \rangle \leq \langle A_2, B \rangle$$

This relation is a partial order over $\mathcal{S}_\mathcal{D}$. In the next theorem, we will show that $\mathcal{S}_\mathcal{D}$ together with the defined partial order forms a dcpo.

**Theorem 3.16.** *The partially ordered set $(\mathcal{S}_\mathcal{D}, \preceq)$ is a dcpo.*

*Proof.* Let $\{A^j\}_{j \in J}$ be a directed family of states. Define $A = \sup_{j \in J} \left(A^j\right)$ to be the pointwise supremum of $\{A^j\}_{j \in J}$. We want to show that $A$ is a state (a linear Scott-continuous function), as well.

1. <u>Linearity</u>:

   Let $B_1, B_2 \in \mathcal{O}_{\mathcal{D}}$ and $\lambda_1, \lambda_2 \in [0,1]$ such that $\lambda_1 + \lambda_2 \leq 1$. First we will show that

   (i) For $k = 1, 2$, $\lambda_k \sup\limits_{j \in J} \big(\langle A^j, B_k\rangle\big) = \sup\limits_{j \in J} \big(\langle A^j, \lambda_k B_k\rangle\big)$

   (ii) $\sup\limits_{i \in J} \big(\langle A^i, B_1\rangle\big) + \sup\limits_{j \in J} \big(\langle A^j, B_2\rangle\big) = \sup\limits_{i,j \in J} \big(\langle A^i, B_1\rangle + \langle A^j, B_2\rangle\big)$.

   For proving (i), first note that since $A^j$ are states, we have $\langle A^j, \lambda_k B_k\rangle = \lambda_k \langle A^j, B_k\rangle$. So we only need to show that $\lambda_k \sup\limits_{j \in J} \big(\langle A^j, B_k\rangle\big) = \sup\limits_{j \in J} \big(\lambda_k \langle A^j, B_k\rangle\big)$. For $k = 1, 2$ and $j \in J$, $\sup\limits_{j \in J} \big(\langle A^j, B_k\rangle\big)$ belongs to $[0, 1]$. So (i) follow from the fact that multiplication on positive real numbers is Scott-continuous. Similarly, (ii) follows from the fact that addition on positive real numbers is Scott-continuous. Therefore, $A = \sup\limits_{j \in J} \big(A^j\big)$ is linear. Next we will show the Scott-continuity of $A = \sup\limits_{j \in J} \big(A^j\big)$.

2. <u>Scott-Continuity</u>:

   For $A$ to be Scott-continuous, it needs to have the two following conditions:

   - Monotonicity:

     Let $B_1, B_2 \in \mathcal{O}_{\mathcal{D}}$ such that $B_1 \sqsubseteq B_2$. So for all $j \in J$, $\langle A^j, B_1\rangle \leq \langle A^j, B_2\rangle$. Therefore, we have

     $$\langle A, B_1\rangle = \sup_{j \in J} \big(\langle A^j, B_1\rangle\big) \leq \sup_{j \in J} \big(\langle A^j, B_2\rangle\big) = \langle A, B_2\rangle$$

     So $\langle A, B_1\rangle \leq \langle A, B_2\rangle$ which implies $A$ is monotone.

   - Preservation of Suprema:

     Let $\{B_k\}_{k \in K}$ be a directed family of observations. Define $B = \sup\limits_{k \in K} (B_k)$. We have

     $$\begin{aligned} \sup_{k \in K} (\langle A, B_k\rangle) &= \sup_{k \in K} \left( \sup_{j \in J} \big(\langle A^j, B_k\rangle\big) \right) \\ &= \sup_{j \in J} \left( \sup_{k \in K} \big(\langle A^j, B_k\rangle\big) \right) \end{aligned}$$

$$
\begin{aligned}
&= \sup_{j \in J} \left( \left\langle A^j, \sup_{k \in K} (B_k) \right\rangle \right) \\
&= \sup_{j \in J} \left( \langle A^j, B \rangle \right) \\
&= \langle A, B \rangle
\end{aligned}
$$

Note that the third equality holds because the $A^j$ are states and therefore are Scott-continuous.

This shows that $A$ is monotone and preserves suprema which implies that $A$ is Scott-continuous.

Therefore, $A = \sup_{j \in J} \left( A^j \right)$ is a state which implies that $(\mathcal{S}_{\mathcal{D}}, \preceq)$ is a dcpo and we are done. □

### 3.4 Finitely Compactly Supported States

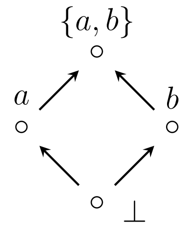
In general, we would like to prove that under suitable conditions, all states on a quantum domain are 'physically realizable'. However, in general, it is not clear what exactly is 'physical'. We therefore define a subset of states, which we call the 'finitely compactly supported states', which are definitely physically realizable, and then we will show that all quantum states arise as directed suprema of finitely compactly supported ones.

#### 3.4.1 Projectable Sets

In this subsection, we will define projectable sets and give some of their properties. In the next subsection, we will use projectable sets to define projection of observations and restricted states which are necessary for the proof of Theorem 3.48.

**Definition 3.17.** Let $D$ be a domain. A finite subset $X \subseteq \mathcal{K}(D)$ is called *projectable* if for all $d \in D$, the set $\{x \in X \mid x \leq d\}$ has a maximum.

*Remark* 3.18. Let $D$ be a dcpo and let $X, Y \subseteq \mathcal{K}(D)$ be projectable. Then $X \cup Y$ is not necessarily projectable. As an example, consider the following domain $D$:




The sets $X = \{\bot, a\} \subset D$ and $Y = \{\bot, b\} \subset D$ are projectable. However,

$$X \cup Y = \{\bot, a, b\}$$

is not projectable, since the set $\{x \in X \cup Y \mid x \leq \{a, b\}\} = \{\bot, a, b\}$ does not have a maximum.

We will now give a definition of projection and inclusion (embedding) maps and discuss some properties of these maps and projectable sets.

**Definition 3.19.** Let $D$ be a bifinite domain and let $X \subset \mathcal{K}(D)$ be projectable. Define the projection map $p_X : D \to X$ by

$$p_X(d) = \max\left(X \cap {\downarrow} d\right) = \max\{x \in X \mid \ x \leq d\}$$

Moreover, let $e_X : X \to D$ be the inclusion map.

**Lemma 3.20.** *Let $M$ be a finite set of compact elements of an algebraic domain $D$. Below each upper bound of $M$, there is a compact upper bound.*

*Proof.* Let $u \in D$ be an upper bound of $M$. Since $D$ is algebraic, the set $\mathcal{K}(D) \cap {\downarrow} u$ is directed, with $u$ as its supremum. We have $M \subseteq \mathcal{K}(D) \cap {\downarrow} u$, so as $\mathcal{K}(D) \cap {\downarrow} u$ is directed, there exists $u' \in \mathcal{K}(D) \cap {\downarrow} u$ such that $u'$ is an upper bound of $M$. So $u'$ is an upper bound of $M$ which is in $\mathcal{K}(D)$ and is below $u$. □

**Proposition 3.21.** *Let $D$ be a bifinite domain and let $M$ be a finite set of compact elements of $D$. We have*

1. *$mc(M)$ is projectable.*

2. *the projection map $p_{mc(M)}$ is Scott-continuous.*

3. *$p_{mc(M)}$ and $e_{mc(M)}$ are adjoints. In other words,*

$$\forall x \in mc(M),\ \forall d \in D,\ x \leq p_{mc(M)}(d) \Leftrightarrow e_{mc(M)}(x) \leq d$$

*Proof.* 1) Let $x \in D$ be given. We need to show that the set

$$A := mc(M) \cap \downarrow x = \{y \in mc(M) \mid y \leq x\}$$

has a maximum. Note that as $D$ is bifinite, $mc(A)$ is finite, which implies that $A$ is finite, as well. Therefore, by Lemma 3.20, there exists a compact upper bound $k$ of $A$, where $k \leq x$ (as $x$ is an upper bound of $A$). Consequently, as $D$ is bifinite, $\mathcal{K}(D)$ is mub-complete, which implies that there exists a minimal compact upper bound $u$ of $A$, below $k$. Since $u$ is minimal upper bound of $A \subseteq mc(M)$, $u \in mc(M)$. Also, we know $u \leq k \leq x$, which implies $u \in \downarrow x$. So, we have $u \in A$. Therefore, $u$ is an upper bound of $A$ which a part of $A$, which implies that $u$ is maximum of $A$.

2) We need to show that $p_{mc(M)}$ is monotone and preserves suprema. Let $x, y \in D$ with $x \leq y$. So $\downarrow x \subseteq \downarrow y$ and thus $mc(M) \cap \downarrow x \subseteq mc(M) \cap \downarrow y$. Therefore, we have

$$p_{mc(M)}(x) = \max\left(mc(M) \cap \downarrow x\right) \leq \max\left(mc(M) \downarrow y\right) = p_{mc(M)}(y).$$

So, $p_{mc(M)}$ is monotone. Next, we will show that $p_{mc(M)}$ preserves directed suprema. Let $A \subseteq D$ be directed. Since $p_{mc(M)}$ is monotone, $p_{mc(M)}(A)$ is directed, as well. So $\sup\left(p_{mc(M)}(A)\right) \leq p_{mc(M)}\left(\sup(A)\right)$. We only need to show that

$$p_{mc(M)}\left(\sup(A)\right) \leq \sup\left(p_{mc(M)}(A)\right).$$

We know that $p_{mc(M)}\left(\sup(A)\right) \leq \sup\left(A\right)$. Also, we know that $p_{mc(M)}\left(\sup(A)\right)$ is compact. Therefore, we have $p_{mc(M)}\left(\sup(A)\right) \leq a$ for some $a \in A$ and thus

$$p_{mc(M)}\left(\sup(A)\right) \leq p(a) \leq \sup\left(p_{mc(M)}(A)\right).$$

3) Let $x \in mc(M)$ and $d \in D$ be given such that $x \leq p_{mc(M)}(d)$. As $p_{mc(M)}(x) \leq x$, for all $x \in D$, we have

$$e_{mc(M)}(x) \leq e_{mc(M)}\left(p_{mc(M)}(d)\right) \leq d.$$

Now assume $x \in mc(M)$ and $d \in D$ such that $e_{mc(M)}(x) \leq d$. We know $x \in mc(M) \cap \downarrow x$, so $x \leq max\left(mc(M) \cap \downarrow x\right) = p_{mc(M)}(x)$. Also, we know $p_{mc(M)}(x) \leq x$. Therefore, we have $p_{mc(M)}(x) = x$ which implies that

$$x = p_{mc(M)}(x) = p_{mc(M)}\left(e_{mc(M)}(x)\right) \leq p_{mc(M)}(d).$$

□

**Corollary 3.22.** *Let $D$ be a bifinite domain and $M$ be a finite mub-closed set. Then $M$ is projectable.*

**Theorem 3.23.** *Let $D$ be a bifinite domain. For every finite set $S \subseteq \mathcal{K}(D)$, there exists a projectable set $X \subseteq \mathcal{K}(D)$ such that $S \subseteq X$.*

*Proof.* From Proposition 3.22, $X = mc(S)$ is projectable and $S \subseteq X$. Now we only need to show that $X \subseteq \mathcal{K}(D)$. We know that $mc(S)$ is finite, since $D$ is bifinite. Also, since $D$ is bifinite, $\mathcal{K}(D)$ is mub-complete. So it contains the minimal upper bounds of finite subsets of $mc(S)$, i.e., it contains $mc(S)$. So $mc(S) \subset \mathcal{K}(D)$.

□

The next Proposition addresses the relation between mub-closed sets and projectable sets.

**Proposition 3.24.** *Let $\mathcal{D} = (D, H)$ be a quantum domain and $M \subseteq \mathcal{K}(D)$. Then $M$ is projectable if and only if $M$ is a finite mub-closed set.*

*Proof.* Assume that $M$ is a finite mub-closed set. So from Corollary 3.22, $M$ is projectable. Now assume $M$ is projectable. Let $S \subset M$. Note that since $M$ is projectable, it is a finite set. So $S$ is finite. This implies that $S$ has a upper bound $d$ in $D$. Therefore, from Lemma 3.20, we have that there exists a compact upper bound $c$ below $d$. Since $D$ is bifinite, there exists a minimal upper bound $u \in \mathcal{K}(D)$ below $c$. So $u$ is a minimal upper bound for $S$.

