HARMONIC ANALYSIS ON AFFINE GROUPS

by

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Abstract

The set of all invertible affine transformations of a two dimensional real vector space forms a locally compact group G_2 that is isomorphic to the semi-direct product group $\mathbb{R}^2 \rtimes \mathrm{GL}_2(\mathbb{R})$, where $\mathrm{GL}_2(\mathbb{R})$ denotes the group of 2×2 real matrices with nonzero determinant. We give an explicit decomposition of the left regular representation of G_2 as a direct sum of infinitely many copies of a single irreducible representation. We also obtain an analogue of the continuous wavelet transform associated to the representation we identify.

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Chapter 1

Introduction

Any affine transformation of the two dimensional real vector space $\mathbb{R}^2$ is of the form $\underline{z} \to A\underline{z} + \underline{x}$, for some linear transformation A of $\mathbb{R}^2$ and some vector $\underline{x} \in \mathbb{R}^2$. We will view $\mathbb{R}^2$ as column vectors and linear transformations of $\mathbb{R}^2$ as given by 2×2 real matrices. Such a transformation is invertible if and only if A is invertible. Let G_2 denote the set of all invertible affine transformations of $\mathbb{R}^2$. Equipped with composition, G_2 is a group and, in fact, is a locally compact group with the natural topology. As a locally compact group, G_2 has a left Haar measure. The main purpose of this thesis is to develop the details of harmonic analysis of functions on G_2 that are square-integrable with respect to its left Haar measure. To be more precise, the left regular representation of G_2 is decomposed as a direct sum of infinitely many copies of a single irreducible unitary representation. A secondary purpose is to derive an analogue of the continuous wavelet transform from the special properties of this irreducible unitary representation of G_2. A new way of transforming and analyzing functions of three variables is obtained.

Basic properties of locally compact groups, left Haar measure, G-spaces, and function spaces are introduced in the first section of Chapter 2. This followed by reviewing Hilbert spaces and operators on Hilbert space, including the unitary group $\mathcal{U}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ and the terminology around unbounded operators on a Hilbert space. The main calculations in this thesis often use isomorphisms from one Hilbert space to another related Hilbert space. In the third section of Chapter 2, the basic form of one of these key Hilbert space isomorphisms is developed. The results in this section are not original, but details are given in a form that is useful for later in the thesis.

The fourth section of Chapter 2 introduces the basic theory of unitary representations. If G is a locally compact group, a unitary representation of G is a homomorphism π of G into $\mathcal{U}(\mathcal{H}_\pi)$, the unitary group of some Hilbert space $\mathcal{H}_\pi$, that is continuous in a natural way. If π is a unitary representation of G and $\eta \in \mathcal{H}_\pi$, define $V_\eta\xi$, for each $\xi \in \mathcal{H}_\pi$, as a complex-valued function on G, by

$$V_\eta\xi(x) = \langle \xi, \pi(x)\eta \rangle_{\mathcal{H}_\pi}, \text{ for all } x \in G.$$

The continuity assumption on unitary representations means that $V_\eta\xi$ is a bounded continuous function on G, for each $\xi \in \mathcal{H}_\pi$. These linear maps $V_\eta : \xi \to V_\eta\xi$ play a key role in this thesis. A unitary representation π is called irreducible if there are no nontrivial closed subspaces of $\mathcal{H}_\pi$ that are invariant under $\pi(x)$, for all $x \in G$. When π is irreducible, the map V_η is one to one, for any nonzero $\eta \in \mathcal{H}_\pi$. The space of all Borel measurable complex-valued functions on G that are square-integrable with respect to the left Haar measure forms a Hilbert space with a natural inner product when functions agreeing almost everywhere are considered equal. This space of square-integrable functions is denoted $L^2(G)$.

For each $x \in G$, define $\lambda_G(x) : L^2(G) \to L^2(G)$ by, for $f \in L^2(G)$,

$$\big(\lambda_G(x)f\big)(y) = f(x^{-1}y), \text{ for all } y \in G.$$

Left invariance of the left Haar measure on G implies that λ_G is a unitary representation of G, called the left regular representation. One of the primary goals in doing harmonic analysis on a particular group is to decompose its left regular representation into irreducible representations in some sense. When G is a compact group, the Peter-Weyl Theorem shows how to decompose λ_G as a direct sum of irreducible representations. Part of the reason why the Peter-Weyl Theorem holds is that Haar measure on a compact group is a finite measure. Therefore, every bounded continuous complex-valued function is in $L^2(G)$ when G is compact. Therefore, if π is irreducible and $\eta \in \mathcal{H}_\pi$, then $V_\eta : \mathcal{H}_\pi \to L^2(G)$. In fact, if the norm of η equals the square root of the dimension of $\mathcal{H}_\pi$ (which is finite when G is compact), then V_η is a unitary map of $\mathcal{H}_\pi$ onto the range of V_η. Moreover, V_η makes π equivalent to a subrepresentation of λ_G.

In general, an irreducible representation π of a locally compact group G is called *square-integrable* if there exist nonzero vectors $\eta, \xi \in \mathcal{H}_\pi$ such that $V_\eta\xi \in L^2(G)$. The properties of square-integrable representations for a general locally compact group were studied by Duflo and Moore in [9]. One of the main theorems from [9] is now called the Duflo-Moore Theorem. This theorem is presented as Theorem 2.5.5 in the fifth section of Chapter 2, along with other basic properties of square-integrable representations. A consequence of the Duflo-Moore Theorem is that, if π is a square-integrable representation of G, then there is a dense subspace $\mathcal{D}_\pi$ of $\mathcal{H}_\pi$ and a positive operator $C_\pi : \mathcal{D}_\pi \to \mathcal{H}_\pi$ such that if $\eta \in \mathcal{D}_\pi$ satisfies $\|C_\pi\eta\|_{\mathcal{H}_\pi} = 1$, then V_η is an isometry of $\mathcal{H}_\pi$ into $L^2(G)$. A direct consequence of V_η being an isometry was recognized in [15]. If V_η is an isometry, then it preserves inner products. That is, for $\xi, \nu \in \mathcal{H}_\pi$,

$$\langle V_\eta\xi, V_\eta\nu\rangle_{L^2(G)} = \langle \xi, \nu\rangle_{\mathcal{H}_\pi}. \tag{1.1}$$

The left hand side of (1.1) can be rearranged using the definition of $V_\eta\nu$. The left Haar measure of G is denoted μ_G.

$$\begin{aligned}\langle V_\eta\xi, V_\eta\nu\rangle_{L^2(G)} &= \int_G V_\eta\xi(x)\overline{\langle \nu, \pi(x)\eta\rangle_{\mathcal{H}_\pi}}\, d\mu_G(x)\\ &= \int_G V_\eta\xi(x)\langle \pi(x)\eta, \nu\rangle_{\mathcal{H}_\pi}\, d\mu_G(x) = \left\langle \int_G V_\eta\xi(x)\pi(x)\eta\, d\mu_G(x), \nu\right\rangle_{\mathcal{H}_\pi},\end{aligned}$$

where the integral of the $\mathcal{H}_\pi$-valued function $V_\eta\xi(x)\pi(x)\eta$ is in the weak sense. Using (1.1) gives, for each $\xi \in \mathcal{H}_\pi$, $\langle \xi, \nu\rangle_{\mathcal{H}_\pi} = \left\langle \int_G V_\eta\xi(x)\pi(x)\eta\, d\mu_G(x), \nu\right\rangle_{\mathcal{H}_\pi}$, for any $\nu \in \mathcal{H}_\pi$. That is,

$$\xi = \int_G V_\eta\xi(x)\pi(x)\eta\, d\mu_G(x), \text{ weakly in } \mathcal{H}_\pi. \tag{1.2}$$

Thus, under the special circumstances of a square-integrable representation π and a distinguished vector $\eta \in \mathcal{H}_\pi$ satisfying $\|C_\pi\eta\|_{\mathcal{H}_\pi} = 1$, any vector $\xi \in \mathcal{H}_\pi$ can be reconstructed from the function $V_\eta\xi$ on G that records the inner product of ξ with $\pi(x)\eta$, for each $x \in G$.

The easiest significant example of a non-compact locally compact group with a square-integrable representation is the group G_1 consisting of affine transformations of the real line. That is, $G_1 = \{[x,a] : x, a \in \mathbb{R}, a \neq 0\}$. The group product in G_1 is given by $[x,a][y,b] = [x+ay, ab]$ and the left Haar integral of any function $f : G_1 \to \mathbb{C}$ for which the integral exists is $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f[x,a]\frac{dx\,da}{a^2}$. There is a natural unitary representation ρ of G_1 on the Hilbert space $L^2(\mathbb{R})$ given by, for any $[x,a] \in G_1$,

$$\rho[x,a]f(t) = |a|^{-1/2} f\left(\frac{t-x}{a}\right), \text{ for a.e. } t \in \mathbb{R},$$

and for any $f \in L^2(\mathbb{R})$. This unitary representation of G_1 has been known to be irreducible for a long time, for example see page 132 of [22] where it is discussed in an equivalent form. It is also relatively easy to show that ρ is square-integrable, so the Duflo-Moore theory applies. A connection was made in [15] with wavelet analysis that was developing as a tool in signal processing at that time. To describe this connection, we need the notation of the Fourier transform, which is introduced in section 7 of Chapter 2. Let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\widehat{\mathbb{R}})$ denote the unitary map such that $\mathcal{F}f = \widehat{f}$, for any integrable $f \in L^2(\mathbb{R})$. Let $\widehat{\mathcal{D}}_\rho = \left\{\xi \in L^2(\widehat{\mathbb{R}}) : \int_{-\infty}^{\infty} \frac{|\xi(\omega)|^2}{|\omega|}\, d\omega < \infty\right\}$. Define a map $T : \widehat{\mathcal{D}}_\rho \to L^2(\widehat{\mathbb{R}})$ by $T\xi(\omega) = |\omega|^{-1/2}\xi(\omega)$, for a.e. $\omega \in \widehat{\mathbb{R}}$ and every $\xi \in \widehat{\mathcal{D}}_\rho$. The dense subspace of $L^2(\mathbb{R})$ given in the Duflo-Moore Theorem is $\mathcal{D}_\rho = \{f \in L^2(\mathbb{R}) : \mathcal{F}f \in \widehat{\mathcal{D}}_\rho\}$. The positive operator C_ρ then takes the form $C_\rho = \mathcal{F}^{-1}T\mathcal{F}$. Abusing notation, write $\widehat{f}$ for $\mathcal{F}f$, for any $f \in L^2(\mathbb{R})$. Then, for any $w \in L^2(\mathbb{R})$, the condition that $\|C_\rho w\|_{L^2(\mathbb{R})} = 1$ becomes

$$\int_{-\infty}^{\infty} \frac{|\widehat{w}(\omega)|^2}{|\omega|}\, d\omega = 1. \tag{1.3}$$

For $[x,a] \in G_1$, $\rho[x,a]w$ is viewed as w "dilated" by a and "translated" by x. It is commonly denoted $w_{x,a}$. That is

$$w_{x,a}(t) = |a|^{-1/2} w\left(\frac{t-x}{a}\right), \text{ for a.e. } t \in \mathbb{R}. \tag{1.4}$$

For a $w \in L^2(\mathbb{R})$ that satisfies (1.3), define $V_w f$, a function on G_1, for each $f \in L^2(\mathbb{R})$, by

$$V_w f[x,a] = \int_{-\infty}^{\infty} f(t)\overline{w_{x,a}(t)}\, dt, \text{ for all } [x,a] \in G_1. \tag{1.5}$$

The reconstruction formula (1.2) becomes, for any $f \in L^2(\mathbb{R})$,

$$f = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V_w f[x,a]\, w_{x,a} \frac{dx\, da}{a^2}, \text{ weakly in } L^2(\mathbb{R}). \tag{1.6}$$

In the language of wavelet theory, a function $w \in L^2(\mathbb{R})$ satisfying (1.3) is called a *wavelet* for the continuous wavelet transform. The map V_w given in (1.5) is called the *continuous wavelet transform* with wavelet w and (1.6) is called the *reconstruction formula*. See [17] and [7] for good introductions to the continuous wavelet transform as well as a presentation of discrete versions of the wavelet transform in one dimension. Note that the transform described in equations (1.3) to (1.6) was known before [15] and (1.6) is also called the Calderón Reproducing Formula in dimension one.

There are many ways in which wavelet theory has been generalized to higher dimensions. One line of research is based on the theory of square-integrable representations and the Duflo-Moore Theorem. In [4], the structure of G_1 was generalized. Let $\mathrm{GL}_n(\mathbb{R})$ denote the locally compact group of invertible $n \times n$ real matrices. Let H be a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$. Then H acts on column vectors in $\mathbb{R}^n$ by matrix multiplication. Let $G = \mathbb{R}^n \rtimes H$, the semi-direct product group. That is,

$$G = \{[\underline{x}, A] : \underline{x} \in \mathbb{R}^n, A \in H\},$$

with group product given by $[\underline{x}, A][\underline{y}, B] = [\underline{x} + A\underline{y}, AB]$. The identity in G is $[\underline{0}, \mathrm{id}_n]$, where id_n denotes the $n \times n$ identity matrix, and $[\underline{x}, A]^{-1} = [-A^{-1}\underline{x}, A^{-1}]$. There is a natural unitary representation ρ of G on the Hilbert space $L^2(\mathbb{R}^n)$. For $[\underline{x}, A] \in G$ and any $f \in L^2(\mathbb{R}^n)$,

$$\rho[\underline{x}, A] f(\underline{y}) = |\det(A)|^{-1/2} f\left(A^{-1}(\underline{y} - \underline{x})\right), \text{ for a.e. } \underline{y} \in \mathbb{R}^n. \tag{1.7}$$

For notational convenience, we use the notation

$$\widehat{\mathbb{R}^n} = \{\underline{\omega} = (\omega_1, \cdots, \omega_n) : \omega_1, \cdots, \omega_n \in \mathbb{R}\}.$$

For $\underline{\omega} \in \widehat{\mathbb{R}^n}$, the H-orbit of $\underline{\omega}$ is $\mathcal{O}_{\underline{\omega}} = \{\underline{\omega}A : A \in H\}$ and $H_{\underline{\omega}} = \{A \in H : \underline{\omega}A = \underline{\omega}\}$ is called the stabilizer of $\underline{\omega}$. When $H_{\underline{\omega}} = \{\mathrm{id}_n\}$, the H-orbit $\mathcal{O}_{\underline{\omega}}$ is called a *free* H-orbit. One consequence of the main theorem in [4] is that, if H is such that there exists an H-orbit $\mathcal{O}_{\underline{\omega}}$ that is free, open, and dense in $\widehat{\mathbb{R}^n}$, then the representation ρ is a square-integrable representation of G. The operator C_ρ of the Duflo-Moore theory was also identified in [4]. Thus any closed subgroup H of $\mathrm{GL}_n(\mathbb{R})$ that has a free, open, and dense orbit in $\widehat{\mathbb{R}^n}$ yields a generalization of the continuous wavelet transform with a reconstruction formula that is a special case of (1.7). In [14], the condition that there exists a free H-orbit is weakened slightly. If $\underline{\omega} \in \widehat{\mathbb{R}^n}$ is such that $\mathcal{O}_{\underline{\omega}}$ is open, dense, and is such that $H_{\underline{\omega}}$ is compact, then ρ, as given by (1.7), is a square-integrable representation of G. One of the main results of this thesis is to develop a generalization of the continuous wavelet transform from a square-integrable representation of G_2 that does not arise as in (1.7). This will be discussed later in the introduction.

In Section 2.6, the Peter-Weyl Theorem is presented in a manner that emphasizes the role of the maps V_η. This theorem gives a method of analyzing functions in $L^2(G)$ when G is a compact group. For compact G, every irreducible representation of G is square-integrable. Every irreducible representation is finite dimensional also. Suppose $\{\pi_j : j \in J\}$ is a list of irreducible representations such that any irreducible representation of G is equivalent to exactly one of the π_j. For $j \in J$, let $d_j = \dim(\mathcal{H}_{\pi_j})$ and let $\{\xi_k^j : 1 \le k \le d_j\}$ be an orthonormal basis of $\mathcal{H}_{\pi_j}$. The subspace $\mathcal{D}_{\pi_j}$ given in the Duflo-Moore theory is simply $\mathcal{H}_{\pi_j}$ itself and the operator C_{π_j} is given by multiplication by the constant $d_j^{-1/2}$. Let $\eta_k^j = d_j^{1/2}\xi_k^j$, for $1 \le k \le d_j$ and for each $j \in J$. Then each η_k^j satisfies the condition $\|C_{\pi_j}\eta_k^j\|_{\mathcal{H}_{\pi_j}} = 1$. Thus $V_{\eta_k^j}$ is a linear isometry of $\mathcal{H}_{\pi_j}$ into $L^2(G)$. Let $\mathcal{K}_{\eta_k^j} = V_{\eta_k^j}\mathcal{H}_{\pi_j}$. The Peter-Weyl Theorem implies that $L^2(G)$ decomposes as an orthogonal direct sum of the closed subspaces $\mathcal{K}_{\eta_k^j}$, for $j \in J$, $1 \le k \le d_j$. In particular, $\{V_{\eta_k^j}\xi_\ell^j : j \in J, 1 \le k, \ell \le d_j\}$ is an orthonormal basis of $L^2(G)$. Another of the main results in this thesis provides an analog of the Peter-Weyl Theorem for the non-compact group G_2.

A brief review of the Fourier transform and the properties used in this thesis is given in the seventh section of Chapter 2.

More details are provided for the process of inducing a representation from a subgroup of a locally compact group and this is done in the eighth section. We use [12] and [20] as sources for induced representations. If π is a unitary representation of a closed subgroup H of a locally compact group G, there is a complicated way of defining a Hilbert space consisting of functions from G to the Hilbert space $\mathcal{H}_\pi$ which satisfy certain conditions along with a unitary representation $\mathrm{ind}_H^G\pi$ of G that acts on that Hilbert space. In certain circumstances, one can show that $\mathrm{ind}_H^G\pi$ is equivalent to a representation acting on a more concrete Hilbert space. In all the cases, where we induce a representation in this thesis, the following situation holds: There is a closed subgroup K of G that is complementary to the closed subgroup H in the sense that $G = KH$ and $K \cap H = \{e\}$, where e is the identity element of G. This situation is discussed in [20], but there is a small error in the treatment in [20]. For that reason, a careful development is presented in Section 2.8 and the correct formula is given in Proposition 2.8.9. There is a representation, denoted σ^π that acts on the Hilbert space $L^2(K; \mathcal{H}_\pi)$ and σ^π is equivalent to $\mathrm{ind}_H^G\pi$. Even though Proposition 2.8.9 is much easier to work with than the abstract definition of an induced representation, there are problems to solve to get a simple form for any particular group.

In [22], George Mackey developed a systematic method of constructing all the irreducible representations of a locally compact group G when $G = N \rtimes H$, where N is an abelian locally compact group and H is a locally compact group that acts on N. For all the groups studied in this thesis, $N = \mathbb{R}^n$ and H is a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$. In Section 2.9, Mackey theory for semi-direct products is summarized for groups of the form $\mathbb{R}^n \rtimes H$. We are particularly interested in the situation where there exists an $\underline{\omega} \in \widehat{\mathbb{R}^n}$ such that the H-orbit $\mathcal{O}_{\underline{\omega}}$ is open and dense in $\widehat{\mathbb{R}^n}$. If π is an irreducible representation of $H_{\underline{\omega}}$, define a representation $\chi_{\underline{\omega}} \otimes \pi$ of $\mathbb{R}^n \rtimes H_{\underline{\omega}}$ by, for

$[\underline{x}, C] \in \mathbb{R}^n \rtimes H_{\underline{\omega}}$ and $\xi \in \mathcal{H}_\pi$,

$$\big(\chi_{\underline{\omega}} \otimes \pi\big)[\underline{x}, C]\xi = e^{2\pi i \underline{\omega x}}\pi(C)\xi.$$

Part of the content of Theorem 2.9.5 is that $\sigma = \mathrm{ind}_{\mathbb{R}^n \rtimes H_{\underline{\omega}}}^{\mathbb{R}^n \rtimes H}(\chi_{\underline{\omega}} \otimes \pi)$ is an irreducible representation of $\mathbb{R}^n \rtimes H$. Moreover, from [21], Corollary 11.1, we know that this induced representation σ will be square-integrable when π is a square-integrable representation of $H_{\underline{\omega}}$. Therefore, the Duflo-Moore theory means that there will be an analog of the continuous wavelet transform associated with such a σ. We know of no case where such a transform has been investigated except when $H_{\underline{\omega}}$ is compact and π in the trivial representation of $H_{\underline{\omega}}$ as presented in [14].

The original work of this thesis is presented in Chapters 3, 4, and 5. In Chapter 3, a group of the form $\mathbb{R}^3 \rtimes H$ is investigated as an illustrative example. The closed subgroup H is selected from one in the list given in [5]. It has the property that there exists an $\underline{\omega} \in \widehat{\mathbb{R}^3}$ such that $\mathcal{O}_{\underline{\omega}}$ is open and dense and $H_{\underline{\omega}}$ is compact. In Theorem 3.1.9 and the material leading up to, a wavelet transform is presented associated with each irreducible representations of $H_{\underline{\omega}}$.

Chapter 4 is devoted to the groups $G_n = \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$. Various algebraic properties are developed as well as the left Haar integration formula for different parametrizations. A key result is Proposition 4.3.3 which says: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$, then A can be uniquely decomposed as $A = M_A C_A$, where

$$M_A = \begin{pmatrix} s & -t \\ t & s \end{pmatrix}, \text{ with } s = \frac{d(ad-bc)}{b^2+d^2}, t = \frac{-b(ad-bc)}{b^2+d^2},$$

and

$$C_A = \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix}, \text{ with } u = \frac{cd+ab}{(ad-bc)}, v = \frac{b^2+d^2}{(ad-bc)}.$$

Thus, we can reparametrize $\mathrm{GL}_2(\mathbb{R})$ as

$$\mathrm{GL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} s & -t \\ t & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} : s, t, u, v \in \mathbb{R}, s^2 + t^2 > 0, v \neq 0 \right\}. \tag{1.8}$$

If $\underline{\omega} = (1, 0) \in \widehat{\mathbb{R}^2}$, then

$$H_{(1,0)} = \{A \in \mathrm{GL}_2(\mathbb{R}) : (1,0)A = (1,0)\} = \left\{ \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} : u, v \in \mathbb{R}, v \neq 0 \right\}.$$

An important observation is that $H_{(1,0)}$ is isomorphic to the one dimensional affine group G_1. The other factors in the parametrization in (1.8) make up another closed subgroup of $\mathrm{GL}_2(\mathbb{R})$. Let

$$K_0 = \left\{ \begin{pmatrix} s & -t \\ t & s \end{pmatrix} : s, t \in \mathbb{R}, s^2 + t^2 > 0 \right\}.$$

Then K_0 is also a closed subgroup of $\mathrm{GL}_2(\mathbb{R})$. Moreover, $\mathrm{GL}_2(\mathbb{R}) = K_0 H_{(1,0)}$ and $K_0 \cap H_{(1,0)} = \{\mathrm{id}_2\}$. This leads to a factorization of G_2. Let

$$K = \{[\underline{0}, M] : M \in K_0\} \quad \text{and} \quad H = \left\{[\underline{x}, C] : \underline{x} \in \mathbb{R}^2, C \in H_{(1,0)}\right\}.$$

Then K and H are closed subgroups of G_2 such that $G_2 = KH$ and $K \cap H = \{[\underline{0}, \mathrm{id}_2]\}$. This factorization of G_2 makes the computations of Chapter 5 feasible.

In the first section of Chapter 5, a unitary representation of the n-dimensional affine group G_n is calculated and realized as a subrepresentation of the left regular representation. Let $N = \{[\underline{x}, \mathrm{id}_n] : \underline{x} \in \mathbb{R}^n\}$. Then N is a closed normal abelian subgroup of G_n. Its dual group is $\widehat{N} = \{\chi_{\underline{\omega}} : \underline{\omega} \in \mathbb{R}^n\}$, where $\chi_{\underline{\omega}}[\underline{x}, \mathrm{id}_n] = e^{2\pi i \underline{\omega}\underline{x}}$, for all $[\underline{x}, \mathrm{id}_n] \in N$. Fix $\underline{\omega}_0 = (1, 0, \cdots, 0) \in \widehat{\mathbb{R}^n}$. The induced representation $\mathrm{ind}_N^{G_n} \chi_{\underline{\omega}_0}$ is unitarily equivalent to a representation $\pi^{\underline{\omega}_0}$ that acts on the Hilbert space $L^2\big(\mathrm{GL}_n(\mathbb{R})\big)$ as follows: For $[\underline{x}, A] \in G_n$ and any $f \in L^2\big(\mathrm{GL}_n(\mathbb{R})\big)$,

$$\pi^{\underline{\omega}_0}[\underline{x}, A] f(B) = e^{2\pi i \underline{\omega}_0 B^{-1} \underline{x}} f(A^{-1}B), \tag{1.9}$$

for almost every $B \in \mathrm{GL}_n(\mathbb{R})$. If $n = 1$, then $\underline{\omega}_0 = 1$ and π^1 is investigated in the second section of Chapter 5, where it is shown to be irreducible and even square-integrable. Also, it turns out to be unitarily equivalent to the natural representation of G_1 on $L^2(\mathbb{R})$ that leads to the continuous wavelet transform in one dimension. However, if $n > 1$, then $\pi^{\underline{\omega}_0}$ is reducible. Nevertheless, we show that $\pi^{\underline{\omega}_0}$ is equivalent to a subrepresentation of the left regular representation of G_n. In fact, Theorem 5.1.4 shows that the left regular representation of G_n is a direct sum of infinitely many copies of $\pi^{\underline{\omega}_0}$.

The third section of Chapter 5 contains the major computation of this work. We focus on the case of $n = 2$ to obtain an analysis of functions in $L^2(G_2)$. The strategy is to exploit the fact that the stability subgroup $H_{(1,0)}$ of $\underline{\omega}_0 = (1, 0)$ is isomorphic to G_1 and π^1 happens to correspond to an irreducible representation of G_1. Mackey theory then tells us that we get an irreducible representation of G_n if we induce $\chi_{(1,0)} \otimes \pi^1$ from $H = \mathbb{R}^2 \rtimes H_{(1,0)}$ up to G_2. Moreover, we know from [21] and [4] that the resulting induced representation will be square-integrable and equivalent to a subrepresentation of the left regular representation of G_2. We realize $\mathrm{ind}_H^{G_2}(\chi_{(1,0)} \otimes \pi^1)$ as a representation we denote by σ acting on the Hilbert space $L^2\big(K; L^2(\mathbb{R}^*)\big)$, where $\mathbb{R}^*$ is the multiplicative group of nonzero real numbers and the measures are the Haar measure of K and $\mathbb{R}^*$. The formula for σ is computed and given in equation (5.8). In steps, σ is moved, using unitary equivalences from $L^2\big(K; L^2(\mathbb{R}^*)\big)$ to a subspace of $L^2\big(K; L^2(H_{(1,0)})\big)$ and then to a subspace of $L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$, where it is equivalent to a subrepresentation of $\pi^{(1,0)}$. Finally, it is moved to $L^2(G_2)$ and a subrepresentation of λ_{G_2}. One of the main theorems of the thesis is Theorem 5.3.7 which establishes σ as a square-integrable representation. We also formulate a slightly weaker version, Theorem 5.3.8, and present a direct proof which is easier to follow.

To make σ easier to understand, it is moved to $L^2\big(\widehat{\mathbb{R}^3}\big)$. The formulas are much easier to write if $\widehat{\mathbb{R}^3}$ is written as $\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}$. That is $\widehat{\mathbb{R}^3} = \{(\underline{\omega}, \omega_3) : \underline{\omega} \in \widehat{\mathbb{R}}, \omega_3 \in \widehat{\mathbb{R}}\}$. We also use the fact that K_0 is homeomorphic to $\mathcal{O} = \mathcal{O}_{(1,0)} = \widehat{\mathbb{R}^2} \setminus \{\underline{0}\}$, which is co-null in $\widehat{\mathbb{R}^2}$. This homeomorphism is given by $\gamma : \mathcal{O} \to K_0$, where $\gamma(\underline{\omega}) = \frac{1}{\|\underline{\omega}\|^2} \begin{pmatrix} \omega_1 & -\omega_2 \\ \omega_2 & \omega_1 \end{pmatrix}$, for all $\underline{\omega} = (\omega_1, \omega_2) \in \mathcal{O}$. For $F \in L^2\big(K, L^2(\mathbb{R}^*)\big)$ and $(\underline{\omega}, \omega_3) \in \widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}$, let

$$(UF)(\underline{\omega}, \omega_3) = \begin{cases} \frac{\big(F[\underline{0}, \gamma(\underline{\omega})]\big)(\omega_3^{-1})}{\|\underline{\omega}\| \cdot |\omega_3|^{1/2}} & \text{for a.e. } \underline{\omega} \in \mathcal{O}, \omega_3 \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then U is a unitary map from $L^2\big(K, L^2(\mathbb{R}^*)\big)$ onto $L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big) = L^2\big(\widehat{\mathbb{R}^3}\big)$. Move σ to $L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$ using U by setting $\sigma_1[\underline{x}, A] = U\sigma[\underline{x}, A]U^{-1}$, for all $[\underline{x}, A] \in G_2$. This gives , for $\xi \in L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$,

$$\big(\sigma_1[\underline{x}, A]\xi\big)(\underline{\omega}, \omega_3) = \tfrac{|\det(A)| \cdot \|\underline{\omega}\|}{\|\underline{\omega}A\|} e^{2\pi i(\underline{\omega}\underline{x} + \omega_3 u_{\underline{\omega},A})} \xi\big(\underline{\omega}A, \omega_3 v_{\underline{\omega},A}\big), \tag{1.10}$$

for a.e. $(\underline{\omega}, \omega_3) \in \widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}$. In (1.10), there are two functions $u_{\underline{\omega},A}$ and $v_{\underline{\omega},A}$. They are rational functions in ω_1, ω_2, a, b, c, and d, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The precise expressions for $u_{\underline{\omega},A}$ and $v_{\underline{\omega},A}$ are given in Proposition 5.3.2. Our derivation of σ_1 shows that it is a square-integrable representation of G_2. We also get explicit ways of showing it is equivalent to a subrepresentation of λ_{G_2}. In fact, we formulate an analog of the Peter-Weyl Theorem for compact groups.

Let $L^2\left(\widehat{\mathbb{R}^2}; \frac{d\omega}{\|\underline{\omega}\|^2}\right)$ denote the weighted L^2-space formed from Borel functions ζ on $\widehat{\mathbb{R}^2}$ such that $\int_{\widehat{\mathbb{R}^2}} |\zeta(\underline{\omega})|^2 \frac{d\omega}{\|\underline{\omega}\|^2} < \infty$. Likewise for $L^2\left(\widehat{\mathbb{R}}; \frac{d\nu}{|\nu|}\right)$. Let $\mathcal{O}_1 = \widehat{\mathbb{R}} \setminus \{0\}$. Note that $C_c(\mathcal{O})$ is dense in $L^2\left(\widehat{\mathbb{R}^2}; \frac{d\omega}{\|\underline{\omega}\|^2}\right)$ and $C_c(\mathcal{O}_1)$ is dense in $L^2\left(\widehat{\mathbb{R}}; \frac{d\nu}{|\nu|}\right)$. Let $\{\zeta_i : i \in I\}$ be an orthonormal basis of $L^2\left(\widehat{\mathbb{R}^2}; \frac{d\omega}{\|\underline{\omega}\|^2}\right)$ consisting of functions in $C_c(\mathcal{O})$ and let $\{\phi_j : j \in J\}$ be an orthonormal basis of $L^2\left(\widehat{\mathbb{R}}; \frac{d\nu}{|\nu|}\right)$ consisting of functions in $C_c(\mathcal{O}_1)$. Define $\psi_{i,j}(\underline{\omega}, \omega_3) = \zeta_i(\underline{\omega})\phi_j(\omega_3)$, for each $(i,j) \in I \times J$. For each $(i,j) \in I \times J$, the function behaves like a wavelet for the representation σ_1. Define

$$V_{\psi_{i,j}}\xi[\underline{x}, A] = \langle \xi, \sigma_1[\underline{x}, A]\psi_{i,j} \rangle_{L^2(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}})},$$

for all $[\underline{x}, A] \in G_2$ and $\xi \in L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$. Let $\mathcal{M}_{i,j} = V_{\psi_{i,j}} L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$, for $(i,j) \in I \times J$. Then each $\mathcal{M}_{i,j}$ is a closed λ_{G_2}-invariant subspace of $L^2(G_2)$ and $V_{\psi_{i,j}}$ is an isometry that intertwines σ_1 with the restriction of λ_{G_2} to $\mathcal{M}_{i,j}$. Moreover

$$L^2(G_2) = \textstyle\sum^{\oplus}_{(i,j) \in I \times J} \mathcal{M}_{i,j}.$$

In addition to this analog of the Peter-Weyl Theorem, the Duflo-Moore operator is easily identified after the calculations carried out in Chapter 5 and analogs of the continuous wavelet transform using σ_1 to move the "wavelets" around are presented in the final section of the thesis.

Chapter 2

General Notation and Background

In this chapter, the notation for basic concepts that will be used in this thesis is established. There are no original results in this chapter; however, a proof may be given if it illustrates a concept that is useful in later chapters. The main sources used are [18], [19], [12], and [20]. When results are taken from other sources, an explicit reference is given.

2.1 Locally Compact Groups

A *group* is a set G equipped with a binary operation $(x, y) \to xy$ called the group product. There is an *identity* element $e \in G$ so that $ex = xe = x$, for each $x \in G$. The group product is associative: that is $x(yz) = (xy)z$, for all $x, y, z \in G$. Also, for each $x \in G$, there is an *inverse*, denoted $x^{-1} \in G$, such that $xx^{-1} = x^{-1}x = e$. The group is called *abelian* if $xy = yx$, for all $x, y \in G$.

A *topological group* is a group G that also has a topology on it such that the maps $(x, y) \to xy$ from $G \times G \to G$ and $x \to x^{-1}$ from G to G are continuous. All the topological groups that come up in this thesis will be Hausdorff. If the topology on a topological group G is Hausdorff and locally compact, then G is called a *locally compact group*.

If G is a group and H is a nonempty subset of G such that xy and x^{-1} are in H, for any $x, y \in H$, then H is called a *subgroup* of G. If G is a locally compact group and H is a closed subgroup of G, then H is a locally compact group when given the topology it inherits as a subset of G. If H is a subgroup of G and $x \in G$, then the set $xHx^{-1} = \{xzx^{-1} : z \in H\}$ is also a subgroup of G. A subgroup H of G is called a *normal subgroup* if $xHx^{-1} = H$, for all $x \in G$.

If H is a subgroup of G and $x \in G$, then the set $xH = \{xz : z \in H\}$ is called a *left coset* of H. Note that $xH = yH$ as sets if and only if $x^{-1}y \in H$. In this thesis, the space of all left cosets will often be important.

Definition 2.1.1. Let H be a subgroup of a group G. The *left coset space* of G modulo H is $G/H = \{xH : x \in G\}$. The map $q : G \to G/H$ given by $q(x) = xH$, for all $x \in G$, is called the *quotient map*. If G is a locally compact group and H is a closed subgroup of G, then G/H is given the strongest topology such that q is continuous. That is, $U \subseteq G/H$ is open if and only if $q^{-1}(U)$ is an open subset of G. This quotient topology on G/H is also locally compact.

When H is a normal subgroup of G, for any $xH, yH \in G/H$,

$$xHyH = (xy)(y^{-1}Hy)H = (xy)HH = (xy)H,$$

so G/H can be made into a group with product given by $(xH)(yH) = (xy)H$, for all $xH, yH \in G/H$. This product is well-defined and satisfies the group axioms with the trivial coset $H = eH$ serving as identity. When G is locally compact and H is a closed normal subgroup of G, G/H is a locally compact group with this product.

If H is a subgroup of G, any $x \in G$ moves around the left cosets of H. For $zH \in G/H$, $x \cdot (zH) = (xz)H \in G/H$. Note that $x \cdot (zH)$ is well-defined and satisfies:

- $e \cdot (zH) = zH$, for all $zH \in G/H$
- $x \cdot \big(y \cdot (zH)\big) = (xy) \cdot (zH)$, for all $x, y \in G$, $zH \in G/H$
- If G is a locally compact group and H is a closed subgroup, then $\big(x, zH\big) \to x \cdot (zH)$ is a continuous map of $G \times G/H$ to G/H.

This action of G on the set G/H is a special case of a group action. In this thesis there will be a number of useful group actions.

Definition 2.1.2. Let G be a group. A *G-space* is a nonempty set Ω and a map from $G \times \Omega$ to Ω denoted by $(x, \omega) \to x \cdot \omega$ satisfying

- $e \cdot \omega = \omega$, for all $\omega \in \Omega$
- $x \cdot (y \cdot \omega) = (xy) \cdot \omega$, for all $x, y \in G$ and $\omega \in \Omega$.

If G is a locally compact group, Ω is a locally compact topological space and $(x, \omega) \to x \cdot \omega$ is a continuous map of $G \times \Omega$ to Ω, then Ω is called a *topological G-space*. We may say G *acts* on Ω to mean that Ω is a topological G-space.

Let Ω be a topological G-space. For $\omega \in \Omega$, the *G-orbit* of ω is $\mathcal{O}_\omega = \{x \cdot \omega : x \in G\}$. The set $H_\omega = \{x \in G : x \cdot \omega = \omega\}$ is a closed subgroup of G called the *isotropy subgroup* of ω. For $x, y \in G$, $x \cdot \omega = y \cdot \omega$ if and only if $x^{-1}y \in H_\omega$; that is, if and only if $xH_\omega = yH_\omega$. Thus, $\theta : G/H_\omega \to \mathcal{O}_\omega$ given by $\theta(xH_\omega) = x \cdot \omega$, for any $xH_\omega \in G/H_\omega$, is well-defined. The map θ is one to one, onto and continuous. Often, θ is a homeomorphism. A topological space X is called *σ-compact* if X is a countable union of compact subsets. For a proof of the following proposition, see Proposition 4.6 of [20].

Proposition 2.1.3. Let G be a σ-compact locally compact group and let Ω be a topological G-space. Let $\omega \in \Omega$ and let $\theta(xH_\omega) = x \cdot \omega$, for all $xH_\omega \in G/H_\omega$. If the orbit $\mathcal{O}_\omega$ is locally compact and Hausdorff, then θ is a homeomorphism of G/H_ω to $\mathcal{O}_\omega$.

A G-space Ω is called *transitive* if $\mathcal{O}_\omega = \Omega$ for some $\omega \in \Omega$. Then $\mathcal{O}_{\omega'} = \Omega$, for all $\omega' \in \Omega$.

Definition 2.1.4. Let Ω be a transitive G-space and fix a point $\omega_0 \in \Omega$. A *cross-section* of the G action on Ω based at ω_0 is a map $\gamma : \Omega \to G$ such that $\gamma(\omega) \cdot \omega_0 = \omega$, for all $\omega \in \Omega$.

The cross-sections used in this thesis will often appear in integrals, so they need to be measurable maps. If X is a locally compact Hausdorff space, $\mathcal{B}_X$ denotes the σ-algebra of Borel subsets of X; that is, the smallest σ-algebra containing all the open subsets of X. Measurability will refer to Borel measurability, which means with respect to $\mathcal{B}_X$ for the appropriate space X. Usually, X is an open subset of some $\mathbb{R}^n$ in this thesis.

Proposition 2.1.5. Let G be a separable locally compact group and assume the topology on G is metrizable. Let Ω be a transitive Hausdorff topological G-space. Let $\omega_0 \in \Omega$. Then there exists a cross-section γ of the G action on Ω based at ω_0 such that γ is Borel measurable.

Proof. Let H_{ω_0} be the isotropy subgroup for ω_0. Since G is separable, it is σ-compact, so G/H_{ω_0} is σ-compact. The hypotheses of Theorem 1 of [10] hold, so there exists a Borel measurable map $\tau : G/H_{\omega_0} \to G$ such $q \circ \tau$ is the identity map on G/H_{ω_0}, where $q : G \to G/H_{\omega_0}$ is the quotient map.

Let $\theta(xH_{\omega_0}) = x \cdot \omega_0$, for each $xH_{\omega_0} \in G/H_{\omega_0}$. Note that $\Omega = \mathcal{O}_{\omega_0}$, so $\theta : G/H_{\omega_0} \to \Omega$ is a homeomorphism by Proposition 2.1.3. Define $\gamma : \Omega \to G$ by $\gamma(\omega) = \tau\left(\theta^{-1}(\omega)\right)$, for all $\omega \in \Omega$. Then γ is a cross-section of the G action on Ω based at ω_0 and γ is Borel measurable. □

Definition 2.1.6. Let X be a locally compact Hausdorff space. A *Radon measure* on X is a Borel measure μ such that

- $\mu(K) < \infty$, for any compact $K \subseteq X$
- $\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ compact}\}$
- $\mu(E) = \inf\{\mu(U) : E \subseteq U \subseteq X, U \text{ open}\}$, for any $E \in \mathcal{B}_X$.

Any locally compact group has a Radon measure on it that is invariant under left translation.

Proposition 2.1.7. Let G be a locally compact group. There exists a nonzero Radon measure μ on G such that $\mu(xE) = \mu(E)$, for all $x \in G$ and any Borel $E \subseteq G$. Moreover, if ν is any nonzero Radon measure on G satisfying $\nu(xE) = \nu(E)$, for all $x \in G$ and any Borel $E \subseteq G$, then there exists a $c > 0$ such that $\nu = c\mu$.

Definition 2.1.8. Let G be a locally compact group. We will fix a Radon measure μ_G on G satisfying the properties of Proposition 2.1.7. This measure μ_G is called the *left Haar measure* of G.

Similarly, G carries a right Haar measure that is unique up to a constant multiple. The homeomorphism $x \to x^{-1}$ interchanges a right Haar measure with a left Haar measure. The modern convention is to work with left Haar measures.

We assume the standard theory of the general Lebesgue integral. In particular, for the measure space $(G, \mathcal{B}_G, \mu_G)$, where $\mathcal{B}_G$ is the σ-algebra of Borel subsets of G, if $f : G \to [0, \infty]$ is a Borel measurable function, then $\int_G f \, d\mu_G = \int_G f(x) \, d\mu_G(x)$ exists in $[0, \infty]$. We have the usual Lebesgue spaces where functions are considered equal if they agree μ_G-almost everywhere (μ_G-a.e.).

Definition 2.1.9. Let G be a locally compact group and $1 \leq p < \infty$. Then

$$L^p(G) = \left\{ f : G \to \mathbb{C} \,\middle|\, f \text{ Borel and } \int_G |f|^p d\mu_G < \infty \right\}.$$

Equipped with the norm $f \to \|f\|_p = \left(\int_G |f|^p d\mu_G \right)^{1/p}$, $L^p(G)$ is a Banach space.

We have a change of variables formula for left translations that is valid for any function f on G for which $\int_G f(x) \, d\mu_G(x)$ has meaning. For any such f and fixed $y \in G$

$$\int_G f(yx) \, d\mu_G(x) = \int_G f(x) \, d\mu_G(x). \tag{2.1}$$

Definition 2.1.10. Let X be a locally compact space. Let $C(X)$ denote the vector space of all continuous complex-valued functions on X. For $f \in C(X)$, the *support* of f is

$$\operatorname{supp}(f) = \overline{\{x \in X : f(x) \neq 0\}}.$$

We say f has *compact support* when $\operatorname{supp}(f)$ is a compact set. Let $C_c(X)$ denote the subspace of $C(X)$ consisting of continuous functions with compact support.

Proposition 2.1.11. Let G be a locally compact group. Then $C_c(G)$ is contained in $L^p(G)$ as a dense subspace, for any $1 \leq p < \infty$.

If $y \in G$ is fixed then $E \to \mu_G(Ey)$, for Borel sets E, is also a left invariant nonzero Radon measure on G. Thus, there exists $\Delta_G(y) \in (0, \infty)$ such that

$$\mu_G(Ey) = \Delta_G(y)\mu_G(E), \text{ for any Borel } E \subseteq G.$$

Proposition 2.1.12. Let G be a locally compact group and let $\mathbb{R}^+$ denote the positive real numbers considered as a locally compact group with multiplication as the group product. There exists a continuous homomorphism $\Delta_G : G \to \mathbb{R}^+$ such that, for any $y \in G$,

$$\mu_G(Ey) = \Delta_G(y)\mu_G(E), \text{ for any Borel } E \subseteq G.$$

Definition 2.1.13. The homomorphism Δ_G is called the *modular function* of G. If $\Delta_G(x) = 1$, for all $x \in G$, then G is called *unimodular*.

Observe that G is unimodular when G is Abelian, compact, or discrete. If G is discrete, then counting measure is the left Haar measure and this is clearly right invariant as well.

We have a change of variables formula for right translation of any function f on G for which $\int_G f(x)\,d\mu_G(x)$ is meaningful. For such an f and any $y \in G$,

$$\Delta_G(y) \int_G f(xy)\,d\mu_G(x) = \int_G f(x)\,d\mu_G(x) \tag{2.2}$$

The change of variables formulas (2.1) and (2.2) will be used frequently.

Here are some examples of left Haar measures on different groups and the corresponding modular function. It is often most convenient to give a formula for $\int_G f\,d\mu_G$, for any $f \in C_c(G)$. By the Riesz Representation Theorem for positive linear functionals on $C_c(X)$, for a locally compact Hausdorff space X, there exists a unique Radon measure determined by that formula for $f \in C_c(G)$.

Example 1. Let $n \in \mathbb{N}$ and $\mathbb{R}^n = \left\{ \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_k \in \mathbb{R}, 1 \leq k \leq n \right\}$, an Abelian locally compact group with addition as group product. Then, for any $f \in C_c(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f\,d\mu_{\mathbb{R}^n} = \int_{\mathbb{R}^n} f(\underline{x})\,d\underline{x} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\underline{x})\,dx_1 \cdots dx_n,$$

where $\int_{-\infty}^{\infty} h(t)\,dt$ simply denotes the Riemann integral, for $h \in C_c(\mathbb{R})$. Left invariance is just the property that $\int_{\mathbb{R}^n} f(\underline{x}-\underline{y})\,d\underline{x} = \int_{\mathbb{R}^n} f(\underline{x})\,d\underline{x}$, for any $\underline{y} \in \mathbb{R}^n$ and $f \in C_c(\mathbb{R}^n)$. Since $\mathbb{R}^n$ is Abelian, it is unimodular.

Example 2. Let $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ and equip it with multiplication of real numbers as group product. Then $\mathbb{R}^*$ is an Abelian locally compact group. For $a \in \mathbb{R}^*$ and $f \in C_c(\mathbb{R}^*)$, a simple change of variables shows

$$\int_{-\infty}^{\infty} f(at)\,\frac{dt}{|t|} = \int_{-\infty}^{\infty} f(t)\,\frac{dt}{|t|}$$

Thus $\int_{\mathbb{R}^*} f\,d\mu_{\mathbb{R}^*} = \int_{-\infty}^{\infty} f(t)\,\frac{dt}{|t|}$, for all $f \in C_c(\mathbb{R}^*)$, defines left Haar measure on $\mathbb{R}^*$. Again, $\mathbb{R}^*$ is unimodular since it is Abelian.

Example 3. The group $\mathbb{R}^*$ acts on the group $\mathbb{R}$ by multiplication. For $a \in \mathbb{R}^*$, $b \to ab$ is an automorphism of $\mathbb{R}$. Let $G_1 = \mathbb{R} \rtimes \mathbb{R}^*$, the semi-direct product for this action. That is,

$$G_1 = \mathbb{R} \rtimes \mathbb{R}^* = \{[b,a] : b \in \mathbb{R}, a \in \mathbb{R}^*\},$$

with product given by $[b_1,a_1][b_2,a_2] = [b_1 + a_1 b_2, a_1 a_2]$, for all $[b_1,a_1],[b_2,a_2] \in G_1$. With this as product, G_1 is a group with identity $[0,1]$ and $[b,a]^{-1} = [-a^{-1}b, a^{-1}]$, for $[b,a] \in G_1$. The product and inversion maps are continuous when G_1 is given the product topology of $\mathbb{R} \times \mathbb{R}^*$. Thus, G_1 is a locally compact group. The group G_1 is called the *affine group* of $\mathbb{R}$. It is sometimes also called the $ax+b$-group since we can view an element $[b,a] \in G_1$ as the transformation $x \to ax + b$ of $\mathbb{R}$.

Fix $[b_0, a_0] \in G_1$. Note that, for $f \in C_c(G_1)$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\big([b_0,a_0][b,a]\big)\frac{db\,da}{a^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f[b_0+a_0b, a_0a]\frac{db\,da}{a^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f[b,a]\frac{db\,da}{a^2}.$$

Therefore $\int_{G_1} f\,d\mu_{G_1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f[b,a]\frac{db\,da}{a^2}$, for all $f \in C_c(G_1)$. This group is not Abelian, so we check a translation of the integral on the right. For $[b_0, a_0] \in G_1$ and $f \in C_c(G_1)$,

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\big([b,a][b_0,a_0]\big)\frac{db\,da}{a^2} &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f[b+ab_0, aa_0]\,db\right)\frac{da}{a^2}\\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f[b, aa_0]\,db\right)\frac{da}{a^2}\\ &= |a_0|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f[b,a]\frac{db\,da}{a^2}\end{aligned}$$

Therefore G_1 is nonunimodular and $\Delta_{G_1}[b_0,a_0] = |a_0|^{-1}$, for all $[b_0,a_0] \in G_1$.

Example 4. Let $\mathrm{GL}_n(\mathbb{R})$ denote the group of all $n\times n$ real matrices A with $\det(A) \neq 0$. For $A \in \mathrm{GL}_n(\mathbb{R})$, let a_{ij} denote the i,j entry of A. So $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}$.

Considering $\mathrm{GL}_n(\mathbb{R})$ as an open subset of $\mathbb{R}^{n^2}$ gives $\mathrm{GL}_n(\mathbb{R})$ a locally compact topology and it is a locally compact group using the product of matrices as the group product. This group is unimodular and the Haar measure $\mu_{\mathrm{GL}_n(\mathbb{R})}$ is such that, for all $f \in C_c(\mathrm{GL}_n(\mathbb{R}))$,

$$\int_{\mathrm{GL}_n(\mathbb{R})} f\,d\mu_{\mathrm{GL}_n(\mathbb{R})} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(A)\,\frac{da_{11}da_{12}\cdots da_{nn}}{|\det(A)|^n}.$$

In particular, $\mathrm{GL}_2(\mathbb{R}) = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} : a,b,c,d \in \mathbb{R}, ad-bc \neq 0\right\}$ and, for any $f \in C_c(\mathrm{GL}_2(\mathbb{R}))$,

$$\int_{\mathrm{GL}_2(\mathbb{R})} f\,d\mu_{\mathrm{GL}_2(\mathbb{R})} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\begin{pmatrix} a & b\\ c & d\end{pmatrix}\frac{da\,db\,dc\,dd}{(ad-bc)^2}.$$

Example 5. This is actually a family of examples generalizing the affine group of $\mathbb{R}$ to higher dimensions. Most of the groups arising later in this thesis are of the following form. Let H be a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$. Form the new group $G = \mathbb{R}^n \rtimes H$ defined by

$$G = \mathbb{R}^n \rtimes H = \{[\underline{x}, A] : x \in \mathbb{R}^n, A \in H\}.$$

The group product is given by $[\underline{x}, A][\underline{y}, B] = [\underline{x} + A\underline{y}, AB]$, for $[\underline{x}, A], [\underline{y}, B] \in G$. The identity in G is $[\underline{0}, \mathrm{id}]$, where id denotes the identity $n \times n$ matrix, and the inverse operation is $[\underline{x}, A]^{-1} = [-A^{-1}\underline{x}, A^{-1}]$, for $[\underline{x}, A] \in G$. It is well-known how to find the left Haar measure for a semi-direct product from the left Haar measures of the factors (see any one of [18], [12], or [20]). For $f \in C_c(G)$,

$$\int_G f\, d\mu_G = \int_H \int_{\mathbb{R}^n} f[\underline{x}, A]\, \frac{d\underline{x}\, d\mu_H(A)}{|\det(A)|}. \tag{2.3}$$

We can verify this with a short calculation. For $[\underline{y}, B] \in G$ and $f \in C_c(G)$,

$$\begin{aligned}
\int_H \int_{\mathbb{R}^n} f\big([\underline{y}, B][\underline{x}, A]\big)\, \frac{d\underline{x}\, d\mu_H(A)}{|\det(A)|} &= \int_H \int_{\mathbb{R}^n} f[\underline{y} + B\underline{x}, BA]\, \frac{d\underline{x}\, d\mu_H(A)}{|\det(A)|} \\
&= \int_H \int_{\mathbb{R}^n} f[\underline{x}, BA]\, \frac{d\underline{x}\, d\mu_H(A)}{|\det(BA)|} \\
&= \int_H \int_{\mathbb{R}^n} f[\underline{x}, A]\, \frac{d\underline{x}\, d\mu_H(A)}{|\det(A)|}.
\end{aligned}$$

Thus, (2.3) does give left Haar measure on G. For right translation let $[\underline{y}, B] \in G$ and $f \in C_c(G)$,

$$\begin{aligned}
\int_H \int_{\mathbb{R}^n} f\big([\underline{x}, A][\underline{y}, B]\big)\, \frac{d\underline{x}\, d\mu_H(A)}{|\det(A)|} &= \int_H \int_{\mathbb{R}^n} f[\underline{x} + A\underline{y}, AB]\, \frac{d\underline{x}\, d\mu_H(A)}{|\det(A)|} \\
&= \int_H \int_{\mathbb{R}^n} f[\underline{x}, AB]\, \frac{d\underline{x}\, d\mu_H(A)}{|\det(A)|} \\
&= |\det(B)| \int_H \int_{\mathbb{R}^n} f[\underline{x}, AB]\, \frac{d\underline{x}\, d\mu_H(A)}{|\det(AB)|} \\
&= \frac{|\det(B)|}{\Delta_H(B)} \int_H \int_{\mathbb{R}^n} f[\underline{x}, A]\, \frac{d\underline{x}\, d\mu_H(A)}{|\det(A)|}.
\end{aligned}$$

Thus, $\Delta_G[\underline{y}, B] = \frac{\Delta_H(B)}{|\det(B)|}$, for all $[\underline{y}, B] \in G$. Note that, if $\Delta_H(B) = |\det(B)|$, for each $B \in H$, then G will be unimodular.

