EVENT HORIZON DETECTION FOR FIVE DIMENSIONAL STATIONARY BLACK HOLES

by

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Abstract

In this thesis we show how to locate the event horizons for five dimensional (5D) stationary black holes. We present the Cartan algorithm in an arbitrary number of dimensions and apply it in 4D and 5D. To facilitate the algorithm in 5D, we classifiy the Weyl tensor using its boost weight decomposition. We also consider the Lorentz frame transformations in 5D. We present the algorithm explicitly for the 4D Kerr metric. For 5D, computations by hand are not feasible. Thus we show how to perform the algorithm on Maple 2016 and illustrate it with four 5D examples: the singly rotating Myers-Perry metric, the Kerr-NUTT-(Anti)-de Sitter metric, the Reissner-Nordstrom-(Anti)-de Sitter metric, and the singly rotating static black ring.

List of Abbreviations and Symbols Used

$\square$	End of proof
$\cap$	Intersection
$\circ$	Function composition
δ^i_j	Kronecker delta
$\emptyset$	Empty set
$\exists$	There exist
$\forall$	For all
$\implies$	Implies
$\in$	Contained in
$\subset$	Subset
$f: X \rightarrow Y$	Function definition
4D	Four dimensions
5D	Five dimensions
GR	General relativity
NP	Newman Penrose
SPI	Scalar polynomial invariant
SR	Special relativity
WAND	Weyl Aligned Null Direction

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Chapter 1

Introduction

One of the most important questions in General Relativity is the following: given a black hole (with mass, angular momentum, electric charge, etc.), how do you find the event horizon? Sometimes knowing the metric of the black hole is not enough to identify the event horizon. One way to locate it is to calculate a set of scalar polynomial invariants (SPI) that vanish on the horizon. However, calculating such invariants can be computationally challenging and the result can be a long expression that is difficult to study.

Instead, one can use Cartan invariants. These can be found by applying the Cartan algorithm which reduces the Riemann tensor and its covariant derivatives into a much simpler form that can then be used to locate event horizons. This has been done successfully by the author and colleagues in [2].

In addition, because of advancements in higher dimensional theories of gravity (such as String Theory), it is now becoming more important to be able to find event horizons for black holes in dimensions greater than four (4D). Fortunately, the Cartan algorithm works for spacetimes of any dimension [3] and there are tools such as boost weight decomposition to extend the techniques used in 4D for finding event horizons in higher dimensional black holes. We show in this thesis how to do this with the aid of Maple 2016.

The focus of this thesis is stationary black holes in five dimensions (5D). We will first review relevant background material. Chapter 2 gives a brief overview of the material in differential geometry needed for this thesis. Chapter 3 reviews spacetime equivalence, the Cartan algorithm, and the classification of the Weyl tensor in

5D. Then, we will briefly go over how the algorithm is done on Maple, as demonstrated with four examples: the Myers-Perry metric, the Kerr-NUTT-Ads metric, the Reissner-Nordstrom-Ads metric, and the rotating black ring.

Chapter 2

Review of Differential Geometry

2.1 Manifolds and Vectors

Before we begin, note that we will use the **projection maps** $\pi^k : \mathbb{R}^m \to \mathbb{R}$ for $k = 1, \ldots, m$. We will also make extensive use of the *Einstein summation convention.* Therefore, $a^i T_i^j b_j$ denotes $\sum_{i=1}^n \sum_{j=1}^n a^i T_i^j b_j$, where n is the dimension of the *manifold* (defined next).

Definition 2.1.1. *A topological space M is a* ***n-dimensional differentiable manifold*** *if there exists a family of pairs $\{(U_i, \phi_i)\}$ that satisfy the following:*

1. *U_i is an open cover of M.*

2. *$\phi_i : U_i \to \mathbb{R}^n$ is a homeomorphic map.*

3. *$\forall (U_i, \phi_i), (U_j, \phi_j)$ with $U_i \cap U_j \neq \emptyset$, the compositon map $\phi_i \circ \phi_j^{-1} : U_i \cap U_j \to U_i \cap U_j$ and its inverse $\phi_j \circ \phi_i^{-1} : U_i \cap U_j \to U_i \cap U_j$ are C^1.*

We call the family $\{(U_i, \phi_i)\}$ an ***atlas*** *of M and each pair (U_i, ϕ_i) a* ***coordinate system*** *(or a* ***coordinate chart****). If, for $k = 1, \ldots, n$, $\pi^k : \mathbb{R}^n \to \mathbb{R}$ is a projection map, then the map $x_i^k = \pi^k \circ \phi_i$ is called a* ***coordinate****.*

In this text, we will refer to a differentiable manifold as simply a manifold. Also, when we pick a single coordinate system, we will drop the subscript index and write simply (U, ϕ) and the corresponding coordinates as $\{x^k\}$.

Definition 2.1.2. *Let (a, b) be an open inteval with $a < 0 < b$ and let M be a n-dimensional manifold.. A* ***smooth curve on*** *M is a map $\lambda : (a, b) \to M$ with the property that for any coordinate system (U, ϕ), the map $\phi_i \circ \lambda : (a, b) \to \mathbb{R}^n$ is differentiable in all orders.*

Definition 2.1.3. *Let M be a n-dimensional manifold. A* ***differentiable function on*** *M is a map $f : M \to \mathbb{R}$ with the property that for any coordimate system (U, ϕ), the map $f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is differentiable in all orders. Denote $C^\infty(M)$ to be the set of differentiable functions on M.*

Definition 2.1.4. *Let M be a n-dimensional manifold and $p \in M$. A map $X_p : C^\infty(M) \to \mathbb{R}$ is called a* ***derivation at*** *p if it satisfies the following:*

1. *$\forall a \in \mathbb{R}, \forall f, g \in C^\infty(M) \implies X_p[af + g] = aX_p[f] + X_p[g]$ (Linearity)*

2. *$\forall f, g \in C^\infty(M), X_p[fg] = X_p[f]g(p) + f(p)X_p[g]$ (Leibniz Rule)*

One can show that for any derivation at p, X_p, and any constant $a \in \mathbb{R}$ (which can be regarded as an element in $C^\infty(M)$ by $a(p) = a$), we have $X_p[a] = 0$. Note that if (U, ϕ) is a coordinate system of M with $p \in U$ and $\{x^i\}$ are the corresponding coordinates, then we can define the **coordinate derivations at** p, $\{(\frac{\partial}{\partial x^i})_p\}$, by:

$$\left(\frac{\partial}{\partial x^i}\right)_p [f] = \frac{\partial}{\partial \pi^i}(f \circ \phi^{-1})|_{\phi(p)} \tag{2.1}$$

Note that the function $f \circ \phi^{-1}$ maps $\mathbb{R}^n$ to $\mathbb{R}$ so the right-hand side of (2.1) is just a partial derivative. Also each of the coordinates $\{x^i\}$ can be regarded as smooth functions on the subset $U \subset M$. One can show that $(\frac{\partial}{\partial x^i})_p[x^j] = \delta_i^j$. If there is no confusion, we will use the notation $(\partial_i)_p$ to mean $(\frac{\partial}{\partial x^i})_p$.

Theorem 2.1.5. *Let M be a n-dimensional manifold and $p \in M$. (A) The set T_pM of derivations at p, called the* ***tangent space at*** *p, is a vector space with the following definitions:*

1. *Addition: $\forall X_p, Y_p \in T_pM, \forall f \in C^\infty(M) \implies (X_p + Y_p)[f] = X_p[f] + Y_p[f]$*

2. *Scalar Multiplication: $\forall X_p \in T_pM, \forall a \in \mathbb{R} \forall f \in C^\infty(M) \implies (aX_p)[f] = a(X_p[f])$*

3. *Zero Derivation at p: Define $0_p \in T_pM$ with the property that $\forall f \in C^\infty(M) \implies 0_p[f] = 0$*

4. *Inverses: $\forall X_p \in T_pM$, define $-X_p \in T_pM$ with the property that $\forall f \in C^\infty(M) \implies (-X_p)[f] = -(X_p[f])$*

(B) If (U, ϕ) is a coordinate system of M with $p \in U$, then $\{(\partial_i)_p\}$ is a basis for T_pM.

The proof of (A) is straightforward so we will only prove (B).

Proof. First check for linear independence. Suppose that $0_p = a^i(\partial_i)_p$ for some constants $a^i \in \mathbb{R}, i = 1, \ldots, n$. Then $0 = 0_p[x^j] = a^i(\partial_i)_p[x^j] = a^i\delta_i^j = a^j$. Thus $\{(\partial_i)_p\}$ is linearly independent.

Now we must show that $\{(\partial_i)_p\}$ spans T_pM. Let $X_p \in T_pM$ and define the numbers $X^i = X_p[x^i]$ for $i = 1, \ldots, n$. Claim: $X_p = X^i(\partial_i)_p$. Recall from advanced calculus that if $F : \mathbb{R}^n \to \mathbb{R}$ is C^∞ and $a \in \mathbb{R}^n$ then $\exists H_i : \mathbb{R}^n \to \mathbb{R}$ which are C^∞ $\forall i = 1, \ldots, n$ such that:

$$H_i(a) = \frac{\partial F}{\partial \pi^i}|_a \tag{2.2}$$

and, for $b \in \mathbb{R}^n$:

$$F(b) = F(a) + (\pi^i(b) - \pi^i(a))H_i(b) \tag{2.3}$$

Let $f \in C^\infty(M)$. Apply (2.2) to the map $F = f \circ \phi^{-1}$ and let $a = \phi(p)$ and $q \in U$ with $b = \phi(q)$:

$$(f \circ \phi^{-1})(b) = (f \circ \phi^{-1})(a) + (\pi^i(b) - \pi^i(a))H_i(b) \tag{2.4}$$

$$\implies f(q) = f(p) + (x^i(q) - x^i(p))H_i(\phi(q)) \tag{2.5}$$

Apply X_p to (2.5) and use the properties of derivations (we are fixing p) we get:

$$X_p[f] = X_p[f(p)] + X_p[x^i(q) - x^i(p)](H_i \circ \phi)|_{q=p} \tag{2.6}$$

$$+ (x^i(q) - x^i(p))|_{q=p} X_p[H_i \circ \phi] \tag{2.7}$$

$f(p)$ is a constant so $X_p[f(p)] = 0$. Also $(x^i(q) - x^i(p))|_{q=p} = x^i(p) - x^i(p) = 0$. Thus:

$$\begin{aligned} X_p[f] &= X_p[x^i(q) - x^i(p)](H_i \circ \phi)|_{q=p} \\ &= (X_p[x^i] - X_p[x^i(p)])H_i(\phi(p)) \\ &= (X^i - 0)H_i(a) \\ &= X^i \frac{\partial (f \circ \phi^{-1})}{\partial \pi^i}|_{\phi(p)} \quad \text{by (2.2)} \\ X_p[f] &= X^i(\partial_i)_p[f] \qquad \text{by (2.1)} \end{aligned}$$

Thus $X_p = X^i(\partial_i)_p$ as claimed. Therefore $\{(\partial_i)_p\}$ spans T_pM. □

Smooth curves on M can be used to define new derivations at p.

Definition 2.1.6. *Let M be a n-dimensional manifold, $p \in M$, and λ be a smooth curve on M with $\lambda(0) = p$. The **tangent vector on** M **at** p is the map $D_{\lambda,p} : C^\infty(M) \to \mathbb{R}$ defined by:*

$$D_{\lambda,p}[f] = \frac{\mathrm{d}(f(\lambda(t)))}{\mathrm{d}t}|_{t=0} \tag{2.8}$$

This map is a derivation at p.

The following theorem shows that tangent vectors at p and derivations at p are the same thing.

Theorem 2.1.7. *Let M be a n-dimensional manifold and $p \in M$. $\forall X_p \in T_pM, \exists$ a smooth curve, λ, such that $\lambda(0) = p$ and $D_{\lambda,p} = X_p$.*

Proof. Let (U, ϕ) be a coordinate system of M with $p \in U$ and $\{x^i\}$ be the corresponding coordinates. Let $a = (a^1, \ldots, a^n) = \phi(p)$. Define $X^i = X_p[x^i]$. Then $X_p = X^i(\partial_i)_p$. Since $\phi(U)$ is open in $\mathbb{R}^n$, $\exists \epsilon > 0$ such that the line $l : (-\epsilon, \epsilon) \to \phi(U)$ defined by $l(t) = (X^1, \ldots, X^n)t + a$. Claim: The smooth curve we are looking for is $\lambda = \phi^{-1} \circ l$.

Note that $\lambda(0) = \phi^{-1}(l(0)) = \phi^{-1}(0 + a) = \phi^{-1}(a) = p$. Now let $f \in C^\infty(M)$.

$$\begin{aligned} D_{\lambda,p}[f] &= \frac{\mathrm{d}(f(\lambda(t)))}{\mathrm{d}t}|_{t=0} \\ &= \frac{\mathrm{d}(f(\phi^{-1}(l(t))))}{\mathrm{d}t}|_{t=0} \\ &= \frac{\partial(f \circ \phi^{-1})}{\partial \pi^i}|_{\phi(p)} \frac{\mathrm{d}\pi^i(l(t))}{\mathrm{d}t}|_{t=0} \\ &= (\partial_i)_p[f] \frac{\mathrm{d}(X^i t)}{\mathrm{d}t}|_{t=0} \\ D_{\lambda,p}[f] &= (\partial_i)_p[f] X^i \end{aligned}$$

Thus $D_{\lambda,p} = X^i(\partial_i)_p = X_p$. □

2.2 Vector Fields and One-Forms

Definition 2.2.1. *Let M be a n-dimensional manifold. A **smooth vector field** (or **derivation**) is a map $X : C^\infty(M) \to C^\infty(M)$ defined by $p \in M, f \in C^\infty(M) \implies (X[f])(p) = X_p[f]$ where $X_p \in T_pM$.*

Definition 2.2.2. *If $O \in M$ is any open set, we define the set $\mathfrak{X}(O)$ of all smooth vector fields on O.*

Given a coordinate chart (U, ϕ), we can define, for $i = 1, \ldots, n$, $\frac{\partial}{\partial x^i} = \partial_i \in \mathfrak{X}(O)$ by $(\partial_i[f])(p) = (\partial_i)_p[f]$ where $p \in M$ and $f \in C^\infty(M)$. Show that $\partial_i[x^j] = \delta_i^j$ If I is an open interval with $0 \in I$ and $\lambda : I \to M$ is a smooth curve on M with $\lambda(0) = p$, then we can define the vector field $D_\lambda \in \mathfrak{X}(\lambda(I))$ by $(D_\lambda[f])(p) = D_{\lambda,p}[f]$.

Theorem 2.2.3. *Let M be a n-dimensional manifold and $O \in M$ open set. $\mathfrak{X}(O)$ is a $C^\infty(M)$-module with the following definitions:*

1. *Addition:* $X, Y \in \mathfrak{X}(O), f \in C^\infty(M) \implies (X+Y)[f] = X[f] + Y[f]$
2. *Function Multiplication:* $X \in \mathfrak{X}(O), f, g \in C^\infty(M) \implies (gX)[f] = gX[f]$
3. *Zero Field:* $0 \in \mathfrak{X}(O)$ *def by* $0[f] = 0\ \forall f \in C^\infty(M)$
4. *Inverse Field:* $\forall X \in \mathfrak{X}(O)$, *def* $-X \in \mathfrak{X}(O)$ *by* $(-X)[f] = -(X[f])\ \forall f \in C^\infty(M)$

Let $X : p \mapsto X_p$. How do we know if X is a smooth vector field? Take a coordinate system (U, ϕ) with coordinates $\{x^i\}$. Then $\forall p \in M$, we know $\exists X_p^1, \ldots, X_p^n \in \mathbb{R}$ such that $X_p = X_p^i(\partial_i)_p$. We can define the maps $X^i : p \mapsto X_p^i$ so that $X = X^i\partial_i$. Thus if $X^1, \ldots, X^n \in C^\infty(U)$, then $X \in \mathfrak{X}(U)$. If this is true for any coordinate chart, then $X \in \mathfrak{X}(M)$.

Definition 2.2.4. *Given $X, Y \in \mathfrak{X}(M)$, the* ***Lie bracket*** *is a map $[X,Y] : C^\infty(M) \to C^\infty(M)$ defined by $f \in C^\infty(M) \implies [X,Y][f] = X[Y[f]] - Y[X[f]]$.*

Proposition 2.2.5. *$[X,Y] \in \mathfrak{X}(M)$ for all $X, Y \in \mathfrak{X}(M)$ and satisfies:*

1. $X, Y, Z \in \mathfrak{X}(M), a \in \mathbb{R} \implies [aX + Y, Z] = a[X,Z] + [Y,Z]$
2. $X, Y \in \mathfrak{X}(M) \implies [X,Y] = -[Y,X]$
3. $X, Y, Z \in \mathfrak{X}(M) \implies [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

If $X = X^i\partial_i$ and $Y = Y^i\partial_i$, then $[X,Y]^i = X^j\partial_j[Y^i] - Y^j\partial_j[X^i]$.

Every vector space V has a corresponding dual space V^*. For $p \in M$, we define $T_p^*M = (T_pM)^*$ and call it the **cotangent space at** p. If (U,ϕ) is a coordinate system with $p \in U$, we can define the dual basis $\{(\mathrm{d}x^i)_p\}$ of $\{(\partial_i)_p\}$. Thus $(\mathrm{d}x^i)_p((\partial_j)_p) = \delta^i_j$.

Definition 2.2.6. *Let M be a n-dimensional manifold. A **smooth 1-form** is a map $\omega : \mathfrak{X}(M) \to C^\infty(M)$ defined by $p \in M, X \in \mathfrak{X}(M) \implies (\omega(X))(p) = \omega_p(X_p)$ where $\omega_p \in T_p^*M$.*

If (U,ϕ) is any chart with coordinates $\{x^i\}$, any smooth 1-form ω can be written as $\omega = \omega_i\mathrm{d}x^i$ where $\omega_1,\ldots,\omega_n \in C^\infty(M)$. We denote $\Omega^1(M)$ to be the set of smooth 1-forms. Like $\mathfrak{X}(M)$, $\Omega^1(M)$ is a $C^\infty(M)$-module.

