APPROXIMATE EQUILIBRIA IN AN ECONOMY WITH INDIVISIBLE GOODS

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ABSTRACT

This thesis examines the general equilibrium allocation of a finite economy where both divisible and indivisible commodities are present. Technical conditions dictate that the number of divisible goods in the economy must be at least one.

It is found that since indivisibility encompasses nonconvexity, the equilibrium in this indivisible economy is plagued by the nonconvexity related problem of infeasibility. On the other hand, indivisibility entails the unique problem that some consumption bundles may not be optimal in equilibrium. The conclusion of possible nonoptimality in the present model is confirmed by similar findings of existing indivisible models. However, the present result reflects an improvement in restricting the potentially nonoptimal situations to a very small set.

The strength of this study is derived from the use of a recent equilibrium existence theorem by Gale and Mas-Colell. This application enables the thesis to produce the above results under fewer and less rigid conditions than existing alternative models.

LIST OF NOTATIONS

Logic

$\forall$	for every
$\exists$	there exists
$\nexists$	there does not exist.
$\Rightarrow$	implies
iff	if and only if

Set Theory

$\{\ldots\}$	set definition
$\mid$	such that
$\in$	is an element of
$\notin$	is not an element of
$\cup$	union
$\cap$	intersection
$\subset$	is a subset of
$\not\subset$	is not a subset of
$\setminus$	set exclusion
$\emptyset$	the empty set

Euclidean Space

R	the set of real numbers
R^n	the Euclidean n-space
Z	the set of integers
Π	Cartesian product
$d(x,y)$	the Euclidean distance between x and y
$N(x,\delta)$	the δ-neighborhood of $\mathring{x}$
Bnd X	the boundary of the set X
Cl X	the closure of the set X
Int X	the interior of the set X
Conv X	the convex hull of the set X, also $\mathring{X}$
$\geq$	$\lambda \geq \delta$, iff λ is at least as great as δ
$>$	$\lambda > \delta$ iff $\lambda \geq \delta$ and $\delta \ngeq \lambda$
$\ngeq$	$\lambda \ngeq \delta$ iff λ is not at least as great as δ
$\geqq$	$x \geqq y$ iff $\forall k \in K$: $x_k \geq y_k$
$\geq$	$x \geq y$ iff $x \geqq y$ and $y \ngeqq x$
$>$	$x > y$ iff $\forall k \in K$; $x_k > y_k$.
$[\lambda,\delta]$	the interval between λ and δ inclusive
θ	the origin $(0,\ldots,0)$
Ω	the non-negative orthant $\{x \in R^n \mid x \geqq \theta\}$
$\alpha\colon \Delta \to P$	a mapping from Δ into P
$\equiv$	identity
$\sum\limits_I$	summation over the index set I

The Set of Proper Points

F	the set of proper points $R^{n_d} \times Z^{n-n_d}$
$R(X)$	the rest of X
$E(X)$	the edge of X
$N(x,\delta)_F$	the δ-neighborhood of x in F
e	unit vector in the divisible direction; $(1,0,\ldots,0)$

Other Symbols in this Thesis

I	the index set of consumers, $\{1,\ldots,m\}$
J	the index set of producers, $\{1,\ldots,\ell\}$
K	the index set of commodities, $\{1,\ldots,n\}$
i	the i-th consumer
j	the j-th producer
k	the k-th commodity
Q	the index set of spanning sets in R^n, $\{1,\ldots,q\}$
$S(x)$	the spanning set of x
X_i	the i-th consumption set
Y	the supply set
Δ	the n-1 unit simplex
$\alpha_i: \Delta \to R$	the income distribution function
$\Pi: \Delta \to R$	the profit function
$C_i(x)$	the not-worse-than x set
E	the basic economy
$\hat{E}$	the convexified economy

W.A.E.	weak approximate equilibrium
O.A.E.	optimal approximate equilibrium
F.A.E.	feasible approximate equilibrium
g_i	the lower bound of X_i
$\bar{g}_i$	the upper bound of X_i
$\bullet$	a binary relation
$\succsim_i$	the preference relation "at least as desired as"
$\succ_i$	"strictly preferred to"
$\sim_i$	"indifferent to"
P_i	the preference relation defined in $\dot{F}$
D(2.1)	Definition (2.1)

ACKNOWLEDGEMENT

I wish to express my deep gratitude to my teacher, Prof. E. Klein, who initiated my interest in general economic equilibrium. He has generously offered his encouragement and guidance without which this thesis could not have been completed.

I would also like to thank the members of my examining committee for sacrificing their precious time in reading this admittedly tedious work.

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I realize that the opportunity costs of all those who have assisted with this thesis are very high. Therefore this acknowledgement may be considered only as a small initial payment of my debt to them.

Chapter 0: Introduction

0.0 General Background

The spectrum of economic systems encompasses a vast range which is bounded at one pole by the centralized system and the other pole by the decentralized system. Briefly, a centralized system is one in which economic activities are planned and manipulated by one group of agents on behalf of all others to reach certain pre-defined social welfare or goals. On the other hand, in a decentralized system economic decisions are undertaken by the individual economic units, each motivated solely by its self-interest. The text book case of perfectly competitive or market economy is an example of a decentralized system. In a market economy, the actions of the individual economic agent (consumer or producer) have no effect on the economic parameters (prices). Each agent takes these parameters as given and behaves accordingly (adjusts his consumption or production bundle) to obtain his selfish objective (maximizing satisfaction or profit).

Numerous writers from Adam Smith on have demonstrated the theoretical superiority, in a specific way, of the decentralized system over the centralized one in the allocation of scarce economic resources. While the question of efficiency in resource distribution is important in its own right, hereafter we will only focus our attention on the other equally interesting issue: that of the logical consistency of a market system. Logical consistency is defined as the mutual compatibility of the actions of all the economic agents in the system; or the ability of the system to allow the actions of its numerous agents to be carried out simultaneously. Certainly the existence of a state of mutual compatibility in a system where there is a large number of different agents each acting out of greed is far from being trivial. The investigation of this state of mutual compatibility of an economic system falls under the area of general equilibrium analysis.

A condensed recount of the early contributions and development of general equilibrium analysis is found in Arrow & Hahn [3, Ch.1]. The most original and notable early contributor in this area is generally considered to be Walras [28] who described the economy as a system of equations where equilibrium is expressed as a state of equality between supply and demand in each and every market. The existence of such a state was first rigorously shown by the mathematician Wald [27]. von Neumann followed shortly with the proof of the existence of general equilibrium using game theory approach in an economy with production only [23]. Of more significance and relevance to the present study are the contributions during the past two and half decades in which mathematical rigor and finesse have been greatly developed. Some of these recent works are represented by McKenzie [22], Arrow & Debreu [2], Gale [15] and Debreu [9], the last being the most formal and complete. A common feature among these modern treatments of the subject is the construction of an axiomatic economic model in which an equilibrium, i.e.

a state of mutual compatibility, is shown to exist as a logical and mathematical deduction of the chosen axioms. As will be explained in more details later in this chapter, the basic objective of this thesis is to analyse the state of mutual compatibility in an economic model where some of the mathematically standard axioms have been relaxed.

0.1 Fixed-point Theorems and the Convexity Assumption

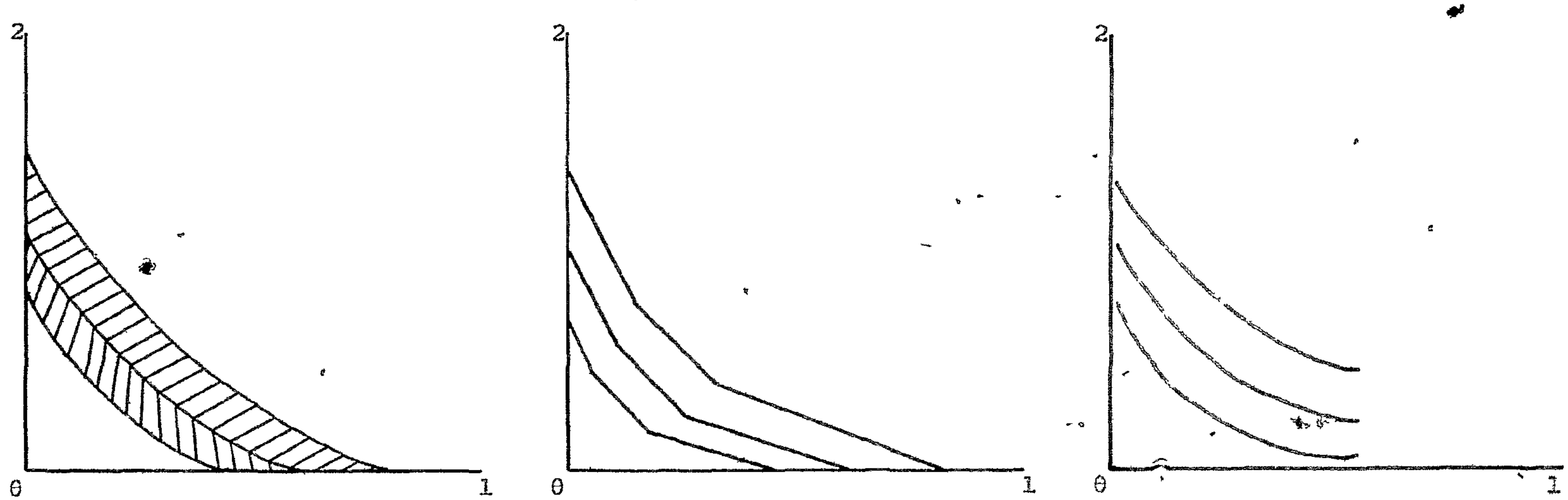
The investigation of the state of general equilibrium and its existence is positive in nature and thus adapts itself readily to mathematical analysis. The predominant mathematical tool employed in the majority of equilibrium economic models is a class of theorems known as the fixed-point theorems (See Berge [6], Kakutani [18], and Klein [19]). The fact that the fixed-point theorems have become the standard modern tool in the analysis of general equilibrium is summarized by Klein: "We do not exaggerate if we say that the problem of the existence of a general equilibrium in a closed economy model is nothing but the problem of the existence of a fixed-point for some suitable mapping, defined in terms of the components of the model." [19, p.122]. This view is further supported by Hildenbrand and Kirman who suggested that their text on equilibrium analysis may be appropriately described by the subtitle: "Variations on themes by Edgeworth and Walras scored for modern instruments, convex sets and fixed-points." [17, p.v].

In order to apply the fixed-point theorem to the proof of existence, equilibrium models have incorporated axioms that are not only a respectable reflection of economic reality, they must also conform to the required mathematical conditions of the theorem. One such axiom , which is well known in many major models, is the axiom of convexity of consumer preference ordering and of the production set. In the discussion immediately below, we will only consider the convexity of the consumer preference ordering.

There are three concepts of convex preferences. The first is referred to as weak convexity which states that if x and y are any two consumption bundles with x being preferred or indifferent to y, then any proper convex combination† of x and y is preferred or indifferent to y, (if $x \succsim y$ then $\lambda x + (1-\lambda) y \succsim y$ for $\lambda \varepsilon]0,1[$). The second concept is called regular convexity which requires that if bundle x is strictly preferred to bundle y then any proper convex combination of x and y is strictly preferred to y, (if $x \succ y$ then $\lambda x + (1-\lambda) y \succ y$ for $\lambda \varepsilon]0,1[$). Lastly, strong convexity says that any convex combination of two indifferent bundles is strictly preferred to both, (if $x \sim y$ then $\lambda x + (1-\lambda) y \succ y$ for $\lambda \varepsilon]0,1[$). Weak convexity allows for the possibility of thick indifference class and the presence of local satiation. Indifference class under regular convexity assumption may include linear segments which in turn imply multi-valued demand mapping. Strict con-

† A proper convex combination of a set of vectors is defined as a weighted sum of these vectors, each weight being positive and the sum of all the weights equals unity. Other technical terms and notations encountered in this section will be formally defined in later chapters.

Figure 0.0



(a) Weak Convexity

(b) Regular Convexity

(c) Strong Convexity

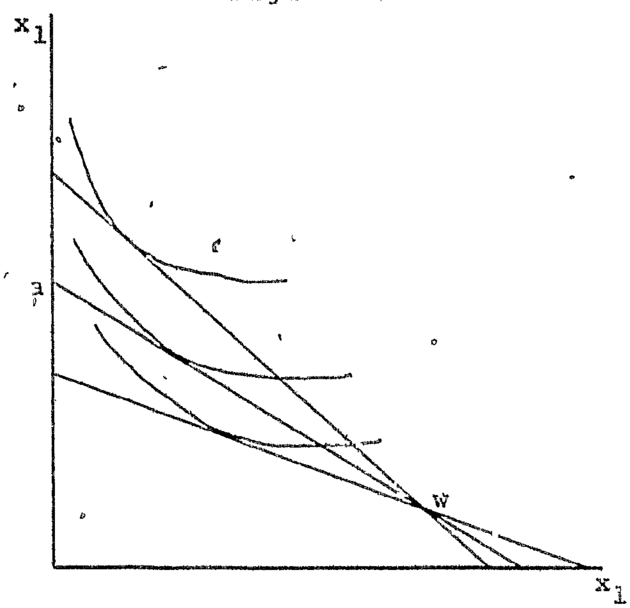
vexity, which is equivalent to the principle of diminishing marginal rate of substitution, results in single-valued demand mapping. These concepts of convexity preferences are illustrated in Figures 0.0 (a), (b), and (c) respectively.

Whatever version of convex preference ordering is assumed, it involves numerous important economic implications. One economic interpretation of the axiom of convex preference is that it prevents the consumer from reacting drastically to infinitesimal change in prices. Even if the consumer response is extreme, as under regular convexity, all commodity bundles between the extremes are possible. Mathematically, this characteristic of consumer behavior generally assures the continuity(upper-semi-continuity) of the demand function (correspondence).' This property of the demand mapping will be seen to be vital to the application of the fixed-point theorems.

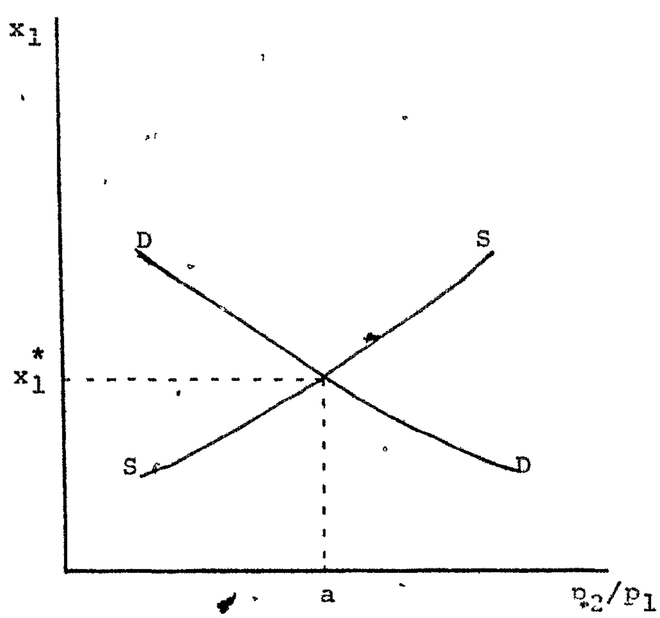
A brief digression at this point is useful to illustrate the effect of the absence of convex preference and the resulting discontinuous demand mapping on the existence of equilibrium.

