INFINITY AND GENERALITY IN THE *TRACTATUS*

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ABSTRACT

There has been a good deal of controversy over the expressive completeness of Wittgenstein's operator N, presented in the *Tractatus*. James Connelly gives an explication of operator N in an attempt to dispel charges against its expressive capacities as the sole operator in Wittgenstein's proposed logical system. Connelly argues that a proper appreciation of infinity as it was then understood by Wittgenstein is fundamental to the exculpation of N. I evaluate Connelly's discussion and demonstration of actual as opposed to potential infinity. I then raise objections regarding Connelly's conflation of actual infinity and finitude, and his introduction and use of notational variables within propositional expressions. I present mathematical induction and recursion as alternative methods for containing infinity. I consider Wittgenstein's differentiation of logical form and logical structure, from which he attempts to, but cannot, justify semantic rule-formation based on syntactic rule-formation. I conclude that N is expressively and functionally incomplete.

LIST OF ABBREVIATIONS AND SYMBOLS USED

TLP	*Tractatus Logico-Philosophicus* (‘*Tractatus*’)
PL	Predicate Logic
UD	Universe of discourse
~	Negation
⊃	Material Conditional
&	Conjunction
v	Disjunction
≡	Material Biconditional
∀	Universal Quantifier
∃	Existential Quantifier
$\bar{p}$	Metavariable: represents all elementary propositions
ξ	Variable whose values are stipulated propositions
$\bar{\xi}$	Metavariable for all propositions of the above kind
ẍ	Geachian class-specifying variable
ÿ	Geachian class-specifying variable
ℕ	Denotes the set of natural numbers
∈	‘is a member of’
$\mathcal{P}$	Denotes a property of interest
ω	Omega: the smallest ordinal number
$\aleph_0$	Aleph null: the cardinality of a countably infinite set

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CHAPTER 1 INTRODUCTION

In his enigmatic and monumental work, the *Tractatus Logico-Philosophicus* (henceforth '*Tractatus*'), Ludwig Wittgenstein presents his logical operator N, over which much controversy has been generated. With the objective of laying out a logical system, the N operator is described at Proposition 6 *et passim* as the sole primitive operator in Wittgenstein's purportedly *expressively complete* logical system. This means that this single operator is able to express the form and content of all meaningful propositions, including general propositions, via its operative application. Some philosophers, including P. T. Geach and Robert Fogelin, have critically assessed the N operator as expressively incomplete, especially with respect to general propositions. They conclude that Wittgenstein overlooked the need for supplementary operators to work in concert with N, and that a class-forming method is necessary in order to fulfill its expressive capacities. But Wittgenstein explicitly proscribed both of these conditions, as James Connelly argues in his defense of operator N. Connelly provides an explication of N with the aim of showing that the charges against this logical operator by both Geach and Fogelin are unfounded. According to Connelly, they each fail in different ways to adhere to certain rules fundamental to a proper application of N. One of these rules is to treat infinity as a bound totality. Connelly urges that N is ultimately expressively complete, and the charges against it cannot be upheld. In what follows, I evaluate Connelly's defense of N, particularly his treatment of infinity and its role in the *Tractatus*.

I begin in Chapter 2 by presenting Wittgenstein's logical atomism. This includes such logical entities as objects, atomic propositions, and facts—the totality of which makes up the world. In Chapter 3 I discuss Connelly's interpretation of

Wittgenstein's notion of infinity in the *Tractatus*. Chapter 4 focuses on the mechanics of the N operator, given in symbolic notation at Proposition 6 as "the general form of a proposition" (TLP 6). I will explain the symbols, and how their respective roles bear largely in the general propositional form and the application of N. I reproduce the worries of both Geach and Fogelin regarding the expressive capacities of the N operator. In Chapter 5, I present my concerns about Connelly's account of infinity, arguing that it more aptly answers to a description of finitude. I then offer alternative techniques for defining and working with infinite series: mathematical induction and recursion. Chapter 6 reveals the logical quantifiers implicit in Connelly's account, which purports to be quantifier-free. In addition to this, I address several problems with Connelly's variable 'i'. Finally, in Chapter 7, I differentiate between logical form and logical structure. Though these terms are often used synonymously by logicians and philosophers, they are discussed discretely in the *Tractatus*; this reveals a problem in Wittgenstein's methodology for the establishment of syntactic and semantic rules in formal language construction. I end by determining, based on reservations over Connelly's account of infinity and Wittgenstein's problematic instructions for syntactic and semantic rule-forming, that N is ultimately expressively, and functionally, incomplete. This supports what has become the standard account of Wittgenstein's logical programme in the *Tractatus*.

CHAPTER 2 WITTGENSTEIN'S LOGICAL ATOMISM

2.1 The Aim of Austerity

The metaphysics of logical atomism is essentially a semantics for classical logic. Metaphysical simples and atomic facts fulfill certain requirements of classical logic, including the *law of the excluded middle* and the *independence thesis* (of elementary propositions). Briefly, the law of the excluded middle dictates that a proposition's truth-value must be either true or false. In classical logic, truth does not admit of any 'degree'. There are two, and only two, truth-valued possibilities, 'true' or 'false', of which a proposition must be one or the other (TLP 5.153). Though both classical logical laws feature prominently in Tractarian semantics, the independence thesis figures especially in James Connelly's (2017a) account of operator N given in Chapter 4. There has been much controversy and doubt about the N operator's expressive completeness; that is, its ability—as the one and only operator in Wittgenstein's logical programme—to successfully express *all* meaningful propositions.

Wittgenstein maintains, and the independence thesis demands, that the truth or falsity of one elementary proposition cannot bear on the truth or falsity of another—an elementary proposition's being logically independent guarantees that its truth-value will have no influence on the truth-value of another (TLP 4.211, 5.134). I will explicate this idea in greater detail in the upcoming discussion of elementary propositions, as well as the logical implication of propositional structure, as these are foundational pieces that bear largely on Connelly's instructions for the correct use of N, discussed in Chapter 4. Connelly claims that the underappreciation of these points, among others, led to the expository errors of both P. T. Geach and Robert Fogelin, who failed, in their own ways, to fully appreciate these fundamental Tractarian commitments.

One of the key aims of the *Tractatus* was to present a rigorously eliminativist, reductionist logical programme, and so jettison the *Platonic penumbra* from logical philosophy. In other words, Wittgenstein disliked, and thought unnecessary, the rather Platonic mathematical and logical posits that Gottlob Frege and Bertrand Russell employed in their logical atomist metaphysics. Both of these philosophers conceived of logical and mathematical notational devices as existent entities—abstract objects that, though non-physical (not comprised of matter), non-mental (not comprised of ideas or beliefs), and non-spatiotemporal (not situated in space and time), are nevertheless thought to exist. Contra Frege and Russell, Wittgenstein wanted to show that logic is exactingly simple—thus revealing the superfluity of positing universals, and dispatching with a perceived overabundance of logical constants, such as quantifiers and connectives. Indeed, at TLP 5.4541, Wittgenstein states that the realm which holds the answers to logical problems is "subject to the law: *Simplex sigillum veri*" (which translates to "*Simplicity is the sign of truth*.")

Thus, Wittgenstein purports to lay out a logical system which is able to completely express the whole of logical semantics; and he claims to show that all Platonic entities and logical constants are therefore eliminable in favour of, as we shall see, a single logical operator (N), a constituent of Wittgenstein's general form of a proposition: $[\bar{p}, (\bar{\xi}), N(\bar{\xi})]$ (TLP 6). The symbolism will be explained in Chapter 4. Indeed, "the description of the most general propositional form is the description of the *one and only* general primitive sign in logic" (TLP 5.472, emphasis added), namely, N. Some philosophers, such as Geach and Fogelin, have been critical of this move, since they think that the elimination of the conventional logical constants (&, ~, v, ⊃, ≡, =, ∀, ∃) would result in an unworkable logical programme. But Wittgenstein is clear in the preface that the *Tractatus* "is not a textbook" (TLP, p. 3). Its objective is not to present notational recommendations in place of the (already perfectly serviceable) classical logical connectives and quantifiers. Instead, Wittgenstein claimed to show that the entirety of logical syntax and semantics could be accounted for via the employment of a single, principal logical operator—specifically, the N operator. Chapter 4 focuses on Connelly's attempt to exculpate Wittgenstein's account by explaining the mechanics of N. I will present several objections to Connelly's treatment in Chapters 5-7.

According to Wittgenstein, there is only one, ultimate analysis of every molecular (or 'complex') proposition (TLP 3.25). That is, everything that is the case (the world), fully analyzed (decomposed), reduces to a unique set of logically elementary (atomic) propositions: hence, 'logical *atomism*'. As Wittgenstein states, the world is made up of facts, which are in turn ultimately made up of objects (TLP 1-2.01). This, very briefly stated, is the whole of Wittgenstein's ontology. Because of the well-known, characteristic complexity and starkness of the *Tractatus*, this ontology requires some unloading. To understand the *Tractatus* is to fully appreciate the logical atomist views which Wittgenstein held. I will here let Wittgenstein speak for himself: "The world is all that is the case […] the world divides into facts […] what is the case—a fact—is the existence of states of affairs. A state of affairs (a state of things) is a combination of objects (things)" (TLP 1, 1.2, 2-2.01). We have, crudely speaking, four 'levels' or 'kinds' of things, beginning with one single but very complex entity (the world), which divides and is further sub-divided first into facts, then atomic facts (states of affairs), and, finally, objects.

2.2 Tractarian Objects

Let us begin with objects, and reverse engineer Wittgenstein's account until we have 'rebuilt' *the world* from the bottom up. By 'world', Wittgenstein is not referring to 'the earth' or cosmopolitan globe, but rather the whole of reality. Tractarian objects are metaphysically simple, and are the referents of simple names. The term 'name' as employed in the *Tractatus* is quite different from its quotidian use. Normally, names denote persons, places, et cetera. Tractarian names uniquely denote simple objects, which are the most metaphysically simple kind of thing—absolutely lacking in complexity. Objects are, logically speaking, the most fundamental components of the world, as they "contain the possibility of all situations" (TLP 2.014), and "make up the substance of the world" (TLP 2.021). The nature of objects is not straightforward: "Objects are […] subsistent; their configuration is what is changing and unstable. The configuration of objects produces states of affairs" (TLP 2.0271-2.0272). In other words, when certain arrangements of objects produce states of affairs, they 'take on' a certain modally real status, in that they 'feature' in what is actually the case. Otherwise, they exist abstractly—subsist—in the realm of modal possibility. Metaphorically, if what is actually the case is an act being played out on a stage, the objects not featured in that scene are waiting in the wings. Their configuration, and their multiplicitous possible configurations, complete and exhaust every logically possible world. Any 'scene' that could be imagined is constrained by the (limited) possible combinations of these logical objects. We could not think of a 'scene' in which this was not the case—to do so would be to go beyond the limits of logic. As Wittgenstein writes in his introduction: "the aim of the [*Tractatus*] is to draw a limit to thought, or rather—not to thought, but to the expression of thoughts: for in order to be able to draw a limit to thought, we should have to find both sides of the limit thinkable (i.e. we should have to be able to think what cannot be thought" (TLP, p. 3). The idea of limits, their exact nature and how they are established, will feature with increasing importance throughout this project.

In order to fulfill certain Tractarian commitments, no kind of thing—even that which is claimed to be 'infinite' in size or amount—can be unending, un-measurable, and un-calculable. Rather, Wittgenstein's programme relies on the 'containment' of an infinity of objects. This claim invites two reactions: [i] this is a confused, untenable claim, since infinity is, by definition, endless; and [ii] this cannot be what Wittgenstein, arguably one of the great philosophical minds of the twentieth century, meant. But there is a wealth of textual evidence supporting the claim that infinity is complete and measurable. Consider the following passages, which suggest that the 'number' of objects is somehow contained, rather than endless: "if I know an object I also know *all its possible occurrences* in states of affairs […] A *new possibility cannot* be discovered later" (TLP 2.0123, emphasis added). Further, "if all objects are given, then at the same time all *possible* states of affairs are also given" (TLP 2.0124). What this means is that as objects combine to configure states of affairs, the number of combinatory possibilities must be somehow restricted. 'Restriction' here simply means that we have a method by which to measure, or calculate, every possibility of an infinite set (of objects). A clearer example may be the game of chess: there are an infinite number of ways in which a chess game can unfold, but a (finite) set of rules allows us to calculate every permissible (possible) move that can occur. It is in this way that we can say of a thing that it is both infinite and measurable, and this is precisely the way in which it is 'contained', or delimited. A new move in chess cannot be discovered; a new combination of objects likewise so.

This idea of a contained infinity has produced much confusion in the literature. Chapter 3 will focus on Connelly's (2017a) account of infinity, and Chapters 5 and 6 aim to dispute his claims while also providing an alternative approach. But for now, let us continue with Wittgenstein's logical atomism.

2.3 Elementary Propositions

When Wittgenstein states that "it is obvious that the analysis of propositions must bring us to elementary propositions" (TLP 4.221), he is saying that all of logic could be performed at the basic level of elementary propositions. These kinds of proposition are, in a sense, simple, i.e. not complex, not made up of parts. They can be written as Pa, Qb, Rc, and so on. P, Q, and R represent predicates; a, b, and c represent names. Semantically simple sentence letter 'p' can be thought of as shorthand for some Pa; q for Qb; 'r' for Rc. Yet, to say they are 'simple' is a bit of a misnomer, for "elementary propositions consist of names. Since, however, we are unable to give the number of names with different meanings, we are also unable to give the composition of elementary propositions" (TLP 5.55). So elementary (atomic) propositions, though constituted by numerous objects in configuration, are propositionally simple, as opposed to propositionally complex.

As molecules are comprised of atoms, so *molecular* propositions have *atomic* propositions as their constituents. Thus we can think of these descriptive terms, 'atomic' and 'molecular', quite literally. Molecular, or complex, propositions are of the following kind: ‘p & q’, ‘~(p v r) & ~(~q ⊃ r)’. It is easily seen that we get molecular propositions when semantically simple sentence letters are joined via sentential connectives:

Symbol:	&	v	~	⊃	≡
Meaning (in English):	‘*and*’	‘*or*’	‘*not*’	‘*if… then…*’	‘*if and only if*’ (‘*iff*’)

A molecular proposition is anything that has a truth-functional structure. A truth-function generates an output (truth-value) based on the truth-values of the inputs (logical arguments). Let us take an arbitrary molecular proposition to illustrate first logical implication, and second the end result of logical analysis.