2.3 Tensor Fields

A review of tensors of vector spaces of finite dimension is presented in appendix A. Since tangent spaces are vector spaces of finite dimension, we can define, for a n-dimensional manifold M and a point $p \in M$, the **valence** (r,s) **tensor at** p to be an element of $\mathcal{T}_{(r,s)}(T_pM)$. This leads naturally to the definition of a tensor field.

Definition 2.3.1. *Let M be a n-dimensional manifold. A **smooth valence** (r,s) **tensor field** is a map $T : \prod_{i=0}^r \Omega^1(M) \times \prod_{i=0}^s \mathfrak{X}(M) \to C^\infty(M)$ defined by $p \in M, \omega_1,\ldots,\omega_r \in \Omega^1(M), X_1,\ldots,X_s \in \mathfrak{X}(M) \implies$ $(T(\omega_1,\ldots,\omega_r,X_1,\ldots,X_s))(p) = T_p(\omega_{1p},\ldots,\omega_{rp},X_{1p},\ldots,X_{sp})$ where $T_p \in \mathcal{T}_{(r,s)}(T_pM)$. The set of smooth valence (r,s) tensor fields is denoted by $\mathcal{T}_{(r,s)}(M)$ and we also let $\mathcal{T}_{(0,0)}(M) = C^\infty(M)$.*

We can naturally extend all of the tensor operations defined in appendix A in a natural way. The only major difference is that scalar multiplication of tensors at a point $p \in M$ now becomes function multiplication on tensor fields. From now on, we will refer to tensor fields as simply tensors, unless there is the possibility of confusion.

Given a coordinate chart (U,ϕ) with coordinates $\{x^i\}$, a tensor can be written in the form $T = T^{i_1i_2\ldots i_r}{}_{j_1j_2\ldots j_s}\partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s}$ where $T^{i_1i_2\ldots i_r}{}_{j_1j_2\ldots j_s} \in$

$C^\infty(M)$. If $\{y^i\}$ is another coordinate system, then:

$$T^{k_1 k_2 \ldots k_r}{}_{l_1 l_2 \ldots l_s} = T^{i_1 i_2 \ldots i_r}{}_{j_1 j_2 \ldots j_s} \frac{\partial y^{k_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{k_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial y^{l_1}} \cdots \frac{\partial x^{j_s}}{\partial y^{l_s}}$$

2.4 Differential Forms

There is a special subset of the valence $(0, s)$ tensors. We call these differential forms.

Definition 2.4.1. *A k-form is a valence $(0, k)$ tensor $T \in \mathcal{T}_{(0,k)}(M)$ such that $\tilde{A}T = T$. The set of k-forms is denoted by $\Omega^k(M)$.*

Note that 0-forms are just elements of $C^\infty(M)$ and 1-forms defined above are the same as the 1-forms defined in section 2.2. Consider this as an extension of the definition of a 1-form.

Definition 2.4.2. *If $\omega \in \Omega^k(M), \gamma \in \Omega^l(M)$, we define **wedge product** (or **exterior product**) of ω and γ to be $\omega \wedge \gamma = \tilde{A}(\omega \otimes \gamma) \in \Omega^{k+l}(m)$.*

Proposition 2.4.3. *For differential forms $\omega \in \Omega^p(M), \beta \in \Omega^q(M), \gamma \in \Omega^r(M)$ and $f \in C^\infty(M)$:*

1. $(f\omega + \beta) \wedge \gamma = (\omega \wedge \gamma)f + \beta \wedge \gamma$.

2. $\omega \wedge \gamma = (-1)^{pq}\gamma \wedge \omega$ *(Wedge products are anti-commutative)*

3. $\omega \wedge \omega = 0$. *(A consequence of property 2)*

4. $(\omega \wedge \beta) \wedge \gamma = \omega \wedge (\beta \wedge \gamma)$ *(Wedge products are associative)*

Property 4 of proposition 2.4.3 gives us the freedom to write the wedge product of multiple differential forms $\omega_1, \ldots, \omega_n$ as $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n$ without parentheses..

Proposition 2.4.4. *If M has dimension n, $\omega \in \Omega^k(M)$ can be written in the form $\omega = \omega_{[i_1 \ldots i_k]} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ for $0 \leq k \leq n$ and $i_1 < i_2 < \cdots < i_k$. Thus $\Omega^k(M)$ has the dimension $\binom{n}{k}$ and no non-zero k-forms exist when $k > n$.*

Definition 2.4.5. *An **exterior derivative** is a map $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ that satisfies the following:*

1. *If* $f \in C^\infty(M)$, *then* $\mathrm{d}f$ *is defined by* $X \in \mathfrak{X}(M) \implies \mathrm{d}f(X) = X[f]$.

2. $\omega \in \Omega^k(M) \implies \mathrm{d}(\mathrm{d}\omega) = 0$

3. $\omega \in \Omega^k(M), \gamma \in \Omega^k(M) \implies \mathrm{d}(\omega + \gamma) = \mathrm{d}\omega + \mathrm{d}\gamma$

4. $\omega \in \Omega^k(M), \gamma \in \Omega^l(M) \implies \mathrm{d}(\omega \wedge \gamma) = \mathrm{d}\omega \wedge \gamma + (-1)^{kl}\omega \wedge \mathrm{d}\gamma$

Proposition 2.4.6. *The exterior derivative is unique. Also if* $\omega = \omega_{[i_1 \dots i_k]}\mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_k}$, *then* $\mathrm{d}\omega = \frac{\partial \omega_{[i_1 \dots i_k]}}{\partial x^\alpha}\mathrm{d}x^\alpha \wedge \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_k}$

Definition 2.4.7. *The* ***interior product*** *is a map* $\iota : \mathfrak{X}(M) \times \Omega^p(M) \to \Omega^{p-1}(M)$ *defined by* $\omega \in \Omega^p(M), Y, X_1, \dots, X_{p-1} \in \mathfrak{X}(M) \implies$
$(\iota_Y \omega)(X_1, \dots, X_{p-1}) = \omega(Y, X_1, \dots, X_{p-1})$.

2.5 Maps between Manifolds

For this section, assume that M is a m dimensional manifold and N is a n dimensional manifold. Also let $h : M \to N$.

Definition 2.5.1. h *is* C^∞ *if for any coordinate charts* (U, ϕ) *on* M *and* (V, ψ) *on* N, *the map* $H = \psi \circ h \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ *is* C^∞. *If in addition* h^{-1} *exists and is also* C^∞, *then* h *is a* ***diffeomorphism***.

For the next 3 definitions, assume that h is C^∞.

Definition 2.5.2. *If* $f \in C^\infty(N)$, *then* $h^*f = f \circ h \in C^\infty(M)$. *This new map* $h^* : C^\infty(N) \to C^\infty(M)$ *is called the* ***pullback***. *If* h *is a diffeomorphism,* $h_* = (h^{-1})^* = (h^*)^{-1}$ *exists and is called the* ***pushfoward***.

Definition 2.5.3. *If* $X \in \mathfrak{X}(M)$, *define* $h_*X \in \mathfrak{X}(N)$ *by* $f \in C^\infty(N) \implies (h_*X)[f] = X[h^*f]$. *This new map* $h_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$ *is called the* ***pushfoward*** *(of vector fields). If* h *is a diffeomorphism,* $h^* = (h^{-1})_* = (h_*)^{-1}$ *exists and is called the* ***pullback*** *(of vector fields).*

Definition 2.5.4. *If* $\omega \in \Omega^1(N)$, *define* $h^*\omega \in \Omega^1(M)$ *by* $X \in \mathfrak{X}(M) \implies (h^*\omega)(X) = \omega(h_*X)$. *This new map* $h^* : \Omega^1(N) \to \Omega^1(M)$ *is called the* ***pullback*** *(of 1-forms). If* h *is a diffeomorphism,* $h_* = (h^{-1})^* = (h^*)^{-1}$ *exists and is called the* ***pushfoward*** *(of 1-forms).*

The definitions of pullback and pushfoward can only be extended to arbitrary tensors if h is a diffeomorphism.

Let $\{x^i\}$ be coordinates of M and $\{y^i\}$ be coordinates of N. Given a C^∞ map $h : M \to N$, how do you relate the coordinate basis $\frac{\partial}{\partial x^i}$ to $\frac{\partial}{\partial y^i}$? Define $h^i = y^i \circ h \in C^\infty(M)$. Then $\left(h_* \frac{\partial}{\partial x^i}\right)[y^j] = \frac{\partial}{\partial x^i}[h^* y^j] = \frac{\partial}{\partial x^i}[y^j \circ h] = \frac{\partial h^j}{\partial x^i}$. Thus:

$$h_* \frac{\partial}{\partial x^i} = \frac{\partial h^j}{\partial x^i} \frac{\partial}{\partial y^j}$$

The notation $\frac{\partial y^j}{\partial x^i}$ is often used to mean $\frac{\partial h^j}{\partial x^i}$. Similarly:

$$h^* \mathrm{d}y^i = \frac{\partial y^i}{\partial x^j} \mathrm{d}x^j$$

Proposition 2.5.5. *Let S, T be tensors, $f \in C^\infty(M)$, and h be a diffeomorpism. The following are true when well-defined:*

1. $h_*(fS + T) = (h_* f)(h_* S) + h_* T$
2. $h_*(S \otimes T) = h_* S \otimes h_* T$
3. $h_*(\mathfrak{C}_{(i,j)}(T)) = \mathfrak{C}_{(i,j)}(h_* T)$

Similar rules apply to h^* as well.

2.6 Flows and Lie Derivatives

Definition 2.6.1. *Let $p \in M$ and $X \in \mathfrak{X}(M)$. An **integral curve of** X **at** p is a smooth curve $\lambda : I_p^X \to M$ (I_p^X is an unknown open inteval with $0 \in I_p^X$) such that $\lambda(0) = p$ and $D_{\lambda,\lambda(t)} = X_{\lambda(t)}, \forall t \in I_p^X$.*

Given coordinates $\{x^i\}$ of M, how do we find λ? Let $X = X^i \partial_i$ and define $X^i(t) = X^i(\lambda(t))$. Also let $p^i = x^i(p)$ and $\lambda^i = x^i \circ \lambda$. Thus $D_{\lambda,\lambda(t)} = X_{\lambda(t)} \implies \frac{\mathrm{d}\lambda^i}{\mathrm{d}t}(t) = X^i(t)$. Thus we must solve the following system of n differential equations:

$$\begin{cases} \frac{\mathrm{d}\lambda^i}{\mathrm{d}t}(t) = X^i(t) \\ \lambda^i(0) = p^i \end{cases} \quad \text{for } i = 1, \dots, n \tag{2.9}$$

For the following, let $I^X = \bigcap_{p \in M} I_p^X$. Note $0 \in I^X$.

Definition 2.6.2. *Let* $X \in \mathfrak{X}(M)$ *and* $t \in I^X$*. The* ***flow of*** X ***by*** t *is a* C^∞ *map* $\phi_t^X : M \to M$ *defined by* $\phi_t^X(p) = \lambda(t)$ *where* λ *is the integral curve of* X *at* p *and given by the solution of (2.9).*

Proposition 2.6.3. ϕ_t^X *is a diffeomorphism* $\forall t \in I^X$ *and:*

1. $\phi_0^X = id$
2. $\phi_s^X \circ \phi_t^X = \phi_{s+t}^X$
3. $\phi_{-t}^X = (\phi_t^X)^{-1}$

We call $\{\phi_t^X\}_{t\in I^X}$ *a* ***one parameter group of diffeomorphisms****.*

Definition 2.6.4. *Let* $X \in \mathfrak{X}(M)$*. The* ***Lie derivative along*** X *is a map* $\mathcal{L}_X : \mathcal{T}_{(r,s)}(M) \to \mathcal{T}_{(r,s)}(M)$ *defined by:*

$$T \in \mathcal{T}_{(r,s)}(M) \implies \mathcal{L}_X T = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \left(\phi_{-t}^X\right)_* T$$

Proposition 2.6.5. *Let* $X \in \mathfrak{X}(M)$ *and* S, T *be tensors of any order. The Lie derivative satisfies the following:*

1. $\mathcal{L}_X(S + T) = \mathcal{L}_X(S) + \mathcal{L}_X(T)$
2. $\mathcal{L}_X(S \otimes T) = \mathcal{L}_X(S) \otimes T + S \otimes \mathcal{L}_X(T)$
3. $\mathcal{L}_X(\mathfrak{C}_{(i,j)}(T)) = \mathfrak{C}_{(i,j)}(\mathcal{L}_X(T))$
4. $h^*(\mathcal{L}_X T) = \mathcal{L}_{h^*X}(h^*T)$ *for a diffeomorphism* h

Some special cases:

1. For a function $f \in C^\infty(M)$, $\mathcal{L}_X(f) = X[f]$.
2. For vectors $X, Y \in \mathfrak{X}(M)$, $\mathcal{L}_X(Y) = [X, Y]$.
3. If $\omega \in \Omega^p(M)$ and $X \in \mathfrak{X}(M)$ then $\mathcal{L}_X\omega = \mathrm{d}(\iota_X\omega) + \iota_X(\mathrm{d}\omega)$. This is known as **Cartan's formula**.

2.7 Brief overview of Lie Groups

Definition 2.7.1. *A* ***Lie Group*** *is a differentiable manifold G with a binary map $G \times G \to G, (g_1, g_2) \mapsto g_1 g_2$, a element $e \in G$, and a map $^{-1} : G \to G, g \mapsto g^{-1}$ that satisfy the following:*

1. *$(G, \cdot, e, ^{-1})$ is a group where e is the identity element and $^{-1}$ denotes the group inverse.*
2. *For a fixed $g \in G$, the map $p \mapsto pg$ is a diffeomorphism.*
3. *For a fixed $g \in G$, the map $p \mapsto gp$ is a diffeomorphism.*
4. *The map $^{-1}$ is a diffeomorphism.*

Definition 2.7.2. *Let M be a n-dimensional manifold and G be a Lie group. An* ***action of*** *G* ***on M*** *is a map $\sigma : G \times M \to M$ that satisfies:*

1. *$p \in M \implies \sigma(e, p) = p$*
2. *$p \in M,\ g_1, g_2 \in G \implies \sigma(g_1, \sigma(g_2, p)) = \sigma(g_1 g_2, p)$*

Definition 2.7.3. *Let M be a n-dimensional manifold, G be a Lie group, and σ be an action of G on M. If $p \in M$, define $G(p) = \{q \in M | \exists g \in G \text{ such that } \sigma(g, p) = q\}$, called the* ***orbit of*** *p* ***by*** *σ.*

Definition 2.7.4. *Let M be a n-dimensional manifold, G be a Lie group, and σ be an action of G on M. If $p \in M$, define $H(p) = \{g \in G | \sigma(g, p) = p\}$, called the* ***isotropy group of*** *p.*

Theorem 2.7.5. *Let M be a n-dimensional manifold, G be a Lie group, and σ be an action of G on M. $\forall p \in M$, $H(p)$ is a subgroup of G and is a Lie group.*

For the special case that $G = \mathbb{R}$, we can define the following:

Definition 2.7.6. *Let M be a n-dimensional manifold and σ be a group action of $\mathbb{R}$ on M. The* ***infinitesimal generator of*** *σ is a vector field $X^\sigma \in \mathfrak{X}(M)$ defined by:*

$$f \in C^\infty(M), p \in M \implies X^\sigma[f](p) = \frac{\mathrm{d} f(\sigma(t, p))}{\mathrm{d}t}|_{t=0} \tag{2.10}$$

Note that in a local coordinate system $\{x^a\}$ and for $\epsilon > 0$, $\sigma^a(\epsilon, p) = p^a + \epsilon (X^\sigma_p)^a + O(\epsilon^2)$. Thus $(X^\sigma_p)^a = \frac{\mathrm{d}\sigma^a(\epsilon, p)}{\mathrm{d}\epsilon}|_{\epsilon=0}$.

2.7.1 Examples

Rotations in $\mathbb{R}^2$: Trivially, the real numbers, $\mathbb{R}$, is a Lie group with the addition operation. We can define the rotation about the origin to be the map $R : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ by $R_\theta(a, b) = (a\cos(\theta) - b\sin(\theta), b\cos(\theta) + a\sin(\theta))$. It is easy to show that R is an action on $\mathbb{R}^2$ by $\mathbb{R}$. The origin is unaffected by any rotation and thus $H((0, 0)) = \mathbb{R}$. For any other point $(a, b) \neq (0, 0)$, we have $H((a, b)) = \{2\pi n | n \in \mathbb{Z}\}$. Orbits are circles: $G((a, b)) = \{(c, d) \in \mathbb{R}^2 | c^2 + d^2 = a^2 + b^2\}$

Translation in $\mathbb{R}^n$ by two fixed elements: The set $\mathbb{R}^2$ is a Lie group with the addition operation $(a, b) + (c, d) = (a + c, b + d)$. Pick two points $p, q \in \mathbb{R}^n$. We can define the translation map $T : \mathbb{R}^2 \times \mathbb{R}^n \to \mathbb{R}^n$ by $T_{(a,b)}(x) = x + ap + bq$. This map is a group action on $\mathbb{R}^n$ by $\mathbb{R}^2$. For $x \in \mathbb{R}^n$, the orbit $G(x)$ is the 2D hyperplane spanned by the elements p and q shifted by x. The isotropy group is just $H(x) = \{(0, 0)\}, \forall x \in \mathbb{R}^n$.

Flows: For a n-dimensional manifold M and a vector field $X \in \mathfrak{X}(M)$, the flow of X can be regarded as a group action on M by $\mathbb{R}$ provided that $I^X = \mathbb{R}$. Given a point $p \in M$, the orbit $G(p)$ is the integral curve of X at p and thus can be found by solving (2.9). The isotropy groups will depend on the nature of X.