Now consider the set $U = \{x \in M \mid x \leq u\}$. Since $M$ is projectable, $U$ has a maximum element $m$. We claim that $m = u$. Now assume to the contrary that $m \neq u$. Then as $u$ is an upper bound for $S$, we have that $S \subset U$ and since $m = \max U$, $m < u$ which implies that $m$ is an upper bound for $S$ which is smaller than $u$. Therefore, $u$ cannot be a minimal upper bound for $S$ which is contradiction. Therefore, $m = u$ and since $m \in M$, the minimal upper bound of $S$ is in $M$. This implies that $M$ contains all the minimal upper bounds of its finite subsets. Therefore, $M$ is a mub-closed set. □

### 3.4.2 Projection Observations and Point Observations

As we discussed before, it is desirable to be able to perform computation by relying on only finitely many elements. The first step is to define a process for getting an observation by relying on finite elements (since the state of the system and thus the result of all other computations will be dependent on observations). In this subsection, for every observation $B$, we will define the notion of projection observation $B$ onto a projectable set of compact elements and we will show that $B$ is the limit of all such projection observations.

**Proposition 3.25.** *Let $\mathcal{D} = (D, H)$ be a quantum domain and let $M$ be a projectable subset of $\mathcal{K}(D)$. For every observation $B \in \mathcal{O}_\mathcal{D}$, there exists a least observation $B|_M$ such that for all $x \in M$, $B_x \sqsubseteq (B|_M)_x$. It is given by*

$$(B|_M)_x = H_{p_M(x),x}\left(B_{p_M(x)}\right)$$

*Moreover, $(B|_M)_m = B_m$ for all $m \in M$, and $B|_M \sqsubseteq B$.*

*Proof.* We want to show that $B|_M$ is an observation. We know that $H_{p_M(x),x}$ is an imiu-map. So since $0 \sqsubseteq B_{p_M(x)} \sqsubseteq I$, we know that $0 \sqsubseteq H_{p_M(x),x}\left(B_{p_M(x)}\right) = (B|_M)_x$ and $(B|_M)_x = H_{p_M(x),x}\left(B_{p_M(x)}\right) \sqsubseteq I$. For all $x, y \in D$, with $x \leq y$, we have

$$\begin{aligned} H_{x,y}\left(B|_M\right)_x &= H_{x,y}\left(H_{p_M(x),x}\left(B_{p_M(x)}\right)\right) \\ &= H_{p_M(y),y}\left(H_{p_M(x),p_M(y)}\left(B_{p_M(x)}\right)\right) \\ &\sqsubseteq H_{p_M(y),y}\left(B_{p_M(y)}\right) \\ &= (B|_M)_y. \end{aligned}$$

Therefore, $B|_M$ is an observation. Also, for all $m \in M$, $p_M(m) = m$. So, we have $(B|_M)_m = H_{p_M(m),m}\left(B_{p_M(m)}\right) = B_m$, which implies that $B|_M$ agrees with $B$ on $M$.

Now we need to show that $B|_M$ is the least such observation. Let $B'$ be an observation such that $B_m \sqsubseteq B'_m$, for all $m \in M$. For all $x \in \mathcal{K}(D)$, we have

$$\begin{aligned} (B|_M)_x &= H_{p_M(x),x}\left(B_{p_M(x)}\right) \\ &\sqsubseteq H_{p_M(x),x}\left(B'_{p_M(x)}\right) \\ &\sqsubseteq B'_x. \end{aligned}$$

Therefore, $B|_M \sqsubseteq B'$. In particular, $B|_M \sqsubseteq B$ □

We call $B|_M$ the 'projection of $B$ onto $M$'. In the next proposition, we will show that the projection of $B$ onto $M$ is monotone in $M$.

**Proposition 3.26.** *Let $N \subset M \subset \mathcal{K}(D)$ be two projectable sets. Then for all observation $B \in \mathcal{O}_\mathcal{D}$, we have*

$$B|_N \sqsubseteq B|_M.$$

*Proof.* For all $x \in N$, $x \in M$ and thus $(B|_N)_x = B_x = (B|_M)_x$. Also, for all $x \in D \backslash M$, $(B|_N)_x \sqsubseteq (B|_M)_x$, since both observations $B|_M$ and $B|_N$ are the projection observations. Since $B|_N \sqsubseteq B$, for all $x \in M \setminus N$, we have $(B|_N)_x \sqsubseteq B_x = (B|_M)_x$. Therefore, we have $B|_N \sqsubseteq B|_M$. □

Now we are ready to show that every observation can be written as a limit of all its projection.

**Theorem 3.27.** *Let $\mathcal{D} = (D, H)$ be a quantum domain and let $B \in \mathcal{O}_{\mathcal{D}}$. Then the set*

$$X = \{B|_M \mid M \subset \mathcal{K}(D) \text{ is a projectable set}\}$$

*is directed and we have*

$$B = \sup(X).$$

*Proof.* Let $M, N \subset \mathcal{K}(D)$ be two projectable sets. So $mc\,(M \cup N)$ is a projectable set as well, and thus $B|_{mc(M\cup N)} \in X$. From Proposition 3.26, we have $B|_N \sqsubseteq B|_{mc(M\cup N)}$ and $B|_M \sqsubseteq B|_{mc(M\cup N)}$. Therefore, $X$ is directed.

Let $M \subset \mathcal{K}(D)$ be a projectable set. We know from Proposition 3.25 that for all observations $B$, $B|_M \sqsubseteq B$. So $B$ is an upper bound for $X$. Now we want to show that $B$ is the least upper bound. Let $B' \in \mathcal{O}_{\mathcal{D}}$ be an upper bound for $X$. So for all $x \in \mathcal{K}(D)$, $B|_{\{x\}} \sqsubseteq B'$. So from Proposition 3.25, we have

$$\forall x \in \mathcal{K}(D),\ B_x = \left(B|_{\{x\}}\right)_x \sqsubseteq (B')_x.$$

Therefore, $B \sqsubseteq B'$ and thus $B = \sup(X)$. □

In Proposition 3.26, we showed that the projection of observations is monotone as a function on $M$. In the next proposition, we will show that it is also monotone as a function of $B$.

**Proposition 3.28.** *Let $\mathcal{D} = (D, H)$ be a quantum domain. Let $B, B' \in \mathcal{O}_{\mathcal{D}}$ such that $B \sqsubseteq B'$ and let $M \subset \mathcal{K}(D)$ be a projectable set. Then $B|_M \sqsubseteq B'|_M$.*

*Proof.* We know that for all $x \in \mathcal{K}(D)$, $H_{p_M(x),x}$ is Scott-continuous. So as $B_{p_M(x)} \sqsubseteq B'_{p_M(x)}$, we have

$$H_{p_M(x),x}\left(B_{p_M(x)}\right) \sqsubseteq H_{p_M(x),x}\left(B'_{p_M(x)}\right).$$

Therefore, based on the definition of projection observations, we have

$$\left(B|_M\right)_x = H_{p_M(x),x}\left(B_{p_M(x)}\right) \sqsubseteq H_{p_M(x),x}\left(B'_{p_M(x)}\right) = \left(B'|_M\right)_x,$$

for all $x \in \mathcal{K}(D)$. Therefore, $B|_M \sqsubseteq B'|_M$. □

The next propositions will cover some of the properties of the projection observations like linearity and preserving suprema.

**Proposition 3.29.** *Let $\mathcal{D} = (D, H)$ be a quantum domain and $M \subset \mathcal{K}(D)$ be a projectable set. Let $B \in \mathcal{O}_\mathcal{D}$ such that $(B)_x = 0$ for all $x \in M$. Then $B|_M = 0$.*

*Proof.* For all $x \in D$, we have $p_M(x) \in M$, since $M$ is projectable. So we have

$$\left(B|_M\right)_x = H_{p_M(x),x}\left(B_{p_M(x)}\right) = H_{p_M(x),x}(0) = 0.$$

□

**Proposition 3.30.** *Let $\mathcal{D} = (D, H)$ be a quantum domain. Let $B^1, B^2 \in \mathcal{O}_\mathcal{D}$ and $\lambda_1, \lambda_2 \in [0,1]$ such that $\lambda_1 + \lambda_2 \leq 1$. Then $(\lambda_1 B^1 + \lambda_2 B^2)|_M = \lambda_1 B^1|_M + \lambda^2 B_2|_M$.*

*Proof.* For all $x \in M$, we have

$$\begin{aligned}\left(\left(\lambda_1 B^1 + \lambda_2 B^2\right)|_M\right)_x &= \left(\lambda_1 B^1 + \lambda_2 B^2\right)_x \\ &= \lambda_1 B^1_x + \lambda_2 B^2_x = \lambda_1\left(B^1|_M\right)_x + \lambda_2\left(B^2|_M\right)_x.\end{aligned}$$

Also, $\left(\lambda_1 B^1 + \lambda_2 B^2\right)|_M$ is the least observation with $B_x \sqsubseteq (B|_M)_x$, for all $x \in M$. Therefore, for all $x \in \mathcal{K}(D)$, we have

$$\left(\left(\lambda_1 B^1 + \lambda_2 B^2\right)|_M\right)_x = \lambda_1\left(B^1|_M\right)_x + \lambda_2\left(B^2|_M\right)_x$$

and thus $\left(\lambda_1 B^1 + \lambda_2 B^2\right)|_M = \lambda_1 B^1|_M + \lambda^2 B_2|_M$. □

**Proposition 3.31.** *Let $\mathcal{D} = (D, H)$ be a quantum domain. Let $\left\{B^k\right\}_{k\in K} \subset \mathcal{O}_\mathcal{D}$ be a directed family of observations and let $M \subset \mathcal{K}(D)$ be a projectable set. If $B = \sup_{k\in K}\left(B^k\right)$, then $B|_M = \sup_{k\in K}\left(B^k|_M\right)$.*

*Proof.* Since $B = \sup_{k\in K}\left(B^k\right)$, we have $B^k \sqsubseteq B$, for all $k \in K$. From Proposition 3.28, we get $B^k|_M \sqsubseteq B|_M$, for all $k \in K$, which implies $\sup_{k\in K}\left(B^k|_M\right) \sqsubseteq B|_M$. Now we need to show $B|_M \sqsubseteq \sup_{k\in K}\left(B^k|_M\right)$. For all $x \in \mathcal{K}(D)$, $p_M(x) \in M$ and thus $B^k_{p_M(x)} = \left(B^k|_M\right)_{p_M(x)}$. Also, we know that $H_{p_M(x),x}$ is Scott-continuous. So for all $x \in \mathcal{K}(D)$, we have

$$\begin{aligned}
(B|_M)_x &= H_{p_M(x),x}\left(B_{p_M(x)}\right) \\
&\sqsubseteq H_{p_M(x),x}\left(\sup_{k\in K}\left(B^k_{p_M(x)}\right)\right) \\
&= H_{p_M(x),x}\left(\sup_{k\in K}\left(\left(B^k|_M\right)_{p_M(x)}\right)\right) \\
&= H_{p_M(x),x}\left(\left(\sup_{k\in K}\left(B^k|_M\right)\right)_{p_M(x)}\right) \\
&\sqsubseteq \left(\sup_{k\in K}\left(B^k|_M\right)\right)_x
\end{aligned}$$

Therefore, $B|_M \sqsubseteq \sup_{k\in K}\left(B^k|_M\right)$ and thus $B|_M = \sup_{k\in K}\left(B^k|_M\right)$.

□

So far, we have defined the projection of observations and proved some of their properties. However, this projection operation need not be restricted to only observations. In the next part, we will give a more general definition for all matrix tuples and will prove that this operation is linear.

**Definition 3.32.** Let $\mathcal{D} = (D, H)$ be a quantum domain, $M \subset \mathcal{K}(D)$ be a projectable set, and $T \in \mathcal{V}_{\mathcal{D}}$ be a matrix tuple. Define a matrix tuple $T|_M$ by

$$(T|_M)_x = H_{p_M(x),x}\left(T_{p_M(x)}\right).$$

for $x \in \mathcal{K}(D)$ (Note that this restriction operation $T \mapsto T|_M$ gives a miu-map on $\mathcal{V}_{\mathcal{D}}$).

**Proposition 3.33.** *Let $\mathcal{D} = (D, H)$ be a quantum domain. Let $T^1, T^2 \in \mathcal{V}_{\mathcal{D}}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. Then $\left(\lambda_1 T^1 + \lambda_2 T^2\right)|_M = \lambda_1 T^1|_M + \lambda_2 T^2|_M$.*

*Proof.* Let $x \in \mathcal{K}(D)$. Then since $H_{p_M(x),x}$ is linear, we have

$$\begin{aligned}
\left(\lambda_1 T^1 + \lambda_2 T^2\right)|_M &= H_{p_M(x),x}\left(\lambda_1 T^1 + \lambda_2 T^2\right) \\
&= \lambda_1 H_{p_M(x),x}\left(T^1\right) + \lambda_2 H_{p_M(x),x}\left(T^2\right) = \lambda_1 T^1|_M + \lambda_2 T^2|_M
\end{aligned}$$ □

In the next part, we will define point observations (observations which will succeed only on or after a certain position in the domain).