The demand curve DD for commodity x_1 expressed as a function of the price ratio P_1 / P_2 in Figure 0.1 (b) is derived from the convex indifference classes in Figure 0.1 (a), w denotes the bundle of initial endowment. Suppose that the supply curve for x_1 is given by SS in Figure 0.1 (b). If single market equilibrium is defined as the price ratio at which demand equals supply, then in Figure 0.1 (b) equilibrium for

Figure 0.1

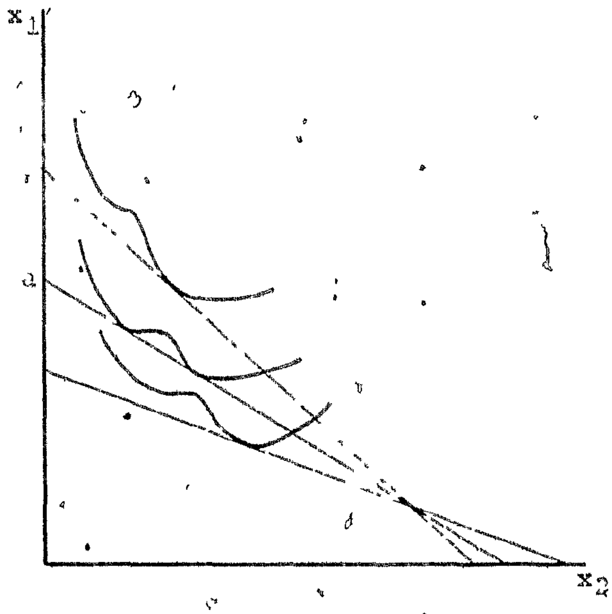


(a)

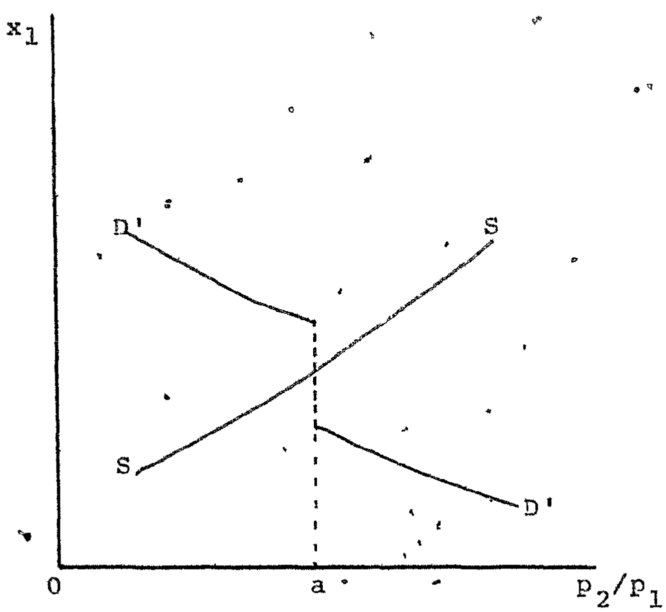


(b)

Figure 0.2

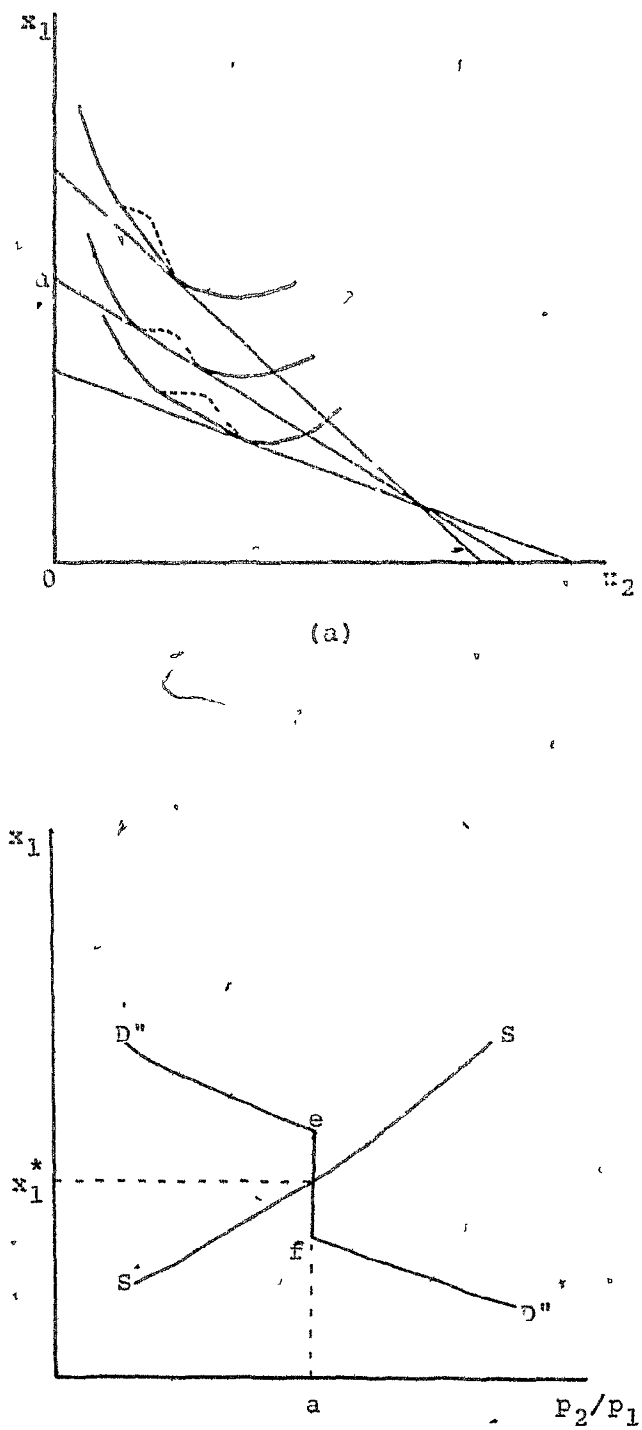


(a)



(b)

Figure 0.3



(a)

(b)

x_1 market is obtained at the price level a where demand and supply coincide at x_1^*.

The same given supply curve SS is drawn in Figure 0.2 (b). The demand curve D'D' in this case is derived from the nonconvex preference classes of Figure 0.2 (a). There is discontinuity at price ratio a and equilibrium as defined no longer exists. That is, there exists no price ratio p_1/p_2 where the corresponding amounts of x_1 demanded and supplied are equal. If partial equilibrium for x_1 market is not obtainable then obviously general equilibrium for the whole system does not exist.

It is interesting to observe that if the nonconvexity in Figure 0.2 (a) is eliminated by bridging the nonconvex gaps with linear line segments then the resulting indifference classes will be convex as shown in Figure 0.3 (a). The corresponding demand curve D"D" is drawn in Fig. 0.3 (b) where all amounts between points e and f are possible thus the discontinuity of the demand curve D'D' in Figure 0.2 (b) is avoided. In this case an equilibrium exists at price ratio a. As will be seen later, this bridging or convexification is the basic method employed in tackling the nonconvex problem.

As shown above, convexity of preference ordering generates the continuity property of demand mapping which is necessary to the application of the fixed-point theorem. Despite this mathematical convenience, the axiom of convex preferences is economically restrictive in that it disregards the classes of anticomplementary commodities and indivisible commodities.

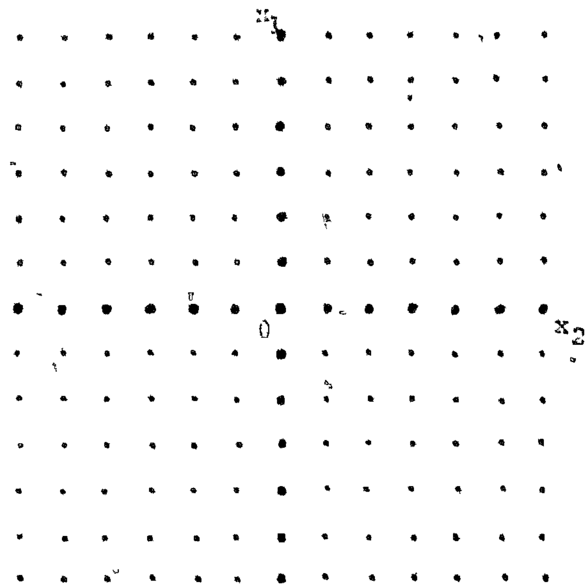
Firstly, anticomplementary commodities refer to certain pairs or groups of commodities, such as sleeping pills and an evening at the theatre, which may be antagonistic in simultaneous consumption. Arrow & Hahn [3, p.173] observed that a proper convex combination of tripe à la mode de Caen and filet de sole Marguéry might not be highly regarded by a gourmet. (They commented, however, that perhaps the dishes of Chinese cuisine may be more suitable for convex preferences.) Secondly, economic reality admits the presence of numerous commodities which are produced and consumed only in whole units. These units might be of such enormous proportion or physical nature that they can hardly be considered divisible, a property subsumed by convexity. Figure 0.4 (a) illustrates a commodity space consisting of all indivisible goods while 0.4 (b) shows a space of mixed indivisible and divisible goods. Obviously, neither of these cases allows convex preference ordering as defined earlier.

For these reasons, it is easily conceived that the possibility of nonconvex preferences existing in reality is very likely. It is desirable, therefore, that theoretical economic models of consumer behavior should incorporate this reflection of reality. As it turns out, a considerable amount of research has been directed toward this area.

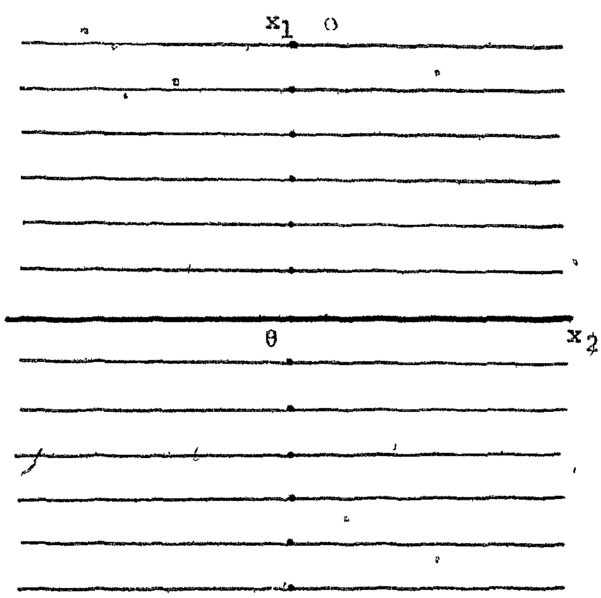
0.2 Literature on Nonconvexity and Indivisibility

One of the first authors to argue that the assumption of convexity is less than absolutely necessary is Farrell [14].

Figure 0.4



(a)



(b)

His paper generated interesting debates on the subject by Bator [5] and Rothenberg [24]. Unfortunately, these discussions on nonconvexity were confined to the flat world of two commodities. The most rigorous analysis of nonconvex preferences in general n-space is by Starr [25]. In this paper the standard proof of existence of equilibrium by McKenzie [22] was applied to an exchange economy with divisible goods where the original nonconvex preferences have been replaced by their convex hulls, or "synthetic" preferences. Using mathematical results on nonconvex sets due to L. S. Shapley and J. H. Folkman, Starr next showed that the discrepancy between this equilibrium and an associated (quasi-) equilibrium in the original nonconvex economy is bounded. The size of this discrepancy depends on the degree of nonconvexity, not on the number of agents in the system. Thus the discrepancy becomes insignificant as the number of agents increases. Arrow & Hahn [3, Ch. 7] restated Starr's results through the concepts of <u>social-approximate</u> and <u>individual-approximate</u> compensated equilibria in an economy with production.

Indivisibility implies the presence of nonconvexity, however, the converse is not necessarily true. Therefore while it is natural that an economic model with indivisible commodities may share some problems in equilibrium with divisible but nonconvex model, it should be expected that indivisibility involves additional theoretical problems unique to itself. For this reason the investigation of indivisibility must be considered quite separately from that of general nonconvexity.

Dierker [12] analysed the state of general equilibrium in an economy with only indivisible commodities. He proved the existence of a near-equilibrium in this model by using the concepts of consumer price insensitivity and "near fixed-point". This solution technique is innovative, however, even when consumer insensitivity to price is decreased, the resulting near-equilibrium still suffers from two major inexactnesses. Firstly, the near-equilibrium allocation is only approximately feasible. Secondly, it may be that the allocated bundles for some consumers are not optimal within their budget constraints. Working in an indivisible environment in which there is at least one divisible good, Broome [7] employed the more conventional method of convexification to create a synthetically continuous 'demand correspondence'. The well-known proof by Debreu [9] is applied to guarantee the existence of an equilibrium in this artificially convex model from which an associated approximate-equilibrium in the original indivisible model is found. Despite the differences in the basic models and mathematical approaches, Broome's approximate-equilibrium is similar to Dierker's in the sense that it also involves infeasibility, and under certain circumstances, nonoptimality. While the first drawback is common to all economies with general nonconvexity, the second seems to be peculiar to those with indivisibility.

It should be noted in passing that one author, Aumann [4], ingeneously circumvented the difficulties of nonconvexity altogether by using an entirely different mathematical approach, that of measure theory. Arguing that the intuitive notion of perfect competition cannot be truly reflected in any system with a finite number of agents, Aumann chose to represent the set of agents by a 'continuum' which of course contains an infinite number of points. Whereas the measure (integral) over the entire continuum is positive, the weight of each agent (point) is nil. This latter property corresponds to the perfect competition notion that individual economic agents have no influence in the market and are simply price-takers. This mathematical technique is elegant and analyses the general equilibrium solution in the ideal limit whereas models with finite number of participants approach the solution from the mathematically less than ideal state and then let the number of agents increase to the limit.

0.3 The Present Study

This thesis is an analysis of the state of general equilibrium in a finite, indivisible economy with at least one divisible commodity and with production. Its aim is twofold. Firstly, it seeks to provide simpler proof of existing results in this type of economy by using different mathematical technique and/or by incorporating less restrictive assumptions. Secondly, it strives to improve present results by introducing certain new assumptions into the model.

In relation to the first objective, the proposed proof of existence will be based on the recent mathematical results by Mas-Colell [21] and Gale and Mas-Colell [16] on general equilibrium models without complete, ordered or transitive preferences. To the best knowledge of this writer, this is the first time these results have been applied to a finite, indivisible economy. Since indivisibility automatically involves nonconvexity, the difficulties of nonconvexity will be treated in this study by the standard technique, à la Scarr. In other words, the discontinuity associated with nonconvexity is temporarily overcome by working with the convex hulls of the relevant sets. Once the existence of equilibrium is obtained in this synthetic environment, the result is related back to the original model through the established properties on nonconvex sets by Shapley and Folkman.

Regarding the second objective, it will be shown that certain concepts of pseudo convexity in an indivisible system may be used to eradicate some weaknesses of the existing results.

0.4. Organization of the Study

The thesis consists of five chapters which are organized as follows. The present chapter, Chapter 0, is a general introduction to this study. It briefly reviews some of the relevant literatures and leads to a discussion of the purpose of this work. Chapter 1 contains the mathematical concepts, properties and their proofs which are to be used in later chapters. It is mathematically self-contained and includes known results by Carathéodory, Shapley-Folkman, and Mas-Colell since these are indispensable to this thesis. Of particular importance to the results in Theorem 4 is the section on integer convexity, its properties and their proofs. In Chapter 2 the basic indivisible economy, its assumptions and the various concepts of equilibria are first defined. The basic economy is then modified by convexifying certain sets and altering the preference ordering. The alteration of the preferences is vital to the application of the Mas-Colell theorem to the present model. The core of the thesis, Chapter 3, contains the mathematical results and their proofs. Economic interpretation of these results are also included in this chapter. The final chapter, Chapter 4, is a comparison of the present findings to those in the related literature.

0.5 Notes

Some chapters are accompanied at the end by a section designated "Notes". This section serves as a general footnote to the entire chapter. It either elaborates on some point made in the body of the chapter and/or relates certain part of the chapter to the relevant references.

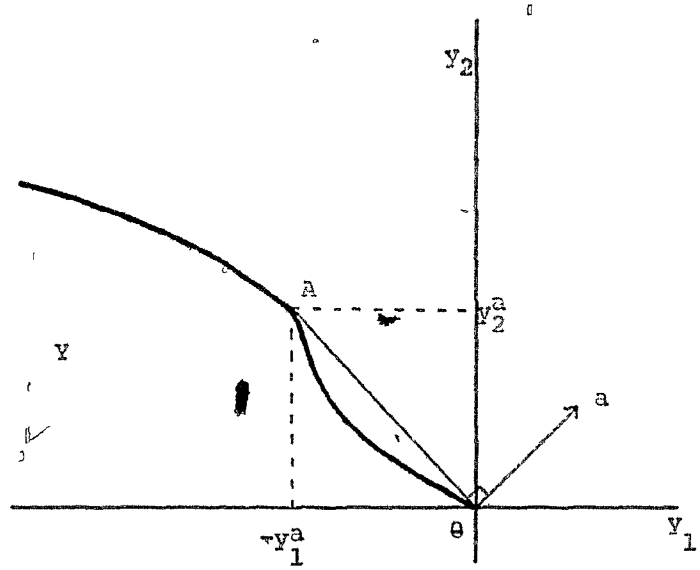
Recent literature in the area of general equilibrium analysis is growing. A general up-to-date introduction to the field is the newly published text by Hildenbrand and Kirman [17] which offers "a simple but formal account of work done to date in that part of economics which we have chosen to call equilibrium analysis".

The discussion of nonconvexity in Section 0.1 was confined to the consumption sector. Nonconvex production set and its problems are elaborated here.