General Linear Groups: We define the **general linear group** $GL(n, \mathbb{R})$ to be the set of $n \times n$ matrices whose elements are real numbers and have a non-zero determinant (i.e. they are invertible). It can be shown that this is a Lie group with respect to matrix multiplication. Here are some special subgroups of $GL(n, \mathbb{R})$ that are also Lie groups:

1. **Orthonormal Group** $O(n) = \{M \in GL(n, \mathbb{R}) | M^T M = MM^T = I_n\}$

2. **Special Linear Group** $SL(n, \mathbb{R}) = \{M \in GL(n, \mathbb{R}) | det(M) = 1\}$

3. **Special Orthonormal Group** $SO(n) = O(n) \cap SL(n, \mathbb{R})$

We can define the group action $\sigma : GL(n, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$ by the matrix-vector multiplication in Linear Algebra: $\sigma(M, p) = Mp$.

Lorentz and Poincaré Groups: This is similar to general linear groups and will be discussed in more depth in section 3.3.

2.8 Covariant Derivatives and the Riemann Curvature Tensor

Before we continue, let us adopt a common notation for partial differentiation. Given a coordinate system $\{x^i\}$ of a n-dimensional manifold M, we will now denote the action of $\frac{\partial}{\partial x^i}$ with a comma followed by the index. For example, if $f \in C^\infty(M)$, then $\frac{\partial f}{\partial x^i} = f_{,i}$ and if $T^{abdc}{}_{ef}$ is some tensor then $\frac{\partial T^{abdc}{}_{ef}}{\partial x^i} = T^{abdc}{}_{ef,i}$

We would like to be able to 'differentiate' a tensor. This can be done in $\mathbb{R}^n$ in a natural way, but not for a general manifold. To see why let us look at vectors as an example. Let $X, Y \in \mathfrak{X}(M)$. Say we want to 'differentiate' a vector field Y at $p \in M$ in the direction of X. Pick a small number $\epsilon > 0$ and let $q(\epsilon) \in M$ be another point ϵ away from p in the direction of X. To calculate the 'derivative' of Y at p along X, we write the following:

$$\lim_{\epsilon \to 0} \frac{Y_{q(\epsilon)} - Y_p}{\epsilon}$$

We can see that this expression makes no sense. The vectors Y_p and $Y_{q(\epsilon)}$ are in two different tangent spaces.

The only way to define a 'derivative' operator on a tensor in any manifold is to add some extra structure to the manifold.

Definition 2.8.1. *Let M be a n-dimensional manifold. An **affine connection** (or **connection**) is a map $\nabla : \mathfrak{X}(M) \times \mathcal{T}_{(r,s)}(M) \to \mathcal{T}_{(r,s)}(M)$ with the following properties:*

1. $X \in \mathfrak{X}(M), f \in C^\infty(M) \implies \nabla_X f = X[f]$
2. $X, Y \in \mathfrak{X}(M), f \in C^\infty(M), T \in \mathcal{T}_{(r,s)}(M) \implies \nabla_{X+fY} T = \nabla_X T + f \nabla_Y T$
3. $X \in \mathfrak{X}(M), S \in \mathcal{T}_{(r,s)}(M), T \in \mathcal{T}_{(r',s')}(M) \implies \nabla_X (S \otimes T) = (\nabla_X S) \otimes T + S \otimes (\nabla_X T)$
4. $X \in \mathfrak{X}(M), T \in \mathcal{T}_{(r,s)}(M), i = 1, \ldots r, j = 1, \ldots, s \implies \nabla_X \mathfrak{C}_{(i,j)}(T) = \mathfrak{C}_{(i,j)}(\nabla_X T)$

*We call $\nabla_X T$ the **covariant derivative** of T along X with respect to ∇.*

Proposition 2.8.2. *Let M be a n-dimensional manifold and ∇ be a connection.*

1. $X \in \mathfrak{X}(M), f \in C^\infty(M), T \in \mathcal{T}_{(r,s)}(M) \implies \nabla_X(fT) = X[f]\nabla_X T + f\nabla_X T$

2. $X, Y \in \mathfrak{X}(M), \omega \in \Omega^1(M) \implies X[\omega(Y)] = (\nabla_X \omega)(Y) + \omega(\nabla_X Y)$

Let $\{x^i\}$ be a coordinate system of M. Then there exists a set of functions ${\Gamma^k}_{ij} \in C^\infty(M)$, called **connection coefficients**, such that $\nabla_{\partial_i}\partial_j = {\Gamma^k}_{ij}\partial_k$. If $\{y^a\}$ is another coordinate system which induces the connection coefficients ${\Gamma^c}_{ab}$ (note the choice if indices), then using the properties of a connection:

$$ {\Gamma^c}_{ab} = {y^c}_{,k}{x^i}_{,a}{x^j}_{,b}{\Gamma^k}_{ij} + {y^c}_{,l}{x^m}_{,a}({x^l}_{,b})_{,m} \tag{2.11} $$

Thus connection coefficients are not the components of a type (1,2) tensor.

Let $X = X^i\partial_i \in \mathfrak{X}(M)$. Using the definition of connection coefficients:

$$ \nabla_{\partial_j} X = \left({X^i}_{,j} + {\Gamma^i}_{jk}X^k\right)\partial_i \tag{2.12} $$

We denote ${X^i}_{;j} = {X^i}_{,j} + {\Gamma^i}_{jk}X^k$ (note the semicolon). Similarly, for $\omega = \omega_i \mathrm{d}x^i$, we can use proposition 2.8.2 to show that:

$$ \nabla_{\partial_j}\omega = \left(\omega_{i,j} - {\Gamma^k}_{ji}\omega_k\right)\mathrm{d}x^i \tag{2.13} $$

We denote $\omega_{i;j} = \omega_{i,j} - {\Gamma^k}_{ji}\omega_k$. We can generalize to any tensor. We get:

$$ \begin{aligned} {T^{i_1 i_2 \ldots i_r}}_{j_1 j_2 \ldots j_s;k} = {T^{i_1 i_2 \ldots i_r}}_{j_1 j_2 \ldots j_s;k} &+ {\Gamma^{i_1}}_{kl}{T^{l i_2 \ldots i_r}}_{j_1 j_2 \ldots j_s} \\ &+ {\Gamma^{i_2}}_{kl}{T^{i_1 l \ldots i_r}}_{j_1 j_2 \ldots j_s} \\ &+ \ldots \\ &+ {\Gamma^{i_r}}_{kl}{T^{i_1 i_2 \ldots l}}_{j_1 j_2 \ldots j_s} \\ &- {\Gamma^l}_{kj_1}{T^{i_1 i_2 \ldots i_r}}_{l j_2 \ldots j_s} \\ &- {\Gamma^l}_{kj_2}{T^{i_1 i_2 \ldots i_r}}_{j_1 l \ldots j_s} \\ &- \ldots \\ &- {\Gamma^l}_{kj_s}{T^{i_1 i_2 \ldots i_r}}_{j_1 j_2 \ldots l} \end{aligned} \tag{2.14} $$

(2.15)

Note that $T^{a_1a_2\ldots a_r}{}_{b_1b_2\ldots b_s;c} = y^{a_1}{}_{,i_1}y^{a_2}{}_{,i_2}\ldots y^{a_r}{}_{,i_r}x^{j_1}{}_{,b_1}x^{j_2}{}_{,b_2}\ldots x^{j_s}{}_{,b_s}x^{k}{}_{,c}T^{i_1i_2\ldots i_r}{}_{j_1j_2\ldots j_s;k}$. This can be regarded as a new $(r, s+1)$ rank tensor, denoted by ∇T.

We can now describe how to compare two tensors at different points on a manifold. This leads to a notion of the 'straightest' possible line on a manifold.

Definition 2.8.3. *Let T be a tensor and λ be a smooth curve. If $\nabla_{D_\lambda}T = 0$ for all points on λ, then T is **parallel** to λ.*

Definition 2.8.4. *Let $p \in M$, λ be a smooth curve with $\lambda(0) = p$ and T_p be a tensor at p. The **parallel transport** of T_p along λ is the tensor field T that is parallel to λ and equals T_p at p.*

Definition 2.8.5. *A **geodesic** on M is a smooth curve μ whose tangent vector is parallel to itself (i.e. $\nabla_{D_\mu}D_\mu = 0$).*

If $\{x^i\}$ are coordinates of M and $\mu^i = x^i \circ \mu$, μ will be a geodesic if it satisfies the following system of ODEs:

$$\frac{\mathrm{d}^2\mu^i(t)}{\mathrm{d}t^2} + \frac{\mathrm{d}\mu^j(t)}{\mathrm{d}t}\frac{\mathrm{d}\mu^k(t)}{\mathrm{d}t}\Gamma^i{}_{jk} = 0 \tag{2.16}$$

We will now define two important tensors that describe the geometric properties of the manifold.

Definition 2.8.6. *The **torsion tensor** is defined by $T^i{}_{jk} = \Gamma^i{}_{jk} - \Gamma^i{}_{kj}$*

This tensor describes the failure of the commutativity of the affine connection. A connection is **symmetric** if $T^i{}_{jk} = 0$.

Definition 2.8.7. *The **Riemann Curvature Tensor** (or **Riemann Tensor** or **curvature tensor**) is defined by $R^i{}_{jkl} = \Gamma^i{}_{lj,k} - \Gamma^i{}_{kj,l} + \Gamma^m{}_{lj}\Gamma^i{}_{km} - \Gamma^m{}_{kj}\Gamma^i{}_{lm}$*

This is the most important tensor in this thesis. It describes the failure of a vector to remain the same once parallel transported along a small closed curve. It also describes how two nearby geodesics that are initially parallel deviates from one another.

2.9 Riemannian Geometry

Definition 2.9.1. *A* ***pseudo-Riemannian Manifold*** *is a manifold with a rank* $(0,2)$ *tensor* g*, called the* **metric tensor***, that satisfies the following:*

1. $X, Y \in \mathfrak{X}(M) \implies g(X,Y) = g(Y,X)$

2. $g(X,Y) = 0$ *for all* $X \in \mathfrak{X}(M) \implies Y = 0$

The components of g with respect to a coordinate system are given by g_{ij}. Property 1 implies that $g_{ij} = g_{ji}$ and property 2 implies that g_{ij} as a matrix is invertible. We denote the inverse of this matrix as g^{ij} (so that $g^{ij}g_{jk} = \delta^i_k$). We also define the scalar $g = \det(g_{ij}) \neq 0$ (the distinction between the tensor g and the scalar g is determined by what context it is being used). Note that we can also define a rank $(2,0)$ tensor g^{-1} that behaves like the metric tensor for 1-forms and its components are g^{ij}. We call this new tensor the **inverse metric**.

The metric tensor and the inverse metric define two maps. The first is the **raising operator** $R : T^{i_1 \dots i_r}{}_{j_1 \dots j_s} \mapsto T^{i_1 \dots i_r j_1}{}_{j_2 \dots j_s} = g^{j_1 k} T^{i_1 \dots i_r}{}_{k j_2 \dots j_s}$. The other is the **lowering operator** $L : T^{i_1 \dots i_r}{}_{j_1 \dots j_s} \mapsto T^{i_1 \dots i_{r-1}}{}_{i_r j_1 \dots j_s} = g_{i_r k} T^{i_1 \dots i_{r-1} k}{}_{j_1 j_2 \dots j_s}$. Note that $R^{-1} = L$ and $L^{-1} = R$.

The reason for the name metric tensor is that it gives a notion of the distance along a curve. If $\lambda : I \to M$ is a smooth function, then $\int_I \sqrt{|g_{\lambda(t)}\left(D_{\lambda(t)}, D_{\lambda(t)}\right)|}\mathrm{d}t$ is a metric function. We can use this formula to reparametrize with respect to the arc-length parameter $s(t)$ such that $\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{|g_{\lambda(t)}\left(D_{\lambda(t)}, D_{\lambda(t)}\right)|}$. This leads to the arc-length notation for a metric $\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$. We use this notation often in this text to define a metric tensor.

A n-dimensional Riemannian manifold (M, g) is **flat** if there exists a coordinate system $\{x^i\}$ such that $g_{ii} = \pm 1$ and $g_{ij} = 0$ whenever $i \neq j$. In particular, we define the **Minkowski metric** η to be $\eta_{11} = -1$, $\eta_{ii} = 1$ for $i = 2, \dots, n$ and $\eta_{ij} = 0$ whenever $i \neq j$. More on this metric in section 3.1.

Metric tensors induce a 'nice' affine connection:

Theorem 2.9.2. *Let (M, g) be a n-dimensional Riemannian manifold. There exists a unique affine connection ∇, called the **Levi-Civita connection**, that satisfies:*

1. *∇ is symmetric (and thus the torsion is zero.)*

2. *$\nabla g = 0$*

The connection coefficients of this connection are given by:

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il} \left(g_{kl,j} + g_{jl,k} - g_{jk,l} \right) \tag{2.17}$$

It is convenient to define $\Gamma_{jkl} = \frac{1}{2} \left(g_{kl,j} + g_{jl,k} - g_{jk,l} \right)$ so that $\Gamma^i{}_{jk} = g^{il} \Gamma_{jkl}$. Given vectors $X, Y \in \mathfrak{X}(M)$, property 2 implies $g_p(X_p, Y_p)$ is the same for any point $p \in M$. Einstein's theory of General Relativity requires this particular choice of connection for reasons that will be clear later. There are other theories of gravity that require a different connection, but we will not consider such theories in this thesis. From now on, we will fix the connection to be the Levi-Civita connection.

In a n-dimensional Riemannian manifold with the Levi-Civita connection, only the Riemann curvature tensor $R^i{}_{jkl}$ describes the intrinsic geometry of the manifold. At first glance, there are n^4 independent entries, but in fact there are much less. The curvature tensor has some nice properties when we set $R_{ijkl} = g_{im} R^m{}_{jkl}$:

$$R_{ijkl} = -R_{ijlk} \tag{2.18}$$

$$R_{ijkl} = -R_{jikl} \tag{2.19}$$

$$R_{ijkl} = R_{klij} \tag{2.20}$$

In addition, we have the **Bianchi identities**:

$$R^i{}_{jkl} + R^i{}_{klj} + R^i{}_{ljk} = 0 \text{ (First Bianchi Identity)} \tag{2.21}$$

$$R^i{}_{jkl;m} + R^i{}_{jlm;k} + R^i{}_{jmk;l} = 0 \text{ (Second Bianchi Identity)} \tag{2.22}$$

These properties reduce the number of independent components of the Riemann tensor to $N_{Riem}(n) = \frac{1}{12} n^2 (n^2 - 1)$. Table 2.1 shows the the values of $N_{Riem}(n)$ for $n = 1, 2, 3, 4, 5$. Note that $N_{Riem}(1) = 0$ which means that all 1-dimensional manifolds are flat. $N_{Riem}(2) = 1$ means that only one component of the Riemann tensor

n	$N_{Riem}(n)$
1	0
2	1
3	6
4	20
5	50

Table 2.1: Table showing the number of independent components of the Riemann tensor in a Riemannian manifold. Here n is the dimension of the manifold and $N_{Riem}(n) = \frac{1}{12}n^2(n^2-1)$ is the corresponding number of independent components.

is required to describe 2-dimensional manifolds (and can be related to the Gaussian curvature from elementary geometry). In General Relativity, $n = 4$ and hence there are 20 components that must be found. In this thesis, we are interested in 5D manifolds and hence we are looking for 50 components.

We can define three special tensors from the Riemann tensor. The first is the **Ricci tensor** $R_{ij} = R^k{}_{ikj}$. This is a symmetric tensor. The second is the **curvature scalar** $R = g^{ij}R_{ij}$ and the third is the **Weyl tensor** (for $n \geq 4$):

$$C_{ijkl} = R_{ijkl} + \frac{1}{n-2}\left(R_{ik}g_{jl} - R_{jk}g_{il} + R_{jl}g_{ik} - R_{il}g_{jk}\right) + \frac{R}{(n-2)(n-1)}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right) \tag{2.23}$$

The Weyl tensor shares the same symmetric properties as the Riemann tensor in addition of being traceless (i.e. if $C^i{}_{jkl} = g^{im}C_{mjkl}$, then $C^m{}_{mkl} = C^m{}_{jml} = C^m{}_{jkm} = 0$). Note that (2.23) says that the the components of the Riemann tensor can be written in terms of the components of the Ricci and Weyl tensors.

2.10 Isometries and Killing Vectors

Definition 2.10.1. *Let (M, g) and $(N, \bar{g})$ be two pseudo-Riemannian manifolds. A map $h : M \to N$ is called an* ***isometry*** *if it is a diffeomorphism and $h^*\bar{g} = g$. We say M and N are* ***isometric*** *if there exists such an isometry.*

We are interested in the case where $N = M$. We say that two metrics g and $\bar{g}$ on M are **equivalent** if there exists an isometry h such that $h^*g = \bar{g}$.

How to find all possible isometries of g? One way to find them is to find Killing vectors:

Definition 2.10.2. *Let (M, g) be a pseudo-Riemannian manifold. A* ***Killing vector*** *is a vector $\xi \in \mathfrak{X}(M)$ such that the flow of ξ by t, ϕ_t^ξ, is an isometry $\forall t \in \mathbb{R}$.*

Proposition 2.10.3. *Let (M, g) be a pseudo-Riemannian manifold. $\xi \in \mathfrak{X}(M)$ is a Killing vector if and only if $\mathcal{L}_\xi g = 0$. With the Levi-Civita connection, we also have the following, called the* ***Killing equation****, to hold true:*

$$(\xi_{;a})_b + (\xi_{;b})_a = 0 \tag{2.24}$$

One can use equation (2.24) to find a local basis of Killing vectors. As an example, for the simple 2D Euclidean space $(\mathbb{R}^2, \delta)$ with Cartesian coordinates (thus $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$), the set of Killing vectors are linear combinations of ∂_x (translations in the x-direction), ∂_y (translations in the y-direction), and $x\partial_y - y\partial_x$ (rotations about the origin).