**Definition 3.34.** Let $\mathcal{D} = (D, H)$ be a quantum domain, $x \in \mathcal{K}(D)$, and $E \in \mathcal{B}(H_x)$ with $0 \sqsubseteq E \sqsubseteq I$. The *closed point observation* $\mathbf{B}^c(x, E)$ is defined by

$$\mathbf{B}^c(x,E)_y = \begin{cases} H_{x,y}(E) & \textit{if } y \geq x \\ 0 & \textit{otherwise.} \end{cases}$$

It is an observation which has chance of succeeding only at or after point $x$. The *open point observation* $\mathbf{B}^\circ(x, E)$ is defined by

$$\mathbf{B}^\circ(x,E)_y = \begin{cases} H_{x,y}(E) & \textit{if } y > x \\ 0 & \textit{otherwise.} \end{cases}$$

It is an observation which has chance of succeeding only after the point $x$.

In the next part, we will explain the relation between point observations and observations of the form $B|_M$, i.e., how the latter can be written as a linear combination of point observations. But before doing so, we are going to show that the value of $B|_M$ depends only on the values of entries of $B$ whose index belong to $M$.

**Lemma 3.35.** *Let $\mathcal{D} = (D, H)$ be a quantum domain, $M \subset \mathcal{K}(D)$ be a projectable set, and $B^1, B^2 \in \mathcal{O}_\mathcal{D}$ be observations. We have*

$$B^1|_M = B^2|_M \quad \Leftrightarrow \quad \forall x \in M, B^1_x = B^2_x.$$

*Proof.* It is clear that if $B^1|_M = B^2|_M$, then for all $x \in M$, $B^1_x = B^2_x$. Now assume for all $x \in M$, $B^1_x = B^2_x$. For all $y \in \mathcal{K}(D)$,

$$(B^1|_M)_y = H_{p_M(y),y}\left(B^1_{p_M(y)}\right) = H_{p_M(y),y}\left(B^2_{p_M(y)}\right) = (B^2|_M)_y$$

since $p_M(y) \in M$. Therefore, we have $B^1|_M = B^2|_M$. □

**Proposition 3.36.** *Let $\mathcal{D} = (D, H)$ be a quantum domain, let $M \subset \mathcal{K}(D)$ be a projectable set, and let $B \in \mathcal{O}_\mathcal{D}$ be an observation. Then*

$$B|_M = \sum_{x \in M} \boldsymbol{B}^c(x, B_x)|_M - \boldsymbol{B}^\circ(x, B_x)|_M$$

*Remark* 3.37. Here, the sum and differences we take are in the space of all matrix tuples, i.e., the intermediate result are not necessarily observations. More specifically, for all $x \in M$, the matrix tuple $\mathbf{B}^c(x, B_x)|_M - \mathbf{B}^\circ(x, B_x)|_M$ is not an observation, but the sum of these matrix tuples is an observation.

*proof of Proposition 3.36.* From linearity of projection operation on matrix tuples (by Proposition 3.33), we have

$$\sum_{x \in M} \mathbf{B}^c(x, B_x)|_M - \mathbf{B}^\circ(x, B_x)|_M = \left( \sum_{x \in M} \mathbf{B}^c(x, B_x) - \mathbf{B}^\circ(x, B_x) \right) \Bigg|_M .$$

So we need to show that

$$B|_M = \left( \sum_{x \in M} \mathbf{B}^c(x, B_x) - \mathbf{B}^\circ(x, B_x) \right) \Bigg|_M .$$

From Lemma 3.35, the equation above holds if and only if for all $y \in M$, we have

$$B_y = \sum_{x \in M} \mathbf{B}^c(x, B_x)_y - \mathbf{B}^\circ(x, B_x)_y.$$

Now let $y \in M$ be given. We have

$$\begin{aligned} \mathbf{B}^c(x, B_x)_y - \mathbf{B}^\circ(x, B_x)_y &= \begin{cases} H_{x,y}(B_x) - H_{x,y}(B_x) & \text{if } x < y \\ B_x - 0 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} B_y & \text{if } y = x \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Therefore, for all $x \in M$, where $x \neq y$, $\mathbf{B}^c(x, B_x)_y - \mathbf{B}^\circ(x, B_x)_y = 0$, which implies that

$$\sum_{x \in M} \mathbf{B}^c(x, B_x)_y - \mathbf{B}^\circ(x, B_x)_y = B_y.$$

Therefore, we have

$$B|_M = \sum_{x \in M} \mathbf{B}^c(x, B_x)|_M - \mathbf{B}^\circ(x, B_x)|_M$$

□

The following example demonstrates the previous proposition for a finite probabilistic domain which is a special case of quantum domains.

**Example 3.38.** Let $\mathcal{D} = (D, H)$ be a finite probabilistic domain, where $D = \{x, y, z\}$. Figure 3.4 shows the diagram of $D$.

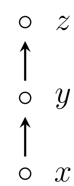



Figure 3.4: A 3-element probabilistic domain

Let $M = \{x, y\}$. Clearly, $M$ is projectable. Let $B = (B_x, B_y, B_z)$ be an observation on $D$ and let $A(B_x, B_y, B_z) = 0.6B_x + 0.3B_y + 0.1B_z$ be a map on observations of $D$. Clearly, $A$ is linear. Monotonicity and preserving directed suprema properties of $A$ follows from that of positive real numbers. So $A$ is a state of $D$.

By definition $B|_M = (B_{p_M(x)}, B_{p_M(y)}, B_{p_M(z)}) = (B_x, B_y, B_y)$. Next, we will calculate the closed and open point observations for every position of $D$.

$$\begin{array}{ll} \mathbf{B}^c(x, B_x) = (B_x, B_x, B_x) & \mathbf{B}^\circ(x, B_x) = (0, B_x, B_x) \\ \mathbf{B}^c(y, B_y) = (0, B_y, B_y) & \mathbf{B}^\circ(y, B_y) = (0, 0, B_y) \\ \mathbf{B}^c(z, B_z) = (0, 0, B_z) & \mathbf{B}^\circ(z, B_z) = (0, 0, 0) \end{array}$$

Now we will find the projection of these point observations:

$$\begin{array}{ll} \mathbf{B}^c(x, B_x)|_M = (B_x, B_x, B_x) & \mathbf{B}^\circ(x, B_x)|_M = (0, B_x, B_x) \\ \mathbf{B}^c(y, B_y)|_M = (0, B_y, B_y) & \mathbf{B}^\circ(y, B_y)|_M = (0, 0, 0) \\ \mathbf{B}^c(z, B_z)|_M = (0, 0, 0) & \mathbf{B}^\circ(z, B_z)|_M = (0, 0, 0) \end{array}$$

So, the subtraction of the closed and open point observations gives:

$$\begin{array}{l} \mathbf{B}^c(x, B_x)|_M - \mathbf{B}^\circ(x, B_x)|_M = (B_x, B_x, B_x) - (0, B_x, B_x) = (B_x, 0, 0) \\ \mathbf{B}^c(y, B_y)|_M - \mathbf{B}^\circ(y, B_y)|_M = (0, B_y, B_y) - (0, 0, 0) = (0, B_y, B_y) \\ \mathbf{B}^c(z, B_z)|_M - \mathbf{B}^\circ(z, B_z)|_M = (0, 0, 0) - (0, 0, 0) = (0, 0, 0) \end{array}$$

This implies that $\sum_{m \in M} \mathbf{B}^c(m, B_m)|_M - \mathbf{B}^\circ(m, B_m)|_M = (B_x, B_y, B_y) = B|_M$.

**Proposition 3.39.** *Let $\mathcal{D} = (D, H)$ be a quantum domain, let $M \subset \mathcal{K}(D)$ be a projectable set, let $x \notin M$, and let $E \in \mathcal{B}(H_x)$. Then we have*

$$\boldsymbol{B}^c(x, E)|_M = \boldsymbol{B}^\circ(x, E)|_M.$$

*Proof.* For all $y \in \mathcal{K}(D)$, we have $(\mathbf{B}^c(x, E)|_M)_y = H_{p_M(y),y}\left(\mathbf{B}^c(x, E)_{p_M(y)}\right)$ and $(\mathbf{B}^\circ(x, E)|_M)_y = H_{p_M(y),y}\left(\mathbf{B}^\circ(x, E)_{p_M(y)}\right)$, where

$$\mathbf{B}^c(x, E)_{p_M(y)} = \begin{cases} H_{x,p_M(y)}(E) & \text{if } x \leq p_M(y) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathbf{B}^\circ(x, E)_{p_M(y)} = \begin{cases} H_{x,p_M(y)}(E) & \text{if } x < p_M(y) \\ 0 & \text{otherwise} \end{cases}.$$

However, since $x \notin M$, $p_M(y) \neq x$. So we have $\mathbf{B}^c(x, E)_{p_M(y)} = \mathbf{B}^\circ(x, E)_{p_M(y)}$ and thus for all $y \in \mathcal{K}(D)$, we have

$$\begin{aligned} (\mathbf{B}^c(x, E)|_M)_y &= H_{p_M(y),y}\left(\mathbf{B}^c(x, E)_{p_M(y)}\right) \\ &= H_{p_M(y),y}\left(\mathbf{B}^\circ(x, E)_{p_M(y)}\right) \\ &= (\mathbf{B}^\circ(x, E)|_M)_y \end{aligned}$$

□

### 3.4.3 Restricted States and Finitely Compactly Supported States

In this subsection, we are going to define the dual concept of projection of observations and we will show that this dual concept is in fact a state (we refer to these states as restricted states). Then we are going to define finitely compactly supported states and discuss the relation between these states and the restricted states. In the last part, we will show that every state can be written as a limit of finitely compactly supported ones.

**Proposition 3.40.** *Let $\mathcal{D} = (D, H)$ be a quantum domain. Let $A \in \mathcal{S}_\mathcal{D}$ be a state of $\mathcal{D}$ and let $M$ be a projectable subset of $\mathcal{K}(D)$. The function $A|_M : \mathcal{O}_\mathcal{D} \to [0, 1]$ defined by*

$$\forall B \in \mathcal{O}_\mathcal{D},\ \langle A|_M, B\rangle = \langle A, B|_M\rangle$$

*is a state.*

*Proof.* We need to show that $A|_M$ is a linear Scott-continuous function. This easily follows from the fact that the restriction operator $B \mapsto B|_M$ is linear and Scott-continuous. Explicitly, the linearity of $A|_M$ comes from the linearity of $A$. In other words, for $B_1, B_2 \in \mathcal{O}_\mathcal{D}$ and $\lambda_1, \lambda_2 \in [0,1]$ such that $\lambda_1 + \lambda_2 \leq 1$, from Proposition 3.30, we have

$$\begin{aligned}
\langle A|_M, \lambda_1 B_1 + \lambda_2 B_2 \rangle &= \langle A, (\lambda_1 B_1 + \lambda_2 B_2)|_M \rangle \\
&= \langle A, \lambda_1 B_1|_M + \lambda_2 B_2|_M \rangle \\
&= \langle A, \lambda_1 B_1|_M \rangle + \langle A, \lambda_2 B_2|_M \rangle \\
&= \lambda_1 \langle A, B_1|_M \rangle + \lambda_2 \langle A, B_2|_M \rangle \\
&= \lambda_1 \langle A|_M, B_1 \rangle + \lambda_2 \langle A|_M, B_2 \rangle
\end{aligned}$$

Therefore $A|_M$ is linear. Next, we will address the monotonicity of $A|_M$. Let $B_1, B_2 \in \mathcal{O}_\mathcal{D}$ such that $B_1 \sqsubseteq B_2$. Therefore, from Proposition 3.28, we have $B_1|_M \sqsubseteq B_2|_M$. From monotonicity of $A$, we get

$$\langle A|_M, B_1 \rangle = \langle A, B_1|_M \rangle \leq \langle A, B_2|_M \rangle = \langle A|_M, B_2 \rangle.$$

So, $A|_M$ is monotone.