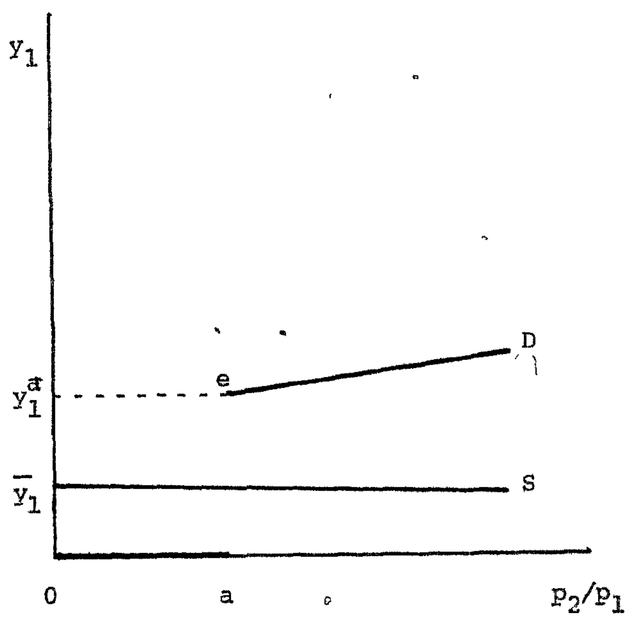
The presence of nonconvex production set in an economy, analogous to the case of nonconvex preferences, implies that the existence of equilibrium is no longer guaranteed. Figure 0.5 (a) shows a nonconvex production set, Y, in a 2-commodity world with y_1 being the input and y_2 the output. At price ratios less than a the price of the input is expensive relative to the price of the output and consequently no production takes place, i.e. the demand for the input y_1 is zero. At price ratio $p_2/p_1 = a$ profit maximization behavior dictates that the producer is indifferent between shutting-down or producing at point A. The demand for the input y_1 corresponding to the price ratio a is then either 0 or y_1^a. The demand curve for y_1 at various price ratios p_2/p_1 is shown as segments (O,A), (e,D) in Figure 0.5 (b). Assume that y_1 is only used for production and its supply is constant at $\bar{y}_1$ with $0 < \bar{y}_1 < y_1^a$, then there is no price ratio at which the demand for the input y_1 equals its supply $\bar{y}_1$. Note, however, that the discrepancy from exact equilibrium at the price ratio a can never exceed $y_1^a/2$.

The lack of exact equilibrium may be artificially eliminated by convexifying the production set Y in Figure 0.5 (a). This is done by bridging the nonconvex part of Y by the linear segment OA, the corresponding demand curve for y_1 is then expressed as (O,a,e,D') in Figure 0.5 (c). Clearly equilibrium occurs at point E with price ratio a and quantity $\bar{y}_1$.

Figure 0.4

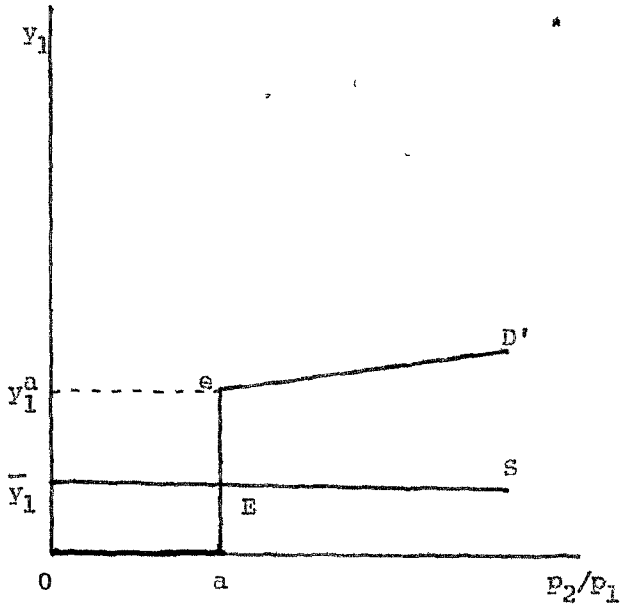


(a)

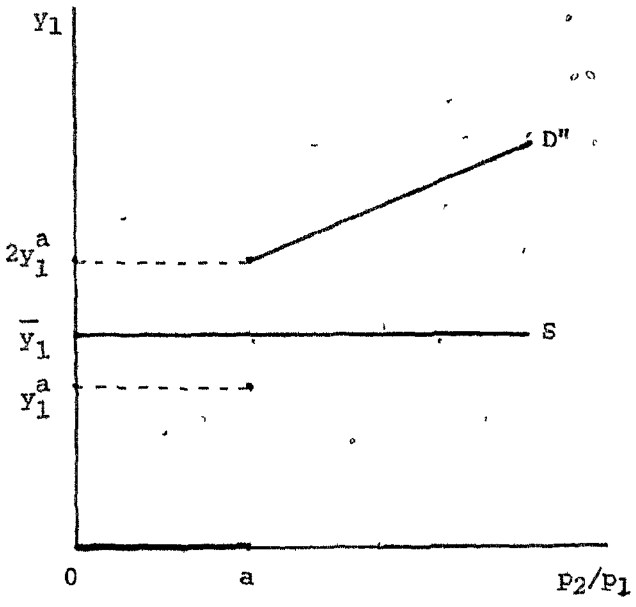


(b)

Figure 0.5



(c)



(d)

To illustrate the fact that the discrepancy due to nonconvexity does not depend on the number of economic agents, assume that there are two identical producers with production sets Y shown in Figure 0.5 (a). At the relative price ratio $p_2/p_1 = a$ either one or both producers could shut down or produce at point A. The corresponding demand for input at price ratio a is 3-valued, $(0, y_1^a, 2y_1^a)$. The entire demand curve is shown in Figure 0.5 (d). If the supply of input is fixed at $\bar{y}_1$ and $0 < \bar{y}_1 < 2y_1^a$ then there exists no exact equilibrium. However, the deviation from exact equilibrium is again at most $y_1^a/2$, the same as in the case of only one producer. It is interesting to note firstly that the increase in number of economic agents does not expand the size of the discrepancy. Secondly, in equilibrium the behaviors of identical agents may be quite different. This discussion of nonconvex production set is adapted from Arrow & Hahn [3, Chapter 7].

Chapter 1: Mathematical Preliminaries

1.0 Some Concepts and Properties in Euclidean n-space

1.0.0 General Commodity Space

All economic models in this thesis consist of a finite number of economic agents as well as a finite number of commodities. The modified economic model in the next chapter further requires a divisible setting. Therefore it is appropriate to examine in some details the properties of the finite Euclidean space R^n.

Let R denote the set of real numbers. It can also be considered as the Euclidean 1-space. A typical element of R is denoted by x, or $x \in R$.

The Cartesian product $\prod^n R = R\times\ldots\times R = R^n$ then denotes the Euclidean n-space with elements being n-tuples $x = (x^1,\ldots,x^n)$ where $x^k \in R$ for $k = 1,\ldots,n$. The Euclidean n-space is a vector space, thus $x \in R^n$ is also known as a <u>vector</u> and n is the <u>dimension</u> of the space.

The operations of <u>vector addition</u>, <u>vector multiplication</u>, and <u>scalar multiplication</u> are defined on R^n respectively as follows:

Let $x \in R^n$, $y \in R^n$, and $\lambda \in R$, then:

$x+y \equiv (x^1+y^1,\ldots,x^n+y^n) \in R^n$;

$xy \equiv (x^1y^1+\ldots+x^ny^n) \in R$;

$\lambda x \equiv (\lambda x^1,\ldots,\lambda x^n) \in R^n$.

Hereafter, superscripts are used to denote the components of a vector whereas subscripts, if any, are used to identify different vectors.

Sometimes the discussion of a particular R^n space may be restricted only to a portion of it, such as the non-negative part, denoted by Ω.

$$\Omega \equiv \{x \in R^n \mid x \geqq \theta\},$$

where $\theta \equiv (0,\ldots,0) \in R^n$ is the element with all components equal to zero, the <u>origin</u> of R^n.

<u>Remark</u>: If there is a finite number, say n, of distinct and divisible goods and services produced and consumed in the economy, then the <u>space of commodities</u> may naturally be represented by the Euclidean n-space R^n. Each vector $x \in R^n$ is then called a <u>bundle</u> of goods where $x^k \in R$ for $k = 1,\ldots,n$ denotes the quantity of the k-th good in the bundle. The commodities in the following economic models are well-defined in the sense of Debreu [9].

1.0.1 Distance and Related Concepts

The notion of distance is basic to the definitions of numerous other metric concepts in a vector space.

Let $x \in R^n$ and $y \in R^n$, the real value $d(x,y)$ defined by:

$$d(x,y) \equiv [\Sigma(x^k - y^k)^2]^{\frac{1}{2}}$$

is called the <u>Euclidean distance</u> between x and y. Clearly it can be shown that $d(x,y)$ satisfies all the axioms of a distance function:

1. $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$;

2. $d(x,y) = d(y,x)$;

3. $d(x,z) \leq d(x,y) + d(y,z)$.

The definition of the neighborhood of a vector follows immediately that of distance. Intuitively, the neighborhood of a vector $x \in R^n$ is the set of vectors which are located within a given distance from x. More formally, let $x \in R^n$, $\delta \in R$ and $\delta > 0$, then the δ-neighborhood of x in R^n, denoted by $N(x,\delta)$, is:

$$N(x,\delta) \equiv \{y \in R^n \mid d(x,y) < \delta\}.$$

Let $X \subset R^n$ and $x \in X$, if there exists a neighborhood of x which is entirely contained in X then x is called an interior point of X. The set of all interior points of X is said to be the interior of X.

Formally, if $X \subset R^n$ then the interior of X, denoted Int X, is:

$$\text{Int } X = \{x \in R^n \mid \exists\ \delta>0:\ N(x,\delta) \subset X\}.$$

A set is open if all its elements are interior points of itself. That is, let $X \subset R^n$, X is an open set if Int $X = X$.

A set is said to be closed if its complement is open. In other words, let $X \subset R^n$ then:

$$X \text{ is closed if Int } (R^n \backslash X) = R^n \backslash X;$$

where $\backslash$ denotes set exclusion and $R^n \backslash X = \{y \in R^n \mid y \notin X\}$.

Remark: The empty set $\emptyset$ and R^n are simultaneously closed and open while $N(x,\delta)$ defined above is open.

A boundary point of $X \subset R^n$ is a point which is neither in the interior of X nor in the interior of $R^n \backslash X$. The set of all boundary points of X is called its boundary and denoted by Bnd X.

Let $X \subset R^n$ then $x \in R^n$ is a boundary point of X if $x \notin \text{Int } X$ and $x \notin \text{Int } (R^n \backslash X)$.

$\text{Bnd } X \equiv \{x \in R^n | x \notin \text{Int } X \text{ and } x \notin \text{Int } (R^n \backslash X)\}$.

Obviously a set $X \subset R^n$ may or may not contain its boundary, depending on whether it is closed or open respectively.

The closure of a set $X \subset R^n$, denoted ClX, is the set composed of all interior and boundary points of X;

$\text{Cl } X \equiv \text{Int } X \cup \text{Bnd } X$.

Properties: Let $X \subset R^n$ and $\delta \in R$,

1. $\text{Bnd } X = \text{Bnd } (R^n \backslash X)$.
2. $\text{Int } X \cap \text{Bnd } X = \emptyset$.
3. $X \cup \text{Bnd } X = \text{Int } X \cup \text{Bnd } X$.
4. X is closed if $X = \text{Cl } X$.
5. $\text{Bnd } X = \{x \in R^n | x \notin \text{Int } X \text{ and } \forall\delta > 0\text{: } N(x,\delta) \cap X \neq \emptyset\}$.
6. $\text{ClX} = \{x \in R^n | \forall\delta > 0\text{: } N(x,\delta) \cap X \neq \emptyset\}$.

Another concept derived from the idea of distance is the boundedness of a set. Intuitively, a set is bounded if the distance between any two points in the set is finite.

Formally, $X \subset R^n$ is said to be bounded if for $x \in X$, a $\delta > 0$ exists such that $X \subset N(x,\delta)$.

A set $X \subset R^n$ is said to be compact if it is closed and bounded.

Examples:

1. The non-negative orthant, Ω, defined earlier is closed but not bounded.

2. Let $X = \{x \in R \mid 1 < x < 10\}$, then:

   $\text{Int } X = X$;

   $\text{Bnd } X = \{1, 10\}$;

   $\text{Cl } X = \{x \in R \mid 1 \leq x \leq 10\}$.

   Obviously X is an open and bounded set.

3. Let $X = \{x \in R \mid 1 \leq x \leq 10\}$, then:

   $\text{Int } X = \{x \in R \mid 1 < x < 10\}$;

   $\text{Bnd } X = \{1, 10\}$;

   $\text{Cl } X = X$.

   Clearly X is closed and bounded, i.e. compact.

1.0.2 Convexity and Nonconvexity

A very important mathematical concept in modern economic theory is that of the convexity of a set. A set is called <u>convex</u> if the line segment connecting any two points of the set is entirely contained in the set.

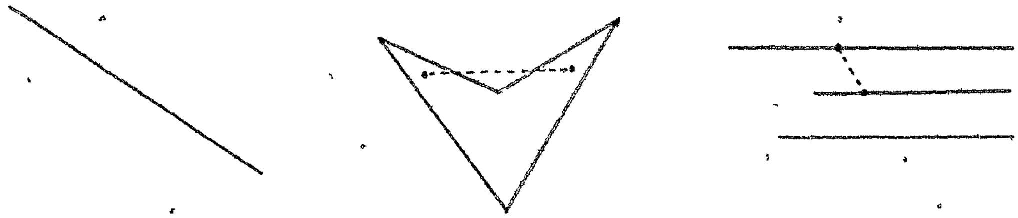
Formally, let $X \subset R^n$ and $x, y \in X$ then X is convex if $\lambda x + (1-\lambda) y \in X$ for $\lambda \in [0,1]$.

Illustrations of convex and nonconvex sets are found in Figures 1.0.

<u>Properties</u>:

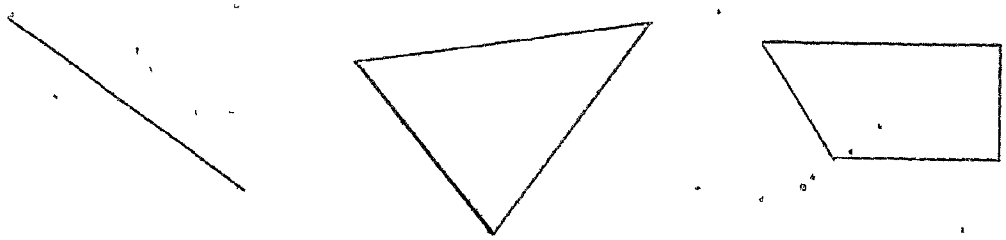
1. The intersection of any number of convex sets is also convex.
2. If $X \subset R^n$ is convex then Int X and Cl X are also convex.
3. If X is closed and convex then it is not possible to partition X into two closed, disjoint subsets.

Figure 1.0



Convex Nonconvex Nonconvex

Figure 1.1



Convex hulls of the above sets

That is, there exist no closed sets X_1 and X_2 such that: $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = \emptyset$.

The convex hull of a set is formed by taking the intersection of all closed convex sets containing the original set. It follows that the convex hull of $X \subset R^n$, denoted by either Conv X or $\dot{X}$, is the smallest closed convex set containing X.

The convex hulls of sets in Figure 1.0 are shown in Figure 1.1.

Let $X \subset R^n$:

Conv $X \equiv \dot{X} \equiv \{x \in \cap Y_i | Y_i$ is closed, convex and $X \subset Y_i\}$.

Property:

If $X \subset R^n$ is closed and convex then Conv $X = X$.

A measure of the degree of nonconvexity of a set is the next topic of discussion. However, the following definitions are needed first.

Let $X \subset R^n$, $x \in$ Conv X and $S = \{x_1, \ldots, x_n\} \subset X$. If $x = \Sigma \lambda_i x_i$ for $\lambda_i \geq 0$ and $\Sigma \lambda_i = 1$ then x is said to be spanned by S or S spans x.

A closed sphere in R^n with center at $\bar{x}$ and radius κ is the set:

$$\{x \in R^n | d(x, \bar{x}) \leq \kappa\} \text{ where } \kappa \in R \text{ and } \kappa > 0.$$

The radius of a set $X \subset R^n$, denoted rad(X), is defined as the radius of the smallest closed sphere containing X.

Let {S} be the collection of finite subsets of X that span the elements of Conv X [The existence of S for every $x \subset$ Conv X is guaranteed by the Carathéodory's Theorem, see Section 1.2].

The inner radius of X, denoted r(X), is defined as

$$r(X) \equiv \sup_{x \in X} \inf_{S \subset \{S\}} \mathrm{rad}(S).$$

The inner radius of the set X is determined by taking for each element $x \in \mathrm{Conv}\, X$ the smallest of rad(S) as S varies over all spanning sets of that point x, then take the largest over all x in Conv X of this infimum.

Remark:

(a) $\mathrm{Conv}\, X = X$ iff $r(X) = 0$

(b) X is nonconvex iff $r(X) > 0$.

Therefore r(X) may be considered as a measure of the degree of nonconvexity of X.