2.11 Non-Coordinate Bases

So far, we have used the coordinate basis as our local frame, but this is not the only option available. If we have a n-dimensional pseudo-Riemannian manifold (M, g), we define a non-coordinate basis $\{e_a\} \subset \mathfrak{X}(M)$ such that $\forall p \in M$:

$$(e_a)_p = e_a{}^\alpha(p)(\partial_\alpha)_p \quad e_a{}^\alpha(p) \in GL(n, \mathbb{R}), \tag{2.25}$$

$$g_p((e_a)_p, (e_b)_p) = e_a{}^\alpha(p) e_b{}^\beta(p) g_{\alpha\beta}(p) = \eta_{ab} \tag{2.26}$$

We call the functions $e_a{}^\alpha$ the **zweibein** (if $n = 4$, then it is called **vielbeins**). Their inverse is denoted by $e^a{}_\alpha$. This gives us the following relation:

$$g_{\alpha\beta} = e^a{}_\alpha e^b{}_\beta \eta_{ab} \tag{2.27}$$

This allows us to transform any vector $X \in \mathfrak{X}(M)$ as follows:

$$X^\alpha = X^a e_a{}^\alpha \qquad X^a = e^a{}_\alpha X^\alpha \tag{2.28}$$

To transform any one-form, define $\{e^a\}$ to be the dual basis of $\{e_a\}$. Then we can show that $e^a = e^a{}_\alpha dx^\alpha$ and hence for any $\omega \in \Omega^1(M)$:

$$\omega_\alpha = \omega_a e^a{}_\alpha \qquad \omega_a = e_a{}^\alpha \omega_\alpha \tag{2.29}$$

Note that $g = \eta_{ab} e^a \otimes e^b$. The Lie bracket $[e_a, e_b]$ is not zero since $\{e_a\}$ is not a coordinate frame. We denote $[e_a, e_b] = C^c{}_{ba} e_c$.

How do the Christoffel symbols in the new non-coordinate frame relate to the those on the ccoordinate frame? Let:

$$\nabla_{e_a} e_b = \Gamma^c{}_{ab} e_c$$

Compare this with $\nabla_{\partial_\alpha} \partial_\beta = \Gamma^\gamma{}_{\alpha\beta} \partial_\gamma$ we get:

$$\Gamma^c{}_{ab} = e^c{}_\gamma e_a{}^\alpha ((e_b{}^\beta)_{,\alpha} + e_b{}^\beta \Gamma^\gamma{}_{\alpha\beta}) \tag{2.30}$$

We can write the torsion and curvature tensors in the non-coordinate frame in terms of the Christoffel symbols in the same frame:

$$T^a{}_{cb} = \Gamma^a{}_{[bc]} - C^a{}_{bc} \tag{2.31}$$

$$R^a{}_{bcd} = \Gamma^a{}_{db,c} - \Gamma^a{}_{cb,d} + \Gamma^e{}_{db} \Gamma^a{}_{ce} - \Gamma^e{}_{cb} \Gamma^a{}_{de} - C^e{}_{cd} \Gamma^a{}_{eb} \tag{2.32}$$

We now define the following one-forms: The first is the **connection one-forms**:

$$\omega^a{}_b = \Gamma^a{}_{bc} e^c \tag{2.33}$$

(Do not let the placements of the indices mislead you. These are indeed one-forms and there are at most n^2 of them.) Next is the **torsion two-form**:

$$T^a = T^a{}_{bc} e^b \wedge e^c \tag{2.34}$$

and finally, the **curvature two-form**:

$$R^a{}_b = R^a{}_{bcd} e^c \wedge e^d \tag{2.35}$$

To find the forms T^a and $R^a{}_b$ given the frame $\{e^a\}$ and the connection one-forms, we can use the **Cartan Structure equations**:

$$T^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b \tag{2.36}$$

$$R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{2.37}$$

By applying the exterior derivative to (2.36) and (2.37), we obtain the non-coordinate version of the Bianchi identities:

$$\mathrm{d}T^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge e^b \tag{2.38a}$$

$$\mathrm{d}R^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0 \tag{2.38b}$$

In the special case where we have the Levi-Civita connection, the torsion vanishes and we have $C^a{}_{bc} = \Gamma^a{}_{bc} - \Gamma^a{}_{cb}$. We also have $\omega_{ab} = -\omega_{ba}$. (2.36) reduces to $\mathrm{d}e^a = -\omega^a{}_b \wedge e^b$ which can now be used to find $\omega^a{}_b$ more efficiently.

Chapter 3

More Background

3.1 Brief Overview of General Relativity

General Relativity (GR) is currently the most accurate model of gravity we have. It proposes that our universe can be modelled by a special kind of pseudo-Riemannian manifold to be defined shortly. While there are theoretical reasons to suggest that GR can not describe everything in our universe (such as the behaviour of gravity at atomic scales), so far GR agrees with all experimental and astrophysical observational tests within uncertainties. To understand GR, we must briefly first look at **Special Relativity (SR)**.

SR is a model of space, time, and the motion of particles and light when no gravity is present. The fundamental postulates of SR are:

1. The laws of physics are the same in all frames where the observer experiences no forces.

2. The speed of light c is the same in all frames.

This tells us that SR can be described by a flat pseudo-Riemannian manifold called a **Minkowski manifold**. An observer is an **inertial observer** if he experiences no forces. He can define a frame by placing rulers at 90 degree angles and hold a stop-watch. His metric tensor will be the Minkowski metric η_{ab}. Another inertial observer will have a different inertial frame but his metric will still be the Minkowski metric. Thus the two frames are related by an element of the Poincare group.

GR is modelled by a more general Lorentzian metric. A **Lorentzian metric** (or a **spacetime metric**) is a pseudo-Riemannian manifold (M, g) where $\forall p \in M$, $\exists\{x^a\}$ coordinates such that $g_{ab}(p) = \eta_{ab}$. This means that GR reduces to SR locally. We will refer to such manifolds as **spacetime manifolds** or just **spacetime**. If M

is n-dimensional, we traditionally index components by $0, 1, 2, \ldots, n-1$ instead of $1, 2, \ldots, n$. The 0 index is often referred as the time components and the others the spatial components.

In this manifold, light will travel along curves with $\mathrm{d}s^2 = 0$ (called **null curves**) and matter (with mass) will travel along any curve where $\mathrm{d}s^2 < 0$ at each point along the curve. Such a curve is called a **worldline** or a **timelike curve**. If this line is a geodesic, the matter experiences no forces other than gravity, which is now incorporated into the structure of the manifold.

One law of physics we would like GR to be consistent with locally is the law of conservation of energy-momentum. This law can be described by the tensor relation ${T^{ab}}_{;b} = 0$ where T^{ab} is the energy-momentum tensor. Comparing this to the Einstein tensor, we get the **Einstein-Field equations**:

$$G_{ab} = 8\pi T_{ab} + g_{ab}\Lambda \tag{3.1}$$

where Λ is a constant, called the **cosmological constant**.

3.2 Event Horizons and Stationary Horizons

When we think of black holes, we imagine objects in space whose gravity is so strong that light close to the object cannot escape its gravitational pull. We can make this intuitive notion of a black hole more precise by looking at the properties of null curves in the vicinity of a black hole.

Definition 3.2.1. *A spacetime manifold (M, g_{ab}) contains a* ***black hole*** *if there exists a region $R \in M$ such that every null curve inside R never reaches spatial infinity in its future. The boundary of such a region is called the* ***event horizon****.*

(A more precise definition of black holes in terms of Penrose-Carter diagrams can be found in [4])

In this thesis, we will only be concerned with black hole spacetimes that are *asymptotically flat* or *(anti)-de Sitter*. We will also only be looking at *stationary black holes*:

Definition 3.2.2. *A* ***stationary spacetime*** *is a spacetime* (M, g_{ab}) *with a timelike Killing vector field (i.e, a vector field* ξ *such that* $\xi^a \xi_a < 0$ *and* $\mathcal{L}_\xi g = 0$*). A* ***stationary horizon*** *is a spatial submanifold* $N \subset M$ *of a stationary spacetime that is tangent and null to the timelike Killing vector.*

In [5] it is proven that event horizons of stationary asymptotically flat black holes must be stationary horizons, but stationary horizons are not in general event horizons. Thus theorem B.0.1 in appendix B can be used to locate all possible candidates for event horizons locally.

3.3 The Lorentz Group

For this section, we will first assume we are in a flat n-dimensional Minkowski manifold $(\mathbb{R}^n, \eta_{ab})$. What is the set of all Killing vectors on this manifold? We need to solve equation (2.24) with the metric η_{ab}. This reduces to $X_{a,b} + X_{b,a} = 0$. One set of solutions are the basis vectors themselves ∂_a. The other set of solutions are $x^i\partial_0 + x^0\partial_i$ for $i = 1, \ldots, n-1$, and $x^j\partial_i - x^i\partial_j$ for $0 < i < j < n$. These vectors generate the **Poincaré group**:

1. *Translations*: $\bar{x}^a = x^a + v^a, v^a \in \mathbb{R}^n$

2. *Time Reversal*: $\bar{x}^0 = -x^0$, $\bar{x}^i = x^i$ for $i = 1, \ldots, n-1$

3. *Spatial Reversal*: $\bar{x}^0 = x^0$, $\bar{x}^i = -x^i$ for $i = 1, \ldots, n-1$

4. *Boost*: $\bar{x}^0 = \cosh(t)x^0 - \sinh(t)x^k, \bar{x}^k = -\sinh(t)x^0 + \cosh(t)x^k, \bar{x}^i = x^i$ for $t \in \mathbb{R}$, $i = 1, \ldots, n-1$ and $i \neq k$.

5. *Spatial Rotations*: $\bar{x}^0 = x^0$, $\bar{x}^i = \Lambda^i_j x^j$ for $i, j = 1, \ldots, n-1$ and $\Lambda^i_j \in SO(n-1)$.

We are only interested in the isotropy group of the Poincaré group since we want to define similar group transformations on a more general Minkowski manifold. We call this subgroup the **Lorentz group**, and it consists of all of the group transformations of the Poincaré group except for translations. We want to restrict the Lorentz group further by not including time and spatial reversals. This new subgroup is usually called the restricted Lorentz group but we will simply refer to it as the Lorentz group.

We can simplify the operations of the Lorentz group on tensors if we change our frame to one of the following special types of non-coordinate orthonormal frames. These special frames $\{e_a\}$ have the property that the norm of two of the n vectors are zero. Hence these are called **null frames**. We denote l and n to be the two null vectors (so $l^a l_a = 0, n^a n_a = 0, l^a n_a = 1, n^a l_a = -1$) and m^i for $i = 2, \ldots, n-1$ are the orthonormal spatial vectors ($l^a (m^i)_a = 0, n^a (m^i)_a = 0, (m^i)^a (m^j)_a = \delta^{ij}$). In five dimensions, the Lorentz frame transformations in this new frame are as follows:

Boost: $$\bar{l} = \lambda l,\ \bar{n} = \lambda^{-1} n,\ \bar{m}^i = m^i \quad (3.2)$$

Null Rotations about l: $$\bar{l} = l,\ \bar{n} = n + z_i m^i - \frac{1}{2} z^j z_j l,\ \bar{m}^i = m^i - z_i l \quad (3.3)$$

Null Rotations about n: $$\bar{l} = l + z_i m^i - \frac{1}{2} z^j z_j n,\ \bar{n} = n,\ \bar{m}^i = m^i - z_i n \quad (3.4)$$

Spatial Rotations: $$\bar{l} = l,\ \bar{n} = n,\ \bar{m}^i = \Lambda^i_j m^j \text{ where } \Lambda^i_j \in SO(n-2) \quad (3.5)$$

When we move to a more general spacetime manifold, we can apply the Lorentz transformations (equations (3.2-3.5) for $n = 5$) locally to find isotropies of this manifold. How to do this and how it relates to finding stationary horizons is the subject of the next few sections.

3.4 Local Equivalence of Spacetimes

Given two Riemannian manifolds (M, g) and $(\bar{M}, \bar{g})$, how can we determine if they are locally isometric? (i.e, whether there exists a diffeomorphism $h : M \to \bar{M}$ such that $h^* g = \bar{g}$?). One way to find out if an isometry exists is to analyse the frame transformations in a coordinate neighbourhood. [6] gives a necessary condition, which uniquely defines a field of one-forms $\{e^a, \omega^a{}_b\}$ given by equation (2.33). Given a point $p \in M$ and a coordinate neighbourhood $\{U, \phi = (x^i)\}$ of p, we pick an orthonormal non-coordinate basis $\{e_a\}$ and then calculate the curvature two-forms using the Cartan Structure equations (2.36, 2.37) with $T^a = 0$. These equations imply that the curvature components must be locally equal.

If we want a sufficient condition, [6] says we must take repeated exterior derivatives of ${R^a}_b$ until no additional functionally independent quantities arises. If the two manifolds are locally equivalent, they must have the same number of functionally independent invariants called the **rank** k. If the dimension of the manifold is n, then $k \leq n$ and if $k < n$, Killing vector fields exist.

Since we assume our manifold is a spacetime manifold, we have the following result [6]:

$$(dR)_{abcd} = R_{abcd;f}e^f + R_{ebcd}{\omega^e}_a + R_{aecd}{\omega^e}_b + R_{abed}{\omega^e}_c + R_{abce}{\omega^e}_d \tag{3.6}$$

This shows that repeated exterior differentiation of R_{abcd} is equivalent to repeated covariant differentiation. Thus a metric is locally characterised by its Riemann tensor and a finite number of covariant derivatives of the Riemann tensor. We thus denote the set R^q by $\{R_{abcd}, R_{abcd;f} \ldots, R_{abcd;f_1 f_2 \ldots f_q}\}$ where q is the **order**. If p is the last order at which a new functionally independent quantity arises, then $R^{p+1} = R^p$. Thus we must find R^{p+1}. This can be done by the Cartan algorithm.

3.5 The Cartan-Karlhede Algorithm and Cartan Invariants

We summarise the algorithm for a spactime of arbitrary dimension n from [3] and apply it in this thesis for $n = 4$ in chapter 4 and $n = 5$ in chapter 6. A rigorous explanation of the Cartan algorithm involves frame bundles, which is outside the scope of this thesis. We want to reduce the set of nonzero invariants as much as possible by expressing the Riemann tensor and its covariant derivatives into a canonical form and only allow frame changes that do not change the curvature tensors. The resulting set R^{p+1} are called **Cartan scalars** or **Cartan invariants**. We will then show how to use the invariants to find stationary horizons

We begin by putting the Weyl tensor into a canonical form. Then, if possible, we use any residual frame freedom to put the Ricci tensor $R_{ab} = {R^c}_{acb}$ into a canonical form. The Riemann tensor can be written in terms of the Weyl and Ricci tensors, where many black hole solutions are in vacuum (which implies $R_{ab} = 0$) so this way is easier. This gives us R^0. To get R^1, we calculate the first covariant derivatives of the Weyl and Ricci tensors and fix the new set of invariants. We repeat for R^2, R^3,

... until no new frame freedom and functionally independent terms occur.

We state the Cartan algorithm here as presented in [3]:

1. Set the order of differentiation q to 0.

2. Calculate the derivatives of the Riemann tensor up to the q^{th} order.

3. Find the canonical form of the Riemann tensor and its covariant derivatives.

4. Fix the frame as much as possible using this canonical form, and note the residual frame freedom (the group of allowed transformations is the *linear isotropy group* H_q). The dimension of H_q is the dimension of the remaining *vertical* freedom of the frame bundle.

5. Find the number t_q of independent functions of space-time position in the components of the Riemann tensor and its covariant derivatives, in the canonical form; this is the the remaining *horizontal* freedom.

6. If the isotropy group and number of independent functions are the same as in the previous step, let $p + 1 = q$, and the algorithm terminates; if they differ (or if $q = 0$), increase q by 1 and go to step 2.

We can now characterize the n-dimensional spacetime manifold by the canonical form used, the successive isotropy group, the number of functionally independent invariants at each order, and the values of the non-zero Cartan invariants. If there are t_p essential spacetime coordinates, the other $n - t_p$ coordinates can be ignored. Thus the dimension of the isotropy group of the spacetime is $s = \dim(H_p)$, and the dimension of the isometry group is $r = s + n - t_p$.

We can compare two spacetime manifolds by comparing discrete properties such as the sequence of isotropy groups or number of functionally independent invariants at each order. This can be used to prove inequivalence; however this is not enough to prove equivalence. We need both of the discrete sequences for each metric match, so that we can compare the forms of the Cartan invariants relative to the same frame.

For the purposes of finding stationary horizons, it is not necessary to complete the full Cartan algorithm. Once the zeroth order is completed, we can use a generalization of Theorem B.0.1 in appendix B presented in [2] to find the stationary horizons:

Theorem 3.5.1. *Let (M, g) be a spacetime manifold with a local cohomogeneity k and which contains a stationary horizon. Let $C^{(1)}, \ldots, C^{(k)}$ be the functionally independent Cartan invariants of M and assume $\mathrm{d}C^{(1)}, \ldots, \mathrm{d}C^{(k)}$ is well-defined. If $W = \mathrm{d}C^{(1)} \wedge \mathrm{d}C^{(2)} \wedge \cdots \wedge \mathrm{d}C^{(k)}$, then at the stationary horizon:*

$$||W||^2 \equiv \frac{1}{k!}\delta^{\alpha_1,\ldots,\alpha_k}_{\beta_1,\ldots,\beta_k} g^{\beta_1\gamma_1}...g^{\beta_k\gamma_k} \times C^{(1)}_{;\alpha_1}...C^{(k)}_{;\alpha_k}C^{(1)}_{;\gamma_1}...C^{(k)}_{;\gamma_k} = 0 \tag{3.7}$$

Once the stationary horizons are found, we manipulate the zeroth and first order Cartan scalars so that they also identify the same stationary horizons. If they do not, we try to manipulate the first order Cartan invariants, then the second order and so on until we have an invariant that identifies the stationary horizons.