Finally, we need to show that $A|_M$ preserves directed suprema. Let $\{B^k\}_{k \in K}$ be a directed family of observations and let $B = \sup_{k \in K} \left( B^k \right)$. We know $A$ preserves suprema, so from Proposition 3.31, we have

$$\begin{aligned}
\sup_{k \in K} \left( \langle A|_M, B^k \rangle \right) &= \sup_{k \in K} \left( \langle A, B^k|_M \rangle \right) \\
&= \left\langle A, \sup_{k \in K} \left( B^k|_M \right) \right\rangle \\
&= \langle A, B|_M \rangle \\
&= \langle A|_M, B \rangle \\
&= \left\langle A|_M, \sup_{k \in K} \left( B^k \right) \right\rangle
\end{aligned}$$

Therefore, $A|_M$ preserves the supremum. So $A|_M$ is a linear Scott-continuous function and thus is a state. □

Now we are going to prove that the restriction of a state $A$ to a projectable set $M$ is monotone in $M$.

**Proposition 3.41.** *Let $\mathcal{D} = (D, H)$ be a quantum domain and let $N \subseteq M \subseteq \mathcal{K}(D)$ be two projectable sets. Then for all $A \in \mathcal{S}_{\mathcal{D}}$, $A|_N \preceq A|_M$.*

*Proof.* For all observations $B \in \mathcal{O}_{\mathcal{D}}$, from Proposition 3.26, we have $B|_N \sqsubseteq B|_M$. Therefore, from Proposition 3.14, we get

$$\langle A|_N, B\rangle = \langle A, B|_N\rangle \leq \langle A, B|_M\rangle = \langle A|_M, B\rangle$$

Since this is true for all observations $B$, we have $A|_N \preceq A|_M$ □

In the next part, for every state $A$ of a quantum domain, we will define an operator with which we can determine the probability of success of the system when it is in a certain position $x$ and the observation made is a point observation. Then we use this operation to find a relation between states and the restricted states.

**Definition 3.42.** Let $\mathcal{D} = (D, H)$ be a quantum domain and let $A \in \mathcal{S}_{\mathcal{D}}$ be a state. For $x \in \mathcal{K}(D)$, we define a positive matrix $A_x \in \mathcal{B}(H_x)$ with $tr(A_x) \leq 1$ as the unique operator such that for all $E \in \mathcal{B}(H_x)$,

$$\begin{aligned} tr\,(A_x E) = \langle A_x, E\rangle &= \text{“}\langle A, \mathbf{B}^c(x,E) - \mathbf{B}^\circ(x,E)\rangle\text{”} \\ &= \langle A, \mathbf{B}^c(x,E)\rangle - \langle A, \mathbf{B}^\circ(x,E)\rangle \in [0,1] \end{aligned}$$

Note that in the notation “$\langle A, \mathbf{B}^c(x,E) - \mathbf{B}^\circ(x,E)\rangle$”, the operation $\mathbf{B}^c(x,E) - \mathbf{B}^\circ(x,E)$ is not an observation. However, for simplicity we use this notation in our calculation and we defined it to be $\langle A, \mathbf{B}^c(x,E)\rangle - \langle A, \mathbf{B}^\circ(x,E)\rangle$.

**Lemma 3.43.** *Let $\mathcal{D} = (D, H)$ be a quantum domain. Let $A \in \mathcal{S}_{\mathcal{D}}$ be a state of $\mathcal{D}$ and $M$ be a projectable subset of $\mathcal{K}(D)$. Then $(A|_M)_x = A_x$, for all $x \in M$.*

*Proof.* Let $E \in \mathcal{B}(H_x)$ and let $x \in M$. Note that

$$(\mathbf{B}^c(x,E) - \mathbf{B}^\circ(x,E))_y = \begin{cases} E & y = x \\ 0 & \text{otherwise} \end{cases}.$$

So, $\mathbf{B}^c(x,E) - \mathbf{B}^\circ(x,E) = (\mathbf{B}^c(x,E) - \mathbf{B}^\circ(x,E))\,|_M$. So we have

$$\begin{aligned}\langle (A|_M)_x, E\rangle &= \langle A|_M, \mathbf{B}^c(x,E) - \mathbf{B}^\circ(x,E)\rangle \\ &= \langle A, (\mathbf{B}^c(x,E) - \mathbf{B}^\circ(x,E))\,|_M\rangle \\ &= \langle A, \mathbf{B}^c(x,E) - \mathbf{B}^\circ(x,E)\rangle \\ &= \langle A_x, E\rangle\end{aligned}$$

Therefore, for all $x \in M$, $(A|_M)_x = A_x$. □

Next, we will show that every map of the form $B \mapsto \sum_{x\in\mathcal{K}(D)} tr\,(A_x B_x)$ is a state. We are also going to give a definition for finitely compactly supported states and show that for any projectable set, the restricted state over that set is a finitely compactly supported state.

**Proposition 3.44.** *Let $\mathcal{D}$ be a quantum domain, let $\mathcal{I}$ be the singleton domain, and let $A \in \mathcal{V}_\mathcal{D}$ with*

- *For $x \in \mathcal{K}(D)$, $0 \sqsubseteq A_x$*
- *$tr(A) \leq 1$*
- *$Supp(A)$ is finite*

*Then the map $S$, given by*

$$\begin{aligned} S : \mathcal{O}_\mathcal{D} &\to \mathcal{O}_\mathcal{I} \\ B &\mapsto \sum_{x\in\mathcal{K}(D)} tr\,(A_x B_x), \end{aligned}$$

*is a state.*

*Proof.* First note that since $tr(A) \leq 1$, for all $B \in \mathcal{O}_\mathcal{D}$, $S(B) \in [0,1]$ and thus $S(B)$ is an observation in domain $\mathcal{I}$. In order to show that the map $S$ is a state, we need to show that $S$ is linear and Scott-continuous. $S$ is linear, as for all $B^1, B^2 \in \mathcal{B}_\mathcal{D}$ and $\lambda_1, \lambda_2 \in [0,1]$, we have

$$\begin{aligned} S\left(\lambda_1 B^1 + \lambda_2 B^2\right) &= \sum_{x\in\mathcal{K}(D)} tr\left(A_x\left(\lambda_1 B^1 + \lambda_2 B^2\right)_x\right) \\ &= \lambda_1 \sum_{x\in\mathcal{K}(D)} tr\big(A_x B^1_x\big) + \lambda_2 \sum_{x\in\mathcal{K}(D)} tr\big(A_x B^2_x\big) \\ &= \lambda_1 S\left(B^1\right) + \lambda_2 S\left(B^2\right) \end{aligned}$$

Now we are going to show that $S$ is monotone. Let $B^1, B^2 \in \mathcal{O}_\mathcal{D}$ such that $B^1 \sqsubseteq B^2$. So for all $x \in \mathcal{K}(D)$, $tr\,(A_x B_x^1) \leq tr\,(A_x B_x^2)$. This implies that

$$S\left(B^1\right) = \sum_{x \in \mathcal{K}(D)} tr\left(A_x B_x^1\right) \leq \sum_{x \in \mathcal{K}(D)} tr\left(A_x B_x^2\right) = S\left(B^2\right).$$

So $S$ is monotone. Next we show that $S$ preserves suprema and thus is Scott-continuous. Let $\{B^k\}_{k \in K}$ be a directed family of observations. We have

$$\begin{aligned} S\left(\sup_{k \in K}(B^k)\right) &= \sum_{x \in \mathcal{K}(D)} tr\left(A_x \left(\sup_{k \in K}\left(B_x^k\right)\right)\right) \\ &= \sup_{k \in K}\left(\sum_{x \in \mathcal{K}(D)} tr\left(A_x B_x^k\right)\right) \\ &= \sup_{k \in K}\left(S\left(B^k\right)\right). \end{aligned}$$

This implies that $S$ is a linear Scott-continuous map and therefore, $S$ is a state. □

Next, we will give a definition for state matrix tuples and finitely compactly supported states.

**Definition 3.45.** Let $\mathcal{D} = (D, H)$ be a quantum domain. A matrix tuple $A \in \mathcal{V}_\mathcal{D}$ is called *state matrix tuple* if the map $B \mapsto \sum_{x \in \mathcal{K}(D)} tr\,(A_x B_x)$, $B \in \mathcal{O}_\mathcal{D}$, is a state. In other words, if $A$ is positive and $tr(A) \leq 1$.

**Definition 3.46.** Let $\mathcal{D}$ be a quantum domain. A state $A$ is *finitely compactly supported* if there exists a state matrix tuple $C \in \mathcal{V}_\mathcal{D}$ with finite support ($Supp(C)$ is finite), such that

$$\forall B \in \mathcal{O}_\mathcal{D},\ \langle A, B\rangle = \sum_{x \in \mathcal{K}(D)} tr\,(C_x B_x)$$

The set of all finitely compactly supported states is given by $\mathcal{FS}_\mathcal{D}$.

**Proposition 3.47.** *Let $\mathcal{D} = (D, H)$ be a quantum domain and let $M$ be a projectable subset of $\mathcal{K}(D)$. Then $A|_M$ is a finitely compactly supported state.*

*Proof.* We write this proof in two steps. Let $A' = A|_M$ and for all $x \in \mathcal{K}(D)$, $A'_x$ be the operator given in Definition 3.42. In the first step we will show that the support of the matrix tuple $(A'_x)_{x \in \mathcal{K}(D)}$ is finite.

**Claim 3.47.1.** *For all $x \notin M$, $A'_x = 0$.*

*Proof of Claim 3.47.1.* Let $x \in \mathcal{K}(D)$ such that $x \notin M$. $A'_x$ by Definition 3.42 is such that for all $E \in \mathcal{B}(H_x)$,

$$\begin{aligned} tr\left(\left(A'_x\right)E\right) &= \langle A', \mathbf{B}^c(x,E)\rangle - \langle A', \mathbf{B}^\circ(x,E)\rangle \\ &= \langle A|_M, \mathbf{B}^c(x,E)\rangle - \langle A|_M, \mathbf{B}^\circ(x,E)\rangle \ . \\ &= \langle A, \mathbf{B}^c(x,E)|_M\rangle - \langle A, \mathbf{B}^\circ(x,E)|_M\rangle \end{aligned}$$

As $x \notin M$, by Proposition 3.39, $\mathbf{B}^c(x,E)|_M = \mathbf{B}^\circ(x,E)|_M$. Therefore, for all $E \in \mathcal{B}(H_x)$, $tr\left(\left(A'_x\right)E\right) = 0$ and thus $A'_x = 0$. [□ Claim 3.47.1]

In the next step, we will show that $A'$ is a finitely compactly supported state.

**Claim 3.47.2.** *For all observation $B \in \mathcal{O}_\mathcal{D}$,*

$$\langle A', B\rangle = \sum_{x \in \mathcal{K}(D)} tr\left(A'_x B_x\right).$$

*Proof of Claim 3.47.2.* Let $B \in \mathcal{O}_\mathcal{D}$ be an observation. We have

$$\begin{aligned} \langle A', B\rangle &= \langle A|_M, B\rangle \\ &= \langle A, B|_M\rangle && \text{(by definition)} \\ &= \sum_{x\in M} \langle A, \mathbf{B}^c(x,B_x)|_M\rangle - \langle A, \mathbf{B}^\circ(x,B_x)|_M\rangle && \text{(from Proposition 3.36)} \\ &= \sum_{x\in M} \langle A|_M, \mathbf{B}^c(x,B_x)\rangle - \langle A|_M, \mathbf{B}^\circ(x,B_x)\rangle && \text{(by definition)} \\ &= \sum_{x\in M} \langle A', \mathbf{B}^c(x,B_x)\rangle - \langle A', \mathbf{B}^\circ(x,B_x)\rangle \\ &= \sum_{x\in M} tr(A'_x B_x) && \text{(by definition 3.42)} \\ &= \sum_{x\in \mathcal{K}(D)} tr(A'_x B_x) && \text{(by Claim 3.47.1)} \end{aligned}$$

[□ Claim 3.47.2]

So we proved that there exists $C \in \mathcal{V}_\mathcal{D}$, with

$$C_x = \begin{cases} A'_x & \text{if } x \in M \\ 0 & \text{if } x \notin M \end{cases}$$

such that $\langle A|_M, B\rangle = \sum_{x\in D} tr(C_x B_x)$. Finally, we will show that $tr(C) \leq 1$. In the previous part, we have shown that there exists $C \in \mathcal{V}_{\mathcal{D}}$ such that for all $B \in \mathcal{O}_{\mathcal{D}}$, $\langle A|_M, B\rangle = \sum_{x\in\mathcal{K}(D)} tr\,(C_x B_x)$. We know that for all $B \in \mathcal{O}_{\mathcal{D}}$, $\langle A|_M, B\rangle \in [0,1]$ (since $A|_M$ is a state). Now set $B = I$. So we get

$$\langle A|_M, I\rangle = \sum_{x\in\mathcal{K}(D)} tr\,(C_x) = tr(C) \leq 1$$

and we are done. □

Now we are ready to discuss the main theorem of this section. In the next theorem, we will show that every state can be written as a limit of its restrictions, each of which we showed (in Proposition 3.47) is a finitely compactly supported state.