1.1 The Set of Proper Points

The Euclidean n-space was found to be suitable in representing the commodity space of an economy with a finite number of divisible goods. It will be assumed later, however, that some commodities may be produced and consumed only in whole units and are thus not divisible. In such an environment economic activities will actually be confined to a subset of "proper points", called F, in R^n.

$$F \equiv \{x \in R^n \mid \forall\, k \in \{1,\ldots,n_d\}: x^k \in R \text{ and } \forall\, k \in \{n_d + 1,\ldots,n\}: x^k \in Z\} = R^{n_d} \times Z^{n-n_d};$$

where n_d denotes the number of divisible commodities in the economy and the remaining commodities, $n-n_d$, are indivisible. The set Z denotes the set of integers. Each vector in F represents a commodity bundle which contains n_d divisible and $n-n_d$ indivisible commodities.

It will further be assumed in the formal model that the number of divisible goods is at least one. However, for the sake of simplification of the notations and expositions of the concepts and properties in F, the specific case of exactly one divisible good, $n_d=1$, will be considered hereafter. Therefore, for the rest of this study the set of proper points is confined strictly to:

$$F \equiv R \times Z^{n-1}.$$

An illustration of F for the case of n= 2 is found in Figure 0.4 (b) of Chapter 0.

An important vector in F is now defined. The <u>unit vector in the divisible direction</u>, denoted by e , is the point:

$$e \equiv (1,0,\ldots,0) \in F.$$

The set of proper points in R^n may be described as a collection of <u>grid lines</u>. The grid line through a point $\bar{x} \in F$ is defined as the following set:

$$\{x \in F | x = \bar{x} + \lambda e, \ \forall \ \lambda \in R\}.$$

Some of the topological concepts defined earlier in R^n are no longer valid in the subset F. These concepts are now modified for F.

Let $X \subset F$:

The <u>δ-neighborhood of $\bar{x} \in X$ in F</u>, denoted by $N(\bar{x},\delta)_F$, is defined by:

$$N(\bar{x},\delta)_F \equiv \{x \in F | d \ (\bar{x},x) < \delta\}.$$

Alternatively, $N(\bar{x},\delta)_F \equiv N(\bar{x},\delta) \cap F$ where $N(\bar{x},\delta)$ is the δ-neighborhood of $\bar{x}$ defined earlier for R^n.

The set $X \subset F$ is said to be <u>closed in F</u> if for every $x \in F$: there exists $\delta > 0$ such that $N(x,\delta)_F \subset X$.

The <u>lower edge of X</u> $\equiv LE(X) \equiv \{x \in X | (x-\lambda e) \notin X, \forall \lambda > 0\}$.

The <u>upper edge of X</u> $\equiv UE(X) \equiv \{x \in X | (x+\lambda e) \notin X, \forall \lambda > 0\}$.

The <u>upper rest of X</u>

$$\equiv UR(X) \equiv X \setminus LE(X) \equiv \{x \in X | x \notin LE(X)\}.$$

The <u>lower rest of X</u>

$$\equiv LR(X) \equiv X \setminus UE(X) \equiv \{x \in X | x \notin UE(X)\}.$$

The <u>hyperplane</u> defined by $p \in R^n$ and $\alpha \in R$ is the set:

$$H(p,\alpha) \equiv \{x \in R^n | px = \alpha\}.$$

The hyperplane $H(p,\alpha)$ is said to <u>support</u> $X \subset F$ <u>from below</u> at point x if:

(1) $x \in H(p,\alpha) \cap X$;

(2) $H(p,\alpha) \cap UR(X) = \emptyset$.

The support is said to be <u>from above</u> if condition (2) is changed to:

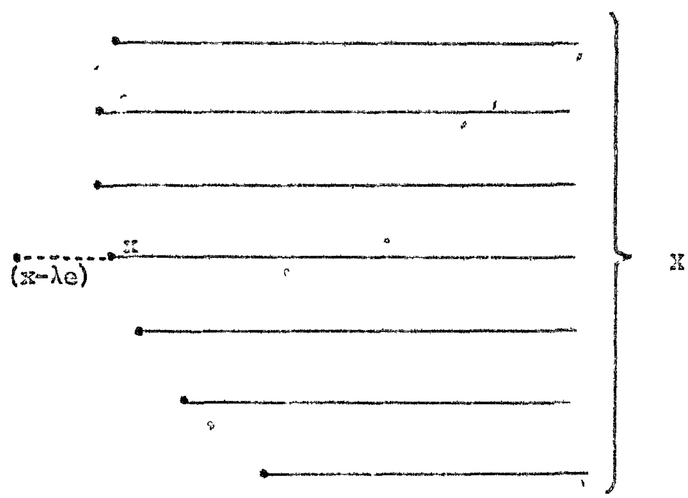
(2') $H(p,\alpha) \cap LR(X) = \emptyset$.

The terms "upper", "lower", "from below" and "from above" are rather cumbersome. Therefore they will be dropped whenever the context is clear. Hereafter the above concepts will be referred to as: the edge of X, E(X); the rest of X, R(X); and the hyperplane $H(p,\alpha)$ "supports X at x".

Figures 1.2 (a) and (b) illustrate the edge and the rest of X respectively.

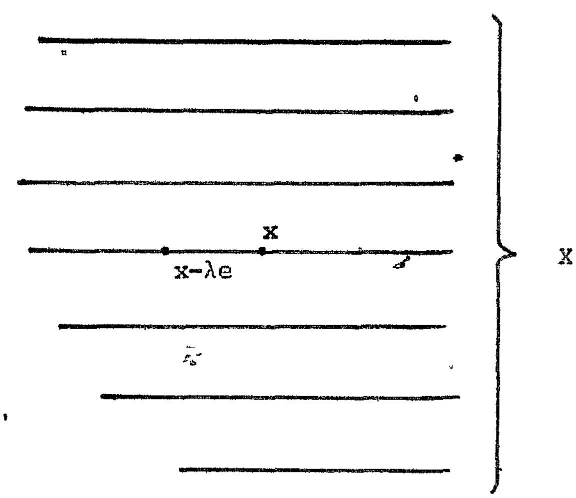
Convexity is clearly not possible in the set of proper

Figure 1.2



(a)

Edge of X



(b)

Rest of X

points due to its construction. A concept of pseudo-convexity is introduced here specifically for sets in F.

A set $X \subset F$ is said to be <u>integer convex</u> if:

$$\text{Conv } X \cap F = X.$$

The concept of integer convexity is illustrated in Figure <u>1.3</u>.

Integer convexity conveys the idea that the nonconvexity involved is attributed solely to the presence of indivisibility, and not to consumer preference or production technology which is the case of nonconvexity in a divisible environment. In other words, if the indivisible goods had been made available in divisible quantities, then integer convex sets would have been convex in the standard sense.

A set of properties of integer convex sets which are useful for later application is given and proved here.

<u>Proposition 1</u>: Let $X \subset F$ be integer convex and $H(p,\alpha)$ support X at $\bar{x}$. If for every $x \in X$: $(x+\lambda e) \in X$ for all $\lambda > 0$ then $p^1 \neq 0$.

<u>Proof</u>: $\bar{x} \subset H(p,\alpha) \cap X$ by hypothesis.

This implies $p\bar{x} = p^1\bar{x}^1 + \ldots + p^n\bar{x}^n$

$$= \alpha.$$

Suppose $p^1 = 0$.

For any $\lambda > 0$: $(\bar{x} + \lambda e) \in R(X)$.

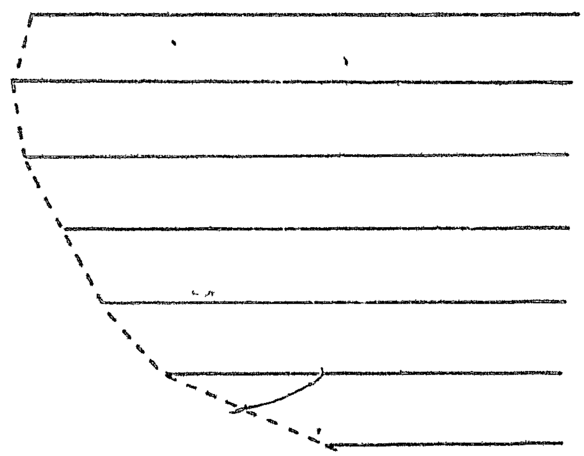
$$p(\bar{x} + \lambda e) = p\bar{x} + \lambda pe$$

$$= \alpha.$$

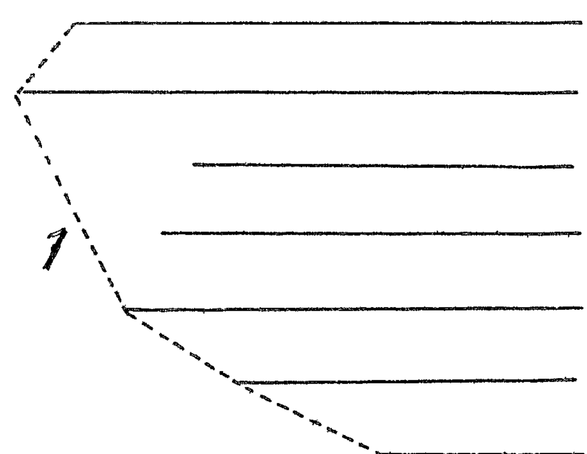
Since $p^1 = e^2 = \ldots, = e^n = 0$.

Thus $(\bar{x} + \lambda e) \in H(p,\alpha) \cap R(X)$ which contradicts the

Figure 1.3



Integer Convex



Integer Nonconvex

condition that $H(p,\alpha) \subset R(X) = \emptyset$. Therefore $p^1 \neq 0$. Q.E.D.

Proposition 2: Let $\{X_i\}_{i \in I}$ be a family of sets in F, then: Conv $(\Sigma X_i) = \Sigma$ Conv X_i.

Proof: This proposition and proof follow directly the results in more general R^n space found in Arrow & Hahn [3, p.387].

Proposition 3: Let $\{X_i\}_{i \in I}$ be a family of sets in F. If X_i is integer convex for every $i \in I$ then ΣX_i is also integer convex.

Proof: (i) Take $x \in (\Sigma \dot{X}_i) \cap F$.

This implies $\exists\, x_i \in \dot{X}_i \quad \forall i \in I$ such that $\Sigma x_i = x$.

Suppose $\exists\, j \in I$ such that $x_j \notin \dot{X}_j \cap F = X_j$, then $\sum_{i \in I} x_i \notin F$.

Therefore $\forall\, i \in I$: $x_i \in \dot{X}_i \cap F = X_i$.

This means $\Sigma x_i = x \in \Sigma X_i$, or : $[(\Sigma \dot{X}_i) \cap F] \subset \Sigma X_i$;

(ii) Take $x \in \Sigma X_i$.

This implies $\exists\, x_i \in X_i \ \forall i \in I$ such that $\Sigma x_i = x$.

But $x_i \in X_i \subset \dot{X}_i \quad \forall i \in I$, which means $\Sigma x_i = x \in \Sigma \dot{X}_i$.

Similarly, $x_i \in X_i \subset F \quad \forall i \in I$, which means $\Sigma x_i = x \in F$.

Thus $\Sigma x_i = x \in \Sigma \dot{X}_i \cap F$, or: $\Sigma X_i \subset (\Sigma \dot{X}_i \cap F)$.

(i) and (ii) together yield: $\Sigma \dot{X}_i \cap F = \Sigma X_i$, which implies that ΣX_i is integer convex. Q.E.D.

Proposition 4: Let $\{X_i\}_{i \in I}$ be a family of integer convex sets in F. For every $x \in E(\Sigma X_i)$ there exists $x_i \in E(X_i)$ for all

$i \in I$ such that $x = \Sigma x_i$.

<u>Proof</u>: Take $x \in E(\Sigma X_i)$.

This implies $\exists\, x_i \in X_i \;\forall i \in I$ such that $x = \Sigma x_i$.

Suppose for some $j \in I$: $x_j \notin E(X_j)$, then; $\exists\, \lambda > 0$ such that $(x_j - \lambda e) \in X_j$ and $\sum_{i \neq j} x_i + (x_j - \lambda e) = \Sigma x_i - \lambda e = (x - \lambda e) \in \Sigma X_i$.

But this implies that $\Sigma x_i = x \notin E(\Sigma X_i)$, a contradiction.

Therefore, for every $i \in I$: $x_i \in E(X_i)$. Q.E.D.

<u>Proposition 5</u>: Let $H(p,\alpha)$ support $\dot{X}$ and $\dot{Y}$ at $\bar{y}$ with $px \geq \alpha$ for all $x \in \dot{X}$ and $py \leq \alpha$ for all $y \in \dot{Y}$. If X and Y are integer convex in F then $H(p,\alpha) \cap X \cap Y \neq \emptyset$.

<u>Proof</u>: $H(p,\alpha) \cap \dot{X} \cap \dot{Y} \neq \emptyset$ by hypothesis.

$\bar{y} \in H \cap \dot{X} \rightarrow \exists$ a spanning set $S_x \subset X$ such that $\bar{y} \in \dot{S}$ or

$\bar{y} = \Sigma \lambda_i x_i$ for $x_i \in S_x$ and $\lambda_i > 0$, $\Sigma \lambda_i = 1$.

This implies $\forall x_i \in S_x$: $x_i \in H(p,\alpha)$.

$\rightarrow H \cap X \neq \emptyset$.

Similarly, $H \cap Y \neq \emptyset$.

Suppose $(H \cap \dot{X} \cap \dot{Y}) \not\subset F$ then the spanning sets $S_x \subset H \cap X$ and $S_y \subset H \cap Y$ of $\bar{y}$ do not exist. This contradicts the statement of the Carathéodory's theorem. Therefore

$(H \cap \dot{X} \cap \dot{Y}) \subset F$ and since $H \cap \dot{X} \cap \dot{Y} \neq \emptyset$, $H \cap \dot{X} \cap \dot{Y} \cap F \neq \emptyset$.

But $\dot{X} \cap F = X$ and $\dot{Y} \cap F = Y$,

Therefore $H \cap \dot{X} \cap \dot{Y} \cap F = H \cap X \cap Y \neq \emptyset$. Q.E.D.

1.2 Binary Relations and Ordering

In the models of later chapters it will be assumed that economic agents make rational decisions. For example, each consumer is assumed to choose the best bundle(s), based on a well-defined preference, from a certain subset of his consumption set. It is relevant, therefore, to discuss here the concepts of binary relation and ordering on set in R^n.

Let $X \subset R^n$. If for every ordered pair $(x,y) \in X \times Y$, a given statement is either true or false then this statement defines a binary relation on X. Denote this relation by $\circledast$, then one writes $x \circledast y$ or $(x,y) \in \circledast$ if the relation holds between x and y, otherwise $x \not\circledast y$ or $(x,y) \notin \circledast$.

If $\circledast$ is a binary relation from X to itself and $x,y,z \in X$. Then $\circledast$ is said to be:

1. transitive if $x \circledast y$ and $y \circledast z \rightarrow x \circledast z$;
2. reflexive if $x \circledast x$ for every $x \in X$;
3. symmetric if $x \circledast y \rightarrow y \circledast x$;
4. a-symmetric if $x \circledast y \rightarrow y \not\circledast x$;
5. anti-symmetric if $x \circledast y$ and $y \circledast x \rightarrow x = y$;
6. complete if either $x \circledast y$ or $y \circledast x$ for all $x,y \in X$.

A set X is said to be partially preordered by the relation $\circledast$ if $\circledast$ is reflexive and transitive. If in addition $\circledast$ is also complete then X is said to be completely preordered by $\circledast$. A partially (completely) preordered set is said to be *partially (completely) ordered* by $\circledast$ if $\circledast$ is also anti-symmetric.

A binary relation $\circledast$ on a set X is said to define an *equivalence relation* if it is transitive, reflexive and symmetric. That is, for any $x, y, z \in X$, $\circledast$ must satisfy:

$$x \circledast y \text{ and } y \circledast z \rightarrow x \circledast z;$$

$$x \circledast x \text{ for every } x \in X;$$

$$x \circledast y \rightarrow y \circledast x.$$

1.3 Other Results

The following well-known results on convex hulls and equilibrium existence will play important roles in later chapters and are therefore grouped here for convenience.

Caratheodory's Theorem: Let $X \subset R^n$. For every $x \in \text{Conv } X$ there exists a spanning set $S \subset X$ of x with at most $n + 1$ elements.

The above theorem is used extensively in several non-convex economic models. Its proof may be found in Eggleston [13, pp. 35-36].

Starr's Extention of Shapley-Folkman Theorem: Let $\{X_i\}_{i \in I}$ be a family of compact sets in R^n with $r(X_i) \leq \kappa$ for all $i \in I$. Then for every $x \in \text{Conv } \Sigma X_i$ there exists a $\bar{x} \in \Sigma X_i$ such that $d(\bar{x}, x) \leq \kappa\sqrt{n}$.

Starr first reported similar result on nonconvex sets by Shapley and Folkman in his paper [25; Appendix] where the above refined version is also given. Arrow & Hahn [3, pp. 399-400] also discussed and proved this theorem. It is mainly used in relating an equilibrium in the convexified economy to an approximate equilibrium in the original nonconvex model.