3.6 Algebraic Classification of Spacetime

We would like some useful information on the Weyl and Ricci tensors if we are to put them into canonical form at the zeroth order of the Cartan algorithm. One such piece of information is the algebraic classification of the curvature tensor. Algebraic classification of spacetimes has been a vital tool for studying four dimensional spacetimes [6]. We only need to be concerned about the Weyl tensor's classification if the Ricci tensor is zero or trivial.

In 4D, there are many tools available for us to classify spacetimes algebraically (e.g. null vectors, 2-spinors, bivectors, SPIs). Generalizations of these approaches to higher dimensions exist [7, 8], but each approach gives a distinct classification [8, 9].

We will use the most well-studied approach [7, 10, 11]. We pick a null frame:

$$\{l, n, m^i\}, \qquad i = 2, \ldots, n-1 \tag{3.8}$$

thus $l^a l_a = n^a n_a = 0$, $l^a n_a = 1$, $(m^i)_a (m^j)^a = \delta^{ij}$. We then look at the null components of the Weyl tensor and study how they are affected by a local Lorentz boost:

$$\ell \mapsto \lambda\ell, \qquad n \mapsto \lambda^{-1}n, \qquad m_i \mapsto m_i \tag{3.9}$$

In general, relative to the basis $\{\theta^a\} = \{l, n, m^i\}$, the components of an arbitrary tensor of rank p transforms under a boost as follows:

$$T'_{a_1 a_2 \ldots a_p} = \lambda^{b_{a_1 a_2 \ldots a_p}} T_{a_1 a_2 \ldots a_p}, \quad b_{a_1 a_2 \ldots a_p} = \sum_{i=1}^{P} (\delta_{a_i 0} - \delta_{a_i 1})$$

where δ_{ab} is the Kronecker delta. The **boost weight** (b.w) of the frame component $T_{a_1 a_2 \ldots a_p}$ is the value $b_{a_1, \cdots a_p}$.

We can use the b.w. decomposition [8, 12, 1] to classify the Weyl tensor. By the symmetries of the Weyl tensor, components of b.w. ± 4 or ± 3 vanish. The remaining components satisfy:

$$\begin{gathered} b.w.\ 2: \ C_0{}^i{}_{0i} = 0;, \quad b.w.\ -2: \ C_1{}^i{}_{1i} = 0 \\ b.w.\ 1: \ C_{010i} = C_0{}^j{}_{ij};, \quad b.w.\ -1: \ C_{101i} = C_1{}^j{}_{ij} \\ b.w.\ 0: \ 2C_{0(ij)1} = C_i{}^k{}_{jk}, \quad 2C_{0[ij]1} = -C_{01ij}, \quad 2C_{0101} = -C^{ij}{}_{ij} = 2C_0{}^i{}_{1i}. \end{gathered}$$

This gives us a way to classify spacetime manifolds of arbitrary dimension [8, 12, 1]. It can be shown that this reproduces the Petrov classification for 4D spacetimes.

For a given null frame $\{\theta^a\} = \{l, n, m^i\}$ and a tensor T, denote the **boost order** $b_T(l)$ to be the largest possible boost weight of T for the frame $\{\theta^a\}$. The boost weight only depends on the null vector l since it is invariant under any Lorentz frame transformation that leaves l fixed. For the Weyl tensor, $|b_C(l)| \leq 2$. We define a **Weyl alligned null direction** (or **WAND**) to be a null vector such that when used as part of a null frame, $b_C \leq 1$. The number of possible WANDs determine the classification of the Weyl tensor [7, 11, 13, 14]. Define $\zeta = \min_l(b_C(l))$. We say that the Weyl tensor is of type **N** if $\zeta = -2$, type **III** if $\zeta = -1$, **II** if $\zeta = 0$, type **I** if $\zeta = 1$, and type **G** if $\zeta = 2$. If, in addition to $\zeta = 0$ there is more than one WAND, then we denote the alignment type as **D** instead of **II**.

If we know in advance the algebraic classification of a given Weyl tensor, we can

b.w.	Constituents	Weyl tensor Components
$+2$	$\hat{H}_{ij}$	$C_{0i0j} = \hat{H}_{ij}$
$+1$	$\hat{n}_{ij},\ \hat{v}_i$	$C_{0ijk} = 2\delta_{i[j}\hat{v}_{k]} + \hat{n}_i{}^l\epsilon_{ljk}$ $C_{010i} = -2\hat{v}_i$
0	$\bar{S}_{ij},\quad \bar{w}_i,\quad \bar{R}$	$C^{ij}{}_{kl} = 4\delta^{[i}{}_{[k}\bar{S}^{j]}{}_{l]} + \frac{1}{3}\bar{R}\delta^{[i}{}_{[k}\delta^{j]}{}_{l]}$ $C_{1i0j} = M_{ij} = -\frac{1}{2}\bar{S}_{ij} - \frac{1}{6}\bar{R}\delta_{ij} - \frac{1}{2}\epsilon_{ijk}\bar{w}^k$ $C_{01ij} = A_{ij} = \epsilon_{ijk}\bar{w}^k$ $C_{0101} = -\frac{1}{2}\bar{R}$
-1	$\check{n}_{ij},\ \check{v}_i$	$C_{1ijk} = 2\delta_{i[j}\check{v}_{k]} + \check{n}_i{}^l\epsilon_{ljk}$ $C_{101i} = -2\check{v}_i$
-2	$\check{H}_{ij}$	$C_{1i1j} = \check{H}_{ij}$

Table 3.1: Constituent parts of the 5D Weyl tensor [1]. Here ϵ_{ijk} is the alternating Levi-Civita symbol for the three dimensional transverse space.

find the Weyl Aligned Null Directions (WANDs) using the following [3]:

$$l^b l^c l_{[c} C_{a]bc[d} l_{f]} = 0 \leftarrow l \text{ is a WAND, at most primary type } \mathbf{I}.$$

$$l^b l^c C_{abc[d} l_{e]} = 0 \leftarrow \ell \text{ is a WAND, at most primary type } \mathbf{II}.$$

$$l^c C_{abc[d} l_{e]} = 0 \leftarrow \ell \text{ is a WAND, at most primary type } \mathbf{III}.$$

$$l^c C_{abcd} = 0 \leftarrow l \text{ is a WAND, at most primary type } \mathbf{N}.$$

When using the Cartan algorithm, we choose to use a null frame with WANDs. This way, the Weyl tensor is simplified in this frame and it is easier to check for isotropy. In particular, if the b.w. 0 components of the Weyl and Ricci tensor are the only nonzero components, we can conclude that there is a boost isotropy.

Table 3.1 shows all of the b.w. components of the Weyl tensor in 5D and its smaller constituent parts [1]. This table can help identify any possible isotropy at zeroth order (especially spatial rotations and boosts). We can apply Lorentz transformations to simplify the smaller constituents and put the Weyl tensor in a canonical form. Some constituents are vectors and matrices, which give geometric information that can be used in the Cartan algorithm.

Chapter 4

The 4D Kerr Metric

As a simple example to illustrate the Cartan algorithm to detect stationary horizons, we will apply it to the 4D Kerr metric.

4.1 Kerr Metric using Cartan Invariants

We will demonstrate the Cartan algorithm by applying it to a simple example: The four-dimensional Kerr metric in Boyer-Lindquist coordinates $\{t, r, \theta, \phi\}$ is given by:

$$
\begin{aligned}
ds^2 = & \left(\frac{2Mr - r^2 - a^2\cos^2\theta}{r^2 + a^2\cos^2\theta}\right) dt^2 - \left(\frac{2Mra\sin^2\theta}{r^2 + a^2\cos^2\theta}\right) dt \otimes d\phi \\
& - \left(\frac{2Mra\sin^2\theta}{r^2 + a^2\cos^2\theta}\right) d\phi \otimes dt - \left(\frac{a^2\cos^2\theta - r^2}{2Mr - a^2 - r^2}\right) dr^2 + \left(r^2 + a^2\cos^2\theta\right) d\theta^2 \\
& - \frac{\sin^2\theta\left(2Mra^2\cos^2\theta - a^2\cos^2\theta - r^2a^2\cos^2\theta - 2Mra^2 - a^2r^2 - r^4\right)}{r^2 + a^2\cos^2\theta} d\phi^2
\end{aligned} \tag{4.1}
$$

where M is a mass parameter, and a is a rotation parameter. We will use the following null frame:

$$
\begin{aligned}
\ell &= dt - \left(\frac{r^2 + a^2\cos^2\theta}{2Mr - a^2 - r^2}\right) dr + a\sin^2\theta d\phi, \qquad (4.2) \\
n &= \left(\frac{2Mr - a^2 - r^2}{2\left(r^2 + a^2\cos^2\theta\right)}\right) dt - \frac{1}{2} dr - \left(\frac{a\sin^2\theta\left(2Mr - a^2 - r^2\right)}{2\left(r^2 + a^2\cos^2\theta\right)}\right) d\phi, \\
m &= \left(-\frac{\imath\sqrt{2}a\sin\theta}{2\left(r + \imath a\cos\theta\right)}\right) dt + \left(\frac{\sqrt{2}\left(r^2 + a^2\cos^2\theta\right)}{2\left(r + \imath a\cos\theta\right)}\right) dt + \left(\frac{\imath\sqrt{2}(a^2 + r^2)\sin\theta}{2\left(r + \imath a\cos\theta\right)}\right) d\phi, \\
\overline{m} &= \left(-\frac{\imath\sqrt{2}a\sin\theta}{2\left(\imath a\cos\theta - r\right)}\right) dt - \left(\frac{\sqrt{2}\left(r^2 + a^2\cos^2\theta\right)}{2\left(\imath a\cos\theta - r\right)}\right) dt + \left(\frac{\imath\sqrt{2}(a^2 + r^2)\sin\theta}{2\left(\imath a\cos\theta - r\right)}\right) d\phi.
\end{aligned}
$$

We begin at zeroth order. We simply calculate the Riemann tensor in the above frame. Since we are in 4D, we can apply the Newman-Penrose (NP) spinor formalism [2] to find the NP curvature scalars. It turns out that the Kerr black hole only has one nonzero curvature scalar:

$$
\Psi_2 = \frac{iM}{\left(a\cos\theta + ir\right)^3} \tag{4.3}
$$

We can use Ψ_2 to define a simpler invariant:

$$C_0 = \left(i\frac{1}{\Psi_2}\right)^{\frac{1}{3}} \tag{4.4}$$

and then take the real and imaginary parts of this invariant to get the following two real valued invariants:

$$\mathrm{Re}(C_0) = \frac{a\cos\theta}{M^{\frac{1}{3}}}, \quad \mathrm{Im}(C_0) = \frac{r}{M^{\frac{1}{3}}} \tag{4.5}$$

These are functionally independent of each other. Thus $t_0 = 2$. Since we have put the Weyl spinor in the canonical form, which only has boost weight zero components, we can refer to the transformation laws in chapter 7 of [6] to find the isotropy group. Only null rotations change the form of the Weyl spinor whereas spatial rotations and boosts do not. Thus $\dim(H_0) = 2$.

We now go to the first order of the 4D Cartan algorithm. We compute the covariant derivative of the Weyl spinor. [15] gives the following components of the Weyl spinor:

$$\begin{aligned}
(D^1\Psi)_{20'} &:= \frac{3M}{(a\cos\theta + ir)^4} \\
(D^1\Psi)_{30'} &:= -\frac{3}{2}\frac{\sqrt{2}Ma\sin\theta}{(a\cos\theta + ir)^5} \\
(D^1\Psi)_{21'} &:= \frac{3}{2}\frac{\sqrt{2}Ma\sin\theta}{(a\cos\theta - ir)(a\cos\theta + ir)^4} \\
(D^1\Psi)_{31'} &:= \frac{3}{2}\frac{(2Mr - a^2 - r^2)M}{(a\cos\theta - ir)(a\cos\theta + ir)^5}
\end{aligned} \tag{4.6}$$

where D^1 is used to denote the first covariant derivative. Since ρ, $\rho' = -\mu$, τ, and $\tau' = -\pi$ are all non-zero, the Kerr metric is a type **D** vacuum spacetime in case III of [15]. We can use the following NP curvature scalars (ρ, μ, τ, π), which can be expressed as ratios of the first and zeroth order Cartan invariants, to fix all remaining isotropy at first order. We follow the recommendation given in [15] by setting the boost such that $|\rho| = |\rho'|$ and the spatial rotations such that either τ or τ' is a real number. We will fix both at $r = 1$ and $\theta' = 0$. We thus have no more isotropy and hence $\dim(H_1) = 0$. Note that it is possible to express (4.6) locally in terms of

(4.3) and thus we do not find any new functionally independent invariants. Therefore $t_1 = 2$. Since $\dim(H_1) \neq \dim(H_0)$, we advance to the second order.

In order to calculate the second covariant derivatives of the Weyl spinor, we refer to formulas $(4.3a') - (4.3i')$ of [15]. We obtain the following non-zero components:

$$(D^2\Psi)_{(20')} \quad (D^2\Psi)_{(21')} \quad (D^2\Psi)_{(30')} \quad (D^2\Psi)_{(31')}$$

$$(D^2\Psi)_{(32')} \quad (D^2\Psi)_{(40')} \quad (D^2\Psi)_{(41')} \quad (D^2\Psi)_{(42')}$$

We trivially have $\dim(H_2) = 0$ since we have fixed our isotropy at first order. Also these components are functionally dependent on (4.3) therefore $t_2 = 0$. Since $\dim(H_2) \neq \dim(H_1)$ and $t_2 = t_1$, the Cartan algorithm terminates.

We can apply theorem 3.5.1 with $J_1 = C_0$ and $J_2 = 2(D^1\Psi)_{31'}C_0^3/3$. In order for $||\mathrm{d}J_1 \wedge \mathrm{d}J_2||^2 = 0$, we require that $r = M \pm \sqrt{M^2 - a^2}$ and $r = M \pm \sqrt{M^2 - a^2\cos^2\theta}$, which are the inner event horizon and the ergosphere, respectively. The ergosphere is not a stationary horizon. Thus theorem 3.5.1 may give extra horizons that are not stationary.

4.2 Kerr using Scalar Polynomial Invariants

Calculations of the scalar polynomial invariants (SPIs, see appendix B) for the Kerr metric were done in [16] where the following Invariants were defined:

$$\begin{aligned}
I_1 &= C^{abcd}C_{abcd} \\
I_2 &= C^{*abcd}C_{abcd} \\
I_3 &= C^{abcd;e}C_{abcd;e} \\
I_4 &= C^{*abcd;e}C_{abcd;e} \\
I_5 &= (I_1)_{;a}(I_1)^{;a} \\
I_6 &= (I_2)_{;a}(I_2)^{;a} \\
I_7 &= (I_1)_{;a}(I_2)^{;a}
\end{aligned} \tag{4.7}$$

where C^{*abcd} is the hodge dual of the Weyl tensor. For the Kerr metric, I_1, I_2, and I_3 were found to be:

$$
\begin{aligned}
I_1 &= -\frac{48M^2(y-r)(y+r)(r^2-4ry+y^2)(r^2+4ry+y^2)}{(r^2+y^2)^6} \\
I_2 &= \frac{2304M^2ry(r^2-3y^2)(3r^2-y^2)}{(r^2+y^2)^6} \\
I_3 &= \frac{720M^2}{(r^2+y^2)^9}(r^4+4r^3y-6r^2y^2-4ry^3+y^4) \\
&\quad (r^4-4r^3y-6r^2y^2+4ry^3+y^4)(2Mr-r^2-y^2)
\end{aligned}
\tag{4.8}
$$

where $y = a\cos\theta$. [16] then defines Q_1 and Q_2 as follows:

$$
\begin{aligned}
Q_1 &= \frac{1}{3\sqrt{3}}\frac{(I_1^2-I_2^2)(I_5-I_6)+4I_1I_2I_7}{(I_1^2+I_2^2)^{\frac{9}{4}}} \\
Q_2 &= \frac{1}{27}\frac{I_5I_6-I_7^2}{(I_1^2+I_2^2)^{\frac{5}{2}}}
\end{aligned}
\tag{4.9}
$$

Substituting (4.8) into (4.9), we get:

$$
\begin{aligned}
Q_1 &= \frac{(r^2-a^2\cos^2\theta)(r^2-2mr+a^2\cos^2\theta)}{m(r^2+a^2\cos^2\theta)^{3/2}} \\
Q_2 &= \frac{a^2\sin^2\theta(r^2-2mr+a^2)}{m^2(r^2+a^2\cos^2\theta)}
\end{aligned}
\tag{4.10}
$$

Q_1 vanishes at the event horizon and Q_2 vanishes at the ergosphere. Hence the reason that the authors of [16] defined Q_1 and Q_2

Therefore, SPI can be used to find stationary horizons. But simply calculating contractions of the curvature tensors does not in general give invariants that vanish at stationary horizons. One must perform algebraic manipulations of the calculated invariants to then obtain the desired ones that do vanish at the horizons. One can use theorem B.0.1 as a guide to know where the horizons are. SPIs also require more computational time for more complex black hole metrics.

Chapter 5

Methods

For five-dimensional spacetimes, performing the Cartan algorithm by hand is not feasible. We thus use Maple 2016 in order to apply the Cartan algorithm on a computer. For each given spacetime manifold, we define its coordinates using the 'DGSetup' command and then input the metric tensor. We then insert the WANDs (or evaluate them if they are not known) and build a null frame from the WANDs using the Gram-Schmidt orthogonalization algorithm. From here, depending on the complexity of the WANDs, we can take two approaches to evaluate the Weyl and Ricci tensors (and their derivatives) in this null frame.

5.1 Maple Frame Data

Maple has a command called 'FrameData' that takes as input a frame and then allows for direct computation of any tensors in the frame using the results from section 2.11. Thus by inserting a null frame with WANDs, the Weyl and Ricci tensors are calculated using the Cartan structure equations (2.37) and then put in a form where we can check for isotropy. At zeroth order, we can directly calculate the quantities in Table 3.1 and study there properties to find any invariance under boosts and spatial rotations. In the first three examples in chapter 6, doing so will reduce the isotropy group to be one-dimensional.