**Theorem 3.48.** *Let $\mathcal{D} = (D, H)$ be a quantum domain and let $A \in \mathcal{S}_{\mathcal{D}}$. Then the set*

$$X = \{A|_M \ \mid M \subset \mathcal{K}(D) \text{ is a projectable set}\}$$

*is directed and we have*

$$A = \sup\,(X).$$

*Proof.* Let $M, N \subset \mathcal{K}(D)$ be two projectable sets. So $mc\,(M \cup N)$ is a projectable set as well, and thus $A|_{mc(M\cup N)} \in X$. From Proposition 3.41, we have $A|_N \preceq A|_{mc(M\cup N)}$ and $A|_M \preceq A|_{mc(M\cup N)}$. Therefore, $X$ is directed.
Let $B \in \mathcal{O}_{\mathcal{D}}$ and let $P = \{M \subset \mathcal{K}(D) \mid M \text{ is projectable}\}$, then we have

$$\begin{aligned}
\sup_{M\in P}\,(\langle A|_M, B\rangle) &= \sup_{M\in P}\,(\langle A, B|_M\rangle) && \text{(based on definition)}\\
&= \left\langle A, \sup_{M\in P}\,(B|_M)\right\rangle && \text{($A$ preserves suprema)}\\
&= \langle A, B\rangle && \text{(from Theorem 3.27)}
\end{aligned}$$

Since this is true for all observations $B$, we have $A = \sup\,(X)$ and we are done. □

So in this section, we showed that every state can be written as a limit of finitely compactly supported states. A finitely compactly supported state is realizable on a physical quantum device, simply by preparing one of the finitely many mixed states according to a probability distribution. Therefore, we have shown that every state is a limit of physically realizable states.

### 3.5 Progressive Superoperators for Finite Quantum Domains

In the previous section, we showed that every state can be written as a limit of finitely compactly supported states. Now in this section we are going to define a progressive superoperator for the finite quantum domains. In a finite quantum domain, every element is a compact element. Therefore, since the set of compact elements is finite, every state is a finitely compactly supported state.

So for every quantum domain $D$ and state $A \in \mathcal{S}_{\mathcal{D}}$, we can write

$$\langle A, B \rangle = \sum_{x \in \mathcal{K}(D)} tr\,(A_x B_x).$$

where $A_x$ is the operator defined in Definition 3.42.

*Notation* 3.49. From now on, for every state $A \in \mathcal{S}_{\mathcal{D}}$, $A$ also denotes the matrix tuple $(A_x)_{x \in \mathcal{K}(D)}$ where $A_x \in \mathcal{B}(H_x)$ is the positive matrix defined in Definition 3.42. Based on this notation, $\mathcal{S}_{\mathcal{D}} \subset {\mathcal{V}_{\mathcal{D}}}^+$.

**Definition 3.50.** Let $\mathcal{D} = (D, H_{\mathcal{D}})$ and $\mathcal{E} = (E, H_{\mathcal{E}})$ be two finite quantum domains. A linear map $F : \mathcal{V}_{\mathcal{D}} \to \mathcal{V}_{\mathcal{E}}$ is *completely positive* if for all $x \in D$ and $y \in E$, the map $F_{yx} : \mathcal{B}(H_x) \to \mathcal{B}(H_y)$ is completely positive, where

$$F\left((A_x)_{x \in \mathcal{K}(D)}\right) = \left(\sum_{x \in \mathcal{K}(D)} F_{yx}\,(A_x)\right)_{y \in \mathcal{K}(E)}.$$

Next we will present a definition for progressive superoperator and we will explain how this definition generalizes concepts like states, observations, and superoperators.

**Definition 3.51.** Let $\mathcal{D}$ and $\mathcal{E}$ be two finite quantum domains. A *progressive superoperator* $F : \mathcal{D} \to \mathcal{E}$ is a linear map from $\mathcal{V}_{\mathcal{D}}$ to $\mathcal{V}_{\mathcal{E}}$ such that

- $F$ is completely positive
- $\forall T \in {\mathcal{V}_{\mathcal{D}}}^+$, $tr\,(F(T)) \leq tr(T)$
- $\forall A_1, A_2 \in \mathcal{S}_{\mathcal{D}}$ with $A_1 \preceq A_2$, $F(A_1) \preceq F(A_2)$.

Note that the set of states $\mathcal{S}_{\mathcal{D}}$ is a subset of the vector space $\mathcal{V}_{\mathcal{D}}$ of all matrix tuples and spans $\mathcal{V}_{\mathcal{D}}$ as a vector space. Therefore, a progressive superoperator, which is a linear map from $\mathcal{V}_{\mathcal{D}}$ to $\mathcal{V}_{\mathcal{E}}$, is completely determined by its action on states. In the following, we will therefore interchangeably refer to $F$ as a map from $\mathcal{V}_{\mathcal{D}}$ to $\mathcal{V}_{\mathcal{E}}$ or as a map from $\mathcal{S}_{\mathcal{D}}$ to $\mathcal{S}_{\mathcal{E}}$.

In the next part, we revisit concepts such as states, observations, etc. as special cases of progressive superoperators. However, before doing so, we will prove some results that we will need later.

**Lemma 3.52.** *Let $H$ be a Hilbert space and let $F : \mathcal{B}(H) \to \mathbb{C}$ be a linear map. Then there exists $B \in \mathcal{B}(H)$ such that for all $A \in \mathcal{B}(H)$, $F(A) = tr\,(AB)$.*

*Proof.* Since $F$ is linear, it is determined by its action on the basis of $\mathcal{B}(H)$. Fix a basis of $H$ and let $E_{ij} \in \mathcal{B}(H)$ such that the entry of $i$th row and $j$th column is 1 and it is 0 everywhere else. Define $b_{ji} = F(E_{ij})$ and let $B = (b_{ij})$. Note that $F(E_{ij}) = tr\,(E_{ij}B)$. Therefore, for all $A \in \mathcal{B}(H)$, $F(A) = tr\,(AB)$. □

*Notation* 3.53. Let $\mathcal{D} = (D, H)$ be a finite quantum domain. For all $A \in \mathcal{S}_{\mathcal{D}}$ and $x \in \mathcal{K}(D)$,

$$\delta_x A = \prod_{y \in \mathcal{K}(D)} \delta_x(y) A_y,$$

where $\delta_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$

**Proposition 3.54.** *Let $\mathcal{D} = (D, H)$ be a finite quantum domain and let $M \in \mathcal{V}_{\mathcal{D}}$ be two matrix tuples. Then the following map is linear.*

$$\begin{aligned} L : \mathcal{V}_{\mathcal{D}} &\to \mathcal{V}_{\mathcal{I}} \\ T &\mapsto \sum_{x \in \mathcal{K}(D)} tr\,(T_x M_x) \end{aligned}$$

*Proof.* Let $T^1, T^2 \in \mathcal{V}_{\mathcal{D}}$ and $\lambda_1, \lambda_2 \in [0, 1]$. So we have

$$L\left(\lambda_1 T^1 + \lambda_2 T^2\right) = \sum_{x \in \mathcal{K}(D)} tr\left(\left(\lambda_1 T^1 + \lambda_2 T^2\right)_x M_x\right)$$

$$= \lambda_1 \sum_{x \in \mathcal{K}(D)} tr\left(T_x^1 M_x\right) + \lambda_2 \sum_{x \in \mathcal{K}(D)} tr\left(T_x^2 M_x\right)$$

$$= \lambda_1 L\left(T^1\right) + \lambda_2 L\left(T^2\right)$$

□

**Lemma 3.55.** *Let $\mathcal{D} = (D, H)$ be a finite quantum domain and let $B \in \mathcal{V_D}^+$. Then, for all $x, y \in \mathcal{K}(D)$ with $x \leq y$,*

$$H_{x,y}\left(B_x\right) \sqsubseteq B_y \quad \Leftrightarrow \quad \forall A \in \mathcal{V_D}^+,\ \langle A^{'}, B\rangle \leq \langle \delta_y A, B\rangle$$

*where $(A')_x = H^{\dagger}_{x,y}(A_y)$ and $A'$ is zero everywhere else.*

*Proof.* First assume that $H_{x,y}\left(B_x\right) \sqsubseteq B_y$. Let $A \in \mathcal{V_D}^+$. We have

$$\langle A', B\rangle = tr\left(H^{\dagger}_{x,y}(A_y)\, B_x\right) = tr\left(A_y H_{x,y}(B_x)\right) \leq tr\left(A_y B_y\right) = \langle \delta_y A, B\rangle.$$

Now assume that for all $A \in \mathcal{V_D}^+$, $\langle A^{'}, B\rangle \leq \langle \delta_y A, B\rangle$ and $x, y \in \mathcal{K}(D)$ with $x \leq y$. So we have $tr\left(A_y H_{x,y}(B_x)\right) \leq tr\left(A_y B_y\right)$ and thus $tr\left(A_y\left(B_y - H_{x,y}(B_x)\right)\right) \geq 0$. Since this holds for all $A \in \mathcal{V_D}^+$ (and thus for all $A_y \in \mathcal{B}\left(H_y\right)^+$), we have $0 \sqsubseteq B_y - H_{x,y}(B_x)$. Therefore, we have $H_{x,y}\left(B_x\right) \sqsubseteq B_y$. □

**Lemma 3.56.** *Let $\mathcal{D} = (D, H)$ be a finite quantum domain and let $A \in \mathcal{V_D}^+$. Then, for all $x, y \in \mathcal{K}(D)$ with $x \leq y$,*

$$A' \preceq \delta_y A$$

*where $(A')_x = H^{\dagger}_{x,y}(A_y)$ and $A'$ is zero everywhere else.*

*Proof.* Let $B \in \mathcal{O_D}$. Since $B$ is an observation, $H_{x,y}\left(B_x\right) \sqsubseteq B_y$. So by Lemma 3.55, we have $\langle A', B\rangle \leq \langle \delta_y A, B\rangle$. Since the choice of $B$ was arbitrary, we have $A' \preceq \delta_y A$. □

In the next part, we present special cases of progressive superoperators. The following theorem shows that observations can be considered as a special case of progressive superoperators.

**Theorem 3.57.** *Let $\mathcal{D} = (D, H)$ be a finite quantum domain, let $\mathcal{I}$ be the singleton domain, and let $B \in \mathcal{O_D}$ be an observation. Then the map $F_B : \mathcal{D} \to \mathcal{I}$ with*

$$\begin{aligned} \mathcal{V}_\mathcal{D} &\to \mathcal{V}_\mathcal{I} \\ A &\mapsto \langle A, B \rangle \end{aligned}$$

*defines a progressive superoperator. Conversely, any progressive superoperator* $F : \mathcal{D} \to \mathcal{I}$ *is of this form.*

*Proof.* First we will show that $F_B$ is a progressive superoperator:

- *Linearity:* From Proposition 3.54, $F_B$ is linear.
- *Non-increasing trace property:* By definition, we have $\langle A, B \rangle = \sum_{x \in \mathcal{K}(D)} tr\,(A_x B_x)$. Also, since $B$ is an observation, $B_x \sqsubseteq I$. Therefore,

  $$\forall x \in \mathcal{K}(D),\ tr(A_x B_x) \leq tr\,(A_x)$$

  which implies $tr\,(F_B\,(A)) \leq tr\,(A)$.
- *Complete positivity:* We know that $F_B$ is linear, so it can be written as

  $$F_B(A) = \sum_{x \in \mathcal{K}(D)} F_x(A_x),$$

  for all $A \in \mathcal{S}_\mathcal{D}$, where $F_x : \mathcal{B}\,(H_x)^+ \to \mathbb{C}$ is how $F_B$ acts on $\delta_x A$. $F_B$ is completely positive if $F_x$ is completely positive, for all $x \in \mathcal{K}(D)$. By simplified Choi's theorem (Theorem 2.12), $F_x$ is completely positive if and only if it is positive. So we only need to show that $F_x$ is positive.

  For all $A \in \mathcal{S}_\mathcal{D}$, $A$ is positive which implies for all $x \in \mathcal{K}(D)$, $0 \sqsubseteq A_x$. Since $B$ is an observation, $0 \sqsubseteq B_x$, for all $x \in \mathcal{K}(D)$. This implies that $F_x(A_x) = tr(A_x B_x) \geq 0$, for all $x \in \mathcal{K}(D)$. This implies that $F_x$ is positive and thus is completely positive. So $F_B$ is completely positive.
- *Monotonicity:* Based on the definition of the order of states, for all states $A_1, A_2 \in \mathcal{S}_\mathcal{D}$ with $A_1 \preceq A_2$,

  $$F_B(A_1) = \langle A_1, B \rangle \leq \langle A_2, B \rangle = F_B(A_2).$$

  So $F_B$ is monotone.