Gale and Mas-Colell Existence Theorem:

The following conditions are sufficient for the existence of equilibrium:

The set Y is closed, convex, contains the negative orthant, and has a bounded intersection with the positive orthant.

The sets X_i are closed, convex, non-empty and bounded below.

The preference mappings P_i are irreflexive [that is, $x_i \notin P_i(x_i)$], have an open graph in $X_i \times X_i$ and their values are non-empty, convex sets.

The functions $\alpha_i(p)$ are continuous and satisfy $\alpha_i(p) > \inf pX_i$ for all p in Δ'.

The proof of this theorem is found in Gale and Mas-Colell [16]. It should be explained briefly that the set Y refers to the production set, X_i is the i-th consumption set, α_i is an income distribution function from Δ' to R where Δ' is the set of unit price vectors which yield finite profits. The equilibrium referred to in the theorem is equivalent to the Walras equilibrium to be defined in Chapter 2.

1.4 Notes

The concept of integer convexity may be considered to be restrictive. However, its inclusion will strengthen the results on the state of equilibrium considerably. For another application of integer convexity, see Conn and Maloy [8].

The definition of convex hull given in this chapter differs slightly from the standard definition. The standard definition of convex hull requires Conv X to be the intersection of all convex, but not necessarily closed, sets which contain X. The stronger definition used in this study simplifies the analysis but does not affect the basic outcome of the model.

Chapter 2: The Basic Model, its Assumptions and Modification

2.0 The Basic Economy

All the activities of the following discussion will take place in a general framework called an economy. Intuitively, an economy consists of a production sector, a consumption sector and a finite collection of well-defined commodities. Each unit in the production sector, referred to as a producer, selects from among the technologically feasible combinations of inputs and outputs one which maximizes his profit. Symmetrically each consumption unit, called a consumer, chooses from his set of affordable bundles one that he most prefers. The optimal decision of each unit is constrained by the prices of the commodities which neither a single producer nor consumer can influence.

Definition (2.0): Formally, an economy with production is defined by the following set of primitive concepts:

1. A total supply set $Y \subset R^n$ of all possible combinations of commodities available for consumption in the economy;
2. A finite number of consumers, indexed by the set $I = \{1,\ldots,m\}$;
3. A consumption set $X_i \subset R^n$ for every $i \in I$;
4. A preference relation $\succsim_i$ defined on X_i for every $i \in I$;

5. An income distribution function
   $\alpha_i : \Delta \to R$ for each $i \in I$ which assigns to the i-th consumer a fraction $\alpha_i(p)$ of the profit $\Pi(p) = \sup pY$. The sum of all shares, $\sum_{i \in I} \alpha_i(p)$, is equal to the total profit $\Pi(p)$. Δ is the n-1 price simplex $\{p \in R^n | p \geq 0, \Sigma p_i = 1\}$.

Notationally, an economy is expressed as:

$$E \equiv \{(X_i, \succsim_i, I), Y, \alpha_i\}.$$

Remark: For the sake of simplicity, the supply side is assumed to consist of only one producer. It is conceivable, however, that the total supply set Y may be treated as the sum of numerous individual production sets and the bundle of initial resources, i.e.

$$Y = \sum_{j \in J} Y_j + \{w\},$$

where J is the finite index set of producers, Y_j is the j-th producer's production set, and $\{w\}$ represents the total amount of all goods available initially.

The preference relation $\succsim_i$ in D(2.0) may be given the verbal interpretation of "at least as desired as". This relation is used to define a very important type of set in each consumption set.

Definition (2.1): In each consumption set the class of not-worse-than sets is defined as a correspondence, C_i, from X_i into its power set:

$$C_i: X_i \to 2^{X_i} \text{ for every } i \in I,$$

where $\quad C_i(\bar{x}_i) \equiv \{x_i \in X_i \mid x_i \succsim_i \bar{x}_i\}$ for every $\bar{x}_i \in X_i$.

The set $C_i(\bar{x}_i)$ consists of all those commodity bundles in X_i which are considered better than or indifferent to the given bundle $\bar{x}_i$ by the i-th consumer. Figure 2.0 illustrates the set $C_i(\bar{x}_i)$ and its convex hull.

Two related relations, $\succ_i$ and $\sim_i$, are derived from $\succsim_i$.

Definition (2.2):

Let $x, y \in X_i$:

(a) $x \succ_i y$ if $x \in C_i(y)$ and $y \notin C_i(x)$;

(b) $x \sim_i y$ if $x \in C_i(y)$ and $y \in C_i(x)$.

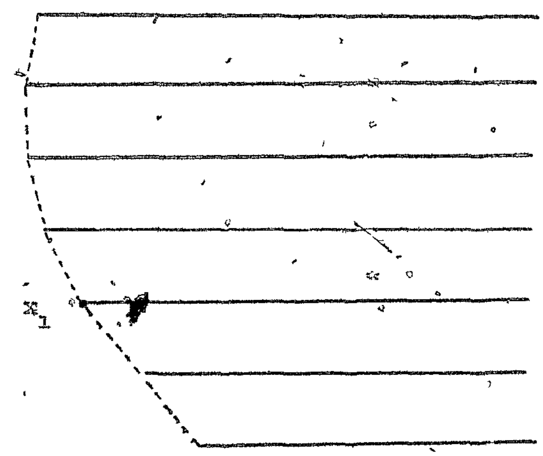
The first relation says that bundle x is considered "strictly preferred to" bundle y by the i-th consumer if x is "at least as desired as" y and y is "not at least as desired" as x. The second relation reads: bundle x is "indifferent to" bundle y in the i-th consumer's view if they are considered "at least as desired as" to each other simultaneously. This definition satisfies the conditions of an "equivalence" relation.

Definition (2.3): (a) An allocation x is an m-tuple of vectors, $x = (x_1, \ldots, x_m)$, where $x_i \in X_i$ for every $i \in \{1, \ldots, m\}$; thus $x \in \prod_{i \in I} X_i \equiv X$.

(b) An allocation is said to be feasible if

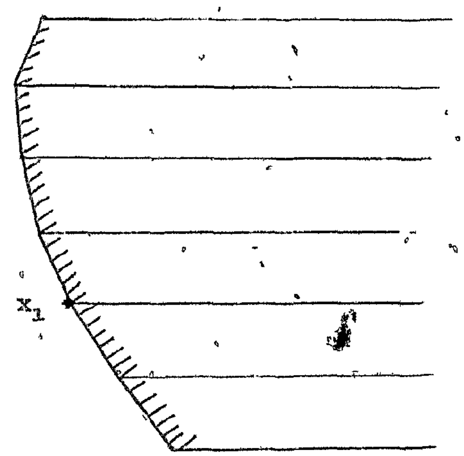
$$\sum_{i \in I} x_i \in Y.$$

Figure 2.0



$C_1(X_1)$

Not-worse-than x_1 set



Conv $C_1^*(X_1)$

An allocation is then nothing but a vector whose components are commodity bundles, each associated with a consumer. A feasible allocation is one whose component sum coincides with some supply bundle.

2.1 Equilibrium Concepts

So far the description of the economy has not indicated whether or not the actions of the individual economic units will bring a situation of mutual satisfaction to the system. The state of simultaneous satisfaction of all individual actions is called a state of equilibrium. The different concepts of equilibrium will be formally defined in this section and the question of existence is answered in the following chapter.

Definition (2.4): A price vector $\bar{p} \in \Delta$, an allocation $\bar{x} = (\bar{x}_1,\ldots,\bar{x}_m) \in X$, and a supply bundle $\bar{y} \in Y$ is said to constitute a Walras equilibrium of the economy if the following conditions are satisfied:

(a) $\bar{p}\bar{y} \geq \bar{p}y$ for every $y \in Y$;

(b) $\bar{p}\bar{x}_i = \alpha_i(\bar{p})$ for every $i \in I$;

(c) For all $i \in I$ and $x_i \in X_i$: $x_i \succ_i \bar{x}_i$ implies $\bar{p}x_i > \bar{p}\bar{x}_i$;

(d) $\Sigma\bar{x}_i = \bar{y}$.

The Walras equilibrium is also known as competitive equilibrium in the literature. It is the equilibrium whose existence is assured by the Gale and Mas-Colell Theorem. It describes the ideal state of compatibility in the economy. Condition (a) states that the equilibrium supply bundle $\bar{y}$ yields the highest net revenue to the producer compared to all other technologically possible bundles. Thus the objective of profit maximization is met in the production sector. The second condition reflects the idea that, for every consumer, the equilibrium consumption bundle $\bar{x}_i$ requires the exhaustion of income. This is due to the implicit assumption in the model that decisions are made over the life spans of the economic units and thus no savings and no future periods are considered. Condition (c) says that for any bundle which is strictly preferred to the chosen bundle, it must be that this preferred bundle can only be bought at a higher level of income than the equilibrium income. This signifies that the chosen bundle is the best bundle that can be purchased given the income and preference. Thus condition (c) satisfies the preference maximizing behavior of the consumer. For this reason, condition (c) is called the condition of "optimality". The last condition requires that the desired consumption of all consumers is exactly equal to the desired supply and is referred to as the "feasibility" condition.

Under less than ideal economic circumstances there may be less than ideal state of equilibrium. The first of such weakened concepts of equilibrium is defined as follows.

Definition (2.5): The tuple $(\bar{p},\bar{x},\bar{y}) \in (\Delta \times X \times Y)$ is called a Weak Approximate Equilibrium (W.A.E.) of magnitude κ if the following conditions are satisfied:

(a) $\bar{p}\bar{y} \geq \bar{p}y$ for every $y \in Y$;

(b) $\bar{p}\bar{x}_i = \alpha_i(\bar{p})$ for every $i \in I$;

(c') For all $i \in I$ and all $x_i \in X_i$: $x_i \succsim_i \bar{x}_i$ implies $\bar{p}x_i \geq \bar{p}\bar{x}_i$;

(d') $d(\Sigma x_i, y) \leq \kappa$, where κ is a given constant.

The concept of Weak Approximate Equilibrium defined above is closely related to the approximate equilibria defined by Broome and Dierker. The first two conditions of profit maximization and income exhaustion are still satisfied. However, condition (c') now states that for any bundle in the consumer's consumption set which is considered to be at least as desired as the chosen bundle, it must cost the same or more to purchase than the equilibrium bundle. Thus it is possible that the income needed to buy the equilibrium bundle may be sufficient to buy another which is strictly preferred. If this is the case then the condition of "optimality" is not met. Furthermore, condition (d') no longer assures exact feasibility. It allows for total consumption to diverge from total supply by a bounded measure. Thus "feasibility" is also absent.

Two other variations of the concept of equilibrium complete this section on definitions of equilibrium.

Definition (2.6): The tuple $(\bar{p},\bar{x},\bar{y}) \in (\Delta \times X\ Y)$ is called an Optimal Approximate Equilibrium (O.A.E.) of magnitude κ if it satisfies the following conditions:

(a) $\bar{p}\bar{y} \geq \bar{p}y$ for every $y \in Y$;

(b) $\bar{p}\bar{x}_i = \alpha_i(p)$ for every $i \in I$;

(c) For all $i \in I$ and $x_i \in X_i$: $x_i \succ_i \bar{x}_i$ implies $\bar{p}x_i > \bar{p}\bar{x}_i$;

(d') $d(\Sigma\bar{x}_i,\bar{y}) \leq \kappa$, where κ is a given scalar.

Definition (2.7): The tuple $(\bar{p},\bar{x},\bar{y},) \in (\Delta^{\vee}X{\times}Y)$ is called a Feasible Approximate Equilibrium if it satisfies the following conditions:

(a) $\bar{p}\bar{y} \geq \bar{p}y$ for every $y \in Y$;

(b) $\bar{p}\bar{x}_i = \alpha_i(\bar{p})$ for every $i \in I$;

(c') For all $i \in I$ and $x_i \in X_i$: $x_i \succsim_i \bar{x}_i$ implies $\bar{p}x_i \geq \bar{p}\bar{x}_i$;

(d) $\Sigma\bar{x}_i = \bar{y}$.

Obviously, an optimal approximate equilibrium is nothing but a Walras equilibrium without the condition of exact feasibility. This equilibrium concept is similar to the quasi-equilibrium found in a divisible but nonconvex economy by Starr. A feasible approximate equilibrium, on the other hand, meets all the conditions of a competitive equilibrium with the exception of optimality.

2.2 Assumptions on E

A set of assumptions and their interpretation is now stated for the economy E defined in D(2.1).

A: Assumptions on the consumption sets and preference relation $\succsim_i$.

For every $i \in I$:

A.1 (i) $X_i \subset F$;

(ii) $n_d \geq 1$.

A.2 X_i is closed in F.

A.3 There exists a $g_i \in F$ such that for every $x_i \in X_i$: $x_i \geqq g_i$.

A.4 For every $x_i \in X_i$ and $\bar{x} \in F$: $\bar{x} \geqq x_i$ implies $\bar{x} \in X_i$.

A.5 $\succsim_i$ is reflexive and transitive.

A.6 For all $x, \bar{x} \in X_i$: $x \succsim_i \bar{x}$ or $\bar{x} \succsim_i x$.

A.7 For every $\bar{x} \in X_i$: $C_i(\bar{x})$ and $\{x \in X_i | \bar{x} \succsim_i x\}$ are closed in F.

A.8 For all $x, \bar{x} \in X_i$: $\bar{x} \geqq x$ implies $\bar{x} \succsim_i x$.

A.9 For every $x \in X_i$ and $\lambda > 0$: $(x+\lambda e) \succ_i x$.

B: Assumptions on the supply set Y.

B.1 $Y \subset F$.

B.2 Y is closed in F.

B.3 $(-\Omega \cap F) \subset Y$.

B.4 $Y \cap R(\Omega \cap F)$ is nonempty and bounded.

C: Other assumptions on E

C.1 $\alpha_i : \Delta' \to R$ is continuous on Δ'.

C2. $\sum_i \alpha_i(p) = \Pi(p)$ for all $p \in \Delta'$.

C.3 $R(X_i) \cap \{x \in F \mid px \leq \alpha_i(p)\} \neq \emptyset$ for all $p \in \Delta'$.

C.4 $Y \cap \sum_i X_i \neq \emptyset$.

With the exception of two, A.1 and A.9, all other assumptions on the consumption sector are basic and may be found in most literature on general equilibrium analysis. The above standard assumptions, however, have been adapted to the present indivisible environment. The "closedness" assumption, A.2, states that if a bundle is in the consumption set then any other bundle on the same grid line very close to it is also in the consumption set. Assumption A.3, "lower boundedness", is a physiological constraint on the consumer's inputs and outputs. It implies the physical condition that one can neither work more than 24 hours in a day nor survive on less than some minimum amount of food. Assumption A.4 states that for a given bundle in the consumption set, then any other bundle (in F) with the same or greater quantity in one or more commodities is also in the consumption set. This is known as the "unlimited consumption" assumption. A.5 is an assumption on the behavior of the consumer. Firstly, the consumer must regard every bundle in his consumption set as "at least as desired" as itself. Secondly, given any three bundles in his consumption set, if the consumer regards the first to be "at least as desired as" the second and the second as "at least as desired as" the third, then the first must also be regarded "at least as desired as" the third.

Another behavioral assumption, A.6, states that all pairs of bundles in the consumption set are comparable in the opinion of the consumer. This is the "completeness" assumption. A.5 and A.6 together imply that $\succsim_1$ defines a complete preordering on the consumption set. The "continuity" assumption, A.7, assures the closedness of the not-better-than and the not-worse-than sets corresponding to any given bundle. That is, if a bundle is "at least as desired as" (or "at most as desired as") a given bundle then any other bundle very close to it is also "at least as desired as" (or "at most as desired as") the given bundle. The "monotonicity" assumption, A.8, reflects the consumer taste of always preferring more to less.

The first nonbasic assumption, A.1, allows the possibility of consumer choice taking place in a mixed environment of divisible and indivisible commodities. However, it restricts the number of divisible goods to be at least one. This makes the present model 'divisible à la Broome'. It is interesting to note that Broome made explicit use of the demand correspondence in his method and thus required the presence of at least one divisible commodity in smoothing this correspondence. As will be evident later, the demand correspondence does not appear in this paper. However, assumption A.1 (ii) is still crucial in asserting certain continuity related properties of the modified preference relation. In order to perform its smoothing function, the indivisible good must have positive valuation. This property is implied by the assumption of "strict monotonicity in the divisible good", A.9, which is also adopted directly from

Broome's model. It is rather strong in requiring that, everything else being equal, a little more of the divisible good is always strictly preferred to a little less.