2.2 Hilbert spaces, Operators, and the Unitary Group

In this section, we recall basic definitions and properties of Hilbert spaces, operators on Hilbert spaces, and the unitary group. We focus on the properties that will be used in later sections. Of particular importance is the definition of a positive operator, when the operator may not be bounded. A good reference for this material is [19].

Definition 2.2.1. A *Hilbert space* is a complex vector space $\mathcal{H}$ equipped with an inner product $(\xi, \eta) \to \langle \xi, \eta \rangle_{\mathcal{H}}$ such that $\mathcal{H}$ is complete with respect to the norm defined by $\|\xi\|_{\mathcal{H}} = \langle \xi, \xi \rangle_{\mathcal{H}}^{1/2}$, for all $\xi \in \mathcal{H}$. If there is no confusion about what Hilbert space is being considered $\langle \xi, \eta \rangle_{\mathcal{H}}$ may be written simply as $\langle \xi, \eta \rangle$.

We frequently need to recover the inner product from the norm.

Proposition 2.2.2. [Polarization Identity] Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Then, for any $\xi, \eta \in \mathcal{H}$,

$$\langle\xi,\eta\rangle = \tfrac{1}{4}\big(\|\xi+\eta\|^2 - \|\xi-\eta\|^2 + i\|\xi+i\eta\|^2 - i\|\xi-i\eta\|^2\big).$$

Definition 2.2.3. Let $\mathcal{H}$ be a Hilbert space. For $\xi, \eta \in \mathcal{H}$, ξ is *orthogonal* to η if $\langle\xi,\eta\rangle = 0$; then we write $\xi \perp \eta$. For $A \subseteq \mathcal{H}$ and $B \subseteq \mathcal{H}$, $A \perp B$ means $\xi \perp \eta$, for all $\xi \in A$ and $\eta \in B$. We also let $A^\perp = \{\eta \in \mathcal{H} : \xi \perp \eta, \text{ for all } \xi \in A\}$.

If $A \subseteq \mathcal{H}$, then $A^\perp$ is a closed subspace of $\mathcal{H}$. Note that $\xi \perp \xi$ implies $\xi = 0$, the zero vector in $\mathcal{H}$. Thus, $A \cap A^\perp = \{0\}$ always.

Proposition 2.2.4. If $\mathcal{K}$ is a closed subspace of a Hilbert space $\mathcal{H}$, then $\mathcal{H} = \mathcal{K} + \mathcal{K}^\perp$. Any $\xi \in \mathcal{H}$ is uniquely written as $\xi = P_\mathcal{K}\xi + P_{\mathcal{K}^\perp}\xi$, with $P_\mathcal{K}\xi \in \mathcal{K}$ and $P_{\mathcal{K}^\perp}\xi \in \mathcal{K}^\perp$.

The map $\xi \to P_\mathcal{K}\xi$ is a linear map of $\mathcal{H}$ into $\mathcal{H}$ with range $\mathcal{K}$. It satisfies:

- $\|\xi - P_\mathcal{K}\xi\| \le \|\xi - \eta\|$, for all $\eta \in \mathcal{K}$.
- $\|P_\mathcal{K}\xi\| \le \|\xi\|$, for all $\xi \in \mathcal{H}$.
- $\|P_\mathcal{K}\xi\| = \|\xi\|$ if and only if $\xi \in \mathcal{K}$.
- $\|P_\mathcal{K}\xi\| = 0$ if and only if $\xi \in \mathcal{K}^\perp$.

Definition 2.2.5. If $\mathcal{K}$ is a closed subspace of $\mathcal{H}$, then $P_\mathcal{K}$ is called the *orthogonal projection* onto $\mathcal{K}$.

Definition 2.2.6. A set of vectors $\{\eta_j : j \in J\}$ is called *orthonormal* if $\eta_j \perp \eta_k$, for $j, k \in J$, $j \ne k$, and $\|\eta_j\| = 1$, for $j \in J$.

Proposition 2.2.7. Let $\mathcal{H}$ be a Hilbert space. Any orthonormal subset of $\mathcal{H}$ is contained in a maximal orthonormal set in $\mathcal{H}$.

Definition 2.2.8. A maximal orthonormal subset of a Hilbert space $\mathcal{H}$ is called an *orthonormal basis* of $\mathcal{H}$.

Proposition 2.2.9. Let $\mathcal{H}$ be a Hilbert space. Let $\{\eta_j : j \in J\}$ be an orthonormal subset of $\mathcal{H}$. Then, the following are equivalent:
(a) $\{\eta_j : j \in J\}$ is an orthonormal basis of $\mathcal{H}$.
(b) $\{\eta_j : j \in J\}^\perp = \{0\}$.
(c) The closed linear span of $\{\eta_j : j \in J\}$ is $\mathcal{H}$.
(d) $\|\xi\|^2 = \sum_{j\in J} |\langle\xi,\eta_j\rangle|^2$, for all $\xi \in \mathcal{H}$.
(e) $\langle\xi,\nu\rangle = \sum_{j\in J}\langle\xi,\eta_j\rangle\langle\eta_j,\nu\rangle$, for all $\xi, \nu \in \mathcal{H}$.
(f) $\xi = \sum_{j\in J}\langle\xi,\eta_j\rangle\eta_j$, for all $\xi \in \mathcal{H}$.

The sum in (e) converges absolutely in $\mathbb{C}$. The convergence in (f) is unconditional norm convergence in $\mathcal{H}$.

Proposition 2.2.10. Let $\mathcal{H}$ be a Hilbert space and let A be an orthonormal subset of $\mathcal{H}$. Then, there exists an orthonormal basis of $\mathcal{H}$ which contains A.

In particular, any Hilbert space has an orthonormal basis. There are many different orthonormal bases in a given Hilbert space, but they will all have the same cardinality.

Proposition 2.2.11. Any two orthonormal bases of a Hilbert space $\mathcal{H}$ have the same cardinality.

Definition 2.2.12. Let $\mathcal{H}$ be a Hilbert space. The cardinality of any orthonormal basis of $\mathcal{H}$ is called the *dimension of* $\mathcal{H}$ and is denoted by $\dim(\mathcal{H})$. If $\dim(\mathcal{H}) < \infty$, then $\mathcal{H}$ is called a finite dimensional Hilbert space.

Definition 2.2.13. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. A map $U : \mathcal{H}_1 \to \mathcal{H}_2$ is called a *unitary map* of $\mathcal{H}_1$ onto $\mathcal{H}_2$ if U is linear, one to one and onto, and satisfies

$$\langle U\xi, U\eta\rangle_{\mathcal{H}_2} = \langle \xi, \eta\rangle_{\mathcal{H}_1}, \text{ for all} \xi, \eta \in \mathcal{H}_1.$$

If U is a unitary map of $\mathcal{H}_1$ onto $\mathcal{H}_2$, then U^{-1} is a unitary map of $\mathcal{H}_2$ onto $\mathcal{H}_1$. When such a unitary exists, $\mathcal{H}_1$ and $\mathcal{H}_2$ are *isomorphic* as Hilbert spaces.

Example 6. Let J be a nonempty set and let

$$\ell^2(J) = \left\{ \underline{\alpha} = (\alpha_j)_{j\in J} : \alpha_j \in \mathbb{C}, \text{ for all } j \in J, \text{ and } \sum_{j\in J} |\alpha_j|^2 < \infty \right\},$$

with coordinate wise vector space operations and inner product

$$\langle \underline{\alpha}, \underline{\beta}\rangle_{\ell^2(J)} = \sum_{j\in J} \alpha_j \overline{\beta_j}, \text{ for all } \underline{\alpha}, \underline{\beta} \in \ell^2(J).$$

Then $\ell^2(J)$ is a Hilbert space. Moreover, if $\mathcal{H}$ is any Hilbert space and $\{\eta_j : j \in J\}$ is an orthonormal basis of $\mathcal{H}$, then the map $U : \ell^2(J) \to \mathcal{H}$ given by $U(\underline{\alpha}) = \sum_{j\in J} \alpha_j \eta_j$, for all $\underline{\alpha} \in \ell^2(J)$, defines a unitary map of $\ell^2(J)$ onto $\mathcal{H}$. This follows easily from (e) and (f) of Proposition 2.2.9. Thus, $\ell^2(J)$ and $\mathcal{H}$ are isomorphic as Hilbert spaces.

A map $W : \mathcal{H}_1 \to \mathcal{H}_2$ is an *isometry* if $\|W\xi\|_{\mathcal{H}_2} = \|\xi\|_{\mathcal{H}_1}$, for all $\xi \in \mathcal{H}_1$. The Polarization Identity implies that, if W is a linear isometry of $\mathcal{H}_1$ onto $\mathcal{H}_2$, then W is a unitary map of $\mathcal{H}_1$ onto $\mathcal{H}_2$.

If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, let $\mathcal{H}_1 \oplus \mathcal{H}_2 = \{(\xi_1, \xi_2) : \xi_1 \in \mathcal{H}_1, \xi_2 \in \mathcal{H}_2\}$ with coordinate wise vector space operations and inner product

$$\langle (\xi_1, \xi_2), (\eta_1, \eta_2)\rangle_{\mathcal{H}_1\oplus\mathcal{H}_2} = \langle \xi_1, \eta_1\rangle_{\mathcal{H}_1} + \langle \xi_2, \eta_2\rangle_{\mathcal{H}_2},$$

for $(\xi_1, \xi_2), (\eta_1, \eta_2) \in \mathcal{H}_1 \oplus \mathcal{H}_2$. If $\mathcal{H}$ is a Hilbert space and $\mathcal{K}$ is a closed subspace of $\mathcal{H}$, then both $\mathcal{K}$ and $\mathcal{K}^\perp$ are Hilbert spaces when given the restriction of the inner product on $\mathcal{H}$. The map $\xi \to (P_{\mathcal{K}}\xi, P_{\mathcal{K}^\perp}\xi)$ is a Hilbert space isomorphism of $\mathcal{H}$ with $\mathcal{K} \oplus \mathcal{K}^\perp$.

Definition 2.2.14. Suppose $\mathcal{H}_j$ is a Hilbert space, for each $j \in J$, where J is a nonempty index set. The *direct sum* of these Hilbert spaces is

$$\sum_{j\in J}^{\oplus} \mathcal{H}_j = \left\{ \underline{\xi} = \left(\xi_j\right)_{j\in J} : \xi_j \in \mathcal{H}_j, \text{ for each } j \in J, \text{ and } \sum_{j\in J} \|\xi_j\|^2 < \infty \right\}.$$

Equipped with coordinate wise vector space operations and inner product

$$\langle \underline{\xi}, \underline{\eta} \rangle_{\sum_{j\in J}^{\oplus} \mathcal{H}_j} = \sum_{j\in J} \langle \xi_j, \eta_j \rangle_{\mathcal{H}_j}, \text{ for } \underline{\xi}, \underline{\eta} \in \sum_{j\in J}^{\oplus} \mathcal{H}_j,$$

$\sum_{j\in J}^{\oplus} \mathcal{H}_j$ is a Hilbert space.

We need to introduce some of the standard concepts for operators on a Hilbert space.

Definition 2.2.15. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. A linear map $A : \mathcal{H}_1 \to \mathcal{H}_2$ is called a *bounded operator* if $\|A\| = \sup\{\|A\xi\|_{\mathcal{H}_2} : \xi \in \mathcal{H}_1, \|\xi\|_{\mathcal{H}_1} \le 1\} < \infty$. Let $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ denote the space of all bounded linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$. When equipped with pointwise defined scalar product and vector space sum, $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a Banach space with the norm $\|\cdot\|$. We denote $\mathcal{B}(\mathcal{H}, \mathcal{H})$ as simply $\mathcal{B}(\mathcal{H})$, for any Hilbert space $\mathcal{H}$. For $A, B \in \mathcal{B}(\mathcal{H})$, the product AB is simply the composition of maps and $\|AB\| \le \|A\| \cdot \|B\|$. The identity map on $\mathcal{H}$ is denoted by I. So $I\xi = \xi$, for all $\xi \in \mathcal{H}$.

Proposition 2.2.16. Let $\mathcal{H}$ be a Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$. Then there exists $A^* \in \mathcal{B}(\mathcal{H})$ such that $\langle A^*\xi, \eta\rangle_{\mathcal{H}} = \langle \xi, A\eta\rangle_{\mathcal{H}}$, for all $\xi, \eta \in \mathcal{H}$. Moreover, for all $A, B \in \mathcal{B}(\mathcal{H})$ and $\alpha \in \mathbb{C}$,
(a) $(A + B)^* = A^* + B^*$.
(b) $(\alpha A)^* = \overline{\alpha} A^*$.
(c) $(AB)^* = B^*A^*$.

Definition 2.2.17. For $A \in \mathcal{B}(\mathcal{H})$, A^* is called the *adjoint* of A. A bounded operator $A \in \mathcal{B}(\mathcal{H})$ is called *self-adjoint* if $A^* = A$.

Proposition 2.2.18. Let $\mathcal{H}$ be a Hilbert space and let $W \in \mathcal{B}(\mathcal{H})$ be such that W is one to one and onto. Then W is a unitary map of $\mathcal{H}$ onto $\mathcal{H}$ if and only if $W^{-1} = W^*$.

Definition 2.2.19. Let $\mathcal{H}$ be a Hilbert space. A *unitary operator* on $\mathcal{H}$ is a unitary map of $\mathcal{H}$ onto $\mathcal{H}$. Let $\mathcal{U}(\mathcal{H})$ denote the set of all unitary operators on $\mathcal{H}$. This is a group with identity I when equipped with composition as the group product. We call $\mathcal{U}(\mathcal{H})$ the *unitary group* of $\mathcal{H}$.

The topology on $\mathcal{B}(\mathcal{H})$ given by the norm $\|\cdot\|$ is called the *norm topology*. Besides the norm topology, there are two other topologies on $\mathcal{B}(\mathcal{H})$ that we will use.

Definition 2.2.20. Let $\mathcal{H}$ be a Hilbert space. The *strong operator topology* (SOT) on $\mathcal{B}(\mathcal{H})$ is the weakest topology such that $A \to \|A\eta\|$ is continuous, for every $\eta \in \mathcal{H}$. The *weak operator topology* (WOT) is the weakest topology on $\mathcal{B}(\mathcal{H})$ such that $A \to$

$\langle A\xi, \eta\rangle_{\mathcal{H}}$ is continuous, for every pair of vectors $\xi, \eta \in \mathcal{H}$. For a subset $\Omega \subseteq \mathcal{B}(\mathcal{H})$, $\overline{\Omega}^{SOT}$ denotes the SOT-topology closure of Ω, $\overline{\Omega}^{WOT}$ denotes the WOT-topology closure of Ω, and $\overline{\Omega}^{\|\cdot\|}$ denotes the norm closure of Ω.

Definition 2.2.21. Let $\mathcal{H}$ be a Hilbert space and let $\Omega \subseteq \mathcal{B}(\mathcal{H})$. We say Ω is *self-adjoint* if $A \in \Omega$ implies $A^* \in \Omega$. We say that Ω is a *subalgebra* of $\mathcal{B}(\mathcal{H})$ if Ω is a vector subspace of $\mathcal{B}(\mathcal{H})$ such that $A, B \in \Omega$ implies $AB \in \Omega$. A self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ is called a $*$-*subalgebra* of $\mathcal{B}(\mathcal{H})$.

Definition 2.2.22. Let $\mathcal{H}$ be a Hilbert space. A *von Neumann algebra* on $\mathcal{H}$ is a $*$-subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ containing I and closed in the weak operator topology.

Definition 2.2.23. Let $\mathcal{H}$ be a Hilbert space and let $\Omega \subseteq \mathcal{B}(\mathcal{H})$. The *commutant* of Ω is

$$\Omega' = \{T \in \mathcal{B}(\mathcal{H}) : TA = AT, \text{ for all } A \in \Omega\}.$$

The maps $A \to TA$, $A \to AT$ and $A \to A^*$ are each WOT-continuous, for any fixed $T \in \mathcal{B}(\mathcal{H})$. Thus, it is easy to show the following.

Proposition 2.2.24. Let $\mathcal{H}$ be a Hilbert space and let $\Omega \subseteq \mathcal{B}(\mathcal{H})$. Suppose that Ω is self-adjoint. Then Ω' is a von Neumann algebra on $\mathcal{H}$.

An important theorem about von Neumann algebras is called the Double Commutant Theorem. For $\Omega \subseteq \mathcal{B}(\mathcal{H})$, $\Omega'' = (\Omega')'$ is the *double commutant* of Ω.

Theorem 2.2.25. [Double Commutant Theorem] Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{A}$ be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ containing I. Then $\mathcal{A}'' = \overline{\mathcal{A}}^{WOT}$.

Thus, if $\Omega \subseteq \mathcal{B}(\mathcal{H})$ is self-adjoint and contains I, then the smallest von Neumann algebra on $\mathcal{H}$ containing Ω is Ω''.

Later in this thesis, we will use operators that are not bounded. These are operators whose domain is a non-closed dense subspace of the Hilbert space.

Definition 2.2.26. Let $\mathcal{H}$ be a Hilbert space. An *operator* on $\mathcal{H}$ is a pair $(\mathcal{D}_T, T)$ where $\mathcal{D}_T$ is a dense subspace of $\mathcal{H}$ and $T : \mathcal{D}_T \to \mathcal{H}$ is a linear map. Sometimes, we simply say T is an operator on $\mathcal{H}$ and leave the domain $\mathcal{D}_T$ understood. If T is an operator on $\mathcal{H}$ with domain $\mathcal{D}_T$, the *graph of* T is

$$\mathcal{G}(T) = \{(\xi, T\xi) : \xi \in \mathcal{D}_T\},$$

a subspace of $\mathcal{H} \oplus \mathcal{H}$. The operator T is called *closed* if $\mathcal{G}(T)$ is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$.

Definition 2.2.27. Let T be an operator on a Hilbert space $\mathcal{H}$. Let $\mathcal{D}_{T^*}$ be the set of all vectors $\eta \in \mathcal{H}$ such that there exists a $\nu \in \mathcal{H}$ with $\langle \xi, \nu\rangle = \langle T\xi, \eta\rangle$, for all $\xi \in \mathcal{D}_T$. Then define $T^*\eta = \nu$. Then $\mathcal{D}_{T^*}$ is a dense subspace of $\mathcal{H}$ and T^* is an operator on $\mathcal{H}$ with domain $\mathcal{D}_{T^*}$ called the *adjoint* of T. If $\mathcal{D}_{T^*} = \mathcal{D}_T$ and $T^* = T$, then T is called *self-adjoint*.

Definition 2.2.28. Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$. If

$$\langle T\xi, \xi \rangle \geq 0, \text{ for all } \xi \in \mathcal{D}_T,$$

then T is called a *positive operator* on $\mathcal{H}$.

Example 7. Let X be a locally compact space and let μ be a Radon measure on X. The space

$$L^2(X, \mu) = \left\{ f : X \to \mathbb{C} : f \text{ is Borel measurable and } \int_X |f|^2 \, d\mu < \infty \right\}.$$

With the usual identification of functions that agree μ-a.e. and inner product given by

$$\langle f, g \rangle = \int_X f(x)\overline{g(x)} \, d\mu(x), \text{ for all } f, g \in L^2(X, \mu),$$

$L^2(X, \mu)$ is a Hilbert space. Let h be a continuous real-valued function on X that is everywhere positive. Let $\mathcal{D}_{T_h} = \{f \in L^2(X, \mu) : hf \in L^2(X, \mu)\}$. Define $T_h f = hf$, for all $f \in \mathcal{D}_{T_h}$. If $C_c(X)$ denotes the space of all continuous complex-valued functions on X with compact support, then $C_c(X)$ is a dense subspace of $L^2(X, \mu)$ and $C_c(X) \subseteq \mathcal{D}_{T_h}$. Then T_h is a self-adjoint operator on $L^2(X, \mu)$ (see Theorem 5.6.4 of [19]). Since $\langle hf, f \rangle = \int_X h(x)|f(x)|^2 d\mu(x) \geq 0$, for all $f \in \mathcal{D}_{T_h}$, T_h is a positive operator on $L^2(X, \mu)$.

2.3 A Hilbert Space Isomorphism

Let X and Y be σ-compact, second countable, locally compact Hausdorff spaces with μ and ν Radon measures on X and Y, respectively. The σ-compact assumption will allow Fubini's Theorem to be used and second countability means $L^2(X, \mu)$ and $L^2(Y, \nu)$ are separable. There are two other Hilbert spaces that use both measure spaces, (X, μ) and (Y, ν), that we will consider. They are $L^2(X \times Y, \mu \times \nu)$ and $L^2\big(X, \mu; L^2(Y, \nu)\big)$. There is a Hilbert space isomorphism between them that is defined in a natural manner. This isomorphism will be used in later calculations without much comment. The purpose of this section is make the details clear. A good reference for the basic integration theory is [13].

We have not yet defined what $L^2\big(X, \mu; L^2(Y, \nu)\big)$ means. It is easier to replace $L^2(Y, \nu)$ with an arbitrary separable Hilbert space $\mathcal{H}$. The argument is modeled on the usual proof that $L^2(X, \mu)$ is complete (see Theorem 6.6 of [13]).

Let $\mathcal{H}$ be a separable Hilbert space. A function $f : X \to \mathcal{H}$ is called *weakly measurable* if, for any $\eta \in \mathcal{H}$, the function $x \to \langle f(x), \eta \rangle_{\mathcal{H}}$ is Borel measurable. If $\{\eta_j : j \in J\}$ is an ONB of $\mathcal{H}$, then $f : X \to \mathcal{H}$ is weakly measurable if and only if $x \to \langle f(x), \eta_j \rangle_{\mathcal{H}}$ is Borel measurable for each $j \in J$, since J is countable when $\mathcal{H}$ is separable. If $f, g : X \to \mathcal{H}$ are weakly measurable, then $x \to \langle f(x), g(x) \rangle_{\mathcal{H}} = \sum_{j \in J} \langle f(x), \eta_j \rangle_{\mathcal{H}} \langle \eta_j, g(x) \rangle_{\mathcal{H}}$ is Borel measurable. In particular, $x \to \|f(x)\|^2$ is Borel measurable. Let

$$L^2(X,\mu;\mathcal{H}) = \left\{ f : X \to \mathcal{H} \,\middle|\, f \text{ weakly measurable and } \int_X \|f(x)\|_{\mathcal{H}}^2 d\mu(x) < \infty \right\}$$

Let $f, g \in L^2(X,\mu;\mathcal{H})$. Note that $x \to \|f(x)\|_{\mathcal{H}}$ is in $L^2(X,\mu)$ and we have, $|\langle f(x), g(x)\rangle_{\mathcal{H}}| \leq \|f(x)\|_{\mathcal{H}} \, \|g(x)\|_{\mathcal{H}}$, for μ-a.e. $x \in X$, by the Cauchy-Schwarz inequality. Since the product of two L^2-functions is integrable, $x \to \langle f(x), g(x)\rangle_{\mathcal{H}}$ is integrable. Also, note that,

$$\begin{aligned} \|f(x)+g(x)\|_{\mathcal{H}}^2 &\leq (\|f(x)\|_{\mathcal{H}} + \|g(x)\|_{\mathcal{H}})^2 \leq (2\max\{\|f(x)\|_{\mathcal{H}}, \|g(x)\|_{\mathcal{H}}\})^2 \\ &= 4\max\{\|f(x)\|_{\mathcal{H}}^2, \|g(x)\|_{\mathcal{H}}^2\} \leq 4\left(\|f(x)\|_{\mathcal{H}}^2 + \|g(x)\|_{\mathcal{H}}^2\right), \end{aligned}$$

for μ-a.e $x \in X$. Thus $f + g \in L^2(X, \mu : \mathcal{H})$. It is now routine to show that $L^2(X, \mu : \mathcal{H})$ is a vector space over $\mathbb{C}$.

Definition 2.3.1. For $f, g \in L^2(X, \mu : \mathcal{H})$, let

$$\langle f, g\rangle = \int_X \langle f(x), g(x)\rangle_{\mathcal{H}} d\mu(x).$$

Then $\langle \cdot, \cdot \rangle$ is an inner product on $L^2(X, \mu : \mathcal{H})$ and $\|f\| = (\langle f, f\rangle)^{1/2}$, for all $f \in L^2(X, \mu : \mathcal{H})$, is a norm when function that agree μ-a.e. are identified.

Theorem 2.3.2. Let μ be a Radon measure on a locally compact Hausdorff space X and let $\mathcal{H}$ be a separable Hilbert space. Then $(L^2(X, \mu : \mathcal{H}), \langle \cdot, \cdot \rangle)$ is a Hilbert space.

Proof. The only nontrivial thing to prove is completeness. We use the fact that a normed vector space $\mathcal{V}$ is complete if and only if every absolutely convergent series converges in $\mathcal{V}$. Suppose $f_k \in L^2(X, \mu : \mathcal{H})$, for $k \in \mathbb{N}$, and $\sum_1^\infty \|f_k\|_{L^2(X,\mu:\mathcal{H})} < \infty$. Let $B = \sum_1^\infty \|f_k\|_{L^2(X,\mu:\mathcal{H})}$. Let $h_k(x) = \|f_k(x)\|_{\mathcal{H}}$, for μ-a.e. $x \in X$. Then $h_k \in L^2(X,\mu)$ and $\|h_k\|_2 = \left(\int_X h_k(x)^2 d\mu(x)\right)^{1/2} = \|f_k\|_{L^2(X,\mu:\mathcal{H})}$.

For $n \in \mathbb{N}$, let $g_n(x) = \sum_{k=1}^n \|f_k(x)\|_{\mathcal{H}} = \sum_{k=1}^n h_k(x)$, for μ-a.e. $x \in X$, and define

$$g(x) = \sum_{k=1}^\infty \|f_k(x)\|_{\mathcal{H}} = \sum_{k=1}^\infty h_k(x) = \lim_{n\to\infty} g_n(x), \text{ for } \mu - \text{a.e.}\, x \in X.$$

So $g_n \in L^2(X,\mu)$ and $\|g_n\|_2 \leq \sum_{k=1}^n \|h_k\|_2 = \sum_{k=1}^n \|f_k\|_{L^2(X,\mu:\mathcal{H})} \leq B$. Thus, $\|g_n\|_2^2 \leq B^2$, for all $n \in \mathbb{N}$. Since $g_n^2 \leq g_{n+1}^2 \leq \cdots \leq g^2$, μ-a.e. By the Monotone Convergence Theorem, $\lim_{n\to\infty} \|g_n\|_2^2 = \lim_{n\to\infty} \int_X |g_n(x)|^2 d\mu(x) = \int_X |g(x)|^2 d\mu(x)$. Therefore, we have $\int_X |g(x)|^2 d\mu(x) \leq B^2$. This implies $g(x) = |g(x)| < \infty$, for μ-a.e. $x \in X$.

Thus, for μ-a.e. $x \in X$, $\sum_{k=1}^\infty \|f_k(x)\|_{\mathcal{H}} < \infty$. Since $\mathcal{H}$ is complete $\sum_{k=1}^\infty f_k(x)$ converges in $\mathcal{H}$, for those x. Let $F(x) = \sum_{k=1}^\infty f_k(x) \in \mathcal{H}$, for any $x \in X$ such that $\sum_{k=1}^\infty \|f_k(x)\|_{\mathcal{H}} < \infty$. Each f_k is weakly measurable, so F is weakly measurable. For μ-a.e $x \in X$, $\|F(x)\|_{\mathcal{H}} \leq \sum_{k=1}^\infty \|f_k(x)\|_{\mathcal{H}} = g(x)$ and $g \in L^2(X,\mu)$. This implies that

$\int_X \|F(x)\|^2_{\mathcal{H}} d\mu(x) \le \int_X |g(x)|^2 d\mu(x) \le B^2$. Thus, $F \in L^2(X, \mu : \mathcal{H})$ and

$$\|F - \sum_{k=1}^{n} f_k\|^2_{L^2(X,\mu:\mathcal{H})} = \int_X \|F(x) - \sum_{k=1}^{n} f_k(x)\|^2_{\mathcal{H}}\, d\mu(x).$$

For μ-a.e. $x \in X$,

$$\begin{aligned}\|F(x) - \sum_{k=1}^{n} f_k(x)\|^2_{\mathcal{H}} &\le \left(\|F(x)\|_{\mathcal{H}} + \sum_{k=1}^{n} \|f_k(x)\|_{\mathcal{H}}\right)^2 \\ &\le \big(g(x) + g(x)\big)^2 = \big(2g(x)\big)^2 = 4g(x)^2.\end{aligned}$$

Since $\int_X 4g(x)^2 d\mu(x) < \infty$. By the Dominated Convergence Theorem and continuity of $\|\cdot\|_{\mathcal{H}}$,

$$\begin{aligned}\lim_{n\to\infty} \int_X \|F(x) - \sum_{k=1}^{n} f_k(x)\|^2_{\mathcal{H}} d\mu(x) &= \int_X \lim_{n\to\infty} \|F(x) - \sum_{k=1}^{n} f_k(x)\|^2_{\mathcal{H}} d\mu(x) \\ &= \int_X \|F(x) - \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x)\|^2_{\mathcal{H}} d\mu(x) \\ &= \int_X \|F(x) - F(x)\|^2_{\mathcal{H}} d\mu(x) = 0.\end{aligned}$$

Therefore, $\sum_1^\infty f_k(x)$ converges in the $L^2(X, \mu : \mathcal{H})$ and $L^2(X, \mu : \mathcal{H})$ is complete. $\square$

Let $\mu \times \nu$ be the product measure on $X \times Y$. Then $\mu \times \nu$ is a Radon measure on $X \times Y$. Our next goal is to construct a Hilbert space isomorphism from $L^2(X \times Y, \mu \times \nu)$ onto $L^2\big(X, \mu : L^2(Y, \nu)\big)$.

For any $f \in L^2(X \times Y, \mu \times \nu)$ we select f from its equivalence class so that f is an everywhere defined Borel function on $X \times Y$. For $x \in X$, define $f_x : Y \to \mathbb{C}$ by $f_x(y) = f(x, y)$, $\forall y \in Y$, and, for $y \in Y$, define f^y by $f^y(x) = f(x, y)$, for all $x \in X$. Then f_x is just $f_{|\{x\} \times Y}$ and f^y is $f_{|X \times \{y\}}$. Thus f_x and f^y are Borel function on Y and X, respectively. Fubini's theorem applied to $|f(x, y)|^2$, says

$$\int_{X\times Y} |f(x,y)|^2 d(\mu \times \nu)(x,y) = \int_X \int_Y |f_x(y)|^2 d\nu(y) d\mu(x)$$

and

$$\int_{X\times Y} |f(x,y)|^2 d(\mu \times \nu)(x,y) = \int_Y \int_X |f^y(x)|^2 d\mu(x) d\nu(y).$$

Since $\int_{X\times Y} |f(x,y)|^2 d(\mu, \nu)(x,y) < \infty$, we must have $\int_Y |f_x(y)|^2 d\nu(y) < \infty$ for μ-a.e. $x \in X$. Let $Wf : X \to L^2(Y, \nu)$ be defined by

$$Wf(x) = \begin{cases} f_x & \text{if } \int_Y |f_x(y)|^2 d\nu(y) < \infty \\ 0 & \text{if } \int_Y |f_x(y)|^2 d\nu(y) = \infty, \end{cases}$$

Then $\|Wf(x)\|^2_{L^2(Y,\nu)} = \int_Y |f_x(y)|^2 d\nu(y)$, for μ-a.e. $x \in X$. Moreover,

$$\int_X \|Wf(x)\|^2_{L^2(Y,\nu)} d\mu(x) = \int_X \int_Y |f_x(y)|^2 d\nu(y) d\mu(x) = \|f\|^2_{L^2(X\times Y,\mu\times\nu)}. \tag{2.4}$$

Thus, $Wf \in L^2(X, \mu : L^2(Y,\nu))$ and $\|Wf\|_{L^2(X,\mu:L^2(Y,\nu))} = \|f\|_{L^2(X,\mu\times\nu)}$.

Theorem 2.3.3. The map $W : L^2(X \times Y, \mu\times\nu) \to L^2(X, \mu : L^2(Y,\nu))$ is a Hilbert space isomorphism. Moreover, $W^{-1} : L^2(X, \mu : L^2(Y,\nu)) \to L^2(X\times Y, \mu\times\nu)$ is given by $W^{-1}F(x,y) = \big(F(x)\big)(y)$, for $(\mu\times\nu)$-a.e. $(x,y) \in X\times Y$, for each $F \in L^2(X, \mu : L^2(Y,\nu))$.

Proof. It is clear that W is linear and (2.4) implies W is an isometry. Using the polarization identity, we see that W preserve inner products. To see that W is onto, suppose $F \in L^2(X, \mu : L^2(Y,\nu))$ is such that $F \perp Wf$, for all $f \in L^2(X\times Y, \mu\times\nu)$. Let $\{ g_j : j \in J\}$ be an ONB of $L^2(Y,\nu)$.

For each $j \in J$ and $x \in X$, let $h_j(x) = \langle F(x), g_j\rangle_{L^2(Y,\nu)}$. Then h_j is a Borel function on X. Also

$$\|F(x)\|^2_{L^2(Y,\nu)} = \sum_{j\in J} |\langle F(x), g_j\rangle_{L^2(Y,\nu)}|^2 = \sum_{j\in J} |h_j(x)|^2.$$

and $\sum_{j\in J}\int_X |h_j(x)|^2\, d\mu(x) = \int_X \|F(x)\|^2_{L^2(Y,\nu)} d\mu(x) = \|F\|^2_{L^2(X,\mu;L^2(Y,\nu))} < \infty$. Therefore, $\int_X |h_j(x)|^2 d\mu(x) < \infty$ and $h_j \in L^2(X,\mu)$, for all $j \in J$.

Let $f_j(x,y) = h_j(x)g_j(y)$ for all $(x,y) \in X\times Y$. Then $f_j \in L^2(X\times Y, \mu\times\nu)$, for each $j \in J$. Notice that $Wf_j(x) = h_j(x)g_j$, for μ-a.e. $x \in X$ and all $j \in J$. However, $F \perp Wf$, for all $f \in L^2(X\times Y, \mu\times\nu)$ Thus,

$$\begin{aligned} 0 = \langle F, Wf_j\rangle_{L^2(X,\mu;L^2(Y,\nu))} &= \int_X \langle F(x), h_j(x)g_j\rangle_{L^2(Y,\nu)} d\mu(x) \\ &= \int_X \overline{h_j(x)}\langle F(x), g_j\rangle_{L^2(Y,\nu)} d\mu(x) = \int_X |h_j(x)|^2 d\mu(x). \end{aligned}$$

Thus, $h_j = 0$ for each $j \in J$. That is $\langle F(x), g_j\rangle_{L^2(Y,\nu)} = 0$, for all $j \in J$ and μ-a.e. $x \in X$. This $F(x) = 0$, for μ-a.e. $x \in X$. Thus $F = 0$ as a member of $L^2(X, \mu; L^2(X, \mu : L^2(Y,\nu))$. Therefore, W is onto and, so, W is a unitary map and a Hilbert space isomorphism of $L^2(X\times Y, \mu\times\nu)$ with $L^2(X, \mu; L^2(Y,\nu))$.

Similar arguments show that W^{-1} as defined in the statement of the theorem maps $L^2(X, \mu; L^2(Y,\nu))$ into $L^2(X\times Y, \mu\times\nu)$ and $W(W^{-1}F) = F$, for all $F \in L^2(X, \mu; L^2(Y,\nu))$. □

Remark. One can also show that $L^2(Y, \nu; L^2(X,\mu))$ is Hilbert space isomorphic to $L^2(X\times Y, \mu\times\nu)$ in the obvious manner.

In the proof of Theorem 2.3.3, we formed a Borel function f_j on $X\times Y$ from a Borel function h_j on X and a Borel function g_j on Y. We will need to do this often in the following chapters, so we introduce a notation for the combined function.

Definition 2.3.4. Let X and Y be nonempty sets. Let $h : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ be functions. The *elementary tensor* function $h \otimes g : X \times Y \to \mathbb{C}$ is defined by

$$(h \otimes g)(x, y) = h(x)g(y), \text{ for all } (x, y) \in X \times Y.$$

See Example 2.6.11 of [19] for a discussion of these elementary tensor functions from which we formulate the following two propositions.

Proposition 2.3.5. Let X and Y be σ-compact, second countable, locally compact Hausdorff spaces. Let $h : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ be Borel functions. Then $h \otimes g$ is a Borel function on $X \times Y$.

Proposition 2.3.6. Let X and Y be σ-compact, second countable, locally compact Hausdorff spaces. Let μ and ν be Radon measures on X and Y, respectively.
(a) $h \in L^2(X, \mu)$ and $g \in L^2(Y, \nu)$ imply $h \otimes g \in L^2(X \times Y, \mu \times \nu)$.
(b) If $h, h' \in L^2(X, \mu)$ and $g, g' \in L^2(Y, \nu)$, then

$$\langle (h \otimes g), (h' \otimes g') \rangle_{L^2(X \times Y, \mu \times \nu)} = \langle h, h' \rangle_{L^2(X,\mu)} \langle g, g' \rangle_{L^2(Y,\nu)}.$$

The Hilbert space tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is carefully defined in Section 2.6 of [19]. We will not repeat this definition here. In Example 2.6.11 of [19], they prove that $L^2(X \times Y, \mu \times \nu)$ is Hilbert space isomorphic with $L^2(X, \mu) \otimes L^2(Y, \nu)$ is a natural way. In fact, combining Example 2.6.11 with Theorem 2.6.4(iii) of [19] gives the following.

Proposition 2.3.7. Let X and Y be σ-compact, second countable, locally compact Hausdorff spaces. Let μ and ν be Radon measures on X and Y, respectively. If $\{h_i : i \in I\}$ is an orthonormal basis of $L^2(X, \mu)$ and $\{g_j : j \in J\}$ is an orthonormal basis of $L^2(Y, \nu)$, then $\{h_i \otimes g_j : (i, j) \in I \times J\}$ is an orthonormal basis of $L^2(X \times Y, \mu \times \nu)$.

2.4 Unitary Representations

In this section, we introduce the definitions and basic properties of unitary representations of locally compact groups. Let G be a locally compact group.

Definition 2.4.1. A *unitary representation* of G is a continuous homomorphism of G into $\mathcal{U}(\mathcal{H}_\pi)$, the unitary group of a Hilbert space $\mathcal{H}_\pi$ equipped with the weak operator topology. The Hilbert space $\mathcal{H}_\pi$ is called the *Hilbert space of* π and $d_\pi = \dim(\mathcal{H}_\pi)$ is called the *dimension of* π

That is, π is a unitary representation of G on the Hilbert space $\mathcal{H}_\pi$ if the following all hold.

1. $\pi(x)$ is a unitary operator on $\mathcal{H}_\pi$, for all $x \in G$.

2. $\pi(xy) = \pi(x)\pi(y)$, for all $x, y \in G$.

3. For any $\xi, \eta \in \mathcal{H}_\pi$, the map $x \to \langle \pi(x)\xi, \eta\rangle$ of G into $\mathbb{C}$ is continuous.

These properties imply that $\pi(e) = I$, where e is the identity in G and I is the identity operator on $\mathcal{H}_\pi$, as well as $\pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$, for any $x \in G$.

Example 8. The *left regular representation* of G acts on the Hilbert space $L^2(G)$. For $x \in G$ and $f \in L^2(G)$, define $\lambda_G(x)f$ on G by,

$$\lambda_G(x)f(y) = f(x^{-1}y), \text{ for a.e. } y \in G.$$

Then $\|\lambda_G(x)f\|_2^2 = \int_G |f(x^{-1}y)|^2\, dy = \int_G |f(y)|^2\, dy = \|f\|_2^2$, by left invariance of left Haar measure. It is clear that $\lambda_G(xy) = \lambda_G(x)\lambda_G(y)$, for all $x, y \in G$, and that $\lambda_G(e) = I$, the identity operator on $L^2(G)$. Therefore, $\lambda_G(x)$ is an invertible isometry; that is $\lambda_G(x) \in \mathcal{U}\big(L^2(G)\big)$. So properties (1) and (2) above hold for λ_G. For property (3), let $f, g \in L^2(G)$. For any $x, y \in G$,

$$|\langle \lambda_G(x)f, g\rangle - \langle \lambda_G(y)f, g\rangle| \le \|\lambda_G(x)f - \lambda_G(y)f\|_2\|g\|_2 = \|\lambda_G(y^{-1}x)f - f\|_2\|g\|_2.$$

So it is sufficient to show that $\lim_{z\to e} \|\lambda_G(z)f - f\|_2 = 0$. See Proposition 2.41 of [12] for a short and clear proof of this.

Fix a unitary representation π of a locally compact group G.

Definition 2.4.2. For $\xi, \eta \in \mathcal{H}_\pi$, let $\varphi^\pi_{\xi,\eta}(x) = \langle \pi(x)\xi, \eta\rangle$, for all $x \in G$. Then $\varphi^\pi_{\xi,\eta}$ is called a *coefficient function* of π.

Proposition 2.4.3. For any $\xi, \eta \in \mathcal{H}_\pi$, $\varphi^\pi_{\xi,\eta} \in C_b(G)$.

Proof. The continuity of each $\varphi^\pi_{\xi,\eta}$ is part of the definition of a unitary representation. Also, $|\varphi^\pi_{\xi,\eta}(x)| = |\langle \pi(x)\xi, \eta\rangle| \le \|\xi\| \cdot \|\eta\|$, for all $x \in G$. Therefore, $\varphi^\pi_{\xi,\eta}$ is a bounded continuous function on G. □

Definition 2.4.4. A subspace $\mathcal{K}$ of $\mathcal{H}_\pi$ is called π-*invariant* if $\pi(x)\xi \in \mathcal{K}$, for all $\xi \in \mathcal{K}$ and all $x \in G$.

If a subspace $\mathcal{K}$ is π-invariant, then $\overline{\mathcal{K}}$ is also π-invariant by the continuity of π. If $\mathcal{K}$ is π-invariant and $\eta \in \mathcal{K}^\perp$, then, for any $\xi \in \mathcal{K}$,

$$\langle \pi(x)\eta, \xi\rangle = \langle \eta, \pi(x^{-1})\xi\rangle = 0,$$

since $\pi(x^{-1})\xi \in \mathcal{K}$, for any $x \in G$. Thus, $\pi(x)\eta \in \mathcal{K}^\perp$. That is, $\mathcal{K}^\perp$ is π-invariant whenever $\mathcal{K}$ is a π-invariant subspace of $\mathcal{H}_\pi$.

Definition 2.4.5. If $\mathcal{K}$ is a closed π-invariant subspace of $\mathcal{H}_\pi$, define $\pi^{\mathcal{K}}$ on G by $\pi^{\mathcal{K}}(x) = \pi(x)_{|\mathcal{K}}$, for each $x \in G$. Then $\pi^{\mathcal{K}}$ is a unitary representation of G on the Hilbert space $\mathcal{K}$. We call $\pi^{\mathcal{K}}$ a *subrepresentation of* π.

Definition 2.4.6. The unitary representation π is called *irreducible* if $\{0\}$ and $\mathcal{H}_\pi$ are the only closed π-invariant subspaces of $\mathcal{H}_\pi$. Otherwise, π is called *reducible.*

Definition 2.4.7. Let π and ρ be unitary representations of G. An *intertwining operator* of π with ρ is a $T \in \mathcal{B}(\mathcal{H}_\pi, \mathcal{H}_\rho)$ such that

$$T\pi(x) = \rho(x)T, \text{ for all } x \in G.$$

Then, we also say T *intertwines* π with ρ. Let $\mathcal{C}(\pi, \rho)$ denote the set of all intertwining operators of π with ρ. Let $\mathcal{C}(\pi) = \mathcal{C}(\pi, \pi)$.

Definition 2.4.8. Let π and ρ be unitary representations of G. If there exists a unitary map $U \in \mathcal{C}(\pi, \rho)$, then we say π is *equivalent* to ρ. This is denoted by $\pi \sim \rho$.

Note that

$$\mathcal{C}(\pi) = \{T \in \mathcal{B}(\mathcal{H}_\pi) : T\pi(x) = \pi(x)T, \text{ for all } x \in G\} = \{\pi(x) : x \in G\}',$$

the *commutant* of $\pi(G)$ in $\mathcal{B}(\mathcal{H}_\pi)$. Let $\mathcal{A}_\pi$ denote the linear span of $\{\pi(x) : x \in G\}$, which is a $*$-subalgebra of $\mathcal{B}(\mathcal{H}_\pi)$. Then $\mathcal{A}'_\pi = \pi(G)' = \mathcal{C}(\pi)$.

Proposition 2.4.9. Let $\mathcal{A}_\pi$ denote the linear span of $\{\pi(x) : x \in G\}$. Then
(a) $\mathcal{C}(\pi)$ is a von Neumann algebra on $\mathcal{H}_\pi$.
(b) $\mathcal{C}(\pi)'$ is the weak operator topology closure of $\mathcal{A}_\pi$.

Proof. Statement (a) is Proposition 2.2.24 and statement (b) is the von Neumann Double Commutant Theorem. See Theorem 5.3.1 of [19]. □

For $\mathcal{K}$ is a closed subspace of $\mathcal{H}_\pi$, $P_\mathcal{K}$ denotes the orthogonal projection of $\mathcal{H}_\pi$ onto $\mathcal{K}$.

Proposition 2.4.10. Let $\mathcal{K}$ be a closed subspace of $\mathcal{H}_\pi$. Then $\mathcal{K}$ is π-invariant if and only if $P_\mathcal{K} \in \mathcal{C}(\pi)$.

Proof. Suppose $\mathcal{K}$ is π-invariant. Then $\mathcal{K}^\perp$ is π-invariant as well. Let $P_{\mathcal{K}^\perp}$ denote the orthogonal projection onto $\mathcal{K}^\perp$. Then $I = P_\mathcal{K} + P_{\mathcal{K}^\perp}$. For any $\xi \in \mathcal{H}_\pi$ and $x \in G$, $\pi(x)P_\mathcal{K}\xi \in \mathcal{K}$ and $\pi(x)P_{\mathcal{K}^\perp}\xi \in \mathcal{K}^\perp$. Thus,

$$P_\mathcal{K}\pi(x)\xi = P_\mathcal{K}\pi(x)\big(P_\mathcal{K}\xi + P_{\mathcal{K}^\perp}\xi\big) = P_\mathcal{K}\pi(x)P_\mathcal{K}\xi + P_\mathcal{K}\pi(x)P_{\mathcal{K}^\perp}\xi = \pi(x)P_\mathcal{K}\xi.$$

Therefore, $P_\mathcal{K}\pi(x) = \pi(x)P_\mathcal{K}$, for all $x \in G$. That is, $P_\mathcal{K} \in \mathcal{C}(\pi)$.

Conversely, suppose $P_\mathcal{K} \in \mathcal{C}(\pi)$. Let $\xi \in \mathcal{K}$. Then, for any $x \in G$,

$$\pi(x)\xi = \pi(x)P_\mathcal{K}\xi = P_\mathcal{K}\pi(x)\xi.$$

Thus $\pi(x)\xi \in \mathcal{K}$, for all $x \in G$. So $\mathcal{K}$ is π-invariant. □

We can now collect together several conditions that are equivalent to the irreducibility of π.

Theorem 2.4.11. Let π be a unitary representation of a locally compact group G. The following are equivalent:

(a) π is irreducible.
(b) $\mathcal{C}(\pi) = \mathbb{C}I$.
(c) $\pi(G)'' = \mathcal{B}(\mathcal{H}_\pi)$.
(d) For $\xi, \eta \in \mathcal{H}_\pi$, if $\varphi^\pi_{\xi,\eta}(x) = 0$, for all $x \in G$, then either $\xi = 0$ or $\eta = 0$.

Proof. (a) $\Leftrightarrow$ (b) By Proposition 2.4.10, π is irreducible if and only if 0 and I are the only projections in the von Neumann algebra $\mathcal{C}(\pi)$. By Proposition 2.2.24, this is equivalent to $\mathcal{C}(\pi) = \mathbb{C}I$.

(b) $\Leftrightarrow$ (c) Since $\pi(G)'' = \mathcal{C}(\pi)'$, $\mathcal{C}(\pi) = \mathbb{C}I$ is equivalent to $\pi(G)'' = \mathcal{B}(\mathcal{H}_\pi)$, by Theorem 5.3.1 of [19].

(d) $\Leftrightarrow$ (a) Assume (d) and suppose $\mathcal{K}$ is a nontrivial closed π-invariant subspace of $\mathcal{H}_\pi$. Let $\xi \in \mathcal{K}$, $\xi \neq 0$, and let $\eta \in \mathcal{K}^\perp$, $\eta \neq 0$. Then $\varphi^\pi_{\xi,\eta}(x) = \langle \pi(x)\xi, \eta\rangle = 0$, for all $x \in G$. This contradiction of (d) implies π must be irreducible when (d) holds. Suppose π is irreducible. Let $\eta \in \mathcal{H}_\pi$, $\eta \neq 0$ and let $\mathcal{K} = \{\pi(x)\eta : x \in G\}^\perp$. If $\xi \in \mathcal{K}$, then, for any $y \in G$,

$$\langle \pi(y)\xi, \pi(x)\eta\rangle = \langle \xi, \pi(y^{-1}x)\eta\rangle = 0,$$

for all $x \in G$. Thus $\pi(y)\xi \in \mathcal{K}$, for all $y \in G$ and any $\xi \in \mathcal{K}$. Thus, $\mathcal{K}$ is a closed π-invariant subspace of $\mathcal{H}_\pi$. Since π is irreducible and $\eta \neq 0$, it must be that $\mathcal{K} = \{0\}$. This implies (d) □

Definition 2.4.12. Let G be a locally compact group. The set of all equivalence classes of irreducible unitary representations of G is denoted $\widehat{G}$.

We will make use of the contragredient of a unitary representation. If $\mathcal{H}$ is any Hilbert space, let $\mathcal{H}^*$ denote the vector space over $\mathbb{C}$ consisting of the same elements as $\mathcal{H}$ and the vector space operations of addition $(\xi, \eta) \to \xi + \eta$ and scalar multiplication $(\alpha, \xi) \to \alpha \cdot \xi = \overline{\alpha}\xi$. Put an inner product on $\mathcal{H}^*$ by defining

$$\langle \xi, \eta\rangle_{\mathcal{H}^*} = \langle \eta, \xi\rangle_{\mathcal{H}}, \text{ for all } \xi, \eta \in \mathcal{H}^*.$$

Then $\mathcal{H}^*$ is a Hilbert space as well. For each $\xi \in \mathcal{H}^*$ define $\varphi_\xi : \mathcal{H} \to \mathbb{C}$ by

$$\varphi_\xi(\nu) = \langle \nu, \xi\rangle_{\mathcal{H}}, \text{ for all } \xi \in \mathcal{H}.$$

Then $\xi \to \varphi_\xi$ is an isometric isomorphism of $\mathcal{H}^*$ with the Banach space dual of $\mathcal{H}$. Thus, we can refer to $\mathcal{H}^*$ as the *dual Hilbert space to* $\mathcal{H}$.

Definition 2.4.13. Let π be a unitary representation of G. For each $x \in G$, define $\overline{\pi}(x)$ on $\mathcal{H}^*_\pi$ by setting $\overline{\pi}(x)\eta = \pi(x)\eta$, for each $\eta \in \mathcal{H}^*$.

Note that $\overline{\pi}(x)$ is additive, since addition is the same in $\mathcal{H}^*_\pi$ as in $\mathcal{H}_\pi$. For $\alpha \in \mathbb{C}$ and $\eta \in \mathcal{H}^*_\pi$, $\overline{\pi}(x)(\alpha \cdot \eta) = \pi(x)(\overline{\alpha}\eta) = \overline{\alpha}\pi(x)\eta = \alpha \cdot \pi(x)\eta = \alpha \cdot \overline{\pi}(x)\eta$. So $\overline{\pi}(x)$ is a linear mapping on $\mathcal{H}^*_\pi$. It is similarly verified that $\overline{\pi}(x)$ is a unitary operator on $\mathcal{H}^*_\pi$ and that $x \to \overline{\pi}(x)$ is a unitary representation.

Definition 2.4.14. If π is a unitary representation of G, then the representation $\overline{\pi}$ is called the *contragredient* of π. We let $\mathcal{H}_{\overline{\pi}} = \mathcal{H}^*_\pi$.

It is clear that $\overline{\pi}$ is irreducible if and only if π is irreducible.

Example 9. Suppose π is a one dimensional representation of G. Fix $\eta \in \mathcal{H}_\pi$ with $\|\eta\| = 1$. Then $\mathcal{H}_\pi = \{\alpha\eta : \alpha \in \mathbb{C}\}$ and $\|\alpha\eta\| = |\alpha|$, for any $\alpha \in \mathbb{C}$. For any $x \in G$, $\|\pi(x)\eta\| = \|\eta\| = 1$. Thus $\pi(x)\eta = \alpha_x\eta$, for some $\alpha_x \in \mathbb{C}$ with $|\alpha_x| = 1$. Note that $x \to \alpha_x$ is then a homomorphism of G into $\mathbb{T}$. On the other hand, $\overline{\pi}(x)\eta = \pi(x)\eta = \alpha_x\eta = \overline{\alpha_x} \cdot \eta$. Thus, while $\pi(x)\xi = \alpha_x\xi$, for all $\xi \in \mathcal{H}_\pi$, we have $\overline{\pi}(x)\xi = \overline{\alpha_x} \cdot \xi$, for any $\xi \in \mathcal{H}_{\overline{\pi}}$.