The following declarative sentence ‘one or the other, or both’ (symbolically written as p v q) exemplifies *inclusive disjunction* in classical logic. p v q can also mean *exclusive* disjunction—which says ‘one or the other, and not both’. I will here be using ‘p v q’ in the exclusive sense. This says that if the truth-value of p is true, the truth-vale of q is false. Likewise, if p is false, then q is true. They cannot both be true, and neither can they both be false, so long as the molecular statement p v q is true. Thus, if the sentence p v q is true, and if p is true, then q must be false. Or if p is false, then q must be true. So p v q and p *logically imply* not-q (~q), and not-p (~p) and p v q *logically imply* q. The truth-value of the one has bearing on the other, given the truth of the compound statement.

A logical analysis performed on a molecular proposition will reveal its elementary components. Let us take the arbitrary complex proposition (p ⊃ q) & r. A logical analysis would reveal the elementary constituents p, q, and r. This is a very simple example, and our arrival at elementary propositions after logical analysis is fairly straightforward. What is less straightforward, and much disputed, is how Wittgenstein claims to deal with logical quantifiers treating of generality—*all* propositions. Thus we move our discussion from the propositional calculus, described above, to the predicate calculus.

The predicate calculus makes use of the above-mentioned symbols, in addition to quantifiers and variables. Predicate logic deals with universal quantification (∀ means 'all') and existential quantification (∃ means 'one', 'some', or 'there is at least one'). Wittgenstein has received much criticism from philosophers such as Geach and Fogelin for his elimination of quantifiers. They urge that no sole, logical operator could completely express all meaningful propositions, especially general propositions. But Russell states in the introduction to the *Tractatus* (TLP, xvi) that the treatment of quantified (general) propositions is *identical* to that of elementary propositions—from which Connelly concludes that this is precisely because the logical analysis performed on propositions containing quantifiers ultimately results in a (very) long truth-functional expansion. The nature of the expansion depends on its being universal, in which case the truth-functional expansion will consist of an infinitely long list of conjunctions—another way of saying 'this *and* this *and* this *and*…' or, simply, 'all'. Or, existentially quantified propositions are analogously expressed with a disjunctive truth-functional expansion—'this *or* this *or* this *or*…' or 'not all not'. I will give Connelly's explanation of how these truth-functional expansions work, and how they are constructed, in greater detail in Chapter 4. Chapters 5 and 6 will aim to show the problems with this account. For now, it is enough to say that, for Wittgenstein, the logical analysis of *every* proposition results in one complete, unique deconstruction, even for singly general propositions, e.g.

$(\exists x)Fx$, multiply general propositions, e.g. $(\exists x)(\exists y)Fxy$, and mixed multiply general propositions, e.g. $(\forall x)(\exists y)Fxy$.

Universal quantification, e.g. $(\forall x)Fx$, says that all x are F. Existentially quantified sentences, e.g. $(\exists x)Fx$, state that there exists at least one x such that x is F. Quantified statements can, according to Wittgenstein and as adopted by Connelly, be logically analyzed into their elementary constituents. But there is a mystery here: if we begin with two different statements, one universally quantified and one existentially quantified, and upon analysis we get Fx, Fx alone says nothing about which statement it originally belonged to. It can be quantified two different ways, as shown above. Fx does not determine the context in which it appears, so it seems as though the difference between universal and existential statements collapses. How, then, can analysis preserve the difference between these two—very different—statements? The solution depends on the treatment of infinity. This is the focus of Chapter 3.

2.4 FACTS AND THE WORLD

From the 'level' of elementary propositions, it is just a matter of the configuration of states of affairs (facts) and the construction of their totality that ultimately determines what is the case. A slightly parodic but helpful analogy is an astronomically large number of puzzle pieces that, assembled correctly, form a whole picture (of reality) as it is at time *t*. How this construction is allegedly possible becomes apparent in Chapter 4, in which is given Connelly's demonstration of how operator N functions.

CHAPTER 3 INFINITY

The *Tractatus* was motivated by a desire to eliminate Platonic entities and allegedly excessive and superfluous logical constants in favour of a more stark metaphysics. In keeping with this austerity, Wittgenstein sought to present a logical system that included *only one* main logical operator. This he presents at Proposition 6 *et passim*: “the general form of a truth-function is $[\bar{p}, (\bar{\xi}), N(\bar{\xi})]$. This is the general form of a proposition.” I will explicate the symbolism and function in detail in Chapter 4, but for now it is enough to know that ‘N’ is the single, principal operator in Wittgenstein’s logical programme. It is to be applied only to elementary propositions and, since it is the only logical operator in Wittgenstein’s system (he is clear on this point), it necessarily must be capable of replacing Russellian quantifiers.

At TLP 5.32, Wittgenstein says that “all truth-functions are the results of successive applications to elementary propositions of a *finite* number of truth-operations” (emphasis added). In other words, the N operator (henceforth sometimes referred to as ‘N’) is to be applied to a list of elementary (atomic) propositions; from here, the resulting truth-functions are ‘N-expressed’, which creates a new list of truth-functions, which is in turn subjected to N, and so on, until we come to the end of the N-expression. ‘*N-expressed*’ propositions refer to those propositions which would otherwise appear in combination with the conventional classical logical operators, since they are intended to be replaced by N. This, in short, is a very simple overview of how the N operator is to be employed. Through these successive iterations, N is claimed by Wittgenstein to be capable of expressing all propositions, without the aid of any auxiliary devices. But the obvious worry is N’s ability to completely express all propositions in an infinite domain without the supplementary aid of e.g. open sentences, quantifiers, or set-forming devices. Since N is to be deployed to account for all propositions and their subsequent truth-functions through a series of successive applications, one would first have to enumerate every relevant proposition, then apply N, then continue reapplying N until all propositions have been expressed. Connelly (2017a) argues that each of these steps necessitates an end point in each catalogue of propositions, because if N has not reached the end of the list of propositions and completed generating all of the truth-functions thereof—which would be impossible when dealing a list that did not end—one could never progress to the next iterative application of N to the resultant truth-functions. This chapter will serve to illustrate Connelly's views of infinity as operating in the *Tractatus*. In Chapters 5 and 6, I will observe several problems with this account, and offer alternative approaches.

Connelly argues that, crucial to the exculpation of N, which P. T. Geach and Robert Fogelin each criticize heavily, is a proper appreciation of W's conception of infinity as *actual* (complete), as opposed to *potential* (incomplete)—one point among several that led, it is claimed, to the expository errors of both of these philosophers. Infinity is most commonly and most obviously predicated of a thing that is unending. But Connelly insists that this is not how Wittgenstein conceived of it during his authorship of the *Tractatus*. Connelly (2017a) notes the high probability that Wittgenstein indirectly got his notion of infinity from Georg Cantor, via Bertrand Russell. Wittgenstein had studied closely under Russell; despite the overt criticisms Wittgenstein made of Russell and some of his ideas in Russell's *Principia Mathematica (*co-written with Alfred North Whitehead) throughout the *Tractatus*, Russell's ideas were tremendously impactful on Wittgenstein (Hintikka & Hintikka, 1986). Russell was, in turn, influenced by Cantor, a German mathematician who subscribed to the notion of many, discrete infinities that differed in size. A focal point of Cantor's work was the distinction between actual (*categorematic*, completed) infinities and absolute, or potential (*syncategorematic*, unending) infinities (Dauben, 1979). Cantor argued that there was nothing mathematically contradictory in the idea of completed infinities, despite the misgivings of doubters who noted that infinite numbers would not 'behave' the way finite numbers would, and that the two kinds were therefore incompatible (Dauben, 1979). This led to the development of transfinite arithmetic, which Cantor spearheaded. Noting the influence Cantor had on Russell, and Wittgenstein's Cambridge days under Russell's tutelage, it is probable that Wittgenstein was familiar with and endorsed the idea of actual infinities. Though he never explicitly states his endorsement, there is much textual evidence supporting this claim. However, I worry about Connelly's construal of the nature of actual, completed infinities, and what it is that renders them complete. For now, I explicate Connelly's interpretation, and express my worries in Chapter 5.

Let us consider the following claim by Wittgenstein:

> The solutions of the problems of logic must be simple, since they set the standard of simplicity. Men have always had an intuition that there must be a sphere in which the answers to questions are symmetrically united—a priori—into a <u>closed regular structure</u>. A <u>sphere</u> in which the proposition, *simplex sigillum veri*, is valid (TLP 5.4541, 2014, my emphasis underlined)

Thus, Connelly infers that in the domain of discourse (logical space), universally and existentially quantified propositions, truth-functionally expanded in their respective series of conjunctions or disjunctions, are enumerable within a 'closed regular sphere'. Wittgenstein describes logical space as an *infinite* whole (TLP 4.463) and also as a *limited* whole (TLP 6.45). Connelly concludes that logical space, as simultaneously infinite and limited, can be thus described as a bound—closed—

infinity. “The facts in logical space are the world” (TLP 1.13). These facts ‘fill’ the whole of logical space, in essence ‘completing the picture’ of reality. For “the world is determined by the facts, and by their being *all* the facts. The *totality* of facts determines what is the case, and also whatever is not the case” (TLP 1.11-1.12, emphasis added). How these facts come to be enumerable is made possible by infinity’s being so bound—for a *totality* of positive and negative facts (TLP 1.12, 2.06) would constitute a list which, Connelly claims, theoretically has a terminus.

If, as Wittgenstein believed, infinity is a bound totality, one could in principle enumerate all of the propositions in a given series. Thus, universal (∀) and existential (∃) quantifiers can be thought of, for Wittgenstein, as shorthand labels indicating truth-functional expansion (Connelly, 2017a, p. 10). Since the formal logical analogue of universal quantification is a conjunctive truth-functional expansion, a universally quantified proposition would be expanded as such: Fa & Fb & Fc &… Fi. Connelly (2017a) uses ‘Fi’ here to indicate ‘infinity’, which marks the terminal member in a propositional series. More will be said on this notational ‘i’ in the subsequent chapters, particularly Chapter 6. Existential quantification is truth-functionally expanded as an infinitely long, though terminating, list of disjunctions, enumerated in the following way: Fa v Fb or Fc v… Fi. Once the truth-functional expansion was complete, according to Connelly, one could in principle view every proposition included in the conjunctive or disjunctive series—for even though the series could be infinitely long, it would, per Connelly’s account of infinity, discontinue.

Connelly states that this is how, for Wittgenstein, the predicate calculus is supposed to reduce to the propositional calculus, as Russell notes in the introduction to the *Tractatus*. As Wittgenstein says, "suppose I am given *all* elementary propositions: then I can simply ask what propositions I can construct out of them. And there I have *all* propositions, and *that* <u>fixes their limits</u>" (TLP 4.51, my emphasis underlined). According to Connelly, even general propositions, once logically analyzed into their truth-functionally expanded form, purportedly constitute a delimited and thus completely 'enumerable' list of elementary propositions. Enumerability will be described in Chapter 5, along with a description which, is, in my opinion, more tractable and consistent with the standard view. Universal and existential quantification, apparently having been shown to be reducible to the propositional calculus, describe the infinite yet limited elementary propositions, combinations of which describe facts, which in turn make up what is the case. Since Wittgenstein claims that "the totality of existing states of affairs is the world" (TLP 2.04), Connelly holds that infinity, for Wittgenstein, must be bound. While I agree that Wittgenstein's account necessitates a kind of contained infinity, my interpretation of a contained infinity is considerably different. I give both in Chapter 5.

CHAPTER 4 THE MECHANICS OF OPERATOR N

4.1 THE SYMBOLISM OF PROPOSITION 6

When Wittgenstein states that "it is obvious that the analysis of propositions must bring us to elementary propositions" (TLP 4.221), he aims to show that, after a reductive logical analysis, all propositions and their truth-functions could be constructed from the basic level of elementary propositions using a single truth-functional operator. For as Wittgenstein asserts, "every proposition is a result of successive applications to elementary propositions of the operation N($\bar{\xi}$)" (TLP 6.001). What this means is that this sole primitive sign, N, via a series of (finite) iterations, can theoretically yield all meaningful propositions. Consequently, all other logical constants (e.g. ⊃, ≡, &, ∀ , ∃) can be eliminated—for "the description of the most general propositional form is the description of the *one and only* general primitive sign in logic" (TLP 5.472, emphasis added). As mentioned in Chapter 3, the much-debated Proposition 6 stipulates "the general form of a truth-function is [$\bar{p}$, ($\bar{\xi}$), N($\bar{\xi}$)]. This is the general form of a proposition." Each participating symbol plays a specific and unique role, and we would do well to 'unpack' this general form of a proposition, the better to appreciate the nature of each symbol and, subsequently, how this general propositional form is intended to work.

First, $\bar{p}$ is a metavariable: it stands proxy for all elementary propositions. $\bar{\xi}$ is also a metavariable, which represents the selected, or relevant, elementary propositions from $\bar{p}$. We will see how 'selection' works in §2. The bar in $\bar{p}$ and $\bar{\xi}$ indicates an infinite series, as per mathematical notation. In my interpretation, this was deliberate on Wittgenstein's part, or at least he felt warranted in his employment of this particular notational apparatus. Wittgenstein had an interest in mathematics, and wrote more on this subject than any other in the post-*Tractatus* period from 1929 through 1944. Wittgenstein noted in no uncertain terms that his “chief contribution has been in the philosophy of mathematics” (Monk, 1990, pp. 326, 466). And mathematical mentions and ideas are threaded through the *Tractatus* itself (TLP 4.04-4.0411, 5.154, 5.43, 5.475, 6.031, 6.2-6.22, 6.2321, 6.233, 6.234-6.4).

The mathematical notation for denoting an infinitely long number, such as a repeating decimal—enumeration being recognizably impractical—can be ‘0.999…’, where the ‘…’ indicates ‘and so on’. Conversely, the number $0.\overline{9}$, with the vinculum (overbar), has the same value. The vinculum signifies the infinitely repeating values in a number which is decimally expressed. As Wittgenstein states, “the bar over the variable [$\bar{\xi}$] indicates that it is the representative of all its values in the brackets” (TLP 5.501). By ‘all’, we understand Wittgenstein to mean an infinite though completed totality. I will discuss how $\bar{\xi}$ comes to represent a specific selection of values momentarily. Given the infinitary amount of elementary propositions, in concert with Wittgenstein’s conception of infinity as contained, his use of the overbar is tenable. Throughout the *Tractatus*, we see the N operator some nine times as applying uniquely to $\bar{\xi}$. At no time does N range over anything else, such as another variable, an open sentence, or molecular (complex) propositions. This is of the utmost importance in the forthcoming discussion of the expository errors of Geach and Fogelin, and especially with respect to Connelly’s variable ‘i’.