With the exception of their adaptation to the indivisible case, assumption B.2 and B.3 are considered basic in most equilibrium models. The closedness of the supply set, B.2, has similar interpretation as assumption A.2 on the consumption set. B.3, the "free disposal" assumption, allows the relevant portion of the non-positive orthant to be included in Y. Its economic interpretation is that outputs may be disposed of without using any inputs. It excludes, therefore, the possibility of "penalties" or negative prices. Assumption B.1 requires the production sector to operate under the identical condition of mixed divisibility and indivisibility as the consumption sector. It should be emphasized that the indivisibility under discussion is due to the physical nature of the commodities rather than due to the technical conditions of production. The last assumption on the supply set, B.4, specifies a nonempty and bounded intersection between Y and the proper portion of the positive orthant. This deviates greatly from the basic assumptions found in many models with production and may appear quite objectionable at first glance. However, the boundedness of the above intersection dictates that only limited outputs may be possible without using any inputs. This is reasonable because by construction the supply set Y includes the vector on initial resources. Thus even if no production activity takes place, the initial resources are still available for consumption. This assumption is necessary for the application of the Gale and Mas-Colell Existence Theorem.

The assumptions on the income distribution function, C.1 and C.2, are taken directly from the Gale and Mas-Colell model. C.1 requires the function to be continuous over the set of relevant prices and C.2 makes certain that the circular flow of income is in equilibrium. That is, total income received by consumers must coincide with the profit generated in the production sector. The set of relevant prices, Δ', includes only those price vectors which yield finite profits.

$$\Delta' = \{p \in \Delta \mid \sup pY < \infty\} \subset \Delta.$$

The above function distributes income with respect to price vectors. However, as noted by Gale and Mas-Colell, any other continuous income distribution scheme (based on the consumers' weights or hair colors for instance) would have been acceptable.

Assumption C.3 states that for all relevant levels of income, the consumer must be able to purchase a bundle in the rest of his consumption set. This will be shown to be equivalent to the Gale and Mas-Colell condition that no consumer will be permitted to starve, regardless of the existing price vectors. As pointed out in the pure exchange models, this condition is met if each trader is assumed to possess a strictly positive "initial endowment". Finally, assumption C.4 is straight forward in making sure that the set of feasible allocation is nonempty. Clearly, economic activity cannot take place at all if this assumption is not satisfied.

2.3 The Convexified Economy with Modified Preference

In this section a new economy defined in the divisible environment of R^n is derived from the indivisible economy by convexifying the supply and consumption sets. The preference relation $\succsim_i$ defined on $X_i \subset F$ must also be modified to cover all points in R^n, not just proper points.

Definition (2.8): The new economy, denoted by $\dot{E}$, consists of the following entities:

1. A total supply set $\dot{Y} \equiv \text{Conv } Y \subset R^n$;
2. A finite number of consumers indexed by $I = \{1,\ldots,n\}$;
3. A consumption set $\dot{X}_i \equiv \text{Conv } X_i \subset R^n$ for all $i \in I$;
4. A preference relation P_i defined on $\dot{X}_i$ for every $i \in I$ where P_i is defined by D(2.9);
5. An income distribution function $\alpha_i : \Delta \to R$ defined identically to the function in D(2.0).

Formally, $\dot{E}$ is expressed by:

$$\dot{E} \equiv \{(\dot{X}_i, P_i, I), \dot{Y}, \alpha_i\}.$$

Most of the components in the new economy are either identical to those in the original economy (α_i, I) or are convex hulls of the original components ($\dot{X}_i$, $\dot{Y}$). The only component which is radically different in the new economy is the preference relation P_i defined on each consumption set $\dot{X}_i$.

This preference plays a vital part in the proof of existence and its construction based on $\succsim_i$ is elaborated in the following definition.

Definition (2.9): The preference relation P_i definded on $\mathring{X}_i$ for every $i \in I$ is a correspondence from $\mathring{X}_i$ to its power set:

$$P_i : \mathring{X}_i \to 2^{\mathring{X}_i}$$

where for all $x_i \in \mathring{X}_i$: $P_i(x_i) \equiv \{x \in \mathring{X}_i | x \in \text{Int}\ \bar{C}_i(x_i)\}$,

with $\bar{C}_i : \mathring{X}_i \to 2^{\mathring{X}_i}$

$$\bar{C}_i(x_i) \equiv \bigcap_{x \in T(x_i)} \mathring{C}_i(x) \equiv \cap\{\mathring{C}_i(x) | x \in T(x_i)\}$$

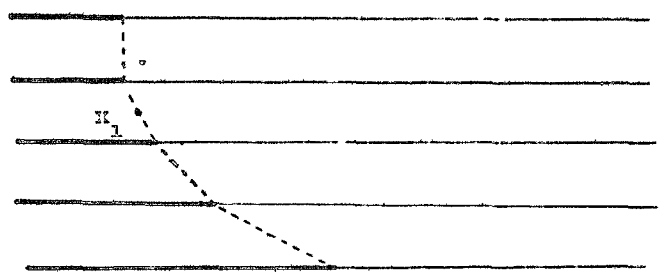
and $T: \mathring{X}_i \to 2^{X_i}$, $T(x_i) \equiv \{x \in X_i | x_i \in \mathring{C}_i(x)\}\ \forall\ x_i \in \mathring{X}_i$.

In general, for every point $x_i \in \mathring{X}_i$ (on or off the grid lines of X_i) the set $P_i(x_i)$ is the interior of the smallest convex hull of the not-worse-than set which contains the point x_i. Of course, in the special case where $x_i \in \mathring{X}_i$ is a proper point (on the grid line) then $P_i(x_i)$ is simply the set:

$$\{x \in \mathring{X}_i | x \in \text{Int}\ \mathring{C}_i(x_i)\}.$$

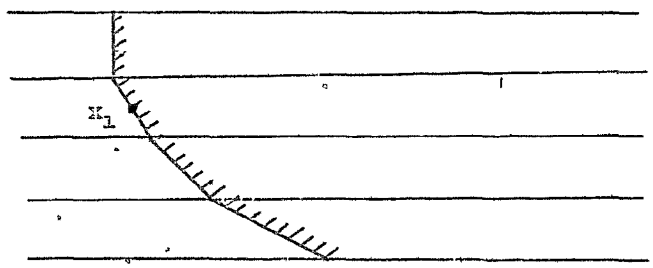
The construction of the set $P_i(x_i)$ for $x_i \in \mathring{X}_i$ is illustrated in Figure 2.1. The heavy grid lines in Figure 2.1 (a) constitute the set $T(x_i)$ which is the set of all proper points in X_i whose not-worse-than convex hulls contains x_i. Figure 2.1 (b) depicts the set $\bar{C}_i(x_i)$ which is equal to the smallest not-worse-than convex hull of all elements in $T(x_i)$. Finally,

Figure 2.1



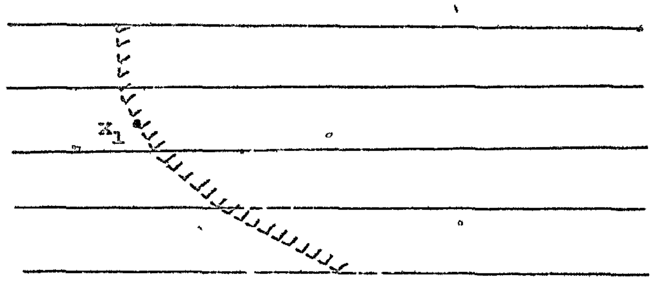
(a)

$T(x_1)$



(b)

$\bar{C}_1(x_1)$



(c)

$P_1(x_1)$

the set $P_1(x_1)$ in Figure 2.1 (c) is the interior of $\bar{C}_1(x_1)$.

2.4 Notes

The set Y is designated the supply set rather than the more common name of production set. This is due to the fact that Y consists not only of the individual production technologies Y_j but also the vector of initial endowment. This construction makes it reasonable to assume B.4.

Dierker [12] suggested that the concept of approximate equilibrium may be defined by relaxing different conditions of exact competitive equilibrium. The Feasible Approximate Equilibrium concept defined by D(2.7) is an attempt in this direction. It weakens optimality while retaining feasibility.

The assumptions on the basic economy are, for the most part, basic to the literature. Even the few not so standard assumptions have appeared in other models. Therefore, with the exception of integer convexity, the other assumptions of the present model are not at all strong.

Chapter 3: The Results

The set of assumptions in Section 2.2 and a few additional ones still to be imposed are sufficient to yield the results of this paper. For the sake of continuity, these results will first be stated and discussed in Sections 3.0 and 3.1, their formal proofs are given separately in Section 3.2.

3.0 Preliminary Results

Lemma (1): Let assumptions A.1, A.2, A.4 and A.9 hold. For all $i \in I$ and $x_i \in X_i$; if $H(p,\alpha)$ supports $C_i(x_i)$ at x_i, then $p^1 \neq 0$ for $p \in \Delta'$.

Remark: It will be shown later [Lemma (5)] that if the tuple $(\bar{p},\bar{x},\bar{y}) \in (\Delta \times X \times Y)$ is an equilibrium then the hyperplane $H(\bar{p},\alpha_i(\bar{p}))$ supports $C_i(\bar{x}_i)$ for every $i \in I$. This property and the result of Lemma (1) imply that any equilibrium price vector must have its first component different from zero. Therefore, herafter the analysis is restricted to Δ'', the set of relevant price vectors whose first component is nonzero.

$$\Delta'' \equiv \{p \in \Delta' \mid p^1 \neq 0\} \subset \Delta.$$

The interpretation of this restriction is that if the divisible good is to play a role in the model, it must have value to the economic units. Otherwise, if the divisible good is free then the model is practically transformed into a case of complete indivisibility.

<u>Lemma (2)</u>: Assumptions A.2 and C.3 imply that

$$\sigma_i(p) > \inf pX_i \text{ for all } p \in \Delta''.$$

<u>Remark</u>: This lemma assures that the wealth assigned to every individual is sufficient to keep him from starving, regardless of the prices. This is one of the several properties which prepare the stage for the application of the Gale and Mas-Colell Theorem.

<u>Lemma (3)</u>: Let assumption A.3 hold and define $\mathring{X} \equiv \prod_{i \in I} \mathring{X}_i$. If $(x_i, \ldots, x_m) = x \in \mathring{X}$ is a feasible allocation for $\mathring{E}$, then there exists a vector $\bar{g}_i$ such that $x_i < \bar{g}_i$ for all $i \in I$.

The existence of an upper bound for all feasible consumption set $\mathring{X}_i$ is guaranteed by the above lemma. Thus the following definition is possible.

<u>Definition (3.0)</u>:

(a) $\hat{X}_i \equiv \{x_i \in \mathring{X}_i \mid x_i \leq \bar{g}_i\}$.

(b) $\hat{P}_i : \hat{X}_i \to 2^{\mathring{X}_i}$

$$\hat{P}_i(x_i) \equiv \{x \in \hat{X}_i \mid x \in \text{Int } \hat{C}_i(x_i)\}$$

where $\hat{C}_i : \hat{X}_i \to 2^{\hat{X}_i}$

$$\hat{C}_i(x_i) = \bigcap_{x \in \hat{T}(x_i)} \mathring{C}_i(x) = \cap\{\mathring{C}_i(x) \mid x \in \hat{T}(x_i)\}$$

for $\hat{T}(x_i) = \{x \in \hat{X}_i \cap X_i \mid x_i \in \mathring{C}_i(x)\}$.

Remark: Substituting $\hat{X}_i$ and $\hat{P}_i$ defined immediately above for $\mathring{X}_i$ and $\mathring{P}_i$ respectively in the economy $\mathring{E}$, a new economy denoted by $\hat{E}$ is derived.

$$\hat{E} \equiv \{(\hat{X}_i, \hat{P}_i, I), \mathring{Y}, \alpha_i\}.$$

Since $\hat{X}_i$ contains all the feasible elements of $\mathring{X}_i$ and $\hat{P}_i$ is a restriction of $\mathring{P}_i$ to $\hat{X}_i$, therefore a Walras equilibrium for $\hat{E}$ is also a Walras equilibrium for $\mathring{E}$.

Lemma (4): Let assumptions A.1 through A.9 hold. Then, for every $i \in I$ and every $x_i \in \hat{X}_i$:

[a] There exists $x_i' \in X_i$ such that $\hat{C}_i(x_i) = \mathring{C}_i(x_i')$;

[b] Int $\hat{C}_i(x_i) \neq \emptyset$;

[c] $x_i \in$ Bnd $\hat{C}_i(x_i)$.

The importance of these results will be evident in the proof of Theorem (1).

Lemma (5): Let $(\bar{p}, \bar{x}, \bar{y})$ be a Walras equilibrium of $\mathring{E}$. Then:

[a] $H(\bar{p}, \Pi(\bar{p}))$ supports both the set $\mathring{Y}$ and Cl $(\Sigma \mathring{P}_i(\bar{x}_i))$ in $\Sigma \bar{x}_i$;

[b] $H(\bar{p}, \alpha_i(\bar{p}))$ supports Cl $(P_i(\bar{x}_i))$ in $\bar{x}_i$.

Geometrically, the hyperplane $H(\bar{p}, H(\bar{p}))$ separates the supply set $\mathring{Y}$ and the closure sum of the individual strictly preferred to sets at the point $\Sigma\bar{x}_i$ which coincides with total supply and total consumption. At the same time the hyperplane $H(\bar{p}, \alpha_i(\bar{p}))$ supports the closure of the individual strictly preferred to set $P_i(\bar{x}_i)$ at the optimal bundle $\bar{x}_i$. This property is used in the proof of Theorem (4).

3.1 The Main Theorems

Theorem (1): (Existence of Walras equilibrium)
Let the set of assumptions A, B, and C in Section 2.2 hold for E. Then its associated convexified economy $\hat{E}$ has a Walras equilibrium.

Remark: The equilibrium allocation stated in Theorem (1) corresponds to the modified economy $\hat{E}$. Clearly, this theorem does not refer to the original economy E. The W.A.E. Existence Theorem immediately below shows a weaker result is possible for the original economy E. However, even with this weaker result the following additional assumptions specifying the degree of nonconvexity of the sets Y and $C_i(x)$ are necessary.

Assumptions on the degree of nonconvexity:

A.10 For all $i \in I$ and all $x \in X_i$: $r(C_i(x)) \leq \kappa$.

B.5 $r(Y) \leq \kappa$.

Theorem (2): (W.A.E. Existence Theorem)
Let the conditions of Theorem (1) and assumptions A.10 and

B.5 hold. Then there exists $(p^*, x^*, y^*) \in (\Delta \times X \times Y)$ such that it is a W.A.E. of magnitude $\kappa\sqrt{n}$ in $\bar{E}$.

The result of Theorem (2) is weak in the sense of lacking both exact optimality and feasibility. The next theorem states the condition which eliminates the possibility of nonoptimality. The result of Theorem (3), therefore, is equivalent to the approximate equilibrium normally associated with a nonconvex but divisible model.

<u>Theorem (3):</u> (Optimality Condition)

Let (p^*, x^*, y^*) be the W.A.E. of economy $\bar{E}$ according to Theorem (2). If $H(p^*, \alpha_i(p^*)) \cap C_i(x_i^*) \cap E(X_i) = \emptyset \;\forall\, i \in I$ then (p^*, x^*, y^*) is an O.A.E. of magnitude $\kappa\sqrt{n}$ in $\bar{E}$.

The last theorem improves the weakness of a W.A.E. in another direction, namely removing infeasibility. In addition to the assumptions stated in Section 2.2, it requires the following pair of assumptions on the <u>type</u> of nonconvexity.

Assumptions on the type of nonconvexity in F:

A.11 For all $i \in I$ and all $x \in X_i$: $\mathring{C}_i(x) \cap F = C_i(x)$.

B.6 $\mathring{Y} \cap F = Y$.

<u>Theorem (4):</u> (F.A.E. Existence Theorem)

Let the conditions of Theorem (1) and assumptions A.11 and B.6 hold. Then there exists a F.A.E., (p^*, x^*, y^*), in the economy $\bar{E}$.

The implication of Theorem (4) is that if the nonconvexity in the original economy E is solely due to the presence of indivisibility rather than due to consumer taste and production technology then exact feasibility is always guaranteed.

3.2 Proofs of the Results

Proof of Lemma (1):

Assumption A.9 => $\forall\lambda > 0$: $(x_1+\lambda e) \succ_1 x_1$.

By definition, $(x_1-\lambda e) \in R(C_1(x_1))$.

Let $\mathring{H}(p,\alpha)$ support $C_1(x_1)$ at x_1 for some $p \in \Delta'$ and $\alpha \in R$, then:

(a) $x_1 \in H(p,\alpha) \cap C_1(x_1)$;

(b) $H(p,\alpha) \cap R(C_1(x_1)) = \emptyset$.

Suppose $p^1 = 0$. Then:

$$p(x_1+\lambda e) = px_1 + \lambda pe$$
$$= \alpha + (p^1e^1+\ldots+p^ne^n)\lambda$$
$$= \alpha + 0$$

since $p^1 = e^2 = \ldots = e^n = 0$.