For a general unitary representation π, the coefficient functions of $\overline{\pi}$ are complex conjugates of the coefficient functions of π.

Proposition 2.4.15. Let π be a unitary representation of a locally compact group G. Let $\eta, \xi \in \mathcal{H}_{\overline{\pi}}$ and consider them as members of $\mathcal{H}_\pi$ as well. Then $\varphi^{\overline{\pi}}_{\eta,\xi} = \overline{\varphi^{\pi}_{\eta,\xi}}$.

Proof. For any $x \in G$,
$$\varphi^{\overline{\pi}}_{\eta,\xi}(x) = \langle \overline{\pi}(x)\eta, \xi\rangle_{\mathcal{H}_{\overline{\pi}}} = \langle \xi, \pi(x)\eta\rangle_{\mathcal{H}_\pi} = \overline{\langle \pi(x)\eta, \xi\rangle_{\mathcal{H}_\pi}} = \overline{\varphi^{\pi}_{\eta,\xi}}(x). \qquad \square$$

The linear map $\xi \to \langle \xi, \pi(x)\eta\rangle_{\mathcal{H}_\pi}$ plays an important role in this study.

Definition 2.4.16. If π is a unitary representation of G and $\eta \in \mathcal{H}_\pi$, let

$$V_\eta\xi(x) = \langle \xi, \pi(x)\eta\rangle_{\mathcal{H}_\pi} = \varphi^{\overline{\pi}}_{\eta,\xi}(x), \text{ for all } x \in G,$$

and each $\xi \in \mathcal{H}_\pi$.

Proposition 2.4.17. Let π be a unitary representation of a locally compact group G. For each $\eta \in \mathcal{H}_\pi$, V_η is a bounded linear map of $\mathcal{H}_\pi$ into $C_b(G)$. The representation π is irreducible if and only if V_η is injective, for every nonzero $\eta \in \mathcal{H}_\pi$.

Proof. For any $\xi \in \mathcal{H}_\pi$, $V_\eta\xi = \varphi^{\overline{\pi}}_{\eta,\xi} \in C_b(G)$ and $\|V_\eta\xi\|_\infty \leq \|\eta\|_{\mathcal{H}_\pi}\|\xi\|_{\mathcal{H}_\pi}$. It is easily checked that V_η is linear. Thus, it is a bounded linear map of $\mathcal{H}_\pi$ into $C_b(G)$. The representation π is irreducible if and only if $\overline{\pi}$ is irreducible and this is equivalent to V_η being injective for each nonzero $\eta \in \mathcal{H}_\pi$ by the equivalence of (a) and (d) in Theorem 2.4.11. $\square$

2.5 Square-integrable Representations

If G is a non-compact locally compact group and π is an irreducible representation of G, the coefficient functions of π or $\overline{\pi}$ are always bounded, but they do not need to be in $L^2(G)$. For example, if $G = \mathbb{R}$ and $\chi_\omega : x \to e^{2\pi i\omega x}$ is a typical irreducible unitary representation of $\mathbb{R}$, then any coefficient function of χ_ω is just a constant multiple of χ_ω, which will not be in $L^2(\mathbb{R})$ unless the constant is 0. If it happens that an irreducible unitary representation has a nonzero coefficient function that lies in $L^2(G)$, then there are significant implications. These properties are introduced in this section.

Definition 2.5.1. Let π be an irreducible representation of a locally compact group G. For $\eta \in \mathcal{H}_\pi$, $V_\eta\xi(x) = \langle \xi, \pi(x)\eta \rangle$, for $x \in G$ and $\xi \in \mathcal{H}_\pi$. If there exist nonzero $\eta, \xi \in \mathcal{H}_\pi$ such that $V_\eta\xi \in L^2(G)$, then π is called a *square-integrable representation* of G.

Note that π is square-integrable if and only if $\overline{\pi}$ is square-integrable and $V_\eta\xi = \varphi^{\overline{\pi}}_{\eta,\xi}$. For a proof of the following proposition, see the proof of part (b) of Theorem 2.25 in [14].

Proposition 2.5.2. If π is a square-integrable representation of a locally compact group G and $\eta \in \mathcal{H}_\pi$ is such that $V_\eta\xi \in L^2(G)$, for one nonzero $\xi \in \mathcal{H}_\pi$, then $V_\eta\xi \in L^2(G)$, for all $\xi \in \mathcal{H}_\pi$.

Definition 2.5.3. Let π be a square-integrable representation of a locally compact group G. Let $\mathcal{D}_\pi$ denote the set of all $\eta \in \mathcal{H}_\pi$ such that $V_\eta\eta \in L^2(G)$ (equivalently, $V_\eta\xi \in L^2(G)$, for all $\xi \in \mathcal{H}_\pi$).

Proposition 2.5.4. Let π be a square-integrable representation of a locally compact group G. Then $\mathcal{D}_\pi$ is a dense subspace of $\mathcal{H}_\pi$.

Proof. Suppose $\eta_1, \eta_2 \in \mathcal{D}_\pi$, then, for any $\xi \in \mathcal{H}_\pi$, $V_{\eta_1+\eta_2}\xi = V_{\eta_1}\xi + V_{\eta_2}\xi \in L^2(G)$, by Proposition 2.5.2. Thus, $\eta_1 + \eta_2 \in \mathcal{D}_\pi$. Likewise, $\alpha\eta_1 \in \mathcal{D}_\pi$, for any $\alpha \in \mathbb{C}$. That is, $\mathcal{D}_\pi$ is a subspace of $\mathcal{H}_\pi$. Let η be a nonzero member of $\mathcal{D}_\pi$. For any $y \in G$ and any $\xi \in \mathcal{H}_\pi$, $V_{\pi(y)\eta}\xi(x) = V_\eta\xi(xy)$, for all $x \in G$. Thus,

$$\int_G \left|V_{\pi(y)\eta}\xi(x)\right|^2 d\mu_G(x) = \int_G \left|V_\eta\xi(xy)\right|^2 d\mu_G(x) = \Delta_G(y)^{-1} \int_G \left|V_\eta\xi(x)\right|^2 d\mu_G(x) < \infty.$$

Therefore, $\mathcal{D}_\pi$ is a π-invariant subspace of $\mathcal{H}_\pi$. Then $\overline{\mathcal{D}_\pi}$ is a π-invariant closed subspace, which must be $\mathcal{H}_\pi$, since π is irreducible. Thus $\mathcal{D}_\pi$ is dense. □

If G is a compact group, $C_b(G) = C(G) \subseteq L^2(G)$. So any irreducible representation π of G is square-integrable. In a later section, we describe the Peter-Weyl theory for compact groups. A part of that theory is a theorem called the *orthogonality relations* for irreducible representations of compact groups. A key relation (see Corollary 2.6.8) states: If π is an irreducible unitary representation of a compact group G and if $\eta, \eta', \xi, \xi' \in \mathcal{H}_\pi$, then

$$\langle V_\eta\xi, V_{\eta'}\xi' \rangle_{L^2(G)} = \frac{1}{d_\pi} \langle \xi, \xi' \rangle_{\mathcal{H}_\pi} \langle \eta', \eta \rangle_{\mathcal{H}_\pi}. \tag{2.5}$$

Duflo and Moore [9] established an important generalization of (2.5). The statement we give is derived from Theorem 2.25 of [14].

Theorem 2.5.5. **[Duflo-Moore]** Let π be a square-integrable representation of a locally compact group G. There exists a positive, self-adjoint, densely defined, operator C_π on $\mathcal{H}_\pi$ with $\mathrm{dom}(C_\pi) = \mathcal{D}_\pi$ and such that, for $\xi, \xi' \in \mathcal{H}_\pi$ and $\eta, \eta' \in \mathcal{D}_\pi$,

$$\langle V_\eta\xi, V_{\eta'}\xi' \rangle_{L^2(G)} = \langle \xi, \xi' \rangle_{\mathcal{H}_\pi} \langle C_\pi\eta', C_\pi\eta \rangle_{\mathcal{H}_\pi}.$$

Moreover, C_π satisfies the *semi-invariance* property $\pi(x)C_\pi\pi(x)^* = \Delta_G(x)^{1/2}C_\pi$, for all $x \in G$.

Definition 2.5.6. Let π be a square-integrable representation of a locally compact group G. The operator C_π of Theorem 2.5.5 is called the *Duflo-Moore operator* of π.

Duflo and Moore (Lemma 1 of [9]) also showed that there is a strong uniqueness of the Duflo-Moore operator of a square-integrable representation that is determined by the semi-invariance property.

Proposition 2.5.7. Let π be a square-integrable representation of a locally compact group G. Suppose T is a positive, self-adjoint, densely defined, operator on $\mathcal{H}_\pi$ that satisfies $\pi(x)T\pi(x)^* = \Delta_G(x)^{1/2}T$, for all $x \in G$. Then $\mathrm{dom}(T) = \mathcal{D}_\pi$ and $T = C_\pi$.

Grossmann, Morlet and Paul [15] showed how the Duflo-Moore Theorem leads to generalizations of the continuous wavelet transform and the associated reconstruction formula. The process follows.

Let π be a square-integrable representation of a locally compact group G. Select $\eta \in \mathcal{D}_\pi$ so that $\|C_\pi\eta\|_{\mathcal{H}_\pi} = 1$. Then Theorem 2.5.5 implies $V_\eta : \mathcal{H}_\pi \to L^2(G)$ is an isometry. Thus $\mathcal{K}_\eta = V_\eta\mathcal{H}_\pi$ is a closed subspace of $L^2(G)$.

Proposition 2.5.8. With the above notation, $\mathcal{K}_\eta$ is a λ_G-invariant closed subspace of $L^2(G)$ and V_η intertwines π with λ_G.

Proof. Let $f \in \mathcal{K}_\eta$ and $y \in G$. There is a unique $\xi \in \mathcal{H}_\pi$ such that $V_\eta\xi = f$. Then

$$V_\eta\pi(y)\xi(x) = \langle\pi(y)\xi, \pi(x)\eta\rangle_{\mathcal{H}_\pi} = \langle\xi, \pi(y^{-1}x)\eta\rangle_{\mathcal{H}_\pi} = V_\eta\xi(y^{-1}x) = \lambda_G(y)f(x),$$

for all $x \in G$. Thus, $\lambda_G(y)f \in \mathcal{K}_\eta$, for each $y \in G$ and $f \in \mathcal{K}_\eta$. That is, $\mathcal{K}_\eta$ is λ_G-invariant. Also, the above calculation shows that V_η intertwines π with λ_G. □

Note that V_η, considered as a unitary map from $\mathcal{H}_\pi$ to $\mathcal{K}_\eta$ intertwining π with $\lambda_G^{\mathcal{K}_\eta}$, shows that $\pi \sim \lambda_G^{\mathcal{K}_\eta}$. An expression for the inverse $V_\eta^* : \mathcal{K}_\eta \to \mathcal{H}_\pi$ is provided by a reconstruction formula.

Proposition 2.5.9. Let π be a square-integrable representation of a locally compact group G and let $\eta \in \mathcal{D}_\pi$ be such that $\|C_\pi\eta\|_{\mathcal{H}_\pi} = 1$. Then, for any $\xi \in \mathcal{H}_\pi$,

$$\xi = \int_G V_\eta\xi(x)\pi(x)\eta \, d\mu_G(x), \text{ weakly in } \mathcal{H}_\pi.$$

Proof. Fix $\xi \in \mathcal{H}_\pi$. For any $\xi' \in \mathcal{H}_\pi$,

$$\begin{aligned}\langle\xi, \xi'\rangle_{\mathcal{H}_\pi} = \langle V_\eta\xi, V_\eta\xi'\rangle_{L^2(G)} &= \int_G V_\eta\xi(x)\overline{\langle\xi', \pi(x)\eta\rangle_{\mathcal{H}_\pi}} d\mu_G(x) \\ &= \int_G V_\eta\xi(x)\langle\pi(x)\eta, \xi'\rangle_{\mathcal{H}_\pi} d\mu_G(x) = \int_G \langle V_\eta\xi(x)\pi(x)\eta, \xi'\rangle_{\mathcal{H}_\pi} \, d\mu_G(x).\end{aligned}$$

Thus, $\xi = \int_G V_\eta\xi(x)\pi(x)\eta \, d\mu_G(x)$, weakly in $\mathcal{H}_\pi$. □

Thus, if $g \in \mathcal{K}_\eta$, then there exists a unique $\xi \in \mathcal{H}_\pi$ such that, for all $\xi' \in \mathcal{H}_\pi$, we have $\langle \xi, \xi' \rangle_{\mathcal{H}_\pi} = \int_G \langle g(x)\pi(x)\eta, \xi' \rangle_{\mathcal{H}_\pi} \, d\mu_G(x)$. Then $V_\eta^* g = \xi$.

Example 10. The group of all invertible affine transformations of $\mathbb{R}$ can be written as $G_1 = \{[x, a] : x \in \mathbb{R}, a \in \mathbb{R}^*\}$ with group product given by $[x, a][x', a'] = [x + ax', aa']$, identity $[0, 1]$, and inverses given by $[x, a]^{-1} = [-a^{-1}x, a^{-1}]$. There is a good discussion of this group in [17] and it will be studied in more detail later in this thesis. Let π be the unitary representation of G_1 on $L^2(\mathbb{R})$ given by, for $[x, a] \in G_1$ and $f \in L^2(\mathbb{R})$,

$$\pi[x, a]f(y) = |a|^{-1/2} f\left(\tfrac{y-x}{a}\right), \text{ for all } y \in \mathbb{R}.$$

It turns out that π is irreducible and, in fact, square-integrable. See [15] or [17] for a proof. Or see [4] for a more general result. To describe the Duflo-Moore operator C_π, we need the Fourier transform, which is introduced carefully in the next section. Here, we will assume it is familiar. Write $\widehat{\mathbb{R}}$ for frequency space and let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\widehat{\mathbb{R}})$ denote the unitary map such that $\mathcal{F}f(\omega) = \widehat{f}(\omega) = \int_{\mathbb{R}} f(y)e^{-2\pi i y\omega} dy$, for all $\omega \in \widehat{\mathbb{R}}$ and $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. The Duflo-Moore operator is a multiplication operator on the Fourier transform side. For any $\xi \in L^2(\widehat{\mathbb{R}})$, let

$$T\xi(\omega) = |\omega|^{-1/2}\xi(\omega), \text{ for a.e. } \omega \in \widehat{\mathbb{R}}.$$

Then T is a positive, self-adjoint, operator on $L^2(\widehat{\mathbb{R}})$ and $C_\pi = \mathcal{F}^{-1}T\mathcal{F}$. Thus, $w \in L^2(\mathbb{R})$ satisfying $\|C_\pi w\|_2 = 1$ means the same as $\int_{\widehat{\mathbb{R}}} |\widehat{w}(\omega)|^2 \frac{d\omega}{|\omega|} = 1$. For such a w, define $V_w f$, for any $f \in L^2(\mathbb{R})$, by

$$V_w f[x, a] = \langle f, \pi[x, a]w \rangle_{L^2(\mathbb{R})}, \text{ for all } [x, a] \in G_1. \tag{2.6}$$

Proposition 2.5.9 tells us that, for any $f \in L^2(\mathbb{R})$,

$$f = \int_{\mathbb{R}} \int_{\mathbb{R}} V_w f[x, a]\pi[x, a]w \tfrac{dx\, da}{a^2}, \text{ weakly in } L^2(\mathbb{R}). \tag{2.7}$$

Starting in the 1980s, the map V_w has been called a *continuous wavelet transform* and (2.7), the *continuous wavelet reconstruction formula*. But (2.7) was known earlier and is version of the *Calderón Reproducing Formula* in one dimension. A function $w \in L^2(\mathbb{R})$ satisfying $\int_{\widehat{\mathbb{R}}} |\widehat{w}(\omega)|^2 \frac{d\omega}{|\omega|} = 1$ is called a *wavelet* for the continuous wavelet transform on $\mathbb{R}$.

In analogy with the situation in the above example, the term wavelet is used more generally.

Definition 2.5.10. Let π be a square-integrable representation of a locally compact group. A π-*wavelet* is a vector $\eta \in \mathcal{H}_\pi$ such that $\|C_\pi \eta\|_{\mathcal{H}_\pi} = 1$, where C_π is the Duflo-Moore operator of π.

Remark. If π is a square-integrable representation of G and $\eta' \in \mathcal{D}_\pi$ and $\eta' \neq 0$, then $\eta = \frac{1}{\|C_\pi \eta'\|_{\mathcal{H}_\pi}} \eta'$ is a π-wavelet. So the linear span of the set of all π-wavelets is the dense subspace $\mathcal{D}_\pi$ of $\mathcal{H}_\pi$.

2.6 Compact Groups and the Peter-Weyl Theorem

In this section we will give a brief summary of the *Peter-Weyl* theorem. This theorem shows exactly how the left regular representation of a compact group is a direct sum of irreducible representations. The presentation draws on [12]. We need some fundamental facts about compact groups.

If G is a compact group, the Haar measure of G is finite. We normalize the Haar measure so that $\mu_G(G) = 1$. Note that G is unimodular when G is compact. Since $\mu_G(G) = 1$, $C(G) \subseteq L^2(G)$. Let π be any unitary representation of G. For $\xi, \eta \in \mathcal{H}_\pi$, $V_\eta \xi \in C(G) \subseteq L^2(G)$, where $V_\eta \xi(x) = \langle \xi, \pi(x)\eta \rangle$, for all $x \in G$. Recall that, if π and ρ are unitary representations of G, then $T \in \mathcal{C}(\pi, \rho)$ that means $T \in \mathcal{B}(\mathcal{H}_\pi, \mathcal{H}_\rho)$ and $T\pi(x) = \rho(x)T$, for all $x \in G$.

Proposition 2.6.1. Let π be a unitary representation of a compact group G and let $\eta \in \mathcal{H}_\pi$.

(a) $V_\eta : \mathcal{H}_\pi \to L^2(G)$ is a bounded linear map.

(b) V_η intertwines π with λ_G. That is, $V_\eta \in \mathcal{C}(\pi, \lambda_G)$.

(c) If π is irreducible and $\eta \neq 0$, then V_η is injective.

Proof. (a) Let $\eta \in \mathcal{H}_\pi$. Then V_η is linear because the inner product is linear in the first argument. For $\xi \in \mathcal{H}_\pi$,

$$\begin{aligned}\int_G |V_\eta \xi(x)|^2 d\mu_G(x) &= \int_G |\langle \xi, \pi(x)\eta \rangle|^2 d\mu_G(x) \\ &\leq \int_G \|\xi\|^2 \|\eta\|^2 d\mu_G(x) = \|\xi\|^2 \|\eta\|^2.\end{aligned}$$

Thus,$V_\eta \in \mathcal{B}\big(\mathcal{H}_\pi, L^2(G)\big)$ and $\|V_\eta\| \leq \|\eta\|$.

(b) For any $y \in G$ and $\xi \in \mathcal{H}_\pi$,

$$\begin{aligned}\big(V_\eta \pi(y)\xi\big)(x) &= \langle \pi(y)\xi, \pi(x)\eta \rangle = \langle \xi, \pi(y)^* \pi(x)\eta \rangle \\ &= \langle \xi, \pi(y^{-1}x)\eta \rangle = V_\eta \xi(y^{-1}x) = \lambda_G(y) V_\eta \xi(x),\end{aligned}$$

for any $x \in G$. Thus, $V_\eta \pi(y) = \lambda_G(y) V_\eta$, for any $y \in G$. That is, V_η intertwines π with λ_G.

(c) This follows from Proposition 2.4.17. □

In particular, any irreducible representation of G is square-integrable. We state this in a proposition together with several other basic properties of representations of compact groups. See Theorem 5.2 of [12] for the proofs of parts (b) and (c).

Proposition 2.6.2. Let G be a compact group.

(a) Any irreducible representation of G is square-integrable.

(b) Any irreducible representation of G is finite dimensional.

(c) Any unitary representation of G is a direct sum of irreducible representations.

Recall that a *coefficient function* of a unitary representation, π, is a function of the form $\varphi^{\pi}_{\eta,\xi}$, for $\eta, \xi \in \mathcal{H}_\pi$, where $\varphi^{\pi}_{\eta,\xi}(x) = \langle \pi(x)\eta, \xi \rangle$, for all $x \in G$. Note that $\varphi^{\pi}_{\eta,\xi} = \overline{V_\eta \xi}$. Let $\mathcal{E}_\pi$ denote the linear span of all coefficient functions of π. That is, $\mathcal{E}_\pi =$ linear span$\{\varphi^{\pi}_{\eta,\xi} : \xi, \eta \in \mathcal{H}_\pi\}$. It is a subspace of $C(G)$ and hence subspace of $L^2(G)$. Suppose that π' is a unitarily equivalent to π. That is, there is a unitary operator $U \in \mathcal{C}(\pi, \pi')$. Then, for $\eta, \xi \in \mathcal{H}_\pi$,

$$\varphi^{\pi}_{\eta,\xi}(x) = \langle \pi(x)\eta, \xi \rangle_{\mathcal{H}_\pi} = \langle U\pi(x)\eta, U\xi \rangle_{\mathcal{H}_{\pi'}} = \langle \pi'(x)U\eta, U\xi \rangle_{\mathcal{H}_{\pi'}} = \varphi^{\pi'}_{U\eta,U\xi}(x),$$

for any $x \in G$. Thus any coefficient function of π is also a coefficient function of π' and conversely. That is, $\mathcal{E}_\pi$ depends only on the unitary equivalence class of π. Moreover, it is invariant under left and right translations.

Proposition 2.6.3. Let π be a unitary representation of a compact group G. The subspace of $C(G)$ of the form $\mathcal{E}_\pi$ is invariant under both left and right translations.

Proof. Let $\varphi \in \mathcal{E}_\pi$. Then, there exist $n \in \mathbb{N}$, $\eta_j, \xi_j \in \mathcal{H}_\pi$ and $\alpha_j \in \mathbb{C}$, for $1 \le j \le n$, such that $\varphi = \sum_{j=1}^n \alpha_j \varphi^{\pi}_{\eta_j,\xi_j}$. For any $y \in G$,

$$\begin{aligned}\lambda_G(y)\varphi(x) = \varphi(y^{-1}x) &= \sum_{j=1}^n \alpha_j \varphi^{\pi}_{\eta_j,\xi_j}(y^{-1}x) = \sum_{j=1}^n \alpha_j \langle \pi(y^{-1}x)\eta_j, \xi_j \rangle \\ &= \sum_{j=1}^n \alpha_j \langle \pi(x)\eta_j, \pi(y)\xi_j \rangle = \sum_{j=1}^n \alpha_j \varphi^{\pi}_{\eta_j,\pi(y)\xi_j}(x),\end{aligned}$$

for all $x \in G$. Similarly, $\rho_G(y)\varphi = \sum_{j=1}^n \alpha_j \varphi^{\pi}_{\pi(y)\eta_j,\xi_j}$. □

The Schur orthogonality relations give information about the inner product of two coefficient functions of irreducible representations of G. The presentation here adapts material from Section 5.2 of [12].

Let π and π' be two representations of G. First, we introduce a method of creating elements of $\mathcal{C}(\pi, \pi')$. Start with any $A \in \mathcal{B}(\mathcal{H}_\pi, \mathcal{H}_{\pi'})$ and let $\widetilde{A} = \int_G \pi'(x^{-1})A\pi(x)\,dx$. This operator valued integral can be interpreted as follows: The operator $\widetilde{A}$ is the unique element of $\mathcal{B}(\mathcal{H}_\pi, \mathcal{H}_{\pi'})$ such that

$$\langle \widetilde{A}\xi, \xi' \rangle = \int_G \langle \pi'(x^{-1})A\pi(x)\xi, \xi' \rangle dx, \quad \text{for all } \xi \in \mathcal{H}_\pi, \xi' \in \mathcal{H}_{\pi'}.$$

Proposition 2.6.4. Let G be a compact group and let π, π' be representations of G. If $A \in \mathcal{B}(\mathcal{H}_\pi, \mathcal{H}_{\pi'})$, then $\widetilde{A} \in \mathcal{C}(\pi, \pi')$.

Proof. Let $y \in G$. For any $\xi \in \mathcal{H}_\pi$ and $\xi' \in \mathcal{H}_{\pi'}$,

$$\begin{aligned}
\langle \widetilde{A}\pi(y)\xi, \xi' \rangle &= \int_G \langle \pi'(x^{-1})A\pi(x)\pi(y)\xi, \xi' \rangle dx = \int_G \langle \pi'(x^{-1})A\pi(xy)\xi, \xi' \rangle dx \\
&= \int_G \langle \pi'((xy^{-1})^{-1})A\pi(x)\xi, \xi' \rangle dx = \int_G \langle \pi'(y)\pi'(x^{-1})A\pi(x)\xi, \xi' \rangle dx \\
&= \int_G \langle \pi'(x^{-1})A\pi(x)\xi, \pi'(y^{-1})\xi' \rangle dx = \langle \widetilde{A}\xi, \pi'(y^{-1})\xi' \rangle = \langle \pi'(y)\widetilde{A}\xi, \xi' \rangle.
\end{aligned}$$

Thus $\widetilde{A}\pi(y) = \pi'(y)\widetilde{A}$, for any $y \in G$. Therefore $\widetilde{A} \in \mathcal{C}(\pi, \pi')$. □

We immediately recognize that, if π and π' are irreducible representations of G such that π is not equivalent π', then $\widetilde{A} = 0$, for any $A \in \mathcal{B}(\mathcal{H}_\pi, \mathcal{H}_{\pi'})$.

In the case where $\pi = \pi'$ is a finite dimensional representation of G, the traces of $\widetilde{A}$ and A agree.

Lemma 2.6.5. Let G be a compact group and let π be a finite dimensional representation of G. If $A \in \mathcal{B}(\mathcal{H}_\pi)$, then $\mathrm{tr}(\widetilde{A}) = \mathrm{tr}(A)$.

Proof. Recall that $d_\pi = \dim(\mathcal{H}_\pi)$. Let $\{\xi_j : 1 \le j \le d_\pi\}$ be an orthonormal basis of $\mathcal{H}_\pi$. For each $x \in G$, $\{\pi(x)\xi_j : 1 \le j \le d_\pi\}$ is an orthonormal basis of $\mathcal{H}_\pi$. Then

$$\begin{aligned}
\mathrm{tr}(\widetilde{A}) &= \sum_{j=1}^{d_\pi} \langle \widetilde{A}\xi_j, \xi_j \rangle = \sum_{j=1}^{d_\pi} \int_G \langle \pi(x^{-1})A\pi(x)\xi_j, \xi_j \rangle dx \\
&= \int_G \left(\sum_{j=1}^{d_\pi} \langle A\pi(x)\xi_j, \pi(x)\xi_j \rangle \right) dx = \int_G \mathrm{tr}(A) dx = \mathrm{tr}(A),
\end{aligned}$$

using the fact that Haar measure on G is normalized. □

Now, let π and π' be any irreducible representations of G. Fix vectors $\eta \in \mathcal{H}_\pi$ and $\eta' \in \mathcal{H}_{\pi'}$ and define $A \in \mathcal{B}(\mathcal{H}_\pi, \mathcal{H}_{\pi'})$ by $A\xi = \langle \xi, \eta \rangle \eta'$, for all $\xi \in \mathcal{H}_\pi$. For $\xi \in \mathcal{H}_\pi$ and $\xi' \in \mathcal{H}_{\pi'}$,

$$\begin{aligned}
\langle \widetilde{A}\xi, \xi' \rangle &= \int_G \langle \pi'(x^{-1})A\pi(x)\xi, \xi' \rangle dx \\
&= \int_G \langle A\pi(x)\xi, \pi'(x)\xi' \rangle dx \\
&= \int_G \langle \langle \pi(x)\xi, \eta \rangle \eta', \pi'(x)\xi' \rangle dx \qquad (2.8) \\
&= \int_G \langle \pi(x)\xi, \eta \rangle \langle \eta', \pi'(x)\xi' \rangle dx \\
&= \int_G \varphi_{\xi,\eta}(x) \overline{\varphi_{\xi',\eta'}(x)}\, dx.
\end{aligned}$$

If π and π' are not equivalent, then $\widetilde{A} = 0$. This leads to the following proposition.

Proposition 2.6.6. Let G be a compact group and let π and π' be irreducible representations of G such that π is not equivalent to π'. Then any coefficient function of π is perpendicular to any coefficient function of π' as members of $L^2(G)$.

Let's look at the implications of (2.8) when $\pi = \pi'$. Thus, let π be an irreducible representation of G and let $\eta, \eta' \in \mathcal{H}_\pi$. Define $A \in \mathcal{B}(\mathcal{H}_\pi)$ by $A\xi = \langle \xi, \eta \rangle \eta'$, for all $\xi \in \mathcal{H}_\pi$. Then, using Lemma 2.6.5,

$$\operatorname{tr}(\widetilde{A}) = \operatorname{tr}(A) = \sum_{j=1}^{d_\pi} \langle A\xi_j, \xi_j \rangle = \sum_{j=1}^{d_\pi} \langle \langle \xi_j, \eta \rangle \eta', \xi_j \rangle = \sum_{j=1}^{d_\pi} \langle \xi_j, \eta \rangle \langle \eta', \xi_j \rangle = \langle \eta', \eta \rangle,$$

by the Parseval Identity. On the other hand, we know by Proposition 2.6.4 that $\widetilde{A} \in \mathcal{C}(\pi)$. Since π is irreducible, there is a constant $c \in \mathbb{C}$ such that $\widetilde{A} = cI$, where I is the identity operator on $\mathcal{H}_\pi$. Therefore, $\operatorname{tr}(\widetilde{A}) = cd_\pi$. Thus, $c = \frac{\langle \eta', \eta \rangle}{d_\pi}$ and $\widetilde{A} = \frac{\langle \eta', \eta \rangle}{d_\pi} I$.

Theorem 2.6.7. Let G be a compact group and let π be an irreducible representation of G. Let $\eta, \eta', \xi, \xi' \in \mathcal{H}_\pi$. Then

$$\int_G \varphi_{\xi,\eta}(x) \overline{\varphi_{\xi',\eta'}(x)}\, dx = \frac{1}{d_\pi} \langle \xi, \xi' \rangle \langle \eta', \eta \rangle.$$

Proof. This follows immediately from (2.8) and the fact that $\widetilde{A} = \frac{\langle \eta', \eta \rangle}{d_\pi} I$ when A is defined by $A\xi = \langle \xi, \eta \rangle \eta'$, for all $\xi \in \mathcal{H}_\pi$. □

The results in Proposition 2.6.6 and Theorem 2.6.7 together are called the *Schur Orthogonality Relations*. For the purpose of this thesis, Theorem 2.6.7 is restated in terms of the V_η maps.

Corollary 2.6.8. Let G be a compact group and let π be an irreducible representation of G. Let $\eta, \eta', \xi, \xi' \in \mathcal{H}_\pi$. Then

$$\langle V_\eta \xi, V_{\eta'} \xi' \rangle_{L^2(G)} = \frac{1}{d_\pi} \langle \xi, \xi' \rangle_{\mathcal{H}_\pi} \langle \eta', \eta \rangle_{\mathcal{H}_\pi}.$$

Proof. In the following calculation the fact that G is unimodular is used in the $x^{-1} \to x$ variable change.

$$\begin{aligned}
\int_G V_\eta \xi(x) \overline{V_{\eta'} \xi'(x)}\, dx &= \int_G \langle \xi, \pi(x)\eta \rangle_{\mathcal{H}_\pi} \overline{\langle \xi', \pi(x)\eta' \rangle_{\mathcal{H}_\pi}} dx \\
&= \int_G \langle \pi(x^{-1})\xi, \eta \rangle_{\mathcal{H}_\pi} \overline{\langle \pi(x^{-1})\xi', \eta' \rangle_{\mathcal{H}_\pi}} dx \\
&= \int_G \langle \pi(x)\xi, \eta \rangle_{\mathcal{H}_\pi} \overline{\langle \pi(x)\xi', \eta' \rangle_{\mathcal{H}_\pi}} dx \\
&= \int_G \varphi_{\xi,\eta}(x) \overline{\varphi_{\xi',\eta'}(x)}\, dx = \frac{1}{d_\pi} \langle \xi, \xi' \rangle_{\mathcal{H}_\pi} \langle \eta', \eta \rangle_{\mathcal{H}_\pi}.
\end{aligned}$$

Thus $\langle V_\eta \xi, V_{\eta'}\xi' \rangle_{L^2(G)} = \frac{1}{d_\pi}\langle \xi, \xi' \rangle_{\mathcal{H}_\pi} \langle \eta', \eta \rangle_{\mathcal{H}_\pi}$. □

Let π be an irreducible representation of G. For $\eta \in \mathcal{H}_\pi$, let $\mathcal{K}_\eta = \{V_\eta \xi : \xi \in \mathcal{H}_\pi\}$. Since $\mathcal{H}_\pi$ is finite dimensional, so is $\mathcal{K}_\eta$. Consider $\mathcal{K}_\eta$ as a subspace of $L^2(G)$. By Proposition 2.6.1, V_η intertwines π with λ_G. Thus $\mathcal{K}_\eta$ is a closed λ_G-invariant subspace of $L^2(G)$.

Corollary 2.6.9. Let G be a compact group and let π be an irreducible representation of G.

(a) If $\eta \in \mathcal{H}_\pi$ satisfies $\|\eta\| = \sqrt{d_\pi}$, then V_η is a unitary map from $\mathcal{H}_\pi$ onto $\mathcal{K}_\eta$.

(b) If $\eta_1, \eta_2 \in \mathcal{H}_\pi$ are orthogonal, then $\mathcal{K}_{\eta_1} \perp \mathcal{K}_{\eta_2}$.

Proof. Both claims follow immediately from Corollary 2.6.8. □

In the language of square-integrable representations and wavelets, we have the following:

Corollary 2.6.10. Let G be a compact group and let π be an irreducible representation of G. The Duflo-Moore operator for π is $d_\pi^{-1/2} I$, where I is the identity operator on $\mathcal{H}_\pi$, $\mathcal{D}_\pi = \mathcal{H}_\pi$, and $\eta \in \mathcal{H}_\pi$ is a π-wavelet if and only if $\|d_\pi^{-1/2}\eta\|_{\mathcal{H}_\pi} = 1$.

If π is an irreducible representation of G and η is a π-wavelet, then $\mathcal{K}_\eta \subseteq \mathcal{E}_{\overline{\pi}}$, the linear span of the coefficient functions of $\overline{\pi}$. Let's fix an orthonormal basis $\{\xi_1, \cdots, \xi_{d_\pi}\}$ of $\mathcal{H}_\pi$. For $1 \le j \le d_\pi$, let $\eta_j = d_\pi^{1/2}\xi_j$. Then each η_j is a π-wavelet and $\mathcal{K}_{\eta_k} \perp \mathcal{K}_{\eta_j}$ if $1 \le k < j \le d_\pi$. Suppose $\varphi^{\overline{\pi}}_{\eta,\xi}$ is any coefficient function of $\overline{\pi}$. Since $\{\eta_1, \cdots, \eta_{d_\pi}\}$ spans $\mathcal{H}_\pi$, we can write $\eta = \sum_{j=1}^{d_\pi} \alpha_j \eta_j$, for some $\alpha_j \in \mathbb{C}$, $1 \le j \le d_\pi$.

$$\begin{aligned}\varphi^{\overline{\pi}}_{\eta,\xi}(x) &= \langle \overline{\pi}(x)\eta, \xi \rangle_{\mathcal{H}_{\overline{\pi}}} = \langle \xi, \pi(x)\eta \rangle_{\mathcal{H}_\pi} \\ &= \sum_{j=1}^{d_\pi} \overline{\alpha_j} \langle \xi, \pi(x)\eta_j \rangle_{\mathcal{H}_\pi} = \sum_{j=1}^{d_\pi} \overline{\alpha_j} V_{\eta_j}\xi(x),\end{aligned}$$

for all $x \in G$. This shows that $\mathcal{E}_{\overline{\pi}} = \mathcal{K}_{\eta_1} \oplus \cdots \oplus \mathcal{K}_{\eta_{d_\pi}}$ and, thus, $\mathcal{E}_{\overline{\pi}}$ is finite dimensional of dimension $d_\pi^2 = d_{\overline{\pi}}^2$. Note that Proposition 2.6.6 implies $\mathcal{E}_\pi \perp \mathcal{E}_{\pi'}$ if π and π' are inequivalent irreducible representations of G. Let $\mathcal{E}$ denote the linear span of $\cup_{\pi \in \widehat{G}} \mathcal{E}_\pi$. Then $\mathcal{E}$ is dense in $L^2(G)$ (see Theorem 5.11 of [12]). Thus, $L^2(G) = \sum^{\oplus}_{\pi \in \widehat{G}} \mathcal{E}_\pi$. Note, we could also write $L^2(G) = \sum^{\oplus}_{\pi \in \widehat{G}} \mathcal{E}_{\overline{\pi}}$. We organize these points in a final theorem for this section.

Theorem 2.6.11. Let G be a compact group.

(a) For each $\pi \in \widehat{G}$, let $\mathcal{E}_\pi$ denote the linear span of the coefficient functions of π. Then $L^2(G) = \sum^{\oplus}_{\pi \in \widehat{G}} \mathcal{E}_{\overline{\pi}}$.

(b) For each $\pi \in \widehat{G}$, let $\{\eta_j : 1 \le j \le d_\pi\}$ be an orthogonal set of vectors in $\mathcal{H}_\pi$ such that $\|d_\pi \eta_j\|_{\mathcal{H}_\pi} = 1$, for $1 \le j \le d_\pi$. Let $\mathcal{K}_{\eta_j} = V_{\eta_j}\mathcal{H}_\pi$, for $1 \le j \le d_\pi$. Then $\mathcal{K}_{\eta_j}$ is a λ_G-invariant subspace of $L^2(G)$ and $\lambda_G^{\mathcal{K}_{\eta_j}} \sim \pi$. Moreover, $\mathcal{E}_{\overline{\pi}} = \mathcal{K}_{\eta_1} \oplus \cdots \oplus \mathcal{K}_{\eta_{d_\pi}}$.

2.7 The Fourier Transform

In this section, we recall the basic properties of Fourier analysis on $\mathbb{R}^n$ and introduce notation that will be used later. There are many references for the material in this section and also different notational conventions. Versions of the results can be found in [23] and [11]

It is useful to distinguish between $\mathbb{R}^n$ and its Fourier dual $\widehat{\mathbb{R}^n}$ by column and row vectors. So

$$\mathbb{R}^n = \left\{ \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_1, \cdots, x_n \in \mathbb{R} \right\}$$

and $\widehat{\mathbb{R}^n} = \{\underline{\omega} = (\omega_1, \cdots, \omega_n) : \omega_1, \cdots, \omega_n \in \mathbb{R}\}$. Note that $\underline{\omega}\underline{x} = \sum_{j=1}^n \omega_j x_j$, for $\underline{\omega} \in \widehat{\mathbb{R}^n}$ and $\underline{x} \in \mathbb{R}^n$. Also, if A is an $n \times n$-real matrix, then $\underline{\omega}(A\underline{x}) = (\underline{\omega}A)\underline{x}$.

Any irreducible unitary representation of $\mathbb{R}^n$ is one dimensional and of the form $\underline{x} \to e^{2\pi i \underline{\omega}\underline{x}}$, for some $\underline{\omega} \in \widehat{\mathbb{R}^n}$.

If $f \in L^1(\mathbb{R}^n)$, the *Fourier transform* of f is a function $\widehat{f} : \widehat{\mathbb{R}^n} \to \mathbb{C}$ defined as follows:

$$\widehat{f}(\underline{\omega}) = \int_{\mathbb{R}^n} f(\underline{x}) e^{-2\pi i \underline{\omega}\underline{x}} d\underline{x}, \text{ for } \underline{\omega} \in \widehat{\mathbb{R}^n}.$$

Then $f \to \widehat{f}$ is a linear map of $L^1(\mathbb{R}^n)$ into $C_0(\widehat{\mathbb{R}^n})$ such that $\|\widehat{f}\|_\infty \leq \|f\|_1$. In this thesis, frequent use is made of a closely related unitary map.

Theorem 2.7.1. [Plancherel] There exists a unitary map $\mathcal{F}$ of $L^2(\mathbb{R}^n)$ onto $L^2(\widehat{\mathbb{R}^n})$ such that $\mathcal{F}f = \widehat{f}$ for all $f \in L^1(\mathbb{R}^n) \cap L^2(\widehat{\mathbb{R}^n})$. Moreover,

$$\mathcal{F}^{-1}\xi(\underline{x}) = \int_{\widehat{\mathbb{R}^n}} \xi(\underline{\omega}) e^{2\pi i \underline{\omega}\underline{x}} d\underline{\omega},$$

for any $\underline{x} \in \mathbb{R}^n$ and $\xi \in L^1(\widehat{\mathbb{R}^n}) \cap L^2(\widehat{\mathbb{R}^n})$.

For $f \in L^2(\mathbb{R}^n)$, the notation $\widehat{f}$ will sometimes be used for $\mathcal{F}f$ even if $f \notin L^1(\mathbb{R}^n)$. Also, $\mathcal{F}^{-1}\xi$ may be denoted $\check{\xi}$. However, we usually use $\mathcal{F}$ to emphasize its importance as a unitary map. To illustrate this, consider the left regular representation $\lambda_{\mathbb{R}^n}$ of the additive group $\mathbb{R}^n$ acting on $L^2(\mathbb{R}^n)$. Define an equivalent unitary representation $\widehat{\lambda_{\mathbb{R}^n}}$ acting on $L^2(\widehat{\mathbb{R}^n})$ by $\widehat{\lambda_{\mathbb{R}^n}}(\underline{x}) = \mathcal{F}\lambda_{\mathbb{R}^n}(\underline{x})\mathcal{F}^{-1}$, for all $\underline{x} \in \mathbb{R}^n$.

Proposition 2.7.2. For $\underline{x} \in \mathbb{R}^n$ and $\xi \in L^2(\widehat{\mathbb{R}^n})$,

$$\left(\widehat{\lambda_{\mathbb{R}^n}}(\underline{x})\xi\right)(\underline{\omega}) = e^{-2\pi \underline{\omega}\underline{x}}\xi(\underline{\omega}), \text{ for a.e. } \underline{\omega} \in \widehat{\mathbb{R}^n}.$$

Proof. Fix $\underline{x} \in \mathbb{R}^n$. First, assume $\xi = \widehat{f}$, for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. So $\xi = \mathcal{F}^{-1}f$.

Then $\lambda_{\mathbb{R}^n}(\underline{x})f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ as well. Thus,

$$\big(\widehat{\lambda_{\mathbb{R}^n}}(\underline{x})\xi\big)(\underline{\omega}) = \mathcal{F}\big(\lambda_{\mathbb{R}^n}(\underline{x})f\big)(\underline{\omega}) = \int_{\mathbb{R}^n} \big(\lambda_{\mathbb{R}^n}(\underline{x})f\big)(\underline{y})e^{-2\pi i\underline{\omega}\underline{y}}d\underline{y} = \int_{\mathbb{R}^n} f(\underline{y}-\underline{x})e^{-2\pi i\underline{\omega}\underline{y}}d\underline{y}$$
$$= \int_{\mathbb{R}^n} f(\underline{y})e^{-2\pi i\underline{\omega}(\underline{y}+\underline{x})}d\underline{y} = e^{-2\pi i\underline{\omega}\underline{x}}\int_{\mathbb{R}^n} f(\underline{y})e^{-2\pi i\underline{\omega}\underline{y}}d\underline{y} = e^{-2\pi i\underline{\omega}\underline{x}}\xi(\underline{\omega}),$$

for all $\underline{\omega} \in \widehat{\mathbb{R}^n}$. Now define $M_{\underline{x}}$ on $L^2(\widehat{\mathbb{R}^n})$ by $\big(M_{\underline{x}}\eta\big)(\underline{\omega}) = e^{-2\pi i\underline{\omega}\underline{x}}\eta(\underline{\omega})$, for a.e. $\underline{\omega} \in \widehat{\mathbb{R}^n}$ and each $\eta \in L^2(\widehat{\mathbb{R}^n})$. Then $M_{\underline{x}} \in \mathcal{U}\big(L^2(\widehat{\mathbb{R}^n})\big)$. Moreover, $M_{\underline{x}}$ agrees with the unitary operator $\widehat{\lambda_{\mathbb{R}^n}}(\underline{x})$ on $\mathcal{F}\big(L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\big)$, a dense subspace of $L^2(\widehat{\mathbb{R}^n})$. Thus, $\widehat{\lambda_{\mathbb{R}^n}}(\underline{x}) = M_{\underline{x}}$. □

Proposition 2.7.2 can be used to identify many closed $\lambda_{\mathbb{R}^n}$-invariant subspaces of $L^2(\mathbb{R}^n)$.

Definition 2.7.3. Let Ω be a Borel subset of $\widehat{\mathbb{R}^n}$. We use the notation $L^2(\Omega)$ for the set of functions in $L^2(\widehat{\mathbb{R}^n})$ that are essentially supported on Ω. That is

$$L^2(\Omega) = \left\{\xi \in L^2(\widehat{\mathbb{R}^n}) : \xi(\underline{\omega}) = 0, \text{ for a.e. } \underline{\omega} \in \widehat{\mathbb{R}^n} \setminus \Omega\right\}.$$

Let $\mathcal{H}^2_\Omega = \{f \in L^2(\mathbb{R}^n) : \mathcal{F}f \in L^2(\Omega)\}$.

Note that $L^2(\Omega)$ is a closed subspace of $L^2(\widehat{\mathbb{R}^n})$. We have $L^2(\Omega) = \{0\}$ if and only if $|\Omega| = 0$, where $|\Omega|$ is the Lebesgue measure of Ω, and $L^2(\Omega) = L^2(\widehat{\mathbb{R}^n})$ if and only if $|\widehat{\mathbb{R}^n} \setminus \Omega| = 0$. Thus, if $|\Omega| > 0$ and $|\widehat{\mathbb{R}^n} \setminus \Omega| > 0$, then $L^2(\Omega)$ is a nontrivial closed subspace of $L^2(\widehat{\mathbb{R}^n})$. For any Borel $\Omega \subseteq \widehat{\mathbb{R}^n}$ and $\underline{x} \in \mathbb{R}^n$, if $\xi \in L^2(\Omega)$, then $M_{\underline{x}}\xi \in L^2(\Omega)$. So $L^2(\Omega)$ is invariant under the action of $\widehat{\lambda_{\mathbb{R}^n}}(\underline{x})$, for any $\underline{x} \in \mathbb{R}^n$. That is, $L^2(\Omega)$ is a $\widehat{\lambda_{\mathbb{R}^n}}$-invariant closed subspace of $L^2(\widehat{\mathbb{R}^n})$. Therefore, $\mathcal{H}^2_\Omega$ is a $\lambda_{\mathbb{R}^n}$-invariant closed subspace of $L^2(\mathbb{R}^n)$. In fact, every $\lambda_{\mathbb{R}^n}$-invariant closed subspace of $L^2(\mathbb{R}^n)$ is of the form $\mathcal{H}^2_\Omega$ for some Borel subset Ω of $\widehat{\mathbb{R}^n}$. See Theorem 9.17 of [23] for a proof when $n = 1$. This proof is easily adapted for general n.

Proposition 2.7.4. If $\mathcal{K}$ is any $\lambda_{\mathbb{R}^n}$-invariant closed subspace of $L^2(\mathbb{R}^n)$ such that $\mathcal{K} \neq \{0\}$, then there exist two $\lambda_{\mathbb{R}^n}$-invariant closed subspaces of $L^2(\mathbb{R}^n)$, say $\mathcal{K}_1$ and $\mathcal{K}_2$, such that $\mathcal{K}_1 \neq \{0\}$, $\mathcal{K}_2 \neq \{0\}$, $\mathcal{K}_1 \perp \mathcal{K}_2$ and $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$.

Proof. By the discussion above, there exists a Borel $\Omega \subseteq \widehat{\mathbb{R}^n}$ such that $\mathcal{K} = \mathcal{H}^2_\Omega$. Since $\mathcal{K} \neq \{0\}$, $|\Omega| > 0$. By properties of Lebesgue measure, we can find Borel subsets Ω_1 and Ω_2 of Ω so that $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega_1 \cup \Omega_2 = \Omega$. Then just take $\mathcal{K}_1 = \mathcal{H}^2_{\Omega_1}$ and $\mathcal{K}_2 = \mathcal{H}^2_{\Omega_2}$. □

One consequence of Proposition 2.7.4 is that $\lambda_{\mathbb{R}^n}$ cannot be written as a direct sum of irreducible representations unlike the case for a compact group.

2.8 Induced Representations

Suppose G is a locally compact group and H is a closed subgroup of G. If π is a unitary representation of H, there is a procedure for building a unitary representation of G by combining the action of G on G/H with π. This new representation is known as an *induced representation*, the representation of G induced by π. A detailed treatment of the theory of induced representations can be found in [20]. The parts of the theory used in this thesis are gathered in this section and adapted for convenient use.

Let $q : G \to G/H$ be the quotient map, $q(x) = xH$, for $x \in G$. The space G/H is locally compact when endowed with the quotient topology. We will assume also that we have chosen a map $p : G/H \to G$ such that $q\big(p(\omega)\big) = \omega$, for all $\omega \in G/H$. That is, we use the axiom of choice to pick a distinguished element in each left H-coset. For all groups and subgroup pairs considered in this thesis, it is easy to find a Borel measurable p, but measurability of p is not needed for this section.

We begin with the existence of a special kind of function on G that relates left Haar integration on G with left Haar integration on H and a quasi-invariant integration over G/H.

Definition 2.8.1. A rho-function for (G, H) is a measurable map $\rho : G \to \mathbb{R}^+$ that satisfies

$$\rho(xh) = \tfrac{\Delta_H(h)}{\Delta_G(h)}\rho(x), \text{ for all } x \in G, h \in H.$$

Proposition 2.8.2. There exists a continuous rho-function ρ for (G, H) such that $\rho(x) > 0$, for all $x \in G$.

See Lemma 1.20 of [20] for the proof.

Example 11. Suppose there exists a closed subgroup K of G such that $K \cap H = \{e\}$, $G = KH$, and the map $(k, h) \to kh$ is a homeomorphism of $K \times H$ with G. Such a complementary subgroup K of H exists in many useful situations. Then, for $x \in G$, there exist unique $k_x \in K$ and $h_x \in H$ such that $x = k_x h_x$ and the map $x \to (k_x, h_x)$ is continuous since it is the inverse of the above homeomorphism. Then $\rho(x) = \frac{\Delta_H(h_x)}{\Delta_G(h_x)}$, for all $x \in G$, defines a continuous and everywhere positive rho-function for (G, H).

Any continuous rho-function determines an associated measure on G/H. The following is a consequence of the Riesz Representation Theorem for positive linear functionals on $C_c(G/H)$ (see Proposition 1.14 in [20]).

Proposition 2.8.3. Let ρ be a continuous rho-function for (G, H). There exists a positive regular Borel measure μ_ρ on G/H such that

$$\int_G f(x)\rho(x)\, d\mu_G(x) = \int_{G/H}\int_H f\big(p(\omega)h\big)\, d\mu_H(h)\, d\mu_\rho(\omega),$$

for all $f \in C_c(G)$.

Recall that G acts on G/H by, for $x \in G$ and $yH \in G/H$, $x \cdot (yH) = (xy)H$. This can also be written neatly as, for $x \in G$ and $\omega \in G/H$, $x \cdot \omega = q\big(xp(\omega)\big)$. For any subset E of G/H, let $x \cdot E = \{x \cdot \omega : \omega \in E\}$. If E is a Borel subset of G/H, then so is $x \cdot E$. For any Borel measure ν on G/H and $x \in G$, define a new Borel measure $x \cdot \nu$ on G/H by $(x \cdot \nu)(E) = \nu(x \cdot E)$, for any Borel $E \subseteq G/H$.

Definition 2.8.4. A regular Borel measure on G/H is quasi-invariant if, for every $x \in G$, $x \cdot \nu$ and ν are mutually absolutely continuous.

Suppose ν is a quasi-invariant regular Borel measure on G/H. For each $x \in G$, $\left[\frac{d(x \cdot \nu)}{d\nu}\right]$ denotes the Radon-Nikodym derivative. That is, $\left[\frac{d(x \cdot \nu)}{d\nu}\right]$ is the non-negative measurable function on G/H such that, for any $\varphi \in C_c(G/H)$,

$$\int_{G/H} \varphi(\omega)\, d(x \cdot \nu)(\omega) = \int_{G/H} \varphi(\omega) \left[\frac{d(x \cdot \nu)}{d\nu}\right](\omega)\, d\nu(\omega).$$

It is also useful to move functions on G/H by elements of G. For $x \in G$ and any function φ on G/H, let $L_x\varphi(\omega) = \varphi(x^{-1} \cdot \omega)$. Elementary integration theory shows that, for a regular Borel measure ν on G/H, $\varphi \in C_c(G/H)$, and $x \in G$,

$$\int_{G/H} \varphi(\omega)\, d(x \cdot \nu)(\omega) = \int_{G/H} \varphi(x^{-1} \cdot \omega)\, d\nu(\omega) = \int_{G/H} L_x\varphi(\omega)\, d\nu(\omega). \tag{2.9}$$

We also use L_x to denote the operation of moving functions on G by x. That is, if Z is any set and $f : G \to Z$, then, for $x \in G$, $L_x f(y) = f(x^{-1}y)$, for all $y \in G$.