4.2 SELECTION—THREE KINDS

Let us now turn to how the propositions to which the N operator is to be applied are *selected*. The metavariable $\bar{p}$ signifies all elementary propositions. The next step is to select from $\bar{p}$ the 'relevant' propositions, which are then represented by metavariable $\bar{\xi}$. 'Relevance' is discussed below. Recall Wittgenstein's instructions that the metavariable $\bar{\xi}$ signifies all of the values within its brackets. To illustrate, he says "e.g. if ξ has the three values P, Q, R, then $(\bar{\xi})$ = (P, Q, R)" (TLP 5.501). We arrive at these values (the *relevant* values) of the metavariable because they are stipulated; this stipulation describes those propositions represented by the metavariable (TLP 3.316, 3.317, 5.501). Wittgenstein gives us three methods by which to describe the bracketed terms at 5.501:

> 1. direct enumeration, in which case we can simply substitute for the variable the constants that are its values; 2. giving a function fx whose values for all values of x are the propositions to be described; 3. giving a formal law that governs the construction of the propositions, in which case the bracketed expression has as its members all the terms of a series of forms

The first method is employed when direct enumeration of the atomic sentences (values) is practicable. As with Wittgenstein's above example, if ξ stands proxy for only three terms, P, Q, R, then $\bar{\xi}$ is easily described by actually enumerating the values that are its terms: P, Q, and R. But we must appreciate that Wittgenstein intended the N operator to completely express all meaningful propositions, and so the variable $\bar{\xi}$ is likely to comprise a very long—indeed infinitely long, though, according to Connelly, "terminating"— catalogue of propositions. This is where the second and third methods feature.

The second method employs the propositional function fx—this open sentence has no truth-value until the variable x is substituted with a simple constant (e.g. a, b, c), at which point the instantiated proposition is truth-valued as true or false. Propositional functions (such as fx) are used in method [2] as a ‘template’ or “logical prototype” (TLP 3.315) to select the relevant propositions—those propositions with a structure corresponding to fx. On Connelly’s account, names are given in place of the variable(s), e.g. fa, fb, fc, …, fi. Then, semantically simple constants (sentence letters that stand for elementary propositions) are assigned in place of the propositions, e.g. p for fa, q for fb, r for fc, …, p^i for fi (Connelly, 2017a, p. 20). The notational ‘i’ in ‘fi’ and ‘p^i’, and its ensuing difficulties, will be discussed in greater detail in Chapter 5. For the present, we may note that Connelly intends for it to mark the ‘infinitieth’, or final, member in a unique infinite catalogue of propositions. All of these now-instantiated elementary propositions (e.g. p, q, r) are symbolized by the metavariable $\bar{\xi}$, to which N is applied. At no point in the *Tractatus* is N applied to anything other than $\bar{\xi}$. The fundamental importance of this will be made shown in the upcoming discussions of Geach and Fogelin in §4, and of Connelly in Chapter 5. Open sentences (propositional functions, e.g. fx) do play a role in the ultimate N-expression of propositions and their truth-functions, but we must appreciate the specific role that these sentences play, and thus where they are not to be engaged. ‘Open’ sentences are such because the variable indicates an empty place in the construct in which constants are instantiated; these sentences are initially required in the collection of relevant propositions—those that are structurally isomorphic to the open sentence. Connelly (2017a) holds that once the variable(s) have been instantiated and the elementary propositions replaced by semantically simple constants, propositional structure becomes irrelevant, as elementary propositions are logically independent—they do not (cannot) entail anything (p. 18). Once propositional structure has been 'dispatched', the relevant list of elementary propositions is represented by $\bar{\xi}$. This dispatching of structure becomes crucial first in the upcoming discussion on Fogelin and Geach, and second in Chapter 7.

The third method is concerned with constructing propositions in accordance with a "formal law," which means that those forms structurally isomorphic to a particular stipulated form will comprise the series for which $\bar{\xi}$ stands proxy. In this way, according to Connelly, method [3] allows for the collection of those propositions which escape the deployment of the first two methods, such as polyadic propositional functions. Consider the dyadic first-order relation xRy. In much the same way that fx was dealt with above, Connelly instructs that the variables x and y in xRy are instantiated with names, such as aRb, bRc, cRd, (which are respectively replaced by semantically simple constants p, q, r) until the relevant list of propositions has been 'completely described'. Upon this complete description, the once-structured polyadic propositional function (e.g. xRy) has been substitutionally exhausted, and the structure therein becomes unimportant. It is in this way that Connelly claims to have shown that those propositions normally expressed in the predicate calculus may be described absolutely. The truth-functionally expansive analysis of any proposition is produced in precisely the same way as it would be in the propositional calculus. As Russell explains,

> Wittgenstein's method of dealing with general propositions [i.e. 'x.(fx)' and '(Ex).fx'] differs from previous methods by the fact that the generality comes only in specifying the set of propositions concerned, and when this has been done the building up of truth-functions proceeds exactly as it would in the case of a finite number of enumerated arguments p, q, r,… (TLP, xvi)

I disagree with Connelly's reading of this passage, but this will be addressed in the subsequent chapters. For now, let us continue with Connelly's approach: all singly general propositions (e.g. (∀x)Fx, (∃x)Fx) may be generated thus: universal quantification reduces to a (closed) series of conjunctions (e.g. Fa & Fb & Fc & … Fi), and likewise existential quantification to a (closed) series of disjunctions (e.g. Fa v Fb v Fc v … Fi). This is also the case in multiply general propositions (e.g. (∃x)(∃y)Fxy), and in mixed multiply general propositions (e.g. (∃x)(∀y)Fxy); all are thus expressible via the propositional calculus. The following demonstration of the equivalence between unmixed and mixed multiply general propositions of the predicate calculus and their logically analogous truth-functional expansions of the propositional calculus is largely inspired by James McGray (2006). I reproduce Connelly's notational i to indicate the 'terminus' of the infinite series but note that McGray's infinite series end with '…'. Unmixed multiply general propositions, for example, (∃x)(∃y)Fxy, may be disjunctively truth-functionally expanded. For the existential quantifier, this would look as follows:

(∃y)Fay v (∃y)Fby v (∃y)Fcy v … (∃y)Fiy

Here, the x in ∃x has been instantiated with simple names for all values of x. Next, we continue the truth-functional expansion by substituting simple names for all values of y in ∃y and creating our lists of disjunctions thus:

(Faa v Fab v Fac v… Fai), (Fba v Fbb v Fbc v… Fbi), (Fca v Fcb v Fcc v… Fci), …

Then, each bracketed set becomes a disjunct, as follows:

(Faa v Fab v Fac v… Fai) v (Fba v Fbb v Fbc v… Fbi) v (Fca v Fcb v Fcc v… Fci) v …

and so on until, according to Connelly, all propositions would be listed. I use '…' here for convenience, not to indicate a non-terminal series. This is thought to create an ultimate list of disjunctions, which has as its disjuncts lists of disjunctions. Regarding the last symbolization, the members of the first, second, and third sets of brackets indicate 'this or this or this or…'; then, the bracketed terms are logically summed against one another: 'this or this or this or…', until there is at least one instance where Fxy is true, just in case there exists an x and a y such that Fxy.

A similar method is followed in the case of mixed multiply general propositions, of the kind (∃x)(∀y)Fxy, except that the truth-functional expansion will first exemplify a logical sum (disjunction), and then a logical product (conjunction). The steps are as follows:

(∀y)Fay v (∀y)Fby v (∀y)Fcy v… (∀y)Fiy

Again, as above, ∃x disappears because the variable x has been substituted for the instantiating constants for all values of x. We continue the expansion by listing the substitution instances for all values of y; each quantified statement (disjunct) results as follows:

(Faa & Fab & Fac &… Fai) v (Fba & Fbb & Fbc &… Fbi) v (Fca & Fcb & Fcc &…Fci) v …

Again, to be consistent with Connelly's account, I use '…' for pragmatic not symbolically suggestive purposes. What this says, simply put, is that it is the case that every proposition in the first set of brackets, (Faa & Fab & Fac &…Fai), is true, *or* it is the case that all propositions in the second set of brackets, (Fba & Fbb & Fbc &…Fbi), are true, *or* … and so on. We may see via this demonstration of truth-functional expansions how the predicate calculus purports to reduce to the propositional calculus, given that propositional functions, general propositions, and molecular (complex) propositions treated with simple constants, and their truth-functions, can be thus expressed. Wittgenstein's all-encompassing reductive method is to apply, as we have seen, to all propositions, even those whose main connective is a quantifier. This, of course, relies entirely on a view of infinity that terminates. I argue that this view is untenable, and propose an alternative, in Chapter 5.

4.3 THE MECHANICS OF N

Now, we come to the actual workings of the N operator. As Russell explains in his introduction to the *Tractatus*:

> It has been shown by Dr Sheffer […] that all truth-functions of a given set of propositions can be constructed out of either of the two functions 'not-p or not-q' or 'not-p and not-q'. Wittgenstein makes use of the latter, assuming a knowledge of Dr Sheffer's work (TLP, xv)

This means that the classical logical connectives (⊃, ≡, &, ~, v), though *practically* useful, are in theory eliminable. Dr. Henry Sheffer showed that all that is necessary to produce a series of propositions and the truth-functions thereof is negation and disjunction, or negation and conjunction. This latter, joint negation, is how the N operator functions. At TLP 5.51, Wittgenstein explains: "if $\bar{\xi}$ has only one value, then $N(\bar{\xi})$ = ~p (not p); if it has two values, then $N(\bar{\xi})$ = ~p.~q (neither p nor q)." N serves to negate and conjoin all propositions, represented by $\bar{\xi}$, occurring within its brackets. Indeed, "if ξ has as its values all the values of a function fx for all values of x, then $N(\bar{\xi})$ = ~∃ x.fx" (TLP 5.52). According to Connelly, what this means is that $\bar{\xi}$, being a metavariable, represents all of the already instantiated instances of fx, such as fa, fb, fc, …, fi. The remainder of this section is an explication of Connelly (2017a), which I will evaluate in the next two chapters. Since these propositions are atomic, they may be substituted for semantically simple constants (e.g. p, q, r… p^i). The result is that p is synonymous with fa, q with fb, r with fc, … p^i with fi. Once the truth-functional expansion is complete, ∃ x (being shorthand for a disjunctive truth-functional expansion) disappears and we are left with an infinitely long yet completed list of elementary propositions. These are the 'values' that are to be placed within the brackets under the scope of the N operator. The result is as follows:

$$N(p, q, r,\ldots , p^i)$$

Since N operates as joint negation, each elementary proposition within the brackets would be negated and conjoined, as follows:

$$\sim p \ \& \sim q \ \& \sim r \ \& \ldots \sim p^i$$

Recall that ∃ is the existential quantifier, which says that (∃ x)fx means that it is true that there exists at least one x such that x is f (satisfies the predicate f). Conversely, ~∃ x.fx means that there does not exist an x such that x is f. This is equivalent the above equation, which says 'not p' and 'not q' and 'not r' for all possible values of x—in other words, not x.

As Connelly (2017a, pp. 20-21) explains, mixed multiply general propositions are completely expressible, as follows: (∃ x)(∀ y)Fxy, truth-functionally expanded, is logically analogous to infinitely long lists of conjunctions serving as the disjuncts in an infinitely long disjunctive series

(Faa & Fab & Fac & … & Fai) v (Fba & Fbb & Fbc & … Fbi) v (Fca & Fcb & Fcc & … Fci) v … v (Fia & … & Fii)

Each atomic proposition is assigned a semantic simple constant:

Faa --- p
Fab --- q
Fac --- r
.
.
.
Fai --- p^{i}

Fba --- p^{1}
Fbb --- q^{1}
Fbc --- r^{1}
.
.
.
Fbi --- p^{1i}

Fca --- p^{2}
Fcb --- q^{2}
Fcc --- r^{2}
.
.
.
Fci --- p^{2i}

Fia --- p^{i1}
.
.
.
Fii --- $p^{i,i}$

$\bar{\xi}$ stands proxy for all of the members on the right. Following Connelly's illustration (2017a, p. 21), applying N as intended (N($\bar{\xi}$)), we may N-express (∃ x)(∀ y)Fxy as follows:

$$N(N(N(N(p), N(q), N(r), \dots, N(p^{i})), N(N(p^{1}), N(q^{1}), N(r^{1}), \dots, N(p^{1i})), N(N(p^{2}),N(q^{2}),N(r^{2}), \dots ,N(p^{2i})), \dots, N(N(p^{i1}), \dots N(p^{i,i}))))$$

This is intended to show the "successive application" Wittgenstein described at TLP 5.32, 5.5, and 6.001, and how it is possible (in principle) to completely express every proposition, generated via a limited number of truth operations. The Ns with the narrowest scope (i.e. ranging over the smallest amount of members) come first in the construction of the N-expression. Then we continue to build outwards until we have the entire catalogue of relevant propositions, which are then subject to a final N operator, which has the widest scope.

N differs from the other classical logical connectives because it can be applied to (infinitely) many propositions. Contrast this with the binary connectives (&, v, ⊃, ≡) which, by definition, connect only and exactly two truth-functional operands (propositions) —e.g. p & q, (p v r) ⊃ (s & q). Negation (~) is a unary connective, which has only one operand (e.g. ~p). Whereas these classical logical connectives are either monadic or dyadic in function, the N operator can serve as the sole truth-functional operator for an indefinite number of arguments. This results in the eliminability of these logical constants, as well as the existential and universal quantifiers. The thrust of Wittgenstein's rigorously austere logical programme is evidenced clearly by this lone, principal operator as deployed to construct all propositions.

This construction is a logical description of the world. The totality of actual elementary propositions are the constituents of all true propositions—those descriptions which successfully 'map' onto reality (TLP 2.1512). "The sum-total of reality is the world" (TLP 2.063), and so the totality of facts that exist (i.e. are true) establishes the totality of facts that does not exist. The former are 'positive facts', the latter 'negative facts' (TLP 2.06). Thus, the enumeration of facts (the complete description of reality) must terminate, such that there are positive and *negative* facts;

the totality of facts says what does exist, and what the totality of facts *does not say* determines those states of affairs which do not obtain. The logical description, therefore, comes to an end once all of the contents that 'fill' the 'metaphysical sphere' have been accounted for. Wittgenstein is able to account for the 'all' of generality precisely because of his notion of bounded infinitary (propositional and objectual) totalities.