At higher orders, in order for implementation in Maple, we can define a valence (1,1) tensor $\Lambda^{a}_{a'}(t)$ that induces one of the ten infinitesimal Lorentz transformations (3.2 - 3.5) with t a small arbitrary parameter ($t = z$ for null rotations, $t = \lambda$ for boosts, and $t = \theta$ for spatial rotations). An arbitrary $(0, n)$ tensor $T_{a_1 a_2 \dots a_n}$ transforms infinitesimally as follows:

$$T_{a'_1 a'_2 \ldots a'_n}(t) = T_{a_1 a_2 \ldots a_n} \Lambda^{a_1}_{a'_1}(t) \Lambda^{a_2}_{a'_2}(t) \ldots \Lambda^{a_n}_{a'_n}(t) \tag{5.1}$$

This applies in the first three examples as we have reduced the isotropy group down to one dimension. As a computational approach, to check for any isotropies, we apply 5.1 to the Weyl and Ricci tensors and their covariant derivatives and then check if $\frac{\mathrm{d}}{\mathrm{d}t} T_{a'_1 a'_2 \ldots a'_n}(t) = 0$. If true, then $T_{a_1 a_2 \ldots a_n}$ is infinitesimally invariant with respect to the frame transformation Λ. If this is true for the curvature tensors up to differentiation order p, then Λ is part of the isotropy group of order p. In order to check for functional independence, we use the algorithm presented in Appendix C where we build a list of the nonzero components of the Weyl and Ricci tensors and reduce it to a list of only functionally independent components.

One advantage of using 'FrameData' is in its simplicity. However, depending on the complexity of the WANDs, it can be computationally challenging. For such cases, the zweilbeins method is recommended. We should also note that there is a Maple package that can be used to determine isotropy directly, but it is computationally more efficient to instead fix the curvature tensor in a canonical form with WANDs to find isotropy (especially in 5D).

5.2 Zweibein

Instead of having Maple calculate all quantities in a given null frame, we stay in the coordinate basis and calculate the zweibein ${e_a}^{\alpha}$ using equation (2.25) as a valence (1,1) tensor in Maple. We then directly compute the Weyl and Ricci tensors in the null frame by applying:

$$C_{abcd} = C_{\alpha\beta\gamma\delta} {e_a}^{\alpha} {e_b}^{\beta} {e_c}^{\gamma} {e_d}^{\delta} \tag{5.2}$$

$$R_{ab} = R_{\alpha\beta} {e_a}^{\alpha} {e_b}^{\beta} \tag{5.3}$$

To apply these equations in Maple, we calculate the Weyl and Ricci tensors in the coordinate frame and then repeatedly apply a tensor contraction command. We can then check for isotropy and functional independence in a similar way to the 'FrameData' method.

One advantage to this method is that it is computationally simpler for spacetime metrics with complicated WANDs. However, it has problems when computing higher order derivatives (even for simple black holes).

Chapter 6

Examples

6.1 The Singly Rotating Myers-Perry Metric

The singly rotating Myers-Perry metric can be regarded as an 5D extension of the 4D Kerr metric and was first introduced in [17]. Note that since this is a vacuum solution, the Ricci tensor is zero. We will use the form given by [18]:

$$\mathrm{d}s^2 = -\sqrt{\frac{1-x}{1-y}}(\mathrm{d}t + R\sqrt{\nu}(1+y)\mathrm{d}\psi)^2 + \frac{R^2}{(x-y)^2}[(x-1)((1-y^2)(1-\nu y)\mathrm{d}\psi^2 + \frac{\mathrm{d}y^2}{(1+y)(1-\nu y)}) + (1-x)^2(\frac{\mathrm{d}x^2}{(1-x^2)(1-\nu x)} + (1+x)(1-\nu x)\mathrm{d}\phi^2)] \tag{6.1}$$

[18] shows that this black hole is of type D and gives the WANDs for this metric:

$$L_\pm = \frac{1}{(x^2-1)(\nu y-1)}\left(\frac{\nu yx - y + \nu x + 1 - 2\nu y}{x-y}R\partial_t - \sqrt{\nu}\partial_\psi\right) \pm \sqrt{\frac{\nu x - 1}{(x-y)(y-1)}}\left(\partial_x + \frac{y^2-1}{x^2-1}\partial_y\right) \tag{6.2}$$

In order to get a null frame, we first define the following 'placeholder' frame $\{L_+, L_-, \partial_\phi, \partial_y, \partial_\psi\}$ and then apply the Gram-Schmidt algorithm to turn it into an orthonormal null frame. The WANDs are simple enough to apply the 'FrameData' method for computing the Weyl tensor, but we also used the vielbein method as well for comparison.

For the zeroth iteration of the Cartan algorithm, we obtain 10 nonzero components of the Weyl tensor:

$$
\begin{aligned}
C_{1010} &= \frac{(x-y)^2(4\nu x+\nu-3)}{4(y-1)^2R^2} \\
C_{1043} = \frac{1}{2}C_{3041} = -\frac{1}{2}C_{4031} &= -\frac{(x-y)^2\sqrt{\nu(x+1)(1-\nu x)}}{(y-1)^2R^2} \\
C_{2021} = -C_{3232} = C_{4242} &= -\frac{(x-y)^2(\nu+1)}{4(y-1)^2R^2} \\
C_{3031} = C_{4041} &= \frac{(x-y)^2(2\nu x+\nu-1)}{4(y-1)^2R^2} \\
C_{4343} &= -\frac{(x-y)^2(4\nu x+3\nu-1)}{4(y-1)^2R^2}
\end{aligned}
\tag{6.3}
$$

However all of these components are functionally dependent on any two independent components (say, for example, C_{1010} and C_{2021}). Thus $t_0 = 2$. Only components with zero boost weight do not vanish. This is to be expected since the metric is of type D and we used both WANDs. Spatial rotations about m_3 do not change the components of the Weyl tensor. To see why, consider the matrices defined by the constituents in Table 3.1:

$$
M_{ij} = C_{1i0j} = \begin{pmatrix}
-\frac{1}{4}\frac{(x-y)^2(\nu-1}{(y-1)^2R^2} & 0 & 0 \\
0 & \frac{1}{4}\frac{(x-y)^2(2\nu x+\nu-1)}{(y-1)^2R^2} & -\frac{1}{2}\frac{\sqrt{(1-\nu x)(\nu)(x+1)}(x-y)^2}{(y-1)^2R^2} \\
0 & \frac{1}{2}\frac{\sqrt{(1-\nu x)(\nu)(x+1)}(x-y)^2}{(y-1)^2R^2} & \frac{1}{4}\frac{(x-y)^2(2\nu x+\nu-1)}{(y-1)^2R^2}
\end{pmatrix}
\tag{6.4}
$$

$$
A_{ij} = C_{01ij} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \frac{\sqrt{(1-\nu x)(\nu)(x+1)}(x-y)^2}{(y-1)^2R^2} \\
0 & -\frac{\sqrt{(1-\nu x)(\nu)(x+1)}(x-y)^2}{(y-1)^2R^2} & 0
\end{pmatrix}
\tag{6.5}
$$

$$
S_{ij} = \begin{pmatrix}
\frac{2(x-y)^2(3\nu x+\nu-2)}{(y-1)^2R^2} & 0 & 0 \\
0 & \frac{(x-y)^2(5\nu x+\nu-4)}{(y-1)^2R^2} & 0 \\
0 & 0 & \frac{(x-y)^2(5\nu x+\nu-4)}{(y-1)^2R^2}
\end{pmatrix}
\tag{6.6}
$$

Since $A_{ij} = \epsilon_{ijk}w^k$, rotations about $w^2 = m^2$ do not change A_{ij}. And since $M_{ij} = -\frac{1}{2}S_{ij} - \frac{1}{6}R\delta_{ij} - \frac{1}{2}A_{ij}$, it follows that M_{ij} is unaffected by spatial rotations about m_3. Therefore $\dim(H_0) = 2$.

For the first order of the Cartan algorithm, we obtain 83 nonzero components of the first covariant derivative of the Weyl tensor. Though they are too numerous to list here, we note that they are all functionally dependent on the following two components:

$$C_{1010;0} = -\frac{3\sqrt{2}}{4}\frac{(2\nu x+\nu-1)(x-y)^{5/2}\sqrt{(1-\nu x)(\nu y-1)(x-1)}}{\sqrt{\nu+1}(y-1)^3R^3}$$

$$C_{1010;3} = -\frac{1}{2}\frac{\nu(6\nu x+\nu-5)(x-y)^{5/2}\sqrt{(x-1)(x+1)(1+y)}}{\sqrt{\nu+1}(y-1)^3R^3} \qquad (6.7)$$

We have displayed $C_{1010;0}$ and $C_{1010;3}$ since they can be used to build a dimensionless event horizon detecting Cartan invariant. However, these components are functionally dependent on any two functionally independent zeroth order invariants found earlier and thus $t_1 = 2$. We also now lose invariance under a boost. Using (5.1), we find that all Lorentz transformations affect $C_{abcd;e}$. Therefore $\dim(H_1) = 0$. The second order does not introduce any new functionally independent invariant. The secondary order Cartan invariants are important for classification of spacetime manifolds, but are not relevant for our interests (which are to detect event horizons) and thus are not shown. Thus $t_2 = 2$ and $\dim(H_2) = 0$ and the algorithm terminates.

Define $J_1 = C_{abcd}C^{abdc}$ and $J_2 = C_{abcd}C^{abef}C^{cd}{}_{ef}$. Using Theorem 3.5.1 with $W = \mathrm{d}J_1 \wedge \mathrm{d}J_2$, we get:

$$||W||^2(y-1)^{24}R^{24} = 2592\nu^2(1+y)(x+1)(x-1)^2(1-\nu x)(1-\nu y)(x-y)^{22}(\nu+1)^3 (8\nu^2x^2+8\nu^2x+3\nu^2-8\nu x-2\nu+3)^2 \qquad (6.8)$$

There are many solutions to $||W||^2 = 0$. However, the only solutions that are stationary horizons are $x = 1/\nu$ and $y = 1/\nu$. Note that $C_{1010;0}$ already detects both horizons. We get similar results when using SPIs. We could define J_1 and J_2 using the Weyl tensor in the original coordinate basis and then find $||W||^2$ to use theorem B.0.1. We get the same $||W||^2$ as in (6.8) and thus get the same horizon $x = y = 1/\nu$. However, the evaluation of J_1 and J_2 are much more difficult as the Weyl tensor is more complicated in the coordinate basis.

6.2 Kerr-NUTT-(Anti)-de Sitter Metric

The Kerr-NUTT-Ads metric is also of type D. Referring to [19], we will use the metric:

$$\begin{aligned}\mathrm{d}s^2 = &\frac{\mathrm{d}x^2}{Q_1} + \frac{\mathrm{d}y^2}{Q_2} + Q_1\left(\mathrm{d}\psi^0 + y^2\mathrm{d}\psi^1\right)^2 + Q_2\left(\mathrm{d}\psi^0 + x^2\mathrm{d}\psi^1\right)^2 \\ &+ \frac{c_0}{x^2y^2}\left[\mathrm{d}\psi^0 + \left(x^2+y^2\right)\mathrm{d}\psi^1 + x^2y^2\mathrm{d}\psi^2\right]^2\end{aligned} \tag{6.9}$$

where $Q_1 = X_1/\left(y^2 - x^2\right)$, $Q_2 = X_2/\left[x^2 - y^2\right]$, $X_1 = c_1x^2 + c_2x^4 - c_0/x^2 - 2b_1$, and $X_2 = c_1y^2 + c_2y^4 + c_0/y^2 - 2b_2$.. c_0, c_1, c_2, b_1, b_2 are free parameters, which are related to the rotation parameters a_1, a_2, the mass and NUT parameters M_1, M_2, and the cosmological constant parameter g as follows:

$$\begin{aligned}c_0 &= -a_1^2a_2^2 \\ c_1 &= 1 + g^2(a_1^2 + a_2^2) \\ c_2 &= -g^2 \\ b_\mu &= \frac{1}{2}(a_1^2 + a_2^2 + a_1^2a_2^2g^2) - M_\mu, \mu = 1, 2\end{aligned} \tag{6.10}$$

Note that this metric is Wick-rotated and thus the null vectors are complex. However the components of the curvature tensor and its covariant derivative are still real-valued.

First define an orthonormal frame:

$$\begin{aligned}e^0 &= \frac{\mathrm{d}x}{\sqrt{Q_1}} \\ e^1 &= \frac{\mathrm{d}y}{\sqrt{Q_2}} \\ e^2 &= \sqrt{Q_1}\left(\mathrm{d}\psi^0 + y^2\mathrm{d}\psi^1\right) \\ e^3 &= \sqrt{Q_4}\left(\mathrm{d}\psi^0 + x^2\mathrm{d}\psi^1\right) \\ e^4 &= \frac{\sqrt{c}}{xy}\left[\mathrm{d}\psi_0 + \left(x^2+y^2\right)\mathrm{d}\psi^1 + x^2y^2\mathrm{d}\psi^2\right]\end{aligned} \tag{6.11}$$

Then, according to [20] and [13] the frame with the two WANDs are simply $\{n, l, m^2, m^3, m^4\}$, where

$$n = \tfrac{i}{\sqrt{2Q_2}}(e^1 + ie^3), l = -i\sqrt{\tfrac{Q_2}{2}}(e^1 - ie^3),$$
$$m^2 = e^0, m^3 = e^2, m^4 = e^4.$$

We will use the 'FrameData' method for this metric. Using this frame, we get the following non-zero components of the Weyl tensor:

$$\begin{aligned} C_{1010} &= \frac{2(x^2+3y^2)(b_1-b_2)}{(x-y)^3(x+y)^3} \\ C_{1032} = \frac{1}{2}C_{2031} &= \frac{8ixy(b_1-b_2)}{(x-y)^3(x+y)^3} \\ C_{2021} = C_{3031} &= \frac{2(x^2+y^2)(b_1-b_2)}{(x-y)^3(x+y)^3} \\ C_{3232} &= -\frac{2(3x^2+y^2)(b_1-b_2)}{(x-y)^3(x+y)^3} \\ C_{4041} = -C_{4242} = -C_{4343} &= \frac{2(b_1-b_2)}{(x-y)^2(x+y)^2} \end{aligned} \tag{6.12}$$

Any of the above components are functionally dependent on a choice of two components. Thus $t_0 = 2$. All components are of boost weight zero and they do not change under a rotation about m_5. To see why, either we can use (5.1) or note the following matrices as defined by Table 3.1

$$M_{ij} = C_{1i0j} = \begin{pmatrix} \frac{2(x^2+y^2)(b_1-b_2)}{(x-y)^3(x+y)^3} & \frac{-4ixy(b_1-b_2)}{(x-y)^3(x+y)^3} & 0 \\ \frac{4ixy(b_1-b_2)}{(x-y)^3(x+y)^3} & \frac{2(x^2+y^2)(b_1-b_2)}{(x-y)^3(x+y)^3} & 0 \\ 0 & 0 & \frac{-2(b_1-b_2)}{(x-y)^3(x+y)^3} \end{pmatrix} \tag{6.13}$$

$$A_{ij} = C_{01ij} = \begin{pmatrix} 0 & \frac{8ixy(b_1-b_2)}{(x-y)^3(x+y)^3} & 0 \\ \frac{-8ixy(b_1-b_2)}{(x-y)^3(x+y)^3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.14}$$

$$S_{ij} = \begin{pmatrix} \frac{8(x^2+4y^2)(b_1-b_2)}{(x-y)^3(x+y)^3} & 0 & 0 \\ 0 & \frac{8(x^2+4y^2)(b_1-b_2)}{(x-y)^3(x+y)^3} & 0 \\ 0 & 0 & \frac{16(x^2+2y^2)(b_1-b_2)}{(x-y)^3(x+y)^3} \end{pmatrix} \tag{6.15}$$

Since $A_{ij} = \epsilon_{ijk}w^k$, rotations about $w^4 = m^4$ do not change A_{ij}. And since $M_{ij} = -\frac{1}{2}S_{ij} - \frac{1}{6}R\delta_{ij} - \frac{1}{2}A_{ij}$, it follows that M_{ij} is unaffected by spatial rotations about m_5. Thus $\dim(H_0) = 2$.

At first iteration, we have 105 components, but they are functionally dependent on the chosen two at zeroth order. Thus $t_1 = 2$. The following invariant detects an event horizon at $X_2 = 0$:

$$C_{1010;0} = -\frac{12i\sqrt{2}(b_1-b_2)yX_2(x^2+y^2)}{(x+y)^5(x-y)^5} \tag{6.16}$$

We lose independence under a boost and rotation about m_5 (this can be verifyed by (5.1)), by setting $C_{0101;1} = 1$ and $C_{1212;4} = 0$. Therefore $\dim(H_1) = 0$. The algorithm would carry on for one more iteration, since $t_1 = t_2 = 2$ and $\dim H_1 = \dim H_2 = 0$. However, we will omit these details.

Define $J_1 = C_{abcd}C^{abdc}$ and $J_2 = C_{abcd}C^{abef}C^{cd}{}_{ef}$. Using Theorem 3.5.1 with $W = \mathrm{d}J_1 \wedge \mathrm{d}J_2$, we get:

$$||W||^2 = \frac{44530220924928x^2y^2X_1X_2(b_1 - b_2)^{10}}{(x-y)^{30}(x+y)^{30}} \tag{6.17}$$

We detect the stationary horizon when $X_1 = 0$ or $X_2 = 0$, which are sixth degree polynomial equations in terms of x and y, respectively. Note that $C_{1010;0}$ already detects the horizon $X_2 = 0$. Like in the case of the singly rotating Myers-Perry metric, $||W||^2$ can be found using SPIs, but the form of the Weyl tensor in the coordinate basis make it computationally more difficult to find $||W||^2$ in this manner.