Now, let ρ be a continuous rho-function for (G, H) and let μ_ρ be the associated regular Borel measure as in Proposition 2.8.3. For $x \in G$, it is clear that $L_{x^{-1}}\rho$ is also a continuous rho-function for (G, H). For any $\varphi \in C_c(G/H)$, there exists a function $f \in C_c(G)$ such that $\varphi(\omega) = \int_H f\big(p(\omega)h\big)\, d\mu_H(h)$, for all $\omega \in G/H$ (see Proposition 1.9 in [20]). Then,

$$L_x\varphi(\omega) = \int_H f\big(p(x^{-1}\cdot\omega)h\big)\, d\mu_H(h) = \int_H f\big(x^{-1}p(\omega)h\big)\, d\mu_H(h) = \int_H L_x f\big(p(\omega)h\big)\, d\mu_H(h),$$

for any $\omega \in G/H$. Using (2.9), Proposition 2.8.3, and left invariance of the Haar integral on G,

$$\begin{aligned}\int_{G/H} \varphi\, d(x \cdot \mu_\rho) &= \int_{G/H} L_x\varphi(\omega)\, d\mu_\rho(\omega) = \int_{G/H}\int_H L_x f\big(p(\omega)h\big)\, d\mu_H(h)\, d\mu_\rho(\omega)\\ &= \int_G L_x f(y)\rho(y)\, d\mu_G(y) = \int_G f(y) L_{x^{-1}}\rho(y)\, d\mu_G(y)\\ &= \int_{G/H}\int_H f\big(p(\omega)h\big)\, d\mu_H(h)\, d\mu_{L_{x^{-1}}\rho}(\omega) = \int_{G/H} \varphi\, d\mu_{L_{x^{-1}}\rho}.\end{aligned} \tag{2.10}$$

Since $\varphi \in C_c(G/H)$ was arbitrary, $x \cdot \mu_\rho = \mu_{L_{x^{-1}}\rho}$.

If ρ is a continuous and everywhere positive rho-function for (G, H) and μ_ρ is the associated regular Borel measure on G/H as in Proposition 2.8.3, notice that, for

$x, y \in G$ and any $h \in H$,

$$\frac{\rho(xyh)}{\rho(yh)} = \frac{\rho(xy)}{\rho(y)},$$

using the defining property of rho-functions. So, with x fixed, $y \to \frac{\rho(xy)}{\rho(y)}$ is constant on left H-cosets. Therefore, we can define $r_\rho : G \times G/H \to (0, \infty)$ by

$$r_\rho(x, \omega) = \frac{\rho\big(xp(\omega)\big)}{\rho\big(p(\omega)\big)}, \text{ for all } (x, \omega) \in G \times G/H.$$

This does not depend on the choice of the cross-section map $p : G/H \to G$. Then r_ρ is a continuous and everywhere positive function on $G \times G/H$. The following is based on Theorem 1.18 in [20].

Theorem 2.8.5. Let H be a closed subgroup of a locally compact group G. Let ρ be a continuous and everywhere positive rho-function for (G, H) and let μ_ρ be the associated regular Borel measure on G/H. Then μ_ρ is a quasi-invariant measure on G/H. Moreover, for any $x \in G$, $\left[\frac{d(x\cdot\mu_\rho)}{d\mu_\rho}\right](\omega) = r_\rho(x, \omega)$, for a.e. $\omega \in G/H$.

Proof. To show that $\left[\frac{d(x\cdot\mu_\rho)}{d\mu_\rho}\right](\omega) = r_\rho(x, \omega)$, for a.e. $\omega \in G/H$, let $\varphi \in C_c(G/H)$ be arbitrary. Select $f \in C_c(G)$ such that $\varphi(\omega) = \int_H f\big(p(\omega)h\big)\, d\mu_H(h)$, for all $\omega \in G/H$. Then, using $x \cdot \mu_\rho = \mu_{L_{x^{-1}}\rho}$ and Proposition 2.8.3,

$$\begin{aligned}
\int_{G/H} \varphi(\omega)\, d(x \cdot \mu_\rho)(\omega) &= \int_{G/H} \int_H f\big(p(\omega)h\big)\, d\mu_H(h)\, d\mu_{L_{x^{-1}}\rho}(\omega) \\
&= \int_G f(y) L_{x^{-1}}\rho(y)\, d\mu_G(y) = \int_G f(y)\rho(xy)\, d\mu_G(y) \\
&= \int_G f(y) \frac{\rho(xy)}{\rho(y)} \rho(y)\, d\mu_G(y) \\
&= \int_{G/H} \int_H f\big(p(\omega)h\big) \frac{\rho(xp(\omega)h)}{\rho(p(\omega)h)}\, d\mu_H(h)\, d\mu_\rho(\omega) \\
&= \int_{G/H} \int_H f\big(p(\omega)h\big)\, d\mu_H(h) r_\rho(x, \omega)\, d\mu_\rho(\omega) \\
&= \int_{G/H} \varphi(\omega) r_\rho(x, \omega)\, d\mu_\rho(\omega),
\end{aligned}$$

since $\frac{\rho(xp(\omega)h)}{\rho(p(\omega)h)} = r_\rho(x, \omega)$, for all $h \in H$, $(x, \omega) \in G \times G/H$. This proves that $r_\rho(x, \omega) = \left[\frac{d(x\cdot\mu_\rho)}{d\mu_\rho}\right](\omega)$, for a.e. $\omega \in G/H$ and all $x \in G$. Moreover, it also shows that $x \cdot \mu_\rho$ is absolutely continuous with respect to μ_ρ, for every $x \in G$. Using the properties of group actions shows these measures are mutually absolutely continuous. Thus, μ_ρ is a quasi-invariant measure on G/H. $\square$

For $\varphi \in C_c(G/H)$ and $x \in G$,

$$\int_{G/H} L_x\varphi(\omega)\, d\mu_\rho(\omega) = \int_{G/H} \varphi(\omega)\, d(x \cdot \mu_\rho)(\omega) = \int_{G/H} \varphi(\omega) r_\rho(x, \omega)\, d\mu_\rho(\omega).$$

Thus, for $\varphi \in C_c(G/H)$ and $x \in G$,

$$\int_{G/H} \varphi(\omega)\, d\mu_\rho(\omega) = \int_{G/H} L_{x^{-1}}\big(L_x\varphi\big)(\omega)\, d\mu_\rho(\omega) = \int_{G/H} \big(L_x\varphi\big)(\omega) r_\rho(x^{-1}, \omega)\, d\mu_\rho(\omega).$$

Integration theory now leads to the following change of variables formula.

Corollary 2.8.6. Let H be a closed subgroup of a locally compact group G. Let ρ be a continuous and everywhere positive rho-function for (G, H) and let μ_ρ be the associated regular Borel measure on G/H. For any Borel function $g : G/H \to [0, \infty]$, or any $g : G/H \to \mathbb{C}$ that is μ_ρ-integrable,

$$\int_{G/H} g(\omega)\, d\mu_\rho(\omega) = \int_{G/H} L_x g(\omega) r_\rho(x^{-1}, \omega)\, d\mu_\rho(\omega),$$

for all $x \in G$.

This change of variables formula for the G action on G/H, when G/H carries the quasi-invariant measure μ_ρ, is what is needed to get a concrete definition of a representation of G induced from a representation of H. There are a variety of ways (all resulting in equivalent representations) of defining induced representations. The most efficient one for the purposes of this thesis is based on Proposition 2.28 of [20]. The same construction is presented in Section 6.1 of [12].

Fix a continuous and everywhere positive rho-function ρ and associated quasi-invariant measure μ_ρ on G/H. Let π be a unitary representation of H on the Hilbert space $\mathcal{H}_\pi$. The first step is to define a Hilbert space consisting of $\mathcal{H}_\pi$-valued functions on G that encode the action of H through π. Recall that a function $\xi : G \to \mathcal{H}_\pi$ is called weakly Borel measurable if $x \to \langle \xi(x), v\rangle$ is Borel measurable, for each $v \in \mathcal{H}_\pi$. We will call such functions measurable for simplicity. There are two properties that measurable functions $\xi : G \to \mathcal{H}_\pi$ could satisfy, the second depending on the first being satisfied.

Property 1: $\xi(xh) = \pi(h^{-1})\xi(x)$, for all $h \in H$ and all $x \in G$.

Note that if ξ satisfies $\xi(xh) = \pi(h^{-1})\xi(x)$, for all $h \in H$ and almost all $x \in G$, then let A denote the set of all $x \in G$ such that $\xi(xh) \neq \pi(h^{-1})\xi(x)$, for some $h \in H$. Then $Ak = A$, for all $k \in H$; so A is a union of left H-cosets. Thus, if we define $\xi' : G \to \mathcal{H}_\pi$ by

$$\xi'(x) = \begin{cases} \xi(x) & \text{if } x \in G \setminus A \\ 0 & \text{if } x \in A, \end{cases}$$

then $\xi' = \xi$ almost everywhere and ξ' satisfies Property 1.

Suppose ξ satisfies Property 1. Then, for any $\omega \in G/H$, $\|\xi\big(p(\omega)h\big)\|_{\mathcal{H}_\pi} = \|\xi\big(p(\omega)\big)\|_{\mathcal{H}_\pi}$, for all $h \in H$, so $\|\xi\big(p(\omega)\big)\|_{\mathcal{H}_\pi}$ does not depend on the choice of p.

Property 2: $\int_{G/H} \|\xi(p(\omega))\|^2_{\mathcal{H}_\pi} d\mu_\rho(\omega) < \infty$.

Standard arguments show that if ξ_1 and ξ_2 are measurable functions from G into $\mathcal{H}_\pi$ that satisfy Properties 1 and 2 and $\alpha \in \mathbb{C}$, then $\xi_1 + \xi_2$ and $\alpha\xi_1$ satisfy Properties 1 and 2. As usual, if $\xi_1(p(\omega)) = \xi_2(p(\omega))$ for μ_ρ-a.e. $\omega \in G/H$, then we say ξ_1 and ξ_2 are equivalent.

Definition 2.8.7. Let $\mathcal{H}^\rho_{\text{ind}\pi}$ denote the vector space of equivalence classes of measurable $\xi : G \to \mathcal{H}_\pi$ that satisfy Properties 1 and 2.

If $\xi_1, \xi_2 \in \mathcal{H}^\rho_{\text{ind}\pi}$, for any $\omega \in G/H$,

$$\left\langle \xi_1(p(\omega)h), \xi_2(p(\omega)h) \right\rangle_{\mathcal{H}_\pi} = \left\langle \xi_1(p(\omega)), \xi_2(p(\omega)) \right\rangle_{\mathcal{H}_\pi}, \text{ for any } h \in H.$$

Thus, $\left\langle \xi_1(p(\omega)h), \xi_2(p(\omega)h) \right\rangle_{\mathcal{H}_\pi}$ does not depend on the choice of p. Also

$$\begin{aligned} \left| \left\langle \xi_1(p(\omega)), \xi_2(p(\omega)) \right\rangle_{\mathcal{H}_\pi} \right| &\le \|\xi_1(p(\omega))\|_{\mathcal{H}_\pi} \|\xi_2(p(\omega))\|_{\mathcal{H}_\pi} \\ &\le \|\xi_1(p(\omega))\|^2_{\mathcal{H}_\pi} + \|\xi_2(p(\omega))\|^2_{\mathcal{H}_\pi}. \end{aligned}$$

Thus, $\omega \to \left\langle \xi_1(p(\omega)), \xi_2(p(\omega)) \right\rangle_{\mathcal{H}_\pi}$ is integrable on G/H with respect to the measure μ_ρ. Let

$$\langle \xi_1, \xi_2 \rangle_{\mathcal{H}^\rho_{\text{ind}\pi}} = \int_{G/H} \left\langle \xi_1(p(\omega)), \xi_2(p(\omega)) \right\rangle_{\mathcal{H}_\pi} d\mu_\rho(\omega).$$

This defines an inner product on $\mathcal{H}^\rho_{\text{ind}\pi}$. Natural modifications of the usual arguments show that $\mathcal{H}^\rho_{\text{ind}\pi}$ is complete in the norm defined by this inner product (see Proposition 2.28 in [20]). The induced representation is defined on this Hilbert space.

Definition 2.8.8. Let π be a unitary representation of a closed subgroup H of a locally compact group G. Let ρ be a continuous and everywhere positive rho-function for (G, H) and let μ_ρ be the associated quasi-invariant measure on G/H. The representation of G induced by π is denoted $\text{ind}^G_H\pi$ and acts on the Hilbert space $\mathcal{H}^\rho_{\text{ind}\pi}$. For $\xi \in \mathcal{H}^\rho_{\text{ind}\pi}$ and $x \in G$,

$$\text{ind}^G_H\pi(x)\xi(y) = \left[\tfrac{\rho(x^{-1}y)}{\rho(y)}\right]^{1/2} \xi(x^{-1}y), \text{ for } y \in G.$$

Let us check that $\operatorname{ind}_H^G\pi(x)$ is a unitary operator. For $\xi_1, \xi_2 \in \mathcal{H}^\rho_{\operatorname{ind}\pi}$,

$$\begin{aligned}
\left\langle \operatorname{ind}_H^G\pi(x)\xi_1, \operatorname{ind}_H^G\pi(x)\xi_2 \right\rangle_{\mathcal{H}^\rho_{\operatorname{ind}\pi}} &= \int_{G/H} \left\langle \operatorname{ind}_H^G\pi(x)\xi_1\big(p(\omega)\big), \operatorname{ind}_H^G\pi(x)\xi_2\big(p(\omega)\big) \right\rangle_{\mathcal{H}_\pi} d\mu_\rho(\omega) \\
&= \int_{G/H} \left\langle \xi_1\big(x^{-1}p(\omega)\big), \xi_2\big(x^{-1}p(\omega)\big) \right\rangle_{\mathcal{H}_\pi} \frac{\rho\big(x^{-1}p(\omega)\big)}{\rho\big(p(\omega)\big)} \, d\mu_\rho(\omega) \\
&= \int_{G/H} \left\langle \xi_1\big(x^{-1}p(\omega)\big), \xi_2\big(x^{-1}p(\omega)\big) \right\rangle_{\mathcal{H}_\pi} r_\rho(x^{-1}, \omega) \, d\mu_\rho(\omega) \\
&= \int_{G/H} \left\langle \xi_1\big(p(\omega)\big), \xi_2\big(p(\omega)\big) \right\rangle_{\mathcal{H}_\pi} d\mu_\rho(\omega) \\
&= \langle \xi_1, \xi_2 \rangle_{\mathcal{H}^\rho_{\operatorname{ind}\pi}} .
\end{aligned}$$

Thus, $\operatorname{ind}_H^G\pi(x)$ preserves inner products. It is clear that $\operatorname{ind}_H^G\pi(x)$ is linear. For $x, z \in G$ and $\xi \in \mathcal{H}^\rho_{\operatorname{ind}\pi}$,

$$\begin{aligned}
\operatorname{ind}_H^G\pi(x)\left(\operatorname{ind}_H^G\pi(z)\xi\right)(y) &= \left[\tfrac{\rho(x^{-1}y)}{\rho(y)}\right]^{1/2} \left(\operatorname{ind}_H^G\pi(z)\xi\right)(x^{-1}y) \\
&= \left[\tfrac{\rho(x^{-1}y)}{\rho(y)}\right]^{1/2} \left[\tfrac{\rho(z^{-1}x^{-1}y)}{\rho(x^{-1}y)}\right]^{1/2} \xi(z^{-1}x^{-1}y) \\
&= \left[\tfrac{\rho((xz)^{-1}y)}{\rho(y)}\right]^{1/2} \xi((xz)^{-1}y) \\
&= \operatorname{ind}_H^G\pi(xz)\xi(y), \text{ for a.e. } y \in G.
\end{aligned}$$

Since $\operatorname{ind}_H^G\pi(e)$ is clearly the identity operator on $\mathcal{H}^\rho_{\operatorname{ind}\pi}$, each $\operatorname{ind}_H^G\pi(x)$ must be a unitary operator and $\operatorname{ind}_H^G\pi : G \to \mathcal{U}\left(\mathcal{H}^\rho_{\operatorname{ind}\pi}\right)$ is a homomorphism. See Section 6.1 of [12] for the proof that $x \to \operatorname{ind}_H^G\pi(x)\xi$ is continuous for each $\xi \in \mathcal{H}^\rho_{\operatorname{ind}\pi}$. Thus, $\operatorname{ind}_H^G\pi$ is a unitary representation of G.

Remark. If ρ' is another continuous and everywhere positive rho-function for (G, H) the resulting induced representation is unitarily equivalent to the one constructed using ρ. Thus, it is common to suppress the dependency on ρ in the notation $\operatorname{ind}_H^G\pi$. (See page 154 in [12].)

In many cases, the closed subgroup H is sitting inside G in a special way and $\operatorname{ind}_H^G\pi$ is unitarily equivalent to a representation with a more transparent structure.

As in Example 11, suppose there exists a closed subgroup K of G such that $K \cap H = \{e\}$, $G = KH$, and the map $(k, h) \to kh$ is a homeomorphism of $K \times H$ with G. Then, for $x \in G$, there exist unique $k_x \in K$ and $h_x \in H$ such that $x = k_x h_x$. Then $\rho(x) = \frac{\Delta_H(h_x)}{\Delta_G(h_x)}$, for all $x \in G$, defines a continuous and everywhere positive rho-function for (G, H). Note that $\rho(kx) = \rho(x)$, for all $x \in G$ and $k \in K$.

If we restrict the quotient map q to K, so $q_{|K}(k) = kH$, for all $k \in K$, then $q_{|K}$ is a homeomorphism. Then there is a very nice cross-section map $p : G/H \to G$ given by $p(kH) = k$, for $k \in K$. If ν is a regular Borel measure on G/H, define a regular Borel measure $\widetilde{\nu}$ on K by $\widetilde{\nu}(E) = \nu\big(q(E)\big)$, for all Borel $E \subseteq K$. Then, for any $\varphi \in C_c(K)$, $\int_K \varphi(k) \, d\widetilde{\nu}(k) = \int_{G/H} \varphi\big(p(\omega)\big) \, d\nu(\omega)$. Let μ_ρ be the measure on G/H associated with

ρ as in Proposition 2.8.3. For any $\varphi \in C_c(K)$, $\varphi \circ p \in C_c(G/H)$. Let $f \in C_c(G)$ be such that

$$\varphi(k) = (\varphi \circ p)(kH) = \int_H f\big(p(kH)h\big)\, d\mu_H(h) = \int_H f(kh)\, d\mu_H(h),$$

for any $k \in K$. Note that, for $\ell \in K$, $L_\ell \varphi(k) = \int_H L_\ell f(kh)\, d\mu_H(h)$. Then Proposition 2.8.3 implies, for any $\ell \in K$,

$$\begin{aligned}
\int_K \varphi(k)\, d\widetilde{\mu_\rho}(k) &= \int_K \int_H f(kh)\, d\mu_H(h)\, d\widetilde{\mu_\rho}(k) = \int_{G/H} \int_H f\big(p(\omega)h\big)\, d\mu_H(h)\, d\mu_\rho(\omega) \\
&= \int_G f(x)\rho(x)\, d\mu_G(x) = \int_G f(\ell^{-1}x)\rho(\ell^{-1}x)\, d\mu_G(x) \\
&= \int_G L_\ell f(x)\rho(x)\, d\mu_G(x) = \int_K \int_H L_\ell f(kh)\, d\mu_H(h)\, d\widetilde{\mu_\rho}(k) \\
&= \int_K L_\ell \varphi(k)\, d\widetilde{\mu_\rho}(k).
\end{aligned}$$

That is $\int_K L_\ell \varphi(k)\, d\widetilde{\mu_\rho}(k) = \int_K \varphi(k)\, d\widetilde{\mu_\rho}(k)$, for any $\ell \in K$ and any $\varphi \in C_c(K)$. This means $\widetilde{\mu_\rho}$ must be a left Haar measure on K. Thus, we can take $\mu_K = \widetilde{\mu_\rho}$.

Define $W : \mathcal{H}^\rho_{\mathrm{ind}\pi} \to L^2(K, \mathcal{H}_\pi)$ as follows: For $\xi \in \mathcal{H}^\rho_{\mathrm{ind}\pi}$, we can select a representative function in the equivalence class of ξ, also denoted ξ, so that Properties 1 and 2 hold. Let $W\xi : K \to \mathcal{H}_\pi$ be given by simply restricting ξ to K. That is

$$W\xi(k) = \xi(k), \text{ for all } k \in K.$$

Then $\int_K \|W\xi(k)\|^2_{\mathcal{H}_\pi} d\mu_K(k) = \int_K \|\xi(k)\|^2_{\mathcal{H}_\pi} d\widetilde{\mu_\rho}(k) = \int_{G/H} \|\xi\big(p(\omega)\big)\|^2_{\mathcal{H}_\pi} d\mu_\rho(\omega) < \infty$, by Property 2. Thus, $W\xi \in L^2(K, \mathcal{H}_\pi)$ and W is an isometry. It is obvious that W is linear. Also, for any $F \in L^2(K, \mathcal{H}_\pi)$, we can chose F so that it is everywhere defined on K. Define $\xi : G \to \mathcal{H}_\pi$ by $\xi(x) = \pi(h_x^{-1})F(k_x)$, for all $x \in G$. If $h \in H$ and $x \in G$, $k_{xh} = k_x$ and $h_{xh} = h_x h$. Thus $\xi(xh) = \pi\big((h_x h)^{-1}\big)F(k_x) = \pi(h^{-1})\pi(h_x^{-1})F(k_x) = \pi(h^{-1})\xi(x)$. Thus ξ satisfies Property 1 and clearly also Property 2. Therefore, $\xi \in \mathcal{H}^\rho_{\mathrm{ind}\pi}$. Moreover, $W\xi = F$. So $W : \mathcal{H}^\rho_{\mathrm{ind}\pi} \to L^2(K, \mathcal{H}_\pi)$ is a unitary map and $W^{-1}F$ is given by $W^{-1}F(x) = \pi(h_x^{-1})F(k_x)$, for all $x \in G$.

In preparation for using W to transfer $\mathrm{ind}_H^G \pi$ from $\mathcal{H}^\rho_{\mathrm{ind}\pi}$ to an equivalent representation on $L^2(K, \mathcal{H}_\pi)$, we make some notational observations. The action of G on G/H transfers to an action of G on K as a topological space homeomorphic to G/H. That is, for $x \in G$ and $k \in K$, $x^{-1} \cdot k = p(x^{-1}kH) = k_{x^{-1}k}$. Since $x^{-1}k = k_{x^{-1}k} h_{x^{-1}k} = (x^{-1} \cdot k) h_{x^{-1}k}$, we have $h_{x^{-1}k} = (x^{-1} \cdot k)^{-1} x^{-1} k$ and $h_{x^{-1}k}^{-1} = k^{-1} x (x^{-1} \cdot k)$. We will also need $\rho(x^{-1}k)/\rho(k)$. But

$$\frac{\rho(x^{-1}k)}{\rho(k)} = \frac{\Delta_H(h_{x^{-1}k})/\Delta_G(h_{x^{-1}k})}{\Delta_H(h_k)/\Delta_G(h_k)} = \frac{\Delta_H(h_{x^{-1}k})/\Delta_G(h_{x^{-1}k})}{\Delta_H(e)/\Delta_G(e)} = \frac{\Delta_H(h_{x^{-1}k})}{\Delta_G(h_{x^{-1}k})}.$$

Let $\sigma^\pi(x) = W \mathrm{ind}_H^G \pi(x) W^{-1}$, for all $x \in G$. For $F \in L^2(K, \mathcal{H}_\pi)$, let $\xi = W^{-1}F$.

Then, for $k \in K$,

$$\begin{aligned}\sigma^{\pi}(x)F(k) &= W\big(\mathrm{ind}_H^G\pi(x)\xi\big)(k) = \mathrm{ind}_H^G\pi(x)\xi(k) = \left[\tfrac{\Delta_H(h_{x^{-1}k})}{\Delta_G(h_{x^{-1}k})}\right]^{1/2}\xi(x^{-1}k)\\ &= \left[\tfrac{\Delta_H(h_{x^{-1}k})}{\Delta_G(h_{x^{-1}k})}\right]^{1/2} W^{-1}F(x^{-1}k) = \left[\tfrac{\Delta_H(h_{x^{-1}k})}{\Delta_G(h_{x^{-1}k})}\right]^{1/2}\pi\left(h_{x^{-1}k}^{-1}\right)F(x^{-1}\cdot k).\end{aligned}$$

The situation just discussed is based on Example 2.29 in [20]. However, there is an error in the definition of the rho-function in the last line of page 74 of [20]. As a result the formula for the representation is incorrect there. Therefore, we state the correct expression in a proposition.

Proposition 2.8.9. Let H and K be closed subgroups of a locally compact group G that satisfy $K \cap H = \{e\}$ and $(k, h) \to kh$ is a homeomorphism of $K \times H$ onto G. Let π be a unitary representation of H. Then $\mathrm{ind}_H^G\pi$ is equivalent to σ^{π} acting on $L^2(K, \mathcal{H}_{\pi})$ by

$$\sigma^{\pi}(x)F(k) = \left[\tfrac{\Delta_H(h_{x^{-1}k})}{\Delta_G(h_{x^{-1}k})}\right]^{1/2}\pi\left(h_{x^{-1}k}^{-1}\right)F(x^{-1}\cdot k), \text{ for a.e. } k \in K,$$

for all $F \in L^2(K, \mathcal{H}_{\pi})$ and for every $x \in G$.

This formula simplifies further when H is an abelian normal subgroup and π is a one dimensional representation of H. When this simplified form is used in this thesis, H is actually an isomorphic copy of $\mathbb{R}^n$. We now work out this simplified form.

Suppose $n \in \mathbb{N}$ and K_0 is a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$. Let

$$G = \mathbb{R}^n \rtimes K_0 = \{[\underline{x}, A] : \underline{x} \in \mathbb{R}^n, A \in K_0\}.$$

Let $H = \{[\underline{x}, \mathrm{id}] : \underline{x} \in \mathbb{R}^n\}$. Then H is an abelian normal closed subgroup of G. Let $K = \{[\underline{0}, A] : A \in K_0\}$, a closed subgroup of G. We have $K \cap H = \{[\underline{0}, \mathrm{id}]\}$ and $\big([\underline{0}, A], [\underline{x}, \mathrm{id}]\big) \to [\underline{0}, A][\underline{x}, \mathrm{id}] = [A\underline{x}, A]$ is a homeomorphism of $K \times H$ with G. Note that

$$k_{[\underline{x},A]} = [\underline{0}, A] \quad \text{and} \quad h_{[\underline{x},A]} = [A^{-1}\underline{x}, \mathrm{id}], \text{ for all } [\underline{x}, A] \in G. \tag{2.11}$$

The modular function of G is given by

$$\Delta_G[\underline{x}, A] = \frac{\Delta_K(A)}{|\det(A)|}, \text{ for all } [\underline{x}, A] \in G.$$

Note that $\Delta_G \equiv 1$ on H and H, itself, is unimodular. So $\left[\tfrac{\Delta_H[\underline{x},\underline{0}]}{\Delta_G[\underline{x},\underline{0}]}\right]^{1/2} = 1$, for all $[\underline{x}, \underline{0}] \in H$.

The irreducible representations of H are all of the form $\chi_{\underline{\omega}}$, for $\underline{\omega} \in \widehat{\mathbb{R}^n}$, where

$$\chi_{\underline{\omega}}[\underline{x}, \mathrm{id}] = e^{2\pi i \underline{\omega}\underline{x}}, \text{ for all } [\underline{x}, \mathrm{id}] \in H.$$

Corollary 2.8.10. Let $G = \mathbb{R}^n \rtimes K_0$, where K_0 is a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$. Let $H = \{[\underline{x}, \mathrm{id}] : \underline{x} \in \mathbb{R}^n\}$ and let $\underline{\omega} \in \widehat{\mathbb{R}^n}$. Then $\mathrm{ind}_H^G\chi_{\underline{\omega}}$ is unitarily equivalent to $\sigma^{\underline{\omega}}$, which acts on $L^2(K_0)$ as follows: For $[\underline{x}, A] \in G$ and $f \in L^2(K_0)$,

$$\sigma^{\underline{\omega}}[\underline{x}, A]f(B) = e^{2\pi i \underline{\omega} B^{-1}\underline{x}} f(A^{-1}B), \text{ for all } B \in K_0.$$

Proof. For $[\underline{x}, A] \in G$ and $[\underline{0}, B] \in K$, $[\underline{x}, A]^{-1}[\underline{0}, B] = [-A^{-1}\underline{x}, A^{-1}B]$, so

$$[\underline{x}, A] \cdot [\underline{0}, B] = k_{[\underline{x},A]^{-1}[\underline{0},B]} = [\underline{0}, A^{-1}B] \quad \text{and} \quad h_{[\underline{x},A]^{-1}[\underline{0},B]} = [-B^{-1}\underline{x}, \mathrm{id}].$$

By Proposition 2.8.9, $\mathrm{ind}_H^G \chi_{\underline{\omega}}$ is unitarily equivalent to $\sigma^{\chi_{\underline{\omega}}}$ acting on $L^2(K)$ by, for $[\underline{x}, A] \in G$ and $f \in L^2(K)$,

$$\sigma^{\chi_{\underline{\omega}}}[\underline{x}, A]f[\underline{0}, B] = \chi_{\underline{\omega}}\left(h^{-1}_{[\underline{x},A]^{-1}[\underline{0},B]}\right) f[\underline{0}, A^{-1}B] = e^{2\pi i \underline{\omega} B^{-1}\underline{x}} f[\underline{0}, A^{-1}B],$$

for all $[\underline{0}, B] \in K$. Let $U : L^2(K) \to L^2(K_0)$ be the obvious unitary map $Uf(B) = f[\underline{0}, B]$, for all $B \in K_0$ and $f \in L^2(K_0)$. Then define $\sigma^{\underline{\omega}}$ acting on $L^2(K_0)$ by $\sigma^{\underline{\omega}}[\underline{x}, A] = U\sigma^{\chi_{\underline{\omega}}}[\underline{x}, A]U^{-1}$, for all $[\underline{x}, A] \in G$. Thus, $\sigma^{\underline{\omega}}[\underline{x}, A]f(B) = e^{2\pi i \underline{\omega} B^{-1}\underline{x}} f(A^{-1}B)$, for all $B \in K_0$ and $\mathrm{ind}_H^G \chi_{\underline{\omega}}$ is unitarily equivalent to both $\sigma^{\chi_{\underline{\omega}}}$ and $\sigma^{\underline{\omega}}$. $\square$

Remark. Continuing with the notation of the Corollary, let $J : L^2(K_0) \to L^2(K_0)$ be given by, for $f \in L^2(K_0)$,

$$Jf(B) = \Delta_{K_0}(B)^{-1/2} f(B^{-1}), \text{ for a.e. } B \in K_0.$$

Then J is a unitary operator on $L^2(K_0)$ with $J^{-1} = J$ and, for any $[\underline{x}, A] \in G$ and $f \in L^2(K_0)$,

$$\begin{aligned} J\sigma^{\underline{\omega}}[\underline{x}, A]J^{-1}f(B) &= \Delta_{K_0}(B)^{-1/2}\sigma^{\underline{\omega}}[\underline{x}, A](J^{-1}f)(B^{-1}) \\ &= \Delta_{K_0}(B)^{-1/2} e^{2\pi i \underline{\omega} B \underline{x}} (J^{-1}f)(A^{-1}B^{-1}) \\ &= \Delta_{K_0}(A)^{1/2} e^{2\pi i \underline{\omega} B \underline{x}} f(BA), \end{aligned}$$

for a.e. $B \in K_0$. This provides another equivalent representation to $\mathrm{ind}_H^G \chi_{\underline{\omega}}$. For now, it will be unnamed.

2.9 Mackey Theory for Semi-direct Products

In [22], George Mackey developed a systematic method of describing $\widehat{G}$ in certain situations where G is a semi-direct product. We will introduce Mackey Theory in this section for the kinds of groups we are interested in. Most of the results are taken from [20].

Throughout this section, H is a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$ and

$$G = \mathbb{R}^n \rtimes H = \{[\underline{x}, A] : \underline{x}, \in \mathbb{R}^n, A \in H\},$$

with group product $[\underline{x}, A][\underline{y}, B] = [\underline{x} + A\underline{y}, AB]$, identity $[\underline{0}, \mathrm{id}]$ and inverse of $[\underline{x}, A]$ given by $[\underline{x}, A]^{-1} = [-A^{-1}\underline{x}, A^{-1}]$. Let $N = \{[\underline{x}, \mathrm{id}] : \underline{x} \in \mathbb{R}^n\}$. a closed normal

Abelian subgroup of G. As in Section 2.7, the space of row vectors is denoted $\widehat{\mathbb{R}^n}$. That is $\widehat{\mathbb{R}^n} = \{\underline{\omega} = (\omega_1, \cdots, \omega_n) : \omega_1, \cdots, \omega_n \in \mathbb{R}\}$. For each $\underline{\omega} \in \widehat{\mathbb{R}^n}$, define a character $\chi_{\underline{\omega}}$ on N by,

$$\chi_{\underline{\omega}}[\underline{x}, \mathrm{id}] = e^{2\pi i \underline{\omega}\underline{x}}, \text{ for all } [\underline{x}, \mathrm{id}] \in N.$$

Then $\widehat{N} = \left\{\chi_{\underline{\omega}} : \underline{\omega} \in \widehat{\mathbb{R}^n}\right\}$. The group G acts on $\widehat{N}$ as follows: If $[\underline{x}, A] \in G$ and $\chi \in \widehat{N}$, then

$$([\underline{x}, A] \cdot \chi)[\underline{y}, \mathrm{id}] = \chi\big([\underline{x}, A]^{-1}[\underline{y}, \mathrm{id}][\underline{x}, A]\big), \text{ for all } [\underline{y}, \mathrm{id}] \in N. \tag{2.12}$$

Compute $[\underline{x}, A]^{-1}[\underline{y}, \mathrm{id}][\underline{x}, A] = [-A^{-1}\underline{x}, A^{-1}][\underline{y} + \underline{x}, A] = [A^{-1}\underline{y}, \mathrm{id}]$. If $\underline{\omega} \in \widehat{\mathbb{R}^n}$ and $\chi = \chi_{\underline{\omega}}$ in (2.12), then

$$([\underline{x}, A] \cdot \chi_{\underline{\omega}})[\underline{y}, \mathrm{id}] = \chi_{\underline{\omega}}[A^{-1}\underline{y}, \mathrm{id}] = e^{2\pi i \underline{\omega} A^{-1}\underline{y}} = \chi_{\underline{\omega}A^{-1}}[\underline{y}, \mathrm{id}],$$

for all $[\underline{y}, \mathrm{id}] \in N$. Thus, $[\underline{x}, A] \cdot \chi_{\underline{\omega}} = \chi_{\underline{\omega}A^{-1}}$. Thus, the action of G on $\widehat{N}$ is completely determined by the action of H on $\widehat{\mathbb{R}^n}$ given by $A \cdot \underline{\omega} = \underline{\omega}A^{-1}$. The G-orbits in $\widehat{N}$ are fundamental to Mackey Theory. Here, we will express the statements in terms of H-orbits in $\widehat{\mathbb{R}^n}$.

Definition 2.9.1. Let $\underline{\omega} \in \widehat{\mathbb{R}^n}$. The *$H$-orbit* of $\underline{\omega}$ is

$$\mathcal{O}_{\underline{\omega}} = \{A \cdot \underline{\omega} : A \in H\} = \{\underline{\omega}A^{-1} : A \in H\} = \{\underline{\omega}A : A \in H\}.$$

The *stability subgroup* for $\underline{\omega}$ is

$$H_{\underline{\omega}} = \{A \in H : A \cdot \underline{\omega} = \underline{\omega}\} = \{A \in H : \underline{\omega}A = \underline{\omega}\}.$$

Note that $H_{\underline{\omega}}$ is a closed subgroup of H. For a given $\underline{\omega} \in \widehat{\mathbb{R}^n}$,

$$\{[\underline{x}, A] \in G : [\underline{x}, A] \cdot \chi_{\underline{\omega}} = \chi_{\underline{\omega}}\} = \mathbb{R}^n \rtimes H_{\underline{\omega}}.$$

Definition 2.9.2. For any unitary representation π of $H_{\underline{\omega}}$ and any $[\underline{x}, A] \in \mathbb{R}^n \rtimes H_{\underline{\omega}}$, let

$$(\chi_{\underline{\omega}} \otimes \pi)[\underline{x}, A] = \chi_{\underline{\omega}}[\underline{x}, \mathrm{id}]\pi(A) = e^{2\pi i \underline{\omega}\underline{x}}\pi(A).$$

Proposition 2.9.3. Let $\underline{\omega} \in \widehat{\mathbb{R}^n}$ and let π be a unitary representation of $H_{\underline{\omega}}$. Then $\chi_{\underline{\omega}} \otimes \pi$ is a unitary representation of $\mathbb{R}^n \rtimes H_{\underline{\omega}}$ also acting on the Hilbert space $\mathcal{H}_\pi$. Moreover, $\chi_{\underline{\omega}} \otimes \pi$ is an irreducible representation of $\mathbb{R}^n \rtimes H_{\underline{\omega}}$ if and only if π is an irreducible representation of $H_{\underline{\omega}}$.

Proof. Clearly, $e^{2\pi i \underline{\omega}\underline{x}}\pi(A)$ is a unitary operator on $\mathcal{H}_\pi$, for each $[\underline{x}, A] \in \mathbb{R}^n \rtimes H_{\underline{\omega}}$. Let $[\underline{x}, A], [\underline{y}, B] \in \mathbb{R}^n \rtimes H_{\underline{\omega}}$. Then, for any $\xi \in \mathcal{H}_\pi$

$$\begin{aligned}(\chi_{\underline{\omega}} \otimes \pi)[\underline{x}, A](\chi_{\underline{\omega}} \otimes \pi)[\underline{y}, B]\xi &= e^{2\pi i \underline{\omega}\underline{x}}\pi(A)\big((\chi_{\underline{\omega}} \otimes \pi)[\underline{y}, B]\xi\big) \\ &= e^{2\pi i \underline{\omega}\underline{x}}\pi(A)e^{2\pi i \underline{\omega}\underline{y}}\pi(B) = e^{2\pi i \underline{\omega}(\underline{x}+\underline{y})}\pi(AB).\end{aligned}$$

On the other hand, $[\underline{x}, A][\underline{y}, B] = [\underline{x} + A\underline{y}, AB]$, so

$$\begin{aligned}(\chi_{\underline{\omega}} \otimes \pi)\big([\underline{x}, A][\underline{y}, B]\big) &= e^{2\pi i \underline{\omega}(\underline{x}+A\underline{y})}\pi(AB) \\ &= e^{2\pi i \underline{\omega}(\underline{x}+\underline{y})}\pi(AB),\end{aligned}$$

since $\underline{\omega}A = \underline{\omega}$, because $A \in H_{\underline{\omega}}$. Thus, $\chi_{\underline{\omega}} \otimes \pi : \mathbb{R}^n \rtimes H_{\underline{\omega}} \to \mathcal{U}(\mathcal{H}_\pi)$ is a homomorphism. The rest of the claims of the proposition are now easy to check. □

We need to make an assumption on the topology of the H-orbits.

Definition 2.9.4. Let X be a topological space. A subset $A \subseteq X$ is called *locally closed* if there exists an open $U \subseteq X$ and a closed $F \subseteq X$ such that $A = U \cap F$.

If each H-orbit in $\widehat{\mathbb{R}^n}$ is locally closed, then Remark 4.26(2) of [20] shows that N is a Mackey compatible subgroup of G in the sense defined in Definition 4.25 of [20]. In this case, there is a systematic method of constructing one member of each equivalence class of irreducible representations of G from the irreducible representations of the stability subgroups, $H_{\underline{\omega}}$. The method is summarized in Theorem 4.29 of [20], which is restated here in our notation.

Theorem 2.9.5. Let H be a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$, let $G = \mathbb{R}^n \rtimes H$, and let $N = \{[\underline{x}, \mathrm{id}] : \underline{x} \in \mathbb{R}^n\}$. Assume that each H-orbit in $\widehat{\mathbb{R}^n}$ is locally closed. Let $X \subseteq \widehat{\mathbb{R}^n}$ be such that $\mathcal{O} \cap X$ is a singleton, for each H-orbit $\mathcal{O}$ in $\widehat{\mathbb{R}^n}$. Then

(a) For each $\underline{\omega} \in X$ and each irreducible representation π of $H_{\underline{\omega}}$, $\mathrm{ind}^G_{\mathbb{R}^n \rtimes H_{\underline{\omega}}}(\chi_{\underline{\omega}} \otimes \pi)$ is an irreducible representation of G.

(b) For each irreducible representation σ of G, there exists a unique $\underline{\omega} \in X$ and, up to equivalence, a unique irreducible representation of $H_{\underline{\omega}}$ such that

$$\sigma \sim \mathrm{ind}^G_{\mathbb{R}^n \rtimes H_{\underline{\omega}}}(\chi_{\underline{\omega}} \otimes \pi).$$

Often notation is abused by using the same symbol for an irreducible representation and its equivalence class. If we do that, the conclusions of Theorem 2.9.5 can be summarized by

$$\widehat{G} = \bigcup_{\underline{\omega} \in X} \left\{ \mathrm{ind}^G_{\mathbb{R}^n \rtimes H_{\underline{\omega}}}(\chi_{\underline{\omega}} \otimes \pi) : \pi \in \widehat{H_{\underline{\omega}}} \right\}. \tag{2.13}$$

Remark. It is easy to check the condition that every H-orbit is locally closed and every example considered in this thesis satisfies this condition. However, there is a closed subgroup H^M of $\mathrm{GL}_4(\mathbb{R})$ such that the H^M-orbits in $\widehat{\mathbb{R}^4}$ are not locally closed. See Example 4.45 of [20]. The group $\mathbb{R}^4 \rtimes H^M$ is called a *Mautner group* and the description given in (2.13) fails for this group.

The main focus of this thesis is on irreducible representations that are square-integrable. So, we might ask when a representation included in (2.13) is square-integrable. There is a clear answer. The following theorem is a consequence of Corollary 11.1 of [21].

Theorem 2.9.6. Let H be a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$, let $G = \mathbb{R}^n \rtimes H$, and let $N = \{[\underline{x}, \mathrm{id}] : \underline{x} \in \mathbb{R}^n\}$. For $\underline{\omega} \in \widehat{\mathbb{R}^n}$ and an irreducible representation of $H_{\underline{\omega}}$, the representation $\mathrm{ind}_{\mathbb{R}^n \rtimes H_{\underline{\omega}}}^G (\chi_{\underline{\omega}} \otimes \pi)$ is square-integrable if and only if the H-orbit $\mathcal{O}_{\underline{\omega}}$ is open and π is a square-integrable representation of $H_{\underline{\omega}}$.

Corollary 2.9.7. Let H be a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$, let $G = \mathbb{R}^n \rtimes H$, and let $N = \{[\underline{x}, \mathrm{id}] : \underline{x} \in \mathbb{R}^n\}$. For $\underline{\omega} \in \widehat{\mathbb{R}^n}$, if the H-orbit $\mathcal{O}_{\underline{\omega}}$ is open and $H_{\underline{\omega}}$ is compact, then $\mathrm{ind}_{\mathbb{R}^n \rtimes H_{\underline{\omega}}}^G (\chi_{\underline{\omega}} \otimes \pi)$ is square-integrable for each $\pi \in \widehat{H_{\underline{\omega}}}$.

The nature of the wavelet transform associated with the square-integrable representations of the form $\mathrm{ind}_{\mathbb{R}^n \rtimes H_{\underline{\omega}}}^G (\chi_{\underline{\omega}} \otimes \pi)$ as in Corollary 2.9.7 has been studied by Führ, see [14], in the case where π is the trivial representation of the compact stability subgroup. In Chapter 3, we work out the details for all square-integrable representations of an example where the orbit is open and the stability subgroup is compact.

Chapter 3

An Example with a Compact Stability Subgroup

In [5], they classified the closed connected subgroups H of $\mathrm{GL}_3(\mathbb{R})$ with an open orbit $\mathcal{O}$ in $\widehat{\mathbb{R}^3}$ and such that the stability subgroup $H_{\underline{\omega}}$ is compact for $\underline{\omega} \in \mathcal{O}$. Since $H_{\underline{\omega}}$ is compact, if π is an irreducible representation of $H_{\underline{\omega}}$, then π is a square-integrable representation of $H_{\underline{\omega}}$. If we induce the representation $\chi_{\underline{\omega}} \otimes \pi$ from $\mathbb{R}^3 \rtimes H_{\underline{\omega}}$ up to $\mathbb{R}^3 \rtimes H$, the result will be a square-integrable representation of $\mathbb{R}^3 \rtimes H$ by Proposition 11.1 of [21] or the first section of [4], but there is value in working out the details.

3.1 The Example

In this section, we work out the details of a square-integrable representation and the associated wavelet theory for an illustrative example selected from the list in [5].

Let $H = \left\{ \begin{pmatrix} a & t_1 & t_2 \\ 0 & \cos(2\pi u) & -\sin(2\pi u) \\ 0 & \sin(2\pi u) & \cos(2\pi u) \end{pmatrix} : a, t_1, t_2 \in \mathbb{R}, a \neq 0, u \in [0,1) \right\}$. In order to keep the formulas compact and easier to read, let

$$R_u = \begin{pmatrix} \cos(2\pi u) & -\sin(2\pi u) \\ \sin(2\pi u) & \cos(2\pi u) \end{pmatrix}, \text{ for } u \in \mathbb{R}.$$

Note that $R_{u+k} = R_u$, for all $k \in \mathbb{Z}$, and $R_{u_1} R_{u_2} = R_{u_1+u_2}$, for $u, u_1, u_2 \in \mathbb{R}$. Also, write $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2 = \left\{ \begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix} : x_1 \in \mathbb{R}, \underline{y} \in \mathbb{R}^2 \right\}$ and let $\underline{t}^{\mathrm{t}} = (t_1, t_2)$, for $\underline{t} \in \mathbb{R}^2$. Then $H = \left\{ \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} : a \in \mathbb{R}^*, \underline{t} \in \mathbb{R}^2, u \in [0,1) \right\}$. For $A = \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix}$ and $B = \begin{pmatrix} b & \underline{s}^{\mathrm{t}} \\ \underline{0} & R_v \end{pmatrix}$ in H,

$$BA = \begin{pmatrix} ba & b\underline{t}^{\mathrm{t}} + \underline{s}^{\mathrm{t}} R_u \\ \underline{0} & R_{v+u} \end{pmatrix} \text{ and } A^{-1} = \begin{pmatrix} a^{-1} & -a^{-1}\underline{t}^{\mathrm{t}} R_{-u} \\ \underline{0} & R_{-u} \end{pmatrix}.$$

One can check that left Haar integration on H is given by, for any $f \in C_c(H)$,

$$\int_H f(A)\, d\mu_H(A) = \int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_0^1 f\begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \frac{du\, d\underline{t}\, da}{|a|^3} \tag{3.1}$$

and the modular function of H is $\Delta_H \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} = a^{-2}$.

Let $G = \{[\underline{x}, A] : \underline{x} \in \mathbb{R}^3, A \in H\}$. The left Haar integral on G is given by

$$\int_H \int_{\mathbb{R}^3} f[\underline{x}, A] \frac{d\underline{x}\, d\mu_H(A)}{|\det(A)|}, \tag{3.2}$$

for any $f \in C_c(G)$. We also have $\Delta_G \left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \right] = |a|^{-3}$.

Let $N = \left\{ \left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \mathrm{id}_3 \right] : x_1 \in \mathbb{R}, \underline{y} \in \mathbb{R}^2 \right\}$, where id_j denotes the $j \times j$ identity matrix, for $j = 2$ or 3. For $(\omega_1, \underline{\omega}) \in \widehat{\mathbb{R}} \times \widehat{\mathbb{R}^2}$, define $\chi_{(\omega_1, \underline{\omega})}$ on N by

$$\chi_{(\omega_1,\underline{\omega})} \left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \mathrm{id}_3 \right] = e^{2\pi i(\omega_1 x_1 + \underline{\omega}\underline{y})}, \text{ for all } \left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \mathrm{id}_3 \right] \in N.$$

Then $\widehat{N} = \{\chi_{(\omega_1,\underline{\omega})} : (\omega_1, \underline{\omega}) \in \widehat{\mathbb{R}} \times \widehat{\mathbb{R}^2}\}$. The action of H on $\widehat{N}$ is given by the action of H on $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}^2}$. For $A = \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \in H$ and $(\omega_1, \underline{\omega}) \in \widehat{\mathbb{R}} \times \widehat{\mathbb{R}^2}$,

$$\begin{aligned} A \cdot (\omega_1, \underline{\omega}) = (\omega_1, \underline{\omega})A^{-1} &= (\omega_1, \underline{\omega}) \begin{pmatrix} a^{-1} & -a^{-1}\underline{t}^{\mathrm{t}} R_{-u} \\ \underline{0} & R_{-u} \end{pmatrix} \\ &= \left(a^{-1}\omega_1, (\underline{\omega} - a^{-1}\omega_1 \underline{t}^{\mathrm{t}})R_{-u}\right). \end{aligned}$$

The H-orbit of $(1, \underline{0}) = (1, 0, 0)$ is

$$\mathcal{O} = \{(a^{-1}, -a^{-1}\underline{t}^{\mathrm{t}} R_{-u}) : a \in \mathbb{R}^*, \underline{t} \in \mathbb{R}^2, u \in [0, 1)\} = \{(\omega_1, \underline{\omega}) : \omega_1 \neq 0, \underline{\omega} \in \widehat{\mathbb{R}^2}\}.$$

Note that $\mathcal{O}$ is co-null in $\widehat{\mathbb{R}^3}$, so $L^2(\widehat{\mathbb{R}^3}) = L^2(\mathcal{O})$. There are other H-orbits, but they are null sets and are not used in the following.

We will compute the irreducible representations of G that are associated with the orbit $\mathcal{O}$, show that they are square-integrable and work out the associated Duflo-Moore operators. The stability subgroup $H_{(1,\underline{0})}$ is found by solving $(1, \underline{0})A^{-1} = (1, \underline{0})$. But

$$(1, \underline{0}) = (a^{-1}, -a^{-1}\underline{t}^{\mathrm{t}} R_{-u}) \text{ implies } a = 1 \text{ and } \underline{t}^{\mathrm{t}} = (0, 0).$$

Thus $H_{(1,\underline{0})} = \left\{ \begin{pmatrix} 1 & \underline{0}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} : u \in [0, 1) \right\}$. This is a compact subgroup of H that is isomorphic to $\mathbb{T}$ via the map $\begin{pmatrix} 1 & \underline{0}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \to e^{2\pi i u}$. Therefore, $\widehat{H_{(1,\underline{0})}} = \{\psi_j : j \in \mathbb{Z}\}$, where

$$\psi_j \begin{pmatrix} 1 & \underline{0}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} = e^{2\pi i j u}, \text{ for } u \in [0,1).$$

For each $j \in \mathbb{Z}$, the representation $\chi_{(1,\underline{0})} \otimes \psi_j$ of $\mathbb{R}^3 \rtimes H_{(1,\underline{0})}$ is given by

$$(\chi_{(1,\underline{0})} \otimes \psi_j) \left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} 1 & \underline{0}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \right] = e^{2\pi i (x_1 + ju)}, \text{ for all } \left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} 1 & \underline{0}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \right] \in \mathbb{R}^3 \rtimes H_{(1,\underline{0})}.$$

Denote $H' = \mathbb{R}^3 \rtimes H_{(1,\underline{0})}$ and $\pi_j = \chi_{(1,\underline{0})} \otimes \psi_j$, an irreducible representation of H'. We can use Proposition 2.8.9 to find the unitary representation σ^{π_j} equivalent to $\mathrm{ind}_{H'}^{G}(\pi_j)$, acting on $L^2(K)$, if we can find a closed subgroup K of G such that $G = KH'$ and $K \cap H' = \{[\underline{0}, \mathrm{id}_3]\}$. This is not difficult in this group. Let

$$K = \left\{ \left[\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \right] : a \neq 0, \underline{t} \in \mathbb{R}^2 \right\}.$$

It is clear that K is a closed subgroup of G and $K \cap H' = \{[\underline{0}, \mathrm{id}_3]\}$. Note that, for any $\left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \right] \in G$, we have

$$\left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \right] = \left[\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} R_{-u} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \right] \left[\begin{pmatrix} a^{-1}(x_1 - \underline{t}^{\mathrm{t}} R_{-u} \underline{y}) \\ \underline{y} \end{pmatrix}, \begin{pmatrix} 1 & \underline{0}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \right]. \quad (3.3)$$

Thus, $G = KH'$ and the hypotheses of Proposition 2.8.9 hold. In order to use the formula in Proposition 2.8.9, we need the modular function of H'. But, it is easy to verify that H' is unimodular and the Haar integral on H' is given by

$$\int_{H'} f \, d\mu_{H'} = \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}^2} f \left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} 1 & \underline{0}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \right] d\underline{y} \, dx_1 \, du,$$

for all $f \in C_c(H')$. Then $\left[\frac{\Delta_{H'}(h)}{\Delta_G(h)} \right]^{1/2} = \Delta_G(h)^{-1/2}$, for $h \in H'$. But any $h \in H'$ is of the form $\left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} 1 & \underline{0}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \right]$, so $\Delta_G(h) = 1$. Thus, the first factor on the right hand side of the formula from Proposition 2.8.9 is 1. To complete the evaluation of the formula in Proposition 2.8.9, we need the following lemma.