Now assume $F : \mathcal{D} \to \mathcal{I}$ is a progressive superoperator. We want to show there exists $B \in \mathcal{O}_{\mathcal{D}}$ such that $F = F_B$. Since $F$ is a progressive superoperator, $F$ is linear. So for all $A \in \mathcal{S}_{\mathcal{D}}$,

$$F(A) = \sum_{x \in \mathcal{K}(D)} F_x\left(A_x\right)$$

where $F_x : \mathcal{B}\left(H_x\right) \to \mathbb{C}$ is how $F$ acts on $\delta_x A$.

From Lemma 3.52, there exists $B_x \in \mathcal{B}\left(H_x\right)$ such that $F_x(A_x) = tr\left(A_x B_x\right)$. This implies that for all $A \in \mathcal{S}_{\mathcal{D}}$, $F\left(A\right) = \sum_{x \in \mathcal{K}(D)} tr\left(A_x B_x\right) = \langle A, B\rangle$.

Next we will show that $B$ defined above is an observation. Let $A_x \in \mathcal{B}\left(H_x\right)$ be a positive matrix. Since $F$ is positive, for all $x \in \mathcal{K}(D)$, $F_x$ is positive, as well. So $F_x\left(A_x\right) = tr(A_x B_x) \geq 0$ which implies that $0 \sqsubseteq B_x$, for all $x \in \mathcal{K}(D)$. As $F$ does not increase the trace, we have

$$F_x\left(A_x\right) = tr\left(A_x B_x\right) \leq tr(A_x)$$

which implies that $0 \leq tr((I - B_x)\, A_x)$. Since this is true for all positive $A_x$, we have that $0 \sqsubseteq I - B_x$ and thus $B_x \sqsubseteq I$.

By Lemma 3.55, $H_{x,y}(B_x) \sqsubseteq B_y$ if and only if for all $A \in {\mathcal{V}_{\mathcal{D}}}^+$,

$$\langle A', B\rangle \leq \langle \delta_y A, B\rangle,$$

where $\left(A'\right)_x = H^{\dagger}_{x,y}(A_y)$ and $A'$ is zero everywhere else. Equivalently, we can write

$$H_{x,y}\left(B_x\right) \sqsubseteq B_y \Leftrightarrow \forall A \in {\mathcal{V}_{\mathcal{D}}}^+,\ F\left(A'\right) \leq F\left(\delta_y A\right)$$

By Lemma 3.56, we know $A' \preceq \delta_y A$. Since $F$ is a progressive superoperator, it is monotone on states. So we have $F(A') \leq F\left(\delta_y A\right)$. This implies that $H_{x,y}\left(B_x\right) \sqsubseteq B_y$ and therefore $B$ is an observation. □

As we have seen in the theorems above, observations can be viewed as progressive superoperators. In the next theorem, we will show that progressive superoperators generalize superoperators.

**Theorem 3.58.** *Let $\mathcal{D} = (D, H_{\mathcal{D}})$ and $\mathcal{E} = (E, H_{\mathcal{E}})$ be two singleton quantum domains with $D = \{x\}$, $E = \{y\}$, $\mathcal{B}\left((H_{\mathcal{D}})_x\right) = \mathbb{C}^{n\times n}$, and $\mathcal{B}\left((H_{\mathcal{E}})_y\right) = \mathbb{C}^{m\times m}$ for some*

*$n, m \in \mathbb{N}$. Let $f : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ be a superoperator. Then $f$ is a progressive superoperator from $\mathcal{D}$ to $\mathcal{E}$. Conversely, every progressive superoperator $F : \mathcal{D} \to \mathcal{E}$ is of this form (a superoperator).*

*Proof.* Since $f$ is a superoperator, it is linear, complete positive, and does not increase the trace. Since $f$ is completely positive, it is monotone. This implies $f$ is a progressive superoperator. Now assume $F : \mathcal{D} \to \mathcal{E}$ is a progressive superoperator. Since $F$ is linear, positive, completely positive, and does not increase the trace, it is clear that $F$ is a superoperator. □

Next, we will address how our definition of progressive superoperators generalizes the definition of stochastic maps given in Definition 2.42. Before doing so, we will cover one of the properties of state matrix tuples.

**Lemma 3.59.** *Let $\mathcal{D} = (D, H)$ be a quantum domain and $A^1, A^2 \in \mathcal{V}_{\mathcal{D}}$ be two state matrix tuples. Then*

$$A^1 = A^2 \;\Leftrightarrow\; \forall B \in \mathcal{O}_{\mathcal{D}}, \sum_{x\in\mathcal{K}(D)} tr\left(A^1_x B_x\right) = \sum_{x\in\mathcal{K}(D)} tr\left(A^2_x B_x\right).$$

*Proof.* If $A^1 = A^2$, then the right hand side holds trivially. Now assume that for all $B \in \mathcal{O}_{\mathcal{D}}$, $\sum_{x\in\mathcal{K}(D)} tr\left(A^1_x B_x\right) = \sum_{x\in\mathcal{K}(D)} tr\left(A^2_x B_x\right)$. Let $x \in \mathcal{K}(D)$ and $E \in \mathcal{B}(H_x)$, then we have

$$\begin{aligned}
\sum_{y\in\mathcal{K}(D)} tr\left(A^1_y \mathbf{B}^c(x,E)_y\right) &= \sum_{y\in\mathcal{K}(D)} tr\left(A^2_y \mathbf{B}^c(x,E)_y\right)\\
\sum_{y\in\mathcal{K}(D)} tr\left(A^1_y \mathbf{B}^\circ(x,E)_y\right) &= \sum_{y\in\mathcal{K}(D)} tr\left(A^2_y \mathbf{B}^\circ(x,E)_y\right).
\end{aligned}$$

So we get

$$\begin{aligned}
tr\left(A^1_x \mathbf{B}^c(x,E)_x\right) &= \sum_{y\in\mathcal{K}(D)} tr\left(A^1_y \mathbf{B}^c(x,E)_y\right) - \sum_{y\in\mathcal{K}(D)} tr\left(A^1_y \mathbf{B}^\circ(x,E)_y\right)\\
&= \sum_{y\in\mathcal{K}(D)} tr\left(A^2_y \mathbf{B}^c(x,E)_y\right) - \sum_{y\in\mathcal{K}(D)} tr\left(A^2_y \mathbf{B}^\circ(x,E)_y\right)\\
&= tr\left(A^2_x \mathbf{B}^c(x,E)_x\right)
\end{aligned}$$

This implies that $tr\left(\left(A^1_x - A^2_x\right) \mathbf{B}^c(x,E)_x\right) = tr\left(\left(A^1_x - A^2_x\right) E\right) = 0$, for all $E \in \mathcal{B}(H_x)$. Therefore, $A^1_x = A^2_x$. As this is true for all $x \in \mathcal{K}(D)$, we have $A^1 = A^2$. □

**Proposition 3.60.** *Let $\mathcal{D}$ and $\mathcal{E}$ be finite quantum domains and let $F : \mathcal{V}_\mathcal{D} \to \mathcal{V}_\mathcal{E}$ be a progressive superoperator. Then $F$ preserves directed suprema of state matrix tuples.*

*Proof.* Let $(A_j)_{j\in J}$ be a directed family of state matrix tuples and let $\sup_{j\in J}(A_j)$ denote the supremum, given by the supremum of the associated states:

$$tr\left(\left(\sup_{j\in J}(A_j)\right)B\right) = \sup_{j\in J}\left(tr(A_jB)\right), \text{ for all } B \in \mathcal{O}_\mathcal{D}.$$

Since $F$ is monotone on the state matrix tuples, $\{F(A_j) : j \in J\}$ is a directed family, as well. We must show that $\sup_{j\in J} F(A_j) = F\left(\sup_{j\in J}(A_j)\right)$.

Note that $\mathcal{D}$ and $\mathcal{E}$ are finite quantum domains, so the trace provides an inner product on the finite dimensional spaces $\mathcal{V}_\mathcal{D}$ and $\mathcal{V}_\mathcal{E}$. The map $F$ is bounded with respect to these trace inner products and so has an adjoint $F^\dagger : \mathcal{V}_\mathcal{E} \to \mathcal{V}_\mathcal{D}$ with respect to them, given by

$$tr\left(F(A)B\right) = tr\left(AF^\dagger(B)\right), \text{ for all } A, B \in \mathcal{V}_\mathcal{D}.$$

Therefore, given an observation $B \in \mathcal{O}_\mathcal{D}$, we have

$$\begin{aligned}
tr\left(F\left(\sup_{j\in J}(A_j)\right)B\right) &= tr\left(\left(\sup_{j\in J}(A_j)\right)F^\dagger(B)\right) && \text{by definition of } F^\dagger \\
&= \sup_{j\in J}\left(tr\left(A_jF^\dagger(B)\right)\right) && \text{by definition of } \sup_{j\in J}(A_j) \\
&= \sup_{j\in J}\left(tr\left(F(A_j)B\right)\right) && \text{by definition of } F^\dagger \\
&= tr\left(\left(\sup_{j\in J} F(A_j)\right)B\right) && \text{by definition of } \sup_{j\in J}(F(A_j)).
\end{aligned}$$

It follows from Lemma 3.59 that $F\left(\sup_{j\in J}(A_j)\right) = \sup_{j\in J}\left(F(A_j)\right)$. □

Now we are ready to address how our definition of progressive superoperator extends the definition of stochastic maps.

**Theorem 3.61.** *Let $\mathcal{D} = (D, H_\mathcal{D})$ and $\mathcal{E} = (E, H_\mathcal{E})$ be two finite probabilistic domains and let $f : \mathcal{S}_\mathcal{D} \to \mathcal{S}_\mathcal{E}$ be a stochastic map. Then $f$ is a progressive superoperator from $\mathcal{D}$ to $\mathcal{E}$. Conversely, every progressive superoperator $F : \mathcal{D} \to \mathcal{E}$ is of this form (a stochastic map).*

*Proof.* To prove the first claim, assume $f : \mathcal{S}_{\mathcal{D}} \to \mathcal{S}_{\mathcal{E}}$ is a stochastic map. We know that $f$ is linear and Scott-continuous (thus it is monotone). By definition, $f$ is positive, as it takes states to states. Again by definition, for all $M \in \mathcal{S}_{\mathcal{D}}$, we have

$$tr(M) \leq 1 \;\Rightarrow\; tr\left(f(M)\right) \leq 1.$$

By scaling, i.e., replacing $M$ by $\frac{1}{x}M$, this also implies

$$tr(M) \leq x \;\Rightarrow\; tr\left(f(M)\right) \leq x.$$

Therefore, we have for all $M \in \mathcal{S}_{\mathcal{D}}$, $tr\left(f(M)\right) \leq tr(M)$. So $f$ does not increase the trace. Lastly, we need to show that $f$ is completely positive. Since $f$ is linear, it can be written as $f = \left(f_{y,x}\right)_{x\in\mathcal{K}(D), y\in\mathcal{K}(E)}$ where

$$f\left(\left(A_x\right)_{x\in\mathcal{K}(D)}\right) = \left(\sum_{x\in\mathcal{K}(D)} f_{yx}\left(A_x\right)\right)_{y\in\mathcal{K}(E)}$$

for all $\left(A_x\right)_{x\in\mathcal{K}(D)} \in \mathcal{S}_{\mathcal{D}}$. We know $f$ is completely positive if and only if all $f_{y,x}$ are completely positive. By definition all $f_{y,x}$ are positive and thus they are completely positive by Theorem 2.12. This implies that $f$ is completely positive, proving the first claim.

To show the converse, assume $F : \mathcal{D} \to \mathcal{E}$ is a progressive superoperator. We want to show that $F$ is a stochastic map. Note that by definition, $F$ is linear and monotone and by Proposition 3.60, $F$ preserves the suprema. Therefore, F is a stochastic map. □

In the next theorem, a definition of states as superoperators is given.