4.4 Geach and Fogelin

We are now in a better position to assess the expository work of Geach and Fogelin, who respectively accused Wittgenstein of either hyperbole or blunder. Geach argues that N *is* expressively complete, but only if it is supplemented with class-forming operators. He writes: "to bring out in full the way Wittgenstein's N operator works, we need (something he does not himself provide) an explicit notation for a class of propositions in which one constituent varies" (1981, p. 169). But Wittgenstein did not provide this class-forming notation precisely because of his belief that "the theory of classes is completely superfluous" (TLP 6.031). We should bear in mind the eliminativist motivations of the *Tractatus*, as Wittgenstein's logical programme champions a single primitive logical operator specifically intended for deployment without recourse to auxiliary measures such as class-theoretic calculation or variable-binding operators.

Geach goes on to say that "Wittgenstein was exaggerating when he said that the theory of classes is altogether superfluous in mathematics (6.031) for he cannot get on without classes of propositions" (1981, p. 169). Indeed, Wittgenstein himself employs the phrase "classes of propositions" (TLP 3.311, 3.315). Some philosophers, James McGray in particular, take this to fully license the use of the variable-binding devices (ẍ, ÿ) that Geach introduces and employs in concert with the N operator, provided they are not used gratuitously (2006, p. 154). McGray agrees with Geach that, with respect to Russellian quantifiers, N requires "significant enhancement" (2006, p. 144), because without the aid of supplementary operators, the Tractarian system is expressively incomplete.

Thus, Geach presents his 'enhancing' strategy: "'(∃ x)fx and 'x(fx)' will come out in a Wittgenstein-style notation as 'N(N(ẍ : fx))' and 'N(ẍ : N(fx))' respectively" (1981, p. 169). According to Geach, this procedure for singly quantified propositions can then be repeated to account for mixed multiply general propositions (1981, p. 169). But, Connelly (2017a) claims, with the aid of McGray's (2006) illustration of truth-functional expansion, that Geach's notation is equivalent to Russellian quantification. Connelly also claims to have shown that Russellian logical constants can be eliminated via the reduction of quantified propositions in the predicate calculus to their truth-functionally expanded analogues in the propositional calculus. McGray's Russellian-Geachian notational equivalence was introduced in the previous section dealing with the mechanics of N—specifically, in the construction of mixed multiply general propositions. Geach himself does not give a demonstrative account of his own notation; I will elaborate on McGray's Geachian account here, to show the logical equivalence. All following symbolic expressions are attributed to McGray, 2006, pp. 154-5. I omit quotation marks for clarity and simplicity.

In order to express (∃x)(∀y)Lxy in Geachian notation, we take it as our first step:

1. (∃x)(∀y)Lxy

Then, we give all substitution instances of the variable x, which eliminates the existential quantifer and gives us the truth-functional expansion:

2. (∀y)Lay v (∀y)Lby v (∀y)Lcy v…

Eliminate Russellian universal quantifier by replacing variable y with all substitution instances:

3. (Laa & Lab & Lac &…) v (Lba & Lbb & Lbc &…) v (Lca & Lcb & Lcc) v …

Apply De Morgan's Law—double negations are eliminated since N is the main operator, as follows:

4. N(~(Laa & Lab & Lac &…) & ~(Lba & Lbb & Lbc &…) & ~ (Lca & Lcb & Lcc &…) &…)

A second N operator with a narrower scope eliminates the joint negation. Though only a, b, and c appear in step [5], McGray explains that all substitution instances of the "class-specifying variable" (p. 154) ẍ, which occupies the leftmost name-place and which will appear in step [12], would be included in the following N-expression:

5. N(N((Laa & Lab & Lac &…), (Lba & Lbb & Lbc &…), (Lca & Lcb & Lcc & …), …))

The next three steps show for each set of brackets the constant of interest, which is located inside the second leftmost bracket. These are a, b, and c, respectively:

6. N(N(a : (Laa & Lab & Lac & …)))

7. N(N(b : (Lba & Lbb & Lbc & …)))

8. N(N(c : (Lca & Lcb & Lcc & …)))

All bracketed expressions included in the N-expression shown in [5] would undergo the following translation. I use [6] as an archetypical example. A third N operator is affixed to the innermost bracketed set. This removes the joint conjunction of the propositions:

9. N(N(a : N(~Laa, ~Lab, ~Lac,…)))

Additional N operators are affixed to each proposition:

10. N(N(a : N(N(Laa), N(Lab), N(Lac),…)))

"The set of unit set N operator propositions" (p. 154), the substitution instances of the variable 'ÿ' which occupy the rightmost name-place of each proposition, are removed, revealing the variable:

11. N(N(a : N(ÿ : N(Lay))))

The name 'a' is removed to reveal the variable 'ẍ'. This results in the expression in Geachian notation:

12. N(N(ẍ : N(ÿ : N(Lxy))))

As we can see, Geach's notation works. But there are two problems here. The first concerns Wittgenstein's claim that "the introduction of any new device into the symbolism of logic is necessarily a momentous event" (TLP 5.452). And yet, if Russellian notation is perfectly serviceable and indeed much more economic in the expression of quantified propositions, why does Wittgenstein introduce the N operator at all? Connelly (2017a) has argued that Wittgenstein did not intend for N to replace the practicality of Russellian notation. Rather, motivated by the maxim *Simplex Sigillum Veri* ('the simple is the sign of the true') with respect to logical problems (TLP 5.4541), his reductionist project was intended to show that logic could, in principle, be performed with the use of only one operator. That it is theoretically but not practically viable is a psychological limitation, not a logical one, according to Connelly's interpretation of Wittgenstein.

The second problem has to do with the above successful display of N working in concert with a class-forming operator, which seems to solidify Geach's claim that the addition of the latter is necessary to the success of the former. There may be a temptation here: Wittgenstein was not forthcoming about the workings of N, and certainly refrained from demonstrating its application; perhaps N does lack fully expressive capacities after all, for where Geach and Fogelin developed it far enough to discover its expressive limits, so too Wittgenstein would have, had he ventured far enough.

As we have seen, and as Wittgenstein stipulates at TLP 5.32, 5.5, and 6.001, N is intended to be successively applied only to elementary propositions (enumerable in a finite amount of steps) and their subsequent truth-functions. It is not intended, as some philosophers have supposed, to operate on open sentences. Both Fogelin and Geach fail in different ways to restrict the application of N solely to elementary propositions. Consequently, their interpretive errors culminate in a stark, yet possibly avoidable, impasse regarding the expressive completeness of operator N. Furthermore, Wittgenstein would, I think, be critical of Geach's notation, since "in logic a new device should not be introduced in brackets or in a footnote with what one might call a completely innocent air" (TLP 5.452). Geach not only introduced his class-forming operators with impunity, but celebrated them as *necessary* devices, which Wittgenstein, for one reason or another, failed to disclose. Geach explicitly contravenes Wittgenstein's overt claim about the full and complete omission and superfluity of class-theoretic operators, accusing Wittgenstein of hyperbole on the subject. McGray (2006) agrees with Geach, stating that "any use of classes or sets in an interpretation of the *Tractatus* should be kept minimal" (p. 154). This is curious, since Wittgenstein decried “the theory of classes [as] *completely* superfluous” (TLP 6.031, emphasis added). It is inconsistent with Tractarian *eliminativism* to *introduce* class-forming operators, let alone operators which are intended to act in concert with N, and only with which N can fulfill its expressive completeness goal. If N indeed required additional operators, it seems pointless, other than as a creative exercise, to devise operators with equivalent capacity as alternatives to Russellian quantifiers for the expression of general propositions (Connelly, 2017a).

Fogelin, like Geach, has “no personal reservations about forming sets (and propositions out of these sets) in just the way that Geach proposes” (1982, p. 126). Thus far, both happily agree on a point which, ironically, Wittgenstein unambiguously rejected (TLP 6.031). Fogelin argues that N cannot work in the way Wittgenstein intended: “there is no way of constructing universal propositions of the form (x)fx unless the system of Tractarian logic is enriched with further truth functional operators such as logical product” (1982, p. 125). This is particularly confusing, since the logical product of e.g. p, q is p & q, and Wittgenstein explicitly proscribes connectives and claims to have shown their dispensability. (The connectives ‘&’ and ‘v’ were used in the above truth-functional expansions, but only as rudimentary demonstrative tools.) Discussion of mixed multiply general propositions brings forth the stalemate between Fogelin and Geach. Fogelin’s concerns are much more damning than Geach’s, as Fogelin opines that “even after such [notational] enrichment there will be no way of constructing such mixed multiply general propositions as (∃ x)(x)(fxy). It is for this […] reason […] that I hold that the logic of the *Tractatus* is fundamentally flawed” (1982, p. 125).

Fogelin is correct in observing that it would be impossible to completely express an infinite series of catalogued propositions via the N operator alone, were that catalogue indeed unending—which is precisely why Connelly explains, but does not justifiably illustrate, the significance of actually infinite, terminating series. An endless series would mean that one could never move from within the brackets to outside the brackets, wherein the application and re-application of N would take place (Connelly, 2017a). N, as the main logical operator, could not possibly range or quantify over an unending series of propositions, because one could never get to the 'next step' of doing so—moving out of the brackets with the narrowest scope to those with wider scope and ultimately to the final iteration of N. As a result, Fogelin, like Geach, is happy to introduce additional logical operators, such as class-forming devices, in order to supplement N and, in his opinion, render it practicable (at least in the case of singly general propositions). But Fogelin is right to point out that this practice "flatly contradicts" (1982, p. 126) one of the fundamental Tractarian theses found at 5.32 which states that "all truth-functions are results of successive applications to elementary propositions of a finite number of truth operations." Fogelin notes that Geach's inflationary notation contravenes Wittgenstein's direction that the applicative steps of N be finite and successive (1982, p. 126). But, Connelly argues, Fogelin and Geach gravely misunderstood Wittgenstein's conception of infinity; were Fogelin to understand its nature as a completed totality (whose members were theoretically enumerable) as opposed to incomplete, his worries of "*finiteness* and *successiveness*" (Fogelin, 1982, p. 126) would cease.

Connelly's view of the importance of structure once again becomes evident. The propositional forms fx and xRy are specifically structured. But once the collection of structurally isomorphic propositions is complete and the open variable(s) have been replaced by substitution instances (fa, fb, fc, … fi), the structure becomes irrelevant, given the independence thesis—viz. that no elementary propositions can follow from or entail another elementary proposition, as they are logically independent. So by the time N is applied, Connelly maintains, open sentences and quantified propositions have been dispatched via the reductive logical analysis process into a manageable list of atomic propositions of the propositional calculus.

When Wittgenstein gives us the three methods of description at TLP 5.501, he is affording us instructions for how to specify the relevant propositions to be placed under the scope of N. He says, "how the description of the propositions is produced is not essential" (TLP 3.317). But *that* we are able to calculate propositional lists of infinite length is indeed essential. The next Chapter will serve to illustrate the problems with Connelly's conception of completed infinites, which subsequently undermine both his notational use of i and attempted vindication of the N operator, discussed in Chapter 6.

CHAPTER 5 DIFFICULTIES IN CONNELLY'S ACCOUNT

5.1 CONCERNS ABOUT CONNELLY'S INFINITY

I will begin by describing the nature of my confusion regarding Connelly's interpretation of the infinite, specifically his use of series that are "infinite but nevertheless terminal" (2017a, pp. 9, 11). In Chapter 3, I explicated Connelly's interpretation of infinity, but did not elaborate on the complications which would inevitably arise from his account. This section will serve to lay bare those complications. A prelude to the task set before us is to specify our vocabulary, the better to make perspicuous any prospective terminological conflations or misuses. In particular, the terms 'finite', 'actually infinite', and 'potentially infinite' as used by Connelly are expounded here:

i) *Finite*: that which has a magnitude that is bound(ed) (limited)
ii) *Actually infinite*: that which forms a countable, complete totality
iii) *Potentially infinite*: that which is boundless, endless, and unmeasurable

While the last definition seems to stand alone, the first two definitions are at risk for conflation—for if a set that is actually infinite is in some way contained, it seems to possess a kind of finiteness. But a little elucidation will show them to be discrete properties. The distinction will, I hope, become manifest in the examples given below.

Connelly (2017a) uses list formulations to exemplify the "infinite but nevertheless terminating" series:

~Fa & *~Fb* & *~Fc* & … & *~Fi* (p. 20)

and

Fab v *Fac* v *Fad* v … v *Fba* v *Fbc* v *Fbd* v … v *Faa* v *Fbb* v *Fcc* v … v *Fii* (p. 23) and

$$\text{N(N(N(N(}p\text{), N(}p^1\text{)), N(N(}q\text{), N(}q^1\text{)), N(N(}r\text{), N(}r^1\text{)), …, N(N(}p^{\,i}\text{), N(}p^{1i}\text{))))} \quad \text{(p. 23)}$$

We may note the notational ‘i’ which prominently appears at the end, or “terminus” (2017a, p. 20), of each respective expiring list formulation. Connelly recounts that

> I have simply supposed the existence of a number, ‘infinity’, which lies at the terminus of various infinitely long series, and have abstracted from the possibility that those series may differ in length or cardinality (2017a, p. 20)

Given this supposition, he notes that “within the [propositional] constructions, the letter ‘i’ is supposed to stand for the “infinitieth” (and thus *final*) iteration of whatever it applies to, be it the infinitieth object in the domain, or the infinitieth proposition in a series” (2017a, p. 20, emphasis added). But the potentially problematic pronouncement of the “final iteration” in an *infinite* series (indicated by i) is liable to mislead. This is, instead, a more apposite description of a (possibly extensive but) *finite* series, which we could enumerate thus, e.g.:

Fa v *Fb* v *Fc* v … v *Fz*

Now, the difference between my example just given and the above-given examples from Connelly (2017a) is the notational ‘i’ tacked onto the end of his series (which indicates the ‘end’ of each infinite series), whereas the final iteration of my (finite) series is indicated by ‘z’. But what is it that makes Connelly’s lists infinite, other than the arbitrary stipulation that they *just are*? One schematic ‘i’ does not an infinite series make. What Connelly’s lists actually seem to typify are *finite* lists under the guise of an ill-fitting and indefensible alias. He does distinguish between *potentially* infinite—as “so unlimited” (2017a, p. 15), and *actually* infinite—as a “so limited totality” (2017a, pp. 10-11), the latter of which he is claiming to make use. However, Connelly's exemplification of "infinite" series does seem to conflate actual infinity and finitude, and his description more aptly suggests the latter.