Lemma 3.1.1. Let $x = \left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \right] \in G$ and $k = \left[\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix}, \begin{pmatrix} b & \underline{s}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \right] \in K$. Then $x^{-1}k = k_{x^{-1}k} h_{x^{-1}k}$, where

$$k_{x^{-1}k} = \left[\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix}, \begin{pmatrix} a^{-1}b & a^{-1}(\underline{s}^{\mathrm{t}} R_u - \underline{t}^{\mathrm{t}}) \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \right] \in K$$

and

$$h_{x^{-1}k} = \left[\begin{pmatrix} b^{-1}(\underline{s}^{\mathrm{t}} \underline{y} - x_1) \\ -R_{-u} \underline{y} \end{pmatrix}, \begin{pmatrix} 1 & \underline{0}^{\mathrm{t}} \\ \underline{0} & R_{-u} \end{pmatrix} \right] \in H'.$$

Proof. Compute $x^{-1} = \left[\begin{pmatrix} -a^{-1}(x_1 - \underline{t}^{\mathrm{t}} R_{-u}\underline{y}) \\ -R_{-u}\underline{y} \end{pmatrix}, \begin{pmatrix} a^{-1} & -a^{-1}\underline{t}^{\mathrm{t}} R_{-u} \\ \underline{0} & R_{-u} \end{pmatrix}\right]$. So

$$x^{-1}k = \left[\begin{pmatrix} -a^{-1}(x_1 - \underline{t}^{\mathrm{t}} R_{-u}\underline{y}) \\ -R_{-u}\underline{y} \end{pmatrix}, \begin{pmatrix} a^{-1}b & a^{-1}(\underline{s}^{\mathrm{t}} - \underline{t}^{\mathrm{t}} R_{-u}) \\ \underline{0} & R_{-u} \end{pmatrix}\right].$$

Now use (3.3) to find $k_{x^{-1}k}$ and $h_{x^{-1}k}$. We check by computing $k_{x^{-1}k}h_{x^{-1}k}$.

$$\left[\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix}, \begin{pmatrix} a^{-1}b & a^{-1}(\underline{s}^{\mathrm{t}} R_u - \underline{t}^{\mathrm{t}}) \\ \underline{0} & \mathrm{id}_2 \end{pmatrix}\right]\left[\begin{pmatrix} b^{-1}(\underline{s}^{\mathrm{t}}\underline{y} - x_1) \\ -R_{-u}\underline{y} \end{pmatrix}, \begin{pmatrix} 1 & \underline{0}^{\mathrm{t}} \\ \underline{0} & R_{-u} \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} -a^{-1}(x_1 - \underline{t}^{\mathrm{t}} R_{-u}\underline{y}) \\ -R_{-u}\underline{y} \end{pmatrix}, \begin{pmatrix} a^{-1}b & a^{-1}(\underline{s}^{\mathrm{t}} - \underline{t}^{\mathrm{t}} R_{-u}) \\ \underline{0} & R_{-u} \end{pmatrix}\right]$$

This verifies the expressions given for $k_{x^{-1}k}$ and $h_{x^{-1}k}$. □

We need $h_{x^{-1}k}^{-1} = \left[\begin{pmatrix} b^{-1}(x_1 - \underline{s}^{\mathrm{t}}\underline{y}) \\ \underline{y} \end{pmatrix}, \begin{pmatrix} 1 & \underline{0}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix}\right]$ and the notation

$$x^{-1} \cdot k = k_{x^{-1}k} = \left[\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix}, \begin{pmatrix} a^{-1}b & a^{-1}(\underline{s}^{\mathrm{t}} R_u - \underline{t}^{\mathrm{t}}) \\ \underline{0} & \mathrm{id}_2 \end{pmatrix}\right].$$

Then $\pi_j(h_{x^{-1}k}^{-1}) = e^{2\pi i\left(b^{-1}(x_1 - \underline{s}^{\mathrm{t}}\underline{y}) + ju\right)}$. Plugging this and the expression for $x^{-1} \cdot k$ into the formula in Proposition 2.8.9 gives the induced representation σ^{π_j}.

Proposition 3.1.2. Let $\left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix}\right] \in G$. Then $\sigma^{\pi_j}\left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix}\right]$ acts on $f \in L^2(K)$ as follows:

$$\left(\sigma^{\pi_j}\left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix}\right] f\right)\left[\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix}, \begin{pmatrix} b & \underline{s}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix}\right]$$

$$= e^{2\pi i\left(b^{-1}(x_1 - \underline{s}^{\mathrm{t}}\underline{y}) + ju\right)} f\left[\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix}, \begin{pmatrix} a^{-1}b & a^{-1}(\underline{s}^{\mathrm{t}} R_u - \underline{t}^{\mathrm{t}}) \\ \underline{0} & \mathrm{id}_2 \end{pmatrix}\right],$$

for all $\left[\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix}, \begin{pmatrix} b & \underline{s}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix}\right] \in K$.

For each $(\omega_1, \underline{\omega}) \in \mathcal{O}$, there is a unique $\gamma(\omega_1, \underline{\omega}) \in K$ such that

$$(\omega_1, \underline{\omega}) = \gamma(\omega_1, \underline{\omega}) \cdot (1, \underline{0}).$$

Let $K_0 = \left\{\begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} : a \in \mathbb{R}^*, \underline{t} \in \mathbb{R}^2\right\}$. So $K = \{[\underline{0}, A] : A \in K_0\}$. Note that

$$\begin{pmatrix} b & \underline{s}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix}\begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} = \begin{pmatrix} ba & b\underline{t}^{\mathrm{t}} + \underline{s}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix}$$

$\begin{pmatrix} b & \underline{s}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \in K_0$. Thus, for $f \in C_c(K_0)$,

$$\begin{aligned}
\int_{\mathbb{R}} \int_{\mathbb{R}^2} f\left(\begin{pmatrix} b & \underline{s}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \right) \frac{d\underline{t}\, da}{|a|^3} &= \int_{\mathbb{R}} \int_{\mathbb{R}^2} f \begin{pmatrix} ba & b\underline{t}^{\mathrm{t}} + \underline{s}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \frac{d\underline{t}\, da}{|a|^3} \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} f \begin{pmatrix} ba & b\underline{t}^{\mathrm{t}} + \underline{s}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \frac{b^2 d\underline{t}}{(ba)^2} \right) \frac{da}{|a|} \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} f \begin{pmatrix} ba & \underline{t}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \frac{d\underline{t}}{(ba)^2} \right) \frac{da}{|a|} \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} f \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \frac{d\underline{t}}{a^2} \right) \frac{da}{|a|} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^2} f \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \frac{d\underline{t}\, da}{|a|^3}.
\end{aligned}$$

Thus, $\int_{K_0} f\, d\mu_{K_0} = \int_{\mathbb{R}} \int_{\mathbb{R}^2} f \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \frac{d\underline{t}\, da}{|a|^3}$, for any integrable function f on K_0. So $\int_K f\, d\mu_K = \int_{\mathbb{R}} \int_{\mathbb{R}^2} f \left[\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \right] \frac{d\underline{t}\, da}{|a|^3}$, for any integrable function f on K. The map $A \to A \cdot (1, \underline{0})$ is a homeomorphism of K_0 with the H-orbit $\mathcal{O}$. Let γ denote the inverse of this homeomorphism. That is, for each $(\omega_1, \underline{\omega}) \in \mathcal{O}$, $\gamma(\omega_1, \underline{\omega}) \cdot (1, \underline{0}) = (\omega_1, \underline{\omega})$. If $\gamma(\omega_1, \underline{\omega}) = \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix}$, then

$$\gamma(\omega_1, \underline{\omega}) \cdot (1, \underline{0}) = (1, \underline{0}) \gamma(\omega_1, \underline{\omega})^{-1} = (1, \underline{0}) = \begin{pmatrix} a^{-1} & -a^{-1}\underline{t}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} = (a^{-1}, -a^{-1}\underline{t}^{\mathrm{t}}).$$

Thus $\gamma(\omega_1, \underline{\omega}) \cdot (1, \underline{0}) = (\omega_1, \underline{\omega})$ implies $\gamma(\omega_1, \underline{\omega})^{-1} = \begin{pmatrix} \omega_1 & \underline{\omega} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix}$ and

$$\gamma(\omega_1, \underline{\omega}) = \begin{pmatrix} \omega_1^{-1} & -\omega_1^{-1}\underline{\omega} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix}. \tag{3.4}$$

Then $(\omega_1, \underline{\omega}) \to [\underline{0}, \gamma(\omega_1, \underline{\omega})]$ is a homeomorphism of $\mathcal{O}$ to K, which can be used to define a unitary map of $L^2(K)$ onto $L^2(\mathcal{O})$.

Definition 3.1.3. For $f \in L^2(K)$ define Uf on $\mathcal{O}$ by $(Uf)(\omega_1, \underline{\omega}) = |\omega_1|^{-1/2} f[\underline{0}, \gamma(\omega_1, \underline{\omega})]$, for a.e. $(\omega_1, \underline{\omega}) \in \mathcal{O}$.

Proposition 3.1.4. For each $f \in L^2(K)$, $Uf \in L^2(\mathcal{O})$. Moreover, U is a unitary map of $L^2(K)$ onto $L^2(\mathcal{O})$.

Proof. For $f \in L^2(K)$,

$$\begin{aligned}
\int_{\widehat{\mathbb{R}}}\int_{\widehat{\mathbb{R}^2}} |Uf(\omega_1,\underline{\omega})|^2 d\underline{\omega}\, d\omega_1 &= \int_{\widehat{\mathbb{R}}}\int_{\widehat{\mathbb{R}^2}} |f[\underline{0},\gamma(\omega_1,\underline{\omega})]|^2 |\omega_1|^{-1} d\underline{\omega}\, d\omega_1 \\
&= \int_{\widehat{\mathbb{R}}} \left(\int_{\mathbb{R}^2} \left| f\left[\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix}, \begin{pmatrix} \omega_1^{-1} & -\omega_1^{-1}\underline{\omega} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \right] \right|^2 d\underline{\omega} \right) |\omega_1|^{-1} d\omega_1 \\
&= \int_{\widehat{\mathbb{R}}} \left(\int_{\widehat{\mathbb{R}^2}} \left| f\left[\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix}, \begin{pmatrix} \omega_1^{-1} & \underline{t}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \right] \right|^2 d\underline{t} \right) |\omega_1|\, d\omega_1 \\
&= \int_{\mathbb{R}}\int_{\widehat{\mathbb{R}^2}} \left| f\left[\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \right] \right|^2 d\underline{t} \frac{da}{|a|^3} \\
&= \int_K |f|^2 d\mu_K.
\end{aligned}$$

Thus $Uf \in L^2(\mathcal{O})$ and U is an isometry. Clearly, U is linear. Moreover, for $\xi \in L^2(\mathcal{O})$, define f on K by $f\left[\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \right] = |a|^{-1/2}\xi(a^{-1}, -a^{-1}\underline{t}^{\mathrm{t}})$, for $\left[\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & \mathrm{id}_2 \end{pmatrix} \right]$ in K. Then a similar change of variables calculation shows $f \in L^2(K)$ and $Uf = \xi$. Thus, U is a unitary map onto $L^2(\mathcal{O})$. □

The unitary U can be used to move σ^{π_j} to an equivalent representation acting on $L^2(\mathcal{O})$.

Definition 3.1.5. For $\left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \right] \in G$, let

$$\sigma_1^{\pi_j}\left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \right] = U\sigma^{\pi_j}\left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \right] U^{-1}.$$

Then $\sigma_1^{\pi_j}$ is a unitary representation of G on $L^2(\mathcal{O})$. To compute an explicit expression for $\sigma_1^{\pi_j}$, let $\xi \in L^2(\mathcal{O})$ and let $f = U^{-1}\xi$. For $\left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \right] \in G$,

$$\begin{aligned}
&\sigma_1^{\pi_j}\left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \right] \xi(\omega_1,\underline{\omega}) = U\left(\sigma^{\pi_j}\left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \right] f \right)(\omega_1,\underline{\omega}) \\
&\quad = |\omega_1|^{-1/2}\sigma^{\pi_j}\left[\begin{pmatrix} x_1 \\ \underline{y} \end{pmatrix}, \begin{pmatrix} a & \underline{t}^{\mathrm{t}} \\ \underline{0} & R_u \end{pmatrix} \right] f\left[\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix}, \begin{pmatrix} \omega_1^{-1} & -\omega_1^{-1}\underline{\omega} \\ \underline{0} & \mathrm{id} \end{pmatrix} \right] \\
&\quad = |\omega_1|^{-1/2} e^{2\pi i(\omega_1 x_1 + \underline{\omega}\underline{y} + ju)} U^{-1}\xi\left[\begin{pmatrix} 0 \\ \underline{0} \end{pmatrix}, \begin{pmatrix} (\omega_1 a)^{-1} & -(\omega_1 a)^{-1}\underline{\omega}R_u - a^{-1}\underline{t}^{\mathrm{t}} \\ \underline{0} & \mathrm{id} \end{pmatrix} \right] \\
&\quad = |a|^{1/2} e^{2\pi i(\omega_1 x_1 + \underline{\omega}\underline{y} + ju)} \xi(a\omega_1, \underline{\omega}R_u + \omega_1\underline{t}^{\mathrm{t}}).
\end{aligned}$$

To make this representation clear in the natural notation of three dimensions, let

$$[\underline{x}, A] = \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} a & t_1 & t_2 \\ 0 & \cos(2\pi u_A) & -\sin(2\pi u_A) \\ 0 & \sin(2\pi u_A) & \cos(2\pi u_A) \end{pmatrix} \right] \in G,$$

where we introduce the notation u_A to keep track of the connection to the matrix A. Then $\sigma_1^{\pi_j}[\underline{x}, A]\xi(\underline{\omega}) = |\det(A)|^{1/2} e^{2\pi i(\underline{\omega}\underline{x} + j u_A)} \xi(\underline{\omega}A)$, for a.e. $\underline{\omega} \in \widehat{\mathbb{R}^3}$ and we are considering $\xi \in L^2(\widehat{\mathbb{R}^3}) = L^2(\mathcal{O})$. It is easy to verify directly that $\sigma_1^{\pi_j}[\underline{x}, A]$ is a unitary operator on $L^2(\widehat{\mathbb{R}^3})$.

Theorem 3.1.6. Let

$$G = \left\{ \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} a & t_1 & t_2 \\ 0 & \cos(2\pi u) & -\sin(2\pi u) \\ 0 & \sin(2\pi u) & \cos(2\pi u) \end{pmatrix} \right] : a \in \mathbb{R}^*, t_1, t_2, u \in \mathbb{R}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \right\}.$$

For each $j \in \mathbb{Z}$, define $\sigma_1^{\pi_j} : G \to \mathcal{U}\big(L^2(\widehat{\mathbb{R}^3})\big)$ by, for

$$[\underline{x}, A] = \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} a & t_1 & t_2 \\ 0 & \cos(2\pi u) & -\sin(2\pi u) \\ 0 & \sin(2\pi u) & \cos(2\pi u) \end{pmatrix} \right] \in G,$$

$\sigma_1^{\pi_j}[\underline{x}, A]\xi(\underline{\omega}) = |\det(A)|^{1/2} e^{2\pi i(\underline{\omega}\underline{x} + j u_A)} \xi(\underline{\omega}A)$, for a.e. $\underline{\omega} \in \widehat{\mathbb{R}^3}$ and each $\xi \in L^2(\widehat{\mathbb{R}^3})$. Then $\sigma_1^{\pi_j}$ is a square-integrable representation of G.

Proof. (Note that this proof closely follows the proof for open and free orbits given in [4].) Standard arguments show that $\sigma_1^{\pi_j}$ is a unitary representation of G. To prove that $\sigma_1^{\pi_j}$ is irreducible, we use property (d) of Theorem 2.4.11. Let $\xi, \eta \in L^2(\widehat{\mathbb{R}^3})$. For simplicity of notation write $\varphi_{\xi,\eta}$ for $\varphi_{\xi,\eta}^{\sigma_1^{\pi_j}}$. Then

$$\begin{aligned} \int_G |\varphi_{\xi,\eta}|^2 \, d\mu_G &= \int_G \left| \langle \sigma_1^{\pi_j}[\underline{x}, A]\xi, \eta \rangle \right|^2 d\mu_G[\underline{x}, A] \\ &= \int_G \left| \int_{\widehat{\mathbb{R}^3}} \sigma_1^{\pi_j}[\underline{x}, A]\xi(\underline{\omega})\overline{\eta(\underline{\omega})} \, d\underline{\omega} \right|^2 d\mu_G[\underline{x}, A] \qquad (3.5) \\ &= \int_G |\det(A)| \left| \int_{\widehat{\mathbb{R}^3}} e^{2\pi i(\underline{\omega}\underline{x} + j u_A)} \xi(\underline{\omega}A)\overline{\eta(\underline{\omega})} \, d\underline{\omega} \right|^2 d\mu_G[\underline{x}, A]. \end{aligned}$$

Note that $e^{2\pi i j u_A}$ factors out of the inner integral and has absolute value 1. Also, let $\phi_A(\underline{\omega}) = \xi(\underline{\omega}A)\overline{\eta(\underline{\omega})}$, for a.e. $\underline{\omega} \in \widehat{\mathbb{R}^3}$ and each $A \in H$. Then $\phi_A \in L^1(\widehat{\mathbb{R}^3})$ and the inverse Fourier transform of ϕ_A is denoted $\check{\phi}_A$. Using the left Haar integration on G

given in (3.2), (3.5) becomes

$$\begin{aligned}\int_G |\varphi_{\xi,\eta}|^2\; d\mu_G &= \int_H \int_{\mathbb{R}^3} |\det(A)| \cdot |\check{\phi_A}(\underline{x})|^2 \frac{d\underline{x},\, d\mu_H(A)}{|\det(A)|} \\ &= \int_H \int_{\mathbb{R}^3} |\check{\phi_A}(\underline{x})|^2 d\underline{x}\, d\mu_H(A) = \int_H \int_{\widehat{\mathbb{R}^3}} |\phi_A(\underline{\omega})|^2 d\underline{\omega}\, d\mu_H(A),\end{aligned} \tag{3.6}$$

where we have used Plancherel's Theorem, which gives ∞ for both sides if ϕ_A is not in $L^2(\widehat{\mathbb{R}^3})$. Recalling what ϕ_A is and changing the order of integration, (3.6) becomes

$$\begin{aligned}\int_G |\varphi_{\xi,\eta}|^2\; d\mu_G &= \int_{\widehat{\mathbb{R}^3}} \left(\int_H |\xi(\underline{\omega}A)|^2 |\eta(\underline{\omega})|^2 d\mu_H(A) \right) d\underline{\omega} \\ &= \int_{\widehat{\mathbb{R}^3}} |\eta(\underline{\omega})|^2 \left(\int_H |\xi(\underline{\omega}A)|^2 d\mu_H(A) \right) d\underline{\omega}.\end{aligned} \tag{3.7}$$

Note that $\int_H |\xi(\underline{\omega}BA)|^2 d\mu_H(A) = \int_H |\xi(\underline{\omega}A)|^2 d\mu_H(A)$, for any $B \in H$. Thus, we can define $c_\xi = \int_H |\xi(\underline{\omega}A)|^2 d\mu_H(A)$, for any $\underline{\omega} \in \mathcal{O}$. Then $\int_H |\xi(\underline{\omega}A)|^2 d\mu_H(A) = c_\xi$, for a.e. $\underline{\omega} \in \widehat{\mathbb{R}^3}$. Thus (3.7) becomes

$$\int_G |\varphi_{\xi,\eta}|^2\; d\mu_G = c_\xi \int_{\widehat{\mathbb{R}^3}} |\eta(\underline{\omega})|^2 d\underline{\omega}. \tag{3.8}$$

Therefore, if $\varphi_{\xi,\eta}[\underline{x}, A] = 0$, for all $[\underline{x}, A] \in G$, then either $\eta = 0$ or $c_\xi = 0$. But $c_\xi = 0$ implies $\xi(\underline{\omega}) = 0$, for a.e. $\underline{\omega} \in \widehat{\mathbb{R}^3}$. By condition (d) of Theorem 2.4.11, $\sigma_1^{\pi_j}$ is irreducible.

On the other hand, if $\xi \in C_c(\mathcal{O})$, $\xi \neq 0$, then $c_\xi < \infty$, so $\varphi_{\xi,\eta} \in L^2(G)$, for all $\eta \in L^2(\widehat{\mathbb{R}^3})$. In the notation of Section 2.5, $V_\xi \eta = \overline{\varphi_{\xi,\eta}} \in L^2(G)$. Therefore, $\sigma_1^{\pi_j}$ is square-integrable. □

The domain of the Duflo-Moore operator for the square-integrable representation $\sigma_1^{\pi_j}$ (see Section 2.5) is $\mathcal{D} = \{\xi \in L^2(\widehat{\mathbb{R}^3}) : V_\xi \xi \in L^2(G)\}$. Since $\int_G |V_\xi \xi|^2 d\mu_G = c_\xi^2$, $\mathcal{D} = \{\xi \in L^2(\widehat{\mathbb{R}^3}) : c_\xi < \infty\}$. A closer look at the condition that $c_\xi < \infty$ will give us a candidate for the Duflo-Moore operator for $\sigma_1^{\pi_j}$. Recall the Haar integral on H. Let $\underline{\omega}_0 = (1, 0, 0)$ be a fixed element of $\mathcal{O}$. If $A = \begin{pmatrix} a & t_1 & t_2 \\ 0 & \cos(2\pi u_A) & -\sin(2\pi u_A) \\ 0 & \sin(2\pi u_A) & \cos(2\pi u_A) \end{pmatrix} \in H$, then $(1, 0, 0)A = (a, t_1, t_2)$. Thus, using (3.1), for $\xi \in L^2(\widehat{\mathbb{R}^3})$,

$$c_\xi = \int_H |\xi\big((1,0,0)A\big)|^2\, d\mu_H(A) = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^1 |\xi(a, t_1, t_2)|^2\, \frac{du\, dt_2\, dt_1\, da}{|a|^3}.$$

Note that the integrand is independent of u. We also can use ω_1, ω_2, and ω_3 as the other variables of integration. Thus, for $\xi \in L^2(\widehat{\mathbb{R}^3})$,

$$\xi \in \mathcal{D} \quad \Leftrightarrow \quad \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \big| |\omega_1|^{-3/2} \xi(\omega_1, \omega_2, \omega_3) \big|^2\, d\omega_3\, d\omega_2\, d\omega_1 < \infty. \tag{3.9}$$

Define $T : \mathcal{D} \to L^2(\widehat{\mathbb{R}^3})$ by

$$(T\xi)(\omega_1, \omega_2, \omega_3) = |\omega_1|^{-3/2}\xi(\omega_1, \omega_2, \omega_3), \text{ for a.e. } (\omega_1, \omega_2, \omega_3) \in \widehat{\mathbb{R}^3},$$

and each $\xi \in L^2(\widehat{\mathbb{R}^3})$. Then T is a densely defined, positive, self-adjoint operator on $L^2(\widehat{\mathbb{R}^3})$.

Lemma 3.1.7. For any $[\underline{x}, A] \in G$, $\sigma_1^{\pi_j}[\underline{x}, A]T\sigma_1^{\pi_j}[\underline{x}, A]^* = \Delta_G[\underline{x}, A]^{1/2}T$.

Proof. If $\underline{\omega} = (\omega_1, \omega_2, \omega_3) \in \widehat{\mathbb{R}^3}$ and $A = \begin{pmatrix} a & t_1 & t_2 \\ 0 & \cos(2\pi u_A) & -\sin(2\pi u_A) \\ 0 & \sin(2\pi u_A) & \cos(2\pi u_A) \end{pmatrix}$, then the first component of $\underline{\omega}A$ is $a\omega_1$. Also, note that $\Delta_G[\underline{x}, A] = |a|^{-3}$, by the observation after (3.2). For any $\xi \in L^2(\widehat{\mathbb{R}^3})$,

$$\begin{aligned}
\sigma_1^{\pi_j}[\underline{x}, A]T\sigma_1^{\pi_j}[\underline{x}, A]^*\xi(\underline{\omega}) &= |\det(A)|^{1/2}e^{2\pi i(\underline{\omega x}+ju_A)}T\sigma_1^{\pi_j}[-A^{-1}\underline{x}, A^{-1}]\xi(\underline{\omega}A) \\
&= |\det(A)|^{1/2}e^{2\pi i(\underline{\omega x}+ju_A)}|a\omega_1|^{-3/2}\sigma_1^{\pi_j}[-A^{-1}\underline{x}, A^{-1}]\xi(\underline{\omega}A) \\
&= e^{2\pi i(\underline{\omega x}+ju_A)}|a\omega_1|^{-3/2}e^{2\pi i\left(\underline{\omega}A(-A^{-1}\underline{x})+j(-u_A)\right)}\xi(\underline{\omega}) \\
&= |a|^{-3/2}|\omega_1|^{-3/2}\xi(\underline{\omega}) = \Delta_G[\underline{x}, A]^{1/2}T\xi(\underline{\omega}),
\end{aligned}$$

for a.e. $\underline{\omega} \in \widehat{\mathbb{R}^3}$. Thus $\sigma_1^{\pi_j}[\underline{x}, A]T\sigma_1^{\pi_j}[\underline{x}, A]^* = \Delta_G[\underline{x}, A]^{1/2}T$. □

By Proposition 2.5.7, T is the Duflo-Moore operator, $C_{\sigma_1^{\pi_j}}$, for $\sigma_1^{\pi_j}$. Then, we can follow the procedure in Proposition 2.5.9 to describe the wavelet transform and reconstruction formula associated with $\sigma_1^{\pi_j}$.

Definition 3.1.8. A function $\eta \in L^2(\widehat{\mathbb{R}^3})$ is called a $\sigma_1^{\pi_j}$-*wavelet* if

$$\int_{\widehat{\mathbb{R}^3}} \frac{|\eta(\underline{\omega})|^2}{|\omega_1|^3}\, d\underline{\omega} = 1.$$

If η is a $\sigma_1^{\pi_j}$-wavelet, then $V_\eta\xi[\underline{x}, A] = \langle \xi, \sigma_1^{\pi_j}[\underline{x}, A]\eta\rangle_{L^2(\widehat{\mathbb{R}^3})}$, for $[\underline{x}, A] \in G$ and $\xi \in L^2(\widehat{\mathbb{R}^3})$, defines the $\sigma_1^{\pi_j}$-*wavelet transform* with $\sigma_1^{\pi_j}$-wavelet η. For $[\underline{x}, A] \in G$, let $\eta_{\underline{x},A} = \sigma_1^{\pi_j}[\underline{x}, A]\eta$, for each $[\underline{x}, A] \in G$.

Theorem 3.1.9. Let $\eta \in L^2(\widehat{\mathbb{R}^3})$ be a $\sigma_1^{\pi_j}$-wavelet. Then, for any $\xi \in L^2(\widehat{\mathbb{R}^3})$,

$$\xi = \int_G V_\eta\xi[\underline{x}, A]\, \eta_{\underline{x},A}\, d\mu_G[\underline{x}, A], \text{ weakly in } L^2(\widehat{\mathbb{R}^3}).$$

Proof. This is just Proposition 2.5.9 in this situation. □

We can move the representation $\sigma_1^{\pi_j}$ using the Fourier unitary map to get an equivalent representation acting on $L^2(\mathbb{R}^3)$. There is a small abuse of notation as we use $\mathcal{F}_1$ to also denote the map from $L^2(\widehat{\mathbb{R}^3}) \to L^2(\mathbb{R}^3)$ such that

$$\mathcal{F}_1\xi(\underline{y}) = \int_{\widehat{\mathbb{R}^3}} \xi(\underline{\omega}) e^{-2\pi i \underline{\omega}\underline{y}} d\underline{\omega}, \text{ for all } \underline{y} \in \mathbb{R}^3,$$

and $\xi \in L^1(\widehat{\mathbb{R}^3}) \cap L^2(\widehat{\mathbb{R}^3})$.

Definition 3.1.10. For $[\underline{x}, A] \in G$, let $\rho^j[\underline{x}, A] = \mathcal{F}_1 \sigma_1^{\pi_j}[\underline{x}, A]\mathcal{F}_1^{-1}$.

Then ρ^j is an irreducible representation of G acting on $L^2(\mathbb{R}^3)$, which is square-integrable, since it is equivalent to $\sigma_1^{\pi_j}$.

Proposition 3.1.11. For any $[\underline{x}, A] \in G$ and $f \in L^2(\mathbb{R}^3)$,

$$\rho^j[\underline{x}, A]f(\underline{y}) = |\det(A)|^{-1/2} e^{2\pi i j u_A} f\left(A^{-1}(\underline{y} - \underline{x})\right),$$

for a.e. $\underline{y} \in \mathbb{R}^3$.

Proof. First, assume f is such that $\mathcal{F}_1^{-1}f$ is integrable. Let $\xi = \mathcal{F}_1^{-1}f$. Then

$$\begin{aligned}
\rho^j[\underline{x}, A]f(\underline{y}) = \mathcal{F}_1\big(\sigma_1^{\pi_j}[\underline{x}, A]\xi\big)(\underline{y}) &= \int_{\widehat{\mathbb{R}^3}} \sigma_1^{\pi_j}[\underline{x}, A]\xi(\underline{\omega}) e^{-2\pi i \underline{\omega}\underline{y}} d\underline{\omega} \\
&= |\det(A)|^{1/2} \int_{\widehat{\mathbb{R}^3}} e^{2\pi i(\underline{\omega}\underline{x} + j u_A)} \xi(\underline{\omega}A) e^{-2\pi i \underline{\omega}\underline{y}} d\underline{\omega} \\
&= |\det(A)|^{1/2} e^{2\pi i j u_A} \int_{\widehat{\mathbb{R}^3}} \xi(\underline{\omega}A) e^{-2\pi i \underline{\omega}(\underline{y} - \underline{x})} d\underline{\omega} \\
&= |\det(A)|^{-1/2} e^{2\pi i j u_A} \int_{\widehat{\mathbb{R}^3}} \xi(\underline{\omega}) e^{2\pi i \underline{\omega} A^{-1}(\underline{y} - \underline{x})} d\underline{\omega} \\
&= |\det(A)|^{-1/2} e^{2\pi i j u_A} f\left(A^{-1}(\underline{y} - \underline{x})\right),
\end{aligned}$$

for any $\underline{y} \in \mathbb{R}^3$. Since $\{f \in L^2(\mathbb{R}^3) : \mathcal{F}_1^{-1}f \in L^1(\widehat{\mathbb{R}^3})\}$ is dense in $L^2(\mathbb{R}^3)$, the formula for $\rho^j[\underline{x}, A]$ given in the proposition holds for any $f \in L^2(\mathbb{R}^3)$. □

Remark. When $j = 0$, ρ^0 is what is sometimes referred as the natural representation of G on $L^2(\mathbb{R}^3)$.

We can now restate the $\sigma_1^{\pi_j}$-wavelet analysis in terms of ρ^j. Note that

Definition 3.1.12. For any $j \in \mathbb{Z}$, $w \in L^2(\mathbb{R}^3)$ is called a ρ^j*-wavelet* if

$$\int_{\widehat{\mathbb{R}^3}} \frac{|\widehat{w}(\underline{\omega})|^2}{|\omega_1|^3} d\underline{\omega} = 1.$$

(Note that this condition is the same for all j.) For a ρ^j-wavelet w and each $[\underline{x}, A] \in G$, let

$$w_{\underline{x},A}(\underline{y}) = |\det(A)|^{-1/2} e^{2\pi i j u_A} w\left(A^{-1}(\underline{y} - \underline{x})\right), \text{ for a.e. } \underline{y} \in \mathbb{R}^3,$$

and let $V_w f[\underline{x}, A] = \langle f, w_{\underline{x},A} \rangle$, for any $[\underline{x}, A] \in G$ and $f \in L^2(\mathbb{R}^3)$. The map $V_w : L^2(\mathbb{R}^3) \to L^2(G)$ is a linear isometry called the ρ^j*-wavelet transform* with ρ^j-wavelet w.

Corollary 3.1.13. Let $j \in \mathbb{Z}$ and let $w \in L^2(\mathbb{R}^3)$ be a ρ^j-wavelet. Then, for any $f \in L^2(\mathbb{R}^3)$,

$$f = \int_H \int_{\mathbb{R}^3} V_w f[\underline{x}, A]\, w_{\underline{x},A}\, \tfrac{d\mu_H(A)\, d\underline{x}}{|\det(A)|}, \text{ weakly in } L^2(\mathbb{R}^3).$$

Chapter 4

The Affine Groups

In this chapter, we establish both algebraic and analytic properties of the groups of affine transformations of $\mathbb{R}^n$ with an emphasis on $n = 1$ and 2.

4.1 The Affine Group of $\mathbb{R}^n$

The affine group of $\mathbb{R}^n$ is

$$G_n = \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R}) = \{[\underline{x}, A] : \underline{x} \in \mathbb{R}^n, A \in \mathrm{GL}_n(\mathbb{R})\},$$

where the group product is, for $[\underline{x}, A], [\underline{y}, B] \in G_n$, given by

$$[\underline{x}, A][\underline{y}, B] = [\underline{x} + A\underline{y}, AB].$$

We described the Haar integral on the unimodular group $\mathrm{GL}_n(\mathbb{R})$ earlier. When there is no chance of confusion, $d\mu_{\mathrm{GL}_n(\mathbb{R})}(A)$ will be denoted simply as dA. Thus, the left Haar integral on G_n is given by, for $f \in C_c(G_n)$,

$$\int_{G_n} f \, d\mu_{G_n} = \int_{\mathrm{GL}_n(\mathbb{R})} \int_{\mathbb{R}^n} f[\underline{x}, A] \, \frac{d\underline{x} \, dA}{|\det(A)|}.$$

In later sections, we will use a unitary map to move the left regular representation of G_n on $L^2(G_n)$ to an equivalent representation on $L^2\big(\mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R})\big)$, where $\mu_{\mathbb{R}^n} \times \mu_{\mathrm{GL}_n(\mathbb{R})}$ is the measure understood to be on $\mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R})$.

Proposition 4.1.1. For $f \in L^2(G_n)$, define Uf on $\mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R})$ by

$$Uf(\underline{y}, B) = f[B\underline{y}, B], \text{ for a.e. } [\underline{y}, B] \in G_n.$$

Then $Uf \in L^2\big(\mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R})\big)$, for all $f \in L^2(G_n)$, and

$$U : L^2(G_n) \to L^2\big(\mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R})\big)$$

is a unitary map.

Proof. First U is an isometry into $L^2\big(\mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R})\big)$ since,

$$\begin{aligned}\int_{\mathrm{GL}_n(\mathbb{R})}\int_{\mathbb{R}^n} |Uf(\underline{y},B)|^2 d\mu_{\mathbb{R}^n}(\underline{y})\, dB &= \int_{\mathrm{GL}_n(\mathbb{R})}\int_{\mathbb{R}^n} |f[B\underline{y},B]|^2\, d\underline{y}\, dB \\ &= \int_{\mathrm{GL}_n(\mathbb{R})}\int_{\mathbb{R}^n} |f[\underline{y},B]|^2\, \frac{d\underline{y}\, dB}{|\det(B)|} \\ &= \int_{\mathrm{GL}_n(\mathbb{R})}\int_{\mathbb{R}^n} |f[\underline{y},B]|^2\, d\mu_{G_n}[\underline{y},B] \\ &= \|f\|^2_{L^2(G_n)}\end{aligned}$$

It is clear that $U : L^2(G_n) \to L^2\big(\mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R})\big)$ is linear. It remains to show that U is onto. Let $g \in L^2\big(\mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R})\big)$. Define f on G_n by $f[\underline{y},B] = g(B^{-1}\underline{y},B)$, for a.e. $[\underline{y},B] \in G_n$. Then a calculation very similar to the above shows that $f \in L^2(G_n)$ and $Uf = g$. Thus, U is a unitary map. □

4.2 The Group G_1

When $n = 1$, $\mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^*$, so $G_1 = \mathbb{R} \rtimes \mathbb{R}^* = \{[x,a] : x \in \mathbb{R}, a \in \mathbb{R}^*\}$. It is called the affine group the real line. As a locally compact space G_1 is identified with the plane with the x-axis removed. Left Haar integration on G_1 is given by

$$\int_{G_1} f\, d\mu_{G_1} = \int_{\mathbb{R}}\int_{\mathbb{R}} f[x,a]\frac{dx\, da}{a^2}, \text{ for all } f \in C_c(G_1).$$

Note that G_1 is non-unimodular and $\Delta_{G_1}[y,b] = |b|^{-1}$, for $[y,b] \in G_1$. This can be verified with some simple changes of variables and order of integration:

$$\begin{aligned}|b|^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} f\big([x,a][y,b]\big)\frac{dx\, da}{a^2} &= |b|^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} f[x+ay,ab]\, dx\frac{da}{a^2} \\ &= |b|^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} f[x,ab]\frac{dx\, da}{a^2} \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f[x,ab]\frac{|b|da}{(ba)^2}dx = \int_{\mathbb{R}}\int_{\mathbb{R}} f[x,a]\frac{dx\, da}{a^2}.\end{aligned}$$

A key part of later arguments is that G_1 sits as a closed subgroup of $\mathrm{GL}_2(\mathbb{R})$. For each $[u,v] \in G_1$, let $\theta[u,v] = \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix}$. We use the parameters $u \in \mathbb{R}$ and $v \in \mathbb{R}^*$ to be consistent with notation used later.

Proposition 4.2.1. The map θ is an injective homomorphism of G_1 into $\mathrm{GL}_2(\mathbb{R})$ and $\theta(G_1)$ is a closed subgroup of $\mathrm{GL}_2(\mathbb{R})$.

Proof. For $[u_1, v_1], [u_2, v_2] \in G_1$,

$$\theta\big([u_1, v_1][u_2, v_2]\big) = \theta[u_1 + v_1 u_2, v_1 v_2] = \begin{pmatrix} 1 & 0 \\ u_1 + v_1 u_2 & v_1 v_2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ u_1 & v_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u_2 & v_2 \end{pmatrix} = \theta[u_1, v_1]\theta[u_2, v_2].$$

Thus, θ is a homomorphism and $\theta[u, v] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if and only if $[u, v] = [0, 1]$, the identity in G_1. Therefore θ is injective.

The image of a homomorphism is a subgroup and the map $\varphi : \mathrm{GL}_2(\mathbb{R}) \to \widehat{\mathbb{R}^2}$ given by $\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a, b)$ is continuous and $\theta(G_1) = \varphi^{-1}\{(1, 0)\}$, so it is a closed subgroup. □

4.3 The Group G_2

For the group G_2, we need to look closely at its structure. We will find different parametrizations of G_2 that will be useful in constructing an induced representation in a later section. We also must identify left Haar integration in the new parametrization.

The natural way of parametrizing G_2 is

$$G_2 = \left\{ \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] : x_1, x_2, a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}.$$

In this parametrization, for $f \in C_c(G_2)$,

$$\int_{G_2} f \, d\mu_{G_2} = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \frac{dx_1 \, dx_2 \, da \, db \, dc \, dd}{|ad - bc|^3}.$$

Our first step in reparametrizing G_2 is to focus on factoring $\mathrm{GL}_2(\mathbb{R})$ as a product of two closed subgroups. Let

$$K_0 = \left\{ \begin{pmatrix} s & -t \\ t & s \end{pmatrix} : s, t \in \mathbb{R}, s^2 + t^2 > 0 \right\}$$

and

$$H_{(1,0)} = \left\{ \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} : u, v \in \mathbb{R}, v \neq 0 \right\}.$$

In Proposition 4.2.1, we showed that $H_{(1,0)}$ is a closed subgroup of $\mathrm{GL}_2(\mathbb{R})$ that is isomorphic to the group G_1. The left Haar measure of G_1 transfers through the isomorphism θ of Proposition 4.2.1. Thus, for $f \in C_c(H_{(1,0)})$,

$$\int_{H_{(1,0)}} f \, d\mu_{H_{(1,0)}} = \int_{\mathbb{R}} \int_{\mathbb{R}} f \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} \frac{du \, dv}{v^2}. \tag{4.1}$$

Let $\mathbb{C}^* = \{z \in \mathbb{C} : z \neq 0\}$, considered as a locally compact Abelian group under multiplication as the group product. If $z \in \mathbb{C}$, let $s = \mathrm{Re}(z)$ and $t = \mathrm{Im}(z)$. Then $z = s + it$ and $z \in \mathbb{C}^*$ if and only if $s^2 + t^2 > 0$. If $z_1 = s_1 + it_1$ and $z_2 = s_2 + it_2$, then $z_1 z_2 = (s_1 s_2 - t_1 t_2) + i(s_1 t_2 + s_2 t_1)$. Define $\phi : \mathbb{C}^* \to K_0$ by

$$\phi(s + it) = \begin{pmatrix} s & -t \\ t & s \end{pmatrix}, \text{ for all } s + it \in \mathbb{C}^*.$$

Proposition 4.3.1. The map $\phi : \mathbb{C}^* \to K_0$ is is an isomorphism of locally compact groups.

Proof. For $z_1 = s_1 + it_1$ and $z_2 = s_2 + it_2$ in $\mathbb{C}^*$,

$$\begin{aligned} \phi(z_1 z_2) = \phi\big((s_1 s_2 - t_1 t_2) + i(s_1 t_2 + s_2 t_1)\big) &= \begin{pmatrix} s_1 s_2 - t_1 t_2 & -s_1 t_2 - s_2 t_1 \\ s_1 t_2 + s_2 t_1 & s_1 s_2 - t_1 t_2 \end{pmatrix} \\ &= \begin{pmatrix} s_1 & -t_1 \\ t_1 & s_1 \end{pmatrix} \begin{pmatrix} s_2 & -t_2 \\ t_2 & s_2 \end{pmatrix} = \phi(z_1)\phi(z_2). \end{aligned}$$

Thus, ϕ is a homomorphism. It is clear that ϕ is a bijection between $\mathbb{C}^*$ and K_0. It is also easy to show that it is a homeomorphism. Thus, ϕ is an isomorphism of locally compact groups. □

Proposition 4.3.2. Left Haar measure μ_{K_0} on K_0 is such that, for any $f \in C_c(K_0)$,

$$\int_{K_o} f \, d\mu_{K_0} = \int_{\mathbb{R}} \int_{\mathbb{R}} f \begin{pmatrix} s & -t \\ t & s \end{pmatrix} \frac{ds\, dt}{s^2 + t^2}.$$

Proof. Fix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in K_0$. Then $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} s & -t \\ t & s \end{pmatrix} = \begin{pmatrix} as - bt & -(at + bs) \\ at + bs & as - bt \end{pmatrix}$. Thus, left multiplication by $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is the same as the map of $\varphi : \mathbb{R}^2 \backslash \{\underline{0}\} \to \mathbb{R}^2 \backslash \{\underline{0}\}$ given by $\varphi \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} as - bt \\ at + bs \end{pmatrix}$. The Jacobian matrix of ϕ is $J_\phi = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, which is constant with determinant $a^2 + b^2$. Using the fact that $B \to |\det(B)|$ is a homomorphism of K_0 into $\mathbb{R}^+$ and the change of variables formula, we have

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} f \left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} s & -t \\ t & s \end{pmatrix} \right) \frac{ds\, dt}{s^2 + t^2} \\ &\qquad = \int_{\mathbb{R}} \int_{\mathbb{R}} f \begin{pmatrix} as - bt & -(at + bs) \\ at + bs & as - bt \end{pmatrix} \frac{(a^2 + b^2)\, ds\, dt}{(as - bt)^2 + (at + bs)^2} \\ &\qquad = \int_{\mathbb{R}} \int_{\mathbb{R}} f \begin{pmatrix} s & -t \\ t & s \end{pmatrix} \frac{ds\, dt}{s^2 + t^2}. \end{aligned}$$

This verifies that $\int_{K_0} f \, d\mu_{K_0} = \int_{\mathbb{R}} \int_{\mathbb{R}} f \begin{pmatrix} s & -t \\ t & s \end{pmatrix} \frac{ds\, dt}{s^2 + t^2}$. □

Proposition 4.3.3. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$, then A can be uniquely decomposed as $A = M_A C_A$, where

$$M_A = \begin{pmatrix} s & -t \\ t & s \end{pmatrix}, \text{ with } s = \frac{d(ad-bc)}{b^2+d^2}, t = \frac{-b(ad-bc)}{b^2+d^2},$$

and

$$C_A = \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix}, \text{ with } u = \frac{cd+ab}{(ad-bc)}, v = \frac{b^2+d^2}{(ad-bc)}.$$

Proof. Fix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ and define $s = \frac{d(ad-bc)}{b^2+d^2}$, $t = \frac{-b(ad-bc)}{b^2+d^2}$, $u = \frac{cd+ab}{(ad-bc)}$, and $v = \frac{b^2+d^2}{(ad-bc)}$. We compute the four entries of $\begin{pmatrix} s & -t \\ t & s \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix}$. The (1,1) entry is

$$\begin{aligned} s - tu &= \frac{d(ad-bc)}{b^2+d^2} - \frac{-b(ad-bc)}{b^2+d^2}\frac{cd+ab}{(ad-bc)} \\ &= \frac{ad^2 - dbc + bcd + ab^2}{b^2+d^2} = a. \end{aligned}$$

Similarly, the (1,2) entry is $-tv = b$, the (2,1) entry is $t + su = c$ and the (2,2) entry is $sv = d$. If we let $M_A = \begin{pmatrix} s & -t \\ t & s \end{pmatrix}$ and $C_A = \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix}$, then $M_A \in K_0$, $C_A \in H_{(1,0)}$ and $M_A C_A = A$. It is clear that $K_0 \cap H_{(1,0)} = \{\mathrm{id}\}$. Thus, if $M \in K_0$ and $C \in H_{(1,0)}$ are such that $MC = A = M_A C_A$, then $M^{-1}M_A = CC_A^{-1}$. Thus, both $M^{-1}M_A$ and CC_A^{-1} are the identity. That is $M = M_A$ and $C = C_A$. This proves the uniqueness of the factorization. □

Recall that $\mathrm{GL}_2(\mathbb{R})$ is a unimodular group and the Haar integral is given by

$$\int_{\mathrm{GL}_2(\mathbb{R})} f \, d\mu_{\mathrm{GL}_2(\mathbb{R})} = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} f\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{da\, db\, dc\, dd}{(ad-bc)^2}.$$

However, the parametrization resulting from factoring $\mathrm{GL}_2(\mathbb{R})$ as $K_0\, H_{(1,0)}$ gives an alternate expression for the Haar integral.

Proposition 4.3.4. Haar integration on $\mathrm{GL}_2(\mathbb{R})$ is given by, for $f \in C_c(\mathrm{GL}_2(\mathbb{R}))$,

$$\int_{\mathrm{GL}_2(\mathbb{R})} f \, d\mu_{\mathrm{GL}_2(\mathbb{R})} = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} f\left(\begin{pmatrix} s & -t \\ t & s \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix}\right) \frac{ds\, dt\, du\, dv}{|v|(s^2+t^2)}.$$

Proof. Let $U = \{\ \underline{z} \in \mathbb{R}^4 : z_1 z_4 - z_2 z_3 > 0\}$, an open subset of $\mathbb{R}^4$. Let $V =$

$\left\{ \begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix} \in \mathbb{R}^4 : v > 0 \right\}$, an open subset of $\mathbb{R}^4$. Note that

$$\begin{pmatrix} s & -t \\ t & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} = \begin{pmatrix} s - tu & -tv \\ t + su & sv \end{pmatrix}$$

and define $\phi : V \to U$ by $\phi \begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix} = \begin{pmatrix} s - tu \\ -tv \\ t + su \\ sv \end{pmatrix}$, for all $\begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix} \in V$. That is, $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$, where $\phi_1 \begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix} = s - tu, \phi_2 \begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix} = -tv, \phi_3 \begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix} = t + su$ and $\phi_4 \begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix} = sv$. Let $J_\phi \begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix}$ denote the Jacobian matrix of ϕ of $\begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix}$. Then $J_\phi \begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -u & -t & 0 \\ 0 & -v & 0 & -t \\ u & 1 & s & 0 \\ v & 0 & 0 & s \end{pmatrix}$ and $\left| \det \left(J_\phi \begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix} \right) \right| = v(s^2 + t^2)$. Define $g \in C_c(U)$ by $g(\underline{z}) = \frac{f\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}}{(z_1 z_4 - z_2 z_3)^2}$ for all $\underline{z} \in U$.

Since ϕ is a continuously differentiable bijection of V and U and $\det(J_\phi)$ vanishes nowhere on V, the change of variables formula holds.

$$\int_{\mathrm{GL}_2(\mathbb{R})} f \, d\mu_{\mathrm{GL}_2(\mathbb{R})} = \int_{\phi(V)} g(\underline{z}) d\underline{z} = \int_V g(\phi(\underline{w})) |\det J_\phi(\underline{w})| d\underline{w}. \tag{4.2}$$

If we write $\underline{w} = \begin{pmatrix} s \\ t \\ u \\ v \end{pmatrix}$, $|\det J_\phi(\underline{w})| = |v|(s^2 + t^2)$. If $\underline{z} = \phi(\underline{w})$, then we need $z_1 z_4 - z_2 z_3 = (s - tu)(sv) - (-tv)(t + su) = s^2 v - tusv + t^2 v + tusv = v(s^2 + t^2)$. Recalling, the definition of g, we have from (4.2) that

$$\begin{aligned} \int_{\mathrm{GL}_2(\mathbb{R})} f \, d\mu_{\mathrm{GL}_2(\mathbb{R})} &= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f \begin{pmatrix} s - tu & -tv \\ t + su & sv \end{pmatrix} \frac{|v|((s^2 + t^2)}{(v(s^2 + t^2))^2} ds \, dt \, du \, dv \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f \left(\begin{pmatrix} s & -t \\ t & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} \right) \frac{ds \, dt \, du \, dv}{|v|(s^2 + t^2)}. \end{aligned}$$

This gives us the Haar integral in this new parametrization. □

Recalling left Haar integration on$H_{(1,0)}$ as given in (4.1) and the Haar integral on K_o from Proposition 4.3.2 and noting that $\det \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} = v$, we get a compact expression for the Haar integral on $\mathrm{GL}_2(\mathbb{R})$ in terms of those for the two factor subgroups.

Proposition 4.3.5. For $f \in C_c(\mathrm{GL}_2(\mathbb{R}))$,

$$\int_{\mathrm{GL}_2(\mathbb{R})} f \, d\mu_{\mathrm{GL}_2(\mathbb{R})} = \int_{K_0} \int_{H_{(1,0)}} f(MC) |\det(C)| \, d\mu_{H_{(1,0)}}(C) \, d\mu_{K_0}(M).$$

We return to G_2 and identify two closed subgroups closely related to the subgroups, K_0 and $H_{(1,0)}$, of $\mathrm{GL}_2(\mathbb{R})$. Define

$$K = \{[\underline{0}, M] : M \in K_0\} \text{ and } H = \{[\underline{x}, C] : \underline{x} \in \mathbb{R}^2, C \in H_{(1,0)}\}.$$

Then K is a closed subgroup of G_2, isomorphic to K_0 under the map $[\underline{0}, M] \to M$ with Haar integration given by $\int_K f \, d\mu_K = \int_{K_0} f[\underline{0}, M] \, d\mu_{K_o}(M)$, for all $f \in C_c(K)$. Also, H is a closed subgroup of G_2 isomorphic to $\mathbb{R}^2 \rtimes H_{(1,0)}$. Recall that left Haar integration on $H_{(1,0)}$ is given by (4.1). Then left Haar integration on H is given by

$$\int_H f \, d\mu_H = \int_{H_{(1,0)}} \int_{\mathbb{R}^2} f[\underline{x}, C] \, \frac{d\underline{x} \, d\mu_{H_{(1,0)}}(C)}{|\det(C)|}, \text{ for all } f \in C_c(H).$$

For $[\underline{x}, A] \in \mathrm{GL}_2(\mathbb{R})$, let $M_A \in K_0$ and $C_A \in H_{(1,0)}$ be as in Proposition 4.3.3 so that $A = M_A C_A$. Then

$$[\underline{x}, A] = [\underline{0}, M_A][M_A^{-1}\underline{x}, C_A], \tag{4.3}$$

where $[\underline{0}, M_A] \in K$ and $[M_A^{-1}\underline{x}, C_A] \in H$. It is clear that K and H only share $[\underline{0}, \mathrm{id}]$ as a common element.

Proposition 4.3.6. The group G_2 factors as $G_2 = KH$ with $K \cap H = \{\mathrm{id}\}$. Moreover, the map $\big([\underline{0}, M], [\underline{x}, C]\big) \to [M\underline{x}, MC]$ is a homeomorphism of $K \times H$ with G_2.

Proof. If $M = \begin{pmatrix} s & -t \\ t & s \end{pmatrix} \in K_0$ and $C = \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} \in H_{(1,0)}$, then

$$MC = \begin{pmatrix} s - ut & -vt \\ t + us & vs \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}).$$

The map $(M, C) \to MC$ is clearly continuous from $K_0 \times H_{(1,0)}$ to $\mathrm{GL}_2(\mathbb{R})$. Its inverse as given by Proposition 4.3.3 is also continuous. Thus $(M, C) \to MC$ is a homeomorphism. Also, the map the map $\big([\underline{0}, M], [\underline{x}, C]\big) \to [M\underline{x}, MC]$ is continuous and its inverse, given in (4.3), is easily seen to be continuous. □

Sometimes it may be convenient to integrate over G_2 in terms of this special factoring. Recall that

$$\int_{G_2} f \, d\mu_{G_2} = \int_{\mathrm{GL}_2(\mathbb{R})} \int_{\mathbb{R}^2} f[\underline{x}, A] \, \frac{d\underline{x} \, d\mu_{\mathrm{GL}_2(\mathbb{R})}(A)}{|\det(A)|}, \text{ for } f \in C_c(G_2).$$

Using Proposition 4.3.5, we get the following expression for left Haar integration over G_2.