6.3 Reissner-Nordström-(Anti)-de Sitter Metric

From [21] the metric for the 5D Reissner-Nordström-(Anti)-de Sitter spacetime is:

$$ds^2 = f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 dS_3 \tag{6.18}$$

where dS_3 is the line element for the unit 3-sphere and $f(r)$ is the function

$$f(r) = 1 - \frac{2M}{r^2} - \frac{2\Lambda r^2}{12} - \frac{Q^2}{r^4}. \tag{6.19}$$

We use the following orthonormal frame:

$$\begin{gathered} e^0 = \sqrt{f(r)}dt, \quad e^1 = \sqrt{\tfrac{dr}{f(r)}}, \\ e^2 = rd\theta, \quad e^3 = r\sin(\theta)d\phi, \quad e^4 = r\sin(\theta)\sin(\phi)d\omega; \end{gathered} \tag{6.20}$$

from which we build the null frame:

$$\begin{gathered} l = \tfrac{1}{\sqrt{2}}(e^0 + e^1), \quad n = \tfrac{1}{\sqrt{2}}(e^0 - e^1) \\ m^2 = e^2, m^3 = e^3, m^4 = e^4. \end{gathered} \tag{6.21}$$

In this frame, l and n are WANDs. Thus the metric is of type D and hence the curvature tensor is invariant under a boost. Note that this metric is not a vacuum spacetime. Therefore, we must compute both the Weyl and Ricci tensors. The simplicity of the metric allows us to use the 'FrameData' method. The nonzero components are:

$$R_{01} = \frac{2(\Lambda r^6 - 6Q^2)}{3r^6}, \quad R_{ii} = \frac{2(\Lambda r^6 + 3Q^2)}{3r^6}, \quad i = 2, 3, 4 \tag{6.22}$$

$$C_{0101} = 3C_{0i1i} = -3C_{ijij} = \frac{3}{2}\frac{4Mr^2 - 5Q^2}{r^6}, \quad i, j = 2, 3, 4, i \neq j. \tag{6.23}$$

We can show that all these terms are functionally dependent on just R_{01}. Thus $t_0 = 1$. Referring to table 3.1, we find that the matrix A_{ij} is zero and M_{ij} is diagonal with $M_{ii} = C_{0i1j}$. We conclude that the Weyl tensor is also invariant under all three spatial rotations. A simple calculation shows that this is also true for the Ricci tensor, Thus dim $H_0 = 4$.

Continuing the Cartan algorithm, we compute the covariant derivative of the Ricci and Weyl tensor:

$$\begin{gathered} R_{01;0} = -4R_{0i;i} = \frac{8Q^2}{8Mr^2 - 15Q^2}, \quad R_{01;1} = -4R_{1i;i} = -\frac{6(8Mr^2 - 15Q^2)6r^4 f(r) Q^2}{r^18} \\ C_{0101;0} = -3C_{0i1i;0} = 3C_{ijij;0} = 1, \\ C_{0101;1} = 3C_{0i1i;1} = -3C_{ijij;1} = -\frac{3}{4}\frac{(8Mr^2 - 15Q^2)6r^4 f(r)}{r^{18}}, \\ C_{010i;i} = -2C_{0iij;j} = -\frac{2}{3}\frac{4Mr^2 - 5Q^2}{8Mr^2 - 15Q^2} \\ C_{011i;i} = 2C_{1iij;j} = \frac{(8Mr^2 - 15Q^2)6r^4 f(r)(4Mr^2 - 5Q^2)}{r^{18}} \end{gathered} \tag{6.24}$$

We lose invariance under a boost, however the spatial rotations are still in the isotropy group. Hence dim $H_1 = 3$. Also all of these terms are a function of R_{01} and hence $t_1 = 1$. At second order, it can be shown that dim $H_2 =$ dim $H_1 = 3$ and $t_2 = t_1 = 1$ and therefore the algorithm terminates. We will omit the nonzero second order Cartan invariants.

To apply theorem 3.5.1, we simply use $J = C_{abcd}C^{abcd}$. We get:

$$||\mathrm{J}||^2 = \frac{32}{3r^{30}}(-2r^4 + 12Mr^2 + 15Q^2)^2 \tag{6.25}$$
$$(-2r^4 + 24Mr^2 + 45Q^2)^2(r^6\Lambda - 6r^4 + 12Mr^4 + 6Q^2)$$

We thus find three candidates for stationary horizons: $-2r^4 + 12Mr^2 + 15Q^2 = 0$, $-2r^4 + 24Mr^2 + 45Q^2 = 0$, and $r^6\Lambda - 6r^4 + 12Mr^4 + 6Q^2 = 0$ (or $f(r) = 0$). The stationary horizon is $f(r) = 0$. Many of the Cartan invariants found, such as $R_{01;1}$, detect this horizon. Finding the SPI that detects stationary horizons in this example takes less time than doing the same in the last two examples thanks to the symmetries in the Reissner-Nordström-(Anti)-de Sitter metric. But using Cartan invariants is still more efficient.

6.4 The Singly Rotating Black Ring (Static) Metric

The rotating black ring metric was discovered by [22]. However, we will refer to the form of the metric given by [18]:

$$\begin{aligned}\mathrm{d}s^2 = &-\frac{F(x)}{F(y)}\left(\mathrm{d}t + R\sqrt{\lambda\nu}\,(1+y)\,\mathrm{d}\psi\right)^2 \\ &+ \frac{R^2}{(x-y)^2}\left[-F(x)\left(G(y)\mathrm{d}\psi^2 + \frac{F(y)}{G(y)}\mathrm{d}y^2\right) + F(y)^2\left(\frac{\mathrm{d}x^2}{G(x)} + \frac{G(x)}{F(x)}\mathrm{d}\phi^2\right)\right]\end{aligned} \tag{6.26}$$

where $F(\xi) = 1 - \lambda\xi$ and $G(\xi) = (1-\xi^2)(1-\nu\xi)$. Note that this metric reduces to Myers-Perry when $\lambda = 1$. This metric is of type $\mathbf{I}_i$. The WANDs of the rotating black ring are much more complicated to find and beyond the scope of this thesis. However, [18] gives a procedure where we denote a null vector $l^a = (\alpha, \beta, \gamma, \delta, \epsilon)$ and then use the following equations to solve for α, β, γ, δ, and ϵ:

$$l_a l^a = 0 \tag{6.27}$$

$$l^b l^c l^{[e} C^{a]}{}_{bc}{}^{[d} l^{f]} = 0 \tag{6.28}$$

[18] obtained a set of polynomials in terns of α, β, γ, δ, and ϵ in which we can solve and store in Maple. Due to the computational complexity of the WANDs, we resort to the zweibein method to compute the Weyl tensor. The components of the Weyl tensor are also large and are thus not presented in this thesis.

For illustration here, we will only consider the static case $\nu = 0$. In this case, $\epsilon = 0$ and β and γ are related by [18]:

$$\begin{aligned} &\lambda(x^2-1)(\lambda x-1)^2\gamma^2 + \lambda(y^2-1)(\lambda y-1)(\lambda x-1)\beta^2 \\ &\qquad + \lambda\beta\gamma\{\lambda(x+y)\left[\lambda x(xy-1)+1-x^2\right] + (x-y)^2(2-\lambda^2) \\ &\qquad + 2(\lambda x-1) + 2xy(1-\lambda y)\} \end{aligned} \tag{6.29}$$

α and δ are then given by:

$$\alpha^2 = \frac{R^2 y(\lambda y-1)^2\left[\gamma(\lambda x-1)+\beta(1-\lambda y)\right]}{(x-y)^2(\lambda x-1)(y^2-1)} \tag{6.30}$$

$$\delta^2 = \frac{\beta(1-\lambda x)\left[\gamma(x^2-1)+\beta(1-y^2)\right]}{(y^2-1)(x^2-1)^2} \tag{6.31}$$

Without loss of generality, we can set $\beta = 1$ and use (6.29) to solve for γ. We then use (6.30) and (6.31) to find α and δ. Let $L = (\alpha, 1, \gamma, \delta, 0)$ and $N = (\alpha, 1, \gamma, -\delta, 0)$. By choosing a null frame with $l \sim L$ and $n \sim N$, we find that all components of the Weyl tensor with b.w. ± 2 are zero. This confirms that the static black ring is of type I_i.

Comparing our results with Table 3.1, in our case the matrix A_{ij} is zero and the matrix M_{ij} is diagonal. So at first glance it seems that the Weyl tensor has significant rotation isotropy. However the spatial rotations affect the forms of C_{010i} and C_{101i} and we can choose rotations so that $C_{0102} = -C_{1012}$, $C_{0103} = C_{1013}$, and $C_{0104} = C_{1014} = 0$. Hence there is no isotropy. Thus $\dim(H_0) = 0$ and $t_0 = 2$. The Cartan algorithm thus terminates at the first order.

As before, we define $J_1 = C_{abcd}C^{abcd}$ and $J_2 = C_{abcd}C^{abef}C^{cd}{}_{ef}$. We then calculate $||W||^2 = ||\mathrm{d}J_1 \wedge \mathrm{d}J_2||^2 = 0$. The result is too complex to explicitly show here, but the solutions to $||W||^2 = 0$ are:

$$x = y \tag{6.32}$$

$$x = 1 \tag{6.33}$$

$$y = 1 \tag{6.34}$$

$$y = \infty \tag{6.35}$$

$$\begin{aligned} 0 = {} & 2\lambda^4 x^5 y - \lambda^4 x^4 y^2 - 2\lambda^4 x^4 - \lambda^4 x^2 y^2 - 2\lambda^3 x^5 - 5\lambda^3 x^4 y + 4\lambda^3 x^3 y^2 - \lambda^4 x^2 + \lambda^4 y^2 \\ & + 7\lambda^3 x^3 + 3\lambda^3 x^2 y + \lambda^3 x y^2 + 8L^2 x^4 - 5\lambda^2 x^2 y^2 + 2\lambda^3 x - 2\lambda^3 y - 9\lambda^2 x^2 - 4\lambda^2 xy \\ & - 2\lambda^2 y^2 - 11\lambda x^3 + 7\lambda x^2 y + 3\lambda x y^2 + 4\lambda x + 5\lambda y + 6x^2 - 6xy - 2 \end{aligned} \tag{6.36}$$

$$0 = 2\lambda x^2 y - \lambda x y^2 + \lambda x - 2\lambda y - 2x^2 + y^2 + 1 \tag{6.37}$$

These thus identify the possible candidates for stationary horizons. The correct horizon is $y = \infty$. This is equivalent the Myers-Perry example when $\nu \to \infty$.

The method using SPIs to locate the stationary horizon actually give the same answer in almost the same amount of computation time in this example. Thus there is no advantage in using either SPIs or Cartan invariants to locate stationary horizons in this example.

Chapter 7

Conclusion

We find that the 5D Cartan algorithm combined with Theorem 3.5.1 and the algebraic classification of the Weyl tensor by its b.w. decomposition is a very effective way to find and detect the candidates for stationary horizon of 5D black holes. For more complex black holes, the computations of the WANDs and the corresponding components of the Weyl tensor in this frame can be challenging. We have used Maple 2016 to perform our calculations. Further exploration is required to determine if computations of the 5D Cartan algorithm are more efficient using other computer software or by using the 'GRTensor' package for Maple 13. Further exploration is also required to determine if any other algebraic classifications mentioned in section 3.6 give simpler results for finding stationary horizons.

The method for finding event horizons can be extended to higher dimensions, but the computations become even more challenging. Every time the dimension increases by one, there is more isotropy to take into account. For example, for 6D spacetimes, the Lorentz group is 21 dimensional. The Cartan algorithm does apply to any dimension and the algebraic classification of the Weyl tensor by its b.w. decomposition has a simple extension.

We would also like to look at a procedure for finding event horizons of non-stationary black holes. The methods in this thesis could be used to find approximate locations of quasi-stationary horizons, but theorem 3.5.1 does not work for more general event horizons.

Appendix A

Tensors

For this section, we will assume that V, is a vector space of dimension n. Recall that if W is also a vector space, then $Lin(V, W)$ is the vector space of all linear functions from V to W. In particular, we define the space $V^* = Lin(V, \mathbb{R})$ the *dual space of* V. Given an (ordered) basis $\{v_a\} \subset V$, there exists a unique basis $\{\theta^a\} \subset V^*$, called the *dual basis*, such that $\theta^a(v_b) = \delta^a_b$. The *double dual of* V is $V^{**} = Lin(V^*, \mathbb{R})$. Since V is of finite dimension, the *double dual isomorphism theorem* says that there is a natural isomorphism from V^{**} to V and hence we can assume that $V^{**} = V$.

Definition A.0.1. *A* ***rank*** (r, s) ***tensor*** *is a map* $T : \prod_{i=0}^{r} V^* \times \prod_{i=0}^{s} V \to \mathbb{R}$ *with linearity in each argument. The set of rank* (r, s) *tensors is denoted as* $\mathcal{T}_{(r,s)}(V)$*. We also define* $\mathcal{T}_{(0,0)}(V) = \mathbb{R}$ *for reasons that will be clear shortly.*

Some special cases:

- $\mathcal{T}_{(0,1)}(V) = V^*$.
- By the double dual isomorphism theorem, $\mathcal{T}_{(1,0)}(V) = V$.
- $\mathcal{T}_{(1,1)}(V)$ is isomorphic to $Lin(V^*, V^*)$. To see why, note that if $\omega \in V^*$ and $T \in \mathcal{T}_{(1,1)}(V)$, we have a new map $\omega T \in V^*$ def by $\omega T : v \mapsto T(\omega, v)$. By the double dual isomorphism, we can similarly show that $\mathcal{T}_{(1,1)}(V)$ is isomorphic to $Lin(V, V)$.

We will now define six operations on tensors. Note that the first two operations tells us that $\mathcal{T}_{(r,s)}(V)$ is a vector space.

Addition of Tensors: If $S, T \in \mathcal{T}_{(r,s)}(V)$, we define $S + T \in \mathcal{T}_{(r,s)}(V)$ by $(S + T)(\omega^1, \ldots \omega^r, u_1, \ldots u_s) = S(\omega^1, \ldots \omega^r, u_1, \ldots u_s) + T(\omega^1, \ldots \omega^r, u_1, \ldots u_s)$ where $\omega^1, \ldots \omega^r \in V^*$ and $u_1, \ldots u_s \in V$.

Scalar Multiplication: If $T \in \mathcal{T}_{(r,s)}(V)$ and $a \in \mathbb{R}$, we define $aT \in \mathcal{T}_{(r,s)}(V)$ by $(aT)(\omega^1, \ldots \omega^r, u_1, \ldots u_s) = aT(\omega^1, \ldots \omega^r, u_1, \ldots u_s)$ where $\omega^1, \ldots \omega^r \in V^*$ and $u_1, \ldots u_s \in V$.

Tensor Product: If $S \in \mathcal{T}_{(r,s)}(V)$ and $T \in \mathcal{T}_{(r',s')}(V)$, we define $S \otimes T \in \mathcal{T}_{(r+r',s+s')}(V)$ by:

$$\begin{aligned}(S \otimes T)(\omega^1, \ldots, \omega^r, \bar{\omega}^1, \ldots, \bar{\omega}^{r'}, u_1, \ldots, u_s, \bar{u}_1, \ldots, \bar{u}_{s'})\\ = S(\omega^1, \ldots, \omega^r, u_1, \ldots, u_s)T(\bar{\omega}^1, \ldots, \bar{\omega}^{r'}, \bar{u}_1, \ldots, \bar{u}_{s'})\end{aligned}$$

where $\omega^1, \ldots \omega^r, \bar{\omega}^1, \ldots, \bar{\omega}^{r'} \in V^*$ and $u_1, \ldots u_s, \bar{u}_1, \ldots, \bar{u}_{s'} \in V$. Note that $a \otimes T = aT$ for any $a \in \mathbb{R}$.

Proposition A.0.2. *Let S, T, U be tensors of any rank and $a \in \mathbb{R}$. The following are true when well-defined:*

1. $(S + T) \otimes U = S \otimes U + T \otimes U$
2. $S \otimes (T + U) = S \otimes T + S \otimes U$
3. $(aS) \otimes T = S \otimes (aT) = a(S \otimes T)$
4. $(S \otimes T) \otimes U = S \otimes (T \otimes U)$ *(Tensor Product is associative)*

Property 4 of proposition A.0.2 gives us the freedom to write the tensor product of multiple tensors $T_1, \ldots, T_n$ as $T_1 \otimes T_2 \otimes \cdots \otimes T_n$ without parentheses. Note that tensor products are NOT in general commutative (i.e. $S \otimes T \neq T \otimes S$).

We can use tensor products to generate a basis for $\mathcal{T}_{(r,s)}(V)$.

Theorem A.0.3. *Let $\{v_a\} \subset V$ be a basis and $\{\theta^a\} \subset V^*$ the dual basis. Then $\{v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes \theta^{j_1} \otimes \cdots \otimes \theta^{j_s}\}$ is a basis of $\mathcal{T}_{(r,s)}(V)$. Hence* $\dim(\mathcal{T}_{(r,s)}(V)) = n^{r+s}$.