Let us examine the following archetypal expressions (Connelly, 2017a, p. 22):

N(N(N(*p*), N(*q*), N(*r*), …, N(p^i)))

This N-expression, Connelly claims, is "equivalent to the following, *infinite*, truth-functional expansion" (2017a, p. 22, emphasis added):

~*Pa* v ~*Pb* v ~*Pc* v … v ~*Pi*

But in what sense do these lists suggest an infinite number of members? Other than the idiosyncratic affixation of 'i' to the 'end' of the 'infinite' list, there is nothing whatsoever to indicate here that the above series are anything but finite. Even though, as Connelly makes clear, these lists should be thought of as very long, so long that they could never be written down in practice, they better exemplify finitely long lists, rather than infinitely long lists. For to say that an infinite list "terminates" is attributing a property to a thing that could not, by definition, exemplify that property. While actual (completed, contained) infinities will be discussed and indeed defended in this and the next chapter, Connelly's contradictory and inexact use of 'terminating infinity' is misleading, as are his mischaracterized examples. There is no tenable way to uphold his attempted way of drawing the distinction between 'finite' and 'infinite', beyond his inclusion of 'i'—which seems, at best, arbitrary. Ironically, it seems to be Connelly's actual infinity which is expressively incomplete.

The advantage (and nature) of 'actual infinity' is not that it is in some way "bound," "terminating," or "limited," in the sense that its enumeration would simply cease—since this is what characterizes finitude. Rather, an infinite set is enumerable just in case it is *equinumerous* (can be placed in one-to-one correspondence) with the natural numbers. More will be said about this shortly. I will be using two terminologies that are closely related but nevertheless distinct. The first is *definition by recursion*: this is a method for defining something into existence. For example, 0 is a natural number, and for any natural number *n*, if *n* is a number, then the successor of *n* is a number. This is a recursive definition of the set of natural numbers—positive integers, sometimes including 0, denoted by the symbol ℕ. The second is *mathematical induction*: a proof technique used to show via a fixed but arbitrary finite number of members of some set A that have a property of interest ($\mathcal{P}$) that for every defined member of set A, that member will have property $\mathcal{P}$. More simply put, if we can prove it for e.g. one member (of a given set), we can prove it for all members (of a given—finite or infinite—set). For example, we use mathematical induction as a procedure for proving that each member of the set of natural numbers has a *hereditary* property, such that any number (*n*), and its immediate successor ($n + 1$) that has the property of interest ($\mathcal{P}$) is a member of the set of natural numbers—for enumeration will inevitably 'reach' this number. "Any property which belongs to 0, and to the successor of any number which has the property, belongs to all the natural numbers" (Russell, 1919, p. 21).

Recursive definition and mathematical induction will be discussed in §2. Any set that is recursively defined is subject to proof by mathematical induction. Thought of in the ways just described, it becomes obvious how the members of infinite set A are countable and contained. We can recall here our definition given at the beginning of this section: *actual* infinities are *countable* totalities (sets). This is a very different, and more tractable, model than that of 'terminal (infinite) lists'.

Connelly is correct that the list of elementary propositions with which we are dealing must be enumerable, since this is what affords Wittgenstein his complete description of the world, up to and including all true and false propositions. But it is not necessary—or even, perhaps, possible in a non-arbitrary way—to claim that a given infinite list is countable purely because it "terminates." For one could rightly ask: at what point does it do this? Why, and how? If a list *terminates*, is it not, by definition, a *finite* list? Rather, more straightforwardly and coherently, an infinite series or set is 'countable' if and only if that set is the same size as (equinumerous with) that of the natural numbers ($\mathbb{N}$).

In fact, the proposal of infinite lists which (somehow) terminate is not particularly helpful either, as Connelly himself makes clear: "of course, infinitely long series of propositions would be practically impossible to enumerate; but then the same thing would be true of many very large, but finitely long series of propositions" (2017a, p. 16). So his oxymoronic posit does not advance us much, if at all, and further still does not help to clarify Connelly's distinction between the finite and the infinite.

Adding to the conflation is Connelly's following claim that

> Wittgenstein did not think there was any essential difference between the finite and the infinite case, because he thought that infinity was actual, as opposed to potential. He thus thought that the terms of an infinitely long series were just as enumerable, in theory, as were those of any finitely long series (2017a, p. 16)

More correctly, Wittgenstein did not differentiate the finite and infinite case because, mathematically, it was inessential to do so. This is well-explained by one of the most highly regarded mathematicians of the twentieth century, Frank Ramsey:

> [The] argument of Hilbert [says] that if the variable has an infinite number of values, if, that is to say, there are an infinite number of things in the world of the logical type in question, we have here an infinite logical sum or product which, like an infinite algebraic sum or product, is initially meaningless and can only be given a meaning in an indirect way. This seems to me to rest on a false analogy; the logical sum of a set of propositions is the proposition that one of the set at least is true, and *it doesn't appear to matter whether the set is finite or infinite. It is not like an algebraic sum to which finitude is essential* (1931, p. 74, emphasis added)

Connelly misconstrues Ramsey's meaning as regards infinite series, for he claims that "Ramsey also characterized Wittgenstein's truth-functional expansions as involving terminal lists of propositions" (2017a, p. 21). This is incorrect. Ramsey noted that, arithmetically, it is immaterial whether a series is finite or infinite, since we are not proceeding *algebraically* (in which case, it is crucial that we deal with finite sets). There is no difference between the finite and the infinite case because, provided we are dealing with an infinity that is small enough so as to be equinumerous with the set of natural numbers, enumerability is equally possible in both cases. It is perfectly coherent to arithmetically describe infinite series without necessitating that an infinite series share finitude's characteristic of being 'terminal'.

I here reproduce Connelly's italics in the following quotation to illustrate how embedded, and indeed essential, 'terminating infinity' is to his vision:

> [After] open sentences are used merely to *describe* the relevant list of elementary propositions, the construction proceeds *exactly* as it would in the case of dealing with a finite number of enumerated sentences letters, because *it too would operate on terminal lists of sentence letters* such as *p*, *q*, and *r*, each assigned to a unique elementary proposition (e.g. *Fab, Fac, Fad,* etc.) (2017a, p. 21)

§2 will serve to illustrate the workings of mathematical induction and recursion, which, as we will see, resemble Connelly's above claim that open sentences describe (serve as the base case) for a given set of elementary propositions. But for the moment it is enough to note the problematic language and conceptual strain of terminal lists of sentence letters, which are purportedly infinite.

The next section will show how we come to construct infinite sets via recursion. More advanced set theory distinguishes *sets* and *classes*, but for our purposes, the distinction is immaterial and thus the terms are henceforth used interchangeably. This reliance on class-construction may invite a worry regarding TLP 6.031, wherein Wittgenstein states in no uncertain terms that "the theory of classes is completely superfluous." I submit that the method of recursion (discussed in §2) need be no more rigorously axiomatic (class-theoretic) than Wittgenstein's three methods of stipulation given at TLP 5.501. Rather, it is simply a 'recipe' for defining lists of relevant propositions—just as are given at TLP 5.501. In fact, I will shortly substantiate the claim that, though never explicitly stated by Wittgenstein, and without contradiction with TLP 6.031, Wittgenstein was working precisely with recursion in mind. For now, let us proceed with the discussion of mathematical induction and recursion and list construction.

5.2 Mathematical Induction and Recursion

The criticism featured in the previous section pertained to Connelly's indefensible method of distinguishing countable (actual) and uncountable (potential) infinities. He claimed that the latter is not enumerable because it is invariably unending, whereas we are afforded enumerability of an actually infinite set precisely because the list of its members would "terminate." Unfortunately, this description applies very well to that of finite lists, and is therefore ambiguous and unsatisfactory. Connelly is right to insist that objects and propositions as discussed in the *Tractatus* must be countable, otherwise the logical programme therein is incapable of yielding any worthwhile result. But there is a more tractable, articulate, and manageable approach than Connelly's conflationary account of 'finite infinity'. This is presented shortly.

As A. W. Moore (1990) notes,

> some infinite sets are bigger than others. Indeed there is no limit to how big an infinite set can be. There is, however, a limit to how small an infinite set can be. Any infinite set is at least as big as ℕ […] Any set that is either finite or the same size as ℕ (that is, as small as its infinitude allows it to be) is said to be *countable*. Any set that is bigger than this is said to be *uncountable* (p. 152)

Whereas Connelly wishes to say that infinite series are (theoretically) enumerable because they terminate at a given point, the standard view—as popularized by Georg Cantor in the nineteenth century—is that infinite sets are countable insofar as they can be placed in one-to-one correspondence with the natural numbers. Bearing in mind that this project is a thesis in philosophy, I will keep discussion of the details of mathematical induction and recursion to the minimum necessary for present purposes.

Recursion is a method by which we can define an infinite set via a fixed but arbitrary number of cases. Mathematical induction is a technique for proving that all members of the defined set have some property of interest. Such are the procedures for 'completing' or 'containing' the infinite. We can recall that "the general form of a truth-function is $[\bar{p}, (\bar{\xi}), N(\bar{\xi})]$. This is the general form of a proposition" (TLP 6). Importantly, Wittgenstein states that "the general propositional form is a variable" (TLP 4.53). The variable nature of the general form of a proposition is crucial, such that "every [mathematically inductive] proof is different, since every proof is designed to establish a different result. But like games of chess or baseball, observation of many leads one to realize that there are patterns, rules of thumb, and tricks of the trade that can be found and exploited over and over again" (Lewis & Papadimitriou, 1998, p. 23). These patterns can be captured via the principle of mathematical induction, which Lewis and Papadimitriou (1998) describe as follows: "any set of natural numbers containing zero, and with the property that it contains $n + 1$ whenever it contains all the numbers up to and including n, must in fact be the set of all natural numbers" (p. 24).

Preliminarily, I emphasize that the inductive proof method with which we are dealing is *mathematical*, and not *philosophical*. Whereas philosophical induction involves a probable conclusion based on a given set of premises, mathematical induction more accurately resembles a deductive process, such that the conclusions reached on the basis of given premises are necessarily and unequivocally true. I also note that the debate continues as to whether 0 or 1 is the first member of the series of natural numbers. The debate is peripheral to our topic, but given the steps below, a first member must be established; thus I will simply assume 0 as the first.

As Lewis and Papadimitriou (1998) note, "in practice, induction is used to prove assertions of the following form: 'For all natural numbers n, property P is true'" (p. 24). We first define our set A: "$A = \{n : \mathcal{P} \text{ is true of } n\}$" (Lewis & Papadimitriou, 1998, p. 24). Then, we prove that each member of set A has property $\mathcal{P}$ as follows:

[1] *base case*: if $0 \in A$, 0 has the property of interest $\mathcal{P}$

[2] *inductive hypothesis*: assume that property $\mathcal{P}$ holds for some fixed but arbitrary number that is greater than or equal to 0

[3] *inductive step*: use [2] to show that $n + 1$ has property $\mathcal{P}$

It is easy to see how the principle of induction affords us our ability to 'go from' zero to the desired natural number by successively adding 1. In this way, the set of natural numbers can be said to be *countable*. Steps [1] to [3] can be used to show that set *A* is equinumerous with $\mathbb{N}$; $\mathcal{P}$ therefore holds for every member of the (countably infinite) series.

Enderton (1977) explains that "a very flexible way of naming a set is the method of *abstraction*. In this method we specify a set by giving the condition—the entrance requirement—that an object must satisfy in order to belong to the set. In this way we obtain the set of all objects x such that x meets the entrance requirement" (p. 4). It should be noted that the abstraction method is not so liberally applicable that it can be used to name any and all sets—it can produce meaningless or paradoxical results depending on the quality of entrance requirements (Enderton, 1977). But this can be avoided as long as the sets are restricted in some way, e.g. avoiding ambiguity in the entrance requirements, and in the terms and concepts being employed (Enderton, 1977, pp. 5-6).

The steps of defining recursively are analogous to those of proving inductively, as shown above. We begin by specifying a base case; the second step is the recursive step, whereby we show that for all members to be defined that those members have the property of interest. The methods of recursion and induction are strikingly similar to several passages in the *Tractatus*. Compare the above description of recursion with TLP 5.501, wherein Wittgenstein instructs that "what the values of the variables are is something that is stipulated. The stipulation is a description of the propositions that have the variable as their representative." The three methods of stipulation, or description, were discussed in the previous chapter. They were 1. enumeration; 2. a propositional function fx where the values of x are the sought-after propositions; 3. a formal law gathering those formally isometric propositions (TLP

5.501). If the list of propositions is small enough, simple enumeration will suffice. However, if the list is finitely or infinitely long (but countable, such that they share an equivalence relation with the natural numbers), the second and third methods can be employed. Indeed, they serve admirably as the specification given in recursion. This is a finitely defined but generally applicative procedure for describing all and only those members of a particular infinite set.

As mentioned above, our proof method of describing the infinite set of natural numbers can be generalized to prove "different results." For example, we need not reference a rule-book each time a move is made from a particular place on a chessboard. Instead, we prove via mathematical induction that our property of interest holds for every application. For example:

[1] base case: all chess pieces in the conventional starting position on the chessboard
[2] inductive hypothesis: if *P* is a chess position, then *P* + *move* is a (allowable) chess position

This gives us our inductively defined set. No two chess games are alike, and though the game permits an infinite number of move-possibilities, we have defined it finitely. Such is the way we get generality. For "if we know the logical syntax of any sign-language, then we have already been given all the propositions of logic" (TLP 6.124). As Wittgenstein notes at TLP 5.522: "what is peculiar to the generality-sign is first, that it indicates a logical prototype, and secondly, that it gives prominence to constants." This means that we provide a typical model of the members of the set, viz. a prototype, by which we denote (give 'prominence to'—identify) the members which answer to that description. It is in this way that infinite sets are countable, and not, contra Connelly, because they are infinite but can be given a final member.