Proposition 4.3.7. For $f \in C_c(G_2)$,

$$\int_{G_2} f \, d\mu_{G_2} = \int_{K_0} \int_{H_{(1,0)}} \int_{\mathbb{R}^2} f[\underline{x}, MC] \, \frac{d\underline{x} \, d\mu_{H_{(1,0)}}(C) \, d\mu_{K_0}(M)}{|\det(M)|}.$$

Chapter 5

Decomposing the Regular Representation of the Full Affine Group

In this chapter, we will study the left regular representation of the group of invertible affine transformations of $\mathbb{R}^n$. First, we obtain a decomposition into subrepresentations, which are irreducible when $n = 1$ but reducible for $n > 1$. Then case of $n = 1$ is explored in more detail and compared to known results for the affine group of the real line and connected to the continuous wavelet transform. The information about the case of $n = 1$ is used to obtain a complete decomposition of the left regular representation for $n = 2$.

5.1 The Affine Group on $\mathbb{R}^n$

Let $G_n = \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R}) = \{[\underline{x}, A] : \underline{x} \in \mathbb{R}^n, A \in \mathrm{GL}_n(\mathbb{R})\}$. Let $N = \{[\underline{y}, \mathrm{id}] : \underline{y} \in \mathbb{R}^n\}$, a closed normal abelian subgroup of G. Note that, for $[\underline{x}, A] \in G_n$ and $[\underline{y}, \mathrm{id}] \in N$,

$$[\underline{x}, A]^{-1}[\underline{y}, \mathrm{id}][\underline{x}, A] = \left[-A^{-1}\underline{x}, A^{-1}\right][\underline{y} + \underline{x}, A] = \left[A^{-1}\underline{y}, \mathrm{id}\right].$$

For $\underline{\omega} \in \widehat{\mathbb{R}^n}$, define a character $\chi_{\underline{\omega}}$ on N by

$$\chi_{\underline{\omega}}[\underline{y}, \mathrm{id}] = e^{2\pi i \underline{\omega}\underline{y}}, \quad \text{for all } [\underline{y}, \mathrm{id}] \in N.$$

Then $\widehat{N} = \{\chi_{\underline{\omega}} : \underline{\omega} \in \widehat{\mathbb{R}^n}\}$. For $[\underline{x}, A] \in G_n$ and $\chi \in \widehat{N}$, $[\underline{x}, A] \cdot \chi \in \widehat{N}$ is defined by

$$\big([\underline{x}, A] \cdot \chi\big)[\underline{y}, \mathrm{id}] = \chi\big([\underline{x}, A]^{-1}[\underline{y}, \mathrm{id}][\underline{x}, A]\big) = \chi\left[A^{-1}\underline{y}, \mathrm{id}\right].$$

Therefore, for $\underline{\omega} \in \widehat{\mathbb{R}^n}$, $[\underline{x}, A] \cdot \chi_{\underline{\omega}} = \chi_{\underline{\omega}A^{-1}}$. It is convenient to work with the action of $\mathrm{GL}_n(\mathbb{R})$ on $\widehat{\mathbb{R}^n}$ given by $(A, \underline{\omega}) \to \underline{\omega}A^{-1}$. There are just two orbits, $\{\underline{0}\}$ and $\mathcal{O} = \widehat{\mathbb{R}^n} \setminus \{\underline{0}\}$. Let $\underline{\omega}_0 = (1, 0, \cdots, 0)$. Then the orbit $\mathcal{O} = \{\underline{\omega}_0 A^{-1} : A \in \mathrm{GL}_n(\mathbb{R})\}$.

We fix a measurable map $\gamma : \mathcal{O} \to \mathrm{GL}_n(\mathbb{R})$ that satisfies

$$\underline{\omega}_0\gamma(\underline{\omega})^{-1} = \underline{\omega} \text{ (equivalently } \underline{\omega}\gamma(\underline{\omega}) = \underline{\omega}_0), \quad \text{for all } \underline{\omega} \in \mathcal{O}.$$

For $\underline{\omega} \in \widehat{\mathbb{R}^n}$, by Corollary 2.8.10, the induced representation $\mathrm{ind}_N^{G_n} \chi_{\underline{\omega}}$ is unitarily equivalent to $\pi^{\underline{\omega}}$ which acts on $L^2\big(\mathrm{GL}_n(\mathbb{R})\big)$ as follows: For $[\underline{x}, A] \in G_n$,

$$\pi^{\underline{\omega}}[\underline{x}, A]f(B) = e^{2\pi i \underline{\omega} B^{-1}\underline{x}} f\left(A^{-1}B\right), \quad \text{for all } B \in \mathrm{GL}_n(\mathbb{R}), f \in L^2\big(\mathrm{GL}_n(\mathbb{R})\big).$$

Proposition 5.1.1. For $\underline{\omega} \in \mathcal{O}$, $\pi^{\underline{\omega}} \sim \pi^{\underline{\omega}_0}$.

Proof. Define $V_{\underline{\omega}} : L^2\big(\mathrm{GL}_n(\mathbb{R})\big) \to L^2\big(\mathrm{GL}_n(\mathbb{R})\big)$ by, for $f \in L^2\big(\mathrm{GL}_n(\mathbb{R})\big)$,

$$V_{\underline{\omega}}f(B) = f\big(B\gamma(\underline{\omega})^{-1}\big), \quad \text{for all } B \in \mathrm{GL}_n(\mathbb{R}).$$

Since $\mathrm{GL}_n(\mathbb{R})$ is unimodular, $\|V_{\underline{\omega}}f\|_2 = \|f\|_2$, for all $f \in L^2\big(\mathrm{GL}_n(\mathbb{R})\big)$. It is also clear that $V_{\underline{\omega}}$ is linear, one to one, and onto. So $V_{\underline{\omega}}$ is a unitary map of $L^2\big(\mathrm{GL}_n(\mathbb{R})\big)$ with itself. For $[\underline{x}, A] \in G_n$, $f \in L^2\big(\mathrm{GL}_n(\mathbb{R})\big)$, and $B \in \mathrm{GL}_n(\mathbb{R})$,

$$\begin{aligned} V_{\underline{\omega}}\pi^{\underline{\omega}}[\underline{x}, A]V_{\underline{\omega}}^{-1}f(B) &= \pi^{\underline{\omega}}[\underline{x}, A]V_{\underline{\omega}}^{-1}f\big(B\gamma(\underline{\omega})^{-1}\big) = e^{2\pi i \underline{\omega}\gamma(\underline{\omega})B^{-1}\underline{x}}V_{\underline{\omega}}^{-1}f\big(A^{-1}B\gamma(\underline{\omega})^{-1}\big) \\ &= e^{2\pi i \underline{\omega}\gamma(\underline{\omega})B^{-1}\underline{x}}f\big(A^{-1}B\big) = e^{2\pi i \underline{\omega}_0 B^{-1}\underline{x}}f\big(A^{-1}B\big) = \pi^{\underline{\omega}_0}[\underline{x}, A]f(B). \end{aligned}$$

This shows that $\pi^{\underline{\omega}} \sim \pi^{\underline{\omega}_0}$. □

Our goal now is to establish an explicit unitary equivalence of the left regular representation, λ_{G_n}, of G_n with an infinite multiple of $\pi^{\underline{\omega}_0}$.

For any $f \in L^2(G_n)$, define Uf on $\mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R})$ by $Uf(\underline{y}, B) = f[B\underline{y}, B]$, for all $(\underline{y}, B) \in \mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R})$. By Proposition 4.1.1, $Uf \in L^2\big(\mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R})\big)$ and U is a unitary map when $\mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R})$ is equipped with the product of Lebesgue measure with Haar measure on $\mathrm{GL}_n(\mathbb{R})$. Moreover, $U^{-1} : L^2\big(\mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R})\big) \to L^2(G_n)$ is given by

$$U^{-1}f[\underline{y}, B] = f(B^{-1}\underline{y}, B), \quad \text{for } [\underline{y}, B] \in G_n, f \in L^2\big(\mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R})\big).$$

Let $\mathcal{F}_1 : L^2\big(\mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R})\big) \to L^2\big(\widehat{\mathbb{R}^n} \times \mathrm{GL}_n(\mathbb{R})\big)$ be the unitary map consisting of taking the Fourier transform in the first variable. That is, for $f \in C_c\big(\mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R})\big)$ and any $(\underline{\omega}, B) \in \widehat{\mathbb{R}^n} \times \mathrm{GL}_n(\mathbb{R})$,

$$\mathcal{F}_1 f(\underline{\omega}, B) = \int_{\mathbb{R}^n} f(\underline{y}, B)e^{2\pi i \underline{\omega}\underline{y}}d\underline{y}.$$

The left regular representation λ_{G_n} of G_n is unitarily equivalent, via $\mathcal{F}_1 \circ U$ to a unitary representation $\widetilde{\lambda_{G_n}}$ acting on $L^2\big(\widehat{\mathbb{R}^n} \times \mathrm{GL}_n(\mathbb{R})\big)$. For $f \in L^2\big(\widehat{\mathbb{R}^n} \times \mathrm{GL}_n(\mathbb{R})\big)$,

$[\underline{x}, A] \in G_n$, and $(\underline{\omega}, B) \in \widehat{\mathbb{R}^n} \times \mathrm{GL}_n(\mathbb{R})$, we have

$$\begin{aligned}
\widetilde{\lambda_{G_n}}[\underline{x}, A]f(\underline{\omega}, B) &= \mathcal{F}_1 U \lambda_{G_n}[\underline{x}, A]\big(U^{-1}\mathcal{F}_1^{-1} f\big)(\underline{\omega}, B) \\
&= \int_{\mathbb{R}^n} U\lambda_{G_n}[\underline{x}, A]\big(U^{-1}\mathcal{F}_1^{-1} f\big)(\underline{y}, B)e^{2\pi i \underline{\omega}\underline{y}} d\underline{y} \\
&= \int_{\mathbb{R}^n} \lambda_{G_n}[\underline{x}, A]\big(U^{-1}\mathcal{F}_1^{-1} f\big)[B\underline{y}, B]e^{2\pi i \underline{\omega}\underline{y}} d\underline{y} \\
&= \int_{\mathbb{R}^n} \left(U^{-1}\mathcal{F}_1^{-1} f\right)\left([\underline{x}, A]^{-1}[B\underline{y}, B]\right) e^{2\pi i \underline{\omega}\underline{y}} d\underline{y} \\
&= \int_{\mathbb{R}^n} \left(U^{-1}\mathcal{F}_1^{-1} f\right)\left([A^{-1}(B\underline{y} - \underline{x}), A^{-1}B]\right) e^{2\pi i \underline{\omega}\underline{y}} d\underline{y} \\
&= \int_{\mathbb{R}^n} \left(\mathcal{F}_1^{-1} f\right)\left(B^{-1}(B\underline{y} - \underline{x}), A^{-1}B\right) e^{2\pi i \underline{\omega}\underline{y}} d\underline{y} \\
&= \int_{\mathbb{R}^n} \left(\mathcal{F}_1^{-1} f\right)\left(B^{-1}(B\underline{y} - \underline{x}), A^{-1}B\right) e^{2\pi i \underline{\omega}\underline{y}} d\underline{y} \\
&= \int_{\mathbb{R}^n} \left(\mathcal{F}_1^{-1} f\right)\left(\underline{y} - B^{-1}\underline{x}), A^{-1}B\right) e^{2\pi i \underline{\omega}\underline{y}} d\underline{y} \\
&= \int_{\mathbb{R}^n} \big(\mathcal{F}_1^{-1} f\big)(\underline{z}, A^{-1}B)e^{2\pi i \underline{\omega}(B^{-1}\underline{x} + \underline{z})} d\underline{z} \\
&= e^{2\pi i \underline{\omega} B^{-1}\underline{x}} \int_{\mathbb{R}^n} \big(\mathcal{F}_1^{-1} f\big)(\underline{z}, A^{-1}B)e^{2\pi i \underline{\omega}\underline{z}} d\underline{z} \\
&= e^{2\pi i \underline{\omega} B^{-1}\underline{x}} f\left(\underline{\omega}, A^{-1}B\right).
\end{aligned}$$

That is, $\widetilde{\lambda_{G_n}}[\underline{x}, A]f(\underline{\omega}, B) = e^{2\pi i \underline{\omega} B^{-1}\underline{x}} f\left(\underline{\omega}, A^{-1}B\right)$. Notice the similarity with the $\pi^{\underline{\omega}}$. This shows how we could write the left regular representation as a direct integral of the $\pi^{\underline{\omega}}$. But Proposition 5.1.1 shows all $\pi^{\underline{\omega}}$, $\underline{\omega} \in \mathcal{O}$, are equivalent to $\pi^{\underline{\omega}_0}$.

Note that $\mathcal{O}$ is a co-null open subset of $\widehat{\mathbb{R}^n}$. We will consider Lebesgue measure on $\mathcal{O}$ as its standard measure, so $L^2(\mathcal{O} \times \mathrm{GL}_n(\mathbb{R}))$ is the same Hilbert space as $L^2\big(\widehat{\mathbb{R}^n} \times \mathrm{GL}_n(\mathbb{R})\big)$. We define W on $L^2(\mathcal{O} \times \mathrm{GL}_n(\mathbb{R}))$ as follows: For $f \in L^2(\mathcal{O} \times \mathrm{GL}_n(\mathbb{R}))$ and $(\underline{\omega}, B) \in \mathcal{O} \times \mathrm{GL}_n(\mathbb{R})$,

$$(Wf)(\underline{\omega}, B) = f\big(\underline{\omega}, B\gamma(\underline{\omega})^{-1}\big).$$

Then Wf is measurable and, using Fubini's Theorem and that $\mathrm{GL}_n(\mathbb{R})$ is unimodular,

$$\begin{aligned}
\int_{\mathrm{GL}_n(\mathbb{R})} \int_{\mathcal{O}} |(Wf)(\underline{\omega}, B)|^2 d\underline{\omega}\, dB &= \int_{\mathcal{O}} \int_{\mathrm{GL}_n(\mathbb{R})} |f\big(\underline{\omega}, B\gamma(\underline{\omega})^{-1}\big)|^2 dB\, d\underline{\omega} \\
&= \int_{\mathrm{GL}_n(\mathbb{R})} \int_{\mathcal{O}} |f\big(\underline{\omega}, B\big)|^2 d\underline{\omega}\, dB = \|f\|_2^2.
\end{aligned}$$

Thus $Wf \in L^2(\mathcal{O} \times \mathrm{GL}_n(\mathbb{R}))$ and W is a linear isometry on $L^2(\mathcal{O} \times \mathrm{GL}_n(\mathbb{R}))$. Clearly, W is onto and W^{-1} is given by $\big(W^{-1}g\big)(\underline{\omega}, B) = g\big(\underline{\omega}, B\gamma(\underline{\omega})\big)$, for all $(\underline{\omega}, B) \in \mathcal{O} \times \mathrm{GL}_n(\mathbb{R})$. So W is a unitary.

Define the unitary representation $\lambda^0_{G_n}$ of G_n on $L^2(\mathcal{O}\times\mathrm{GL}_n(\mathbb{R}))$ by, for $[\underline{x}, A] \in G_n$,

$$\lambda^0_{G_n}[\underline{x}, A] = W\widetilde{\lambda_{G_n}}[\underline{x}, A]W^{-1}.$$

So, for $f \in L^2\big(\mathcal{O} \times \mathrm{GL}_n(\mathbb{R})\big)$ and $(\underline{\omega}, B) \in \mathcal{O} \times \mathrm{GL}_n(\mathbb{R})$,

$$\lambda^0_{G_n}[\underline{x}, A]f(\underline{\omega}, B) = W\widetilde{\lambda_{G_n}}[\underline{x}, A]W^{-1}f(\underline{\omega}, B) = \widetilde{\lambda_{G_n}}[\underline{x}, A]W^{-1}f\big(\underline{\omega}, B\gamma(\underline{\omega})^{-1}\big)$$

$$= e^{2\pi i\underline{\omega}\gamma(\underline{\omega})B^{-1}\underline{x}}\left(W^{-1}f\right)\left(\underline{\omega}, A^{-1}B\gamma(\underline{\omega})^{-1}\right) = e^{2\pi i\underline{\omega}_0 B^{-1}\underline{x}}f\big(\underline{\omega}, A^{-1}B\big).$$

So we have $\lambda^0_{G_n}[\underline{x}, A]f(\underline{\omega}, B) = e^{2\pi i\underline{\omega}_0 B^{-1}\underline{x}}f\big(\underline{\omega}, A^{-1}B\big)$, for a.e. $(\underline{\omega}, B) \in \mathcal{O} \times \mathrm{GL}_n(\mathbb{R})$, for all $f \in L^2\big(\mathcal{O} \times \mathrm{GL}_n(\mathbb{R})\big)$, and all $[\underline{x}, A] \in G_n$. The unitary representation $\lambda^0_{G_n}$ is equivalent to λ_{G_n}.

Let $\eta \in L^2(\mathcal{O})$ be such that $\|\eta\|_2 = 1$. Let $\mathcal{H}_\eta = \{\eta\otimes f : f \in L^2\big(\mathrm{GL}_n(\mathbb{R})\big)\}$, where $\eta\otimes f$ is defined by $(\eta\otimes f)(\underline{\omega}, B) = \eta(\underline{\omega})f(B)$, for all $(\underline{\omega}, B) \in \mathcal{O}\times\mathrm{GL}_n(\mathbb{R})$. Then $\mathcal{H}_\eta$ is a closed subspace of $L^2\big(\mathcal{O} \times \mathrm{GL}_n(\mathbb{R})\big)$. Define $W_\eta : L^2\big(\mathrm{GL}_n(\mathbb{R})\big) \to L^2\big(\mathcal{O} \times \mathrm{GL}_n(\mathbb{R})\big)$ by, for $f \in L^2\big(\mathrm{GL}_n(\mathbb{R})\big)$,

$$W_\eta f(\underline{\omega}, B) = (\eta \otimes f)(\underline{\omega}, B) = \eta(\underline{\omega})f(B), \quad \text{for a.e. } (\underline{\omega}, B) \in \mathcal{O} \times \mathrm{GL}_n(\mathbb{R}).$$

Proposition 5.1.2. Let $\{\eta_j : j \in J\}$ be an orthonormal basis of $L^2(\mathcal{O})$. For each $j \in J$, $\mathcal{H}_{\eta_j}$ is a closed $\lambda^0_{G_n}$-invariant subspace of $L^2(\mathcal{O} \times \mathrm{GL}_n(\mathbb{R}))$ and the map W_{η_j} intertwines $\pi^{\underline{\omega}_0}$ with the restriction of $\lambda^0_{G_n}$ to $\mathcal{H}_{\eta_j}$. Moreover,

$$L^2(\mathcal{O} \times \mathrm{GL}_n(\mathbb{R})) = \textstyle\sum^{\oplus}_{j\in J} \mathcal{H}_{\eta_j}.$$

Proof. Since $\{\eta_j : j \in J\}$ is an orthonormal basis of $L^2(\mathcal{O})$, it follows from Proposition 2.3.7 that $L^2(\mathcal{O} \times \mathrm{GL}_n(\mathbb{R})) = \sum^{\oplus}_{j\in J} \mathcal{H}_{\eta_j}$. For each $j \in J$, it is clear that W_{η_j} is a unitary map onto $\mathcal{H}_{\eta_j}$. For $[\underline{x}, A] \in G_n$ and $f \in L^2(\mathrm{GL}_n(\mathbb{R}))$,

$$\begin{aligned}
W_{\eta_j}\pi^{\underline{\omega}_0}[\underline{x}, A]W_{\eta_j}^{-1}(\eta_j \otimes f)(\underline{\omega}, B) &= W_{\eta_j}\big(\pi^{\underline{\omega}_0}[\underline{x}, A]f\big)(\underline{\omega}, B)\\
&= \eta_j(\underline{\omega})\big(\pi^{\underline{\omega}_0}[\underline{x}, A]f\big)(B)\\
&= \eta_j(\underline{\omega})e^{2\pi i\underline{\omega}_0 B^{-1}\underline{x}}f\big(A^{-1}B\big)\\
&= e^{2\pi i\underline{\omega}_0 B^{-1}\underline{x}}(\eta_j \otimes f)\big(\underline{\omega}, A^{-1}B\big)\\
&= \lambda^0_{G_n}[\underline{x}, A](\eta_j \otimes f)(\underline{\omega}, B),
\end{aligned}$$

for all $(\underline{\omega}, B) \in \mathcal{O} \times \mathrm{GL}_n(\mathbb{R})$. Therefore $\mathcal{H}_{\eta_j}$ is a $\lambda^0_{G_n}$-invariant subspace of $L^2(\mathcal{O} \times \mathrm{GL}_n(\mathbb{R}))$ and the map W_{η_j} intertwines $\pi^{\underline{\omega}_0}$ with the restriction of $\lambda^0_{G_n}$ to $\mathcal{H}_{\eta_j}$. □

Returning to a single $\eta \in L^2(\mathcal{O})$ with $\|\eta\|_2 = 1$, let us see where $\eta \otimes f$ goes as we map it with the above unitaries back into $L^2(G_n)$.

First, we have $W^{-1}(\eta \otimes f)(\underline{\omega}, B) = \eta(\underline{\omega})f\big(B\gamma(\underline{\omega})\big)$, for $(\underline{\omega}, B) \in \mathcal{O} \times \mathrm{GL}_n(\mathbb{R})$. Next,

$$\mathcal{F}_1^{-1}W^{-1}(\eta \otimes f)(\underline{y}, B) = \int_{\widehat{\mathbb{R}^n}} \eta(\underline{\omega})f\big(B\gamma(\underline{\omega})\big)e^{-2\pi i\underline{\omega}\underline{y}}d\underline{\omega}.$$

Finally,

$$U^{-1}\mathcal{F}_1^{-1}W^{-1}(\eta\otimes f)[\underline{y},B]=\mathcal{F}_1^{-1}W^{-1}(\eta\otimes f)(B^{-1}\underline{y},B)$$
$$=\int_{\widehat{\mathbb{R}^n}}\eta(\underline{\omega})f\big(B\gamma(\underline{\omega})\big)e^{-2\pi i\underline{\omega}B^{-1}\underline{y}}d\underline{\omega}.$$

We can define $U_\eta : L^2(\mathrm{GL}_n(\mathbb{R}))\to L^2(G_n)$ by

$$U_\eta f[\underline{y},B]=\int_{\widehat{\mathbb{R}^n}}\eta(\underline{\omega})f\big(B\gamma(\underline{\omega})\big)e^{-2\pi i\underline{\omega}B^{-1}\underline{y}}d\underline{\omega}, \tag{5.1}$$

for a.e. $[\underline{y},B]\in G_n$, $f\in L^2(\mathrm{GL}_n(\mathbb{R}))$. Thus, we obtain the following result:

Proposition 5.1.3. Let $\eta\in L^2(\mathcal{O})$ satisfy $\|\eta\|_2=1$. Then U_η is an isometric linear map of $L^2(\mathrm{GL}_n(\mathbb{R}))$ into $L^2(G_n)$ that intertwines $\pi^{\underline{\omega}_0}$ with λ_{G_n}.

Now, fix an orthonormal basis $\{\eta_j : j\in J\}$ of $L^2(\mathcal{O})$. For each $j\in J$, let $\mathcal{L}_{\eta_j}=U^{-1}\mathcal{F}_1^{-1}W^{-1}\mathcal{H}_{\eta_j}$. Since $U^{-1}\mathcal{F}_1^{-1}W^{-1}$ is a unitary map of $L^2(\mathcal{O}\times\mathrm{GL}_n(\mathbb{R}))$ onto $L^2(G_n)$,

$$L^2(G_n)=\textstyle\sum_{j\in J}^{\oplus}\mathcal{L}_{\eta_j}.$$

Thus, we have a decomposition of the left regular representation of G_n. Note that $L^2(\mathcal{O})$ is identified with $L^2(\widehat{\mathbb{R}^n})$.

Theorem 5.1.4. Let $\mathcal{O}=\widehat{\mathbb{R}^n}\setminus\{\underline{0}\}$. Let $\{\eta_j : j\in J\}$ be an orthonormal basis of $L^2(\widehat{\mathbb{R}^n})$. For each $j\in J$, define $U_{\eta_j} : L^2(\mathrm{GL}_n(\mathbb{R}))\to L^2(G_n)$ by

$$U_{\eta_j}f[\underline{y},B]=\int_{\widehat{\mathbb{R}^n}}\eta_j(\underline{\omega})f\big(B\gamma(\underline{\omega})\big)e^{-2\pi i\underline{\omega}B^{-1}\underline{y}}d\underline{\omega},$$

for $[\underline{y},B]\in G_n$, and $f\in L^2(\mathrm{GL}_n(\mathbb{R}))$. Let $\mathcal{L}_{\eta_j}=U_{\eta_j}L^2(\mathrm{GL}_n(\mathbb{R}))$. Then $\mathcal{L}_{\eta_j}$ is a closed λ_{G_n}-invariant subspace of $L^2(G_n)$ and U_{η_j} intertwines $\pi^{\underline{\omega}_0}$ with the restriction of λ_{G_n} to $\mathcal{L}_{\eta_j}$. Moreover, $L^2(G_n)=\sum_{j\in J}^{\oplus}\mathcal{L}_{\eta_j}$

5.2 The Affine Group on $\mathbb{R}$

When $n=1$, $G_1=\mathbb{R}\rtimes\mathbb{R}^*$. We recall that

$$\int_{\mathbb{R}^*}f\,d\mu_{\mathbb{R}^*}=\int_{\mathbb{R}}f(b)\frac{db}{|b|},$$

where the integral on the right hand side is the Lebesgue integral on $\mathbb{R}$, and

$$\int_{G_1}f\,d\mu_{G_1}=\int_{\mathbb{R}}\int_{\mathbb{R}}f[y,b]\frac{dy\,db}{b^2}.$$

We continue to write $N=\{[y,1] : y\in\mathbb{R}\}$ and $\widehat{N}=\{\chi_\omega : \omega\in\mathbb{R}\}$, where $\chi_\omega[y,1]=e^{2\pi i\omega y}$, for $[y,1]\in N$. Select $\underline{\omega}_0=\omega_0=1$. Then $\pi^{\underline{\omega}_0}=\pi^1$ acts on $L^2(\mathbb{R}^*)$ via, for

$[x,a] \in G_1$, $f \in L^2(\mathbb{R}^*)$, and a.e. $b \in \mathbb{R}^*$,

$$\pi^1[x,a]f(b) = e^{2\pi i b^{-1}x} f\big(a^{-1}b\big).$$

For any $g \in L^2(\mathbb{R}^*)$, define $V_g : L^2(\mathbb{R}^*) \to C(G_1)$ by $V_g f[x,a] = \langle f, \pi^1[x,a]g\rangle$, for $[x,a] \in G_1$ and $f \in L^2(\mathbb{R}^*)$.

Proposition 5.2.1. For $g \in L^2(\mathbb{R}^*)$, if there exists a nonzero $f_0 \in L^2(\mathbb{R}^*)$ such that $V_g f_0 \in L^2(G_1)$, then $\int_{\mathbb{R}} \big||u|^{1/2} g(u)\big|^2 \frac{du}{|u|} < \infty$. Moreover, if $\int_{\mathbb{R}} \big||u|^{1/2} g(u)\big|^2 \frac{du}{|u|} < \infty$, then $V_g f \in L^2(G_1)$, for all $f \in L^2(\mathbb{R}^*)$.

Proof. This proof follows the pattern of derivation of the admissiblility condition for the continuous wavelet transform (see [4], for example). Let $f, g \in L^2(\mathbb{R}^*)$. Then

$$\begin{aligned}
\int_{G_1} |V_g f[x,a]|^2 d\mu_{G_1}\big([x,a]\big) &= \int_{\mathbb{R}} \int_{\mathbb{R}} |\langle f, \pi^1[x,a]g\rangle|^2 \frac{dx\, da}{a^2} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(b) \overline{\pi^1[x,a]g(b)}\, \frac{db}{|b|} \right|^2 \frac{dx\, da}{a^2} \qquad (5.2)\\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(b^{-1}) \overline{g}(a^{-1}b^{-1}) e^{-2\pi i b x}\, \frac{db}{|b|} \right|^2 \frac{dx\, da}{a^2}
\end{aligned}$$

Let $\varphi_a(b) = |b^{-1}| f(b^{-1}) \overline{g}(a^{-1}b^{-1})$, for $a, b \in \mathbb{R}^*$. Note that

$$\begin{aligned}
\int_{\mathbb{R}} |\varphi_a(b)|\, db = \int_{\mathbb{R}} |f(b^{-1}) \overline{g}(a^{-1}b^{-1})|\, \frac{db}{|b|} &= \int_{\mathbb{R}^*} |f(b^{-1}) g(a^{-1}b^{-1})|\, d\mu_{\mathbb{R}^*}(b) \\
= \int_{\mathbb{R}^*} |f(b)| \cdot |g(a^{-1}b)|\, d\mu_{\mathbb{R}^*}(b) &= \langle |f|, \lambda_{\mathbb{R}^*}(a)|g|\rangle_{L^2(\mathbb{R}^*)}.
\end{aligned}$$

Thus, $\int_{\mathbb{R}} |\varphi_a(b)|\, db = \langle |f|, \lambda_{\mathbb{R}^*}(a)|g|\rangle_{L^2(\mathbb{R}^*)} \le \|f\|_2 \|g\|_2 < \infty$, so $\varphi_a \in L^1(\widehat{\mathbb{R}})$. Let $\varphi_a^\vee \in C_0(\mathbb{R})$ denote the inverse Fourier transform of φ_a. Then (5.2) becomes

$$\int_{G_1} |V_g f[x,a]|^2 d\mu_{G_1}\big([x,a]\big) = \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \varphi_a(b) e^{-2\pi i b x}\, db \right|^2 \frac{dx\, da}{a^2} = \int_{\mathbb{R}} \int_{\mathbb{R}} |\varphi_a^\vee(x)|^2 \frac{dx\, da}{a^2}.$$

Plancherel's Theorem implies $\|\varphi_a^\vee\|_2^2 = \|\varphi_a\|_2^2$ even when either side is ∞. Thus

$$\begin{aligned}
\int_{G_1} |V_g f[x,a]|^2 d\mu_{G_1}\big([x,a]\big) &= \int_{\mathbb{R}} \int_{\widehat{\mathbb{R}}} |\varphi_a(\omega)|^2 \frac{d\omega\, da}{a^2} \\
&= \int_{\mathbb{R}} \int_{\widehat{\mathbb{R}}} \left| \frac{1}{|\omega|} f(\omega^{-1}) \overline{g}(a^{-1}\omega^{-1}) \right|^2 \frac{d\omega\, da}{a^2} \qquad (5.3)\\
&= \int_{\mathbb{R}} \int_{\widehat{\mathbb{R}}} \big||\omega^{-1}|^{1/2} f(\omega^{-1}) \overline{g}(a^{-1}\omega^{-1})\big|^2 \frac{d\omega\, da}{|\omega| a^2} \\
&= \int_{\mathbb{R}} \int_{\widehat{\mathbb{R}}} \big||\omega|^{1/2} f(\omega) \overline{g}(a^{-1}\omega)\big|^2 \frac{d\omega\, da}{|\omega| a^2}
\end{aligned}$$

Note that the integration with respect to $\frac{d\omega}{|\omega|}$ can be viewed as the Haar integral on $\mathbb{R}^*$. Thus,

$$
\begin{aligned}
\int_{G_1} |V_g f[x,a]|^2 d\mu_{G_1}\big([x,a]\big) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^*} \left| |\omega|^{1/2} f(\omega) \overline{g}(a^{-1}\omega) \right|^2 d\mu_{\mathbb{R}^*}(\omega) \right) \frac{da}{a^2} \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^*} \left| |a\omega|^{1/2} f(a\omega) \overline{g}(\omega) \right|^2 d\mu_{\mathbb{R}^*}(\omega) \right) \frac{da}{a^2} \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^*} |f(a\omega)|^2 \left| |\omega|^{1/2} g(\omega) \right|^2 d\mu_{\mathbb{R}^*}(\omega) \right) \frac{da}{|a|} \\
&= \int_{\mathbb{R}^*} \left(\int_{\mathbb{R}} |f(\omega a)|^2 \frac{da}{|a|} \right) \left| |\omega|^{1/2} g(\omega) \right|^2 d\mu_{\mathbb{R}^*}(\omega) \qquad (5.4) \\
&= \int_{\mathbb{R}^*} \left(\int_{\mathbb{R}} |f(a)|^2 \frac{da}{|a|} \right) \left| |\omega|^{1/2} g(\omega) \right|^2 d\mu_{\mathbb{R}^*}(\omega) \\
&= \int_{\mathbb{R}^*} |f(a)|^2 d\mu_{\mathbb{R}^*}(a) \int_{\mathbb{R}^*} \left| |\omega|^{1/2} g(\omega) \right|^2 d\mu_{\mathbb{R}^*}(\omega) \\
&= \|f\|_2^2 \int_{\mathbb{R}^*} \left| |u|^{1/2} g(u) \right|^2 d\mu_{\mathbb{R}^*}(u).
\end{aligned}
$$

Therefore, for any $f, g \in L^2(\mathbb{R}^*)$,

$$\int_{G_1} |V_g f[x,a]|^2 d\mu_{G_1}\big([x,a]\big) = \|f\|_2^2 \int_{\mathbb{R}^*} \left| |u|^{1/2} g(u) \right|^2 d\mu_{\mathbb{R}^*}(u).$$

If the exists one nonzero $f_0 \in L^2(\mathbb{R}^*)$ such that $V_g f_0 \in L^2(G_1)$, then we must have

$$\int_{\mathbb{R}^*} \left| |u|^{1/2} g(u) \right|^2 d\mu_{\mathbb{R}^*}(u) < \infty.$$

On the other hand, if $\int_{\mathbb{R}^*} \left| |u|^{1/2} g(u) \right|^2 d\mu_{\mathbb{R}^*}(u) < \infty$, then $V_g f \in L^2(G_1)$, for every $f \in L^2(\mathbb{R}^*)$. □

Let $\mathcal{D} = \left\{ g \in L^2(\mathbb{R}^*) : \int_{\mathbb{R}^*} \left| |u|^{1/2} g(u) \right|^2 d\mu_{\mathbb{R}^*}(u) < \infty \right\}$. This is a subspace of $L^2(\mathbb{R}^*)$. It is not a closed subspace, but it is dense in $L^2(\mathbb{R}^*)$.

Remark. The $\mathbb{R}^*$-orbit of 1 in $\widehat{\mathbb{R}}$ is $\mathcal{O}_1 = \widehat{\mathbb{R}} \setminus \{0\}$. Writing $L^2(\mathcal{O}_1)$ indicates functions square-integrable with respect to Lebesgue measure on $\mathcal{O}_1$ while $L^2(\mathbb{R}^*)$ is formed with respect to the Haar measure of $\mathbb{R}^*$. Both $\mathcal{O}_1$ and $\mathbb{R}^*$ are parametrized by the nonzero real numbers. So we can think of $\mathcal{D}$ as $L^2(\mathcal{O}_1) \cap L^2(\mathbb{R}^*)$. If it is necessary to distinguish which norm or inner product is in use, a subscript of either $L^2(\mathcal{O}_1)$ or $L^2(\mathbb{R}^*)$ will be used.

Proposition 5.2.2. Let $f_1, f_2 \in L^2(\mathbb{R}^*)$ and $g_1, g_2 \in \mathcal{D}$. Then

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(G_1)} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^*)} \langle g_2, g_1 \rangle_{L^2(\mathcal{O}_1)}.$$

Proof. This can be established with a lengthy calculation similar to that in the proof of Proposition 5.2.1. However, there is a more efficient proof. By (5.4), for $g \in \mathcal{D}$ and $f \in L^2(\mathbb{R}^*)$,

$$\int_{G_1} |V_g f[x,a]|^2 d\mu_{G_1}([x,a]) = \|f\|^2_{L^2(\mathbb{R}^*)} \|g\|^2_{L^2(\mathcal{O}_1)}$$

If we fix a $g \in \mathcal{D}$, $g \neq 0$, and let $C_g = \|g\|_{L^2(\mathcal{O}_1)}$, then $B_g(f_1, f_2) = C_g^{-2} \langle V_g f_1, V_g f_2 \rangle_{L^2(G_1)}$, for all $f_1, f_2 \in L^2(\mathbb{R}^*)$, defines a sesquilinear form on $L^2(\mathbb{R}^*)$. Since $B_g(f,f) = \|f\|^2_{L^2(\mathbb{R}^*)}$, for all $f \in L^2(\mathbb{R}^*)$, the polarization identity implies $B_g(f_1,f_2) = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^*)}$. Thus,

$$\langle V_g f_1, V_g f_2 \rangle_{L^2(G_1)} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^*)} \|g\|^2_{L^2(\mathcal{O}_1)}, \text{ for } f_1, f_2 \in L^2(\mathbb{R}^*), g \in \mathcal{D}. \tag{5.5}$$

Now, fix $f_1, f_2 \in L^2(\mathbb{R}^*)$ such that $C_{f_1,f_2} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^*)} \neq 0$. Define a sesquilinear form B_{f_1,f_2} on $\mathcal{D}$ now, by

$$B_{f_1,f_2}(g_1,g_2) = \overline{C_{f_1,f_2}^{-1} \langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(G_1)}}, \text{ for all } g_1, g_2 \in \mathcal{D}.$$

Then (5.5) implies $B_{f_1,f_2}(g,g) = \|g\|^2_{L^2(\mathcal{O}_1)}$, for all $g \in \mathcal{D}$. By polarization again $B_{f_1,f_2}(g_1,g_2) = \langle g_1, g_2 \rangle_{L^2(\mathcal{O}_1)}$, for all $g_1, g_2 \in \mathcal{D}$. This implies

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(G_1)} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^*)} \langle g_2, g_1 \rangle_{L^2(\mathcal{O}_1)},$$

for all $g_1, g_2 \in \mathcal{D}$, when $\langle f_1, f_2 \rangle_{L^2(\mathbb{R}^*)} \neq 0$. Of course, this identity trivially holds if $\langle f_1, f_2 \rangle_{L^2(\mathbb{R}^*)} = 0$. □

Theorem 5.2.3. The representation π^1 is irreducible and square-integrable. Moreover, if $g \in L^2(\mathbb{R}^*)$ satisfies $\int_{\mathbb{R}} \left| |u|^{1/2} g(u) \right|^2 \frac{du}{|u|} = 1$, V_g is a linear isometry of $L^2(\mathbb{R}^*)$ into $L^2(G_1)$ that intertwines π^1 with a subrepresentation of the left regular representation of G_1.

Proof. The identity in

$$\int_{G_1} |V_g f[x,a]|^2 d\mu_{G_1}\big([x,a]\big) = \|f\|_2^2 \int_{\mathbb{R}^*} \left| |u|^{1/2} g(u) \right|^2 d\mu_{\mathbb{R}^*}(u)$$

shows that $V_g f = 0$ implies either f or g is 0. This implies that π^1 is irreducible. Moreover, if $g \in \mathcal{D}$, then $V_g f$ is square-integrable and if $\int_{\mathbb{R}^*} \left| |u|^{1/2} g(u) \right|^2 d\mu_{\mathbb{R}^*}(u) = 1$, then $V_g : L^2(\mathbb{R}^*) \to L^2(G_1)$ is an isometry. Also, if $\int_{\mathbb{R}^*} \left| |u|^{1/2} |g(u) \right|^2 d\mu_{\mathbb{R}^*}(u) = 1$ and $f \in L^2(\mathbb{R}^*)$, then, for $[x,a] \in G_1$,

$$\begin{aligned} V_g \pi^1[x,a] f[y,b] &= \langle \pi^1[x,a] f, \pi^1[y,b] g \rangle = \langle f, \pi^1[x,a]^* \pi^1[y,b] g \rangle \\ &= \langle f, \pi^1\big([x,a]^{-1}[y,b]\big) g \rangle = \lambda_{G_1}[x,a] V_g f[y,b], \end{aligned}$$

for all $[y,b] \in G_1$. This shows that the range of V_g is a λ_{G_1}-invariant subspace of $L^2(G_1)$ and V_g intertwines π^1 with the restriction of λ_{G_1} to this subspace. □

If $g \in L^2(\mathbb{R}^*)$ satisfies $\int_{\mathbb{R}} \left||u|^{1/2} g(u)\right|^2 \frac{du}{|u|} = 1$, then V_g is a version of the continuous wavelet transform on $\mathbb{R}$. To see this recall that $\mathcal{O}_1 = \{\omega \in \widehat{\mathbb{R}} : \omega \neq 0\}$ and $\gamma : \mathcal{O}_1 \to \mathbb{R}^*$ is simply the map $\gamma(\omega) = \omega^{-1}$. For $f \in L^2(\mathbb{R}^*)$, define $U_\gamma f$ on $\mathcal{O}_1$ by

$$\big(U_\gamma f\big)(\omega) = |\omega|^{-1/2} f\big(\gamma(\omega)\big), \text{ for all } \omega \in \mathcal{O}_1 \text{ and } \big(U_\gamma f\big)(0) = 0.$$

Then

$$\int_{\widehat{\mathbb{R}}} |\big(U_\gamma f\big)(\omega)|^2 d\omega = \int_{\widehat{\mathbb{R}}} \left||\omega|^{-1/2} f(\omega^{-1})\right|^2 d\omega = \int_{\mathbb{R}} \left|f(u^{-1})\right|^2 \frac{du}{|u|} = \int_{\widehat{\mathbb{R}}} \left|f(u)\right|^2 \frac{du}{|u|} = \|f\|_2^2.$$

That is, $U_\gamma f \in L^2(\widehat{\mathbb{R}})$ and U_γ is an isometry. It is easily verified that U_γ is a unitary map. Now, if $g \in L^2(\mathbb{R}^*)$ satisfies $\int_{\mathbb{R}^*} \left||u|^{1/2} |g(u)\right|^2 d\mu_{\mathbb{R}^*}(u) = 1$, let $\zeta = U_\gamma g$. Then

$$\begin{aligned}\int_{\widehat{\mathbb{R}}} \frac{|\zeta(\omega)|^2}{|\omega|} d\omega &= \int_{\widehat{\mathbb{R}}} \left||\omega|^{-1/2} g(\omega^{-1})\right|^2 \frac{d\omega}{|\omega|} = \int_{\mathbb{R}} \left||u|^{1/2} g(u)\right|^2 \frac{du}{|u|} \\ &= \int_{\mathbb{R}^*} \left||u|^{1/2} g(u)\right|^2 d\mu_{\mathbb{R}^*}(u) = 1.\end{aligned}$$

What does π^1 become when conjugated by U_γ. Let $\pi[x,a] = U_\gamma \pi^1[x,a] U_\gamma^{-1}$, for all $[x,a] \in G_1$. For any $\xi \in L^2(\widehat{\mathbb{R}})$, we have (writing $f = U_\gamma^{-1}\xi$)

$$\begin{aligned}\pi[x,a]\xi(\omega) &= U_\gamma \pi^1[x,a] f(\omega) = |\omega|^{-1/2} \pi^1[x,a] f\big(\omega^{-1}\big) = |\omega|^{-1/2} e^{2\pi i \omega x} f\big(a^{-1}\omega^{-1}\big) \\ &= |a|^{1/2} e^{2\pi i \omega x} \big|\omega a\big|^{-1/2} f\big((\omega a)^{-1}\big) = |a|^{1/2} e^{2\pi i \omega x} \xi(\omega a), \text{ for all } \omega \in \widehat{\mathbb{R}}.\end{aligned}$$

Thus, $\pi[x,a]\xi(\omega) = |a|^{1/2} e^{2\pi i \omega x} \xi(\omega a)$, for all $\omega \in \widehat{\mathbb{R}}, \xi \in L^2(\widehat{\mathbb{R}})$, and $[x,a] \in G_1$. Finally, we use the inverse Fourier transform to move π to a representation on $L^2(\mathbb{R})$. Let $\rho[x,a] = \mathcal{F}^{-1}\pi[x,a]\mathcal{F}$, for all $[x,a] \in G_1$. Then, for $f \in L^2(\mathbb{R})$, $\xi = \mathcal{F}f$, and $t \in \mathbb{R}$,

$$\begin{aligned}\rho[x,a]f(t) = \mathcal{F}^{-1}\pi[x,a]\xi(t) &= \int_{\widehat{\mathbb{R}}} \pi[x,a]\xi(\omega) e^{-2\pi i \omega t} d\omega \\ &= \int_{\widehat{\mathbb{R}}} |a|^{1/2} e^{2\pi i \omega x} \xi(\omega a) e^{-2\pi i \omega t} d\omega = \int_{\widehat{\mathbb{R}}} |a|^{1/2} \xi(\omega a) e^{-2\pi i \omega (t-x)} d\omega \\ &= \int_{\widehat{\mathbb{R}}} |a|^{-1/2} \xi(\nu) e^{-2\pi i \nu a^{-1}(t-x)} d\nu = |a|^{-1/2} f\big(a^{-1}(t-x)\big).\end{aligned}$$

Thus, ρ is the natural representation of G_1 on $L^2(\mathbb{R})$ and $w \in L^2(\mathbb{R})$ is admissible if $\widehat{w} = \zeta$, where ζ is as above. That is, if

$$\int_{\widehat{\mathbb{R}}} \frac{|\widehat{w}(\omega)|^2}{|\omega|} d\omega = 1.$$

Therefore $V_g : L^2(\mathbb{R}^*) \to L^2(G_1)$ above is just the standard continuous wavelet transform in disguise, converted using $\mathcal{F}$ and U_γ. Note that V_g intertwines π^1 with the left regular representation of G_1 restricted to the image of V_g.

Comparing U_η and V_g

Let's simplify U_η from Proposition 5.1.3 when $n = 1$. Then $\eta \in L^2(\mathcal{O}_1)$ and $\int_{\widehat{\mathbb{R}}} |\eta(\omega)|^2 d\omega = 1$. For $f \in L^2(\mathbb{R}^*)$, we have

$$U_\eta f[y,b] = \int_{\widehat{\mathbb{R}}} \eta(\omega) f\big(b\omega^{-1}\big) e^{-2\pi i \omega b^{-1} y}\, d\omega, \text{ for any } [y,b] \in G_1.$$

For $[x,a] \in G_1$, $f \in L^2(\mathbb{R}^*)$, and $[y,b] \in G_1$,

$$\begin{aligned} U_\eta \pi^1[x,a] f[y,b] &= \int_{\widehat{\mathbb{R}}} \eta(\omega) \pi^1[x,a] f\big(b\omega^{-1}\big) e^{-2\pi i \omega b^{-1} y} d\omega \\ &= \int_{\widehat{\mathbb{R}}} \eta(\omega) e^{2\pi i \omega b^{-1} x} f\big(a^{-1} b \omega^{-1}\big) e^{-2\pi i \omega b^{-1} y} d\omega \\ &= \int_{\widehat{\mathbb{R}}} \eta(\omega) f\big((a^{-1}b)\omega^{-1}\big) e^{-2\pi i \omega (a^{-1}b)^{-1} a^{-1}(y-x)} d\omega \\ &= U_\eta f\big([a^{-1}(y-x), a^{-1}b]\big) = U_\eta f\big([-a^{-1}x, a^{-1}][y,b]\big) \\ &= \lambda_{G_1}[x,a] U_\eta f[y,b]. \end{aligned}$$

On the other hand, if $g \in L^2(\mathbb{R}^*)$ satisfies $\int_{\mathbb{R}^*} \big||u|^{1/2} g(u)\|^2 d\mu_{\mathbb{R}^*}(u) = 1$, then

$$V_g f[y,b] = \int_{\mathbb{R}^*} f(u)\overline{g}(b^{-1}u) e^{-2\pi i u^{-1} y} d\mu_{\mathbb{R}^*}(u) = \int_{\mathbb{R}} f(u^{-1}) \overline{g}(b^{-1}u^{-1}) e^{-2\pi i u y} \frac{du}{|u|}, \quad (5.6)$$

for $[y,b] \in G_1$ and $f \in L^2(\mathbb{R}^*)$. Change variables in (5.6). Let $\omega = bu$. So $u = b^{-1}\omega$ and $du = |b|^{-1} d\omega$. Then

$$V_g f[y,b] = \int_{\widehat{\mathbb{R}}} \frac{\overline{g}(\omega^{-1})}{|\omega|} f(b\omega^{-1}) e^{-2\pi i b^{-1} \omega y}\, d\omega.$$

Thus, if $\eta(\omega) = \frac{\overline{g}(\omega^{-1})}{|\omega|}$, for all $\omega \in \mathcal{O}_1$, then $U_\eta = V_g$. Note that

$$\begin{aligned} \int_{\widehat{\mathbb{R}}} |\eta(\omega)|^2 d\omega &= \int_{\widehat{\mathbb{R}}} \left| \frac{\overline{g}(\omega^{-1})}{|\omega|} \right|^2 d\omega = \int_{\mathbb{R}} \left| \frac{g(\omega^{-1})}{|\omega|^{1/2}} \right|^2 \frac{d\omega}{|\omega|} = \int_{\mathbb{R}} \big||u|^{1/2} g(u)\big|^2 \frac{du}{|u|} \\ &= \int_{\mathbb{R}^*} \big||u|^{1/2} g(u)\big|^2 d\mu_{\mathbb{R}^*}(u) = \int_{\mathcal{O}_1} |g(u)|^2 du = 1. \end{aligned} \quad (5.7)$$

Proposition 5.2.4. If $g \in L^2(\mathbb{R}^*)$ satisfies $\int_{\mathbb{R}^*} \big||u|^{1/2} g(u)\|^2 d\mu_{\mathbb{R}^*}(u) = 1$ and $\eta(\omega) = \frac{\overline{g}(\omega^{-1})}{|\omega|}$, for all $\omega \in \mathcal{O}_1$, then $\eta \in L^2(\mathcal{O}_1)$, $\|\eta\|_{L^2(\mathcal{O}_1)} = 1$ and $U_\eta = V_g$.

More generally, if $g, g' \in \mathcal{D}$, $\eta(\omega) = \frac{\overline{g}(\omega^{-1})}{|\omega|}$ and $\eta'(\omega) = \frac{\overline{g'}(\omega^{-1})}{|\omega|}$, for all $\omega \in \mathcal{O}_1$, then the same change of variables as in (5.7) shows that $\langle \eta, \eta' \rangle_{L^2(\mathcal{O}_1)} = \langle g', g \rangle_{L^2(\mathcal{O}_1)}$. This means that if J is an index set and $\{g_j : j \in J\}$ is a set of members of $L^2(\mathbb{R}^*)$

such that each $g_j \in \mathcal{D} = L^2(\mathbb{R}^*) \cap L^2(\mathcal{O}_1)$ and $\{g_j : j \in J\}$ is an orthonormal set in $L^2(\mathcal{O}_1)$, then letting $\eta_j(\omega) = \frac{\overline{g_j}(\omega^{-1})}{|\omega|}$, for all $\omega \in \mathcal{O}_1$ and $j \in J$ gives an orthonormal set $\{\eta_j : j \in J\}$ in $L^2(\mathcal{O}_1)$. Combining this observation with Theorem 5.1.4 in the case of $n = 1$, we get the following result.

Theorem 5.2.5. Let $\mathcal{O}_1 = \widehat{\mathbb{R}} \setminus \{0\}$ be equipped with the restriction of Lebesgue measure and let $\mathcal{D} = L^2(\mathcal{O}_1) \cap L^2(\mathbb{R}^*)$. Suppose $\{g_j : j \in J\}$ is a set in $\mathcal{D}$ that forms an orthonormal basis in $L^2(\mathcal{O}_1)$. For each $j \in J$, define

$$V_{g_j} f[x, a] = \langle f, \pi^1[x, a] g_j \rangle_{L^2(\mathbb{R}^*)}, \text{ for } [x, a] \in G_1, f \in L^2(\mathbb{R}^*).$$

Let $\mathcal{K}_{g_j} = V_{g_j} L^2(\mathbb{R}^*)$. Then $\mathcal{K}_{g_j}$ is a closed λ_{G_1}-invariant subspace of $L^2(G_1)$ and V_{g_j} is a unitary map of $L^2(\mathbb{R}^*)$ onto $\mathcal{K}_{g_j}$ that intertwines π^1 with the restriction of λ_{G_1} to $\mathcal{K}_{g_j}$. Moreover, $L^2(G_1) = \sum_{j \in J}^{\oplus} \mathcal{K}_{g_j}$.

5.3 The Affine Group of the Plane

Throughout this section, let G denote $G_2 = \mathbb{R}^2 \rtimes \mathrm{GL}_2(\mathbb{R})$. The closed subgroup $N = \{[\underline{y}, \mathrm{id}] : \underline{y} \in \mathbb{R}^2\}$ is normal and abelian. As we saw for G_n there is just one nontrivial orbit for the action of G on $\widehat{N}$. In this section, we will follow the procedure in Section 2.8 to find a distinguished representation of G. Then a sequence of unitary maps between different Hilbert spaces will be used to move this distinguished representation into a subrepresentation of the left regular representation of G.

The nontrivial orbit in $\widehat{N}$ is $\{\chi_{\underline{\omega}} : \underline{\omega} \in \mathcal{O}\}$, where $\mathcal{O} = \widehat{\mathbb{R}^2} \setminus \{(0,0)\}$. Recall that $[\underline{x}, A] \in G$ acts on $\chi_{\underline{\omega}}$ so that $[\underline{x}, A] \cdot \chi_{\underline{\omega}} = \chi_{\underline{\omega} A^{-1}}$. The point $\underline{\omega}_0 = (1, 0)$ will serve as a representative point in $\mathcal{O}$. The stability subgroup for this point is

$$H_{(1,0)} = \big\{A \in \mathrm{GL}_2(\mathbb{R}) : (1,0)A = (1,0)\big\} = \left\{ \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} : u, v \in \mathbb{R}, v \neq 0 \right\}.$$

Recall from Section 4.3 that $K_0 = \left\{ \begin{pmatrix} s & -t \\ t & s \end{pmatrix} : s, t \in \mathbb{R}, s^2 + t^2 > 0 \right\}$ is a subgroup in $\mathrm{GL}_2(\mathbb{R})$ that is complementary to $H_{(1,0)}$ in the sense that $K_0 \cap H_{(1,0)} = \{\mathrm{id}\}$ and $\mathrm{GL}_2(\mathbb{R}) = K_0 H_{(1,0)}$. For each $\underline{\omega} = (\omega_1, \omega_2) \in \mathcal{O}$, there is a unique matrix $\gamma(\underline{\omega}) \in K_0$ such that $\gamma(\underline{\omega}) \cdot (1,0) = \underline{\omega}$. Since $\underline{\omega} = \gamma(\underline{\omega}) \cdot (1,0) = (1,0)\gamma(\underline{\omega})^{-1}$, the top row of $\gamma(\underline{\omega})^{-1}$ must be $(\omega_1 \quad \omega_2)$. Thus

$$\gamma(\underline{\omega})^{-1} = \begin{pmatrix} \omega_1 & \omega_2 \\ -\omega_2 & \omega_1 \end{pmatrix} \quad \text{and} \quad \gamma(\underline{\omega}) = \frac{1}{\|\underline{\omega}\|^2} \begin{pmatrix} \omega_1 & -\omega_2 \\ \omega_2 & \omega_1 \end{pmatrix}.$$

We will frequently use that $(1,0)\gamma(\underline{\omega})^{-1} = \underline{\omega}$ and $\underline{\omega}\gamma(\underline{\omega}) = (1,0)$, for any $\underline{\omega} \in \mathcal{O}$.