This implies that $(x_1+\lambda e) \in H(p,\alpha)$ which contradicts the condition that $H(p,\alpha) \cap R(C_1(x_1)) = \emptyset$.

Therefore $p^1 \neq 0$. Q.E.D.

Proof of Lemma (2):

Take $x \in R(X_1) \cap \{x \in F \mid px \leq \alpha_1(p)\}$ for some $p \in \Delta''$.
The closedness of X_1 => $\exists\lambda > 0$ such that $(x-\lambda e) \in X_1$.
Clearly $p(x-\lambda e) = px - \lambda pe < px \leq \alpha_1(p)$
because $p^1 > 0$, $e^1 > 0$, and $\lambda > 0$.

Therefore $\inf p\mathring{X}_i \leq p(x-\lambda e) < \alpha_i(p)$. Q.E.D.

Proof of Lemma (3):

By A.3: $\exists\ g_i \in F$ such that $x_i \geq g_i$ for every $x_i \in X_i$, i.e. every X_i is bounded from below by $g_i \in F$.

Thus, every $\mathring{X}_i$ is also bounded from below by $g_i \in F$. It follows then, that there exists a vector $u \in R^n$ such that :

$$u < \sum_{i\in A} \mathring{X}_i \text{ for any } A \subset I.$$

Define: $\hat{Y} \equiv \{y \in \mathring{Y} | y \geq u\}$.

$\hat{Y}$, which is now bounded from below as well as above (because $\mathring{Y} \cap \Omega \neq \emptyset$ is bounded, $\mathring{Y}$ is convex, and $\mathring{Y} \cap \Sigma\mathring{X}_i \neq \emptyset$) contains all feasible allocations.

Thus, there exists a vector f such that $f > \hat{Y}$. Hence, for any feasible allocation $x = (x_1, \ldots, x_n) \in \mathring{X}$:

$$\Sigma x_i = y < f.$$

This implies: $x_i < f - \sum_{j\neq i} x_j < f - u \equiv \bar{g}_i$. Q.E.D.

Proof of Lemma (4).

[a] Take any $x_i \in \hat{X}_i = \{x \in \mathring{X}_i | x \leq \bar{g}_i\}$.

By the Carathéodory's theorem, there exists a finite set of at most n+1 elements in $\hat{X}_i \cap X_i$ that spans x_i, i.e.

$\exists\ S(x_i) \subset (\hat{X}_i \cap X_i)$ such that $x_i = \sum_{h\in Q} \lambda_h s_h$ for $\lambda_h > 0$,

$\sum_{h\in Q} \lambda_h = 1$, $s_h \in S(x_i)$ for every $h \in Q$ and $Q \equiv \{1,\ldots,q\}$ with $q \leq n+1$.

Observe that $S(x_i)$ is compact (finite), and that $\succsim_i$ is continuous and complete on X_i. Hence $\exists\ s_{h'} \in S(x_i)$ such that

$s_h \succsim_1 s_{h'}$ for every $s_h \in S(x_1)$.

By D(2.1): $S(x_1) \subset C_1(s_{h'})$ which implies that $\mathring{S}(x_1) \subset \mathring{C}_1(s_{h'})$.

Furthermore, $x_1 = \sum_{h \in \Omega} \lambda_h s_h \subset \mathring{S}(x_1)$, thus it follows that

$$x_1 \subset \mathring{C}_1(s_{h'}).$$

Now $x_1 \subset \mathring{C}_1(s_{h'})$ implies that $s_{h'} \in \hat{T}(x_1)$ so that $\hat{T}(x_1) \neq \emptyset$,

and by D(3.0): $\hat{C}_1(x_1) \subset \mathring{C}_1(s_{h'})$.

By continuity, transitivity and completeness of $\succsim_i$, the collection $\{\mathring{C}_i(x) \,|\, x \in \hat{T}(x_1)\}$ is totally ordered by inclusion. $\hat{T}(x_1)$ is compact. $\hat{C}_1(x_1)$ is closed being an intersection of closed sets, and thus equals to the smallest set in the collection, i.e.

$$\hat{C}_1(x_1) = \min \{\mathring{C}_1(x) \,|\, x \in \hat{T}(x_1)\}.$$

Define: $\bar{\lambda} \equiv \max \lambda > 0$ such that $x_1 \subset \mathring{C}_i(s_{h'} + \bar{\lambda} e)$.

Since by construction $\mathring{C}_1(s_{h'} + \bar{\lambda} e) = \min \{\mathring{C}_1(x) \,|\, x \in \hat{T}(x_1)\}$,

it follows that $\hat{C}_1(x_1) = \mathring{C}_i(s_{h'} + \bar{\lambda} e) = \mathring{C}_1(x_1')$ with $x_1' \equiv s_{h'} + \bar{\lambda} e$.

Q.E.D.

[b] Take the vector $s_{h'} + \bar{\lambda} e = x_1'$ of result [a] above for which $\hat{C}_1(x_1) = \mathring{C}_1(x_1')$. From the reflexivity of $\succsim_1$: $x_1' \succsim_1 x_1'$, it follows that $x_1' \in C_1(x_1')$ so that $\mathring{C}_1(x_1') \neq \emptyset$.

By A.4 and A.9: $x_1' + \lambda e \in X_1$ and $x_1' + \lambda e \succ_i x_1'$ for $\lambda > 0$.

Hence $x_1' + \lambda e \in R(C_1(x_1'))$.

This implies that Int $\mathring{C}_1(x_1') \neq \emptyset$. Therefore Int $\hat{C}_1(x_1) \neq \emptyset$.

Q.E.D.

[c] Let $x_1 \in \hat{X}_1$. By the result of [a] above, $\exists\, x_1' \in X_1$ such that $\hat{C}_1(x_1) = \mathring{C}_1(x_1')$ where $x_1' = s_{h'} + \bar{\lambda} e$ for

$$\bar{\lambda} \equiv \max\{\lambda \geq 0 \mid x_1 \in \mathring{C}_1(s_{h'} + \lambda e)\}.$$

Suppose $x_1 \notin$ Bnd $\hat{C}_1(x_1) =$ Bnd $\mathring{C}_1(x_1')$. Then $\exists\, \lambda' > \bar{\lambda}$ such that $x_1 \in \mathring{C}_1(s_{h'} + \lambda' e)$ contradicting the definition of $\bar{\lambda}$. Therefore $x_1 \in$ Bnd $\hat{C}_1(x_1)$. Q.E.D.

<u>Proof of Lemma (5)</u>:

[a] Since $(\bar{p}, \bar{x}, \bar{y})$ is a Walras equilibrium, one has:

$\Sigma\bar{x}_1 = \bar{y} \in \mathring{Y}$ by D(2.4)-(d).

$$\bar{p}\,\Sigma\bar{x}_1 = \bar{p}\bar{y} = \bar{p}\bar{x}_1 + \ldots + \bar{p}\bar{x}_m$$
$$= \alpha_1(\bar{p}) + \ldots + \alpha_m(\bar{p})$$
$$= \Sigma\alpha_1(\bar{p}) = \Pi(\bar{p})$$

This implies $\Sigma\bar{x}_1 \in H(\bar{p}, \Pi(\bar{p}))$

Thus $\Sigma\bar{x}_1 = \bar{y} \in H(\bar{p}, \Pi(\bar{p})) \cap \mathring{Y}$.

(1) First, one shows that $H(\bar{p}, \Pi(\bar{p}))$ supports $\mathring{Y}$ at $\Sigma\bar{x}_1 = \bar{y}$.

Suppose not, i.e. suppose $H(\bar{p}, \Pi(\bar{p})) \cap$ Int $\mathring{Y} \neq \emptyset$, and choose $z \in H(\bar{p}, \Pi(\bar{p})) \cap$ Int $\mathring{Y}$.

One has:

$\bar{p}z = \bar{p}\Sigma\bar{x}_i = \Pi(\bar{p}) = \sup \bar{p}\mathring{Y}$

$\bar{p}z \geq \bar{p}y$ for every $y \in Y$

But if $z \in \text{Int } \mathring{Y}$ then there must exist $\delta > 1$ such that $\delta z \in \mathring{Y}$.

However, $\bar{p}(\delta z) = \delta(\bar{p}z) > \bar{p}z = \Pi(\bar{p})$ which contradicts the fact

$\bar{p}z = \sup \bar{p}\mathring{Y}$. Hence, $H(\bar{p},\Pi(\bar{p})) \cap \text{Int } \mathring{Y} = \emptyset$.

Therefore $H(\bar{p},\Pi(\bar{p}))$ supports $\mathring{Y}$ at $\Sigma\bar{x}_i = \bar{y}$.

(2) Next one shows that $H(\bar{p},\Pi(\bar{p}))$ supports Cl $(\Sigma P_i(\bar{x}_i))$ at $\Sigma\bar{x}_i$.

Note that Cl $(P_i(\bar{x}_i) = \hat{C}_i(\bar{x}_i)$ and $\bar{x}_i \in \text{Bnd } \hat{C}_i(\bar{x}_i)$. This

implies that $\bar{x}_i \in \text{Cl}(P_i(\bar{x}_i))$ and thus $\Sigma\bar{x}_i \in \text{Cl}(\Sigma P_i(\bar{x}_i))$.

Hence $\Sigma\bar{x}_i \in H(\bar{p},\Pi(\bar{p})) \cap \text{Cl}(\Sigma P_i(\bar{x}_i))$.

For all $i \in I$, let $x_i \in P_i(\bar{x}_i)$. Then $\bar{p}x_i > \bar{p}\bar{x}_i = \alpha_i(\bar{p})$.

Hence $\bar{p}\Sigma x_i = \bar{p}x_1 + \ldots + \bar{p}x_m > \bar{p}\Sigma\bar{x}_i = \Pi(\bar{p})$

$\Rightarrow x_i \notin H(\bar{p},\Pi(\bar{p}))$

But $x_i \in P_i(\bar{x}_i)$ implies $x_i \in \text{Int } [\text{Cl}(P_i(\bar{x}_i))]$.

Thus $H(\bar{p},\Pi(\bar{p})) \cap \Sigma P_i(\bar{x}_i) = H(\bar{p},\Pi(\bar{p})) \cap \text{Int } [\text{Cl}(\Sigma P_i(\bar{x}_i))] = \emptyset$

Therefore $H(\bar{p},\Pi(\bar{p}))$ supports $\text{Cl}[\Sigma P_i(\bar{x}_i))]$ at $\Sigma\bar{x}_i$.

[b] For every $i \in I$ one has that

$\bar{p}\bar{x}_i = \alpha_i(\bar{p})$ implies $\bar{x}_i \in H(\bar{p},\alpha_i(\bar{p}))$

Furthermore, $\bar{x}_i \in \text{Cl}(P_i(\bar{x}_i))$ as shown earlier. Thus:

$\bar{x}_i \in H(\bar{p},\alpha_i(\bar{p})) \cap \mathrm{Cl}\,(P_i(\bar{x}_i))$.

Consider any $x_i \in P_i(\bar{x}_i) = \mathrm{Int}\,\mathrm{Cl}\,(P_i(\bar{x}_i))$. It follows D(2,4)-(c): $\bar{p}x_i > \bar{p}\bar{x}_i = \alpha_i(\bar{p})$. Thus for all $i \in I$ and for every $x_i \in P_i(\bar{x}_i)$: $x_i \notin H(\bar{p},\alpha_i(\bar{p}))$, or:

$$H(\bar{p},\alpha_i(\bar{p})) \cap P_i(\bar{x}_i) = H(\bar{p},\alpha_i(\bar{p})) \cap \mathrm{Int}\,[\mathrm{Cl}(P_i(\bar{x}_i))] = \emptyset.$$

Therefore $H(\bar{p},\alpha_i(\bar{p}))$ supports $\mathrm{Cl}\,(P_i(\bar{x}_i))$ in $\bar{x}_i$. Q.E.D.

Proof of Theorem (1):

It is sufficient to show that economy $\hat{E}$ satisfies the conditions stated in the Gale and Mas-Colell Existence Theorem.

1. Supply set:

$\hat{Y}$ is closed and convex by the definition of convex hull

B.3: $(-\Omega \cap F) \subset Y \Rightarrow -\Omega \subset \hat{Y}$

B.4: $Y \cap R(\Omega \cap F)$ nonempty and bounded $\Rightarrow \hat{Y} \cap \Omega$ nonempty and bounded

2. Consumption set: $\forall\, i \in I$

$\hat{X}_i$ is closed and convex by D(3.0) and convex hull definition

A.3, A.4 and Lemma (3) $\Rightarrow \hat{X}_i \neq \emptyset$

X_i is lower bounded $\Rightarrow \hat{X}_i$ is also lower bounded

3. Preference relation: $\forall\, i \in I$

D(3.0) and Lemma (4)-[c] $\Rightarrow \forall\, x_i \in \hat{X}_i$; $x_i \notin \hat{P}_i(x_i)$

Lemma (4)-[b]: $\forall\, x_i \in \hat{X}_i$ $\mathrm{Int}\,\hat{C}_i(x_i) \neq \emptyset \Rightarrow \hat{P}_i(x_i) \neq \emptyset$

Property 2 [Section 1.0.2] $\Rightarrow \hat{P}_i(x_i)$ is convex

D(3.0) $\Rightarrow$ open graph

4. Income distribution function: $\forall\ i \in I$

C.1: α_i is continuous

C.3 and Lemma (2) $\Rightarrow \alpha_i(p) > \inf p\hat{X}_i$. Q.E.D.

<u>Proof of Theorem (2):</u>

Let $(\bar{p},\bar{x},\bar{y}) \in (\Delta \times X \times Y)$ be a Walras equilibrium in economy $\mathring{E}$ as guaranteed by Theorem (1).

One has $\bar{y} \in \mathring{Y}$. Hence by the Carathéodory's theorem, there existsa smallest spanning set $S(\bar{y}) \subset Y$ of $\bar{y}$ such that

$\bar{y} = \Sigma\lambda_h y_h$, where $y_h \in S(\bar{y})$ and $\Sigma\ \lambda_h = 1,\ \lambda_h > 0$

with $h \in Q \equiv \{1,\ldots,q\}$ and $q \le n+1$.

For each $y_h \in S(\bar{y}) \subset Y \subset \mathring{Y} \Rightarrow \bar{p}y_h \le \bar{p}\bar{y}$ by D(2.4)-(a).

Substituting $\Sigma\lambda_h y_h$ for $\bar{y}$:

$\bar{p}\bar{y} = \bar{p}(\Sigma\lambda_h y_h) = \Sigma\ \lambda_h \bar{p} y_h$

This implies that with $\lambda_h > 0$: $\bar{p}y_h = \bar{p}\bar{y}$ for every $h \in Q$.

Similarly, for every $i \in I$.

One has $\bar{x}_i \in Cl(P_i(\bar{x}_i)) = \mathring{C}_i(x_i') $ for some $x_i' \in X_i$ by Lemma (4)-[a].

Hence by the Carathéodory's theorem, there exists a smallest spanning set $S(\bar{x}_i) \subset C_i(x_i')$ of $\bar{x}_i$.

By the same argument as above, $\bar{p}x_h = \bar{p}\bar{x}_i = \alpha_i(\bar{p})$, $\forall\ i \in I$.

Now consider the set $\{\Sigma\mathring{S}(\bar{x}_i) - \mathring{S}(\bar{y})\}$: By the (extended) Shapley and Folkman theorem:

There exists a point $(\Sigma x_i^* - y^*) \in \{\Sigma S(\bar{x}_i) - S(\bar{y})\}$ such that:

$$d\ [(\Sigma\bar{x}_i - \bar{y}), (\Sigma x_i^* - y^*)] \le \kappa\sqrt{n}.$$

Since $\Sigma\bar{x}_i = \bar{y}$, this last inequality implies:

$$d(\Sigma x_i^*, y^*) \le \kappa\sqrt{n} \qquad D(2.5)-(d')$$

Since $y^* \in S(\bar{y})$: $\bar{p}\bar{y} = \bar{p}y^* \ge \bar{p}y$, $\forall\ y \in Y$ $\qquad D(2.5)-(a)$

and $x_i^* \in S(\bar{x}_i)$: $\bar{p}x_i^* = \bar{p}\bar{x}_i = \alpha_i(\bar{p})$ $\qquad D(2.5)-(b)$

Lastly, $\forall\ x_i \in X_i$: if $x_i \underset{\sim}{>}_i x_i^* \Rightarrow x_i \in C_i(x_i^*)$

and $C_i(x_i^*) \subset Cl\ (P_i(\bar{x}_i))$.

Also, $H(\bar{p}, \alpha_i(\bar{p}))$ supports $Cl\ (P_i(\bar{x}_i))$ in $\bar{x}_i$

and $\bar{p}\bar{x}_i = \bar{p}x_i^* \Rightarrow \bar{p}x_i \ge \bar{p}x^*$, $\forall\ x_i \in C_i(x_i^*)$. $\qquad D(2.5)-(c')$

Q.E.D.