Proof. First, lets us check for linear independence. Assume that $0 = A^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} v_{i_1} \otimes$

$\cdots \otimes v_{i_r} \otimes \theta^{j_1} \otimes \cdots \otimes \theta^{j_s}$. Then:

$$\begin{aligned}
0 &= 0(\theta^{k_1}, \ldots, \theta^{k_r}, v_{l_1}, \ldots, v_{l_s}) \\
&= (A^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes \theta^{j_1} \otimes \cdots \otimes \theta^{j_s})(\theta^{k_1}, \ldots, \theta^{k_r}, v_{l_1}, \ldots, v_{l_s}) \\
&= A^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} \theta^{k_1}(v_{i_1}) \ldots \theta^{k_r}(v_{i_r}) \theta^{j_1}(v_{l_1}) \ldots \theta^{j_s}(v_{l_s}) \\
&= A^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} \delta^{k_1}_{i_1} \ldots \delta^{k_r}_{i_r} \delta^{j_1}_{l_1} \ldots \delta^{j_s}_{l_s} \\
0 &= A^{k_1 \ldots k_r}{}_{l_1 \ldots l_s}
\end{aligned}$$

Now check that $\{v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes \theta^{j_1} \otimes \cdots \otimes \theta^{j_s}\}$ spans $\mathcal{T}_{(r,s)}(V)$. Let $T \in \mathcal{T}_{(r,s)}(V)$. Define: $T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} = T(\theta^{i_1}, \ldots, \theta^{i_r}, v_{j_1}, \ldots, v_{j_s})$, called the *components of T relative to* $\{v_a\}$. Claim:

$$T = T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes \theta^{j_1} \otimes \cdots \otimes \theta^{j_s}$$

Verifying this claim is easy. Simply evaluate T at one of the basis:

$$\begin{aligned}
T^{k_1 \ldots k_r}{}_{l_1 \ldots l_s} &= T(\theta^{k_1}, \ldots, \theta^{k_r}, v_{l_1}, \ldots, v_{l_s}) \\
&= (T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes \theta^{j_1} \otimes \cdots \otimes \theta^{j_s})((\theta^{k_1}, \ldots, \theta^{k_r}, v_{l_1}, \ldots, v_{l_s}) \\
&= T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} \delta^{k_1}_{i_1} \ldots \delta^{k_r}_{i_r} \delta^{j_1}_{l_1} \ldots \delta^{j_s}_{l_s} \\
&= T^{k_1 \ldots k_r}{}_{l_1 \ldots l_s}
\end{aligned}$$

□

Thus a tensor is uniquely defined by specifying its components relative to a basis in V. If $T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s}$ and $T^{k_1 \ldots k_r}{}_{l_1 \ldots l_s}$ are the components of a tensor $T \in \mathcal{T}_{(r,s)}(V)$ relative to bases $\{v_a\}$ and $\{\bar{v}_a\}$ respectively and $\bar{v}_b = A^a_b v_a$, then:

$$T^{k_1 \ldots k_r}{}_{l_1 \ldots l_s} = T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} A^{k_1}_{i_1} \ldots A^{k_r}_{i_r} (A^{-1})^{j_1}_{l_1} \ldots (A^{-1})^{j_s}_{l_s}$$

For this reason, in physics and relativity, tensors are identified by their components relative to a given basis. In this text, we will often do so as well. We can express the previous three tensor operations in terms of the components relative to a given basis of V:

- *Addition*: $(S+T)^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} = S^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} + T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s}$

- *Scalar Multiplication*: $(aT)^{i_1 \dots i_r}{}_{j_1 \dots j_s} = aT^{i_1 \dots i_r}{}_{j_1 \dots j_s}$

- *Tensor Product*: $(S \otimes T)^{i_1 \dots i_r k_1 \dots k_{r'}}{}_{j_1 \dots j_s l_1 \dots l_{s'}} = S^{i_1 \dots i_r}{}_{j_1 \dots j_s} T^{k_1 \dots k_{r'}}{}_{l_1 \dots l_{s'}}$

Contraction: It is easiest to define a contraction relative to a basis of V, and then show that it is a basis-independent operation. Let $\{v_a\} \subset V$ be a basis, $T \in \mathcal{T}_{(r,s)}(V)$, $i = 1, \dots, r$, and $j = 1, \dots, s$. We define $\mathfrak{C}_{(i,j)}(T) \in \mathcal{T}_{(r-1,s-1)}(V)$ by:

$$\begin{aligned}
&\mathfrak{C}_{(i,j)}(T)(\omega^1, \dots \omega^{i-1}, \omega^{i+1}, \dots \omega^r, u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_s) \\
&\qquad = T(\omega^1, \dots \omega^{i-1}, \theta^k, \omega^{i+1}, \dots \omega^r, u_1, \dots, u_{j-1}, v_k, u_{j+1}, \dots, u_s)
\end{aligned}$$

where $\omega^1, \dots \omega^{i-1}, \omega^{i+1}, \dots \omega^r \in V^*$ and $u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_s \in V$. Now let $\{\bar{v}_a\} \in V$ be another basis and $\{\bar{\theta}^a\}$ be its dual. Let $\bar{v}_b = A^a_b v_a$ and thus $\bar{\theta}^b = (A^{-1})^a_b \theta^a$. Then:

$$\begin{aligned}
&T(\omega^1, \dots \omega^{i-1}, \bar{\theta}^b, \omega^{i+1}, \dots \omega^r, u_1, \dots, u_{j-1}, \bar{v}_b, u_{j+1}, \dots, u_s) \\
&\quad = T(\omega^1, \dots \omega^{i-1}, (A^{-1})^b_{a'} \theta^{a'}, \omega^{i+1}, \dots \omega^r, u_1, \dots, u_{j-1}, A^a_b v_a, u_{j+1}, \dots, u_s) \\
&\quad = (A^{-1})^b_{a'} A^a_b T(\omega^1, \dots \omega^{i-1}, \theta^{a'}, \omega^{i+1}, \dots \omega^r, u_1, \dots, u_{j-1}, v_a, u_{j+1}, \dots, u_s) \\
&\quad = \delta^a_{a'} T(\omega^1, \dots \omega^{i-1}, \theta^{a'}, \omega^{i+1}, \dots \omega^r, u_1, \dots, u_{j-1}, v_a, u_{j+1}, \dots, u_s) \\
&\quad = T(\omega^1, \dots \omega^{i-1}, \theta^a, \omega^{i+1}, \dots \omega^r, u_1, \dots, u_{j-1}, v_a, u_{j+1}, \dots, u_s) \\
&\quad = \mathfrak{C}_{(i,j)}(T)(\omega^1, \dots \omega^{i-1}, \omega^{i+1}, \dots \omega^r, u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_s)
\end{aligned}$$

Thus contractions are independent of the basis used. The components of a contraction relative to a given basis are:

$$(\mathfrak{C}_{(\alpha,\beta)}(T))^{i_1 \dots i_{\alpha-1} i_{\alpha+1} \dots i_r}{}_{j_1 \dots j_{\beta-1} j_{\beta+1} \dots j_s} = T^{i_1 \dots i_{\alpha-1} \gamma i_{\alpha+1} \dots i_r}{}_{j_1 \dots j_{\beta-1} \gamma j_{\beta+1} \dots j_s}$$

Note that if $A \in \mathcal{T}_{(1,1)}(V)$, then $\mathfrak{C}_{(1,1)}(A) \in \mathcal{T}_{(0,0)}(V)$ with the components given by A^k_k. This is why $\mathcal{T}_{(0,0)}(V) = \mathbb{R}$.

Proposition A.0.4. *Let $S, T \in \mathcal{T}_{(r,s)}(V)$, $a \in \mathbb{R}$, $i = 1, \dots, r$, and $j = 1, \dots, s$. Then $\mathfrak{C}_{(i,j)}(aS + T) = a\mathfrak{C}_{(i,j)}(S) + \mathfrak{C}_{(i,j)}(T)$.*

Using tensor products and contractions, we can make sense of what it means to contract two tensors together. For example, if $A^{ab}{}_c$ and $B^{de}{}_{fgh}$ are the components of two tensors, then $A^{a\alpha}{}_{\beta}B^{\beta b}{}_{cd\alpha}$ are the components of the tensor $\mathfrak{C}_{(3,1)}\left(\mathfrak{C}_{(2,4)}\left(A \otimes B\right)\right)$. A consequence of this is the following: If $u = u^a v_a \in V$ and $\omega = \omega_a \theta^a \in V^*$, then $\omega(u) = u^a \omega_a = \mathfrak{C}_{(1,1)}\left(u \otimes \theta\right)$. Thus evaluating ω at u is a special case of a contraction of tensors.

Symmetrization: Recall from algebra that a *permutation of* n *elements* is an element of the set $S_n = \{\tau : \{1, \ldots, n\} \to \{1, \ldots, n\} | \tau \text{ is a bijection}\}$. If $i, j = 1, \ldots, n$ and $i \neq j$, define a *transposition of* n *elements* $(i, j) \in S_n$ by:

$$(i,j)(x) = \begin{cases} j & , x = i \\ i & , x = j \\ x & , \text{otherwise} \end{cases}$$

Every permutation of n can be written as a finite composition of transpositions of n elements. This composition is not unique but the number of compositions will always be an even or odd number. Thus we can define for any $\tau \in S_n$:

$$sign(\tau) = \begin{cases} 1 & , \tau \text{ is a composition of an even number of transpositions} \\ -1 & , \tau \text{ is a composition of an odd number of transpositions} \end{cases}$$

We can now define the symmetrisation of a tensor. Let $T \in \mathcal{T}_{(0,r)}\left(V\right)$. We define $\tilde{S}T \in \mathcal{T}_{(0,r)}\left(V\right)$ by $((\tilde{S}T)(u_1, \ldots u_r) = \frac{1}{r!}\sum_{\tau \in S_r} T(u_{\tau(1)}, \ldots u_{\tau(r)})$. We use the following notation to denote the components:

$$T_{(i_1 \ldots i_r)} = (\tilde{S}T)_{i_1 \ldots i_r} = \frac{1}{r!}\sum_{\tau \in S_r} T_{i_{\tau(1)} \ldots i_{\tau(r)}}$$

We can of course define a symmertrization on some of the indices and on tensors of arbitrary order. This is easiest to define using components. As an example, if $T^{abd}{}_{de}$ is a tensor, then $T^{a(bc)}{}_{de}$ means $\frac{1}{2}(T^{abc}{}_{de} + T^{acb}{}_{de})$.

Antisymmetrization: If $T \in \mathcal{T}_{(0,r)}\left(V\right)$, let $\tilde{A}T \in \mathcal{T}_{(0,r)}\left(V\right)$ by $((\tilde{A}T)(u_1, \ldots u_r) = \frac{1}{r!}\sum_{\tau \in S_r} sign(\tau)T(u_{\tau(1)}, \ldots u_{\tau(r)})$. We use the following notation to denote the components:

$$T_{[i_1 \ldots i_r]} = (\tilde{A}T)_{i_1 \ldots i_r} = \frac{1}{r!}\sum_{\tau \in S_r} sign(\tau) T_{i_{\tau(1)} \ldots i_{\tau(r)}}$$

We can define the antisymmetrisation of an arbitrary tensor in a similar way to the symmetrisation and we can choose to only apply the antisymmetrisation on some of the indices.

Proposition A.0.5. *Let T, T' be tensors and $a \in \mathbb{R}$. The following are true when well-defined:*

1. $\tilde{S}(aT + T') = a\tilde{S}T + \tilde{S}T'$
2. $\tilde{A}(aT + T') = a\tilde{A}T + \tilde{A}T'$
3. $\tilde{S}(\tilde{S}T) = \tilde{S}T$
4. $\tilde{A}(\tilde{A}T) = \tilde{A}T$
5. $\tilde{S}(\tilde{A}T) = \tilde{A}(\tilde{S}T) = 0$

A tensor T is called **totally-symmetric** if $\tilde{S}T = T$ and **totally-antisymmetric** if $\tilde{A}T = T$.

For $T \in \mathcal{T}_{(0,2)}(V)$, we can represent the components of T as an $n \times n$ matrix (n is the dimension of the vector space V). Denote this matrix as $[T_{ij}]$ where T_{ij} are the components relative to some basis in V. If $\tilde{S}T = T$, then $T_{ij} = T_{ji}$ and hence $[T_{ij}]$ is a symmetric matrix. Simmilarly, if $\tilde{A}T = T$, then $T_{ij} = -T_{ji}$ and hence $[T_{ij}]$ is a skew-symmetric matrix and hence the diagonal entries are zero.

Appendix B

Scalar Polynomial Invariants

In 1869, Christoffel showed that any scalar function on a n-dimensional Riemannian (or pseudo-Riemannian) manifold (M, g) constructed from the metric g must be a function of $R_{abcd}, R_{abcd;e}$ and higher order covariant derivatives [23]. The simplest of such scalar functions are scalar polynomial invariants. The **scalar polynomial invariance** (or SPI) of a given spacetime metric g is the set of functions generated by operations on (contractions of) the curvature tensors (such as the Riemann or the Weyl tensors) such as $R_{ab}R^{ab}, C_{abcd}C^{abef}C^{cd}_{ef}$, $R_{ab;c}R^{ab;c}, C_{abcd;e}C^{abcd;e}$, etc.

Let M be of dimension n. The number of functionally independent SPIs is n. However, the number of algebraically independent SPIs (i.e; SPIs not satisfying any polynomial relation) constructed from the metric and its derivatives up to order p is [23]:

$$N(n,p) = \begin{cases} 0 & \text{if } p = 0 \text{ or } 1 \\ \frac{n(n+1)(n+p)!}{2n!p!} - \frac{(n+p+1)!}{(n-1)!(p+1)!} + n & \text{if } p \geq 2 \end{cases} \tag{B.1}$$

SPIs can be used to find the event horizon of a black hole. [24] provides a formula that can be used to calculate a new SPI, W, from the wedge product of n gradients of functionally independent SPIs (where n is the local cohomogeneity of the metric). The norm of W will vanish at the stationary horizons.

Theorem B.0.1. *Let (M, g) be a spacetime manifold with a local cohomogeneity n and contains a stationary horizon. Let $S^{(i)}$, $i = 1, ..., n$ be the functionally independent SPIs of M and $\mathrm{d}S^{(i)}$ is well-defined. If $W = \mathrm{d}S^{(1)} \wedge \mathrm{d}S^{(2)} \wedge \cdots \wedge \mathrm{d}S^{(n)}$, then at the stationary horizon:*

$$||W||^2 \equiv \frac{1}{n!}\delta^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_n} g^{\beta_1\gamma_1}\ldots g^{\beta_n\gamma_n} \times S^{(1)}_{;\alpha_1}\ldots S^{(n)}_{;\alpha_n} S^{(1)}_{;\gamma_1}\ldots S^{(n)}_{;\gamma_n} = 0 \tag{B.2}$$

SPIs are a good way to find the event horizons of a black hole since they are conceptually simple. However they do not necessarily uniquely characterize the spacetime

[25] and, more importantly, they are a challenge to compute (even for computers). Unless the spacetime metric is very simple or the dimension is small, finding event horizons using SPIs is not feasible since the SPIs can be very large polynomials. Therefore, we resort to Cartan invariants, found by applying the Cartan algorithm.

Appendix C

Test for Functional Independence

Before we discuss functional dependence, we need a preliminary fact:

Theorem C.0.1. *Let V be a n-dim vector space and let $\theta_1, \ldots, \theta_r \in V^*$ ($r \leq n$). Then $\theta_1, \ldots, \theta_r$ are linearly dependent if and only if $\theta_1 \wedge \cdots \wedge \theta_r = 0$.*

Proof. ($\Rightarrow$) Assume, without loss of generality, that $\theta_1 = \sum_{i=2}^{r} a^i \theta_i$. Then $\theta_1 \wedge \cdots \wedge \theta_r = a^2\theta_2 \wedge \theta_2 \wedge \cdots \wedge \theta_r + \cdots + a^r\theta_r \wedge \theta_2 \wedge \cdots \wedge \theta_r = 0 + \cdots + 0 = 0$.

($\Leftarrow$) We will prove the converse by proving that if $\theta_1, \ldots, \theta_r$ are linearly independent then $\theta_1 \wedge \cdots \wedge \theta_r \neq 0$. Since V^* is of dimension n, we can find $\theta_{r+1}, \ldots, \theta_n$ such that $\{\theta_1, \ldots, \theta_n\} \subset V^*$ is a basis. Thus $S = \{\theta_{i_1} \wedge \cdots \wedge \theta_{i_r}\}_{0<i_1<i_2<\cdots<i_r \leq n} \subset \Omega^1(V)$ a basis. But $\theta_1 \wedge \cdots \wedge \theta_r \in S$ thus $\theta_1 \wedge \cdots \wedge \theta_r \neq 0$. ☐

We define local functional dependence as follows:

Definition C.0.2. *Let M be a n-dimensional manifold and $p \in M$. We say that a collection of r functions $f^1, \ldots, f^r \in C^\infty(M)$ are **functionally dependent near** p if there exist a open neighborhood $p \in U \subset M$ and a map $H : \mathbb{R}^r \to \mathbb{R}$ such that $\forall q \in U$, $H(f^1(q), \ldots, f^r(q)) = 0$. If no such map H exists, we say that $f^1, \ldots, f^r$ are **functionally independent**.*

The following theorem informs us that local functional dependence of a set of functions is related to the local linear dependence of their differentials. The proof can be found in [26]:

Theorem C.0.3. *Let M be a n-dimensional manifold, $p \in M$, and $f^1, \ldots, f^r \in C^\infty(M)$. Then $f^1, \ldots, f^r$ are functionally dependent near p if and only if $(\mathrm{d}f^1)_p, \ldots (\mathrm{d}f^r)_p$ are linearly dependent.*

We can combine the two above theorems into one:

Theorem C.0.4. *Let M be a n-dimensional manifold, $p \in M$, and $f^1, \ldots, f^r \in C^\infty(M)$. Then $f^1, \ldots, f^r$ are functionally dependent near p if and only if $\mathrm{d}f^1 \wedge \cdots \wedge \mathrm{d}f^r = 0$ at p.*

Say you are given a list of smooth functions $g^1, \ldots, g^r \in C^\infty(M)$ and you want to reduce the list to only include functionally independent functions at any point $p \in M$. Define an empty list S and add g^1 to S. Then check if $\mathrm{d}g^1 \wedge \mathrm{d}g^2 = 0$. If false, include g^2 into S. Then check if $(\wedge_{f \in S} \mathrm{d}f) \wedge \mathrm{d}g^3 = 0$. If false, include g^3 into S. Repeat until all functions are considered. To summarize:

1. Insert g^1 into S.

2. For i from 2 to r do:

   (a) Evaluate $\omega = \wedge_{f \in S} \mathrm{d}f$

   (b) If $\omega \wedge \mathrm{d}g^i = 0$, insert g^i into S.

   end do.

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