We also have, for each $A \in \mathrm{GL}_2(\mathbb{R})$, unique matrices $M_A \in K_0$ and $C_A \in H_{(1,0)}$ such that $A = M_A C_A$. In our calculations later, various matrices related to $A \in \mathrm{GL}_2(\mathbb{R})$ and $\underline{\omega} \in \mathcal{O}$ arise and there are a number of identities involving these matrices that are useful. We also use the entries of the matrix $C_{A^{-1}\gamma(\underline{\omega})}^{-1}$ and need a notation for these entries. Let $u_{\underline{\omega},A} = (0,1)C_{A^{-1}\gamma(\underline{\omega})}^{-1}\begin{pmatrix}1\\0\end{pmatrix}$ and $v_{\underline{\omega},A} = (0,1)C_{A^{-1}\gamma(\underline{\omega})}^{-1}\begin{pmatrix}0\\1\end{pmatrix}$. Then $C_{A^{-1}\gamma(\underline{\omega})}^{-1} = \begin{pmatrix}1 & 0\\ u_{\underline{\omega},A} & v_{\underline{\omega},A}\end{pmatrix}$. The identities we need are collected into a proposition.

Proposition 5.3.1. Let $A, B \in \mathrm{GL}_2(\mathbb{R})$ and $\underline{\omega} \in \mathcal{O}$. Then

(a) $M_A = \gamma\big((1,0)A^{-1}\big)$ and $C_A = \gamma\big((1,0)A^{-1}\big)^{-1}A$,

(b) $M_{A\gamma(\underline{\omega})} = \gamma\big(\underline{\omega}A^{-1}\big)$ and $C_{A\gamma(\underline{\omega})} = \gamma\big(\underline{\omega}A^{-1}\big)^{-1}A\gamma(\underline{\omega})$,

(c) $M_{A^{-1}\gamma(\underline{\omega})} = \gamma\big(\underline{\omega}A\big)$ and $C_{A^{-1}\gamma(\underline{\omega})} = \gamma\big(\underline{\omega}A\big)^{-1}A^{-1}\gamma(\underline{\omega})$,

(d) $u_{\underline{\omega},AB} = u_{\underline{\omega},A} + v_{\underline{\omega},A}u_{\underline{\omega}A,B}$ and $v_{\underline{\omega},AB} = v_{\underline{\omega},A}v_{\underline{\omega}A,B}$,

(e) $\det\left(\gamma(\underline{\omega})^{-1}\right) = \|\underline{\omega}\|^2$ and $\det\left(\gamma(\underline{\omega})\right) = \|\underline{\omega}\|^{-2}$,

(f) $\det\left(C_{A^{-1}\gamma(\underline{\omega})}\right) = \frac{\|\underline{\omega}A\|^2}{\det(A)\|\underline{\omega}\|^2}$, and

(g) $v_{\underline{\omega},A} = \det\left(C_{A^{-1}\gamma(\underline{\omega})}^{-1}\right) = \frac{\det(A)\|\underline{\omega}\|^2}{\|\underline{\omega}A\|^2}$.

Proof. (a) and (c) follow from (b), since $\gamma\big((1,0)\big)$ is the identity matrix. For any $\underline{\omega} \in \mathcal{O}$, $\gamma\big(\underline{\omega}A^{-1}\big) \in K_0$ by definition of γ. On the other hand,

$$\begin{aligned}(1,0)\left(\gamma\big(\underline{\omega}A^{-1}\big)^{-1}A\gamma(\underline{\omega})\right) &= \left((1,0)\gamma\big(\underline{\omega}A^{-1}\big)^{-1}\right)A\gamma(\underline{\omega})\\ &= \underline{\omega}A^{-1}A\gamma(\underline{\omega}) = \underline{\omega}\gamma(\underline{\omega}) = (1,0).\end{aligned}$$

Thus $C_{A\gamma(\underline{\omega})} = \gamma\big(\underline{\omega}A^{-1}\big)^{-1}A\gamma(\underline{\omega})$ and $M_{A\gamma(\underline{\omega})} = \gamma\big(\underline{\omega}A^{-1}\big)$, by uniqueness. Clearly (e) is true while (f) and (g) follow from (c) and (e). It remains to verify (d).

By (c), $C_{A^{-1}\gamma(\underline{\omega})}^{-1} = \gamma(\underline{\omega})^{-1}A\gamma(\underline{\omega}A)$. Thus

$$\begin{aligned}C_{(AB)^{-1}\gamma(\underline{\omega})}^{-1} &= \gamma(\underline{\omega})^{-1}(AB)\gamma(\underline{\omega}AB) = \gamma(\underline{\omega})^{-1}A\gamma(\underline{\omega}A)\gamma(\underline{\omega}A)^{-1}B\gamma(\underline{\omega}AB)\\ &= C_{A^{-1}\gamma(\underline{\omega})}^{-1}C_{B^{-1}\gamma(\underline{\omega}A)}^{-1}.\end{aligned}$$

That is,

$$\begin{aligned}\begin{pmatrix}1 & 0\\ u_{\underline{\omega},AB} & v_{\underline{\omega},AB}\end{pmatrix} &= \begin{pmatrix}1 & 0\\ u_{\underline{\omega},A} & v_{\underline{\omega},A}\end{pmatrix}\begin{pmatrix}1 & 0\\ u_{\underline{\omega}A,B} & v_{\underline{\omega}A,B}\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ u_{\underline{\omega},A} + v_{\underline{\omega},A}u_{\underline{\omega}A,B} & v_{\underline{\omega},A}v_{\underline{\omega}A,B}\end{pmatrix},\end{aligned}$$

which establishes (d). □

The detailed values of $u_{\underline{\omega},A}$ and $v_{\underline{\omega},A}$ are usually not needed, but may sometimes be useful.

Proposition 5.3.2. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ and $\underline{\omega} = (\omega_1, \omega_2) \in \mathcal{O}$. Then

$$u_{\underline{\omega},A} = \frac{(ac+bd)(\omega_1^2 - \omega_2^2) - (a^2+b^2-c^2-d^2)\omega_1\omega_2}{(a\omega_1 + c\omega_2)^2 + (b\omega_1 + d\omega_2)^2}$$

and

$$v_{\underline{\omega},A} = \frac{(ad-bc)(\omega_1^2 + \omega_2^2)}{(a\omega_1 + c\omega_2)^2 + (b\omega_1 + d\omega_2)^2}.$$

Proof. These are both obtained by straightforward calculation from the definitions of $u_{\underline{\omega},A}$ and $v_{\underline{\omega},A}$. □

The map $[u,v] \to \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix}$ is an isomorphism of the group $G_1 = \mathbb{R} \rtimes \mathbb{R}^*$ with $H_{(1,0)}$. Recall the representation π^1 of G_1 acts on the Hilbert space $L^2(\mathbb{R}^*)$ by, for $[u,v] \in G_1$ and $f \in L^2(\mathbb{R}^*)$,

$$\pi^1[u,v]f(t) = e^{2\pi i t^{-1}u} f(v^{-1}t), \quad \text{for } t \in \mathbb{R}^*.$$

By Theorem 5.2.3, π^1 is an irreducible representation of G_1. We will simplify notation by considering π^1 as an irreducible representation of $H_{(1,0)}$. That is, if $C = \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} \in H_{(1,0)}$, then we let $\pi^1(C) = \pi^1[u,v]$.

Let $H = \mathbb{R}^2 \rtimes H_{(1,0)}$. The representation $\chi_{(1,0)} \otimes \pi^1$ of H given by

$$\left(\chi_{(1,0)} \otimes \pi^1\right)[\underline{x}, B] = \chi_{(1,0)}(\underline{x})\pi^1(B), \quad \text{for } [\underline{x}, B] \in H,$$

is an irreducible representation of H on $L^2(\mathbb{R}^*)$. Then $\mathrm{ind}_H^G\left(\chi_{(1,0)} \otimes \pi^1\right)$ is an irreducible representation of G by Theorem 2.9.5. Let $K = \{[\underline{0}, L] : L \in K_0\}$, a closed subgroup of G that is complementary to H. That is $K \cap H = \{[\underline{0}, \mathrm{id}]\}$ and $G = KH$. By Proposition 2.8.9, $\mathrm{ind}_H^G\left(\chi_{(1,0)} \otimes \pi^1\right)$ is equivalent to a representation σ acting on $L^2\big(K, L^2(\mathbb{R}^*)\big)$. In preparation to defining σ, take $[\underline{x}, A] \in G$, and $[\underline{0}, L] \in K$ and compute $[\underline{x}, A]^{-1}[\underline{0}, L] = [-A^{-1}\underline{x}, A^{-1}L]$. Now factor

$$[-A^{-1}\underline{x}, A^{-1}L] = [\underline{0}, M_{A^{-1}L}][-M_{A^{-1}L}^{-1}A^{-1}\underline{x}, C_{A^{-1}L}],$$

with the elements $[\underline{0}, M_{A^{-1}L}] \in K$ and $[-M_{A^{-1}L}^{-1}A^{-1}\underline{x}, C_{A^{-1}L}] \in H$. Observe that $C_{A^{-1}L}^{-1}M_{A^{-1}L}^{-1}A^{-1} = \left(M_{A^{-1}L}C_{A^{-1}L}\right)^{-1}A^{-1} = L^{-1}$. Therefore,

$$[-M_{A^{-1}L}^{-1}A^{-1}\underline{x}, C_{A^{-1}L}]^{-1} = [L^{-1}\underline{x}, C_{A^{-1}L}^{-1}].$$

Thus, Proposition 2.8.9 gives, for $F \in L^2\big(K, L^2(\mathbb{R}^*)\big)$, $[\underline{x}, A] \in G$, and $[\underline{0}, L] \in K$,

$$\begin{aligned} \sigma[\underline{x}, A]F[\underline{0}, L] &= \left|\det\left(C_{A^{-1}L}\right)\right|^{-1/2}\left(\chi_{(1,0)} \otimes \pi^1\right)\left[L^{-1}\underline{x}, C_{A^{-1}L}^{-1}\right]F[\underline{0}, M_{A^{-1}L}] \\ &= \left|\det\left(C_{A^{-1}L}\right)\right|^{-1/2} e^{2\pi i (1,0) L^{-1}\underline{x}}\, \pi^1\Big(C_{A^{-1}L}^{-1}\Big)F[\underline{0}, M_{A^{-1}L}]. \end{aligned} \tag{5.8}$$

To make σ more concrete, use a unitary map to move to the Hilbert space $L^2(\widehat{\mathbb{R}^3})$. The formulas are simplified if we write $\widehat{\mathbb{R}^3}$ as $\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}$. Define $U : L^2\big(K, L^2(\mathbb{R}^*)\big) \to L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$ by, for $F \in L^2\big(K, L^2(\mathbb{R}^*)\big)$ and a.e. $(\underline{\omega}, \omega_3) \in \widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}$, let

$$(UF)(\underline{\omega}, \omega_3) = \begin{cases} \frac{\big(F[\underline{0}, \gamma(\underline{\omega})]\big)(\omega_3^{-1})}{\|\underline{\omega}\| \cdot |\omega_3|^{1/2}} & \text{for } \underline{\omega} \in \mathcal{O}, \omega_3 \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.3.3. The map $U : L^2\big(K, L^2(\mathbb{R}^*)\big) \to L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$ is unitary with inverse U^{-1} given by, for $f \in L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$ and $[\underline{0}, L] \in K$,

$$\big(U^{-1} f[\underline{0}, L]\big)(\nu) = \begin{cases} \frac{\|(1,0)L^{-1}\| \, f\big((1,0)L^{-1}, \nu^{-1}\big)}{|\nu|^{1/2}} & \text{if } \nu \neq 0 \\ 0 & \text{if } \nu = 0. \end{cases}$$

Proof. For $F \in L^2\big(K, L^2(\mathbb{R}^*)\big)$,

$$\begin{aligned} \|UF\|^2_{L^2(\widehat{\mathbb{R}^2} \times \widehat{R})} &= \int_{\widehat{\mathbb{R}^2}} \int_{\widehat{\mathbb{R}}} |UF(\underline{\omega}, \omega_3)|^2 d\omega_3 \, d\underline{\omega} \\ &= \int_{\widehat{\mathbb{R}^2}} \int_{\widehat{\mathbb{R}}} \left| \frac{(F[\underline{0}, \gamma(\underline{\omega})])\,(\omega_3^{-1})}{\|\underline{\omega}\| \cdot |\omega_3|^{1/2}} \right|^2 d\omega_3 \, d\underline{\omega} \\ &= \int_{\widehat{\mathbb{R}^2}} \int_{\widehat{\mathbb{R}}} \big|(F[\underline{0}, \gamma(\underline{\omega})])\,(\omega_3^{-1})\big|^2 \frac{d\omega_3}{|\omega_3|} \frac{d\underline{\omega}}{\|\underline{\omega}\|^2} \\ &= \int_K \int_{\mathbb{R}^*} \big|(F[\underline{0}, L])\,(\omega_3^{-1})\big|^2 d\mu_{\mathbb{R}^*}(\omega_3) \, d\mu_K([\underline{0}, L] \\ &= \|F\|_{L^2(K, L^2(\mathbb{R}^*))} \end{aligned}$$

This shows that U is an isometry. Then, U is one-to-one. Moreover, U is onto since for $f \in L^2(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}})$, we will show $U^{-1} f \in L^2(K, L^2(\mathbb{R}^*))$. Then

$$\begin{aligned} U(U^{-1} f)(\underline{\omega}, \omega_3) &= \frac{U^{-1} f[\underline{0}, \gamma(\underline{\omega})](\omega_3^{-1})}{\|\underline{\omega}\| \cdot |\omega_3|^{1/2}} \\ &= \frac{1}{\|\underline{\omega}\| \cdot |\omega_3|^{1/2}} \frac{\|\underline{\omega}\| \, f(\underline{\omega}, \omega_3)}{|\omega_3^{-1}|^{1/2}} \\ &= f(\underline{\omega}, \omega_3), \end{aligned}$$

for a.e. $(\underline{\omega},\omega_3)\in\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}$. Now, for any $g\in L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})$,

$$\begin{aligned}
\|U^{-1}g\|^2_{L^2(K,L^2(\mathbb{R}^*))} &= \int_K\int_{\mathbb{R}^*}|U^{-1}g[\underline{0},L](\nu)|^2 d\mu_{\mathbb{R}^*}(\nu)\,d\mu_K([\underline{0},L]) \\
&= \int_K\int_{\mathbb{R}}\left|\frac{\|(1,0)L^{-1}\|\,g\big((1,0)L^{-1},\nu^{-1}\big)}{|\nu|^{1/2}}\right|^2\frac{d\nu}{|\nu|}d\mu_K([\underline{0},L]) \\
&= \int_{\widehat{\mathbb{R}^2}}\int_{\mathbb{R}^*}\|\underline{\omega}\|^2|g(\underline{\omega},\nu^{-1})|^2\frac{d\nu}{|\nu|^2}\,\frac{d\underline{\omega}}{\|\underline{\omega}\|^2} \\
&= \int_{\mathbb{R}^*}\int_{\widehat{\mathbb{R}^2}}|g(\underline{\omega},\nu^{-1})\,|^2\,d\underline{\omega}\,\frac{d\nu}{|\nu|^2} \\
&= \int_{\widehat{\mathbb{R}^2}}\int_{\widehat{\mathbb{R}}}|\,g(\underline{\omega},\omega_3)\,|^2 d\omega_3\,d\underline{\omega} \\
&= \|g\|^2_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})}
\end{aligned}$$

Note that, $L=\begin{pmatrix}\omega_1/\|\underline{\omega}\|^2 & -\omega_2/\|\underline{\omega}\|^2\\ \omega_2/\|\underline{\omega}\|^2 & \omega_1/\|\underline{\omega}\|^2\end{pmatrix}$ if $(1,0)L^{-1}=\underline{\omega}$ for any $\underline{\omega}\in\mathcal{O}$. □

For $[\underline{x},A]\in G$, let $\sigma_1[\underline{x},A]=U\sigma[\underline{x},A]U^{-1}$. Then σ_1 is an irreducible representation of G on $L^2\big(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}\big)$. To find a formula for σ_1, let $\xi\in L^2\big(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}\big)$ and set $F=U^{-1}\xi\in L^2\big(K,L^2(\mathbb{R}^*)\big)$. For $[\underline{x},A]\in G$ and a.e. $(\underline{\omega},\omega_3)\in\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}$,

$$\begin{aligned}
\big(\sigma_1[\underline{x},A]\xi\big)(\underline{\omega},\omega_3) &= \big(U\sigma[\underline{x},A]F\big)(\underline{\omega},\omega_3)=\frac{\big(\sigma[\underline{x},A]F[\underline{0},\gamma(\underline{\omega})]\big)(\omega_3^{-1})}{\|\underline{\omega}\|\cdot|\omega_3|^{1/2}} \\
&= \frac{|\det(C_{A^{-1}\gamma(\underline{\omega})})|^{-1/2}}{\|\underline{\omega}\|\cdot|\omega_3|^{1/2}}\,e^{2\pi i\underline{\omega}\underline{x}}\left(\pi^1\left(C^{-1}_{A^{-1}\gamma(\underline{\omega})}\right)F[\underline{0},M_{A^{-1}\gamma(\underline{\omega})}]\right)(\omega_3^{-1}).
\end{aligned}\tag{5.9}$$

By Proposition 5.3.1 (f), $\frac{|\det(C_{A^{-1}\gamma(\underline{\omega})})|^{-1/2}}{\|\underline{\omega}\|\cdot|\omega_3|^{1/2}}=\frac{|\det(A)|^{1/2}}{\|\underline{\omega}A\|\cdot|\omega_3|^{1/2}}$. By Prop. 5.3.1 (c), $M_{A^{-1}\gamma(\underline{\omega})}=\gamma(\underline{\omega}A)$. Also, recall that $C^{-1}_{A^{-1}\gamma(\underline{\omega})}=\begin{pmatrix}1 & 0\\ u_{\underline{\omega},A} & v_{\underline{\omega},A}\end{pmatrix}$. Thus, (5.9) implies

$$\begin{aligned}
\big(\sigma_1[\underline{x},A]f\big)(\underline{\omega},\omega_3) &= \frac{|\det(A)|^{1/2}}{\|\underline{\omega}A\|\cdot|\omega_3|^{1/2}}e^{2\pi i\underline{\omega}\underline{x}}\left(\pi^1[u_{\underline{\omega},A},v_{\underline{\omega},A}]F[\underline{0},\gamma(\underline{\omega}A)]\right)(\omega_3^{-1}) \\
&= \frac{|\det(A)|^{1/2}}{\|\underline{\omega}A\|\cdot|\omega_3|^{1/2}}e^{2\pi i\underline{\omega}\underline{x}}e^{2\pi i\omega_3u_{\underline{\omega},A}}\big(U^{-1}\xi[\underline{0},\gamma(\underline{\omega}A)]\big)(v^{-1}_{\underline{\omega},A}\omega_3^{-1})
\end{aligned}\tag{5.10}$$

Before applying U^{-1}, recall $(1,0)\gamma(\underline{\omega})^{-1}=\underline{\omega}$, for any $\underline{\omega}\in\mathcal{O}$, so $(1,0)\gamma(\underline{\omega}A)^{-1}=\underline{\omega}A$. Thus,

$$\begin{aligned}
\big(\sigma_1[\underline{x},A]\xi\big)(\underline{\omega},\omega_3) &= \frac{|\det(A)|^{1/2}}{\|\underline{\omega}A\|\cdot|\omega_3|^{1/2}}e^{2\pi i\underline{\omega}\underline{x}}e^{2\pi i\omega_3u_{\underline{\omega},A}}\frac{\|\underline{\omega}A\|\xi\big(\underline{\omega}A,\omega_3v_{\underline{\omega},A}\big)}{|v^{-1}_{\underline{\omega},A}\omega_3^{-1}|^{1/2}} \\
&= |\det(A)|^{1/2}|v_{\underline{\omega},A}|^{1/2}e^{2\pi i\underline{\omega}\underline{x}}e^{2\pi i\omega_3u_{\underline{\omega},A}}\xi\big(\underline{\omega}A,\omega_3v_{\underline{\omega},A}\big).
\end{aligned}$$

Finally, Proposition 5.3.1 (g) says $v_{\underline{\omega},A} = \frac{\det(A)\|\underline{\omega}\|^2}{\|\underline{\omega}A\|^2}$. Therefore,

$$\big(\sigma_1[\underline{x},A]\xi\big)(\underline{\omega},\omega_3) = \tfrac{|\det(A)|\cdot\|\underline{\omega}\|}{\|\underline{\omega}A\|} e^{2\pi i(\underline{\omega}\underline{x}+\omega_3 u_{\underline{\omega},A})}\xi\big(\underline{\omega}A,\omega_3 v_{\underline{\omega},A}\big). \tag{5.11}$$

Note that (5.11) is for almost every $(\underline{\omega},\omega_3) \in \widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}$, for any $\xi \in L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})$, and for any $[\underline{x},A] \in G$.

Remark. As a check on the accuracy of the calculations leading to (5.11), we verify that

(a) $\|\sigma_1[\underline{x},A]\xi\|^2_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})} = \|\xi\|^2_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})}$, for all $\xi \in L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})$, and for any $[\underline{x},A]\in G$.

(b) $\sigma_1[\underline{x},A]\sigma_1[\underline{y},B] = \sigma_1[\underline{x}+A\underline{y},AB]$, for all $[\underline{x},A],[\underline{y},B]\in G$.

Proof. First, we prove (a). Let $\xi \in L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})$. Then

$$\begin{aligned}\|\sigma_1[\underline{x},A]\xi\|^2_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})} &= \int_{\widehat{\mathbb{R}^2}}\int_{\widehat{\mathbb{R}}} \left|\tfrac{|\det(A)|\cdot\|\underline{\omega}\|}{\|\underline{\omega}A\|}\right|^2 \ \left|e^{2\pi i(\underline{\omega}\underline{x}+\omega_3 u_{\underline{\omega},A})}\xi\big(\underline{\omega}A,\omega_3 v_{\underline{\omega},A}\big)\right|^2 d\underline{\omega}_3\ d\omega \\ &= \int_{\widehat{\mathbb{R}^2}}\int_{\widehat{\mathbb{R}}} \tfrac{|\det(A)|^2\cdot\|\underline{\omega}\|^2}{\|\underline{\omega}A\|^2}\ \left|\xi\big(\underline{\omega}A,\omega_3 v_{\underline{\omega},A}\big)\right|^2 d\underline{\omega}_3\ d\omega\end{aligned}$$

by change the variables, let $t = \omega_3\, v_{\underline{\omega},A}$; $dt = v_{\underline{\omega},A}\, d\omega_3$. Thus

$$\|\sigma_1[\underline{x},A]\xi\|^2_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})} = \int_{\widehat{\mathbb{R}^2}}\int_{\widehat{\mathbb{R}}} |\det(A)|\ \left|\xi\big(\underline{\omega}A,t\big)\right|^2 dt\ d\underline{\omega}$$

by change the variables again, let $\underline{\omega}' = \underline{\omega}A$; $d\underline{\omega}' = |\det(A)|\, d\underline{\omega}$. So

$$\|\sigma_1[\underline{x},A]\xi\|^2_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})} = \int_{\widehat{\mathbb{R}}}\int_{\widehat{\mathbb{R}^2}}\ |\xi\big(\underline{\omega}',t)|^2 d\underline{\omega}'\ dt = \|\xi\|^2_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})}.$$

Second, to prove (b), let $\xi \in L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})$. For any $[\underline{y},B]\in G$, let $\eta = \sigma_1[\underline{y},B]\xi$. Then, for any $[\underline{x},A]\in G$.

$$\begin{aligned}\sigma_1[\underline{x},A]\ \big(\sigma_1[\underline{y},B]\xi\big)\ (\underline{\omega},\omega_3) &= \sigma_1[\underline{x},A]\eta(\underline{\omega},\omega_3) \\ &= \tfrac{|\det(A)|\cdot\|\underline{\omega}\|}{\|\underline{\omega}A\|} e^{2\pi i(\underline{\omega}\underline{x}+\omega_3 u_{\underline{\omega},A})}\eta\big(\underline{\omega}A,\omega_3 v_{\underline{\omega},A}\big) \\ &= \tfrac{|\det(A)|\,|\det(B)|\cdot\|\underline{\omega}\|}{\|\underline{\omega}AB\|} e^{2\pi i\underline{\omega}(\underline{x}+A\underline{y})}\ e^{2\pi i\omega_3(u_{\underline{\omega},A}+v_{\underline{\omega},A}u_{\underline{\omega}A,B})}\xi\big(\underline{\omega}AB,\omega_3 v_{\underline{\omega},A}v_{\underline{\omega}A,B}\big),\end{aligned}$$

using, $\eta(\underline{\omega}A,\omega_3 v_{\underline{\omega},A}) = \frac{|\det(B)|\cdot\|\underline{\omega}A\|}{\|\underline{\omega}AB\|} e^{2\pi i(\underline{\omega}A\underline{y}+\omega_3 v_{\underline{\omega},A}u_{\underline{\omega}A,B})} f(\underline{\omega}AB,\omega_3 v_{\underline{\omega},A}v_{\underline{\omega}A,B})$. On the other hand, the right hand side of (b) is given by

$\sigma_1[\underline{x}+A\underline{y},AB]f(\underline{\omega},\omega_3) = \frac{|\det(AB)|\cdot\|\underline{\omega}\|}{\|\underline{\omega}AB\|} e^{2\pi i\big(\underline{\omega}(\underline{x}+A\underline{y})+\omega_3 u_{\underline{\omega},AB}\big)} f(\underline{\omega}AB,\omega_3 v_{\underline{\omega},AB})$.

By proposition 5.3.1 (d), we have that $\nu_{\underline{\omega},A}\nu_{\underline{\omega}A,B} = \nu_{\underline{\omega},AB}$ and $u_{\underline{\omega},AB} = u_{\underline{\omega},A} + \nu_{\underline{\omega},A}u_{\underline{\omega}A,B}$. By these we get the equality in (b). $\square$

Now, return to the representation σ on the Hilbert space $L^2\big(K,L^2(\mathbb{R}^*)\big)$. Using the homeomorphism $\gamma : \mathcal{O} \to K_0$ and the identities collected in Proposition 5.3.1 help make the expression given in (5.8) easier to read. For $F \in L^2\big(K,L^2(\mathbb{R}^*)\big)$, $[\underline{x},A]\in G$, and a.e. $\underline{\omega} \in \mathcal{O}$,

$$\sigma[\underline{x}, A]F[\underline{0}, \gamma(\underline{\omega})] = \tfrac{|\det(A)|^{1/2}\|\underline{\omega}\|}{\|\underline{\omega}A\|} e^{2\pi i \underline{\omega}\underline{x}}\, \pi^1\big(\gamma(\underline{\omega})^{-1}A\gamma(\underline{\omega}A)\big)F[\underline{0}, \gamma(\underline{\omega}A)]. \tag{5.12}$$

Theorem 5.2.3 says that, if $g \in L^2(\mathbb{R}^*)$ satisfies $\int_{\mathbb{R}^*} \big||\nu|^{1/2}g(\nu)\big|^2 d\mu_{\mathbb{R}^*}(\nu) = 1$, then $V_g : L^2(\mathbb{R}^*) \to L^2(H_{(1,0)})$ is an isometry that intertwines π^1 with $\lambda_{H_{(1,0)}}$. Recall that

$$V_g f(D) = \langle f, \pi^1(D)g\rangle_{L^2(\mathbb{R}^*)}, \text{ for all } D \in H_{(1,0)}, f \in L^2(\mathbb{R}^*).$$

Let $\mathcal{K}_g = V_g L^2(\mathbb{R}^*)$, a closed $\lambda_{H_{(1,0)}}$-invariant subspace of $L^2(H_{(1,0)})$.

Let $V_g' : L^2\big(K, L^2(\mathbb{R}^*)\big) \to L^2\big(K, L^2(H_{(1,0)})\big)$ be given by $\big(V_g'F\big)[\underline{0}, L] = V_g\big(F[\underline{0}, L]\big)$, for all $[\underline{0}, L] \in K$ and $F \in L^2\big(K, L^2(\mathbb{R}^*)\big)$. Since V_g is an isometry, so is V_g' and the range of V_g' is $L^2(K, \mathcal{K}_g)$. For $[\underline{x}, A] \in G$, $F \in L^2\big(K, L^2(\mathbb{R}^*)\big)$, and $\underline{\omega} \in \mathcal{O}$,

$$\begin{aligned}(V_g'\sigma[\underline{x}, A]F)[\underline{0}, \gamma(\underline{\omega})] &= \tfrac{|\det(A)|^{1/2}\|\underline{\omega}\|}{\|\underline{\omega}A\|} e^{2\pi i \underline{\omega}\underline{x}}\, V_g\Big(\pi^1\big(\gamma(\underline{\omega})^{-1}A\gamma(\underline{\omega}A)\big)F[\underline{0}, \gamma(\underline{\omega}A)]\Big)\\ &= \tfrac{|\det(A)|^{1/2}\|\underline{\omega}\|}{\|\underline{\omega}A\|} e^{2\pi i \underline{\omega}\underline{x}}\, \lambda_{H_{(1,0)}}\big(\gamma(\underline{\omega})^{-1}A\gamma(\underline{\omega}A)\big)V_g\big(F[\underline{0}, \gamma(\underline{\omega}A)]\big).\end{aligned}$$

Thus, σ is equivalent to a representation σ_2 acting on $L^2(K, \mathcal{K}_g)$ as follows: For $[\underline{x}, A] \in G$, $\varphi \in L^2(K, \mathcal{K}_g)$, and $\underline{\omega} \in \mathcal{O}$,

$$\big(\sigma_2[\underline{x}, A]\varphi\big)[\underline{0}, \gamma(\underline{\omega})] = \tfrac{|\det(A)|^{1/2}\|\underline{\omega}\|}{\|\underline{\omega}A\|} e^{2\pi i \underline{\omega}\underline{x}}\, \lambda_{H_{(1,0)}}\big(\gamma(\underline{\omega})^{-1}A\gamma(\underline{\omega}A)\big)\varphi[\underline{0}, \gamma(\underline{\omega}A)]. \tag{5.13}$$

The next step is to map $L^2(K, \mathcal{K}_g)$ isometrically into $L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$.

Let $W_1 : L^2\big(\mathrm{GL}_2(\mathbb{R})\big) \to L^2\big(K, L^2(H_{(1,0)})\big)$ be given by

$$\big(W_1 f[\underline{0}, M]\big)(C) = |\det(C)|^{1/2} f(MC),$$

for all $C \in H_{(1,0)}$, $[\underline{0}, M] \in K$.

Proposition 5.3.4. The map W_1 is a unitary map onto $L^2\big(K, L^2(H_{(1,0)})\big)$ and its inverse is given by, for $F \in L^2\big(K, L^2(H_{(1,0)})\big)$ and $B \in \mathrm{GL}_2(\mathbb{R})$,

$$\big(W_1^{-1}F\big)(B) = |\det(C_B)|^{-1/2}(F[\underline{0}, M_B])(C_B).$$

Proof. For any integrable h on $\mathrm{GL}_2(\mathbb{R})$, the Haar integral on $\mathrm{GL}_2(\mathbb{R})$ can be expressed as

$$\int_{\mathrm{GL}_2(\mathbb{R})} h\, d\mu_{\mathrm{GL}_2(\mathbb{R})} = \int_{K_0}\int_{H_{(1,0)}} h(MC)\,|\det(C)|\, d\mu_{H_{(1,0)}}(C)\, d\mu_{K_0}(M).$$

Thus, for any $f \in L^2(\mathrm{GL}_2(\mathbb{R}))$,

$$\begin{aligned}\int_{K_0} \|W_1 f[\underline{0}, M]\|^2_{L^2(H_{(1,0)})} d\mu_{K_0}(M) &= \int_{K_0}\int_{H_{(1,0)}} |(W_1 f[\underline{0}, M])(C)|^2 d\mu_{H_{(1,0)}} d\mu_{K_0}(M) \\ &= \int_{K_0}\int_{H_{(1,0)}} |f(MC)|^2 |\det(C)|\, d\mu_{H_{(1,0)}}\, d\mu_{K_0}(M) \\ &= \int_{\mathrm{GL}_2(\mathbb{R})} |f|^2\, d\mu_{\mathrm{GL}_2(\mathbb{R})} < \infty.\end{aligned}$$

Thus, $W_1 f \in L^2\big(K, L^2(H_{(1,0)})\big)$ and W_1 is an isometry. It is clear W_1 is linear. Thus, the range of W_1 is a closed subspace of $L^2\big(K, L^2(H_{(1,0)})\big)$. We need to show that the range of W_1 is all of $L^2\big(K, L^2(H_{(1,0)})\big)$. For any $k \in C_c(K_0)$ and $h \in C_c(H_{(1,0)})$, define $F_{k,h} \in L^2\big(K, L^2(H_{(1,0)})\big)$ by $\big(F_{k,h}[\underline{0}, M]\big)(C) = k(M)h(C)$. The linear span of $\{F_{k,h} : k \in C_c(K_0), h \in C_c(H_{(1,0)})\}$ is dense in $L^2\big(K, L^2(H_{(1,0)})\big)$. So we just need to show each $F_{k,h}$ is in $W_1 L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$. For $k \in C_c(K_0)$ and $h \in C_c(H_{(1,0)})$, let $f_{k,h}(B) = |\det(C_B)|^{-1/2} k(M_B)h(C_B)$, for all $B \in \mathrm{GL}_2(\mathbb{R})$. Since $B \to (M_B, C_B)$ is a homeomorphism of $\mathrm{GL}_2(\mathbb{R})$ with $K_0 \times H_{(1,0)}$ and $B \to |\det(C_B)|^{-1/2}$ is continuous, $f_{k,h} \in C_c\big(\mathrm{GL}_2(\mathbb{R})\big) \subseteq L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$. Moreover, since $M_{MC} = M$ and $C_{MC} = C$,

$$\begin{aligned}\big(W_1 f_{k,h}[\underline{0}, M]\big)(C) &= |\det(C)|^{1/2} f_{k,h}(MC) = |\det(C)|^{1/2} |\det(C)|^{-1/2} k(M)h(C) \\ &= \big(F_{k,h}[\underline{0}, M]\big)(C),\end{aligned}$$

for any $C \in H_{(1,0)}$ and $[\underline{0}, M] \in K$. Thus, $F_{k,h} \in W_1 L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$, for any $k \in C_c(K_0)$ and $h \in C_c(H_{(1,0)})$. This implies W_1 is a unitary map onto $L^2\big(K, L^2(H_{(1,0)})\big)$. Also, $W_1^{-1} F_{k,h} = f_{k,h}$ so, for any $B \in \mathrm{GL}_2(\mathbb{R})$,

$$\begin{aligned}W_1^{-1} F_{k,h}(B) = f_{k,h}(B) &= |\det(C_B)|^{-1/2} k(M_B)h(C_B) \\ &= |\det(C_B)|^{-1/2} (F_{k,h}[\underline{0}, M_B])(C_B).\end{aligned}$$

Since $\{F_{k,h} : k \in C_c(K_0), h \in C_c(H_{(1,0)}\}$ is total in $L^2\big(K, L^2(H_{(1,0)})\big)$,

$$\big(W_1^{-1} F\big)(B) = |\det(C_B)|^{-1/2} (F[\underline{0}, M_B])(C_B),$$

for all $B \in \mathrm{GL}_2(\mathbb{R})$ and $F \in L^2\big(K, L^2(H_{(1,0)})\big)$. □

Continuing with a fixed $g \in L^2(\mathbb{R}^*)$ satisfying $\int_{\mathbb{R}^*} \big||\nu|^{1/2} g(\nu)\big|^2 d\mu_{\mathbb{R}^*}(\nu) = 1$, let

$$\mathcal{H}_g = W_1^{-1} L^2(K, \mathcal{K}_g) \subseteq L^2\big(\mathrm{GL}_2(\mathbb{R})\big).$$

Then $\mathcal{H}_g$ is a closed subspace of $L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$ and $W_1 : \mathcal{H}_g \to L^2(K, \mathcal{K}_g)$ is a unitary map. Note that we use the same notation for W_1 and its restriction to $\mathcal{H}_g$. Recall the representation $\pi^{(1,0)}$ of G acting on the Hilbert space $L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$. For $[\underline{x}, A] \in G$, $f \in L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$,

$$\pi^{(1,0)}[\underline{x}, A] f(B) = e^{2\pi i (1,0) B^{-1} \underline{x}} f(A^{-1}B), \text{ for all } B \in \mathrm{GL}_2(\mathbb{R}).$$

Proposition 5.3.5. Let $g \in L^2(\mathbb{R}^*)$ satisfy $\int_{\mathbb{R}^*} \left||\nu|^{1/2} g(\nu)\right|^2 d\mu_{\mathbb{R}^*}(\nu) = 1$. The subspace $\mathcal{H}_g$ of $L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$ is $\pi^{(1,0)}$-invariant and the restriction of $\pi^{(1,0)}$ to $\mathcal{H}_g$ is equivalent to σ_2 via the unitary map $W_1 : \mathcal{H}_g \to L^2(K, \mathcal{K}_g)$.

Proof. Let $[\underline{x}, A] \in G$. For $f \in \mathcal{H}_g$, let $F = W_1 f \in L^2(K, \mathcal{K}_g)$ and $L^2(K, \mathcal{K}_g)$ is σ_2-invariant. Then $\sigma_2[\underline{x}, A]F \in L^2(K, \mathcal{K}_g)$ as well. Thus $W_1^{-1}\sigma_2[\underline{x}, A]F = W_1^{-1}\sigma_2[\underline{x}, A]W_1 f \in \mathcal{H}_g$.

For any $B \in \mathrm{GL}_2(\mathbb{R})$, let $\underline{\omega} = (1,0)B^{-1}$. By Proposition 5.3.1 (a) $M_B = \gamma(\underline{\omega})$ and $C_B = \gamma(\underline{\omega})^{-1}B$. Then, using (5.13) and $|\det\big(\gamma(\underline{\omega})\big)|^{1/2} = \|\underline{\omega}\|^{-1}$,

$$\begin{aligned} W_1^{-1}\sigma_2[\underline{x}, A]F(B) &= |\det(C_B)|^{-1/2}\big(\sigma_2[\underline{x}, A]F[\underline{0}, M_B]\big)(C_B) \\ &= \tfrac{|\det(\gamma(\underline{\omega}))|^{1/2}}{|\det(B)|^{1/2}}\big(\sigma_2[\underline{x}, A]F[\underline{0}, \gamma(\underline{\omega})]\big)(\gamma(\underline{\omega})^{-1}B) \\ &= \tfrac{|\det(A)|^{1/2}}{|\det(B)|^{1/2}\|\underline{\omega}A\|} e^{2\pi i \underline{\omega}\underline{x}}\, \lambda_{H_{(1,0)}}\big(\gamma(\underline{\omega})^{-1}A\gamma(\underline{\omega}A)\big)\big(F[\underline{0}, \gamma(\underline{\omega}A)]\big)(\gamma(\underline{\omega})^{-1}B) \\ &= \tfrac{|\det(A)|^{1/2}}{|\det(B)|^{1/2}\|\underline{\omega}A\|} e^{2\pi i \underline{\omega}\underline{x}}\, \big(F[\underline{0}, \gamma(\underline{\omega}A)]\big)(\gamma(\underline{\omega}A)^{-1}A^{-1}B) \\ &= \tfrac{|\det(A)|^{1/2}}{|\det(B)|^{1/2}\|\underline{\omega}A\|} e^{2\pi i \underline{\omega}\underline{x}}\, \big(W_1 f[\underline{0}, \gamma(\underline{\omega}A)]\big)\big(\gamma(\underline{\omega}A)^{-1}A^{-1}B\big). \end{aligned}$$

Before applying W_1, note that $|\det\big(\gamma(\underline{\omega}A)^{-1}A^{-1}B\big)|^{1/2} = \frac{|\det(B)|^{1/2}\|\underline{\omega}A\|}{|\det(A)|^{1/2}}$, which will cancel the first factor in the previous expression. Therefore, recalling that $\underline{\omega} = (1,0)B^{-1}$,

$$W_1^{-1}\sigma_2[\underline{x}, A]W_1 f(B) = e^{2\pi i \underline{\omega}\underline{x}} f(A^{-1}B) = e^{2\pi i (1,0)B^{-1}\underline{x}} f(A^{-1}B) = \pi^{(1,0)}[\underline{x}, A]f(B).$$

This implies that $\mathcal{H}_g$ is $\pi^{(1,0)}$-invariant and the restriction of $\pi^{(1,0)}$ to $\mathcal{H}_g$ is equivalent to σ_2. □

Recall Theorem 5.2.5. The nontrivial orbit in the one dimensional case is $\mathcal{O}_1 = \widehat{R} \setminus \{0\}$, which is naturally identified with $\mathbb{R}^*$ and $\mathcal{D} = L^2(\mathcal{O}_1) \cap L^2(\mathbb{R}^*)$. Fix $\{g_j : j \in J\} \subseteq \mathcal{D}$ such that $\{g_j : j \in J\}$ is an orthonormal basis in $L^2(\mathcal{O}_1)$. For any $j \in J$,

$$\int_{\mathbb{R}^*} \left||\nu|^{1/2} g_j(\nu)\right|^2 d\mu_{\mathbb{R}^*}(\nu) = \int_{\mathcal{O}_1} |g_j(\nu)|^2 d\nu = \|g_j\|_{L^2(\mathcal{O}_1)} = 1.$$

Identifying $H_{(1,0)}$ with G_1, Theorem 5.2.5 says that $L^2(H_{(1,0)}) = \sum_{j\in J}^{\oplus} \mathcal{K}_{g_j}$. Therefore,

$$L^2\big(K, L^2(H_{(1,0)})\big) = \textstyle\sum_{j\in J}^{\oplus} L^2(K, \mathcal{K}_{g_j}).$$

Applying W_1^{-1}, now considered as a unitary map of $L^2\big(K, L^2(H_{(1,0)})\big)$ onto $L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$, we get a decomposition of $L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$.

Proposition 5.3.6. Let $\{g_j : j \in J\} \subseteq \mathcal{D}$ be an orthonormal basis in $L^2(\mathcal{O}_1)$. Then each $\mathcal{H}_{g_j}$ is a closed $\pi^{(1,0)}$-invariant subspace of $L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$ and the restriction of $\pi^{(1,0)}$ to $\mathcal{H}_{g_j}$ is equivalent to σ. Moreover, $L^2\big(\mathrm{GL}_2(\mathbb{R})\big) = \sum_{j\in J}^{\oplus} \mathcal{H}_{g_j}$.

Recall from Proposition 5.1.3, if $\eta \in L^2(\mathcal{O})$ satisfies $\|\eta\|_{L^2(\mathcal{O})} = 1$, then there is an isometric linear map $U_\eta : L^2\big(\mathrm{GL}_2(\mathbb{R})\big) \to L^2(G)$ that intertwines $\pi^{(1,0)}$ with λ_G. The map U_η is defined by

$$U_\eta f[\underline{y}, B] = \int_{\widehat{\mathbb{R}^2}} \eta(\underline{\omega}) f\big(B\gamma(\underline{\omega})\big) e^{-2\pi i \underline{\omega} B^{-1} \underline{y}} d\underline{\omega}, \text{ for all } [\underline{y}, B] \in G, f \in L^2\big(\mathrm{GL}_2(\mathbb{R})\big).$$

The steps we have taken to move from $L^2\big(K, L^2(\mathbb{R}^*)\big)$ to $L^2(G)$ are summarized in the following diagram.

$$\begin{array}{c} \big(L^2\big(K, L^2(\mathbb{R}^*)\big); \sigma\big) \\ \big\downarrow V_g' \\ (L^2(K, \mathcal{K}_g); \sigma_2) \\ \big\downarrow W_1^{-1} \\ \big(L^2\big(\mathrm{GL}_2(\mathbb{R})\big); \pi^{(1,0)}\big) \\ \big\downarrow U_\eta \\ (L^2(G); \lambda_G) \end{array}$$

The vertical maps are linear isometries from the upper Hilbert space into the lower Hilbert space intertwining the corresponding unitary representations. Thus, $\Phi_{\eta,g} = U_\eta \circ W_1^{-1} \circ V_g'$ is a linear isometry of $L^2\big(K, L^2(\mathbb{R}^*)\big)$ into $L^2(G)$ that intertwines σ with λ_G.

Let $F \in L^2\big(K, L^2(\mathbb{R}^*)\big)$. For $[\underline{x}, A] \in G$,

$$\begin{aligned} \Phi_{\eta,g} F[\underline{x}, A] &= \int_{\widehat{\mathbb{R}^2}} \eta(\underline{\omega}) \big(W_1^{-1} V_g' F\big)\big) \big(A\gamma(\underline{\omega})\big) e^{-2\pi i \underline{\omega} A^{-1} \underline{x}} d\underline{\omega} \\ &= |\det(A)| \int_{\widehat{\mathbb{R}^2}} \eta(\underline{\omega} A) \big(W_1^{-1} V_g' F\big)\big) \big(A\gamma(\underline{\omega} A)\big) e^{-2\pi i \underline{\omega} \underline{x}} d\underline{\omega}. \end{aligned} \tag{5.14}$$

Observe that $M_{A\gamma(\underline{\omega}A)} = \gamma(\underline{\omega})$ and $C_{A\gamma(\underline{\omega}A)} = \gamma(\underline{\omega})^{-1} A\gamma(\underline{\omega}A)$. So

$$\det(C_{A\gamma(\underline{\omega}A)}) = \det\big(\gamma(\underline{\omega})^{-1} A\gamma(\underline{\omega}A)\big) = \frac{\|\underline{\omega}\|^2 \det(A)}{\|\underline{\omega}A\|^2}$$

Thus,

$$\begin{aligned} \big(W_1^{-1} V_g' F\big)\big) \big(A\gamma(\underline{\omega}A)\big) &= \tfrac{\|\underline{\omega}A\|}{\|\underline{\omega}\| \cdot |\det(A)|^{1/2}} V_g(F[\underline{0}, \gamma(\underline{\omega})]) \big(\gamma(\underline{\omega})^{-1} A\gamma(\underline{\omega}A)\big) \\ &= \tfrac{\|\underline{\omega}A\|}{\|\underline{\omega}\| \cdot |\det(A)|^{1/2}} \langle F[\underline{0}, \gamma(\underline{\omega})], \pi^1\big(\gamma(\underline{\omega})^{-1} A\gamma(\underline{\omega}A)\big) g\rangle_{L^2(\mathbb{R}^*)} \\ &= \tfrac{\|\underline{\omega}A\|}{\|\underline{\omega}\| \cdot |\det(A)|^{1/2}} \int_{\mathbb{R}^*} (F[\underline{0}, \gamma(\underline{\omega})])(\nu) \overline{\pi^1\big(\gamma(\underline{\omega})^{-1} A\gamma(\underline{\omega}A)\big) g(\nu)}\, d\mu_{\mathbb{R}^*}(\nu). \end{aligned}$$

Inserting this into (5.14) gives

$$\Phi_{\eta,g} F[\underline{x}, A] = \int_{\widehat{\mathbb{R}^2}} \int_{\mathbb{R}^*} \tfrac{|\det(A)|^{1/2} \|\underline{\omega}A\|}{\|\underline{\omega}\|} \eta(\underline{\omega}A) e^{-2\pi i \underline{\omega}\underline{x}} (F[\underline{0}, \gamma(\underline{\omega})])(\nu) \overline{\pi^1\big(\gamma(\underline{\omega})^{-1} A\gamma(\underline{\omega}A)\big) g(\nu)}\, d\mu_{\mathbb{R}^*}(\nu)\, d\underline{\omega}.$$

We will compare the expression for $\Phi_{\eta,g} F$ with a coefficient function of the irreducible representation σ.

If $E \in L^2\big(K, L^2(\mathbb{R}^*)\big)$ is fixed, then, for any $F \in L^2\big(K, L^2(\mathbb{R}^*)\big)$, $V_E F$ is the continuous function on G defined by $V_E F[\underline{x}, A] = \langle F, \sigma[\underline{x}, A]E\rangle_{L^2(K,L^2(\mathbb{R}^*))}$, for all $[\underline{x}, A] \in G$. Recall that the Haar integral over K can be expressed using the parametrization $\underline{\omega} \to \gamma(\underline{\omega})$ by $\mathcal{O}$, which is co-null in $\widehat{\mathbb{R}^2}$. Then, for $[\underline{x}, A] \in G$,

$$\begin{aligned} V_E F[\underline{x}, A] &= \langle F, \sigma[\underline{x}, A]E\rangle_{L^2(K,L^2(\mathbb{R}^*))} \\ &= \int_{\widehat{\mathbb{R}^2}} \int_{\mathbb{R}^*} (F[\underline{0}, \gamma(\underline{\omega})])(\nu)\overline{\sigma[\underline{x}, A](E[\underline{0}, \gamma(\underline{\omega})])(\nu)}\, d\mu_{\mathbb{R}^*}(\nu) \frac{d\underline{\omega}}{\|\underline{\omega}\|^2}. \end{aligned} \tag{5.15}$$

Note that

$$\overline{\sigma[\underline{x}, A](E[\underline{0}, \gamma(\underline{\omega})])(\nu)} = \frac{|\det(A)|^{1/2}\|\underline{\omega}\|}{\|\underline{\omega}A\|} e^{-2\pi i \underline{\omega x}}\, \overline{\pi^1\big(\gamma(\underline{\omega})^{-1}A\gamma(\underline{\omega}A)\big)E[\underline{0}, \gamma(\underline{\omega}A)](\nu)}.$$

If we select E as a function built from η and g, then we can make the expression for $V_E F$ coincide with that for $\Phi_{\eta,g}$. Define E as follows: For $[\underline{0}, L] \in K$ and $\nu \in \mathbb{R}^*$,

$$E[\underline{0}, L](\nu) = \tfrac{\overline{\eta}((1,0)L^{-1})}{|\det(L)|}\, g(\nu).$$

If $L = \gamma(\underline{\omega}A)$, then $E[\underline{0}, \gamma(\underline{\omega}A)](\nu) = \frac{\overline{\eta}(\underline{\omega}A)}{|\det(\gamma(\underline{\omega}A))|}\, g(\nu) = \|\underline{\omega}A\|^2\overline{\eta}(\underline{\omega}A)\, g(\nu)$. Thus

$$\overline{\sigma[\underline{x}, A](E[\underline{0}, \gamma(\underline{\omega})])(\nu)}\frac{1}{\|\underline{\omega}\|^2} = \frac{|\det(A)|^{1/2}\|\underline{\omega}A\|}{\|\underline{\omega}\|}\eta(\underline{\omega}A)\, e^{-2\pi i \underline{\omega x}}\, \overline{\pi^1\big(\gamma(\underline{\omega})^{-1}A\gamma(\underline{\omega}A)\big)g(\nu)}.$$

Substituting into (5.15), there is a perfect match with the expression for $\Phi_{\eta,g}F$. Thus, we have the following theorem.

Theorem 5.3.7. Let $g \in L^2(\mathbb{R}^*)$ satisfy $\int_{\mathbb{R}^*} \big||\nu|^{1/2}g(\nu)\big|^2 d\mu_{\mathbb{R}^*}(\nu) = 1$ and let $\eta \in L^2(\widehat{\mathbb{R}^2})$ satisfy $\|\eta\|_{L^2(\widehat{\mathbb{R}^2})} = 1$. Let $E \in L^2\big(K, L^2(\mathbb{R}^*)\big)$ be defined as

$$E[\underline{0}, L](\nu) = \frac{\overline{\eta}((1,0)L^{-1})}{|\det(L)|}\, g(\nu), \text{ for each } [\underline{0}, L] \in K \text{ and } \nu \in \mathbb{R}^*.$$

Define $V_E F[\underline{x}, A] = \langle F, \sigma[\underline{x}, A]E\rangle_{L^2(K,L^2(\mathbb{R}^*))}$, for $[\underline{x}, A] \in G$ and $F \in L^2\big(K, L^2(\mathbb{R}^*)\big)$. Then V_E is a linear isometry of $L^2\big(K, L^2(\mathbb{R}^*)\big)$ into $L^2(G)$ that intertwines σ with λ_G. In particular, σ is a square-integrable representation of G.

Remark. It is perhaps useful to formulate a more direct proof of a weaker version of Theorem 5.3.7 that still implies the full content of Theorem 5.3.7 using Duflo-Moore theory. The arguments below show the connection with the continuous wavelet transform in one dimension clearly.