Proof of Theorem (3):

We need to show only that the tuple (p^*, x^*, y^*) satisfies condition (c) of D(2.4).

Take some $x_i' \in H(p^*, \alpha_i(p^*)) \cap C_i(x_i^*)$ and assume that x_i' is an anomalous point with $x_i' >_i x_i^*$.

By the statement of the theorem, $x_i' \notin E(X_i)$. This means $x_i' \in R(X_i)$ and that there exists $\lambda > 0$ such that $(x_i' - \lambda e) \in X_i$ with $x_i' >_i (x_i' - \lambda e)$ and $(x_i' - \lambda e) \sim_i x_i^*$.

but $p^*(x_i' - \lambda e) = (p^*x_i' - \lambda p^*e) < p^*x_i' = p^*x^*$

This last inequality contradicts condition (c') of D(2.5) which requires that:

$$\forall\ x_i \succsim_i x_i^* :\ p^*x_i \geq p^*x_i^*.$$

Therefore, $x^* \succsim_i x \quad \forall\ x_i \in H(p^*, \alpha_i(p^*)) \cap C_i(x_i^*)$ and if $x_i' \in C_i(x_i^*)$ and $x_i' \succ_i x_i^*$ then $p^*x_i' > p^*x_i^*$. Q.E.D.

Proof of Theorem (4):

Let $(\bar{p}, \bar{x}, \bar{y}) \in (\Delta \times \dot{X} \times \dot{Y})$ be a Walras equilibrium in the economy $\mathcal{E}$ as guaranteed by Theorem (1).

By Lemma (5): $H(\bar{p}, \Pi(\bar{p}))$ supports $\dot{Y}$ and $Cl[\Sigma P_i(\bar{x}_i)]$ in $\bar{y}$

and $H(\bar{p}, \alpha_i(\bar{p}))$ supports $Cl\ (P_i(\bar{x}_i))$ in $\bar{x}_i$.

This implies $H(\bar{p}, \Pi(\bar{p})) \cap \dot{Y} \cap Cl[\Sigma P_i(\bar{x}_i)] \neq \emptyset$

$\Rightarrow H(\bar{p}, \Pi(\bar{p})) \cap \dot{Y} \cap \Sigma\hat{C}_i(\bar{x}_i) \neq \emptyset$

$\Rightarrow H(\bar{p}, \Pi(\bar{p})) \cap \dot{Y} \cap \Sigma\hat{C}_i(x_i') \neq \emptyset$

where $x_i' \in \dot{X}_i$ is that of Lemma (4) - a .

Since $\dot{Y}$ is integer convex by assumption B.6., &

Assumption A.11 on the integer convexity of $C_i(x_i')\ \forall\ i \in I$ and Proposition 3 of Section (1.1) imply $\Sigma C_i(x_i')$ is also integer convex.

The result of Proposition 5 of Section (1.1) then yields the following:

$$\{H(\bar{p},\Pi(\bar{p})) \cap \mathring{Y} \cap \Sigma\mathring{C}_i(x_i') \cap F\} \neq \emptyset$$

$$\Rightarrow \{H(\bar{p},\Pi(\bar{p})) \cap Y \cap \Sigma C_i(x_i')\} \neq \emptyset.$$

Take an element $y^* \in \{H(\bar{p},\Pi(\bar{p})) \cap Y \cap \Sigma C_i(x_i')\}$:

Clearly, $\bar{p}y^* = \bar{p}\bar{y} \geq \bar{p}y, \quad \forall\, y \in Y.$ D(2.4)-(a)

Since $y^* \in E(\Sigma C_i(x_i')) \Rightarrow \forall\, i \in I$: there exists

$x_i^* \in E(C_i(x_i'))$ such that $\Sigma x_i^* = y^*$ D(2.4)-(d)

Furthermore, $\Pi(\bar{p}) = \bar{p}y^* = \bar{p}\Sigma x_i^*$

$$= \bar{p}x_1^* + \ldots + \bar{p}x_m^*$$

$$= \alpha_1(\bar{p}) + \ldots + \alpha_m(\bar{p})$$

Since $\bar{p}x_i^* \geq \alpha_i(\bar{p})$, this implies:

$\bar{p}x_i^* = \alpha_i(\bar{p})$ for every $i \in I$ D(2.4)-(b)

Lastly, $x_i^* \in C_i(x_i')$ implies $C_i(x_i^*)$ in contained

in $C_i(x_i')$. Thus:

$$\forall\, x_i \succsim_i x_i^*: \quad x_i \in C_i(x_i^*) \subset C_i(x_i')$$

but $C_i(x_i') = \hat{C}_i(\bar{x}_i) = \mathrm{Cl}\,(P_i(\bar{x}_i))$

Thus $\bar{p}x_i \geq \bar{p}x_i' = \bar{p}x_i^*$ D(2.5)-(c')

Let $\bar{p} = p^*$, then certainly (p^*, x^*, y^*) is a F.A.C. in the economy. Q.E.D.

Chapter 4. Relations with the Literature

The connection of this thesis to several published works is evident throughout the preceding chapters. As is the case with many of the recent papers on equilibrium analysis, the general flavor, style and formulation of the present problem may be traced back to the influential works of Debreu. His modern axiomatic treatment of general equilibrium analysis stimulates a flood of investigations of which this paper is but a small, specialized part. The thesis has also benefited from the works of Weddepohl. This is reflected by the present choice of notations and the formal definition of the economy which are similar to those found in Weddepohl [29] and [31]. Incidentally, Weddepohl's results on dual sets and dual correspondences and their application to equilibrium theory were initially considered as a potential solution to the present problem of nonconvexity and indivisibility. However, preliminary investigation indicated that the process of dualization does not satisfactorily eliminate discontinuity and the attempt was aborted.

The treatment of general nonconvexity in this thesis is directly related to the technique initiated by Starr [25] and elaborated by Arrow and Hahn [3, Ch. 7]. It involves the result by Shapley and Folkman which intuitively states that for every point in the sum of the convex hulls of a collection of compact sets there is a point in the sum of the original sets located close to it. The proximity of these two points depends on the degree of nonconvexity of the sets involved. This property was used to establish the results on approximately feasible equilibria in Theorem (2) and Theorem (3).

The objectives of this thesis could not have been reached without the application of the Gale and Mas-Colell Existence Theorem. It is clear from Chapter 2 and Chapter 3 that the modified economy, particularly the redefined preference, had been molded to satisfy the sufficient conditions of this theorem. This method of proving equilibrium existence for the indivisible model is much more simple and direct than the existing alternatives. The desirability of this theorem in the present context will be discussed in more details below.

There are few studies of general (approximate) equilibrium in finite economies with indivisibility. One of these, a recent paper entitled "Exchange Equilibrium in an Economy with Indivisible Commodities" by Alexander, Lloyd and Rowcroft [1] turns out to be quite different and incompatible with the present thesis. It stipulates consumer choice in a commodity space involving future time periods and different methods of payments which do not conform with the commodity space of the present model.

Another indivisible model which appears to be more similar to this thesis is by Dierker [12]. Working in an economy with all commodities being indivisible, Dierker showed mathematical originality in establishing a quasi equilibrium state which lacks both exact feasibility and optimality. However, that the results of Dierker's model seem to be "alienated" and "irreconcilable" with those of a model with at least one divisible commodity has been discussed at length by Broome [7].

The present model of indivisibility which includes at least one divisible commodity is more parallel to that of Broome. This similarity enables a more direct and meaningful comparison.

Using the proof of existence by Debreu and the results on convex hulls by Shapley and Folkman, Broome obtained a "Near Equilibrium" existence theorem which is reproduced here for discussion, using his notations.

Broome [7, pp. 241–242] 4.11. Theorem. "Near Equilibrium". Let Assumptions 2.1, 2.2, 2.4, 2.5, 2.6, 2.7, and 2.8 be satisfied. Write $\kappa^b = \max \{r^i \mid i \in I\}$. $\exists\ p^* \geq 0$, $\exists\ (x^{**1}, x^{**2}, \ldots, x^{**m})$:

(a) $[[\forall i \in I: \quad x^{**i} \in X^i \quad]$ &

(b) $[\forall i \in I: p^* \cdot x^{**i} \leqq p^* \cdot w^i]$ &

(c) $[\forall i \in I\ \forall x \in X^i: [[p^* \cdot x \leqq p^* \cdot w^i$ &

$x \notin \text{edge } X^i \cap \{x \mid p^* \cdot x = p^* \cdot w^i\}] \Rightarrow x^{**i}\ \ {}^{i}x]]$ &

(d) $[\exists x^{*t} \leqq w^t: |x^{*t} - \sum_{i \in I} x^{**i}| \leqq \kappa^b \sqrt{n}]]$.

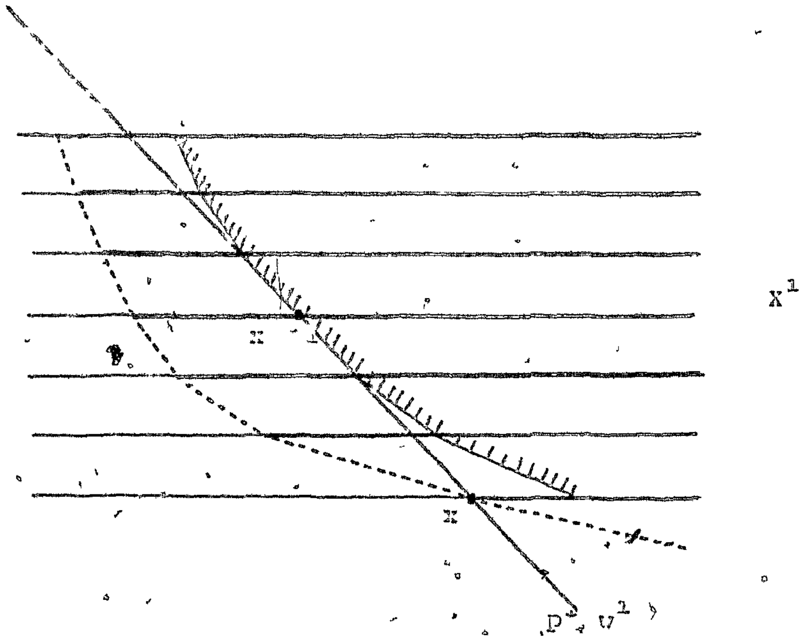
The near equilibrium in the above theorem involves two weaknesses. Firstly, the desired aggregate consumption may not coincide with the desired aggregate supply. The size of the deviation is determined by the structure of the model. This is exhibited by condition (d) of the theorem. Broome concluded that this possibility of infeasibility is expected whenever nonconvexity is present. This finding does not conflict with that established by Starr in a divisible setting. Secondly, the bundle allocated to the consumer in a near equilibrium may not be his optimal choice under the circumstances. That is, there might exist another strictly preferred bundle which he can purchase with the equilibrium income. This possibility is shown by condition (c). Broome attributed this problem solely to the presence of indivisible commodities in the system. However, he showed that this problem is very unlikely to occur since it happens only in the intersection of the income hyperplane and the edge of the consumption set. This exceptional case of nonoptimality is called the "problem of the edge".

The Weak Approximate Equilibrium established by Theorem (2) of this thesis is very similar to the near equilibrium concept above. Except for the different notations, condition (d') of D(2.5) expresses an identical infeasibility to that found by Broome. Condition (c') of D(2.5) also refers to potential nonoptimality. It allows the possibility that in Weak Approximate Equilibrium some consumers may be able to improve their satisfaction using the same equilibrium income. However, the result of Theorem (2) is somewhat weaker than Broome's because it only acknowledges the possibility of nonoptimality without pinpointing the area of occurence. Despite this inability to specify the circumstances of non-optimality, Theorem (2) is not as fruitless as it seems. Its strength lies in the fact that it yields results which are almost as strong as Broome's theorem yet requiring fewer and simpler assumptions. Disregarding the set of assumptions on the income distribution function and the supply set (Broome worked with a pure exchange system and did not involve production), the other assumptions of the present model are both basic and similar to those used by Broome. Furthermore, Theorem (2) of this thesis is proved without the following two nonbasic assumptions. Firstly, Broome required Assumption 2.5 on the "overriding desirability of the divisible commodity" to demonstrate the upper-semi-continuity of the demand correspondence. This is the second of two assumptions made on the desirability of the divisible good, and in Broome's own words, it seems an "unfortunate superfluity". Secondly, Broome made Assumption 2.6 to make sure "there is always some spanning set with a significant member in rest X^1." This assumption "is not only complicated to state, but also appears to have little contact with intuition". [7, p.229].

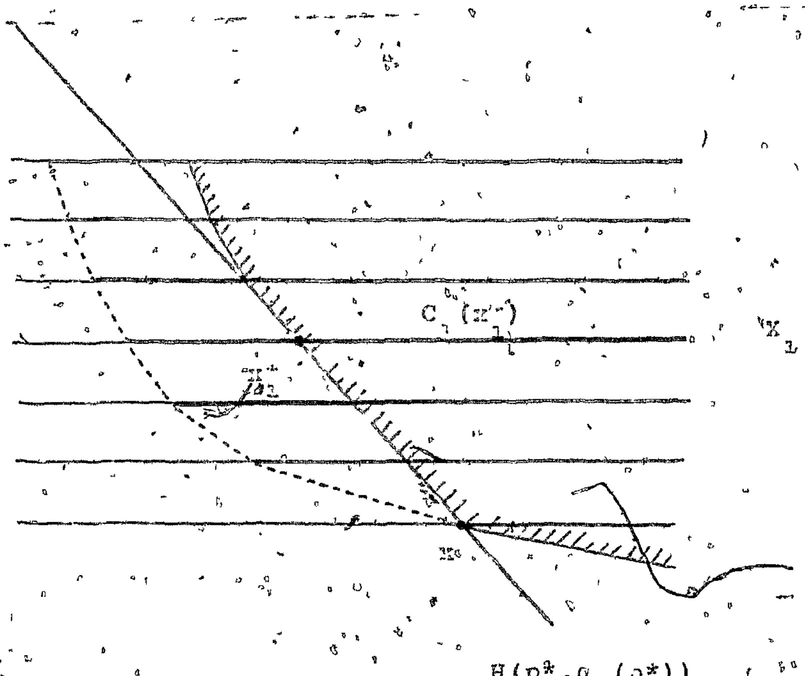
Theorem (3) of this thesis specifies the condition under which optimal choice of every consumer is guaranteed in an economy with indivisibility. According to condition (c) of Broome's theorem, all the bundles in the intersection of the budget plane and the edge X^i are anomalous points. The present Theorem (3), however, restricts the set of anomalous points further to the intersection of the income hyperplane, the edge of X_i, and the not-worse-than set corresponding to the equilibrium bundle. Obviously the second intersection is a proper subset of the first intersection and therefore it contains fewer anomalous points. This implies the non-optimality in Theorem (3) of this thesis is less probable to occur than Broome's "problem of the edge". The difference between these two results is illustrated in Figure 4.0.

The introduction of integer convex preference and production sets enables this thesis to expand the discussion on indivisibility in another direction. Recall that integer convexity may be interpreted as a special type of nonconvexity which is caused strictly by the indivisible nature of the commodities rather than by consumer preference structure or production technology (e.g. increasing returns). In other words, the assumption of integer convexity conveys the idea that ceteris paribus, if complete divisibility could somehow be introduced into the system, then all relevant production and consumption sets would have been convex. Under this assumption, Theorem (4) guarantees the existence of a Feasible Approximate Equilibrium in an economy with indivisible goods. This approxiamte equilibrium concept

Figure 4.0



x - Broome's anomalous point



x - Theorem (3)'s anomalous point

satisfies all conditions of a Walras equilibrium except one, that of optimal choice for every individual consumer. The results of Theorem (4) presents an interesting contrast to existing results in completely divisible models with nonconvexity. That is, a divisible environment with nonconvexity yields optimality but lacks exact feasibility whereas an indivisible model with nonconvexity purged (i.e. integer convexity assumed) will have exact feasibility but suffers from non - optimality. Therefore it is expected that nonoptimality is associated with indivisibility and infeasibility is associated with nonconvexity.

It may be concluded from the foregoing discussion that this thesis has achieved the objectives stated in Section 0.3. Firstly, the mathematical technique employed in this study is relatively less complex than those in the existing literature. At the same time, the present results have been derived under fewer and more relaxed conditions than the other models in the field. The findings of the thesis further confirm Broome's conclusion that his "problem of the edge", or the possibility of nonoptimal consumption for some individuals in equilibrium, is a problem specifically associated with the presence of indivisibility and seems to be ineradicable. However, the thesis succeeds in reducing the probability of this occurence by confining the anomalous points to a smaller set. It appears, therefore, that while the likelihood of nonoptimal equilibrium consumption may be decreased, the possibility of nonoptimality cannot be eliminated as long as some commodities are indivisible.

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