5.3 INDUCTION AND RECURSION IN THE *TRACTATUS*

This account is, I think, extremely felicitous as regards the Tractarian programme. Recall, particularly, the above modus ponens argument concerning chess moves (step [2]: the inductive hypothesis). Wittgenstein states that "in logic every proposition is the form of a proof. Every proposition of logic is a modus ponens represented in signs. (And one cannot express the modus ponens by means of a proposition.)" (TLP 6.1264). One could object that we are in violation of the bracketed statement by describing the modus ponens (proof method); but that need not cause us any worry. Indeed, it bolsters the exacting austerity which characterizes the *Tractatus*: mathematical induction is a *syntactic* proof, not a semantic one. And Wittgenstein tells us that "mathematics is a logical method. The propositions of mathematics are equations, and therefore pseudo-propositions. A proposition of mathematics does not express a thought" (TLP 6.2-6.21). Rather, the recursive and inductive methods respectively establish and verify all and only those propositions which make up the complete description of reality.

Just as Lewis and Papadimitriou (1998) described above, we can generalize well beyond the natural numbers by constructing different proofs intended to generate different results. TLP 6.126 suggests that Wittgenstein was using such a procedure: "one can calculate whether a proposition belongs to logic, by calculating the logical properties of the *symbol*. And this is what we do when we 'prove' a logical proposition. For, without bothering about sense or meaning, we construct the logical proposition out of others using only *rules that deal with signs*" (TLP 6.126). We avoid propositional errors by making use of "a sign-language that is governed by *logical* grammar—by logical syntax" (TLP 3.325). Wittgenstein's focus on syntax, and neglect of sense or meaning, are the focus Chapter 7.

5.4 Classes and Ontology

We saw previously Geach's claim that Wittgenstein's banishment of class theory was an exaggeration. Geach thus felt entitled to supplement the N operator with class-forming devices. Connelly is particularly worried about this move. As he notes, "it is hard to see how classes can fit in to this ontology. It is hard, in particular, to see how classes can either be identified with facts, or how facts can have classes as constituents (especially since Wittgenstein stipulates that facts only have objects as constituents; TLP 2.01)" (2017a, p. 14). Connelly's worry seems apt: Wittgenstein's main driver is to lay out an eliminativist logical programme, precisely designed to eliminate anything that is not essential to logic (e.g. logical constants and class-theoretic supplements).

But we can contrast Connelly's claim with TLP 6.2321, wherein Wittgenstein states that "the possibility of proving the propositions of mathematics means simply that their correctness can be perceived without its being necessary that what they express should itself be compared with the facts in order to determine its correctness." What this means is that we are not bound by any ontological commitment to the mathematical propositions that we are using. Rather, we use recursion to describe, then induction to prove (substantiate), statements about infinitely many propositions in logic. We do not need to worry, as Connelly does, that we are giving too much metaphysical credence to sets and means of proofs, just as we do not—and Connelly certainly does not—give ontological status to the N operator. Just as N features in a methodology for describing propositions, "mathematics is a method of logic" (TLP

6.234). We can employ mathematical propositions (recursive definitions and inductive proofs) without needing a full-blown, ontologically endowed theory of classes (explicitly condemned at TLP 6.031) alongside facts; we can simply appeal to the methodological stipulations that Wittgenstein provides us at TLP 5.501—of which Connelly himself readily approves. “It is […] enough to say that the list of elementary propositions required to express a general proposition *via* successive applications of N, may simply be stipulated to be the values of a particular propositional function. Moreover, Wittgenstein is clear that this is only one possible method of stipulating the relevant propositions and is not in any way essential” (Connelly, 2017a, pp. 16-17).

This quote is illuminating for two reasons: first, we can see the analogical relation of Wittgenstein’s “stipulation” and the “specification” in the recursive and inductive steps. Their meaning is interchangeable. Second, Connelly points out that the method of stipulation is in no way essential. This permits us to employ mathematical procedures to allow for the countability of sets, without imbuing the sets with any kind of ontological status or essentiality. Our ‘set-forming’ (‘propositional list constructing’) techniques are nothing more than mathematical steps using the “logical prototypes” (TLP [1961] 3.24, 5.522) which Connelly endorses “as a means of describing the lists of elementary propositions to which one would apply the N operator” (2017a, pp. 19-20).

Our account is perfectly consistent with the *Grundgedanke* (TLP 4.0312) that Connelly is so determined to obey—and rightly so. For “the logic of the world, which is shown in tautologies by the propositions of logic, is shown in equations by mathematics” (TLP 6.22). Mathematics, like logic, is a formal system. We do not need to posit the existence of logical constants (for which Wittgenstein criticized Russell and Frege, TLP 5.4), or give any ontological reification to our mathematical procedures.

5.5 Infinite (and Limited) Logical Space

Wittgenstein describes logical space both as "the infinite whole" (TLP 4.463) and "a limited whole" (TLP 6.45). In the previous discussion of recursive and inductive methods, we saw how it is possible to completely describe and contain an infinite set. The example used was that of a chess game, which sanctions an infinite number of moves within a finite framework. We can see that e.g. the moves of chess are both infinite and limited. An infinite set is completed insofar as it shares an equivalence relation of equinumerosity with the natural numbers. Again at TLP 5.5262, we are told of the breadth of logical space: "the range that the *totality* of elementary propositions leaves open for its construction is exactly the same as that which is *delimited* by entirely general propositions" (emphasis added).

In one of his Cambridge lectures given between 1930-32, Wittgenstein says of TLP 1.13: "'the facts in logical space are the world'. Logical space has the same meaning as grammatical space. Geometry is a kind of grammar: there is an analogy between grammar and geometry. Grammatical space includes all possibilities. 'Logic treats of every possibility' 2.0121" (King & Lee, 1980, p. 119). There is an important and illustrative analogy between chess, logic, grammar, and geometry, and other rule-governed formulaic systems—of which I will speak in greater detail in Chapter 7. Though they are infinite, they are restricted such that they can determine allowable 'moves' and exclude the unallowable. Chess treats of every possible (allowable) move, grammar treats of every possible (allowable) statement, and so forth. Thus far, I have tried to evidence the probability that Wittgenstein developed his logical system with recursive definitions and inductive proofs in mind—affording him his infinite but simultaneously enumerable totalities. I charitably suppose that Connelly thinks that it is a property of infinity whether it is countable or uncountable, that there is something prior to mathematical investigation that determines (or has already determined) whether it is actually or potentially infinite. Given the untenability of Connelly's account of infinity, I submit the above as an alternative.

CHAPTER 6 QUANTIFIERS AND NOTATION

6.1 IMPLICIT QUANTIFIERS

Connelly has set out to demonstrate and legitimize Wittgenstein's logical programme in its complete elimination of quantifiers. As such, Connelly proposes that the predicate calculus reduces to Wittgenstein's treatment of the propositional calculus, in which case quantifiers are eliminable such that all propositions (including general propositions) can be constructed out of the propositional calculus. In this case, general propositions are to be expressed in their 'logically analogous' truth-functional expansions, as presented in Chapter 4.

We can, tentatively, identify three distinct logical calculi that have been thus far discussed. The first is the propositional calculus; the second the predicate calculus; and the third is Wittgenstein's schematic approach which, in a way, occupies a place between the first two calculi. This is so because the 'logically analogous' truth-functional expansions of the propositional calculus are said to be capable of expressing generality but in the absence of quantifiers. In this case, Wittgenstein's calculus differs from the calculus of predicate logic, but is claimed to have greater expressive power than does the propositional calculus. Connelly has offered an explication and defense of Wittgenstein's attempt to eliminate quantifiers while still accounting for infinite domains, specifically via the purported logically analogous, actually infinite truth-functional expansions. But there is a large stumbling block here: strings of propositions in propositional logic are required to be finite in length. There must be a method whereby we can identify and distinguish well-formed formulae in a logical system. Determining whether a logical expression is syntactically well-formed requires that the expression is finite.

Connelly claims to have shown how to construct infinitely long, though terminal, series of propositions. If that is correct, then the truth-functional expansions could be shown like so:

Fa & *Fb* & *Fc* & … & *Fi*

But we have seen in the preceding chapter that his account of actual infinity is more aptly an account of finiteness. As such, the truth-functional expansions of the propositional calculus, like the one above, are (must be) finitely long—something required for their expressive completeness. But how, then, are we to get generality from this proposed treatment? If we are instead to consider that the propositional strings are infinite—that is, ongoing—then working solely within the confines of the propositional calculus will not work because of the finiteness constraint. The propositional calculus is not designed for, nor capable of, dealing in domains of any kind. This is precisely why there is a separate calculus, viz. the predicate calculus, intended to deal with finite and infinite domains. Now, if we still wished to account for infinite domains in the absence of quantifiers, we could consider a restrictive predicate logic—the restriction being that there are no quantifiers. But here again, a problem arises. Consider the following:

If Socrates is human, then Socrates is mortal, which we can symbolize as:

$$Hs \supset Ms$$

Though this seems prima facie to circumvent the need for quantifiers, there lingers here an implicit quantifier:

$$(\forall x)(Hx \supset Mx)$$

This says that for all x, if x is human, then x is mortal. *Given* this statement, we can say that if Socrates is human (Hs), then Socrates is mortal (Ms). x must satisfy certain conditions in order to contribute to the meaningful expression of a statement—denoting particular object terms that participate in a domain of discourse. If this quantifier is absented, there can be no talk of domains, infinite or otherwise. Hs ⊃ Ms yields only a singular meaningful statement, unless we define our terms, and the context or domain in which these terms feature. Connelly tried to give a method for establishing disclosed and complete domains through his use of logically analogous, infinite truth-functional expansions, but there is no way to account for an infinite domain in this way—despite his attempt to demonstrate the 'finite' (terminating) quality of 'actually infinite' lists.

6.2 The Inseparability of Variables and Quantifiers

In order to make explicit the notational confusions occurring in Connelly's account, let us first take a brief look at the mechanics of predicate logic. Predicate logic (henceforth PL) differs from propositional (sentential) logic insofar as the former employs quantifiers and variables in addition to propositions and logical operators, of which the latter only makes use. The importance of the sentential character of propositional logic will be discussed shortly. 'x' is what is known as a *free* variable in PL, which means that it is an open (un-instantiated) place-holder for substitution instances (i.e., for objectual constants, such as a, b, c). In order to show proofs in PL, x must be bound by a quantifier (universal or existential), like so (∃x)Fx. (∃x)Fx is an example of a w*ell-formed* statement in PL, whereas, e.g., Fx is not a well-formed statement in PL, since it contains a variable that is not quantified. Without a quantifier, there is no way to tell what is being said of the variable x in Fx, and it therefore bears no meaningful information. Quantifiers are deployed for the purpose of referring to the *universe of discourse* (UD)—in other words, the domain in which objects and relations are being discussed. Variables and quantifiers must work in concert, if well-formed statements in PL are to be given.

Since this is the case, Connelly's variable 'i' incontrovertibly implies a quantifier. For i (in e.g. Fi, Fai, Fii) cannot be an unbound (free) variable—unless one could, inconceivably, be satisfied with its being communicatively barren. On the other hand, if i is indeed a bound variable (conveying objectual information within an established UD), the rules of PL require that it work in concert with a quantifier (∀ or ∃) in order to establish the domain in which the propositional terms can be meaningfully discussed. Connelly proposes that he has defensibly shown how to operate in the absence of quantifiers; thus he is not constrained by the rules of the predicate calculus and is licensed to make various notational uses of i. But the following sections will show, respectively, that though quantifiers do not feature in his examples, they are nevertheless contained within them, and that his notational usage results in an ambiguity that renders his treatment confusing and incomplete.

6.3 Connelly's Notational 'i'

As Connelly states,

> on my reading, N should be understood as a *sentential* operator, which operates successively first on selections of elementary propositions, and then, in turn, on their truth-functions. Constructing a proposition in N notation (or 'N-expressing' a proposition) is thus a matter of first applying the N operator to the relevant selection of elementary propositions, and in turn applying the N operator to these 'N-expressions,' and so on, until the desired expression is achieved (2017a, p. 19, emphasis added)

Connelly insists that considering N a "sentential as opposed to a quantificational operator" (2017a, p. 1) resolves recalcitrant and inevitable problems that arise from conceiving it as the latter. But this is problematic, because only quantificational operators are capable of establishing meaningful reference to domains. A sentential operator therefore could not supply us with the infinitary feature of logical expression we are seeking. An alternative, as discussed in the preceding chapter, is the use of recursion. Then, mathematical induction can be used to substantiate statements regarding particular recursively defined infinite sets. I will use ‘sets’ and ‘series’ here interchangeably. Though ‘series of propositions’ is more analogous to Connelly’s ‘list formulations’, ‘sets’ and ‘series’ have the same meaning in what follows.

Let us recall Connelly’s (2017a) list formulations; some typical examples include:

i) *~Fa* & *~Fb* & *~Fc* & … & *~Fi* (p. 20)

and

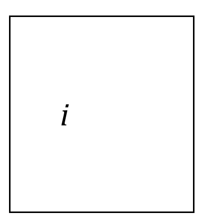
ii) *Fab* v *Fac* v *Fad* v…v *Fba* v *Fbc* v *Fbd* v … v *Faa* v *Fbb* v *Fcc* v … v *Fii* (p. 23)

and

iii)

$$N(N(N(N(p), N(p^{1})), N(N(q), N(q^{1})), N(N(r), N(r^{1})), \ldots, N(N(p^{i}), N(p^{1i})))) \quad \text{(p. 23)}$$

To make the following argument more apparent, we can reformulate our enclosed list formulation, of the kind above, into an equally enclosed box formulation, wherein all propositions, including i, would necessarily feature, like so:



For the purposes of clarity, I omit all other propositions (p, q, r, …), but we can easily imagine their enumeration in box-form rather than list-form. The box, like the first and last propositions in Connelly's series, indicates the limit on the domain of discourse, since it must be, in some way, delimited. Recall Connelly's supposition that "a number, 'infinity' […] lies at the terminus of various infinitely long series, [where he then] abstracted from the possibility that those series may differ in length or cardinality" (2017a, p. 20). According to the countability-requirement of infinite sets, those sets must be identical in size (equinumerous) to the set of natural numbers. Implicitly, then, Connelly is committed to the claim that $\forall$ i $\in \mathbb{N}$, which says that for all i, i is a member of the set of natural numbers. Further, Connelly must indirectly hold that $(\exists x)x$ = i, which says that there exists an x such that x is identical to i: i is a variable whose definition requires quantification over the natural numbers. If i does not satisfy this description, it cannot be said to meaningfully indicate a numeric value, and certainly could not be deployed as a number for the purposes of enumerating an infinite set, as Connelly purports.

Even more perplexing is Connelly's use of i on both the left- and right-hand sides of the illustration below:

$$p^{i} : Pi$$

The i to the left of the colon is a superscript which is conventionally used to index the members of a series. The left i in Connelly's schema is, presumably, a variable that is to be instantiated with a member of the set of natural numbers, appreciating that Connelly has "simply supposed the existence of a number, 'infinity', which lies at the terminus of various infinitely long series, and [has] abstracted from the possibility that *those series may differ in length or cardinality*" (2017a, p. 20, emphasis added).