**Theorem 3.62.** *Let $\mathcal{D} = (D, H)$ be a finite quantum domain, let $\mathcal{I}$ be the singleton domain, and let $A \in \mathcal{S}_{\mathcal{D}}$ be a state. Then the map $F_A : \mathcal{I} \to \mathcal{D}$ with*

$$\begin{array}{rcl} \mathcal{S}_{\mathcal{I}} & \to & \mathcal{S}_{\mathcal{D}} \\ x & \mapsto & xA \end{array}$$

*defines a progressive superoperator. Conversely, every progressive superoperator $F : \mathcal{I} \to \mathcal{D}$ is of this form.*

*Proof.* By definition, $F_A$ is positive. Also, $F_A$ is linear, since for all $x, y \in \mathcal{S}_\mathcal{I}$ and $\lambda_1, \lambda_2 \in [0,1]$, we have

$$F_A(\lambda_1 x + \lambda_2 y) = (\lambda_1 x + \lambda_2 y) A = \lambda_1 x A + \lambda_2 y A = \lambda_1 F_A(x) + \lambda_2 F_A(y).$$

We know $A \in \mathcal{S}_\mathcal{D}$, so $tr(A) \leq 1$. This implies that for all $x \in \mathcal{S}_\mathcal{I}$, $tr(F_A(x)) = tr(xA) = xtr(A) \leq x = tr(x)$. Therefore, $F_A$ does not increase the trace.

Next, we will address the monotonicity of $F_A$. For all $x, y \in \mathcal{S}_\mathcal{I}$ with $x \leq y$, $F_A(x) = xA \leq yA = F_A(y)$, so $F_A$ is monotone. To show that $F_A$ is completely positive, note that $F_A$ can be written as $F_A = (F_y)_{y \in \mathcal{K}(D)}$ where $F_y : \mathbb{C}^+ \to \mathcal{B}(H_y)$ is $F_y(x) = (F(x))_y = xA_y$, for all $x \in \mathcal{S}_\mathcal{I}$. $F_A$ is completely positive if and only if all $F_y$ are completely positive. By definition, each $F_y$ is positive. So by Lemma 2.12, all $F_y$ are completely positive and therefore, $F_A$ is completely positive.

For the converse, assume $F : \mathcal{D} \to \mathcal{E}$ is a progressive superoperator. We want to show that $F$ is a state (linear Scott-continuous map). By definition, $F$ is linear and monotone and by Proposition 3.60, it preserves suprema. Therefore, $F$ is a state. □

### 3.6 Dual Definition of Progressive Superoperator

In the previous section, we have shown that our definition of progressive superoperator generalizes several concepts. However, this definition does not generalize well to infinite domains. For generalizing the special cases, such as observations and states, we had to define the progressive superoperators on states. But in the case of infinite domains, states themselves are defined indirectly via observations (see Definition 3.13), whereas the observations are directly defined on the compact elements of the domain (see Definition 3.10). In the infinite case, it is therefore more convenient to define progressive superoperators as functions from observations to observations rather than from states to states.

Therefore, in order to have a more natural definition for our progressive superoperators (one that hopefully can be extended to non-finite domains in future work), in this section, we will re-define progressive superoperators by dualizing the previous definition. We will show that the two definitions of progressive superoperators are equivalent for finite quantum domains, and thus all the special cases of progressive superoperators we considered in in the previous section, can also be written using this new definition.

*Notation* 3.63. Let $\mathcal{D}$ be a finite quantum domain, then $\langle -, - \rangle_{\mathcal{D}} : \mathcal{V}_{\mathcal{D}} \times \mathcal{V}_{\mathcal{D}} \to \mathbb{C}$ is the map defined by

$$\langle A, B \rangle_{\mathcal{D}} = \sum_{x \in \mathcal{K}(D)} tr\,(A_x B_x), \text{ for all } A, B \in \mathcal{V}_{\mathcal{D}}.$$

**Lemma 3.64.** *Let $\mathcal{D} = (D, H)$ be a finite quantum domain and let $B, B' \in \mathcal{V}_{\mathcal{D}}$ such that for all $A \in \mathcal{V}_{\mathcal{D}}$, $\langle A, B \rangle = \langle A, B' \rangle$. Then $B = B'$.*

*Proof.* Since $\langle A, B \rangle = \langle A, B' \rangle$, for all $A \in \mathcal{V}_{\mathcal{D}}$, we have

$$\langle \delta_x A, B \rangle = \langle \delta_x A, B' \rangle,$$

for all $x \in \mathcal{K}(D)$. This implies that $tr\,(A_x B_x) = tr\,(A_x B'_x)$. As a result, we have $tr\,(A_x(B_x - B'_x)) = 0$, for all $A_x \in \mathcal{B}\,(H_x)$ and thus $B_x = B'_x$. Since this is true for all $x \in \mathcal{K}(D)$, we get $B = B'$. □

**Theorem 3.65.** *Let $\mathcal{D} = (D, H_{\mathcal{D}})$ and $\mathcal{E} = (E, H_{\mathcal{E}})$ be two finite quantum domains. For all completely positive $F : \mathcal{V}_{\mathcal{D}} \to \mathcal{V}_{\mathcal{E}}$, there exists a unique completely positive $G : \mathcal{V}_{\mathcal{E}} \to \mathcal{V}_{\mathcal{D}}$ such that*

$$\forall A \in \mathcal{V}_{\mathcal{D}}, B \in \mathcal{V}_{\mathcal{E}},\ \langle F(A), B \rangle_{\mathcal{E}} = \langle A, G(B) \rangle_{\mathcal{D}}$$

*Proof.* Since $F$ is linear, it can be written as $F = (F_{xy})_{x \in \mathcal{K}(E), y \in \mathcal{K}(D)}$, where each $F_{xy} : \mathcal{B}\,(H_y) \to \mathcal{B}\,(H_x)$ is linear and completely positive (since $F$ is). This implies that for all $x \in \mathcal{K}(E)$ and $A \in \mathcal{V}_{\mathcal{D}}$, $F(A)_x = \sum_{y \in \mathcal{K}(D)} F_{xy}(A_y)$. Also, since $F_{xy}$ is linear and completely positive, by Theorem 2.10, it has a Kraus representation, i.e., $F_{xy}(A_y) = \sum_{j \in J_{xy}} S_j A_y S_j^\dagger$. Now define $G = (G_{yx})_{x \in \mathcal{K}(E), y \in \mathcal{K}(D)}$, where $G_{yx} : \mathcal{B}\,(H_x) \to \mathcal{B}\,(H_y)$, with

$$G_{yx}(B_x) = \sum_{j \in J_{xy}} S_j^\dagger B_x S_j.$$

Therefore, we can write

$$
\begin{aligned}
\langle F(A), B\rangle &= \sum_{x\in\mathcal{K}(E)} tr\left(F(A)_x B_x\right) \\
&= \sum_{x\in\mathcal{K}(E)} tr\left(\left(\sum_{y\in\mathcal{K}(D)} F_{xy}(A_y)\right) B_x\right) \\
&= \sum_{x\in\mathcal{K}(E)} \sum_{y\in\mathcal{K}(D)} tr\left(F_{xy}(A_y) B_x\right) \\
&= \sum_{x\in\mathcal{K}(E)} \sum_{y\in\mathcal{K}(D)} tr\left(\left(\sum_{j\in J_{xy}} S_j A_y S_j^\dagger\right) B_x\right) \\
&= \sum_{x\in\mathcal{K}(E)} \sum_{y\in\mathcal{K}(D)} tr\left(\sum_{j\in J_{xy}} S_j A_y S_j^\dagger B_x\right) \\
&= \sum_{x\in\mathcal{K}(E)} \sum_{y\in\mathcal{K}(D)} \sum_{j\in J_{xy}} tr\left(S_j A_y S_j^\dagger B_x\right) \\
&= \sum_{x\in\mathcal{K}(E)} \sum_{y\in\mathcal{K}(D)} \sum_{j\in J_{xy}} tr\left(A_y S_j^\dagger B_x S_j\right) \\
&= \sum_{x\in\mathcal{K}(E)} \sum_{y\in\mathcal{K}(D)} tr\left(A_y \left(\sum_{j\in J_{xy}} S_j^\dagger B_x S_j\right)\right) \\
&= \sum_{y\in\mathcal{K}(D)} \sum_{x\in\mathcal{K}(E)} tr\left(A_y G_{yx}(B_x)\right) \\
&= \sum_{y\in\mathcal{K}(D)} tr\left(A_y \sum_{x\in\mathcal{K}(E)} G_{yx}(B_x)\right) \\
&= \sum_{y\in\mathcal{K}(D)} tr\left(A_y G(B)_y\right) \\
&= \langle A, G(B)\rangle
\end{aligned}
$$

By definition, $G$ is completely positive.

Next we will show that $G$ is unique. Assume there exists another map $G' : \mathcal{V}_\mathcal{E} \to \mathcal{V}_\mathcal{D}$ such that

$$\forall A \in \mathcal{V}_\mathcal{D}, B \in \mathcal{V}_\mathcal{E},\ \langle F(A), B\rangle_\mathcal{E} = \langle A, G'(B)\rangle_\mathcal{D}.$$

Fix $B \in \mathcal{V}_\mathcal{E}$. So for all $A \in \mathcal{V}_\mathcal{D}$, $\langle A, G(B)\rangle_\mathcal{D} = \langle F(A), B\rangle_\mathcal{E} = \langle A, G'(B)\rangle_\mathcal{D}$, which implies by Lemma 3.64 that $G(B) = G'(B)$. Since this holds for all $B \in \mathcal{V}_\mathcal{E}$, we have $G = G'$. Therefore, $G$ is unique. □

*Remark* 3.66. The map $F$ and $G$ in Theorem 3.65 are called each others *adjoint*.

**Lemma 3.67.** *Let $\mathcal{D}$ be a finite quantum domain and $B^1, B^2 \in \mathcal{O}_{\mathcal{D}}$. Then,*

$$\forall A \in \mathcal{V_D}^+,\ \langle A, B^1\rangle \leq \langle A, B^2\rangle \ \Leftrightarrow\ B^1 \sqsubseteq B^2$$

*Proof.* First assume that for all $A \in \mathcal{V_D}^+$, $\langle A, B^1\rangle \leq \langle A, B^2\rangle$. In particular, given $x \in \mathcal{D}$, we get $\langle \delta_x A, B^1\rangle \leq \langle \delta_x A, B^2\rangle$. Therefore, we have

$$\begin{aligned} tr\left(A_x B_x^1\right) &= \sum_{y\in\mathcal{K}(D)} tr\Big(\left(\delta_x A\right)_y B_y^1\Big) \\ &= \left\langle \delta_x A, B^1\right\rangle \\ &\leq \left\langle \delta_x A, B^2\right\rangle \\ &= \sum_{y\in\mathcal{K}(D)} tr\Big(\left(\delta_x A\right)_y B_y^2\Big) = tr\left(A_x B_x^2\right) \end{aligned}$$

So $tr\left(A_x\left(B_x^2 - B_x^1\right)\right) \geq 0$. This holds for all positive $A_x$, which implies that $0 \sqsubseteq B_x^2 - B_x^1$ and thus $B_x^1 \sqsubseteq B_x^2$. Since this is true for all $x \in \mathcal{K}(D)$, we have $B^1 \sqsubseteq B^2$.

Now assume $B^1 \sqsubseteq B^2$. Let $A \in \mathcal{V_D}^+$. We have

$$\left\langle A, B^1\right\rangle = \sum_{y\in\mathcal{K}(D)} tr\left(A_y B_y^1\right) \leq \sum_{y\in\mathcal{K}(D)} tr\left(A_y B_y^2\right) = \left\langle A, B^2\right\rangle$$

Therefore, for all $A \in \mathcal{V_D}^+$, $\left\langle A, B^1\right\rangle \leq \left\langle A, B^2\right\rangle$. □

The next theorem will cover the relation between properties of adjoints $F$ and $G$, given in Theorem 3.65.

**Theorem 3.68.** *Let $F$ and $G$ be as in Theorem 3.65. Then we have:*

*(i) $F$ is trace non-increasing, i.e., $\forall A \in \mathcal{V_D}^+$, $tr\left(F(A)\right) \leq tr(A)$, if and only if $G\left(I_{\mathcal{V_E}}\right) \sqsubseteq I_{\mathcal{V_D}}$*

*(ii) $F$ is a progressive superoperator $\Leftrightarrow$ $\forall B \in \mathcal{O_E}$, $G(B) \in \mathcal{O_D}$.*

*Proof.* *(i):* Assume that for all $A \in \mathcal{V_D}^+$, $tr\left(F(A)\right) \leq tr(A)$. So for all $A \in \mathcal{V_D}^+$, we have

$$\left\langle A, G\left(I_{\mathcal{V_E}}\right)\right\rangle = \left\langle F(A), I_{\mathcal{V_E}}\right\rangle = tr\left(F(A)\right) \leq tr\left(A\right) = \left\langle A, I_{\mathcal{V_D}}\right\rangle.$$

So by Lemma 3.67, $G\left(I_{\mathcal{V_E}}\right) \sqsubseteq I_{\mathcal{V_D}}$.