Let $\eta \in L^2(\mathcal{O})$ satisfy $\|\eta\|_{L^2(\mathcal{O})} = 1$. Let $g \in L^2(\mathbb{R}^*)$ satisfy $\int_{\mathbb{R}} \big||\nu|^{1/2}g(\nu)\big|^2 \frac{d\nu}{|\nu|} = 1$. Let $E \in L^2\left(K, L^2(\mathbb{R}^*)\right)$ be defined by

$$E[\underline{0}, L](\nu) = |\det(L)|^{-1}\overline{\eta}\big((1,0)L^{-1}\big)\, g(\nu),$$

for a.e $\nu \in \mathbb{R}^*$ and $[\underline{0}, L] \in K$. Fix nonzero $\xi \in L^2(K_0)$ and nonzero $h \in L^2(\mathbb{R}^*)$. Define $F : K \to L^2(\mathbb{R}^*)$ by $F[\underline{0}, L] = \xi(L)h$, for a.e. $L \in K_0$. Then $F \in L^2\big(K, L^2(\mathbb{R}^*)\big)$ and $\|F\|^2_{L^2(K,L^2(\mathbb{R}^*))} = \|\xi\|^2_{L^2(K_0)}\|h\|^2_{L^2(\mathbb{R}^*)}$.

Theorem 5.3.8. Let $E, F \in L^2\big(K, L^2(\mathbb{R}^*)\big)$ be defined as above. Then $V_E F \in L^2(G)$ and, thus, σ is a square-integrable representation of G. Moreover, $V_E : L^2\big(K, L^2(\mathbb{R}^*)\big) \to L^2(G)$ is an isometry.

Proof. Our main goal is to show that: $\|V_E F\|^2_{L^2(G)} = \|F\|^2_{L^2(K,L^2(\mathbb{R}^*))}$. Recall the maps U_η and V_g. From (5.1), for $f \in L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$,

$$U_\eta f[\underline{y}, B] = \int_{\widehat{\mathbb{R}^2}} \eta(\underline{\omega}) f\big(B\gamma(\underline{\omega})\big) e^{-2\pi i \underline{\omega} B^{-1}\underline{y}}\, d\underline{\omega}, \text{ for a.e. } [\underline{y}, B] \in G_2. \tag{5.16}$$

Since $\eta \in L^2(\mathcal{O})$ and $\|\eta\|_{L^2(\mathcal{O})} = 1$, Proposition 5.1.3 implies U_η is a linear isometry of $L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$ into $L^2(G)$.

We also have the isometry $W_1^{-1} : L^2\big(K, L^2(H_{(1,0)}\big) \to L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$ given by

$$W_1^{-1} F_0(B) = |\det(C_B)|^{-1/2}\big(F_0[\underline{0}, M_B]\big)(C_B), \tag{5.17}$$

for a.e. $B \in \mathrm{GL}_2(\mathbb{R})$ and $F_0 \in L^2\big(K, L^2(H_{(1,0)}\big)$.

Since $g \in L^2(\mathbb{R}^*)$ satisfies $\int_{\mathbb{R}} \big||\nu|^{1/2} g(\nu)\big|^2 \frac{d\nu}{|\nu|} = 1$, Theorem 5.2.3 implies V_g is a linear isometry of $L^2(\mathbb{R}^*)$ into $L^2(H_{(1,0)})$, where, for $f \in L^2(\mathbb{R}^*)$,

$$V_g f(C) = \langle f, \pi^1(C) g\rangle_{L^2(\mathbb{R}^*)} = \int_{\mathbb{R}} f(\nu)\overline{\pi^1(C)g(\nu)}\,\frac{d\nu}{|\nu|}, \text{ for } C \in H_{(1,0)}.$$

Note that, since $F[\underline{0}, L](\nu) = \xi(L)h(\nu)$, for a.e. $\nu \in \mathbb{R}^*$ and $[\underline{0}, L] \in K$,

$$\begin{aligned} V_E F[\underline{x}, A] &= \int_K \int_{\mathbb{R}} \xi(L)h(\nu)\overline{\big(\sigma[\underline{x}, A]E[\underline{0}, L]\big)(\nu)}\frac{d\nu}{|\nu|} d\mu_K([\underline{0}, L]) \\ &= \int_{\widehat{\mathbb{R}^2}} \int_{\mathbb{R}} \xi\big(\gamma(\underline{\omega})\big)h(\nu)\overline{\big(\sigma[\underline{x}, A]E[\underline{0}, \gamma(\underline{\omega})]\big)(\nu)}\frac{d\nu}{|\nu|}\frac{d\underline{\omega}}{\|\underline{\omega}\|^2}. \end{aligned} \tag{5.18}$$

Now

$$\begin{aligned} &\big(\sigma[\underline{x}, A]E[\underline{0}, \gamma(\underline{\omega})]\big)(\nu) \\ &\qquad = \left|\det\left(C_{A^{-1}\gamma(\underline{\omega})}\right)\right|^{-1/2} e^{2\pi i \underline{\omega}\,\underline{x}} \pi^1\left(C^{-1}_{A^{-1}\gamma(\underline{\omega})}\right) E\left[\underline{0}, M_{A^{-1}\gamma(\underline{\omega})}\right](\nu) \end{aligned} \tag{5.19}$$

Note

$$E\left[\underline{0}, M_{A^{-1}\gamma(\underline{\omega})}\right](\nu) = \frac{\overline{\eta}\big((1,0)M^{-1}_{A^{-1}\gamma(\underline{\omega})}\big)}{\left|\det\left(M_{A^{-1}\gamma(\underline{\omega})}\right)\right|} g(\nu) = \frac{\overline{\eta}(\underline{\omega}A)}{\left|\det\left(M_{A^{-1}\gamma(\underline{\omega})}\right)\right|} g(\nu).$$

So

$$\begin{aligned}&\big(\sigma[\underline{x}, A]E[\underline{0}, \gamma(\underline{\omega})]\big)(\nu)\\&\qquad= \left|\det\left(C_{A^{-1}\gamma(\underline{\omega})}\right)\right|^{-1/2} e^{2\pi i \underline{\omega}\,\underline{x}} \frac{\overline{\eta}(\underline{\omega}A)}{\left|\det\left(M_{A^{-1}\gamma(\underline{\omega})}\right)\right|}\pi^1\left(C^{-1}_{A^{-1}\gamma(\underline{\omega})}\right)g(\nu)\end{aligned} \tag{5.20}$$

Insert this into (5.18) and rearrange to get

$$\begin{aligned}&V_EF[\underline{x}, A] =\\&\int_{\widehat{\mathbb{R}^2}} \left|\det\left(C_{A^{-1}\gamma(\underline{\omega})}\right)\right|^{-1/2} e^{-2\pi i\underline{\omega}\underline{x}}\eta(\underline{\omega}A)\frac{\xi\big(\gamma(\underline{\omega})\big)}{\left|\det\left(M_{A^{-1}\gamma(\underline{\omega})}\right)\right|}\int_{\mathbb{R}} h(\nu)\overline{\pi^1\left(C^{-1}_{A^{-1}\gamma(\underline{\omega})}\right)g(\nu)}\frac{d\nu}{|\nu|}\frac{d\underline{\omega}}{\|\underline{\omega}\|^2}\\&= \int_{\widehat{\mathbb{R}^2}} \left|\det\left(C_{A^{-1}\gamma(\underline{\omega})}\right)\right|^{-1/2} e^{-2\pi i\underline{\omega}\underline{x}}\eta(\underline{\omega}A)\frac{\xi\big(\gamma(\underline{\omega})\big)}{\left|\det\left(M_{A^{-1}\gamma(\underline{\omega})}\right)\right|}V_gh\left(C^{-1}_{A^{-1}\gamma(\underline{\omega})}\right)\frac{d\underline{\omega}}{\|\underline{\omega}\|^2}.\end{aligned}$$

Make the change of variables $\underline{\omega} \to \underline{\omega}A^{-1}$. From Proposition 5.3.1, we have that $\gamma(\underline{\omega}A^{-1}) = M_{A\gamma(\underline{\omega})}$ and $C^{-1}_{A^{-1}\gamma(\underline{\omega}A^{-1})} = C_{A\gamma(\underline{\omega})}$. Thus

$$V_EF[\underline{x}, A] = \int_{\widehat{\mathbb{R}^2}} \frac{|\det(C_{A\gamma(\underline{\omega})}|^{1/2}}{|\det(A)|}e^{-2\pi i\underline{\omega}A^{-1}\underline{x}}\eta(\underline{\omega})\frac{\xi(M_{A\gamma(\underline{\omega})})}{|\det\big(\gamma(\underline{\omega})\big)|}V_gh\big(C_{A\gamma(\underline{\omega})}\big)\frac{d\underline{\omega}}{\|\underline{\omega}A^{-1}\|^2}.$$

But $\frac{|\det(C_{A\gamma(\underline{\omega})})|^{1/2}}{|\det(A)|\cdot|\det\big(\gamma(\underline{\omega})\big)|\cdot\|\underline{\omega}A^{-1}\|^2} = |\det(C_{A\gamma(\underline{\omega})})|^{-1/2}$, which can be verified using the identities in Proposition 5.3.1. Thus,

$$V_EF[\underline{x}, A] = \int_{\widehat{\mathbb{R}^2}} \eta(\underline{\omega})e^{-2\pi i\underline{\omega}A^{-1}\underline{x}}|\det(C_{A\gamma(\underline{\omega})})|^{-1/2}\xi(M_{A\gamma(\underline{\omega})})V_gh\big(C_{A\gamma(\underline{\omega})}\big)\,d\underline{\omega}.$$

Now, using (5.17),

$$|\det(C_{A\gamma(\underline{\omega})})|^{-1/2}\xi(M_{A\gamma(\underline{\omega})})V_gh\big(C_{A\gamma(\underline{\omega})}\big) = W_1^{-1}(V_g'F)\big(A\gamma(\underline{\omega})\big).$$

Then

$$V_EF[\underline{x}, A] = \int_{\widehat{\mathbb{R}^2}} \eta(\underline{\omega})(W_1^{-1}(V_g'F))\big(A\gamma(\underline{\omega})\big)e^{-2\pi i\underline{\omega}\underline{x}}\,d\underline{\omega} = U_\eta\big(W_1^{-1}(V_g'F)\big)[\underline{x}, A].$$

using (5.16). Since U_η, W_1^{-1}, and V_g' are all linear isometries,

$$\|V_EF\|^2_{L^2(G)} = \|F\|^2_{L^2(K,L^2(\mathbb{R}^*))}.$$

Therefore, $V_EF \in L^2(G)$ and thus σ is a square-integrable representation (see Definition 2.5.1). Moreover E is a nonzero vector in $\mathcal{D}_\sigma$ and F is a nonzero vector in $L^2\big(K, L^2(\mathbb{R}^*)\big)$ with $\|V_EF\|_{L^2(G)} = \|F\|_{L^2(K,L^2(\mathbb{R}^*))}$. It follows from Theorem 2.5.5 that V_E is a linear isometry of the Hilbert space of σ into $L^2(G)$. □

It is useful to formulate the content of Theorem 5.3.7 for the representation σ_1 on the Hilbert space $L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$. Recall the unitary map $U : L^2\big(K, L^2(\mathbb{R}^*)\big) \to$

$L^2\big(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}\big)$ given by, for $F \in L^2\big(K, L^2(\mathbb{R}^*)\big)$ and $(\underline{\omega},\omega_3) \in \widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}$, let

$$(UF)(\underline{\omega},\omega_3) = \begin{cases} \frac{\left(F[\underline{0},\gamma(\underline{\omega})]\right)(\omega_3^{-1})}{\|\underline{\omega}\|\cdot|\omega_3|^{1/2}} & \text{for } \underline{\omega}\in\mathcal{O}, \omega_3\neq 0 \\ 0 & \text{otherwise.}\end{cases}$$

For E as in Theorem 5.15, $\xi \in L^2\big(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}\big)$, let $F = U^{-1}\xi$. Then, for $[\underline{x}, A] \in G$,

$$\begin{aligned}\langle \xi, \sigma_1[\underline{x},A]UE\rangle_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})} &= \langle UF, U\sigma[\underline{x},A]E\rangle_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})} \\ &= \langle F, \sigma[\underline{x},A]E\rangle_{L^2(K,L^2(\mathbb{R}^*))} = V_E F[\underline{x},A].\end{aligned}$$

Let $\psi = UE$ and define $V_\psi\xi[\underline{x},A] = \langle \xi, \sigma_1[\underline{x},A]\psi\rangle_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})}$, for $[\underline{x},A]\in G$ and $\xi \in L^2\big(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}\big)$. Then V_ψ is a linear isometry of $L^2\big(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}\big)$ into $L^2(G)$ that intertwines σ_1 with λ_G. Note that $\det\big(\gamma(\underline{\omega})\big) = \|\underline{\omega}\|^{-2}$, so

$$\begin{aligned}\psi(\underline{\omega},\omega_3) &= UE(\underline{\omega},\omega_3) = \big(E[\underline{0},\gamma(\underline{\omega})]\big)(\omega_3^{-1})\|\underline{\omega}\|^{-1}|\omega_3|^{-1/2} \\ &= \frac{\overline{\eta}\big((1,0)\gamma(\underline{\omega})^{-1}\big)}{|\det\big(\gamma(\underline{\omega})\big)|} g(\omega_3^{-1})\|\underline{\omega}\|^{-1}|\omega_3|^{-1/2} = \|\underline{\omega}\|\overline{\eta}(\underline{\omega})|\omega_3|^{-1/2}g\big(\omega_3^{-1}\big),\end{aligned} \tag{5.21}$$

for a.e. $(\underline{\omega},\omega_3)\in\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}$. Define $\zeta(\underline{\omega}) = \|\underline{\omega}\|\overline{\eta}(\underline{\omega})$, for a.e. $\underline{\omega}\in\widehat{\mathbb{R}^2}$, and $\phi(\nu) = |\nu|^{-1/2}g\big(\nu^{-1}\big)$, for a.e. $\nu\in\widehat{\mathbb{R}}$. Then $1 = \int_{\widehat{\mathbb{R}^2}}|\eta(\underline{\omega})|^2 d\underline{\omega} = \int_{\widehat{\mathbb{R}^2}}|\zeta(\underline{\omega})|^2\frac{d\underline{\omega}}{\|\underline{\omega}\|^2}$ and

$$1 = \int_{\mathbb{R}^*}\big||\nu|^{1/2}g(\nu)\big|^2 d\mu_{\mathbb{R}^*}(\nu) = \int_{\mathbb{R}^*}\big||\nu|^{-1/2}g(\nu^{-1})\big|^2 d\mu_{\mathbb{R}^*}(\nu) = \int_{\widehat{\mathbb{R}}}|\phi(\nu)|^2\tfrac{d\nu}{|\nu|}.$$

Recall the notation for elementary tensor products of functions on the factors of a product space. For $\kappa\in L^2\big(\widehat{\mathbb{R}^2}\big)$ and $\theta\in L^2\big(\widehat{\mathbb{R}}\big)$, $\kappa\otimes\theta\in L^2\big(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}\big)$ is defined by

$$(\kappa\otimes\theta)(\underline{\omega},\omega_3) = \kappa(\underline{\omega})\theta(\omega_3), \text{ for all } (\underline{\omega},\omega_3)\in\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}.$$

Thus, we can restate Theorem 5.3.7 in terms of the equivalent representation σ_1 as a corollary.

Corollary 5.3.9. Let $\zeta \in L^2\big(\widehat{\mathbb{R}^2}\big)$ satisfy the condition that $\int_{\widehat{\mathbb{R}^2}}|\zeta(\underline{\omega})|^2\frac{d\underline{\omega}}{\|\underline{\omega}\|^2} = 1$ and $\phi\in L^2\big(\widehat{\mathbb{R}}\big)$ satisfy $\int_{\widehat{\mathbb{R}}}|\phi(\omega_3)|^2\frac{d\omega_3}{|\omega_3|} = 1$. Let $\psi = \zeta\otimes\phi$ and define $V_\psi\xi[\underline{x},A] = \langle\xi,\sigma_1[\underline{x},A]\psi\rangle_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})}$, for all $[\underline{x},A]\in G$ and each $\xi\in L^2\big(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}\big)$. Then V_ψ is a linear isometry of $L^2\big(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}\big)$ into $L^2(G)$ intertwining σ_1 with λ_G.

Combining Theorem 5.1.4 with Proposition 5.3.6 will now provide a decomposition of the regular representation, λ_G on $L^2(G)$, into infinitely many copies of σ. Recall that $\mathcal{O}_1 = \widehat{\mathbb{R}}\setminus\{0\}$ is the orbit of 1 in $\widehat{\mathbb{R}}$ under the action of $\mathbb{R}^* = \mathrm{GL}_1(\mathbb{R})$ and it is endowed with Lebesgue measure. When the nonzero real numbers are considered as a group, $\mathbb{R}^*$, it is endowed with Haar measure, $\mu_{\mathbb{R}^*}$. The vector space $\mathcal{D} =$

$\left\{h \in L^2(\mathbb{R}^*) : \int_{\mathbb{R}^*} \left||u|^{1/2}h(u)\right|^2 d\mu_{\mathbb{R}^*}(u) < \infty\right\}$ can be thought of as $L^2(\mathcal{O}_1) \cap L^2(\mathbb{R}^*)$. Fix an orthonormal basis $\{g_j : j \in J\}$ of $L^2(\mathcal{O}_1)$ consisting of functions in $\mathcal{D}$ as in Proposition 5.3.6. For each $j \in J$, $\mathcal{K}_{g_j} = V_{g_j} L^2(\mathbb{R}^*)$ and $\mathcal{H}_{g_j} = W_1^{-1} L^2(K, \mathcal{K}_{g_j})$. Then $\mathcal{H}_{g_j}$ is a closed $\pi^{(1,0)}$-invariant subspace of $L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$ and the restriction of $\pi^{(1,0)}$ to $\mathcal{H}_{g_j}$ is equivalent to σ via $W_1^{-1} \circ V_g'$. Moreover, Proposition 5.3.6 says $L^2\big(\mathrm{GL}_2(\mathbb{R})\big) = \sum_{j\in J}^{\oplus} \mathcal{H}_{g_j}$.

On the other hand, $\mathcal{O} = \widehat{\mathbb{R}^2} \setminus \{(0,0)\}$ is the orbit of $(1,0)$ in $\widehat{\mathbb{R}^2}$ under the action of $\mathrm{GL}_2(\mathbb{R})$. It is equipped with the Lebesgue measure of $\widehat{\mathbb{R}^2}$. Fix an orthonormal basis $\{\eta_i : i \in I\}$ of $L^2(\widehat{\mathbb{R}^2}) = L^2(\mathcal{O})$. For each $i \in I$, $U_{\eta_i} : L^2(\mathrm{GL}_2(\mathbb{R})) \to L^2(G)$ is given by

$$U_{\eta_i} f[\underline{y}, B] = \int_{\widehat{\mathbb{R}^2}} \eta_i(\underline{\omega}) f\big(B\gamma(\underline{\omega})\big) e^{-2\pi i \underline{\omega} B^{-1} \underline{y}}\, d\underline{\omega},$$

for $[\underline{y}, B] \in G$ and $f \in L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$. By Theorem 5.1.4, each $U_{\eta_i} L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$ is a λ_G-invariant closed subspace of $L^2(G)$ and U_{η_i} intertwines $\pi^{(1,0)}$ with the restriction of λ_G to $U_{\eta_i} L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$ and $L^2(G) = \sum_{i\in I}^{\oplus} U_{\eta_i} L^2\big(\mathrm{GL}_2(\mathbb{R})\big)$.

For each $(i,j) \in I \times J$, form $E_{i,j} \in L^2\big(K, L^2(\mathbb{R}^*)\big)$ by

$$E_{i,j}[\underline{0}, L](\nu) = \tfrac{\overline{\eta_i}((1,0)L^{-1})}{|\det(L)|}\, g_j(\nu), \tag{5.22}$$

for $\nu \in \mathbb{R}^*$ and $[\underline{0}, L] \in K$. The orthogonal decompositions just recalled imply the following theorem.

Theorem 5.3.10. Let $\{\eta_i : i \in I\}$ be an orthonormal basis of $L^2(\mathcal{O})$ and let $\{g_j : j \in J\}$ be an orthonormal basis of $L^2(\mathcal{O}_1)$ consisting of functions in $L^2(\mathbb{R}^*)$. For each $(i,j) \in I \times J$, form $E_{i,j} \in L^2\big(K, L^2(\mathbb{R}^*)\big)$ as in (5.22) and let $\mathcal{M}_{i,j} = V_{E_{i,j}} L^2\big(K, L^2(\mathbb{R}^*)\big)$. Then each $\mathcal{M}_{i,j}$ is a closed λ_G-invariant subspace of $L^2(G)$ and $V_{E_{i,j}}$ is an isometry that intertwines σ with the restriction of λ_G to $\mathcal{M}_{i,j}$. Moreover $L^2(G) = \sum_{(i,j)\in I\times J}^{\oplus} \mathcal{M}_{i,j}$.

Again, we consider this for the representation σ_1 acting on $L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$. First, let

$$L^2\left(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}; \tfrac{d\underline{\omega}\, d\nu}{\|\underline{\omega}\|^2|\nu|}\right), \quad L^2\left(\widehat{\mathbb{R}^2}; \tfrac{d\underline{\omega}}{\|\underline{\omega}\|^2}\right), \text{ and } L^2\left(\widehat{\mathbb{R}}; \tfrac{d\nu}{|\nu|}\right)$$

denote the weighted L^2-spaces with respect to the indicated weights. Recall that

$$L^2\left(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}; \tfrac{d\underline{\omega}\, d\nu}{\|\underline{\omega}\|^2|\nu|}\right) \text{ is naturally isomorphic to } L^2\left(\widehat{\mathbb{R}^2}; \tfrac{d\underline{\omega}}{\|\underline{\omega}\|^2}\right) \otimes L^2\left(\widehat{\mathbb{R}}; \tfrac{d\nu}{|\nu|}\right).$$

For $\kappa \in L^2\left(\widehat{\mathbb{R}^2}; \tfrac{d\underline{\omega}}{\|\underline{\omega}\|^2}\right)$, define $U_1\kappa(\underline{\omega}) = \|\underline{\omega}\|^{-1}\overline{\kappa}(\underline{\omega})$, for a.e. $\underline{\omega} \in \widehat{\mathbb{R}^2}$. For $\theta \in L^2\left(\widehat{\mathbb{R}}; \tfrac{d\nu}{|\nu|}\right)$, define $U_2\theta(\nu) = |\nu|^{-1/2}\theta(\nu^{-1})$, for a.e. $\nu \in \widehat{\mathbb{R}}$.

Lemma 5.3.11. The map U_1 is a unitary map of $L^2\left(\widehat{\mathbb{R}^2}; \tfrac{d\underline{\omega}}{\|\underline{\omega}\|^2}\right)$ onto $L^2\big(\widehat{\mathbb{R}^2}\big)$ and U_2 is a unitary map of $L^2\left(\widehat{\mathbb{R}}; \tfrac{d\nu}{|\nu|}\right)$ onto $L^2\big(\widehat{\mathbb{R}}\big)$.

Proof. We show the calculation for U_2, which is a little more complicated than that for U_1. Let $\theta \in L^2\left(\widehat{\mathbb{R}}; \frac{d\nu}{|\nu|}\right)$. Then

$$\begin{aligned}
\|U_2\theta\|^2_{L^2(\widehat{\mathbb{R}})} &= \int_{\widehat{\mathbb{R}}} \left||\nu|^{-1/2}\theta(\nu^{-1})\right|^2 d\nu \qquad (\omega = \nu^{-1}, d\nu = \omega^{-2}d\omega) \\
&= \int_{\widehat{\mathbb{R}}} \left||\omega|^{1/2}\theta(\omega)\right|^2 \omega^{-2} d\omega \\
&= \int_{\widehat{\mathbb{R}}} |\theta(\omega)|^2 \tfrac{d\omega}{|\omega|} = \|\theta\|^2_{L^2(\widehat{\mathbb{R}}; |\nu|^{-1}d\nu)}.
\end{aligned}$$

Thus $U_2\theta \in L^2\big(\widehat{\mathbb{R}}\big)$ and U_2 is an isometry. It is clear that U_2 is linear. Also, the fact that U_2 is onto $L^2\big(\widehat{\mathbb{R}}\big)$ and $U_2^{-1}g(\nu) = |\nu|^{-1/2}g(\nu^{-1})$, for a.e. $\nu \in \widehat{\mathbb{R}}$, $g \in L^2(\widehat{\mathbb{R}})$. Thus, U_2 is a unitary map of $L^2\left(\widehat{\mathbb{R}}; \frac{d\nu}{|\nu|}\right)$ onto $L^2\big(\widehat{\mathbb{R}}\big)$. ☐

It is useful to have an orthonormal basis for $L^2\left(\widehat{\mathbb{R}^2}; \frac{d\omega}{\|\omega\|^2}\right)$ consisting of functions which also lie in $L^2\big(\widehat{\mathbb{R}^2}\big)$ as well as for the pair $L^2\left(\widehat{\mathbb{R}}; \frac{d\nu}{|\nu|}\right)$ and $L^2\big(\widehat{\mathbb{R}}\big)$.

Lemma 5.3.12. (a) There exists a countable set $\{\zeta_i : i \in I\}$ in $C_c(\mathcal{O})$ that is an orthonormal basis of $L^2\left(\widehat{\mathbb{R}^2}; \frac{d\omega}{\|\omega\|^2}\right)$. (b) There exists a countable set $\{\phi_j : j \in J\}$ in $C_c(\mathcal{O}_1)$ that is an orthonormal basis of $L^2\left(\widehat{\mathbb{R}}; \frac{d\nu}{|\nu|}\right)$.

Proof. We start with the proof for (b). Because the unitary U_2 of Lemma 5.3.11 maps $C_c(\mathcal{O}_1)$ to $C_c(\mathcal{O}_1)$, it suffices to find an orthonormal basis of $L^2(\widehat{\mathbb{R}})$ consisting of functions in $C_c(\mathcal{O}_1)$. Start with an orthonormal basis, $\{g_k : k \in \mathbb{N}\}$, of $L^2(\widehat{\mathbb{R}})$ consisting of functions in $C_c(\widehat{\mathbb{R}})$ such as the basis given by a Daubechies wavelet. Let

$$\mathcal{C} = \left\{ \sum_{j=1}^{n} a_j g_{k_j} : n \in \mathbb{N}, k_1, \cdots, k_n \in \mathbb{N}, a_j \in \mathbb{Q} + i\mathbb{Q}, 1 \le j \le n \right\}.$$

Then $\mathcal{C}$ is a countable dense subset of $L^2(\widehat{\mathbb{R}})$ consisting of continuous functions of compact support. Our next step is to replace each member of $\mathcal{C}$ with a sequence of from $C_c(\mathcal{O}_1)$. For each $k \in \mathbb{N}$, let

$$s_k(\omega) = \begin{cases} 1 & \text{if } |\omega| > 1/k \\ k(k+1)\big(|\omega| - 1/(k+1)\big) & \text{if } 1/(k+1) \le |\omega| \le 1/k \\ 0 & \text{if } |\omega| < 1/(k+1), \end{cases}$$

for all $\omega \in \widehat{\mathbb{R}}$. Then, for any $h \in \mathcal{C}$, $\lim_{k\to\infty} \|h - s_k h\|_{L^2(\widehat{\mathbb{R}})} = 0$. Moreover, $s_k h \in C_c(\mathcal{O}_1)$, for all $h \in \mathcal{C}$ and $k \in \mathbb{N}$. Then $\mathcal{C}' = \{s_k h : h \in \mathcal{C}, k \in \mathbb{N}\}$ is a countable dense sunset of $L^2\big(\widehat{\mathbb{R}}\big)$ consisting of members of $C_c(\mathcal{O}_1)$. Now, index the members of $\mathcal{C}'$ by $\mathbb{N}$. Thus, write $\mathcal{C}'$ as a sequence $h_1, h_2, h_3, \cdots$. Apply the Gram-Schmidt process to this sequence to generate an orthonormal basis $\mathcal{B}$ of $L^2\big(\widehat{\mathbb{R}}\big)$ consisting of finite linear combinations of members from $\mathcal{C}'$. This proves (b).

To prove (a), for $g, h \in \mathcal{B}$, let $g \otimes h$ be defined on $\widehat{\mathbb{R}^2}$ by $(g \otimes h)(\omega_1, \omega_2) = g(\omega_1)h(\omega_2)$, for all $(\omega_1, \omega_2) \in \widehat{\mathbb{R}^2}$. Then $\{g \otimes h : (g, h) \in \mathcal{B} \times \mathcal{B}\}$ is countable set in $C_c(\mathcal{O})$ that is an orthonormal basis of $L^2\left(\widehat{\mathbb{R}^2}; \frac{d\underline{\omega}}{\|\underline{\omega}\|^2}\right)$. □

Fix countable sets $\{\zeta_i : i \in I\}$ in $C_c(\mathcal{O})$ and $\{\phi_j : i \in J\}$ in $C_c(\mathcal{O}_1)$ as in Lemma 5.3.12. For each $i \in I$, let $\eta_i = U_1\zeta_i$. Since $U_1 : L^2\left(\widehat{\mathbb{R}^2}; \frac{d\underline{\omega}}{\|\underline{\omega}\|^2}\right) \to L^2\big(\widehat{\mathbb{R}^2}\big)$ is a unitary map, $\{\eta_i : i \in I\}$ is an orthonormal basis of $L^2\big(\widehat{\mathbb{R}^2}\big) = L^2(\mathcal{O})$. Likewise, let $g_j = U_2\phi_j$, for each $j \in J$, to get an orthonormal basis of $L^2\big(\widehat{\mathbb{R}}\big) = L^2(\mathcal{O}_1)$. If $\psi_{i,j} = \zeta_i \otimes \phi_j$, then $\psi_{i,j} \in L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$. If $E_{i,j}$ is defined from η_i and g_j as in equation (5.22), for each $(i, j) \in I \times J$, then $\psi_{i,j} = UE_{i,j}$, as in equation (5.21). Combining Corollary 5.3.9 with Theorem 5.3.10, gives a corollary of Theorem 5.3.10. Note that, for $\xi \in L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$ and $F = U^{-1}\xi$,

$$V_{\psi_{i,j}}\xi[\underline{x}, A] = \langle \xi, \sigma_1[\underline{x}, A]\psi_{i,j}\rangle = \langle F, \sigma[\underline{x}, A]E_{i,j}\rangle = V_{E_{i,j}}F[\underline{x}, A],$$

for all $[\underline{x}, A] \in G$. Therefore, the closed λ_G-invariant subspaces in Theorem 5.3.10 can also be given as $\mathcal{M}_{i,j} = V_{\psi_{i,j}}L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$, for $(i, j) \in I \times J$.

Corollary 5.3.13. Let $\{\zeta_i : i \in I\}$ be an orthonormal basis of $L^2\left(\widehat{\mathbb{R}^2}; \frac{d\underline{\omega}}{\|\underline{\omega}\|^2}\right)$ consisting of functions in $C_c(\mathcal{O})$ and let $\{\phi_j : j \in J\}$ be an orthonormal basis of $L^2\left(\widehat{\mathbb{R}}; \frac{d\nu}{|\nu|}\right)$ consisting of functions in $C_c(\mathcal{O}_1)$. Let $\psi_{i,j} = \zeta_i \otimes \phi_j$ and let $\mathcal{M}_{i,j} = V_{\psi_{i,j}}L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$, for $(i, j) \in I \times J$. Then each $\mathcal{M}_{i,j}$ is a closed λ_G-invariant subspace of $L^2(G)$ and $V_{\psi_{i,j}}$ is an isometry that intertwines σ_1 with the restriction of λ_G to $\mathcal{M}_{i,j}$. Moreover $L^2(G) = \sum^{\oplus}_{(i,j)\in I\times J} \mathcal{M}_{i,j}$.

5.3.1 The Duflo-Moore Operator

Recall that, for $\eta \in L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$, the linear map $V_\eta : L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big) \to C_b(G)$ is defined by

$$V_\eta\xi[\underline{x}, A] = \langle \xi, \sigma_1[\underline{x}, A]\eta\rangle_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})},$$

for any $[\underline{x}, A] \in G$ and any $\xi \in L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$.

The steps leading to Corollary 5.3.13 indicate that the space

$$\begin{aligned}\mathcal{D}_T &= L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big) \bigcap L^2\left(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}; \tfrac{d\underline{\omega}\, d\nu}{\|\underline{\omega}\|^2|\nu|}\right) \\ &= \left\{\eta \in L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big) : \int_{\widehat{\mathbb{R}}}\int_{\widehat{\mathbb{R}^2}} |\eta(\underline{\omega}, \nu)|^2 \tfrac{d\underline{\omega}\, d\nu}{\|\underline{\omega}\|^2|\nu|} < \infty\right\}\end{aligned}$$

is important. It is a dense subspace of $L^2\big(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}\big)$. We define a *multiplication operator* T on $L^2\big(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}\big)$ by, for $\xi \in L^2\big(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}\big)$,

$$T\xi(\underline{\omega},\nu) = \|\underline{\omega}\|^{-1}|\nu|^{-1/2}\xi(\underline{\omega},\nu), \text{ for a.e. } (\underline{\omega},\nu) \in \widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}.$$

Then $T\xi$ is a Borel measurable function on $\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}$, for any $\xi \in L^2\big(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}\big)$. However $T\xi \in L^2\big(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}\big)$ if and only if $\xi \in \mathcal{D}_T$. Thus, T is an unbounded operator. For any $\xi \in \mathcal{D}_T$, $\langle T\xi,\xi\rangle_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})} \geq 0$. That is, T is a *positive* operator.

Lemma 5.3.14. For any $\xi \in \mathcal{D}_T$, $\sigma_1[\underline{x},A]\xi \in \mathcal{D}_T$, for all $[\underline{x},A] \in G$.

Proof. Fix $[\underline{x},A] \in \mathcal{D}_T$. For any $\xi \in \mathcal{D}_T$ and a.e. $(\underline{\omega},\omega_3) \in \widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}$, by (5.11),

$$\begin{aligned}
\int_{\widehat{\mathbb{R}}}\int_{\widehat{\mathbb{R}^2}} |\sigma_1[\underline{x},A]\xi(\underline{\omega},\omega_3)|^2\frac{d\underline{\omega}\,d\omega_3}{\|\underline{\omega}\|^2|\omega_3|} &= |\det(A)|^2\int_{\widehat{\mathbb{R}}}\int_{\widehat{\mathbb{R}^2}} \tfrac{\|\underline{\omega}\|^2}{\|\underline{\omega}A\|^2}|\xi(\underline{\omega}A,\omega_3 v_{\underline{\omega},A})|^2\tfrac{d\underline{\omega}\,d\omega_3}{\|\underline{\omega}\|^2|\omega_3|}\\
&= |\det(A)|\int_{\widehat{\mathbb{R}}}\int_{\widehat{\mathbb{R}^2}} \tfrac{1}{\|\underline{\omega}\|^2}|\xi(\underline{\omega},\omega_3 v_{\underline{\omega}A^{-1},A})|^2\tfrac{d\underline{\omega}\,d\omega_3}{|\omega_3|}\\
&= |\det(A)|\int_{\widehat{\mathbb{R}^2}}\int_{\widehat{\mathbb{R}}} \tfrac{1}{\|\underline{\omega}\|^2}|\xi(\underline{\omega},\omega_3 v_{\underline{\omega}A^{-1},A})|^2\tfrac{d\omega_3}{|\omega_3|}d\underline{\omega}\\
&= |\det(A)|\int_{\widehat{\mathbb{R}}}\int_{\widehat{\mathbb{R}^2}} |\xi(\underline{\omega},\omega_3)|^2\frac{d\underline{\omega}\,d\omega_3}{\|\underline{\omega}\|^2|\omega_3|} < \infty
\end{aligned}$$

Thus, $\sigma_1[\underline{x},A]\xi \in \mathcal{D}_T$. □

So $\mathcal{D}_T$ is a (non-closed) σ_1-invariant subspace of $L^2\big(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}\big)$. For any $[\underline{x},A] \in G$, it is easy to see that $\sigma_1[\underline{x},A]T\sigma_1[\underline{x},A]^*$ is a self-adjoint positive operator with domain $\mathcal{D}_T$. It is an important fact that $\sigma_1[\underline{x},A]T\sigma_1[\underline{x},A]^*$ is just a special multiple of the operator T.

Proposition 5.3.15. For any $[\underline{x},A] \in G$, $\sigma_1[\underline{x},A]T\sigma_1[\underline{x},A]^* = \Delta_G[\underline{x},A]^{1/2}T$.

Proof. Fix $[\underline{x},A] \in G$ and recall that $\Delta_G[\underline{x},A] = |\det(A)|^{-1}$. For any $\xi \in \mathcal{D}_T$, let $\xi' = \sigma_1[\underline{x},A]^{-1}\xi = \sigma_1[-A^{-1}\underline{x},A^{-1}]\xi$. Using (5.11),

$$\begin{aligned}
\sigma_1[\underline{x},A]T\xi'(\underline{\omega},\omega_3) &= \tfrac{|\det(A)|\cdot\|\underline{\omega}\|}{\|\underline{\omega}A\|}e^{2\pi i(\underline{\omega}\underline{x}+\omega_3 u_{\underline{\omega},A})}\big(T\xi'\big)\big(\underline{\omega}A,\omega_3 v_{\underline{\omega},A}\big)\\
&= \tfrac{|\det(A)|\cdot\|\underline{\omega}\|}{\|\underline{\omega}A\|^2|\omega_3 v_{\underline{\omega},A}|^{1/2}}e^{2\pi i(\underline{\omega}\underline{x}+\omega_3 u_{\underline{\omega},A})}\xi'\big(\underline{\omega}A,\omega_3 v_{\underline{\omega},A}\big),
\end{aligned} \tag{5.23}$$

for a.e. $(\underline{\omega},\omega_3) \in L^2\big(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}\big)$. On the other hand,

$$\begin{aligned}
\xi'\big(\underline{\omega}A,\omega_3 v_{\underline{\omega},A}\big) &= \sigma_1[-A^{-1}\underline{x},A^{-1}]\xi\big(\underline{\omega}A,\omega_3 v_{\underline{\omega},A}\big)\\
&= \tfrac{|\det(A^{-1})|\cdot\|\underline{\omega}A\|}{\|\underline{\omega}AA^{-1}\|}e^{2\pi i(\underline{\omega}A(-A^{-1}\underline{x})+\omega_3 v_{\underline{\omega},A}u_{\underline{\omega}A,A^{-1}})}\xi\big(\underline{\omega},\omega_3\big)\\
&= \tfrac{|\det(A)|^{-1}\cdot\|\underline{\omega}A\|}{\|\underline{\omega}\|}e^{-2\pi i(\underline{\omega}\underline{x}+\omega_3 u_{\underline{\omega},A})}\xi\big(\underline{\omega},\omega_3\big)
\end{aligned} \tag{5.24}$$

using $v_{\underline{\omega},A}v_{\underline{\omega}A,A^{-1}} = 1$ and $v_{\underline{\omega},A}u_{\underline{\omega}A,A^{-1}} = -u_{\underline{\omega},A}$, which follow from Proposition 5.3.1 (d) and the observation that $C_{\gamma(\underline{\omega})} = \text{id}$, for any $\underline{\omega} \in \mathcal{O}$. Inserting (5.24) in (5.23), using $|v_{\underline{\omega},A}|^{1/2} = \frac{|\det(A)|^{1/2}\|\underline{\omega}\|}{\|\underline{\omega}A\|}$ from Proposition 5.3.1 (g), and reducing gives

$$\begin{aligned}\sigma_1[\underline{x},A]T\sigma_1[\underline{x},A]^*\xi(\underline{\omega},\omega_3) &= |\det(A)|^{-1/2}\|\underline{\omega}\|^{-1}|\omega_3|^{-1/2}\xi(\underline{\omega},\omega_3)\\ &= \Delta_G[\underline{x},A]^{1/2}T\xi(\underline{\omega},\omega_3),\end{aligned}$$

for a.e. $(\underline{\omega},\omega_3)\in\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}$. This shows that $\sigma_1[\underline{x},A]T\sigma_1[\underline{x},A]^*$ and $\Delta_G[\underline{x},A]^{1/2}T$ agree as self-adjoint positive operators on $L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})$. □

In the terminology of [9], Proposition 5.3.15 says T is semi-invariant, with weight $\Delta_G^{1/2}$, with respect to the irreducible representation σ_1. However, Corollary 5.3.9 implies that σ_1 is a square-integrable representation. Let C_{σ_1} denote the Duflo-Moore operator associated with σ_1 as described in Theorem 2.25 of [14]. In particular, C_{σ_1} is semi-invariant, with weight $\Delta_G^{1/2}$, with respect to the irreducible representation σ_1 as well. By Lemma 1 of [9], there is a positive constant c such that $C_{\sigma_1}=cT$.

Theorem 5.3.16. The Duflo-Moore operator C_{σ_1} associated with σ_1 is given by, for any $\xi\in L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})$, $C_{\sigma_1}\xi(\underline{\omega},\omega_3)=\|\underline{\omega}\|^{-1}|\omega_3|^{-1/2}\xi(\underline{\omega},\omega_3)$, for a.e. $(\underline{\omega},\omega_3)\in\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}}$.

Proof. Let $c>0$ be such that $C_{\sigma_1}=cT$. As in Corollary 5.3.9, let $\zeta\in L^2(\widehat{\mathbb{R}^2})$ satisfy the condition that $\int_{\widehat{\mathbb{R}^2}}|\zeta(\underline{\omega})|^2\frac{d\underline{\omega}}{\|\underline{\omega}\|^2}=1$ and $\phi\in L^2(\widehat{\mathbb{R}})$ satisfy $\int_{\widehat{\mathbb{R}}}|\phi(\omega_3)|^2\frac{d\omega_3}{|\omega_3|}=1$. Let $\psi=\zeta\otimes\phi$. Then

$$\begin{aligned}\|T\psi\|^2_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})} &= \int_{\widehat{\mathbb{R}}}\int_{\widehat{\mathbb{R}^2}}\left|\|\underline{\omega}\|^{-1}|\omega_3|^{-1/2}\psi(\underline{\omega},\omega_3)\right|^2 d\underline{\omega}\,d\omega_3\\ &= \int_{\widehat{\mathbb{R}}}\left(\int_{\widehat{\mathbb{R}^2}}|\zeta(\underline{\omega})|^2\frac{d\underline{\omega}}{\|\underline{\omega}\|^2}\right)|\phi(\omega_3)|^2\frac{d\omega_3}{|\omega_3|}=1.\end{aligned}$$

Moreover, V_ψ is an isometry of $L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})$ into $L^2(G)$. On the other hand, from Theorem 2.25 of [14] we see that V_ψ is an isometry of $L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})$ into $L^2(G)$ exactly when

$$1=\|C_{\sigma_1}\psi\|_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})}=\|cT\psi\|_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})}=c\|T\psi\|_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})}=c.$$

Therefore, $C_{\sigma_1}=T$. □

The Duflo-Moore orthogonality relations for the square-integrable representation σ_1 can now be stated.

Corollary 5.3.17. Let $\xi_1,\xi_2\in L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})$ and $\psi_1,\psi_2\in\mathcal{D}_T$. Then

$$\langle V_{\psi_1}\xi_1,V_{\psi_2}\xi_2\rangle_{L^2(G)}=\langle\xi_1,\xi_2\rangle_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})}\langle T\psi_2,T\psi_1\rangle_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})}.$$

Corollary 5.3.18. Let $\psi\in L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})$ satisfy

$$\int_{\widehat{\mathbb{R}}}\int_{\widehat{\mathbb{R}^2}}\frac{|\psi(\underline{\omega},\omega_3)|^2}{\|\underline{\omega}\|^2|\omega_3|}d\underline{\omega}\,d\omega_3=1. \tag{5.25}$$

Then $V_\psi : L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big) \to L^2(G)$ is an isometry that intertwines σ_1 with λ_G.

5.4 The Generalized Wavelet Transforms

Corollaries 5.3.17 and 5.3.18 are the critical parts of forming a continuous wavelet transform and reconstruction formula arising from the square-integrable representation σ_1 of G.

Definition 5.4.1. A function $\psi \in L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$ is called a σ_1-*wavelet* if

$$\int_{\widehat{\mathbb{R}^2}} \int_{\widehat{\mathbb{R}}} \frac{|\psi(\underline{\omega}, \omega_3)|^2}{\|\underline{\omega}\|^2 |\omega_3|} d\underline{\omega}\, d\omega_3 = 1.$$

For each $\underline{x} \in \mathbb{R}^2$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$, define $\psi_{\underline{x},A}$ on $\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}$ by

$$\psi_{\underline{x},A}(\underline{\omega}, \omega_3) = \tfrac{|\det(A)| \cdot \|\underline{\omega}\|}{\|\underline{\omega} A\|} e^{2\pi i (\underline{\omega}\underline{x} + \omega_3 u_{\underline{\omega},A})} \psi\big(\underline{\omega} A, \omega_3 v_{\underline{\omega},A}\big), \text{ for a.e. } (\underline{\omega}, \omega_3) \in \widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}},$$

where

$$u_{\underline{\omega},A} = \frac{(ac + bd)(\omega_1^2 - \omega_2^2) - (a^2 + b^2 - c^2 - d^2)\omega_1\omega_2}{(a\omega_1 + c\omega_2)^2 + (b\omega_1 + d\omega_2)^2}$$

and

$$v_{\underline{\omega},A} = \frac{(ad - bc)(\omega_1^2 + \omega_2^2)}{(a\omega_1 + c\omega_2)^2 + (b\omega_1 + d\omega_2)^2}.$$

Then $\psi_{\underline{x},A} \in L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$.

Definition 5.4.2. For each $\xi \in L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$, let

$$V_\psi \xi[\underline{x}, A] = \langle \xi, \psi_{\underline{x},A} \rangle_{L^2(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}})}, \text{ for all } \underline{x} \in \mathbb{R}^2, A \in \mathrm{GL}_2(\mathbb{R}).$$

Then V_ψ is called the σ_1-*wavelet transform* with σ_1-wavelet ψ.

The reconstruction formula given in Proposition 2.5.9 can now be stated for the σ_1-wavelet transform.

Theorem 5.4.3. Let $\psi \in L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$ be a σ_1-wavelet. Then, for any $\xi \in L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$,

$$\xi = \int_{\mathrm{GL}_2(\mathbb{R})} \int_{\mathbb{R}^2} V_\psi \xi[\underline{x}, A]\, \psi_{\underline{x},A}\, \frac{d\underline{x}\, d\mu_{\mathrm{GL}_2(\mathbb{R})}(A)}{|\det(A)|},$$

weakly in $L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$.

Recall that the integral in Theorem 5.4.3 is over the 6 real variables x_1, x_2, a, b, c, d, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\frac{d\underline{x}\, d\mu_{\mathrm{GL}_2(\mathbb{R})}(A)}{|\det(A)|} = \frac{dx_1\, dx_2\, da\, db\, dc\, dd}{|ad - bc|^3}$.

This transform can also be expressed in a related manner by taking a Fourier transform in the distinguished third variable.

Let $\mathcal{F}_3 : L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big) \to L^2\big(\widehat{\mathbb{R}^2} \times \mathbb{R}\big)$ be the unitary map such that

$$\mathcal{F}_3\xi(\underline{\omega}, t) = \int_{\widehat{\mathbb{R}}} \xi(\underline{\omega}, \omega_3)e^{-2\pi i\omega_3 t}d\omega_3, \text{ for any } (\underline{\omega}, t) \in \widehat{\mathbb{R}^2} \times \mathbb{R},$$

and $\xi \in C_c\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$. Use $\mathcal{F}_3$ to move σ_1 to an equivalent representation acting on $L^2\big(\widehat{\mathbb{R}^2} \times \mathbb{R}\big)$. Then $\mathcal{F}_3^{-1} : L^2\big(\widehat{\mathbb{R}^2} \times \mathbb{R}\big) \to L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$ is such that

$$\mathcal{F}_3^{-1}f(\underline{\omega}, \omega_3) = \int_{\mathbb{R}} f(\underline{\omega}, t)e^{2\pi i\omega_3 t}dt, \text{ for all } (\underline{\omega}, t) \in \widehat{\mathbb{R}^2} \times \mathbb{R}$$

and $f \in C_c\big(\widehat{\mathbb{R}^2} \times \mathbb{R}\big)$.

Definition 5.4.4. For each $[\underline{x}, A] \in G_2$, let $\rho[\underline{x}, A] = \mathcal{F}_3\sigma_1[\underline{x}, A]\mathcal{F}_3^{-1}$.

Proposition 5.4.5. The map ρ is a square-integrable representation of the affine group $G = \mathbb{R}^2 \rtimes \mathrm{GL}_2(\mathbb{R})$ on the Hilbert space $L^2\big(\widehat{\mathbb{R}^2} \times \mathbb{R}\big)$. For $f \in L^2\big(\widehat{\mathbb{R}^2} \times \mathbb{R}\big)$ and $[\underline{x}, A] \in G$,

$$\rho[\underline{x}, A]f(\underline{\omega}, t) = \tfrac{\|\underline{\omega}A\|}{\|\underline{\omega}\|}e^{2\pi i\underline{\omega}\underline{x}}f\left(\underline{\omega}A, \tfrac{t-u_{\underline{\omega},A}}{v_{\underline{\omega},A}}\right),$$

for a.e. $(\underline{\omega}, t) \in \widehat{\mathbb{R}^2} \times \mathbb{R}$.

Proof. Since ρ is equivalent to σ_1 (and σ), it is an irreducible representation that is square-integrable. To verify its formula, let $\xi = \mathcal{F}_3^{-1}f$. Then

$$\begin{aligned}\rho[\underline{x}, A]f(\underline{\omega}, t) = \mathcal{F}_3\sigma_1[\underline{x}, A]\xi(\underline{\omega}, t) &= \int_{\widehat{\mathbb{R}}} \sigma_1[\underline{x}, A]\xi(\underline{\omega}, \omega_3)e^{-2\pi i\omega_3 t}d\omega_3 \\ &= \int_{\widehat{\mathbb{R}}} \tfrac{|\det(A)|\cdot\|\underline{\omega}\|}{\|\underline{\omega}A\|}e^{2\pi i(\underline{\omega}\underline{x}+\omega_3 u_{\underline{\omega},A})}\xi(\underline{\omega}A, \omega_3 v_{\underline{\omega},A})e^{-2\pi i\omega_3 t}d\omega_3 \qquad (5.26)\\ &= \tfrac{|\det(A)|\cdot\|\underline{\omega}\|}{\|\underline{\omega}A\|}e^{2\pi i\underline{\omega}\underline{x}}\int_{\widehat{\mathbb{R}}} \xi(\underline{\omega}A, \omega_3 v_{\underline{\omega},A})e^{-2\pi i\omega_3(t-u_{\underline{\omega},A})}d\omega_3.\end{aligned}$$

Make the change of variables $\omega_3' = \omega_3 v_{\underline{\omega},A}$. So $d\omega_3' = |v_{\underline{\omega},A}|d\omega_3 = \frac{|\det(A)|\cdot\|\underline{\omega}\|^2}{\|\underline{\omega}A\|^2}d\omega_3$. Thus, (5.26) becomes

$$\begin{aligned}\rho[\underline{x}, A]f(\underline{\omega}, t) &= \tfrac{\|\underline{\omega}A\|}{\|\underline{\omega}\|}e^{2\pi i\underline{\omega}\underline{x}}\int_{\widehat{\mathbb{R}}} \xi(\underline{\omega}A, \omega_3')e^{-2\pi i\omega_3' v_{\underline{\omega},A}^{-1}(t-u_{\underline{\omega},A})}d\omega_3' \\ &= \tfrac{\|\underline{\omega}A\|}{\|\underline{\omega}\|}e^{2\pi i\underline{\omega}\underline{x}}f\left(\underline{\omega}A, \tfrac{t-u_{\underline{\omega},A}}{v_{\underline{\omega},A}}\right),\end{aligned}$$

for a.e. $(\underline{\omega}, t) \in \widehat{\mathbb{R}^2} \times \mathbb{R}$. □

Definition 5.4.6. A function $w \in L^2\big(\widehat{\mathbb{R}^2} \times \mathbb{R}\big)$ is called a *ρ-wavelet* if $\mathcal{F}_3^{-1}w$ is a σ_1-wavelet.

If w is a ρ-wavelet, for each $[\underline{x}, A] \in G$, define $w_{\underline{x},A} = \rho[\underline{x}, A]w$. That is,

$$w_{\underline{x},A}(\underline{\omega}, t) = \frac{\|\underline{\omega}A\|}{\|\underline{\omega}\|} e^{2\pi i \underline{\omega} \underline{x}} w\left(\underline{\omega}A, \frac{t - u_{\underline{\omega},A}}{v_{\underline{\omega},A}}\right), \text{ for a.e. } (\underline{\omega}, t) \in \widehat{\mathbb{R}^2} \times \mathbb{R},$$

where $u_{\underline{\omega},A}$ and $v_{\underline{\omega},A}$ are as above.

Definition 5.4.7. For each $f \in L^2\big(\widehat{\mathbb{R}^2} \times \mathbb{R}\big)$, let

$$V_w f[\underline{x}, A] = \langle f, w_{\underline{\omega},A} \rangle_{L^2(\widehat{\mathbb{R}^2} \times \mathbb{R})}, \text{ for all } \underline{x} \in \mathbb{R}^2, A \in \mathrm{GL}_2(\mathbb{R}).$$

Then V_w is the ρ-*wavelet transform* with ρ-wavelet w.

Theorem 5.4.8. Let $w \in L^2\big(\widehat{\mathbb{R}^2} \times \mathbb{R}\big)$ be a ρ-wavelet. Then, for any $f \in L^2\big(\widehat{\mathbb{R}^2} \times \mathbb{R}\big)$,

$$f = \int_{\mathrm{GL}_2(\mathbb{R})} \int_{\mathbb{R}^2} V_w f[\underline{x}, A]\, w_{\underline{x},A} \frac{d\underline{x}\, d\mu_{\mathrm{GL}_2(\mathbb{R})}(A)}{|\det(A)|},$$

weakly in $L^2\big(\widehat{\mathbb{R}^2} \times \mathbb{R}\big)$.

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