In other words, i ranges over the natural numbers, such that i is—must be—a member of ℕ. This is exemplified in iii) above.

The superscript use of i reveals the built-in quantifier working in concert with i. For i ranges over the natural numbers, such that it is a variable which, as Connelly intends, stands for the "last" proposition in a (infinite) series. We can see, then, that Connelly's proposal implicitly involves the following argument:

$(\exists x)(x \in \mathbb{N} \ \& \ x = i)$

This says that there exists an x such that x is a natural number and x is identical to i. If Connelly wishes to refute this claim, he cannot continue to hold that his infinite series are countable—which is paramount to his project. The whole purpose of the *Tractatus*, and Connelly's explication, was to show the eliminability of quantifiers – but Connelly's notational i does not achieve this.

Conversely, the i appearing to the right of the colon in

$p^{i} : Pi$

is a subscript variable for which objectual names (e.g. a, b, c) are substituted. This is shown above in i) and ii). It is here that the ambiguity and duplicitous use of Connelly's notational i, which appears on both sides of the schema, becomes evident. The same inscription is employed to represent two (very different) kinds of thing: numbers (indices), and names.

Consider the series

$\sim Fa \ \& \sim Fb \ \& \sim Fc \ \& \ \ldots \ \& \sim Fi$ (p. 20)

Since i is intended by Connelly to represent "a number, 'infinity', which lies at the terminus of variously infinitely long series" (2017a, p. 20), i does not make sense in this series, because here, a, b, and c are substitution instances (constants) of the variable x in the propositional function Fx. But i, as we have seen, is not a constant—it is a variable. So in the above series, Connelly has substituted a variable (x) with another variable (i). And Connelly himself has noted that *no* variables, other than $\bar{\xi}$, may appear under the scope of N—though this plainly occurs in iii). Further, if i is a number, how does it occupy a name-place in the propositional function? What is the function Fx to mean when it is instantiated with a number (much less an index variable), viz. Fi? The above series exemplifies elementary propositions, and Wittgenstein reports that "an elementary proposition consists of names. It is a nexus, a concatenation, of names" (TLP 4.22). It seems incongruous, then, to conceive of i as a simple constant, conventionally represented by names, e.g. a, b, c. It is clear that i does not signify a name, as do a, b, and c—since the numeric variable i does not denote an object like a name, a, does. Though Connelly wishes to use Fi as the i-th proposition in a series, he has not made clear how this is.

6.4 Sets Versus Numbers

It is with Connelly's kind permission that I am able to discuss his as yet unpublished paper (under review by *Belgrade Philosophical Annual*), in which he seems to have updated his notation from i to ω, as follows: "N(fa, fb, fc, fd,…,fω) (where 'ω' stands for the infinitieth and final constant on the list)" (Connelly, 2017b, p. 10); "fa v fb v fc v fd v … v fω (where 'ω' again, stands for the infinitieth and final individual constant on the list)" (Connelly, 2017b, p. 10). Though Connelly does not explain the notational switch, perhaps it is an attempt to exemplify the countability of infinite series with greater clarity, using Cantor's transfinite arithmetic to more precisely measure the size of an infinite set. But this is no less problematic than his use of i. As A. W. Moore (1990) notes, “ $\aleph_0$ itself is just omega [ω], the first infinite ordinal. We can now see that ‘$\aleph_0$’, ‘ω’ and ‘ℕ’ are three names for one and the same entity, to wit the Set [*sic*] of natural numbers” (p. 153).

It seems, then, that Connelly’s use of ω, that he uses interchangeably with i, denotes a *set* rather than a number. But superscripts (such as the ‘1’ in p^1) are customarily employed as indices that exemplify the natural numbers indicating numeric membership in ℕ. And the indices must necessarily belong to the set of natural numbers, as per the requirement that all and only infinite sets that are the same size (that is, no larger) than the set of natural numbers are countable.

Set symbols (e.g. ℵ, ω, ℕ, and, in Connelly’s original treatment, i) are not typically employed in superscript. The reason for this is easy to see: the first proposition on a list is written as p^1, the second as p^2, the third as p^3, and so on. Bearing in mind the apparent synonymy of i and ω as illustrated by Connelly, we would not write p^i, for since i and ω are indicative of sets, it seems that *p* would then morph into a metavariable standing proxy for the set indicated. How else could it be conceived? It is unlikely that Connelly is using i as indicative of a set; but if he is not, then the interchangeability of i and ω is unclear—for the former was used by Connelly (2017a) to indicate the final number in a series and/or as an objectual constant, whereas the latter is used exclusively to denote sets. “*Ordinals* (or *ordinal numbers*) [such as ω] are used to measure length, or shape of a well-ordered infinite series” (Moore, 1990, p. 124). A *well-ordered series* is a set on which a certain order or organization has been imposed, such that the first member of the set is identified, barring the empty (null) set, which has no members; the second member of the set is identified, barring a set of just one member; the third member of the set is identified, barring a set of only two members; and so on (Moore, 1990, p. 123).

Concomitantly, for each member identified, another member is identified as its direct successor: that which comes next on the list of members, granting that there are members remaining to be identified in the series. And since, as stated above, ordinals measure the length (shape) of well-ordered finite or infinite series, "there must be an ordinal which is the first to succeed all the natural numbers. It is referred to as ω. We must beware of thinking of ω as an extremely big natural number – so big that it lies infinitely far along the progression of natural numbers" (Moore, 1990, p. 125). Reading Connelly charitably, this is, I think, precisely how he conceives of both i and ω. For he indicates several times throughout both papers (2017a; 2017b) that i (or ω) lies at "the terminus" of an infinitely long (but so ending) series.

Importantly, ordinal numbers (those used for ordering) differ from cardinal numbers (those used for counting). Ordinals do not 'behave' the same way that cardinals do—arithmetical operations on ordinals display this. For example, the size (cardinality) of ω is $\aleph_0$; the cardinality of ω + 1 is $\aleph_0$; the cardinality of ω + ω is $\aleph_0$; and so on. Ordinal numbers are not used to count, but rather indicate the well-ordering of a series, where each member of the series stands in relative position to the other members of that series. Whereas cardinal numbers are used for *counting* the members (indicating the size) of a set, ordinals are used instead to describe the well-ordering (structure) of a set. "The two different symbols ω and $\aleph_0$ are two different names for the same set. Hence ω = $\aleph_0$. And both are just signs for the set of natural numbers. So $\omega = \aleph_0 = \mathbb{N}$" (Steinhart, 2009, pp. 166-7). Connelly wishes to use the first ordinal ω to indicate the end of a countably infinite series (like so: $f\omega$). But this is a misuse of ordinals.

Connelly does note that

> greater clarity could perhaps be achieved *via* the use of transfinite numerals. However, as discussed above, Wittgenstein himself did not deploy such numerals, nor the distinctions that they embody, within his characterization of the infinite in the *Tractatus*. To be faithful to the text, therefore, I have simply supposed the existence of a number, 'infinity' which lies at the terminus of various infinitely long series (2017a, p. 20)

But Connelly's number i 'behaves' very differently than do transfinite arithmetical numbers. Perhaps Connelly wishes to say that i is used in the same way that '$\aleph_0$', 'ω' and '$\mathbb{N}$' are used. But as discussed above, '$\aleph_0$', 'ω' and '$\mathbb{N}$' indicate *sets* of infinite size, not numbers—especially not in the context of superscript indices, which deal primarily with numbers (of the set $\mathbb{N}$).

CHAPTER 7 FORMAL LANGUAGE CONSTRUCTION (SYNTAX VERSUS SEMANTICS)

7.1 COMPARING FORM AND STRUCTURE

Currently, the terms ‘logical form’ and ‘logical structure’ are used fairly interchangeably. But there is a discreet and crucial difference in the way Wittgenstein employs these terms in the *Tractatus*. Connelly (2017a) maintains that “elementary propositions *have structure*, but are nevertheless logically independent. This entails that N may be applied, in the basis case, to semantically atomic sentence letters which lack any internal structure, just as Wittgenstein indicates within his symbol for the general form of a truth-function: [$\bar{p}$, ($\bar{\xi}$), N($\bar{\xi}$)]. (TLP 6)” (p. 1). For my part, I see no explicit indication at Proposition 6 that N is to apply specifically to unstructured elementary (atomic) propositions. Rather, consider the following claims by Wittgenstein: “the way in which objects hang together in the atomic fact is the structure of the atomic fact” (TLP 2.032, 2014); “the structure of the fact consists of the structures of the atomic facts” (TLP 2.034, 2014); “in the atomic fact the objects are combined in a definite way” (TLP 2.031, 2014).

Taking these claims together with Connelly’s instructions that propositional lists are constructed according to logical “prototypes” (2017a, pp. 19, 20, 21, 24) of the kinds e.g. Fx or xRy, wherein the variables x (and y) are instantiated with names (a, b, c) that denote objects, and that these objects are necessarily configured in a particular, determinate way, it seems incongruous to submit that elementary propositions are devoid of internal structure. It seems more accurate to consider the objects comprising atomic facts as internally structuring them. This entails that the propositions of interest match the configuration of the propositional template, or "logical prototype" (TLP 3.24, 3.315, 5.522; Connelly, 2017, pp. 19, 20, 21, 24) used to collect the intended propositions—those that are structurally isomorphic.

On Connelly's account, the second method of stipulation at TLP 5.501, discussed in the preceding chapter, gives us the propositional function 'Fx' for the purpose of collecting propositions that are structurally isomorphic, such as Fa, Fb, Fc. Though these are atomic (elementary) propositions, they are objectually constituted (structured) such that the object-names that compose them are configured in such a way as to "assert the existence of a state of affairs" (TLP 4.21), for "an elementary proposition consists of names. It is a nexus, a concatenation, of names" (TLP 4.22). In the case of two-place predicates, xRy (given, according to Connelly, at the third method), the propositions of interest are collected based specifically and exclusively on their structural isomorphism to the specified logical prototype (template), such that their objects are comparably configured to the stipulated function. It is unclear how Connelly dispatches with internal structure (even if the propositions are represented by semantically simple constants such as p, q, and r), especially considering his discussion of propositional collection relative to the stipulated functional template given for the purpose of identifying and assembling the so structured propositions of interest.

Let us consider Wittgenstein's following claim:

> I call a series that is ordered by an *internal* relation a series of forms. The order of the number-series is not governed by an external relation but by an internal relation. The same is true of the series of propositions
>
> 'aRb',
> '(∃x):aRx.xRb',
> '(∃x, y):aRx.xRy.yRb',
> and so forth
>
> (If b stands in one of these relations to a, I call b a successor of a.) (TLP 4.1252)

We may recall that a series generated via mathematical induction is well-ordered; that is, a 'first' member is identified, as well as its successor, granting that there is more than one member to the series. "The internal relation by which a series is ordered is equivalent to the operation that produces one term from another" (TLP 5.232), and this is what structures infinite series: the inductive step in a mathematical proof assumes a property of interest for all defined members of the set based on the standard definition of the base case. A well-ordered set shares an isomorphism with the natural numbers. The number-series, generated via a recursive definition, affords us talk of infinitary propositions: we abstract from this method for defining a contained infinite number series to a contained infinite propositional series. The propositional series is, too, ordered by an internal relation. The following instructions from the *Tractatus* for the construction of propositional series are, on my reading, strongly suggestive of offering a recursive definition. They are not, however, operationally straightforward and unproblematic when employed by Wittgenstein—this will be discussed in the subsequent section.

> The structures of propositions stand in internal relations to one another. In order to give prominence to these internal relations we can adopt the following mode of expression: we can represent a proposition as the result of an operation that produces it out of other propositions (which are the bases of the operation). An operation is the expression of a relation between the structures of its results and of its bases. The operation is what has to be done to the one proposition in order to make the other out of it. And that will, of course, depend on their formal properties, on the internal similarity of their forms (TLP 5.2-5.231)

"If we are given the general form according to which propositions are constructed, then with it we are also given the general form according to which one proposition can be generated out of another by means of an operation" (TLP 6.002). At first glance, it seems that operations on our specified base case result in the construction of all desired propositions. I attempted to justify this sort of recursive operation with respect to defining infinite sets. However, the infinity of syntax is different from the infinity of semantics—a difference that Wittgenstein and Connelly fail to heed, which results in an insuperable problem for the Tractarian programme.

7.2 Syntax Versus Semantics

Let us first begin by recollecting TLP 5.501 wherein we are told the three methods of stipulating the relevant (bracketed) variables, represented by metavariable $\bar{\xi}$. They are "1. direct enumeration, in which case we can simply substitute for the variable the constants that are its values; 2. giving a function fx whose values for all values of x are the propositions to be described; 3. giving a formal law that governs the construction of the propositions, in which case the bracketed expression has as its members all the terms in a series of forms." The first method is straightforward enough as it deals with finite lists that are easily enumerable, and bears no salient information about the construction of propositional series—it is therefore inconsequential to our purpose and omitted in what follows. I will focus on the second and third methods (henceforth referred to as '[2]' and '[3]', respectively). Relative to [2] and [3], Wittgenstein states that "we can speak in a certain sense of formal properties of objects and atomic facts, or of properties of the structure of facts, and in the same sense of formal relations and relations of structures" (TLP 4.122, 2014). To make explicit the relation between propositions 5.501 and 4.122, and to show the distinctness of the two classes being described, let us disassemble the individual sentences in these two propositions and place them into the distinct groups to which they belong:

a)

- i) [2] giving a function fx whose values for all values of x are the propositions to be described
- ii) formal properties [of objects and atomic facts]
- iii) formal relations

b)

- i) [3] giving a formal law that governs the construction of the propositions, in which case the bracketed expression has as its members all the terms in a series of forms
- ii) properties of the structure [of facts]
- iii) relations of structure

To group a) belong the syntactic rules of a formal language construction, as given by Wittgenstein. He describes a purely formal—symbolic—method employed for collecting those elementary propositions to be described. Wittgenstein then establishes group b), giving a semantic account that is based on the syntactical, recursive feature of propositional logic. It is in this step that Wittgenstein attempts to vindicate his elimination of quantifiers by proposing construction of propositional lists akin to Connelly's truth-functional expansions. This, it seems, would show and delimit the infinite domain (normally established via quantifiers) in actual constructions of statements without the use of quantificational notation. Accordingly, [2] specifies the syntax of propositional logic, and [3] (given [2]) purportedly licenses the elimination of quantifiers by constructing their logically analogous truth-functional expansions out of the propositional calculus. Wittgenstein assumes a similarity between [2] and [3], namely that whatever allows for the infinitary account of syntax (formal relations) allows the same for semantics (structural relations), and thus gives a principled way to gain [3] from [2]. But the seamless move from [2] to [3] is impermissible—the rules governing [2] (syntactical features) cannot be equally applied to [3] (semantics).