Now assume $G(I_{\mathcal{V_E}^+}) \sqsubseteq I_{\mathcal{V_D}^+}$. For all $A \in \mathcal{V_D}^+$, we have

$$\begin{aligned} tr\left(F(A)\right) &= \sum_{x\in\mathcal{K}(E)} tr\left(F(A)_x\right) \\ &= \sum_{x\in\mathcal{K}(E)} tr\left(F(A)_x\left(I_{\mathcal{V}_{\mathcal{E}}+}\right)_x\right) \\ &= \left\langle F(A), I_{\mathcal{V}_{\mathcal{E}}+}\right\rangle \\ &= \left\langle A, G(I_{\mathcal{V}_{\mathcal{E}}+})\right\rangle \\ &= \sum_{y\in\mathcal{K}(E)} tr\left(A_y G\left(I_{\mathcal{V}_{\mathcal{E}}+}\right)_y\right) \\ &\le \sum_{y\in\mathcal{K}(E)} tr\left(A_y\right) = tr(A) \end{aligned}$$

*(ii):* First assume that for all $B \in \mathcal{O}_{\mathcal{E}}$, $G(B) \in \mathcal{O}_{\mathcal{D}}$. Since in particular, $I_{\mathcal{V}_{\mathcal{E}}+} \in \mathcal{O}_{\mathcal{E}}$, it follows that $G\left(I_{\mathcal{V}_{\mathcal{E}}+}\right) \in \mathcal{O}_{\mathcal{D}}$, hence $G\left(I_{\mathcal{V}_{\mathcal{E}}+}\right) \sqsubseteq I_{\mathcal{V}_{\mathcal{D}}+}$. Therefore, $F$ is trace non-increasing by part (i). In particular, $F$ maps states to states. What is left to show is that for all $A_1, A_2 \in \mathcal{S}_{\mathcal{D}}$, $A_1 \preceq A_2$ implies $F(A_1) \preceq F(A_2)$. So assume that $A_1 \preceq A_2$ and let $B \in \mathcal{O}_{\mathcal{D}}$ be an arbitrary observation. Then we have

$$\langle F(A_1), B\rangle = \langle A_1, G\left(B\right)\rangle \le \langle A_2, G\left(B\right)\rangle = \langle F(A_2), B\rangle.$$

Therefore, $F(A_1) \preceq F(A_2)$ as claimed.

For the converse direction, assume that $F$ is a progressive superoperator. Let $B \in \mathcal{O}_{\mathcal{E}}$. So we need to show that $G(B) \in \mathcal{O}_{\mathcal{D}}$. For all $A \in \mathcal{S}_{\mathcal{D}}$, $\langle A, G(B)\rangle = \langle F(A), B\rangle \ge 0$ which implies that $0 \sqsubseteq G(B)$, for all $x \in \mathcal{K}(D)$. Next we need to show that $G(B) \sqsubseteq I_{\mathcal{O}_{\mathcal{D}}}$. Since $B \in \mathcal{O}_{\mathcal{E}}$, $B \sqsubseteq I_{\mathcal{V}_{\mathcal{E}}}$. Also, since $F$ is a progressive superoperator, $F$ does not increase the trace. So from part (i), we have that $G(B) \sqsubseteq G(I_{\mathcal{V}_{\mathcal{E}}}) \sqsubseteq I_{\mathcal{V}_{\mathcal{D}}}$.

Finally, we must show that $H_{x,y}\left(G(B)_x\right) \sqsubseteq G(B)_y$ holds for all $x \le y$. So fix $x$ and $y$ and consider any $A \in \mathcal{B}\left(H_y\right)^+$. Let $\delta_y A$ denote a state which has $A$ in its $y$ position and is zero everywhere else. So by Lemma 3.56, we have $\delta_x\left(H^\dagger_{x,y}A\right) \preceq \delta_y A$ and therefore, since $F$ preserves the order on states, also $F\left(\delta_x\left(H^\dagger_{x,y}A\right)\right) \preceq F(\delta_y A)$. Since $B$ is an observation, we have

$$\begin{aligned} \left\langle \delta_x\left(H^\dagger_{x,y}A\right), G(B)\right\rangle &= \left\langle F\left(\delta_x\left(H^\dagger_{x,y}A\right)\right), B\right\rangle \\ &\le \langle F(\delta_y A), B\rangle \\ &= \langle \delta_y A, G(B)\rangle \end{aligned}$$

$\delta_y A$ and $\delta_x\left(H_{x,y}^{\dagger}A\right)$ are states which has $A$ (resp. $H_{x,y}^{\dagger}A$) in its $y$ position (resp. $x$ position) and are zero everywhere else. So the pairing maps above can be simplified as follows:

$$tr\left(\left(H_{x,y}^{\dagger}A\right)G(B)_x\right)=\left\langle\delta_x\left(H_{x,y}^{\dagger}A\right),G(B)\right\rangle\leq\langle\delta_y A,G(B)\rangle=tr\left(A\,G(B)_y\right)$$

We know that $tr\left(\left(H_{x,y}^{\dagger}A\right)G(B)_x\right)=tr\left(A\,H_{x,y}\left(G(B)_x\right)\right)$. So we get

$$tr\left(A\,H_{x,y}\left(G(B)_x\right)\right)\leq tr\left(A\,G(B)_y\right)$$

and thus $tr\Big(A\Big(G(B)_y-H_{x,y}\left(G(B)_x\right)\Big)\Big)\geq 0$. Since this is true for all $A\in\mathcal{B}\left(H_y\right)^+$, we have $H_{x,y}\left(G(B)_x\right)\sqsubseteq G(B)_y$, which implies that $G(B)$ is an observation. □

*Remark* 3.69. Note that in the case (ii), the condition $G(B)\in\mathcal{O}_{\mathcal{D}}$ implies that $F_{G(B)}=F_B\circ F$, where $F_B$ is the progressive superoperator defined in Theorem 3.57.

Now we are ready to give a formal dual definition for progressive superoperators. We will show that the two definitions we have given for progressive superoperators are equivalent.

**Definition 3.70.** (Dual Definition of Progressive Superoperator) Let $\mathcal{D}$ and $\mathcal{E}$ be two finite quantum domains. A *progressive superoperator* $G:\mathcal{D}\to\mathcal{E}$ is a linear map from $\mathcal{V}_{\mathcal{E}}$ to $\mathcal{V}_{\mathcal{D}}$ such that

- $G$ is completely positive
- $G\left(I_{\mathcal{V}_{\mathcal{E}}}\right)\sqsubseteq I_{\mathcal{V}_{\mathcal{D}}}$
- $\forall B\in\mathcal{O}_{\mathcal{E}},\ G(B)\in\mathcal{O}_{\mathcal{D}}$

**Theorem 3.71.** *Definitions 3.51 and 3.70 for progressive superoperators are equivalent.*

*Proof.* This follows from Theorem 3.65. □

As a result of Theorem 3.71, it follows that the special cases of progressive superoperators we discussed in the previous section, can be defined using the dual definition.

**Corollary 3.72.** *Let $\mathcal{D}$ be a finite quantum domain, let $\mathcal{I}$ be the singleton domain, and let $B \in \mathcal{O}_{\mathcal{D}}$ be an observation then the map $G_B : \mathcal{D} \to \mathcal{I}$ with*

$$\begin{array}{rcl} \mathcal{V}_{\mathcal{I}} & \to & \mathcal{V}_{\mathcal{D}} \\ x & \mapsto & xB \end{array}$$

*defines a progressive superoperator. Conversely, every progressive superoperator $G : \mathcal{D} \to \mathcal{I}$ is of this form.*

**Corollary 3.73.** *Let $\mathcal{D}$ and $\mathcal{E}$ be two finite probabilistic domains and let $f : \mathcal{S}_{\mathcal{D}} \to \mathcal{S}_{\mathcal{E}}$ be a stochastic map. Let $g : \mathcal{V}_{\mathcal{E}} \to \mathcal{V}_{\mathcal{D}}$ be the map in Theorem 3.65 with*

$$\langle f(A), B \rangle_{\mathcal{E}} = \langle A, g(B) \rangle_{\mathcal{D}},$$

*for all $A \in \mathcal{S}_{\mathcal{D}}$ and $B \in \mathcal{O}_{\mathcal{E}}$. Then the map $G_f : \mathcal{D} \to \mathcal{E}$ with*

$$\begin{array}{rcl} \mathcal{V}_{\mathcal{E}} & \to & \mathcal{V}_{\mathcal{D}} \\ B & \mapsto & g(B) \end{array}$$

*defines a progressive superoperator. Conversely, every progressive superoperator $G : \mathcal{D} \to \mathcal{E}$ is of this form (a stochastic map).*

**Corollary 3.74.** *Let $\mathcal{D} = (D, H_{\mathcal{D}})$ and $\mathcal{E} = (E, H_{\mathcal{E}})$ be two singleton quantum domains with $D = \{x\}$, $E = \{y\}$, $\mathcal{B}((H_{\mathcal{D}})_x) = \mathbb{C}^{n \times n}$, and $\mathcal{B}((H_{\mathcal{E}})_y) = \mathbb{C}^{m \times m}$ for some $n, m \in \mathbb{N}$. Let $f : \mathbb{C}^{n \times n} \to \mathbb{C}^{m \times m}$ be a superoperator. Let $g : \mathbb{C}^{m \times m} \to \mathbb{C}^{n \times n}$ be the map in Theorem 3.65 with*

$$\langle f(A), B \rangle_{\mathcal{E}} = \langle A, g(B) \rangle_{\mathcal{D}},$$

*for all $A \in \mathbb{C}^{n \times n}$ and $B \in \mathcal{O}_{\mathcal{E}}$. Then the map $G_f : \mathcal{D} \to \mathcal{E}$ with*

$$\begin{array}{rcl} \mathcal{V}_{\mathcal{E}} & \to & \mathcal{V}_{\mathcal{D}} \\ B & \mapsto & g(B) \end{array}$$

*defines a progressive superoperator. Conversely, every progressive superoperator $G : \mathcal{D} \to \mathcal{E}$ is of this form (a superoperator).*

**Corollary 3.75.** *Let $\mathcal{D}$ be a finite quantum domain, let $\mathcal{I}$ be the singleton domain, and let $A \in \mathcal{S}_{\mathcal{D}}$ be a state. Then the map $G_A : \mathcal{I} \to \mathcal{D}$ with*

$$\begin{aligned} \mathcal{V}_{\mathcal{D}} &\to \mathcal{V}_{\mathcal{I}} \\ B &\mapsto \langle A, B \rangle \end{aligned}$$

*defines a progressive superoperator. Conversely, any progressive superoperator* $G : \mathcal{I} \to \mathcal{D}$ *is of this form.*

# Chapter 4

# Conclusions and Future Work

In this thesis, we gave a new definition of quantum domains. This definition is much simpler than what was proposed in [11, 13]. We defined observations, states, and finitely compactly supported states and we proved that every state can be viewed as a limit of finitely compactly supported ones.

The simplicity of our definition enabled us to define progressive superoperators for finite domains, and to show that observations, states, and stochastic maps are special cases, which is something that has not been done before. Then we discussed that this definition cannot be generalized to infinite domains, because it was defined on states. We solved this problem by giving a second, dual definition of progressive superoperators. We proved that the two definitions are equivalent, but the second definition, which is based on observations instead of states, creates a foundation for defining a progressive superoperator for infinite domains, in the future.

## 4.1 Future Work

Having defined an abstract notion of the progressive superoperators, the next step will be to show that this definition is physically realizable, i.e., every progressive superoperator is physically implementable and every physical operation can be modelled by a progressive superoperator.

The progressive superoperators defined in this thesis were only defined on finite quantum domains. As mentioned in the introduction, it is of interest to see how our definition of progressive superoperators can be extended to the infinite case. Specifically, the main goal of future work on this topic is to define a suitable *category* of quantum domains and to investigate its properties. Such a category has (finite and infinite) quantum domains as its objects and progressive superoperators as its morphisms. One of the most important questions that arise after this category is defined is that whether this category can serve as a model for quantum programming languages. Also, we are interested in studying this category in order to determine the types of limits, colimits, and other structure it has and to find out whether there are any functors of interest on this category.

Another line of questions that can be pursued in future work is to find the connections between the category of quantum domains and that of Von Neumann algebras. Also, it would be of interest to find out whether domain equations have a solution in the category of quantum domains.

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