7.3 Syntactic and Semantic Rules

Wittgenstein has presented and justified the *syntactical* (formal, symbolic) rules for series construction [2], and transitioned (without justification) to constructions of series with semantic content [3] (dealing with facts and relations). As Wittgenstein employs a single logical operator (N), he provides instructions for collecting elementary propositions, out of which facts (whose sum is the world [TLP 2.063]) can be constructed via successive applications of this sole operator:

> We can represent a proposition as the result of an operation that produces it out of other propositions (which are the bases of the operation). An operation is the expression of a relation between the structures of its results and of its bases. The operation is what has to be done to the one proposition in order to make the other out of it (TLP 5.21-5.23)

We can see Wittgenstein's intention of constructing facts (that deal in semantics) on the basis of purely symbolic functions (syntax). He presupposes the meaning of names and the sense of elementary propositions (TLP 6.124, 2014), and as such, presumes this takes care of the semantic component of all subsequent constructed complex propositions. However, the standard view is that establishing the syntax of a language and establishing the semantics of a language are two distinct, dissimilar ventures, each of which requires a different set of rules. Syntax *generates* well-formed sentences in a formal system, whereas semantics *interprets* these sentences with respect to the domain in which they do or do not obtain. As previously discussed, the (syntax of) propositional calculus—of which Wittgenstein makes exclusive use—is not designed to deal with domain or content, its being purely symbolic (communicatively inert prior to interpretation). Syntax governs form—symbols that have no content or context (domain); semantics manages facts—structures with content, that must feature in a specified context (domain).

We can observe the endeavor to give an infinite (general) account of both syntactical strings of propositions, and of the semantic accounts of quantification:

> Suppose that I am given *all* elementary propositions: then I can simply ask what propositions I can construct out of them. And there I have *all* propositions, and *that* fixes their limits. Propositions comprise all that follows from the totality of all elementary propositions (and, of course, from its being the *totality* of them *all*). (Thus, in a certain sense, it could be said that all propositions were generalizations of elementary propositions.) (TLP 4.51-2)

Wittgenstein purports to give the form of generality ('all'), which is intended to validate the content of generality. We can reverse engineer Wittgenstein's constructive intentions by considering the following: "it is obvious that the analysis of propositions must bring us to elementary propositions which consist of names in immediate combination. This raises the question how such combination into propositions comes about" (TLP 4.221); "it immediately strikes one as probable that the introduction of elementary propositions provides the basis for understanding all other kinds of proposition. Indeed the understanding of general propositions *palpably* depends on the understanding of elementary propositions" (TLP 4.411); "an operation manifests itself in a variable; it shows how we can get from one form of proposition to another. It gives expression to the difference between the forms. (And what the bases of an operation and its result have in common is just the bases themselves.)" (TLP 5.24).

But the result of an operation is a series of descriptive facts (assertoric statements), whereas the bases are purely formal. However, Wittgenstein maintains that "propositions comprise all that follows from the totality of all elementary propositions (and, of course, from its being the *totality* of them *all*.) (Thus, in a certain sense, it could be said that *all* propositions were generalizations of elementary propositions.)" (TLP 4.52). It becomes clear that Wittgenstein has given instructions for syntactic rule-forming, while presuming it is equivalent to, or sufficient for, semantic rule-forming: “it is obvious that we can easily express how propositions may be constructed with this operation, and how they may not be constructed with it; so it must be possible to find an exact expression for this” (TLP 5.503). This anticipates the general form of a proposition. “The existence of a general propositional form is proved by the fact that there cannot be a proposition whose form could not have been foreseen (i.e. constructed). The general form of a proposition is: *This is how things stand*” (TLP 4.5, emphasis added); “if all true elementary propositions are given, the result is a complete description of the world. The world is completely described by giving all elementary propositions, and adding which of them are true and which false” (TLP 4.26). But what these “things” (terms) are, and where they “stand” (domain), is never established. The propositional calculus with which Wittgenstein is generating his “complete description of the world” (TLP 4.26) deals with well-formedness, which we define recursively. The difficulty is that well-formed formulae, merely as such, do not have content and cannot establish domains—they are purely formal (symbolic). At [2], Wittgenstein gives a recursive definition of sentences that comply with the well-formedness rule; in addition to well-formedness, the *completeness principle* dictates that in order for a formal language to be complete, every formula of that system that possesses a property of interest must be generable based on the mechanics of the system. In my interpretation, Wittgenstein has satisfied these requirements at [2]. However, finding justification with [3] on the basis of [2] is impermissible; [2] allows the generation of propositional series, but [3] requires an interpretive rule and domain establishment—neither of which are provided and cannot be admissibly presupposed.

We require a sort of ‘mapping guidance’ from which the truth or falsity of propositions (assertoric statements about the world) is determined. Wittgenstein’s *picture theory*, which states, roughly, that a proposition’s truth is established if it accurately maps onto the actual world, tries to provide an infinitary account of semantics in addition to (or concurrently with) an infinitary account of syntax, without establishing a methodology for doing so—the justification is presumed to be self-evident. “Propositions show what they say” (TLP 4.461). [2] deals with well-formedness, but then [3] moves into realm of assertions. We can recursively define those propositions which satisfy the stipulated logical prototype, but this does not automatically warrant claims about which conditions obtain. There is a difference between collections of propositions which satisfy the logical prototype, and whether these conditions obtain. The difference between mathematical procedures and the content of facts is made clear: “the possibility of proving the propositions of mathematics means simply that their correctness can be perceived without its being necessary that what they express should itself be compared with the facts in order to determine its correctness” (TLP 6.2321).

[2] gives us “the specification of all true elementary propositions [which] describes the world completely. The world is completely described by the specification of all elementary propositions *plus the specification*, which of them are true and which false” (TLP 4.26, 2014, emphasis added). Wittgenstein never provides, but simply presupposes, this specification.

7.4 A Question of Completeness

The advantage of Wittgenstein’s proposed system is that it supposedly does not require metalinguistic devices (such as quantifiers) to work with natural language.

Connelly (2017a) seeks to demonstrate and justify the expressive completeness of the N operator by introducing his truth-functional conjunctive expansions which doubly eliminate quantifiers and fully express generality, but let us consider two distinct kinds of completeness. Formal systems are said to be *expressively* complete if and only if they can fully express all the objects of concern—in this case, the form *and* content of the catalogued propositions. Connelly purportedly shows this by offering his "infinitely long (but nevertheless terminal) conjunctions or disjunctions of the propositional calculus, which correspond to quantification over infinite domains within the predicate calculus. In this case, no facts would be left over, and thus there would be nothing left over to say" (2017a, p. 11). But given the above considerations that a general semantic account is presupposed but unjustified, his account of N shows that it is expressively incomplete.

On the other hand, we may turn our attention to *functional* completeness, which requires that a system's logical connectives (in Wittgenstein's case, solely and primarily the N operator) can give full expression to all proposed propositional functions. Given that Wittgenstein drew inspiration from Dr. Henry Sheffer (as Russell presumes at TLP, xv), who showed "that all truth-functions of a given set of propositions can be constructed out of […] 'not-p and not-q'" (TLP, xv), it seems plausible that Wittgenstein's logical system employing exclusively the N operator could, perhaps, be said to be functionally complete. However, G. E. M. Anscombe notes that though the set of elementary propositions is denumerable (their size being $\aleph_0$), "the truth-functions of an infinite set of elementary propositions form a non-denumerable set. This is so, because the number of different assignments of truth-values to n propositions is 2^n. The number of different assignments of truth-values to $\aleph_0$ propositions (i.e. to a denumerably infinite set of propositions) is therefore $2^{\aleph_0}$" (1959, p. 136). As Anscombe observes, "this has been proved by Cantor to be greater than $\aleph_0$; that is to say, you could not find a one-one correlation between a set whose number was $2^{\aleph_0}$ and a set whose number was $\aleph_0$" (1959, p. 136). Further, "the truth-functions of $\aleph_0$ propositions must be *at least* as many as the possible ways of assigning truth-values to them. Therefore an account which correlates the series of truth-functions of an infinite set of elementary propositions with the series of natural numbers, as Wittgenstein's does, must be wrong if Cantor is right" (Anscombe, 1959, p. 136). It seems, then, that Wittgenstein—and Connelly—loses the successiveness requirement integral to N's expressive and functional completeness, for the series of truth-values is non-denumerable, so one cannot work from the sets of inner brackets to the outer sets, as Connelly intends.

Anscombe's observations also illustrate clearly how an infinitary semantic (truth-valued) account could not be given: the set of truth value assignments is non-denumerable insofar as it does not share an equivalence relation with the set of natural numbers. Given this, [3], in order to produce anything meaningful, requires further specification.

CHAPTER 8 CONCLUSION

The N operator, given at Proposition 6 in Wittgenstein's *Tractatus*, is intended by Wittgenstein to act as the only operator in his logical system, and is claimed to be capable of expressing all (even complex and general) propositions. The brevity with which Wittgenstein wrote leaves much that is unexplained and unexplored with respect to the nature and expressive capabilities of the N operator. Philosophers such as Geach and Fogelin have attempted to determine both the workings of N, and its success as the lone operator with which all propositions can be expressed. They ultimately concluded that Wittgenstein's operator N cannot fulfill its duties as regards general propositions, and that N is required to work in concert with auxiliary devices, such as class-forming operators. Connelly has argued against Geach and Fogelin that their criticisms of operator N as expressively incomplete are the result of their own misunderstandings of certain Tractarian commitments, and subsequently their own misapplication of the operator. Had they appreciated certain fundamental conditions, such as treating infinity as terminating, and employing N as a sentential as opposed to quantificational operator, Connelly insists that their allegations would not be conclusive. Connelly offers a detailed account of N that is intended to show its expressive completeness, and free Wittgenstein's logical system from the disparaging and, according to Connelly, inaccurate evaluations given by Geach and Fogelin.

In order to address these matters, I first described logical atomism as featured in the *Tractatus*, the better to lay bare the kinds of logical entities which are being discussed throughout Wittgenstein's work and this project. These logical entities were objects, atomic propositions, and facts—the totality of which is the world. Chapter 3 focused on Connelly's explanation of infinity as conceived by Wittgenstein when writing the *Tractatus*. In Chapter 4, I explained the nature and role of each symbol as contributing to the mechanics of "the general form of a proposition" (TLP 6). I then demonstrated how Connelly intends the N operator to work, which relies on his idea of terminating infinity. The concerns of Geach and Fogelin, specifically regarding N's ability to express general propositions, were explained and offset against Connelly's account. Chapter 5 presented an evaluation and critique of Connelly's justification of actual infinity, followed by my proposal of mathematical induction and recursion for completing infinity. In Chapter 6, I showed that though Connelly claims to be working with propositions that do not involve or require quantifiers, as Wittgenstein intended, there are quantifiers implicit in his account. I also raised concerns regarding certain notations of which he makes use, specifically the variable 'i', which is ambiguous and misappropriated. The seventh and final Chapter explicated Wittgenstein's treatments of logical form and logical structure, with which he attempts to account for syntactic and semantic construction of formal languages, respectively. This led Wittgenstein to presuppose the justification of the latter based on the justification of the former, though, as I hope to have shown, he is not licensed to do this. I ended Chapter 7 by concluding that, despite Connelly's alleged completion of infinity via truth-functional expansions, coupled with Wittgenstein's illicit move from syntactic rule-forming to semantic rule-forming, the N operator is both expressively and functionally incomplete.

REFERENCE LIST

Anscombe, G. E. M. (1959). *An introduction to Wittgenstein's* Tractatus*: Themes in the philosophy of Wittgenstein*. South Bend, IN: St. Augustine's Press.

Connelly, J. R. (2017a). On operator N and Wittgenstein's logical philosophy. *Journal for the History of Analytic Philosophy, 5*(4), 1-27.

Connelly, J. R. (2017b). On Wittgenstein's transcendental deductions. Manuscript under review by *Belgrade Philosophical Annual*, 1-31.

Dauben, J. W. (1979). *Georg Cantor: His mathematics and philosophy of the infinite*. Cambridge, MA: Harvard University Press.

Enderton, H. B. (1977). *Elements of set theory*. San Diego, CA: Academic Press, Inc.

Fogelin, R. (1982). Wittgenstein's operator N. *Analysis, 42*, 124-127.

Geach, P. T. (1981). Wittgenstein's operator N. *Analysis*, *41*, 168-170.

Hintikka, M. B., & Hintikka, J. (1986). *Investigating Wittgenstein*. Oxford, UK: Basil Blackwell Ltd.

King, J., & Lee, D. (Eds.). (1980). *Wittgenstein's Lectures Cambridge, 1930–32*. Chicago, IL: The University of Chicago Press.

Lewis, H. R., & Papadimitriou, C. H. (1998). *Elements of the theory of computation* (2nd ed.). Upper Saddle River, NJ: Prentice Hall, Inc.

McGray, J. W. (2006). The power and the limits of Wittgenstein's N operator. *History of Philosophy and Logic*, *27*, 143-169.

Monk, R. (1990). *Ludwig Wittgenstein: The duty of genius*. London, UK: Vintage Books.

Moore, A. W. (1990). *The Infinite*. New York, NY: Routledge.

Ramsey, F. P. (2013). *The foundations of mathematics and other logical essays*. R. B. Braithwaite (Ed.). Mansfield Centre, CT: Martino Publishing.

Russell, B. (n.d.). *Introduction to mathematical philosophy*. Middletown, DE. (Original work published in 1919)

Steinhart, E. (2009). *More precisely: The math you need to do philosophy*. Peterborough, ON: Broadview Press.

Wittgenstein, L. (1961). *Tractatus Logico-Philosophicus*. (D. F. Pears & B. F. McGuinness, Trans.). New York, NY: Routledge Classics. (Original work published in 1921)

Wittgenstein, L. (2014). *Tractatus Logico-Philosophicus*. (C. K. Ogden, Trans.). New York, NY: Seven Treasures Publications. (Original work published in 1921)